Calculo Infinitesimal I - 2ª Lista de Exercícios - Prof. Felipe Acker

1 - Seja $P(x) = a_n x^n + \dots + a_0$, $Q(x) = b_m x^m + \dots + b_0$

Se $n > m$:

$$\frac{P(x)}{Q(x)} = K(x) + \frac{r(x)}{q(x)}, \quad K(x) = \frac{a_n}{b_m} x^{n-m} + \dots + \frac{a_{n-m}}{b_0}$$ e $r(x)$ com

grau menor que $q(x)$

Assim, $$\lim_{x\to\infty} \frac{P(x)}{Q(x)} = \lim_{x\to\infty} \left[K(x) + \frac{r(x)}{q(x)}\right] = \lim_{x\to\infty} K(x) + \underbrace{\lim_{x\to\infty} \frac{r(x)}{q(x)}}_{\to 0}$$

$$= \lim_{x\to\infty} K(x) = \lim_{x\to\infty} \frac{a_n}{b_m} x^{n-m} + \dots + \frac{a_{n-m}}{b_0}$$

Se $n-m$ é par, $\lim_{x\to\infty} K(x) = +\infty$

Se $n-m$ é impar, $\lim_{x\to\infty} K(x) = +\infty$

E $$\lim_{x\to-\infty} \frac{P(x)}{Q(x)} = \lim_{x\to-\infty} \left[K(x) + \frac{r(x)}{q(x)}\right] = \lim_{x\to-\infty} K(x) + \underbrace{\lim_{x\to-\infty} \frac{r(x)}{q(x)}}_{\to 0}$$

$$= \lim_{x\to-\infty} K(x) = \lim_{x\to-\infty} \frac{a_n}{b_m} x^{n-m} + \dots + \frac{a_{n-m}}{b_0} =$$

Se $n-m$ é par, $\lim_{x\to-\infty} K(x) = +\infty$

Se $n-m$ é impar $\lim_{x\to-\infty} K(x) = -\infty$

Se $n < m$:

$$\lim_{x\to\infty} \frac{P(x)}{Q(x)} = 0, \quad \lim_{x\to-\infty} \frac{P(x)}{Q(x)} = 0$$

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Se $n=m$, $\frac{P(x)}{Q(x)} = \frac{a_n}{b_m} + \frac{r}{Q(x)}$, $r \in \mathbb{R}$

$$\lim_{x\to\infty} \frac{P(x)}{Q(x)} = \lim_{x\to-\infty} \frac{P(x)}{Q(x)} = \frac{a_n}{b_m}$$

Se $Q(0) \neq 0$, $\lim_{x\to 0+} \frac{P(x)}{Q(x)} = \lim_{x\to 0} \frac{P(x)}{Q(x)} = \frac{a_0}{b_0}$

Se $Q(0)=0$, $n>m$, $\frac{P(x)}{Q(x)} = K(x) + \frac{r(x)}{Q(x)}$, $K(x)$ tem todas as parcelas dependentes de x

$\lim_{x\to 0+} \frac{P(x)}{Q(x)} = +\infty$, se $\lim_{x\to 0+} r(x) > 0$ e $b_m > 0$

$\lim_{x\to 0+} \frac{P(x)}{Q(x)} = -\infty$, sem $\lim_{x\to 0+} r(x) > 0$ e $b_m < 0$

$\lim_{x\to 0+} \frac{P(x)}{Q(x)} = -\infty$, se $\lim_{x\to 0+} r(x) < 0$ e $b_m > 0$

$\lim_{x\to 0+} \frac{P(x)}{Q(x)} = +\infty$, se $\lim_{x\to 0+} r(x) < 0$ e $b_m < 0$

2 – Seja $S(n) = \sum_{k=n+1}^{2n} \frac{1}{k}$

Assim $S(n) = \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n-1} + \frac{1}{2n}$

$$S(n+1) = \frac{1}{n+2} + \frac{1}{n+3} + \dots + \frac{1}{2(n+1)-2} + \frac{1}{2(n+1)-1} + \frac{1}{2(n+1)}$$

$$= \frac{1}{n+2} + \frac{1}{n+3} + \dots + \frac{1}{2n} + \frac{1}{2n+1} + \frac{1}{2n+2}$$

Assim $S(n+1) - S(n) = \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1}$

$$\Rightarrow S(n+1) = \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1} + S(n)$$

$$= \frac{1}{2n+1} + \frac{1}{2(n+1)} - \frac{2}{2(n+1)} + S(n)$$

$$= \frac{1}{2n+1} + \frac{1-2}{2(n+1)} + S(n)$$

$$= \frac{1}{2n+1} - \frac{1}{2(n+1)} + S(n) = \frac{1}{(2n+2)-1} - \frac{1}{2(n+1)} + S(n) =$$

$$= \frac{1}{2(n+1)-1} - \frac{1}{2(n+1)} + S(n)$$

$$\Rightarrow \quad S(n+1) = \frac{1}{2(n+1)-1} - \frac{1}{2(n+1)} + S(n)$$

$\Rightarrow$ Se $n+1 = m$ $\Rightarrow$ $\boxed{S(m) = \frac{1}{2m-1} - \frac{1}{2m} + S(m-1)}$

Como $S(m) = \sum_{k=m+1}^{2m} \frac{1}{k}$, o valor mínimo aceitável para m é $m=1$, pois $2m$ tem que ser maior que zero, senão a somatória não faz sentido.

Assim $S(m) = \frac{1}{2m-1} - \frac{1}{2m} + \frac{1}{2(m-1)-1} - \frac{1}{2(m-1)} + S(m-2)$

$$= \frac{1}{2m-1} - \frac{1}{2m} + \frac{1}{2m-3} - \frac{1}{2m-2} + S(m-2)$$

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$$S(m) = \frac{1}{2m-1} - \frac{1}{2m} + \frac{1}{2m-3} - \frac{1}{2m-2} + \dots + \frac{1}{3} - \frac{1}{4} + S(1)$$

$$= \frac{1}{2m-1} - \frac{1}{2m} + \frac{1}{2m-3} - \frac{1}{2m-2} + \dots + \frac{1}{3} - \frac{1}{4} + \frac{1}{1} - \frac{1}{2}$$

$$= -\frac{1}{2m} + \frac{1}{2m-1} - \frac{1}{2m-2} + \frac{1}{2m-3} - \dots - \frac{1}{2} + \frac{1}{1}$$

$$= \frac{(-1)^{2m+1}}{2m} + \frac{(-1)^{(2m-1)+1}}{2m-1} + \frac{(-1)^{(2m-2)+1}}{2m-2} + \frac{(-1)^{(2m-3)+1}}{2m-3} + \dots + \frac{(-1)^{2+1}}{2} + \frac{(-1)^{1+1}}{1}$$

$$= \sum_{k=1}^{2m} \frac{(-1)^{k+1}}{k}$$

$$\therefore\ S(m) = \sum_{k=1}^{2m} \frac{(-1)^{k+1}}{k}, \text{ se } m=n$$

$$\Rightarrow \boxed{S(n) = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}} \quad \text{c.q.d } \square$$

3_ $\sum_{k=0}^{n-1} e^{ik\theta} = e^0 + e^{i\theta} + e^{2i\theta} + e^{3i\theta} + \dots + e^{(n-1)i\theta} \Rightarrow$ p.g, $a_0 = e^0 = 1$, $q = e^{i\theta}$

$$\Rightarrow e^{i\theta} \cdot \sum_{k=0}^{n-1} e^{ik\theta} = e^{i\theta} + e^{2i\theta} + e^{3i\theta} + e^{4i\theta} + \dots + e^{ni\theta}$$

$$\Rightarrow e^{i\theta} \cdot \sum_{k=0}^{n-1} e^{ik\theta} - \sum_{k=0}^{n-1} e^{ik\theta} = e^{ni\theta} - e^0 = e^{ni\theta} - 1$$

$$\Rightarrow \left(\sum_{k=0}^{n-1} e^{ik\theta}\right)\left(e^{i\theta} - 1\right) = e^{ni\theta} - 1 \Rightarrow \sum_{k=0}^{n-1} e^{ik\theta} = \frac{e^{ni\theta} - 1}{e^{i\theta} - 1}$$

Das fórmulas de Newton (Taylor) de Seno e Cosseno, no conjunto dos complexos e, também, da fórmula de Newton da função exponencial, sabe-se:

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} \quad e \quad \operatorname{sen}\theta = \frac{e^{i\theta}-e^{-i\theta}}{2}$$

Assim:

$$\sum_{k=0}^{n-1}\cos(k\theta) = \sum_{k=0}^{n-1}\left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right) = \frac{1}{2}\sum_{k=0}^{n-1}\left(e^{ik\theta}+e^{-ik\theta}\right) = \frac{1}{2}\left(\sum_{k=0}^{n-1}e^{ik\theta}+\sum_{k=0}^{n-1}e^{-ik\theta}\right)$$

$$= \frac{1}{2}\left(\frac{e^{ni\theta}-1}{e^{i\theta}-1}+\frac{e^{ni(-\theta)}-1}{e^{i(-\theta)}-1}\right) = \frac{1}{2}\left(\frac{e^{ni\theta}-1}{e^{i\theta}-1}+\frac{\frac{1}{e^{ni\theta}}-1}{\frac{1}{e^{i\theta}}-1}\right)$$

$$= \frac{1}{2}\left(\frac{e^{ni\theta}-1}{e^{i\theta}-1}+\frac{1-e^{ni\theta}}{e^{ni\theta}}\cdot\frac{e^{i\theta}}{1-e^{i\theta}}\right)$$

$$= \frac{1}{2}\left(\frac{e^{ni\theta}-1}{e^{i\theta}-1}+\frac{e^{ni\theta}-1}{e^{i\theta}-1}\cdot\frac{e^{i\theta}}{e^{ni\theta}}\right)$$

$$\sum_{k=0}^{n-1}\cos(k\theta) = \frac{1}{2}\left(\frac{e^{ni\theta}-1}{e^{i\theta}-1}\right)\left(1+\frac{e^{i\theta}}{e^{ni\theta}}\right)$$

$$\sum_{k=0}^{n-1}\operatorname{sen}(k\theta) = \frac{1}{2}\left(\frac{e^{ni\theta}-1}{e^{i\theta}-1}\right)\left(1-\frac{e^{i\theta}}{e^{ni\theta}}\right)$$

4- a) $\lim\limits_{n\to\infty}\sqrt[n]{n} = \lim\limits_{n\to\infty} n^{\frac{1}{n}} = \lim\limits_{n\to\infty} e^{\ln n^{\frac{1}{n}}} = \lim\limits_{n\to\infty} e^{\frac{\ln n}{n}} = e^{\lim\limits_{n\to\infty}\frac{\ln n}{n}} = e^0 = 1$

b) $\lim\limits_{n\to\infty}\sqrt{n+1}-\sqrt{n} = \lim\limits_{n\to\infty}\frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}\left(\sqrt{n+1}+\sqrt{n}\right) = \lim\limits_{n\to\infty}\frac{(\sqrt{n+1})^2-(\sqrt{n})^2}{\sqrt{n+1}+\sqrt{n}} =$

$$= \lim_{n\to\infty}\frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} = \lim_{n\to\infty}\frac{1}{\sqrt{n+1}+\sqrt{n}} = 0$$

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c) $\lim\limits_{n\to\infty} \dfrac{a^n}{n^k} = \lim\limits_{n\to\infty} e^{\ln \frac{a^n}{n^k}} = \lim\limits_{n\to\infty} e^{\ln a^n - \ln n^k} = \lim\limits_{n\to\infty} e^{n\ln a - k\ln n} = \lim\limits_{n\to\infty} e^{\overset{\infty}{n}\left(\ln a - k\overset{0}{\frac{\ln n}{n}}\right)} = \infty$

d) $\lim\limits_{n\to\infty} \dfrac{a^n}{n!} = \lim\limits_{n\to\infty} \dfrac{a^n}{n^n\left(1+\cancel{\frac{\alpha_1}{n}}+\cancel{\frac{\alpha_2}{n^2}}+\dots+\cancel{\frac{\alpha_{n-1}}{n^{n-1}}}\right)} = \lim\limits_{n\to\infty} \dfrac{a^n}{n^n} = \lim\limits_{n\to\infty} \left(\dfrac{a}{n}\right)^n = 0$

e) $\lim\limits_{n\to\infty} \dfrac{n!}{n^n} = \lim\limits_{n\to\infty} \dfrac{n(n-1)(n-2)\dots2\cdot1}{n^n} = \lim\limits_{n\to\infty} \dfrac{n^n\left(1+\cancel{\frac{\alpha_1}{n}}+\cancel{\frac{\alpha_2}{n^2}}+\dots+\cancel{\frac{\alpha_{n-1}}{n^{n-1}}}\right)}{n^n} =$

$= \lim\limits_{n\to\infty} \dfrac{n^n}{n^n} = 1$

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