CS208: Mathematical Foundations of CS **Fall 2018**

Lecture 24: October 9

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24.1 Approximating Sums

It is not always possible to find a closed-form expression for a sum. *For Example*

$$S = \sum_{i=1}^{N} \sqrt{x}$$

No closed form is known for S .So, for this we need to find approximation for S. We need to find closed-form upper and lower bounds for S.

Theorem 1 *Let f: be an increasing function*

$$S = \sum_{i=1}^{n} f(i)$$

and

$$I = \int_{1}^{n} f(x)dx$$

then

$$I + f(1) \leq S \leq I + f(n)$$

Similarly, if f is weakly decreasing, then

$$I + f(n) \leq S \leq I + f(1)$$

Example 1

$$S = \sum_{i=1}^{n} \sqrt{x}$$

$$I = \int_{1}^{n} \sqrt{x}dx$$

$$I = \frac{2}{3}(n^{\frac{3}{2}} - 1)$$

therefore,

$$\frac{2}{3}n^{\frac{3}{2}} + \frac{1}{3} \leq S \leq \frac{2}{3}n^{\frac{3}{2}} + \sqrt{n} - \frac{2}{3}$$

24.1.1 Harmonic Numbers

The nth *harmonic number is*

$$H_n = \sum_{i=1}^{n} \frac{1}{i}$$

or

$$H_n = \int_1^n \frac{1}{n} dx = ln(n),$$

therefore,

$$ln(n) + \frac{1}{n} \leq H_n \leq ln(n) + 1$$

24.2 Products

We can convert any product into a sum by taking a logarithm.

$$P = \prod_{i=1}^{n} f(i)$$

or

$$ln(P) = \sum_{i=1}^{n} ln(f(i)$$

Example 2

$$P = n!$$

$$ln(P) = \sum_{i=1}^{n} ln(i)$$

or

$$ln(P) = \int_1 nln(i)di = nln(n) - n + 1$$

therefore,

$$nln(n) - n + 1 \leq ln(P) \leq nln(n) - n + 1 + ln(n)$$

and,

$$\frac{n^n}{en-1} \leq P \leq \frac{n^{n+1}}{en-1}$$

24.2.1 Stirling's Formula

Theorem 2 *(Stirlings Formula). For all n $\geq$ 1*

$$n! = \sqrt{2\pi n}(\frac{n}{e})^n e^{\epsilon(n)}$$

where ,

$$\frac{1}{12n+1} \leq \epsilon(n) \leq \frac{1}{12n}$$

24.3 Floor and Ceil

In mathematics and computer science, the floor and ceiling functions map a real number to the greatest preceding or the least succeeding integer, respectively.

$$\lfloor x \rfloor = LargestInteger \leqslant x$$

$$\lceil x \rceil = LeastInteger \geqslant x$$

24.3.1 Properties

$$\begin{gathered}
\lfloor -x \rfloor = \lceil x \rceil \\
\lfloor -x \rfloor = -\lceil x \rceil \\
n < x \iff n < \lceil x \rceil \\
x \leqslant n \iff \lceil x \rceil \leqslant n \\
n \leqslant x \iff x \leqslant \lceil x \rceil \\
x - 1 < \lfloor -x \rfloor \leqslant x \leqslant \lceil x \rceil < x + 1
\end{gathered}$$

Example 3

$$\left\lfloor \sqrt{\lfloor x \rfloor} \right\rfloor = \lfloor \sqrt{x} \rfloor$$

Proof:

$$m = \left\lfloor \sqrt{\lfloor x \rfloor} \right\rfloor$$

$$m \leqslant \left\lfloor \sqrt{\lfloor x \rfloor} \right\rfloor \leqslant m + 1$$

$$m^2 \leqslant \lfloor x \rfloor \leqslant (m+1)^2$$

$$m^2 \leqslant x \leqslant (m+1)^2$$

$$m \leqslant \sqrt{x} \leqslant (m+1)$$

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Example 4 *Compute the following considering m as a perfect square*

$$\sum_{k=0}^{m} \left\lfloor \sqrt{k} \right\rfloor$$

Solution: Suppose that $n^2 \leq k < (n+1)^2$ *, then* $\left\lfloor \sqrt{k} \right\rfloor = n$*. There are 2n + 1 integers k in this interval, so if* $m \leq n^2+2n$*, they contribute n(2n+1) to the total. Thus, if* $n^2 \leq m <$ *(n+1)2, following equation holds true.*

$$\sum_{k=0}^{m} \left\lfloor \sqrt{k} \right\rfloor = \sum_{k=1}^{n-1} (2k^2 + k) + (m - n^2 + 1)n$$

Now since $m=n^2$ *the above equation reduces to*

$$\sum_{k=0}^{m} \left\lfloor \sqrt{k} \right\rfloor = \sum_{k=1}^{\sqrt{m}-1} (2k^2 + k) + \sqrt{m}$$

$$\sum_{k=0}^{m} \left\lfloor \sqrt{k} \right\rfloor = \frac{\sqrt{m}(\sqrt{m}-1)(2\sqrt{m}-1)}{3} + \frac{\sqrt{m}(\sqrt{m}-1)}{2} + \sqrt{m}$$

Answer:

$$\frac{\sqrt{m}(\sqrt{m}-1)(2\sqrt{m}-1)}{3} + \frac{\sqrt{m}(\sqrt{m}-1)}{2} + \sqrt{m}$$

References

E. LEHMAN, F. T. LEIGHTON AND A. R. MEYER, Mathematics for Computer Science, 2013 , Chapter 14