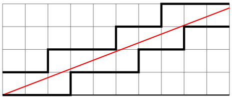
Change of basis in Numeration Systems

Pablo Rotondo

LIGM, Université Gustave Eiffel

Based on joint work with

Valérie Berthé, Eda Cesaratto and Martín D. Safe



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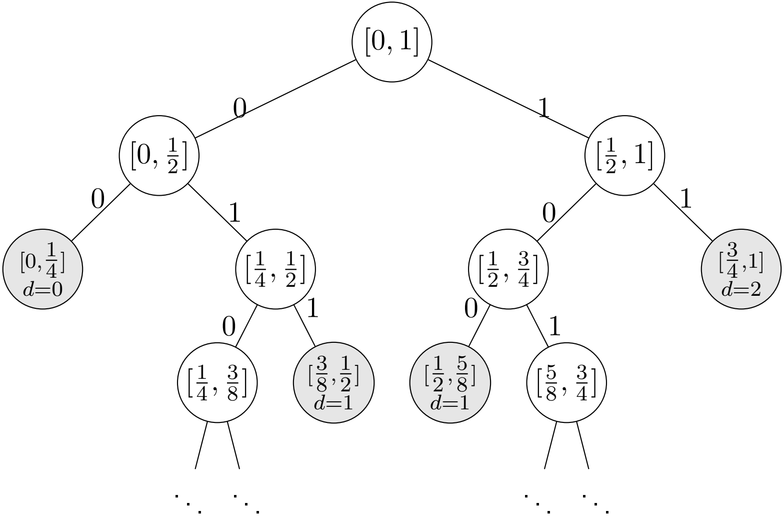
Meeting EPA!,

Buenos Aires, 23 October, 2024.

Example: first digit d in base 3 of $x \in [0, 1)$ from binary $x = (0.b_1 b_2 \dots)_2$?



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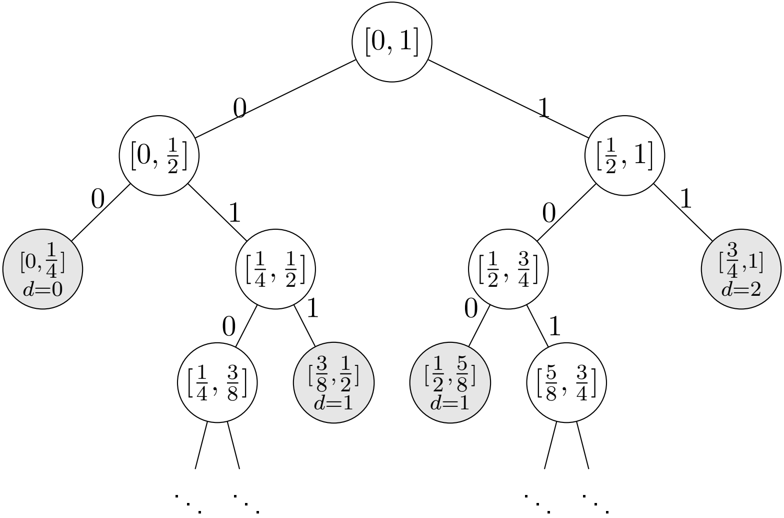

$\Rightarrow$ Only intervals containing $\frac{1}{3}$ and $\frac{2}{3}$ remain. Expected cost

$$\mathbb{E}[C] = \sum_{k\geq 0} \Pr(C \geq k) = 1 + \sum_{k \geq 1} \frac{2}{2^k} = 3 \texttt{ bits}\,.$$

Change of basis: binary to d-ary

- Given n binary digits $b_1, b_2, \ldots, b_n \in \{0, 1\}$ of
$$x = (0.b_1 b_2 \ldots)_2 \in [0, 1] .$$
- Number $L = L_n(x)$ of d-ary digits $0 \leq d_1, \ldots, d_L < d$ deduced?
$$x = (0.d_1 d_2 \ldots)_d \in [0, 1] .$$

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$$L_n(x) = n/A \,,$$
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One digit in base d^L "corresponds" to one in base 2^n if $d^L \approx 2^n$.

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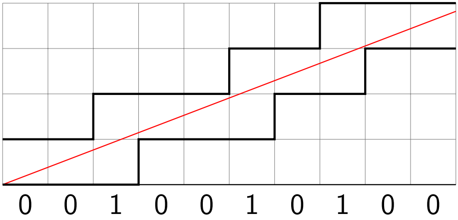
Motivation: simulating Sturmian words

Sturmian words. discrete coding of lines: horizontal (0), vertical (1) steps



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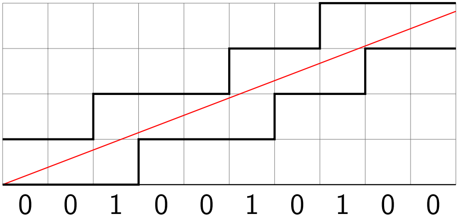

Theorem (Morse, Hedlund '40)

Binary sequence (u_k) *is Sturmian iff there is an irrational* $\alpha \in (0,1)$ *and* $\beta \in [0,1)$ *such that for all* $k \geq 0$*,*

$$u_k = \lfloor (k+1)\alpha + \beta \rfloor - \lfloor k\alpha + \beta \rfloor .$$

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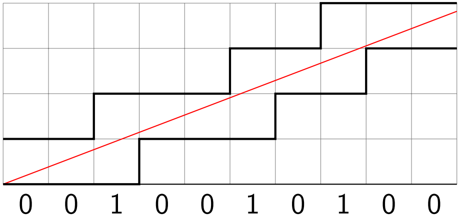

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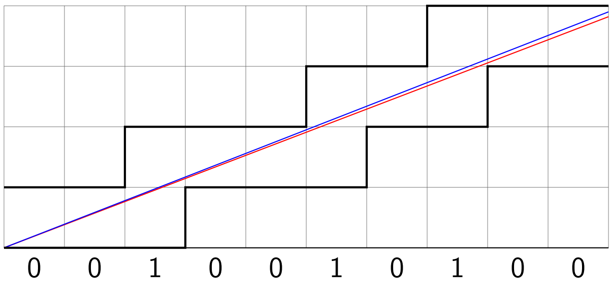
Remark The parameters $\alpha \in [0,1) \setminus \mathbb{Q}$ and $\beta \in [0,1)$ are unique !

Question. if we have approximation of α, and $\beta = 0$[†], how many *Sturm* digits (u_k) of α are deduced?



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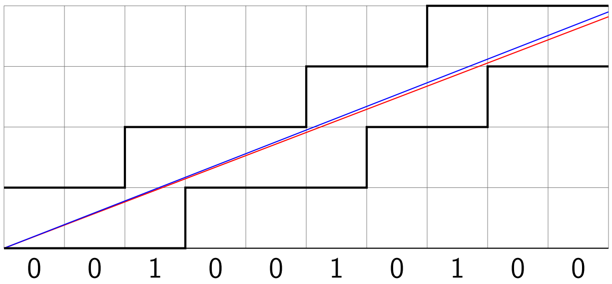

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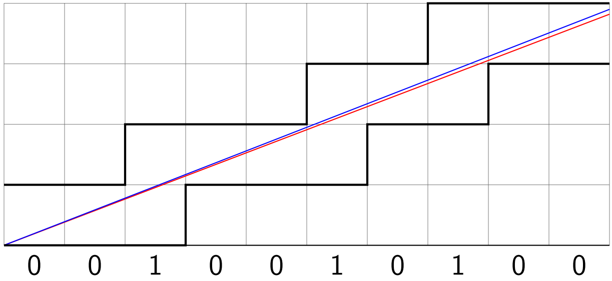

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Remark. First difference: one line above $(a, b) \in \mathbb{Z}^2$ while other below:

$$\implies \text{rational } a/b \in [\alpha_2, \alpha_1] \text{ implies } u_{b-1}^{\langle\alpha_2\rangle} = 0, u_{b-1}^{\langle\alpha_1\rangle} = 1.$$

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Plan of the talk

1. Unidimensional partitions of positive entropy
2. Undimensional partitions with zero entropy
3. Farey partition: zero entropy partitions for Sturmian digits
4. Bidimensional partitions
5. Conclusions and other work

Section

First historical results: Lochs' Theorem

- Given n decimal digits $d_1, d_2, \ldots, d_n$ of $x \in [0, 1]$,

$$x = (0.d_1 d_2 \ldots)_{10} \in [0, 1] \, .$$

- Number $L_n(x)$ of CFE-digits (partial quotients) deduced without error ?

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}} \, .$$

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Theorem (Lochs '64)

The rate of CF-digits per decimal given satisfies

$$\lim_{\mathtt{d}\to\infty} \frac{L_n(x)}{n} = \frac{6 \log 2 \log 10}{\pi^2} \doteq 0.9702701\ldots\,,$$

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"Example". The first 1000 decimals of π determine exactly 968 partial quotients of π.

Systems of partitions: a model for numeration

Example: decimal expansion

Associated partitions $\mathcal{D} = (\mathcal{D}_n)$ for the decimal expansion:

$$\mathcal{D}_n = \left\{ \left(\frac{k}{10^n}, \frac{k+1}{10^n} \right) : k \in \{0, 1, \ldots, 10^n - 1\} \right\}.$$

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Intervals determine expansion up to depth n:

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implies that expansion is $x = (0.d_1 d_2 \ldots d_n c_{n+1} c_{n+2} \ldots)_{10}$.

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Definition (System of interval partitions)

Sequence of (open) interval partitions $\mathcal{P} = (\mathcal{P}_n)$ of $[0, 1]$

- $\mathcal{P}_{n+1}$ refinement of $\mathcal{P}_n$ for every n.
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Model for numeration systems: more generally,

- notation $I_n^{\mathcal{P}}(x) = I \in \mathcal{P}_n$ such that $x \in I$,
- first n symbols for x determine $I_n^{\mathcal{P}}(x)$ and conversely.

Entropy of a partition

Entropy dictates size of intervals

- *Shannon entropy*[‡]:

$$H(\mathcal{P}) = -\lim_{k\to\infty} \frac{1}{k} \sum_{I\in\mathcal{P}_k} |I| \log |I| \ .$$

- *Point-wise entropy*: for almost every x

$$h(\mathcal{P}) = -\lim_{k\to\infty} \frac{1}{k} \log \left| I_k^{\mathcal{P}}(x) \right| \ .$$

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Remark. By Fatou's Lemma $h(\mathcal{P}) \leq H(\mathcal{P})$ if both exist.

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Existence of point-wise entropy

Systems of partitions associated with good (positive entropy) dynamical systems have point-wise entropy:

Theorem (Shannon,McMillan,Breiman)

Let T be an ergodic measure preserving transformation on a probability space $(\Omega, \mathcal{B}, \mu)$ and let P be a finite or countable generating partition for T for which $H_\mu(P) < \infty$. Then for μ-a.e. x,

$$\lim_{n\to\infty} -\frac{\log \mu\left(P_n(x)\right)}{n} = h_\mu(T)\,.$$

Here $H_\mu(P)$ denotes the entropy of the partition P, $h_\mu(T)$ the entropy of T and $P_n(x)$ denotes the element of the partition $\bigvee_{i=0}^{n-1} T^{-i}P$ containing x.

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Lochs' index for systems of partitions $\mathcal{P}^1, \mathcal{P}^2$

$$L_n(x; \mathcal{P}^1, \mathcal{P}^2) := \sup\{m \geq 0 : I_n^{\mathcal{P}^1}(x) \subset I_m^{\mathcal{P}^2}(x)\},$$

depth in $\mathcal{P}^2$ deduced from depth n in $\mathcal{P}^1$.

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Explanation

If $I_n^{\mathcal{P}^1}(x)$ splits over (intersects) several $J \in \mathcal{P}^2_m$,

$\Longrightarrow$ we cannot yet decide on $I_m^{\mathcal{P}^2}(x)$

Unidimensional partitions of positive entropy

Theorem (Dajani, Fieldsteel, 2001)

Consider systems of partitions $\mathcal{P}^1$ and $\mathcal{P}^2$, with positive point-wise entropies $h(\mathcal{P}^1)$ and $h(\mathcal{P}^2)$. Then

$$\lim_{n\to\infty} \frac{1}{n} L_n(x; \mathcal{P}^1, \mathcal{P}^2) = \frac{h(\mathcal{P}^1)}{h(\mathcal{P}^2)}$$

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We deduce Lochs' Theorem and result for d-ary basis:

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- **Base d.** Since $|I_n^{\mathcal{D}}(x)| = d^{-n}$, $h(\mathcal{D}) = \log d$.
- **Continued fractions.** Entropy $h(\mathcal{C}) = \frac{\pi^2}{6 \log 2}$

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What if $h(\mathcal{P}_1) = 0$ or $h(\mathcal{P}_2) = 0$? e.g., Sturm digits (u_k)

- If $h(\mathcal{P}_2) = 0$ and $h(\mathcal{P}_1) > 0$, almost surely $L/t \to \infty$.
- If $h(\mathcal{P}_2) > 0$ and $h(\mathcal{P}_1) = 0$, almost surely $L/t \to 0$.

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In our work [BCRS'23] we generalize this result to zero entropy...

Section

Log-balancedness and weight function

Definition (Weight function)

A system of partitions $\mathcal{P} = (\mathcal{P}_n)$ is *log-balanced* a.e. (resp. in measure) with *weight function* $f\colon \mathbb{N} \to \mathbb{R}_{>0}$, $f(n) \to \infty$, if

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Example

- For positive entropy $h = h(\mathcal{P}) > 0$

$$f(n) = h \times n\,.$$

- If partition is log-balanced, entropy 0 corresponds to

$$f(n) = o(n)\,.$$

Result for zero entropy

Theorem (Berthé,Cesaratto,R.,Safe, 2023)

Consider systems of partitions $\mathcal{P}^1$ and $\mathcal{P}^2$, with a.e. weight functions f_1 and f_2. Then, under certain technical conditions

$$\lim_{n\to\infty} \frac{f_2\left(L_n(x;\mathcal{P}^1,\mathcal{P}^2)\right)}{f_1(n)} = 1\,,$$

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The conditions are:

- $\sum e^{-\delta f_1(n)} < \infty$ for every $\delta > 0$;
- f_2 is non decreasing ;
- $f_2(n+1) - f_2(n) = o(f_2(n))$ as $n \to \infty$.

Discussion: conditions of our main result

We recall the conditions:

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Intuitively, the first condition is the most constraining one:

- Condition (b) reflects the fact that $\mathcal{P}_2$ is refining ;
- Condition (c) means that $f_2(n+1) \sim f_2(n)$;
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Examples

– Condition (a) not satisfied when $f_1(n) = \log n$,
– Condition (a) satisfied for $f_1(n) \geq (\log n)^2$.

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– Condition (a) satisfied for $f_1(n) \geq (\log n)^2$.
– Condition (c) not satisfied when $f_2(n) = \exp(n)$,
– Condition (c) is satisfied when $f_2(n) = \exp(\sqrt{n})$.

Discussion: conditions of our result for zero entropy

Example: appropriate output partitions $\mathcal{P}_2$

Subexponential weight functions of the form

$$f_2(n) = \exp(g(n))\,,$$

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Superlogarithmic weight functions

$$f_1(n) = (\log n) \cdot g(n)\,,$$

with $g(t) \to \infty$.

Section

A zero entropy system for Sturmian digits

Farey partition (Sturm source) is built by splitting intervals at *mediant*

$$\texttt{mediant}(a/b, c/d) := (a+b)/(c+d)\,.$$

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Construction of the *Farey partition* $\mathcal{F}_n$:

- Base case: $\mathcal{F}_0 = \{[0,1]\}$.
- Building $\mathcal{F}_n$: split $\left[\frac{a}{b}, \frac{c}{d}\right] \in \mathcal{F}_{n-1}$ at mediant $\frac{a+c}{b+d}$, if $b+d \leq n+1$.

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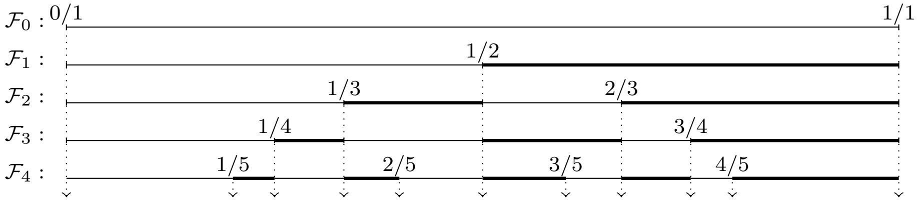

Farey partition: natural zero entropy system with weight

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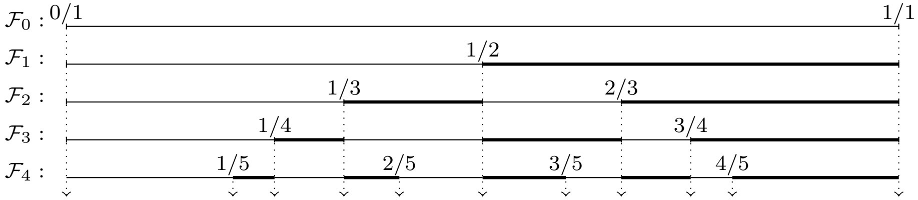

Properties:

- $\mathcal{F}_k$ determines[§] char. Sturmian word up to u_k: prefix $u_0 \ldots u_k$.
- The end-points $\mathcal{F}_k$ are exactly $\{\frac{a}{b} \in \mathbb{Q} : 0 \leq a \leq b \leq k+1\}$.
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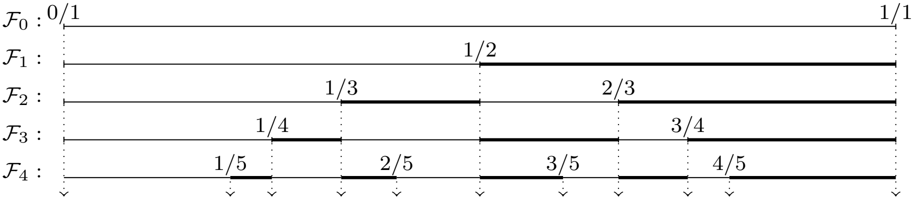

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Weight of the Farey partition

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Farey intervals have comparable size almost everywhere:

Lemma

For almost every x*, for large* $n \geq n_0(x)$

$$\frac{1}{n^2} \leq \left|I_n^{\mathcal{F}}(x)\right| \leq \frac{(\log n)(\log\log n)}{n^2}$$

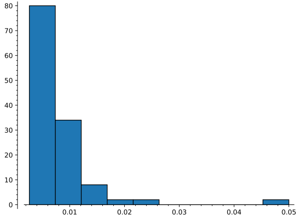

Figure. Histogram of interval sizes for $n = 20$. $\frac{1}{20^2} = 0.0025$, $\frac{1}{20} = 0.05$.

Consequences: producing digits of Sturmian word

From n digits of the slope α, we deduce *exponentially many*:

Corollary: from binary to Farey

Let $\mathcal{F}$ be the Farey partition, then

$$\log L_n(x;\mathcal{B},\mathcal{F}) \sim \frac{\log 2}{2} \times n \,,$$

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Corollary

Let $\mathcal{P}$ with $h(\mathcal{P}) > 0$ and $\mathcal{F}$ be the Farey partition, then

$$\log L_n(x; \mathcal{P}, \mathcal{F}) \sim \frac{h(\mathcal{P})}{2} \times n\,,$$

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Section

Two dimensions: joint coding of a pair

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Theorem (Dajani, De Vries, Johnson 2005)

Consider systems of partitions $\mathcal{P}^1$ and $\mathcal{P}^2$ of the square $[0,1]^2$ satisfying

(i) *$\mathcal{P}^1$ is made out of squares.*

(ii) *$\mathcal{P}^2$ consisting of convex polygons, of pointwise entropy $h(\mathcal{P}^2) > 0$.*

(iii) *There are constants $\beta, c_0, c_1 > 0$ so that, for every I from a partition in $\mathcal{P}^2$, $c_0\lambda(I) \leq (\mathrm{diam}(I))^\beta \leq c_1\lambda(I)$.*

Then, for a.e. $(x,y) \in [0,1]^2$,

$$\lim_{n\to\infty} \frac{1}{n} L_n(x,y;\mathcal{P}^1,\mathcal{P}^2) = \frac{\beta}{2(\beta-1)} \frac{h(\mathcal{P}^1)}{h(\mathcal{P}^2)} .$$

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Under suitable balance conditions we can go from $\mathcal{P}^1$ to a partition $\mathcal{P}^2$ made out of squares. $\Longrightarrow$ Log-balanced measures and diameters.

Example of interest: Ostrowski expansion

Ostrowski transformation

Given irrationals $x, y \in [0,1]$ define

$$S(x,y) = (\{1/x\}, \{y/x\}) ,$$

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Digits are produced at each iteration $i \geq 1$ by (x_i, y_i)

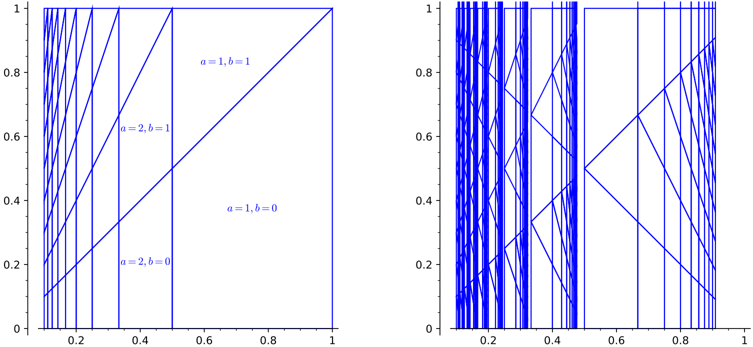
$$a_i = \lfloor 1/x_i \rfloor , \qquad b_i = \lfloor y_i/x_i \rfloor .$$

We retrieve (x, y) by

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}}, \qquad y = \sum_{i=1}^{\infty} b_i \cdot x_0 \ldots x_{i-1} .$$

Partitions: Ostrowski expansion

Partition $\mathcal{P}_1$ according to (a_1, b_1) and $\mathcal{P}_2$ according to (a_1, b_1, a_2, b_2).



Section

1. Unidimensional partitions of positive entropy
2. Undimensional partitions with zero entropy
3. Farey partition: zero entropy partitions for Sturmian digits
4. Bidimensional partitions
5. Conclusions and other work

Final remarks and conclusions

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References

Thank you for your attention!

V. Berthé, E. Cesaratto, P. Rotondo, and M. Safe,
Lochs-type theorems beyond positive entropy ,
Monatsh Math, vol 200, pp. 737–779, 2023.

G. Lochs,
Die ersten 968 Kettenbruchnenner von π,
Monatsh. Math. 67, pp. 311–316, 1963.

K. Dajani, and A. Fieldsteel,
Equipartition of Interval Partitions and an Application to Number Theory,
Proceedings of the American Mathematical Society, vol 129, n. 12, pp. 3453–3460, 2001.

K. Dajani, A. Johnson, and M. de Vries,
The relative growth of information in two-dimensional partitions.,
Real Anal. Exchange 31, (2) 397–408, 2005/2006.