1 Maclaurin Series

Sometimes we need to approximate complicated functions as polynomials. Maclaurin series provides a nice tool to accomplish this. In essence, it provides us an infinite series. However, we choose a couple of first terms to approximate the function. Before talking about the technique, let's have an example. The Maclaurin series representation of the function $f(x) = e^x$ is as follows:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

If I expand this infinite series, and write some first terms, I have

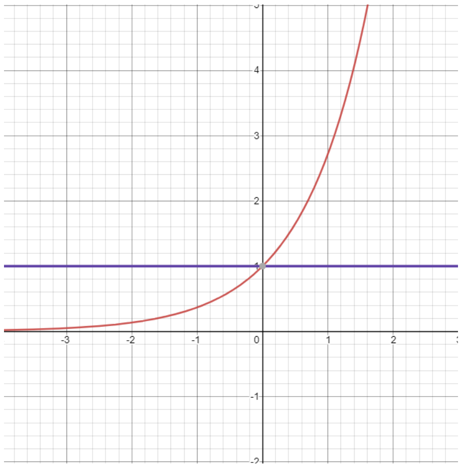
$$e^x = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots$$

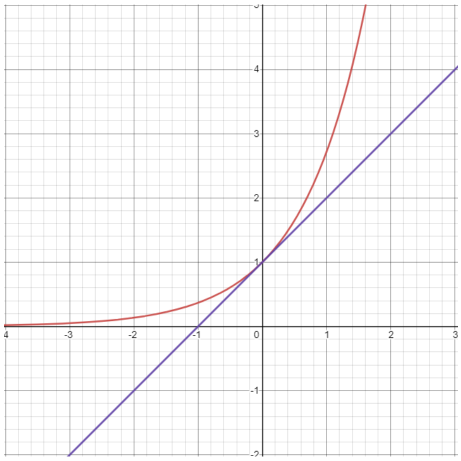
This expansion, gives us a tool to approximate e^x. Here are some approximations:

- A constant approximation by choosing only the first term: $e^x \approx 1$
- A linear approximation by choosing the first two terms: $e^x \approx 1 + x$
- A quadratic approximation by choosing the first three terms: $e^x \approx 1 + x + \frac{x^2}{2}$
- A cubic approximation by choosing the first four terms: $e^x \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$
- and so forth.

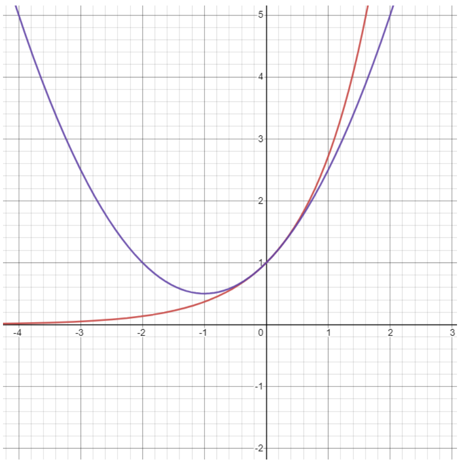
Let's see what are these approximations.Pay attention how good is the approximation wehen we are close to zero. The red curve is $y = e^x$, and the the blue curve is the approximation.



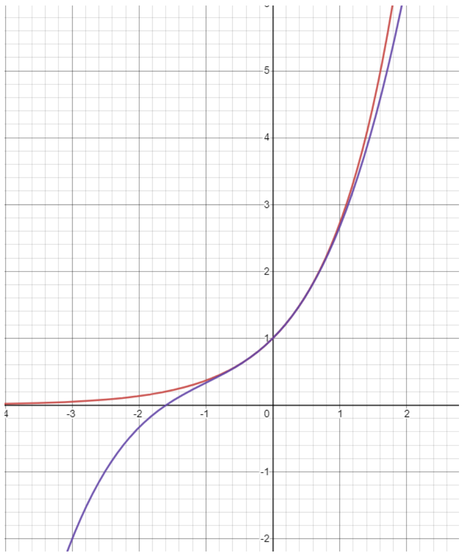
(a) Constant approximation



(b) Linear approximation



(c) Quadratic approximation



(d) Cubic

Figure 1

For a function $y = f(x)$, we use the following formula to find its Maclaurin series:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

where $f^{(n)}(0)$ is the n^{th} derivative of the function calculated at 0. In other words,

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \dots$$

Let's find the Maclaurin series of $y = e^x$ as an example. If I plug these values in the formula:

Order of derivative	derivative	derivative at zero
0	$f(x) = e^x$	$f(0) = 1$
1	$f'(x) = e^x$	$f'(0) = 1$
2	$f''(x) = e^x$	$f''(0) = 1$
3	$f^{(3)}(x) = e^x$	$f^{(3)}(0) = 1$

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \dots$$

$$e^x = 1 + \frac{1}{1}x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots$$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots$$

Let's do another example and find the Maclaurin series of $f(x) = \sin(x)$.

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \dots$$

Order of derivative	derivative	derivative at zero
0	$f(x) = \sin(x)$	$f(0) = 0$
1	$f'(x) = \cos(x)$	$f'(0) = 1$
2	$f''(x) = -\sin(x)$	$f''(0) = 0$
3	$f^{(3)}(x) = -\cos(x)$	$f^{(3)}(0) = -1$
4	$f^{(4)}(x) = \sin(x)$	$f^{(4)}(0) = 0$
5	$f^{(5)}(x) = \cos(x)$	$f^{(5)}(0) = 1$
6	$f^{(6)}(x) = -\sin(x)$	$f^{(6)}(0) = 0$
7	$f^{(7)}(x) = -\cos(x)$	$f^{(7)}(0) = -1$

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \dots$$

$$\sin(x) = 0 + \frac{1}{1!}x + \frac{0}{2!}x^2 + \frac{(-1)}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 + \frac{0}{6!}x^6 + \frac{(-1)}{7!}x^7 + \dots$$

$$\sin(x) = 0 + x + 0x^2 - \frac{x^3}{3!} + 0x^4 + \frac{x^5}{5!} + 0x^6 - \frac{x^7}{7!} + \dots$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

If you want to approximate $\sin(x)$ using three first nonzero terms, then we have

$$\sin(x) \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}$$