Mathematical Foundations of CS (CS 208)

Assignment 4

Due: October 9, 2018

Note: Each question will be graded as per the following criteria:

(a) 5 - Thorough understanding of the topic and conceptual clarity

(b) 4 - Adequate understanding of the topic and conceptual clarity

(c) 3 - Some understanding of the topic and conceptual clarity

(d) 2 or 1 - Poor understanding of the topic and messed up concepts

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1. The *reversal* of a string is the string written backwards, for example, $rev(abcd) = dcba$.
   (a) Give a simple recursive definition of $rev(s)$ based on the recursive definition of $s \in A^\star$ and of concatenation operation discussed in the class.
   (b) Prove that $rev(s \cdot t) = rev(t) \cdot rev(s)$ for all strings $s, t \in A^\star$. You may assume that concatenation is associative: $(r \cdot s) \cdot t = r \cdot (s \cdot t)$ for all strings $r, s, t \in A^\star$. If you are not convinced of associativity of concatenation, then you may prove it by structural induction on the recursive definition of string discussed in the class.

2. Provide simple recursive definitions of the following sets:
   (a) $S = \{2^k 3^m 5^n \in \mathbb{N} | k, m, n \in \mathbb{N}\}$
   (b) $S = \{2^k 3^{2k+m} 5^{m+n} \in \mathbb{N} | k, m, n \in \mathbb{N}\}$
   (c) $S = \{(a, b) \in \mathbb{Z}^2 | (a - b)$ is a multiple of 3.$\}$

3. Define the number $\#_c(s)$ of occurrences of the character $c \in A$ in the string s recursively on the definition of $s \in A^\star$:
   Base case: $\#_c(\lambda) := 0$.
   Constructor case:
   $$\#_c(\langle a, s \rangle) = \begin{cases} \#_c(s) & a \neq c \\ 1 + \#_c(s) & a = c \end{cases}$$
   Prove by structural induction that for all $s, t \in A^\star$ and $c \in A$
   $$\#_c(s \cdot t) = \#_c(s) + \#_c(t).$$

4. Evaluate the sums $S_n = \sum_{k=0}^{n} (-1)^{n-k}$, $T_n = \sum_{k=0}^{n} (-1)^{n-k} k$, and $U_n = \sum_{k=0}^{n} (-1)^{n-k} k^2$ by the perturbation method, assuming that $n \geq 0$.

5. Evaluate the sum $\sum_{k=1}^{n} (-1)^k k / (4k^2 - 1)$.

6. Prove the Dirichlet box principle: if n objects are put into m boxes, some box must contain $\geq \lceil n/m \rceil$ objects, and some box must contain $\leq \lfloor n/m \rfloor$.

7. Show that the expression
   $$\left\lceil \frac{2x+1}{2} \right\rceil - \left\lceil \frac{2x+1}{4} \right\rceil + \left\lfloor \frac{2x+1}{4} \right\rfloor$$
   is always either $\lfloor x \rfloor$ or $\lceil x \rceil$. In what circumstances does each case arise?

8. Prove or disprove: $\lfloor x \rfloor + \lfloor y \rfloor + \lfloor x + y \rfloor \leq \lfloor 2x \rfloor + \lfloor 2y \rfloor$.