CS208: Mathematical Foundations of CS **Fall 2018**

Lecture 17: September 18

Lecturer: Samar *Scribes: Vipul Sharma (B17069), Neeraj Kumar Sharma (B17050)*

17.1 Infinite Cardinality

- *A* strict B *iff* NOT(*A* surj *B*)
- *A* surj *B* iff *B* inj *A*.
- If *A* surj *B* and *B* surj C, then *A* surj C.
- If *A* bij *B* and *B* bij C, then *A* bij C.
- *A* bij *B* iff *B* bij *A*.
- For any sets *A*, *B*, if *A* surj *B* and *B* surj *A*, then *A* bij *B*.
- For all sets *A* and *B*, *A* surj *B* OR *B* surj *A*.

17.1.1 Proving a set infinite

Let *A* be a set and $b \notin A$, then *A is infinite iff A* bij $A \cup \{\mathrm{B}\}$.

17.1.2 Countable Sets

- A set *C* is countable *iff* its elements can be listed in order, that is, the elements in *C* are precisely the elements in the sequence c_0, c_1, ... c_n
- This is same as saying that the function, *f* : N $\rightarrow$ C defined by the rule that $f(i) = c_i$ is a bijection.
- **Definition:** A set C is countably infinite *iff* N bij *C*. A set is countable *iff* it is finite or countably infinite. A set is uncountable *iff* it is not countable.
  The most fundamental countably infinite set is the set N itself. But the set Z of all integers is also countably infinite, because the integers can be listed in the order:

  0, -1, 1, -2, 2, -3, 3......

  In this case, there is a simple formula for the nth element of this list. That is, the bijection *f* : N $\rightarrow$ Z such that *f(n)* is the nth element of the list can be written as:

$$f(n) = \begin{cases} \text{n}/2 & \text{if n is even} \\ \text{-}(n+1)/2 & \text{if n is odd} \end{cases}$$

- If A is an infinite set, and B is countable, then A surj B.
- **Theorem:** The Cartesian product N x N is a countably infinite set.
  Proof: Consider the elements of N x N put in a table in the following way:

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6) ...
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6) ...
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6) ...
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6) ...
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6) ...
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  So, the pair (a,b) in N x N occupies the row a and the column b. First we visit the first element in the first row (1,1). Then we visit the first element in the second row (2,1) and the second element in the first row (1,2). Then we visit the first element in the third row (3,1), the second element in the second row (2,2) and then the third element in the first row (3,1)... and so on. Thus we show that N x N is countably infinite. This process of counting is called **dovetailing**. What matters here is that N is countably infinite (by its definition - this is the set of all natural numbers) so we can order its element and build a table with rows and columns corresponding to that order.

17.1.3 Power Sets are much bigger

Cantors astonishing discovery was that not all infinite sets are the same size. In particular, he proved that for any set A the power set pow(A) is strictly bigger than A.
Theorem : For any set A, A strict pow(A).

- pow(N) is uncountable
- $\{ 0, 1 \}^{\omega}$ is uncountable.
- If U is an uncountable set and A surj U, then A is uncountable.
- If C is a countable set and C surj A, then A is countable.
- The set R of real numbers is uncountable.
- N strict pow(N) strict pow(pow(N)) strict pow(pow(pow(N))) strict ....