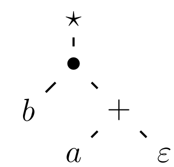
Analytic Combinatorics of Unlabeled Objects

Set of exercises II: responses

1. Consider the combinatorial class $\mathcal{R}$ of regular expressions as unary-binary trees:

   - leaves are decorated by the letters "a", "b" or the empty word "ε".
   - internal binary nodes are decorated by the operators union "$+$" or concatenation "$\bullet$",
   - internal unary-nodes are decorated by the Kleene-star "$\star$".

   $$(b \cdot (a + \varepsilon))^\star$$

   Give a combinatorial specification. Determine the ordinary generating function supposing that the size is the total number of nodes.

   Response. The specification is

   $$\mathcal{R} = \{a, b, \varepsilon\} + \{\star\} \times \mathcal{R} + \{+\} \times \mathcal{R} \times \mathcal{R} + \{\bullet\} \times \mathcal{R} \times \mathcal{R}\,.$$

   This translates into an equation for the OGF $R(z)$,

   $$R(z) = 3z + zR(z) + 2z(R(z))^2\,,$$

   since here every node (even the leaves) have size 1.

   Solving the equation we obtain:

   $$R(z) = \frac{1 - z - \sqrt{1 - 2z - 23z^2}}{4z}\,,$$

   noting that the other root does not make sense at $z = 0$.

2. Postponed to TD3.

3. We recall that integer compositions are specified by

   $$\mathcal{C} = \mathtt{Seq}(\{\mathbf{1}, \mathbf{2}, \mathbf{3}, \ldots\}) \simeq \mathtt{Seq}(\mathtt{Seq}_{\geq 1}(\mathcal{Z}))\,.$$

   During class, we used OGFs to prove that the average number of terms in a composition of n is $\sim n/2$.

   (a) What if we restricted the compositions to have an even number of terms? Find the total number of compositions of n into an even number of terms.

   (b) Let us consider now the number of even terms in a general composition of n. Show that the average number of even terms is $\sim n/6$.

   (c) Now we want to fix the number m of even terms. Find the generating function.

Responses.

(a) This can be done by simply considering a constrained "Seq" constructor in which we just consider the even powers. We obtain

$$\mathcal{C}_2 = \mathtt{Seq}_{\mathtt{even}}(\{\mathbf{1},\mathbf{2},\mathbf{3},\ldots\}) \Longrightarrow C_2(z) = \frac{1}{1-\left(\frac{z}{1-z}\right)^2} = \frac{(1-z)^2}{1-2z} = 1 + \frac{z^2}{1-2z}\,.$$

Another possible way to see this is to mark the number of terms and extract the even terms, or even considering that $\mathcal{C}_2 = \mathtt{Seq}(\{\mathbf{1},\mathbf{2},\mathbf{3},\ldots\} \times \{\mathbf{1},\mathbf{2},\mathbf{3},\ldots\})$, a sequence of pairs.

For the total number of partitions into even parts we obtain, for $n \geq 2$

$$[z^n]C_2(z) = [z^n]\frac{z^2}{1-2z} = [z^{n-2}]\frac{1}{1-2z} = 2^{n-2}\,.$$

We note that $[z^0]C_2(z) = 1$ and $[z^1]C_2(z) = 0$.

(b) We mark the even terms

$$\mathcal{C} = \mathtt{Seq}\left(\mathcal{U} \times \{\mathbf{2},\mathbf{4},\mathbf{6},\ldots\} + \{\mathbf{1},\mathbf{3},\mathbf{5},\ldots\}\right) \Longrightarrow C(z,u) = \frac{1}{1 - u\frac{z^2}{1-z^2} - \frac{z}{1-z^2}} = \frac{1-z^2}{1-z-(1+u)\,z^2}\,.$$

Differentiating in u and then setting $u=1$ we obtain

$$\partial_u C(z,1) = z^2 \frac{1-z}{(1+z)(1-2z)^2}\,.$$

Apply partial fractions

$$\frac{1-z}{(1+z)(1-2z)^2} = \frac{1/3}{(1-2z)^2} + \frac{-4/9}{1-2z} + \frac{2/9}{1+z}\,.$$

Extracting coefficients and noting that the dominant terms come from $\frac{1/3}{(1-2z)^2}$,

$$[z^n]\partial_u C(z,1) = [z^{n-2}]\frac{1-z}{(1+z)(1-2z)^2} \sim [z^{n-2}]\frac{1/3}{(1-2z)^2} = \tfrac{1}{3}(n-1)2^{n-2}\,.$$

Since $[z^n]C(z,1) = 2^{n-1}$, dividing we obtain $\frac{[z^n]\partial_u C(z,1)}{[z^n]C(z,1)} \sim \frac{1}{6}n$.

(c) The generating function we look for is $F_m(z) = [u^m]C(z,u)$. Thus, expanding the binomials and summing the contributions

$$F_m(z) = [u^m]\sum_{k=0}^{\infty}\left(u\tfrac{z^2}{1-z^2} + \tfrac{z}{1-z^2}\right)^k = \sum_{k=0}^{\infty}\binom{k}{m}\tfrac{z^{2m}}{(1-z^2)^m}\left(\tfrac{z}{1-z^2}\right)^{k-m}\,.$$

We simplify the latter expression by using $\sum\binom{k}{m}u^k = \frac{u^m}{(1-u)^{m+1}}$,

$$F_m(z) = z^m\sum_{k=0}^{\infty}\binom{k}{m}\left(\tfrac{z}{1-z^2}\right)^k = z^m\frac{\left(\frac{z}{1-z^2}\right)^m}{\left(1-\left(\frac{z}{1-z^2}\right)\right)^{m+1}} = \frac{z^{2m}}{(1-z-z^2)^{m+1}}\,.$$

4. We consider the binary trees

$$\mathcal{B} = \mathcal{E} + \mathcal{Z} \times \mathcal{B} \times \mathcal{B}\,,$$

where the node $\mathcal{E}$ is the empty-node, while $\mathcal{Z}$ is an atom of weight 1.

(a) Give a specification for the binary trees $\mathcal{B}^\circ$ marking the length of the left-most branch.

(b) Solve the equation for $B(z,u)$ and show $\partial_u B(z,1) = z(B(z))^3$.

(c) Let $A(z) = zB(z)$. Apply the Lagrange Inversion Formula to compute $[z^n](A(z))^3$.

(d) Deduce that the average length of the left-most branch[1] tends to 3 as $n \to \infty$.

(e) Generalize. Find the exact number of binary trees of size n with left-most branch of length m.

Responses.

(a) We consider $\mathcal{B}$ as given, and we have $\mathcal{B}^\circ = \mathcal{E} + \mathcal{Z} \times \mathcal{U} \times \mathcal{B}^\circ \times \mathcal{B}$.
Another specification is the following:

$$\mathcal{B}^\circ = \mathrm{Seq}(\mathcal{Z} \times \mathcal{U} \times \mathcal{B})\,,$$

by seeing the left-most branch as a list with binary trees pending on the right of each node.

(b) We have $B(z,u) = \frac{1}{1-zuB(z)}$. Differentiating we find $\partial_z B(z,1) = \frac{zB(z)}{(1-zB(z))^2}$.
Recall that $B(z) = 1 + z(B(z))^2$, thus we derive $B(z) = \frac{1}{1-zB(z)}$. We deduce from this that $\partial_z B(z,1) = z(B(z))^3$.

(c) We have $A(z) = z\phi(A(z))$ with $\phi(u) = \frac{1}{1-u}$. Indeed $B(z) = \frac{1}{1-zB(z)}$. We deduce, applying the LIF with $f(u) = u^3$ and $\phi(u) = \frac{1}{1-u}$,

$$[z^n]A(z)^3 = \tfrac{[u^{n-1}]}{n}\{3u^2(\phi(u))^n\} = \tfrac{3}{n}[u^{n-3}]\{\tfrac{1}{(1-u)^n}\} = \tfrac{3}{n}\binom{n-1+n-3}{n-3}\,.$$

Thus $[z^n]A(z)^3 = \frac{3}{n}\binom{2n-4}{n-3}$.

(d) We have $[z^n]\partial_u B(z,1) = [z^n]zB(z)^3 = [z^{n+2}]A(z)^3 = \frac{3}{n+2}\binom{2n}{n-1}$ while $[z^n]B(z) = \binom{2n}{n}\frac{1}{n+1}$. Dividing we obtain the average length:

$$3\frac{n+1}{n+2}\frac{\binom{2n}{n-1}}{\binom{2n}{n}} = 3\frac{n+1}{n+2} \times \frac{n}{n+1} = 3\frac{n}{n+2} \to 3\,.$$

(e) We extract $[u^m]B(z,u) = [u^m]\frac{1}{1-zuB(z)} = (zB(z))^m$, which is the generating function of the trees having left-most branch of length m. We apply the same argument to $A(z)$.
The total number of trees of size n with left-most branch of length m is:

$$[z^n]A(z)^m = \frac{m}{n}[u^{n-m}]\{\tfrac{1}{(1-u)^n}\} = \frac{m}{n}\binom{n-1+n-m}{n-m}$$

[1] We will later see in class that the average-depth of a random binary tree is rather $\Theta(\sqrt{n})$.