Lecture 22: October 3

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22.1 Sum and Asymptotics

Sums and products arise regularly in the analysis of algorithms, financial applications, physical problems, and probabilistic systems. For example,

$$1 + 2 + 3 + 4 + ... + n = n(n-1)/2 \tag{22.1}$$

The following summation formula can be calculated using Gauss's method which he applied when he was in school.

Theorem 1 $\Sigma i = n(n-1)/2$, $\forall i \in N$

Proof: We will prove this using Gauss's method.

Let $S = \Sigma i$, $\forall i \in N$

$S = 1 + 2 + 3 + ... + n$

$S = n + (n-1) + (n-2) + ... + 1$ Adding both the equations we get,

$\Rightarrow 2S = n(n+1)$

$\Rightarrow S = n(n+1)/2$

Hence proved

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22.1.1 The Perturbation Method

Given a sum that has a nice structure, it is often useful to "perturb" the sum so that we can somehow combine the sum with the perturbation to get something much simpler. For example, suppose

$$S = 1 + x + x^2 + x^3 + ... + x^n \tag{22.2}$$

An example of a perturbation would be:

$$xS = x + x^2 + x^3 + x^4 + ... + x^{n+1} \tag{22.3}$$

On subtracting equation 10.3 from equation 10.2, we get

$$S - Sx = 1 - x^{n+1} \tag{22.4}$$

$\Rightarrow S = (1 - x^{n+1})/(1 - x)$

These methods are used in generating functions.

22.2 Approximating Sums

Defining increasing and decreasing functions,

1. A function $f : R^+ \to R^+$ is strictly increasing if, $x < y \Rightarrow f(x) < f(y)$
2. and it is weakly increasing when, $x < y \Rightarrow f(x) \leq f(y)$
3. Similarly, f is strictly decreasing when, $x < y \Rightarrow f(x) > f(y)$
4. and it is weakly decreasing when, $x < y \Rightarrow f(x) \geq f(y)$

Theorem 2 *Let f $f : R^+ \to R^+$ be a weakly increasing function, define*

$$S = \sum_{i=1}^{n} f(i) \tag{22.5}$$

and,

$$I = \int_{i=1}^{n} f(i)dx \tag{22.6}$$

Then,

$$I + f(1) \leq S \leq I + f(n) \tag{22.7}$$

References

[1] Eric Lehman , F Thomson Leighton, Albert R Meyer "Mathematics for Computer Science," *Chapter 14,Sums and Asymptotics*, 2017, pp. 563–573