

Analytic Combinatorics of Unlabeled Objects

Set of exercises I: responses

1. Using generating functions prove the following:

- (a) $\sum_{j=0}^n \binom{j}{p} = \binom{n+1}{p+1}$. [Hockey-stick identity]
- (b) $\sum_{j=1}^n j^2 = \frac{n(n+1)(2n+1)}{6}$.
- (c) $\sum_{j=1}^n j^p \sim n^{p+1}/(p+1)$ for every integer $p \geq 0$.
- (d) $\sum_j \binom{n}{j} \binom{m}{k-j} = \binom{n+m}{k}$. [Vandermonde's identity]

Responses.

- (a) Recalling that if $A(z) = \sum a_n z^n$ we have $\frac{1}{1-z} A(z) = \sum_n (\sum_{j \leq n} a_j) z^n$, the left-hand side is the application to $A(z) = \frac{z^p}{(1-z)^{p+1}}$, the OGF of $\left(\binom{n}{p}\right)_n$.

Thus we obtain $[z^n] \frac{1}{1-z} \frac{z^p}{(1-z)^{p+1}} = [z^n] \frac{z^p}{(1-z)^{p+2}} = [z^{n+1}] \frac{z^{p+1}}{(1-z)^{p+2}} = \binom{n+1}{p+1}$.

- (b) We recall that if $A(z) = \sum a_n z^n$, then $z \partial_z A(z) = \sum a_n n z^n$. Thus, applying this twice to $\frac{1}{1-z} = \sum z^n$, we have

$$\sum_{n=0}^{\infty} n^2 z^n = (z \partial_z)^2 \frac{1}{1-z} = \frac{z + z^2}{(1-z)^3}.$$

Now, by the same argument of (a), the OGF of the partial sums is:

$$\sum_{n=0}^{\infty} \left(\sum_{j=0}^n j^2 \right) z^n = \frac{z + z^2}{(1-z)^4}.$$

Now we remark that $[z^n] \frac{z}{(1-z)^4} = [z^{n+2}] \frac{z^3}{(1-z)^4} = \binom{n+2}{3}$ and $[z^n] \frac{z^2}{(1-z)^4} = [z^{n+1}] \frac{z^3}{(1-z)^4} = \binom{n+1}{3}$. We obtain $\sum_{j=0}^n j^2 = \binom{n+2}{3} + \binom{n+1}{3}$.

Optionally, one can also do: $j^2 = 2\binom{j}{2} + \binom{j}{1}$ and then use $\sum \binom{n}{p} z^n = \frac{z^p}{(1-z)^{p+1}}$.

- (c) We proceed by induction. This is clearly true for $p = 0$. Now assume this to hold for all $p < q$. We show this for q .

Notice that $\binom{j}{q} = \frac{j(j-1)\dots(j-q+1)}{q!} = \frac{j^q}{q!} + O(j^{q-1})$. Summing over j we have, by part (a),

$$\binom{n+1}{q+1} = \sum_{j=0}^n \binom{j}{q} = \frac{1}{q!} \sum_{j=0}^n j^q + O\left(\sum_{j=0}^n j^{q-1}\right).$$

By the inductive hypothesis $\sum_{j=0}^n j^{q-1} \sim n^q/q$. Thus the remainder is actually $O(n^q)$ and we deduce

$$\binom{n+1}{q+1} = \sum_{j=0}^n \binom{j}{q} = \frac{1}{q!} \sum_{j=0}^n j^q + O(n^q).$$

Since $\binom{n+1}{q+1} = \frac{n^{q+1}}{(q+1)!} + O(n^q)$ by the same argument, we obtain

$$\sum_{j=0}^n j^q = \frac{q!}{(q+1)!} n^{q+1} + O(n^q) = \frac{1}{q+1} n^{q+1} + O(n^q).$$

- (d) Multiply through by z^k and sum over k . We use the usual convention (in order to avoid the limits of summation) that we sum over $k \in \mathbb{Z}$, $j \in \mathbb{Z}$. Note that $\binom{a}{b} = 0$ for $b > a$ or $b < 0$. Thus the expressions make sense.

We obtain:

$$\begin{aligned} \sum_k \left(\sum_j \binom{n}{j} \binom{m}{k-j} \right) z^k &= \sum_j \binom{n}{j} \sum_k \binom{m}{k-j} z^k \\ &= \sum_j \binom{n}{j} z^j \sum_k \binom{m}{k-j} z^{k-j} \\ &= \sum_j \binom{n}{j} z^j (1+z)^m \\ &= (1+z)^{n+m} \\ &= \sum_k \binom{n+m}{k} z^k. \end{aligned}$$

2. Consider the question:

Given an OGF $F(z) = \sum_{n \geq 0} a(n) z^n$, and $q \in \mathbb{Z}_{\geq 1}$
how to obtain an OGF for $\sum_{n \geq 0} a(nq) z^{nq}$?

- (a) Let $\omega = \exp(2\pi i/q)$. Prove that

$$\sum_{n \geq 0} a(nq) z^{nq} = \frac{1}{q} \sum_{k=0}^{q-1} F(z\omega^k). \quad (1)$$

- (b) Using (1), prove that if F has radius of convergence R_F , for $0 \leq c < R_F$,

$$a(0) = \int_0^1 F(c e^{2\pi i t}) dt.$$

- (c) Obtain a formula for $\sum_{n \geq 0} a(nq+r) z^{nq+r}$ with $r \in \{0, \dots, q-1\}$.

Responses.

- (a) We remark that

$$\begin{aligned} \sum_{k=0}^{q-1} F(z\omega^k) &= \sum_{k=0}^{q-1} \sum_n a_n \omega^{kn} z^n \\ &= \sum_n a_n z^n \left(\sum_{k=0}^{q-1} \omega^{kn} \right). \end{aligned}$$

Here, by the Geometric sum,

$$\sum_{k=0}^{q-1} \omega^{kn} = \sum_{k=0}^{q-1} (\omega^n)^k = \begin{cases} \frac{1-(\omega^n)^q}{1-\omega^n}, & \text{if } \omega^n \neq 1, \\ q, & \text{if } \omega^n = 1. \end{cases}$$

Being ω a q root of unity, $(\omega^n)^q = 1$. On the other hand, $\omega^n = 1$ iff $n \equiv 1 \pmod{q}$.

Therefore we deduce that $\sum_{k=0}^{q-1} \omega^{kn} = q \times \mathbf{1}_{n \equiv 0 \pmod{q}}$, and part (a) follows.

(b) The right-hand side of (1) is actually a Riemann sum:

$$\frac{1}{q} \sum_{k=0}^{q-1} F(z\omega^k) = \frac{1}{q} \sum_{k=0}^{q-1} F(z \exp(2\pi i k/q)).$$

If $z = c < R_F$, within its radius of convergence F is continuous and therefore $t \mapsto F(c \exp(2\pi i t))$ is Riemann-integrable. Taking $q \rightarrow \infty$ we obtain $\frac{1}{q} \sum_{k=0}^{q-1} F(c\omega^k) \rightarrow \int_0^1 F(c e^{2\pi i t}) dt$.

Now, from part (a) we remark that $\frac{1}{q} \sum_{k=0}^{q-1} F(c\omega^k) = a(0) + a(q)c^q + \dots$. When $q \rightarrow \infty$ we have that $a(q)c^q + a(2q)c^{2q} + \dots$ is bounded by the tail of a convergence series $\sum_n a(n)c^n$, thus it tends to 0. We deduce $\frac{1}{q} \sum_{k=0}^{q-1} F(c\omega^k) \rightarrow a(0)$.

(c) For $r > 0$, consider $G(z) = F(z)z^{q-r}$ and apply (a) to $G(z)$.

3. The Stirling numbers of the second kind $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$ count the number of partitions of a set of n elements into k non-empty subsets. Without loss of generality, we suppose the set of n elements is $[n] = \{1, \dots, n\}$.

Prove the following identities

(a) $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} = \left\{ \begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix} \right\} + k \left\{ \begin{smallmatrix} n-1 \\ k \end{smallmatrix} \right\}$ for all¹ $n, k \geq 0$.

(b) $\sum_{n \geq 0} \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} z^n = \frac{z^k}{(1-z)(1-2z)\dots(1-kz)}.$

Find a formula for $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$ by applying partial fractions.

Response.

(a) In a partition of $[n] = \{1, \dots, n\}$ into k parts, either n is alone making its own part, or not.

– The total number of partitions in which n is alone is $\left\{ \begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix} \right\}$, as the rest is a partition of $[n-1]$ into $k-1$ parts.

– The total number of partitions in which n is not alone is $k \times \left\{ \begin{smallmatrix} n-1 \\ k \end{smallmatrix} \right\}$. Indeed, erasing n we have a partition of $[n-1]$ into k parts. Then we must decide to which of these k parts n belongs to, hence the factor k .

¹We define $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} = 0$ if every $n < 0$, $k < 0$ or $n < k$.

(b) Multiply both sides of the recurrence by k and sum over $k \in \mathbb{Z}$, with the convention $\begin{Bmatrix} a \\ b \end{Bmatrix} = 0$ if $a < b$ or $a < 0$. We obtain

$$\begin{aligned} \sum_n \begin{Bmatrix} n \\ k \end{Bmatrix} z^n &= \sum_n \begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix} z^n + k \sum_n \begin{Bmatrix} n-1 \\ k \end{Bmatrix} z^n \\ &= z \sum_n \begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix} z^{n-1} + kz \sum_n \begin{Bmatrix} n-1 \\ k \end{Bmatrix} z^{n-1}. \end{aligned}$$

Letting $A_k(z) = \sum_n \begin{Bmatrix} n \\ k \end{Bmatrix} z^k$, we obtain

$$A_k(z) = zA_{k-1}(z) + kzA_k(z) \implies A_k(z) = \frac{z}{1-kz} A_{k-1}(z).$$

Iterating until $k = 0$, $A_0(z) = 1$ and we have

$$A_k(z) = \frac{z}{1-z} \frac{z}{1-2z} \cdots \frac{z}{1-kz},$$

as desired.

Now we apply partial fractions:

$$A_k(z) = \sum_{j=1}^k \frac{\alpha_j}{1-jz},$$

for some coefficients α_j . We compute α_j by multiplying through by $1-jz$ and making $j \rightarrow 1/z$

$$\begin{aligned} \alpha_j &= \lim_{z \rightarrow 1/j} (1-jz)A_k(z) \\ &= (1/j)^k \prod_{r: 1 \leq r \leq k, r \neq j} \frac{1}{1-r/j} \\ &= (1/j)^k j^{k-1} \prod_{r=1}^{j-1} \frac{1}{j-r} \times \prod_{r=j+1}^k \frac{1}{r-j} \\ &= (-1)^{k-j} (1/j)^k j^{k-1} \frac{1}{(j-1)!(k-j)!} \\ &= (-1)^{k-j} (1/j)^k j^k \frac{1}{j!(k-j)!} \\ &= (-1)^{k-j} \frac{1}{j!(k-j)!}. \end{aligned}$$

Thus

$$\begin{aligned}
\begin{Bmatrix} n \\ k \end{Bmatrix} &= [z^n] A_k(z) \\
&= \sum_{j=1}^k \alpha_j [z^n] \frac{1}{1-jz} \\
&= \sum_{j=1}^k \alpha_j j^n \\
&= \sum_{j=1}^k (-1)^{k-j} \frac{j^n}{j!(k-j)!}.
\end{aligned}$$

4. In this exercise we give a combinatorial interpretation to

$$\sum_{n \geq 0} \begin{Bmatrix} n \\ k \end{Bmatrix} z^n = \frac{z^k}{(1-z)(1-2z) \dots (1-kz)}.$$

We define an algorithm. Consider a partition $P = \{S_1, \dots, S_k\}$:

- We keep an list L of the *known* parts from P . Initially $L = []$.
- We iterate $j = 1, \dots, n$. For iteration j , let $S_j \in P$ with $j \in S_j$. If S_j appears in L , write its index. If not, append it and write $|V| + 1$.

The *numbers written* belong to $[k]$. They constitute the *backbone*

$$P = \{\{4, 6, 7\}, \{1, 3\}, \{2, 5\}\} \mapsto L(P) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \mathbf{1} & \mathbf{2} & \mathbf{1} & \mathbf{3} & \mathbf{2} & \mathbf{3} & \mathbf{3} \end{pmatrix}$$

Prove that this yields a bijection. Find a combinatorial specification and deduce the OGF of the partitions into k parts, k fixed.

Response. Essentially, reading the numbers in $L(P)$ gives back the partition. We remark that we have numbered the elements of the partition according to the order given by the minimum element in each part. A valid backbone $L(P)$, then, must have the first appearance of j before the appearance of $j+1$, for each $j \in [k]$. Given any sequence with this properties, it corresponds to a unique partition and vice-versa. Thus we obtain a bijection. Put another way, the numbers in $L(P)$ label the parts of the partition by their order of discovery when reading $j = 1, 2, \dots, n$.

We note that $L(P)$ can be specified as

$$\{1\} \times \text{Seq}(\{1\}) \times \{2\} \times \text{Seq}(\{1, 2\}) \times \dots \times \{k\} \times \text{Seq}(\{1, \dots, k\}),$$

where each number has weight 1.

The corresponding OGF is precisely

$$\frac{z^k}{(1-z)(1-2z) \dots (1-kz)},$$

as desired.