SIT787 -Mathematics for AI

Trimester 2, 2021

Due: no later than the end of Week 11, Sunday 3 October 2021, 8:00pm AEST

Note:

- A proper way of presenting your solutions is part of the assessment. Please follow the order of questions in your submission.
- Your submission can be handwritten but it must be legible. Please write neatly. If I cannot read your solution, I cannot mark it.
- Provide the way you solve the questions. All steps (workings) to arrive at the answer must be clearly shown. I need to see your thoughts.
- For a final answer without a proper justification no score will be given.
- Only (scanned) electronic submission would be accepted via the unit site (Deakin Sync).
- Your submission must be in ONE pdf file. Multiple files and/or in different file format, e.g. .jpg, will NOT be accepted. If you need to change the format of your submission, it will be subject to the late submission penalty.

Question 1) **Probability, Distributions:** Let X be a discrete random variable that takes values in $\{-2, -1, 0, 1, 2\}$ each with the probability of $\frac{1}{5}$. Also, Y is another discrete random variable defined as $Y = X^2$.

(i) Construct the joint probability distribution table.

(ii) Are X and Y independent? Justify.

(iii) Find $\text{Corr}(X, Y)$.

(iv) Based on your answer to part (b), can you explain the result in part (c)?

[3+3+3+3= 12 marks]

Question 2) **Probability, Bayes' Theorem:** There is a corona virus test with 90% accuracy. Given that only 2% of the population is infected, answer the following questions.

(i) What does it mean that the accuracy of the test is 90%?

(ii) If a person tests positive, what is the probability that the person actually has the disease.

(iii) What is the probability that a patient is missclassified?

(iv) If the person decides to repeat the test again (independent of the first one) what is the probability of being infected given that both tests are positive?

(v) If the person decides instead of repeating the same test, he takes independently a cheaper test with accuracy of 80%. What is the probability of being infected given that both tests are positive?

[3+3+3+3+4=16 marks]

Question 3) **Probability, Distributions:** Let X be a continuous random variable with a probability density function given by

$$f(x) = \begin{cases} kx^2 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(i) Find the value of k.

(ii) Calculate $P(X \leq \frac{1}{2})$.

(iii) Find the cumulative distribution of X.

(iv) Compute $P(X \leq \frac{1}{2})$ using the cumulative distribution function you found in part (c). If there is any difference, can you explain the reason?

[2+2+2+2=8 marks]

Question 4) Multivariate functions and Optimisation. The idea of local and global minimum and maximum exists in studying functions with several variables. For a function of two variables $z = f(x, y)$, if f has a local maximum or minimum at a point (a, b), and the first order partial derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$. You can imagine that if $f_x(a, b) = 0$ and $f_y(a, b) = 0$, then there will be a flat tangent plane, similar to a horizontal tangent line in single variable functions.

Formally, a point (a, b) is a critical point (or stationary point) of $z = f(x, y)$ either if $f_x(a, b) = 0$ and $f_y(a, b) = 0$, or if one of these partial derivatives does not exist. Also, we need to be careful that at a critical point, a multivariate function could have a local maximum, a local minimum or neither.

Second derivative test for a local max or min for functions with 2 variables: Consider $z = f(x, y)$. We need to be able to determine whether a function has an extreme value at a critical point. For a critical point (a, b), let

$$D = D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

be the discriminant.

- If $D > 0$ and $f_{xx}(a, b) > 0$, then f has a local minimum at (a, b).
- If $D > 0$ and $f_{xx}(a, b) < 0$, then f has a local maximum at (a, b).
- If $D < 0$, then f does not have a local max or a local min at (a, b). It has a saddle point there.
- If $D = 0$, the test gives no information. We need to use other mathematical techniques to check the situation.

To find a local minimum, a local maximum, or a saddle point of multivariate functions with more than 2 variables, we need to compute its Hessian matrix at all the critical points. Consider the multivariate function $w = f(x, y, z)$. You see that the number of independent variables is more than 2. The Hessian of this matrix is defined as

$$H(x, y, z) = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y \partial z} \\ \frac{\partial^2 f}{\partial z \partial x} & \frac{\partial^2 f}{\partial z \partial y} & \frac{\partial^2 f}{\partial z^2} \end{bmatrix}$$

Generally, $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$, $\frac{\partial^2 f}{\partial x \partial z} = \frac{\partial^2 f}{\partial z \partial x}$, and $\frac{\partial^2 f}{\partial z \partial y} = \frac{\partial^2 f}{\partial y \partial z}$, hence, the Hessian matrix is always symmetric. Also, note that the Hessian is a function of x, y, and z. The critical points of this function is either $\nabla f = \overrightarrow{\mathbf{0}}$, or the partial derivative does not exist. To find a local minimum, a local maximum, or a saddle point, we plug the critical points in the Hessian matrix, compute its eigenvalues, and we make a decision based on signs of the eigenvalues, as described in the second derivative test.

Second derivative test: Find all critical points, plug them in the Hessian matrix, and compute its eigenvalues.

- If **all** eigenvalues are strictly positive, then the critical point is a local minimum.
- If **all** the eigenvalues are strictly negative, the critical point is a local maximum point.
- If eigenvalues do not have the same sign and **all** are non-zero, then the critical point is a saddle point.
- If any of the eigenvalues is zero, then no conclusion can be drawn without further information.

It is important to note that by this extension, you can find local maximum, local minimum, or saddle points of a function with any number of variables. In its most general form, to find critical points of a multivariate function with n variables $y = f(x_1, x_2, \ldots, x_n)$, you need to solve the system $\nabla f = \overrightarrow{\mathbf{0}}$, or find the points that any of the partial derivatives does not exist. Then, you need to from the Hessian matrix at the critical points, find eigenvalues, and make a decision based on the signs of the eigenvalues.

So far what we had was unconstrained optimisation. In other words, we wanted to find a local maximum or minimum, or a saddle point of a multivariate function without any restriction or constraints. However, sometimes we want to find a maximum or a minimum of $f(x, y)$ subject to a constraint $g(x, y) = k$, where k is a constant. To solve this problem, we use the Lagrange Multiplier technique. To find maximum and minimum values of $f(x, y)$ subject to the constraint $g(x, y) = k$ (assuming that these max and min values exist and $\nabla g \neq 0$ on $g(x, y) = k$):

(a) Find all values of x, y, and λ such that

$$\nabla f(x, y) = \lambda \nabla g(x, y) \text{ and}$$

$$g(x, y) = k$$

(b) Evaluate f at all the points (x, y) that result from step (a). The largest of these values is the global maximum value and the smallest is the global minimum value of f.

[**Note:** Please pay attention to the way the λ notation is used here. It is not an eigenvalue. Here, it is a nonzero multiplier – we are not interested in its value, but it will facilitate the finding of the critical points.]

This technique could be applied for functions having more than two variables. Imagine, you want to find the maximum and minimum of $w = f(x, y, z)$ subject to constraint $g(x, y, z) = 0$. Then you need to solve the following system:

$$\begin{cases} \nabla f(x, y, z) = \lambda \nabla g(x, y, z) \\ g(x, y, z) = 0 \end{cases}$$

This system has four equations and 4 unknowns. However, generally it is not linear. Sometimes, you may use some mathematical tricks to find the solution of the system.

A further extension is when there are two constraints (you can see that we can easily generalise the idea to many constraints). To find the maximum and minimum of a function $w = f(x, y, z)$

subject to two constraints $g(x, y, z) = 0$ and $h(x, y, z) = 0$, we need to introduce two Lagrange multipliers λ and μ to obtain the following system of equations:

$$\begin{cases} \nabla f(x, y, z) = \lambda \nabla g(x, y, z) + \mu \nabla h(x, y, z) \\ g(x, y, z) = 0 \\ h(x, y, z) = 0 \end{cases}$$

To summarise, you need to be familiar with the following optimisation problems, and how we can solve them. Also, you need to be extremely careful about using local or global adjectives in your communications.

- Optimisation problems
  - Unconstrained Optimisation
    - Single-variable function: (covered in assignment 1)
    - Multivariate functions
      - Multivariate with two variables (use the discriminant)
      - Multivariate with more than two variables (use eigenvalues)
  - Constraint Optimisation
    - Global max and min of a single variable function in an interval (covered in assignment 1)
    - Global max and min of a multivariate functions with some constraints (Lagrange multipliers)

Now, let's solve some optimisation problems.

(i) Find and classify all the critical points of the function

$$f(x, y) = x^3 + y^3 + 3x^2 - 3y^2 - 8.$$

(ii) Find all values for k so that $f(x, y) = x^2 + kxy + y^2$ has a local minimum at $(0, 0)$. Give your answer in the form of an interval.

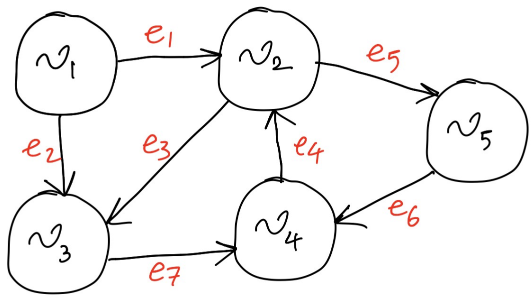
Now solve the constrained optimisation problems below using Lagrange multipliers.

(iii) Find the minimum, maximum, both, or neither, value of $f(x, y) = 4x + 4y - x^2 - y^2$ subject to the constraint $x^2 + y^2 = 2$.

(iv) Find the extreme value of $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraints $2x + 2y - 4z = 0$ and $4x - 2y + 2z = 0$, and determine whether the extreme point is a maximum or minimum point.

[7+10+7+10=34 marks]

Question 5) Graph Theory. Consider the following directed Graph $G(V, E)$.



(i) Find the adjacency matrix (call it A) and the incidence matrix (call it B) of this graph.

(ii) Remove the direction of the arcs in the graph, and find the adjacency matrix of the undirected version of this graph. Call the matrix C.

(iii) Find rank(B).

(iv) Show that $B^T\overrightarrow{\mathbf{1}} = \overrightarrow{\mathbf{0}}$, where $\overrightarrow{\mathbf{1}}$ is the (column) vector whose components are all equal to 1, and $\overrightarrow{\mathbf{0}}$ is the (column) vector whose components are all equal to 0. Based on $B^T\overrightarrow{\mathbf{1}} = \overrightarrow{\mathbf{0}}$, what can you conclude about eigenvalues and eigenvectors of BB^T?

(v) Set $L = BB^T$. Show that $L = D - C$, where $D = \text{diag}(\deg(v_1), \deg(v_2), \dots, \deg(v_5))$, namely, the degrees of vertices are on the main diagonal of D.

(vi) Find $\det(L)$ without direct calculation.

(vii) Without direct calculation, show that all the eigenvalues of L are nonnegative.

(viii) Without direct calculation, show that one of the eigenvalues of L is zero and the corresponding eigenvector is $\overrightarrow{\mathbf{1}}$.

[4+3+3+4+4+4+4+4=30 marks]

Question 1) **Probability, Distributions:** Let X be a discrete random variable that takes values in $\{-2, -1, 0, 1, 2\}$ each with the probability of $\frac{1}{5}$. Also, Y is another discrete random variable defined as $Y = X^2$.

(i) Construct the joint probability distribution table.

(ii) Are X and Y independent? Justify.

(iii) Find $\mathrm{Corr}(X, Y)$.

(iv) Based on your answer to part (b), can you explain the result in part (c)?

(i)

$X \in \{-2, -1, 0, 1, 2\}$

$P(X=-2) = P(X=-1) = P(X=0) = P(X=1) = P(X=2) = \frac{1}{5}$

$Y = X^2 \implies Y \in \{0, 1, 4\}$

$P(Y=0) = P(X^2=0) = P(X=0) = \frac{1}{5}$

$P(Y=1) = P(X^2=1) = P(X=1) + P(X=-1) = \frac{2}{5}$

$P(Y=4) = P(X^2=4) = P(X=2) + P(X=-2) = \frac{2}{5}$

Y \ X	-2	-1	0	1	2	marginal
0	P(-2,0)	P(-1,0)	P(0,0)	P(1,0)	P(2,0)	P(Y=0)
1	P(-2,1)	P(-1,1)	P(0,1)	P(1,1)	P(2,1)	P(Y=1)
4	P(-2,4)	P(-1,4)	P(0,4)	P(1,4)	P(2,4)	P(Y=4)
marginal	P(X=-2)	P(X=-1)	P(X=0)	P(X=1)	P(X=2)	

Y \ X	−2	−1	0	1	2	marginal
0	0	0	1/5	0	0	1/5
1	0	1/5	0	1/5	0	2/5
4	1/5	0	0	0	1/5	2/5
marginal	1/5	1/5	1/5	1/5	1/5	1

$P(-2,0) = P(X=-2,\ Y=0) = 0$ impossible

$P(-2,1) = P(X=-2,\ Y=1) = 0$ impossible

$P(-2,4) = P(X=-2,\ Y=4) = P(X=-2) = \frac{1}{5}$

$P(-1,0) = P(X=-1,\ Y=0) = 0$ impossible

$P(-1,1) = P(X=-1,\ Y=1) = P(X=-1) = \frac{1}{5}$

$P(-1,4) = P(X=-1,\ Y=4) = 0$ impossible

(ii) For X and Y to be independent, we need

$$P(X=x \cap Y=y) = P(X=x)P(Y=y) \text{ for all } x \text{ and } y$$

$x \in S_X = \{-2,-1,0,1,2\}$ $\quad$ $y \in S_Y = \{0,1,4\}$

$P(-2,0) = 0$ $\quad$ $P(X=-2) = \frac{1}{5}$ $\quad$ $P(Y=0) = \frac{1}{5}$

$$0 \neq \left(\tfrac{1}{5}\right)\left(\tfrac{1}{5}\right)$$

Therefore X and Y are not independent.

(iii) $\text{Corr}(X,Y) = \dfrac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)}\sqrt{\text{Var}(Y)}}$

$\text{Cov}(X,Y) = E[XY] - \left(E[X]E[Y]\right)$

So, we need to compute $E[X]$, $E[Y]$, $E[X^2]$, $E[Y^2]$, and $E[XY]$.

$$E[X] = \sum_x x\, p(x) = \sum_x x\, P(X=x)$$
$$= (-2)(\tfrac{1}{5}) + (-1)(\tfrac{1}{5}) + (0)(\tfrac{1}{5}) + (1)(\tfrac{1}{5}) + (2)(\tfrac{1}{5}) = 0$$

$$E[Y] = \sum_y y\, P(Y=y) = (0)(1/5) + (1)(2/5) + (4)(2/5) = 2$$

$$E[X^2] = \sum_x x^2 P(X=x) = (-2)^2(\tfrac{1}{5}) + (-1)^2(\tfrac{1}{5}) + (0)^2(\tfrac{1}{5}) + (1)^2(\tfrac{1}{5}) + (2)^2(\tfrac{1}{5})$$
$$= 2$$

$$E[Y^2] = \sum_y y^2 P(Y=y) = (0)^2(1/5) + (1)^2(2/5) + (4)^2(2/5) = \frac{34}{5}$$

$$\text{Var}(X) = E[X^2] - \left(E[X]\right)^2 = 2 - (0)^2 = 2$$
$$\text{Var}(Y) = E[Y^2] - \left(E[Y]\right)^2 = \frac{34}{5} - (2)^2 = \frac{34}{5} - 4 = \frac{34-20}{5} = \frac{14}{5}$$

$$E[XY] = \sum_x \sum_y xy\, P(X=x \text{ and } Y=y)$$
$$= (-2)(4)(1/5) + (-1)(1)(\tfrac{1}{5}) + (0)(0)(1/5) + (1)(1)(1/5)$$
$$+ (2)(4)(1/5) = 0$$

(I just wrote the pairs that $P(x,y) \neq 0$)

$$Cov(X,Y) = E[XY] - E[X]E[Y] = 0 - (0)(2) = 0$$

$$Corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)\,Var(Y)}} = \frac{0}{\sqrt{(2)(\frac{14}{5})}} = 0$$

(iv) in part (ii) we concluded that X and Y are dependent. We would expect this as we defined $Y = X^2$. The values of Y depends on the values of X. However, in part (iii), we saw that the $Corr(X,Y) = 0$.

We know that if X and Y are two independent random variables, then $Cov(X,Y) = Corr(X,Y) = 0$. Also, if $Corr(X,Y) = 0$ we cannot conclude that X and Y are independent. The only thing that we can say is that X and Y are uncorrelated.

The answer that I expect for this part of the question 1, is noticing the following items:

- X and Y are dependent
- $Corr(X, Y) = 0$
- We know that $Corr(X, Y)$ represent the strength of a linear association between X and Y. We see that X and Y do not have a linear relationship. Therefore, their $Corr(X, Y) = 0$.
- However, X and Y are related to each other with a quadratic relationship, and this make them dependent.

Question 2) Probability, Bayes' Theorem: There is a corona virus test with 90% accuracy. Given that only 2% of the population is infected, answer the following questions.

(i) What does it mean that the accuracy of the test is 90%?

(ii) If a person tests positive, what is the probability that the person actually has the disease.

(iii) What is the probability that a patient is missclassified?

(iv) If the person decides to repeat the test again (independent of the first one) what is the probability of being infected given that both tests are positive?

(v) If the person decides instead of repeating the same test, he takes independently a cheaper test with accuracy of 80%. What is the probability of being infected given that both tests are positive?

Let's define the following events:

D: a person infected

$\bar{D}$: a person is not infected

pos: the test result is positive

$neg = \overline{pos}$: the test result is negative.

We are given that $P(D) = 2\% = \frac{2}{100} = 0.02$

(i) When a test is 90% accurate, it means that the test is capable of positively detecting infected people and it is capable of negatively detecting healthy people. Therefore

$$P(pos|D) = 90\% = \frac{90}{100} = 0.9$$

$$P(neg|\bar{D}) = 90\% = \frac{90}{100} = 0.9$$

Also, we can conclude that

$$P(neg \mid D) = 1 - P(pos \mid D) = 1 - 0.9 = 0.1$$

$$P(pos \mid \bar{D}) = 1 - P(neg \mid \bar{D}) = 1 - 0.9 = 0.1$$

(ii) We are asked about $P(D \mid pos)$.

$$P(D \mid pos) = \frac{P(pos \mid D)\, P(D)}{P(pos \mid D)\, P(D) + P(pos \mid \bar{D})\, P(\bar{D})}$$

$\boxed{* \; P(\bar{D}) = 1 - P(D) = 1 - 0.02}$

$$= \frac{(0.9)(0.02)}{(0.9)(0.02) + (0.1)(0.98)}$$

$$= \frac{0.018}{0.018 + 0.098} = \frac{0.018}{0.116} \simeq 0.16$$

(iii) Someone is misclassified if a healthy person tests positive or an infected person tests negative.

$$P(\text{misclassified}) = P(D \cap neg) + P(\bar{D} \cap pos)$$

$$= P(neg \mid D)\, P(D) + P(pos \mid \bar{D})\, P(\bar{D})$$

$$= (0.1)(0.02) + (0.1)(0.98)$$

$$= 0.002 + 0.098 = 0.1$$

iv

(iv) If the person decides to repeat the test again (independent of the first one) what is the probability of being infected given that both tests are positive?

Let's assume T_1 and T_2 are events that the result of the first and repeated test is positive.

$$P(T_1 | D) = 90\% = \frac{90}{100} = 0.9$$
$$P(T_2 | D) = 0.9$$

We are interested in $P(D \mid T_1 \text{ and } T_2)$.

$$P(D | T_1 \cap T_2) = \frac{P(T_1 \cap T_2 | D)\, P(D)}{P(T_1 \cap T_2 | D) P(D) + P(T_1 \cap T_2 | \bar{D})\, P(\bar{D})}$$

$$\boxed{\text{because } T_1 \text{ and } T_2 \text{ are independent} \quad P(T_1 \cap T_2 | D) = P(T_1 | D)\, P(T_2 | D)}$$

$$= \frac{P(T_1 | D) P(T_2 | D) P(D)}{P(T_1 | D)\, P(T_2 | D) P(D) + P(T_1 | \bar{D})\, P(T_2 | \bar{D})\, P(\bar{D})}$$

$$= \frac{(0.9)^2 (0.02)}{(0.9)^2 (0.02) + (0.1)^2 (0.98)} = \frac{0.0162}{0.026}$$

$$\simeq 0.62$$

(v) If the person decides instead of repeating the same test, he takes independently a cheaper test with accuracy of 80%. What is the probability of being infected given that both tests are positive?

T_1 the original test is positive

$P(T_1|D) = 0.9$

$P(T_1|\bar{D}) = 0.1$

T_2 the cheaper test with accuracy 80%

$P(T_2|D) = 0.8$

$P(T_2|\bar{D}) = 0.2$

We are interested in $P(D|T_1 \cap T_2)$.

$$P(D|T_1 \cap T_2) = \frac{P(T_1 \cap T_2|D)\,P(D)}{P(T_1 \cap T_2|D)P(D) + P(T_1 \cap T_2|\bar{D})\,P(\bar{D})}$$

because T_1 and T_2 are independent

$$\boxed{P(T_1 \cap T_2|D) = P(T_1|D)\,P(T_2|D)}$$

$$= \frac{P(T_1|D)P(T_2|D)\,P(D)}{P(T_1|D)\,P(T_2|D)P(D) + P(T_1|\bar{D})\,P(T_2|\bar{D})\,P(\bar{D})}$$

$$= \frac{(0.9)(0.8)(0.02)}{(0.9)(0.8)(0.02) + (0.1)(0.2)(0.98)}$$

$$= \frac{0.0144}{0.034} \simeq 0.42$$

Question 3) Probability, Distributions: Let X be a continuous random variable with a probability density function given by

$$f(x) = \begin{cases} kx^2 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(i) Find the value of k.

(ii) Calculate $P(X \le \frac{1}{2})$.

(iii) Find the cumulative distribution of X.

(iv) Compute $P(X \le \frac{1}{2})$ using the cumulative distribution function you found in part (c). If there is any difference, can you explain the reason?

(i) $\int_{-\infty}^{+\infty} f(x)\,dx = 1$ and $f(x) \geq 0 \quad x \in [0,1]$

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_0^1 kx^2\,dx = k\left(\tfrac{1}{3}x^3\right]_0^1 = k\left(\tfrac{1}{3}\right) = 1 \longrightarrow \boxed{k=3}$$

$$f(x) = \begin{cases} 3x^2 & 0 \le x \le 1 \\ 0 & \text{else} \end{cases} \qquad f(x) \geq 0 \text{ for } x \in [0,1]$$

(ii) $$P(X \le \tfrac{1}{2}) = \int_{-\infty}^{1/2} f(x)\,dx = \int_0^{1/2} 3x^2\,dx = 3\left(\tfrac{1}{3}x^3\right)_0^{1/2}$$

$$= 3\left(\tfrac{1}{24} - 0\right) = \tfrac{1}{8}$$

(iii) To find CDF: $F(a)$

$F(a) = P(X \le a) \qquad a \in \mathbb{R}$

if $a<0$

$$\Rightarrow F(a) = \int_{-\infty}^{0} f(x)\,dx = \int_{0}^{0} 0\,dx = 0$$

if $0 \le a \le 1 \Rightarrow F(a) = \int_{-\infty}^{a} f(x)\,dx$

$$= \int_{0}^{a} 3x^2\,dx = 3\left(\frac{x^3}{3}\right]_0^a = 3\left(\frac{a^3}{3}\right) = a^3$$

if $a>1 \Rightarrow F(a) = \int_{-\infty}^{a} f(x)\,dx = \int_{0}^{1} f(x)\,dx = 1$

$$\boxed{F(a) = \begin{cases} 0 & a<0 \\ a^3 & 0 \le a \le 1 \\ 1 & a>1 \end{cases}}$$

(iv) $F\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$

both results are the same.

Now, let's solve some optimisation problems.

(i) Find and classify all the critical points of the function

$$f(x, y) = x^3 + y^3 + 3x^2 - 3y^2 - 8.$$

(ii) Find all values for k so that $f(x, y) = x^2 + kxy + y^2$ has a local minimum at $(0, 0)$. Give your answer in the form of an interval.

Now solve the constrained optimisation problems below using Lagrange multipliers.

(iii) Find the minimum, maximum, both, or neither, value of $f(x, y) = 4x+4y-x^2-y^2$ subject to the constraint $x^2 + y^2 = 2$.

(iv) Find the extreme value of $f(x, y, z) = x^2+y^2+z^2$ subject to the constraints $2x+2y-4z = 0$ and $4x-2y+2z = 0$, and determine whether the extreme point is a maximum or minimum point.

(i) $f(x,y) = x^3 + y^3 + 3x^2 - 3y^2 - 8$

stationary points: $f_x = 0$, $f_y = 0$

$$\begin{aligned} f_x &= 3x^2 + 6x = 0 \\ f_y &= 3y^2 - 6y = 0 \end{aligned} \Rightarrow \begin{cases} 3x^2 + 6x = 0 \rightarrow x = 0 \text{ or } x = -2 \\ 3y^2 - 6y = 0 \rightarrow y = 0 \text{ or } y = 2 \end{cases}$$

$\Rightarrow$ the stationary points $= \{(0,0), (0,2), (-2,0), (-2,2)\}$

We need to find 2^{nd} derivatives and the discriminant.

$f_{xx} = 6x + 6 \quad f_{xy} = 0 \quad f_{yy} = 6y - 6$

$$D(x,y) = f_{xx} f_{yy} - [f_{xy}]^2 = (6x+6)(6y-6)$$

$D(0,0) = (6)(-6) = -36 < 0$ $\quad$ (0,0) is a saddle point

$\left.\begin{array}{l} D(0,2) = (6)(12-6) = 36 > 0 \\ f_{xx}(0,2) = 6 > 0 \end{array}\right\} \Rightarrow$ (0,2) is a local min

$\left.\begin{array}{l} D(-2,0) = (-6)(-6) = 36 > 0 \\ f_{xx}(-2,0) = -6 < 0 \end{array}\right\} \Rightarrow$ (-2,0) is a local max

$D(-2,2) = (-6)(6) = -36 \longrightarrow$ (-2,2) is a saddle point

(ii) Find all values for k so that $f(x, y) = x^2 + kxy + y^2$ has a local minimum at $(0, 0)$. Give your answer in the form of an interval.

$f(x,y) = x^2 + kxy + y^2$ has local min at $(0,0)$

Then, $(0,0)$ should satisfy in $f_x(0,0) = f_y(0,0) = 0$.

$f_x(x,y) = 2x + ky \rightarrow f_x(0,0) = 0$ ✓

$f_y(x,y) = kx + 2y \rightarrow f_y(0,0) = 0$ ✓

To have $(0,0)$ as a local minimum, we need to have $D(0,0) > 0$ and $f_{xx}(0,0) > 0$

$f_{xx} = 2 \quad f_{xy} = k \quad f_{yy} = 2$

$D(x,y) = f_{xx}(x,y)\, f_{yy}(x,y) - \left[f_{xy}(x,y)\right]^2$

$D(0,0) = (2)(2) - k^2 = 4 - k^2$

$f_{xx}(0,0) = 2 > 0$ ✓

We need to have $D(0,0) > 0 \Rightarrow 4 - k^2 > 0$

$\Rightarrow k^2 < 4 \Rightarrow -2 < k < 2$

Also, please not that when $k=2$ or $k=-2$ $D(0,0) = 0$. So this test is not conclusive when $k=2$ or $k=-2$. We need to analyse these cases separately.

When $k=2 \Rightarrow f(x,y) = x^2+2xy+y^2$

$$= (x+y)^2 \geq 0$$

then $f(x,y) \geq f(0,0) = (0+0)^2 = 0$

then $(0,0)$ is a local minimum when $k=2$.

When $k=-2$, $f(x,y) = x^2-2xy+y^2$

$$= (x-y)^2 \geq 0$$

then $f(x,y) \geq f(0,0) = (0-0)^2 = 0$

therefore, $(0,0)$ is a local minimum when $k=-2$.

The solution for this problem is

$$\boxed{k \in [-2, 2]}.$$

(iii) Find the minimum, maximum, both, or neither, value of $f(x, y) = 4x + 4y - x^2 - y^2$ subject to the constraint $x^2 + y^2 = 2$.

We need to construct this system

$$\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y) = K \end{cases}$$

$$\nabla f = \begin{bmatrix} f_x \\ f_y \end{bmatrix} = \begin{bmatrix} 4-2x \\ 4-2y \end{bmatrix}$$

$$\nabla g = \begin{bmatrix} g_x \\ g_y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

$$\begin{cases} 4-2x = 2\lambda x \Rightarrow 2x(\lambda+1) = 4 \rightarrow x = \dfrac{2}{\lambda+1}. \\ 4-2y = 2\lambda y \rightarrow 2y(\lambda+1) = 4 \rightarrow y = \dfrac{2}{\lambda+1} \\ x^2+y^2 = 2 \end{cases}$$

now we plug this in the 3rd equation

$$\frac{4}{(\lambda+1)^2} + \frac{4}{(\lambda+1)^2} = 2 \rightarrow \frac{8}{(\lambda+1)^2} = 2$$

$$\Rightarrow (\lambda+1)^2 = 4 \rightarrow \lambda+1 = \pm 2 \rightarrow \begin{cases} \lambda = 1 \\ \lambda = -3 \end{cases}$$

if $\lambda = 1 \Rightarrow x = 1, y = 1 \Rightarrow (1,1)$

if $\lambda = -3 \Rightarrow x = -1 \quad y = -1 \Rightarrow (-1,-1)$

$$f(1,1) = 4(1) + 4(1) - (1)^2 - (1)^2 = \boxed{6 \text{ max value}}$$

$$f(-1,-1) = 4(-1) + 4(-1) - (-1)^2 - (-1)^2 = \boxed{-10 \text{ min value}}$$

(iv) Find the extreme value of $f(x, y, z) = x^2+y^2+z^2$ subject to the constraints $2x+2y-4z = 0$ and $4x-2y+2z = 0$, and determine whether the extreme point is a maximum or minimum point.

We need to construct this system

$$\begin{cases} \nabla f = \lambda \nabla g + \mu \nabla h \\ g(x,y) = c \\ h(x,y) = k \end{cases}$$

$$\nabla f = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix} \quad \nabla g = \begin{bmatrix} 2 \\ 2 \\ -4 \end{bmatrix} \quad \nabla h = \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix}$$

$$\begin{cases} 2x = 2\lambda + 4\mu \\ 2y = 2\lambda - 2\mu \\ 2z = -4\lambda + 2\mu \\ 2x + 2y - 4z = 0 \\ 4x - 2y + 2z = 0 \end{cases} \rightarrow$$

$$\begin{array}{llllll} 2x & & & -2\lambda - 4\mu & = 0 \\ & 2y & & -2\lambda + 2\mu & = 0 \\ & & 2z & +4\lambda - 2\mu & = 0 \\ 2x & + 2y & - 4z & & = 0 \\ 4x & - 2y & + 2z & & = 0 \end{array}$$

$$\left[\begin{array}{ccccc|c} 2 & 0 & 0 & -2 & -4 & 0 \\ 0 & 2 & 0 & -2 & 2 & 0 \\ 0 & 0 & 2 & 4 & -2 & 0 \\ 2 & 2 & -4 & 0 & 0 & 0 \\ 4 & -2 & 2 & 0 & 0 & 0 \end{array}\right]$$

$R_4 \leftarrow R_4 - R_1$

$R_5 \leftarrow R_5 - 2R_1$

$$\left[\begin{array}{ccccc|c} 2 & 0 & 0 & -2 & -4 & 0 \\ 0 & 2 & 0 & -2 & 2 & 0 \\ 0 & 0 & 2 & 4 & -2 & 0 \\ 0 & 2 & -4 & 2 & 4 & 0 \\ 0 & -2 & 2 & 4 & 8 & 0 \end{array}\right]$$

$R_4 \leftarrow R_4 - R_2$

$R_5 \leftarrow R_5 + R_2$

$$\left[\begin{array}{ccccc|c} 2 & 0 & 0 & -2 & -4 & 0 \\ 0 & 2 & 0 & -2 & 2 & 0 \\ 0 & 0 & 2 & 4 & -2 & 0 \\ 0 & 0 & -4 & 4 & 2 & 0 \\ 0 & 0 & 2 & 2 & 10 & 0 \end{array}\right]$$

$$R_4 \leftarrow R_4 + 2R_3$$
$$R_5 \leftarrow R_5 - R_3$$

$$\left[\begin{array}{ccccc|c} 2 & 0 & 0 & -2 & -4 & 0 \\ 0 & 2 & 0 & -2 & 2 & 0 \\ 0 & 0 & 2 & 4 & -2 & 0 \\ 0 & 0 & 0 & 12 & -2 & 0 \\ 0 & 0 & 0 & -2 & 12 & 0 \end{array}\right]$$

$$R_5 \leftarrow R_5 + \frac{1}{6}R_4$$

$$\left[\begin{array}{ccccc|c} 2 & 0 & 0 & -2 & -4 & 0 \\ 0 & 2 & 0 & -2 & 2 & 0 \\ 0 & 0 & 2 & 4 & -2 & 0 \\ 0 & 0 & 0 & 12 & -2 & 0 \\ 0 & 0 & 0 & 0 & \frac{35}{3} & 0 \end{array}\right]$$

$$\frac{35}{3}\mu = 0 \Longrightarrow \mu = 0$$

$$12\lambda - 2(0) = 0 \longrightarrow \lambda = 0$$

$$2z + 4(0) - 2(0) = 0 \longrightarrow z = 0$$

$$2y - 2(0) + 2(0) = 0 \longrightarrow y = 0$$

$$2x - 2(0) - 4(0) = 0 \longrightarrow x = 0$$

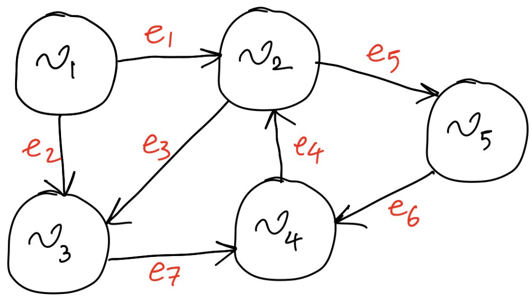
$\longrightarrow$ the only point is $(0,0,0)$.

$f(0,0,0) = 0$

this is a minimum point a

$f(x,y,z) = x^2 + y^2 + z^2 \geq 0.$

Question 5) Graph Theory. Consider the following directed Graph $G(V, E)$.



(i) Find the adjacency matrix (call it A) and the incidence matrix (call it B) of this graph.

(ii) Remove the direction of the arcs in the graph, and find the adjacency matrix of the undirected version of this graph. Call the matrix C.

(iii) Find rank(B).

(iv) Show that $B^T\vec{\mathbf{1}} = \vec{\mathbf{0}}$, where $\vec{\mathbf{1}}$ is the (column) vector whose components are all equal to 1, and $\vec{\mathbf{0}}$ is the (column) vector whose components are all equal to 0. Based on $B^T\vec{\mathbf{1}} = \vec{\mathbf{0}}$, what can you conclude about eigenvalues and eigenvectors of BB^T?

(v) Set $L = BB^T$. Show that $L = D - C$, where $D = \text{diag}(\deg(v_1), \deg(v_2), \ldots, \deg(v_5))$, namely, the degrees of vertices are on the main diagonal of D.

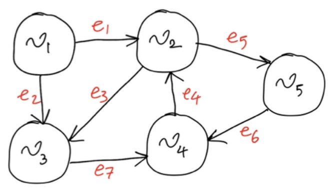
(vi) Find $\det(L)$ without direct calculation.

(vii) Without direct calculation, show that all the eigenvalues of L are nonnegative.

(viii) Without direct calculation, show that one of the eigenvalues of L is zero and the corresponding eigenvector is $\vec{\mathbf{1}}$.

[4+3+3+4+4+4+4+4=30 marks]

Question 5) Graph Theory. Consider the following directed Graph $G(V, E)$.



(i) Find the adjacency matrix (call it A) and the incidence matrix (call it B) of this graph.

(ii) Remove the direction of the arcs in the graph, and find the adjacency matrix of the undirected version of this graph. Call the matrix C.

(iii) Find $\text{rank}(B)$.

(iv) Show that $B^T\vec{\mathbf{1}} = \vec{\mathbf{0}}$, where $\vec{\mathbf{1}}$ is the (column) vector whose components are all equal to 1, and $\vec{\mathbf{0}}$ is the (column) vector whose components are all equal to 0. Based on $B^T\vec{\mathbf{1}} = \vec{\mathbf{0}}$, what can you conclude about eigenvalues and eigenvectors of BB^T?

(v) Set $L = BB^T$. Show that $L = D - C$, where $D = \text{diag}(\deg(v_1), \deg(v_2), \ldots, \deg(v_5))$, namely, the degrees of vertices are on the main diagonal of D.

(vi) Find $\det(L)$ without direct calculation.

(vii) Without direct calculation, show that all the eigenvalues of L are nonnegative.

(viii) Without direct calculation, show that one of the eigenvalues of L is zero and the corresponding eigenvector is $\vec{\mathbf{1}}$.

[4+3+3+4+4+4+4+4=30 marks]

G is a simple directed graph

$G = (V, E)$

$V = \{v_1, v_2, v_3, v_4, v_5\}$

$E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$

A is a 5×5 matrix

B is a 5×7 matrix

(i)

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

$$B = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{array} \\ \begin{bmatrix} -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \end{bmatrix} \end{array}$$

(ii)

$$C = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

$$C = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}_{5 \times 5}$$

(iii) To find the rank of a matrix, we use Gaussian elimination.

$$\begin{bmatrix} -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \end{bmatrix} \quad R_2 \leftarrow R_2 + R_1$$

$$\begin{bmatrix} -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \end{bmatrix} \quad R_3 \leftarrow R_3 + R_2$$

$$\begin{bmatrix} -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \end{bmatrix} \quad R_4 \leftarrow R_4 + R_3$$

$$\begin{bmatrix} -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \end{bmatrix} \quad R_5 \leftarrow R_5 + R_4$$

$$\begin{bmatrix} -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

B is in row echelon form now. This implies $\operatorname{rank}(B) = 4$ because of the non~~z~~zero rows in REF.

(iv) Show that $B^T\vec{\mathbf{1}} = \vec{\mathbf{0}}$, where $\vec{\mathbf{1}}$ is the (column) vector whose components are all equal to 1, and $\vec{\mathbf{0}}$ is the (column) vector whose components are all equal to 0. Based on $B^T\vec{\mathbf{1}} = \vec{\mathbf{0}}$, what can you conclude about eigenvalues and eigenvectors of BB^T?

$$B = \begin{bmatrix} -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \end{bmatrix}_{5\times 7}$$

$$B^T\vec{1} = \underbrace{\begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 & 0 \end{bmatrix}}_{7\times 5} \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}_{5\times 1} = \begin{bmatrix} -1+1 \\ -1+1 \\ -1+1 \\ 1-1 \\ -1+1 \\ 1-1 \\ -1+1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{7\times 1} = \vec{0}$$

So, $B^T\vec{1} = \vec{0}$. Multiply B to the both sides, we have

$$\underset{5\times 7}{B}\,\underset{7\times 5}{B^T}\,\underset{5\times 1}{\vec{1}} = \underset{5\times 7}{B}\,\underset{7\times 1}{\vec{0}} \rightarrow \underbrace{(BB^T)}_{5\times 5}\underbrace{\vec{1}}_{5\times 1} = \underset{5\times 1}{\vec{0}}$$

$$\rightarrow (BB^T)\vec{1} = (\underset{\text{Scalar}}{0})\underset{5\times 1}{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}$$

Therefor, $\lambda = 0$ is an eigenvalue, and $\vec{1}_{5\times 1}$ is an eigenvector of BB^T.

(v) Set $L = BB^T$. Show that $L = D - C$, where $D = \text{diag}(\deg(v_1), \deg(v_2), \ldots, \deg(v_5))$, namely, the degrees of vertices are on the main diagonal of D.

$$B = \begin{bmatrix} -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \end{bmatrix}_{5\times 7} \qquad B^T = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 & 0 \end{bmatrix}_{7\times 5}$$

$\deg(v_1) = \text{in-degree} + \text{out-degree} = \deg^-(v_1) + \deg^+(v_1)$

$\deg(v_1) = 0 + 2 = 2 \qquad \deg(v_2) = 2 + 2 = 4 \qquad \deg(v_3) = 1 + 2 = 3$

$\deg(v_4) = 2 + 1 = 3 \qquad \deg(v_5) = 1 + 1 = 2$

$$D = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}_{5\times 5}$$

$$L = BB^T = \begin{bmatrix} -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \end{bmatrix}_{5\times 7} \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 & 0 \end{bmatrix}_{7\times 5}$$

$$L = \begin{bmatrix} 2 & -1 & -1 & 0 & 0 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 3 & -1 & 0 \\ 0 & -1 & -1 & 3 & -1 \\ 0 & -1 & 0 & -1 & 2 \end{bmatrix}_{5\times 5}$$

$$D - C = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}_{5\times 5} - \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}_{5\times 5}$$

$$= \begin{bmatrix} 2 & -1 & -1 & 0 & 0 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 3 & -1 & 0 \\ 0 & -1 & -1 & 3 & -1 \\ 0 & -1 & 0 & -1 & 2 \end{bmatrix}_{5\times 5}$$

Therefore, $L = D - C$.

(vi) Find $\det(L)$ without direct calculation.

you may decide to find the det using the definition. But it needs a lot work!

$$\det(L) = 2 \det \begin{bmatrix} 4 & -1 & -1 & -1 \\ -1 & 3 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$$

$$- (-1) \det \begin{bmatrix} -1 & -1 & -1 & -1 \\ -1 & 3 & -1 & 0 \\ 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$$+ (-1) \det \begin{bmatrix} -1 & 4 & -1 & -1 \\ -1 & -1 & -1 & 0 \\ 0 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix} \quad \cdots$$

(not efficient)

Or, you may decide to use Gaussian elimination,

$$\begin{bmatrix} 2 & -1 & -1 & 0 & 0 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 3 & -1 & 0 \\ 0 & -1 & -1 & 3 & -1 \\ 0 & -1 & 0 & -1 & 2 \end{bmatrix} \qquad \begin{aligned} R_2 &\leftarrow R_2 + \tfrac{1}{2} R_1 \\ R_3 &\leftarrow R_3 + \tfrac{1}{2} R_1 \end{aligned}$$

$$\begin{bmatrix} 2 & -1 & -1 & 0 & 0 \\ 0 & 7/2 & \frac{-3}{2} & -1 & -1 \\ 0 & -3/2 & 5/2 & -1 & 0 \\ 0 & -1 & -1 & 3 & -1 \\ 0 & -1 & 0 & -1 & 2 \end{bmatrix} \qquad \begin{aligned} R_3 &\leftarrow R_3 + \tfrac{3}{7} R_2 \\ R_4 &\leftarrow R_4 + \tfrac{2}{7} R_2 \\ R_5 &\leftarrow R_5 + \tfrac{2}{7} R_2 \end{aligned}$$

$$\begin{bmatrix} 2 & -1 & -1 & 0 & 0 \\ 0 & 7/2 & \frac{-3}{2} & -1 & -1 \\ 0 & 0 & 13/7 & -10/7 & -3/7 \\ 0 & 0 & -10/7 & 19/7 & -\frac{9}{7} \\ 0 & 0 & -3/7 & -\frac{9}{7} & 12/7 \end{bmatrix} \qquad \begin{aligned} R_4 &\leftarrow R_4 + \tfrac{10}{13} R_3 \\ R_5 &\leftarrow R_5 + \tfrac{3}{13} R_3 \end{aligned}$$

$$\begin{bmatrix} 2 & -1 & -1 & 0 & 0 \\ 0 & 7/2 & \frac{-3}{2} & -1 & -1 \\ 0 & 0 & 13/7 & -10/7 & -3/7 \\ 0 & 0 & 0 & 21/13 & -21/13 \\ 0 & 0 & 0 & -21/13 & 21/13 \end{bmatrix} \qquad R_5 \leftarrow R_5 + R_4$$

$$\begin{bmatrix} 2 & -1 & -1 & 0 & 0 \\ 0 & 7/2 & \frac{-3}{2} & -1 & -1 \\ 0 & 0 & 13/7 & -10/7 & -3/7 \\ 0 & 0 & 0 & 21/13 & -21/13 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

det(L) is the product of pivots in its row echelon form. Therefore, det(L) = 0.

You see that this approach is time consuming!

However, we know that $B^T\vec{1} = \vec{0}$ from (iv)

multiply B to the both sides

$$\underset{5\times7}{B}\,\underset{7\times5}{B^T}\,\underset{5\times1}{\vec{1}} = \underset{5\times7}{B}\,\underset{7\times1}{\vec{0}}$$

$$\underset{5\times5}{L}\,\underset{5\times1}{\vec{1}} = \underset{5\times1}{\vec{0}}$$

which can be written as

$$\underset{5\times5}{L}\,\underset{5\times1}{\vec{1}} = (0)\,\underset{5\times1}{\vec{1}}$$

so zero is an eigenvalue of L too.

Knowing that det(L) is the product of its eigenvalues, so, det(L) = 0.

(vii) Show that all the eigenvalues of L are nonnegative.

We can use three approaches to prove this.

1. using the theory of eigenvalues and eigenvectors
2. using the concept of a positive semidefinite matrix. If a matrix is positive semidefinite, then all its eigenvalues are non-negative.
3. using this fact that

the number of positive pivots =

the number of positive eigenvalues

1. $L = BB^T$

Let's λ is an eigenvalue of L and

$\vec{v}$ is the corresponding eigenvector.

$$L\vec{v} = \lambda\vec{v}$$

note that if I divide both side by $\|\vec{v}\|$

I have: $\frac{1}{\|\vec{v}\|} L\vec{v} = \frac{1}{\|\vec{v}\|} \lambda\vec{v}$

$\rightarrow L\left(\frac{\vec{v}}{\|\vec{v}\|}\right) = \lambda \frac{\vec{v}}{\|\vec{v}\|} \Rightarrow L\hat{v} = \lambda\hat{v}.$

In other words, the unit vector $\hat{v}$ is an eigenvector corresponding to λ.

Here you need to consider some facts:

- $\|\hat{v}\| = 1$
- $\|\hat{v}\|^2 = 1$
- $\|\vec{x}\|^2 = \vec{x}^T \cdot \vec{x}$ $\quad$ $\vec{x}$ is a vector
- $L = BB^T \Rightarrow \boxed{BB^T\hat{v} = \lambda\hat{v}}$

Now:

$$\lambda = \|\hat{v}\|^2 \lambda$$

$$= (\hat{v}^T \cdot \hat{v})\lambda = \hat{v}^T \cdot (\lambda \hat{v})$$

$$= \hat{v}^T \cdot (\boxed{BB^T \hat{v}}) = \hat{v}^T \cdot B B^T \hat{v}$$

$$= (B^T \hat{v})^T (B^T \hat{v}) = \|B^T \hat{v}\|^2 \geq 0$$

2

Definition of positive semidefinite (PSD) matrices: $M_{n\times n}$ is PSD if and only if

For all $\vec{0} \neq \vec{x} \in \mathbb{R}^n$ $\vec{x}^T M \vec{x} \geq 0$.

If a matrix is PSD, then all its eigenvalues are non-negative.

consider

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \in \mathbb{R}^5.$$

Let's see whether L is a PSD matrix.

I will compute $L\vec{x}$ first. then I multiply $\vec{x}^T$ to it from the left.

$$L\vec{x} = \begin{bmatrix} 2 & -1 & -1 & 0 & 0 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 3 & -1 & 0 \\ 0 & -1 & -1 & 3 & -1 \\ 0 & -1 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

$$= \begin{bmatrix} 2x_1 - x_2 - x_3 \\ -x_1 + 4x_2 - x_3 - x_4 - x_5 \\ -x_1 - x_2 + 3x_3 - x_4 \\ -x_2 - x_3 + 3x_4 - x_5 \\ -x_2 - x_4 + 2x_5 \end{bmatrix}$$

Then

$$\vec{x}^T(L\vec{x}) = [x_1\ x_2\ x_3\ x_4\ x_5]\begin{bmatrix} 2x_1 - x_2 - x_3 \\ -x_1 + 4x_2 - x_3 - x_4 - x_5 \\ -x_1 - x_2 + 3x_3 - x_4 \\ -x_2 - x_3 + 3x_4 - x_5 \\ -x_2 - x_4 + 2x_5 \end{bmatrix}$$

$$= 2x_1^2 - x_1x_2 - x_1x_3 - x_1x_2 + 4x_2^2 - x_2x_3 - x_2x_4 - x_2x_5 - x_3x_1 - x_3x_2 + 3x_3^2 - x_3x_4 - x_4x_2 - x_4x_3 + 3x_4^2 - x_4x_5 - x_5x_2 - x_4x_5 + 2x_5^2$$

$$\begin{aligned} = {} & x_1^2 + x_2^2 - 2x_1x_2 \\ & x_1^2 + x_3^2 - 2x_1x_3 \\ & x_2^2 + x_3^2 - 2x_2x_3 \\ & x_2^2 + x_4^2 - 2x_2x_4 \end{aligned}$$

$$x_2^2 + x_5^2 - 2x_2x_5$$

$$x_3^2 + x_4^2 - 2x_3x_4$$

$$x_4^2 + x_5^2 - 2x_4x_5$$

$$= (x_1-x_2)^2 + (x_1-x_3)^2 + (x_2-x_3)^2 + (x_2-x_4)^2 +$$
$$(x_2-x_5)^2 + (x_3-x_4)^2 + (x_4-x_5)^2 \geq 0$$

$\rightarrow$ L is a positive semidefinite matrix $\Longrightarrow$ all $\lambda_i \geq 0$.

3. we have seen the row echlon form of L as

$$\begin{bmatrix} 2 & -1 & -1 & 0 & 0 \\ 0 & 7/2 & -\frac{3}{2} & -1 & -1 \\ 0 & 0 & 13/7 & -10/7 & -3/7 \\ 0 & 0 & 0 & 21/13 & -\frac{21}{13} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

all pivots ≥ 0, then all $\lambda_i \geq 0$.

(*) Also, we can show this using the concept of leading determinants (n upper-left determinants), but we don't cover it.

(viii) Without direct calculation, show that one of the eigenvalues of L is zero and the corresponding eigenvector is $\vec{\mathbf{1}}$.

We have already seen that

$$B^T \vec{1} = \vec{0}$$

Multiply from left by B we have

$$B(B^T\vec{1}) = B(\vec{0}) = \vec{0}$$

$$L\vec{1} = \vec{0} = (0)\vec{1}$$

So, $\lambda = 0$ is an eigenvalue of L

and $\vec{1}$ is an eigenvector.

* Be careful with the sizes of matrices:

$$\underset{5\times 7}{B}, \quad \underset{7\times 5}{B^T}, \quad \underset{5\times 5}{L}$$

$$\underset{7\times 5}{B^T}\underset{5\times 1}{\vec{1}} = \underset{7\times 1}{\vec{0}} = (0)\underset{7\times 1}{\vec{1}}$$

$$\underset{5\times 5}{L}\underset{5\times 1}{\vec{1}} = \underset{5\times 1}{\vec{0}} = (0)\underset{5\times 1}{\vec{1}}$$

$\vec{1}_{7\times 1}$ and $\vec{1}_{5\times 1}$ are different vectors.

However, they are vectors with 1s.