ON-SHELL RECURSION RELATIONS AT TREE LEVEL

JESSE STRYKER
20 Jan. 2016

- Review
- Motivate complex deformation of amplitudes
- Formulate " " in the spinor-helicity formalism
- Introduce BCFW recursion rel'ns

Last week:

- Introduced the spinor-helicity formalism for HEP amplitudes.
- Fields → massless → do calculations with helicity basis of Dirac spinors for fermionic fields
- Basic object of study: $A_n = A_n(\{p_i \,;\, \text{type } i\}_{i=1}^{n}) \in \mathbb{C}$

  "n-point amplitude"

  $\longrightarrow A_n(\{p_i, \text{type } i, \text{helicity } i\}_{i=1}^{n})$

  "helicity amplitude"

- Use helicity states $|p\rangle$, $\langle p|$, $|p]$, $[p|$ for the particles.

  Squares $\sim h > 0$

  Angles $\sim h < 0$

By convention, we regard all external momenta as outgoing.

For spin-$\frac{1}{2}$ fermion,

$$\bar{u}_+(p) = ([p|^{a}, 0)$$
$$\bar{u}_-(p) = (0, \langle p|_{\dot{a}})$$
$$v_+(p) = \begin{pmatrix} |p]_a \\ 0 \end{pmatrix}$$
$$v_-(p) = \begin{pmatrix} 0 \\ |p\rangle^{\dot{a}} \end{pmatrix}$$

External line rule for spin-1 massless vector is to "dot-in" polarization vec. Can be expressed using the spinor-helicity notation

$$\epsilon_-^\mu(p;q) = \frac{-\langle p|\gamma^\mu|q]}{\sqrt{2}\,[q\,p]}$$

$$\epsilon_+^\mu(p;q) = -\frac{\langle q|\gamma^\mu|p]}{\sqrt{2}\,\langle q\,p\rangle}$$

$q \neq p$ is arbitrary reference spinor ; reflects gauge invariance

- Introduced spinor-helicity formalism applied to $\mathcal{L}_{YM} = -\frac{1}{4}\,\mathrm{Tr}\,F_{\mu\nu}F^{\mu\nu}$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \frac{ig}{\sqrt{2}}[A_\mu, A_\nu]$

$A_\mu = A_\mu^a T^a$

with gauge gp $G = SU(3)$.

→ Choose <u>Gervais-Neveu gauge</u> to gauge-fix.

→ Feyn. rules give
- gluon propagator $\boxed{\delta^{ab}\frac{\eta_{\mu\nu}}{p^2}}$
- 3-g vertex $\ni$ ~~...~~ $\tilde{f}^{abc}$
- 4-g vertex $\ni$ ~~...~~ $\tilde{f}^{abx}\tilde{f}^{xcd}$ + perms

and kinematic factors

- Important result:

gluon state ; fixed for the diagram.

$$A_n^{\text{full, tree}} = g^{n-2}\sum_{\substack{\text{perms}\\ \sigma}} \underbrace{A_n[1\,\sigma(2\ldots n)]}_{\text{partial amplitudes : "color-ordered" amp's}}\,\mathrm{Tr}\left(T^{a_1}T^{\sigma(a_2}\cdots T^{a_n)}\right)$$

• MHV amp: Parke-Taylor Fla: For n-gluons at tree-level,

$$A_n[1^- 2^- 3^+ \cdots n^+] = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}$$

or

$$\boxed{A_n[1^+ \cdots i^- \cdots j^- \cdots n^+] = \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}}$$

~~• Today~~

• Computed $A_3[1^- 2^- 3^+] = \dfrac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle}$

and $A_4[1^- 2^- 3^+ 4^+] = \dfrac{\langle 12\rangle}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}$

• Today: • Want to learn how the "subamplitudes" can be used to compute larger tree diagrams.

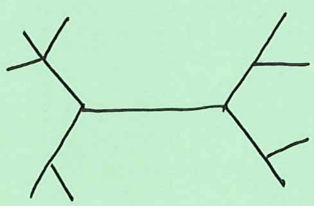
• Understand what the BCFW (Britto, Cachazo, Feng, Witten) recursion rel'ns (really, BCFW shift) are.

DEFORMATION OF $\{p_i\}$

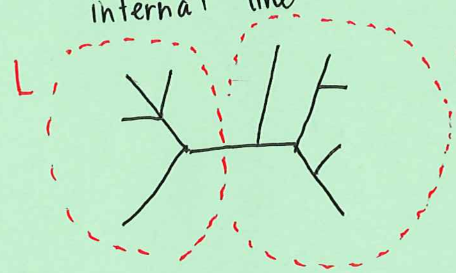
- Key idea: Use comp. analysis and exploit analytic props of on-shell (OS) amplitudes
  - Most famous O.S. rec. relns are BCFW, but $\exists$ others.

AMPLITUDES

- $A_n = A_n(\{p_i\}_{i=1}^N, \{\text{types } i\}, \{h_i\})$ where $p_i^2 = 0 \; \forall i$ and momentum is conserved: $\sum_{i=1}^n p_i^\mu = 0$.
- Picture to have in mind:

$A_n =$ 

Can imagine any diagram being separated into blobs through an internal line:

$A_n =$ 

~~E the~~ Want $A_n = A^L \frac{1}{P_I^2} A^R$

but for this to be useful, the propagator needs to go on-shell, that way the amplitudes of the sub diagrams are themselves O.S. amplitudes.

- We're going to complexify the momenta to achieve this factorization.

FORMAL DEVELOPMENT

- Introduce n complex vec's, $\{r_i\}_{i=1}^n$, some possibly zero, with following properties:
  - (i) $\sum_{i=1}^n r_i^\mu = 0$
  - (ii) $r_i \cdot r_j = 0 \quad \forall$ pairs (i,j) (so r_i are all null)
  - (iii) $p_i \cdot r_i = 0$ (no sum) $\forall i$

- Then, using these shifts, define the shifted momenta:

$$\boxed{\hat{p}_i^\mu \equiv p_i^\mu + z\, r_i^\mu \quad , \quad z \in \mathbb{C}.}$$

- $A_n \xrightarrow[\text{at } z]{\text{complex deformation}} \hat{A}_n(z)$, holomorphic fn s.t. $A_n = \hat{A}_n(0)$.

Properties of $\{\hat{p}_i\}$

(A) $\sum_i \hat{p}_i^\mu = 0 \sim$ "momentum cons."

(B) $\hat{p}_i^2 = 0 \sim$ momenta are on-shell still

(C) We define nontrivial subset of generic momenta by $\pi_I = \{p_i\}_{i\in I}$, where $I \subset \{1,2,\dots,n\}$, by

$$\begin{cases} \bullet\ 2 \leq \#(\pi_I) \leq n-2 \\ \bullet\ \left(\sum_{i\in I} p_i\right)^2 \neq 0 \end{cases}.$$

$\Rightarrow$ Define $P_I^\mu \equiv \sum_{i\in I} p_i^\mu$ and $R_I^\mu \equiv \sum_{i\in I} r_i^\mu$.

$\Rightarrow$ $\hat{P}_I^2 = P_I^2 + z\, 2\, P_I \cdot R_I$

$$\Rightarrow \boxed{\hat{P}_I^2 = -\frac{P_I^2}{z_I}(z - z_I) \quad \text{where} \quad z_I \equiv -\frac{P_I^2}{2 P_I \cdot R_I}}$$

- Easy to see analytic structure of shifted prop.
- $\exists\, z_I$ for all possible nontrivial subsets π_I.

- So far nothing has depended on tree/loop structure of a diagram.

Now let's specialize to tree-level amplitudes.

- Simple analytic structure: no branch cuts, only poles, and all the poles are order one poles for generic momenta.
- The poles come in as $\frac{1}{\hat{P}_I^2}$ and earlier we wrote

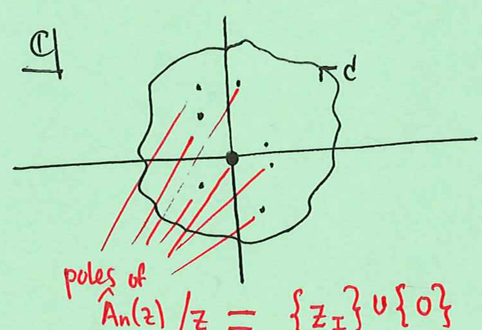
$$\hat{P}_I^2 = -\frac{P_I^2}{z_I}(z - z_I) \qquad \left(z_I = -\frac{P_I^2}{2\, P_I \cdot R_I} \right)$$

so it is easy to see the pole locations.

- By construction, $P_I^2 \neq 0$, so we further know all poles are off the origin in $\mathbb{C}$.

- A related fn, $\frac{\hat{A}_n(z)}{z}$ therefore has the same analytic structure up to the additional pole at the origin, whose residue is $A_n = \hat{A}_n(0)$.

- Suppose we apply Cauchy's residue theorem to a contour C enclosing all these poles:



$$\oint dz \frac{\hat{A}_n(z)}{z} = 2\pi i \sum (\text{residues of } \tfrac{\hat{A}_n(z)}{z}\text{'s poles})$$
$$= 2\pi i \left(A_n + \sum_{\{z_I\}} \mathrm{Res}_{z=z_I} \frac{\hat{A}_n(z)}{z} \right)$$

But also, $\oint dz \frac{\hat{A}_n(z)}{z} = -2\pi i (\text{residue at } \infty)$

$$\equiv -2\pi i\, B_n \qquad B_n \text{ is } \mathcal{O}(z^0) \text{ term in large } z \text{ expansion of } \hat{A}_n(z).$$

$$\Rightarrow \boxed{A_n = -\sum_{\{z_I\}} \mathrm{Res}_{z=z_I} \frac{\hat{A}_n(z)}{z} - B_n}$$

FACTORIZATION

- We will assume $B_n = 0$; sometimes this can be proven. Whether or not B_n vanishes generally depends on the choice of shift we made. A "valid" shift means $B_n = 0$.

- So, our recipe for computing A_n is to add up all the residues of $\frac{\hat{A}_n(z)}{z}$ from the poles mentioned previously. (Obviously, if this task were not easier, we would not be developing this formalism.)

- A straightforward evaluation shows how these residues reduce to products of amplitudes of smaller on-shell diagrams:

$$\mathrm{Res}_{z=z_I}\left[\frac{\hat{A}_n(z)}{z}\right] = \lim_{z\to z_I} \frac{\hat{A}_n(z)}{z}(z-z_I)$$

$$= \lim_{z\to z_I} \frac{\hat{A}_n(z)}{\cancel{z}}\left(-\frac{\hat{P}_I^2\ \cancel{z_I}}{P_I^2}\right)$$

$$= \lim_{z\to z_I}\left(-\frac{\cancel{\hat{P}_I^2}}{P_I^2}\ \hat{A}^L(z)\ \frac{1}{\cancel{\hat{P}_I^2}}\ \hat{A}^R(z)\right)$$

AS the $\hat{P}_I$ goes on-shell, the Amplitude is dominated by $\hat{A}_L(z_I)\frac{1}{\hat{P}_I^2}\hat{A}_R(z_I)$.

$$\boxed{\mathrm{Res}_{z=z_I}\left[\frac{\hat{A}_n(z)}{z}\right] = -\ \hat{A}^L(z_I)\ \frac{1}{P_I^2}\ \hat{A}^R(z_I)}$$

unshifted value

- Evidently we can build up bigger amplitudes out of smaller subamplitudes which have presumably been determined already. In this way, we have the recursive formula

$$\boxed{A_n = +\sum_{\text{diagrams } I} \hat{A}_L(z_I)\ \frac{1}{P_I}\ \hat{A}_R(z_I)}$$

BCFW Recursion

- The Britto, Cachazo, Feng, Witten on-shell rec. rel'ns are a particular form of the complex deformation procedure we have considered. Namely, we use the BCFW shift.

- Recall that some of the $\{r_i\}_{i=1}^{n}$ may be vanishing. The BCFW shift affects exactly two of the momenta, say $r_i \neq 0$ and $r_j \neq 0$ with $i \neq j$ and all other $r_k = 0$. Then the shift of momenta p_i and p_j results in the following transformation on the helicity spionors:

$$\boxed{\begin{aligned} |\hat{i}] &= |i] + z|j] \\ |\hat{j}] &= |j] \\ |\hat{i}\rangle &= |i\rangle \\ |\hat{j}\rangle &= |j\rangle - z|i\rangle \end{aligned}}$$

- This is referred to as an $[i, j\rangle$-shift.

- Importantly, we have the following identities:

$$\langle \hat{i}\hat{j}\rangle = \langle ij\rangle,$$
$$[\hat{i}\hat{j}] = [ij];$$

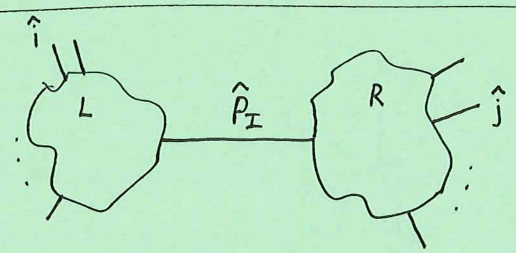
For $k \neq i$ or j,

$$[\hat{i}k] = [ik] + [jk]z,$$
$$\langle \hat{j}k\rangle = \langle jk\rangle - \langle ik\rangle z,$$
$$\langle \hat{i}k\rangle = \langle ik\rangle,$$
$$[\hat{j}k] = [jk].$$

Many spinor bracket combinations are unaffected. Only the $[\hat{i}k]$ and $\langle \hat{j}k\rangle$ are z-dependent, and they are both linear in z.

· With the BCFW shift, the recursive formula can be expressed as

$$A_n = \sum_{\text{diagrams } I}$$



Note that each blob contains one of the shifted momenta's external legs.

<u>Claim:</u> The Parke-Taylor Fla can be proven using the BCFW rec. rel'ns. (details in Elvang & Huang)

- Let's look at one example for concreteness: $|-,-\rangle$ shift.

The notation here means $i=1$ and $j=2$ and we have in mind a MHV amplitude $A_n(1^- 2^- 3^+ \cdots n^+)$.

- We'll take the answer, apply the shift, and show that the recipe gives back the expression for A_n.

- Parke-Taylor says
$$A_n = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}.$$

- If we take this and apply a shift to the expression, then the only spinor bracket that is nontrivially affected by the transformation is
$$\langle \hat{2}\hat{3}\rangle = \langle 23\rangle - z\langle 13\rangle$$
and all others stay the same.

$$\Rightarrow \quad \hat{A}_n(z) = \frac{\langle 12\rangle^4}{\langle 12\rangle\big(\langle 23\rangle - z\langle 13\rangle\big)\cdots\langle n1\rangle}$$

$$= \frac{\langle 12\rangle^3}{\big(-\langle 13\rangle\big)\left(z - \frac{\langle 23\rangle}{\langle 13\rangle}\right)\cdots\langle n1\rangle}$$

- Now we see that there is only one possible z_I, namely $z_I = \frac{\langle 23\rangle}{\langle 13\rangle}$, so we evaluate the residue of $\frac{\hat{A}_n(z)}{z}$ at z_I:

$$\mathrm{Res}_{z=z_I} \frac{\hat{A}_n(z)}{z} = \left(\frac{1}{z_I} \frac{\langle 12\rangle^3}{(-\langle 13\rangle)\cancel{\left(z - \frac{\langle 23\rangle}{\langle 13\rangle}\right)} \cdots \langle n1\rangle} \right) \cancel{\left(z - \frac{\langle 23\rangle}{\langle 13\rangle}\right)}$$

$$= \frac{\langle 13\rangle}{\langle 23\rangle} \frac{\langle 12\rangle^3}{(-\langle 13\rangle)\langle 34\rangle \cdots \langle n1\rangle}$$

$$= \frac{-\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle \cdots \langle n1\rangle}.$$

The "summation" here is trivial, and it is obvious that the result is identical to the exact, unshifted expression we started with.

- Didn't actually do recursion here. Rather, just showed how the application of the shift and the described procedure results in the right answer. "consistency check"

- Next week: Applications of BCFW