SIT787 -Mathematics for AI

Trimester 2, 2021

Due: no later than the end of Week 7, Sunday 5 September 2021, 8:00pm AEST

Note:

- A proper way of presenting your solutions is part of the assessment. Please follow the order of questions in your submission.
- Your submission can be handwritten but it must be legible. Please write neatly. If I cannot read your solution, I cannot mark it.
- Provide the way you solve the questions. All steps (workings) to arrive at the answer must be clearly shown. I need to see your thoughts.
- For a final answer without a proper justification no score will be given.
- Only (scanned) electronic submission would be accepted via the unit site (Deakin Sync).
- Your submission must be in ONE pdf file. Multiple files and/or in different file format, e.g. .jpg, will NOT be accepted. If you need to change the format of your submission, it will be subject to the late submission penalty.

Question 1) Consider these vectors:

$$\boldsymbol{u} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \boldsymbol{v} = \begin{bmatrix} 2 \\ 2 \\ -2 \\ 4 \end{bmatrix}, \text{ and } \boldsymbol{w} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -2 \end{bmatrix}$$

(i) Determine which two vectors are most similar to each other based on these norms:

(a) ℓ_2 norm:

$$\text{dist}(\boldsymbol{x}, \boldsymbol{y}) = ||\boldsymbol{x} - \boldsymbol{y}||_2 = ||\boldsymbol{y} - \boldsymbol{x}||_2 = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

(b) ℓ_1 norm:

$$\text{dist}(\boldsymbol{x}, \boldsymbol{y}) = ||\boldsymbol{x} - \boldsymbol{y}||_1 = ||\boldsymbol{y} - \boldsymbol{x}||_1 = \sum_{i=1}^{n} |x_i - y_i|$$

(c) ℓ_∞ norm:

$$\text{dist}(\boldsymbol{x}, \boldsymbol{y}) = ||\boldsymbol{x} - \boldsymbol{y}||_\infty = ||\boldsymbol{y} - \boldsymbol{x}||_\infty = \max_{i=1,\ldots,n} |x_i - y_i|$$

(ii) Determine which two vectors are most similar to each other based on the cosine similarity measure (you can use decimals here):

$$\cos(\theta) = \frac{\boldsymbol{x} \cdot \boldsymbol{y}}{||\boldsymbol{x}||||\boldsymbol{y}||}$$

(iii) Explain the reason behind the difference in result between (i) and (ii).

(iv) Can the difference be resolved? Give details of your suggestion, if you have any, and explain the outcome if your suggestions are applied. (you can use decimals here).

[2+2+3+8= 15 marks]

Question 2) Consider the following set of vectors:

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ h \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ k \end{bmatrix} \right\}$$

Based on different values of h and k, discuss the number of independent vectors in this set.

[10 marks]

Question 3) You are tasked with uncovering information about an incomplete matrix, some of whose entries are unknown and denoted as $a, b, c,$ and d :

$$A = \begin{bmatrix} -1 & 0 & a \\ b & 4 & c \\ d & 0 & 0 \end{bmatrix}$$

Given that rank$(A) = 2$, how many distinct eigenvalues does A have? **[10 marks]**

Question 4) Consider the following matrix:

$$A = \begin{bmatrix} 3 & -3 & 0 \\ 3 & -1 & 2 \\ b & 0 & 2 \end{bmatrix}$$

Find the values for b (if possible) so that:

(i) The determinant of A is 4.

(ii) The rank of A is 2.

(iii) $\frac{1}{2}$ is an eigenvalue of A^{-1}.

(iv) The system $A\boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ has no solutions.

(v) The system $A\boldsymbol{x} = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$ has infinitely many solutions.

[5 + 5 + 5 + 5+ 5 = 25 marks]

Now consider

$$A = \begin{bmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix}$$

for questions 5, 6, and 7.

Question 5) Find $S = A^T A$ matrix and

(a) Find the characteristic polynomial of S

(b) Find the eigenvalues of S, and call them λ_1 and λ_2 where $\lambda_1 \geq \lambda_2$.

(c) Find the eigenvectors S.

(d) Are the eigenvectors orthonormal? If they are not, convert them into orthonormal vectors using the Gram-Schmidt process. Call them $\boldsymbol{v_1}, \boldsymbol{v_2}$ after orthonormalisation.

(e) Using the eigenvalues of S, $\lambda_1 \geq \lambda_2$, set $\sigma_1 = \sqrt{\lambda_1}$ and $\sigma_2 = \sqrt{\lambda_2}$ and make this matrix

$$D = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{bmatrix}$$

We will use this matrix later.
Make a matrix V using the orthonormal vectors you obtained from the previous part. The ordering of the columns of V should be the same as the ordering of the eigenvalues, that is $V = \begin{bmatrix} \boldsymbol{v_1} & \boldsymbol{v_2} \end{bmatrix}$. Show that V is an orthogonal matrix.

[3+3+3+3+3= 15 marks]

Question 6) Find $T = AA^T$ matrix and

(a) Find the characteristic polynomial of T

(b) Find the eigenvalues of T, and order them as $\lambda_1 \geq \lambda_2 \geq \lambda_3$

(c) Find the eigenvectors T.

(d) Are the eigenvectors orthonormal? If they are not, convert them into orthonormal vectors using the Gram-Schmidt process. Call them $\boldsymbol{x_1}, \boldsymbol{x_2}, \boldsymbol{x_3}$ according to the order of their corresponding eigenvalues.

[3+3+3+3= 12 marks]

Question 7) Consider two orthonormal vectors you obtained in question 6.

(a) Using them we want to make three orthonormal vectors $\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}$ such that

$$\boldsymbol{u_1} = \frac{1}{\sigma_1} A\boldsymbol{v_1}$$

$$\boldsymbol{u_2} = \frac{1}{\sigma_2} A\boldsymbol{v_2}$$

Also, find the third vector $\boldsymbol{u_3}$,

$$\boldsymbol{u_3} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

such that $\boldsymbol{u_3}$ is a unit vector that is orthogonal to $\boldsymbol{u_1}$ and $\boldsymbol{u_2}$. In other words

$$||\boldsymbol{u_3}||_2 = 1 \text{ and } \boldsymbol{u_3} \perp \boldsymbol{u_1} \text{ and } \boldsymbol{u_3} \perp \boldsymbol{u_2}.$$

Put these three vectors as columns in a matrix $U = \begin{bmatrix} \boldsymbol{u_1} & \boldsymbol{u_2} & \boldsymbol{u_3} \end{bmatrix}$.

(b) Show that U is an orthogonal matrix.

(c) Compute UDV^T.

(d) Explain the relationship between $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$ and $\{\boldsymbol{x_1}, \boldsymbol{x_2}, \boldsymbol{x_3}\}$.

[5+2+3+3= 13 marks]