Mathematical Foundations of CS (CS 208)

Assignment 3

Due: September 25, 2018

Note: Each question will be graded as per the following criteria:

(a) 5 - Thorough understanding of the topic and conceptual clarity

(b) 4 - Adequate understanding of the topic and conceptual clarity

(c) 3 - Some understanding of the topic and conceptual clarity

(d) 2 or 1 - Poor understanding of the topic and messed up concepts

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1. Prove $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ for all sets A, B, and C, by using a chain of iff's to show that $x \in A \cup (B \cap C)$ IFF $x \in (A \cup B) \cap (A \cup C)$ for all elements x.

2. Let A and B be sets. Prove that

   (a) $pow(A \cap B) = pow(A) \cap pow(B)$.

   (b) $(pow(A) \cup pow(B)) \subseteq pow(A \cup B)$.

3. Prove that for any sets A, B, C and D, if the Cartesian products $A \times B$ and $C \times D$ are disjoint, then either A and C are disjoint or B and D are disjoint.

4. For each of the following real-valued functions on the real numbers, indicate whether it is a bijection, a surjection but not a bijection, an injection but not a bijection, or neither an injection nor a surjection.

   (a) $x \to x^2$

   (b) $x \to x^3$

   (c) $x \to \sin x$

   (d) $x \to x \sin x$

   (e) $x \to e^x$

5. Give an example of a relation R that is a total injective function from a set A to itself but is not a bijection.

6. Let $f : A \to B$ and $g : B \to C$ be function.

   (a) Prove that if the composition $g \circ f$ is a bijection, then f is a total injection and g is a surjection.

   (b) Show there is a total injection f and a bijection, g, such that $g \circ f$ is not a bijection.

7. Give an example of two uncountable sets A and B such that there is no bijection between them.

8. For each of the following sets, indicate whether it is finite, countably infinite, or uncountable. Justify your answers.

   (a) The set of even integers greater than 10^{100}.

   (b) The power set of integer interval $[10, \ldots, 10^{10}]$.

   (c) $U \cup D$, where U is an uncountable set and D be a countably infinite set.

   (d) $U - D$, where U and D are as in part (d).

9. A function $f : \mathbb{N} \to \mathbb{N}$ is said to be rapidly increasing if $f(n+1) \geq 2^{f(n)}$ for all $n \in \mathbb{N}$. Let RAPID be the set of rapidly increasing functions. Use a diagonalization argument to prove that RAPID is uncountable.