

2.) Let E_1 is the case of vaccinated
 a) E_2 is the case of partially vaccinated
 E_3 is the case of not vaccinated.

$$\Rightarrow P(E_1) = 30\% = 0.3 \quad P(\bar{E}_1) = 0.7$$

$$P(E_2) = 20\% = 0.2 \quad P(\bar{E}_2) = 0.8$$

$$P(E_3) = 50\% = 0.5 \quad P(\bar{E}_3) = 0.5$$

~~(a)~~ Randomly selected person is completely vaccinated. $\rightarrow \cancel{30\%} = P(E_1) = 0.3$.

b) we need to find

$$P(E_1 | \text{Pos}) = \frac{P(\text{Pos} | E_1) P(E_1)}{P(\text{Pos} | E_1) P(E_1) + P(\text{Pos} | E_2) P(E_2) + P(\text{Pos} | E_3) P(E_3)}$$

Finding missing probabilities based on given %:

$$P(\cancel{\text{Pos}} | E_1) = 0.05 \quad P(\text{Neg} | E_1) = 0.95$$

$$P(\text{Pos} | E_2) = 0.25 \quad P(\text{Neg} | E_2) = 0.75$$

$$P(\text{Pos} | E_3) = 0.45 \quad P(\text{Neg} | E_3) = 0.55$$

7) fitting in i)

$$P(B_1 | \text{Pos}) = \frac{(0.05)(0.3)}{(0.05)(0.3) + (0.25)(0.2) + (0.45)(0.5)}$$
$$= 0.051$$

c) Two independent test for turning out to be positive.

$$P(T_1 | B_1) = 0.05 \quad P(T_2 | B_1) = 0.05$$

As T_1 & T_2 are independent.

$$P(T_1 \cap T_2 | B_1) = \frac{P(T_1 | B_1) P(T_2 | B_1) P(B_1)}{\left(P(T_1 | B_1) P(T_2 | B_1) P(B_1) + P(T_1 | \bar{B}_1) P(T_2 | \bar{B}_1) P(\bar{B}_1) \right)}$$

$$= \frac{(0.05)^2 (0.3)}{(0.05)^2 (0.3) + (0.95)^2 (0.7)}$$

= 0.001 is the required probability
that says the person belongs to the completely vaccinated group (truest)

d) Given that there is another test which is more accurate:

$$P(T_2 | E_1) = 0.15$$

$$P(T_2 | E_2) = 0.25$$

$$P(T_2 | E_3) = 0.35$$

⇒ We need to find the probability that a person who has tested positive belongs to ~~vaccinated~~ ^{partially} vaccinated group.

⇒ As the tests are independent we need to find all the cases where both tests were positive. This will be our total cases. — 1)

⇒ Favorable cases would be the number in the partially vaccinated group. — 2)

$$P(T_1 \cap T_2 | E_2) \\ = \frac{P(T_1 | E_2) P(T_2 | E_2) P(E_2)}{\left(P(T_1 | E_1) P(T_2 | E_1) P(E_1) + P(T_1 | E_2) P(T_2 | E_2) P(E_2) \right.} \\ \left. + P(T_1 | E_3) P(T_2 | E_3) P(E_3) \right)}$$

$$2 \frac{(0.25)(0.25)(0.2)}{\left((0.05)(0.15)(0.3) + (0.5)(0.35)(0.45) \right.} \\ \left. + (0.25)(0.25)(0.2) \right)}$$

$$= 0.133$$

Q.3.) $\lambda_1 = 3$, $\lambda_2 = 2$ & $\lambda_3 = 0$

i) The linear system

$Ax = b$ is solvable or has a solution
only when b is in columnspace of A .

Now, as $\lambda_3 = 0$.

$$\Rightarrow \det(A) = 0$$

We also know that ~~the~~ b should be of
form

$$b \in \underbrace{\begin{pmatrix} a_1 \\ a_2 \\ 0 \end{pmatrix}}_{(\alpha_1, \alpha_2 \in \mathbb{R}^3)}$$

ii) The solution $Ax = b$ is only unique
when all the rows are linearly independent
but here

$$\det(A) = 0 \text{ & } \text{Rank}(A) = 2.$$

\Rightarrow The solution $Ax = b$ linear system
will not be unique.

$$Q.U) \quad A = \begin{bmatrix} 0.6 & 1-c \\ 0.4 & c \end{bmatrix}$$

To find eigenvalues we need to find all λ for which

$$(A - I\lambda) v = 0$$

finding the characteristic polynomial.

$$\Rightarrow \begin{pmatrix} 0.6 & 1-c \\ 0.4 & c \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\lambda$$

$$\Rightarrow \begin{pmatrix} -\lambda + 0.6 & 1-c \\ 0.4 & -\lambda + c \end{pmatrix} = 0$$

$$\Rightarrow \lambda^2 - c\lambda - 0.6\lambda + c - 0.4 = 0$$

$$\Rightarrow \lambda_1, \lambda_2 = c - 0.4$$

Equating $(A - I\lambda)v = 0$, we get the eigenvectors for some λ .

for $\lambda = 1$

$$\begin{bmatrix} -\lambda + 0.6 & -c+1 \\ 0.4 & c-\lambda \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow 0.4n_1 + n_2(1-c) = 0$$

$$\Rightarrow \cancel{n_1}^2 - \frac{0.4}{1-c} \cancel{n_2}$$

$$\Rightarrow n_2^2 + \cancel{n_1}^2 = \left(\frac{0.4}{1-c}\right)^2$$

$$\Rightarrow n_1 = \begin{pmatrix} e^{-0.4t}/1-c \\ 0 \end{pmatrix} \quad \text{for } t \geq 0$$

$$n_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \left(-\frac{0.4}{1-c}\right)$$

$$\text{for } \lambda = c - 0.4$$

Decrease

$$\begin{bmatrix} (c-0.4)+0.6 & 1-c \\ 0.6 & c-c+0.4 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1-c & 1-c \\ 0.6 & 0.4 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow n_1 = -n_2$$

$$\Rightarrow \text{For } n_2 \neq 0 \text{ we have } n_2 = \begin{bmatrix} -t \\ t \end{bmatrix}$$

$$\Rightarrow V_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Eigenvalues.

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix} \text{ for } \lambda = c - 0.4$$

$$-\begin{pmatrix} 0.4 \\ 1-c \end{pmatrix} \text{ for } \lambda = 1$$

b) for $c = 1.4$.

$$\text{the eigenvalue will be } \lambda = c - 0.4 \\ = 1.4 - 0.4 \\ = 1$$

The eigenvector for this will be same.

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

for $\lambda = 1$

The eigenvector will be

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

2) for $c = 1.4$ we have only one eigenvalue
 $\lambda = 1$ & one eigenvector $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

Q8.) a)

$$f(x,y) = x^2 e^{-xy}, \text{ All directions } \vec{u} = \begin{bmatrix} a \\ b \end{bmatrix}$$

Solv-

→ We need to solve the following equation.

$$D_{\vec{u}} f(a,b) = \nabla f(a,b)^T \cdot \hat{u}$$

Now, finding partial derivatives w.r.t. x and y

$$\nabla f(x,y) = \begin{bmatrix} f_x(x,y) \\ f_y(x,y) \end{bmatrix}$$

To find w.r.t x, using product rule.

$$\frac{\partial}{\partial x} (uv) = v \frac{\partial u}{\partial x} + u \frac{\partial v}{\partial x} \quad \text{where } u = e^{-xy} \text{ & } v = x^2$$

$$\Rightarrow x^2 \frac{\partial}{\partial x} e^{-xy} + e^{-xy} \frac{\partial}{\partial x} x^2$$

$$x^2 \frac{\partial f(x,y)}{\partial x} = -x^2 y e^{-xy} + 2x e^{-xy}$$

Now finding partial derivative w.r.t. y

$$\frac{\partial}{\partial y} (x^2 e^{-xy})$$

Taking constant out:

$$\Rightarrow x^2 \left(\frac{\partial}{\partial y} e^{-ny} \right)$$

Using chain rule $\frac{\partial}{\partial y} (e^{-ny}) = \frac{\partial e^u}{\partial u} \frac{\partial u}{\partial y}$

where

$$u = -ny \quad \& \quad \frac{\partial}{\partial u} (e^u) = e^u$$

$$\therefore e^{-ny} \left(\frac{\partial}{\partial y} (-ny) \right) x^2$$

$$\Rightarrow f_y = \frac{\partial f(n,y)}{\partial y} = -x^3 e^{-ny}$$

Now finding $\nabla f(n,y)$ using calculated values:

$$\nabla f(n,y) = \begin{bmatrix} f_n(n,y) \\ f_y(n,y) \end{bmatrix} = \begin{bmatrix} -e^{-ny} x^2 y + \frac{2x}{e^{-ny}} \\ -x^3 e^{-ny} \end{bmatrix}$$

at $(1,0)$

$$\nabla f(1,0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Now as given \vec{v} is not a unit vector,

we need to divide it by $\|\vec{v}\|$ to convert it into a unit vector.

$$\|\vec{u}\| = \sqrt{a^2 + b^2}$$

$$\Rightarrow \hat{u} = \begin{bmatrix} \frac{a}{\sqrt{a^2+b^2}} \\ \frac{b}{\sqrt{a^2+b^2}} \end{bmatrix}$$

We know that directional derivative at point $(1,0)$
 $= 1$

$$\Rightarrow \begin{bmatrix} 2 \\ -1 \end{bmatrix}^T \cdot \begin{bmatrix} \frac{a}{\sqrt{a^2+b^2}} \\ \frac{b}{\sqrt{a^2+b^2}} \end{bmatrix} = D_{\vec{u}}(1,0) = 1$$

$$\Rightarrow \frac{2a}{\sqrt{a^2+b^2}} - \frac{b}{\sqrt{a^2+b^2}} = 1$$

$$\Rightarrow 2a - b = \sqrt{a^2 + b^2}$$

$$\Rightarrow b^2 = (2a - \sqrt{a^2+b^2})^2$$

$$\Rightarrow b^2 = 4a^2 + a^2 + b^2 - 4a\sqrt{a^2+b^2}$$

$$\Rightarrow b^2 = \frac{3a}{4}, \text{ for } a > 0$$

$$b \leq 0, \text{ for } a \geq 0$$

As b is not positive for $a \leq 0$,

ii) The equation values for b is only valid in
 $a \in (0, \infty)$

iii) All directions defined by

$$\vec{u} = \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ 3a/4 \end{bmatrix} \quad \forall a \in (0, \infty)$$

S.b.) $z = f(x, y) = x^3 + y^3 + x^2 + ay^2 + 2$ at $(1, 2)$
goes through origin.

Also The tangent plane is defined by

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Here, $z_0 = 0$

Finding partial derivatives:

$$\frac{\partial z}{\partial x} = 3x^2 + 2x$$

$$\frac{\partial z}{\partial x}(1, 2) = 3 + 2 \\ = 5$$

$$\frac{\partial z}{\partial y} = 3y^2 + 2ay$$

$$\frac{\partial z}{\partial y}(1, 2) = 12 + 8a \\ 12 + 4a$$

∴ The tangent plane ~~at~~ to the surface of z
which goes through origin $(0, 0, 0)$

$$z - 0 = 5(x - 1) + (12 + 4a)(y - 2)$$

$$\Rightarrow z = 5x - 5 + 12y + 4ay - 24 - 8a - i)$$

As this goes through the origin:

$$\Rightarrow (n_1 y_1, z) = (0, 0, 0) \text{ in } i)$$

$$8a^2 - 2^9 \Rightarrow a^2 - 2^9 / 8$$

$$\therefore f(x, y, z) = x^2 + y^2 + z^2 + xyz$$

i) finding stationary points:

$$\frac{\partial f}{\partial x} = 2x \quad \frac{\partial f}{\partial y} = 2y + xz$$

$$\frac{\partial f}{\partial z} = 2z + xy$$

Equating these partial derivatives to 0 we get

$$2x = 0 \quad \text{--- i)}$$

$$2y + xz = 0 \quad \text{--- ii)}$$

$$2z + xy = 0 \quad \text{--- iii)}$$

Solving these eqn we get.

$$x = 0, y = 0, z = 0.$$

The only stationary point for given $f(x, y, z)$ is $(0, 0, 0)$.

ii) finding Hessian Matrix.

To find hessian we need second partial derivatives.

$$\begin{array}{l} f_{xx}=2 \quad f_{yy}=2 \quad f_{zz}=2 \\ f_{xy}=0 \quad f_{yz}=1 \quad f_{xz}=0 \\ f_{yz}=0 \quad f_{zz}=2 \end{array}$$

Putting these values in matrix we get

$$H = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

iii) finding eigenvectors & eigenvalues

To find eigenvalues & eigenvectors we need the characteristic polynomial for H

$$\Rightarrow \det(H - \lambda I) = 0$$

$$\Rightarrow -\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0$$

$$\Rightarrow (\lambda - 3)(\lambda - 2)(\lambda - 1) = 0$$

$\Rightarrow \lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$ are the

eigenvalues.

Now finding eigenvectors corresponding to these eigenvalues:

forall v such that $(H - I\lambda)v = 0$ we get one eigenvector.

for $\lambda = 1$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array} \right]$$

$$R_3 \leftarrow R_3 - R_2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\Rightarrow v_1 = 0 \quad \text{--- i)}$$

$$v_2 + v_3 = 0 \quad \text{--- ii)}$$

from i) & ii)

$$v^2 \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -v_3 \\ v_3 \end{pmatrix}$$

$$\boxed{\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}} \quad (\lambda = 1)$$

$\lambda = 2$;

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow v_2 = 0$$

$$v_3 = 0$$

$$\Rightarrow v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_1 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \boxed{v = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ for } v_1 = 1} \quad (\lambda = 2)$$

$\lambda = 3$

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left[\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array} \right]$$

$$R_3 \leftarrow R_3 + R_2$$

$$R_2 \leftarrow -R_2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\Rightarrow v_1 = 0$$

$$v_2 - v_3 = 0 \quad v_2 = v_3$$

for $v_3 = 1$

$$\begin{bmatrix} v^2 & (1) \end{bmatrix} \quad (\lambda = 3)$$

iv) Now as we know that all the eigen values are non-negative

\Rightarrow The stationary point $(0,0,0)$ is a local minimum.

d) $f(x,y) = e^{(xy)^2}$ subject to
 $x+2y = 3$
 $x-y=0$.

Soln. $f(x,y)$ can be written as

$$(e^{x^2}) \cdot (e^{xy})$$

Using Lagrange's. we need to solve.

$$\nabla f = \lambda \nabla g + \mu \nabla h$$

$$g(n,y) = K$$

$$h(n,y) = C.$$

$$\nabla f = \begin{bmatrix} f_n \\ f_y \end{bmatrix} = \begin{bmatrix} 2ny^2 e^{xy^2} \\ 2n^2y e^{xy^2} \end{bmatrix}$$

$$\nabla g = \begin{bmatrix} g_n \\ g_y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\nabla h = \begin{bmatrix} h_n \\ h_y \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\Rightarrow 2ny^2 e^{(ny)^2} = \lambda + \mu \quad \text{--- i)}$$

$$2n^2y e^{(ny)^2} = 2\lambda - \mu \quad \text{--- ii)}$$

$$n+2y-3=0 \quad \text{--- iii)}$$

$$n-y=0 \quad \text{--- iv)}$$

Solving this we get

$$\lambda = \frac{4e}{3}, \quad \mu = \frac{2e}{3} \quad n=1, y=1$$

$$n=0, y=\frac{3}{2}$$

$$y=0, n=0$$

$$\& y=0, n=3.$$

2) Checking the value at

$$\begin{cases} f(3,0) = 1 & \end{cases} \quad [\text{local min}]$$

$$\begin{cases} f(1,1) = e \approx 2.7 & \end{cases} \quad [\text{local max}]$$

$$\begin{cases} f(0,3/2) = 1 & \end{cases} \quad [\text{local min}]$$

$$\begin{cases} f(0,0) = 1 & \end{cases} \quad [\text{local min}].$$

~~Step 3 (2)~~

* We get multiple local minima for either $x=0$ or $y=0$ or both $x=y=0$.