PTJC Talk #3: Wilson Lines & Confinement

Sam Kowash

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Over the last few talks, we've approached the topic of confinement and phases in gauge theories (and Yang–Mills in particular) in broad strokes by analyzing RG flows and by considering toy theories like the abelian Higgs model. Today, we'll delve a little deeper into our toolbox for actually diagnosing phase behavior in Yang–Mills. In particular, we'd like to find evidence for the presence or absence of a linear confining potential like that produced by the flux tubes we saw last week.

(1) Regge trajectories

Why are we looking for a linear potential? Certainly this is not the only way to confine quarks; why not, say, harmonic? The flux tube picture is nice, but why should we prefer it over another? The answer is that this picture is supported by experiment on a very intuitive level. Consider a pair of relativistic point particles bound by a linear potential in the CM frame: their energy will be

$$E = p + \sigma r, \tag{1.1}$$

where p is their relative momentum, r their separation, and σ the strength of the potential or the spring tension. We can rewrite this in terms of angular momentum $J = pr$ as

$$E = \frac{J}{r} + \sigma r, \tag{1.2}$$

whose minimum is attained at $r = \sqrt{J/\sigma}$ so that

$$E_{\text{min}}^2 = \left(\sqrt{\sigma J} + \sqrt{\sigma J}\right) = \sqrt{2}\sigma J, \tag{1.3}$$

where we should understand this CM energy as the rest mass of the composite particle or resonance. This is a (rather simple) example of a Regge trajectory, which in general identifies the poles of the S-matrix in the complex J plane for a given energy. By itself this continuous relation doesn't buy us a lot of phenomenology, but we know that in the real world the angular momentum is *quantized* and we should require $J \in \mathbb{Z}$. This gives us a set of *discrete* predictions for the energies of bound states, and more sophisticated analyses find impressive experimental agreement both for light mesons and for baryons, with an estimated string tension $\sigma \sim 1.2\ \text{GeV}^2$. This encourages us to examine YM for evidence of a linear confining potential, which we will do after a brief recap of the essential ingredients of gauge theories.

(2) Gauge field as connection

What are the essential ingredients of the pure Yang-Mills theory that we are considering? The elementary field is the gauge potential $A_\mu(x)$, which takes values in the Lie algebra $\mathfrak{g}$

corresponding to the gauge group G of the theory. It appears in the action through the field strength,

$$F_{\mu\nu} = \partial_{[\mu} A_{\nu]} - i[A_\mu, A\nu]$$

which defines the kinetic term

$$S_{\text{YM}} = -\frac{1}{2g^2} \int \mathrm{d}^4x \ \mathrm{tr}\, F_{\mu\nu} F^{\mu\nu},$$

where we've scaled the coupling out in front, which will be pleasant for some expansions. However, the gauge field is not *only* a dynamical degree of freedom in our system, but carries information about the realization of the gauge symmetry. When we wish to add matter to a gauge theory, we couple it through the covariant derivative

$$\mathcal{D}_\mu = \partial_\mu - iA_\mu,$$

which both reveals the gauge field as a force carrier *and* as carrying geometrical data relating fields at different points on the spacetime manifold, which is to say that the gauge field is a *connection*. The field strength can then be understood as a curvature corresponding to that connection, and we come away with a view of gauge theories as being very like general relativity, where there is a dynamical interplay between curvature and charges.

(3) Wilson loops

In particular, we can use the gauge connection to describe the parallel transport of gauge data, which is to say the color degree of freedom. Consider a heavy test particle that we drag along some worldline $C = \{x(\tau)\}_\tau$ in a background field A_μ. Attach to that particle a vector w in a representation $R(G)$ of constant length $w^\dagger w = \kappa$, and insist that it obey

$$i\frac{\mathrm{d}w}{\mathrm{d}\tau} = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} A_\mu(x) w,$$

where A_μ is in the appropriate representation to act on w. We can integrate this along some finite segment from x_i to x_f to get an operator $w(x_f) = U[x_i, x_f; C]w(x_i)$, which comes out to be

$$U[x_i, x_f; C] = \mathcal{P} \exp\left[i \int_{x_i}^{x_f} A_\mu \mathrm{d}x^\mu\right].$$

Note that this takes us from an algebra element to a group element; the gauge group tells its charges how to evolve under transport. This operator should be very familiar: in the case that $G = U(1)$, it is exactly the Aharonov–Bohm phase! We will call this more general construction a *Wilson line*. It has fine gauge covariance,

$$U \to \Omega(x_i) U \Omega^\dagger(x_f),$$

but we'd really rather work with something gauge *invariant*, which we can achieve by sewing the ends together and taking a trace, yielding the Wilson *loop*

$$W[C] = \mathrm{tr}\, \mathcal{P} \exp\left[i \oint_C A_\mu \mathrm{d}x^\mu\right]. \tag{3.1}$$

Mathematically, we can also recognize this as nontrivial *holonomy*, the hallmark of parallel transport on curved manifolds.

We now have a nice gauge-invariant quantity telling us *something* about the shape of our gauge configuration, which seems useful. How do we actually use it?

(4) Wilson loop as an electric probe

Parts of the last few talks have described phases of gauge theory principally in terms of the profile of the quark–antiquark potential: Coulomb ($1/r$), confined (r), Higgs (constant). This is sane enough, but how do we actually determine what that potential is? The general procedure is as follows:

1. Insert heavy $q, \bar{q}$ separated by R in a gauge-covariant way: $Q(0) = q(x)U[x, y; C]\bar{q}(y)$ (we really should be attaching a Wilson line as a string to any quark insertion to make a gauge-invariant object)
2. Evolve pair in Euclidean time for some T
3. Destroy pair with $Q^\dagger(T)$

Integrating out the color degrees of freedom in our static quarks makes this into a Wilson loop (cf. Tong), of which we then take an expectation! At long T, the partition function becomes dominated by the ground state contribution $\exp(-E_0 T)$, and the ground state energy in the presence of our $q\bar{q}$ pair is exactly the static quark potential.

Thus, we may define

$$V(R) \equiv \lim_{T\to\infty} -\frac{\ln \langle W[C] \rangle}{T},$$

showing that if we can compute the Wilson loop for any system, we can compute the potential and thereby diagnose the phase of the system. We will now see how confinement emerges in a particular limit.

(5) Ex: strong coupling limit on the lattice

Let us take our theory to a lattice Λ with spacing a. Our gauge information is now carried by *link variables* $U_\mu(n) = \exp\left[iaA_\mu(n)\right]$ which are exactly analogous to the finite transporters we thought about in the continuum. The correct Euclidean action is not trivial to see, but gauge invariance and hermeticity suggest a form that turns out to be the right one:

$$S_G = \underbrace{-\frac{2N}{g^2}}_{-\beta} \sum_P \frac{1}{2N} \operatorname{tr}\left(U_P + U_P^\dagger\right) + \text{const.},$$

where U_P is a path-ordered product of link variables around an elementary plaquette P. This turns out to be exactly equivalent to a kinetic term in a lattice field strength, but this formulation is much more usable. The group-valued link variables are our dynamical degrees of freedom, so we will need to consider in a moment how to integrate over a group (Haar measure).

It's simple enough to build up our Wilson loop as an ordered product $W_C[U]$ of link variables and take its expectation in the familiar way:

$$\langle W_C[U] \rangle = \frac{\int \mathcal{D}U\, W_C[U] \prod_P \exp\left[\frac{\beta}{2N} \operatorname{tr}\left(U_P + U_P^\dagger\right)\right]}{\int \mathcal{D}U \prod_P \exp(\beta S_P)}.$$

We wish to consider the strong-coupling limit of this expectation, in which β becomes small and we may expand the exponential upstairs in its powers (the downstairs expansion gives only higher-order corrections):

$$\exp\left[\beta \sum_P S_P\right] = \prod_P \left[\sum_{n=0} \frac{\beta^n}{n!} (S_P)^n\right].$$

The expansion lets us pop out as many copies of each plaquette of each orientation as we like, at the cost of a β for each one, so we want to find the leading term by the smallest number of plaquettes providing a nonvanishing integral against the Wilson loop. The relevant link integration rules are

$$\int \mathrm{d}U U^{ab} = 0$$
$$\int \mathrm{d}U U^{ab} U^{cd} = 0$$
$$\int \mathrm{d}U U^{ab} (U^\dagger)^{cd} = \frac{1}{N} \delta^{ac} \delta^{bd}.$$

This shows us that we get the lowest order contribution exactly by tiling the loop with plaquettes oriented so that every link is traversed once in each direction. This then tells us that

$$\langle W_C[U] \rangle \propto \left(\frac{\beta}{2N}\right)^{RT}$$

and accordingly that

$$V(R) \propto \left(-\ln \frac{\beta}{2N}\right) R,$$

a linear potential!