Billiards Mathematics in the Unit Square and Equilateral Triangle

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Abstract

We study mathematical billiards in the square and equilateral triangle. For the square, we characterize when a billiards trajectory is periodic and find an explicit formula for the k^{th} bounce of the billiards sequence. For the equilateral triangle, we prove that for every even integer $k \geq 4$, there exists a k-periodic billiards trajectory.

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1 Introduction

Billiards mathematics refers to a dynamical system consisting of a massless and infinitesimally small ball that moves in a broken line defined by bounces against the sides of some region. The segments between these bounces are known as legs, and each leg makes an incidence angle and exit angle as the ball approaches and departs a side of the region, respectively. Since the bounces are friction-less, by Snell's law of equal reflection, the incidence angle equals the exit angle. Unless the ball bounces off a corner, which causes the billiards trajectory to become undefined, the ball's initial position and angle completely determine its future path. We call this path a *billiards trajectory*, shown below in Figure 1.

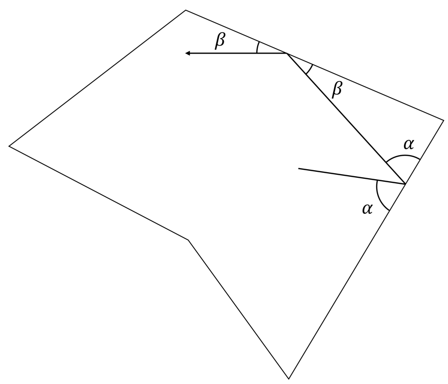


Figure 1: A billiards trajectory in a polygon

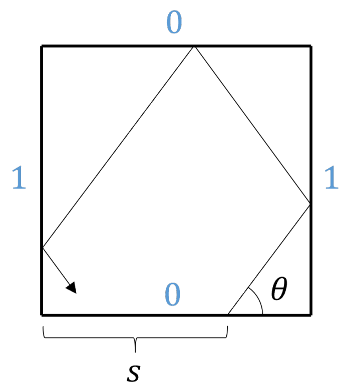
In this paper, we explore various properties of billiards trajectories within a square and an equilateral triangle. One natural question that can be asked is under what conditions a billiards trajectory is periodic. A billiards trajectory is said to be *periodic* if the ball returns to its same initial location and initial angle (s, θ), and thus the trajectory repeats. In the case of the square, we show the following:

Theorem 1. *Consider the billiards trajectory with equation $y = mx + n$. The trajectory is periodic if and only if the slope $m \in \mathbb{Q}$.*

Additionally, in the case of the equilateral triangle, we show:

Theorem 2. *For every even $k \in \{6n + 4m\}$ for $n, m \in \mathbb{N}_0$ with $n + m \geq 1$, there exists a k-periodic billiards trajectory in the equilateral triangle.*

A second property of billiards trajectories in the square arises when we keep track of which sides of the square the ball bounces off. To do so, we label each horizontal side of the square with a 0 and each vertical side with a 1. Then, we keep track of the sequence labels $\{b_1, b_2, \ldots\}$, where each label $b_i \in \{0, 1\}$ corresponds to the side that the i^{th} bounce of the billiards trajectory occurs on. We call this sequence the *billiards sequence*, shown below in Figure 2.



$$\{b_1, b_2, b_3, \ldots\} = \{1, 0, 1, \ldots\}$$

Figure 2: The billiards sequence

We will show that there is a way to derive an explicit formula for which side the k-th bounce is on:

Theorem 3. *The k-th label of the billiards sequence, b_k, is given by the explicit formula*

$$f(k) = 1 - \lceil b(k-s) + b \rceil + \lceil b(k-s) \rceil$$

where $b = \frac{m}{m+1}$ and $m = \tan\theta$, the slope of the billiard trajectory line.

While describing the k-th bounce in a billiards sequence may initially seem purely geometric, a critical step in proving them is to transform the billiards trajectory into an algebraic representation. In particular, as opposed to working with the billiards trajectory within the square or triangle, we instead describe an equivalent representation of a billiards trajectory within a tiled Euclidean plane. This transition allows us to directly leverage the initial location and angle of the ball to characterize the future bounces in terms of these initial conditions.

For the square, we first describe this equivalent representation then proceed to characterize the initial conditions that will result in a periodic trajectory. We then work towards finding an explicit formula describing the k-th label in a billiards sequence. We conclude with the equilateral triangle by studying for which k there exists a k-periodic trajectory.

2 Billiards trajectories in the square

2.1 Unfolding the square billiards trajectory

To more easily analyze how the billiards ball behaves in the unit square, we first introduce the concept of unfolding the billiards trajectory using the Square Unfolding Method described below. This method indeed relies on "unfolding" the billiards trajectory to obtain an equivalent representation of a billiards trajectory that lives within the Euclidean plane tiled with squares.

Method 1 (Square Unfolding Method)**.** *Suppose that a billiards trajectory has n bounces, and let the variable s_i denote the side of the square that the i^{th} bounce occurs on. For $i = 1, \dots, n$ reflect the square and the trajectory from bounce i onward over side s_i. In particular, a trajectory of n bounces has n legs, and when reflecting the square over the side that the i^{th} bounce occurs on, also reflect leg $i+1, \dots, n$ over s_i.*

We refer to this representation of a billiards trajectory as the *unfolded billiards trajectory*. Additionally, define a bounce of the unfolded billiards trajectory as where it intersects a side and define a leg as the trajectory between two subsequent bounces. Refer to Figure 3 for a single step of the Square Unfolding Method, and refer to Figure 4 for the resulting unfolded billiards trajectory.

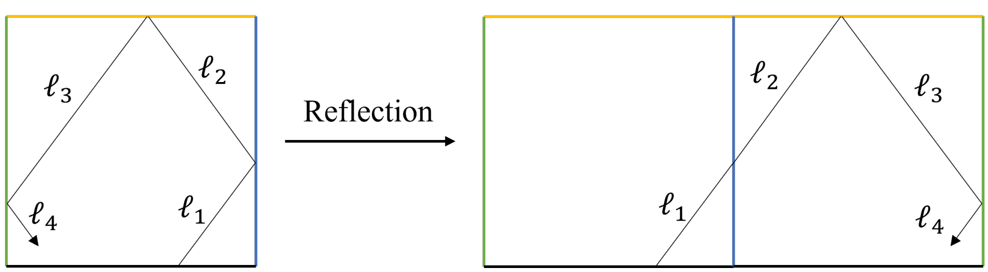


Figure 3: A reflection of the square

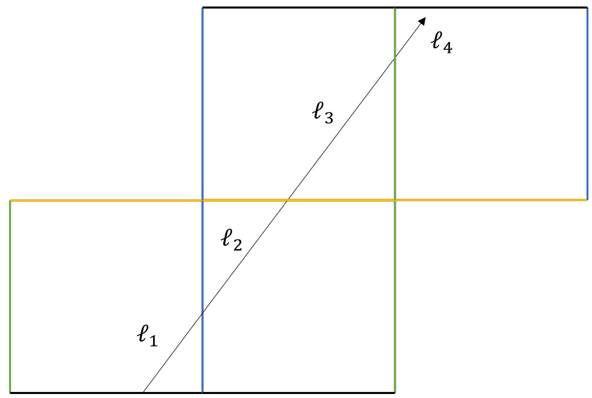


Figure 4: The unfolded billiards trajectory

To make use of the unfolded billiards trajectory, we prove two main properties. We first claim that the unfolded trajectory will be straight. In particular, the unfolded trajectory will be a ray.

Lemma 1. *The unfolded billiards trajectory obtained using the Square Unfolding Method is a ray in the tiled Euclidean plane.*

Proof. It suffices to prove that each pair of subsequent legs in the unfolded billiards trajectory are co-linear. As shown in Figure 5 below, l_1 forms an angle of α with s_1. By Snell's law of equal reflection, $\alpha = \beta$. Since we reflect l_2 over s_1 in the unfolding method to obtain l_2', then l_2' also makes an angle of $\gamma = \beta$ with s_1, and thus $\gamma = \alpha$ by transitivity. As l_1 and l_1' intersect s_1 on opposite sides both with angle α, they are co-linear. Because the trajectory is co-linear at each bounce, and because squares tile the Euclidean Plane, the unfolded trajectory will form a ray in the Euclidean Plane. □

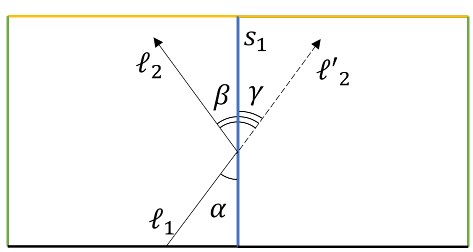


Figure 5: Unfolding the square

The above lemma establishes a connection between the geometric representation of a billiards trajectory and an equivalent algebraic representation. In particular, if the billiards trajectory originates at location s units along the bottom side of the square with angle θ, first introduced in Figure 2, then by Lemma 1 we can think of the unfolded billiards trajectory as a ray in $\mathbb{R}^2$ with equation

$$y = \tan(\theta) \cdot (x - s). \tag{1}$$

A second property of the unfolded billiards trajectory concerns the billiards sequence in the tiled Euclidean Plane. This property will be particularly helpful in algebraically deriving equations to describe the billiards sequence in Sections 3.2 and 3.3.

Claim 1. *The billiards sequence is equivalent between the unfolded billiards trajectory and the original billiards trajectory.*

Proof. Note that each bounce of the unfolded billiards trajectory is the result of a single unfolding iteration of the Square Unfolding Method, and that each iteration will always produce a bounce in the unfolded billiards trajectory. As we only perform a reflection across side s_i if the billiards trajectory bounced against s_i, then the only time that the unfolded trajectory will cross a side is if the ball bounced off that side, and thus the two sequences will be equivalent.

Additionally, note that the Square Unfolding Method is completely reversible. In particular, given an unfolded billiards trajectory we can "refold" it into the same billiards trajectory that produced the unfolded billiards trajectory. Since unfolding and then refolding a given billiards trajectory will bring the billiards trajectory back to itself, it must follow that the billiards sequence is equivalent between both representations. □

This process of unfolding a billiards trajectory is not particular to the square; it can be applied to any shape that can tile the plane using reflections through the edges. We will use it again when analyzing billiards on an equilateral triangle in Section 4.

2.2 Periodic trajectories

Recall that a billiards trajectory is periodic if the ball returns to its initial conditions (s, θ), and thus retraces its path ad infinitum. Given that (s, θ) completely determines a billiards trajectory, a natural question is what (s, θ) results in a periodic trajectory. To answer this question, we use the unfolded billiards trajectory representation.

For the purpose of describing periodic trajectories, the orientation of the squares in the unfolded representation matters. A periodic trajectory will be a line joining the same point in two distinct squares with the same orientation. Figure 6 illustrates this. This is true because the billiards ball will be returning to the same initial position s with the same outgoing angle θ, which by definition is a periodic trajectory.

More precisely, a periodic trajectory is defined by a line that goes from the initial position $(s, 0)$ to the same position in a translation of the original square, namely, $(s + 2p, 2q) : p, q \in \mathbb{Z}, q \neq 0$.

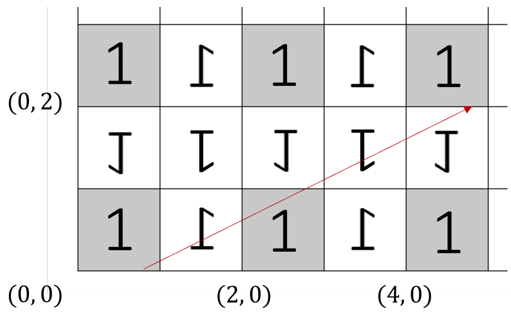


Figure 6: Periodic trajectory in the lattice representation. The 1's represent the orientation of each square. The squares with the same orientation as the original are shaded.

Theorem 4. *2.2 Consider the* billiards trajectory *with equation* $y = mx + n$. *The trajectory is periodic if and only if the slope* $m \in \mathbb{Q}$.

Proof. Suppose m is rational, then $m = \frac{y_0}{x_0}$ such that $x_0, y_0 \in \mathbb{Z}$. Note that in the tiled plane, the line connecting the starting point, $(s, 0)$, with the same starting point in the translation, $(s+2x_0, 2y_0)$, represents a periodic *billiards trajectory* with slope $m = \frac{y_0}{x_0}$, which is a number in $\mathbb{Q}$ by definition.

Conversely, assume that a given trajectory is periodic. In that case, the line representing the trajectory connects $(s, 0)$ with the same position in a translation $(s + 2x_0, 2y_0) : x_0, y_0 \in \mathbb{Z}$ of the original square. Then, the slope of the line is $m = y_0/x_0 \in \mathbb{Q}$.

Therefore, a trajectory is periodic if and only if the slope $m \in \mathbb{Q}$. $\square$

3 Billiards sequences in the square

Not only can we study billiards trajectories, but we can also study the corresponding billiards sequences. Recall that the billiards sequence is a sequence of 0s and 1s corresponding to the sides of the square that the billiards trajectory bounces off of, where bouncing off a horizontal side yields a 0 and bouncing off a vertical side yields a 1.

In this section, we will be exploring different ways of describing the billiards sequence. To start, we will be showing what types of restrictions there are on any billiards sequence.

3.1 Restrictions on the billiards sequence

Claim 2. *In a billiards sequence, after any* 0*, to then record exactly* k 1*'s in a row, then*

$$\frac{1}{k+1} < \tan(\theta) < \frac{1}{k-1}.$$

Proof. We will show this claim holds by proving an upper and lower bound on $\tan\theta$. To record exactly k 1s after any 0, geometrically, this means the billiards trajectory needs to intersect k vertical lines while only moving upwards by one square in the lattice grid.

To obtain an upper bound on the trajectory angle, in the billiards trajectory, we want to minimize the horizontal displacement needed to move upwards by one square in the lattice grid.

Note that the scenario where this occurs is when the bottom point is just barely left of the vertical line $x = a$ and the top point is just barely right of another vertical line $x = a + k - 1$. In the figure below, the red dots are intersections with vertical lines, which represent the k 1s.

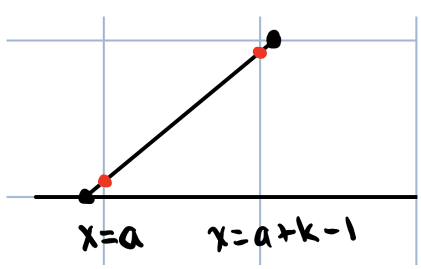


Figure 7: Greatest possible θ yielding exactly k 1s after a 0

To obtain the least possible angle, we want to maximize the length of the line from one side of of the square to another, from just right of the line $x = a - 1$ to just left of the line $x = a + k$, as shown below in Figure 8. Similar to Figure 7, the red dots represent intersections with vertical lines in the grid, which give the k 1s in the billiards sequence.

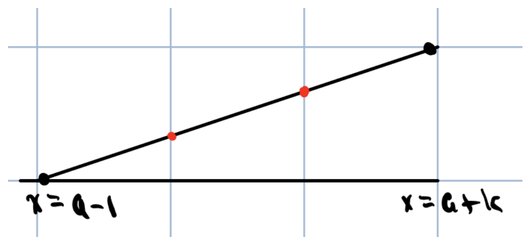


Figure 8: Least possible θ yielding exactly k 1s after a 0

Thus, to yield exactly k 1s after any 0 in the billiards sequence, the initial condition $\tan\theta$ must be at least $\frac{1}{k+1}$ and at most $\frac{1}{k-1}$.

A result that follows is that the length of sequences of consecutive numbers can only differ by one; if we know there are k 1s in a row, there can only be either $k-1$ or $k+1$ 1s in a row elsewhere in the sequence, but not both. Otherwise, the relation between $\tan\theta$ and k would not be satisfied.

For instance, a sequence can have three 0s in a row and two 0s in a row but not three 0s in a row and one 0 in a row. □

3.2 Location of the n-th 0

In Section 3.1, we gave a restriction on the billiards sequence. However, to describe the properties of billiards sequences in more detail, we seek to algebraically represent the sequence. In this section, we will find a formula that yields the location of the n-th 0. The location refers to the index of the n-th 0 in the sequence, where the sequence is indexed starting from 1.

Theorem 5. *The location of the n-th 0 is at index $\lfloor na+s \rfloor$, where $a = \frac{1}{\tan\theta}+1$.*

Proof. First, note that the number of bounces is in one-to-one correspondence with how many times the unfolded billiards trajectory intersects all the horizontal and vertical lines in the lattice plane. In particular, the sum of all intersections through $y = e$, $e \in \mathbb{Z}$ and all intersections through $x = f$, $f \in \mathbb{Z}$ is the total number of bounces in the billiards sequence.

Also, recall that the horizontal sides of the square are labeled with 0s and the vertical sides are labeled with 1s. So, an intersection with any horizontal line adds a 0 to the billiards sequence, and similarly, an intersection with any vertical line adds a 1 to the billiards sequence. Thus, to determine the location of the n-th 0, we want to count the number of horizontal and vertical line crossings.

To obtain the number of horizontal line crossings, we first observe that by the n-th 0 in the sequence, there must have been n horizontal line crossings. So, there are n horizontal line crossings.

To obtain the number of vertical line crossings, we use Equation (1), which represents the unfolded billiards trajectory:

$$y = (x - s) \cdot \tan\theta.$$

By plugging in $y = n$ and solving for x, we obtain the number of vertical lines the billiards trajectory intersects before the n-th 0, $\lfloor \frac{n+s\tan\theta}{\tan\theta} \rfloor$.

Adding the number of horizontal and vertical line crossings together, we get that the location of the n-th 0 is

$$n + \lfloor \frac{n + s\tan\theta}{\tan\theta} \rfloor.$$

Simplifying this expression yields

$$\lfloor na + s \rfloor, a = \frac{1}{\tan\theta} + 1.$$

Note that $a > 1$, which will be a useful fact in Section 3.3. □

3.3 An explicit formula for the k-th bounce

Now we will use the equation for the location of the n-th 0 to find an explicit formula $f(k)$, which describes which side of the square the k-th bounce is on. This is equivalent to finding the k-th digit of the billiards sequence. If the k-th bounce is on a horizontal side, the digit will be 0, and if the k-th bounce is on a vertical side, the digit will be 1.

Theorem 6. *The k-th digit of the billiards sequence is given by the explicit formula*

$$f(k) = 1 - \lceil b(k - s) + b \rceil + \lceil b(k - s) \rceil$$

where $b = \frac{m}{m+1}$ and $m = \tan\theta$, the slope of the billiard trajectory line.

Proof. The intuition behind the proof of this theorem is finding the possible indices of all 0s in the billiards sequence, then using this observation to determine if a given element of the sequence, b_k, is 0 or not. To achieve this, we will prove the following three results,

1. The kth element of the billiards sequence $b_k = 0$ if and only if $\exists n: \ k = \lfloor na + s \rfloor$.

2. $\exists n: \ k = \lfloor na + s \rfloor$ if and only if there is an integer in the interval $[b(k-s), b(k-s)+b)$, where $b = \frac{m}{m+1} \in (0,1)$.

3. There is an integer in the interval $[b(k-s), b(k-s)+b)$, where $b \in (0,1)$, if and only if $\lceil b(k-s)+b \rceil - \lceil b(k-s) \rceil = 1$.

From here, by transitivity, we will conclude that the k-th element of the billiards sequence b_k is a zero if and only if $\lceil b(k-s)+b \rceil - \lceil b(k-s) \rceil = 1$. Then, the explicit formula given in the theorem will follow directly from this.

Now that we have laid out the idea for the proof, we will proceed to show the three results listed above.

Lemma 2. *The kth element of the billiards sequence is* 0 *if and only if there exists an* $n \in \mathbb{N}$ *such that* $k = \lfloor na + s \rfloor$.

Proof. From section 3.2, we know that in the billiards sequence, the location of the n-th 0 is $\lfloor na + s \rfloor$ where $a = \frac{1}{\tan\theta} + 1$. Thus, the k-th digit of the billiards sequence is 0 if and only if there exists an $n \in \mathbb{N}$ such that $k = \lfloor na + s \rfloor$. So

$$f(k) = 0 \iff \exists n: \ k = \lfloor na + s \rfloor.$$

$\square$

Lemma 3. $\exists n: \ k = \lfloor na + s \rfloor$ *if and only if we can fit an integer in the interval* $[b(k-s), b(k-s)+b)$, *where* $b = \frac{m}{m+1} \in (0,1)$

Proof. We will use the following fact to show inequalities that hold if and only if there exists such an $n \in \mathbb{N}$ such that $k = \lfloor na + s \rfloor$.

Fact: $(\forall x \in \mathbb{R}) \ x - 1 < \lfloor x \rfloor \leq x$.

Plugging in the quantity $k = \lfloor na + s \rfloor$ for x, we obtain

$$na + s - 1 < k \le na + s.$$

Solving for n in both branches of this inequality yields

$$\frac{1}{a}(k - s) \le n < \frac{1}{a}(k - s + 1).$$

To simplify the notation, let $b = \frac{1}{a}$, then

$$b(k - s) \le n < b(k - s) + b.$$

where $b \in (0, 1)$, since $a > 1$.

If there exists an $n \in \mathbb{N}$ such that $k = \lfloor na + s \rfloor$, it must satisfy these bounds. So, these bounds hold if and only if there is an integer in the interval $[b(k - s), b(k - s) + b)$, where $b \in (0, 1)$. □

Lemma 4. *There is an integer in the interval* $[b(k - s), b(k - s) + b)$*, where* $b \in (0, 1)$*,* $\iff \lceil b(k - s) + b \rceil - \lceil b(k - s) \rceil = 1$.

Proof. Note that this is equivalent to saying that there is an integer in the interval $[x, x + b)$, for some $x, b \in \mathbb{R} : b \in (0, 1)$ if and only if $\lceil x + b \rceil - \lceil x \rceil = 1$. We can divide this into two cases. First, if x is an integer, it is not hard to see that the statement is true. If x is not an integer, the only way there is an integer in the interval $[x, x + b)$ is if $x + b > \lceil x \rceil$, which means that $\lceil x + b \rceil - \lceil x \rceil > 0$. But since $b \in (0, 1)$, we also have that $\lceil x + b \rceil - \lceil x \rceil < 2$. Therefore, it must be that if there is an integer in $[x, x+b)$, $\lceil x+b \rceil - \lceil x \rceil = 1$. □

As we have just shown a series of if and only if statements in Lemmas 2, 3, and 4, by the transitivity of if and only if statements, the k-th digit of the billiards sequence is 0 if and only if $\lceil b(k - s) + b \rceil - \lceil b(k - s) \rceil = 1$.

We can rewrite this statement as a piece-wise function $f(k)$, where $f(k)$ represents the kth digit in the sequence:

$$f(k) = \begin{cases} 0, \lceil b(k - s) + b \rceil - \lceil b(k - s) \rceil = 1 \\ 1, \lceil b(k - s) + b \rceil - \lceil b(k - s) \rceil = 0. \end{cases}$$

Finally, to simplify even further, note that we can write $f(k)$ without cases as

$$f(k) = 1 - \lceil b(k-s) + b \rceil + \lceil b(k-s) \rceil,$$

where $b = \frac{m}{m+1}$ and $m = \tan\theta$. $\square$

This explicit equation describing the digit of the k-th bounce can be used in a variety of ways. One of these is showing that if a billiards trajectory has a rational slope, its sequence is periodic. Earlier, in Section 2.2, we gave a geometric proof showing the relationship between rational slopes and periodic trajectories. Now, armed with the explicit equation for the k-th digit, we can give an algebraic proof.

Remark. *If the slope $m \in \mathbb{Q}$, then the billiards trajectory is periodic.*

Proof. First, note that $m \in \mathbb{Q}$ implies that there exist $p, q \in \mathbb{Z}$, $\gcd(p,q) = 1$ such that $m = p/q$. We claim that the period is $L = p + q$.

In the explicit formula $f(k)$, we had represented b as $\frac{m}{m+1}$. Now, we substitute in $\frac{p}{q}$ for m, obtaining:

$$b = \frac{m}{m+1} = \frac{\frac{p}{q}}{\frac{p}{q}+1} = \frac{1}{1+\frac{p}{q}} = \frac{p}{p+q} = \frac{p}{L}$$

where $p, L \in \mathbb{Z}$.

Then, $f(k)$ can be rewritten as

$$f(k) = 1 - \left\lceil \frac{p}{L}(k-s) + b \right\rceil + \left\lceil \frac{p}{L}(k-s) \right\rceil.$$

To prove L is a period, it suffices to show that $(\forall k)\ f(k+L) = f(k)$. Note that

$$\begin{aligned} f(k+L) &= 1 - \left\lceil \frac{p}{L}(k+L-s) + b \right\rceil + \left\lceil \frac{p}{L}(k+L-s) \right\rceil \\ &= 1 - \left\lceil \frac{p}{L}(k-s) + b + p \right\rceil + \left\lceil \frac{p}{L}(k-s) + p \right\rceil. \end{aligned}$$

Since $p \in \mathbb{Z}$, we can pull them out of the ceiling functions and simplify further:

$$\begin{aligned}
&= 1 - \left\lceil \frac{p}{L}(k-s) + b \right\rceil - p + \left\lceil \frac{p}{L}(k-s) + p \right\rceil + p \\
&= 1 - \left\lceil \frac{p}{L}(k-s) + b \right\rceil + \left\lceil \frac{p}{L}(k-s) + p \right\rceil \\
&= f(k).
\end{aligned}$$

Thus, we have successfully shown that if the slope of the periodic trajectory is rational, then the billiards sequence is periodic. □

4 Equilateral triangle

Similarly to the case of the square, periodic trajectories exist within the equilateral triangle too. However, while we did not previously specify the number of bounces in the periodic trajectory in the case of the unit square, we now search for $k-$periodic trajectories, where a billiards trajectory is $k-$*periodic* if after exactly k bounces it returns to its initial conditions (s, θ) and thus repeats.

Before proving there exists $k-$periodic trajectories in the triangle for certain $k \in \mathbb{N}$, we proceed to describe the Equilateral Triangle Unfolding Method, which similarly to the square, will allow us to represent billiards trajectories as a ray in the Euclidean plane tiled with equilateral triangles.

4.1 Unfolding the triangular billiards trajectory

As demonstrated on the square, we can use an unfolding method to transform a billiards trajectory in a shape that tiles the Euclidean plane to a ray in the tiled Euclidean plane, which we call the unfolded billiards trajectory. Since equilateral triangles tile the Euclidean plane, as depicted in Figure **??** below, we can also use an unfolding method on the equilateral triangle.

Method 2 (Triangle Unfolding Method)**.** *Suppose that a billiards trajectory has n bounces, and let the variable s_i denote the side of the triangle that the i^{th} bounce occurs on. For $i = 1, \dots, n$ reflect the triangle and the trajectory from bounce i onward over side s_i. In particular, a trajectory of n bounces has n legs, and when reflecting the square over the side that the i^{th} bounce occurs on, also reflect leg $i+1, \dots, n$ over s_i.*

A single reflection iteration of the Triangle Unfolding Method, which is identical to the Square Unfolding Method, is shown below in Figure 9.

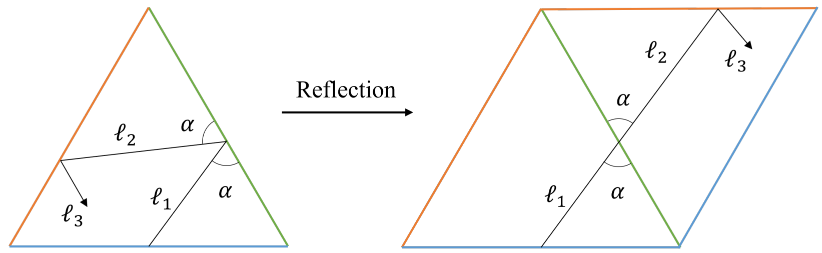


Figure 9: A unfolding of the triangle

Lemma 5. *The unfolded billiards trajectory obtained using Triangle Unfolding Method is a line segment in the tiled Euclidean plane.*

Proof. Using Figure 9 above, we use a similar argument as in the corresponding proof of the unfolded trajectory of the square in Lemma 1. In particular, because both l_1 and the reflected l_2 intersect s_1 with an angle of α, they are co-linear. □

4.2 Unfolded trajectories in the triangle

We proceed towards our goal of finding certain k-periodic trajectories by first showing a correspondence between periodic trajectories in the original and unfolded representations.

Consider an equilateral triangle tiling of the Euclidean space where triangles in the same orientation are shaded, as shown in Figure 10 below, where the orientation refers to the relative position of the colored sides of the triangle. We call this the *oriented* tiled Euclidean plane.

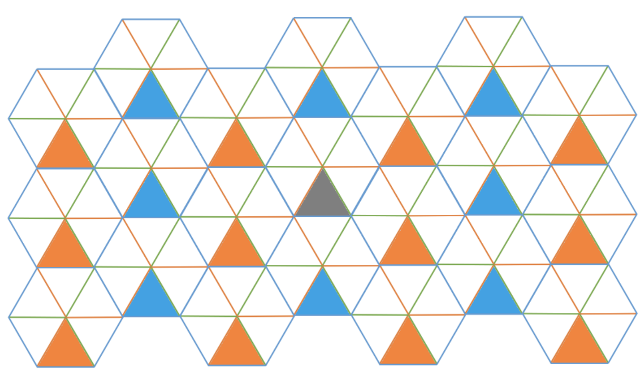


Figure 10: Decomposing the oriented tiled Euclidean plane where V_1 covers shifts to blue triangles, and V_2 covers shifts to orange triangles.

We now introduce a definition that will formalize this notion of equivalent points between triangles of the same orientation, as this will allow us to later prove that trajectories between equivalent points are periodic.

Definition 1 (Equivalent points)**.** *Given a point (x_0, y_0) in the tiled Euclidean plane, then a point (x, y) is said to be equivalent if $(x, y) \in V_1 \cup V_2$ where*

$$V_1 = \{(x_0 + 3n,\ y_0 + \sqrt{3}m) \in \mathbb{R}^2 \mid n, m \in \mathbb{Z}\}$$

$$V_2 = \{(x_0 + 3n + \frac{3}{2},\ y_0 + \sqrt{3}m + \frac{\sqrt{3}}{2}) \in \mathbb{R}^2 \mid n, m \in \mathbb{Z}\}.$$

To convince oneself that Definition 1 is in agreement with Figure 10, note that V_1 and V_2 covers shifts to blue triangles, and V_2 covers shifts to orange triangles, respectively.

We now use this notion of equivalent points to argue that unfolded trajectories connecting equivalent points correspond to periodic billiards trajectories.

Claim 3. *A ray between any two equivalent points in an oriented tiled Euclidean plane represents a periodic billiards trajectory.*

Proof. As the Triangle Unfolding Method produces a straight line between the starting position and the location of the n^{th} bounce, because there is exactly one line connecting any two points, then every billiards trajectory will have exactly one unfolded trajectory. Similarly, each unfolded billiards trajectory will be refolded into exactly one billiards trajectory. Therefore, there is a 1-1 correspondence between billiard trajectories in the triangle and unfolded billiards trajectories in the tiled Euclidean plane.

As a ray between two equivalent points will by definition return to triangle in the same orientation with the same initial angle, the trajectory it represents is periodic. □

This result allows us to search for periodic billiards trajectories in the triangle by instead finding periodic unfolded trajectories, which are rays connecting equivalent points of the oriented tiled Euclidean plane.

4.3 A construction of k-periodic trajectories for certain k

When searching for k-periodic trajectories, notice from Figure 10 that a natural search direction is along the direction that translates shaded triangles to other shaded triangles. In particular, there are two principle directions whose length as measured in number of bounces will be used to prove the following result.

Theorem 7. *For every even $k \in \{6n + 4m\}$ for $n, m \in \mathbb{N}_0$ with $n + m \geq 1$, there exists a k-periodic billiards trajectory in the equilateral triangle.*

Proof. There are two possible directions that result in a translation from equivalent point to equivalent point, one with a length of 4 and the other with a length of 6. We show that we can take combinations of such translations to achieve the desired result for $k \in \{6n + 4m\}$.

Direction 1 involves moving from (x_0, y_0) to $(x_0 + 3, y_0)$, and Direction 2 involves moving from (x_0, y_0) to $(x_0, y_0 + \sqrt{3})$, as shown by in Figure 11 below. As shown, Direction 1 yields 6 bounces in the corresponding trajectory since it intersects 6 lines, and Direction 2 yields 4 bounces in the corresponding trajectory for the same reason.

Given some starting point (x_0, y_0), all combinations of Direction 1 and Direction 2 to $(x, y) = (x_0 + 3n, y_0 + \sqrt{3}m)$ for $n, m \in \mathbb{Z}$ and $n + m \geq 1$ will translate to another distinct equivalent point because $(x_0 + 3m, y_0 + \sqrt{3}m) \in V_1 \cup V_2$ from Definition 1.

It now remains to be shown that the resultant vector of combinations of Direction 1 and Direction 2, $(x_0 + 3n, y_0 + \sqrt{3}m)$, is the sum of the number of bounces from each direction individually.

First note that Direction 1 is orthogonal to Direction 2, as the first only translates the x-coordinate while the second only translates the y-coordinate. Secondly, note that the number of bounces of each direction is simply the number of lines that the direction crosses. By orthogonality, any line crossed by Direction 1 *must* not be crossed by Direction 2, and similarly for Direction 2 with respect to Direction 1. Therefore, the number of lines crossed by a n translations of Direction 1 and m translations of Direction 2 is simply $6n + 4m$. Therefore, for every $k \in \{6n + 4m\}$ there exists a k-periodic trajectory between (x_0, y_0) and $(x_0 + 3n, y_0 + \sqrt{3}m)$ for $n, m \in \mathbb{Z}$ where $n + m \geq 1$. $\square$

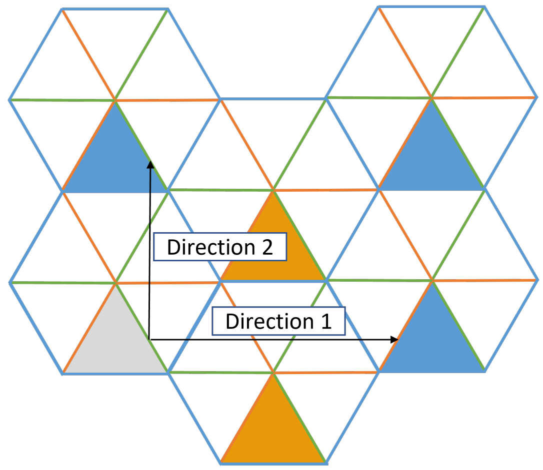


Figure 11: Direction 1 of length 6 bounces and Direction 2 of length 4 bounces translate between equivalent points

5 Who did what

Abstract and Introduction: mostly Peter, others helped edit/proofread

2.1: Peter

2.2: Oscar

3.1: mostly Cathy, Peter helped rephrase

3.2: Cathy

3.3: mostly Cathy, Oscar helped to reformat section.

3.3 Remark: Oscar proved rational implies periodic algebraically using explicit formula.

4.1: Peter

4.2: Peter

4.3: Peter

All group members helped each other proofread and edit. Oscar created all of the figures.

6 Reflection

One area of the paper where we worked especially hard was fine-tuning the introduction. Our initial introduction was very confusing to read, and we struggled with balancing what types of topics to formally define in the introduction, as well as where to introduce different concepts. We worked with both mentors and gathered their feedback to eventually have the introduction format we do now.

Another major feedback point was concerning all the math and text leading up to the explicit equation for the k-th bounce in the square. Initially, we struggled with even how to mathematically obtain this equation. Then, after we did reach an explicit formula, we had to figure out how to explain it in the paper in a way that was easy and organized to read. The presentation helped out with this, since we had to figure out how to organize the ideas in way that was easy to follow for the audience.

Finally, a major feedback point was making our language more concise and not having too many redundant figures. To address this, we have fewer figures and reference them in multiple sections. We also worked to proofread our language to make it more concise.