**CS208: Mathematical Foundations of CS** **Fall 2018**

# Lecture 14: September 12

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## 14.1 Binary Relations

A binary relation is a mapping between objects of one set to the object of another set. For example, ”Divides” is a relation between every *a* from set of integers and *b* from set of integers when $a \mid b$.

**Definition 1** *A binary relation, R, consists of a set, A, called the domain of R, a set, B, called the codomain of R, and a subset of $A \times B$ called the graph of R.*

To indicate that a relation *R* is from $\mathbb{A}$ to $\mathbb{B}$, we write $R : \mathbb{A} \to \mathbb{B}$.

**Example 14.1.1** $R : \mathbb{R} \to \mathbb{R} = \{ (x, y) : (x, y) \in \mathbb{R}^2 \mid x<y \}$
In this example $x \in \mathbb{R}$ : Domain, $y \in \mathbb{R}$ : Co-domain and $(x, y)$ : Graph of R $\in \mathbb{R}^2$

**Note:** A $relation$ is a $function$ if **at most one** pair exists in graph corresponding to each element of the domain.

### 14.1.1 Relational Diagrams

We can visualize a relation in terms of a diagram as two sets of elements: one such that arrows come out of its elements, and the other such that arrows sink into its elements. The first is the domain of the relation and the second is the codomain of the relation.

**Definition 2** *A Binary Relation R is :*

- a $function$ when each element of the domain has [$\leq$ **1 arrow out**].
- a $total$ when each element of the domain has [$\geq$ **1 arrow out**].
- a $total function$ when each element of the domain has [= **1 arrow out**].
- a $injective$ when each element of the codomain has [$\leq$ **1 arrow in**].
- a $surjective$ when each element of the codomain has [$\geq$ **1 arrow in**].
- a $bijective$ when each element of the codomain has [= **1 arrow in**] and [= **1 arrow out**].

**Example 14.2.1** $R : \mathbb{R}^+ \to \mathbb{R} = \{\ (x, y) : x \in \mathbb{R}^+, y \in \mathbb{R} \mid y = logx\ \}$ :
This is a bijective total function as there are [= **1 arrow out**] and [= **1 arrow in**].
**Example 14.2.2** $R : \mathbb{R}^+ \to \mathbb{R} = \{\ (x, y) : x \in \mathbb{R}^+, y \in \mathbb{R} \mid y = e^{sin(x)}\ \}$ : This is a total function because there are [= **1 arrow out**] but neither injective, nor surjective because for some elements, there are [$<$ **1 arrow in**] and for some elements there are [$>$ 1 arrows in] since $y$ is bounded between $e^{-1}$ and e.
**Example 14.2.3** $R : \mathbb{R}^+ \to \mathbb{R} = \{\ (x, y) : x \in \mathbb{R}^+, y \in \mathbb{R} \mid y = \sqrt{x}\ \}$ :
This is an injective total as there are [$\geq$ **1 arrow out**] and [= **1 arrow in**].

### 14.1.2 Relational Image

**Definition 3** *The image of a set, Y , under a relation, R, written R.Y /, is the set of elements of the codomain, B, of R that are related to some element in Y . In terms of the relation diagram, R.Y / is the set of points with an arrow coming in that starts from some point in Y.*

### 14.1.3 Inverse Relations and Images

**Definition 4** *The inverse, $R^{-1}$ of a relation $R : A \to B$ is the relation from B to A such that for $a \in A$ and $b \in B$*

$$b\ R^{-1}\ a \Leftrightarrow a\ R\ b$$

In case of inverse relation, the arrows heads in the relational diagrams are reversed, i.e. the arrows coming out of an element in relation R will sink at that element in $R^{-1}$.

**Definition 5** *The image of a set under the relation, $R^{-1}$, is called the inverse image of the set. That is, the inverse image of a set, X, under the relation, R, is defined to be $R^{-1}(X)$.*

## References

[1] E. LEHMAN, F. T. LEIGHTON and S. WINOGRAD, *Mathematics for Computer Science*, 2015.