Mathematical Foundations of CS (CS 208)

Assignment 1

Due: August 28, 2018

Note: Each question will be graded as per the following criteria:

(a) 5 - Thorough understanding of the topic and conceptual clarity

(b) 4 - Adequate understanding of the topic and conceptual clarity

(c) 3 - Some understanding of the topic and conceptual clarity

(d) 2 or 1 - Poor understanding of the topic and messed up concepts

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1. Identify where the bug is in the following bogus proof.

   Bogus Claim: If a and b are two equal real numbers, then $a = 0$.

   Bogus Proof:

$$\begin{aligned} a &= b \\ a^2 &= ab \\ a^2 - b^2 &= ab - b^2 \\ (a-b)(a+b) &= (a-b)b \\ a+b &= b \\ a &= 0. \end{aligned}$$

2. It's known that the Arithmetic Mean is at least as large as the Geometric Mean, namely,

$$\frac{a+b}{2} \geq \sqrt{ab}$$

   for all nonnegative real numbers a and b. A student came up with the following proof for this fact. Is it correct? If not, then what is your objection and how would you fix it?

   Purported Proof:

$$\begin{aligned} \frac{a+b}{2} &\overset{?}{\geq} \sqrt{ab} \\ a+b &\overset{?}{\geq} 2\sqrt{ab} \\ a^2 + 2ab + b^2 &\overset{?}{\geq} 4ab \\ a^2 - 2ab + b^2 &\overset{?}{\geq} 0 \\ (a-b)^2 &\geq 0 \quad \text{which is known to be true.} \end{aligned}$$

3. Using proof by contradiction, prove that for any $n > 0$, if a^n is even, then a is even.

4. Prove that $\log_4 6$ is irrational.

5. Use Well Ordering Principle to prove that:

    (a) there is no solution over the positive integers to the equation: $4a^3 + 2b^3 = c^3$.

    (b) any integer greater than or equal to 8 can be represented as the sum of nonnegative integer multiples of 3 and 5.

6. Prove by induction on n that

$$1 + r + r^2 + \ldots + r^n = \frac{r^{n+1} - 1}{r - 1}$$

for all $n \in \mathbb{N}$ and numbers $r \neq 1$.

Remember to formally

(a) Declare proof by induction.

(b) Identify the induction hypothesis $P(n)$.

(c) Establish the base case.

(d) Prove that $P(n) \implies P(n+1)$.

(e) Conclude that $P(n)$ holds for all $n \geq 1$.

as in the five part template.

7. In the class we discussed Fibonacci numbers. Prove by induction that for all $n \geq 1$,

$$F(n-1) \cdot F(n+1) - F(n)^2 = (-1)^n.$$

8. You are given envelopes, numbered $0, 1, \ldots, n-1$. Envelope 0 contains $2^0 = 1$ rupees. Envelope 1 contains $2^1 = 2$ rupees, ..., and Envelope $n-1$ contains 2^{n-1} rupees. Let $P(n)$ be the assertion that:

For all nonnegative integers $k < 2^n$, there is a subset of the n envelopes whose contents total to exactly k rupees.

Prove by induction that $P(n)$ holds for all integers $n \geq 1$.

9. A group of $n \geq 1$ people can be divided into teams, each containing either 4 or 7 people. What are the possible values of n? Use strong induction to prove that your answer is correct.

10. Use strong induction to prove that $n \leq 3^{n/3}$ for every integer $n \geq 0$.