CS208: Midsem 1

Solutions

Problem 1

a. The food is good but the service is poor.
b. The food is poor and so is the service.
c. Either the food is poor or the service is poor, but the price is low.
d. Either the food is good or the service is excellent.
e. The price is high but either the food is poor or the service is poor.

Problem 3

a. valid: there is an x in the domain with property A says it is false that everything in the domain fails to have property A.

b. not valid: domain is the integers, $P(x)$ is "x is even", $Q(x)$ is "x is prime". Because there are prime integers, $(\exists x)Q(x)$ and therefore $(\forall x)P(x) \vee (\exists x)Q(x)$ is true. But it is false that every integer is even or prime, so the implication is false.

Problem 4(a)

We first prove that if x and y are odd, so is xy. A direct proof will work. Suppose that both x and y are odd. Then $x = 2n + 1$ and $y = 2m + 1$, where m and n are integers. Then $xy = (2n + 1)(2m + 1) = 4nm + 2m + 2n + 1 = 2(2nm + m + n) + 1$. This has the form $2k + 1$, where k is an integer, so xy is odd.

Next we prove that if xy is odd, both x and y must be odd, or

$$xy \text{ odd} \rightarrow x \text{ odd and } y \text{ odd}$$

A direct proof would begin with the hypothesis that xy is odd, which leaves us little more to say. A proof by contraposition works well because we'll get more useful information as hypotheses. So we will prove

$$(x \text{ odd and } y \text{ odd})' \rightarrow (xy \text{ odd})'$$

By De Morgan's law $(A \wedge B)' \Leftrightarrow A' \vee B'$, we see that this can be written as

$$x \text{ even or } y \text{ even} \rightarrow xy \text{ even} \qquad (1)$$

The hypothesis "x even or y even" breaks down into three cases. We consider each case in turn.

1. x even, y odd: Here $x = 2m$, $y = 2n + 1$, and then $xy = (2m)(2n + 1) = 2(2mn + m)$, which is even.
2. x odd, y even: This works just like case 1.
3. x even, y even: Then xy is even by Example 5.

This completes the proof of (1) and thus of the theorem. ●

Problem 5

The basis step is to show $P(1)$, that $2^{2(1)} - 1 = 4 - 1 = 3$ is divisible by 3. Clearly this is true.

We assume that $2^{2k} - 1$ is divisible by 3, which means that $2^{2k} - 1 = 3m$ for some integer m, or $2^{2k} = 3m + 1$ (this little rewriting trick is the key to these "divisibility" problems). We want to show that $2^{2(k+1)} - 1$ is divisible by 3.

$$
\begin{aligned}
2^{2(k+1)} - 1 &= 2^{2k+2} - 1 \\
&= 2^2 \cdot 2^{2k} - 1 \\
&= 2^2(3m + 1) - 1 && \text{(by the inductive hypothesis)} \\
&= 12m + 4 - 1 \\
&= 12m + 3 \\
&= 3(4m + 1) && \text{where } 4m + 1 \text{ is an integer}
\end{aligned}
$$

Thus $2^{2(k+1)} - 1$ is divisible by 3.

Problem 6

For the base case, a 1-piece puzzle requires 0 steps to assemble. Assume that any block of r pieces, $1 \leq r \leq k$, requires $r - 1$ steps to assemble. Now consider a puzzle with $k + 1$ pieces. The last step in assembling the puzzle is to fit together two blocks of size r_1 and r_2 with $1 \leq r_1 \leq k$, $1 \leq r_2 \leq k$, and $r_1 + r_2 = k + 1$. By the inductive hypothesis these blocks required $r_1 - 1$ and $r_2 - 1$ steps to assemble, so with the final step, the total number of steps required is $(r_1 - 1) + (r_2 - 1) + 1 = (r_1 + r_2) - 1 = k$.