Week 13: Hierarchical models II

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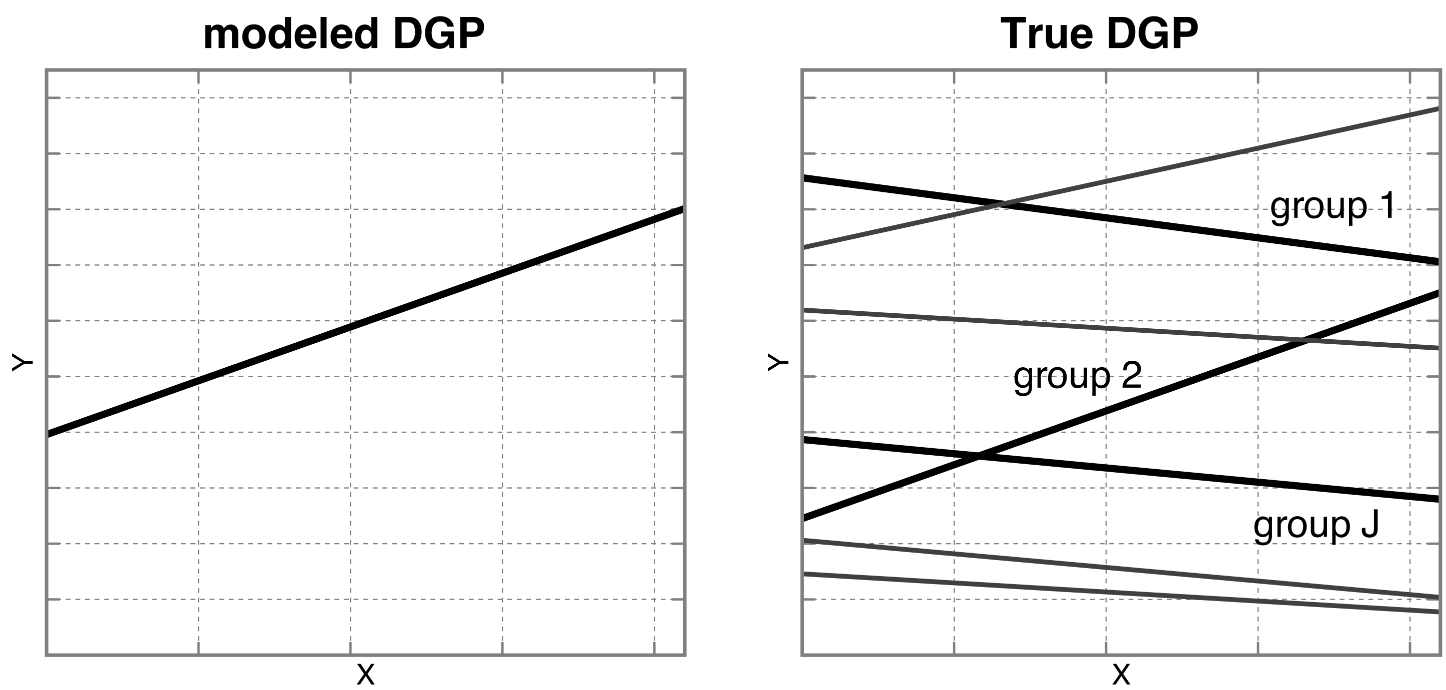
Effect heterogeneity between groups/clusters

- Unobserved/unmodeled differences between groups



Random coefficient models

- Model defined by the interaction between x and groups.
- Group-specific intercepts and group-specific effect coefficients

$$y_{ij} = \alpha_j + \mathbf{x}_i \boldsymbol{\beta}_j + \epsilon_{ij}$$
$$\alpha_j = \gamma_0 + \xi_j$$

- Where unit i has intercept α_j and a covariate effect coefficient β_j
- Group level effect model

$$\boldsymbol{\beta_j} = \boldsymbol{\gamma}_1 + \boldsymbol{\zeta}_j$$

- Reduced form

$$y_{ij} = \gamma_0 + \mathbf{x}_i(\boldsymbol{\gamma}_1 + \boldsymbol{\zeta}_j) + \xi_j + \epsilon_{ij}$$

  i.e., we have heteroscedastic errors

$$y_{ij} = \gamma_0 + \mathbf{x}_i \boldsymbol{\gamma_1} + \mathbf{x}_i \boldsymbol{\zeta_j} + \xi_j + \epsilon_{ij}$$

Random coefficient models

- Assuming, as before, uncorrelated level-1 and level-2 residuals

$$\text{Cov}(\epsilon, \xi_j) = 0 \text{ and } \text{Cov}(\epsilon, \zeta_j) = 0$$

- New distributional assumption for ζ_j:

$$\zeta_j \sim N(0, \sigma^2_\zeta)$$

- Generally, allow for dependence between intercept and effect heterogeneity, i.e.,

$$\text{Cov}(\zeta_j, \xi_j) \neq 0$$

Alternative notation, explicitly hierarchical

- First, rewrite our well-known unit level equation

$$y_{ij} \sim N(\alpha_j + \mathbf{x}_i \boldsymbol{\beta}_j, \sigma_\epsilon^2)$$

- Model for heterogeneity in intercepts α_j:

$$\alpha_j \sim N(\mu_\alpha, \sigma_\alpha^2)$$

- Model for β_j

$$\beta_j \sim N(\mu_\beta, \sigma_\beta^2)$$

- Random effects & random coefficients distribution:

$$\begin{bmatrix} \alpha_j \\ \beta_j \end{bmatrix} \sim N\left(\begin{bmatrix} \mu_\alpha \\ \mu_\beta \end{bmatrix}, \begin{bmatrix} \sigma_\alpha^2 & \sigma_\alpha \sigma_\beta \\ \sigma_\alpha \sigma_\beta & \sigma_\beta^2 \end{bmatrix} \right)$$

Varying intercept & varying slope

- In lmer syntax simply add x to ‘random’ part:

```
M3 <- lmer(y ~ 1 + x + (1 + x | county))
```

```
...
Fixed effects:
            Estimate Std. Error t value
(Intercept)   1.4628     0.0539   27.15
x            -0.6811     0.0876   -7.78
```

- Estimated $\mu_\alpha = 1.463$
- Estimated $\mu_\beta = -0.681$

Varying intercept & varying slope

```
...
Random effects:
 Groups   Name        Variance Std.Dev. Corr
 county   (Intercept) 0.122    0.349
          x           0.118    0.344    -0.337
 Residual             0.557    0.746
Number of obs: 919, groups: county, 85
```

- Estimated $\sigma_\epsilon^2 = 0.557$
- Estimated $\sigma_\alpha^2 = 0.122$
- Estimated $\sigma_\beta^2 = 0.118$
- Estimated $\rho = -0.337$

County level estimates

- County-specific estimates α_j and β_j

```
R> coef(M3)
$county
   (Intercept)       x
1       1.1445 -0.5406
2       0.9334 -0.7709
3       1.4717 -0.6689
...
82      1.6003 -0.7268
83      1.6943 -1.1511
84      1.5991 -0.7327
85      1.3788 -0.6532
```

Extracting estimates/predictions

- Covariate effects

```
fixef(M3)
(Intercept)           x
     1.4628     -0.6811
```

- Random effects

```
ranef(M3)
$county
   (Intercept)          x
1     -0.318246  0.140485
2     -0.529388 -0.089754
...
83     0.231498 -0.470005
84     0.136351 -0.051624
85    -0.083977  0.027921
```

Let's inspect county 2 again

- Let ξ_j and ζ_j be estimated random intercept and slope for county j
- The regression for a specific country j is given by:

$$\hat{y}_j = (\mu_\alpha + \xi_j) + (\mu_\beta + \zeta_j)x$$

- Thus for a house in county 2 we have

$$\begin{aligned}\hat{y}_2 &= (1.463 - 0.529) + (-0.681 - 0.09)x \\ &= 0.933 - 0.771\end{aligned}$$

- Calculate from lmer output:

```
fixef(M3) + ranef(M3)[[1]][2,]
```

or

```
coef(M3)$county[2,]
```

Explaining effect heterogeneity

- Add covariate to county level which explains variation in floor effect

$$\begin{bmatrix} \alpha_j \\ \beta_j \end{bmatrix} \sim N\left(\begin{bmatrix} \gamma_0^\alpha + \gamma_1^\alpha u_j \\ \gamma_0^\beta + \gamma_1^\beta u_j \end{bmatrix}, \begin{bmatrix} \sigma_\alpha^2 & \rho\sigma_\alpha\sigma_\beta \\ \rho\sigma_\alpha\sigma_\beta & \sigma_\beta^2 \end{bmatrix} \right),$$

- Thus our second level 'regressions' are:

$$E[\alpha_j] = \gamma_0^\alpha + \gamma_1^\alpha u_j$$
$$E[\beta_j] = \gamma_0^\beta + \gamma_1^\beta u_j$$

Explaining effect heterogeneity

- Again, using 'pseudo regression' notation ...

$$
\begin{aligned}
y_{ij} &= \alpha_j + \beta_j x_i + \epsilon_{ij} \\
\alpha_j &= \gamma_0^\alpha + \gamma_1^\alpha u_j + \eta_j^\alpha \\
\beta_j &= \gamma_0^\beta + \gamma_1^\beta u_j + \eta_j^\beta
\end{aligned}
$$

- ... we see that this is a macro-micro interaction (a.k.a. 'cross-level interaction'):

$$
\begin{aligned}
y_{ij} &= \left[\gamma_0^\alpha + \gamma_1^\alpha u_j + \eta_j^\alpha\right] + \left[\gamma_0^\beta + \gamma_1^\beta u_j + \eta_j^\beta\right] x_i + \epsilon_{ij} \\
&= \gamma_0^\alpha + \gamma_1^\alpha u_j + \eta_j^\alpha + \gamma_0^\beta x_i + \gamma_1^\beta x_i u_j + \eta_j^\beta x_i + \epsilon_{ij}
\end{aligned}
$$

- In R note the following relationships:

γ_0^α	`Intercept`
γ_0^β	`x`
γ_1^α	`u`
γ_1^β	`x:u`

Explaining effect heterogeneity

- Thus to specify the 'cross-level interaction' model say:

```
M4 <- lmer(y ~ 1 + x + u + x:u + (1 + x | county))

Random effects:
 Groups   Name        Variance Std.Dev. Corr
 county   (Intercept) 0.0155   0.124
          x           0.0943   0.307    0.410
 Residual             0.5617   0.749
Number of obs: 919, groups: county, 85

Fixed effects:
            Estimate Std. Error t value
(Intercept)   1.4686     0.0352    41.7
x            -0.6709     0.0844    -7.9
u        0.8081     0.0906     8.9
x:u     -0.4195     0.2271    -1.8
```

Extract information from output objects

- County specific estimates via `coef()`:

```
$county
   (Intercept)        x        u       x:u
1        1.459  -0.6469   0.8081   -0.4195
2        1.496  -0.8890   0.8081   -0.4195
3        1.477  -0.6466   0.8081   -0.4195
...
85       1.439  -0.7011   0.8081   -0.4195
```

Extract information from output objects

- Calculate J county specific intercepts

$$\hat{\alpha}_j = \gamma_0^{\alpha} + \gamma_1^{\alpha} u_j$$

```
coef(M4)$county[,1] + coef(M4)$county[,3]*u
```

```
 [1] 0.9018 0.8109 1.3852 1.0480 1.3637 1.7609 1.8009 1.7288 ...
```

- Calculate J county specific slopes

$$\hat{\beta}_j = \gamma_0^{\beta} + \gamma_1^{\beta} u_j$$

```
coef(M4)$county[,2] + coef(M4)$county[,4]*u
```

```
[1] -0.3578 -0.5335 -0.5990 -0.3500 -0.5633 -0.8541 -0.4352 ...
```

Plot of estimated intercepts and slopes by u_j

- Estimated $\hat{\alpha}_j$ and $\hat{\beta}_j \pm 1$ s.e.