Recall that $\mathbb{OP}^2$ may be defined as a subset of the exceptional Jordan algebra $\mathfrak{H}(3,\mathbb{O})$. Specifically, a *point* of $\mathbb{OP}^2$ is a trace 1 idempotent,

$$p \in \mathfrak{H}(3,\mathbb{O}) \qquad \text{s.t.} \qquad p^2 = p, \qquad \operatorname{tr} p = 1,$$

while a *line* of $\mathbb{OP}^2$ is a trace 2 idempotent,

$$\ell \in \mathfrak{H}(3,\mathbb{O}) \qquad \text{s.t.} \qquad \ell^2 = \ell, \qquad \operatorname{tr} \ell = 2.$$

We say a point $p$ is *incident* with a line $\ell$ if $p \circ \ell = p$. The set of points incident with a given line form an $\mathbb{OP}^1$.

Every idempotent has to have trace 0, 1, 2 or 3, corresponding to $0 \in \mathfrak{H}(3,\mathbb{O})$, points, lines, and $1 \in \mathfrak{H}(3,\mathbb{O})$, respectively. More generally, the trace of an idempotent in $\mathfrak{H}(n,\mathbb{A})$ can lie in $\{0,\dots,n\}$, with only the unit having trace $n$.

For every idempotent $u \in \mathfrak{H}(3,\mathbb{O})$ there is an orthogonal *Peirce decomposition*

$$\mathfrak{H}(3,\mathbb{O}) = E_0(u) \oplus E_{1/2}(u) \oplus E_1(u)$$

into eigenspaces of multiplication with $u$,

$$E_\lambda(u) = \{X \in \mathfrak{H}(3,\mathbb{O}) \mid X \circ u = \lambda X\}.$$

We denote $p_0 = \operatorname{diag}(1,0,0)$ and $\ell_0 = \operatorname{diag}(0,1,1)$. Then one can check that

$$E_0(p_0) = E_1(\ell_0) = \begin{pmatrix} 0 & 0 \\ 0 & \mathfrak{H}_2(\mathbb{O}) \end{pmatrix}.$$

**Theorem 1.** *Let $u \in \mathfrak{H}(3,\mathbb{O})$ be an idempotent.*

(i) *The space $E_1(u)$ is a subalgebra.*

(ii) *The map $u \mapsto E_1(u)$ is injective.*

(iii) *An automorphism of $\mathfrak{H}(3,\mathbb{O})$ preserves $E_1(u)$ if and only if it fixes $u$.*

*Proof.* (i) Recall the Jordan identity

$$a^2 \circ (a \circ b) = a \circ (a^2 \circ b).$$

Linearizing it once gives

$$2(a \circ c) \circ (a \circ b) + a^2 \circ (c \circ b) = c \circ (a^2 \circ b) + 2a \circ ((a \circ c) \circ b).$$

Now let $X, Y \in E_1(u)$. Thus we have $X^2 = X \circ u = X$, $Y^2 = Y \circ u = Y$, $u^2 = u$. Plugging in $a = u$, $b = X$ and $c = Y$ yields

$$\begin{aligned}(X \circ Y) \circ u = u^2 \circ (X \circ Y) &= Y \circ (u^2 \circ X) + 2u \circ ((u \circ Y) \circ X) - 2(u \circ Y) \circ (u \circ X) \\ &= X \circ Y + 2(X \circ Y) \circ u - 2X \circ Y \\ &= -X \circ Y + 2(X \circ Y) \circ u\end{aligned}$$

and thus $(X \circ Y) \circ u = X \circ Y$, i.e. $X \circ Y \in E_1(u)$.

(ii) Suppose there are idempotents $u, v$ with $E_1(u) = E_1(v)$. The identities $u^2 = u$ and $v^2 = v$ respectively imply that $u \in E_1(u)$ and $v \in E_1(v)$. But then also $u \in E_1(v)$ and $v \in E_1(u)$, so

$$u = u \circ v = v.$$

(iii) For any $X \in \mathfrak{H}(3, \mathbb{O})$ and $f \in \mathrm{F}_4$, we have

$$\begin{aligned} X \in f(E_1(u)) &\iff f^{-1}(X) \circ u = f^{-1}(X) \\ &\iff f^{-1}(X \circ f(u)) = f^{-1}(X) \\ &\iff X \circ f(u) = X \\ &\iff X \in E_1(f(u)). \end{aligned}$$

Clearly, an automorphism $f$ fixing $u$ also preserves $E_1(u)$. Conversely, if $f(E_1(u)) = E_1(u)$, then $E_1(u) = E_1(f(u))$, and by (ii) this implies $f(u) = u$.

□

**Theorem 2.** *If $\ell \in \mathfrak{H}(3, \mathbb{O})$ is a line, then $E_1(\ell) \cong \mathfrak{H}(2, \mathbb{O})$.*

*Proof.* Since $\mathrm{F}_4$ acts transitively on the set of lines, we can assume that $\ell = \ell_0$. Then

$$E_1(\ell) = \begin{pmatrix} 0 & 0 \\ 0 & \mathfrak{H}_2(\mathbb{O}) \end{pmatrix}.$$

□

**Lemma 3.** *Let $\varphi : \mathfrak{H}(2, \mathbb{O}) \to \mathfrak{H}(3, \mathbb{O})$ be an injective algebra homomorphism. Then $\varphi(1)$ is a line.*

*Proof.* Clearly the image of an idempotent is again an idempotent. The Jordan algebra $\mathfrak{H}_2(\mathbb{O})$ is spanned by elements

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \xi(x) = \begin{pmatrix} 0 & x \\ \bar{x} & 0 \end{pmatrix} \quad (x \in \mathbb{O})$$

for which the multiplication table looks as follows:

$$\begin{aligned} &e_1^2 = e_1, && e_2^2 = e_2, && e_1 \circ e_2 = 0, \\ &e_1 \circ \xi(x) = \frac{1}{2}\xi(x), && e_2 \circ \xi(x) = \frac{1}{2}\xi(x), && \xi(x) \circ \xi(y) = \langle x, y \rangle (e_1 + e_2). \end{aligned}$$

In particular, the unit is $1 = e_1 + e_2$.

Let now $u = \varphi(e_1)$ and $v = \varphi(e_2)$. These are nonzero idempotents satisfying $u \circ v = 0$, so their sum is also an idempotent. Since the trace of an idempotent can only be 0, 1, 2 or 3, this leaves us with two possibilities.

- $\mathrm{tr}(u) = \mathrm{tr}(v) = 1$, so $\mathrm{tr}(u + v) = 2$. Then $u + v = \varphi(1)$ is a line.

- $\mathrm{tr}(u+v) = 3$. Without restriction, assume $\mathrm{tr}(u) = 1$ and $\mathrm{tr}(v) = 2$. Using the action of $\mathrm{F}_4$, we can assume that $u = p_0$. So

  $$v \in E_0(u) = \begin{pmatrix} 0 & 0 \\ 0 & \mathfrak{H}_2(\mathbb{O}) \end{pmatrix},$$

  but then $\mathrm{tr}(v) = 2$ forces $v = \ell_0$.

  By the multiplication rules above, we must have $\varphi(\xi(x)) \in E_{1/2}(p_0) \cap E_{1/2}(\ell_0)$. In fact,

  $$E_{1/2}(p_0) = E_{1/2}(\ell_0) = \left\{ X_{a,b} = \begin{pmatrix} 0 & a & b \\ \bar{a} & 0 & 0 \\ \bar{b} & 0 & 0 \end{pmatrix} \,\middle|\, a, b \in \mathbb{O} \right\}.$$

  Also, $\varphi(\xi(x))^2 = |x|^2(u+v) = |x|^2 \cdot 1$. But one may check that

  $$X_{a,b}^2 = \begin{pmatrix} |a|^2 + |b|^2 & 0 & 0 \\ 0 & |a|^2 & \bar{a}b \\ 0 & \bar{b}a & |b|^2 \end{pmatrix},$$

  which is proportional to the identity matrix only if $a = b = 0$. Thus $\varphi(\xi(x)) = 0$, which is a contradiction and rules out this case.

□

**Theorem 4.** *Every subalgebra of $\mathfrak{H}(3,\mathbb{O})$ isomorphic to $\mathfrak{H}(2,\mathbb{O})$ is of the form $E_1(\ell)$ for a unique line $\ell$.*

*Proof.* Let $\varphi : \mathfrak{H}(2,\mathbb{O}) \to \mathfrak{H}(3,\mathbb{O})$ be an injective algebra homomorphism, and denote $\ell = \varphi(1)$. We show that $\operatorname{im}\varphi = E_1(\ell)$.

For any $X \in \mathfrak{H}(2,\mathbb{O})$, we see that

$$\varphi(X) \circ \ell = \varphi(X) \circ \varphi(1) = \varphi(X \circ 1) = \varphi(X),$$

thus $\operatorname{im}\varphi \subseteq E_1(\ell)$. By Lemma 3, $\ell$ is a line, and with Theorem 2 we have

$$\dim \operatorname{im}\varphi = \dim \mathfrak{H}(2,\mathbb{O}) = \dim E_1(\ell),$$

so in fact $\operatorname{im}\varphi = E_1(\ell)$. Theorem 1 (ii) now guarantees that $\ell$ is the unique idempotent with this property. □

With Theorem 1 (iii) it follows:

**Corollary 5.** *The subgroup of $\mathrm{F}_4$ preserving a subalgebra $\mathfrak{H}(2,\mathbb{O}) \subset \mathfrak{H}(3,\mathbb{O})$ is the subgroup preserving a line $\ell$, or preserving a point $p = 1 - \ell$, and it is conjugate to $\mathrm{Stab}_{\mathrm{F}_4}(p_0) \cong \mathrm{Spin}(9)$.*