Assignment 1 - Prateek Singh - 221218743

Q1 $x^{1-\frac{1}{3}\ln(x^2)} = \frac{1}{\sqrt[3]{e^2}}$. Find x.

Assuming that $x \in R$ we can write this as:

$$x^{1-\frac{1}{3}\ln(x^2)} = \frac{1}{e^{2/3}}$$

Taking ~~log~~ ln both sides to eliminate the exponents

$$\Rightarrow \ln(x)\left(1-\frac{\ln(x^2)}{3}\right) = -\frac{2}{3}$$

$$\Rightarrow \ln(x) - \left(\ln(x) \times \frac{\ln(x^2)}{3}\right) = -2/3$$

$$\Rightarrow \ln(x) - \frac{2\ln^2(x)}{3} = -2/3$$

$$\Rightarrow \ln^2(x) - \frac{3\ln(x)}{2} = 1$$

Completing the square on the left hand side.

$$\ln^2(x) + \frac{9}{16} - \frac{3\ln(x)}{2} = \frac{25}{16}$$

$$\left(\ln(x) - \frac{3}{4}\right)^2 = \frac{25}{16}$$

$$\ln(x) - \frac{3}{4} = \frac{5}{4} \qquad \ln(x) - \frac{3}{4} = -\frac{5}{4}$$

$\rightarrow (1)$ $\qquad$ $\rightarrow (2)$

Using (1)

$\ln(x) = 2$

Taking e of both sides

$\boxed{x = e^2}$

Using (2)

$\log(x) = -1/2$

$\boxed{x = \frac{1}{\sqrt{e}}}$ ⇐ (Taking e of both sides)

We can say that x takes values of

$x = e^2$ or $x = \frac{1}{\sqrt{e}}$

Defined for $\forall x \in R$

ii) $a + (a - a^{-1})^{-1}$

We can write this as

$$a + \frac{1}{(a - a^{-1})}$$

Further

$$= a + \frac{1}{a - \frac{1}{a}}$$

We can see that for values $a = 0, a = -1, a = 1$ the expression becomes not defined

$\Rightarrow \{a \in R : x \neq \{-1, 0, 1\}\}$

Rough

$$\frac{a}{a^2 - 1} + a$$

$$\frac{a}{(a+1)(a-1)} + a$$

$\Rightarrow a \neq -1$

$a \neq 1$

$a \neq 0$

Further simplifying:

$$a + \frac{1}{\left(\frac{a^2-1}{a}\right)}$$

$$= a + \frac{a}{a^2-1}$$

$$= a + \frac{a}{a^2-1^2}$$

[using the standard expansion of

$$x^2 - y^2 = (x+y)(x-y)$$]

$$= a + \frac{a}{(a-1)(a+1)}$$

Taking LCM and combining the fractions into one term

$$= \frac{a(a-1)(a+1) + a}{(a-1)(a+1)}$$

$\Rightarrow$ Taking a common and grouping we get:

$$\frac{a^3 + (a-a)}{(a-1)(a+1)}$$

$$= \frac{a^3}{(a-1)(a+1)}$$

Q2.) $y = x^3 - 2x^2$

i) x & y intercepts

finding x intercept, we put $y = x^3 - 2x^2 = 0$

$\Rightarrow x^3 - 2x^2 = 0$

Taking x^2 common

$$\underbrace{x^2}\underbrace{(x-2)} = 0$$

Either

$$\underbrace{x^2 = 0}_{i)} \quad \text{or} \quad \underbrace{x - 2 = 0}_{ii)}$$

In ii) we can see that $\boxed{x = 2}$

In i): we can see that $\boxed{x^2 = 0 \Rightarrow x = 0}$

Plugging this in $y = x^3 - 2x^2$ we get $\boxed{(0,0)\ \&\ (2,0)}$.

For y intercepts we put $x = 0$ x-Intercepts.

$\Rightarrow y = 0^3 - 20^2$

$\Rightarrow y = 0$ & plugging this in $y = x^3 - 2x^2$ we have $x = 0$

$\boxed{y\text{ intercept} = (0,0)}$

ii) Find all the stationary points and classify them.

Stationary points are points on curve $y = f(x)$ where the slope is 0. i.e $y = f(x)$ & if $f'(c) = 0$ then c is stationary.

Therefore to find stationary points we find $f'(x)$

$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(x^3 - 2x^2)$

$= 3x^2 - 4x$

$f'(x) = x(3x-4)$

Stationary points where $f'(x) = 0$

$\Rightarrow x(3x-4) = 0$ $\boxed{\Rightarrow x = 0 \text{ or } x = \frac{4}{3}}$

As $x(3x-4)$ is a polynomial so it is continuous. It is also defined $\forall x \in R$ $(-\infty, \infty)$

For classification we use first derivative test on our intervals $(-\infty, 0)$, $(0, 4/3)$ & $(\frac{4}{3}, \infty)$ for points $x = \{-1, 1, 2\}$

iii) Express the intervals for which the fun^n is inc or dec

$x = -1$, $f'(x) = 7$ $(-\infty, 0) \rightarrow$ increasing in this

$x = 1$, $f'(x) = -1$ $(0, 4/3) \rightarrow$ decreasing

$x = 2$, $f'(x) = 4$ $(4/3, \infty) \rightarrow$ increasing

* This tells us at $x = 0$ we have maxima

& at $x = 4/3$ we have minima

iv) sketch the function by hand

$x = 0$, maxima $y = 0$

$x = 4/3 \approx 1.33$ minima $y = \frac{64}{729} - \frac{16 \times 2}{9} \approx \underline{-3.5}$

$x = -1$, $x = 1$, $x = 2$ [Test points]

Pluging in $y = x^3 - 2x^2$, for all test points

$y = -1 - 2 = -3$ $(x = -1)$

$y = 1 - 2 = -1$ $(x = 1)$

$y = 8 - 8 = 0$ $(x = 2)$

Point of intersects

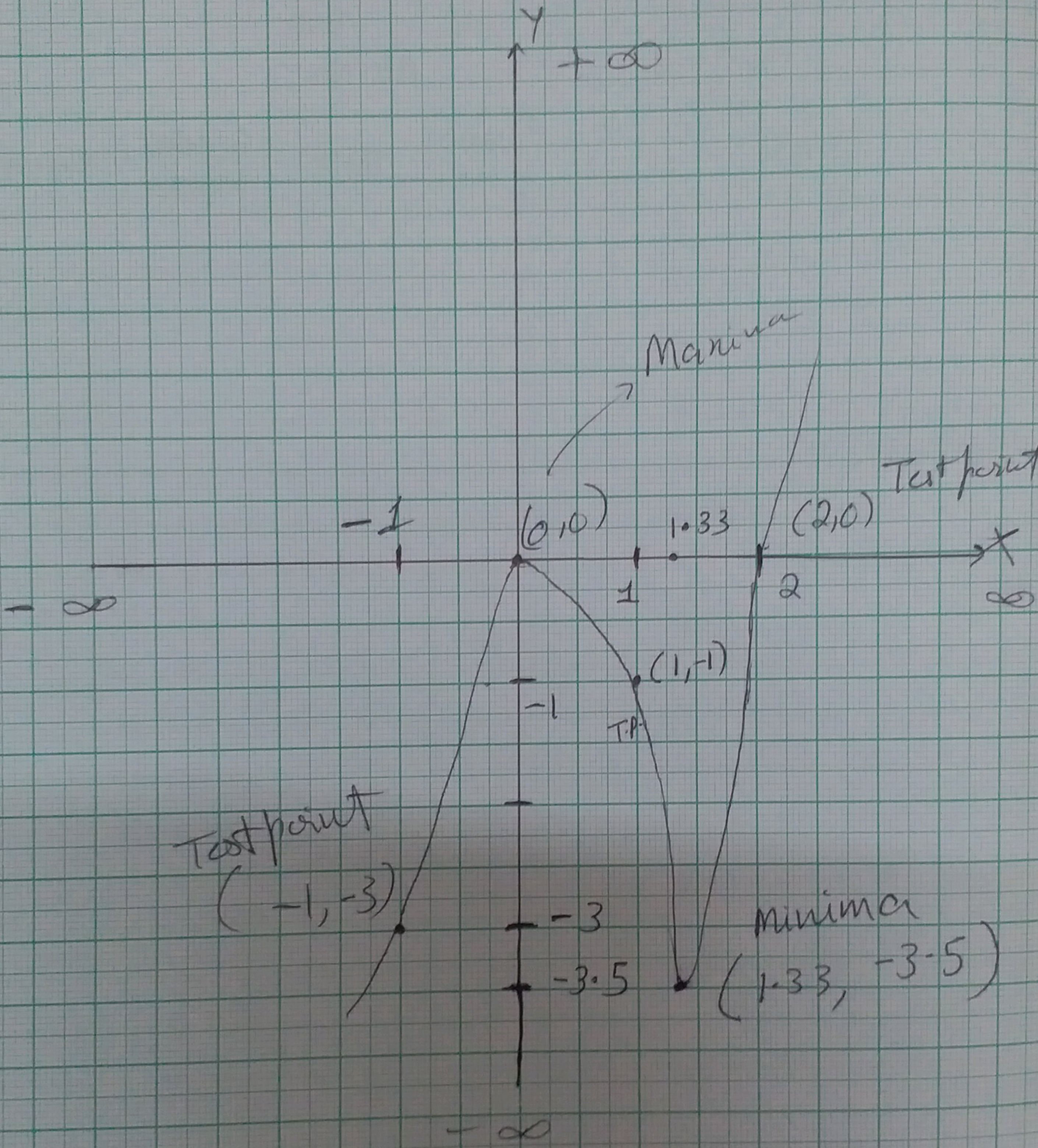
$(x_1 y_1) = (0, 0) \longrightarrow$ Maxima

$(x_2 y_2) = (4/3, -3.5) \longrightarrow$ Minima $(1.33, -3.5)$

$(x_3 y_3) = (-1, -3) \longrightarrow$ Test point

$(x_4 y_4) = (1, -1) \longrightarrow$ Test point

$(x_5 y_5) = (2, 0) \longrightarrow$ Test point



Y
+∞
Maxima
Test point
(0,0)
-1
1.33
(2,0)
X
-∞
1
2
∞
(1,-1)
-1
T.P.
Test point
(-1, -3)
-3
minima
-3.5
(1.33, -3.5)
-∞

3.) Find the derivatives of the following functions!

i) $f(x) = e^{x\ln(a)} + e^{a\ln(x)} + e^{a\ln(a)}$

Rewriting this expression as:

$\left[e^{a\ln(x)} = x^a,\ e^{x\ln(a)} = a^x,\ e^{a\ln(a)} = a^a \right.$

$\left. \text{as } e^{\ln(x)} = x \right]$

$f(x) = x^a + a^x + a^a = y$ (let)

$\frac{d}{dx}(x^a + a^x + a^a) = \frac{dy}{dx}$

$= \frac{d}{dx}x^a + \frac{d}{dx}a^x + \frac{d}{dx}a^a$

$\because \frac{d}{dx}(\text{constant}) = 0$

$$= \frac{d}{dx} x^a + \frac{d}{dx} a^x + 0$$

$$= a x^{a-1} + \cancel{a^x \log a} \; a^x \ln(a)$$

$$\boxed{\Rightarrow \frac{d}{dx}\left(e^{x\ln(a)} + e^{a\ln(x)} + e^{a\ln(a)}\right) = a x^{a-1} + a^x \ln a}$$

~~(ii)~~ $\cancel{\frac{d}{dx}(\sqrt{2x}}$

ii) $y = \sqrt{2x}\,(x - \sin(2x))$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left(\sqrt{2x}\,(x - \sin(2x)\right)$$

$$= \frac{d}{dx}\left(\sqrt{2}\sqrt{x}\,(x - \sin(2x)\right)$$

$$= \sqrt{2}\left(\frac{d}{dx}\left(\sqrt{x}\,(x - \sin(2x)\right)\right)$$

Using product rule, $\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$, where

$u = \sqrt{x}$ & ~~[illegible]~~ $v = x - \sin(2x)$

$$= \left(\sqrt{x}\left(\frac{d}{dx}(x - \sin(2x))\right) + \left(\frac{d}{dx}\sqrt{x}\right)(x - \sin(2x))\right)$$

$$\times \sqrt{2}$$

$$= \sqrt{2}\left(\left(\frac{d}{dx}\sqrt{x}\right)(x-\sin(2x)) + \left(\frac{d}{dx}(x) - \frac{d}{dx}(\sin(2x))\right) \cdot \sqrt{x}\right)$$

Derivative of $x = 1$

$$= \sqrt{2}\left(\left(\frac{d}{dx}(\sqrt{x})\right)(x-\sin(2x)) + \sqrt{x}\left(-\frac{d}{dx}\sin 2x\right)+1\right)$$

Using chain rule $\frac{d}{dx}(\sin 2x) = \frac{d\sin u}{du}\frac{du}{dx}$, where $u = 2x$

& $\frac{d}{du}(\sin(u)) = \cos u$

$$= \sqrt{2}\left(\left(\frac{d}{dx}\sqrt{x}\right)(x-\sin 2x) + \sqrt{x}\left(1-\cos 2x\left(\frac{d}{dx}(2x)\right)\right)\right)$$

$$= \sqrt{2}\left(\left(\frac{d}{dx}\sqrt{x}\right)(x-\sin 2x)\right) + \sqrt{x}\left(1-2\left(\frac{d}{dx}(x)\right)\cos 2x\right)\right)$$

$$= \sqrt{2}\left(\sqrt{x}\left(1-2\cos 2x\frac{d}{dx}x\right) + \left(\frac{d}{dx}\sqrt{x}\right)(x-\sin 2x)\right)$$

$$= \sqrt{2}\left(\sqrt{x}\left(1-2\cos 2x + (x-\sin 2x)\frac{1}{2\sqrt{x}}\right)\right)$$

$$\boxed{\therefore \frac{dy}{dx} = \frac{4x\cos 2x + \sin 2x - 3x}{\sqrt{2}\sqrt{x}}}$$

iii) $y = e^{3\sin(2x)}$

$$\frac{dy}{dx} = \frac{d}{dx} e^{3\sin(2x)}$$

Using chain rule, $\frac{d}{dx}\left(e^{3\sin 2x}\right) = \frac{de^{u}}{du}\frac{du}{dx}$

where $u = 3\sin(2x)$ & $\frac{d}{du}e^{u} = e^{u}$

$$= e^{3\sin 2x}\left(\frac{d}{dx} 3\sin 2x\right)$$

Using chain rule $\frac{d}{dx}\sin 2x = \frac{d\sin u}{du}\frac{du}{dx}$ where $u = 2x$ & $\frac{d}{du}\sin u = \cos u$

$$= \left(\cos 2x \frac{d}{dx} 2x\right)\left(3e^{3\sin 2x}\right)$$

$$\boxed{\frac{dy}{dx} = 6e^{3\sin 2x}\cos 2x}$$

(iv) $y = \dfrac{1}{\ln\left(x + \sqrt{1+x^2}\right)}$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{\ln\left(x+\sqrt{1+x^2}\right)}\right)$$

$$\cancel{\frac{d}{dx}\left(\frac{1}{\ln(1+x+x^2)}\right)}$$

Using the chain rule, $\dfrac{d}{dx}\left(\dfrac{1}{\log\left(\sqrt{x^2+1}+x\right)}\right)$

$= \dfrac{d}{du}\dfrac{1}{u}\dfrac{du}{dx}$, where $u = \log\left(\sqrt{x^2+1}+x\right)$

& $\dfrac{d}{du}\dfrac{1}{u} = \dfrac{1}{u^2}$

$$= -\frac{\frac{d}{dx}\left(\log\left(x+\sqrt{1+x^2}\right)\right)}{\log^2\left(x+\sqrt{1+x^2}\right)}$$

Using chain rule again $\dfrac{d}{dx}\left(\left(\log\sqrt{x^2+1}\right)+x\right)$

$= \dfrac{d\log u}{du}\dfrac{du}{dx}$ where $u = \sqrt{x^2+1}+x$ & $\dfrac{d\log u}{du} = -$

$$= -\frac{1}{\log^2\left(x+\sqrt{1+x^2}\right)}\left(\frac{\frac{d}{dx}\left(x+\sqrt{1+x^2}\right)}{x+\sqrt{1+x^2}}\right)$$

$$= -\frac{1}{\log^2\left(x+\sqrt{1+x^2}\right)\left(x+\sqrt{1+x^2}\right)} \left(\frac{d}{dx}\left(\sqrt{1+x^2}\right)+1\right)$$

Using the chain rule

$$\frac{d}{dx}\left(\sqrt{x^2+1}\right) = \frac{d\sqrt{u}}{du}\frac{du}{dx}, \text{ where } u = x^2+1$$

$$\& \ \frac{d}{du}\sqrt{u} = \frac{1}{2\sqrt{u}}$$

$$= -\frac{1+\left(\frac{\frac{d}{dx}(1+x^2)}{2\sqrt{1+x^2}}\right)}{\left(x+\sqrt{1+x^2}\right)\log^2\left(x+\sqrt{1+x^2}\right)}$$

$$= -\frac{1+\left(\frac{d}{dx}(1)+\frac{d}{dx}x^2\right)\frac{1}{2\sqrt{1+x^2}}}{\left(x+\sqrt{1+x^2}\right)\log^2\left(x+\sqrt{1+x^2}\right)}$$

$$= -\frac{1+\left(\frac{d}{dx}x^2 / 2\sqrt{1+x^2}\right)}{\left(x+\sqrt{1+x^2}\right)\log^2\left(x+\sqrt{1+x^2}\right)}$$

$$= -\left(1+\frac{x}{\sqrt{1+x^2}}\right) \Big/ \left(x+\sqrt{1+x^2}\right)\log^2\left(x+\sqrt{1+x^2}\right)$$

Answer:

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1+x^2}\,\log^2\left(x+\sqrt{1+x^2}\right)}$$

4 i) $f(x) = x^2 \sin(x)$, first three non-zero terms $= \tilde{f}(x)$

For a function $y = f(x)$, we use following formula for Maclaurin series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

where $f^{(n)}(0)$ is the n^{th} derivative calculated at 0. Using this condition, we find first three non-zero terms

$\Rightarrow$ for $f(x) = x^2 \sin(x)$

$$f'(x) = \frac{d}{dx}(x^2 \sin(x))$$

Using product rule:

$$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$$, where $u = x^2$ & $v = \sin x$

$$= x^2\left(\frac{d}{dx}\sin x\right) + \left(\frac{d}{dx}x^2\right)\sin x$$

$$= x^2 \cos x + \cancel{2x\cos x}\ 2x\sin x$$

$f'(0) = 0$

$$f''(x) = \frac{d}{dx}(f'(x))$$

$$= \frac{d}{dx}(x^2\cos x + 2x\sin x)$$

$$= \frac{d}{dx}(x^2 \cos x) + 2\frac{d}{dx} x\sin x$$

[=> Using product rule

$$\frac{d(uv)}{dx} = \frac{v\,du}{dx} + u\frac{dv}{dx}, \text{ where } u = x^2 \ \& \ v = \cos x$$

$$= 2\left(\frac{d}{dx} x\sin x\right) + \cos x \frac{d}{dx} x^2 + x^2 \frac{d}{dx}\cos x$$

$$= 2\left(\frac{d}{dx} x\sin x\right) + 2x\cos x - x^2\sin x$$

=> Using product rule again

$$\frac{d(uv)}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}, \text{ where } u = x \ \& \ v = \sin(x)$$

$$= 2x\cos x - x^2\sin x + 2\left(x\frac{d}{dx}\sin x + \frac{d}{dx}x \times \sin x\right)$$

$$= 2x\cos x - x^2\sin x + 2\left(x\cos x + \sin x\right)$$

$$f''(x) = 4x\cos x - x^2\sin x + 2\sin x$$

$$f''(0) = (4 \times 0 \times 1) - (0 \times 0) + (2 \times 0)$$

$$= 0$$

$$f'''(x) = \frac{d}{dx} f''(x)$$

$$= \frac{d}{dx}\left(4x\cos x - x^2 \sin x + 2\sin x\right)$$

$$= \underbrace{4\left(\frac{d}{dx} x\cos x\right)}_{\text{Using product rule}} - \frac{d}{dx} x^2 \sin x + 2\left(\frac{d}{dx}\sin x\right)$$

$$= 4\left(\cos x \frac{d}{dx} x + x \frac{d}{dx}(\cos x)\right) - \left(\frac{d}{dx} x^2 \sin x\right) + 2\left(\frac{d}{dx}\sin x\right)$$

$$= \left(4(\cos x - x\sin x)\right) - \underbrace{\left(\frac{d}{dx} x^2 \sin x\right)}_{} + 2\frac{d}{dx}\sin x$$

Using ~~chain~~ product rule $\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$

where $u = x^2$ & $v = \sin(x)$

$$= \left(4(\cos x - x\sin x)\right) - \left(x^2 \frac{d}{dx}\sin x + \left(\frac{d}{dx}x^2\right)\sin x\right) + 2\frac{d}{dx}\sin x$$

$$= 4(\cos x - x\sin x) - x^2\cos x - 2x\,\cancel{\cos x}\,\sin x + 2\cos x$$

$$= 6\cos x - 2x\,\cancel{\cos}\,\sin x - 4x\sin x - x^2\cos x$$

$$= (6 - x^2)\cos x - 6x\sin(x)$$

$$f'''(0) = 6 - 0 = 6$$

$$f^4(x) = \frac{d}{d(x)} f^3(x)$$

$$= \frac{d}{dx} (6-x^2)\cos x - 6x\sin x$$

Using product rule

$$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}, \text{ where } u = 6-x^2$$
$$\& \; v = \cos x$$

$$= -6\left(\frac{d}{dx} x\sin x\right) + \left(\cos x\left(\frac{d}{dx}(6-x^2)\right)\right) + \left((6-x)^2 \frac{d}{dx}\cos x\right)$$

$$- (6-x)^2 \sin x$$

$$= -6\left(\frac{d}{dx} x\sin x\right) - 2x\cos x$$

$\Rightarrow$ Using product rule $\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$

where $u = x$ & $v = \sin x$

$$= -2x\cos x - (6-x^2)\sin(x) - 6\left(x\left(\frac{d}{dx}(\sin x)\right) + \frac{dx}{dx}\sin x\right)$$

$$= -2x\cos x - (6-x^2)\sin(x) - 6(x\cos x + \sin x)$$

$$= -2x\cos x - (6-x^2)\sin x - 6x\cos x - 6\sin x$$

$$f^4(0) = 0 - 0 - 0 - 0 = 0$$

$$f^5(x) = \frac{d}{dx} f^4(x)$$

$$= \frac{d}{dx}\left(-8x\cos x - (6-x^2)\sin x - 6\sin x\right)$$

$$= -8\frac{d}{dx}x\cos x - \frac{d}{dx}\left((6-x^2)\sin x\right) - 6\frac{d}{dx}\sin x$$

$$= -8(\cos x - x\sin x) - \frac{d}{dx}\left(6\sin x - x^2\sin x\right) - 6\frac{d}{dx}\sin x$$

[Using previously calculated values]

$$= -8\cos x + 8x\sin x - 6\cos x - (x^2\cos x + 2x\sin x)$$
$$-6\cos x$$

$$= -20\cos x + 8x\sin x - x^2\cos x + 2x\sin x$$
$$= (x^2 - 20)\cos x + 10x\sin x$$

$$f^5(0) = -20\cos 0 + 0$$
$$= -20$$

$$f^6(x) = \frac{d}{dx} f^5(x)$$

$$= \frac{d}{dx}(x^2-20)\cos x + \frac{d}{dx} 10x\sin x$$

$$= \frac{d}{dx} x^2\cos x - \frac{d}{dx} 20\cos x + \frac{d}{dx} 10x\sin x$$

Using previously calculated values:

$$= 2x\cos x - x^2\sin x + 20\sin x + 10(x\cos x + \sin x)$$

$$= 2x\cos x - x^2\sin x + 30\sin x + 10x\cos x$$

$$= 12x\cos x - x^2\sin x + 30\sin x$$

$$f^6(0) = 0 - 0 + 0 = 0$$

$$f^7(x) = \frac{d}{dx} f^6(x)$$

$$= \frac{d}{dx} 12x\cos x - \frac{d}{dx} x^2\sin x + \frac{d}{dx} 30\sin x$$

$$= 12(\cos x - x\sin x) - x^2\cos x - 2x\sin x + 30\cos$$

$$= 42\cos x - 14\sin x - x^2\cos x$$

$$f^7(0) = 42 - 0 - 0$$

After $f^7(x)$ at $x = 0$, we have our first 3 non zero terms.

$$\Rightarrow \tilde{f}(x) = x^3 - \frac{x^5}{6} + \frac{x^7}{120}$$

ii) Use expression in i) to find an approximation of

$$I(a) = \int_1^a x^2 \sin x \, dx \approx \int_1^a \tilde{f}(x) \, dx$$

Q.5.)

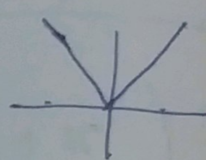
i) $f(x) = |2x-3|$

$y = |-3+2x|$

$\Rightarrow$ Equating it to 0, to find the corner poin of mod curve

$2x+3 = 0$

$x=1.5$

$\Rightarrow x = -\frac{3}{2} \Rightarrow x = \frac{3}{2}$ as $|x| =$ 

& y is always +ve.

$\Rightarrow y = |2x-3|$ is differentiable when $x \neq 3/2$

$$\frac{dy}{dx} = \frac{d}{dx}|2x-3|$$

using the chain rule, $\frac{d}{dx}|2x-3| = \frac{d|u|}{du}\frac{du}{dx}$

where $u = 2x-3$ & $\frac{d|u|}{dx} = \frac{u}{|u|}$

$$= \frac{(2x-3)\frac{d}{dx}(2x-3)}{|2x-3|}$$

$$= \left(2\frac{d}{dx}x + 0\right)\left(\frac{2x-3}{|2x-3|}\right)$$

$$\boxed{= \frac{4x-6}{|2x-3|} = \frac{dy}{dx}}$$

(iii) $f(x) = 2|x-3| - 3|x-2| = y$ (let)

$$\frac{dy}{dx} = \frac{d}{dx}\left(2|x-3| - 3|x-2|\right)$$

Equating mod terms to 0 to check where are the corner points for

$|x-3| \Rightarrow x = 3$ & $|x-2| \Rightarrow x = 2$

$\Rightarrow 2|x-3| - 3|x-2|$ is differentiable over R

when $x \neq 2$ & $x \neq 3$

$$\frac{dy}{dx} = 2\left(\frac{d}{dx}|x-3|\right) - 3\left(\frac{d}{dx}|x-2|\right)$$

Using chain rule, $\frac{d}{dx}(|x-3|) = \frac{d|u|}{du}\frac{du}{dx}$, where

$u = x-3$ & $\frac{d}{du}(|u|) = \frac{u}{|u|}$

$$= -3\left(\frac{d}{dx}|x-2|\right) + 2\left(\frac{(x-3)\left(\frac{d}{dx}(x-3)\right)}{|x-3|}\right)$$

$$= -3\left(\frac{d}{dx}|x-2|\right) + \frac{2(x-3)}{|x-3|}$$

Using chain rule $\frac{d}{dx}(|x-2|) = \frac{d|u|}{du}\frac{du}{dx}$, where

$u = x-2$ & $\frac{d}{du}|u| = \frac{u}{|u|}$

$$= \frac{2(-3+x)}{|-3+x|} - 3\left(\frac{(x-2)\left(\frac{d}{dx}(x-2)\right)}{|x-2|}\right)$$

$$= \frac{2(x-3)}{|x-3|} - \frac{3(x-2)\frac{d}{dx}x}{|x-2|}$$

$$\boxed{\frac{dy}{dx} = \frac{2(x-3)}{|x-3|} - \frac{3(x-2)}{|x-2|}}$$

6.i) $f(x) = \begin{cases} ax+b & \text{if } x<1 \\ x^4+x+1, & \text{if } x \geq 1 \end{cases}$

Let $ax+b = g(x)$ & $x^4+x+1 = h(x)$

$\Rightarrow f(x) = \begin{cases} g(x) & \text{if } x<1 \\ h(x), & \text{if } x \geq 1 \end{cases}$

for a piecewise function to be continuous there are two neccessary conditions:

a) The function should be continuous everywhere and at the point of joining

b) The derivative of ~~both~~ the components must have same value at the point of joining.

$\Rightarrow$ f(x) is continuous only if $g(x) = h(x)$ at $x=1$

& $g'(x) = h'(x)$ at $x=1$

$\Rightarrow$ At $x=1$

$h(x) = x^4+x+1 = 3$

$\Rightarrow g(x) = 3$ for $f(x)$ to be continuous

$\boxed{\Rightarrow \quad ax+b = 3 \text{ \& as } x=1 \Rightarrow a+b=3}$ —i)

According to 2nd rule

$$g'(x) = h'(x)$$

$$h'(x) = \frac{d}{dx}(x^4 + x + 1)$$

$$= 4x^3 + 1$$

$$g'(x) = \frac{d}{dx}(ax + b)$$

$$= a$$

At $x = 1$

$$h'(1) = 4 + 1$$
$$= 5$$

$$g'(x) = a = 5$$

$\Rightarrow$ substituting $a = 5$ in i) we get

$$5 + b = 3$$
$$\Rightarrow b = -2$$

Therefore we can say that for $a = 5$ and $b = -2$, $f(x)$ is differentiable at every $x \in R$.

6 ii) $f(x) = \begin{cases} \sqrt{x^2-2} & \text{if } x \le 0 \\ x^3 - x & \text{if } x > 0 \end{cases}$

Find derivative & points where this is not differentiable.

For $x \le 0$, $f(x) = \sqrt{x^2-2}$

$\Rightarrow$ $f(x)$ will not be defined when $\sqrt{x^2-2} < 0$

because it will be imaginary.

$\Rightarrow \quad x^2 - 2 > 0 \quad \Rightarrow \quad x^2 > 2$

$\Rightarrow x > \sqrt{2}$ & $x < -\sqrt{2}$

$\Rightarrow \forall x \in R$, $f(x)$ will be differentiable when $x > \sqrt{2}$ & $x < -\sqrt{2}$, for all other x, $f(x)$ not differentiable.

for $x > 0$, $f(x) = x^3 - x$

Since $x^3 - x$ is a polynomial, it is continuous everywhere and is also defined for all $\mathbb{R}$

$\Rightarrow$ In this domain $f(x)$ is differentiable every where.

for finding derivative, we find the derivative over the two domains

For $x \leq 0$, $f(x) = \sqrt{x^2-2}$

$$\Rightarrow \frac{d}{dx}(\sqrt{x^2-2}) = \frac{\frac{d}{dx}(x^2-2)}{2\sqrt{x^2-2}}$$

[Using chain rule

$$\frac{d}{dx}(\sqrt{x^2-2}) = \frac{d\sqrt{u}}{du} \times \frac{du}{dx}$$

where $u = x^2-2$ & $\frac{d\sqrt{u}}{du} = \frac{1}{2\sqrt{u}}$]

$$= \frac{\frac{d}{dx}(x^2)+0}{2\sqrt{x^2-2}}$$

$$= \frac{2x}{2\sqrt{x^2-2}}$$

$$\boxed{f'(x) \text{ for } x \leq 0 = \frac{x}{x^2-2}}$$

$f'(x)$ for $x > 0$

$$= \frac{d}{dx}(x^3 - x) = 3x^2 - 1$$

$$\boxed{\Rightarrow f'(x) = \begin{cases} \frac{x}{x^2-2} & , x \leq 0 \\ 3x^2 - 1 & , x > 0 \end{cases}}$$

7i) Absolute maximum & absolute minimum points of $f(x) = x^{1/2} - x^{3/2}$ in $[0, 4]$

for finding maximum & minimum, we first find the ~~stationary~~ critical points

$$\Rightarrow f'(x) = \frac{1}{2} x^{\frac{1}{2}-1} - \frac{3}{2} x^{3/2 - 1}$$

$$= \frac{1}{2\sqrt{x}} - \frac{3\sqrt{x}}{2}$$

$$= \frac{1 - 3x}{2\sqrt{x}}$$

To find where $f'(x) = 0$

$$\Rightarrow \frac{1-3x}{2\sqrt{x}} = 0 \Rightarrow x = 1/3$$

We only have one critical point at $x = \frac{1}{3}$ for $f(x) = \sqrt{x} - x^{3/2}$

Domain is $\{x \in \mathbb{R}, \text{ where } x \geq 0\}$

$$= [0, \infty)$$

So we have three points $\{0, 1/3, 4\}$

At $x = 1/3$, $f(x) = \frac{2}{3\sqrt{3}}$

$\quad \& \; f'(x) = 0$

$$f''(x) = \frac{d}{dx}(f'(x))$$

$$= \frac{d}{dx}\left(\frac{1-3x}{2\sqrt{x}}\right)$$

Using product rule $\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$

where $u = 1-3x$ & $v = \frac{1}{\sqrt{x}}$

$$= \frac{1}{2}\left(\frac{\frac{d}{dx}(1-3x)}{\sqrt{x}} + (1-3x)\frac{d\frac{1}{\sqrt{x}}}{dx}\right)$$

$$= \frac{1}{2}\left((1-3x)\frac{d(1/\sqrt{x})}{dx} + \left(\frac{-3\frac{d}{dx}x + 0}{\sqrt{x}}\right)\right)$$

$$= \frac{1}{2}\left((1-3x)\times\left(-\frac{1}{2x^{3/2}}\right) - \frac{3}{\sqrt{x}}\right)$$

$$= \frac{1}{2}\left(-\frac{1-3x}{2x^{3/2}} - \frac{3}{\sqrt{x}}\right)$$

$$= \frac{-1-3x}{4x^{3/2}} = f''(x)$$

At $x = 1/3$, $f''(x) = -\frac{3\sqrt{3}}{2}$

As the sign is -ve then according to the second derivative test $f''(c) > 0$, $x = c$ is maxima

We can say

$f(x) = x^{1/2} - x^{3/2}$ has maxima at $x = 1/3$

At $x=0$ $f(x) = 0$

At $x = 4$ $f(x) = -6$

2) in interval $[0,4]$ x has a minima at $x=4$ and maxima at $x = 1/3$

7ii) Find all local max & local min points of

$$f(x) = \begin{cases} 0 & \text{if } x \geq 0 \\ 1-\sqrt{1-x^2} & \text{otherwise} \end{cases}$$

To find the local minimum and local maximum points we find the ~~stationary~~ ~~pointes~~ critical poi

for $x \geq 0$, $f(x) = 0$, $f'(x) = 0 \ \forall x \geq 0$

$\Rightarrow$ This function has a single value for all $x \geq 0$

for $x<0$, $f(x) = 1-\sqrt{1-x^2}$
[domain $x \in R : -1 \leq x < 0$] ∀

$f'(x) =$

$$\frac{d}{dx}\left(1-\sqrt{1-x^2}\right)$$

$$= -\frac{d}{dx}\left(\sqrt{1-x^2}\right) + 0$$

$\Rightarrow$ Using chain rule

$$\frac{d}{dx}\left(\sqrt{1-x^2}\right) = \frac{d\sqrt{u}}{du}\frac{du}{dx}$$ where

$u = 1-x^2$

$$\frac{d}{du}(\sqrt{u}) = \frac{1}{2\sqrt{u}} \Rightarrow \frac{dy}{dx} = -\frac{\frac{d}{dx}(1-x^2)}{2\sqrt{1-x^2}}$$

$$= \frac{\frac{d}{dx}(1) - \frac{d}{dx}(x^2)}{2\sqrt{1-x^2}}$$

$$= \frac{2x}{2\sqrt{1-x^2}} = \frac{x}{\sqrt{1-x^2}} = \frac{dy}{dx}$$

Equating $f'(x) = 0$

$$\frac{x}{\sqrt{1-x^2}} = 0 \Rightarrow x = 0$$

$\Rightarrow$ $x=0$ is the only critical point

* Using first derivative as second derivative will yield x term in denominator only.

Check $f(x)$ at end points & critical points.

$$\text{domain of } f(x) = \begin{cases} 0 & , x \geq 0 \\ 1-\sqrt{1-x^2} & , x < 0 \end{cases}$$

$$= \begin{cases} R \\ [-1, 0) \end{cases}$$

$$\boxed{\Rightarrow \{\text{Domain} \in R : -1 \leq x < \infty\}}$$

$\Rightarrow$ Checking ~~for~~ $f(x)$ at $\{-1, 0\}$

At

$x = -1, \quad f(x) = 1$

$x = 0, \quad f(x) = 0$

$\Rightarrow$ $f(x)$ has maxima at $x = -1$ with value 1

$f(x)$ has ~~minima~~ no one minimum point

as $\forall x \geq 0 \quad f(x) = 0$

$\Rightarrow$ It can be said that $\forall x \geq 0$ $f(x)$ is minimum.