Analytic Combinatorics of Unlabeled Objects

Set of exercises 4

October 13, 2025

1. Meromorphic asymptotics [warm up]

For each of the following functions $f(z)$, find the radius of convergence and an equivalent (asymptotic) for the coefficients of the power-series expansion $f(z) = \sum a_n z^n$ at $z = 0$.

1. $f(z) = \frac{1}{(1-2z)^2} \log\left(\frac{1}{1-z}\right)$.
2. $f(z) = \frac{e^{2z}-1}{(1-z)(1-2z)^3}$.
3. $f(z) = \frac{1}{2-e^z}$.

2. The Kraft-McMillan inequality

In this exercise we prove a key inequality in Coding and Information Theory by using the radius of convergence.

Let $\mathcal{A} = \{a_1, \dots, a_k\}$ and $\mathcal{B} = \{\mathbf{0}, \mathbf{1}\}$. A **binary code** $c\colon \mathcal{A}^\star \to \mathcal{B}^\star$ is a morphism, i.e., $c(xy) = c(x)c(y)$. The code c is *uniquely decodable* iff it is injective.

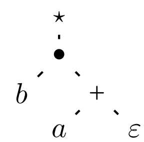
1. Let[1] $A(z) = \sum_{a \in \mathcal{A}} z^{|c(a)|}$. Show $C(z) = \sum_{v \in c(\mathcal{A}^\star)} z^{|v|} = \frac{1}{1-A(z)}$.
2. [**Kraft-McMillan inequality**] Prove that if c is uniquely decodable, then
$$\sum_{a \in \mathcal{A}} 2^{-|c(a)|} \leq 1\,.$$
Hint. Reason by contradiction. Note that power-series are continuous within their radius of convergence.
3. [**Shannon bound**] (Optional) Consider a distribution $\mathbf{p} = (p_1, \dots, p_k)$ over $(a_1, \dots, a_k)$. Prove that, for a random symbol X,
$$H(\mathbf{p}) = \sum_{j=1}^{k} p_j \log_2(1/p_j) \leq \sum_{j=1}^{k} p_j |c(a_j)| = \mathbb{E}[c(X)]\,.$$
Hint. Prove that $\log x \leq x - 1$ for all $x \in \mathbb{R}_{>0}$. Use the Kraft-McMillan inequality.

[1] Here $|w|$ is the length of the word, i.e., the number of symbols.

3. Asymptotics number of regular expressions [Transfer Theorem]

Consider the combinatorial class $\mathcal{R}$ of regular expressions as unary-binary trees:

- leaves are decorated by the letters "a", "b" or the empty word "ε".
- internal binary nodes are decorated by the operators union "$+$" or concatenation "$\bullet$",
- internal unary-nodes are decorated by the Kleene-star "$\star$".



$$(b \cdot (a + \varepsilon))^\star$$

Determine the asymptotics (an equivalent) for the number of regular expressions with n symbols (nodes).

4. Daffodil Lemma [Localization of singularities]

a) Prove the following classical lemma:

Lemma 1 (Daffodil Lemma). *Let $f(z)$ be analytic on $|z| < \rho$, with non-negative coefficients at $z = 0$. If the expansion does not reduce to a single monomial and, for some $w \neq 0$, $|w| < \rho$, we have $f(|w|) = |f(w)|$, then*

1. *we must have $w = Re^{i\theta}$ with $\theta/(2\pi) = \frac{r}{p}$ a rational, $\gcd(r,p) = 1$,*
2. *the non-zero coefficients $a_n = [z^n]f(z)$ occur at positions n all belonging to $a + p\mathbb{Z}_{\geq 0}$ for some fixed a.*

b) Consider the class $\mathcal{C}$ of binary words with at most K consecutive ones, with $K \in \mathbb{Z}_{\geq 2}$ fixed.

1. Prove that $C(z) = \frac{1}{1 - z\frac{1-z^K}{1-z}}$.
2. Give a polynomial equation characterizing the radius of convergence $z = \rho_C$. Prove that ρ_C is a simple pole of $C(z)$.
3. Prove that there are no other singularities on $|z| = \rho_C$.
4. Conclude that $[z^n]C(z) \sim c \cdot \rho_C^{-n}$ and give an expression for c in terms of ρ_C and K.

Note. The Fibonacci numbers correspond to the case $K = 2$, then $[z^n]C(z) = f_{n+1}$ and $\rho_C = 1/\phi$, $\phi = (1 + \sqrt{5})/2$.