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Q1.) i) $\{-2, -1, 0, 1, 2\}$ Prob = $1/5$, Construct joint Prob distribution table: $Y = X^2$

Soln →

X \ Y	4	1	0	1	4	Prob (X)
−2	$1/25$	$1/25$	$1/25$	$1/25$	$1/25$	$1/5$
−1	$1/25$	$1/25$	$1/25$	$1/25$	$1/25$	$1/5$
0	$1/25$	$1/25$	$1/25$	$1/25$	$1/25$	$1/5$
1	$1/25$	$1/25$	$1/25$	$1/25$	$1/25$	$1/5$
2	$1/25$	$1/25$	$1/25$	$1/25$	$1/25$	$1/5$
Prob(Y)	$5/25$	$5/25$	$5/25$	$5/25$	$5/25$	1

1 $\Sigma P(X)$

$\Sigma P(Y)$

ii) X & Y independent?

For X & Y to be independent $P(X = x \cap Y = y) = P(X = x)\, P(Y = Y)$

$\Rightarrow P(-2, 0) = 1/25$

$P(X = -2) = 1/5$ $P(Y = 0) = 1/5$

$\Rightarrow 1/5 \times 1/5 = 1/25 \Rightarrow P(X = x \cap Y = y) = P(X = x)\, P(Y = Y)$

$\Rightarrow$ X and Y are independent.

Q.2) i)

When we say that a test is 90% accurate, then we are talking about the accuracy of the test i.e. how often it can correctly predict true responses as true and false responses as false. Eg:

Given that

$P(\text{Test}) = 90\% \Rightarrow P'(\text{Test})$ [Inaccurate] $= 100-90 = 10\%$

Given that 2% of the population is infected: $\frac{2}{100} = 0.02$

This also means that 98% of pop^n is not infected $= \frac{98}{100} = 0.98$ ← Healthy

This means that the test was accurate on 90% of both these cases. i.e.

~~True Negative Case~~ Test is -ve and person is -ve $= 90\%$ of $98\% = 88.2\%$ (TN)

True Positive Case $= 90\%$ of 2% of pop^n $= 1.8\%$ (TP)
↳ Test is +ve & person is +ve

These two are called True Negative and True Positive respectively. From this we can get False Positive (for Tests) and false [wrong] Negative.

FP = When Test is +ve but person is not infected (-ve)
$= 98 - 88.2 = 9.8\%$

FN = When Test is -ve but person is infected $= 2 - 1.8 = 0.2\%$

So a 90% accurate test has [FP: 9.8%, FN: 0.2%, TP: 1.8%, TN: 88.2%]

2 ii) If a person tests positive then the probability of the person actually having the disease can be denoted as

P(Infected | Positive Test) or P (Infected | Test)

Then according to Bayes Theorem :

$$P(\text{Infected} \mid \text{Positive Test}) \text{ or } P(\text{Infected} \mid \text{Test}) = \frac{P(\text{Positive Test} \mid \text{Infected}) \times P(\text{Infected})}{P(\text{Test}) \times P(\text{Infected}) + P'(\text{Test}') \times P(\text{Healthy})}$$

we know :

P (Positive Test | Infected) = 90% = 0.9

P (Infected) = 2% = 0.02 P (Healthy) = 98% = 0.98

P (Test) = 90% , P' (Test) = 10% = 0.01

Same = 0.9 of tests | of ~~population~~ tests

The denominator can be written as the sum of

FP + TP = (9.8 + 1.8)% = 11.6% [P (Positive Test)]

Now plugging in the values

$$= \left(\frac{90 \times 2}{11.6} \right)\% = \frac{180}{11.6}\%$$

= 15.5% probability of actually having the disease when test is positive. [0.155 probability]

iii) Probability that a patient is misclassified.
Misclassification is when Test is negative and person is infected and Test is positive and person is healthy.
This is nothing but the inaccuracy of the tests.
or $P'(\text{Test})$ ⇒ $P(\text{Test}) = 90\%$

⇒ $P'(\text{Test}) = 100 - 90 = 10\%$.

$P'(\text{Test}) = 0.1$

iv) Given that both the tests are +ve, probability that ~~the~~ the person is infected is.

$0.15 \rightarrow$ Probability of first positive.

As the two events are independent

⇒ The total probability = ~~2 × P(In~~

$= 2 \times P(\text{Infected} \mid \text{Positive Test})$

$= 2 \times 15.5\%$

$= 31\%$

v) Cheaper test Accuracy = 80%.

TP = 1.6% [80% of 2%]

FP = 20% of 98 = 19.6%.

$P(\text{Infected} \mid \text{Cheap Test}) = \frac{80 \times 2}{21.2} = \frac{160}{21.2} = 7.5\%$

⇒ Total prob = ~~0.~~15.5 + 7.5 = 23 %

3.) $f(x) = \begin{cases} kx^2 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$

i) Since $f(x)$ is a Probability density function

we can say, $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_0^1 Kx^2\,dx = 1$$

$$\Rightarrow K \frac{x^3}{3}\Big]_0^1 = 1$$

$$\Rightarrow \frac{1}{3}K = 1$$

$$\Rightarrow K = 3$$

ii) Calculate, $P(X \le 1/2)$

$$\Rightarrow P(X \le 1/2) = \int_{-\infty}^{1/2} 3x^2\,dx$$

$$= \int_0^{1/2} 3x^2\,dx$$

$$= 3\int_0^{1/2} x^2\,dx$$

$$= 3\left[\frac{x^3}{3}\right]_0^{1/2} = 3 \times \frac{1}{3}\left[x^3\right]_0^{1/2}$$

$$= 1/8$$

(m) Cumulative distribution of X?

for a continuous random variable.

$$F_x(a) = P(X \le a) = \int_{-\infty}^{a} f_x(x)\, dx$$

for $x < 0$

$$F(x) = \int_{-\infty}^{x} 0\, dt = 0$$

for $x > 1$

$$F(x) = \int_{-\infty}^{x} f(t)\, dt = 1$$

for $0 \le x \le 1$

$$F(x) = \int_{0}^{x} 3t^2\, dt = 3\frac{t^3}{3}\Big]_0^x = \underline{\underline{x^3}}$$

$$\therefore F(x) = \begin{cases} 0 & \text{for } x < 0 \\ x^3 & \text{for } 0 \le x \le 1 \\ 1 & \text{for } x > 1 \end{cases}$$

iv) $f(1/2) = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$

The value from CDF is same as that of PDF.

4.i) Classify: $f(x,y) = x^3 + y^3 + 3x^2 - 3y^2 - 8$ critical points.

Soln: To classify & finding critical points we need partial derivatives.

$$\frac{\partial}{\partial x}(x^3 + y^3 + 3x^2 - 3y^2 - 8)$$

$$= 6x + 3x^2 \quad \text{—— i)}$$

$$\frac{\partial}{\partial y}(x^3 + y^3 + 3x^2 - 3y^2 - 8)$$

$$= -6y + 3y^2 \quad \text{—— ii)}$$

Equating i) & ii) with 0

$$3x^2 + 6x = 0$$
$$3y^2 - 6y = 0$$

$\Rightarrow 3x(x + 2) = 0 \Rightarrow x = 0$ or $x = -2$

$\Rightarrow 3y(y - 2) = 0 \Rightarrow y = 0$ or $y = 2$

for all combinations of x & y, we have the critical points as:

$(0,0), (0,2), (-2,0), (-2,2)$

To classify these points we need to find the 2nd partial derivatives

$f_{xx} = 6x+6 \qquad f_{xy} = 0$

$f_{yy} = 6y-6$

According to 2nd derivative test:

$$D = D(a,b) = f_{xx}(a,b)\, f_{yy}(a,b) - [f(x,y)(a,b)]^2$$

$$D = (6x+6)(6y-6) - 0 \qquad -\text{iii)}$$

$$= 6xy - 36x + 36y - 36$$

Plugging critical points into D.

~~a) at $(0,0)$ $D(0,0) = -36 < 0$ saddle point.~~

~~b) at $(0,2)$ $D(0,2) = 36 > 0$ min point.~~

~~c) at $(-2,0)$ $D(-2,0) = 36 > 0$ min point~~

~~d) at $(-2,2)$ $D(-2,2) = -24 + 72 + 72 - 36$~~

a) at $(0,0)$ $D(0,0) = -36 < 0$ $\boxed{\text{saddle point}}$

f doesn't have max/min at this point

b) at $(0,2)$ $D(0,2) = 36 > 0$ $\boxed{\text{min point}}$.

$f_{xx}(a,b) = f_{xx}(0,2) = 6 > 0$

c) at $(-2,0)$ $D(-2,0) = 36$ $\boxed{\text{max point}}$

$f_{xx}(a,b) = f_{xx}(-2,0) = -6 < 0$

d) at $(-2, 2)$

$D(-2,2) = -36 < 0$ saddle point.

ii) $x^2 + kxy + y^2$ has a local minimum at $(0,0)$

finding partial derivatives.

$$\frac{\partial}{\partial x}(x^2 + kxy + y^2)$$

$$= 2x + ky$$

$$\frac{\partial}{\partial y}(x^2 + kxy + y^2) = 2y + kx$$

As we know that we have to find min. $\Rightarrow$ finding 2nd order derivatives.

$f_{xx} = 2$, $f_{yy} = 2$ $f_{xy} = k$

Given : local minimum at $(0,0)$

$\Rightarrow D(0,0) > 0$ & $f_{xx} > 0$

Now $D = f_{xx}(a,b)\, f_{yy}(a,b) - [f_{xy}(a,b)]^2$

$= 4 - k^2$

$\Rightarrow 4 - k^2 > 0$

$$\boxed{\Rightarrow k^2 < 4 \Rightarrow k \in (-2, 2)}$$

$f(x,y) = 4x + 4y - x^2 - y^2$ subject to constraint $x^2 + y^2 = 2$.

finding points which satisfy

$$\nabla f(x,y) = \lambda \nabla g(x,y)$$

$$\& \; g(x,y) = K.$$

Here $g(x,y) = x^2 + y^2$ & $K = 2$.

Now, let us consider the following system:

$$\begin{matrix} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g(x,y) = K \end{matrix} \longrightarrow \begin{cases} -2x + 4 = 2\lambda x & - 1) \\ -2y + 4 = 2\lambda y & - 2) \\ x^2 + y^2 = 2 & - 3) \end{cases}$$

$\Rightarrow 2x + 2\lambda x = 4$

$\Rightarrow x + \lambda x = 2 \Rightarrow \cancel{x} \, x(1+\lambda) = 2$

$\Rightarrow \lambda = 1$

In (2)

$-2y + 4 = 2y$

$\Rightarrow 4y = 4 \Rightarrow y = 1$

In (3)

$x^2 + 1 = 2$

$\Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

$x = -1$ in (3)

$1 + y^2 = 2$

$\Rightarrow y = \pm 1$

$x = 1$ in (3)

$1 + y^2 = 2$

$\Rightarrow y^{\cancel{2}} = \pm 1$

The ~~critical~~ extreme points are : $(1,1)$ $(-1,-1)$, $(1,-1)$, $(-1,1)$

At $(1,1)$ $f(x,y) = 6$ [Max]

At $(1,-1)$ $f(x,y) = -2$ [Neither min/max]

At $(-1,-1)$ $f(x,y) = -10$ [Min]

At $(-1,1)$ $f(x,y) = -2$ [Neither min/max]

iv) $f(x,y,z) = x^2+y^2+z^2$

$$2x + 2y - 4z = 0$$
$$4x - 2y + 2z = 0$$

As this is a 3 variable function, we need to solve

$$\begin{cases} \nabla f(x,y,z) = \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z) \\ g(x,y,z) = 0 \\ h(x,y,z) = 0 \end{cases}$$

where $g(x,y,z) = 2x + 2y - 4z$

$h(x,y,z) = 4x - 2y + 2z$

From this we get following system

$$f_x = \lambda g_x + \mu h_x$$
$$f_y = \lambda g_y + \mu h_y$$
$$f_z = \lambda g_z + \mu h_z \longrightarrow$$
$$g(x,y,z) = 0$$
$$h(x,y,z) = 0$$

$$\begin{cases} 2x = 2\lambda + 4\mu & \text{— 1} \\ 2y = 2\lambda - 2\mu & \text{— 2} \\ 2z = -4\lambda + 2\mu & \text{— 3} \\ 2x + 2y - 2z = 0 & \text{— 4} \\ 4x - 2y + 2z = 0 & \text{— 5} \end{cases}$$

From 1

$$x = \lambda + 2\mu$$

From 2

$$y = \lambda - 2\mu$$

From 3

$$z = -2\lambda + \mu$$

In 4 $x = \lambda + 2\mu$, $y = \lambda - 2\mu$, $z = -2\lambda + \mu$

$$2\lambda + 4\mu + 2\lambda - 4\mu + 4\lambda - 2\mu = 0 \quad \text{— 6}$$

In 5 $x = \lambda + 2\mu$, $y = \lambda - 2\mu$, $z = -2\lambda + \mu$

$$4\lambda + 8\mu - 2\lambda + 4\mu - 4\lambda + 2\mu = 0 \quad \text{— 7}$$

From 6 & 7

$$8\lambda - 2\mu = 0$$
$$-2\lambda + 14\mu = 0$$

[Multiply 7 and add]

$$54\lambda = 0 \Rightarrow \boxed{\lambda = 0}$$

$$\Rightarrow 8\times 0 - 2\mu = 0 \Rightarrow \boxed{\mu = 0}$$

$\Rightarrow$ We have,

$\lambda = 0, \mu = 0, x = 0, y = 0, z = 0.$

Only one extreme point $(x, y, z) = (0, 0, 0)$.

At $(0,0,0)$ $f(x, y, z) = 0$

Now as we know,

$x^2 + y^2 + z^2$ is an equation a sphere.

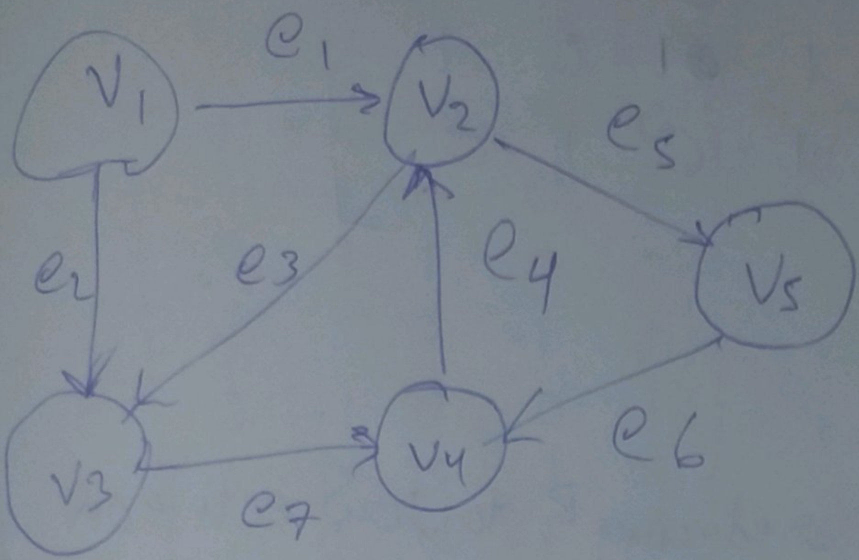
We also can say that from a general equation of sphere

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$$

(a, b, c) represent the center of the sphere.

$\Rightarrow$ Here $a, b, c = 0$, then the extreme point here is the center of sphere defined by $f(x, y, z)$

$$\boxed{\text{Also, since there is no upper bound for the function } f(x,y,z) \Rightarrow \text{the extreme point is a Minimum.}}$$



Adjacency matrix

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Incidence matrix

$$B = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{array} \\ \begin{bmatrix} -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \end{bmatrix} \end{array}$$

ii)

$$C = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

iii) Rank (B)

To find rank, reducing B to row echleon form

$R_2 \leftarrow R_2 + R_1$

$R_3 \leftarrow R_2 + R_3$

$$\begin{bmatrix} -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \end{bmatrix}$$

$R_4 \leftarrow R_3 + R_4$

$R_5 \leftarrow R_4 + R_5$

$$\begin{bmatrix} -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

To find rank we check for non-zero matrix in REF, in this case we have 4 non-zero rows so rank(B) = 4.

iv) $B^T = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 & 0 \end{bmatrix}$ $\vec{1} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

multiplying these two matrices we get

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \vec{0}$$

As we get $B^T \vec{1} = \vec{0}$

$\Rightarrow BB^T\vec{1} = 0$

from this, we can see that $BB^T\vec{1} = \vec{0}$ is in form of $A\vec{x} = \lambda\vec{x} \Rightarrow (A - \lambda I)\vec{x} = \vec{0}$

from this, we can also say that:

$\vec{1}$ is an eigenvector of BB^T and eigenvalue for this would be $= 0$. $(\lambda = 0)$. This also means that this eigenvector is in null space.

v) $L = BB^T$

$$L = \begin{bmatrix} -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 & 0 \end{bmatrix}$$

$$L = \begin{bmatrix} 2 & -1 & -1 & 0 & 0 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 3 & -1 & 0 \\ 0 & -1 & -1 & 3 & -1 \\ 0 & -1 & 0 & -1 & 2 \end{bmatrix}$$

Now

$L = D - C$,

Finding D: [As we have to subtract then $\dim(C) = \dim(D)$]

$$D = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

Now

$$D - C = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 & -1 & 0 & 0 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 3 & -1 & 0 \\ 0 & -1 & -1 & 3 & -1 \\ 0 & -1 & 0 & -1 & 2 \end{bmatrix}$$ which is $= L$

vi) $\det(L) \Rightarrow$

We know $L = BB^T$

Now $Rank(B) = 4 \Rightarrow Rank(B^T) = 4$.

Now, According to the property that:

$$\boxed{\begin{aligned} &rank(A,B) \leq \min(rank(A), rank(B)) \\ \Rightarrow\ &Rank(L) \leq \min(Rank(B), Rank(B^T)) \end{aligned}}$$

$Rank(L) \leq \min(4, 4)$.

$Rank(L) \leq 4$, this means that matrix L is not a full rank matrix.

$$\boxed{\Rightarrow \det(L) = 0.}$$

vii) As the rank $(L) \leq 4$ and one eigenvalue is 0 then to have a Rank of 4 it has to have 4 non zero eigenvalues. Also the matrix being a semidefinite Matrix shows that all the eigenvalues are non-negative as well with atleast one eigenvalue $= 0$.

viii) Show one of eigen values of L is 0 & eigenvector is $\vec{1}$.

Similar to part iv)

We know,

$$L = BB^T$$

$\Rightarrow$ For eigenvectors we need to write the matrix in equation $A\vec{x} = \lambda\vec{x}$

Here, we know

$$BB^T\vec{1} = \vec{0}$$

$$\Rightarrow L\vec{1} = \vec{0}$$

This is in form of $A\vec{x} = \lambda\vec{x}$ [$(A-\lambda)\vec{x} = 0$.

$$\Rightarrow (L-\lambda)\vec{1} = \vec{0}. \quad \text{— i)}$$

We can see that if $\lambda = 0$ then, the equation i) is satisfied.

$\Rightarrow \lambda$ is an eigenvalue & $\vec{1}$ is the corresponding eigenvector.