ALGEBRA FLIGHT MANUAL

A QUICK AND INCOMPLETE INTRODUCTION
TO ALGEBRA I AND ALGEBRA II

BY

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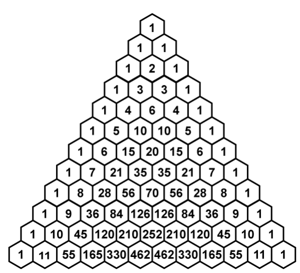
AUTHOR OF JUST THIS BOOK



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Part I

Algebra 1

Introduction

Brief History of Algebra

The first guy to think of using variables was Diophantus, a greek mathematician, who lived in Alexandria (Egypt) in the 3rd century A.D. He then influenced some Arab mathematicians, which furthermore dug into the idea of balancing the equations and actually inventing the word "algebra" (from arabic "al-jabr": "the reunion of broken parts"). Al-Khwarizmi wrote his *Al-kitab al-mukhatasar fi hisab al-gabr wa'l-muqabala* ("The Compedious Book on Calculation by Completion and Balancing") somewhere in the 9th century. The guy, who introduced x, y and z for the variables was Rene Descartes in the 16th century. He chose them for typographic reasons.

The Core Idea of Algebra

The core idea of algebra is to investigate the behavior of numbers in different combinations, thus abstracting and revealing the combination itself. You may find out that combining any numbers in a certain way always gives the same answer. Rewriting the problem with letters and solving it algebraically shows exactly how the pattern is born and where it starts to work. The best example, that I can think of is from the Alex Bellos's book *Alex's Adventure in Numberland*.

Choose any 3-digit number, where the difference between the first and the last digit is at least 2. For example, *753*. Then rewrite this number backwards: 357. And subtract it from the initial number:

$$753 - 357 = 396$$

Add the result and it's inverse:

$$396 + 693 = 1089$$

Let's try other number, for example *421*:

$$421 - 124 = 297$$

$$297 + 792 = 1089$$

In fact, if you take *any* number, such that $|a - c| \geq 2$ you'll get *1089*.

In order to really understand, what's happening behind this magic, we can use algebra. Let's rewrite our digits as abc. Our number then could be written as $100a+10b+c$. The inverse gives us cba and $100c+10b+a$. Subtracting $abc-cba$ then could be written as:

$$(100a + 10b + c) - (100c + 10b + a)$$

$$100a + \cancel{10b} + c - 100c - \cancel{10b} - a$$

$$100a - a - 100c + c$$

$$\cancelto{99a}{100a - a} \quad \cancelto{99c}{-100c + c}$$

$$99a - 99c$$

$$99(a - c)$$

In the last step we've only used the algebraic rule of distributive property, because we can use it to further simplify the expression. But accidentally (or actually not that accidentally) we ended up with our restriction or *domain* as part of the expression:

$$99(a - c) \rightarrow |a - c| \geq 2$$

Because of this restriction and mapping we can deduct that $(a - c)$ must be equal to one of these numbers: *2, 3, 4, 5, 6, 7* or *8*. So $99(a - c)$ must be equal to one of these numbers: *198*(99×2), *297*(99×3), *396*(99×4), *495*(99×5), *594*(99×6), *693*(99×7) or *792*(99×8).

We can see, that no matter what number we start with, we'll end up with one of these intermediate numbers.

Now let's add them with their inverses using the same technique. We can express the intermediates' digits as def and their inverses as fed. So the sum of these numbers would look like:

$$100d + 10e + f + 100f + 10e + d$$

Simplifying it algebraically gives us another expression:

$$100d + 10e + f + 100f + 10e + d$$

$$100d+100f+10e+10e+d+f$$
$$100(d+f)+20e+(d+f)$$

Now the last step is to figure out, how this expression always ends up with *1089*. We can look carefully at our intermediary numbers (*198, 297, 396...*) and notice some patterns: their middle digit or e is always *9*, and the sum of the first and the last digit or $(d+f)$ is also *9*. To understand, how exactly this happens we can look at the multiplication process as a repetitive addition:

$$99(a-c)$$
$$99\times 2$$
$$99+99$$
$$90+9+90+9$$
$$90+90+9+9$$
$$180+18$$
$$198$$

Or for another number:

$$99\times 3$$
$$99+99+99$$
$$90+9+90+9+90+9$$
$$90+90+90+9+9+9$$
$$270+27$$
$$297$$

We can see that 99×3 is the same as $(90\times 3)+(9\times 3)$, which clearly shows us, that the first number is the second number multiplied by 10. Because of this shift, the tenth's place digits add up to *9*, as well as hundredth's and one's place digits.

Knowing this feature, we can rearrange our $100d+10e+f+100f+10e+d$ expression, such that $(d+f)$ would show up:

$$100d+10e+f+100f+10e+d$$
$$100d+100f+10e+10e+d+f$$
$$100(d+f)+20e+(d+f)$$

Now plug in 9 instead of $(d+f)$ and e:

$$(100\times 9)+(20\times 9)+9$$

$$900+180+9$$

$$1089$$

If we had some mathematical intuition, we could have logically deducted that the expression must contain *constants* and then set an equation, which gives us the same result:

$$100(d+f)+20e+(d+f)$$

$$100x+20x+x=1089$$

$$121x=1089$$

$$x=9$$

So the core idea of algebra is all about finding such patterns and actually examining the behavior of these patterns, regardless of the numbers used in them. Patterns maybe written in the form of *functions* and then *graphed* in order to visualize their behavior on a large scale and with different inputs. A graph can reveal other features, like *structure, form* and *symmetry*.

Then algebra studies equations, that could be solved, have one or several solutions, no solutions at all or solutions, that lie beyond the real number system (*imaginary numbers*). Examining different objects and their features, like polynomials, vectors, matrices (*linear algebra*), groups, rings and fields (*abstract algebra*) helps to find solutions to even more complex real life problems.

Also algebra takes some arithmetic operations like *exponents* and *fractions* to another level by introducing *rational exponents* and *logarithms*.

Equations

One of the basic ideas, that algebra deals with are *equations*. An equation is a *statement*, that declares an equality of two things. The best way to understand this concept is to see it as scales and balances. People used those in the old days to find out, how much something weighs. You put an object with unknown mass on one part of the scales and on the opposite part you put weights with known labeled mass (like 1mg, 5mg or 1kg, 2kg etc.). Then you add or subtract these weights until you reach a balance state of the scales.

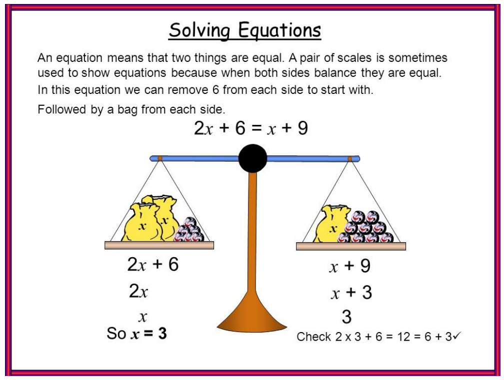

Figure 1: The concept of equations becomes much clearer, when introduced through scales

You can even put two unknowns on each side and balance them with weights. Then in order to find out the mass of one bag (see Figure 1), you would like to collect all the bags on one side of the scales and all the weights on the other. Then just divide the total mass of the weights by the number of bags and get your result.

This idea goes deeply through algebra, and it's always good to remember, what you're really doing with equations. For example, it's easy to forget to add a number to *both sides* of the equation, when *completing the square* with quadratic equations. More on that later.

You have to also remember to collect only identical numbers or equal bags, otherwise the balance won't occur. Speaking algebraically, you can do $2x + x = 3x$, but $2x + y = 3x$ doesn't make sense.

Chapter 1

Pre-Algebra

Inequalities

You solve inequalities the same way as you solve equations, except, that multiplying one side by a negative number, changes the sign.

$$2x+5 \geq 10$$

$$(-1) \times 2x+5 \geq 10 \times (-1)$$

$$5-2x \leq -10$$

$$-2x \leq -5$$

$$x \geq 2.5$$

Another good way to think about inequalities is to visualize the range of numbers. For example, $4 < x < 4$ has no solutions, $4 \leq x \leq 10$ has finite number of solutions or $\{4, 5, 6, 7, 8, 9, 10\}$ and $(0, \infty) = \{x \in \mathbb{R} : x > 0 \text{ and } x > 1\}$ has infinite number of solutions where $x > 0$. These concepts are visualized in Figure 1.1.

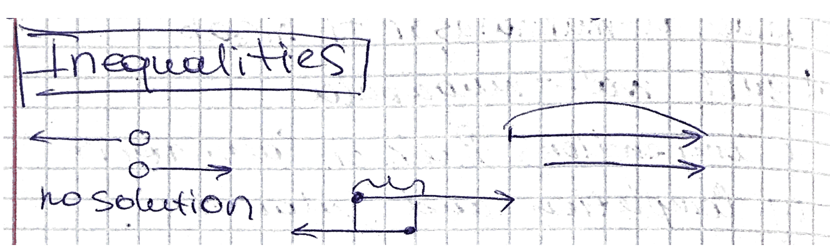

Figure 1.1: The range of x in different situations

Units

Working with units usually requires some knowledge of *dimensional analysis*. You'll find a lot of this in physics classes, especially in astrophysics.

For example, let's find out what distance someone would travel for an hour with a speed of 5 *meters per second*:

$$5\frac{m}{s} \times 1h = 5\frac{mh}{s} = 5\frac{m\cancel{h}}{\cancel{s}} \times \frac{3600}{1}\frac{\cancel{s}}{\cancel{h}} = \frac{18000}{1}\frac{\cancel{m}}{1} \times \frac{1}{1000}\frac{km}{\cancel{m}} = 18km$$

Or we can translate 50 *kilometers per hour* into *meters per second*:

$$50\frac{km}{\cancel{h}} \times \frac{1}{3600}\frac{\cancel{h}}{s} = \frac{50 \times 1}{3600}\frac{km}{s} = \frac{1}{72}\frac{\cancel{km}}{s} \times \frac{1000}{1}\frac{m}{\cancel{km}} = \frac{1000}{72}\frac{m}{s} \approx 13.9\frac{m}{s}$$

The conversion happens by multiplying by the equivalent ratio in other units, such that the units could be cross-multiplied and canceled. For example, if you forget the correct formula of the *degrees* to *radians* conversion, you use dimensional analysis to figure it out:

$$90^\circ = \frac{90\cancel{^\circ}}{1} \times \frac{\pi}{180\cancel{^\circ}} = \frac{\pi}{2}$$

Ratios, Rates and Proportions

A phrase *4 monkeys for every 5 bananas* means $\frac{4}{5}$ or $4:5$. A ratio is a relationship between two numbers indicating how many times the first number contains the second. It could be illustrated explicitly as a fraction:

$$\frac{1}{2} = \frac{2}{4} = \frac{4}{8}$$

or as a number:

$$\frac{1}{2} = 0.5, \ \frac{5}{2} = 2.5$$

The phrase *4 monkeys for every 5 bananas* can be viewed as how many monkeys does 1 banana have:

$$\frac{4}{5} = 0.8$$

Or the other way around: how many bananas does 1 monkey have:

$$\frac{5}{4} = 1.25 \times 4 = 5$$

If ratio compares two quantities of the same unit, *rates* compare quantities of different units.

Problem

It takes 36 minutes for 7 people to paint 4 walls. How many minutes does it take 9 people to paint 7 walls?

First of all you have to find out how much time does it take for 1 person to paint 1 wall. It takes $\frac{1}{4}$ of the time to paint 1 wall, so it takes 9 minutes for 7 people to paint 1 wall, (1.1). For 1 person it'll take 7 times as long or 63 minutes, (1.2). To paint 7 walls for 1 person would require 7 times as much time or 441 minutes, (1.3). At last, for 9 people to paint 7 walls it'll take 9 times less time or 49 minutes, (1.4).

$$
\begin{aligned}
36 \times \frac{1}{4} &= 9 && (1.1)\\
9 \times 7 &= 63 && (1.2)\\
63 \times 7 &= 441 && (1.3)\\
\frac{441}{9} &= 49 && (1.4)
\end{aligned}
$$

The idea behind this problem is based on understanding, that $\frac{7}{4}$ is *7 painters for 4 walls* and $\frac{7}{1}$ is *7 painters for 1 wall*. The difference here, is that the second statement is *reduced* by 4, so you have to reduce *36 minutes* by the same amount. And then you should realize, that $\frac{1}{7}$ is *1 painter for 7 walls*, and the difference with $\frac{7}{1}$ is that the number of people was *reduced* by 7 or that the time was *increased* by 7. You always want to find the ratio or the quantity of something *per unit*:

$$7:4 \rightarrow 7:1 \rightarrow 1:1 \rightarrow 1:7 \rightarrow 9:7$$

Another way to look at this problem is to find out how many painters are there for each wall:

$$\frac{7}{4} = 1.75$$

and then multiply this ratio by 36 minutes:

$$1.75 \times 36 = 63$$

which directly gets you to how much time it takes for 1 person to paint 1 wall. In my opinion, this is a much more intuitive approach.

Percents

A *percent* means something *divided by a hundred*:

$$35\% = 35 \text{ per hundred } = \frac{35}{100} = 35 : 100 = 0.35$$

To convert a fraction into percent, you need to bring both the numerator and the denominator to a 100:

$$\frac{4}{5} = \frac{4 \times 20}{5 \times 20} = \frac{80}{100} = 0.8 = 80\%$$

To increase a number by a certain percent, you need to add to that number the percent of that number:

$$305 + 9\% = 305 + (305 \times 0.09) = 305 + 27.45 = 332.45$$

this is the same as taking 109% of 305:

$$305 \times 109\% = 305 \times 1.09 = 332.45$$

Chapter 2

Linear Equations

Linearity is a is the property of a mathematical relationship or function which means that it can be graphically represented as a straight line. Examples are the relationship of voltage and current across a resistor (Ohm's law), or the mass and weight of an object. Proportionality implies linearity, but linearity does not imply proportionality.

In other words, when we talk about linearity, we talk about a specific *relationship* between two variables, one of which is dependent on the other. This relationship could be written in different forms, such as *equation* or a *function*. It all depends, what idea you want to emphasize.

Slope-Intercept Form

The slope-intercept form of a linear equation is the following:

$$y = mx + b$$

where m is the *slope* of the graph and b is some *constant*.

In a linear equation y is always produced by using some basic arithmetic properties on x, like multiplication or division and/or addition or subtraction. That way all of the y outputs increase or decrease with a constant rate, thus forming a line on a graph. This rate is called the *slope* and represented mathematically as:

$$\frac{\Delta y}{\Delta x} = \frac{\text{change in } y}{\text{change in } x} = \frac{\text{rise}}{\text{run}}$$

In physical world this rate or slope could take a form of speed, for example, where distance is y and time is x.

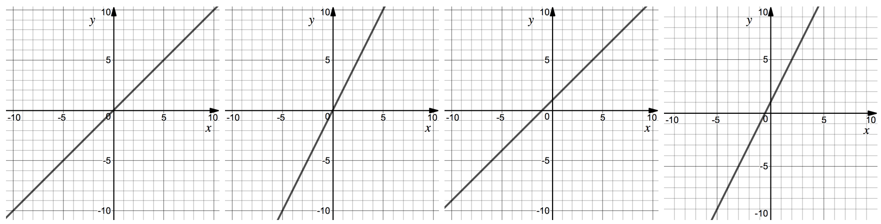

Figure 2.1: Linear equation behavior. From left to right: $y = x$, $y = 2x$, $y = x + 1$, $y = 2x + 1$.

Shifting Linear Functions

Positive change in slope (multiplying x) increases the steepness of the line, the opposite operation make the line more narrow. Adding or subtracting a constant shifts the line upwards or downwards, changing the y-intercept (Figure 2.1).

To graph a linear function from it's equation in the slope-intercept form, you need to take the slope or m and add the Δy to the y coordinate and Δx to the x coordinate of the y-intercept of the line, which is the constant b:

$$y = mx + b \to y = 2x + 1$$

$$\frac{\Delta y}{\Delta x} = \frac{2}{1}$$

$$(0, 1) \to (0 + 1, 1 + 2) \to (1, 3)$$

If given 2 points, but you need the equation, then first find the slope and then substitute one of the points into the equation and solve for b: Let the two points be (2, 5) and (4, 9). Then the slope would be:

$$\frac{\Delta y}{\Delta x} = \frac{9-5}{4-2} = \frac{4}{2} = 2$$

Solve for b:

$$y = mx + b$$

$$y = 2x + b$$

$$5 = 2 \cdot 2 + b$$

$$5 = 4 + b$$

$$5 - 4 = b$$

$$b = 1$$

$$y = 2x + 1$$

Point-Slope Form

If give 2 points it is more easier to write a linear equation in the point-slope form:

$$y - b = m(x - a)$$

where b is the y coordinate in the second point, a is the x coordinate in the second point and m is the slope. It basically says, that change in y is m times the change in x:

$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y - b}{y - a} = m$$

$$\cancel{(x - a)} \cdot \frac{y - b}{\cancel{x - a}} = m \cdot (x - a)$$

$$y - b = m(x - a)$$

The key feature of the equation written in point-slope form, is that you can find a point just from setting the difference or deltas equal to zero. For example:

$$y - 1 = 2(x - 3)$$

$$y - 1 = 0 \rightarrow y = 1$$

$$x - 3 = 0 \rightarrow x = 3$$

So our first point is (3, 1). To find the second point, apply to slope:

$$(3, 1) \rightarrow (3 + 1, 1 + 2) \rightarrow (4, 3)$$

Standard Form

A linear equation written in standard form may look like this:

$$ax + by = c$$

It's a combination of x and y values, that equal to some constant. You can find x and y intercepts by substituting x and y with 0 and solving for one of the variables:

$$2x + 3y = 12$$

$$2x + 3 \cdot 0 = 12$$

$$2x = 12$$

$$x = 6$$

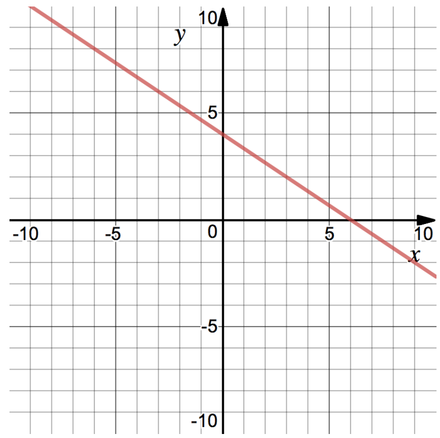

Figure 2.2: The graph of $2x+3y=12$ intercepts x axis at 6 and y axis at 4

$$2 \cdot 0 + 3y = 12$$
$$3y = 12$$
$$y = 4$$

In this example our two points are (6, 0) and (0, 4).

Age Word Problems

Some world problems could be written in such a way, that you can apply one of the forms of linear equations to model it. For example, the slope-intercept form is more suited for questions concerning dependencies, like distance and time relationship or profit and expenses. It's quite easy to capture, because the problem would usually be talking about some kind of output. The standard form is mostly about a known amount of something, a constant, that is formed by some unknown values. The point-slope form, on the other hand, is a bit trickier. Such problems usually talk about some kind of change in time between unknown objects, and should find the values of these objects. The key to solving these kind of problems is to figure out a way in which one object could be expressed in terms of the other object.

Problem

Ben is 12 years older, than Isaac. Three years ago Ben was 3 times as old as Isaac. How old is Ben and Isaac?

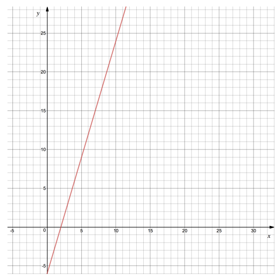

Figure 2.3: The graph of $y-3=3(x-3)$ has a point (21, 9) which is the answer to our problem.

So first of all we can model some variables. Let Ben be b and Isaac be i. We know, that Ben is 12 years older, than Isaac, so we can express either of them in terms of each other, such that $b=i+12$ or $i=b-12$. Then we could try to look for a suitable linear equation form. Because we have change in time involved, we should definitely give a point-slope form a go:

$$y-b=m(x-a)$$

If the first point is the present age of Ben and Isaac, then the second point could be their age three years ago or $b-3$ and $i-3$. We know, that Ben 3 years ago was 3 times as old as Isaac, so we can write our equation:

$$b-3=3(i-3)$$

In order to solve for Ben, we have to substitute i with $b-12$:

$$b-3=3(b-12-3)$$

$$b-3=3b-36-9$$

$$b-3b=-36-9+3$$

$$-2b=-42$$

$$b=21$$

So Ben is 21 years old and Isaac is *12 years younger* or 9 years old. You can also solve this problem graphically by finding two points. The first one is (3, 3) and the second one we can find from the slope of 3: (3+1, 3+3)→ (4, 6). Plot a line and find a point, where the difference between x and y is 12. Or find two points where a y coordinate from one point divided by an x coordinate from another point gives 3 and then add 3 to both of them to get the final answer (Figure 2.3).

Chapter 3

Sequences

A sequence can be thought of as a list of elements with a particular order. More formally a sequence is an enumerated collection of objects in which repetitions are allowed. Unlike a set, it has to be ordered in some way and same elements may appear more, than once. The underlying idea of sequence makes it more appear as a function, where the *term* is the output and the *element number* is the input.

Arithmetic Sequences

An arithmetic sequence or *progression* is a type of mathematical sequence, such that the *common difference* between the terms is constant.

The explicit formula for arithmetic sequence may look like this:

$$a(n) = b + m(n-1)$$

where n is the element of the sequence, b is the first term or starting point and m is the common difference. For example,

$$a(n) = 5 + 2(n-1)$$

describes the following sequence:

$$(5, 7, 9, 11, 13, 15, 17, ...)$$

With this formula you can find n^{th} elements of the sequence.

Arithmetic sequences could also be describes recursively:

$$\begin{cases} a(1) = 5 \\ a(n) = a(n-1) + 2 \end{cases}$$

In order to solve this, you have to start from the required n^{th} element and work backwards until you reach the first element or the *base case*. Then plug in the first element to the equation of the second element, that element to the equation of the third element and so on until you reach the desired n^{th} element:

$$a(5) = \overset{13}{\cancel{?}}$$

$$\begin{aligned} a(5) &= \overset{11}{\cancel{a(4)}} + 2 \\ a(4) &= \overset{9}{\cancel{a(3)}} + 2 \\ a(3) &= \overset{7}{\cancel{a(2)}} + 2 \\ a(2) &= \overset{5}{\cancel{a(1)}} + 2 \\ a(1) &= 5 \end{aligned}$$

Geometric Sequences

A geometric sequence is a type of mathematical sequence, such that the *common ratio* between the terms is constant. It means, that the next term is the previous term multiplied by a certain number, that is not equal to 0 or 1, and it's always the same.

An explicit formula for a geometric sequence may look like this:

$$a(n) = k \cdot r^{n-1}$$

where k is first term, r is the common ratio and n is the element number.

It also has a recursive definition:

$$\begin{cases} a(1) = k \\ a(n) = a(n-1) \cdot r \end{cases}$$

Chapter 4

Functions

Functions are pretty much the same as we know them in programming, except that in mathematics we pay more attention to their properties and other details. In Algebra I you get familiar with the *domain* and *range* of functions, their basic notation and word problems, expressed as functions. More details would be learned in Algebra II course, like *end behavior*, *inverse functions*, *function arithmetic* and other interesting aspects. However, the real engagement with functions begins at college, where along with sets, they are the central object of investigation in higher mathematics, especially in *set theory*.

Domain of a function is a set of all inputs, for which a function is defined. For example, $\{x \in \mathbb{R} \mid x \geq 0\}$ means, that the domain of some function is a *set of all real numbers greater than or equal to zero*. The domain can be expressed as a finite set: $\{5, 2, 1\}$.

The *range* or the *codomain* of a function is a set of all possible outputs. It could be written as $\{f(x) \in \mathbb{R} \mid x \neq 0\}$.

Functions are usually written in the form of an equation, starting with $f(x)$, which means *function of* x. Here x is a *variable, input* or an *argument* of a function. $f(x)$ is the output or the *image* of x *by* f. In word problems it is usually stated in a form, like *the position of a planet is a function of time*. It could written as $P(t)$ or $P = f(t)$, where P is the position of the planet and t is time.

Functions could also have *if* statements in definition. For example:

$$f(x) : \begin{cases} 1 & \text{if } x = 20 \\ \pi & \text{if } x = 5 \end{cases}$$

Chapter 5

Systems of Equations

A *system of equations* is a collection of equations, that share the same x and y solutions.

There are two methods for solving systems of equations: *elimination method* and *substitution method*.

Elimination Method

The idea behind this method is to form one equation out of two, by adding them together and eliminating one of the variables:

$$\begin{cases} 2y + \cancel{7x} = -5 \\ 5y - \cancel{7x} = 12 \end{cases}$$

$$7y = 7$$

$$y = 1$$

Then you can plug in the y into one of the equations and get x:

$$2 \cdot 1 + 7x = -5$$

$$7x = -5 + 2$$

$$7x = -7$$

$$x = -1$$

You can "massage" the equations by multiplying them by some number in order to reach a condition, where one of the variables could cancel out.

Substitution Method

For this method, you have to rewrite one of the equations in terms of y or x and then substitute that equation in the second one:

$$\begin{cases} 7x + 10y = 36 \\ -2x + y = 9 \end{cases}$$

$$y = 2x + 9$$

$$7x + 10(2x + 9) = 36$$

$$7x + 20x + 90 = 36$$

$$27x = -54$$

$$x = -2$$

Then plug in the x value into one of the equations:

$$y = 2(-2) + 9$$

$$y = -4 + 9$$

$$y = 5$$

Equivalent Systems of Equations

Sometimes you may be asked to consider where some systems of equations are equivalent. There are two conditions: *a sum of 2 equations* and a *multiple of itself.* These conditions could be visualized graphically (Figures 5.1-5.3).

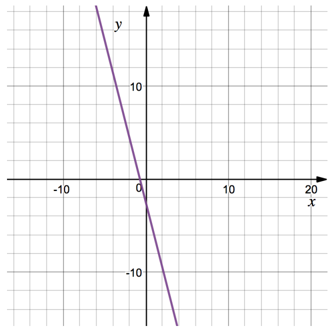

Figure 5.1: Two equations $2y+7x=-5$ and $4y+14x=-10$ have infinite number of solutions, because they are multiples of each other and therefore share the same line.

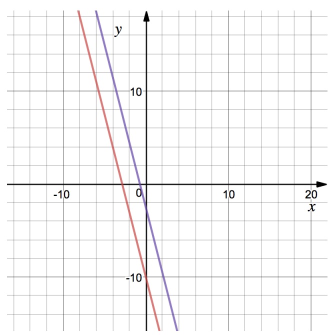

Figure 5.2: Two equations $y=-\frac{7}{2}x-2.5$ and $y=-\frac{7}{2}x-10$ have no solutions, because they have the same slope, but different y-intercepts. One the graph they are parallel to each other.

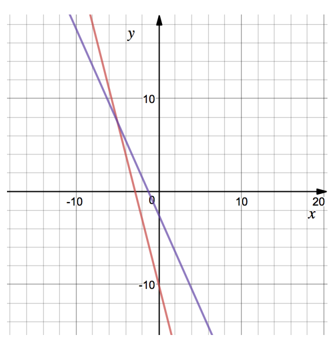

Figure 5.3: Two equations $y=-\frac{7}{2}x-10$ and $y=-2x-2.5$ have exactly one solution, because they are totally different equations.

Chapter 6

Piecewise Functions

Piecewise functions are defined on different intervals and form line segments on a graph.

For example, a piecewise function may be defined like this:

$$f(x) : \begin{cases} 6 & -6 < x \leq -2 \\ -3 & -2 < x \leq 5 \end{cases}$$

The graph of such function is shown on Figure 6.1.

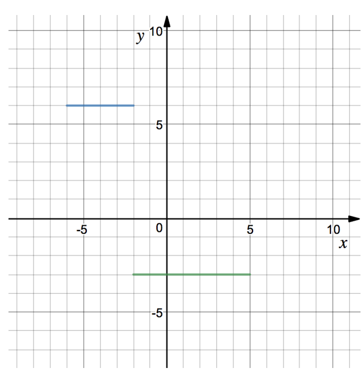

Figure 6.1: Piecewise functions

Chapter 7

Exponential Growth

Exponential growth may be described by a function, where the argument x is an exponent, to which a certain expression is raised. The only thing that changes is the exponent. It grows really fast, even faster, than a quadratic function, with which it could be accidentally confused. An example:

$$h(x) = 27 \cdot (\frac{1}{3})^x$$

Complex percents are usually shown as an example of exponential growth:

$$100 \cdot (1.3)^n$$

Here is an illustration of the growth of exponential function:

$$2^1, 2^2, 2^3, 2^4, 2^5, 2^6, 2^7, 2^8, 2^9, 2^{10}$$

$$2, 4, 8, 16, 32, 64, 128, 256, 512, 1024$$

Compare to a quadratic function:

$$1^2, 2^2, 3^2, 4^2, 5^2, 6^2, 7^2, 8^2, 9^2, 10^2$$

$$2, 4, 9, 16, 25, 36, 49, 64, 81, 100$$

The graphs of $y = 2^x$ and $y = x^2$ are shown on Figure 7.1.

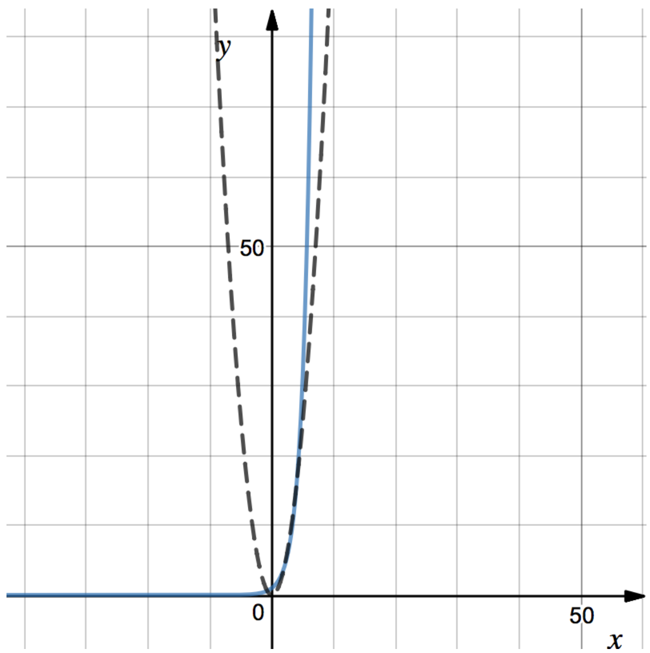

Figure 7.1: Function $y = 2^x$ quickly surpasses the positive value of $y = x^2$ at the point (3, 16).

Chapter 8

Polynomials

On polynomials you spend probably most of the time in Algebra I. There's a lot of stuff you can do with them, and they'll certainly stick with you throughout the college math courses.

So a *polynomial* is an *expression* consisting of variables and coefficients, that involves only addition, subtraction, multiplication and non-negative integer exponents of variables:

$$x^2 - 4x + 7$$

$$x^3 + 2xyz^2 + 1$$

With these kind of polynomials you can construct equations, that encode a wide range of scientific and mathematical problems. In Algebra I you take a closer look at *quadratic equations*.

The *coefficient* of a polynomial is the number, that stands in front of a variable. The coefficient along with a variable are called the *term* of a polynomial.

There could be different forms of polynomials, like *monomials*, that consist only of a term ($6x$ or 6), *bynomials*, that have two terms ($9x^2 - 5$), *trinomials* with three terms ($7y^2 - 3y + \pi$), etc. Usually polynomials with more, than 3 terms are just called polynomials. But occasionally you can see some people referring to them as quadrinomials and similar, but these are rarely used, although exist.

Polynomials in standard form are written as a sum of terms with exponents in descending order. The term with the highest exponent is written first, and that exponent is called the *degree* of a polynomial, because it determines the behavior of the polynomial function. Or in other words, it has the greatest impact on the result of a polynomial equation.

Basic Arithmetic Operations with Polynomials

You can all of the basic arithmetic operations with polynomials, just like with numbers. Some rules, though, good to remember.

Only terms with the same degree and variables could be added. For example, you can add $4x$ with $2x$ or $2xy^2$ with $\frac{1}{2}xy^2$.

Group constants and variables together when multiplying:

$$(-4x^2)(7x^3) = -4 \cdot 7 \cdot x^2 \cdot x^3 = -28 \cdot x^{2+3} = -28x^5$$

When multiplying a polynomial by a monomial, multiply each term of the polynomial by that monomial and add the resulting expressions (distribution property):

$$x(x^2 - 5x - 6) = (x \cdot x^2) + (x \cdot (-5x)) + (x \cdot (-6)) = x^3 - 5x^2 - 6x$$

When multiplying two binomials, multiply the second binomial by each term of the first binomial, apply the distributive property and add the resulting expressions:

$$(x-2)(x-6) = x(x-6) - 2(x-6) = x^2 - 6x - 2x + 12 = x^2 - 8x + 12$$

If you add some terms to the second binomial resulting in a greater polynomial, the technique remains the same:

$$(1+x)(x^2 - 5x - 6) = 1(x^2 - 5x + 6) + x(x^2 - 5x - 6) = x^2 - 5x - 6 + x^3 - 5x^2 - 6x$$

$$x^3 - 4x^2 - 11x - 6$$

At this point you may have heard of the so-called FOIL technique, which is the same thing, but with expansion operations skipped. Such techniques are bad, because you don't understand the underlying processes, and this will be a great pain in the neck in the future.

Polynomial Arithmetic Patterns

There are some patterns, recognizing which will help you skip all the arithmetic operations.

Difference of squares:

$$(a+b)(a-b) = a^2 - b^2$$

Perfect squares:

$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a-b)^2 = a^2 - 2ab + b^2$$

For example, $(c-5)(c+5)$ could be expanded as $c^2 - 25$ and $(m+7)^2$ as $m^2 + 14m + 49$.

Although it's could to remember these patterns, you can (and actually should while first time learning them) always multiply them as any other polynomial using distributive property described in the previous subsection.

Factorization

Factorization is the main technique used in solving quadratic equations. It's the inverse operation of expanding polynomials, so some of the patterns you already know, like *difference of squares* and *perfect squares*. Factoring out other polynomials might be tedious at first.

Sum-product pattern can be used, when a polynomial is in form of $x^2 + bx + c$ and factors of c add up to b:

$$x^2 + 7x + 12$$
$$x^2 + (4+3)x + (4 \cdot 3)$$
$$(x+3)(x+4)$$

The grouping method can be used, when a polynomial is in form $ax^2 + bx + c$ and there are no common factors. Again you can find factors of ac, that add up to b, but instead of factoring them out directly, rewrite the middle term bx as a sum of these factors. Group terms, that share common factors and rewrite the expression as a product of the sum of these common factors and a resulting shared binomial:

$$2x^2 + 7x + 3$$
$$3 \cdot 2 = 6$$
$$7 = 6 + 1$$
$$6 = 6 \cdot 1$$
$$2x^2 + 6x + x + 3$$
$$2x(x+3) + 1(x+3)$$
$$(x+1)(2x+1)$$

On the next page you can find a table, that summarizes all the factoring techniques.

Method	**Example**	**When is it applicable?**
Factoring out common factors	$6x^2+3x=3x(2x+1)$	If each term in the polynomial shares a common factor
Difference of squares	$x^2-9=(x-3)(x+3)$	If expression represents difference of squares
Perfect square trinomial	$x^2+10x+25=(x+5)^2$	First and last term are perfect squares and middle term is twice the product of their square roots
Sum-product pattern	$x^2+7x+12=(x+3)(x+4)$ $4\cdot 3=12$ $4+3=7$	If in form x^2+bx+c and has factors of c, that add up to b
Grouping method	$2x^2+7x+3$ $3\cdot 2=6$ $6\cdot 1=6$ $6+1=7$ $2x^2+6x+x+3$ $2x(x+3)+1(x+3)$ $(x+3)(2x+1)$	If in form ax^2+bx+c and has factors of ac, that add up to b

CHECK LIST

Before any manipulations put to standard form.

1. **Is there a common factor?**
   If no, move to question 2. If yes, factor out GCF and continue to question 2.

2. **Is there a difference of squares?**
   If yes, factor out $a^2 - b^2$ as $(a-b)(a+b)$. If not, move to question 3.

3. **Is there a perfect square?**
   $x^2 - 10x + 25$ or $4x^2 + 12x + 9$
   If yes, factor out like $a^2 \pm 2ab + b^2 = (a)^2$. If not, move to question 4.

4. **a) Is there an expression in form $x^2 + bx + c$?**
   If not, move to question 5. If yes, move to **b)**.

   b) Are there factors of c, that add up to b?
   If yes, use sum-product, else → **stop**.

5. **Are there factors of ac, that add up to b?**
   Must be in form $ax^2 + bx + c$. Use grouping method. If not possible → **stop**, no factorization possible.

Chapter 9

Quadratics

Quadratic equations are constructed of an even-degree polynomial and their graph is forming a *parabola*. It looks somewhat similar to an exponential equation, but is less steep, has a vertex and is symmetric, because solving it requires to take a square root, which has both positive and negative values.

Quadratic equations could model problems, that describe a trajectory of an object, like a ball or a rocket. For example, the two x-intercepts are points, where a rocket will hit the ground, y-intercept is it's starting point, and *vertex* is it's maximum point elevation. Or it could be used to describe a situation of two prices, at which a company would receive no profit.

Forms of Quadratic Equations

There are 3 forms of quadratic equations.

Standard form:

$$x^2 - x - 6$$

reveals the y-intercept, which is -6 in this case.

Factored form:

$$2(x-3)(x+4)$$

reveals the x's or "zeros" or "roots" of the equation.

$$x - 3 = 0$$

$$x = 3$$

$$x + 4 = 0$$

$$x = -4$$

Vertex form:

$$-2(x+5)^2+4$$

reveals the vertex of a parabola, which is -5 and 4 in this case.

$$x+5=0$$

$$x=-5$$

$$y=4$$

Graphing Quadratic Equations

To graph a quadratic equation written in *standard form*, you first have to find the x coordinate of the vertex by using the $\frac{-b}{2ac}$ part of the *quadratic formula*. Then plug it in the equation and solve for y. The second point is the y-intercept, which could be seen from the equation itself.

To graph a quadratic equation written in *factored form* you have add the "zeros" or x's and divide the sum by 2, which will be the x coordinate of the vertex. Plug it in the equation and solve for y. Pick one of the "zeros" for the other point.

To graph a quadratic equation written in *vertex form* you have to take the vertex from the equation and then choose a point close to it, such that $(x+5)^2$ would equal to 1 or other perfect square. Then find y by plugging in that x value into the equation.

Solving Quadratic Equations

The main difference from linear equations is that you end up with a $\sqrt{x^2}$ or $(x+b)^2$ at the end. There's no point of taking the square root of an expression, like x^2+2x+b^2, because you end up with a $\sqrt{x}$, which is not solvable. And of course, you have two $\pm$ solutions.

Some basic examples of solving quadratic equations:

$$x^2=36$$

$$\sqrt{x^2}=\pm\sqrt{36}$$

$$x=\pm 6$$

And more complex:

$$(x-2)^2=49$$

$$\sqrt{(x-2)^2}=\pm\sqrt{49}$$

$$x - 2 = \pm 7$$

$$x = \pm 7 + 2$$

$$x_1 = 7 + 2 = 9$$

$$x_2 = -7 + 2 = -5$$

So this means, that you have to always factor equations, when possible (more on how to do that on page 30). It won't necessarily always be a perfect square as in the previous example. More often you'll use the sum-product pattern:

$$x^2 - 3x - 10 = 0$$

$$(x - 5)(x + 2) = 0$$

Now you need to find the "zeros" of this equation. In order for the whole equation to be equal to 0, one or both of the factors must be equal to 0:

$$\cancel{(x - 5)}^{\,0} \times (x + 2) = 0$$

$$(x - 5) \times \cancel{(x + 2)}^{\,0} = 0$$

You can set both factors equal to zero and solve these separate equations:

$$x - 5 = 0$$

$$x_1 = 5$$

$$x + 2 = 0$$

$$x_2 = -2$$

So the two possible solutions for $x^2 - 3x - 10 = 0$ are 5 and -2.

Completing the Square

When you've gone through the check list (page 30), and realized, that factoring is not possible, you can use the *completing the square* technique. Let's take:

$$x^2 - 10x + 12 = 0$$

this equation cannot be factored using our basic tools from the check list. So completing the square might be helpful:

$$x^2 - 10x = -12$$

$$x^2 - 10x + 25 = -12 + 25$$
$$(x-5)^2 = 13$$

The idea here is that you isolate the $x^2 + bx$ of the $x^2 + bx + c$ equation on one side and pick a number, by adding which you can get it in the form of a perfect square expression of type $(a \pm b)^2$. To find that number, you have to divide b by 2 and square the result:

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = c + \left(\frac{b}{2}\right)^2$$

And don't forget to add this number to *both sides of the equation*. The rest of the solving will be the same as with perfect square equations, except, that you'll end up answer for x's containing non-perfect square roots:

$$(x-5)^2 = 13$$
$$\sqrt{(x-5)^2} = \pm\sqrt{13}$$
$$x - 5 = \pm\sqrt{13}$$
$$x = \pm\sqrt{13} + 5$$

Completing the square gets a bit tricky, when you deal with equations in form $ax^2 + bx + c$ and you have to factor out some common factors first:

$$4x^2 + 4x - 5 = 0$$

there's actually no common factor for the whole equation, but you can factor only the $ax^2 + bx$ part:

$$4(x^2 + x) - 5 = 0$$
$$4(x^2 + x) = 5$$

Now you can proceed to completing the square:

$$4\left(x^2 + x + \left(\frac{x}{2}\right)^2\right) = 5 + 4\left(\frac{x}{2}\right)^2$$
$$4\left(x^2 + x + \frac{1}{4}\right) = 5 + 4 \cdot \frac{1}{4}$$

Notice how we added not only $\frac{1}{4}$, but $4 \cdot \frac{1}{4}$ in order for both sides of the equation to be absolutely equivalent. From this point you can solve the rest of the equation as usual:

$$4\left(x + \frac{1}{2}\right)^2 = 6$$

$$\left(x+\frac{1}{2}\right)^2 = \frac{6}{4}$$

$$\sqrt{\left(x+\frac{1}{2}\right)^2} = \pm\sqrt{\frac{6}{4}}$$

$$x+\frac{1}{2} = \pm\frac{\sqrt{6}}{2}$$

$$x = \pm\frac{\sqrt{6}}{2}-\frac{1}{2}$$

$$x = \pm\frac{1}{2}\sqrt{6}-\frac{1}{2}$$

Quadratic Formula

It can get really tough, when completing the square hits some hairy fractions. In this last worst scenario the *quadratic formula* may be used:

$$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

You may be surprised, by it's exactly the same as the completing the square technique and is derived directly from it. Here's where the beauty of algebra comes into play. The same equation or technique, expressed in variables, can give you a pretty interesting shortcut for solving some complex problems. This is the proof of quadratic formula:

$$ax^2+bx+c=0$$

$$ax^2+bx=-c$$

$$x^2+\frac{b}{a}x=-\frac{c}{a}$$

$$x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=-\frac{c}{a}+\left(\frac{b}{2a}\right)^2$$

$$x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=-\frac{c}{a}+\frac{b^2}{4a^2}$$

$$\left(x+\frac{b}{2a}\right)^2=-\frac{c}{a}+\frac{b^2}{4a^2}$$

$$\left(x+\frac{b}{2a}\right)^2=-\frac{4a\cdot c}{4a^2}+\frac{b^2}{4a^2}$$

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$$

$$\sqrt{\left(x+\frac{b}{2a}\right)^2}=\sqrt{\frac{b^2-4ac}{4a^2}}$$

$$x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$x=-\frac{b\pm\sqrt{b^2-4ac}}{2a}$$

Vertex and Discriminant

You can quickly find the x coordinate of a vertex of a parabola from $-\frac{b}{2a}$ part of the quadratic formula. It is possible to prove this, by knowing, that vertex occurs on the vertical line of symmetry. Suppose we have the same equation:

$$ax^2+bx+c=0$$

Shifting up or down has no effect on the symmetry and vertex, so we can exclude c from the equation:

$$ax^2+bx=0$$

$$x(ax+b)=0$$

In order for the equation to be equal to 0, either x or $(ax+b)$ must be equal to 0. Thus, our "zeros" are the following:

$$x_1=0$$

$$ax+b=0$$

$$ax=-b$$

$$x_2=-\frac{b}{a}$$

The vertex would be halfway between the "zeros", so our final answer is:

$$x_v=-\frac{b}{2a}$$

There's another algebraic proof for mapping $-\frac{b}{2a}$ to the x of the vertex. You can add the "zeros" directly from the quadratic formula:

$$\left(-\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a}\right)+\left(-\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}\right)$$

$$-\frac{b}{2a}+\cancel{\frac{\sqrt{b^2-4ac}}{2a}}-\frac{b}{2a}-\cancel{\frac{\sqrt{b^2-4ac}}{2a}}$$

$$-\frac{\cancel{2}b}{\cancel{2}a}=-\frac{b}{a}$$

Then divide the sum by 2 to get the middle or the vertex:

$$-\frac{b}{2a}$$

A *discriminant* is the part under the square root of the quadratic formula:

$$b^2-4ac$$

When it's *equal to 0*, then the equation has only *one solution* and you're dealing with a perfect square:

$$(x+7)^2=0$$

$$(x^2+7x+49)=0$$

$$\frac{-14\pm\sqrt{196-4\cdot 49}}{2}$$

$$\frac{-14\cancel{\pm}\sqrt{0}}{2}$$

$$-\frac{14}{2}=-7$$

The graph of such equation will be touching, but not crossing the x axis at (-7, 0).

When a discriminant is *positive*, then you're dealing with an equation, that has *two solutions*, because you're actually taking the square root. It could be $(x+7)^2=36$, $(x+7)(x-7)=0$ or $(x-3)(x+4)=0$. The graph will intersect the x axis at exactly two points.

Finally, when a discriminant is *negative*, then there are *no solutions* to the equation, because taking a square root of a negative number is not defined in the real number system. An equation might look like $(x+7)^2=-36$. In this case, the graph wouldn't intersect or touch the x axis at all, it'll be above or below it.

You can view these behaviors on Figures 9.1-9.3.

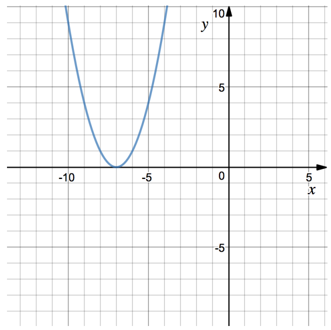

Figure 9.1: The graph of $(x+7)^2 = 0$ has one solution and the curve touches the x axis

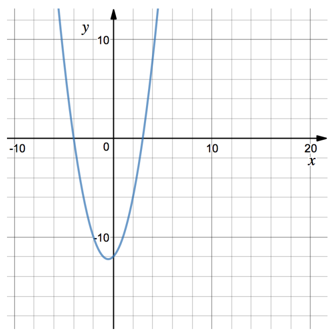

Figure 9.2: The graph of $(x-3)(x+4) = 0$ has two solutions and the curve intersects the x axis at two points

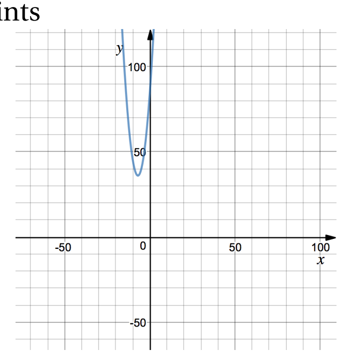

Figure 9.3: The graph of $(x+7)^2 = -36$ has one solution and the curve does not touch the x axis

Conclusion

This is the end of Algebra I. In the next class – Algebra II – most of these concepts will be investigated at a more deep level, especially those related to functions and their behavior. You'll see more operations with polynomials and get introduced to inverse functions and inverse exponentiation operations called *logarithms*. Although you might be tempted to used quadratic formula straight away for all the equations, try sticking more to completing the square, just to always remember, where the formula is derived from. This is true for all formulas and memorization.

Part II

Algebra 2

Introduction

The main focus of Algebra II is to introduce new types of *functions*, their *features* and *specific behaviors*. You'll learn about *radical, rational, exponential, logarithmic, sinusodial* and some other types of advanced functions. These ideas make up about 70% of the whole course.

In Algebra I you were introduced to equations and some new types of mathematical objects, like polynomials and functions. Algebra II, on the other hand, is more concerned teaching you, how to apply arithmetic properties to these objects, rather than introducing new ones. For example, you'll deal a lot with composing, adding, subtracting, multiplying and dividing functions, which would seem a bit odd in Algebra I. Although you've had an opportunity to manipulate polynomials arithmetically, Algebra II goes further and shows you a way to divide polynomials using long division.

However, there will still be some room to solve more complex forms of equations. For example, you'll learn to expand polynomials of type $(a \pm b)^n$ beyond the second degree, like $(a \pm b)^3$ and $(a \pm b)^4$ using some really advanced math techniques, such as *combinatorics*, taught mostly at *probability and statistics* classes. At the end, you'll be introduced to *series* and *sum notation* $\left(\sum\right)$, which is really cool *Pre-Calculus* stuff.

Inversion is another idea heavily taught in Algebra II. You'll learn about *inverse functions* and inverse exponential operations, called *logarithms*, especially how to graph them, how to apply arithmetic operations on them and solve equations, consisting of unknown exponents and logarithms. In this section, a "magic" irrational number e will show up, that makes mortgage calculations much easier.

Also, you'll dig deeper in *trigonometry* by studying the *circle definition* of sin, cos, tan, and how it helps to graph them on a coordinate plane. You'll discover real life problems, that could be solved using sinusodial functions.

Finally, you'll expand your math universe going beyond real numbers ($\mathbb{R}$) to *complex and imaginary numbers* ($\mathbb{C}$), which help in solving equations, consisting of square roots of *negative numbers*, like $\sqrt{-1}$.

Chapter 1

Functions

You can add, subtract, multiply and divide functions, just like numbers.

Combining Functions

Combining functions means *adding* them together. If you have two functions $f(x) = x + 1$ and $g(x) = 2x$, a combined function $h(x)$ would look like this:

$$h(x) = f(x) + g(x)$$

$$h(x) = (x + 1) + (2x)$$

$$h(x) = 3x + 1$$

To evaluate a combined function, you can plug in some value directly to $h(x)$ or evaluate $f(x)$ and $g(x)$ separately and then add the outputs.

A graphical connection could be viewed as a sum of y outputs of two functions. Same idea with subtraction, multiplication and division. If one of the functions is shifted, the combined graph will also be shifted.

You can subtract functions the same way:

$$h(x) = f(x) - g(x)$$

$$h(x) = (x + 1) - (2x)$$

$$h(x) = -x + 1$$

multiply

$$h(x) = f(x) \cdot g(x)$$

$$h(x) = (x + 1)(2x)$$

$$h(x) = 2x^2 + 2x$$

or divide

$$h(x) = \frac{f(x)}{g(x)}$$

$$h(x) = \frac{x+1}{2x}$$

In the case of division, you should always state, that the resulting function would be undefined at a certain value, that makes the whole expression be divisible by 0. For example,

$$h(x) = \frac{x+1}{2x}$$

is defined in range: $\{x \mid x \in \mathbb{R} \text{ and } x \neq 0\}$.

Composing Functions

A *composite* function could be formed by taking the output of some function and using it as an input for another function. The resulting function is called a composite.

There are two ways to do this. Either evaluate some function separately, and plug in the output to another function. Or take the whole function and plug it in place of a independent variable, like x. In other words, use the expression itself as an input.

For example, you have two functions $f(x) = 3x - 1$ and $g(x) = x^3 + 2$. Let's find, what $f(g(3))$ is:

$$g(x) = x^3 + 2$$

$$g(3) = 3^3 + 2$$

$$g(3) = 29$$

$$f(29) = 3(29) - 1$$

$$f(29) = 86$$

or

$$f(x) = 3x - 1$$

$$f(g(x)) = 3(x^3 + 2) - 1$$

$$f(g(x)) = 3x^3 + 6 - 1$$

$$f(g(x)) = 3x^3 + 5$$

a new composite function is born:

$$f(g(x)) = 3x^3 + 5$$

input 3 into this new function:

$$f(g(3)) = 3(3)^3 + 5$$

$$f(g(x)) = 86$$

Formal mathematical notation for a composite function is:

$$(f \circ g)(x) = f(g(x))$$

Transforming Functions

One of the most non-intuitive subjects in algebra. *Shifting* a function *up or down* just means that you have to add some number to it. For example, shifting a function up looks like this:

$$f(x) = g(x) + 2$$

Same idea of you want to compress or stretch a function *up or down*:

$$f(x) = 2g(x)$$

$$f(x) = \frac{1}{2}g(x)$$

Reflecting is also not that hard, just take the negative:

$$f(x) = -g(x)$$

The hardest thing is to transform a function along the x axis, it's always the opposite operation. For example, shifting *left* by 2 would look like this:

$$f(x) = g(x + 2)$$

right

$$f(x) = g(x - 2)$$

Why we are doing the opposite, when shifting functions along the x axis? Probably, because we are thinking differently now. The question here is: *What should we do to x in order to get it back?* The best analogy here are timezones. Perhaps, you live in Toronto and decide to visit Vancouver. You know, that there's a 3 hour time difference between the cities. So arriving in Vancouver you might want to translate your watch backwards by 3 hours. To keep track of the Toronto time you have to *add* 3 hours to local Vancouver time. That's why we can say, that Toronto time expressed *in terms of* Vancouver time is $x+3$, where x is Vancouver time.

The compression of a function by 2 would look like this:

$$f(x) = g\left(\frac{x}{2}\right)$$

and the other way around

$$g(x) = f(2x)$$

The difference here is in the question. When you transform along the x axis, you ask *What should be done to the current input in order to get the same output?* When you transform along the y axis, you ask *What should be done to the output to get another output?*

Inverse Functions

Inverse functions take the output of a function as input and vice versa:

$$f(a) = b \Leftrightarrow f^{-1}(b) = a$$

Graphically they will we *reflections* of each other along $y = x$. When we *isolate* variables while solving equations, we make use of the same inversion idea.

To find inverse functions, you have to solve for x using y instead of the function output ($f(x)$) and then substitute x for y:

$$f(x) = 3x + 2$$

$$y = 3x + 2$$
$$y - 2 = 3x$$
$$x = \frac{y-2}{3}$$
$$f^{-1}(y) = \frac{y-2}{3}$$
$$f^{-1}(x) = \frac{x-2}{3}$$

To find out, whether two functions are inverses of each other, we can use *the inverse composition rule*:

$$f(g(x)) = x \text{ for all } x \text{ in the domain of } g$$

$$g(f(x)) = x \text{ for all } x \text{ in the domain of } f$$

Substituting one function as input to another function should give x as the result, if they are inverses:

$$f(x) = \frac{x+1}{3}$$
$$g(x) = 3x - 1$$
$$f(g(x)) = \frac{g(x)+1}{3}$$
$$f(g(x)) = \frac{3x-1+1}{3}$$
$$f(g(x)) = \frac{\cancel{3}x}{\cancel{3}}$$
$$f(g(x)) = x$$

Generally, a function is invertible, if for every input it has exactly one output. So quadratic functions are non-invertible. Graphically, it should pass the horizontal line test.

Chapter 2

Complex Numbers

Imaginary unit or number i is the backbone of the *complex number system*:

$$i = \sqrt{-1}$$

thus

$$i^2 = -1$$

We can express any negative square roots in terms of imaginary numbers:

$$\text{For } a > 0,\ \sqrt{-a} = i\sqrt{a}$$

If we keep on taking higher powers of i, we get interesting results:

$$i^3 = i^2 \cdot i = (-1) \cdot i = -i$$

$$i^4 = i^3 \cdot i = -i \cdot i = -(\sqrt{-1}) \cdot \sqrt{-1} = -(\sqrt{-1})^2 = -(-1) = 1$$

or

$$i^4 = i^2 \cdot i^2 = (-1) \cdot (-1) = 1$$

$$i^5 = i^4 \cdot i = 1 \cdot i = i$$

So the continuous pattern will look like this:

i^1	i^2	i^3	i^4	i^5	i^6	i^7	i^8
i	-1	$-i$	1	i	-1	$-i$	1

To find high exponents of i you can use several techniques – finding a close *multiple of 4* or *modulo*:

$$i^{138} = \ ?$$

find factors of i^{138}, such that one of them is i^n, where n is a multiple of 4 (last two digits must be a multiple of 4)

$$i^{138} = i^{136} \cdot i^2$$

$$(i^4)^{34} \cdot i^2$$

$$1^{34} \cdot i^2$$

$$1 \cdot i^2$$

$$1 \cdot (-1)$$

$$-1$$

Modulo is more straightforward – divide 138 into 4, and the remainder is your power of i. 4 goes into 138 exactly 34 times without a remainder, and then you have $138 - (34 \cdot 4) = 138 - 136 = 2$ left.

Complex numbers are formed by adding real numbers to imaginary numbers:

$$a + bi$$

where a is a real number and b is the real number coefficient of the pure imaginary number.

The bi part is called *pure imaginary* number. Any real number could also be complex, as well as pure imaginary by adding 0.

There's a *complex plane*, y axis of which represents imaginary part of the complex number and x axis – the real part.

Arithmetic operations with complex numbers are the same as with real numbers, except that we group pure imaginary parts and make use of $i^2 = -1$.

You can solve quadratic equations, that have no solutions in the real number system. Just change the negative square root part of the answer to imaginary number.

You can also factor polynomials using complex numbers, which takes us beyond our check list in Algebra I. The technique here is to find a *sum of squares* pattern and then change it to the *difference of squares* using complex numbers:

$$36a^8 + 2b^6$$

$$(6a^4)^2 + (\sqrt{2}b^3)^2$$

we can express addition as subtraction of a negative number

$$(6a^4)^2 - (-(\sqrt{2}b^3)^2)$$

which literally means there's a -1 in front of the second expression

$$(6a^4)^2 - (-1(\sqrt{2}b^3)^2)$$

and we know, that -1 is i^2

$$(6a^4)^2 - i^2(\sqrt{2}b^3)^2$$

we can transfer it inside the second square expression, by taking the square root of it

$$(6a^4)^2 - (i\sqrt{2}b^3)^2$$

$$(6a^4 - i\sqrt{2}b^3)(6a^4 + i\sqrt{2}b^3)$$

Chapter 3

Arithmetic with Polynomials

Long Division

In Algebra II we're introduced to another arithmetic property concerning polynomials – division. As well as numbers, you can also use long division to divide polynomials:

$$\frac{x^2+3x+6}{x+1} = x+2+\frac{4}{x+1}$$

$$\begin{array}{r|l} & \quad\;\; x+2 \\ \cline{2-2} x+1) & x^2+3x+6 \\ & \underline{-x^2\;\;-x\quad} \\ & \qquad 2x+6 \\ & \qquad \underline{-2x-2} \\ & \qquad\qquad 4 \end{array}$$

You take the *biggest* term from a divisor $x+1$, which is x and see how many times it goes into the *biggest* term of the dividend x^2+3x+6, which is x^2, and ignore everything else. In our example it goes exactly x times, so you write x on top of the x column or $3x$ in this case. Then you multiply the divisor $x+1$ by x and write the answers beneath the corresponding terms and subtract from them. Notice: *you're subtracting the whole expression* $x(x+1) = x^2 + x$, *so you need to change the signs.*

$$x^2+3x-(x^2+x) = \cancel{x^2}+3x-\cancel{x^2}-x = 2x$$

Then you bring in the term, that's left and repeat all the operations until you get to an expression, that is not divisible by $x+1$. How many times x goes into $2x$?

Exactly 2 times. Write 2 on top of 6 and multiply $x+1$ by 2. Then subtract the results, changing the signs along the way.

Polynomial Remainder Theorem

With this theorem you can easily find, whether a linear expression is a factor of some polynomial:

$$\frac{f(x)}{x-a}$$

$$f(a) = \text{ remainder}$$

For example,

$$\frac{3x^2-4x+7}{x-1}$$

$$\begin{array}{r|l} & \quad\;\; 3x-1 \\ \cline{2-2} x-1 & 3x^2-4x+7 \\ & \underline{-3x^2+3x} \\ & \qquad -x+7 \\ & \qquad\;\; \underline{x-1} \\ & \qquad\qquad 6 \end{array}$$

If you substitute 1 instead of x, you'll get the same result:

$$3x^2-4x+7$$

$$3(1)^2-4(1)+7$$

$$3-4+7=-1+7=6$$

You can also find missing coefficients. *For what c is $x-5$ a factor of $p(x)$?*

$$p(x) = x^3+2x^2+cx+10$$

first of all notice, that $x-5$ is a factor of $p(5)$, because 5 turns $x-5$ into 0, and if there's a zero factor, the whole expression would equal to 0, so substitute 5 in place of x, and solve for c

$$p(5) = 5^3+2(5)^2+5c+10$$

$$p(5) = 125+50+5c+10$$

$$185+5c=0$$

$$5c = -185$$

$$c = -37$$

We can prove the theorem by expressing $3x^2 - 4x + 7$ as a multiple of $x - 1$ and $3x - 1$ with the remainder 6 added:

$$3x^2 - 4x + 7 = (3x - 1)(x - 1) + 6$$

Let $3x^2 - 4x + 7$ be $f(x)$, $(3x - 1)$ be $q(x)$, $(x - 1)$ be $(x - a)$ and r be the remainder 6:

$$f(x) = q(x)(x - a) + r$$

input a in the function $f(x)$:

$$f(a) = q(a)(a - a) + r$$

$$f(a) = q(a) \cdot 0 + r$$

$$f(a) = r$$

Chapter 4

Polynomials

The Binomial Theorem

The *binomial theorem* allows you to easily expand binomial expressions of type $(a+b)^n$, where n is any positive integer.

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Example:

$$(a+b)^4 = \sum_{k=0}^{4} \binom{4}{k} a^{4-k} b^k$$

$$\binom{4}{0} a^4 + \binom{4}{1} a^3 b^1 + \binom{4}{2} a^2 b^2 + \binom{4}{3} a^1 b^3 + \binom{4}{4} b^4$$

$$\binom{4}{0} = \frac{4!}{0!(4-0)!} = \frac{\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}{1\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}} = 1$$

$$\binom{4}{1} = \frac{4!}{1!(4-1)!} = \frac{4\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}{1\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}} = 4$$

$$\binom{4}{2} = \frac{4!}{2!(4-2)!} = \frac{4\cdot 3\cdot\cancel{2}\cdot\cancel{1}}{2\cdot 1\cdot\cancel{2}\cdot\cancel{1}} = \frac{12}{2} = 6$$

$$\binom{4}{3} = \frac{4!}{3!(4-3)!} = \frac{4\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}{\cancel{3}\cdot\cancel{2}\cdot\cancel{1}\cdot 1} = 4$$

$$\binom{4}{4} = \frac{4!}{4!(4-4)!} = \frac{\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}{\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}\cdot 1} = 1$$

$$\cancelto{1}{\binom{4}{0}} a^4 + \cancelto{4}{\binom{4}{1}} a^3 b^1 + \cancelto{6}{\binom{4}{2}} a^2 b^2 + \cancelto{4}{\binom{4}{3}} a^1 b^3 + \cancelto{1}{\binom{4}{4}} b^4$$

Result:

$$a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

Of course, you can (hopefully) get the same result just by doing some polynomial arithmetic:

$$(a+b)^4 = (a+b)^2 \cdot (a+b)^2$$

$$(a^2 + 2ab + b^2)(a^2 + 2ab + b^2)$$

$$(a^2)(a^2 + 2ab + b^2) + 2ab(a^2 + 2ab + b^2) + b^2(a^2 + 2ab + b^2)$$

$$(a^4 + 2a^3b + a^2b^2) + (2a^3b + 4a^2b^2 + 2ab^3) + (a^2b^2 + 2ab^3 + b^4)$$

$$a^4 + 2a^3b + 2a^3b + a^2b^2 + a^2b^2 + 4a^2b^2 + 2ab^3 + 2ab^3 + b^4$$

$$a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

Although it requires only basic arithmetic operations, the example above is much hairier. Just imagine what a nightmare you'll go through with $(a+b)^{20}$:

$$(a+b)^{20} = (a+b)^4 \cdot (a+b)^4 \cdot (a+b)^4 \cdot (a+b)^4 \cdot (a+b)^4$$

you can make use of the previous calculations,

$$(a^4+4a^3b+6a^2b^2+4ab^3+b^4)(a^4+4a^3b+6a^2b^2+4ab^3+b^4)(a^4+4a^3b+6a^2b^2+4ab^3+b^4)$$

$$\times$$

$$(a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4)(a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4)$$

but it won't help much.

On the other hand, you might noticed a pattern in the answer

$$a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

First and last terms are a and b to the power of the original expression $(a+b)^4$. Then the power of a in subsequent terms is reducing by one and the power of

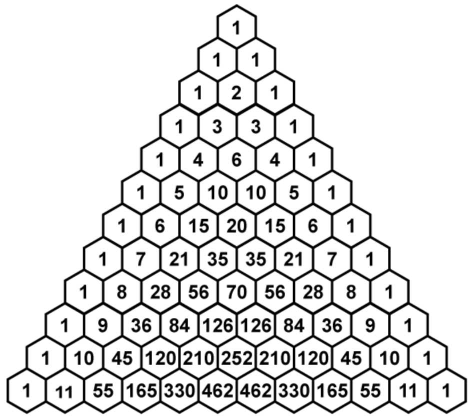

Figure 4.1: Number of unique ways to get to each node – a Pascal triangle

b is increasing (we can assume, that the first and last terms have b and a to the zero power). And the coefficients also increase, until they reach a sort of a vertex in the middle, and then decrease, repeating themselves, or reflect each other.

The first example makes use of this pattern by means of *combinatorics*, which is not part of Algebra classes, but Statistics and Probability or even Pre-Calculus.

It involves combinatorics, because you're choosing a certain amount of b's from a set of 4 expressions and figuring out how many different ways or combinations are there to achieve the same result. On small problems, like $(a+b)^3$, you can literally count these combinations. The idea is that the product of powers of a and b must be equal to 4. So when you're choosing one b, the power of a must equal to 3 in order to complement this rule. When you choose two b's, the power of a must also be 2. I guess, it derives from the idea of multiplying all the terms 4 times by each other.

Pascal Triangle

Another way to solve combinatorics problems is the *pascal triangle*. It's basically a graphical interpretation of the number of unique ways or routes you can get to each node. You can count these routes by hand or take the sum of numbers inside the upper left and upper right nodes. The routes are the coefficients, that we write in expanded binomials. The Pascal triangle shown on Figure 1 can help you expand binomials up to 11^{th} power.

Factoring Higher-Degree Polynomials

You can use grouping method to factor polynomials with a higher degree. For example:

$$2x^5 + x^4 - 2x - 1$$

$$x^4(2x+1) - 1(2x+1)$$

$$(2x+1)(x^4-1)$$

Notice the difference of squares:

$$(2x+1)(x^2-1)(x^2+1)$$

and another one:

$$(2x+1)(x-1)(x+1)(x^2+1)$$

Now you can find the "zeros": $x = -\frac{1}{2}$, $x = 1$, $x = -1$, $x^2 = -1$.

Sum of Cubes

If you can rewrite a polynomial expression as a sum of cubes $a^3 + b^3$, the factoring is possible using this formula:

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$

Example:

$$27x^6 + 125 = (3x^2)^3 + 5^3$$

$$(3x^2+5)(9x^4 - 15x^2 + 25)$$

Difference of Cubes

Difference of cubes

$$a^3 - b^3$$

can be factored using a formula similar to the sum of cubes:

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

The Fundamental Theorem of Algebra

The *fundamental theorem of algebra* states, that a polynomial of type

$$p(x) = ax^n + bx^{n-1} + \cdots k$$

has exactly n roots, either real or complex. And complex roots always come in pairs, so a third-degree polynomial could not have three complex roots.

To find a number of real roots, that a 7-degree polynomial could have, we can form a table:

real roots	complex roots
7	**0**
6	1
5	**2**
4	3
3	**4**
2	5
1	**6**
0	7

Because complex roots show up in pairs, the possible combinations are (7,0), (5, 2), (3, 4) and (1, 6). For an odd-degree polynomial you can take the degree and work backwards, taking every second number, except 0. For an even-degree polynomial it's the same drill, but including 0.

Vocabulary and Multiplicity

Some vocabulary:

$$g(x) = (x-3)(x+2)$$

$(x-3)$ and $(x+2)$ are *linear factors* of $g(x)$

$x = 3$ and $x = -2$ are *roots* or solutions of the equation $g(x) = 0$

3 and -2 are *zeros* of the function g

(3, 0) and (-2, 0) are the x-intercepts of the graph $y = g(x)$

If a function has linear factors, raised to some power, we say, that it's zero has a *multiplicity* of that power:

$$g(x) = (x-3)^2(x+2)$$

It means, that there are two identical zeros of -3. On the graph it'll touch the x-axis at that point. It's called a *double zero*.

If a zero has even multiplicity, it'll *touch* the x-axis at that point. If a zero has odd multiplicity, it'll *cross* the x-axis at that point.

Positive and Negative Intervals of Polynomials

To sketch a graph of a polynomial function, it's not enough to find the zeros, you'll need to understand, how the function is *behaving* on different intervals.

$$g(x) = (x-3)^2(x+2)$$

First of all, you have to find the zeros, which are 3 and -2. Now let's write out some intervals, using these zeros:

$$-\infty < x < -2$$

$$-2 < x < 3$$

$$3 < x < \infty$$

We might notice, that 3 is a *double zero*, so around this x the function will behave by touching the x axis. So probably, the last two intervals will be positive and the first one – negative. But the best way is to check some values.

Let's pick -3 for $-\infty < x < -2$, then:

$$g(-3) = (-3-3)^2(-3+2)$$

you don't have to necessarily do the math

$$g(-3) = (-)^2(-) = (+)(-) = (-)$$

Let's pick 1 for $-2 < x < 3$, then:

$$g(1) = (1-3)^2(1+2)$$

$$g(1) = (-)^2(+) = (+)(+) = (+)$$

Finally, let's pick 5 for $3 < x < \infty$, then:

$$g(5) = (5-3)^2(5+2)$$

$$g(5) = (+)^2(+) = (+)(+) = (+)$$

So we have a combination of $(-)(+)(+)$ intervals and we guessed correctly. You can view the final graph on Figure 13.2.

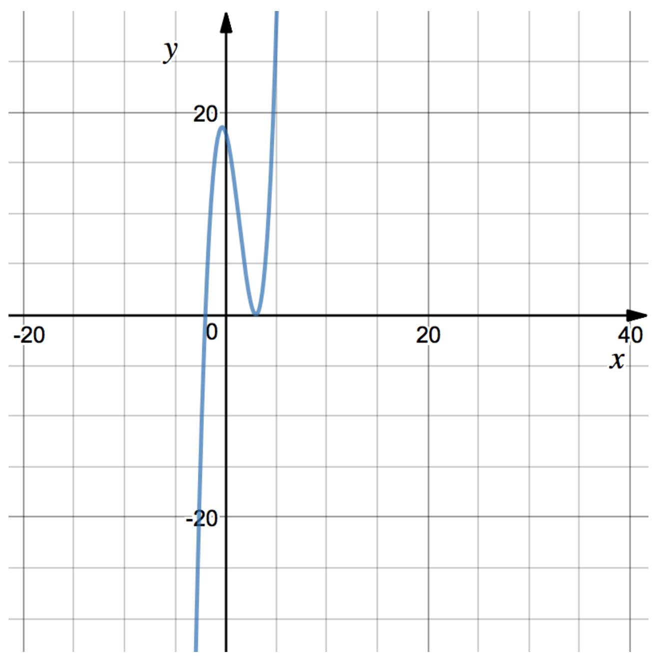

Figure 4.2: A graph of $y = (x-3)^2(x+2)$ shows three intervals, first of which is negative and the other two – positive

End Behavior of Polynomials

Finding out the end behavior of polynomial functions is pretty straightforward. Here are several rules:

$$f(x) = ax^n$$

where a is the coefficient and n is the power of our monomial, then if

$$n \text{ is } \textbf{even} \text{ and } a > 0$$

$$2x^2$$

$$x \to -\infty,\ f(x) \to +\infty, \text{ and } x \to +\infty,\ f(x) \to +\infty$$

$$n \text{ is } \textbf{even} \text{ and } a < 0$$

$$-2x^2$$

$$x \to -\infty,\ f(x) \to -\infty, \text{ and } x \to +\infty,\ f(x) \to -\infty$$

$$n \text{ is } \textbf{odd} \text{ and } a > 0$$

$$2x^3$$

$$x \to -\infty,\ f(x) \to -\infty, \text{ and } x \to +\infty,\ f(x) \to +\infty$$

$$n \text{ is } \textbf{even} \text{ and } a < 0$$

$$-2x^3$$

$$x \to -\infty,\ f(x) \to +\infty, \text{ and } x \to +\infty,\ f(x) \to -\infty$$

The main rule here is that the leading term in a polynomial describes the end behavior of the whole polynomial function, because it has the greatest power and has the greatest impact on the outputs of the function. Eventually at really large inputs it'll overcome any other terms.

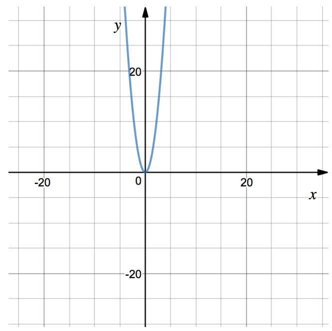

Figure 4.3: $y = 2x^2$: $x \to -\infty$, $f(x) \to +\infty$, and $x \to +\infty$, $f(x) \to +\infty$

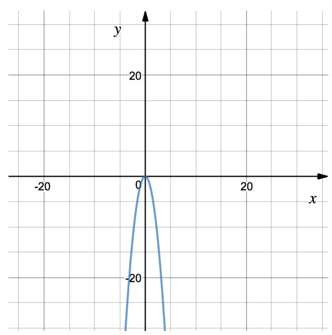

Figure 4.4: $y = -2x^2$: $x \to -\infty$, $f(x) \to -\infty$, and $x \to +\infty$, $f(x) \to -\infty$

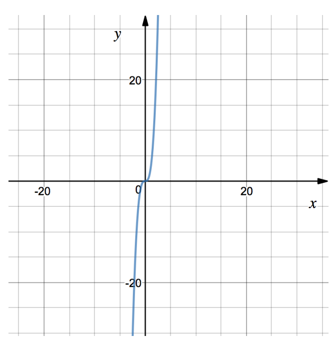

Figure 4.5: $y = 2x^3$: $x \to -\infty$, $f(x) \to -\infty$, and $x \to +\infty$, $f(x) \to +\infty$

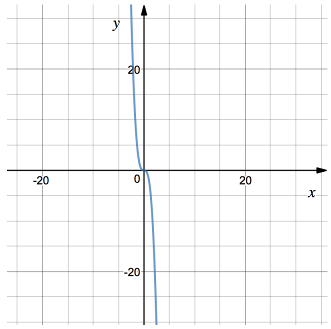

Figure 4.6: $y = -2x^3$: $x \to -\infty$, $f(x) \to +\infty$, and $x \to +\infty$, $f(x) \to -\infty$

Graphing Polynomial Functions

We know have to graph parabolas or quadratic functions, now let's graph polynomials of a higher degree. For example,

$$f(x) = (3x-2)(x+2)^2$$

1. *Find the y-intercept* of the graph by substituting 0 for x:

$$f(0) = (3 \cdot 0 - 2)(0+2)^2$$

$$f(0) = -2 \cdot 4 = -8$$

   So our y-intercept is (0, -8).

2. *Find the zeros*:

$$3x - 2 = 0$$

$$3x = 2$$

$$x = \frac{2}{3}$$

$$x + 2 = 0$$

$$x = -2$$

   So our zeros are $\frac{2}{3}$ and -2 (a *double zero*).

3. *Find the end behavior* by rewriting the polynomial in standard form:

$$f(x) = (3x-2)(x+2)^2$$

$$f(x) = 3x^3 + 10x^2 + 4x - 8$$

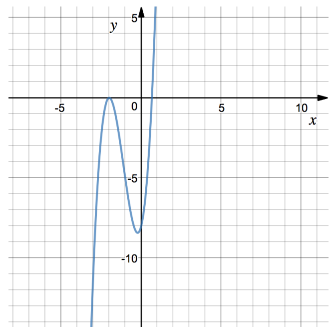

Figure 4.7: A graph of $f(x) = (3x-2)(x+2)^2$.

Now examine the first term $3x^3$. Is it's power *even* or *odd*? Is it's coefficient *negative* or *positive*? In our example the degree is odd and coefficient is positive, so $x \to -\infty$, $f(x) \to -\infty$, and $x \to +\infty$, $f(x) \to +\infty$.

4. *Sketch the graph.* Draw the end behaviors first. Then mark the zeros on the x axis. Notice, that -2 is a *double zero*, so it'll touch the x axis, that's why you have to draw an arc there facing downwards. At another zero draw a line, crossing the x axis. Finally, mark the y-intercept and fill in the spaces with curves. The resulting graph is shown on Figure 13.7.

Symmetry of Functions

A shape has a *reflective symmetry* if it remains unchanged after a reflection across a line. So according to this symmetric property functions could be *even* or *odd*.

If a function is symmetric with respect to the y-axis, it's considered to be *even*. Quadratic functions are even.

If a function is symmetric with respect to the *origin*, it's considered to be *odd*. Cubic functions are odd. Visually it means, that the line could be rotated 180° about the origin and it remains unchanged.

Algebraically a function f is even if $f(-x) = f(x)$ and odd if $f(-x) = -f(x)$. The second equation could interpreted as reflection about the y-axis and then another reflection about the x-axis.

So a polynomial is considered *even* if *all* of it's terms are of *even* degree. And it's considered *odd* if *all* of it's terms are of *odd* degree. If it's a mix of even and odd degree terms, then a function is considered neither even nor odd.

Chapter 5

Radical Relationships

Solving Radical Equations

To solve radical equations, you have to square both sides in order to get rid of the square roots:

$$\sqrt{x+1} = 2x+5$$

$$(\sqrt{x+1})^2 = (2x+5)^2$$

$$x+1 = 4x^2+20x+25$$

$$4x^2+19x+24 = 0$$

$$x = \frac{-19 \pm \sqrt{19^2 - 4\cdot 4\cdot 24}}{2\cdot 4}$$

$$x = \frac{-19 \pm \sqrt{-63}}{8}$$

Radical equations could have one solution, two solutions or no solutions. To figure that out it's not enough to just solve for x's. In our example above we ended up with no real solutions, because the answer contains a square root of a negative number. Although even if we had some real results for x's, it's not enough to say, that the *equation* has any solutions: you have to plug the answer back into the original equation and see if it works out.

This situation happens, because of the nature of radicals: when squaring, you can loose the negative part of it, and the original equation usually states a principal square root condition. You think of it this way: by squaring, you return the equation back to a double solution state, and you check only for the principal square root state (only positive solutions). That's why, you can end up

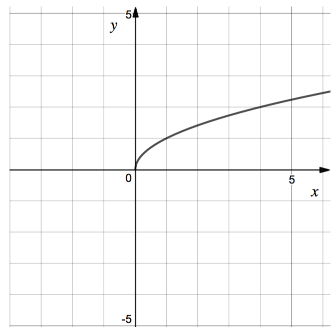

Figure 5.1: A graph of $y = \sqrt{x}$.

with *extraneous* solutions – some negative x answers, that are reflections of the other part of the equation, for example: $-3 = 3$.

To solve *cube root equations*, you have to raise both sides of an equation to the *third* power.

The domain of a radical function is the part underneath the square root greater than or equal to zero. You can form an inequality to solve for x. In other words, it shouldn't be negative.

Graphs of Radical Functions

Graphs of *radical functions* look like a half parabola rotated 90° clockwise or $-90°$. It's because we're dealing with principal roots, so only positive values, and a radical function is an inverse of a quadratic function, that's where it get's it's rotation from.

You can manipulate radical functions the same way as other functions. To shift upwards add values *to the radical*: $\sqrt{x}+2$. To shift left or right, add or subtract values *to the x inside the the radical*: $\sqrt{x+2}$. To reflect along the y-axis, take the negative of x *inside the radical*: $\sqrt{-x}$. To reflect along the x-axis, take the negative of *the radical itself*: $-\sqrt{x}$.

Chapter 6

Rational Relationships

Simplifying Rational Expressions

Rational equations or functions are those expressed as a fraction:

$$\frac{x^2+3x+2}{x^2-1}$$

You can simplify them by factoring and canceling the common factors:

$$\frac{(x+2)\cancel{(x+1)}}{(x-1)\cancel{(x+1)}}$$

Notice, that the function will be *undefined*, if the denominator is equal to zero. So in our example, either $x \neq 1$ or $x \neq -1$. We must state one of the restrictions (the one, that disappeared) after simplification:

$$\frac{x+2}{x-1},\ x \neq -1$$

The easiest way to deal with more tricky rationals is to factor everything out as much as possible, even monomials, like $10x^2 = 2 \cdot 5 \cdot x \cdot x$.

Other useful idea is the rule:

$$a-b \text{ and } b-a \to -1$$

Or in other words you can put -1 in front of an expression to reflect it, like $(a-b) = -1(b-a)$. This gives us an opportunity to construct a common factor. Then just don't forget to multiply the remaining part by -1 in the final answer.

$$\frac{(3-x)(x-1)}{(x-3)(x+1)}$$

$$\frac{-1(-3+x)(x-1)}{(x-3)(x+1)}$$
$$\frac{-1\cancel{(x-3)}(x-1)}{\cancel{(x-3)}(x+1)}$$
$$\frac{-1(x-1)}{(x+1)}$$
$$\frac{1-x}{x+1},\ x \neq 3$$

Multiplying and Dividing Rational Expressions

While multiplying rationals is pretty easy (basically the same process with cancellation, described in the previous subsection), dividing is bit tricky. First of all, you have to take the reciprocal of the divisor and turn the whole thing into multiplication. Now the restrictions include *both the numerator and the denominator* of the divisor:

$$\frac{x^2+x-6}{x^2+3x-10} \div \frac{x+3}{x-5}$$
$$\frac{\cancel{(x+3)}\cancel{(x-2)}}{(x+5)\cancel{(x-2)}} \cdot \frac{x-5}{\cancel{x+3}}$$
$$\frac{x-5}{x+5},\ x \neq -5, -3, 5$$

Adding and Subtracting Rational Expressions

You do this the same way as with numerical fractions, bringing expressions to a common denominator. Usually, you just multiply each fraction by a whole expressed in terms of the denominator of the opposite fraction:

$$\frac{3}{x-2} - \frac{2}{x+1}$$
$$\left(\frac{x+1}{x+1}\right) \cdot \frac{3}{x-2} - \frac{2}{x+1} \cdot \left(\frac{x-2}{x-2}\right)$$
$$\frac{3(x+1)}{(x+1)(x-2)} - \frac{2(x-2)}{(x+1)(x-2)}$$
$$\frac{(3x+3)-(2x-4)}{(x+1)(x-2)}$$
$$\frac{3x+3-2x+4}{(x+1)(x-2)}$$

$$\frac{x+7}{(x+1)(x-2)}$$

Sometimes it's better to find the *least common multiple* of the denominators:

$$6 \to 3 \cdot 2$$

$$4 \to 2 \cdot \cancel{2}$$

$$3 \cdot 2 \cdot 2 = 12$$

With variable expressions it's almost the same:

$$\frac{2}{(x-2)(x+1)} + \frac{3}{(x+1)(x+3)}$$

$$(x-2)(x+1)$$

$$\cancel{(x+1)}(x+3)$$

$$lcm = (x-2)(x+1)(x+3)$$

$$\left(\frac{x+3}{x+3}\right) \cdot \frac{2}{(x-2)(x+1)} + \frac{3}{(x+1)(x+3)} \cdot \left(\frac{x-2}{x-2}\right)$$

$$\frac{2(x+3)+3(x-2)}{(x-2)(x+1)(x+3)}$$

$$\frac{5x}{(x-2)(x+1)(x+3)}$$

Solving Rational Equations

The key technique in solving rational equations is to get rid of the fraction by multiplying each side by some value, such that the denominators would cancel out:

$$\frac{x-2}{x+1} = \frac{x-4}{x+2}$$

$$\cancel{(x+1)} \cdot \frac{x-2}{\cancel{x+1}} = \frac{x-4}{x+2} \cdot (x+1)$$

$$x-2 = \frac{(x+1)(x-4)}{x+2}$$

$$(x+2) \cdot (x-2) = \frac{(x+1)(x-4)}{\cancel{x+2}} \cdot (\cancel{x+2})$$

$$(x-2)(x+2) = (x+1)(x-4)$$

$$\cancel{x^2} \cancel{-4} = \cancel{x^2} - 3x \cancel{-4}$$

$$-3x = 0$$

$$x = 0$$

When you have an expression, like $(\frac{3}{x-1} - 1)$ and you multiply it by $(x-1)(x+1)$, don't forget to multiply *each* part of the first expression *separately* by the second:

$$\left(\frac{3}{x-1} - 1\right) \cdot (x-1)(x+1)$$

$$\left(\left(\frac{3}{\cancel{x-1}}\right) \cdot \cancel{(x-1)}(x+1)\right) - (1 \cdot (x-1)(x+1)$$

$$3(x+1) - (x-1)(x+1)$$

$$3x + 3 - x^2 - 1$$

$$-x^2 + 3x + 2$$

$$x^2 - 3x - 2$$

Direct and Inverse Variations

We say, that y varies *directly* with x, when both variables change in the same direction. For example, if we multiply x by 2, our y would also be multiplied by 2:

$$y = kx$$

We say, that y varies *inversely* with x, when both variables change in the opposite directions. For example, if we multiply x by 2, our y would be divided by 2:

$$y = k \cdot \frac{1}{x}$$

Asymptotes

When we deal with radical functions, it's the first time (well, almost) the *asymptotes* show up. What's an asymptote? It's a line, which a graph of some function approaches, but never actually reaches. The name comes from the greek word meaning "never falling together", the author of which is a Greek geometer and astronomer Appollonius of Perga who lived in the beginning of the 2$^{\text{nd}}$ century BC. The home of the term is analytic geometry and is used as a property of a curve. Asymptotes and *limits* of functions seem to be a first taste of *differential calculus*.

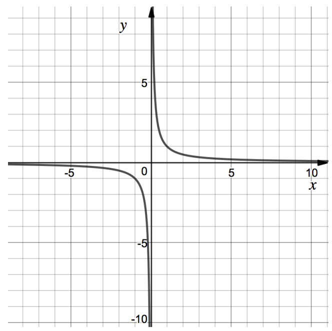

Figure 6.1: A graph of $y = \frac{1}{x}$.

Vertical Asymptotes of Rational Functions

Why do rational functions have asymptotes in the first place? Why there's no asymptotes in linear, quadratic, polynomial and radical functions? Well, because of the unique property of the rational function – the *undefined division by zero*. It's graph must do something weird, behave really strangely at those points. And it actually does.

One possible behavior at an undefined point, is just excluding that point from the game. It won't do anything. There will literally be a "hole" (∘), after which the curve will continue it's desired way. An we've seen such behavior in Algebra I, while studying *piecewise functions*. In the context of rational functions, we call this point a *removable discontinuity*. It's called removable, because we can remove it, while simplifying (see *Simplifying Rational Expressions* from Chapter 6 of Algebra 1):

$$f(x) = \frac{(x+3)\cancel{(x+1)}}{(x+2)\cancel{(x+1)}}$$

$$f(x) = \frac{(x+3)}{(x+2)},\ x \neq -1$$

The reason, why this point doesn't transform into an asymptote, is probably, because the expression, in which it's enclosed $(x+1)$, doesn't have the superior influence on the end behavior of the function. It's just not big enough, or doesn't have a higher degree in the expanded version. So the function continues to it's destination. Another possible behavior is a situation, where a graph "hits the wall". You can view it as hitting some magnetic field around the wall, penetrating that field as hard as it can, but never actually touching the wall itself. The asymptotic behavior comes from the idea, that there are infinite real numbers between 1 and 0. The decimals would just go on forever. Then we have zero.

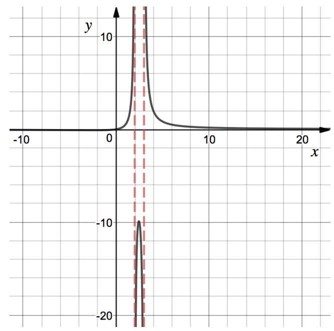

Figure 6.2: A graph of $y = \frac{x}{x^2+5x+6}$.

And a mirror situation on the other, negative, side. Looks like a "black hole" of some sort.

So if we find an x value, that makes a denominator equal to zero, and we cannot cancel it out, we're dealing with a *vertical asymptote*. You can try values for yourself, like -1.99, -1.999, -1.9999, that's the function approaching from the positive side, or -2.01, -2.001, -2.0001 (from the negative side), and see what happens to $f(x)$ or y. It should become a really large positive or negative number. That's the way you figure out the behavior of the function near the asymptotes. Or you can just try one value and watch the sign:

$$f(x) = \frac{(-2.01+3)}{(-2.01+2)} = \frac{+}{-} = -, x \neq -1$$

$$f(x) = \frac{(-1.99+3)}{(-1.99+2)} = \frac{+}{+} = +, x \neq -1$$

Mathematically it could be written this way:

$$\text{As } x \to -2^-, f(x) \to -\infty, \text{ and as } x \to -2^+, f(x) \to +\infty$$

There could be more, than one asymptote, if there are two x's in the denominator, that are not canceled out. This means, that something interesting would happen in between these asymptotes as well. More on this later.

End Behavior of Rational Functions

In the last subsection we've seen a "middle" behavior of rational functions, caused by division by zero, but what about their *end behavior*? The idea is the same, as with polynomials (see *End Behavior of Polynomials* from Chapter 4 of Algebra

2). We have to analyze, what will the function approach, when x become really positive or really negative, in this case, a rational function. Let's look at an example:

$$f(x) = \frac{x^2 + 3x + 10}{x^2 + 1}$$

What value $f(x)$ or y will approach, if $x \to +\infty$ and $x \to -\infty$? We can leave only the highest degree terms, as we know, that they have the greatest impact on the resulting behavior:

$$f(x) = \frac{x^2 \cancel{+3x+10}}{x^2 \cancel{+1}} = \frac{x^2}{x^2}$$

So let's see:

$$f(x) = \frac{1000^2}{1000^2} = 1$$

If we go with the negatives, we get the same result:

$$f(x) = \frac{(-1000)^2}{(-1000)^2} = 1$$

We could've just solved the equation:

$$f(x) = \frac{x^2}{x^2} = 1$$

This is our end behavior – 1. Again, because it's a rational function, and there's always something to be divided by, the function doesn't approach ∞, instead it approaches a certain value. More to speak, in the example below, there could've been different coefficients:

$$f(x) = \frac{2x^2}{3x^2} = \frac{2}{3}$$

so the function will approach $\frac{2}{3}$ instead.

We can state a **general rule** here:

If the numerator and the denominator have *same degree terms*, then our function would approach *the coefficients*.

$$f(x) = \frac{2x^2}{x^2} = 2$$

If the *denominator has a greater degree term, than the numerator*, then our function would *approach zero*.

$$f(x) = \frac{2x}{x^2} = \frac{2}{x}$$

If the *numerator has a greater degree term, than the denominator,* then our function's end behavior *wouldn't be an asymptote* at all. It would be a linear graph approaching $\pm\infty$. Are they called oblique asymptotes?

$$f(x) = \frac{2x^2}{x} = 2x$$

Such end behavior is called a *horizontal asymptote*.

Zeros and Graphs of Rational Functions

Zeros of a rational function are x values at which the *numerator* will equal to zero. These are the points, where a function intercepts the x axis.

There also could be more, than one vertical asymptote, if the *denominator* is a quadratic expression with two real solutions. The function then will have end behavior, some behavior near the outside of two asymptotes and some behavior in between the two asymptotes (a function can cross the horizontal asymptote here, if the y intercept doesn't cause the function to be undefined) (see Figure 17.1). And the middle behavior is actually not a parabola, it's not symmetric. In some graphs, it looks like the middle behavior copies the outside behavior with some possible reflection.

Chapter 7

Exponential Growth and Decay

Main takeaway of this section are models of rational exponential functions, solving exponential equations and rewriting exponents in different ways.

Rewriting Exponential Expressions

We can use exponential properties to simplify expressions and rewrite them in different forms, such as $A \cdot B^t$:

$$10 \cdot 9^{\frac{t}{2}+2} \cdot 5^{3t}$$

$$10 \cdot (9^{\frac{1}{2}})^t \cdot 9^2 \cdot (5^3)^t$$

$$10 \cdot (3)^t \cdot 81 \cdot (125)^t$$

$$10 \cdot 81 \cdot 3^t \cdot 125^t$$

$$810 \cdot (3 \cdot 125)^t$$

$$810 \cdot 375^t$$

If our goal is to end up with an expression of t power, we must isolate t in the expression by breaking up the exponents. In other words, you have to do something with the coefficient to get to the pure t power, shrink it or increase it. In word problems you're often are asked to covert some rate from per decade to per day, so this technique is helpful.

Another idea is that you can express:

$$9^{\frac{t}{2}+2}$$

as

$$9^{\frac{t}{2}} \cdot 9^2$$

because adding something to an exponent is the same thing as multiplying by the same base to the power of the added number.

In other situations, you might want to manipulate or "massage" the expression by adding and subtracting some numbers or multiplying by a fraction of some form in order to get desired results:

$$\frac{1}{32} \cdot 2^t$$

We need to rewrite in a form $A \cdot B^{\frac{t}{10}-1}$. Because we want $\frac{t}{10}$ as an exponent, let's make it up:

$$t = \frac{t}{\cancel{10}} \cdot \frac{\cancel{10}}{1} = t$$

$$\frac{1}{32} \cdot 2^{\frac{t}{10} \cdot 10}$$

$$\frac{1}{32} \cdot (2^{10})^{\frac{t}{10}}$$

$$\frac{1}{32} \cdot 1024^{\frac{t}{10}}$$

We can stop right here, if want the expression in form $A \cdot B^{\frac{t}{10}}$, but we want it in $A \cdot B^{\frac{t}{10}-1}$, so let's construct that -1:

$$\frac{1}{32} \cdot 1024^{\frac{t}{10}-1+1}$$

$$\frac{1}{32} \cdot 1024^{\frac{t}{10}-1} \cdot 1024^1$$

$$\frac{1024}{32} \cdot 1024^{\frac{t}{10}-1}$$

$$32 \cdot 1024^{\frac{t}{10}-1}$$

Solving Exponential Equations

To solve exponential equations, you must first bring all of the terms to the same base, and then solve the exponents as an equation:

$$26^{9x+5} = 1$$

$$26^{9x+5} = 26^0$$
$$9x + 5 = 0$$
$$x = -\frac{5}{9}$$

Another example:

$$2^{3x+5} = 64^{x-7}$$
$$2^{3x+5} = (2^6)^{x-7}$$
$$3x + 5 = 6(x - 7)$$
$$3x + 5 = 6x - 42$$
$$-3x = -47$$
$$x = \frac{47}{3}$$

So here we just expressed 64 as 2^6. Most of the equation problems will exploit only this feature. It's like vacation after rational functions!

Exponential Models

Several things you have to remember. A *factor* is something you multiply by another number. In the context of exponential growth problems, like

$$750 \cdot (1.85)^t$$

1.85 is a factor. If you want percents, just subtract one – 85%. If you want to shrink something, multiply by a factor, that's less, than one:

$$750 \cdot (0.85)^t$$

Notice, that this doesn't mean shrinking by 85% though, but by 15%. Don't believe? Subtract the one. It's not subtracting, it's taking a less amount each time. Same drill with fractions:

$$750 \cdot \left(\frac{2}{3}\right)^t$$

$\frac{2}{3}$ is a factor, but you're shrinking by $\left|\frac{2}{3} - 1\right| = \left|\frac{2}{3} - \frac{3}{3}\right| = \frac{1}{3}$.

Now time. When you see something like this:

$$1000 \cdot \left(\frac{1}{2}\right)^{\frac{t}{5.5}}$$

it means, that you're shrinking whatever you're shrinking by half every 5.5 whatever t is, seconds, for example. If you want to convert it to *every second*, then take the half to the power of 5.5:

$$1000 \cdot \left(\frac{1}{2}^{\frac{1}{5.5}}\right)^t$$

$$1000 \cdot (0.88)^t$$

Now you're shrinking by a factor of 0.88 or by 12% every second.

If you have a model, such that:

$$(1.35)^{\frac{t}{6}+5}$$

then you just have to convert that 5 into 1.35^5, and that'll be your starting point:

$$1.35^5 \cdot (1.35)^{\frac{t}{6}}$$

$$4.5 \cdot (1.35)^{\frac{t}{6}}$$

Want per t? No problem:

$$4.5 \cdot (1.35^{\frac{1}{6}})^t$$

$$4.5 \cdot (1.05)^t$$

Last thing to talk about in this section is how to differ between linear and exponential models. It might seem a rather dumb question to be asked at the 36$^{\text{th}}$ page of Algebra II, but here's why it's not. In real life you won't see perfect linear growth, the common difference will vary up or down a bit, which by no means a reason not to call it linear. But on the other hand, if you check the common ratio, it could turn out to be more constant, and the seemed linearity would be just a coincidence. I won't give examples here, just check the both all the time to be sure. Yay! Logarithms!

Chapter 8

Logarithms

Definition, Restrictions and Special Cases

What's a *logarithm*? It's like an *integral*, a thing that makes you feel smart, but pretty dumb at it's nature. Seriously, it's an inverse of exponents.

$$2^4 = \log_2(16)$$

Really, you should be worrying about trigonometry now, that's waiting for ya in the next section.

Some restrictions. The base must be positive, because logarithms could be used to calculate square roots:

$$\log_{16}(4) = \frac{1}{2}$$

$$16^{\frac{1}{2}} = 4$$

$$\sqrt{16} = 4$$

It can't be $\sqrt{-16}$.

The base most not also be equal to 1, because again that makes no sense, when you rewrite the expression as an exponent:

$$\log_1(3) = x$$

$$1^x = 3$$

One raised to any power is one, it could not equal 3. Lastly, the argument must be greater, than zero, because the base is greater, than zero.

Logarithm of base is usually written without a base:

$$\log_{10}(x) = \log(x)$$

And logarithm of base e as:

$$\log_e(x) = \ln(x)$$

First one is called the *common* logarithm, the other one – the *natural* logarithm.

The Constant e and Compound Interest

The logarithms themselves a easy to grasp, but the ideas behind logarithmic scales and especially compound interest with that e could be quite daunting and mind blowing. It's pretty sure, you'll want to take some time and think about them. So trigonometry can wait a little bit.

So what is e? It's an *irrational* number called *Euler's number*, just like π or ϕ. If π shows up in circles and ϕ in the golden ratio, e is number, that a compound interest approaches (it's used in many other fields, which are outside of Algebra II for now).

$$e \approx 2,718281828459045...$$

A compound interest is formed by *compounding* interests for certain periods. Instead of taking interest once, you take a reduced interest (divided by the number of periods), but after each period. This way your taking interest of the initial sum increased by the interest from the previous term, which ends up with paying more or earning more, if it's a bank deposit. That's where the compounding comes from.

$$1 + 100\% = 1 + (100\% \div 100) = 1 + 1 = 2$$

$$1 + (50\% + 50\%) = (1 + 0.5) + (1 + 0.5)(0.5) = 1.5 + 1.5 \cdot 0.5 = 1.5 + 0.75 = 2.25$$

This pattern could be written as:

$$1 \cdot \left(1 + \frac{100\%}{2}\right)^2$$

$$1 \cdot (1 + 50\%) \cdot (1 + 50\%)$$

$$1 \cdot (1 + 0.5) \cdot (1 + 0.5)$$

$$1 \cdot (1.5) \cdot (1.5)$$

$$2.25$$

So $(1.5 + 50\% = 1.5 + 1.5 \cdot 0.5 = 1.5 + 0.75)$ is the same as $(1.5 \cdot 1.5)$, as $(100 + 30\% = 100 + 100 \cdot 0.3 = 100 + 30)$ is the same as $(100 \cdot 1.3)$.

If you keep on increasing the periods, your interest approach e:

$$1 \cdot \left(1 + \frac{100\%}{6}\right)^6 \approx 2.52$$

$$1 \cdot \left(1 + \frac{100\%}{12}\right)^{12} \approx 2.61$$

$$1 \cdot \left(1 + \frac{100\%}{365}\right)^{365} \approx 2.7145674820219727...$$

$$1 \cdot \left(1 + \frac{100\%}{1000}\right)^{1000} \approx 2.7169239322355936...$$

$$1 \cdot \left(1 + \frac{100\%}{n}\right)^{n} \approx 2.718281828459045...$$

Now we can rewrite it in simpler form:

$$\cancel{1} \cdot \left(1 + \frac{\cancel{100\%} \rightarrow 1}{n}\right)^{n} = \left(1 + \frac{1}{n}\right)^{n} \approx 2.718281828459045...$$

Or even in calculus notation:

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^{n} \approx 2.718281828459045... = e$$

We can play around with e and see, that you can calculate any interest. For example, let's take 10%, which is more realistic for a loan. If we split this percent into 12 months, we'll get:

$$1 \cdot \left(1 + \frac{10\%}{12}\right)^{12} = \left(1 + \frac{0.1}{12}\right)^{12} \approx 1.105170918075648$$

So approximately you'll pay 10.52%, instead of 10%. If you're borrowing \$500,000, that's an extra \$2,600.

Notice, that e is still at the heart of this problem, only raised the power of 0.1, instead of 1:

$$e^{0.1} \approx 2.718281828459045...^{0.1} \approx 1.105170918075648...$$

You can go the other way around and find out what's your interest rate, based on the compound interest 10.52%:

$$\log_e = \ln(1.105170918075648) = 0.1$$

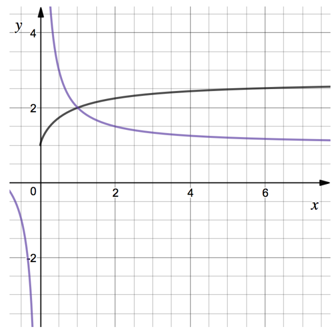

Figure 8.1: Graphs of $\left(1+\frac{1}{x}\right) \to 1$ and $\left(1+\frac{1}{x}\right)^x \to e$.

Logarithmic Scale

All of this e relationship is *logarithmic* by it's nature.

If you closely examine the formula $\left(1+\frac{1}{x}\right)^x$ it breaks down to one essential part, which is $\frac{1}{x}$. As $x \to \infty$, $\frac{1}{x} \to 0$, because you're taking a smaller fraction every time. If $\frac{1}{x} \to 0$, then $\left(1+\frac{1}{x}\right) \to 1$. When we compound, we increase each term by a certain percent $\frac{1}{x}$, which itself approaches 0. We compound more, but we use smaller percents. It's like shifting gears on a bicycle: you make more revolutions, but your speed is decreasing to a point, where you can barely move. In the case of interest compounding, the limit of $\frac{1}{x}$ is sort of translated into the positive direction, with some modifications made along the way, that prevent them from being perfectly reflected. If you put $\left(1+\frac{1}{x}\right)$ and $\left(1+\frac{1}{x}\right)^x$ on a graph, you'll see, that they almost mirror each other after $x = 1$ (see Figure 11). So I can guess, that approach to e is a modified approach to 1. For a really deep understanding you probably need some knowledge of calculus.

We'll try to describe a *logarithmic scale*, without drawing one. On a linear scale 10 is divided into 10 equal parts. So one step is $\frac{1}{10}$ or 1: you get from 1 to 2 by adding 1.

A logarithmic scale is totally different. Each step is a *multiple* of 10 or a *power* to which you raise 10. For example, step one is not 1, but 10^1, step two is not 2, but 10^2 and so on. Going upwards is pretty clear: $1, 10, 100, 1000 \ldots$. But what's happening in between? Where does 2 belong on this scale? To what power do we have to raise 10 to get 2? That sounds very much like a logarithm problem!

And we shouldn't be afraid of going backwards, because we know how to take fractional exponents:

$$2^2 = 4$$

$$4^{\frac{1}{2}} = 4^{0.5} = 2$$

$$\sqrt{4} = 2$$

Now let's try to plot our 2 on the logarithmic scale. We can divide one step into 10 equal parts, but remember, that we are dividing a power! So each little step will be $10^{\frac{1}{10}}$. Let's use a calculator to figure out the values:

$$10^x = 2$$

$$\log(2) \approx 0.3 = \frac{3}{10}$$

We have to take 10 to the power $\frac{3}{10}$ to get 2. Count three steps and plot 2 there.

Here's the calculations for other numbers:

$$10^x = 3$$

$$\log(3) \approx 0.477 = \frac{1 \cdot 5}{2 \cdot 5} = \frac{5}{10}$$

$$10^x = 4$$

$$\log(4) \approx 0.602$$

$$10^x = 5$$

$$\log(5) \approx 0.698$$

$$10^x = 6$$

$$\log(6) \approx 0.778$$

$$10^x = 7$$

$$\log(7) \approx 0.845$$

$$10^x = 8$$

$$\log(8) \approx 0.903$$

$$10^x = 9$$

$$\log(9) \approx 0.954$$

$$10^x = 10$$

$$\log(10) = 1$$

Numbers are not lined equally, because the closer you get to 10, the less power you need to raise the number to get to 10. And because equal steps on a logarithmic scale represent equal percentage growth, the numbers are kind of pilling at the end: from 1 to 2 it's 50%, but from 2 to 3 it's only 30% and so on. The logarithmic meaning of this scale, is that you have to raise 3 a little bit more than to a power of 2 in order to get to 10, and that's why it's in the middle. Position of the number on a logarithmic scale is the distance of that number from 10 in terms of multiplying by itself. So 2 is a third of the way from 10, 3 is halfway, etc.

Whatever the connection, just remember: one scale is $10^{\frac{1}{10}}$, so to plot any number on it you have to take the $\log(n)$ of that number. That's it. And equal distance is equal percentage change.

Properties of Logarithms

The product rule:

$$\log_b(MN) = \log_b(M) + \log_b(N)$$

Proof:

$$\log_2(4 \cdot 8) = \log_2(2^2 \cdot 2^3) = \log_2(2^{2+3}) = 2 + 3 = \log_2(4) + \log_2(8)$$

The quotient rule:

$$\log_b\left(\frac{M}{N}\right) = \log_b(M) - \log_b(N)$$

Proof:

$$\log_2\left(\frac{4}{8}\right) = \log_2\left(\frac{2^2}{2^3}\right) = \log_2(2^{2-3}) = 2 - 3 = \log_2(4) - \log_2(8)$$

The power rule:

$$\log_b(M^p) = p\log_b(M)$$

Proof:

$$\log_2(4^2) = \log_2((2^2)^2) = \log_2(2^{2\cdot 2}) = 2 \cdot 2 = \log_2(4) \cdot 2 = 2\log_2(4)$$

Change of Base Rule

The *change of base rule*:

$$\log_2(50) = \left(\frac{\log_{10}(50)}{\log_{10}(2)}\right)$$

Instead of log you can take a logarithm of any base, but this one is used more often, because it's on the calculators.

Proof:

$$\log_2(50) = n$$

$$2^n = 50$$

$$\log_x(2^n) = \log_x(50)$$

$$n\log_x(2) = \log_x(50)$$

$$n = \frac{\log_x(50)}{\log_x(2)}$$

Solving Exponential Equations with Logarithms

It's much easier, than solving exponential equations using exponential properties:

$$5 \cdot 2^x = 240$$

$$2^x = 48$$

Now instead of converting 48 to 2 of some power (which won't be a whole number), we can use a logarithm:

$$\log_2(48) = x$$

$$x = \frac{\log(48)}{\log(2)}$$

$$x \approx 5.585$$

Another example:

$$6 \cdot 10^{2x} = 48$$

$$10^{2x} = 8$$

$$\log(8) = 2x$$

$$x = \frac{\log(8)}{2}$$

$$x \approx 0.452$$

Graphs of Exponential and Logarithmic Functions

To graph an exponential function, we have to remember, that the number you add or subtract to y is the asymptote, if you have a negative sign in front of y, then the function will point downwards, and if the exponent is negative, then it'll be reflected:

$$y = -2 \cdot 3^x + 5$$

A logarithmic function is a reflection of an exponential function by the origin. The asymptote will be vertical, so it is formed by adding or subtracting to x instead of y as in exponentials:

$$y = -\log_2(x+2)$$

The asymptote will be -2, the function will be reflected over x axis and shifted left by 2.

Chapter 9

Trigonometry

Most of the trigonometry was in Geometry classes, but in Algebra II we'll check out new features, such as the *circle definition* of trig functions, which gives us the power to work with more complicated angles and build graphs of sin and cos. You would have to be familiar with *radians*.

Circle Definition of Trigonometric Functions

It states, that:

$$\cos(\theta) = x$$

$$\sin(\theta) = y$$

This is derived from the triangle definition of trigs. Cosine is the adjacent over hypotenuse, so it'll be the x coordinate. Sine is the opposite over hypotenuse,

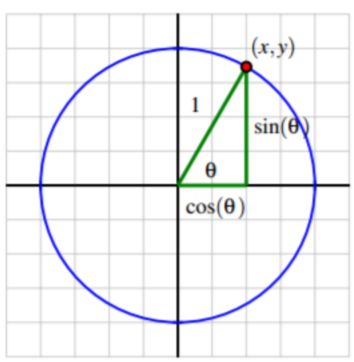

Figure 9.1: The circle definition of trigonometric functions.

so it's the y coordinate. You can see that from the Figure 18.1.

Graphs of Trigonometric Functions

Because we're rotating while taking angles, the graph will be a *sinusodial* or like a wave. The x axis are the angle measures in radians, while the y axis are the outputs of sin or cos functions.

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$y = \sin(x)$	0	1	0	-1	0

The sin function goes up, reaches a peak at 90°, then goes back to initial state at π or 180°, then goes down at 270° and returns back make one full revolution. Upward and downward movement is influenced by the quadrants of the Cartesian coordinate plane.

The range is $-1 \leq \sin(\theta) \leq 1$ and domain are all real numbers.

The cosine starts at 1, not 0, because $\cos(0) = 1$:

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$y = \sin(x)$	1	0	-1	0	1

Trigonometric Identities

Symmetric identities of sine and cosine:

$$(\cos(\theta), \sin(\theta)) = (\cos(-\theta), \sin(-\theta)) = (\cos(\pi-\theta), \sin(\pi\theta)) = (\cos(\pi+\theta), \sin(\pi+\theta))$$

You can view this separately:

$$\sin(\theta) = \sin(\theta - \pi)$$

$$-\sin(\theta) = \sin(\theta + \pi) = \sin(-\theta)$$

$$\cos(\theta) = \cos(-\theta) = -cos(\pi - \theta)$$

$$-\cos(\theta) = \cos(\theta + \pi)$$

Periodicity:

$$\cos(\theta) = \sin\left(\theta + \frac{\pi}{2}\right)$$

Pythagorean identity:

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

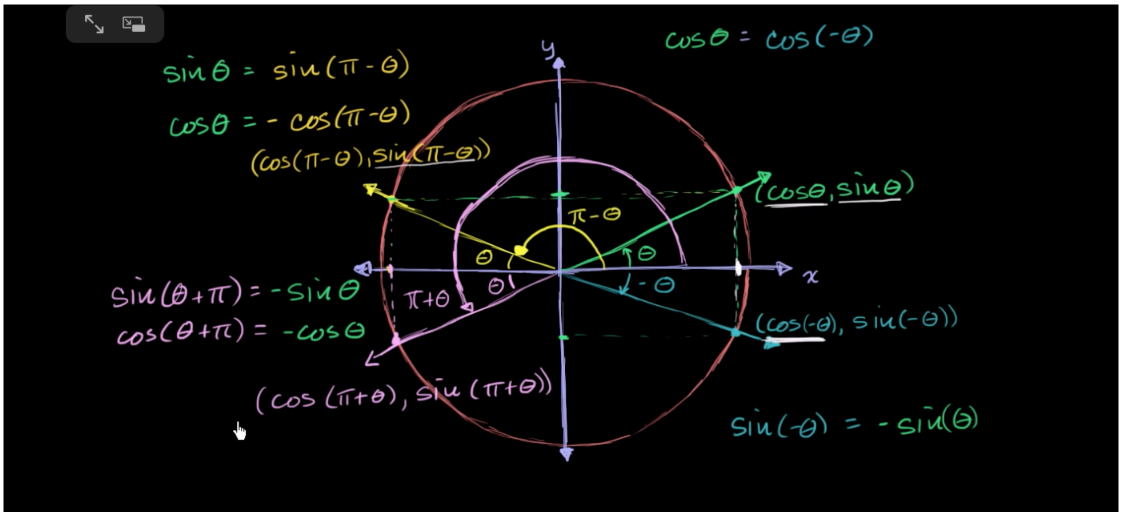

Figure 9.2: Sine and cosine symmetry.

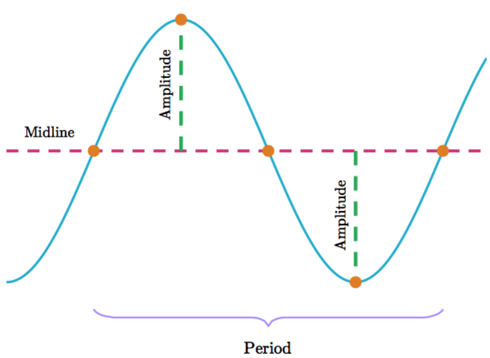

Figure 9.3: Midline, amplitude and period of sinusodial function.

Features of Sinusodial Functions

Period is when a function make one full revolution. *Midline* is the line, that passes in the middle of two *extremum* points. *Amplitude* is the vertical distance from extremum to the midline (Figure 18.3).

$$y = -\frac{1}{2}\cos 3x$$

Amplitude is the $\left|-\frac{1}{2}\right|$ part. Period is 2π divided by the coefficient of x or 3 $\left(\frac{2\pi}{3}\right)$.

Graphing Sinusodial Functions

For example:

$$y = 2\sin(-x)$$

The amplitude will be 2, the midline will be 0 and the function will be reflected (start going downwards from 0) without stretching (a period of 2π). You can divide the period by 2 to get the segment from on extremum to another. Or you can divide it by 4 to get the halfway between extremums, which will cross the midline.

The cosine function starts at 1 (or the positive amplitude) and goes down. A negative cosine will be reflected: it'll start at -1 and go upward (or the negative amplitude).

The midline is 0 by default, by can be modified by adding of subtraction to y.

To calculate the period from a graph, that has real numbers on the x axis instead of radians, just find the point where it makes one revolution and set an equation:

$$\frac{2\pi}{k} = 8$$

. Everything else is quite easy: midline is the middle between two extremums (add that to y), amplitude is the distance from that midline to extremums (coefficient of sin or cos) and period is the coefficient of x.

Chapter 10

Series

Remember the arithmetic and geometric *sequences* from Algebra I? Same drill, but adding all the terms together.

Arithmetic Series

The formula for *arithmetic series*:

$$S_n = \frac{a_1 + a_i}{2} \cdot n$$

The proof of this formula is based on addition of two arithmetic sequences, one of which is reversed. The sum will be always the same. To to find the sum of one of the sequence you just take the first and the last term, add them, divide by 2 and multiply by the number of terms.

There could be different problems with series.

When n is unknown:

Usually you'll have a full sequence from the start:

$$-50 + (-44) + (-38) + \ldots + 2038 + 2044$$

Take the difference between the first and last terms:

$$2044 - (-50) = 2044 + 50 = 2094$$

Divide by the common ratio or n, which is $50 - 44 = 6$:

$$\frac{2094}{6} = 349 \text{ times}$$

You get 349 times numbers in the sequence were increase by 6. Because it's the number of additions, you have to include the first term:

$$349 + 1 = 350 \text{ terms}$$

Then use the formula:

$$S_{350} = \frac{-50 + 2044}{2} \cdot 350 = 348950$$

When a_i is unknown:

Usually you'll have a recursive definition of a sequence and the number of terms (335 in this example):

$$a_1 = 2$$

$$a_i = a_{i-1} - 3$$

Subtract one from the number of terms to get the number of subtractions:

$$335 - 1 = 334$$

Then find the total subtracted sum:

$$334 \cdot 3 = 1002$$

And subtract that sum from the first term to find the last term:

$$2 - 1002 = -1000$$

Finally, plug all the values into the formula:

$$S_{335} = \frac{2 + (-1000)}{2} \cdot 335 = -167165$$

When all values are present:

Usually you'll have a sum notation involved:

$$\sum_{k=1}^{275} (-5k + 12)$$

The sequence will look like this:

$$(-5 \cdot 1 + 12) + (-5 \cdot 2 + 12) + ... + (-5 \cdot 275 + 12)$$

If we start from $k = 0$, then the number of terms will 274. You also have to subtract one from the number of terms, when you're dealing with the number of additions and vice versa. Here you just plug in the values:

$$S_{275} = \frac{7 - 1363}{2} \cdot 275 = -186450$$

Geometric Series

Sum notation:

$$\sum_{k=0}^{n} ar^k = ar^0 + ar^1 + ar^2 + ... + ar^n$$

Geometric series formula:

$$S_n = \frac{a(1-r^n)}{1-r}$$

The proof of this formula is based on the idea, that you multiply one sequence by r and then subtract them (without reversing), all the terms will cancel out, except the two: a and ar^n. From there on you can do the sum:

$$S_n - rS_n = a - ar^n$$

$$S_n(1-r) = a(1-r^n)$$

$$S_n = \frac{a(1-r^n)}{1-r}$$

Geometric series are good at solving problems like calculating mortgage payments or medicine concentration in blood after some period of time.

Let's take a look at the mortgage problem. For example, your bank gives you a \$200,000 loan, and you'll have to pay 6% yearly interest (0.5% monthly) for 30 years (360 months). Assume, that your monthly payment will be \$1,200. Then the sequence of payments will look like this:

$$(((20000(1.005) - 1200)1.005 - 1200)1.005 - 1200)\ldots \text{ 366 times} = 0$$

How to figure out the payment? Let L be the loan amount, i be monthly interest, n be the number of months and P – your monthly payment. Then

$$(((L(1+i) - P)(1+i) - P)(1+i) - P)\ldots \text{ 366 times} = 0$$

Let's look at the equation at $n = 1$:

$$L(1+i) - P = 0$$

$$P = L(1+i)$$

$$L = \frac{P}{1+i}$$

Now at $n = 2$:

$$(L(1+i) - P)(1+i) - P = 0$$

$$P = (L(1+i) - P)(1+i)$$
$$\frac{P}{1+i} = L(1+i) - P$$
$$P + \frac{P}{1+i} = L(1+i)$$
$$L = \frac{P}{1+i} + \frac{P}{(1+i)^2}$$

So you can see a pattern:

$$L = \frac{P}{1+i} + \frac{P}{(1+i)^2} + \ldots + \frac{P}{(1+i)^n}$$
$$L = P\left(\frac{1}{1+i} + \frac{1}{(1+i)^2} + \ldots + \frac{1}{(1+i)^n}\right)$$

And the part in the parentheses is a geometric series:

$$\sum_{n=1}^{n} \frac{1}{(1+i)^n}$$

Let's further abstract:

$$S = \frac{1}{1+i} + \frac{1}{(1+i)^2} + \ldots + \frac{1}{(1+i)^n}$$
$$r = \frac{1}{1+i}$$

then

$$S = r^1 + r^2 + r^3 + \ldots + r^n$$
$$rS = r(r^1) + r(r^2) + r(r^3) + \ldots + r(r^n)$$
$$S - rS = r - r^{n+1}$$
$$S(1-r) = r - r^{n+1}$$
$$S = \frac{r - r^{n+1}}{1-r}$$

Now we can rewrite the formula for our loan:

$$L = P\left(\frac{r - r^{n+1}}{1-r}\right)$$

And the payment would be:

$$P = L\left(\frac{1-r}{r - r^{n+1}}\right)$$

Finally, we can calculate the payment:

$$r = \frac{1}{1+0.05} = 0.995$$

$$P = 200000\left(\frac{1-0.995}{0.995-0.995^{361}}\right) = 1200$$

Chapter 11

Conic Sections

By conic section we usually mean graphs, that are formed by slicing a cone in different ways. One of them is a parabola, another – hyperbola. In this section we'll take a look at some new features of a parabola.

Parabolas are commonly known as the graphs of quadratic functions. They can also be viewed as the set of all points whose distance from a certain point (the *focus*) is equal to their distance from a certain line (the *directrix*).

The equation of a parabola looks like this:

$$\sqrt{(y-3)^2} = \sqrt{(x+2)^2 + (y-5)^2}$$

where $\sqrt{(y-3)^2}$ is the distance between (x, y) and the directrix at $y = 3$, and $\sqrt{(x+2)^2 + (y-5)^2}$ is the distance between (x, y) and the focus.

We can rewrite this equation in terms of y:

$$y = \frac{1}{2(b-k)}(x-a)^2 + \frac{1}{2}(b+k)$$

where (a, b) is our focus and k is the directrix.

In terms of x:

$$x = \frac{1}{2(a-k)}(y-b)^2 + \frac{1}{2}(a+k)$$

The main takeaway is that the *vertex* of a parabola is halfway between the focus and the directrix. And the distance from the focus to the curve of a parabola is the same as the distance from that point on the curve to the directrix (a perpendicular line).

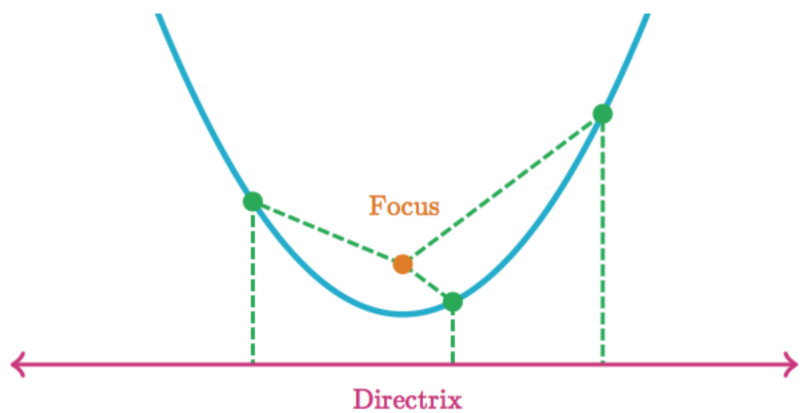

Figure 11.1: Directrix and focus of a parabola

Conclusion

Algebra II is about twice as large as Algebra I. The main focus of this class is to teach you the ideas, that will be heavily used in calculus, like symmetry, end behavior of functions and their limits. The hardest part is probably the *rational functions* with the ideas of asymptotes. The next hardest part is *logarithms*, and here you have to sort of rewire your brain to think inversely all the time. *Logarithmic* scales along with *compound interest and Euler's number (e)* is a subject, that you'll completely understand only learning calculus. Inversion is I guess the motto of Algebra II. *Polynomials and combinatorics* is another tough part, but *trigonometry* is not that scary.