SIT787 -Mathematics for AI

Trimester 1, 2021

Due: no later than the end of Week 3, , Sunday 2 August 2021, 8:00pm AEST

Note:

- A proper way of presenting your solutions is part of the assessment. Please follow the order of questions in your submission.
- Your submission can be handwritten but it must be legible. Please write neatly. If I cannot read your solution, I cannot mark it.
- Provide the way you solve the questions. All steps (workings) to arrive at the answer must be clearly shown. I need to see your thoughts.
- For a final answer without a proper justification no score will be given.
- Only (scanned) electronic submission would be accepted via the unit site (Deakin Sync).
- Your submission must be in ONE pdf file. Multiple files and/or in different file format, e.g. .jpg, will NOT be accepted. If you need to change the format of your submission, it will be subject to the late submission penalty.

[Q1] Evaluate the following expressions to give the simplest possible form. Also, determine for what values of the variables the expressions are not defined. (do not convert the coefficients into decimals.)

(i) Find x (You need to use index rules and logarithmic rules properly):

$$x^{1-\frac{1}{3}\ln(x^2)} = \frac{1}{\sqrt[3]{e^2}}$$

(ii) $a + (a - a^{-1})^{-1}$

[5+5 = 10 marks]

[Q2] For the function $y = x^3 - 2x^2$

(i) Find all the x- and y-intercepts.

(ii) Find all the stationary points and classify them. (you may convert the coordinates of the stationary point(s) into decimal for drawing purposes.)

(iii) Express the intervals for which the function is increasing or decreasing.

(iv) Sketch the function by hand. Label all the important points on the graph of the function.

[2 + 5 + 5 + 3 = 15 marks]

[Q3] Find the derivatives of the following functions:

(i) $f(x) = e^{x\ln(a)} + e^{a\ln(x)} + e^{a\ln(a)}$, a is a constant.

(ii) $y = \sqrt{2x}(x - \sin(2x))$

(iii) $y = e^{3\sin(2x)}$

(iv) $y = \frac{1}{\ln(x+\sqrt{1+x^2})}$

[5 + 5 + 5 + 5 = 20 marks]

[Q4] Find

(i) Use established Maclaurin series to find the first three non-zero terms of $f(x) = x^2\sin(x)$. Call the function $\tilde{f}(x)$ and keep all coefficients as fractions (no decimals).

(ii) Use your expression in (i) above to find an approximation of

$$I(a) = \int_1^a x^2\sin(x)dx \approx \int_1^a \tilde{f}(x)dx$$

as a function of a without using any calculators.

(iii) Consider the following function

$$f(x) = \begin{cases} ax + \frac{b}{x} & \text{if } 1 \le x \le e \\ 0 & \text{otherwise.} \end{cases}$$

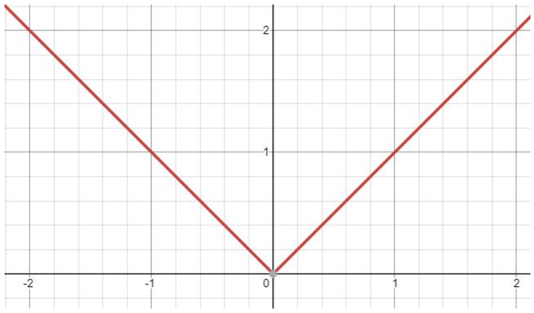
e is the Euler number $e \approx 2.71828$. Find a and b so that $\int_{-\infty}^{+\infty} f(x)dx = 1$ and $\int_{-\infty}^{+\infty} xf(x)dx = 0$. Do not convert any coefficient into decimals. You should not use any calculators.

[5 + 5 + 5 = 15 marks]

[Q5] Sometimes a function is not differentiable at all points. For example, consider the absolute value function

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

This function is not differentiable at $x = 0$. As you can see, the graph of the function changes

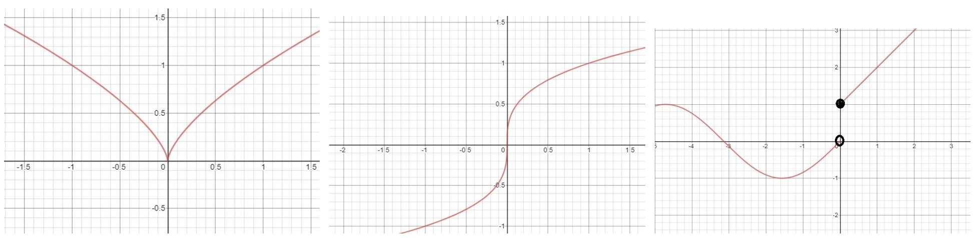


direction abruptly when $x = 0$. In general, if the graph of a function f has a "corner" or "kink" in it, then the graph of the function does not have a unique tangent at this point and f is not differentiable there. In addition, if a function is not continuous at a point $x = a$, then f is not differentiable at a. So, at any discontinuity, f fails to be differentiable. A third possibility is that the curve has a vertical tangent line when $x = a$. Let's see some examples with their plots.

(a) $f(x) = x^{\frac{2}{3}}$

(b) $g(x) = x^{\frac{1}{3}}$

(c) $h(x) = \begin{cases} x + 1 & \text{if } x \geq 0 \\ \sin(x) & \text{if } x < 0. \end{cases}$



(a) (b) (c)

Figure 1: (a) $f(x)$ (b) $g(x)$ (c) $h(x)$

Let's see why these functions are not differentiable.

(a) $f(x) = x^{\frac{2}{3}}$. $f'(x) = \frac{2}{3\sqrt[3]{x}}$. The derivative function is not defined at $x = 0$. In Figure 1 (a), you see that the original function has a corner at $x = 0$, and the tangent is vertical. This function is differentiable everywhere except $x = 0$.

(b) $g(x) = x^{\frac{1}{3}}$. $g'(x) = \frac{1}{3\sqrt[3]{x^2}}$. The derivative function is not defined at $x = 0$. In Figure 1 (b), you see that the tangent is vertical at $x = 0$. This function is differentiable everywhere except $x = 0$.

(c) $h(x) = \begin{cases} x + 1 & \text{if } x \geq 0 \\ \sin(x) & \text{if } x < 0. \end{cases}$ This function is not continuous at $x = 0$ as you can see in Figure 1 (c). How can you check whether the function is not continuous at $x = 0$? well, if you plug $x = 0$ in the upper and lower rules, you get different values. $(0) + 1 \neq \sin(0)$, or $1 \neq 0$. If a function is not continuous at a point, it is not differentiable there. When $x \geq 0$, $h(x) = x + 1$. Its derivative for $x > 0$ is $h'(x) = 1$. When $x < 0$, $h(x) = \sin(x)$. On this interval, $h'(x) = \cos(x)$. Therefore, the derivative of this functions is

$$h'(x) = \begin{cases} 1 & \text{if } x > 0 \\ \cos(x) & \text{if } x < 0. \end{cases}$$

This function is differentiable everywhere except $x = 0$.

The derivative of the absolute value function $f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$ is computed as follows:

$$f'(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

You see that the derivative of the function just to the left of $x = 0$ is -1 and just to the right of $x = 0$ is $+1$. The absolute value function is differentiable everywhere except for $x = 0$. You always need to be careful with the break points. We will learn how to find derivetive of this kind of function soon.

Find the derivative of the functions, and demonstrate where the function is not differentiable.

(i) $f(x) = |2x - 3|$

(ii) $f(x) = 2|x - 3| - 3|x - 2|$

[5 + 10 = 15 marks]

[Q6] Piecewise defined functions: Generally, you see function defined as $y = f(x)$. However, sometimes functions are defined by different rules in different parts of their domain. Such functions are called piecewise defined functions. For example,

$$f(x) = \begin{cases} 1 - x & \text{if } x \leq -1 \\ x^2 & \text{if } x > -1. \end{cases} \qquad g(x) = \begin{cases} \frac{1}{2} - \frac{x}{2} & \text{if } x \leq -1 \\ x^2 & \text{if } x > -1. \end{cases}$$

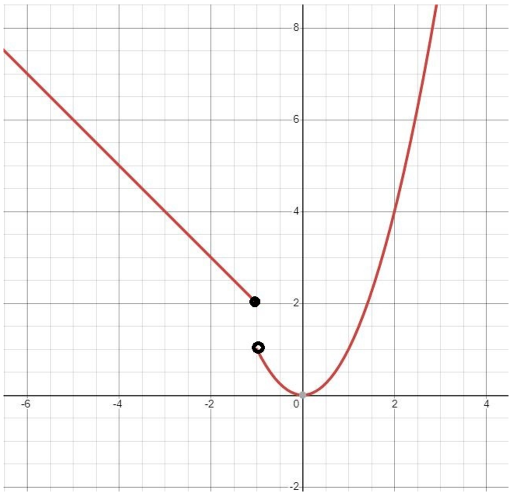


Figure 2: (a)

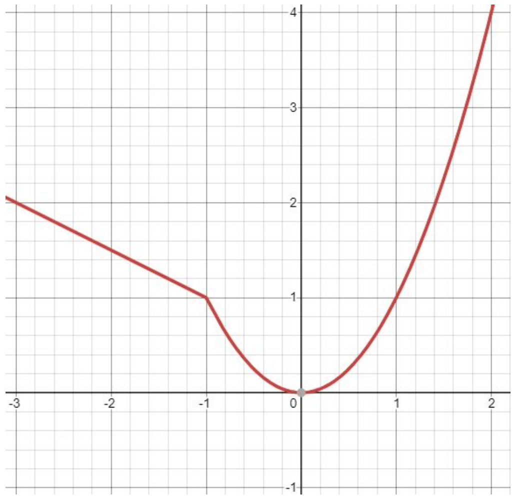
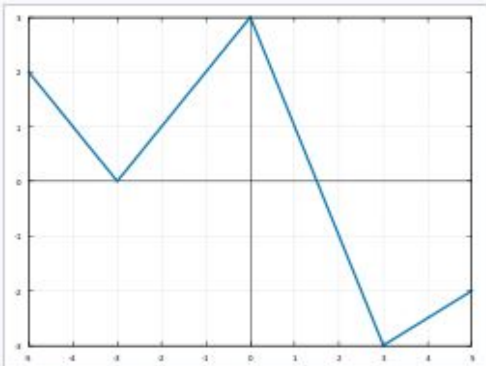


Figure 3: (b)

The most famous function of this type is the absolute value functions.

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

They even can have more than two rules on different parts of their domains (from wikipedia).



Plot of the piecewise linear function

$$f(x) = \begin{cases} -3 - x & \text{if } x \leq -3 \\ x + 3 & \text{if } -3 \leq x \leq 0 \\ 3 - 2x & \text{if } 0 \leq x \leq 3 \\ 0.5x - 4.5 & \text{if } 3 \leq x \end{cases}$$

In its general form, a piecewise defined function is represented as

$$y = f(x) = \begin{cases} f_1(x) & \text{if } a \leq x < b \\ f_2(x) & \text{if } b \leq x < c \\ f_3(x) & \text{if } c \leq x < d \end{cases}$$

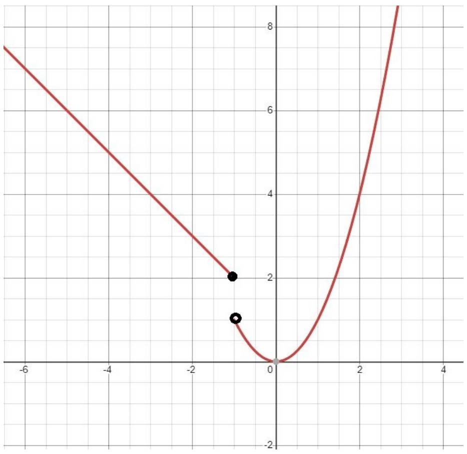
A function of this type is continous if its constituent functions are continuous on the corresponding intervals and there is no discontinuity at each breakpoint. In other words, $f_1(x)$ should be continuous on $[a, b)$, $f_2(x)$ should be continuous on $[b, c)$, and $f_3(x)$ should be continuous on $[c, d)$. In addition, there should not be a discontinouty on the breaking points $x = b$ and $x = c$.

For example the function in Figure 2:(a) is discontinuous, and the function in Figure 3:(b) is continuous.

A piecewise function is differentiable on a given interval in its domain if the following conditions have to be satisfied in addition to those for continuity mentioned above:

- its constituent functions are differentiable on the corresponding open intervals,
- at the points where two subintervals touch, the corresponding derivatives of the two neighboring subintervals should match.

For example, for $f(x) = \begin{cases} 1 - x & \text{if } x \leq -1 \\ x^2 & \text{if } x > -1. \end{cases}$



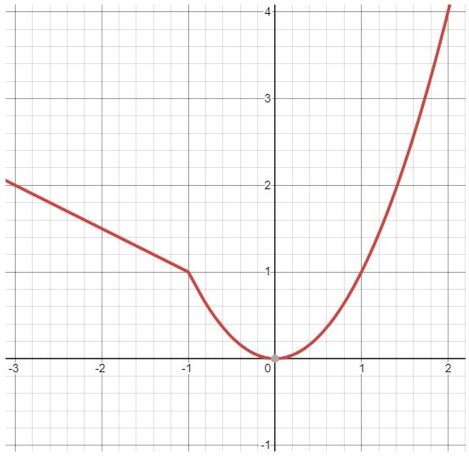
its derivative can be found as

- On the open interval $x < -1$ or $(-\infty, -1)$, $f(x) = 1 - x$, and $f'(x) = -1$.
- On the open interval $x > 1$ or $(-1, \infty)$, $f(x) = x^2$, and $f'(x) = 2x$.
- We visually see that the function is not continuous at $x = -1$, therefore it is not differentiable there. In other words, if you plug $x = -1$ in the upper and lower rules, you get different values. In addition, at $x = -1$, the derivative of the first rule is -1 and of second rule is $2(-1)$. As $-1 \neq -2$ it is not differentiable at $x = -1$. The derivative of this function is

  $$f'(x) = \begin{cases} -1 & \text{if } x < -1 \\ 2x & \text{if } x > -1. \end{cases}$$

  We observe that the derivative function is not defined at $x = -1$.

Another example, For $g(x) = \begin{cases} \frac{1}{2} - \frac{x}{2} & \text{if } x \leq -1 \\ x^2 & \text{if } x > -1. \end{cases}$



- On the open interval $x < -1$ or $(-\infty, -1)$, $g(x) = \frac{1}{2} - \frac{x}{2}$, and $g'(x) = \frac{-1}{2}$.
- On the open interval $x > -1$ or $(-1, \infty)$, $g(x) = x^2$, and $g'(x) = 2x$
- at $x = -1$, the function is continuous (as $\frac{1}{2} - \frac{-1}{2} = (-1)^2$), but using the first rule, $g'(-1) = \frac{-1}{2}$ and using the second rule $g'(-1) = 2(-1) = -2$. However, $\frac{-1}{2} \neq -2$. This function is not differentiable at $x = -1$. Then, the derivative of this function is

$$g'(x) = \begin{cases} \frac{-1}{2} & \text{if } x < -1 \\ 2x & \text{if } x > -1. \end{cases}$$

There are some piecewise defined functions that are continuous and differentiable everywhere. For example

$$f(x) = \begin{cases} x - \frac{1}{4} & \text{if } x < \frac{1}{2} \\ x^2 & \text{if } x \geq \frac{1}{2}. \end{cases}$$

Now, let's solve some problems.

(i) Consider the function defined by

$$f(x) = \begin{cases} ax + b & \text{if } x < 1 \\ x^4 + x + 1 & \text{if } x \geq 1. \end{cases}$$

for what value(s) of $a, b \in \mathbb{R}$ is the function f differentiable at every $x \in \mathbb{R}$?

Find the derivative of the following function, and explain on what points the function is not differentiable:

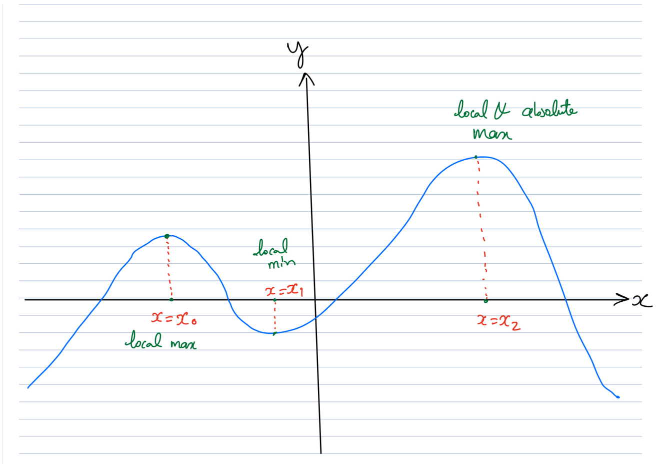
(ii) $f(x) = \begin{cases} \sqrt{x^2 - 2} & \text{if } x \leq 0 \\ x^3 - x & \text{if } x > 0. \end{cases}$

[8 + 7 = 15 marks]

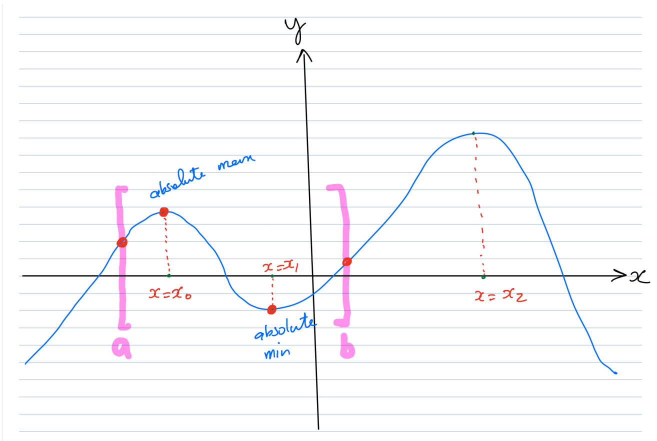
[Q7] **Optimisation**: With optimisation, we want to find the maximum and minimum of a function. We have two types of maximum and minimum points. Absolute maximum and absolute minimum, and local minimum and local maximum. Let's define them properly for a function $y = f(x)$.

- Absolute (or global) maximum: A point $x = c$ is called an absolute maximum of f, if for any value in its domain ($x \in \text{Dom}(f)$), $f(x) \leq f(c)$
- Absolute (or global) minimum: A point $x = c$ is called an absolute minimum of f, if for any value in its domain ($x \in \text{Dom}(f)$), $f(x) \geq f(c)$
- Local maximum: A point $x = c$ is called a local maximum of f, if for every value x near c, $f(x) \leq f(c)$
- Local minimum: A point $x = c$ is called a local minimum of f, if for every value x near c, $f(x) \geq f(c)$.

In the follwoing plot, the function has a local maximum at $x = x_0$, a local minimum at $x = x_1$, and a local maximum at $x = x_2$. The point $x = x_2$ is a global maximum as well. There is not an absolute minimum for this function.



However, if we are given a closed interval, we can find the absolute maximum and minimum of the function in that interval. For example, in the following plot, in the closed interval $[a, b]$, the function has an absolute maximum at $x = x_0$ and an absolute minimum at $x = x_1$.



To find all these interesting points, we need to find critical points. When we find them, then we need to classify each critical point as a global/local maximum/minimum. The critical points for a given function $y = f(x)$ are

- the points $x = c$ where $f'(c) = 0$, or
- the points $x = c$ where $f'(c)$ does not exist.

Optimation problems type 1: Find absolute maximum and minimum of $y = f(x)$ in a given closed interval $[a, b]$:
To solve this problem,

(a) Find critical points of $f(x)$ and evaluate f in them

(b) Find $f(a)$ and $f(b)$

The maximum of value of items in (a) and (b) is the absolute maximum of $f(x)$ in $[a, b]$, and the minimum value of items in (a) and (b) is the absolute minimum of $f(x)$ in $[a, b]$.

Optimation problems type 2 (First derivative test): Find local maximum and minimum of $y = f(x)$ using first derivative:

- Find all critical points of $f(x)$
- If for a critical point $x = c$, f' changes from positive to negative (f changes from increasing to decreasing), $x = c$ is a local maximum point.
- If for a critical point $x = c$, f' changes from negative to positive (f changes from decreasing to increasing), $x = c$ is a local minimum point.

Optimation problems type 2 (Second derivative test): Find local maximum and minimum of $y = f(x)$ using second derivative:

- Find all critical points of $f(x)$
- For a critical point $x = c$, if $f'(c) = 0$ and $f''(c) > 0$, $x = c$ is a local minimum.
- For a critical point $x = c$, if $f'(c) = 0$ and $f''(c) < 0$, $x = c$ is a local maximum.
- if $f''(c) = 0$, the test is inconclusive. It does not give any useful information, and we need to use other techniques to decide on the type of the stationary point.

Now solve the following problems:

(i) Find absolute maximum and absolute minimum points of $f(x) = x^{\frac{1}{2}} - x^{\frac{3}{2}}$ in the closed interval $[0, 4]$.

(ii) Find all the local maximum and local minimum points of

$$f(x) = \begin{cases} 0 & \text{if } x \geq 0 \\ 1 - \sqrt{1 - x^2} & \text{otherwise} \end{cases}$$

[5 + 5 = 10 marks]

[Q1] Evaluate the following expressions to give the simplest possible form. Also, determine for what values of the variables the expressions are not defined. (do not convert the coefficients into decimals.)

(i) Find x (You need to use index rules and logarithmic rules properly):

$$x^{1-\frac{1}{3}\ln(x^2)} = \frac{1}{\sqrt[3]{e^2}}$$

(ii) $a + (a - a^{-1})^{-1}$

[5+5 = 10 marks]

(i) $x^{1-\frac{1}{3}\mathrm{Ln}(x^2)} = \frac{1}{\sqrt[3]{e^2}}$

as we know for $\mathrm{Ln}(x)$, the argument of Ln should be positive. Therefore, $x^2 > 0$, which means $\boxed{x \in \mathbb{R} \setminus \{0\}}$. The expression is not defined at $x = 0$.

Also, we know that $\sqrt[3]{e^2} = e^{\frac{2}{3}} \Rightarrow \frac{1}{\sqrt[3]{e^2}} = e^{-2/3}$

Based on index rules, $x^{m+n} = x^m x^n$.

Based on logarithmic rules $\mathrm{Ln}(x^m) = m\,\mathrm{Ln}(x)$.

Let's take Ln from both sides of the equation:

$$\mathrm{Ln}\left(x^{1-\frac{1}{3}\mathrm{Ln}(x^2)}\right) = \mathrm{Ln}\left(e^{-2/3}\right)$$

$$\left(1-\frac{1}{3}\ln(x^2)\right)\ln(x)=-\frac{2}{3}\ln(e)=-\frac{2}{3}$$

$\boxed{\ln(e)=1}$

$$\left(1-\frac{2}{3}\ln(x)\right)\ln(x)=-\frac{2}{3}$$

Let's consider $\ln(x)=t$. The equation by plugging t instead of $\ln(x)$ will be

$$\left(1-\frac{2}{3}t\right)t=-\frac{2}{3}$$

$$t-\frac{2}{3}t^2+\frac{2}{3}=0 \quad \text{(multiply both sides by } -3\text{)}$$

$$-\frac{2}{3}t^2+t+\frac{2}{3}=0$$

$$\Rightarrow -2t^2+3t+2=0$$

$$\boxed{ax^2+bx+c=0 \Rightarrow x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}}$$

$$t=\frac{-3\pm\sqrt{9+16}}{2(-2)}=\frac{-3\pm 5}{-4}=\begin{cases}\frac{-3-5}{-4}=2\\[4pt] \frac{-3+5}{-4}=-\frac{1}{2}\end{cases}$$

if $t=2 \Rightarrow \ln(x)=2 \Rightarrow \boxed{x=e^2}$

if $t=-\frac{1}{2} \Rightarrow \ln(x)=-\frac{1}{2} \Rightarrow x=e^{-1/2}=\frac{1}{e^{1/2}}=\boxed{\frac{1}{\sqrt{e}}}$

(ii) $a + (a - a^{-1})^{-1}$

$$a + (a - a^{-1})^{-1} = a + \left(a - \frac{1}{a}\right)^{-1} = a + \frac{1}{a - \boxed{\frac{1}{a}}}$$

because of $\boxed{\frac{1}{a}}$: $a \neq 0$

also, $a - \frac{1}{a} \neq 0$, which means $\frac{a^2 - 1}{a} \neq 0$

$\Rightarrow a^2 - 1 \neq 0$ or $a \neq \pm 1$

Therefore, to have this expression valid a should not be in $\boxed{\{-1, 0, 1\}}$

$$a + \frac{1}{a - \frac{1}{a}} = a + \frac{1}{\frac{a^2 - 1}{a}} = a + \frac{a}{a^2 - 1}$$

$$= \frac{a(a^2 - 1) + a}{a^2 - 1} = \frac{a^3 - a + a}{a^2 - 1} = \boxed{\frac{a^3}{a^2 - 1}}.$$

[5 + 5 = 10 marks]

[Q2] For the function $y = x^3 - 2x^2$

(i) Find all the x- and y-intercepts.

(ii) Find all the stationary points and classify them. (you may convert the coordinates of the stationary point(s) into decimal for drawing purposes.)

(iii) Express the intervals for which the function is increasing or decreasing.

(iv) Sketch the function by hand. Label all the important points on the graph of the function.

[2 + 5 + 5 + 3 = 15 marks]

[Q3] Find the derivatives of the following functions:

$y = x^3 - 2x^2$

(i)

x-intercept: put $y=0 \Rightarrow x^3 - 2x^2 = 0$

$\Rightarrow x^2(x-2) = 0 \Rightarrow \begin{cases} x^2 = 0 \rightarrow x = 0 \\ x - 2 = 0 \rightarrow x = 2 \end{cases}$

The x-intercepts are $(0,0)$ and $(2,0)$.

y-intercept: put $x = 0 \Rightarrow y = (0)^3 - 2(0)^2 = 0$

The y-intercept is $(0,0)$.

(ii) To find stationary points of $y = f(x)$:

put $f'(x) = 0$ and check where it is not differentiable.

$y = x^3 - 2x^2$

$y' = 3x^2 - 4x = 0 \Rightarrow x(3x-4) = 0 \rightarrow \begin{cases} x = 0 \\ x = \frac{4}{3} \end{cases}$

We need to find the y-coordinate of these points.

When $x=0 \Rightarrow y=0 \longrightarrow (0,0)$

when $x=\frac{4}{3} \Rightarrow y=\left(\frac{4}{3}\right)^3-2\left(\frac{4}{3}\right)^2=\frac{64}{27}-\frac{32}{9}=\frac{128-192}{54}$

$=-\frac{32}{27} \longrightarrow \left(\frac{4}{3},-\frac{32}{27}\right)$

Also, the function is differentiable at all points.

The set of stationary points is $\boxed{\left\{(0,0),\left(\frac{4}{3},-\frac{32}{27}\right)\right\}}$

To classify the stationary points we use the second derivative test:

$$f''(x)=6x-4$$

$f''(0)=6(0)-4=-4<0 \Rightarrow$ $\boxed{(0,0) \text{ is a local max point}}$

$f''\left(\frac{4}{3}\right)=6\left(\frac{4}{3}\right)-4=4>0 \Rightarrow$ $\boxed{\left(\frac{4}{3},-\frac{32}{27}\right) \text{ is a local min point}}$

(iii) where the function is increasing and decreasing

x	$-\infty$	$x=0$		$x=\frac{4}{3}$	$+\infty$
$f'(x)=3x^2-4x$	try $x=-1$ $f'(-1)=3+4=7>0$	0	try $x=1$ $f'(1)=3-4=-1<0$	0	try $x=2$ $f'(2)=12-8=4<0$

The function is increasing at $\boxed{(-\infty, 0) \text{ and } \left(\tfrac{4}{3}, +\infty\right)}$.

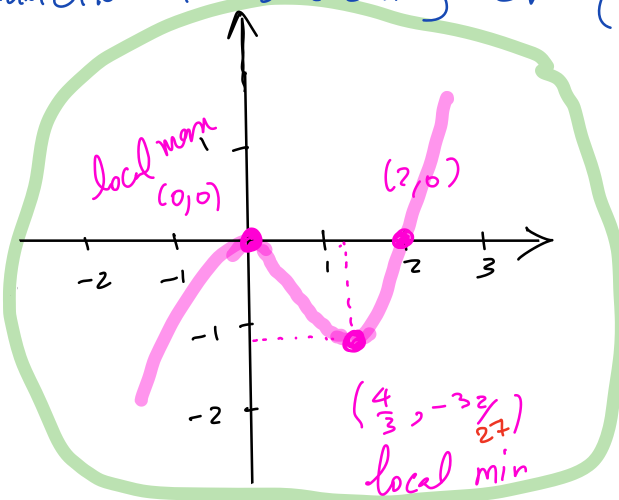
Also, the function is decreasing $\boxed{\left(0, \tfrac{4}{3}\right)}$.

(iv) Sketch the function: What we know

- y-intercept $(0,0)$
- x-intercept $(0,0)$, $(2,0)$
- stationary points $\left\{ \underbrace{(0,0)}_{\text{local max}}, \underbrace{\left(\tfrac{4}{3}, -\tfrac{32}{27}\right)}_{\text{local min}} \right\}$

$\left(\frac{4}{3}, -\frac{32}{27}\right)$ approximately is $(1.3, -1.2)$

- the function is increasing on $(-\infty, 0)$ and $\left(\frac{4}{3}, +\infty\right)$
  the function is decreasing on $(0, 4/3)$



[Q3] Find the derivatives of the following functions:

(i) $f(x) = e^{x \ln(a)} + e^{a \ln(x)} + e^{a \ln(a)}$, a is a constant.

(ii) $y = \sqrt{2x}(x - \sin(2x))$

(iii) $y = e^{3 \sin(2x)}$

(iv) $y = \frac{1}{\ln(x+\sqrt{1+x^2})}$

(i) $f(x) = e^{x\ln(a)} + e^{a\ln(x)} + e^{a\ln(a)}$

we know that $n\ln(x) = \ln(x^n)$ and

$e^{\ln(x)} = x$

$f(x) = e^{\ln(a^x)} + e^{\ln(x^a)} + e^{\ln(a^a)}$

$= a^x + x^a + a^a$

$(a^x)' = a^x \ln(a)$

$$\boxed{f'(x) = a^x \ln(a) + a x^{a-1}.}$$

(ii) $y = \sqrt{2x}\,(x - \sin(2x))$

we use the product rule here:

$$y' = \frac{1}{\sqrt{2x}}\left(x - \sin(2x)\right) + \sqrt{2x}\left(1 - 2\cos(2x)\right)$$

$$\boxed{= \frac{x - \sin(2x)}{\sqrt{2x}} + \sqrt{2x}\left(1 - 2\cos(2x)\right)}$$

(iii) $y = e^{3\sin(2x)}$

$$\boxed{y' = 6\cos(2x)\, e^{3\sin(2x)}}$$

(iv) $y = \dfrac{1}{\ln\left(x + \sqrt{1 + x^2}\right)}$

Let's find derivative of $\ln(x+\sqrt{1+x^2})$

first. $[\ln(u)]' = \frac{u'}{u}$

$$[\ln(x+\sqrt{1+x^2})]' = \frac{1+\frac{2x}{2\sqrt{1+x^2}}}{x+\sqrt{1+x^2}}$$

$$= \frac{1+\frac{x}{\sqrt{1+x^2}}}{x+\sqrt{1+x^2}} = \frac{\frac{x+\sqrt{1+x^2}}{\sqrt{1+x^2}}}{x+\sqrt{1+x^2}}$$

$$= \frac{1}{\sqrt{1+x^2}}$$

now let's get back to the original function:

$$y = \frac{1}{\ln(x+\sqrt{1+x^2})}$$

using the quotient rule:

$$y' = \frac{\overset{\text{zero}}{(1)'}\left(\ln\left(x+\sqrt{1+x^2}\right)\right) - (1)\left(\ln\left(x+\sqrt{1+x^2}\right)\right)'}{\left[\ln\left(x+\sqrt{1+x^2}\right)\right]^2}$$

$$= \frac{-\left[\ln\left(x+\sqrt{1+x^2}\right)\right]'}{\left[\ln\left(x+\sqrt{1+x^2}\right)\right]^2} = \frac{\frac{-1}{\sqrt{1+x^2}}}{\left[\ln\left(x+\sqrt{1+x^2}\right)\right]^2}$$

$$= \boxed{\frac{-1}{\sqrt{1+x^2}\left[\ln\left(x+\sqrt{1+x^2}\right)\right]^2}}$$

[Q4] Find

(i) Use established Maclaurin series to find the first three non-zero terms of $f(x) = x^2 \sin(x)$. Call the function $\tilde{f}(x)$ and keep all coefficients as fractions (no decimals).

(ii) Use your expression in (i) above to find an approximation of

$$I(a) = \int_1^a x^2 \sin(x) dx \approx \int_1^a \tilde{f}(x) dx$$

as a function of a without using any calculators.

(iii) Consider the following function

$$f(x) = \begin{cases} ax + \frac{b}{x} & \text{if } 1 \leq x \leq e \\ 0 & \text{otherwise.} \end{cases}$$

e is the Euler number $e \approx 2.71828$. Find a and b so that $\int_{-\infty}^{+\infty} f(x)dx = 1$ and $\int_{-\infty}^{+\infty} x f(x)dx = 0$. Do not convert any coefficient into decimals. You should not use any calculators.

[5 + 5 + 5 = 15 marks]

(i) first three nonzero terms of $f(x) = x^2 \sin(x)$

based on the Maclaurin series expansion

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)\, x^n}{n!}$$

$$= f(0) + f'(0)x + \frac{1}{2} f''(0) x^2 + \cdots$$

We need to find several derivatives of $f(x)$ at $x = 0$

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$f(x) = x^2 \sin(x)$	$f(0) = 0$
1	$f'(x) = 2x\sin(x) + x^2\cos(x)$	$f'(0) = 0$
2	$f''(x) = 2\sin(x) + 2x\cos(x) + 2x\cos(x) - x^2\sin(x)$	$f''(0) = 0$
3	$f^{(3)}(x) = 2\cos(x) + 2\cos(x) - 2x\sin(x) + 2\cos(x) - 2x\sin(x) - 2x\sin(x) - x^2\cos(x)$ $= 6\cos(x) - 6x\sin(x) - x^2\cos(x)$	$f^{(3)}(0) = 6$
4	$f^{(4)}(x) = -6\sin(x) - 6\sin(x) - 6x\cos(x) - 2x\cos(x) + x^2\sin(x)$ $= -12\sin(x) - 8x\cos(x) + x^2\sin(x)$	$f^{(4)}(0) = 0$
5	$f^{(5)}(x) = -12\cos(x) - 8\cos(x) + 8x\sin(x) + 2x\sin(x) + x^2\cos(x)$ $= -20\cos(x) + 10x\sin(x) + x^2\cos(x)$	$f^{(5)}(0) = -20$
6	$f^{(6)}(x) = 20\sin(x) + 10x\cos(x) + 10\sin(x) + 2x\cos(x) - x^2\sin(x)$ $= 30\sin(x) + 12x\cos(x) - x^2\sin(x)$	$f^{(6)}(0) = 0$

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$$\left| \begin{aligned} f^{(7)}(x) &= 30\cos(x) + 12\cos(x) - 12x\sin(x) \\ &- 2x\sin(x) - x^2\cos(x) \\ &= 42\cos(x) - 14x\sin(x) - x^2\cos(x) \end{aligned} \right| \quad f^{(7)}(0) = 42$$

$$f(x) \simeq \hat{f}(x) = \frac{1}{3!} f^{(3)}(0) x^3 + \frac{1}{5!} f^{(5)}(0) x^5$$

$$+ \frac{1}{7!} f^{(7)}(0) x^7$$

$$\hat{f}(x) = \frac{6x^3}{6} + \frac{(-20)x^5}{120} + \frac{(7)(6)x^7}{(7)(6)5!}$$

$$= x^3 - \frac{1}{6}x^5 + \frac{1}{120}x^7$$

$$f(x) \simeq x^3 - \frac{1}{6}x^5 + \frac{1}{120}x^7$$

$$I(a) = \int_1^a x^2 \sin(x)\, dx$$

$$\simeq \int_1^a \left(x^3 - \frac{1}{6}x^5 + \frac{1}{120}x^7\right) dx$$

$$= \left[\frac{1}{4}x^4 - \frac{1}{36}x^6 + \frac{1}{960}x^8\right]_1^a$$

$$= \frac{a^4}{4} - \frac{a^6}{36} + \frac{a^8}{960} - \frac{1}{4} + \frac{1}{36} - \frac{1}{960}$$

$$= \frac{a^4}{4} - \frac{a^6}{36} + \frac{a^8}{960} - \frac{643}{2880}$$

$$\boxed{I(a) \simeq \frac{a^4}{4} - \frac{a^6}{36} + \frac{a^8}{960} - \frac{643}{2880}}$$

$$f(x) = \begin{cases} ax + \frac{b}{x} & \text{if } 1 \leq x \leq e \\ 0 & \text{otherwise.} \end{cases}$$

$$\int_{-\infty}^{+\infty} f(x)\,dx = 1$$

$$\int_{-\infty}^{+\infty} x f(x)\,dx = 0$$

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_1^e \left(ax + \frac{b}{x}\right)dx = \left[\frac{1}{2}ax^2 + b\operatorname{Ln}(x)\right]_1^e$$

$$= \frac{1}{2}ae^2 + b\operatorname{Ln}(e) - \frac{1}{2}a - b\ln(1) = 1$$

$$\Rightarrow \frac{e^2}{2}a + b - \frac{a}{2} = 1$$

$$\Rightarrow \left(\frac{e^2}{2} - \frac{1}{2}\right)a + b = 1 \qquad \mathrm{I}$$

$$\int_{-\infty}^{+\infty} x f(x)\,dx = \int_1^e x\left(ax + \frac{b}{x}\right)dx = \int_1 (ax^2 + b)\,dx$$

$$= \left[\frac{1}{3}ax^3 + bx\right]_1^e = \frac{a}{3}e^3 + be - \frac{1}{3}a - b = 0$$

$$\Rightarrow \left(\frac{e^3}{3} - \frac{1}{3}\right)a + (e-1)b = 0 \qquad \mathrm{II}$$

by solving this system we find a and b:

$$\begin{cases} \left(\frac{e^2}{2} - \frac{1}{2}\right) a + b = 1 \\ \left(\frac{e^3}{3} - \frac{1}{3}\right) a + (e-1) b = 0 \end{cases}$$

$$\begin{cases} \frac{1}{2}(e^2-1)\, a + b = 1 \\ \frac{1}{3}(e^3-1)\, a + (e-1)\, b = 0 \end{cases}$$

from the first equation:

$$b = 1 - \frac{1}{2}(e^2-1)\, a$$

put this in the second equation:

$$\frac{1}{3}(e^3-1)\, a + (e-1) - \frac{1}{2}\underbrace{(e^2-1)(e-1)}_{e^3-e^2-e+1} a = 0$$

$$a\left[\frac{1}{3}e^3 - \frac{1}{3} - \frac{1}{2}e^3 + \frac{1}{2}e^2 + \frac{1}{2}e - \frac{1}{2}\right) = 1-e$$

$$a\left[-\frac{1}{6}e^3+\frac{1}{2}e^2+\frac{1}{2}e-\frac{5}{6}\right]=1-e$$

$$\boxed{\begin{aligned} a &= \frac{1-e}{-\frac{1}{6}e^3+\frac{1}{2}e^2+\frac{1}{2}e-\frac{5}{6}} \\ b &= 1-\frac{(1-e)(e^2-1)}{-\frac{1}{3}e^3+e^2+e-\frac{5}{3}} \end{aligned}}$$

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derivative of this kind of function soon.

Find the derivative of the functions, and demonstrate where the function is not differentiable.

(i) $f(x) = |2x - 3|$

(ii) $f(x) = 2|x - 3| - 3|x - 2|$

[5 + 10 = 15 marks]

(i) $f(x) = |2x-3|$

$2x - 3 = 0 \Longrightarrow x = \frac{3}{2}$

$$f(x) = \begin{cases} 2x-3 & 2x-3 \geq 0 \\ -(2x-3) & 2x-3 < 0 \end{cases}$$

$$\Longrightarrow f(x) = \begin{cases} 2x-3 & x \geq \frac{3}{2} \\ 3-2x & x < \frac{3}{2} \end{cases}$$

$$f'(x) = \begin{cases} 2 & x > 3/2 \\ -2 & x < \frac{3}{2} \end{cases}$$

at $x = \frac{3}{2}$, the two derivatives do not agree, therefore f is not differentiable at $x = \frac{3}{2}$.

(ii) $f(x) = 2|x-3| - 3|x-2|$

we need to see when each expression inside the absolute value signs are positive or negative.

$x - 3 = 0 \longrightarrow x = 3$

$x - 2 = 0 \longrightarrow x = 2$

x	$-\infty$ 2	2 3	3 $+\infty$
$(x-3)$	$-$ $2\lvert x-3\rvert = -2(x-3)$	$-$ $2\lvert x-3\rvert = -2(x-3)$	0 $+$ $2\lvert x-3\rvert = 2(x-3)$
$(x-2)$	$-$ $(-3)\lvert x-2\rvert = (-3)(2-x)$	0 $+$ $-3\lvert x-2\rvert = (-3)(x-2)$	$+$ $-3\lvert x-2\rvert = (-3)(x-2)$
$2\lvert x-3\rvert - 3\lvert x-2\rvert$			

$-2x+6-6+3x = x \quad\Big|\quad -2x+6-3x+6 = -5x+12 \quad\Big|\quad 2x-6-3x+6 = -x$

$$f(x) = 2|x-3| - 3|x-2| = \begin{cases} x & x<2 \\ -5x+12 & 2 \le x \le 3 \\ -x & x>3 \end{cases}$$

$$\boxed{f'(x) = \begin{cases} 1 & x<2 \\ -5 & 2<x<3 \\ -1 & x>3 \end{cases}}$$

at $x=2$ and $x=3$ the function is not differentiable.

6 Now, let's solve some problems.

(i) Consider the function defined by

$$f(x) = \begin{cases} ax + b & \text{if } x < 1 \\ x^4 + x + 1 & \text{if } x \geq 1. \end{cases}$$

for what value(s) of $a, b \in \mathbb{R}$ is the function f differentiable at every $x \in \mathbb{R}$?

Find the derivative of the following function, and explain on what points the function is not differentiable:

(ii) $f(x) = \begin{cases} \sqrt{x^2 - 2} & \text{if } x \leq 0 \\ x^3 - x & \text{if } x > 0. \end{cases}$

[8 + 7 = 15 marks]

We need to have the function continuous and differentiable at $x=1$. in all other points the function is differentable.

to be continuous at $x=1$:

$$a(1) + b = (1)^4 + (1) + 1$$

$$\boxed{a + b = 3}$$

to be differentiable at $x=1$:

$$f'(x) = \begin{cases} a & x < 1 \\ 4x^3 + 1 & x > 1 \end{cases}$$

at $x=1$, we should have $a = 4(1)^3 + 1$

$\Longrightarrow \boxed{a = 5} \Longrightarrow \boxed{b = -2}$

ii $f(x) = \begin{cases} \sqrt{x^2 - 2} & x \leq 0 \\ x^3 - x & x > 0 \end{cases}$ ($x^2 - 2 \geq 0$ for this function to be defined) or $x \leq \sqrt{2}$

$$f(x) = \begin{cases} \sqrt{x^2-2} & x \leq \sqrt{2} \\ x^3 - x & x > 0 \end{cases}$$

$$f'(x) = \begin{cases} \dfrac{x}{\sqrt{x^2-2}} & x \leq \sqrt{2} \\ 3x^2 - 1 & x > 0 \end{cases}$$

at $x = 0$, the upper rule is not defined. Essentially, the function is not continuous at $x = 0$, therefore it is not differentiable at $x = 0$.

$$f'(x) = \begin{cases} \dfrac{x}{\sqrt{x^2-2}} & x < -\sqrt{2} \\ 3x^2 - 1 & x > 0 \end{cases}$$

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Now solve the following problems:

(i) Find absolute maximum and absolute minimum points of $f(x) = x^{\frac{1}{2}} - x^{\frac{3}{2}}$ in the closed interval $[0, 4]$.

(ii) Find all the local maximum and local minimum points of

$$f(x) = \begin{cases} 0 & \text{if } x \geq 0 \\ 1 - \sqrt{1 - x^2} & \text{otherwise} \end{cases}$$

[5 + 5 = 10 marks]

(i) $f(x) = x^{1/2} - x^{3/2} = \sqrt{x} - (\sqrt{x})^3$ $\quad \boxed{x \geqslant 0}$

stationary points : when $f'(x) = 0$ or $f'(x)$ does not exist.

$f'(x) = \frac{1}{2} x^{-1/2} - \frac{3}{2} x^{1/2} = \frac{1}{2\sqrt{x}} - \frac{3\sqrt{x}}{2}$

at $x = 0$ the function is not differentiable

$f'(x) = 0 \rightarrow \frac{1 - 3x}{2\sqrt{x}} = 0 \rightarrow 1 - 3x = 0 \rightarrow x = 1/3$

stationary points $= \{0, 1/3\}$

We need to find the function values at $0, 1/3, 4$

$f(0) = 0$

$f(1/3) = \frac{1}{\sqrt{3}} - \frac{1}{(\sqrt{3})^3} \simeq 0.577 - 0.19 = 0.387$ absolute max

$f(4) = \sqrt{4} - (\sqrt{4})^3 = 2 - 8 = -6$ absolute min

$$f(x) = \begin{cases} 0 & \text{if } x \geq 0 \\ 1 - \sqrt{1 - x^2} & \text{otherwise} \end{cases}$$

$$f(x) = \begin{cases} 0 & x \geq 0 \\ 1 - \sqrt{1-x^2} & x < 0 \end{cases}$$

but $1 - x^2 \geq 0$ or $x^2 \leq 1$ or $-1 \leq x \leq +1$

for the lower rule to be defined.

As x also should be less than zero for this rule, the function is

$$f(x) = \begin{cases} 0 & x \geq 0 \\ 1 - \sqrt{1-x^2} & -1 \leq x < 0 \end{cases}$$

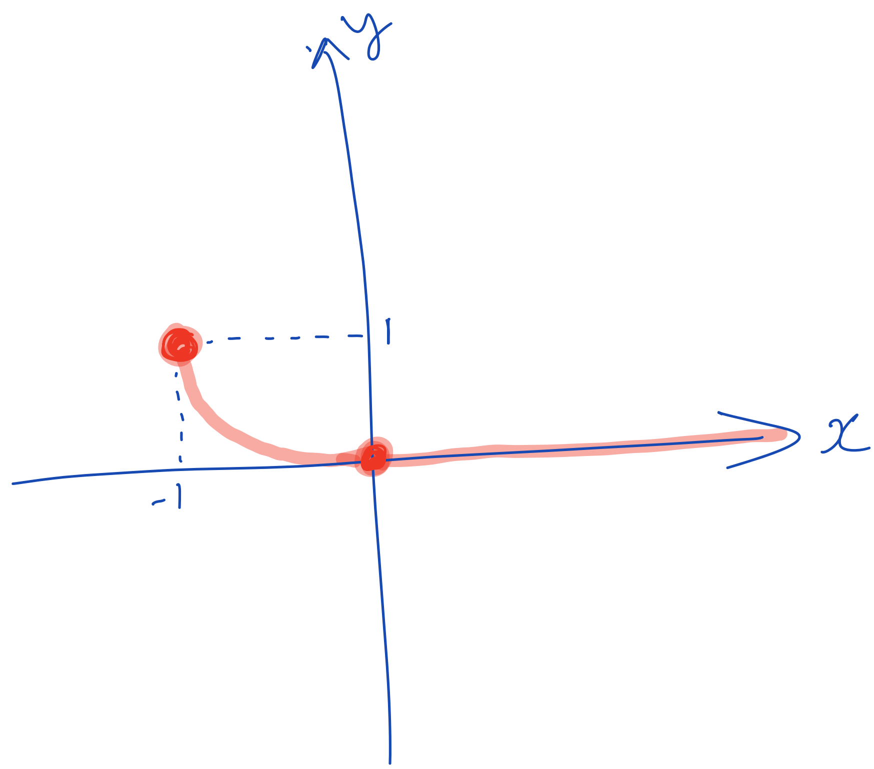
$$f'(x) = \begin{cases} 0 & x > 0 \\ \dfrac{x}{\sqrt{1-x^2}} & -1 < x < 0 \end{cases}$$

we need to check at $x = 0$.

at $x=0$ both rules agree, so the function is differentiable at $x=0$

$$f'(x) = \begin{cases} 0 & x \geq 0 \\ \dfrac{x}{\sqrt{1-x^2}} & -1 < x < 0 \end{cases}$$

Here is a plot of the function.



We need to find all stationary points

where $\begin{cases} f'(x)=0 \\ f'(x) \text{ does not exist.} \end{cases}$

If $f'(x)=0 \implies x \geq 0$

all points in $[0, +\infty)$ are stationary points.

at $x=-1$, the original function is defined and $f(-1)=1$.

But, this function is not differentiable at $x=-1$.

The set of stationary points

$= \{-1, \text{ all } x \in [0, \infty)\}$

The function is decreasing before $x=0$, as the value of $f'(x) < 0$ at this range. from $x=0$ to $+\infty$, the derivative is zero, so the function is flat. So all the points in $[0, +\infty)$ are local minimums.

At $x=-1$, the function is only defined to the right of this function, and $f'(x) < 0$ at the interval $(0, -1)$. So, $x=-1$ the function reaches to its local maxima, which is a global maxima.