Recurrence of substitutive Sturmian words

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Joint work with

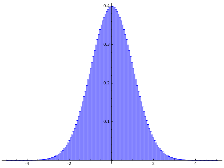
Brigitte Vallée

Journée de l'axe AlgoComb, Normastic,

Caen, 28 May, 2019.

Context

- Probabilistic analysis



Object/experiment/execution?

$\Rightarrow$ Models, averages, distribution?

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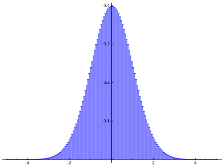

- Word Combinatorics

Study of *words*
$\Rightarrow$ subwords (factors), frequencies

Thue-Morse
$\sigma: 0 \mapsto 01,\ 1 \mapsto 10$
$01101001\ldots$

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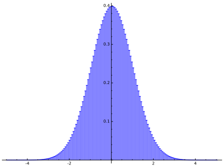

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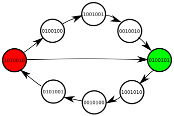
- Sturmian Words

simplest not eventually periodic.
$\Rightarrow$ recurrence: worst case, average?



Plan of the talk

1. Sturmian words
   - General Sturmian words
   - Substitutive words
2. Recurrence function
   - Definition and classical results
   - Our models and results
3. Quadratic Irrational slope
   - Model
   - Main result
4. Toolbox for the proofs
5. Conclusion

Complexity and Sturmian wordsm

Definition

Complexity function of an infinite word $\boldsymbol{u} \in \mathcal{A}^{\mathbb{N}}$

$$p_{\boldsymbol{u}} \colon \mathbb{N} \to \mathbb{N}\,, \qquad p_{\boldsymbol{u}}(n) = \#\{\text{factors of length } n \text{ in } \boldsymbol{u}\}\,.$$

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Important property

$$\boldsymbol{u} \in \mathcal{A}^{\mathbb{N}} \text{ is not eventually periodic}$$
$$\iff p_{\boldsymbol{u}}(n+1) > p_{\boldsymbol{u}}(n) \text{ for all } n \in \mathbb{N}$$

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Sturmian words are the "simplest" that are not eventually periodic.

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Definition

$\boldsymbol{u} \in \{0,1\}^{\mathbb{N}}$ is Sturmian $\Longleftrightarrow p_{\boldsymbol{u}}(n) = n+1$ for each $n \geq 0$.

Explicit construction

Given $\alpha, \beta \in [0,1)$ we define

$$\underline{\mathfrak{S}}_{\alpha,\beta}(n) = \lfloor (n+1)\,\alpha + \beta \rfloor - \lfloor n\,\alpha + \beta \rfloor \ ,$$
$$\overline{\mathfrak{S}}_{\alpha,\beta}(n) = \lceil (n+1)\,\alpha + \beta \rceil - \lceil n\,\alpha + \beta \rceil \ ,$$

for $n \geq 0$.

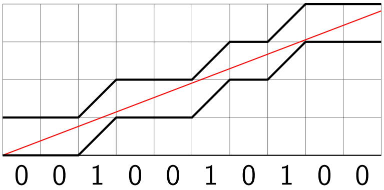

Figure : Sequences $\underline{\mathfrak{S}}_{\alpha,\beta}$ and $\overline{\mathfrak{S}}_{\alpha,\beta}$ are discrete codings of $y = \alpha x + \beta$.

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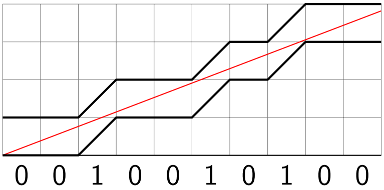

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Theorem [Morse & Hedlund '40]

▶ $\boldsymbol{u}$ is Sturmian $\iff$ there are $\alpha, \beta \in [0,1)$, α irrational, such that

$$u_i = \underline{\mathfrak{S}}_{\alpha,\beta}(i)\,, \quad \text{for all} \quad i \geq 0\,, \text{ or } u_i = \overline{\mathfrak{S}}_{\alpha,\beta}(i)\,, \quad \text{for all} \quad i \geq 0\,.$$

Substitutive words

Definition (Substitutive word)

A word $\boldsymbol{u}$ is substitutive iff $\sigma(\boldsymbol{u}) = \boldsymbol{u}$ for a primitive morphism σ.

Primitivity: σ is primitive iff the associated *matrix* M_σ is primitive

$$M_\sigma = \begin{array}{cc} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} |\sigma(0)|_0 & |\sigma(0)|_1 \\ |\sigma(1)|_0 & |\sigma(1)|_1 \end{bmatrix} \end{array}$$

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its *fixed point* $\boldsymbol{f}_\infty$ can be constructed by iteration

$$\boldsymbol{f}_0 = \mathbf{0},\ \boldsymbol{f}_1 = \mathbf{01},\ \boldsymbol{f}_2 = \mathbf{010},\ \boldsymbol{f}_3 = \mathbf{01001}, \ldots\ \boldsymbol{f}_\infty = \mathbf{0100101001001}\ldots.$$

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Slope α of a *substitutive* Sturmian word is a quadratic irrational,

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Reminder for CFEs

$$\alpha = [m_1, m_2, \ldots] := \cfrac{1}{m_1 + \cfrac{1}{m_2 + \ddots}}$$

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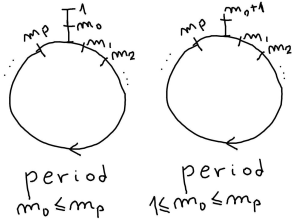
Theorem (Characterization by continued fractions)

The Sturmian word $\underline{\mathfrak{S}}(\alpha, \alpha)$ is substitutive

$$\Longleftrightarrow$$

α is qi and preperiod is of form given here.



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Consider an infinite word $\boldsymbol{u}$. Its recurrence function is:

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- Inequality relating the functions,

$$R_{\boldsymbol{u}}(n) \geq n + p_{\boldsymbol{u}}(n) - 1.$$

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$$R_{\boldsymbol{u}}(n) \geq \underbrace{n}_{\text{first factor}} + \underbrace{p_{\boldsymbol{u}}(n) - 1}_{\substack{\text{count } +1 \\ \text{for every other factor}}}.$$

Recurrence of Sturmian words: a link to arithmetic

Theorem (Morse, Hedlund, 1940)

The recurrence function is piecewise affine and satisfies

$$R_\alpha(n) = n - 1 + q_{k-1}(\alpha) + q_k(\alpha)\,, \qquad \text{for } q_{k-1}(\alpha) \le n < q_k(\alpha).$$

Truncating the expansion at depth k we get a convergent

$$\frac{p_k(\alpha)}{q_k(\alpha)} = \cfrac{1}{m_1 + \cfrac{1}{m_2 + \ddots \frac{1}{m_k}}}.$$

The denominators $q_k(\alpha)$ are called the continuants of α.

Recurrence quotient

$$S(\alpha, n) := \frac{R_\alpha(n) + 1}{n} = 1 + \frac{q_{k-1}(\alpha) + q_k(\alpha)}{n}\,, \quad q_{k-1}(\alpha) \leq n < q_k(\alpha)\,.$$

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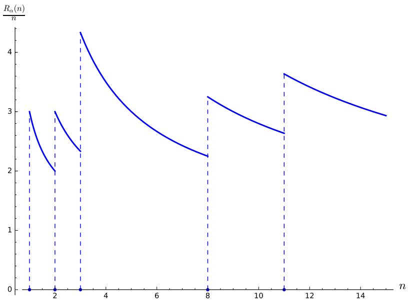
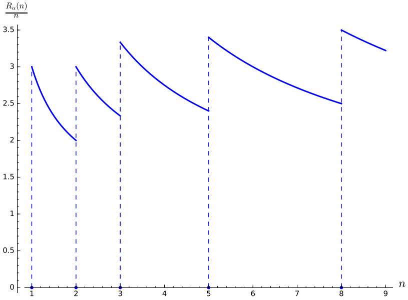


Recurrence quotient $\alpha = e^{-1}$.



Recurrence quotient $\alpha = \phi^{-2}$.

Studies of the recurrence function

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  - results for almost all α.

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- In our probabilistic setting we
  - fix an integer n (we want $n \to \infty$ ...)
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    (1) the "generic" reals from $[0,1]$
    (2) quadratic irrationals of *"size"* $\leq D$ and let $D \to \infty$.
  - study expectations $\mathbb{E}_\alpha[S(\alpha,n)]$, distributions $\mathbb{P}_\alpha\left(S(\alpha,n) \leq \lambda\right)$

Theorem (uniform $\alpha \in (0,1)$, [R.,Vallée,17])

The random variable $\alpha \mapsto S(\alpha, n)$ *admits a limiting distribution*

$$\lim_{n\to\infty} \mathbb{P}(\alpha : S(\alpha,n) \le \lambda) = \int_{[2,\lambda]} g(y)dy \,,$$

for $\lambda \ge 2$ *(and* 0 *otherwise), where the density* g *equals*

$$g(\lambda) = \begin{cases} \frac{12}{\pi^2}\,\frac{1}{\lambda-1}\,\log(1+\frac{\lambda-2}{1}) & \textit{if } \lambda \in [2,3] \\ \frac{12}{\pi^2}\,\frac{1}{\lambda-1}\,\log(1+\frac{1}{\lambda-2}) & \textit{if } \lambda \in [3,\infty) \end{cases} .$$

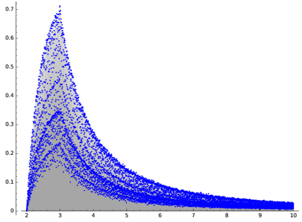

Figure : Histogram with $\epsilon = 1/n$.

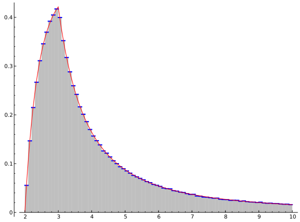

Figure : Limit density.

For $q_{k-1}(\alpha) \leq n < q_k(\alpha)$,

$$S(\alpha, n) = 1 + \frac{q_{k-1}(\alpha)+q_k(\alpha)}{n} = 1 + \frac{q_k(\alpha)}{n}\left(\frac{q_{k-1}(\alpha)}{q_k(\alpha)} + 1\right)$$
$$= f\left(\frac{q_{k-1}(\alpha)}{q_k(\alpha)}, \frac{q_k(\alpha)}{n}\right),$$

with

$$f(x, y) = 1 + y(1+x)\,, \quad (x, y) \in \mathcal{D} := \{(x, y) \in \mathbb{R}_{\geq 0} \colon xy \leq 1 < y\}\,.$$

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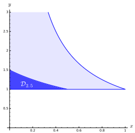
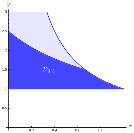
Limit distribution for $\alpha \mapsto S(\alpha, n)$ (+ more general class) given by

$$\lim_{n\to\infty} \mathbb{P}\left(\alpha : S(\alpha,n) \le \lambda\right) = \frac{6}{\pi^2}\iint_{\mathcal{D}_\lambda} \omega(x,y)\,dxdy,$$

$$\mathcal{D}_\lambda = \{(x,y) \in \mathcal{D} : f(x,y) \le \lambda\}, \quad \omega(x,y) = \frac{1}{y(1+x)}.$$



The domain $\mathcal{D}_\lambda$
$\lambda = 2.5, \quad \lambda = 3.5.$



The index of tours $\ell(\alpha, n)$

Simplifying assumptions for the talk.

- Slopes α that are reduced quadratic irrationals, i.e., corresponding to purely periodic expansions.
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Definition (ℓ-th tour)

The interval

$$\mathit{\Gamma}_\ell(\alpha) := \Big(q_{\ell p}(\alpha), q_{(\ell+1)p}(\alpha)\Big]$$

is called the ℓ-th tour of α, and

$$q_{\ell p+1}(\alpha), \ldots, q_{(\ell+1)p}(\alpha)$$

are said to be the continuants of the ℓ-th tour.

Theorem

Fix $\alpha = \left[\underline{m_1, \ldots, m_p}\right]$, *i.e., period* $(m_1, \ldots, m_p)$.

Then for every fixed r *the following limit exists*

$$Q_r(\alpha) := \lim_{\ell \to \infty} \frac{q_{\ell p + r}(\alpha)}{q_{\ell p}(\alpha)},$$

furthermore convergence is exponential in ℓ.

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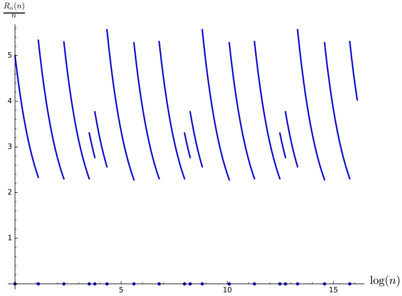

Figure : Logarithmic plot of the recurrence quotient $S(\alpha, n)$ for $\alpha = [\underline{3, 3, 3, 1, 1}] = \frac{5\sqrt{317}-63}{86}$

Model for quadratic irrationals

Quadratic irrationals present two *striking features*

- Countable and dense subset of $[0, 1]$.
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$$\mathcal{S}_D := \Big\{ \alpha \text{ quadratic irrational } : \varrho(\alpha) \leq D \Big\},$$

- Restriction to ℓ-th tour $\mathit{\Gamma}_\ell(\alpha)$

$$S_\ell(\alpha, n) = [\![n \in \mathit{\Gamma}_\ell(\alpha)]\!] \, S(\alpha, n) \,.$$

Main result for substitutive Sturmian words

For quadratic irrationals

probabilities are discrete and defined from

$$R_D(\ell,\lambda) := \Big\{(\alpha,n) : \alpha \in \mathcal{S}_D,\ n \in \Gamma_\ell(\alpha),\ S(\alpha,n) \le \lambda\Big\},$$

Main result (?) (R.,Vallée,19)

Limit distribution for $\alpha \mapsto S(\alpha,n)$ over quadratic irrationals

$$\lim_{D,u,\ell\to\infty} \frac{\Big|R_D(\ell,\lambda) \cap \{\frac{n}{q_{\ell p}} \in u\cdot(1,\theta)\}\Big|}{|\mathcal{S}_D|\cdot u\cdot(\theta-1)} = \frac{6}{\pi^2}\iint_{\mathcal{D}_\lambda} \omega(x,y)dxdy\,,$$

$$\mathcal{D}_\lambda = \{(x,y)\in\mathcal{D} : f(x,y)\le\lambda\}, \qquad \omega(x,y) = \tfrac{1}{y(1+x)}.$$

An analytic "dictionary"

A prefix $(m_1, \ldots, m_k)$ of the CFE defines an homography $g \in \mathcal{H}^k$

$$g(x) := \cfrac{1}{m_1 + \cfrac{1}{\ddots + \cfrac{1}{m_k + x}}}$$

associated with an operator, its generating function,

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$\Rightarrow$ For *generic* α:

Mellin transform of distribution yields $(\mathbf{I}-\mathbf{H}_{\rho/2+1})^{-1}[G_\rho](0)$.

Principles of the proofs

For *quadratic irrational* α, additional steps:

- α is fixed point $x_g^\star \in (0,1)$

$$x_g^\star = g(x_g^\star) = \cfrac{1}{m_1 + \cfrac{1}{\ddots + \cfrac{1}{m_k + x_g^\star}}}$$

  of some $g \in \mathcal{H}^\star$.

- generating function related to *trace* of operators

$$f \mapsto (\mathbf{I} - \mathbf{H}_{s/2})^{-1}\left[L_{\lambda,\rho} \cdot (\mathbf{I} - \mathbf{H}_{(s+\rho)/2})^{-1}[f]\right].$$

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- Multidimensional analogs?

Thank you!