Absorbing patterns in BST-like expression-trees

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Work with

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Séminaire Algo, LIGM, Univ. Gustave Eiffel,
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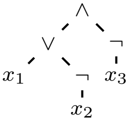
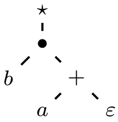
Introduction

- Expression trees



$(x_1 \vee \neg x_2) \wedge \neg x_3$



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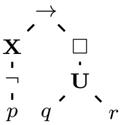

$\mathbf{X}(\neg p) \to \Box(q\mathbf{U}r)$

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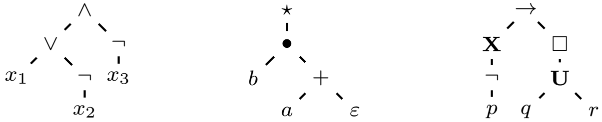

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- Automated testing, benchmark testing
    - Correctness and performance of algorithms

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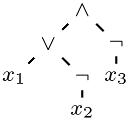

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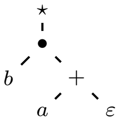

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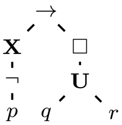

$\mathbf{X}(\neg p) \rightarrow \Box(q\mathbf{U}r)$

- Automated testing, benchmark testing
  - Correctness and performance of algorithms
- Randomly generated input
  - Realistic distribution
  - Simple implementation, possibility of theoretical analysis.

BST-like trees: a natural construction algorithm

Idea to draw tree of size n:

– if drawn operator is binary, choose *size* of branches uniformly.

– build subtrees recursively and independently.

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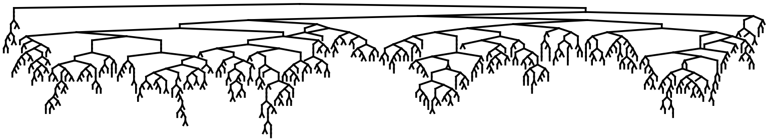
Code used in tool `lbtt` (from `TCS`) to draw an LTL formula:

function RandomFormula(n):
if $n = 1$ **then**
  p := random symbol in $AP \cup \{\top, \bot\}$;
  return p;
else if $n = 2$ **then**
  op := random unary operator in $\{\neg, \mathbf{X}, \Box, \Diamond\}$;
  f := RandomFormula(1);
  return $op\ f$;
else
  op := random operator in $\{\neg, \mathbf{X}, \Box, \Diamond, \wedge, \vee, \rightarrow, \leftrightarrow, \mathbf{U}, \mathbf{R}\}$;
  if op *in* $\{\neg, \mathbf{X}, \Box, \Diamond\}$ **then**
    f := RandomFormula($n-1$);
    return $op\ f$;
  else
    x := uniform integer in $[1, n-2]$;
    f_1 := RandomFormula(x);
    f_2 := RandomFormula($n-x-1$);
    return $(f_1\ op\ f_2)$;

BST-like trees: distribution over unary-binary trees



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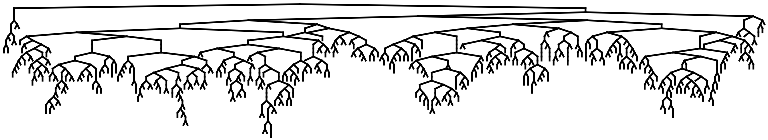


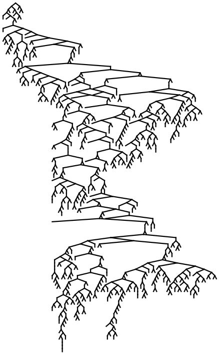
BST-like tree distribution is not uniform.

- Binary nodes ≈ balanced $\frac{n}{2}$–$\frac{n}{2}$.
  not for uniform trees

$$\mathbb{E}_n[\min(|T_L|, |T_R|)] \sim c_0\sqrt{n}\,.$$

- Expected height of different order

$$\Theta(\log n) \text{ vs } \Theta(\sqrt{n})\,.$$



Uniform and BST-like distributions

The uniform distribution:

- naturally maximizes entropy.
- can be sampled efficiently (Boltzmann,Recursive,Devroye's constrainted GW).
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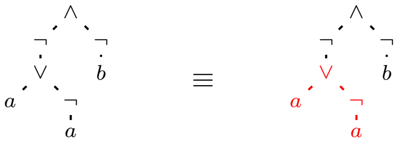
Let us see what happens with uniform **expressions** first...

Semantic simplification

Number of nodes may not reflect true complexity of tree



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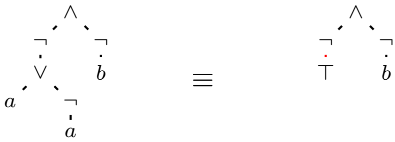

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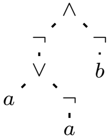

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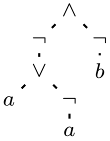

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Universal result for uniform tree model:

Theorem (Koechlin,Nicaud,R,'20)

Expected size of reduction of uniform tree bounded, as size $\to \infty$.

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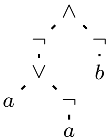

$\equiv$ $\perp$

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- Idea based on absorbing pattern $\mathcal{P}$, e.g., `false` $\wedge (\ldots) \equiv$ `false`,

$$\begin{array}{c}\circledast \\ / \ \backslash \\ \mathcal{P} \quad T\end{array} \rightsquigarrow \mathcal{P} \qquad \begin{array}{c}\circledast \\ / \ \backslash \\ T \quad \mathcal{P}\end{array} \rightsquigarrow \mathcal{P}$$

- For regular expressions on two letters, `constant bound` ≈ 77.8 .

In our work we

- draw an random BST-like tree expression of size n.
- study expected size of reduced expressions as $n \to \infty$.
- answer the question: do BST-like distributions present the same flaw as the uniform one?

[1] Left to right
$(p_\star, p_\bullet, p_+) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$,$(p_\star, p_\bullet, p_+) = (\frac{5}{29}, \frac{5}{29}, \frac{19}{29})$,$(p_\star, p_\bullet, p_+) = (\frac{1}{10}, \frac{1}{10}, \frac{8}{10})$

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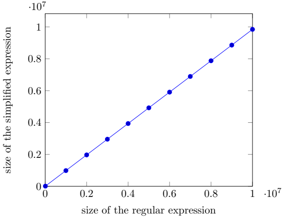
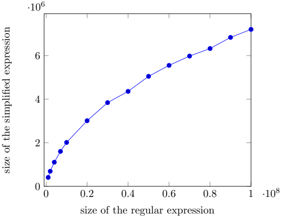
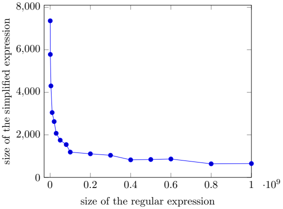
Experimental expected size (10 000 samples)[1] on regular expressions $(+, \bullet, \star)$ on two letters a, b:

$\mathcal{P} = (a+b)^\star$ absorbing for union $\circledast = +$







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Plan of the talk

1. Model: BST-like trees and absorbing patterns

2. Main Theorem and outline of the proof

3. Conclusions

Expression trees and BST-like model

Let $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2$ be non-empty finite sets of labels:
the *leaves*, *unary*, and *binary* operators respectively.

Definition

The family $\mathcal{E}(\mathcal{A})$ of expression trees over $\mathcal{A} = (\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2)$ is defined inductively from the leaves.

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return $\overset{op}{\underset{T_L \;\; T_R}{/\backslash}}$.

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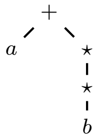
If $n = 2$: pick op. of arity 1...

Example: BST-like distribution

Consider the regular expressions $(+, \bullet, \star)$ on two letters a, b

(i)



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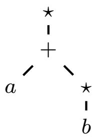

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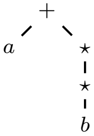
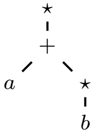
(i)  (ii) 

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- Distribution not uniform for any choice of parameters.

Absorbing patters: simplifying the trees

Definition (Simplification, absorbing pattern)

Let $\mathcal{E}(\mathcal{A})$ be the family of tree expression over the family of labels (operators) $\mathcal{A} = (\mathcal{A}_i)$, consider

- an "operation" $\circledast \in \mathcal{A}_a$ with arity $a = 2$,
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We simplify by applying bottom-up the rule:

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Denote by $\sigma(T) = \sigma(T, \mathcal{P}, \circledast)$ the simplification of $T \in \mathcal{E}(\mathcal{A})$.

Theorem. *Consider a family of expression trees defined from unary and binary operators with an absorbing pattern* $\mathcal{P}$ *for an operator* $\circledast$ *of arity* 2.

Take the simplification consisting in inductively changing a $\circledast$*-node by* $\mathcal{P}$ *whenever one of its children simplifies to* $\mathcal{P}$.

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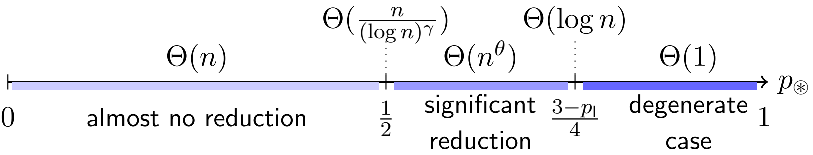

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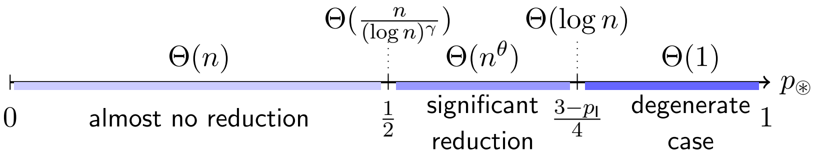
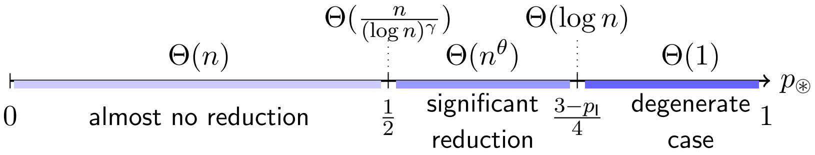
Then the *expected size* *of the* *simplification* *of a random BST-like tree has an asymptotic behaviour given by the following cases, depending on the probability* $p_\circledast$ *of the absorbing operator:*



- Probability $p_\circledast$ of $\circledast$, and p_l of picking unary operator.
- Two critical points $p_\circledast = 1/2$ and $p_\circledast = (3 - p_\mathsf{l})/4$
- Regimes from no reduction $\Theta(n)$ to complete reduction $\Theta(1)$

The main regimes experimentally



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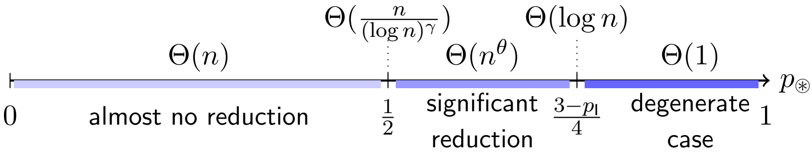

Experimental plots (10 000 samples)for regular expressions on two letters a, b: $\mathcal{P} = (a+b)^\star$ absorbing for union $\circledast = +$

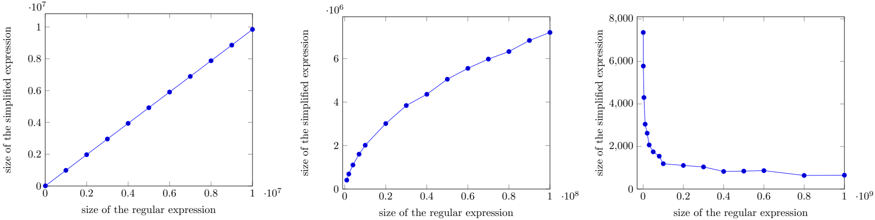

Figure: Left to right: linear ($p_+ = p_\star = p_. = \frac{1}{3}$), sublinear ($p_+ = \frac{19}{29}$, $p_\star = p_. = \frac{5}{29}$) and constant ($p_+ = \frac{8}{10}$, $p_\star = p_. = \frac{1}{10}$).

Scheme for the proof

We employ Analytic Combinatorics to study the expectation,

- Ordinary generating function

$$E(z) := \sum_{n=0}^{\infty} e_n z^n \,,$$

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- **Analytic Step.** We look at $E(z)$ over the complex $z \in \mathbb{C}$.

  A *Transfer Theorem* links the behaviour of $E(z)$ at its dominant singularity to asymptotics of $e_n \Rightarrow$ Study singularities

$$E(z) \sim_{z\to 1} \lambda(1-z)^{-\alpha} \Longrightarrow e_n \sim \lambda n^{\alpha-1}/\Gamma(\alpha)$$

Symbolic step: recurrence

We consider a fundamental sequence

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The recurrence for e_n involves γ_n,

$$e_{n+1} = 1+(s-1)\gamma_{n+1}\mathbf{1}_{n+1\neq s} + p_{\mathsf{I}} e_n + \frac{2p_{\mathsf{II}}}{n-1}\sum_{j=1}^{n-1} e_j + \frac{2p_{\circledast}}{n-1}\sum_{j=1}^{n-1}(e_j - s\gamma_j)(1-\gamma_{n-j})\,,$$

here $p_{\mathsf{II}} := 1 - p_{\mathsf{I}} - p_{\circledast}$ and $s = |\mathcal{P}|$.

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Proof.

Consider $\texttt{root} = \circledast$. Then $\sigma(T) = \mathcal{P}$ with proba γ_{n+1}. Else subtract the cases of reduced branches $\sigma(T_L) = \mathcal{P}$ or $\sigma(T_R) = \mathcal{P}$. □

Symbolic step: differential equation

Recurrence yields first order differential equation

$$E'(z) = F(z, A(z)) + \frac{1}{1-p_{\mathsf{I}}z}\left(\frac{2}{z} - p_{\mathsf{I}} + 2\,(1-p_{\mathsf{I}})\,\frac{z}{1-z} - 2p_{\circledast}A(z)\right) \cdot E(z)\,,$$

in terms of $A(z) = \sum_n \gamma_n z^n$.

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First order differential equations can be solved explicitly

Proposition

The equation $U'(z) = f(z) + g(z)U(z)$ where f, g are analytic functions on Ω has a unique solution analytic on Ω, satisfying $U(0) = u_0$,

$$U(z) = \exp\left(\int_0^z g(\zeta)d\zeta\right)\left(u_0 + \int_0^z f(\zeta)\exp\left(-\int_0^\zeta g(w)dw\right)d\zeta\right).$$

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Our coefficients depend on z and the unkown generating function $A(z)$.

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The generating functions $A(z)$ and $E(z)$

- are analytic for $|z| < 1$, as the series converge absolutely.

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Solution of ODE gives asymptotics

$$E(z) \sim \frac{c}{(1-z)^2} \left(2 + \int_0^z F(w, A(w)) I(w) dw \right) (I(z))^{-1}, \quad z \to 1,$$

where $I(z) := \exp\left(2p_{\circledast} \int_0^z \frac{A(w)}{1-p_{\mathsf{I}} w} dw \right)$.

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To apply the Transfer Theorem and complete the proof:

- we require precise asymptotics for $A(z)$ at $z = 1$,
- we show that $A(z)$ and $E(z)$ are analytic over $\Omega = \mathbb{C} \setminus [1, \infty)$.

Fully reducible trees: probabilities

We study the generating function $A(z) = \sum \gamma_n z^n$

Proposition

The probabilities $\gamma_n = \Pr_n \{\sigma(T) = \mathcal{P}\}$ satisfy, for $n \geq |\mathcal{P}|$,

$$\gamma_{n+1} = \frac{p_{\circledast}}{n-1} \sum_{k=1}^{n-1} (\gamma_k + \gamma_{n-k} - \gamma_k \gamma_{n-k}) \, .$$

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Recurrence translates into Riccati differential equation

$$A'(z) = (s-2)\gamma_s z^{s-1} + \left(\frac{2}{z} + 2p_{\circledast} \frac{z}{1-z} \right) A(z) - p_{\circledast} \cdot (A(z))^2 \,,$$

where $s = |\mathcal{P}|$.

Analytic step: linearization of Riccati

Considering $v(z)$ such that $p_{\circledast} A(z) = v'(z)/v(z)$,

Riccati equation becomes linear

$$v''(z) = p_{\circledast} \cdot (s-2)\gamma_s z^{s-1} v(z) + \left(\frac{2}{z} + 2p_{\circledast} \frac{z}{1-z} \right) v'(z) \,.$$

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For linear ODEs:

- domain of analyticity well-understood.
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The generating function $A(z)$ satisfies, $z \to 1$

- For $p_{\circledast} > \frac{1}{2}$, $A(z) = \frac{\gamma_\infty}{1-z} + O((1-z)^{2p_{\circledast}-2})$,
- For $p_{\circledast} = \frac{1}{2}$, $A(z) = \frac{2}{1-z}\left(\log\left(\frac{1}{1-z}\right)\right)^{-1}\left(1 + O\left(\log\left(\frac{1}{1-z}\right)^{-1}\right)\right)$
- For $p_{\circledast} < \frac{1}{2}$, $A(z) \sim \frac{D}{(1-z)^{2p_{\circledast}}}$,

where $\gamma_\infty := (2p_{\circledast}-1)/p_{\circledast}$ and $D > 0$ is a constant.

Probability of full reduction

Theorem

The probability γ_n of being fully reducible tends to the constant $\gamma_\infty := (2p_\circledast - 1)/p_\circledast$ for $p_\circledast > \frac{1}{2}$ and to zero otherwise.

Moreover,

- *for $p_\circledast = \frac{1}{2}$ we have $\gamma_n \sim \frac{2}{\log n}$,*
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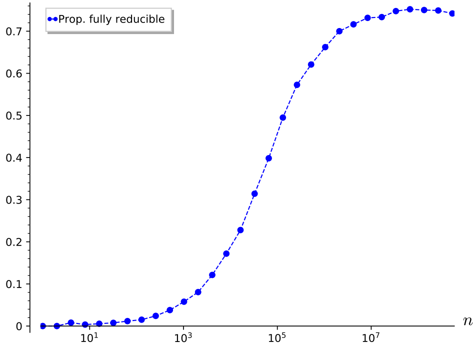
Experimental plot: regular expressions on two letters with

$$(p_+, p_\bullet, p_\star) = \left(\tfrac{8}{10}, \tfrac{1}{10}, \tfrac{1}{10}\right).$$

Then

$$\lim_{n\to\infty} \gamma_n = 3/4\,.$$



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Recall. From the Riccati equation, we have a 2nd order linear ODE

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When $\alpha_1 - \alpha_2 \in \mathbb{Z}$, the factor $(1-z)^{|\alpha_1-\alpha_2|}$ is polynomial.
$\Rightarrow$ we obtain an independent solution by $\times \log(1-z)$.

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2. Absorbing pattern model is general
   $\Rightarrow$ consider interactions between operators?
3. Take a concrete case: LTL formulas.

Thank you!