Mathematical Foundations of CS (CS 208)

Assignment 5

Due: October 23, 2018

Note: Each question will be graded as per the following criteria:

(a) 5 - Thorough understanding of the topic and conceptual clarity

(b) 4 - Adequate understanding of the topic and conceptual clarity

(c) 3 - Some understanding of the topic and conceptual clarity

(d) 2 or 1 - Poor understanding of the topic and messed up concepts

---

1. Either prove or disprove each of the following statements:
   (a) $n! = O((n+1)!)$ (b) $(n+1)! = O(n!)$ (c) $n! = \Theta((n+1)!)$ (d) $n! = o((n+1)!)$ (e) $(n+1)! = o(n!)$

2. For each pair of functions below, indicate the smallest $c \in \mathbb{Z}^+$, and for that smallest c, the smallest $n_0 \in \mathbb{Z}^+$, that would establish $f = O(g)$ as per the definition of "big-Oh" introduced in the class. If there is no such c, then write ∞.
   (a) $f(n) = \frac{1}{2}n^2, g(n) = n.$
   (b) $f(n) = n, g(n) = n \ln n.$
   (c) $f(n) = 2^n, g(n) = n^4 \ln n.$
   (d) $f(n) = 3 \sin\left(\frac{\pi(n-1)}{100}\right) + 2, g(n) = 0.2.$

3. How many of the billion numbers in the integer interval $[1, \ldots, 10^9]$ contain the digit 1?

4. There are 20 books arranged in a row on a shelf. Describe a bijection between ways of choosing 6 of these books so that no two adjacent books are selected, and 15-bit strings with exactly 6 ones.

5. How many ways are there to pick three distinct numbers from the integer interval $[1, \ldots, 15]$ such that the sum of the numbers is divisible by three?

6. Suppose $2n + 1$ numbers are selected from $\{1, 2, \ldots, 4n\}$. Using the Pigeonhole Principle, show that there must be two selected numbers whose difference is 2. Clearly indicate the pigeons, holes, and rules for assigning a pigeon to a hole.

7. Let $k_1, k_2, \ldots, k_{101}$ be a sequence of 101 integers. A sequence $k_{m+1}, k_{m+2}, \ldots, k_n$, where $0 \leq m < n \leq 101$ is called a subsequence. Prove that there is a subsequence whose elements sum to a number divisible by 100. (Hint: you may use Pigeonhole Principle.)

8. The working days in the next year can be numbered $1, 2, 3, \ldots, 300$. I would like to avoid as many as possible.
   - On even-numbered days, I'll say Im sick.
   - On days that are a multiple of 3, I'll say I was stuck in traffic.
   - On days that are a multiple of 5, I'll refuse to come out from under the blankets.

   In total, how many work days will I avoid in the coming year? (Hint: you may use Inclusion-Exclusion Principle.)

9. Give a combinatorial proof for the identity:

$$\sum_{r-0}^{n} \binom{n}{r}\binom{m}{k-r} = \binom{n+m}{k}.$$