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Contrived Analogy : Intentions 1.) Give perspective on Macroscopic Goals of next hour
2.) Entertain Me

- My Instrument, the Marimba, is 8ft long and Approx. 300 lbs. It takes some physicality and clever Solutions to perform it is slightly different than the tasks we have evolved to do.

- As it has been quickly becoming a western art instrument, Composers who are not also performers are writing music for this instrument. And this is great. because it means we are no longer a self contained community and it is good to have external interactions in your intellectual endeavours.

- But these compositions tend to contain a larger number of un-playable passages. The physical information is there, but can be obscured by technically complicated instructions that may or may not be contributing to the moment the composer had in mind.

We don't Go crying to the composer at this point. We allow these passages to remain because 1.) It is good to be aware of your inadequacies and 2.) It clearly depicts goals for the next ambitious generation of performers to surpass us.

So this just means that every time you pick up a new composition, there may be some necessary translation to extract the physically meaningful parts and throw out/modify the extraneous parts.

In this talk we will solidify what has been anticipated for the past two weeks which is constructing BCFW recursion relations. We will find an expression of Amplitudes in terms of Shifted momenta that need to be translated back to physical momenta. We will find poles that should be present that don't look to be present as well as poles that seem to be present that should not be. Through All of this we will have the goal of making the physical characteristics manifest, removing the Extraneous bits.

So let's get Started

Inductive Proof of Parke-Taylor

$$A_n[1^-2^-3^+] = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 31\rangle} \sim t_1^{-2h_1} t_2^{-2h_2} t_3^{-2h_3}$$ Set by • little group scaling

$$\sim t_1^{+2} t_2^{+2} t_3^{-2}$$ • Locality

$|i\rangle \to t_i |i\rangle$

Just like in Conformal theories where 3-point correlation functions set by 1

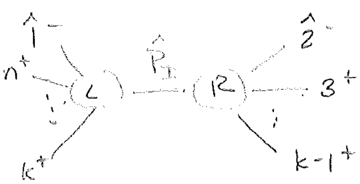
$$A_n^{\text{full, tree}} = g^{n-2} \sum_{\text{perms } \sigma} A_n[1\,\sigma(2\ldots n)]\, \mathrm{Tr}[T^{a_1} T^{\sigma(a_2 \ldots a_n)}]$$

$A_n[1^-2^-3^+ \ldots n^+] = ?$ Shifts $|--\rangle$, $|-+\rangle$, $|+-\rangle$, $|++\rangle$

large z behavior $\hat{A}_n^{\text{tree}}(z) \sim \frac{1}{z}$ $\frac{1}{z}$ z^3 $\frac{1}{z}$

(adjacent shifts) - non-adjacent gets extra $\frac{1}{z}$

Choose $|--\rangle$ with $[1,2\rangle$

$$A_n[1^-, 2^-, 3^+, \ldots n^+] = \sum_{\text{diagrams}} \sum_{h_I}$$ 

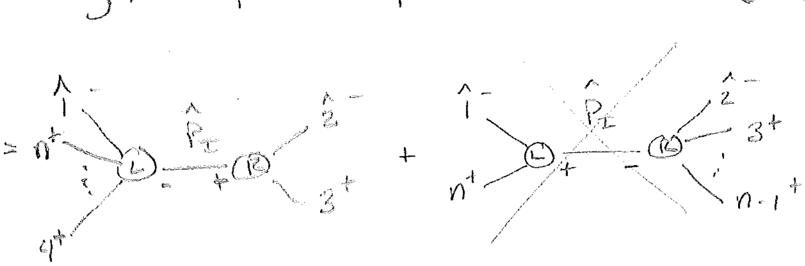
$$= \sum_{\text{diagrams}} A_n[\hat{1}^- \hat{P}_I^{h_I} k^+ \ldots n^+] \frac{1}{P_I} A_n[\hat{2}^- 3^+ \ldots k-1^+ \hat{P}_I^{h_I}]$$

Properties:
* Amplitudes are cyclic (from the trace structure) (2.82)
  Indicate clockwise - good to go
  counterclockwise $(-1)^n$
* $A_n(1^+2^+\ldots n^+) = 0$ and $A_n(1^-2^+\ldots n^+) = 0$ At tree level,
  by looking @ combinations of polarization vectors (2.7)
* $A_n(1,\ldots n) = (-1)^n A(n \ldots 1)$
  helicity flips with Parity.

$\hat{P}_I$ helicity: outgoing from (L) with $h_I \Rightarrow$ outgoing from (R) with $-h_I$

Result of calculating All On-shell Amplitudes with outgoing lines $\sum_i P_i = 0$

$\sum_i \langle qi\rangle[ik] = 0$

$= $ 

$\langle q|P|k] = -\langle qP\rangle[Pk]$

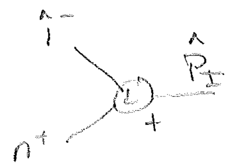
2.34

$\langle k|P|k] = -\langle kP\rangle[Pk]$
$= \langle Pk\rangle[Pk]$
$= 2P.k$

$\hat{P}_I$ evaluated at Residue $z = z_I$ s.t. $\hat{P}_I^2 = 0$.

On-Shell Amplitudes built out of On-Shell Amplitudes!

Now, Look at piece



know from fixed 3-point Amplitudes: $\hat{P}_{in} = \hat{1} + n$

$$A_3[\hat{1}^- -\hat{P}_{in}^+ \; n^+] = \frac{[\hat{P}_{in}\, n]^4}{[\hat{P}_{in} n][n \hat{1}][\hat{1}\hat{P}_{in}]}$$

(flip all helicities : $|\rangle \to |]$)

On-shell shift condition $\quad [1,2\rangle$

$$\hat{P}_{in}^2 = 0 = \langle \hat{1} n\rangle[\hat{1} n] \underset{\text{defining } z}{=} \langle 1 n\rangle[\hat{1} n] = 0$$

$$\Rightarrow \quad [\hat{1} n] = 0$$

See if the numerator saves us:

$$|\hat{P}_{in}\rangle[\hat{P}_{in} n] = -\hat{P}_{in}|n] = -(\hat{P}_1 + \overbrace{P_n}^{\text{solution to the massless Weyl Eqn}})|n]$$

$$= -\hat{P}_1|n]$$

$$= |\hat{1}\rangle[\hat{1} n] = 0$$

$$\text{So } A_3[\hat{1}^- -\hat{P}_{in}^+ \; n^+] = \frac{0^4}{0^3} = 0$$

Amplitude vanishes and we are down to 1 diagram. Before we go on, Identify why this happened so we can exclude graphs more quickly in the future.

Anti-MHV $[\,]$ diagram, with $[1$-shifted momenta
will happen for
MHV $\langle\,\rangle$ diagram, with $\langle 1$-shifted momentum

} shift bracket must be <u>different</u> from Amplitude bracket

Left with 1 MHV × anti-MHV gluon tree

$$A_n[\hat{1}^- \hat{2}^- 3^+ \ldots n^+] = \hat{A}_{n-1}[\hat{1}^- \hat{P}_{23}^- \, 4^+ \ldots n^+] \frac{1}{P_{23}^2} A_3[\hat{2}^- 3^+ -\hat{P}_{23}^+]$$

Inductive Statement that Parke-Taylor holds for $(n-1)$-point Amplitudes.
Demonstrating power of BCFW recursion in particle number

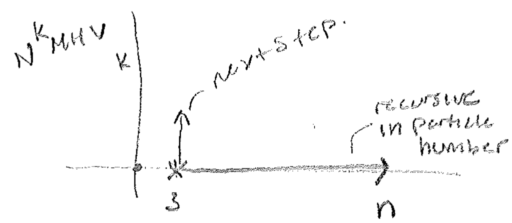
$$A_n[\hat 1^- \hat 2^- 3^+ \cdots n^+] = \frac{\langle \hat 1 \quad \hat P_{23}\rangle^4}{\langle \hat 1 \hat P_{23}\rangle\langle \hat P_{23} 4\rangle\cdots\langle n 1\rangle} \quad \frac{1}{\langle 23\rangle[23]} \quad \frac{[3\hat P_{23}]^4}{[3\hat P_{23}][\hat P_{23}\hat 2][\hat 2 3]}$$

Now some tricks to make this Evidently the n-point parke-Taylor

$[12\rangle$

$$\langle \hat 1 \hat P_{23}\rangle[3\hat P_{23}] = +\langle \hat 1|\hat P_{23}|3]$$
$$= \langle \hat 1|\hat 2|3] = -\langle \hat 1\hat 2\rangle[\hat 2 3]$$
$$= -\langle 12\rangle[23]$$

$$\langle \hat P_{23} 4\rangle[\hat P_{23}\hat 2] = \langle 4|\hat P_2 + P_3|\hat 2]$$
$$= -\langle 43\rangle[32]$$
$$= \langle 34\rangle[32]$$



Now rid of all $\hat P_{23}$ and ^'s in general.

$$= \frac{\langle 12\rangle^3[23]^3}{\langle 34\rangle\cancel{[32]}\langle 45\rangle\cdots\langle n1\rangle} \quad \frac{1}{\langle 23\rangle\cancel{[23]}} \quad \frac{1}{\cancel{[23]}}$$

Cancelling $[23]^3$ $= \dfrac{\langle 12\rangle^3}{\langle 23\rangle\langle 34\rangle\langle 45\rangle\cdots\langle n1\rangle}$ the n-point Parke-Taylor from only n-1 point Amplitudes!

Recursion in n.

Adjacent. If want 1 succinct eq for any MHV helicity Mess → Supersymmetry.

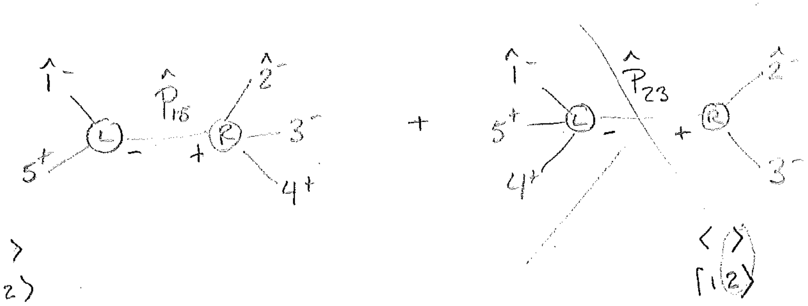
Now. The BCFW recursion is not only for building in Particle #, but also in $N^{K}MHV$

Stay mostly in our comfort zone with an Anti-MHV that we will treat as N^1MHV

$A_5[1^-2^-3^-4^+5^+]$ with 1--> so as to challenge ourselves with an NMHV rather than an Anti-MHV.

$[12\rangle$

$A_5[1^-2^-3^-4^+5^+] =$



$$A_5[1^- 2^- 3^- 4^+ 5^+] = \frac{\langle \hat{1} \hat{P}_{15}\rangle^4}{\langle \hat{1}\hat{P}_{15}\rangle\langle \hat{P}_{15} 5\rangle\langle 5 \hat{1}\rangle} \frac{1}{\langle 15\rangle[15]} \frac{\langle \hat{2} 3\rangle^4}{\langle \hat{2} 3\rangle\langle 34\rangle\langle 4 \hat{P}_{15}\rangle\langle \hat{P}_{15}\hat{2}\rangle}$$

Would like to get rid of the λ's, $\hat{P}_{15}^2 = 0$. Remember, we expect a ratio full of [] brackets if it is to look like the Anti-MHV Parke-Taylor

Trick: get $|P\rangle^3$ out of Top + Bottom with $\frac{[\hat{P}X]^3}{[\hat{P}X]^3} \cdot \frac{[\hat{P}2]^3}{[\hat{P}2]^3}$

Why is this choice convenient?

$X = \hat{1} \Rightarrow \frac{0^3}{0^3}$

$[12\rangle$

$$\langle \hat{1}\hat{P}\rangle[\hat{P}2] = -\langle \hat{1}|\hat{1}+5|2] = \langle \hat{1}5\rangle[52]$$
$$= \langle 15\rangle[52]$$

$$\langle \hat{P}5\rangle[\hat{P}2] = \langle 5|\hat{1}+5|2] = -\langle 5\hat{1}\rangle[\hat{1}2]$$
$$= -\langle 51\rangle[12]$$

$[\hat{1}2] = [12] + z\underbrace{[22]}_{0}$

$$\langle 4\hat{P}\rangle[\hat{P}2] = -\langle 4|\hat{1}+5|2] = \langle 4|\hat{2}+3+4|2]$$
$$= \langle 4|3|2]$$
$$= \langle 34\rangle[32]$$

$$\langle \hat{2}\hat{P}\rangle[\hat{P}2] = -2\hat{p}\cdot\hat{p}_2 = 2\hat{p}_2\cdot(\hat{p}_2+p_3+p_4) = (\hat{p}_2+p_3+p_4)^2 - (p_3+p_4)^2$$
$$= \hat{P}_{15}^2 - \langle 34\rangle[34] = -\langle 34\rangle[34]$$

Putting this together:

$$A_5[---++] = -\frac{\langle 15\rangle^3[52]^3\langle \hat{2}3\rangle^3}{\langle 15\rangle[12]\langle 51\rangle\langle 15\rangle[15]\langle 34\rangle\langle 34\rangle[32]\langle 34\rangle[34]}$$

$$= -\frac{[25]^3\langle \hat{2}3\rangle^3}{[12][23][34][15]\langle 34\rangle^3}$$

Bracket Evaluated At residue value of $z = z_{15}$ s.t. $\hat{P}_{15}^2 = 0$

$$\hat{P}_{15}^2 = \langle \hat{1}5\rangle[\hat{1}5] = \langle 15\rangle[\hat{1}5] = 0$$

z chose s.t. $[\hat{1}5] = 0$

$$[15] + z_{15}[25] = 0 \quad \Rightarrow \quad z_{15} = -\frac{[15]}{[25]}$$

$$\begin{aligned} \langle \hat{2}3\rangle &= \langle 23\rangle - z_{15}\langle 13\rangle \\ &= \langle 23\rangle + \frac{[15]\langle 13\rangle}{[25]} \\ &= \frac{-\langle 32\rangle[25] - \langle 31\rangle[15]}{[25]} \\ &= \frac{\langle 34\rangle[45]}{[25]} \end{aligned} \qquad \sum_i \langle pi\rangle[ik] = 0$$

Replacing $\langle \hat{2}3\rangle$:

$$\frac{[25]^3 \cancel{\langle 34\rangle^3} [45]^3}{[12][23][34][25]^3[51] \cancel{\langle 34\rangle^3}} = \frac{[45]^4}{[12][23][34][45][51]}$$

Need Reflective Statement here

Venturing Further: This time into the Unknown with the 6-point split-helicity Amp.

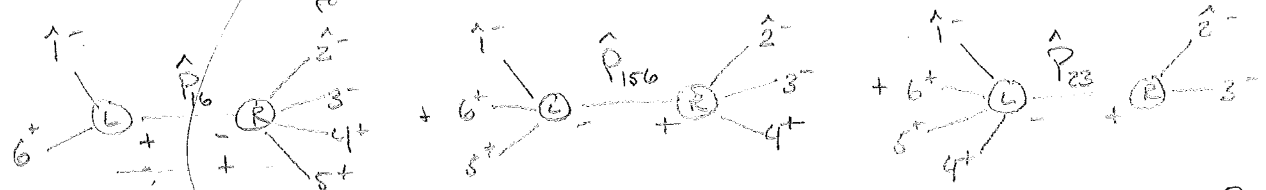
$$A_6[1^- 2^- 3^- 4^+ 5^+ 6^+]$$

Diagrams first:

Saw earlier how BCFW is recursive for particle #, now see how it is recursive for $N^K MHV$ level K.

$[21\rangle$ - shift

lower-point NMHV shows up in recursion relations.



Anti-MHV x NMHV

$[12\rangle$ - shift

$-$ $+$ ✓ X

2 Diagrams Compared to 38 Feynman Diagrams

Non-trivial, but true that $A + B + C = A' + B'$. Often Easiest to check numerically rather than venturing through the thicket of identities and mom. conservation as above.

Before we dive into this Amplitude, let's talk about poles.

Singularities only arise from scalar invariants à là Feynman

from week 1: for color-ordered Amplitudes, expect poles only from adjacent momenta going collinear — propagator momenta is only sum of adjacent momenta

$A(z)$ should be meromorphic (Feynman Rules Analytic)

MHV: only 2-particle poles

NMHV: 3 particle poles + 2 particle poles

⋮ ⋮

See poles at $P_{156}^2 = 0$ and $P_{16}^2 = 0$

$|-++\rangle$

why not $P_{345}^2 = P_{126}^2 = 0$

$$A_6[---+++] = \frac{\langle \hat{1}\hat{P}_{16}\rangle^3}{\langle \hat{P}_{16} 6\rangle\langle 6\hat{1}\rangle} \frac{1}{\langle 16\rangle[16]} \frac{\langle \hat{2}3\rangle^3}{\langle 34\rangle\langle 45\rangle\langle 5\hat{P}_{16}\rangle\langle \hat{P}_{16}\hat{2}\rangle} \frac{[\hat{P}_{16}3]}{[\hat{P}_{16}3]}$$

$\langle 12\rangle$

denom:

$$\langle \hat{P}_{16}\hat{2}\rangle[\hat{P}_{16}3] = \langle \hat{2}|\hat{1}+6|3] = -\langle 21\rangle[\hat{1}3] - \langle \hat{2}6\rangle[63]$$

$$\hat{P}_{16}^2 = \langle \hat{1}6\rangle[\hat{1}6] = \langle 16\rangle[\hat{1}6]$$

$$[\hat{1}6] = 0 = [16] + z[26]$$

$$z = -\frac{[16]}{[26]}$$

$$[\hat{1}3] = \frac{[13][26] - [16][23]}{[26]} \xrightarrow[\text{Identity}]{\text{Shouten}} -\frac{[12][63]}{[26]} = \frac{[12][36]}{[26]}$$

$+ [16][32]$

$\langle r\,\sigma(ijk)\rangle = 0$

$$\langle \hat{2}6\rangle = \frac{\langle 26\rangle[26] + [16]\langle 16\rangle}{[26]}$$

$$\frac{\langle 12\rangle}{[26]}\Big(\underbrace{[13][26] + [16][32]}_{\text{schouten}\; -[12][63]}\Big) + \frac{[36]}{[26]}\big(\langle 16\rangle[16] + \langle 26\rangle[26]\big)$$

$$\frac{[36]}{[26]}\big(\langle 12\rangle[12] + \ldots$$

$P_{126}^2 = P_{345}^2$

denominator:

$$\langle \hat{P}_{16}\hat{2}\rangle[\hat{P}_{16}3] = \langle \hat{2}|\hat{1}+6|3] = -\langle 21\rangle[\hat{1}3] - \langle \hat{2}6\rangle[63]$$

$$= \frac{[36]}{[26]}\big(\langle 12\rangle[12] + \langle 26\rangle[26] + \langle 16\rangle[16]\big)$$

$$= \frac{[36]}{[26]}\big(2P_1\cdot P_2 + 2P_2\cdot P_6 + 2P_1\cdot P_6\big) = \frac{[36]}{[26]}P_{126}^2$$

And there is our 126-pole!

Now that we have hit the Schooten Identity, I believe we have covered the entire basis of tricks needed for working with gluon Amplitudes

now I don't feel bad Sending you off on your own to Simplify $A_6[---+++]$ with the hint that $\frac{[\hat{P}_{16}\hat{2}]^3}{[\hat{P}_{16}\hat{2}]^3}$ factor is useful and Schematically.

$$A_6[---+++] = \frac{\langle 3|1+2|6]^3}{P_{126}^2 [12][61]\langle 34\rangle\langle 45\rangle\langle 5|1+6|2]} + \frac{1}{P_{156}^2}\left\{\frac{\sim}{\sim \langle 5|1+6|2]}\right\}$$

1.) Spurious poles. $\langle 5|1+6|2]$ bit odd as contains [ ] with non-adjacent lines... In both of them...

$$\frac{1}{(z-w)(z-v)} = \frac{1}{(z-v)(v-w)} + \frac{1}{(z-w)(w-v)}$$

algebraically true $v \neq w$ s.t. both are simple poles, Residues @ $z = v, w$ and $z \to \infty$ behaviour match.

Cancellation of Spurious poles = consistency condition. often checked w/ computational Algebra

Active Area of Research: how to make symmetries manifest and eliminate Spurious poles.

proof of Spurious-ity often involves gauge-dependent terms

- ruining the excitement of gauge invariance of these methods (avoiding gauge choice or field redefinitions.

2.) Little Group Scaling. $|i\rangle \to t_i |i\rangle$ $\quad |i] \to t_i^{-1}|i]$

$$A_n[1\ldots n] \to \prod_{i=1}^{n} t_i^{-2h_i} A_n[1\ldots n] \qquad \text{gluons: } \pm 1 \text{ helicity.}$$

$\underbrace{\langle 3 1\rangle [16]}_{0 \text{ factors of } t_1}$

$$P_{126}^2 = (1+2+6)^2 = 2p_1 \cdot p_2 + 2p_2 \cdot p_6 + 2p_1 \cdot p_6$$
$$= \langle 12\rangle[12] + \langle 26\rangle[26] + \ldots$$

similar Story 1 of each so no Scaling.

expectation $\sim t_1^2 t_2^2 t_3^2 t_4^{-2} t_5^{-2} t_6^{-2}$ ✓

Before we wrap up, I want to atleast make you aware of some other shift-choices that are explored in the literature

recall: $\sum_{i=1}^{n} r_i^\mu = 0$

$r_i \cdot r_j = 0$

$p_i \cdot r_i = 0$

diff. recursive structure for square-spinor shift

$|\hat{i}] = |i] + z c_i |X]$ with $\sum_i c_i |i\rangle = 0$

Recall Schouten: $c_1 = \langle 23 \rangle \quad c_2 = \langle 31 \rangle \quad c_3 = \langle 12 \rangle$

$c_{i>3} = 0$

satisfies momentum conservation

= Risager-shift.

also the all-line shift. $N^K MHV \sim \frac{1}{z^K}$ for large z

- Tower of MHV = bricks of $N^K MHV$ amplitudes

$A[N^K MHV] \to \sum_{K+1} MHV$ — 2004 MHV vertex expansion = CSW expansion — before recursion from complex shifts.

think about how one might win: MHV consist of $\langle\ \rangle$ only

shift on $|\ ]$ only.

diagrams experience shift only through

$$|\hat{P}\rangle \frac{[\hat{P} X]}{[\hat{P} X]} \propto \hat{P}_I |X] \frac{1}{[\hat{P} X]} \propto P_I |X] \frac{1}{[\hat{P} X]}$$ cancels top + bottom.

CSW Prescription $|\hat{P}_I\rangle \to P_I |X]$ check numerical independence of X.

Seem to be many representations for the same amplitude. Connections promised in future.

Reflect: We have constructed <u>all</u> higher-point gluon tree amplitudes (both in particle # n and N^kMHV level k) from the input of the

$\underbrace{\text{3-point}}_{\text{Little group Scaling, Locality}}$ gluon amplitude.

Is this true for All theories? Unfortunately, No.

Can only recurse what you put in.

Here: put in 4d, Local, $m=0$ $s=1$, $[g]=0$

fixed entire S_{gluon} matrix

$\mathrm{tr}F_{\mu\nu}F^{\mu\nu} \to A^2\partial A + A^4$

A^4 in Yang-Mills contains no new on-shell info: Set by $A^2\partial A$ off-shell gI

But, in theories with added ϕ^4 - Exists gauge-invariant information in $\lambda|\phi|^4$

BCFW will return solution for λ set s.t. $B_n = 0$

One way to get around this is if symmetries fix the 4-scalar contact term

Amplitunists?

Enter $N=4$ SYM.. all trees defined by 3-point gluon vertex.

Enter Gravity S-matrix determined by 3-vertex interaction gravitons

- ∞ interactions from expansion of Einstein Hilbert about flat-space metric fixed by diffeomorphism invariance of off-shell Lagrangian

- not needed for onshell tree diagrams

Additional Comments on including Scalar-ϕ^4 Interaction into Scalar QED

Scalar QED:

$$\mathcal{L} \supset e(A^\mu \partial_\mu \varphi^*)\varphi - A^\mu \varphi^* (\partial_\mu \varphi)) - e^2 A^\mu A_\mu \varphi^* \varphi$$

$A[\varphi \varphi^* \gamma \gamma]$ - constructable by BCFW

only 3-pt vertices needed

$A^\mu A_\mu \varphi^* \varphi$ - sets off-shell gauge invariance

Just like A^4 in YM.

$A[\varphi \varphi^* \varphi \varphi^*]$, however $\overset{\sqrt{2}e}{\swarrow}$

BCFW $\rightarrow$ $\tilde{e}^2 \langle \ \rangle$

Feynman $\rightarrow$ $\tilde{e}^2 (1 + \langle \ \rangle)$

What happened?

can only curse what you input.

Input: 4D, local, massless spin 1 + massless, charged spin 0, $[\tilde{e}]=0$

$\Downarrow$

Popped up: $\lambda |\varphi|^4$

$\hookrightarrow$ BCFW $\rightarrow$ $-\lambda + \tilde{e}^2 (1 + \langle \ \rangle)$

Family of Scalar QED models by λ

would understand how it fails when it does fail.

BCFW picked out one tuned to $\lambda = \tilde{e}^2$ as this is the condition to eliminate boundary term B_n $\rightarrow$ required for 'valid' recursion

Be wary of $\lambda |\phi|^4$ interactions... unless there is additional symmetries to fix the necessary gauge-independent information.

eg. 1) Gravity

2) SuperSymmetry - Next week ← favorite of Amplitugicians :)