$Q_1)\ x^2+y^2=45$

$f(x,y)=\|(x,y)-(x_0,y_0)\|=\|(x,y)-(1,2)\|=\|(x,y)\|=\sqrt{(x-1)^2+(y-2)^2}$, como $(x-1)^2+(y-2)^2>0,\ \forall x,\forall y\in\mathbb{R}$, maximizar/minimizar $f(x,y)$ é maximizar/minimizar $f^2(x,y)=(x-1)^2+(y-2)^2=F(x,y)$

Assim, devemos maximizar/minimizar:

$$\begin{cases} F(x,y)=(x-1)^2+(y-2)^2 \\ R:\ x^2+y^2=45 \end{cases}$$

$\Rightarrow$ Se $g(x,y)=x^2+y^2$, $\nabla F=\lambda\nabla g \Rightarrow \begin{bmatrix}\frac{\partial F}{\partial x}(x,y)\\ \frac{\partial F}{\partial y}(x,y)\end{bmatrix}=\lambda\begin{bmatrix}\frac{\partial g}{\partial x}(x,y)\\ \frac{\partial g}{\partial y}(x,y)\end{bmatrix}$

$$\Rightarrow \begin{bmatrix}2(x-1)\\ 2(y-2)\end{bmatrix}=\lambda\begin{bmatrix}2x\\ 2y\end{bmatrix} \Rightarrow \begin{cases}2x-1=2\lambda x\\ 2y-4=2\lambda y\end{cases} \text{ e } x^2+y^2=45$$

$$2\lambda x-2x=-1$$
$$\Rightarrow 2x(\lambda-1)=-1$$
$$2x=\frac{-1}{\lambda-1}=\frac{1}{1-\lambda}$$
$$x=\frac{1}{2(1-\lambda)}$$

$$2\lambda y-2y=-4$$
$$\cancel{2}y(\lambda-1)=-\cancel{4}$$
$$y(\lambda-1)=-2$$
$$y=\frac{-2}{\lambda-1}=\frac{2}{1-\lambda}$$

Portanto: $\begin{cases}x=\frac{1}{2(1-\lambda)}\\ y=\frac{2}{1-\lambda}\\ x^2+y^2=45\end{cases} \Rightarrow \left[\frac{1}{2(1-\lambda)}\right]^2+\left[\frac{2}{1-\lambda}\right]^2=45 \Rightarrow \left[\frac{1/2}{1-\lambda}\right]^2+\left[\frac{2}{1-\lambda}\right]^2=45$

$$\Rightarrow \frac{\frac{1}{4}}{(1-\lambda)^2}+\frac{2^2}{(1-\lambda)^2}=45 \Rightarrow \frac{\frac{1}{4}+4}{(1-\lambda)^2}=45 \Rightarrow (1-\lambda)^2=\frac{\frac{1+16}{4}}{45}$$

$$\Rightarrow (1-\lambda)^2=\frac{15}{4}\cdot\frac{1}{45}=\frac{1}{4\cdot 3}=\frac{1}{12} \Rightarrow 1-2\lambda+\lambda^2=\frac{1}{12} \Rightarrow \lambda^2-2\lambda+\frac{11}{12}=0$$

$$\Delta=4-4\cdot 1\cdot\frac{11}{12}=4-\frac{11}{3}=\frac{12-11}{3}=\frac{1}{3} \qquad \lambda=\frac{2\pm\sqrt{\frac{1}{3}}}{2}=\frac{2\pm\frac{\sqrt{3}}{3}}{2}=\frac{\frac{6\pm\sqrt{3}}{3}}{2}=\frac{6\pm\sqrt{3}}{6}$$

$$x=\frac{1}{2\left(\pm\frac{\sqrt{3}}{6}\right)}=\frac{1}{\pm\frac{\sqrt{3}}{3}}=\pm\frac{3}{\sqrt{3}}=\pm\sqrt{3}$$

$$y=\frac{2}{\pm\frac{\sqrt{3}}{6}}=\pm\frac{2\cdot 6}{\sqrt{3}}=\pm\frac{2\cdot 6\sqrt{3}}{\sqrt{3}\cdot\sqrt{3}}=\pm\frac{12\sqrt{3}}{3}=\pm 4\sqrt{3} \quad \therefore$$

$P_1=(\sqrt{3},4\sqrt{3})$ ou

$P_1=(-\sqrt{3},-4\sqrt{3})$

$$F(P_1)=F(\sqrt{3},4\sqrt{3})=(\sqrt{3}-1)^2+(4\sqrt{3}-2)^2<(-\sqrt{3}-1)^2+(-4\sqrt{3}-2)^2=(\sqrt{3}+1)^2+(4\sqrt{3}+2)^2=F(-\sqrt{3},-4\sqrt{3})=F(P_2)$$

$\therefore P_1=(\sqrt{3},4\sqrt{3})$ é o ponto mais próximo e $P_2(-\sqrt{3},-4\sqrt{3})$ é o ponto mais distante.

$Q_2)$ $f(x,y) = x^2 - 2xy + 2y^2$, $f \in C^\infty \Rightarrow$ Infinitamente diferenciável

a) $\dfrac{\partial f}{\partial x}(x,y) = 2x - 2y$ $\quad \dfrac{\partial^2 f}{\partial x^2}(x,y) = 2$ $\quad \dfrac{\partial^2 f}{\partial y \partial x}(x,y) = -2 = \dfrac{\partial^2 f}{\partial x \partial y}(x,y)$

$\dfrac{\partial f}{\partial y}(x,y) = -2x + 4y$ $\quad \dfrac{\partial^2 f}{\partial y^2}(x,y) = 4$

Pontos críticos: $\left.\begin{array}{l} \dfrac{\partial f}{\partial x}(x,y) = 0 \Rightarrow 2x - 2y = 0 \Rightarrow 2x = 2y \Rightarrow x = y \\ \dfrac{\partial f}{\partial y}(x,y) = 0 \Rightarrow -2x + 4y = 0 \Rightarrow -2x = -4y \Rightarrow x = 2y \end{array}\right\} \Rightarrow y = 2y \Rightarrow y = 0$

$\Rightarrow x = 0$

Ponto crítico: $P = (0,0)$

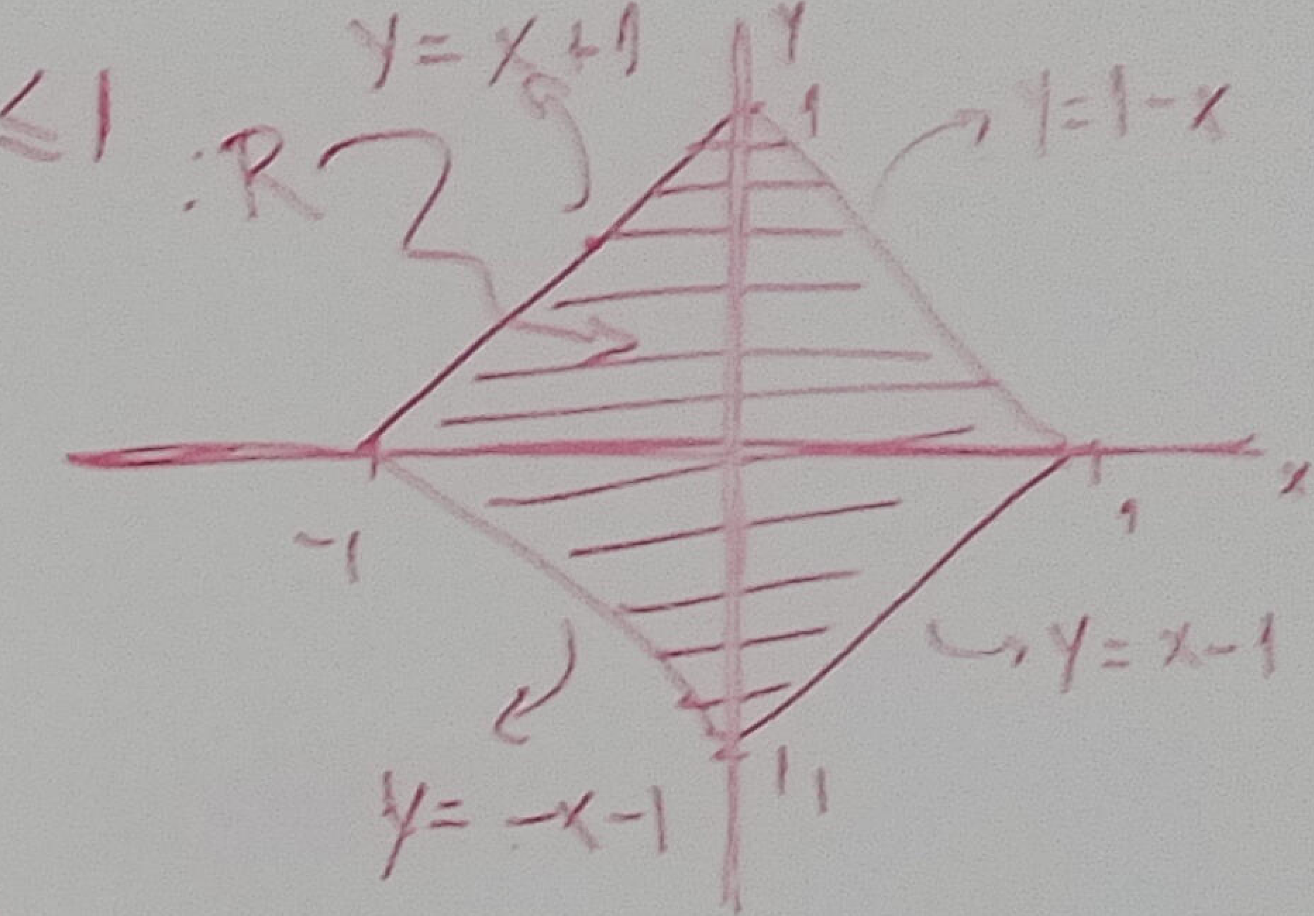
Hessiana: $\mathcal{H} = \begin{bmatrix} f_{xx}(0,0) & f_{xy}(0,0) \\ f_{yx}(0,0) & f_{yy}(0,0) \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix}$ $\quad \det \mathcal{H} = 2 \cdot 4 - (-2)(-2) = 8 - 4 = 4 > 0$,

Portanto a Hessiana é definida.

Como $\mathcal{H}$ é definida e $\dfrac{\partial^2 f}{\partial^2 x}(0,0) > 0 \Rightarrow P(0,0)$ é ponto de mínimo local, com $\boxed{f(0,0) = 0}$

$\hookrightarrow P(0,0) \in D \setminus \partial D$ (Interior a D)

$|x| + |y| \leq 1 : \mathbb{R}^2$



Correspondente a:

$\left.\begin{array}{l} y \leq x+1 \\ y \leq 1-x \\ y \geq -x-1 \\ y \geq x-1 \end{array}\right\}$ Define D

Estudo de fronteira:

$y = x+1 \Rightarrow f(x,x+1) = x^2 - 2x(x+1) + 2(x+1)^2 = x^2 - 2x^2 - 2x - 2x^2 + 4x + 2 = -3x^2 + 2x + 2 = f_1(x)$, $x \in [-1,0]$

$f_1'(x) = -6x + 2 = 0 \Rightarrow x = \frac{1}{3}$, $f(\frac{1}{3}) = -3 \cdot \frac{1}{9} + 2 \cdot \frac{1}{3} + 2 = -\frac{1}{3} + \frac{2}{3} + 2 = 2 + \frac{1}{3} = \frac{7}{3}$ $\therefore \boxed{f(\frac{1}{3}, \frac{4}{3}) = \frac{7}{3}}$, $\boxed{f(-1,0) = -3 + 2 + 2 = 1}$

$y = -x-1 \Rightarrow f(x,-x-1) = x^2 - 2x(-x-1) + 2(-x-1)^2 = x^2 + 2x^2 + 2x + 2x^2 + 4x + 2 = 5x^2 + 6x + 2 = f_2(x)$, $x \in [-1,0]$

$f_2'(x) = 10x + 6 = 0 \Rightarrow x = \frac{-6}{10} = \frac{-3}{5}$ $\therefore f_2(\frac{-3}{5}) = 5 \cdot \frac{9}{25} - 6 \cdot \frac{3}{5} + 2 = \frac{9}{5} - \frac{18}{5} + 2 = \frac{1}{5}$ $\therefore \boxed{f(\frac{-3}{5}, \frac{-2}{5}) = \frac{1}{5}}$, $\boxed{f(0,-1) = 2}$

$y = x-1 \Rightarrow f(x,x-1) = x^2 - 2x(x-1) + 2(x-1)^2 = x^2 - \cancel{2x^2} + 2x + \cancel{2x^2} - 4x + 2 = x^2 - 2x + 2 = f_3(x)$, $x \in [0,1]$

$f_3'(x) = 2x - 2 = 0 \Rightarrow x = 1$ $\therefore f_3(1) = 1 - \cancel{2} + \cancel{2} = 1$ $\therefore \boxed{f(1,0) = 1}$

$y = 1-x \Rightarrow f(x,1-x) = x^2 - 2x(1-x) + 2(1-x)^2 = x^2 - 2x + 2x^2 + 2 - 4x + 2x^2 = 5x^2 - 6x + 2 = f_4(x)$, $x \in [0,1]$

$f_4'(x) = 10x - 6 = 0 \Rightarrow x = \frac{3}{5}$ $\therefore f_4(\frac{3}{5}) = \frac{5 \cdot 9}{25} - 6 \cdot \frac{3}{5} + 2 = \frac{9}{5} - \frac{18}{5} + 2 = \frac{-9+10}{5} = \frac{1}{5}$ $\therefore \boxed{f(\frac{3}{5}, \frac{2}{5}) = \frac{1}{5}}$, $\boxed{f(0,1) = 2}$

Portanto, $P(0,0)$ é mínimo absoluto, com $f(0,0) = 0$ e $P(0,-1)$ e $P(0,1)$ são máximos absolutos com $f(0,-1) = f(0,1) = 2$ //

Bonus 1 _ a) $f(x,y) = e^{x^2+xy}$ → Aproximação quadrática: Fórmula de Taylor de grau 2 – $\underbrace{\text{Erro} \in O(3)}$ → Não calcular.

f é infinitamente diferenciável ∴ $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$

$\frac{\partial f}{\partial x}(x,y) = e^{x^2+xy}\cdot(2x+y)$, $\frac{\partial^2 f}{\partial x^2}(x,y) = e^{x^2+xy}(2x+y)^2 + e^{x^2+xy}\cdot 2 = e^{x^2+xy}\left((2x+y)^2+2\right)$

$\frac{\partial f}{\partial y}(x,y) = e^{x^2+xy}\cdot x$, $\frac{\partial^2 f}{\partial y^2}(x,y) = e^{x^2+xy}\cdot x^2$

$\frac{\partial^2 f}{\partial y \partial x}(x,y) = e^{x^2+xy}\cdot x(2x+y) + e^{x^2+xy} = e^{x^2+xy}(2x^2+xy+1)$

$\frac{\partial^2 f}{\partial x \partial y}(x,y) = e^{x^2+xy}\cdot x(2x+y) + e^{x^2+xy}\cdot 1 = e^{x^2+xy}(2x^2+xy+1)$

Iguais!

Assim, é válida a fórmula de Taylor:

$$f(x+\Delta x, y+\Delta y) = f(x,y) + \frac{\partial f}{\partial x}(x,y)\Delta x + \frac{\partial f}{\partial y}(x,y)\Delta y + \frac{1}{2!}\left[\frac{\partial^2 f}{\partial x^2}(x,y)\Delta x^2 + 2\frac{\partial^2 f}{\partial y \partial x}(x,y)\Delta x \Delta y + \frac{\partial^2 f}{\partial y^2}(x,y)\Delta y^2\right] + \underbrace{O(3)}_{\text{Ignorar o Erro!}}$$

$$\begin{array}{r} 100 \\ 15 \\ \hline 85 \end{array}$$

$$\Rightarrow f(0{,}5; -0{,}8) = f(0+0{,}5; 0-0{,}8) = f(0,0) + \frac{\partial f}{\partial x}(0,0)\cdot 0{,}5 + \frac{\partial f}{\partial y}(0,0)\cdot(-0{,}8) + \frac{1}{2}\left[\frac{\partial^2 f}{\partial x^2}(0,0)(0{,}5)^2 + 2\frac{\partial^2 f}{\partial y \partial x}(0,0)\cdot 0{,}5\cdot(-0{,}8) + \frac{\partial^2 f}{\partial y^2}(0,0)(-0{,}8)^2\right] = (*)$$

$f(0,0) = e^0 = 1$

$\frac{\partial f}{\partial x}(0,0) = e^0\cdot 0 = 0$

$\frac{\partial f}{\partial y}(0,0) = e^0\cdot 0 = 0$

$\frac{\partial^2 f}{\partial x^2}(0,0) = e^0\cdot 2 = 2$

$\frac{\partial^2 f}{\partial y^2}(0,0) = e^0\cdot 0 = 0$

$\frac{\partial^2 f}{\partial y \partial x}(0,0) = e^0\cdot 1 = 1$

$(*) = 1 + 0 + 0 + \frac{1}{2}\left[2\cdot 0{,}25 - 2\cdot 1\cdot 0{,}5\cdot 0{,}8 + 0\right] = 1 + \frac{1}{2}\left[0{,}5 - 0{,}8\right] = 1 - \frac{1}{2}\cdot 0{,}3 = 1 - 0{,}15 = 0{,}85$

∴ $f(0{,}5; -0{,}8) \simeq 0{,}85$

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b) $f(0{,}5; -0{,}8) = 0{,}86070797642505 78$ (Calculado no Python)

Erro $= f(0{,}5; -0{,}8) - 0{,}85 = 0{,}01070797642505 78$ (Calculado no Python)

Erro% $= \dfrac{f(0{,}5; -0{,}8) - 0{,}85}{f(0{,}5; -0{,}8)} \simeq 1{,}24\,\%$
