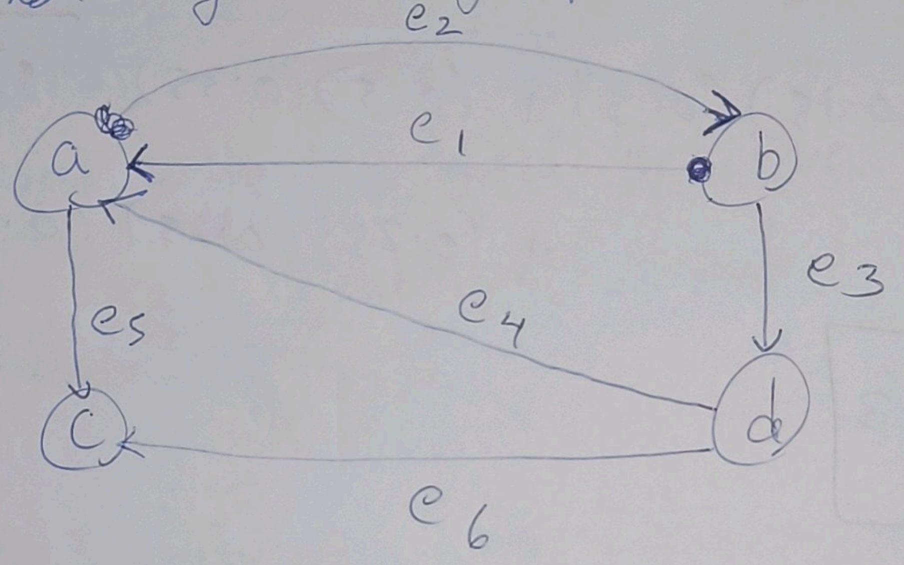
1.)

$$M = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{pmatrix} 1 & -1 & 0 & 1 & -1 & 0 \\ -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 0 & -1 \end{pmatrix} \end{array}$$

c.) Drawing the graph.



a.) Based on this we can say that M is a connected graph and a directed graph as well.

b) Adjacency matrix.

$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

d) Length 2 paths we need to find

$$A^2 = A \cdot A \cdot = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

we can see that there is only 1 path of length 2 between nodes b and c [1 from b → c] [0 from c to b]

c) In terms of connectivity we can say that there is atleast one vertex from which we can not go to any other vertex. [Vertex C]. All zeroes in the matrix denote that. Also we can see that $R_2 = R_4 \Rightarrow$ [Trace = 2]. Only one independent row.

2.) Let E_1 is the case of vaccinated

a.) E_2 is the case of partially vaccinated

E_3 is the case of not vaccinated.

$\Rightarrow P(E_1) = 30\% = 0.3$ $\quad P(\bar{E_1}) = 0.7$

$P(E_2) = 20\% = 0.2$ $\quad P(\bar{E_2}) = 0.8$

$P(E_3) = 50\% = 0.5$ $\quad P(\bar{E_3}) = 0.5$

Randomly selected person is completely vaccinated. $\rightarrow$ $30\% = P(E_1) = 0.3$.

b.) We need to find

$$P(E_1 | Pos) = \frac{P(Pos|E_1) \cdot P(E_1)}{P(Pos|E_1)P(E_1) + P(Pos|E_2)P(E_2) + P(Pos|E_3)P(E_3)} \quad -(i)$$

Finding missing probabilities based on given %.

$P(Pos|E_1) = 0.05$ $\quad P(Neg|E_1) = 0.95$

$P(Pos|E_2) = 0.25$ $\quad P(Neg|E_2) = 0.75$

$P(Pos|E_3) = 0.45$ $\quad P(Neg|E_3) = 0.55$

$\Rightarrow$ fitting in i)

$$P(E_1 | Pos) = \frac{(0.05)(0.3)}{(0.05)(0.3) + (0.25)(0.2) + (0.45)(0.5)}$$

$= 0.051$

c) Two independent test. for turning out to be positive.

$P(T_1 | E_1) = 0.05$ $P(T_2 | E_1) = 0.05$

As T_1 & T_2 are independent.

$$P(T_1 \cap T_2 | E_1) = \frac{P(T_1 | E_1)\, P(T_2 | E_1)\, P(E_1)}{\left(P(T_1 | E_1)\, P(T_2 | E_1)\, P(E_1) + P(T_1 | \overline{E}_1)\, P(T_2 | \overline{E}_1)\, P(\overline{E}_1) \right)}$$

$$= \frac{(0.05)^2 (0.3)}{(0.05)^2 (0.3) + (0.95)^2 (0.7)}$$

$=$ 0.001 is the required probability (tvetgt)
that says the person belongs to the completely vaccinated group

d.) Given that there is another test which is more accurate:

$$P(T_2 \mid E_1) = 0.15$$

$$P(T_2 \mid E_2) = 0.25$$

$$P(T_2 \mid E_3) = 0.35.$$

$\Rightarrow$ We need to find the probability that a person who has tested positive belongs to partially vaccinated group.

$\Rightarrow$ As the tests are independent we need to find all the cases where both tests were positive. This will be our total cases. — 1)

$\Rightarrow$ Favorable cases would be the number in the partially vaccinated group. — 2)

$$P(T_1 \cap T_2 \mid E_2)$$

$$= \frac{P(T_1 \mid E_2)\, P(T_2 \mid E_2)\, P(E_2)}{\left(P(T_1 \mid E_1)\, P(T_2 \mid E_1)\, P(E_1) + P(T_1 \mid E_2)\, P(T_2 \mid E_2)\, P(E_2) + P(T_1 \mid E_3)\, P(T_2 \mid E_3)\, P(E_3) \right)}$$

$$= \frac{(0.25)(0.25)(0.2)}{\left((0.05)(0.15)(0.3) + (0.5)(0.35)(0.45) + (0.25)(0.25)(0.2) \right)}$$

$$\boxed{= 0.133}$$

Q.3.) $\lambda_1 = 3$, $\lambda_2 = 2$ & $\lambda_3 = 0$

i) The lineal system

$Ax = b$ is solvable or has a solution only when b is in columnspace of A.

Now, as $\lambda_3 = 0$.

$\Rightarrow \det(A) = 0$

We also know that ~~the~~ b should be of form

$$b = \begin{pmatrix} a_1 \\ a_2 \\ 0 \end{pmatrix} \quad (a_1, a_2 \in R^3)$$

ii) The solution $Ax = b$ is only unique when all the rows ~~are~~ linearly independent but here

$$\det(A) = 0 \quad \& \quad Rank(A) = 2.$$

$\Rightarrow$ The solution $Ax = b$ linear system will not be unique.

Q.4.) $A = \begin{bmatrix} 0.6 & 1-c \\ 0.4 & c \end{bmatrix}$

Soln To find eigenvalues we need to find all λ for which

$$(A - I\lambda)\,v = 0$$

finding the characteristic polynomial.

$$\Rightarrow \begin{pmatrix} 0.6 & 1-c \\ 0.4 & c \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\lambda$$

$$= \begin{pmatrix} -\lambda + 0.6 & 1-c \\ 0.4 & -\lambda + c \end{pmatrix} = 0$$

$$\Rightarrow \lambda^2 - c\lambda - 0.6\lambda + c - 0.4 = 0$$

$$\Rightarrow \lambda_1 = 1 \;,\; \lambda_2 = c - 0.4$$

Equating $(A - I\lambda)v = 0$, we get the eigenvectors for some λ.

for $\lambda = 1$

$$\begin{bmatrix} -\lambda + 0.6 & -c+1 \\ 0.4 & c-\lambda \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow 0.4x_1 + x_2(1-c) = 0$

$\Rightarrow x_1 = -\dfrac{0.4}{1-c}\, x_2$

$\Rightarrow x_2 = t$ & $x_1 = -\left(\dfrac{0.4}{1-c}\right) t$

$\Rightarrow x_1 = \begin{pmatrix} -0.4t/1-c \\ 1 \end{pmatrix}$ for $t = 1$

$x_1 = \begin{pmatrix} -\dfrac{0.4}{1-c} \\ 1 \end{pmatrix}$

for $\lambda = c - 0.4$

$$\begin{bmatrix} -(c-0.4)+0.6 & 1-c \\ 0.6 & c-c+0.4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1-c & 1-c \\ 0.6 & 0.4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x_1 = -x_2$

$\Rightarrow$ for $x_2 = t$ we have $x_2 = \begin{bmatrix} -t \\ t \end{bmatrix}$

$\Rightarrow v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Eigenvectors.

$$\begin{bmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \text{ for } \lambda = c - 0.4 \\ \begin{pmatrix} -\left(\frac{0.4}{1-c}\right) \\ 1 \end{pmatrix} \text{ for } \lambda = 1 \end{bmatrix}$$

b.) for $c = 1.4$.

the eigenvalue will be $\lambda = c - 0.4$

$= 1.4 - 0.4$

$= 1$

the eigenvector for this will be same.

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

for $\lambda = 1$

the eigenvector will be

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

$\Rightarrow$ for $c = 1.4$ we have only one eigenvalue $\lambda = 1$ & one eigenvector $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

Q5.) a.)

$$f(x,y) = x^2 e^{-xy}, \text{ All directions } \vec{u} = \begin{bmatrix} a \\ b \end{bmatrix}$$

Solⁿ.

→ We need to solve the following equation.

$$D_{\vec{u}} f(a,b) = \nabla f(a,b)^T \cdot \hat{u}$$

Now, finding partial derivatives w.r.t. x and y

$$\nabla f(x,y) = \begin{bmatrix} f_x(x,y) \\ f_y(x,y) \end{bmatrix}$$

To find w.r.t x, using product rule.

$$\frac{\partial}{\partial x}(uv) = v\frac{\partial u}{\partial x} + u\frac{\partial v}{\partial x} \quad \text{where } u = e^{-xy} \text{ \& } v = x^2$$

$$\Rightarrow \quad x^2 \frac{\partial}{\partial x} e^{-xy} + e^{-xy} \frac{\partial}{\partial x} x^2$$

$$f_x = \frac{\partial f(x,y)}{\partial x} = -x^2 y e^{xy} + 2x e^{-xy}$$

~~Here~~ Finding partial derivative w.r.t. y

$$\frac{\partial}{\partial y}(x^2 e^{-xy})$$

Taking constant out:

$$\Rightarrow x^2 \left(\frac{\partial}{\partial y} e^{-xy} \right)$$

Using chain rule $\frac{\partial}{\partial y}(e^{-xy}) = \frac{\partial e^u}{\partial u} \frac{\partial u}{\partial y}$

where

$u = -xy$ & $\frac{\partial}{\partial u}(e^u) = e^u$

$$= e^{-xy} \left(\frac{\partial}{\partial y}(-xy) \right) x^2$$

$$\Rightarrow f_y = \frac{\partial f(x,y)}{\partial y} = -x^3 e^{-xy}$$

Now finding $\nabla f(x,y)$ using calculated values:

$$\nabla f(x,y) = \begin{bmatrix} f_x(x,y) \\ f_y(x,y) \end{bmatrix} = \begin{bmatrix} -e^{-xy} x^2 y + 2x e^{-xy} \\ -x^3 e^{-xy} \end{bmatrix}$$

at $(1,0)$

$$\nabla f(1,0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Now as given $\vec{v}$ is not a unit vector,
we need to divide it by $\|\vec{v}\|$ to convert it into a unit vector.

$\|\vec{u}\| = \sqrt{a^2+b^2}$

$\Rightarrow \hat{u} = \begin{bmatrix} \frac{a}{\sqrt{a^2+b^2}} \\ \frac{b}{\sqrt{a^2+b^2}} \end{bmatrix}$

We know that directional derivative at point $(1,0)$ $= 1$

$\Rightarrow \begin{bmatrix} 2 \\ -1 \end{bmatrix}^T \cdot \begin{bmatrix} \frac{a}{\sqrt{a^2+b^2}} \\ \frac{b}{\sqrt{a^2+b^2}} \end{bmatrix} = D_{\vec{u}}(1,0) = 1$

$\Rightarrow \frac{2a}{\sqrt{a^2+b^2}} - \frac{b}{\sqrt{a^2+b^2}} = 1$

$\Rightarrow 2a - b = \sqrt{a^2+b^2}$

$\Rightarrow b^2 = (2a - \sqrt{a^2+b^2})^2$

$\Rightarrow b^2 = 4a^2 + a^2+b^2 - 4a\sqrt{a^2+b^2}$

$\Rightarrow b = \frac{3a}{4}$, for $a > 0$

$b \leq 0$, for $a = 0$

As b is not ~~a~~ positive for $a \leq 0$,

$\Rightarrow$ The ~~equation lies~~ values for b is only valid in $a \in (0, \infty)$

$\Rightarrow$ All directions defined by

$$\vec{u} = \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ 3a/4 \end{bmatrix} \quad \forall a \in (0, \infty)$$

5.b.) $z = f(x,y) = x^3 + y^3 + x^2 + ay^2 + 2$ at $(1,2)$ goes through origin.

The tangent plane is defined by

$$z - z_0 = f_x(x_0 y_0)(x - x_0) + f_y(x_0 y_0)(y - y_0)$$

Here, $z_0 = 0$

Finding partial derivatives:

$$\frac{\partial z}{\partial x} = 3x^2 + 2x \qquad \frac{\partial z}{\partial x}(1,2) = 3 + 2 = 5$$

$$\frac{\partial z}{\partial y} = 3y^2 + 2ay \qquad \frac{\partial z}{\partial y}(1,2) = \cancel{12 + 8a}$$
$$12 + 4a$$

⇒ The tangent plane to the surface of z which goes through origin $(0,0,0)$

$$z - 0 = 5(x-1) + (12 + 4a)(y-2)$$

$$\Rightarrow z = 5x - 5 + 12y + 4ay - 24 - 8a \quad \text{— i)}$$

As this goes through the origin:

$\Rightarrow (x, y, z) = (0, 0, 0)$. in i)

$8a = -29 \Rightarrow a = -29/8$

c.) $f(x,y,z) = x^2 + y^2 + z^2 + yz$

i) finding stationary points:

$f_x = \frac{\partial f(x,y,z)}{\partial x} = 2x$ $\qquad$ $f_y = 2y + z$

$f_z = 2z + y$

Equating these partial derivatives to 0 we get

$2x = 0$ —— i)

$2y + z = 0$ —— ii)

$2z + y = 0$ —— iii)

Solving these eqn we get.

$x = 0, y = 0, z = 0.$

The only stationary point for given $f(x,y,z)$ is $(0,0,0)$.

ii) finding Hessian Matrix.

To find hessian we need second partial derivatives.

$f_{xx} = 2$ $\quad f_{yx} = 0$ $\quad f_{zx} = 0$

$f_{xy} = 0$ $\quad f_{yy} = 2$ $\quad f_{zy} = 1$

$f_{xz} = 0$ $\quad f_{yz} = 1$ $\quad f_{zz} = 2$

Putting these values in matrix we get

$$H = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

iii) finding eigenvectors & eigenvalues

To find eigenvalues & eigenvectors we need the characteristic polynomial for H

$\Rightarrow \det(H - \lambda I) = 0$

$\Rightarrow -\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0$

$\Rightarrow (\lambda - 3)(\lambda - 2)(\lambda - 1) = 0$

$\Rightarrow \lambda_1 = 1, \lambda_2 = 2 \quad \lambda_3 = 3$ are the eigenvalues.

Now finding eigenvectors corresponding to these eigen values:

$\forall$ all v such that $(H - I\lambda) v = 0$ we get one eigenvector.

for $\lambda = 1$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$R_3 \leftarrow R_3 - R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow v_1 = 0$ — i)

$v_2 + v_3 = 0$ — ii)

from i) & ii)

$$v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -v_3 \\ v_3 \end{pmatrix}$$

$$\boxed{\text{for } v_3 = 1 \quad \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}} \quad (\lambda = 1)$$

$\lambda = 2$;

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow v_2 = 0$

$v_3 = 0$

$$\Rightarrow v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_1 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \boxed{v = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{for } v_1 = 1} \qquad (\lambda = 2)$$

$\lambda = 3$

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left[\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

$R_3 \leftarrow R_3 + R_2$

$R_2 \leftarrow -R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow v_1 = 0$

$v_2 - v_3 = 0 \qquad v_2 = v_3$

for $v_3 = 1$

$$\boxed{v_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}} \qquad (\lambda = 3)$$

iii) Now as we know that all the eigen values are non-negative

$\Rightarrow$ The stationary point $(0,0,0)$ is a local minimum.

d.) $f(x,y) = e^{(xy)^2}$ subject to

$x + 2y = 3$

$x - y = 0$.

Soln. $\rightarrow$ $f(x,y)$ can be written as

$(e^{xy}) \cdot (e^{xy})$

Using Lagrange's. we need to solve.

$$\nabla f = \lambda \nabla g + \mu \nabla h$$

$g(x,y) = k$

$h(x,y) = c.$

$$\nabla f = \begin{bmatrix} f_x \\ f_y \end{bmatrix} = \begin{bmatrix} 2xy^2 e^{xy^2} \\ 2x^2y\, e^{xy^2} \end{bmatrix}$$

$$\nabla g = \begin{bmatrix} g_x \\ g_y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\nabla h = \begin{bmatrix} h_x \\ h_y \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\Rightarrow 2xy^2 e^{(xy)^2} = \lambda + \mu$ — i)

$2x^2y\, e^{(xy)^2} = 2\lambda - \mu$ — ii)

$x + 2y - 3 = 0$ — iii)

$x - y = 0$ — iv)

Solving this we get

$\lambda = \dfrac{4e}{3}$, $\mu = \dfrac{2e}{3}$

& $y = 0$, $x = 3$.

$x = 1, y = 1$

$x = 0, y = 3/2$

$y = 0, x = 0$

2) Checking the value at

$f(3,0) = 1$ [Local min]

$f(1,1) = e \approx 2.7$ [local max]

$f(0,3/2) = 1$ [local min]

$f(0,0) = 1$ [local min].

* We get multiple local minima for either $x=0$ or $y=0$ or both x & $y = 0$.