Grimoire of curvature sign conventions, conditions, operators, and examples

Because conventions are confusing, I state mine, how they relate to other conventions and what happens on spaces of constant sectional curvature.

Throughout this note we fix a Riemannian manifold (M, g) of dimension n and denote by (e_i) an orthonormal basis of $T := T_pM$ for some $p \in M$.

Some definitions

The Riemannian curvature tensor. Given any affine connection ∇ on a vector bundle $EM \to M$, its *curvature* is the section $R^\nabla \in \Omega^2(M, \operatorname{End} EM)$ given by

$$R^\nabla(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.$$

In particular, if ∇ is the Levi-Civita connection on T of a Riemannian metric g, then $R = R^\nabla$ is the *Riemannian curvature tensor*. We may lower its indices via the metric as

$$R(X, Y, Z, W) = g(R(X, Y)Z, W).$$

We call this convention for R the *forward convention* and shall use it unless otherwise stated. The other common convention, which we call the *backward convention*[1], is $R^\nabla(X, Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$, which we see for example in [1, 2, 3, 7, 14].

Sectional curvature. The *sectional curvature* of a Riemannian metric is determined by R by

$$\sec(X \wedge Y) = \frac{R(X, Y, Y, X)}{g(X, X)g(Y, Y) - g(X, Y)^2}$$

and depends only on the two-plane spanned by X and Y. There is only one convention for sec, namely the one where the round sphere $S^n(r)$ of radius r has *positive* constant sectional curvature $\sec \equiv r^{-2}$. (Correspondingly, hyperbolic space has *negative* constant sectional curvature). The Riemannian curvature tensor of a metric of constant sectional curvature $\sec \equiv k$ has the form

$$R(X, Y)Z = k(g(Y, Z)X - g(X, Z)Y).$$

For the backward convention, the correct formulae read

$$\sec(X \wedge Y) = \frac{R(X, Y, X, Y)}{g(X, X)g(Y, Y) - g(X, Y)^2}$$

as well as

$$R(X, Y)Z = k(g(X, Z)Y - g(Y, Z)X).$$

Note that the formula in [1, Prop. 1.88] has a sign error.

[1] For lack of a better name. This is not meant to be derogatory.

The Kulkarni–Nomizu product. For symmetric 2-tensors $h_1, h_2 \in \mathrm{Sym}^2$, their *Kulkarni–Nomizu product* is the 4-tensor $h_1 \owedge h_2 \in \mathrm{Sym}^2 \Lambda^2$ given by

$$\begin{aligned}(h_1 \owedge h_2)(X, Y, Z, W) = h_1(X, Z)h_2(Y, W) + h_1(Y, W)h_2(X, Z) \\ - h_1(X, W)h_2(Y, Z) - h_1(Y, Z)h_2(X, W).\end{aligned}$$

The same convention is used in [1, 3]. $h_1 \owedge h_2$ is always an algebraic curvature tensor (that is, it satisfies the first Bianchi identity) – moreover, on a space of $\sec \equiv k$, the curvature tensor has the form

$$R = -\frac{k}{2} g \owedge g,$$

or in the backward convention

$$R = \frac{k}{2} g \owedge g.$$

Ricci and scalar curvature. One may contract the Riemannian curvature tensor R to obtain the *Ricci tensor*

$$\mathrm{Ric}(X, Y) = \sum_i R(e_i, X, Y, e_i) = \mathrm{tr}(Z \mapsto R(Z, X)Y).$$

Contracting with the metric yields the *scalar curvature*

$$\mathrm{scal} = \mathrm{tr}_g \mathrm{Ric} = \sum_i \mathrm{Ric}(e_i, e_i) = \sum_{i,j} R(e_i, e_j, e_j, e_i).$$

As for sectional curvature, there is really only one convention for Ric and scal. A space with $\sec \equiv k$ has constant Ricci curvature $\mathrm{Ric} = (n-1)kg$ and scalar curvature $\mathrm{scal} = n(n-1)k$.

In the backward convention for R, one has of course

$$\mathrm{Ric}(X, Y) = \sum_i R(X, e_i, Y, e_i) = \mathrm{tr}(Z \mapsto R(X, Z)Y)$$

The Ricci tensor is often turned into an endomorphism $\mathrm{Ric} \in \mathrm{End}(T)$ using the metric.

Curvature operator of the first kind. The Riemannian curvature tensor R gives rise to a symmetric endomorphism $\widehat{R} : \Lambda^2 T \to \Lambda^2 T$ via

$$\langle \widehat{R}(X \wedge Y), Z \wedge W \rangle = R(X, Y, Z, W),$$

where the inner product on 2-forms is given by

$$|X \wedge Y|^2 = g(X, X)g(Y, Y) - g(X, Y)^2.$$

$\widehat{R}$ is called the *curvature operator of the first kind.* This way of defining $\widehat{R}$ is what we call the *negative convention*, used for example by Friedrich, Semmelmann, etc. and we shall also use it unless otherwise stated.

In contrast, there is also the *positive convention* which differs by a sign. In this convention, $\widehat{R}$ is related to sectional curvature simply by

$$\sec(\sigma) = \frac{\langle \widehat{R}\sigma, \sigma \rangle}{\langle \sigma, \sigma \rangle}$$

for decomposable bivectors σ, so $\widehat{R}$ is positive whenever sec is positive. The positive convention is used for example in [1, 2, 7].

We return to the negative convention. $\widehat{R}$ may also directly defined by

$$\widehat{R}(X \wedge Y) = \frac{1}{2} \sum_i e_i \wedge R(X, Y)e_i.$$

Another formula is

$$(\widehat{R}\sigma)(X, Y) = \sum_{i<j} R(e_i, e_j, X, Y)\sigma(e_i, e_j) = \frac{1}{2} \sum_{i,j} R(e_i, e_j, X, Y)\sigma(e_i, e_j).$$

Here we see that this is $-\frac{1}{2}$ of the convention for $\widehat{R}$ used in [3, 14, 15] (which we may call the *doubly positive convention*).

In the negative convention, the curvature operator operator on a space of $\sec \equiv k$ is

$$\widehat{R} = -k \operatorname{Id}_{\Lambda^2}.$$

Using the inner product on Λ^2 defined above, one has

$$\widehat{g \owedge g} = 2 \operatorname{Id}_{\Lambda^2},$$

so we recover the formula $R = -\frac{k}{2} g \owedge g$ from above.

Curvature operator of the second kind. The *curvature operator of the second kind* is another symmetric endomorphism $\mathring{R} : \operatorname{Sym}^2 T \to \operatorname{Sym}^2 T$ derived from R via

$$(\mathring{R}h)(X, Y) = \sum_i h(R(e_i, X)Y, e_i) = \sum_{i,j} R(X, e_i, e_j, Y)h(e_i, e_j).$$

(using the forward convention for R). This immediately implies

$$\mathring{R}g = \operatorname{Ric}.$$

In the case of $\sec \equiv k$, we have

$$\mathring{R} = (n-1)k \operatorname{Id}_{\mathbb{R}g} - k \operatorname{Id}_{\operatorname{Sym}^2_0 T}.$$

We call this convention for $\mathring{R}$ the *Ricci-like convention* or the *almost negative convention.* It is used by [1, 3]. The opposite sign convention (where $\langle \mathring{R}\cdot, \cdot \rangle$ is positive on $\operatorname{Sym}^2_0 T$)

is called the *anti-Ricci-like convention* or *almost positive convention*, used for example in [4, 14, 15].

$\mathring{R}$ preserves the space $\mathrm{Sym}^2_0 T$ of *trace-free* tensors if and only if g is Einstein. Just like for sec or $\widehat{R}$, the curvature tensor R may be reconstructed from $\mathring{R}$. Other possible ways to contract R with 2-tensors are discussed in [3].

Because the operator $\mathring{R}$ acts differently on g than on trace-free tensors (with different sign even for constant sectional curvature!), one sometimes considers instead the operator

$$\mathrm{pr}_{\mathrm{Sym}^2_0} \circ \mathring{R}\big|_{\mathrm{Sym}^2_0}$$

and calls this the curvature operator of the second kind [12].

Other contractions with the curvature tensor. One may also contract a two-tensor $\alpha \in T \otimes T$ with other slots of the curvature tensor. If R^{ab} denotes the operator defined by contracting the a, b-slots of R with α (where $1 \leq a, b \leq 4$), then by the symmetries of the curvature tensors only R^{12} and $R^{23} = -R^{13}$ are actually of interest. If $h \in \mathrm{Sym}^2 T$, then clearly

$$R^{12}h = 0, \qquad R^{23}h = \mathring{R}h.$$

For $\sigma \in \Lambda^2 T$ on the other hand, we have

$$R^{12}\sigma = 2\widehat{R}\sigma, \qquad R^{23}\sigma = -\widehat{R}\sigma.$$

using the first Bianchi identity for the second part.

The standard curvature endomorphism. Let (ω_k) be any orthonormal basis of $\Lambda^2 T$, for example $(e_i \wedge e_j)_{i<j}$. We identify $\Lambda^2 T \cong \mathfrak{so}(T)$ using the metric, i.e. via

$$(X \wedge Y)(Z) = g(X, Z)Y - g(Y, Z)X.$$

If $EM \to M$ is any vector bundle associated to the orthonormal frame bundle P, i.e. $EM = P \times_\rho E$ for some representation $\rho : \mathrm{O}(n) \to \mathfrak{gl}(E)$, then the *standard-/Weitzenböck curvature endomorphism* on EM is the fibrewise (symmetric) endomorphism $\mathcal{K}(R, E)$ on EM given by

$$\mathcal{K}(R, E) = \sum_k \rho_*(\omega_k)\rho_*(\widehat{R}\omega_k).$$

This endomorphism[2] is also sometimes denotes $q(R)$ (Semmelmann) is also sometimes denoted $\mathcal{K}(R, EM)$ [2], or $-K$ [7], or Ric [16]. On a space with sec $\equiv k$, we have

$$\mathcal{K}(R, E) = k \,\mathrm{Cas}_E^{\mathfrak{so}(n)}$$

[2] [1, §1.139,§1.143] introduces similar operators $c^2_\rho(R)$ and Γ (note the sign change in the second term of Γ according to differing sign conventions for R) and claims that $\Gamma = -2c^2_\rho(R)$. The mysterious factor of 2 probably comes from the inner product in Λ^2.

where $\mathrm{Cas}_E^{\mathfrak{so}(n)}$ is the (nonnegative) Casimir constant of the $\mathrm{O}(n)$-representation E. On the bundle of covariant p-tensors, $EM = \bigotimes^p T^*M$, we may succinctly write

$$(\mathcal{K}(R,E)\alpha)(X_1,\dots,X_p) = \sum_{i,j} (R(e_j,X_i)\alpha)(X_1,\dots,X_{i-1},e_j,X_{i+1},\dots,X_p).$$

There is an uglier formula

$$(\mathcal{K}(R,E)\alpha)_{i_1\dots i_p} = \sum_k \mathrm{Ric}_{i_k j}\alpha_{i_1\dots}{}^{j}{}_{\dots i_p} + \sum_{k\neq l} R_{i_k j i_l m}\alpha_{i_1\dots}{}^{j}{}_{\dots}{}^{m}{}_{\dots i_p},$$

using Einstein summation convention. We have the following identities (it's possible to show them directly, but they also follow from the formula above):

$$\begin{aligned}
\mathcal{K}(R,T) &= \mathrm{Ric},\\
\mathcal{K}(R,T\otimes T) &= \mathrm{Ric}_* + 2R^{13},\\
\mathcal{K}(R,\Lambda^2) &= \mathrm{Ric}_* + 2\widehat{R},\\
\mathcal{K}(R,\mathrm{Sym}^2 T) &= \mathrm{Ric}_* - 2\mathring{R},
\end{aligned}$$

where the Ricci endomorphism acts on tensors through the specific[3] $\mathfrak{gl}(T)$-representation

$$\begin{aligned}
(A,v) &\mapsto Av, && A\in\mathfrak{gl}(T),\ v\in T,\\
(A,\alpha) &\mapsto \alpha\circ A^\top, && A\in\mathfrak{gl}(T),\ v\in T,
\end{aligned}$$

with extension to higher tensors as derivation. In particular for $\alpha\in\mathrm{End}(T)$, we have

$$\mathrm{Ric}_*\alpha = \mathrm{Ric}\circ\alpha + \alpha\circ\mathrm{Ric},$$

and if g is Einstein with $\mathrm{Ric} = Eg$, we have $\mathrm{Ric}_* = p\,\mathrm{Id}$ on $\bigotimes^p T^*$. More relations are available in [2, Thm. B]. Specializing to p-forms or symmetric tensors, we have

$$\begin{aligned}
\mathcal{K}(R,\Lambda^p T^*)\alpha &= \sum_{i,j} e^j \wedge (e_i \lrcorner R(e_i,e_j)\alpha), && \alpha\in\Lambda^p T^*,\\
\mathcal{K}(R,\mathrm{Sym}^p T^*)\alpha &= \sum_{i,j} e^j \odot (e_i \lrcorner R(e_i,e_j)\alpha), && \alpha\in\mathrm{Sym}^p T^*.
\end{aligned}$$

The endomorphism $\mathcal{K}(R,E)$ is precisely the curvature term appearing in the *Lichnerowicz Laplacian*

$$\Delta_{\mathrm{L}} = \nabla^*\nabla + \mathcal{K}(R,E).$$

Some authors also consider *Lichnerowicz-type Laplacians* where this curvature term is scaled by a positive constant [16]. The reason for this is the occurence of terms of the type $\nabla^*\nabla + c\mathcal{K}(R,E)$ in various Weitzenböck formulae.

[3] Note that the representation on covectors is by the transpose

The quantization map. Let q denote the *quantization map*

$$q : \operatorname{Sym}^{\leq \bullet} \mathfrak{so}(T) \longrightarrow \mathfrak{U}^{\leq \bullet}\mathfrak{so}(T), \qquad X^{\odot k} \mapsto X^k$$

which is an isomorphism of filtered $\mathrm{O}(n)$-modules. Using the metric duality, we understand the curvature tensor R as an element of $\operatorname{Sym}^2 \mathfrak{so}(T)$ and write

$$R = \frac{1}{2}\sum_{k,l} \langle R, \omega_k \odot \omega_l \rangle \omega_k \odot \omega_l = \frac{1}{2}\sum_k \omega_k \odot \widehat{R}(\omega_k)$$

for an orthonormal basis (ω_k) of $\mathfrak{so}(T)$. Thus, for a representation $\rho : \mathrm{O}(n) \to \mathfrak{gl}(E)$, we have

$$\mathcal{K}(R, E) = 2\rho_*(q(R)),$$

where ρ_* is extended to $\mathfrak{U}\mathfrak{so}(T)$. This factor of two is responsible for some clash of notation between papers by Semmelmann (who uses $q \sim \mathcal{K}$) resp. Weingart (who uses q for the quantization map).

Irreducible decomposition of the curvature tensor. The space of algebraic curvature tensors, i.e. the kernel of the Bianchi operator $b : \operatorname{Sym}^2 \Lambda^2 T^* \to \Lambda^4 T^*$ with

$$b(R)(X, Y, Z, W) = R(X, Y, Z, W) + R(Y, Z, X, W) + R(Z, X, Y, W),$$

decomposes for $n \geq 5$ into three irreducible parts: the scalar part $\mathbb{R} g \owedge g$, the traceless Ricci part $\operatorname{Sym}^2_0 T^* \owedge g$, and the Weyl part. (For $n = 4$, the Weyl part splits further into self-dual and anti-self-dual part. For $n = 3$, the Weyl part vanishes. For $n = 2$, both traceless Ricci and Weyl part vanish. For $n = 1$, there is no curvature at all.) The projections of a curvature tensor R to these parts are respectively given by

$$R = U + Z + W, \qquad U = \frac{\mathrm{scal}}{2n(n-1)} g \owedge g, \qquad Z = \frac{1}{n-2}\mathrm{Ric}^0 \owedge g.$$

The Weyl tensor W is annihilated by all contractions with g.

Parallel skew torsion If ∇^τ is a metric connection with parallel and totally skew-symmetric torsion T^τ, then we can write

$$\nabla^\tau = \nabla^g + \tau, \qquad T^\tau = 2\tau$$

for some ∇^τ-parallel $\tau \in \Omega^3(M)$. The curvature R^τ does not generally satisfy the Bianchi identity, but is still pair-symmetric, i.e. $R^\tau \in \operatorname{Sym}^2 \Lambda^2 T$. The difference to the Riemannian curvature is given by [5]

$$(R - R^\tau)_{X,Y} Z = [\tau_X, \tau_Y]Z - 2\tau_{\tau_X Y} Z,$$

$X, Y, Z \in T$, where we write

$$g(\tau_X Y, Z) := \tau(X, Y, Z).$$

Equivalently, with $\tau^2(X,Y) := -\tau_{\tau_X Y} \in \operatorname{Sym}^2 \Lambda^2 T^*$, we have

$$R - R^\tau = \tau^2 + b(\tau^2).$$

Sometimes one might want to use the standard curvature endomorphism associated to such a connection. In [17, Lem. 3.1] it is stated that

$$\mathcal{K}(R,E) - \mathcal{K}(R^\tau, E) = \mathcal{K}(\tau^2, E)$$

on $E = \operatorname{Sym}^p T$ any symmetric tensor power, but actually this is more general: since

$$\rho_* \circ q : \operatorname{Sym}^2 \Lambda^2 T \longrightarrow \operatorname{Sym}^2 E$$

is $\mathrm{O}(n)$-equivariant, the contribution of $b(\tau^2) \in \Lambda^4 T$ must vanish for any E such that

$$\operatorname{Hom}_{\mathrm{O}(n)}(\Lambda^4 T, \operatorname{Sym}^2 E) = 0.$$

In turn, the formulas for the Lichnerowicz Laplacian on normal homogeneous spaces given in [18] generalizes to these bundles.

A digression on inner products and tensors

Recall that

$$\langle X \wedge Y, \alpha \rangle_{\Lambda^2 T} = \langle Y, X \lrcorner \alpha \rangle = \alpha(X,Y).$$

$$\langle e_i \wedge e_j, e_k \wedge e_l \rangle_{\Lambda^2 T} = \langle e_j, e_i \lrcorner (e_k \wedge e_l) \rangle = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}.$$

The two summands cannot be 1 simultaneously since $e_i \wedge e_i = 0$. If the first summand is 1, this means $\langle e_i \wedge e_j, e_i \wedge e_j \rangle = 1$. If the second summand is 1, this means that $\langle e_i \wedge e_j, e_j \wedge e_i \rangle = -1$. Thus $(e_i \wedge e_j)_{i<j}$ is an ONB of $\Lambda^2 T$.

For the symmetric square, we stipulate in the same vein

$$\langle X \odot Y, h \rangle_{\operatorname{Sym}^2 T} \stackrel{!}{=} \langle Y, X \lrcorner h \rangle = h(X,Y).$$

$$\langle e_i \odot e_j, e_k \odot e_l \rangle_{\operatorname{Sym}^2 T} \stackrel{!}{=} \langle e_j, e_i \lrcorner (e_k \odot e_l) \rangle = \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}.$$

The two summands are both 1 if $i = j = k = l$ and we get $\langle e_i \odot e_i, e_i \odot e_i \rangle = 2$. If only one of them is 1, this means $\langle e_i \odot e_j, e_i \odot e_j \rangle = \langle e_i \odot e_j, e_j \odot e_i \rangle = 1$, $i \neq j$. Thus $(\frac{1}{\sqrt{2}} e_i \odot e_i)_i \cup (e_i \odot e_j)_{i<j}$ is an ONB of $\operatorname{Sym}^2 T$.

Similarly we define the inner product on $\operatorname{Sym}^2 \mathfrak{so}(T)$. Interpreting R as an element of $\operatorname{Sym}^2 \mathfrak{so}(T)^*$, we obtain

$$\sum_l \langle R, \omega_k \odot \omega_l \rangle \omega_l = \sum_l R(\omega_k, \omega_l)\omega_l = \sum_l \langle \widehat{R}(\omega_k), \omega_l \rangle \omega_l = \widehat{R}(\omega_k).$$

Thus

$$\sum_{k,l} \langle R, \omega_k \odot \omega_l \rangle \omega_k \odot \omega_l = \sum_k \omega_k \odot \widehat{R}(\omega_k).$$

On the other hand, since $(\frac{1}{\sqrt{2}}\omega_k \odot \omega_k)_k \cup (\omega_k \odot \omega_l)_{k<l}$ is an ONB of $\mathrm{Sym}^2 \mathfrak{so}(T)$, we actually have

$$R = \frac{1}{2}\sum_k \langle R, \omega_k \odot \omega_k\rangle \omega_k \odot \omega_k + \sum_{k<l}\langle R, \omega_k \odot \omega_l\rangle \omega_k \odot \omega_l = \frac{1}{2}\sum_{k,l}\langle R, \omega_k \odot \omega_l\rangle \omega_k \odot \omega_l$$
$$= \frac{1}{2}\sum_k \omega_k \odot \widehat{R}(\omega_k).$$

This reminds us of a similar calculation rule for Λ^2, namely

$$\alpha = \frac{1}{2}\sum_i e_i \wedge \alpha(e_i).$$

Weitzenböck formulae

Having introduced the curvature endomorphism $q(R)$, it is time to show where it appears. Let ∇ denote the Levi-Civita connection of a Riemannian manifold (M, g). We also denote with ∇ its extension to tensor bundles, as well as the connection on some generic vector bundle EM.

- On $\Omega^p(M)$,
$$d^*d + dd^* = \nabla^*\nabla + q(R).$$
- On $\mathscr{S}^p(M)$,
$$\delta\delta^* - \delta^*\delta = \nabla^*\nabla - q(R),$$
where $\delta^* h = \sum_i e^i \odot \nabla_{e_i} h$ (so that $L_{\alpha^\sharp} g = \delta^*\alpha$ for $\alpha \in \Omega^1(M)$).
- On $\Omega^p(M, EM)$,
$$(d^\nabla)^* d^\nabla + d^\nabla (d^\nabla)^* = \nabla^*\nabla + q(R)_{\Lambda^p T^*} + \sum_k (\omega_k)_{\Lambda^p T^*} \otimes \widehat{R}(\omega_k)_E$$
$$= \nabla^*\nabla + q(R)_{\Lambda^p T^* \otimes E} - q(R)_E - \sum_k \widehat{R}(\omega_k)_{\Lambda^p T^*} \otimes (\omega_k)_E,$$
where $d^\nabla(\alpha \otimes v) = \sum_i (e^i \wedge \nabla_{e_i}\alpha \otimes v + e^i \wedge \alpha \otimes \nabla_{e_i} v)$.
- In particular, for $\alpha \in \Omega^1(M, TM)$,
$$((d^\nabla)^* d^\nabla + d^\nabla (d^\nabla)^*)\alpha = \nabla^*\nabla\alpha + \alpha \circ \mathrm{Ric} + R^{13}\alpha.$$
- If (M, g) is Einstein, we recover on $\Omega^1(M, T^*M)$
$$(d^\nabla)^* d^\nabla + d^\nabla (d^\nabla)^* = \nabla^*\nabla\alpha + \frac{1}{2} q(R).$$

Weitzenböck formulae for *double forms* $\Omega^p(M, \Lambda^q T^*M)$ are available in [11].

Some examples

We have already seen spaces of constant sectional curvature (spherical or hyperbolic). Let us have a look at other symmetric spaces.

Complex projective space. The following is taken from [3, §5] and adapted to our conventions. Let $M = \mathbb{CP}^n$ with its standard complex structure J and the Fubini–Study metric (normalized so that $1 \le \sec \le 4$).

The curvature operator of the first kind (positive convention) may be written as

$$\widehat{R}\sigma = \sigma - J \circ \sigma \circ J - \langle J, \sigma \rangle J,$$

or, utilizing the decomposition $\Lambda^2 T^* = \mathbb{R}J \oplus \Lambda_0^{2,+} \oplus \Lambda^{2,-}$,

$$\widehat{R} = 2(n+1)\,\mathrm{Id}_{\mathbb{R}J} + 2\,\mathrm{Id}_{\Lambda_0^{2,+}}.$$

For the curvature operator of the second kind (almost negative convention), we have in turn

$$\mathring{R}h = -\frac{1}{2}h + \frac{1}{2}\operatorname{tr}(h)g - \frac{3}{2}J \circ h \circ J$$

and $\mathrm{Sym}^2 T^* = \mathbb{R}g \oplus \mathrm{Sym}_0^{2,+} \oplus \mathrm{Sym}^{2,-}$, thus

$$\mathring{R} = (n+1)\,\mathrm{Id}_{\mathbb{R}g} + \mathrm{Id}_{\mathrm{Sym}_0^{2,+}} - 2\,\mathrm{Id}_{\mathrm{Sym}^{2,-}}.$$

Here the superscript $\pm$ indicates the subspace of tensors commuting (resp. anticommuting) with J. We note that $\Lambda_0^{2,+} \cong \mathrm{Sym}_0^{2,+} \cong \mathfrak{su}(n)$.

$\mathbb{CP}^n$ is Einstein, i.e. $Z = 0$. For the scalar curvature, we have $U = \frac{n+1}{2n-1} g \owedge g$, so

$$\widehat{U} = \frac{2(n+1)}{2n-1}\,\mathrm{Id}_{\Lambda^2}, \qquad \mathring{U} = \frac{2(n^2-1)}{2n-1}\,\mathrm{Id}_{\mathbb{R}g} - \frac{2(n+1)}{2n-1}\,\mathrm{Id}_{\mathrm{Sym}_0^2}.$$

Hence the Weyl parts are given by

$$\begin{aligned} \widehat{W} &= \frac{4(n^2-1)}{2n-1}\,\mathrm{Id}_{\mathbb{R}J} + \frac{2(n-2)}{2n-1}\,\mathrm{Id}_{\Lambda_0^{2,+}} - \frac{2(n+1)}{2n-1}\,\mathrm{Id}_{\Lambda^{2,-}}, \\ \mathring{W} &= \frac{n+1}{2n-1}\,\mathrm{Id}_{\mathbb{R}g} - \frac{3}{2n-1}\,\mathrm{Id}_{\mathrm{Sym}_0^{2,+}} - \frac{6n}{2n-1}\,\mathrm{Id}_{\mathrm{Sym}^{2,-}}. \end{aligned}$$

Cubing and tracing just for fun, we obtain

$$\begin{aligned} (2n-1)^3 \operatorname{tr}(\widehat{W}^3) &= 8(n-1)(n+1)n(2n-1)(4n^2+2n-11), \\ (2n-1)^3 \operatorname{tr}(\mathring{W}^3) &= -(n+1)(216n^4 - n^2 + 25n - 28). \end{aligned}$$

For $n = 3$ we obtain $\operatorname{tr}(\widehat{W}^3) = \frac{5952}{25}$ and $\operatorname{tr}(\mathring{W}^3) = -\frac{70136}{125}$.

The same analysis holds true on the dual (complex hyperbolic space $\mathbb{CH}^n$), but with signs of the curvature reversed.

Some important results and references

- If $\mathring{R} > 0$ on $\mathrm{Sym}^2_0 T$ and $\delta R = 0$ (harmonic curvature), then $\sec > 0$ [8]. (alm. pos. conv.)
- If $\mathring{R} \geq 0$ on $\mathrm{Sym}^2_0 T$ and (M, g) is Einstein, then $\sec \equiv k$ [6]. (alm. pos. conv.)
- $q(R) \geq 0$ on all $\mathfrak{so}(n)$-representations if and only if $\widehat{R} \geq 0$ [7, §4].
- If $\mathring{R} > 0$ on $\mathrm{Sym}^2_0 T$, then $\sec > 0$ [4]. (alm. pos. conv.)
- $\mathring{R}$ preserves $\mathrm{Sym}^2_0 T$ if and only if (M, g) is Einstein.
- If (M, g) is compact, connected, orientable, with $\mathring{R} > 0$ on $\mathrm{Sym}^2_0 T$, then M is a real homology sphere [15]. Even better, it is diffeomorphic to a spherical space form [4].
- If (M, g) is compact, connected, orientable, with $\widehat{R} > 0$, then M is a real homology sphere [13].
- If $\widehat{R} \geq \delta$, then $\sec \geq \delta/2$ [14, 15]. (doubly pos. conv.)
- If $\mathring{R} \geq \delta$ on $\mathrm{Sym}^2_0 T$, then $\sec \geq \delta$ [15]. (alm. pos. conv.)
- If $\mathring{R} \geq 0$ on $\mathrm{Sym}^2 T$, then g is flat [15]. (alm. pos. conv.)
- If $\widehat{R} > 0$, then the Gauß–Bonnet integrand is positive [9].

Recheck these and clear up!!! especially for conventions.

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