# Chapter 2: Power system models

## ECE-692 Advanced power system modeling and analysis

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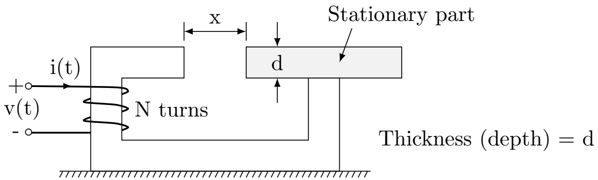
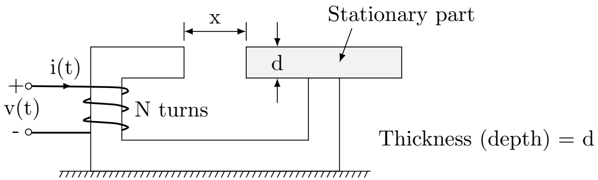
# Short overview of electromechanical systems

# 1. Electromagnetic systems



# 1. Electromagnetic systems




Assume the magnetic core has infinite permeability ($\mu = \infty$), then the total magnetomotive force falls completely in the air gap. Thus,

$$H = \frac{Ni}{x} \quad \Rightarrow \quad \phi = \int_S \mu_0 H dS = \frac{\mu_0 d^2 Ni}{x}$$

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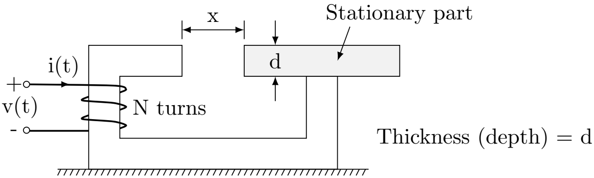


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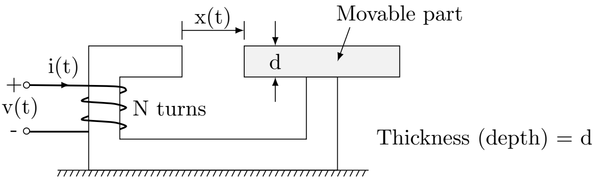
$$H = \frac{Ni}{x} \quad \Rightarrow \quad \phi = \int_S \mu_0 H dS = \frac{\mu_0 d^2 Ni}{x}$$

The flux linkage (total flux) and the induced voltage in the coil become:

$$\lambda = N\phi \qquad v = \frac{d\lambda}{dt} = \frac{d\lambda}{di}\frac{di}{dt} = \frac{\mu_0 d^2 N^2}{x}\frac{di}{dt} = L\frac{di}{dt}$$

# 1. Electromechanical systems

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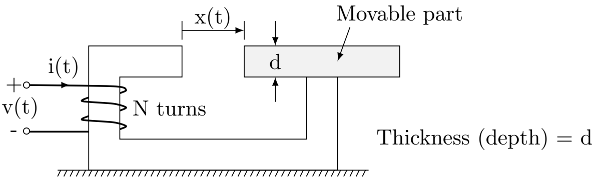


Now, a force will be induced in the movable part. Assuming the magnetic field is conservative, then the change in the energy stored at the magnetic field is given by:

$$\frac{dW_m}{dt} = vi - f^e \frac{dx}{dt} = i\frac{d\lambda}{dt} - f^e \frac{dx}{dt}$$

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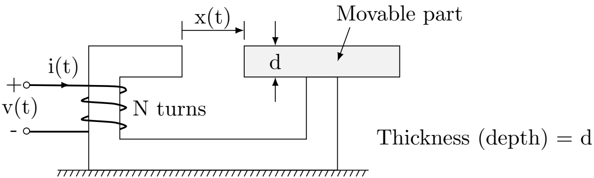


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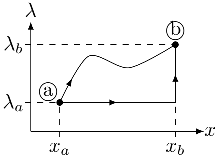
$$\frac{dW_m}{dt} = vi - f^e \frac{dx}{dt} = i\frac{d\lambda}{dt} - f^e \frac{dx}{dt}$$

$$\Rightarrow dW_m = i d\lambda - f^e dx$$

# 1. Electromechanical systems




As $dW_m = id\lambda - f^e dx$, the integral through the orthogonal path becomes:

$$W_m(\lambda_b, x_b) - W_m(\lambda_a, x_a) = -\int_{x_a}^{x_b} f^e(\lambda_a, x)dx + \int_{\lambda_a}^{\lambda_b} i(\lambda, x_b)d\lambda$$

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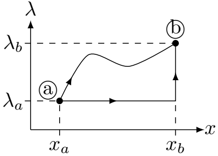


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By setting the point ⓐ at the origin ($x_a = 0$,$\lambda_a = 0$), and making point ⓑ a general point ($x$, $\lambda$), then:

$$W_m(\lambda, x) = \int_0^{\lambda} i(\lambda, x)d\lambda$$

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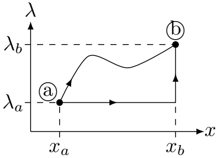


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$$W_m(\lambda, x) = \int_0^{\lambda} i(\lambda, x)d\lambda$$

Why is this important? When $W_m$ is known, the force is determined as $f^e = -\partial W_m(\lambda, x)/\partial x$

# 1. Electromechanical systems

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Co-energy ($W'_m$): As $d(\lambda i) = \lambda di + id\lambda$, then $id\lambda = d(\lambda i) - \lambda di$. By replacing this in $dW_m = id\lambda - f^e dx$, we obtain:

$$\begin{aligned} dW_m &= id\lambda - f^e dx \\ &= d(\lambda i) - \lambda di - f^e dx \\ \Rightarrow d(\underbrace{\lambda i - W_m}_{W'_m}) &= \lambda di + f^e dx \end{aligned}$$

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\end{aligned}
$$

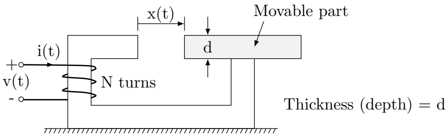
With this new definition, we can prove that:

$$W'_m = \int_0^i \lambda(i, x) di$$

$$f^e = \frac{\partial W'_m}{\partial x}$$

# 1. Electromechanical systems

Back to our system:




$$\lambda = \frac{\mu_0 d^2 N^2 i}{x}$$

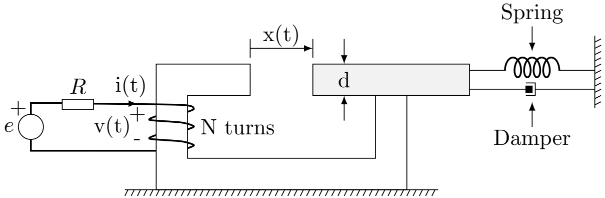
$$v = \frac{d\lambda}{dt} = \underbrace{\frac{\partial \lambda}{\partial i} \frac{di}{dt}}_{\text{transforming voltage}} + \underbrace{\frac{\partial \lambda}{\partial x} \frac{dx}{dt}}_{\text{speed voltage}}$$

$$W'_m = \int \lambda(x,i) di = \frac{\mu_0 d^2 N^2}{2} \frac{i^2}{x}$$

$$f^e = \frac{\partial W'_m}{\partial x} = -\frac{\mu_0 d^2 N^2}{2} \left(\frac{i}{x}\right)^2$$

# 1. Electromechanical systems

Let's add some electrical and mechanical dynamics:



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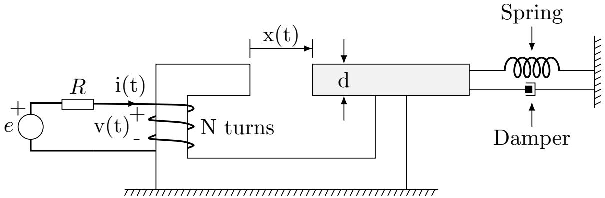



By applying Newton's law:

$$m\frac{d^2x}{dt^2} = f^e - f_s - f_d = -\frac{\mu_0 d^2 N^2}{2}\left(\frac{i}{x}\right)^2 - K_s(x - x_0) - K_d\dot{x}$$

By applying Kirchhoff's voltage law:

$$e = Ri + \frac{d\lambda}{dt}$$

This is an electromechanical system which involves electric, magnetic and mechanical variables.

# 1. Electromechanical systems

By defining $v = \dot{x}$, the system is modeled by the following set of DAEs:

$$\begin{aligned}
\dot{x} &= v \\
m\dot{v} &= -\frac{\mu_0 d^2 N^2}{2}\left(\frac{i}{x}\right)^2 - K_s(x - x_0) - K_d\dot{x} \\
\dot{\lambda} &= -Ri + e \\
0 &= -\lambda + \mu_0 d^2 N^2 \frac{i}{x}
\end{aligned}$$

or by the following set of DEs if $i$ is explicitly solved in terms of $\lambda$ and $x$:

$$\begin{aligned}
\dot{x} &= v \\
m\dot{v} &= -\frac{1}{2\mu_0 d^2 N^2}\lambda^2 - K_s(x - x_0) - K_d\dot{x} \\
\dot{\lambda} &= -\frac{R}{\mu_0 d^2 N^2}\lambda x + e
\end{aligned}$$

# Synchronous generators

# 2. Sketch of a synchronous generator

Note that: (a) there are three dampers, (b) the d-q axes rotate at synchronous speed, and (c) the q-axis leads the d-axis by 90°.

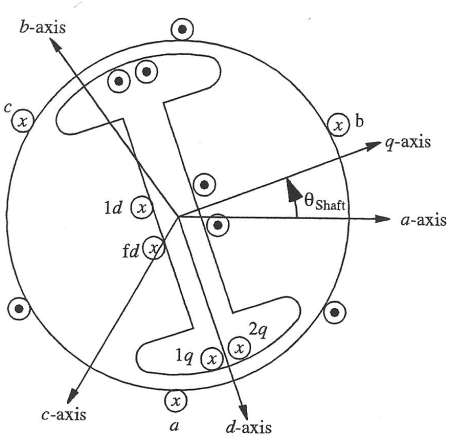
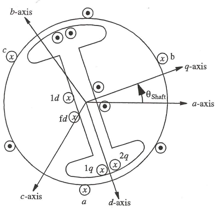



*Figure obtained from [Sauer & Pai]*

# 2. Model of a synchronous generator




$$v_a = r_s i_a + \frac{d\lambda_a}{dt}$$
$$v_b = r_s i_b + \frac{d\lambda_b}{dt}$$
$$v_c = r_s i_c + \frac{d\lambda_c}{dt}$$
$$v_{fd} = r_{fd} i_{fd} + \frac{d\lambda_{fd}}{dt}$$
$$v_{1d} = r_{1d} i_{1d} + \frac{d\lambda_{1d}}{dt}$$
$$v_{1q} = r_{1q} i_{1q} + \frac{d\lambda_{1q}}{dt}$$
$$v_{2q} = r_{2q} i_{2q} + \frac{d\lambda_{2q}}{dt}$$
$$\frac{d\theta_{shaft}}{dt} = \frac{2}{p}\omega$$
$$J\frac{2}{p}\frac{d\omega}{dt} = T_m - T_e - T_{fw}$$

where:

$$i_{rotor} = [i_{fd}, i_{1d}, i_{1q}, i_{2q}]^T, \quad \lambda_{rotor} = [\lambda_{fd}, \lambda_{1d}, \lambda_{1q}, \lambda_{2q}]^T$$

In this formulation there are pending relationships for: flux linkages and the electric torque. Note the model is formulated using "motor" convention.

# 2. Park's transformation

$$v_{dqo} = T_{dqo} v_{abc}, \quad i_{dqo} = T_{dqo} i_{abc}, \quad \lambda_{dqo} = T_{dqo} \lambda_{abc}$$

where

$$T = \frac{2}{3} \begin{bmatrix} \sin\left(\frac{p}{2}\theta_{shaft}\right) & \sin\left(\frac{p}{2}\theta_{shaft} - \frac{2\pi}{3}\right) & \sin\left(\frac{p}{2}\theta_{shaft} + \frac{2\pi}{3}\right) \\ \cos\left(\frac{p}{2}\theta_{shaft}\right) & \cos\left(\frac{p}{2}\theta_{shaft} - \frac{2\pi}{3}\right) & \cos\left(\frac{p}{2}\theta_{shaft} + \frac{2\pi}{3}\right) \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

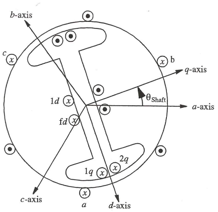
$$T^{-1} = \begin{bmatrix} \sin\left(\frac{p}{2}\theta_{shaft}\right) & \cos\left(\frac{p}{2}\theta_{shaft}\right) & 1 \\ \sin\left(\frac{p}{2}\theta_{shaft} - \frac{2\pi}{3}\right) & \cos\left(\frac{p}{2}\theta_{shaft} - \frac{2\pi}{3}\right) & 1 \\ \sin\left(\frac{p}{2}\theta_{shaft} + \frac{2\pi}{3}\right) & \cos\left(\frac{p}{2}\theta_{shaft} + \frac{2\pi}{3}\right) & 1 \end{bmatrix}$$

Thus,

$$\begin{aligned} v_{abc} &= r_s i_{abc} + \frac{d}{dt}(\lambda_{abc}) \\ \Rightarrow T_{dqo} v_{abc} &= r_s T_{dqo} i_{abc} + T_{dqo} \frac{d}{dt}(\lambda_{abc}) \\ \Rightarrow v_{dqo} &= r_s i_{dqo} + T_{dqo} \frac{d}{dt}(T_{dqo}^{-1} \lambda_{dqo}) \end{aligned}$$

# 2. Park's transformation

Using the Park's transformation, the model in dqo frame becomes:

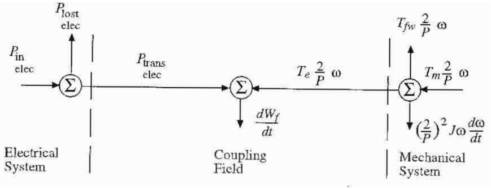



$$
\begin{aligned}
v_d &= r_s i_d - \omega\lambda_q + \frac{d\lambda_d}{dt} \\
v_q &= r_s i_q + \omega\lambda_d + \frac{d\lambda_q}{dt} \\
v_o &= r_s i_o + \frac{d\lambda_o}{dt} \\
v_{fd} &= r_{fd} i_{fd} + \frac{d\lambda_{fd}}{dt} \\
v_{1d} &= r_{1d} i_{1d} + \frac{d\lambda_{1d}}{dt} \\
v_{1q} &= r_{1q} i_{1q} + \frac{d\lambda_{1q}}{dt} \\
v_{2q} &= r_{2q} i_{2q} + \frac{d\lambda_{2q}}{dt} \\
\frac{d\theta_{shaft}}{dt} &= \frac{2}{p}\omega \\
J\frac{2}{p}\frac{d\omega}{dt} &= T_m - T_e - T_{fw}
\end{aligned}
$$

The transformed dqo variables ($v_d$, $v_q$, $v_o$, $i_d$, $i_q$, $i_o$, $\lambda_d$, $\lambda_q$, and $\lambda_o$) are constant states and not sinusoidal as in the abc frame.

# 2. Finding an expression for $T_e$

First of all, we need expressions for the power balance and stored energy in the magnetic field:




$$P_{in_{elec}} = v_a i_a + v_b i_b + v_c i_c + v_{fd} i_{fd} + v_{1d} i_{1d} + v_{1q} i_{1q} + v_{2q} i_{2q}$$

$$P_{lost_{elec}} = r_s(i_a^2 + i_b^2 + i_c^2) + r_{fd} i_{fd}^2 + r_{1d} i_{1d}^2 + r_{1q} i_{1q}^2 + r_{2q} i_{2q}^2$$

$$P_{trans_{elec}} = i_a \frac{d\lambda_a}{dt} + i_b \frac{d\lambda_b}{dt} + i_c \frac{d\lambda_c}{dt} + i_{fd} \frac{d\lambda_{fd}}{dt} + i_{1d} \frac{d\lambda_{1d}}{dt} + i_{1q} \frac{d\lambda_{1q}}{dt} + i_{2q} \frac{d\lambda_{2q}}{dt}$$

# 2. Finding an expression for $T_e$

In terms of the transformed variables:

$$
\begin{aligned}
P_{in_{elec}} &= \tfrac{3}{2}v_d i_d + \frac{3}{2}v_q i_q + 3v_o i_o + v_{fd}i_{fd} + v_{1d}i_{1d} + v_{1q}i_{1q} + v_{2q}i_{2q} \\
P_{lost_{elec}} &= \tfrac{3}{2}r_s(i_d^2 + i_q^2 + 2i_o^2) + r_{fd}i_{fd}^2 + r_{1d}i_{1d}^2 + r_{1q}i_{1q}^2 + r_{2q}i_{2q}^2 \\
P_{trans_{elec}} &= -\tfrac{3}{2}\tfrac{p}{2}\frac{d\theta_{shaft}}{dt}\lambda_q i_d + \tfrac{3}{2}i_d\frac{d\lambda_d}{dt} + \tfrac{3}{2}\tfrac{p}{2}\frac{d\theta_{shaft}}{dt}\lambda_d i_q + \tfrac{3}{2}i_q\frac{d\lambda_q}{dt} \\
&\quad + 3i_o\frac{d\lambda_o}{dt} + i_{fd}\frac{d\lambda_{fd}}{dt} + i_{1d}\frac{d\lambda_{1d}}{dt} + i_{1q}\frac{d\lambda_{1q}}{dt} + i_{2q}\frac{d\lambda_{2q}}{dt}
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\frac{dW_f}{dt} &= P_{trans_{elec}} + T_e\tfrac{2}{p}\omega \\
&= \left[\tfrac{3}{2}\tfrac{p}{2}(\lambda_d i_q - \lambda_q i_d) + T_e\right]\frac{d\theta_{shaft}}{dt} + \tfrac{3}{2}i_d\frac{d\lambda_d}{dt} + \tfrac{3}{2}i_q\frac{d\lambda_q}{dt} \\
&\quad + 3i_o\frac{d\lambda_o}{dt} + i_{fd}\frac{d\lambda_{fd}}{dt} + i_{1d}\frac{d\lambda_{1d}}{dt} + i_{1q}\frac{d\lambda_{1q}}{dt} + i_{2q}\frac{d\lambda_{2q}}{dt}
\end{aligned}
$$

Note the coupling field is conservative, thus, $W_f$ is determined by integrating the previous expression independently of the chosen integration path.

# 2. Finding an expression for $T_e$

Let's consider the following arbitrary path from the de-energized ($W_f^o = 0$) condition to the energized condition:

- **Step 1** Integrate rotor position to some arbitrary $\theta_{shaft}$ while all sources are de-energized. This adds zero to $W_f$ since $\lambda_d$, $\lambda_q$ and $T_e$ must be zero.
- **Step 2** Integrate each source in sequence while maintaining $\theta_{shaft}$ at its arbitrary position. Thus, $W_f$ will be the sum of seven integrals for the seven independent sources $\lambda_d$, $\lambda_q$, $\lambda_o$, $\lambda_{fd}$, $\lambda_{1d}$, $\lambda_{1q}$ and $\lambda_{2q}$

Let's assume as well that the relationships between the flux linkages and the currents are independent of $\theta_{shaft}$, thus

$$\frac{\partial W_f}{d\theta_{shaft}} = \frac{3}{2}\frac{p}{2}(\lambda_d i_q - \lambda_q i_d) + T_e = 0 \quad \Rightarrow \quad \boxed{T_e = \frac{3}{2}\frac{p}{2}(\lambda_d i_q - \lambda_q i_d)}$$

# 2. Definition of the loading angle $\delta$

It is convenient to define an angle that is constant for constant speed shaft:

$$\delta = \frac{p}{2}\theta_{shaft} - \omega_s t$$

where $\omega_s = 2\pi f = 120\pi \approx 376.9911$ rad/s.

For a given loading condition, when the system is in steady-state, this angle will be constant. If the loading changes, then in the steady-state this angle will achieve a different constant value. In general, the larger the loading, the larger the angle $\delta$.

The time derivative of $\delta$ is:

$$\frac{d\delta}{dt} = \frac{p}{2}\frac{\theta_{shaft}}{dt} - \omega_s$$

and as $\frac{d\theta_{shaft}}{dt} = \frac{2}{p}\omega$, this becomes:

$$\frac{d\delta}{dt} = \omega - \omega_s$$

# 2. Model in dqo frame

By using the dqo transformation, the torque and angle definitions, the synchronous machine model becomes:

$$\begin{aligned}
\frac{d\lambda_d}{dt} &= -r_s i_d + \omega\lambda_q + v_d \\
\frac{d\lambda_q}{dt} &= -r_s i_q - \omega\lambda_d + v_q \\
\frac{d\lambda_o}{dt} &= -r_s i_o + v_o \\
\frac{d\lambda_{fd}}{dt} &= -r_{fd} i_{fd} + v_{fd} \\
\frac{d\lambda_{1d}}{dt} &= -r_{1d} i_{1d} + v_{1d} \\
\frac{d\lambda_{1q}}{dt} &= -r_{1q} i_{1q} + v_{1q} \\
\frac{d\lambda_{2q}}{dt} &= -r_{2q} i_{2q} + v_{2q} \\
\frac{d\delta}{dt} &= \omega - \omega_s \\
J\frac{2}{p}\frac{d\omega}{dt} &= T_m - T_e - T_{fw} \\
\text{where} \quad T_e &= -\frac{3}{2}\frac{p}{2}(\lambda_q i_d - \lambda_d i_q)
\end{aligned}$$

# 2. Per unit system

The per unit system is usually used in power system analysis, as this simplifies models and analysis. The per unit system is a scaling process that we can see as a change of variables and a change of parameters.

$$V_d = \frac{v_d}{V_{base}^{dqo}}, \quad V_q = \frac{v_q}{V_{base}^{dqo}}, \quad V_o = \frac{v_o}{V_{base}^{dqo}}$$
$$I_d = \frac{-i_d}{I_{base}^{dqo}}, \quad I_q = \frac{-i_q}{I_{base}^{dqo}}, \quad I_o = \frac{-i_o}{I_{base}^{dqo}}$$
$$\psi_d = \frac{\lambda_d}{\Lambda_{base}^{dqo}}, \quad \psi_q = \frac{\lambda_q}{\Lambda_{base}^{dqo}}, \quad \psi_o = \frac{\lambda_o}{\Lambda_{base}^{dqo}}$$

where $V_{base}^{dqo}$ is the peak rated line-to-neutral voltage ($\sqrt{2}V_{RMS}$). Note that a "generator convention" is employed (observe negative sign in currents).

Considering $S_{base}$ is the three-phase nominal power and $\omega_{base}$ is the nominal electrical speed in rad/s (synchronous speed), then:

$$I_{base}^{dqo} = \frac{2}{3}\frac{S_{base}}{V_{base}^{dqo}}, \quad \Lambda_{base}^{dqo} = \frac{V_{base}^{dqo}}{\omega_{base}}$$

# 2. Per unit system

The variables in the rotor circuits are defined as:

$$V_{fd} = \frac{v_{fd}}{V_{base}^{fd}}, \quad V_{1d} = \frac{v_{1d}}{V_{base}^{1d}}, \quad V_{1q} = \frac{v_{1q}}{V_{base}^{1q}}, \quad V_{2q} = \frac{v_{2q}}{V_{base}^{2q}}$$

$$I_{fd} = \frac{i_{fd}}{I_{base}^{fd}}, \quad I_{1d} = \frac{i_{1d}}{I_{base}^{1d}}, \quad I_{1q} = \frac{i_{1q}}{I_{base}^{1q}}, \quad I_{2q} = \frac{i_{2q}}{I_{base}^{2q}}$$

$$\psi_{fd} = \frac{\lambda_{fd}}{\Lambda_{base}^{fd}}, \quad \psi_{1d} = \frac{\lambda_{1d}}{\Lambda_{base}^{1d}}, \quad \psi_{1q} = \frac{\lambda_{1q}}{\Lambda_{base}^{1q}}, \quad \psi_{2q} = \frac{\lambda_{2q}}{\Lambda_{base}^{2q}}$$

The parameters in per unit are:

$$R_s = \frac{r_s}{Z_{base}^{dqo}}, \quad R_{fd} = \frac{r_{fd}}{Z_{base}^{fd}}, \quad R_{1d} = \frac{r_{1d}}{Z_{base}^{1d}}$$

$$R_{1q} = \frac{r_{1q}}{Z_{base}^{1q}}, \quad R_{2q} = \frac{r_{2q}}{Z_{base}^{2q}}$$

Further details of the bases are in P.W. Sauer, M.A. Pai, Power system dynamics and stability, Prentice Hall, 1998.

# 2. The H-constant of inertia

In power system, it is customary to use a scaled version of the constant of inertia. By definition, the H-constant of inertia is:

$$H = \frac{\frac{1}{2} J \left(\omega_{base} \frac{2}{p}\right)^2}{S_{base}}$$

The advantage of this definition is that it offers a more intuitive interpretation of its value: The H-constant of inertia corresponds to half the time that would require to take the rotor of a synchronous generator from rest to nominal speed at nominal torque. The H-constant of inertia is measured in seconds.

# 2. The dqo model in per unit

$$
\begin{aligned}
\frac{1}{\omega_s}\frac{d\psi_d}{dt} &= R_s I_d + \frac{\omega}{\omega_s}\psi_q + V_d \\
\frac{1}{\omega_s}\frac{d\psi_q}{dt} &= R_s I_q - \frac{\omega}{\omega_s}\psi_d + V_q \\
\frac{1}{\omega_s}\frac{d\psi_o}{dt} &= R_s I_o + V_o \\
\frac{1}{\omega_s}\frac{d\psi_{fd}}{dt} &= -R_{fd} I_{fd} + V_{fd} \\
\frac{1}{\omega_s}\frac{d\psi_{1d}}{dt} &= -R_{1d} I_{1d} + V_{1d} \\
\frac{1}{\omega_s}\frac{d\psi_{1q}}{dt} &= -R_{1q} I_{1q} + V_{1q} \\
\frac{1}{\omega_s}\frac{d\psi_{2q}}{dt} &= -R_{2q} I_{2q} + V_{2q} \\
\frac{d\delta}{dt} &= \omega - \omega_s \\
\frac{2H}{\omega_s}\frac{d\omega}{dt} &= T_M - T_E - T_{FW} \\
\text{where} \quad & T_E = \psi_d I_q - \psi_q I_d
\end{aligned}
$$

All variables and parameters are in per unit, except $\omega$ in rad/s, $\omega_s \approx 376.9911$ rad/s, $\delta$ in radians and $H$ in seconds.

# 2. Relation of voltage/current in dqo with grid phasors

Assume three-phase voltages and currents are perfectly sinusoidal and balanced:

$$
\begin{aligned}
v_a &= \sqrt{2}V_S \cos\left(\omega_s t + \theta_S\right) & i_a &= \sqrt{2}I_S \cos\left(\phi_s t + \theta_S\right)\\
v_b &= \sqrt{2}V_S \cos\left(\omega_s t + \theta_S - \frac{2\pi}{3}\right) & i_b &= \sqrt{2}I_S \cos\left(\omega_s t + \phi_S - \frac{2\pi}{3}\right)\\
v_c &= \sqrt{2}V_S \cos\left(\omega_s t + \theta_S + \frac{2\pi}{3}\right) & i_c &= \sqrt{2}I_S \cos\left(\omega_s t + \phi_S + \frac{2\pi}{3}\right)
\end{aligned}
$$

By Park's transformation:

$$
\begin{aligned}
v_d &= V_S \sin\left(\frac{p}{2}\theta_{shaft} - \omega_s t - \theta_S\right) & i_d &= I_S \sin\left(\frac{p}{2}\theta_{shaft} - \omega_s t - \phi_S\right)\\
v_q &= V_S \cos\left(\frac{p}{2}\theta_{shaft} - \omega_s t - \theta_S\right) & i_q &= I_S \cos\left(\frac{p}{2}\theta_{shaft} - \omega_s t - \phi_S\right)\\
v_o &= 0 & i_o &= 0
\end{aligned}
$$

# 2. Relation of voltage/current in dqo with grid phasors

As $\delta = \frac{p}{2}\theta_{\text{shaft}} - \omega_s t$, then

$$v_d = V_S \sin(\delta - \theta_S) \qquad v_q = V_S \cos(\delta - \theta_S)$$

Finally, the relation of voltage and current in dqo with grid phasors is:

$$\begin{aligned}
(v_q - jv_d) &= V_S \cos(\delta - \theta_S) - jV_S \sin\ (\delta - \theta_S) \\
&= V_S \cos(\theta_S - \delta) + jV_S \sin\ (\theta_S - \delta) \\
&= V_S e^{j(\theta_S - \delta)} = V_S e^{j\theta_S} e^{-j\delta} \\
\Rightarrow V_S e^{j\theta_S} &= (v_q - jv_d)\, e^{j\delta} \\
V_S e^{j\theta_S} &= -j\,(v_d + jv_q)\, e^{j\delta} \\
&\boxed{V_S e^{j\theta_S} = (v_d + jv_q)\, e^{j\left(\delta - \frac{\pi}{2}\right)}}
\end{aligned}$$

In the case of stator currents: $\boxed{I_S e^{j\phi_S} = (i_d + ji_q)\, e^{j\left(\delta - \frac{\pi}{2}\right)}}$

# 2. Linear magnetic circuit

$$\lambda_{abc} = L_{ss}\left(\theta_{shaft}\right) i_{abc} + L_{sr}\left(\theta_{shaft}\right) i_{rotor}$$
$$\lambda_{rotor} = L_{rs}\left(\theta_{shaft}\right) i_{abc} + L_{rr}\left(\theta_{shaft}\right) i_{rotor}$$

where $i_{rotor} = \left[i_{fd}, i_{1d}, i_{1q}, i_{2q}\right]^T$ and $\lambda_{rotor} = \left[\lambda_{fd}, \lambda_{1d}, \lambda_{1q}, \lambda_{2q}\right]^T$. The inductance matrices $L_{ss} \in \mathbb{R}^{3\times 3}$ and $L_{sr} = L_{rs}^T \in \mathbb{R}^{3\times 4}$ depend on the angle $\theta_{shaft}$, however, the matrix $L_{rr} \in \mathbb{R}^{4\times 4}$ is independent of $\theta_{shaft}$. By applying Park's transformation:

$$\lambda_{abc} = L_{ss} i_{abc} + L_{sr} i_{rotor}$$
$$\Rightarrow T\lambda_{abc} = T L_{ss} i_{abc} + T L_{sr} i_{rotor}$$
$$\Rightarrow \boxed{\lambda_{dqo} = T L_{ss} T^{-1} i_{dqo} + T L_{sr} i_{rotor}}$$
$$\lambda_{rotor} = L_{rs} i_{abc} + L_{rr} i_{rotor}$$
$$\Rightarrow \boxed{\lambda_{rotor} = L_{rs} T^{-1} i_{dqo} + L_{rr} i_{rotor}}$$

We are assuming that $T L_{ss} T^{-1}$, $T L_{sr}$ and $L_{rs} T^{-1}$ are all independent of $\theta_{shaft}$.

# 2. Linear magnetic circuit

$$\lambda_d = (L_{\ell s} + L_{md})i_d + L_{sfd}i_{fd} + L_{s1d}i_{1d}$$
$$\lambda_{fd} = \frac{3}{2}L_{sfd}i_d + L_{fdfd}i_{fd} + L_{fd1s}i_{1d}$$
$$\lambda_{1d} = \frac{3}{2}L_{s1d}i_d + L_{fd1d}i_{fd} + L_{1d1d}i_{1d}$$

$$\lambda_q = (L_{\ell s} + L_{mq})i_q + L_{s1q}i_{1q} + L_{s2q}i_{2q}$$
$$\lambda_{1q} = \frac{3}{2}L_{s1q}i_q + L_{1q1q}i_{1q} + L_{1q2q}i_{2q}$$
$$\lambda_{2q} = \frac{3}{2}L_{s2q}i_q + L_{1q2q}i_{1q} + L_{2q2q}i_{2q}$$
$$\lambda_o = L_{\ell s}i_o$$

The definition of inductance is as usual, except for $L_{md}$ y $L_{mq}$. Let's consider a linear approximation for the self and mutual stator inductance:

$$L_{aa} = L_{\ell s} + L_A - L_B \cos\left(\frac{p}{2}\theta_{shaft}\right)$$
$$L_{ab} = -\frac{1}{2}L_A - L_B \cos\left(\frac{p}{2}\theta_{shaft} - \frac{2\pi}{3}\right)$$
$$L_{ac} = -\frac{1}{2}L_A - L_B \cos\left(\frac{p}{2}\theta_{shaft} + \frac{2\pi}{3}\right)$$

Then, after the Park's transformation, $L_{md} = \frac{3}{2}(L_A + L_B)$ y $L_{mq} = \frac{3}{2}(L_A - L_B)$

# 2. Linear magnetic circuit

Further considerations:

1. Adequate bases must be chosen for the rotor circuits (bases that make the equivalent circuit as symmetrical as possible are sought)
2. Apply terminal conditions to eliminate rotor currents from the model.
3. Several parameters and variables definitions are required to make the model compact.

Some definitions are:

$$E'_q = \frac{X_{md}}{X_{fd}}\psi_{fd} \qquad E'_d = -\frac{X_{mq}}{X_{1q}}\psi_{1q} \qquad E_{fd} = \frac{X_{md}}{R_{fd}}V_{fd}$$

where

$$X_{md} = \frac{\omega_s L_{md}}{Z_{base}^{dqo}} \qquad X_{mq} = \frac{\omega_s L_{mq}}{Z_{base}^{dqo}}$$

$$X_{fd} = \frac{\omega_s L_{fdfd}}{Z_{base}^{fd}} \qquad X_{1q} = \frac{\omega_s L_{1q1q}}{Z_{base}^{1q}}$$

# 2. Resulting SG dynamic model

$$\frac{1}{\omega_s}\frac{d\psi_d}{dt} = R_s I_d + \frac{\omega}{\omega_s}\psi_q + V_d$$
$$\frac{1}{\omega_s}\frac{d\psi_q}{dt} = R_s I_q - \frac{\omega}{\omega_s}\psi_d + V_q$$
$$\frac{1}{\omega_s}\frac{d\psi_o}{dt} = R_s I_o + V_o$$
$$\frac{d\delta}{dt} = \omega - \omega_s$$
$$\frac{2H}{\omega_s}\frac{d\omega}{dt} = T_M - T_E - T_{FW}$$

with $T_E = \psi_d I_q - \psi_q I_d$

$$T'_{do}\frac{dE'_q}{dt} = -E'_q - (X_d - X'_d)\Big[I_d - \frac{(X'_d - X''_d)}{(X'_d - X_{\ell s})^2}(\psi_{1d} + (X'_d - X_{\ell s})I_d - E'_q)\Big] + E_{fd}$$
$$T'_{qo}\frac{dE'_d}{dt} = -E'_d + (X_q - X'_q)\Big[I_q - \frac{(X'_q - X''_q)}{(X'_q - X_{\ell s})^2}(\psi_{2q} + (X'_q - X_{\ell s})I_q + E'_d)\Big]$$
$$T''_{do}\frac{\psi_{1d}}{dt} = -\psi_{1d} + E'_q - (X'_d - X_{\ell s})I_d$$
$$T''_{qo}\frac{\psi_{2q}}{dt} = -\psi_{2q} - E'_d - (X'_q - X_{\ell s})I_q$$

$$\psi_d = -X''_d I_d + \frac{(X''_d - X_{\ell s})}{(X'_d - X_{\ell s})}E'_q + \frac{(X'_d - X''_d)}{(X'_d - X_{\ell s})}\psi_{1d}$$
$$\psi_q = -X''_q I_q - \frac{(X''_q - X_{\ell s})}{(X'_q - X_{\ell s})}E'_d + \frac{(X'_q - X''_q)}{(X'_q - X_{\ell s})}\psi_{2q}$$
$$\psi_o = -X_{\ell s} I_o$$

# 2. SG steady-state model

We must set to zero all differential equations and find the fundamental relationships among the remaining variables. By inspection of the previous dynamic model, we have:

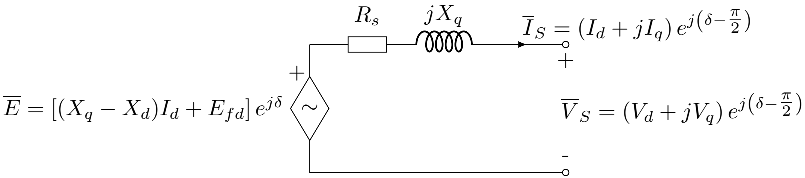
$$
\begin{aligned}
V_d &= -R_s I_d - \psi_q & \psi_d &= -X'_d I_d + E'_q \\
V_q &= -R_s I_q + \psi_d & \psi_q &= -X'_q I_q - E'_d \\
E'_q &= -(X_d - X'_d) I_d + E_{fd} & \Rightarrow V_d &= -R_s I_d + X'_q I_q + E'_d \\
E'_d &= (X_q - X'_q) I_q & V_q &= -R_s I_q - X'_d I_d + E'_q
\end{aligned}
$$

By using the relationship between dqo variables and phasor, the following complex equation is obtained:

$$
\begin{aligned}
(V_d + jV_q) e^{j(\delta - \frac{\pi}{2})} &= -(R_s + jX_q)(I_d + jI_q) e^{j(\delta - \frac{\pi}{2})} + \overline{E} \\
\overline{E} &= j\left[(X_q - X'_d) I_d + E'_q\right] e^{j(\delta - \frac{\pi}{2})} \\
&= \left[(X_q - X_d) I_d + E_{fd}\right] e^{j\delta}
\end{aligned}
$$

# 2. SG steady-state model

When it comes to determine the steady-state condition, it is customary to use the term "finding the angle of the voltage behind the quadrature reactance" (assuming the stator resistance is zero)




If $\overline{I}_S$ and $\overline{V}_S$ are known (after solving a power flow), the variables $\delta$, $I_d$, $I_q$ and $E_{fd}$ are calculated as:

$$\delta = \sphericalangle \overline{E}$$

$$I_q - jI_d = \overline{I}_S e^{-j\delta} \quad \Rightarrow \quad I_q = \mathrm{Re}\left(\overline{I}_S e^{-j\delta}\right) \text{ and } I_d = -\,\mathrm{Imag}\left(\overline{I}_S e^{-j\delta}\right)$$

$$E_{fd} = |\overline{E}| - (X_q - X_d)I_d$$

# Automatic voltage regulator (AVR)

# 3. IEEE type-1 exciter

There are different types of exciters that can be classified for their performance, response, and physical characteristics. In this section, we will review the IEEE type-1 exciter, which uses d.c. generators as an amplifier and exciter.

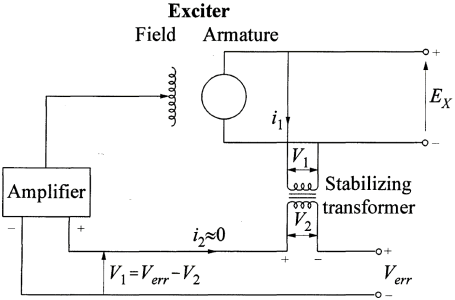
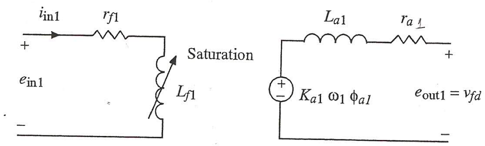



*Figure obtained from [Kundur]*

# 3. IEEE type-1 exciter

## Exciter

This is a d.c. rotatory exciter. Consider a generator represented through its equivalent circuit:




Neglecting $L_{a1}$ and $r_{a1}$, the following relationships are obtained:

$$e_{in1} = i_{in1} r_{f1} + \frac{d\lambda_{f1}}{dt} \qquad \lambda_{f1} = N_{f1}\phi_{f1}$$

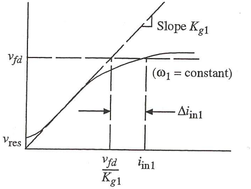
$$v_{fd} = K_{a1}\omega_1\phi_{a1} \qquad \phi_{f1} = \sigma_1\phi_{a1}$$

where $\sigma_1$ is a coefficient of dispersion.

*Figure obtained from [Sauer & Pai]*

# 3. IEEE type-1 exciter

As $\omega_1$ is constant, then $\lambda_{f1} = \frac{N_{f1}\sigma_1}{K_{a1}\omega_1} v_{fd}$. Considering the core magnetic characteristic, the following relationship between voltage and current is obtained:




Thus, $v_{fd} = K_{g1} i_{in1} = \frac{K_{a1}\omega_1}{N_{f1}\sigma_1}\lambda_{f1} = \underbrace{\frac{K_{a1}\omega_1}{N_{f1}\sigma_1} L_{f1us}}_{K_{g1}} i_{in1} = \frac{K_{g1}}{L_{f1}}\lambda_{f1}$—the subscript $us$ refers to the unsaturated characteristic. To represent saturation, consider $f_{sat}(v_{fd}) = \frac{\Delta i_{in1}}{v_{fd}}$. As a result,

$$i_{in1} = \frac{v_{fd}}{K_{g1}} + f_{sat}(v_{fd}) v_{fd}$$

*Figure obtained from [Sauer & Pai]*

# 3. IEEE type-1 exciter

Replacing these definitions in the exciter fundamental equation:

$$e_{in1} = \frac{r_{f1}}{K_{g1}} v_{fd} + r_{f1} f_{sat}(v_{fd}) v_{fd} + \frac{L_{f1us}}{K_{g1}} \frac{dv_{fd}}{dt}$$

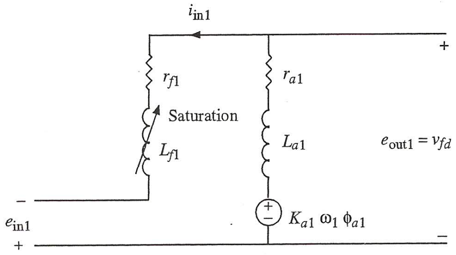
Using the base of the field circuit, this fundamental equation becomes:

$$\underbrace{\frac{X_{md}}{R_{fd}} \frac{e_{in1}}{V_{base}^{fd}}}_{V_R} = \underbrace{\frac{r_{f1}}{K_{g1}}}_{K_E^{sep}} \underbrace{\frac{X_{md}}{R_{fd}} \frac{v_{fd}}{V_{base}^{fd}}}_{E_{fd}} + \underbrace{r_{f1} f_{sat}\left(\frac{R_{fd} V_{base}^{fd}}{X_{md}} E_{fd}\right)}_{S_E(E_{fd})} E_{fd} + \underbrace{\frac{L_{f1us}}{K_{g1}}}_{T_E} \frac{dE_{fd}}{dt}$$

$$\Rightarrow T_E \frac{dE_{fd}}{dt} = -\left(K_E^{sep} + S_E(E_{fd})\right) E_{fd} + V_R$$

# 3. IEEE type-1 exciter

The field of the exciter can be self-excited:




It can be proved that the model is similar to the case of separately excited exciter, except that $K_E^{self} = K_E^{sep} - 1$

$$T_E \frac{dE_{fd}}{dt} = -\left(K_E^{self} + S_E(E_{fd})\right) E_{fd} + V_R$$

*Figure obtained from [Sauer & Pai]*

# 3. IEEE type-1 exciter

## Voltage regulator

| | |
|---|---|
| Exciter | Control voltage at the terminals of the synchronous generator by applying a proper excitation voltage ($E_{fd}$) |
| Regulator | Allows the automatic voltage control by comparing the signals with a reference and by amplifying the error signal. |

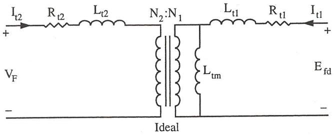
A measure of the terminal voltage is compared with a reference; then the error is amplified and sent as input to the exciter ($V_R$). The amplification can be done through a pilot exciter (another d.c. generator) or a static converter. Either way, this amplification is traditionally represented through a first order model, as it was done during the exciter modeling.

$$T_A \frac{V_R}{dt} = -V_R + K_A V_{in}$$

$$V_R^{min} \leq V_R \leq V_R^{max}$$

# 3. IEEE type-1 exciter

It is customary to use an stabilizing transformer that is connected between the output of the exciter and the input of the amplifier (comparing stage). The error voltage sent as input to the amplifier results in $V_{error} = V_t - V_{ref} - V_{stab}$. The model of the stabilizing loops is:




Assume that $L_{t2}$ is large and $I_{t2} = 0$ initially, thus, $I_{t2}$ remains close to zero. As a result, the model of the stabilizing loop becomes:

$$E_{fd} = R_{t1} I_{t1} + (L_{t1} + L_{tm}) \frac{dI_{t1}}{dt}$$

$$V_F = \frac{N_2}{N_1} L_{tm} \frac{dI_{t1}}{dt}$$

*Figure obtained from [Sauer & Pai]*

# 3. IEEE type-1 exciter

By combining these equations and differentiating $V_F$ with respect to time:

$$\frac{dV_F}{dt} = \frac{N_2}{N_1} L_{tm} \left( \frac{1}{(L_{t1} + L_{tm})} \left( \frac{dE_{fd}}{dt} - \frac{R_{t1}}{L_{tm}} \frac{N_2}{N_1} V_F \right) \right)$$

Define:

$$T_F = \frac{L_{t1} + L_{tm}}{R_{t1}} \qquad K_F = \frac{N_1}{N_2} \frac{L_{tm}}{R_{t1}}$$

Thus,

$$T_F \frac{dV_F}{dt} = -V_F + K_F \left( -\frac{K_E + S_E(E_{fd})}{T_E} E_{fd} + \frac{V_R}{T_E} \right)$$

By using $R_f = \frac{K_F}{T_F} E_{fd} - V_F$ as an intermediate variable, we obtain:

$$T_F \frac{dR_f}{dt} = -R_f + \frac{K_F}{T_F} E_{fd}$$

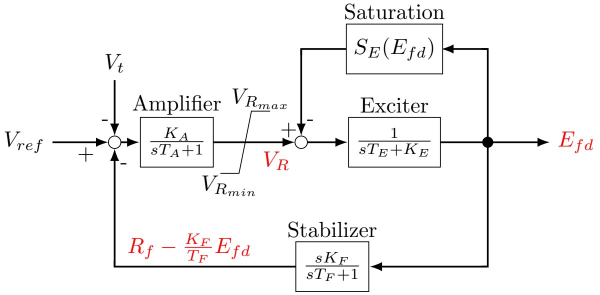
# 3. IEEE type-1 exciter

In summary:

$$T_E \frac{dE_{fd}}{dt} = -\left(K_E + S_E(E_{fd})\right) E_{fd} + V_R$$

$$T_A \frac{V_R}{dt} = -V_R + +K_A R_f - \frac{K_A K_F}{T_F} E_{fd} + K_A \left(V_{ref} - V_t\right)$$

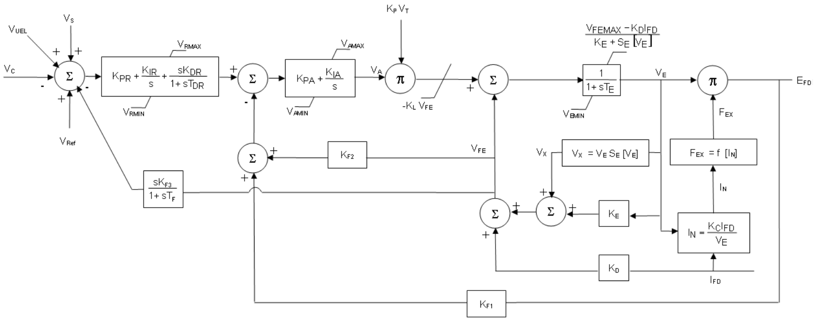
$$T_F \frac{dR_f}{dt} = -R_f + \frac{K_F}{T_F} E_{fd} \quad \text{with} \quad V_R^{min} \leq V_R \leq V_R^{max}$$



# 3. Other exciter models

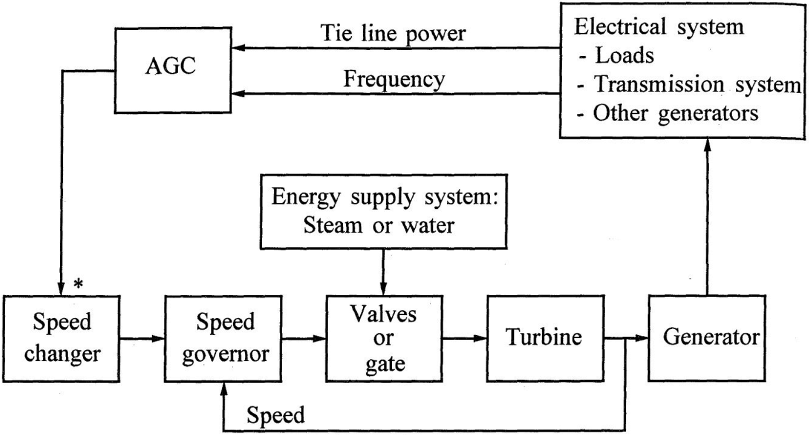
Other types of exciters can be found in "IEEE Recommended Practice for Excitation System Models for Power System Stability Studies," in IEEE Std 421.5-2016 (Revision of IEEE Std 421.5-2005), 2006. For example:

**Exciter type-AC7B**:



# Speed control

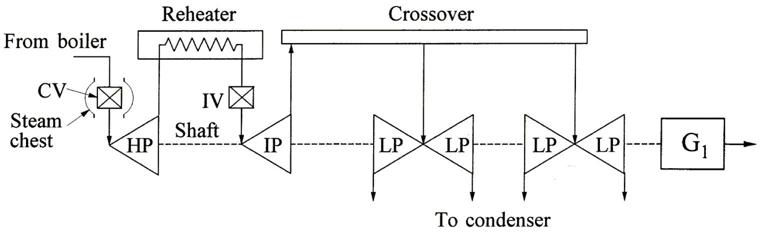
# 4. Sketch of power generation and control




* AGC applied only to selected units

# 4. Steam turbine

Sketch of a steam turbine system:




The system can be classified in two types: (a) *tandem compound*, all turbines are coupled to the same shaft, and (b) *cross compound*, the individual turbines are separated based on their pressure levels and are connected to different generators.

*Figure obtained from [Kundur]*

# 4. Steam turbine

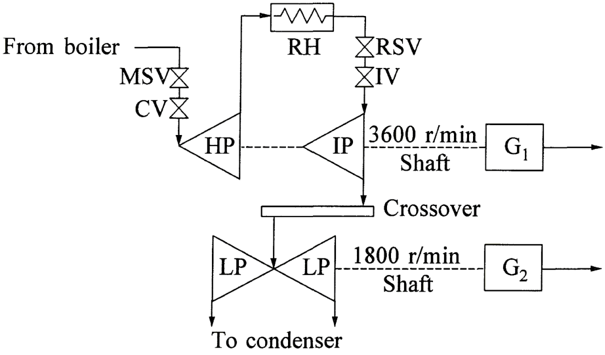



*Figure obtained from [Kundur]*

# 4. Steam turbine—A model

Let's consider a high and low pressure steam turbines in tandem compound, including a reheating stage. For simplicity, assume a linear model:

*Steam chest*:

$$T_{CH}\frac{dP_{CH}}{dt} = -P_{CH} + P_{SV}$$

where $P_{SV}$ is the position of the steam valve at the steam chest, and $P_{CH}$ and $T_{CH}$ are the power output and time constant of the steam chest. Consider that a fraction of the power output is converted into torque through the high pressure turbine, $T_{HP} = K_{HP}P_{CH}$, and the remaining power is sent to the reheater, $(1 - K_{HP})P_{CH}$

# 4. Steam turbine—A model

*HP turbine*:

$$\frac{d\delta_{HP}}{dt} = \omega_{HP} - \omega_s$$
$$\frac{2H_{HP}}{\omega_s}\frac{d\omega_{HP}}{dt} = T_{HP} - T_{HL}$$

where $T_{HL}$ is the torque transmitted though the shaft to the low pressure turbine, $H_{HP}$ is the H-constant of inertia of the turbine, $\omega_{HP}$ is the angular speed of the turbine, and $\delta_{HP}$ is the corresponding angle. The connection of the shaft has some degree of flexibility, which is represented by $T_{HL} = -K_{HL}(\delta_{LP} - \delta_{HP})$

*Reheat stage*:

$$T_{RH}\frac{dP_{RH}}{dt} = -P_{RH} + (1 - K_{HP})P_{CH}$$

where $P_{RH}$ is the power output of the reheater, and $T_{RH}$ is its corresponding time constant.

# 4. Steam turbine—A model

*LP turbine*:

$$\frac{d\delta_{LP}}{dt} = \omega_{LP} - \omega_s$$
$$\frac{2H_{LP}}{\omega_s}\frac{d\omega_{LP}}{dt} = T_{HL} + T_{LP} - T_M$$

where $T_{LP}$ is the torque from the low pressure turbine, $H_{LP}$ is the H-constant of inertia of the low pressure turbine, $\omega_{LP}$ is the angular speed of the turbine, and $\delta_{LP}$ is its corresponding angle. Assume that the power output of the reheater, $P_{RH}$, is converted completely in torque, thus, $T_{LP} = P_{RH}$. Here, we are assuming again there is a flexible shaft between the low pressure turbine and the electrical generator. Thus,

$$T_M = -K_{LM}(\delta - \delta_{LP})$$

# 4. Steam turbine—A model

In summary,

$$
\begin{aligned}
T_{CH}\frac{dP_{CH}}{dt} &= -P_{CH} + P_{SV} \\
\frac{d\delta_{HP}}{dt} &= \omega_{HP} - \omega_s \\
\frac{2H_{HP}}{\omega_s}\frac{d\omega_{HP}}{dt} &= T_{HP} + K_{HL}(\delta_{LP} - \delta_{HP}) \\
T_{RH}\frac{dP_{RH}}{dt} &= -P_{RH} + (1 - K_{HP})P_{CH} \\
\frac{d\delta_{LP}}{dt} &= \omega_{LP} - \omega_s \\
\frac{2H_{LP}}{\omega_s}\frac{d\omega_{LP}}{dt} &= -K_{HL}(\delta_{LP} - \delta_{HP}) + P_{RH} + K_{LM}(\delta - \delta_{LP})
\end{aligned}
$$

# 4. Steam turbine—Simplified model

If we assume that the shaft is rigid, the model becomes

$$\begin{aligned}
&\delta_{HP} = \delta_{LP} = \delta \\
&K_{HP}P_{CH} = T_{HP} = T_{HL} \\
&T_{HL} + T_{LP} = T_{HL} + P_{RH} = T_M \\
&\Rightarrow K_{HP}P_{CH} = T_M - P_{RH} \\
&\Rightarrow K_{HP}\frac{dP_{CH}}{dt} = \frac{dT_M}{dt} - \frac{dP_{RH}}{dt}
\end{aligned}$$

Substituting,

$$\begin{aligned}
T_{RH}\frac{dP_{RH}}{dt} &= -P_{RH} + (1 - K_{HP})P_{CH} \\
T_{RH}\frac{dT_M}{dt} - T_{RH}K_{HP}\frac{dP_{CH}}{dt} &= -T_M + K_{HP}P_{CH} + P_{CH} - K_{HP}P_{CH} \\
T_{RH}\frac{dT_M}{dt} &= -T_M + P_{CH} + T_{RH}K_{HP}\underbrace{\left[\frac{1}{T_{CH}}(-P_{CH} + P_{SV})\right]}_{\frac{dP_{CH}}{dt}}
\end{aligned}$$

# 4. Steam turbine—Simplified model

$$T_{RH}\frac{dT_M}{dt} = -T_M + \left(1 - \frac{T_{RH}K_{HP}}{T_{CH}}\right)P_{CH} + \frac{T_{RH}K_{HP}}{T_{CH}}P_{SV}$$
$$T_{CH}\frac{dP_{CH}}{dt} = -P_{CH} + P_{SV}$$

When there is no reheater, simply make $T_{RH} = 0$ to obtain $T_M = P_{CH}$. Thus

$$T_{CH}\frac{dT_M}{dt} = -T_M + P_{SV}$$

# 4. Governor (speed control)

Now, we study how to control $P_{SV}$.

$$\Delta y_b = K_{ba}\Delta y_a + K_{bc}\Delta y_c$$
$$\Delta y_d = K_{dc}\Delta y_c + K_{de}\Delta y_e$$

$$\Delta P_C = K_a \Delta y_a$$
$$\Delta \omega = \omega_s K_b \Delta y_b$$

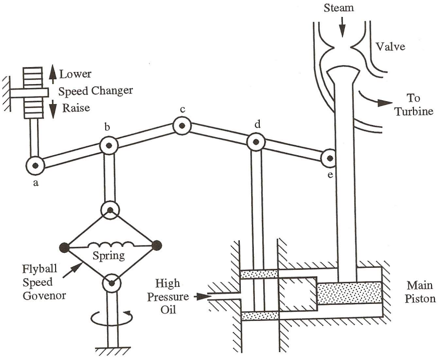



*Figure obtained from [Sauer & Pai]*

# 4. Governor (speed control)

Considering a time delay associated with the servo-mechanism:

$$\frac{d\Delta y_e}{dt} = -K_e \Delta y_d$$

$$\Rightarrow \frac{d\Delta y_e}{dt} = -K_e \left(K_{dc}\Delta y_c + K_{de}\Delta y_e\right)$$

$$= -K_e K_{dc}\left(\frac{\Delta y_b - K_{ba}\Delta y_a}{K_{bc}}\right) - K_e K_{de}\Delta y_e$$

$$= -\frac{K_e K_{dc}}{K_{bc}K_b}\frac{\Delta\omega}{\omega_s} + \frac{K_e K_{dc} K_{ba}}{K_{bc}K_a}\Delta P_C - K_e K_{de}\Delta y_e$$

Define $\Delta P_{SV}$ to be proportional to $\Delta y_e$ as:

$$\Delta y_e = \frac{K_{dc}K_{ba}}{K_{de}K_{bc}K_a}\Delta P_{SV}$$

# 4. Governor (speed control)

$$\frac{K_{dc}K_{ba}}{K_{de}K_{bc}K_a}\frac{d\Delta P_{SV}}{dt} = -\frac{K_e K_{dc}}{K_{bc}K_b}\frac{\Delta\omega}{\omega_s} + \frac{K_e K_{dc}K_{ba}}{K_{bc}K_a}\Delta P_C - \frac{K_e K_{de}K_{dc}K_{ba}}{K_{de}K_{bc}K_a}\Delta P_{SV}$$

Define $T_{SV} = \frac{1}{K_e K_{de}}$, thus

$$\left(\frac{1}{K_e K_{de}}\right)\frac{K_{ba}}{K_a}\frac{d\Delta P_{SV}}{dt} = -\frac{1}{K_b}\frac{\Delta\omega}{\omega_s} + \frac{K_{ba}}{K_a}\Delta P_C - \frac{K_{ba}}{K_a}\Delta P_{SV}$$

$$\Rightarrow T_{SV}\frac{d\Delta P_{SV}}{dt} = -\frac{K_a}{K_b K_{ba}}\frac{\Delta\omega}{\omega_s} + \Delta P_C - \Delta P_{SV}$$

Define $droop = \frac{K_{ba}K_b}{K_a}\frac{\omega_s}{2\pi}\left[\frac{\mathrm{Hz}}{\mathrm{p.u.-MVA}}\right]$

$$T_{SV}\frac{d\Delta P_{SV}}{dt} = -\Delta P_{SV} + \Delta P_C - \frac{\omega_s}{2\pi droop}\frac{\Delta\omega}{\omega_s}$$

# 4. Governor (speed control)

Finally, define $R_D = \frac{2\pi droop}{\omega_s}$, thus

$$T_{SV}\frac{d\Delta P_{SV}}{dt} = -\Delta P_{SV} + \Delta P_C - \frac{1}{R_D}\frac{\Delta\omega}{\omega_s}$$
$$\Rightarrow T_{SV}\frac{dP_{SV}}{dt} = -P_{SV} + P_C - \frac{1}{R_D}\left(\frac{\omega}{\omega_s} - 1\right)$$

The coefficient $R_D$ is defined as the speed regulation constant. For example, consider that $R_D = 0.05$ (5% of regulation) and, moreover, consider that the generator is at no-load, $P_{SV} = P_C = 0$. Now, if $P_C$ (*set-point*) is set to zero, when the generator is loaded at nominal conditions, $P_{SV} = 1$ p.u., the speed of the generator will be:

$$T_{SV}\frac{dP_{SV}}{dt} = 0 = -1 + 0 - \frac{1}{R_D}\left(\frac{\omega}{\omega_s} - 1\right)$$
$$\Rightarrow R_D = 1 - \frac{\omega}{\omega_s} \;\Rightarrow\; \omega = (1 - R_D)\omega_s$$

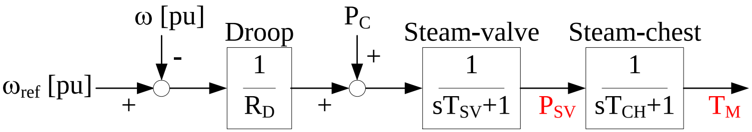
# 4. Simplified governor & steam turbine model

In summary, a single steam turbine (without reheater, and intermediate and low pressure turbines) with a linear controller is modeled through the following set of equations:

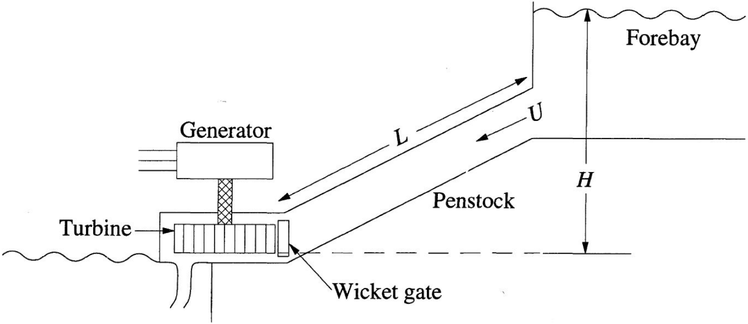
$$T_{CH}\frac{dT_M}{dt} = -T_M + P_{SV}$$

$$T_{SV}\frac{dP_{SV}}{dt} = -P_{SV} + P_C - \frac{1}{R_D}\left(\frac{\omega}{\omega_s} - 1\right)$$

$$\text{with } \; 0 \leq P_{SV} \leq P_{SV}^{max}$$



# 4. Hydro turbine




$$U = K_u G \sqrt{H}$$

*Figure obtained from [Kundur]*

- $U$ Water speed
- $G$ Gate position
- $H$ Hydraulic head at gate
- $K_u$ Constant of proportionality

# 4. Hydro turbine

Consider an operating point of reference $U_0 = K_U G_0 \sqrt{H_0}$. Linearizing around this point:

$$\Delta U = \left.\frac{\partial U}{\partial H}\right|_0 \Delta H + \left.\frac{\partial U}{\partial G}\right|_0 \Delta G$$

where:

$$\left.\frac{\partial U}{\partial H}\right|_0 = \frac{K_u G_0}{2\sqrt{H_0}} = \frac{U_0}{2H_0}$$

$$\left.\frac{\partial U}{\partial G}\right|_0 = K_U\sqrt{H_0} = \frac{U_0}{G_0}$$

$$\Rightarrow \Delta U = \frac{U_0}{2H_0}\Delta H + \frac{U_0}{G_0}\Delta G \quad \Rightarrow \quad \frac{\Delta U}{U_0} = \frac{1}{2}\frac{\Delta H}{H_0} + \frac{\Delta G}{G_0}$$

$$\Rightarrow \boxed{\Delta \overline{U} = \frac{1}{2}\Delta \overline{H} + \Delta \overline{G}} \quad \text{(first equation)}$$

# 4. Hydro turbine

Around the operating point, the mechanical power at the turbine is $P_{m0} = K_p H_0 U_0$. Linearizing:

$$\Delta P_m = \left.\frac{\partial P_m}{\partial H}\right|_0 \Delta H + \left.\frac{\partial P_m}{\partial U}\right|_0 \Delta U$$

where:

$$\left.\frac{\partial P_m}{\partial H}\right|_0 = K_p U_0 = \frac{P_{m0}}{H_0}$$

$$\left.\frac{\partial P_m}{\partial G}\right|_0 = K_p H_0 = \frac{P_{m0}}{U_0}$$

$$\Rightarrow \Delta P_m = \frac{P_{m0}}{H_0}\Delta H + \frac{P_{m0}}{U_0}\Delta U \Rightarrow \frac{\Delta P_m}{P_{m0}} = \frac{\Delta H}{H_0} + \frac{\Delta U}{U_0}$$

$$\Delta \overline{P}_m = \Delta \overline{H} + \Delta \overline{U} = \Delta \overline{H} + \frac{1}{2}\Delta \overline{H} + \Delta \overline{G}$$

$$\boxed{\Delta \overline{P}_m = \frac{3}{2}\Delta \overline{H} + \Delta \overline{G}} \quad \text{(second equation)}$$

# 4. Hydro turbine

We must consider the mass of water that is passing through the penstock. By Newton's law, the acceleration of the mass of water due to a change in the hydraulic head is given by:

$$\underbrace{\rho L A}_{\text{mass}} \frac{d\Delta U}{dt} = -A\rho g \Delta H$$

donde

| | |
|---|---|
| $\rho$ | Water density |
| $A$ | Pipe area (cross section of the penstock) |
| $g$ | Gravitational constant |
| $\rho g \Delta H$ | Change in pressure at the turbine gate |

# 4. Hydro turbine

$$\rho L A \frac{d\Delta U}{dt} = -A\rho g \Delta H \Rightarrow \frac{L}{g}\frac{d\Delta U}{dt} = -\Delta H \Rightarrow \frac{LU_0}{gH_0}\frac{d\Delta \overline{U}}{dt} = -\Delta\overline{H}$$

$$\Rightarrow \boxed{T_w \frac{d\Delta U}{dt} = -\Delta \overline{H}} \quad \text{(third equation)} \qquad T_w = \frac{LU_0}{gH_0}$$

By Laplace's transformation:

$$\Delta\overline{H} = 2\Delta\overline{U} - 2\Delta\overline{G} \tag{1}$$

$$T_w s \Delta\overline{U} = -\Delta\overline{H} \tag{2}$$

$$\Delta\overline{P}_m = \frac{3}{2}\Delta\overline{H} + \Delta\overline{G} \tag{3}$$

Substituting (1) in (2) and (3), we have:

$$\Delta\overline{P}_m = 3\Delta\overline{U} - 2\Delta\overline{G} \qquad \Delta\overline{U} = \frac{2}{2+T_w s}\Delta\overline{G}$$

# 4. Hydro turbine

Combining the previous two equations, the following linear model of a hydro turbine is obtained:

$$\boxed{\Delta\overline{P}_m = \frac{(1 - T_w s)}{\left(1 + \frac{T_w}{2} s\right)} \Delta\overline{G}}$$

Nonlinear model. With $\Delta H = H - H_0$, we have:

$$U = K_u G \sqrt{H}$$

$$P = K_p H U$$

$$\frac{L}{g}\frac{d\Delta U}{dt} = \frac{L}{g}\frac{dU}{dt} = -\Delta H \Rightarrow \frac{dU}{dt} = -\frac{g}{L}\Delta H$$

# 4. Hydro turbine

The nominal speed and nominal power of the turbine are given by: $U_r = K_u G_n \sqrt{H_r}$ and $P_r = K_p H_r U_r$.

Referring all variables to their nominal values, we have:

$$\overline{U} = \frac{U}{U_r} = \frac{K_u G \sqrt{H}}{K_u G_r \sqrt{H_r}} = \frac{G}{G_r}\sqrt{\frac{H}{H_r}} = \overline{G}\sqrt{\overline{H}}$$

$$\overline{P} = \frac{P}{P_r} = \frac{K_p H U}{K_p H_r U_r} = \frac{H}{H_r}\frac{U}{U_r} = \overline{H}\,\overline{U}$$

Thus, the dynamic equation in per unit results in:

$$\frac{dU}{dt} = -\frac{g}{L}\Delta H \;\Rightarrow\; \frac{d\overline{U}}{dt} = -\frac{gH_r}{LU_r}(\overline{H} - \overline{H}_0) = -\frac{1}{T_w}(\overline{H} - \overline{H}_0)$$

By Laplace's transformation:

$$\frac{\overline{U}}{\overline{H} - \overline{H}_0} = \frac{-1}{T_w s}$$

# 4. Hydro turbine

The mechanical power output is $P_m = P - P_L$, where $P_L = K_p U_{NL} H$ is the loss power, and $U_{NL}$ the water speed without load. Thus,

$$\overline{P}_m = \frac{P_m}{P_r} = \frac{P}{P_r} - \frac{P_L}{P_r} = \overline{P} - \overline{U}_{NL}\ \overline{H}$$
$$= \overline{U}\ \overline{H} - \overline{U}_{NL}\ \overline{H} = (\overline{U} - \overline{U}_{NL})\ \overline{H}$$

**Torque equation**: In the case that the base power of the turbine, $P_r$, is different to the base power of the system, $P_b$, the mechanical torque must be expressed in the system base. Therefore,

$$\overline{T}_m = \frac{T_m}{T_b} = \frac{\overline{P}_m \frac{P_r}{P_b}}{\frac{\omega}{\omega_b}} = \frac{1}{\omega}(\overline{U} - \overline{U}_{NL})\overline{H}\ \frac{P_r}{P_b} = \frac{1}{\overline{\omega}}\overline{P}_m\ \frac{P_r}{P_b}$$

# 4. Hydro turbine

**Gate model**: Consider the following relationship between the ideal ($\overline{G}$) and real gate opening ($\overline{g}$)

$$\overline{G} = A_t \overline{g}$$

$$A_t = \frac{1}{\overline{g}_{FL} - \overline{g}_{NL}}$$

donde

- $\overline{g}_{FL}$ Gate opening at full load
- $\overline{g}_{NL}$ Gate opening at no load

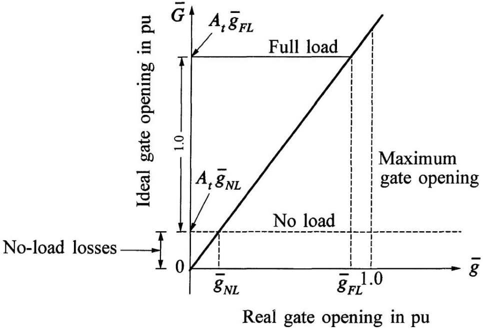



*Figure obtained from [Kundur]*

# 4. Hydro turbine

In summary,

$$\overline{H} = \left(\frac{\overline{U}}{\overline{G}}\right)^2$$
$$\overline{P}_m = (\overline{U} - \overline{U}_{NL})\overline{H}$$

$$\overline{U} = -\left(\frac{1}{T_w s}\right)(\overline{H} - \overline{H}_0) = \left(\frac{1}{T_w s}\right)(\overline{H}_0 - \overline{H})$$

$$\overline{G} = A_t \overline{g}$$
$$\overline{T}_m = \frac{1}{\overline{\omega}} \overline{P}_m \frac{P_r}{P_b}$$

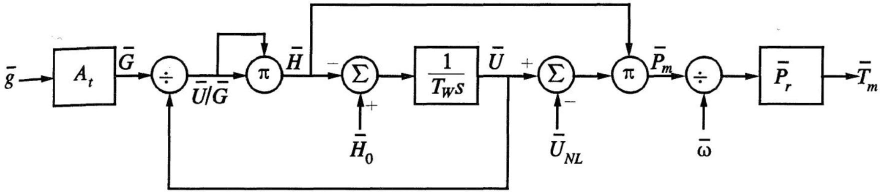
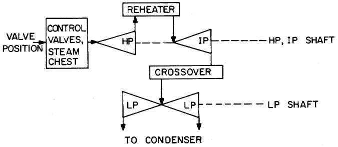



*Figure obtained from [Kundur]*

# 4. Other models

Other turbine models can be consulted in "Dynamic Models for Steam and Hydro Turbines in Power System Studies," Power Apparatus and Systems, IEEE Transactions on , vol.PAS-92, no.6, pp.1904,1915, Nov. 1973. For example:

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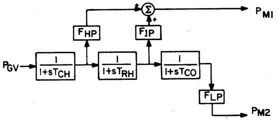


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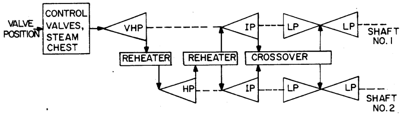


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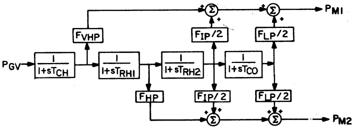
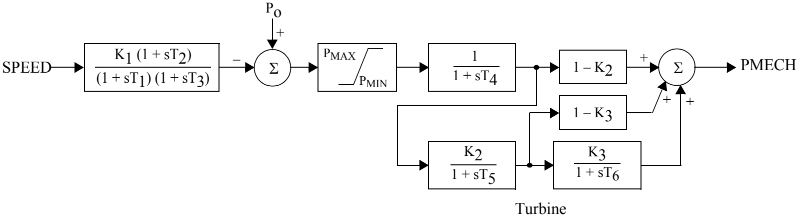



(F)

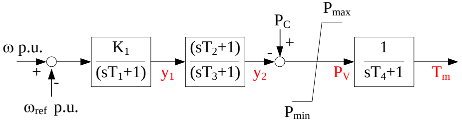
# 4. The IEESGO model

The IEESGO is a general-purpose turbine-governor model. By choosing proper parameters, this model gives a good representation of a steam turbine with reheat stage or an approximate representation of hydro turbine of simple configuration (Source: PSS/E-32.0 Model Library)



# 4. IEESGO—Set of DAEs

Consider a simplified version with $K_2 = K_3 = 0$.




DAEs mode with $\omega_{ref} = 1$ p.u.

$$y_1 = \frac{K_1}{(T_1 s + 1)}(\omega - \omega_{ref})$$

$$\Rightarrow \boxed{T_1 \dot{y}_1 = -y_1 + K_1 (\omega - 1)}$$

$$y_2 = \frac{(T_2 s + 1)}{(T_3 s + 1)} y_1 := (T_2 s + 1) y_3$$

where, $$y_3 = \frac{y_1}{(T_3 s + 1)} \Rightarrow \boxed{T_3 \dot{y}_3 = -y_3 + y_1}$$

# 4. IEESGO—Set of DAEs

Thus,

$$
\begin{aligned}
y_2 = (T_2 s + 1) y_3 \Rightarrow y_2 &= y_3 + T_2 \dot{y}_3 \\
&= y_3 + T_2 \left( \frac{-y_3}{T_3} + \frac{y_1}{T_3} \right) \\
\Rightarrow & \boxed{y_2 = \frac{T_2}{T_3} y_1 + \left( 1 - \frac{T_2}{T_3} \right) y_3}
\end{aligned}
$$

From the last block, we obtain:

$$
T_m = \frac{P_V}{(T_4 s + 1)} \Rightarrow \boxed{T_4 \dot{T}_m = -T_m + P_V}
$$

# 4. IEESGO—Set of DAEs

In summary, the simplified IEESGO model has 3 differential equations and 2 algebraic equations:

$$
\begin{aligned}
T_1\dot{y}_1 &= -y_1 + K_1(\omega - 1) \\
T_3\dot{y}_3 &= -y_3 + y_1 \\
T_4\dot{T}_m &= -T_m + P_V \\
y_2 &= \left(1 - \frac{T_2}{T_3}\right) y_3 + \frac{T_2}{T_3} y_1
\end{aligned}
$$

where:

$$
PV = \begin{cases} P_C - y_2, & P_{min} \leq P_C - y_2 \leq P_{max} \\ P_{max}, & P_C - y_2 \geq P_{max} \\ P_{min}, & P_C - y_2 \leq P_{min} \end{cases}
$$

# 4. IEESGO—Set of DAEs

The variable $P_V$ must consider power limits. However, when $P_V$ hits a limit is a nuisance from the modeling point of view (modifying a differential equation). An alternative to this, it is to move the limits and apply them to an algebraic equation as follow:

Define,

$$P_V = P_C - y_2$$

$$y_{2i} = \left(1 - \frac{T_2}{T_3}\right) y_3 + \frac{T_2}{T_3} y_1 \text{ (intermediate variable)}$$

# 4. IEESGO—Set of DAEs

Thus:

$$
\begin{aligned}
T_1 \dot{y}_1 &= -y_1 + K_1 (\omega - 1) \\
T_3 \dot{y}_3 &= -y_3 + y_1 \\
T_4 \dot{T}_m &= -T_m + P_C - y_2 \\
y_{2i} &= \left(1 - \frac{T_2}{T_3}\right) y_3 + \frac{T_2}{T_3} y_1
\end{aligned}
$$

Applying limits over the algebraic variable $y_2$

$$
y_2 = \begin{cases} P_C - P_{max}, & P_{min} > P_C - y_{2i} \\ P_C - P_{max}, & P_{max} < P_C - y_{2i} \\ y_{2i}, & P_{min} \leq P_C - y_{2i} \leq P_{max} \end{cases}
$$

# Standard models

# 5. Need for simplified models

1. Power systems are large scale.
2. Power systems are dynamic stiff systems: while electromagnetic dynamics are typically in the range of milliseconds, the electromechanical dynamics are in the range of seconds.
3. Stiff systems are hard to solve due to the need of very small time steps, that can considerably increase the simulation time.
4. We use time scale separation, we focus on the electromechanical dynamics and represent fast dynamics as infinitely fast dynamics (algebraic variables that can change their values instantaneously if required)
5. Stator flux dynamics are represented by algebraic equations
6. The rest of the grid (lines, transformer, others) are represented through algebraic equations (their dynamics are assumed infinitely fast)

# Standard models

## (Brief description of timescale separation)

# 5. Timescale separation

General concept. Consider the following generic system,

$$\dot{x} = f(x, z) \qquad x(0) = x^o$$
$$\dot{z} = g(x, z) \qquad z(0) = z^o$$

where $x \in \mathbb{R}^{n\times 1}$, and $z \in \mathbb{R}^{m\times 1}$. Assume that the dynamics of $x$ and $z$ occur at different timescales, e.g., milliseconds and seconds.

To decouple these dynamics, we seek for an integral manifold for $z = h(x)$ which satisfies the differential equation of $z$. Thus,

$$\dot{z} = \frac{\partial z}{\partial x}\dot{x} = \frac{\partial h}{\partial x} f\left(x, h(x)\right) = g\left(x, h(x)\right)$$

If the initial condition belongs to the manifold, $z^o = h(x^o)$, we say that the integral manifold is an exact solution of $\dot{z} = g(x, z)$, and the following reduced order model is an exact model:

$$\boxed{\dot{x} = f\left(x, h(x)\right)}$$

# 5. Timescale separation

A complete linear example. Consider the following second order system (assume that $\varepsilon$ is a small parameter)

$$\begin{aligned} \dot{x} &= -x + z & \qquad x(0) &= x^o \\ \varepsilon\dot{z} &= -x - z & \qquad z(0) &= z^o \end{aligned}$$

# 5. Timescale separation

A complete linear example. Consider the following second order system (assume that $\varepsilon$ is a small parameter)

$$\begin{aligned} \dot{x} &= -x + z & x(0) &= x^o \\ \varepsilon \dot{z} &= -x - z & z(0) &= z^o \end{aligned}$$

The system eigenvalues are:

$$\lambda_{1,2} = \frac{-\epsilon - 1 \mp \sqrt{\epsilon^2 - 6\epsilon + 1}}{2\epsilon}$$

In the limit,

$$\lambda_1 = \lim_{\epsilon \to 0} \frac{-\epsilon - 1 - \sqrt{\epsilon^2 - 6\epsilon + 1}}{2\epsilon} = \frac{-1-1}{2\epsilon} = \frac{-1}{\epsilon} = -\infty$$

$$\lambda_2 = \lim_{\epsilon \to 0} \frac{-\epsilon - 1 + \sqrt{\epsilon^2 - 6\epsilon + 1}}{2\epsilon} = \frac{0}{0} \quad \text{indeterminate!}$$

Applying L'Hopital's rule we can show that $\lambda_2 \to -2$. As a result, this system has a very fast mode $e^{-t/\epsilon}$ and a slow mode $e^{-2t}$—we have two distinctive dynamics!

# 5. Timescale separation

We propose a linear manifold of the form $z = hx$, where $h$ is a real constant. This manifold must satisfy the differential equation of $z$, thus

$$
\begin{aligned}
\epsilon \dot{z} = -x - z \quad & \Rightarrow \varepsilon \frac{\partial z}{\partial x} \dot{x} = -x - hx \\
& \Rightarrow \varepsilon h \left(-x + hx\right) = -x - hx \\
& \Rightarrow \left(-\varepsilon h + \varepsilon h^2\right) x = -(1 + h)x \\
& \Rightarrow \left(-\varepsilon h + \varepsilon h^2\right) = -(1 + h) \\
& \Rightarrow \boxed{\varepsilon h^2 + (1 - \varepsilon)h + 1 = 0}
\end{aligned}
$$

If there exists a solution for h, then the manifold $z = hx$ exists.

# 5. Timescale separation

We propose a linear manifold of the form $z = hx$, where $h$ is a real constant. This manifold must satisfy the differential equation of $z$, thus

$$
\begin{aligned}
\epsilon \dot{z} = -x - z \quad & \Rightarrow \varepsilon \frac{\partial z}{\partial x} \dot{x} = -x - hx \\
& \Rightarrow \varepsilon h \left(-x + hx\right) = -x - hx \\
& \Rightarrow \left(-\varepsilon h + \varepsilon h^2\right) x = -(1+h)x \\
& \Rightarrow \left(-\varepsilon h + \varepsilon h^2\right) = -(1+h) \\
& \Rightarrow \boxed{\varepsilon h^2 + (1-\varepsilon)h + 1 = 0}
\end{aligned}
$$

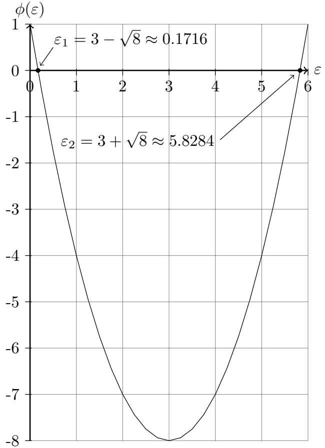
If there exists a solution for h, then the manifold $z = hx$ exists.

The solution for $h$ is given by:

$$
h(\varepsilon) = \frac{-(1-\varepsilon) \pm \sqrt{(1-\varepsilon)^2 - 4\varepsilon}}{2\varepsilon} \in \mathbb{R} \Leftrightarrow \phi(\epsilon) = (1-\varepsilon)^2 - 4\varepsilon \geq 0
$$

# 5. Timescale separation

Roots of $\phi(\varepsilon)$ are given by:
$\phi(\varepsilon) = (1-\varepsilon)^2 - 4\varepsilon = 0$
$\Rightarrow \varepsilon_{1,2} = 3 \mp \sqrt{8}$



# 5. Timescale separation

Roots of $\phi(\varepsilon)$ are given by:
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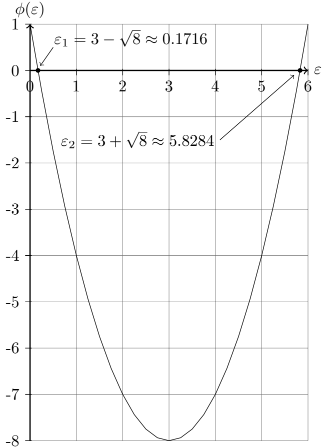
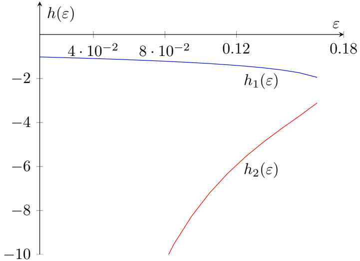



If $\varepsilon < \varepsilon_1$, then we have two solutions for $h$ given by:

$$h_1(\varepsilon) = \frac{-(1-\varepsilon)+\sqrt{(1+\varepsilon)^2-4\varepsilon}}{2\varepsilon}$$

$$h_2(\varepsilon) = \frac{-(1-\varepsilon)-\sqrt{(1-\varepsilon)^2-4\varepsilon}}{2\varepsilon}$$




In general, manifolds may not exist and may not be unique

# 5. Timescale separation

With $\varepsilon \ll \varepsilon_1$, $x$ is the variable associated with the slow mode, while $z$ is associated with the fast mode. As we are interested in the slow mode, we choose the manifold $z = h_1 x$.

In the case $\varepsilon = 0$,

$$\lim_{\varepsilon \to 0} h_1(\varepsilon) = \lim_{\varepsilon \to 0} \frac{-(1-\varepsilon) + \sqrt{(1+\varepsilon)^2 - 4\varepsilon}}{2\varepsilon} = -1$$

In the case $\varepsilon \neq 0$, we use a Taylor expansion series to represent $h$ around $-1$. Therefore, the manifold becomes:

$$z = h(\varepsilon)x$$
$$\text{where} \quad h(\varepsilon) = h_0 + h_1\varepsilon + h_2\varepsilon^2 + \ldots$$

By using this manifold in the differential equation of $z$, we get:

$$\varepsilon \dot{z} = -x - z$$
$$\Rightarrow \varepsilon \frac{\partial z}{\partial x}\dot{x} = -x - h(\varepsilon)x = -(1 + h(\varepsilon))x$$

# 5. Timescale separation

$$\Rightarrow \varepsilon\left(h_0 + h_1\varepsilon + h_2\varepsilon^2 + \ldots\right)\left(-x + \left[h_0 + h_1\varepsilon + h_2\varepsilon^2 + \ldots\right]x\right) = \\ -\left(1 + h_0 + h_1\varepsilon + h_2\varepsilon^2 + \ldots\right)x$$

$$\Rightarrow \left(h_0 + h_1\varepsilon + h_2\varepsilon^2 + \ldots\right)\left((-1 + h_0)\varepsilon + h_1\varepsilon^2 + h_2\varepsilon^3 + \ldots\right) = \\ -\left(1 + h_0 + h_1\varepsilon + h_2\varepsilon^2 + \ldots\right)$$

By equating terms based on the powers of $\varepsilon$, we obtain:

$$\varepsilon^0 : 1 + h_0 = 0 \Rightarrow h_0 = -1$$

$$\varepsilon^1 : h_0(-1 + h_0) = -h_1 \Rightarrow h_1 = -2$$

$$\varepsilon^2 : h_0 h_1 + h_1(-1 + h_0) = -h_2 \Rightarrow h_2 = -6 \quad \text{, and so on}$$

To sum up, we obtain the following different approximations:

| | |
|---|---|
| $z \approx -x$ | Zero-order manifold |
| $z \approx -(1 + 2\varepsilon)x$ | First-order manifold |
| $z \approx -(1 + 2\varepsilon + 6\varepsilon^2)x$ | Second-order manifold |

# 5. Timescale separation

The reduced-order model becomes:

$$\begin{aligned} \dot{x} &= -x + z \\ z &= hx \\ x(0) &= x_0, z(0) = z_0 \end{aligned} \quad \Rightarrow \dot{x} = -(1-h)x \Rightarrow \boxed{x(t) = x_0 e^{-(1-h)t}}$$

Considering $\varepsilon = 0.1$, we have:

| | | |
|---|---|---|
| Zero-order manifold | : | $x(t) = x_0 e^{-2t}$ |
| First-order manifold | : | $x(t) = x_0 e^{-(2+2\varepsilon)t} = x_0 e^{-2.2t}$ |
| Second-order manifold | : | $x(t) = x_0 e^{-(2+2\varepsilon+6\varepsilon^2)t} = x_0 e^{-2.26t}$ |
| Exact solution | : | $x(t) = \left(\frac{7.7}{6.4}x_0 + \frac{z_0}{6.4}\right) e^{-2.3t} - \left(\frac{1.3}{6.4}x_0 + \frac{z_0}{6.4}\right) e^{-8.7t}$ |

# 5. Timescale separation

A general linear example. In an n-dimensional linear case we have:

$$\dot{x} = Ax + Bz$$
$$\varepsilon\dot{z} = Cx + Dz$$

where $\varepsilon$ is a small constant; $x \in \mathbb{R}^{n\times 1}$; $z \in \mathbb{R}^{m\times 1}$; and $A$, $B$, $C$, and $D$ are all matrices with proper dimensions.

A zero-order manifold is obtained for $z$ by letting $\varepsilon$ be equal to zero and solving in terms of $x$ as follows:

$$\varepsilon\dot{z} = 0 = Cx + Dz \Rightarrow \boxed{z = -D^{-1}Cx}$$
$$\Rightarrow \dot{x} = \left(A - BD^{-1}C\right)x$$

We assume that $D$ is full rank and, therefore, invertible.

# 5. Timescale separation

A graphical nonlinear example. Initial value problem with $\varepsilon = 0.05$

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_3 \\ \dot{x}_2 &= -x_2 - x_3 \\ \varepsilon \dot{x}_3 &= \tan^{-1}(1 - x_1 + x_2 - x_3) \end{aligned} \qquad \Rightarrow \dot{x}_3 = \frac{\tan^{-1}(1-x_1+x_2-x_3)}{\varepsilon}$$

# 5. Timescale separation

A graphical nonlinear example. Initial value problem with $\varepsilon = 0.05$

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_3 \\ \dot{x}_2 &= -x_2 - x_3 \\ \varepsilon \dot{x}_3 &= \tan^{-1}\left(1 - x_1 + x_2 - x_3\right) \end{aligned} \qquad \Rightarrow \dot{x}_3 = \frac{\tan^{-1}(1-x_1+x_2-x_3)}{\varepsilon}$$

Note there exists $\delta$, small, such that if $|1 - x_1 + x_2 - x_3| >> \delta$, then $|\dot{x}_3|$ gets large.

# 5. Timescale separation

A graphical nonlinear example. Initial value problem with $\varepsilon = 0.05$

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_3 \\ \dot{x}_2 &= -x_2 - x_3 \\ \varepsilon \dot{x}_3 &= \tan^{-1}\left(1 - x_1 + x_2 - x_3\right) \end{aligned} \qquad \Rightarrow \dot{x}_3 = \frac{\tan^{-1}(1-x_1+x_2-x_3)}{\varepsilon}$$

Note there exists $\delta$, small, such that if $|1 - x_1 + x_2 - x_3| >> \delta$, then $|\dot{x}_3|$ gets large.

Let M be the set defined by $\{x_1, x_2, x_3 \in \mathbb{R} : 1 - x_1 + x_2 - x_3 = 0\}$

**In state space, we say:**

If the distance between the system trajectory and M is much larger than $\delta$, then $x_3$ will exhibit a fast response

# 5. Timescale separation

Example (cont.). Dynamic simulation with $\varepsilon = 0.05$

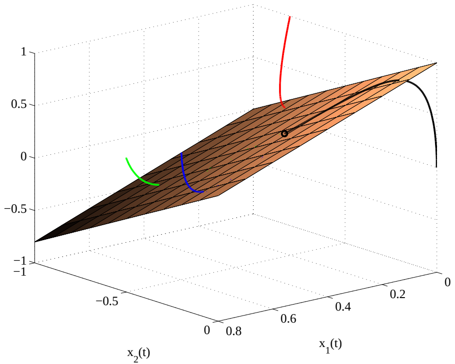
State-space trajectories from 4 different initial points:

Point 1 $x_1 = 0.8,\ x_2 = -0.2,\ x_3 = 0.5$

Point 2 $x_1 = 0.0,\ x_2 = 0.0,\ x_3 = 0.0$

Point 3 $x_1 = 0.6,\ x_2 = -0.8,\ x_3 = 0.0$

Point 4 $x_1 = 0.0,\ x_2 = -0.8,\ x_3 = 1.0$



# 5. Timescale separation

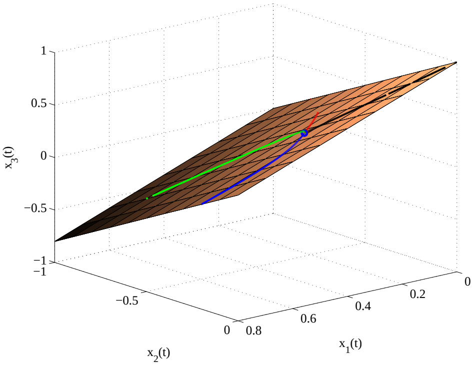
Example (cont.). Considering a zero-order manifold for $z$ ($\varepsilon = 0$)

$$M : \{x_1, x_2, x_3 \in \mathbb{R} : 1 - x_1 + x_2 - x_3 = 0\}$$

Simplified model:

$$\dot{x}_1 = -x_1 + x_3$$
$$\dot{x}_2 = -x_2 - x_3$$
$$0 = 1 - x_1 + x_2 - x_3$$

| | |
|---|---|
| Outside M | Dynamic of $x_3$ infinitely fast, i.e., $\dot{x}_3 = \infty$ |
| Inside M | Dynamic of $x_3$ in the same time scale of $x_1$ and $x_2$ |



# Standard models

## (back to SG models for power system simulations)

# 5. Two-axis model for the SG

The dynamics associated to the dampers $2q$ y $1d$ are represented through a zero-order manifold (set $T''_{do}$ and $T''_{qo}$ to zero) to obtain:

$$
\begin{aligned}
0 &= -\psi_{1d} + E'_q - (X'_d - X_{\ell s})I_d \\
0 &= -\psi_{2q} - E'_d - (X'_q - X_{\ell s})I_q
\end{aligned}
$$

Replacing the expressions above in the flux equations for $\psi_d$ and $\psi_q$, we have:

$$
\begin{aligned}
\psi_d &= -X'_d I_d + E'_q \\
\psi_q &= -X'_q I_q - E'_d
\end{aligned}
$$

Moreover, the dynamics associated to the stator flux linkages are also represented through a zero-order manifold (assume $\frac{1}{\omega_s} \approx 0$) to obtain:

$$
\begin{aligned}
0 &= R_s I_d + \frac{\omega}{\omega_s}\left(-X'_q I_q - E'_d\right) + V_d \\
0 &= R_s I_q + \frac{\omega}{\omega_s}\left(X'_d I_d - E'_q\right) + V_q
\end{aligned}
$$

# 5. Two-axis model for the SG

By assuming further that $\omega = \omega_s$,

$$E'_d = V_d + R_s I_d - X'_q I_q$$
$$E'_q = V_q + R_s I_q + X'_d I_d$$

By combining these two into a complex equation, we have:

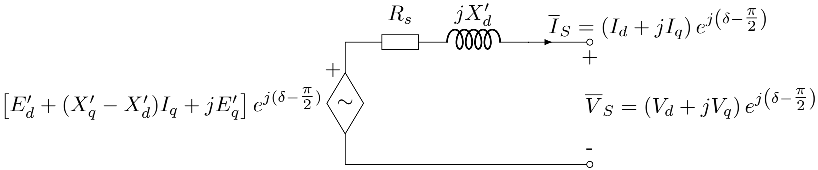
$$\begin{aligned} E'_d + jE'_q &= (V_d + jV_q) + R_s(I_d + jI_q) - X'_q I_q + jX'_d I_d \\ &= (V_d + jV_q) + R_s(I_d + jI_q) - X'_q I_q + jX'_d(I_d + jI_q - jI_q) \\ &= (V_d + jV_q) + R_s(I_d + jI_q) + jX'_d(I_d + jI_q) - X'_q I_q + X'_d I_q \end{aligned}$$

$$\Rightarrow E'_d + (X'_q - X'_d)I_q + jE'_q = (V_d + jV_q) + (R_s + jX'_d)(I_d + jI_q)$$

$$\Rightarrow \left[E'_d + (X'_q - X'_d)I_q + jE'_q\right] e^{(\delta - \frac{\pi}{2})} = \underbrace{V_S e^{j\theta_S}}_{(V_d + jV_q)e^{(\delta - \frac{\pi}{2})}} + (R_s + jX'_d)(I_d + jI_q)e^{(\delta - \frac{\pi}{2})}$$

# 5. Two-axis model for the SG

The previous circuital relationship is described by the following equivalent circuit:




The two-axis model is defined by this equivalent circuit (which describes the algebraic variables $I_d$ and $I_q$) and the following differential equations:

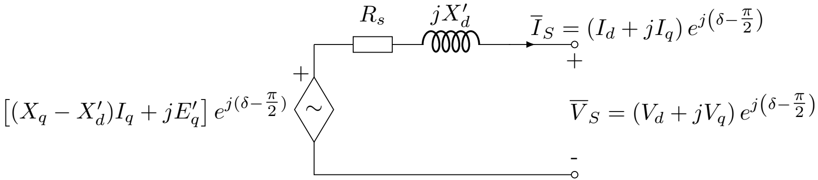
$$T'_{do}\dot{E}'_q = -E'_q - (X_d - X'_d)I_d + E_{fd}$$
$$T'_{qo}\dot{E}'_d = -E'_d + (X_q - X'_q)I_q$$

# 5. One-axis model for the SG

In this model, the dynamics of the damper $1q$ are also represented through a zero-order manifold (set $T'_{qo}$ to zero). Thus,

$$0 = -E'_d + (X_q - X'_q)I_q$$

By expressing $E'_d$ in terms of the current $I_q$, two things change in the model: (a) the internal voltage of the equivalent circuit, and (b) the torque equation.




The one-axis model is defined by this equivalent circuit (which describes the algebraic variables $I_d$ and $I_q$) and the following differential equation:

$$T'_{do}\dot{E}'_q = -E'_q - (X_d - X'_d)I_d + E_{fd}$$

# Complete models
## (adding AVR and governor/turbine models)

# 5. SG two-axis model, IEEE Type 1 Exciter, and 2nd order governor/turbine

$$T'_{do}\dot{E}'_q = -E'_q - (X_d - X'_d)I_d + E_{fd}$$

$$T'_{qo}\dot{E}'_d = -E'_d + (X_q - X'_q)I_q$$

$$\dot{\delta} = \omega - \omega_s$$

$$2H\dot{\omega} = T_M - E'_d I_d - E'_q I_q - (X'_q - X'_d)I_d I_q - T_{FW}$$

$$T_E\dot{E}_{fd} = -\left(K_E + S_E(E_{fd})\right)E_{fd} + V_R$$

$$T_A\dot{V}_R = -V_R + +K_A R_f - \frac{K_A K_F}{T_F}E_{fd} + K_A\left(V_{ref} - V_t\right)$$

$$T_F\dot{R}_f = -R_f + \frac{K_F}{T_F}E_{fd}$$

$$T_{CH}\dot{T}_M = -T_M + P_{SV}$$

$$T_{SV}\dot{P}_{SV} = -P_{SV} + P_C - \frac{1}{R_D}\left(\frac{\omega}{\omega_s} - 1\right)$$

with $V_R^{min} \leq V_R \leq V_R^{max}$, $V_t = \sqrt{V_d^2 + V_q^2}$, $0 \leq P_{SV} \leq P_{SV}^{max}$

# 5. SG one-axis model, IEEE Type 1 Exciter, and 2nd order governor/turbine

$$
\begin{aligned}
T'_{do}\dot{E}'_q &= -E'_q - (X_d - X'_d)I_d + E_{fd} \\
\dot{\delta} &= \omega - \omega_s \\
2H\dot{\omega} &= T_M - E'_q I_q - (X_q - X'_d)I_d I_q - T_{FW} \\
T_E\dot{E}_{fd} &= -\left(K_E + S_E(E_{fd})\right)E_{fd} + V_R \\
T_A\dot{V}_R &= -V_R + +K_A R_f - \frac{K_A K_F}{T_F}E_{fd} + K_A\left(V_{ref} - V_t\right) \\
T_F\dot{R}_f &= -R_f + \frac{K_F}{T_F}E_{fd} \\
T_{CH}\dot{T}_M &= -T_M + P_{SV} \\
T_{SV}\dot{P}_{SV} &= -P_{SV} + P_C - \frac{1}{R_D}\left(\frac{\omega}{\omega_s} - 1\right)
\end{aligned}
$$

with $V_R^{min} \leq V_R \leq V_R^{max}, \quad V_t = \sqrt{V_d^2 + V_q^2}, \quad 0 \leq P_{SV} \leq P_{SV}^{max}$

# 5. SG two-axis model, IEEE Type 1 Exciter, and 3rd order governor/turbine

$$
\begin{aligned}
T'_{do}\dot{E}'_q &= -E'_q - (X_d - X'_d)I_d + E_{fd}\\
T'_{qo}\dot{E}'_d &= -E'_d + (X_q - X'_q)I_q\\
\dot{\delta} &= \omega - \omega_s\\
2H\dot{\omega} &= T_M - E'_d I_d - E'_q I_q - (X'_q - X'_d)I_d I_q - T_{FW}\\
T_E\dot{E}_{fd} &= -\left(K_E + S_E(E_{fd})\right)E_{fd} + V_R\\
T_A\dot{V}_R &= -V_R + +K_A R_f - \frac{K_A K_F}{T_F}E_{fd} + K_A\left(V_{ref} - V_t\right)\\
T_F\dot{R}_f &= -R_f + \frac{K_F}{T_F}E_{fd}\\
T_1\dot{y}_1 &= -y_1 + K_1(\omega - 1)\\
T_3\dot{y}_3 &= -y_3 + y_1\\
T_4\dot{T}_m &= -T_m + P_C - y_2\\
0 &= -y_{2i} + \left(1 - \frac{T_2}{T_3}\right)y_3 + \frac{T_2}{T_3}y_1
\end{aligned}
$$

$$
y_2 = \begin{cases} P_C - P_{min}, & P_{min} > P_C - y_{2i}\\ P_C - P_{max}, & P_{max} < P_C - y_{2i}\\ y_{2i}, & \text{otherwise}\end{cases}
$$

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