MATH 1830 NOTES

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UNIT 4: INTEGRATION

4.1 Area by Sums

Introduction

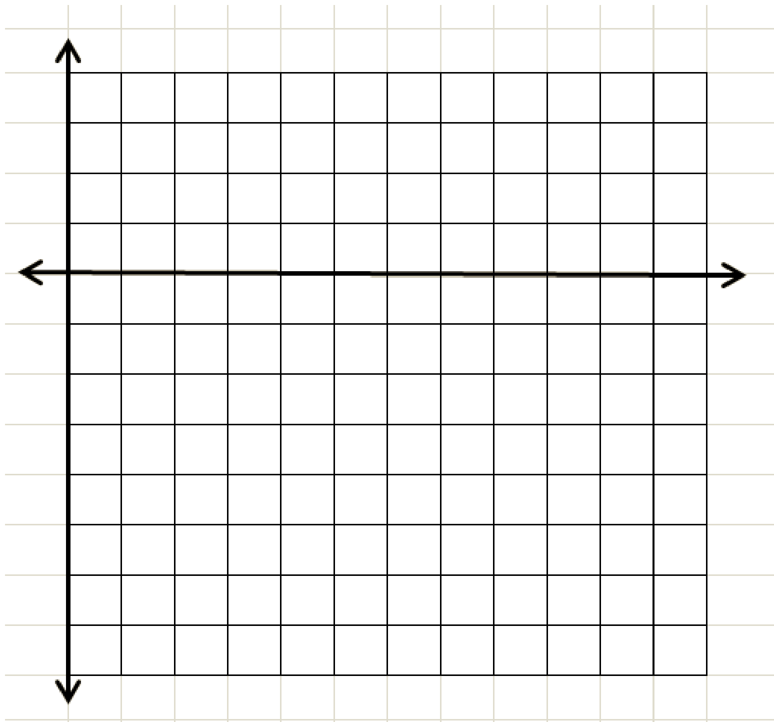
1. The rate of change of the population (in thousands of people per year) of North Dakota between 1985 and 1996 can be modeled by

$$p(t) = \begin{cases} -7.35 & 0 \le t \le 6 \\ 2.5 & 6 < t \le 11 \end{cases}$$

Where t represents the number of years since 1985.

(Source: Statistical Abstract, 1998)

   a. Sketch a graph of the rate of change function.



b. Find the area of the region between the graph of p and the horizontal axis from 0 to 6. Interpret your answer.

c. Find the area of the region between the graph of p and the horizontal axis from 6 to 11. Interpret your answer.

d. Was the population of North Dakota in 1996 greater or less than the population in 1985? By how much did the population change between 1985 and 1996?

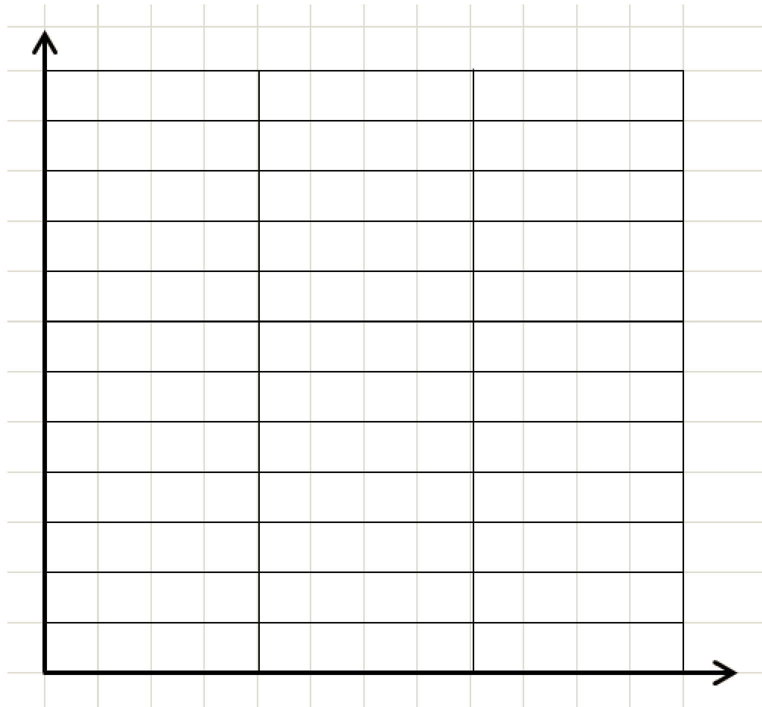
e. What information would you need to determine the population of North Dakota in 1996?

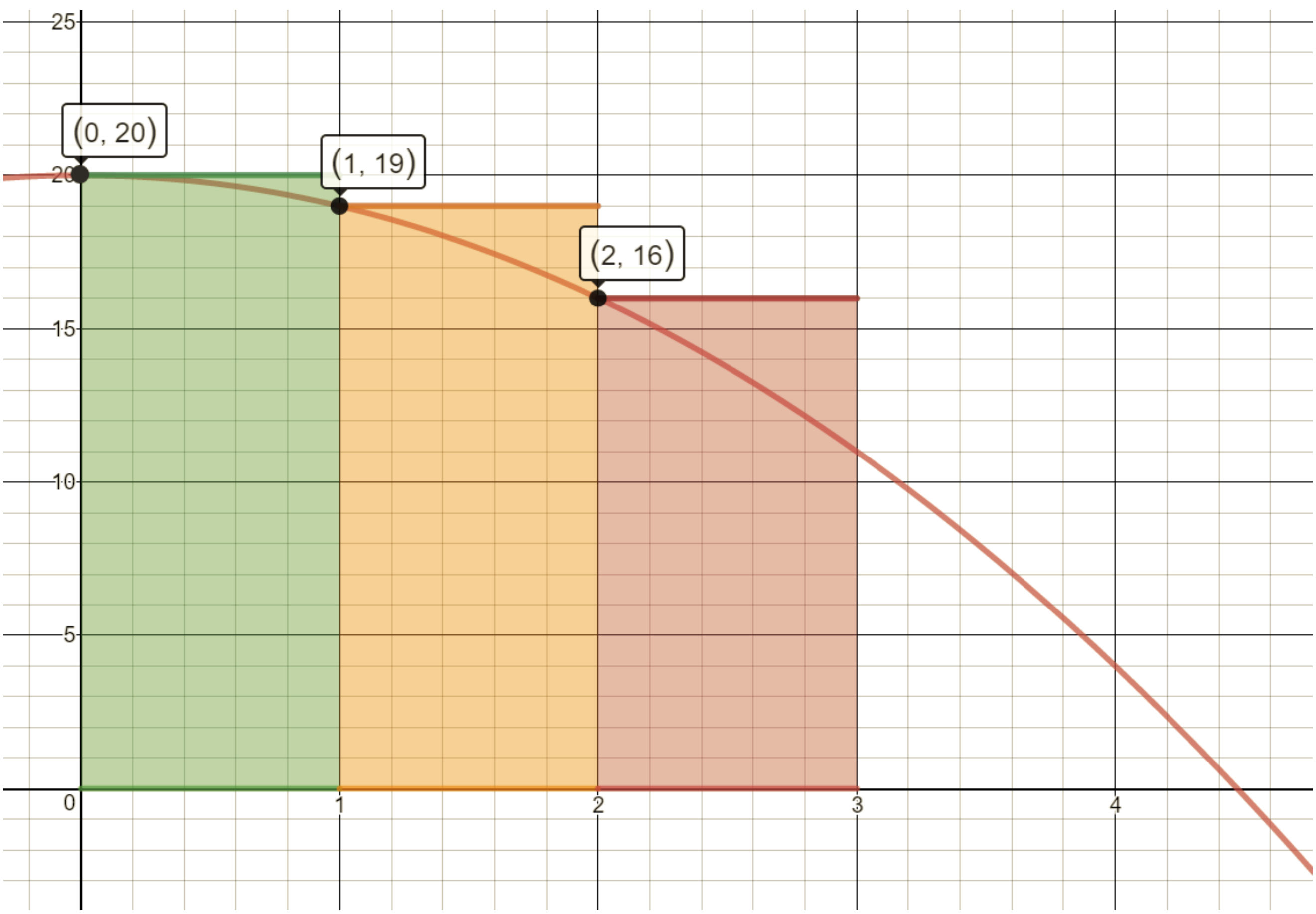
f. What is the relationship between the area of the regions and the population of North Dakota?

2. An office worker assembles advertising portfolios. As fatigue sets in, the number of portfolios he can assemble per hour decreases. Using regression, it is determined that he can assemble $f(t) = 20 - t^2$ portfolios per hours t hours after he begins work.

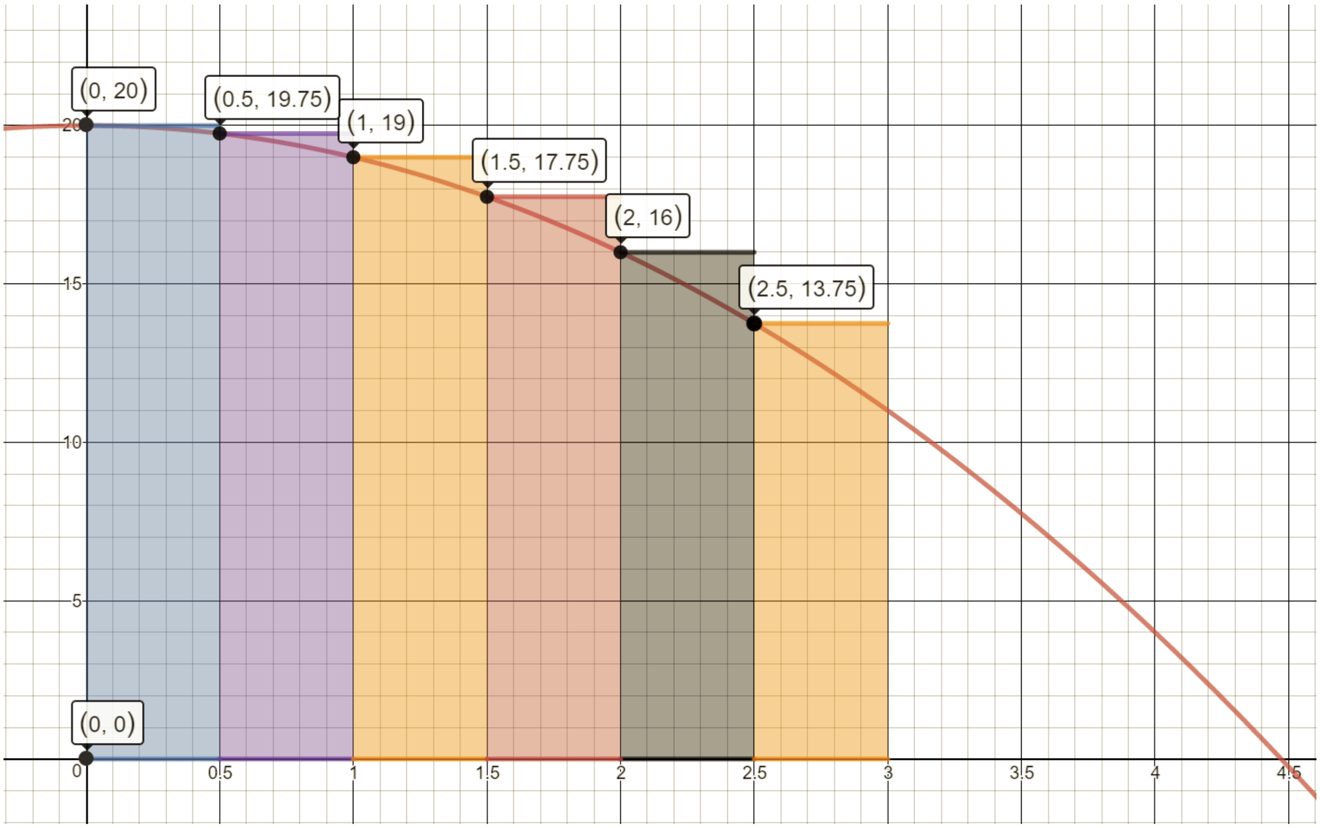
    a. How many portfolios can he assemble in the third hour?

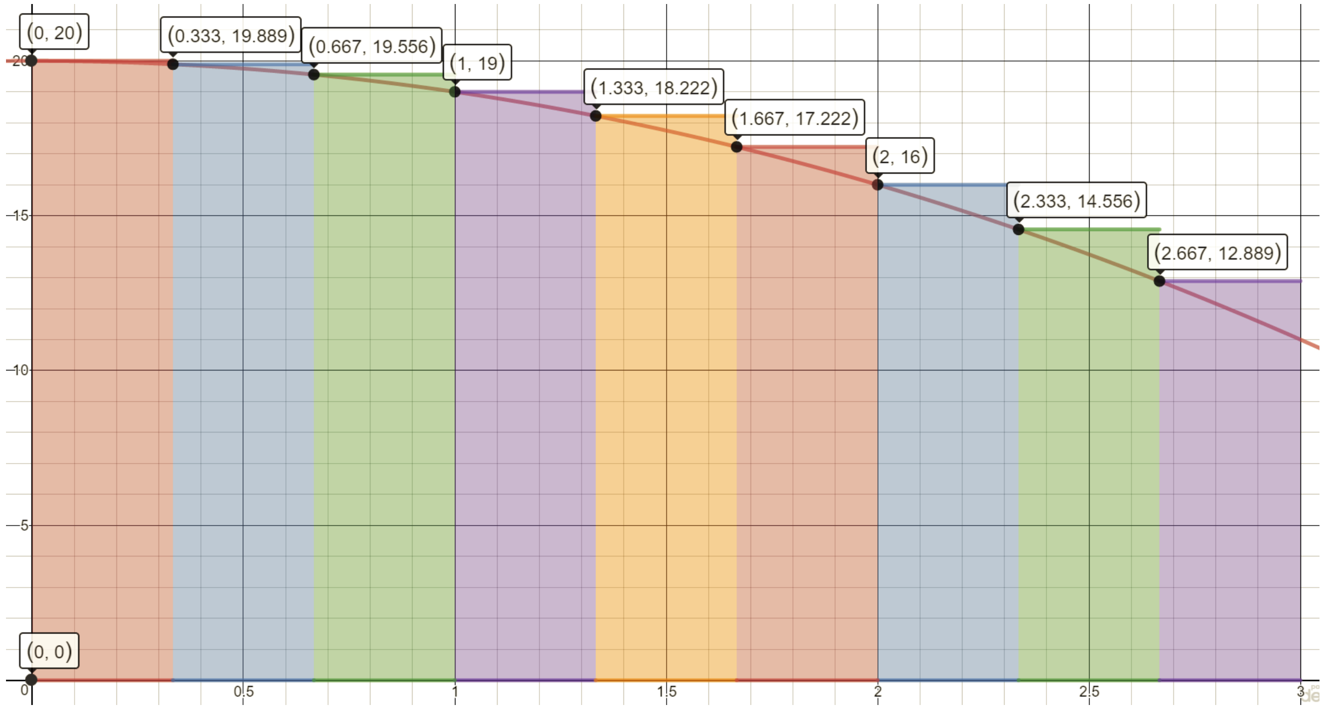
b. Graph the equation on the interval [0,3] on the graph below using three left rectangles.





c. Find the area under the curve using 6 left rectangles and then 9 left rectangles.



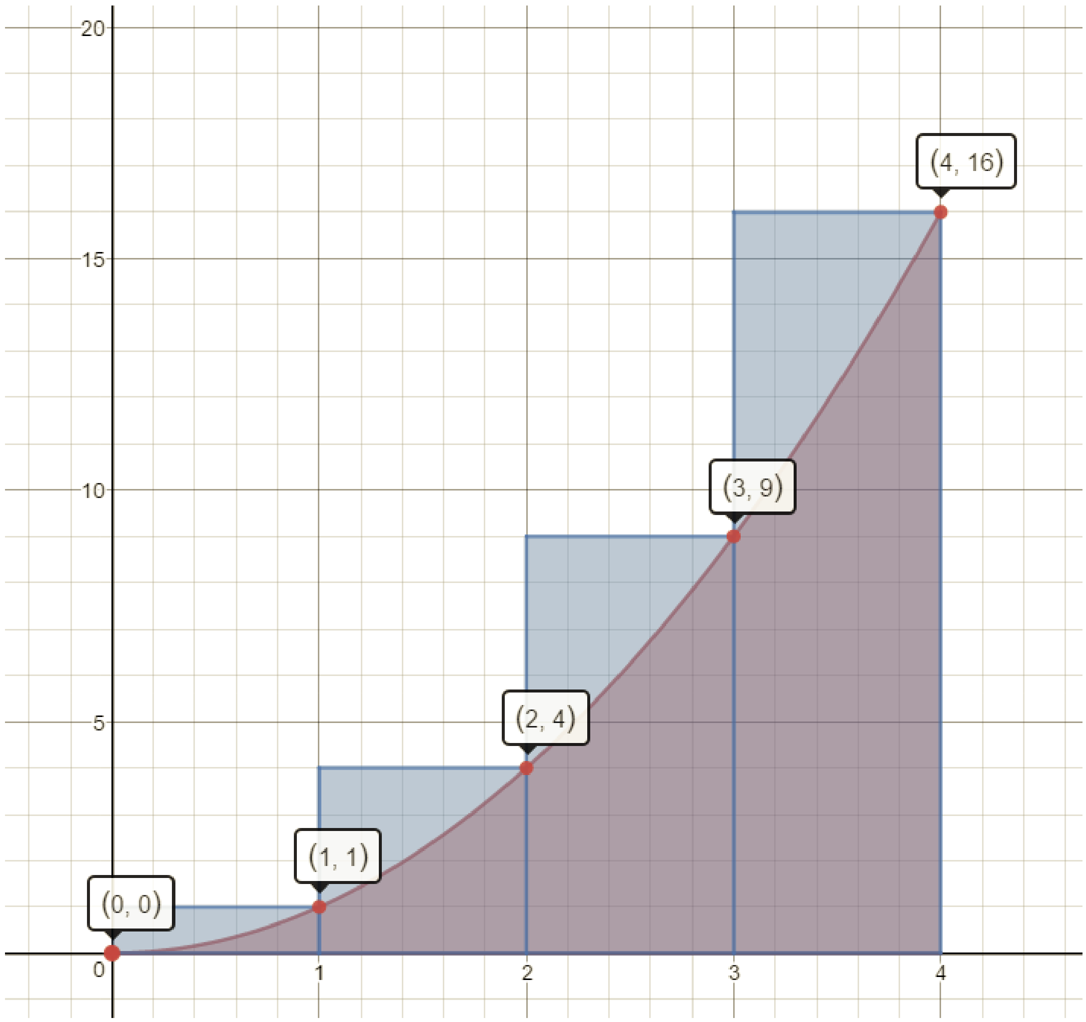


d. Do you think using 3 or 6 or 9 rectangles is a more accurate measure of the area? What could you do to get an even better measure?

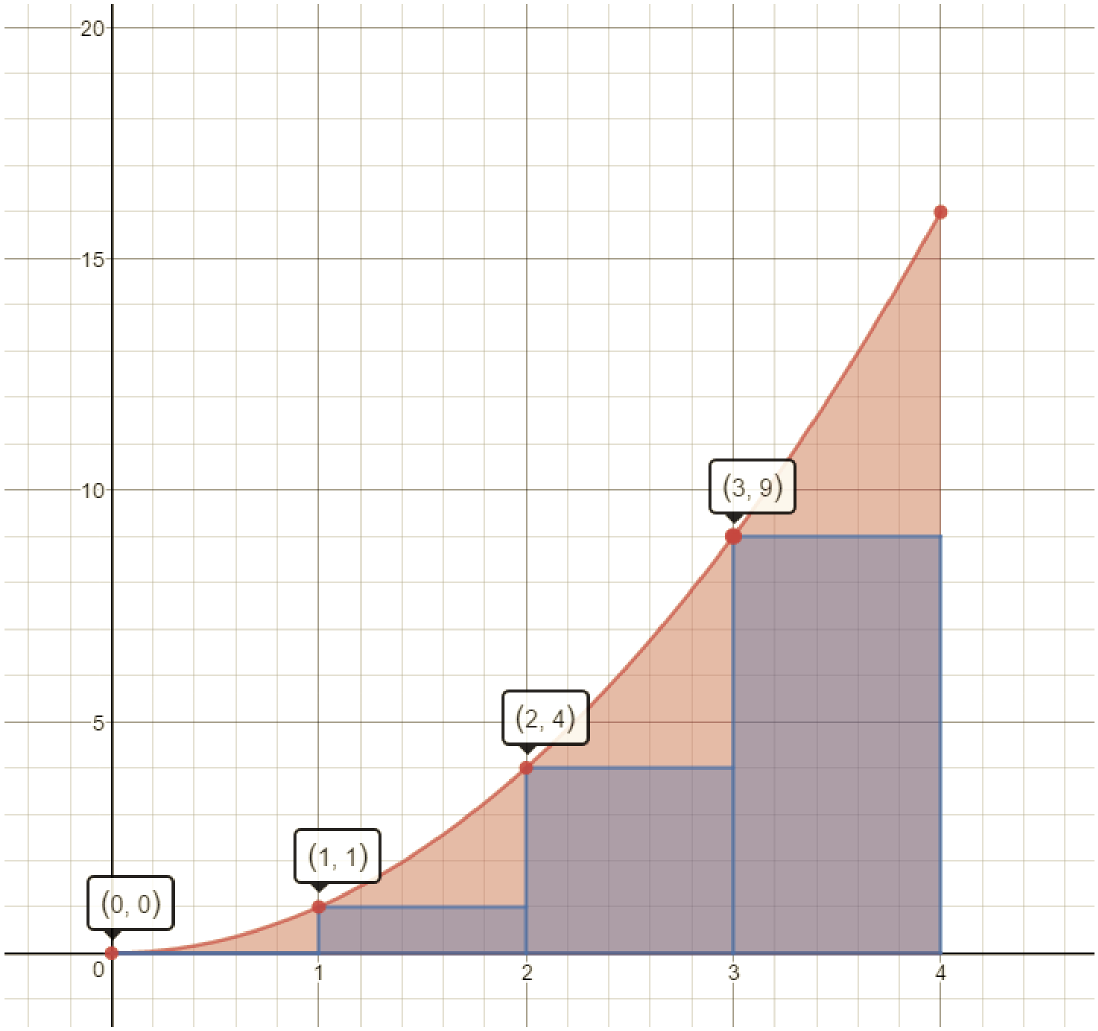
e. What does the area under the curve represent?

Notes

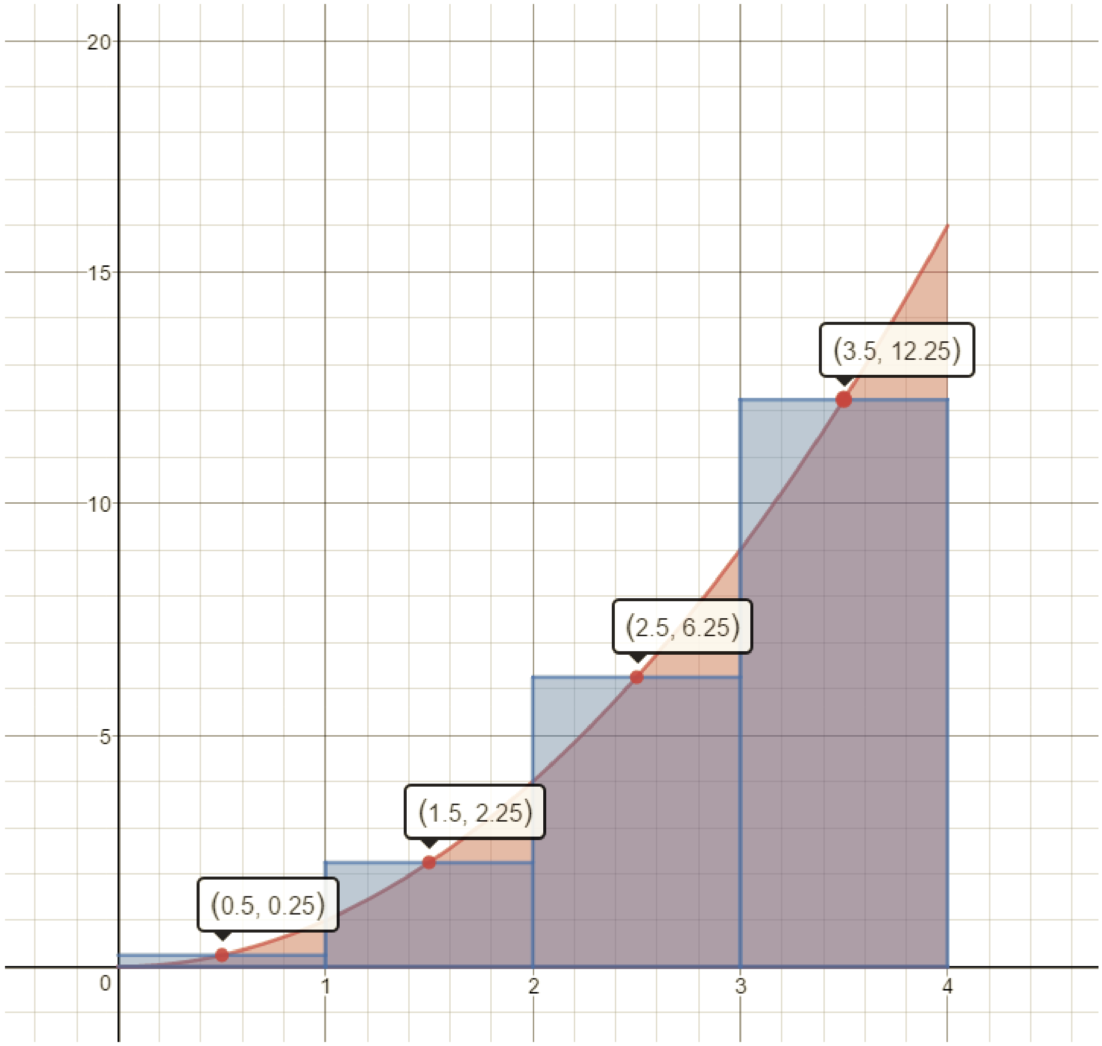
1. Estimate the area under the curve $f(x) = x^2$ on the domain [0,4] by summing the areas of the four rectangles. Is your estimate greater than the actual area or less than the actual area?



Right Hand Rectangles

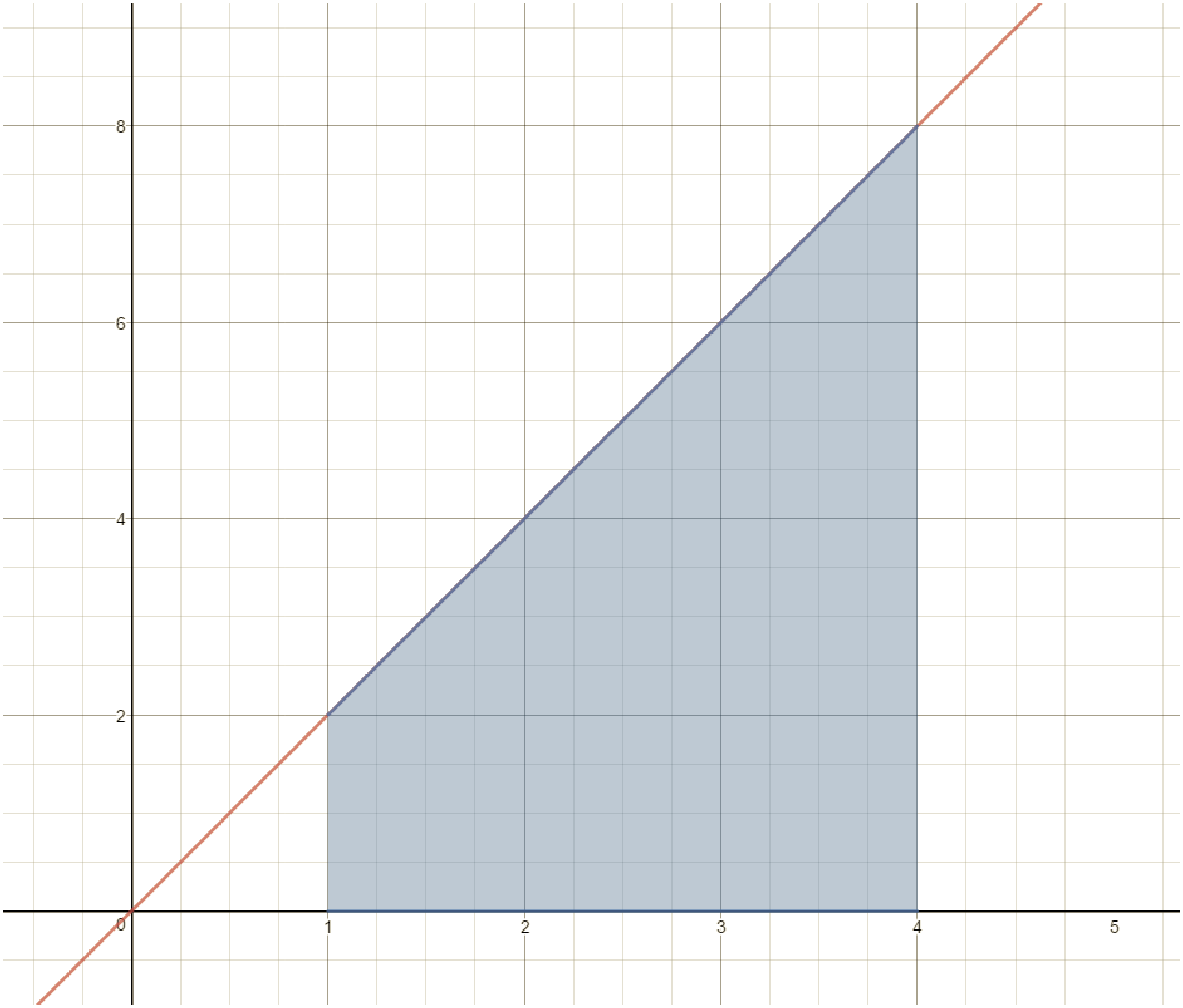


Left Hand Rectangle



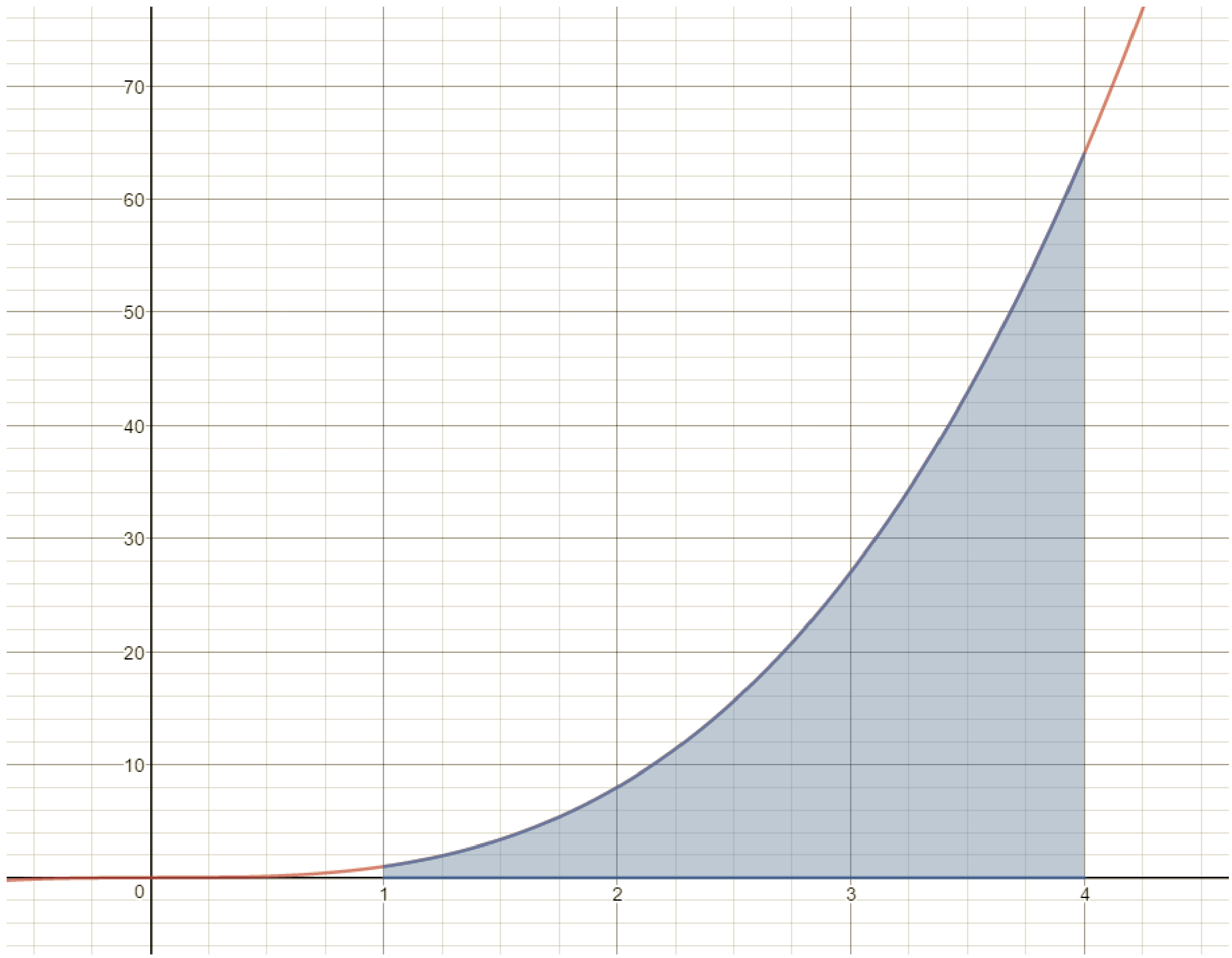
Mid-Point Rectangles

2. Use Right Hand Rectangles to estimate the area under the curve $f(x) = 2x$ on the domain [1,4].



Is this estimate less than the actual area or greater than the actual area?

3. Use Mid-Point Rectangles to estimate the area under the curve $f(x) = x^3$ on the domain [1,4].



Is this estimate less than the actual area or greater than the actual area?

4. Use Left Hand Rectangles to estimate the area under the curve $f(x) = \frac{1}{x}$ on the domain [0.5, 2.0].

Is this estimate less than the actual area or greater than the actual area?

4.2 Fundamental Theorem of Calculus

Introduction

From 4.1 homework problem #2:

Given the function $f(x) = 4 - 0.16x^2$

Approximate the area under the curve on the interval [0, 6] using 6 right rectangles.

Riemann Sum Exploration

Notes

GENERAL ANTIDERIVATIVE FORMULAS

$$\int E'\left[I(x)\right]I'(x)\ dx\ = E\left[I(x)\right] + C$$

$$\int \left[f(x)\right]^n f(x)\ dx\ = \tfrac{\left[f(x)\right]^{n+1}}{n+1} + C \quad n \neq -1$$

$$\int e^{f(x)} \cdot f'(x)\ dx\ = e^{f(x)} + C$$

$$\int \frac{1}{f\left(x\right)}\ f'\left(x\right)\ dx\ \ =\ \ \ln[f\left(x\right)] + C$$

$$\int u^n\ du\ \ =\ \ \tfrac{u^{n+1}}{n+1} + C\ \ n \neq -1$$

$$\int e^u\ du\ \ =\ \ e^u + C$$

$$\int \frac{1}{u}\ du\ \ =\ \ \ln|u| + C$$

THE FUNDAMENTAL THEOREM OF CALCULUS

Let f be continuous on [a, b]. If F is any antiderivative for f on [a,b], then

$$\int_a^b f\left(x\right)\mathrm{d}\,x\ \ = F(b) - F(a)$$

Evaluate the Definite Integral: Compare your answer to the area you calculated in 4.1 notes.

1. $S(x) = \displaystyle\int_0^4 x^2 dx$

2. $S(x) = \int_1^4 2x dx$

3. $S(x) = \int_1^4 x^3 dx$

4. $S(x) = \int_{0.5}^2 \frac{1}{x} dx$

5. $\int_0^7 8x \ dx$

6. $\int_2^3 4x^3\ dx$

7. $\int_0^3 5e^x\ dx$

8. $\int_1^4 \frac{7}{x}\ dx$

9. $\int_3^7 \left(2-4x^2\right)\ dx$

10. $\int_{4}^{25} \frac{5}{\sqrt{x}}\, dx$

11. Cost: A company manufactures mountaineering 75-liter backpacks. The research department produced the marginal cost function

$$B'(x) = 400 - \frac{x}{5} \qquad 0 \le x \le 1000$$

Where $B'(x)$ is in dollars and x is the number of backpacks produced per week. Compute the increase in cost when production level increases from 0 backpacks per week to 600 backpacks per week. Set up a definite integral and evaluate it.

12. Costs of Upkeep of a Marina: Maintenance costs for a marina generally increase as the structures at the marina age. The rate of increase in maintenance costs (in dollars per year) for a particular marina is given approximately by

$$M'(x) = 30x^2 + 2000$$

where x is the age of marina, in years, and M(x) is the total accumulated costs of maintenance for x years. Write a definite integral that gives the total maintenance costs from the third through the seventh year, and evaluate the integral.

4.3 Indefinite Integrals

Introduction

The marginal cost of producing x units of a commodity is given by

$$C'(x) = 3x^2 + 2x.$$

1. Find the cost function $C(x)$ for this commodity.

2. If the fixed costs are $2000, find the cost of producing 20 units.

Notes

GENERAL INTEGRAL FORMULAS

$$\int E'[I(x)]\,I'(x)\;dx\;=E[I(x)]+C$$

$$\int [f(x)]^n f(x)\;dx\;=\tfrac{[f(x)]^{n+1}}{n+1}+C\quad n\neq -1$$

$$\int e^{f(x)}\cdot f'(x)\;dx\;=e^{f(x)}+C$$

$$\int \frac{1}{f(x)}\;f'(x)\;dx\;=\;\ln[f(x)]+C$$

$$\int u^n\;du\;=\;\tfrac{u^{n+1}}{n+1}+C\;\;n\neq -1$$

$$\int e^u\;du\;=\;e^u+C$$

$$\int \frac{1}{u}\;du\;=\;\ln|u|+C$$

Find the indefinite integral. Check by differentiating.

1. $\int 10\;dx$

2. $\int 15x^2\;\;dx$

3. $\int x^{-4}\;\;dx$

4. $\int 8\;x^{1/3}\;\;\;dx$

5. $\int x^2 \ (1 + x^3) \ dx$

6. $\int \left(4x^3 \ + \ \frac{2}{x^3}\right) \ dx$

7. $\displaystyle \int \frac{1 - xe^x}{x}\, dx$

Find the particular antiderivative of each derivative that satisfies the given condition.

8. $R'(x) = 600 - 0.6x$ when $R(0) = 0$

9. $\frac{dQ}{dt} = \frac{100}{t^2}$ when $Q(1) = 400$

10. $\frac{dy}{dt} = 5e^t - 4$ when $y(0) = -1$

11. Renewable Energy: According to the Energy Research Institute, in 2012, US consumption of renewable energy was 8.45 quadrillion Btu (or 8.45×10^{15} Btu). Since the 1960's, consumption has been growing at a rate (in quadrillion Btu per year) given by

$$f'(t) = 0.004t + 0.062$$

where t is in years after 1960. Find $f(t)$ and estimate US consumption of renewable energy in 2024.

12. Sales Analysis: The rate of change of the monthly sales of a newly released video game is given by

$$S'(t) = 400t^{1/3}$$

$$S(0) = 0$$

where t is the number of months since the game was released and $S(t)$ is the number of games sold each month (in thousands). Find $S(t)$. When will monthly sales reach 20,000,000 games?

13. Efficiency of a Machine Operator: The rate at which a machine operator's efficiency Q (in percent) changes with respect to time on the floor without a break is modeled by the function

$$\frac{dQ}{dt} = 0.3t - 7 \qquad 0 \leq t \leq 16 \ hrs$$

where t is the number of house the operator has been working. Find $Q(t)$ given that the operator's efficiency after working 2 hours is 82%.

Find the operators efficiency after 4 hours. After 8 hours.

4.4 Integration by Substitution

Introduction

Find the derivative.

1. $f(x) = \left(3x^5 + 5x\right)^4$

2. $f(x) = \sqrt[5]{\left(3x^2 + 7\right)^4}$

3. $y = \ln\left(3x^2 + 5x\right)$

4. $f(x) = e^{5x^3 - 6x + 1}$

Notes

General Indefinite Integral Formulas

General Antiderivative Formulas

$$\int E'\left[I(x)\right] I'(x)\ dx\ = E\left[I(x)\right] + C$$

$$\int \left[f(x)\right]^n f(x)\ dx\ = \frac{\left[f(x)\right]^{n+1}}{n+1} + C \quad n \neq -1$$

$$\int e^{f(x)} \cdot f'(x)\ dx\ = e^{f(x)} + C$$

$$\int \frac{1}{f\left(x\right)}\ f'\left(x\right)\ dx\ \ =\ \ln[f\left(x\right)] + C$$

$$\int u^n\ du\ \ =\ \ \frac{u^{n+1}}{n+1} + C\ \ n \neq -1$$

$$\int e^u\ du\ \ =\ \ e^u + C$$

$$\int \frac{1}{u}\ du\ \ =\ \ \ln|u| + C$$

Find each indefinite integral and check the result by differentiating.

1. $\int \left(x^9 + 1\right)^4 \left(9x^8\right) dx$

2. $\int \left(6x^3 - 7\right)^{-6} \left(18x^2\right) \, dx$

3. $\int \frac{2}{9x-4} (9) \, dx$

4. $\displaystyle\int (3t+5)^4 \, dt$

5. $\displaystyle\int e^{-2x} \, dx$

6. $\displaystyle\int \frac{x}{5+x^2} \, dx$

7. $\int \frac{t^2}{(t^3 - 1)^4} \, dt$

8. $\int \frac{x - 1}{x^2 - 2x + 5} \, dx$

9. $\displaystyle\int \frac{1}{x\ln x}\ dx$

10. $\displaystyle\int \frac{x}{\sqrt{x+3}}\,dx$

11. $\int x(x+2)^5 \; dx$

12. Continuous Money Flow: Suppose money is flowing continuously into a savings account at a rate of \$1000 per year at interest rate of 2%, compounded continuously. The amount that is paid over time, dt, is

$$A'(t) = 1000e^{.02t}.$$

What is the accumulation of the amount over the first 5 years?

4.5 Area between Curves

Introduction

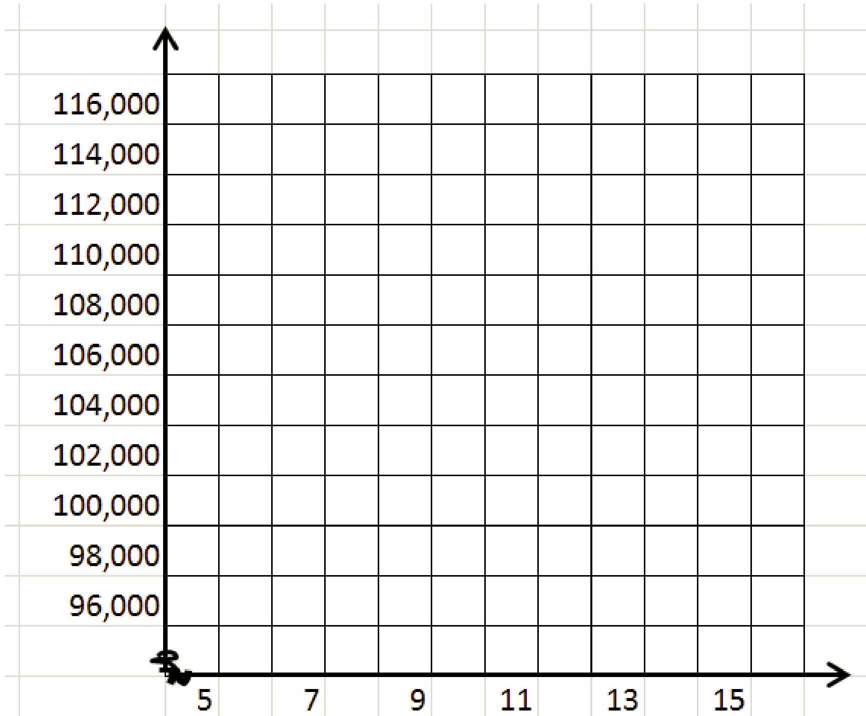
Before 1995, the U. S. Census Bureau used the model below to project the number (in thousands) of households in the United States, where t is the number of years after 1990.

$$N_1 = 1.35t^2 + 1078.4t + 92,323$$

For the years 1995-2005, the actual number of households N in the United States can be modeled by

$$N_2 = 18.32t^2 + 1178.3t + 92,099$$

1. Graph N_1 on the interval $5 \leq t \leq 15$ and shade the area under the curve. What does this shaded area represent?



2. Now graph N_2 on the interval $5 \leq t \leq 15$ on the same graph and shade the area under the curve. What does this shaded area represent?

3. Did the projection model over-project or under-project the number of households?

4. How would you determine the difference in the number of households given by each model?

5. Find the difference in the number of households for the two models.

Notes

AREA BETWEEN CURVES

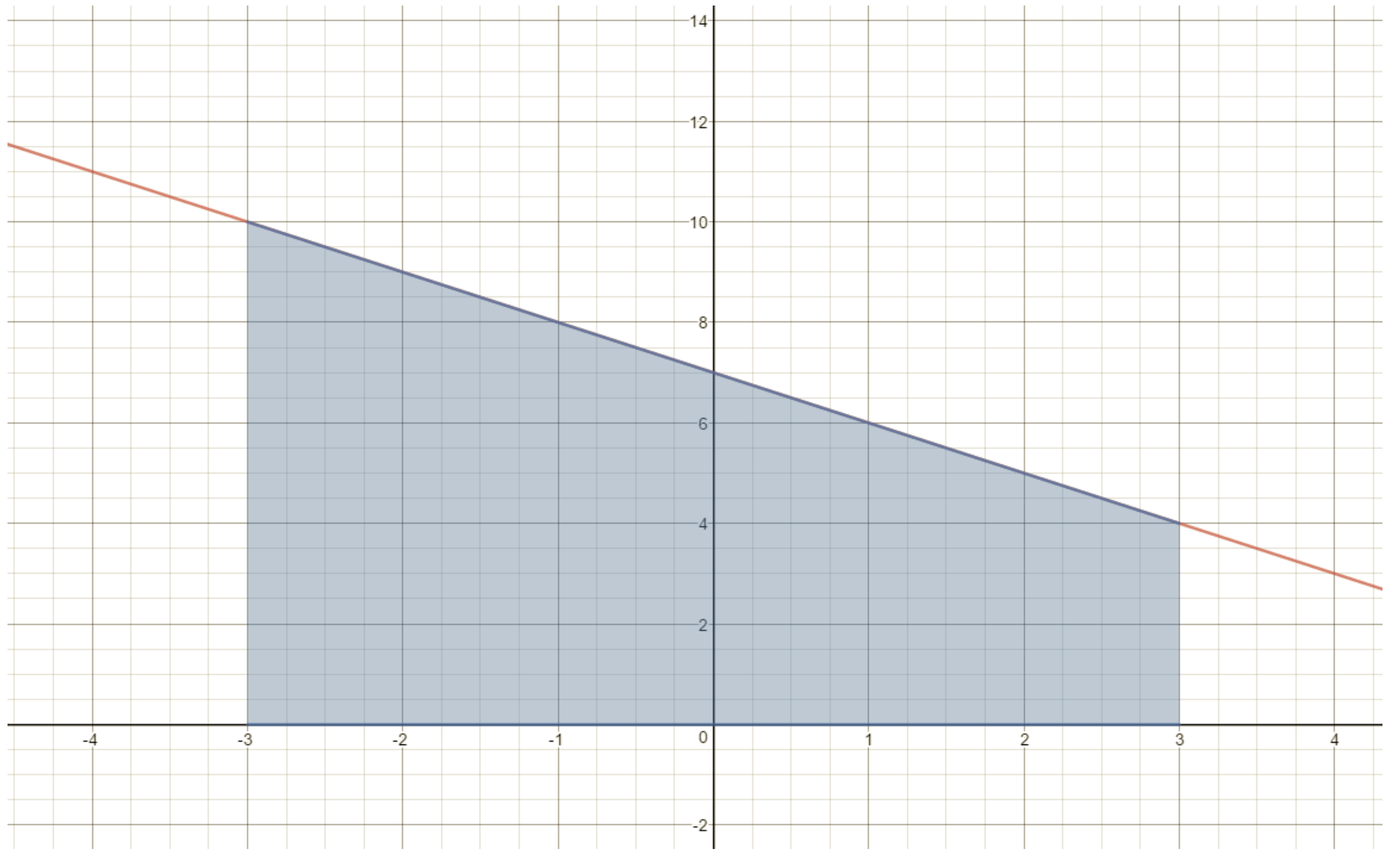
If f and g are continuous and $f(x) > g(x)$ over the interval $[a, b]$, then the area bounded by $y = f(x)$ and $y = g(x)$for $a \le x \le b$ is given exactly by

$$A = \int_a^b [f(x) - g(x)]\ dx$$

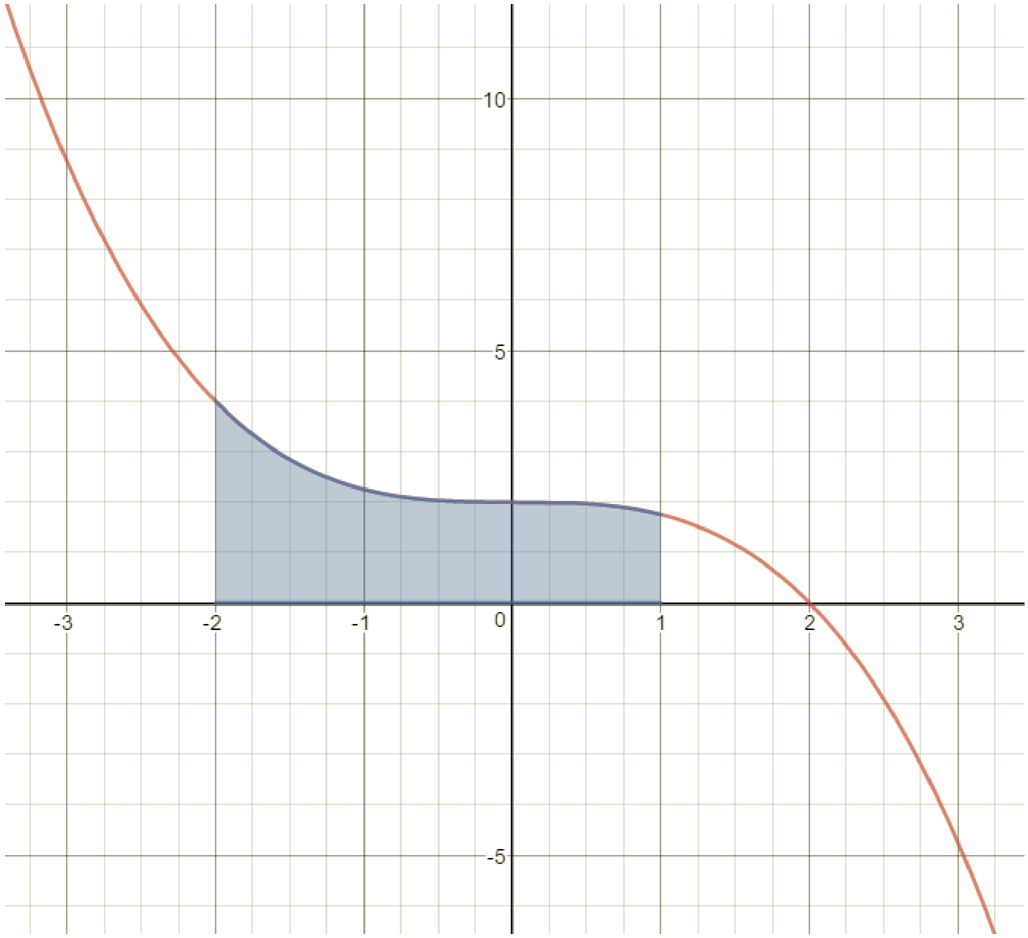
Area Bounded by an Interval

Find the area bounded by the graphs of the indicated equations over the given interval. Compute answers to three decimal places.

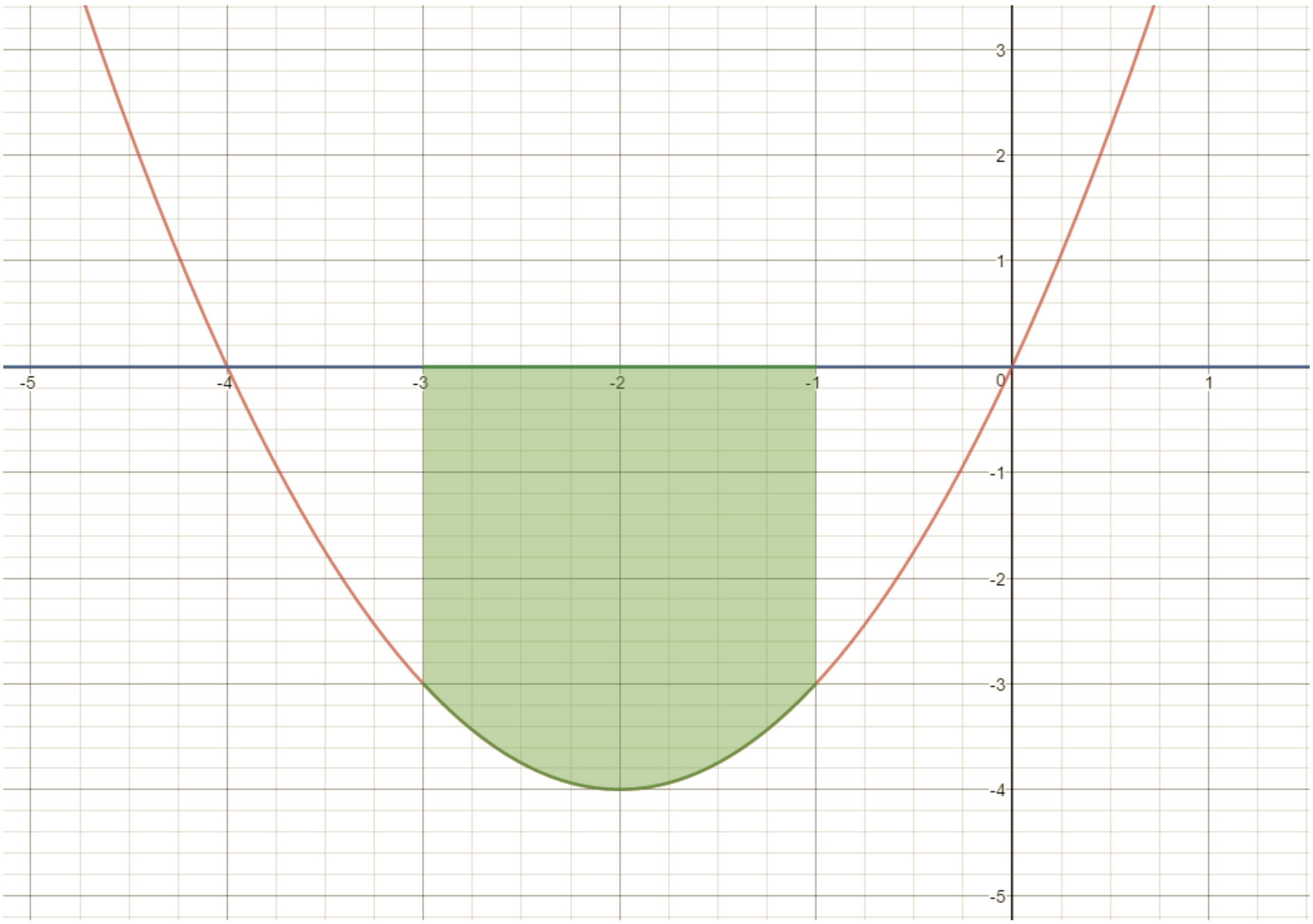
1. $y_1 = -x + 7$ and $y_2 = 0$ on the interval $-3 \leq x \leq 3$



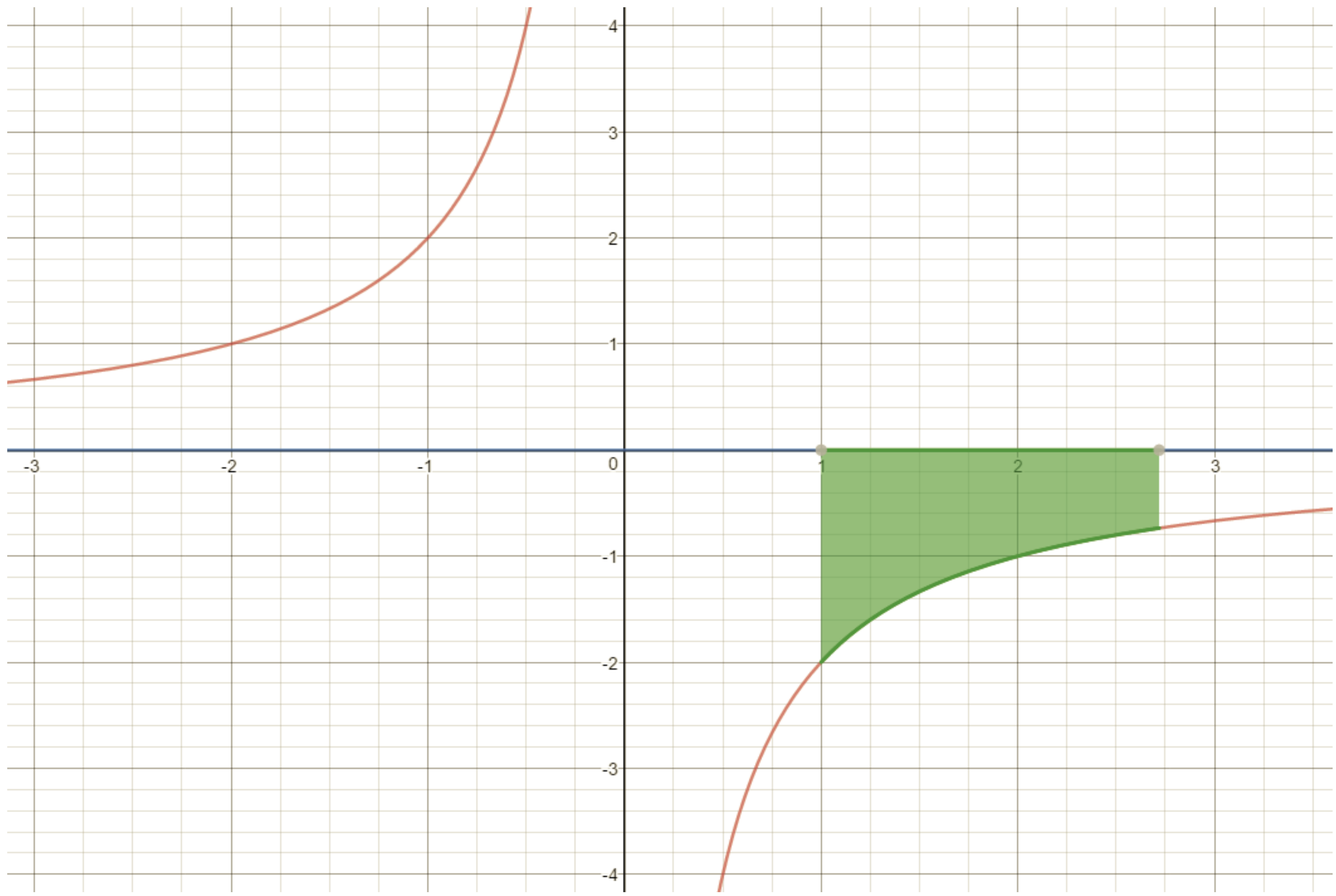
2. $y_1 = -\frac{1}{4}x^3 + 2$ and $y_2 = 0$ on the interval $-2 \le x \le 1$



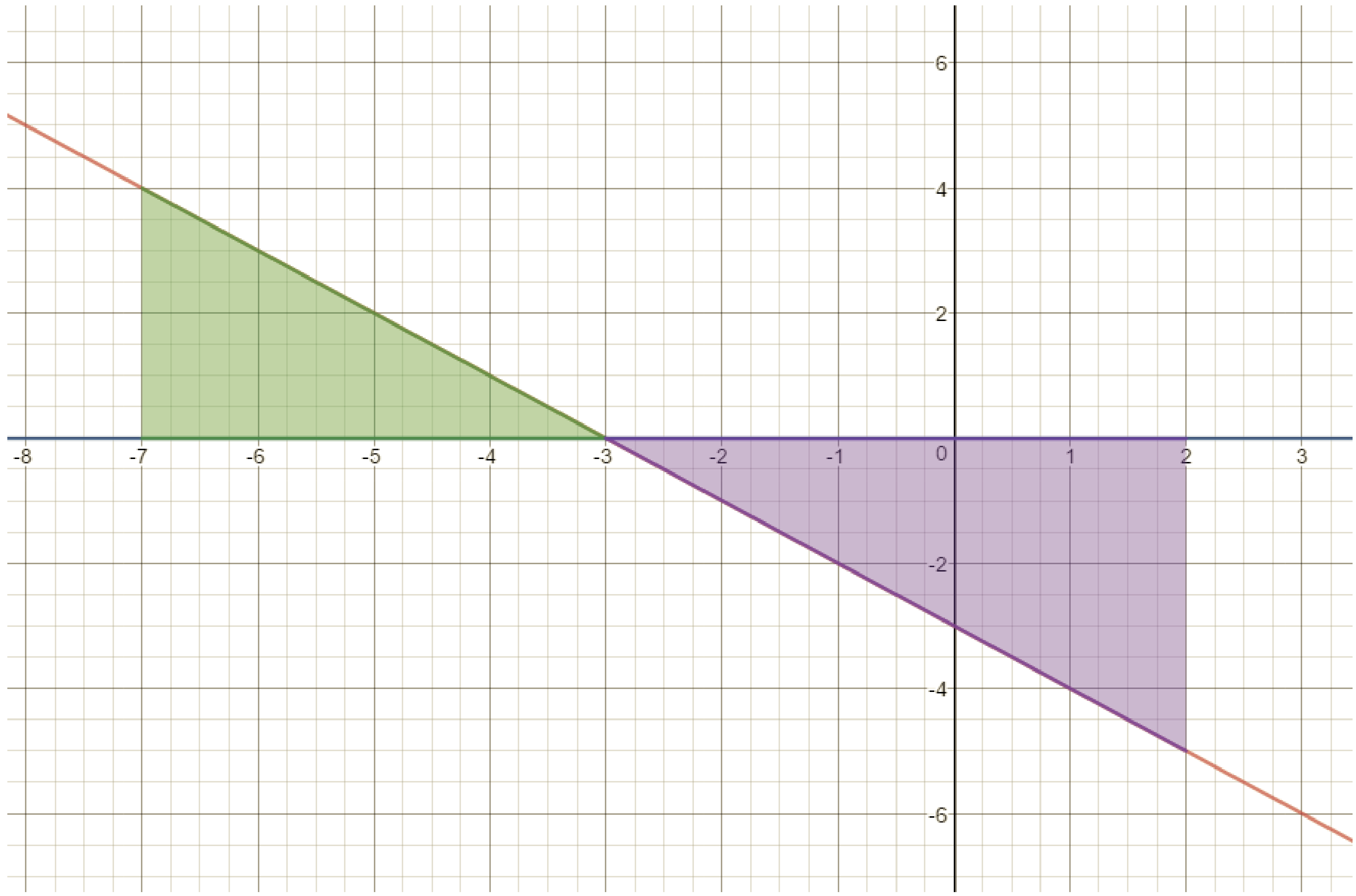
3. $y_1 = \ x\,(4+x)$ and $y_2 = 0$ on the interval $-3 \le x \le -1$



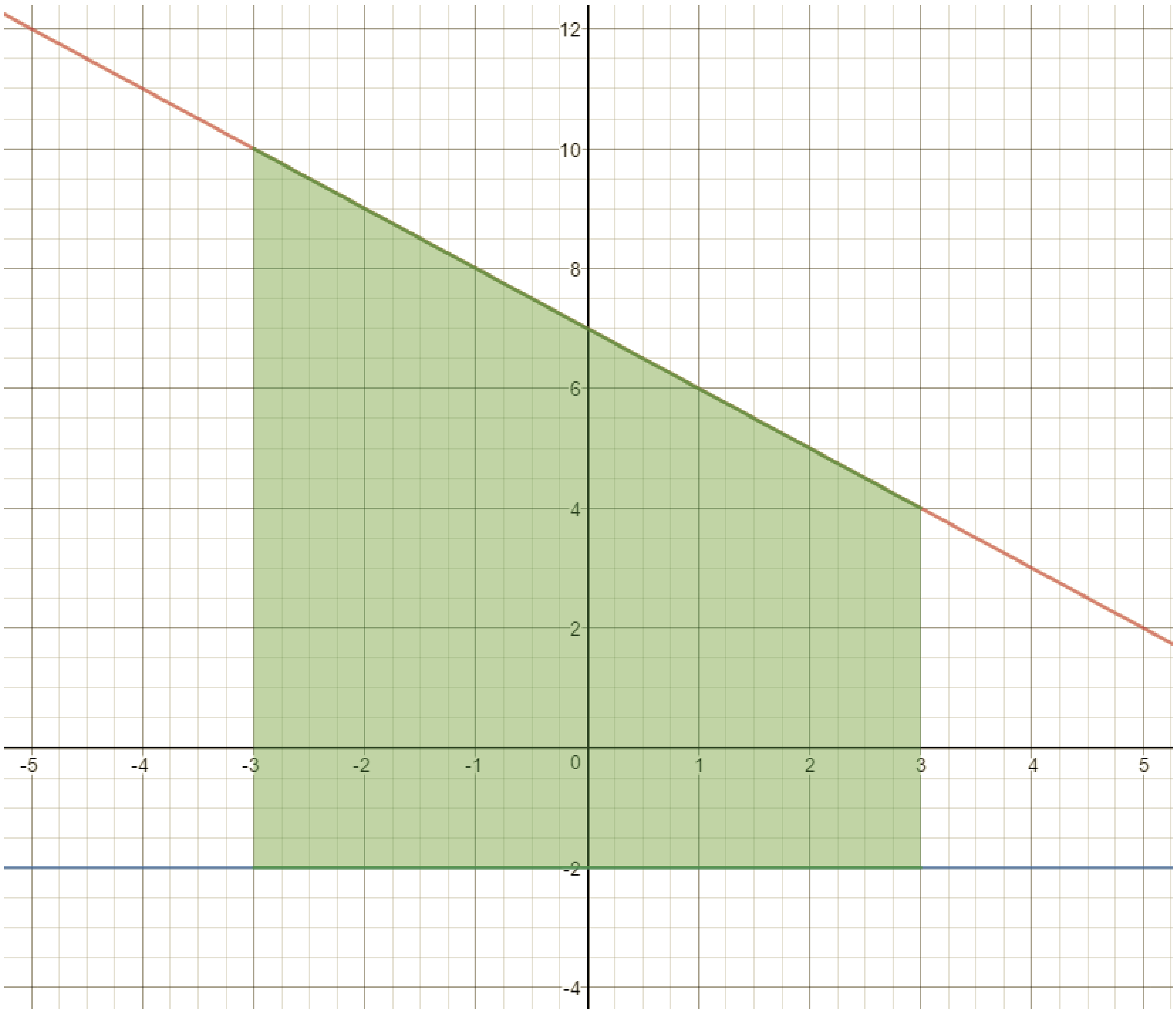
4. $y_1 = -\frac{2}{x}$ and $y_2 = 0$ on the interval $1 \le x \le e$



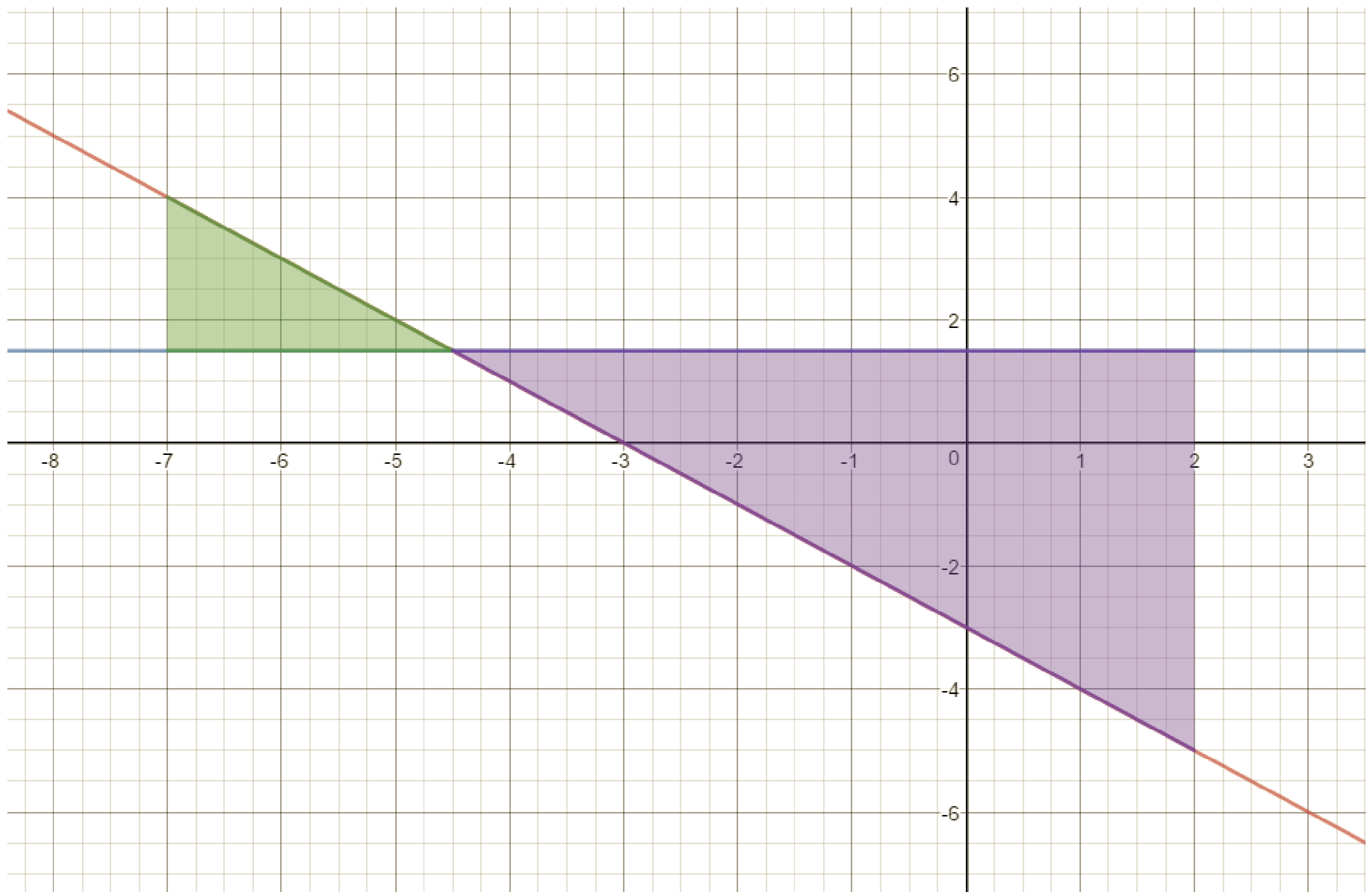
5. $y_1 = \ -x - 3$ and $y_2 = 0$ on the interval $-7 \leq x \leq 2$



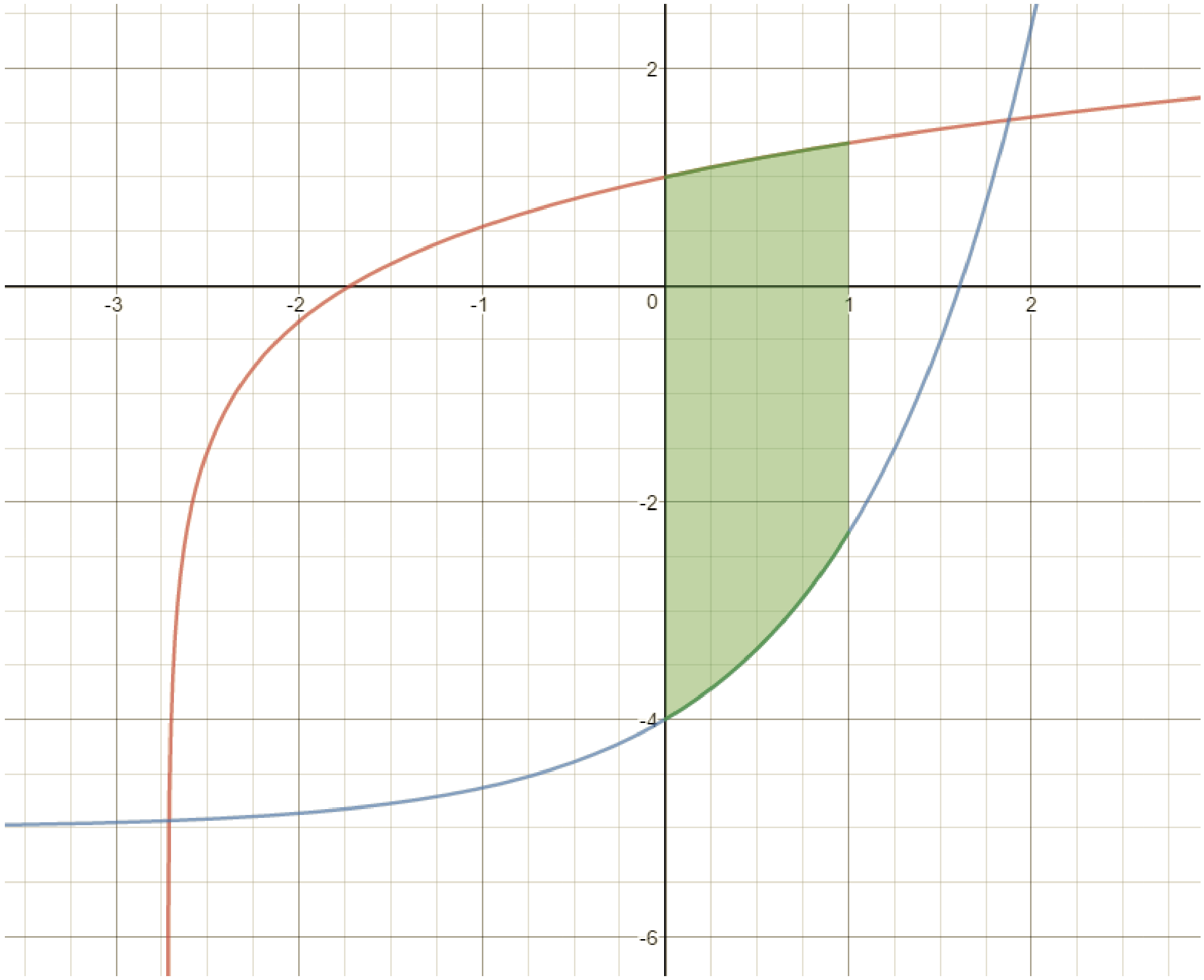
6. $y_1 = -x+7$ and $y_2 = -2$ on the interval $-3 \leq x \leq 3$



7. $y_1 = \ -x - 3$ and $y_2 = 1.5$ for the interval $-7 \leq x \leq 2$



8. $y_1 = \ln(x+e)$ and $y_2 = e^x - 5$ on the interval $0 \leq x \leq 1$

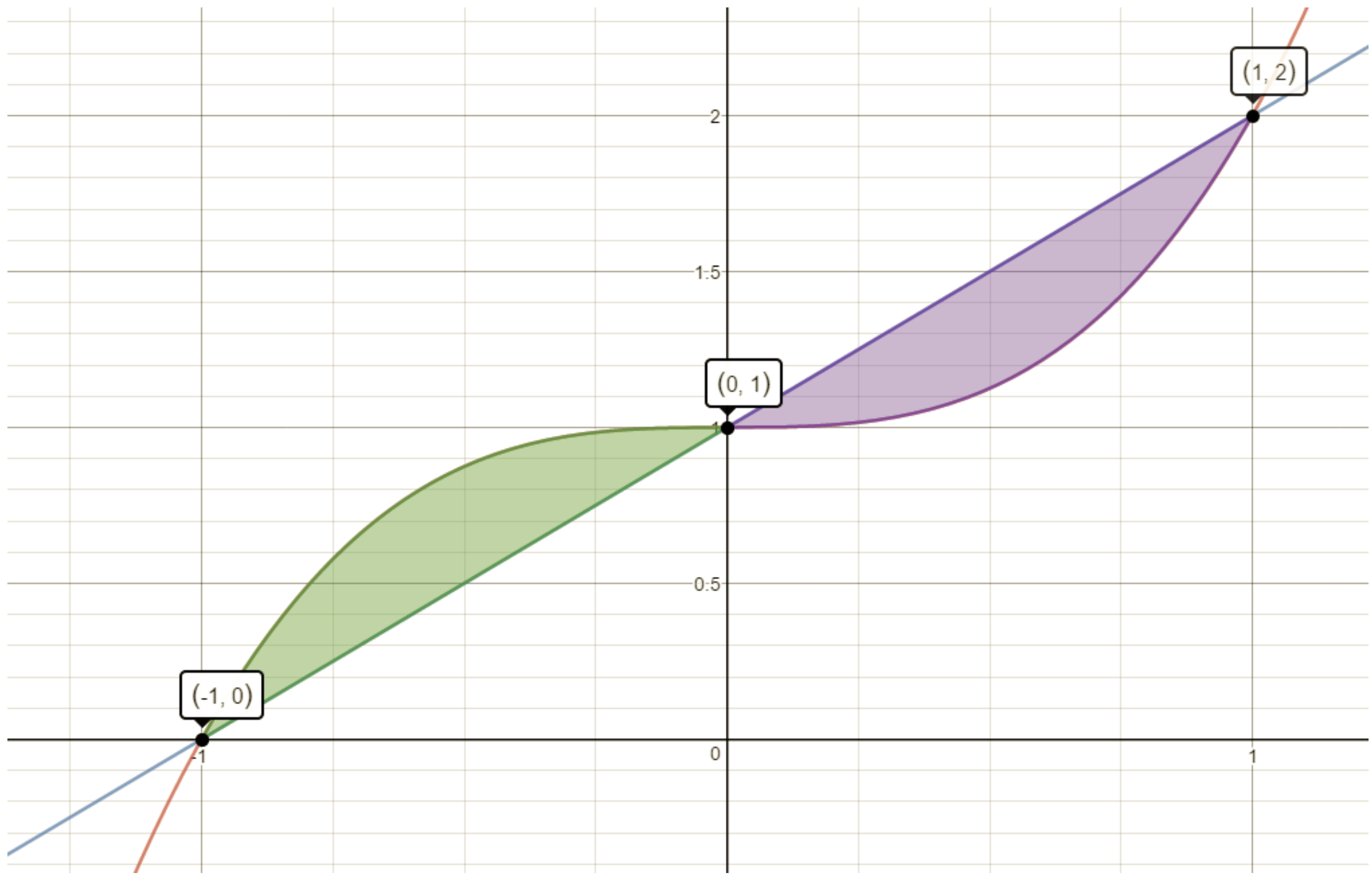


Define the total area bounded by the two functions. How would you determine the interval, if you were asked to calculate the total area bounded between the two functions?

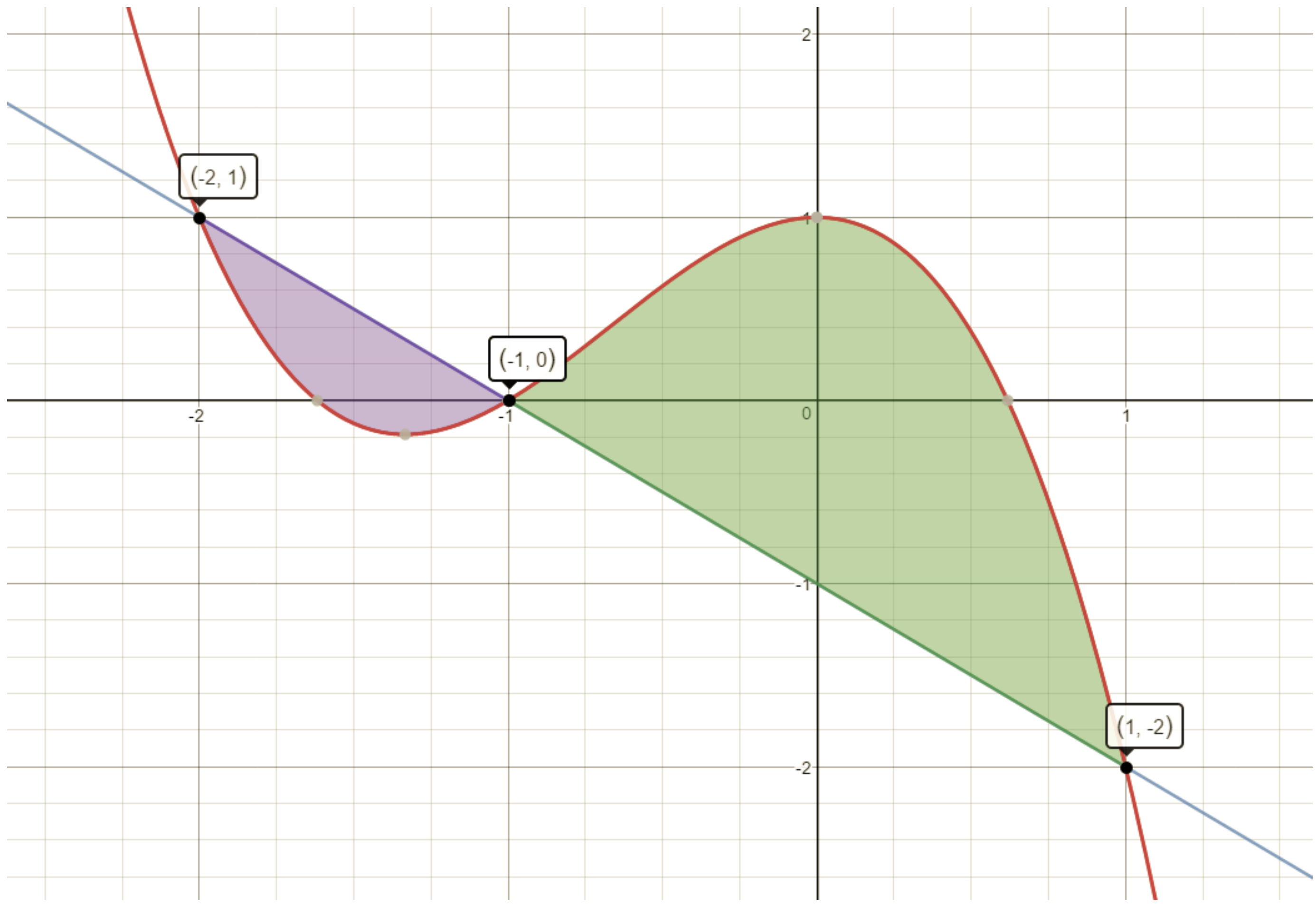
Area Not Bounded by an Interval

Find the area bounded by the graphs of the indicated equations

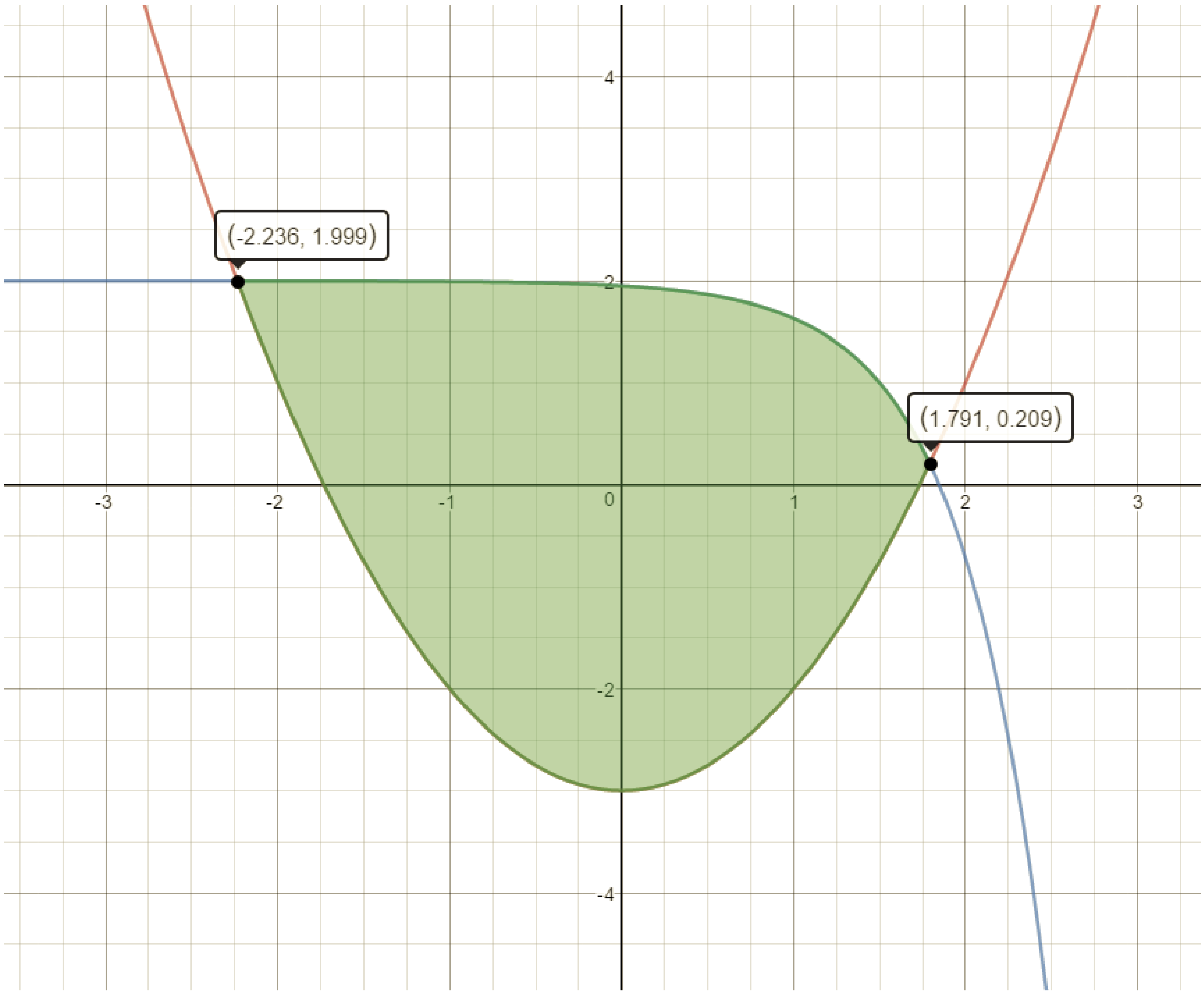
9. $y_1 = x^3 + 1$ and $y_2 = x + 1$



10. $y_1 = -x^3 - 2x^2 + 1$ and $y_2 = -x - 1$



11. $y_1 = -3 + x^2$ and $y_2 = -e^{2x-3} + 2$



12. The useful life of a piece of rental equipment is the duration for which the equipment will be profitable to the rental business, and not how long the equipment will actually last. Many factors affect a piece of equipment's useful life, including the frequency of use, the age when acquired and the repair policy and certain environmental conditions. The change in revenue for renting the equipment, over time, is modeled by

$$R'(t) = 4te^{-0.2t^2},$$

in thousands of dollars per year. The change in cost for routine maintenance of the equipment is modeled by

$$C'(t) = .5t,$$

also in thousands of dollars per year. Find the area between the graphs of C' and R' over the interval from the time the equipment is purchased until the equipment reaches the end of its useful life. Interpret the results.

<u>Note: The end of the useful life is t where</u> $C'(t) = R'(t)$