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Assignment 2

Q.1: $u = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$ $v = \begin{bmatrix} 2 \\ 2 \\ -2 \\ 4 \end{bmatrix}$ & $w = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -2 \end{bmatrix}$

i) l_2 norm

$$\sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} = dist(x, y)$$

$$dist(\vec{u}, \vec{v}) = \|\vec{v} - \vec{u}\| = \left\| \begin{matrix} 2-1 \\ 2-2 \\ -2-0 \\ 4-1 \end{matrix} \right\| = \left\| \begin{bmatrix} 1 \\ 0 \\ -2 \\ 3 \end{bmatrix} \right\|$$

$$= \sqrt{1^2 + 0^2 + (-2)^2 + (3^2)} = \sqrt{14} \simeq 3.7$$

~~dist(u, w)~~ $dist(\vec{v}, \vec{w}) = \|\vec{w} - \vec{v}\| = \left\| \begin{matrix} 1-2 \\ 2-2 \\ 0-(-2) \\ -2-4 \end{matrix} \right\|$

$$= \left\| \begin{bmatrix} -1 \\ 0 \\ 2 \\ -6 \end{bmatrix} \right\|$$

$$= \sqrt{-1^2 + 0^2 + 2^2 + (-6)^2} = \sqrt{41} \simeq 6.4$$

$dist(\vec{u}, \vec{w}) = \|\vec{w} - \vec{u}\| = 3 \rightarrow$ Most similar (u and w)

ii) l_1 norm

$$dist(x,y) = \sum_{i=1}^{n} |x_i - y_i|$$

$$dist(\vec{u},\vec{v}) = |\vec{v} - \vec{u}|_1$$

$$= |3| + |-2| + |0| + |1| = 6$$

$$dist(\vec{v},\vec{w}) = |\vec{w} - \vec{v}|_1$$

$$= |-6| + |2| + |0| + |-1| = 9$$

$$dist(\vec{u},\vec{w}) = |\vec{w} - \vec{u}|_1$$

$= 3 \rightarrow$ most similar. (u and w).

iii) l_∞ norm $= \max_{i=1,\dots n} |x_i - y_i|$

$dist(\vec{u},\vec{v}) = 3$

$dist(\vec{v},\vec{w}) = 6$

$dist(\vec{u},\vec{w}) = 3$

* In this norm ($\vec{u}$ & $\vec{v}$) and ($\vec{u}$ & $\vec{w}$) are equally si

ii) Cosine similarity

$$\cos(\theta) = \frac{x \cdot y}{\|x\| \|y\|}$$

$$\cos(\theta_{u,v}) = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} = \frac{10}{2\sqrt{7}\sqrt{6}} = \frac{5}{\sqrt{42}} \approx 0.22$$

$$\theta_{u,v} \simeq 76.7^\circ$$

$$\cos(\theta_{v,w}) = \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \|\vec{w}\|} = \frac{14}{2\sqrt{7} \cdot 3} = 0.88$$

$$\theta_{v,w} = 28^\circ$$

$$\cos(\theta_{u,w}) = \frac{\vec{u} \cdot \vec{w}}{\|\vec{w}\| \|\vec{u}\|} = \frac{1}{\sqrt{6}} = 0.59$$

$$\theta_{u,w} = 93^\circ$$

∴ Vectors <u>v</u> & <u>w</u> are most similar as the angle with each other is smaller than others.

iii) Cosine similarity considers unit length vector to calculate dot product, whereas the other distances consider the magnitude of the vectors $|x|$ as well. This results in difference between results from cosine similarity and results from other methods.

iv) Yes. The difference can be resolved if we take magnitude out of the picture. i.e. If we normalize the input vectors then the diffe between cosine ~~sta~~ similarity and other dista will be minimal.

For eg. if we consider L_2 normalized versions of the given vectors then:

$$\vec{u}_{L_2 norm} = \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 0 \\ 1/\sqrt{6} \end{bmatrix} \simeq \begin{bmatrix} 0.4 \\ 0.8 \\ 0 \\ 0.4 \end{bmatrix}$$

$$\vec{V}_{L_2 norm} = \begin{bmatrix} \cancel{1/2}\sqrt{7} \\ 1/\sqrt{7} \\ -1/7 \\ 2/\sqrt{7} \end{bmatrix} \simeq \begin{bmatrix} 0.37 \\ 0.37 \\ -0.37 \\ 0.75 \end{bmatrix}$$

$$\vec{\omega}_{L_2 norm} = \begin{bmatrix} 1/3 \\ 2/3 \\ 0 \\ -2/3 \end{bmatrix} \simeq \begin{bmatrix} 0.33 \\ 0.66 \\ 0 \\ -0.66 \end{bmatrix}$$

Now if we compute euclidean (L2) distance:

$\underline{dist(\vec{u},\vec{v})} = 0.66$ $\underline{dist(\vec{u},\vec{\omega})} = 1.07$

$\underline{dist(\vec{v},\vec{\omega})} = 1.48$

we can see that the distances are now much less apart. This same behavior holds true for other norms as well.

2.) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ h \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ K \end{bmatrix} \right\}$

Soln. For the vectors to be independent all linear combination of the vectors must sum up to 0.

i.e.

$$c_1 \vec{v_1} + c_2 \vec{v_2} + c_3 \vec{v_3} = \vec{0}$$

Now if $c_1 = c_2 = c_3 = 0$ then the set of vectors are linealy independent.

$$c_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ h \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 1 \\ K \end{bmatrix} = \vec{0}$$

$$\Rightarrow \begin{bmatrix} c_1 + c_2 + 0 \cdot c_3 \\ 2c_1 + hc_2 + c_3 \\ c_1 + c_2 + Kc_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \begin{array}{l} \text{—— i)} \\ \text{—— ii)} \\ \text{—— iii)} \end{array}$$

Now from i) we have

$$c_1 + c_2 = 0$$

putting this value in iii) we get

$$Kc_3 = 0 \Rightarrow \boxed{c_3 = 0}$$

putting $C_3 = 0$ in ii)

$$2C_1 + hC_2 = 0$$

$$\Rightarrow 2C_1 = -hC_2$$

putting $C_1 = -C_2$ from i) in this

we have,

$$C_2(h-2) = 0$$

for $h = 2$ we will have $0 = 0$

for all other h we have $\boxed{C_2 = 0}$

$\Rightarrow$ for $h \neq 2$ we can say $C_2 = 0$

putting in i)

$$\left.\begin{array}{l} C_1 + C_2 = 0 \\ C_1 + 0 = 0 \\ \boxed{C_1 = 0} \end{array}\right\} h \neq 2$$

Now for all $\{k \in \mathbb{R} \ \& \ h \in \mathbb{R} / h \neq 2\}$

we can say

$$C_1 = C_2 = C_3 = 0$$

For this condition all the combinations of vectors will be linealy independent.

3.) $A = \begin{bmatrix} -1 & 0 & a \\ b & 4 & c \\ d & 0 & 0 \end{bmatrix}$

Solⁿ: Given rank(A) = 2.

As we have rank (A) = 2, we have two independent vectors in rowspace/columnspace.

Also, we have two non-zero ~~eigenvectors~~ eigenvalues & one zero eigenvalue.

This will result in 3 distinct eigenvalues [Two nonzero & 1 zero]

4.) $A = \begin{bmatrix} 3 & -3 & 0 \\ 3 & -1 & 2 \\ b & 0 & 2 \end{bmatrix}$

i) find b when determinant of A is 4.

For determinant along first row, we have,

$$\text{Det}(A) = 3\begin{vmatrix} -1 & 2 \\ 0 & 2 \end{vmatrix} + 3\begin{vmatrix} 3 & 2 \\ b & 2 \end{vmatrix} + 0$$

$$= -6 + 18 - 6b = 12 - 6b$$

Now as Det (A) = 4

$\Rightarrow$ $12 - 6b = 4$

$$\boxed{b = \frac{8}{6} = \frac{4}{3}}$$

ii) Rank (A) = 2

We know that rank (A) = no. of pivots in row echleon form.

$\Rightarrow$ for REF we do

$$R_2 \leftarrow R_2 - R_1$$
$$R_3 \leftarrow R_3 + \left(-\frac{b}{3} R_1\right)$$

$$= \begin{bmatrix} 3 & -3 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & -b+2 \end{bmatrix}$$

$\because$ Rank (A) = 2 & we already have two non-zero pivots

$\Rightarrow$ $-b+2 = 0$

$$\boxed{\Rightarrow b = 2.}$$

iii) $\frac{1}{2}$ is an eigenvalue of A^{-1}

$A\vec{x} = \lambda\vec{x}$ is the equation which represents λ as an eigenvalue of A & x is the corresponding eigenvector

from this we can say,

$\vec{x} = A^{-1}(\lambda\vec{x})$ $\quad [A^{-1}Ax = \lambda A^{-1}\vec{x}]$

$\Rightarrow$ $A^{-1}\vec{x} = \frac{1}{\lambda}\vec{x}$

From this, we can say that for inverse of a matrix we have eigen value λ $\Rightarrow$ for the matrix the eigen value will be λ^{-1}.

$\Rightarrow$ 2 is an eigenvalue of A

$(A - \lambda I)\vec{x} = 0$ form:

$$\begin{bmatrix} 3 & -3 & 0 \\ 3 & -1 & 2 \\ b & 0 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\lambda = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -\lambda+3 & -3 & 0 \\ 3 & \lambda-1 & 2 \\ b & 0 & -\lambda+2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Finding the det $(A - \lambda I)$ & putting $\lambda = 2$ we get:

$$-6b - \lambda^3 + 4\lambda^2 - 10 + 12 \quad (\lambda = 2)$$

$$-6b = 0$$

$$\boxed{\Rightarrow b = 0}$$

iv) $Ax = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ has no solutions.

$$\Rightarrow \begin{bmatrix} 3 & -3 & 0 \\ 3 & -1 & 2 \\ b & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Transforming into augmented matrix form for Gaussian elimination.

$$\left[\begin{array}{ccc|c} 3 & -3 & 0 & 1 \\ 3 & -1 & 2 & 0 \\ b & 0 & 2 & 0 \end{array}\right]$$

$R_2 \leftarrow R_2 - R_1$

$$\left[\begin{array}{ccc|c} 3 & -3 & 0 & 1 \\ 0 & 2 & 2 & -1 \\ b & 0 & 2 & 0 \end{array}\right]$$

$R_3 \leftarrow R_3 - R_1 * \frac{b}{3}$

$$\left[\begin{array}{ccc|c} 3 & -3 & 0 & 1 \\ 0 & 2 & 2 & -1 \\ 0 & b & 2 & -b/3 \end{array}\right]$$

$R_3 \leftarrow 3R_3$

$R_3 \leftarrow \cancel{\frac{3b}{2} R_2}\ R_3 - \frac{3b}{2} R_2$

$$\left[\begin{array}{ccc|c} 3 & -3 & 0 & 1 \\ 0 & 2 & 2 & -1 \\ 0 & 0 & -3b+6 & \frac{b}{2} \end{array}\right]$$

$R_1 \leftarrow R_1/3 \; ; \; R_2 \leftarrow R_2/2$

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 1/3 \\ 0 & 1 & 1 & -1/2 \\ 0 & 0 & -3b+6 & b/2 \end{array}\right]$$

for this system to have no solution we must have.

$-3b + 6 = 0$

$\Rightarrow b = 6/3 = 2$

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 1/3 \\ 0 & 1 & 1 & -1/2 \\ 0 & 0 & 0 & 1 \end{array}\right] \quad \therefore \underline{\text{No solution}}$$

v) $Ax = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$ has infine solutions.

Solⁿ. →

$$\left[\begin{array}{ccc|c} 3 & -3 & 0 & -3 \\ 3 & -1 & 2 & 1 \\ b & 0 & 2 & 2 \end{array}\right]$$

(Using augmented matrix form for Gaussian ~~subst~~ elimination)

$$R_2 \leftarrow R_2 - R_1$$

$$R_1 \leftarrow R_1/3, \quad R_2 \leftarrow R_2/2$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & -1 \\ 0 & 1 & 1 & 2 \\ b & 0 & 2 & 2 \end{array}\right]$$

$$R_3 \leftarrow R_3 - bR_1$$

$$R_3 \leftarrow R_3 - bR_2$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -b+2 & -b+2 \end{array}\right] \Leftarrow \left(\left[\begin{array}{ccc|c} \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & d \end{array}\right] \right.$$

This form will give inf

To have infinite solutions we must have

$$-b+2=0 \Rightarrow b=2$$

5)

$$A = \begin{bmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix}$$

$$S = A^T A$$

$$= \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 1 \\ 1 & 11 \end{bmatrix}$$

i) For finding characteristic polynomial we take,

$$\det(S - \lambda I)$$

$$= \begin{vmatrix} 11 & 1 \\ 1 & 11 \end{vmatrix} - \lambda \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} -\lambda + 11 & 1 \\ 1 & -\lambda + 11 \end{vmatrix}$$

$$\boxed{p(\lambda) = \lambda^2 - 22\lambda + 120}$$

ii) Finding eigenvalues

$$|S - \lambda I| = 0$$

(Using the calculated value)

$$\begin{vmatrix} -\lambda + 11 & 1 \\ 1 & -\lambda + 11 \end{vmatrix} = 0$$

$$\lambda^2 - 22\lambda + 120 = 0$$

$$(\lambda - 12)(\lambda - 10) = 0$$

$$\Rightarrow \lambda = 12 \text{ or } \lambda = 10$$

As $\lambda_1 \geq \lambda_2$

$$\Rightarrow \lambda_1 = 12 \ \& \ \lambda_2 = 10$$

iii) Eigenvectors

Substituting λ_1 in $|S - \lambda I| \vec{x} = 0$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right]$$ (Augmented matrix form)

$$R_2 \leftarrow R_2 + R_1$$

$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad x_1 = x_2$

Let $x_1 = 1$

we have

$x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightarrow$ Eigenvector corresponding to 12

Similarly putting λ_2 (10) in the equation we get

$x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \rightarrow$ Eigenvector for $\lambda_2 = 10$

iv) The eigenvectors are ~~orthonorm~~ orthogonal if their dot product is zero.

$x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Dot product $= \begin{bmatrix} -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = -1 \times 1 + 1 \times 1 = 0$

$\Rightarrow$ They are orthogonal.

for making them orthonormal we just normalize them:

$$V_1 = \frac{\vec{x_1}}{||\vec{x_1}||} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$v_2 = \frac{\vec{x_2}}{\|\vec{x_2}\|} = \frac{1}{\sqrt{2}}\begin{vmatrix} -1 \\ 1 \end{vmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

5.e.) Set $\sigma_1 = \sqrt{\lambda_1}$ $\quad \sigma_2 = \sqrt{\lambda_2}$

$$D = \begin{bmatrix} \sqrt{12} & 0 \\ 0 & \sqrt{10} \\ 0 & 0 \end{bmatrix}$$

$$V = [v_1\ v_2] = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$V^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

If the matrix is orthogonal, then

$VV^T = I$

$$\Rightarrow \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ $\Rightarrow$ The matrix is orthogonal.

6. i) $T = A A^T$

$$= \begin{bmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 0 & 2 \\ 0 & 10 & 4 \\ 2 & 4 & 2 \end{bmatrix}$$

a) Characteristic polynomial of T

$P(\lambda)$ [Characteristic polynomial]

$$= |T - \lambda I|$$

$$= \begin{vmatrix} 10 & 0 & 2 \\ 0 & 10 & 4 \\ 2 & 4 & 2 \end{vmatrix} - \lambda \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} -\lambda+10 & 0 & 2 \\ 0 & \lambda+10 & 4 \\ 2 & 4 & -\lambda+2 \end{vmatrix}$$

$R_1 \leftrightarrow R_3$

$R_3 \leftarrow \frac{1}{2}(\lambda-10)R_1 + R_3$

$$= -\begin{vmatrix} 2 & 4 & -\lambda+2 \\ 0 & -\lambda+10 & 4 \\ 0 & 2(\lambda-10) & -\frac{\lambda^2}{2}+6\lambda-8 \end{vmatrix}$$

$R_3 \leftarrow R_3 + R_2 \times \frac{2\lambda - 20}{10 - \lambda}$

$$= -\begin{vmatrix} 2 & 4 & -\lambda + 2 \\ 0 & -\lambda + 10 & 4 \\ 0 & 0 & -\frac{1}{2}\lambda(\lambda - 12) \end{vmatrix}$$

$$\underline{P}(\lambda) = -\frac{1}{2} \times 2\left(-\lambda(\lambda - 12)\right)(10 - \lambda))$$

$$= -\lambda^3 + 22\lambda^2 - 120\lambda$$

b.) find the eigenvalues of T, $\lambda_1 \geq \lambda_2 \geq \lambda_3$

for eigenvalues

$$\underline{P}(\lambda) = 0$$

$$\Rightarrow \lambda^3 + 22\lambda^2 - 120\lambda = 0$$

$$= -\lambda(\lambda - 12)(\lambda - 10) = 0$$

$$= \lambda(\lambda - 12)(\lambda - 10) = 0$$

$\Rightarrow \lambda_1 = 12$, $\lambda_2 = 10$ and $\lambda_3 = 0$

c) find the eigenvectors of T

$$(T - I\lambda)\vec{x} = 0$$

Substitute 12. in $(T - I\lambda)\vec{x} = 0$

$$\begin{bmatrix} -\lambda+10 & 0 & 2 \\ 0 & -\lambda+10 & 4 \\ 2 & 4 & -\lambda+2 \end{bmatrix} \vec{x} = 0$$

we get,

$$\begin{bmatrix} -2 & 0 & 2 \\ 0 & -2 & 4 \\ 2 & 4 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{— i)}$$

Substituting 10

$$\begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 4 \\ 2 & 4 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{— ii)}$$

Substituting 0

$$\begin{bmatrix} 10 & 0 & 2 \\ 0 & 10 & 4 \\ 2 & 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{— iii)}$$

from i) we get

$x_2 = 2x_1$ & $x_3 = x_1$

Using $x_1 = 1$ the eigenvector is $= \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

from ii) we get

$$x_1 = -2x_2$$

$$x_3 = 0$$

we can write

$$\begin{bmatrix} -2x_2 \\ x_2 \\ 0 \end{bmatrix}$$

$\Rightarrow$ for $x_2 = 1$

we have eigenvector for eigenvalue 10 as

$$\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

from iii) we get

$$x_1 = -\frac{x_3}{5}$$

$$x_2 = -\frac{2}{5}x_3$$

$$\Rightarrow \begin{bmatrix} -\frac{x_3}{5} \\ -\frac{2}{5}x_3 \\ x_3 \end{bmatrix}$$

for $x_3 = 5$, the eigenvector corresponding to 0

$$= \cancel{\begin{bmatrix} -1/5 \\ -2/5 \\ 1 \end{bmatrix}} \quad \begin{bmatrix} -1 \\ -2 \\ 5 \end{bmatrix}$$

$\Rightarrow$ for $\cancel{\lambda}_1 = 12 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

$\cancel{\lambda}_2 = 10 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

$\cancel{\lambda}_3 = 0 = \cancel{\begin{bmatrix} -1/5 \\ -2/5 \\ 1 \end{bmatrix}} \begin{bmatrix} -1 \\ -2 \\ 5 \end{bmatrix}$

(d.)

For the eigenvectors to be ortonormal, they have to be orthogonal

$$x_1 \cdot x_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = -2 + 2 = 0$$

$$x_2 \cdot x_3 = 2/5 - 2/5 + 0 = 0$$

$$x_3 \cdot x_1 = \quad 5 - 5 = 0$$

$\Rightarrow$ The eigenvectors are ortogonal [As dot product is zero]

$$\hat{x}_1 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$$

$$\hat{x}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix}$$

$$\hat{x}_3 = \frac{1}{\sqrt{30}} \begin{bmatrix} -1 \\ -2 \\ 5 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{30} \\ -2/\sqrt{30} \\ 5/\sqrt{30} \end{bmatrix}$$

7.) a.) $u_1 = \frac{1}{\sigma_1} A v_1 \quad , \quad u_2 = \frac{1}{\sigma_2} A v_2$

$\Rightarrow$ As $\sigma_1 = \sqrt{12}$ & $\sigma_2 = \sqrt{10}$

$$\Rightarrow u_1 = \frac{1}{\sqrt{12}} \begin{bmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$\Rightarrow u_2 = \frac{1}{\sqrt{10}} \begin{bmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$\Rightarrow U_1 = \begin{bmatrix} \sqrt{2} \\ 2\sqrt{2} \\ \sqrt{2} \end{bmatrix} \frac{1}{\sqrt{12}} , \quad U_2 = \begin{bmatrix} -2\sqrt{2} \\ \sqrt{2} \\ 0 \end{bmatrix} \times \frac{1}{\sqrt{10}}$$

$$\Rightarrow U_1 = \begin{bmatrix} \sqrt{6}/6 \\ \sqrt{6}/3 \\ \sqrt{6}/6 \end{bmatrix} , \quad U_2 = \begin{bmatrix} -2\sqrt{5}/5 \\ \sqrt{5}/5 \\ 0 \end{bmatrix}$$

Given $\|U_3\|_2 = 1$ & $U_3 \perp U_1$ $U_3 \perp U_2$

$\Rightarrow U_3 \cdot U_1 = 0$ & $U_3 \cdot U_2 = 0$

Let $U_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ $\because U_3 \cdot U_1 = 0$

$$\Rightarrow \begin{bmatrix} a \\ b \\ c \end{bmatrix} \begin{bmatrix} \sqrt{6}/6 \\ \sqrt{6}/3 \\ \sqrt{6}/6 \end{bmatrix} = 0$$

$$\Rightarrow \frac{a}{\sqrt{6}} + \sqrt{\frac{2}{3}}\, b + \frac{c}{\sqrt{6}} = 0 \quad \text{—— i)}$$

$\because U_3 \cdot U_2 = 0$

$$\begin{bmatrix} -2\sqrt{5}/5 \\ \sqrt{5}/5 \\ 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

$$\Rightarrow -\frac{b}{\sqrt{5}} - \frac{2a}{\sqrt{5}} = 0 \quad \text{—— ii)}$$

Using i) & ii)

$b = -2a$ & $c = 3a$ ⟵ iii)

We also know that $\|U_3\| = 1$

$$\sqrt{a^2+b^2+c^2} = 1$$

$$\Rightarrow \sqrt{a^2+(-2a)^2+(3a)^2} = 1$$

$$\Rightarrow a^2+4a^2+9a^2 = 1$$ [Squaring sides]

$$\Rightarrow 14a^2 = 1$$

$$\Rightarrow a = \frac{1}{\sqrt{14}} \text{ or } a = -1/\sqrt{14}$$

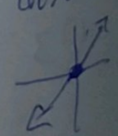
$\Rightarrow$ Using iii) for $a = 1/\sqrt{14}$

$b = -\sqrt{2/7}$ & $c = 3/\sqrt{14}$

using iii) for $a = -1/\sqrt{14}$

$b = \sqrt{2/7}$ & $c = -3/\sqrt{14}$

These vectors are in opposite direction



Using +ve [Assuming], we have $U_3 = \begin{bmatrix} 1/\sqrt{14} \\ -\sqrt{2/7} \\ 3/\sqrt{14} \end{bmatrix}$

$U = [U_1, U_2, U_3]$ or $U_3 = \begin{bmatrix} -1/\sqrt{14} \\ \sqrt{2}/\sqrt{7} \\ -3/\sqrt{14} \end{bmatrix}$

$$\Rightarrow U = \begin{bmatrix} \sqrt{6}/6 & -2\sqrt{5}/5 & 1/\sqrt{14} \\ \sqrt{6}/3 & \sqrt{5}/5 & -\sqrt{2}/7 \\ \sqrt{6}/6 & 0 & 3/\sqrt{14} \end{bmatrix}$$

$$\begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ u_1 & u_2 & u_3 + \end{array}$$

b.) Show that U is an orthogonal matrix.

We already know that $u_3 \perp u_1$ & $u_3 \perp u_2$

Check for $u_1 \perp u_2$, [If their dot product is 0]

$$= \begin{bmatrix} \sqrt{6}/6 \\ \sqrt{6}/3 \\ \sqrt{6}/\sqrt{6} \end{bmatrix} \cdot \begin{bmatrix} -2\sqrt{5}/5 \\ \sqrt{5}/5 \\ 0 \end{bmatrix}$$

$= 0$

$\Rightarrow$ The two vectors are $\perp$ar $\Rightarrow u_1 \perp u_2 \perp u_3$

$\Rightarrow$ The matrix U is an orthogonal matrix.

c.) Compute UDV^T

$$U = \begin{bmatrix} \sqrt{6}/6 & -2\sqrt{5}/5 & 1/\sqrt{14} \\ \sqrt{6}/3 & \sqrt{5}/5 & -\sqrt{2/7} \\ \sqrt{6}/6 & 0 & 3/\sqrt{14} \end{bmatrix}$$

$$D = \begin{bmatrix} \sqrt{12} & 0 \\ 0 & \sqrt{10} \\ 0 & 0 \end{bmatrix}$$

$$V^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$\Rightarrow UDV^T = \begin{bmatrix} \sqrt{6}/6 & -2\sqrt{5}/5 & 1/\sqrt{14} \\ \sqrt{6}/3 & \sqrt{5}/5 & -\sqrt{2/7} \\ \sqrt{6}/6 & 0 & 3/\sqrt{14} \end{bmatrix}$$

$$\times \begin{bmatrix} \sqrt{12} & 0 \\ 0 & \sqrt{10} \\ 0 & 0 \end{bmatrix} \times \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix}$$

(Too lengthy calculation, so ommitting steps

d.) Explain the relationship between $\{u_1, u_2, u_3\}$ & $\{x_1, x_2, x_3\}$

$$\begin{bmatrix} \sqrt{6}/6 & -2\sqrt{5}/5 & 1/\sqrt{14} \\ \sqrt{6}/3 & \sqrt{5}/5 & -\sqrt{2/7} \\ \sqrt{6}/6 & 0 & 3/\sqrt{14} \end{bmatrix} \qquad \begin{bmatrix} 1/\sqrt{6} & -2/\sqrt{5} & -1/\sqrt{30} \\ 2/\sqrt{6} & 1/\sqrt{5} & -2/\sqrt{30} \\ 1/\sqrt{6} & 0 & 5/\sqrt{30} \end{bmatrix}$$

U (Let) $\qquad$ X (Let)

Here matrix X is having eigenvectors as columns which are orthonormal.

Matrix V contains orthogonal vectors itself with V_3 also being orthonormal to v_1 & v_2.

These V vectors along with Eigenvectors decompose A into it's rank one components. [Not sure].