CS208: Mathematical Foundations of CS **Fall 2018**

Lecture 25: October 10

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25.1 Problem on Floor function

Problem 1. *Evaluate* $\sum_{k=0}^{n} \lfloor \sqrt{k} \rfloor$

Solution 1. *Our idea is to introduce the variable n =* $\lfloor \sqrt{k} \rfloor$.

$$= \sum_{m,k \geq 0} m[k < n][m = \left\lfloor \sqrt{k} \right\rfloor]$$

$$= \sum_{m,k \geq 0} m[k < n][m^2 \leq k \leq (m+1)^2]$$

$$= \sum_{m,k \geq 0} m[k < n][m^2 \leq k < (m+1)^2 \leq n] + \sum_{m,k \geq 0} m[k < n][m^2 \leq k < n < (m+1)^2]$$

Here, the second term will be 0 when n is a perfect square as no integer lies between m and m+1.
Case 1: *n is a perfect square*
Let n = a^2

$$\sum_{m,k \geq 0} m[k < n][m^2 \leq k < (m+1)^2 \leq n]$$

$$= \sum_{m,k \geq 0} m[k < n][m^2 \leq k < (m+1)^2 \leq a^2]$$

$$= \sum_{m \geq 0} m((m+1)^2 - m^2)[m+1 \leq a]$$

$$= \sum_{m \geq 0} m(2m+1)[m < a]$$

$$= \sum_{m \geq 0} (2m^{\underline{2}} + 3m^{\underline{1}})\delta m \quad [1]$$

$$= \frac{2}{3}a(a-1)(a-2) + \frac{3}{2}a(a-1) \quad [2]$$

$$= \frac{1}{6}(4a+1)(a)(a-1)$$

Case 2: *n is not a perfect square*
Let a = $\lfloor \sqrt{n} \rfloor$ *, then we merely need to add the terms for* $a^2 \leq k < n$ *; which are all equal to a.*
So,

$$\sum m[m^2 \leq k < n < (m+1)^2] = a(n - a^2)$$

So,

$$\sum_{k=0}^{n} \left\lfloor \sqrt{k} \right\rfloor = na - \frac{1}{3}a^3 - \frac{1}{2}a^2 - \frac{1}{6}a$$

25.2 Asymptotic Notation

There are 5 types of asymptotic notation Θ , O, Ω, o and ω .

$$\Theta(g(n)) = \{f(n) \mid \exists\, c_1, c_2, n_o \; s.t. \; c_1 g(n) \leq f(n) \leq c_2 g(n) \; \forall\, n \geq n_o\}$$
$$f(n) = \frac{1}{2}n^2 - \frac{3}{n} = \Theta(n^2)$$
$$c_1 n^2 \leq \frac{1}{2}n^2 - \frac{3}{n} \leq c_2 n^2$$
$$c_1 \leq \frac{1}{2} - \frac{3}{n^3} \leq c_2$$

A valid set of values for c_1,c_2 and n_o are $c_1 = \frac{1}{2}$, $c_2 = \frac{1}{6}$ and $n_o = \sqrt[3]{9}$.
So , $f(n) = \Theta(n^2)$.

Notes

1. Falling powers $n^{\underline{k}}$ (read n to the falling k) is defined as follows :

   $$n^{\underline{k}} = n(n-1)(n-2)....(n-(k-1))$$

   Similarly, Rising powers $n^{\overline{k}}$ (read n to the rising k) is defined as follows :

   $$n^{\overline{k}} = n(n+1)(n+2)....(n+(k-1))$$

2.
$$\begin{aligned}\sum_{x=a}^{b} x^{\underline{n}} &= \sum_{x=a}^{b} [x(x-1)(x-2)...(x-(n-1))] \\ &= \sum_{x=a}^{b} (x^n + \sum_{k=1}^{n-1} a_k x^k) \\ &= \begin{cases} \frac{x^{n+1}}{n+1}\Big|_a^b & n \neq -1 \\ H_x\Big|_a^b & n = -1 \end{cases}\end{aligned}$$