① Intro, χral Symmetry

QCD matrix elements can be viewed as ~~functions~~ of the Dirac operator and quark masses, averaged over all possible background gluon fields. They are of course averaged over a very particular distribution that encodes all the spacetime and charge correlations of the theory.

In general this average is not "quenched," that is the distribution being averaged over depends on the Dirac operator and quark masses.

We can, however bury all of this complication in notation, and express the QCD partition fn as

$$Z_{QCD} = \left\langle \int \prod_f \mathcal{D}\psi_f \mathcal{D}\bar{\psi}_f \, e^{-\int d^4x\, \bar{\psi}_f (\not{D}+m_f)\psi_f} \right\rangle_{YM} = \left\langle \prod_f \det(\not{D}+m_f) \right\rangle_{YM}$$

Our topic this quarter is of course RMT. ~~[illegible]~~ Notice partition function is expressed as determinant of random matrix.

Universality – are there some properties that are independent of the distribution used?

universality – generally true for matrices w/ iid elements and some fixed symmetry class

- QCD $\not{D}$ does not have iid matrix elements $\langle \not{D} \rangle_{S(A)}$, correlations is spacetime/spin/color
- dynamical correlations are not universal across QFTs

Amazing result of χRMT (Shuryak & Verbaarschot '92, building on spectral analysis of QCD by Leutwyler & Smilga '92) is that some ~~[illegible]~~ static properties of ~~[illegible]~~ QCD are!

χral Symmetry

working definition (ignores several PTJCs worth of physics)

$$\exists\ \gamma_5 \text{ s.t. } \{\not{D}, \gamma_5\} = 0 \quad (\gamma_5^2 = 1,\ \not{D}^\dagger = -\not{D})$$

~~finite dim matrix~~

This has consequences ~~for~~ for spectrum of $\not{D}$

Treat $\not{D}$ as discrete spectrum (finite volume)

$$\not{D}\psi_n = i\lambda_n\psi_n \Rightarrow \not{D}(\gamma_5\psi_n) = -i\lambda_n(\gamma_5\psi_n) \qquad \int dx\, \psi_n^\dagger(x)\psi_m(x) = \delta_{mn}$$

choose a χral basis where

insensitive to UV reg & renorm details

$$\gamma_5\psi_n^\pm = \pm\gamma_5\psi_n^\pm, \qquad \not{D}_{mn}^{++} = \int dx\, \psi_m^+(x)\not{D}\psi_n^+ = 0 = \not{D}_{mn}^{--}$$

in this basis

$$\not{D} = \begin{pmatrix} 0 & iW & 0 \\ iW^\dagger & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{matrix} \}N_+ \\ \}N_- \\ \}N \end{matrix} \qquad \det(\not{D}+m_f) = \prod_n (i\lambda_n + m_f) = \left(\prod_{n,\lambda_n=0}(m_f)\right)\prod_{n,\lambda_n>0}(\lambda_n^2+m_f^2)$$

what is the rank of W, and how many zero modes are there? ~~dependence of partition fn~~

Topological Charge

Non Abelian gauge fields can be ~~classified~~ can be classified by an integer ν called the topological charge.

In an intuitive sense, there are semiclassical gluon configs localized in space & time called "instantons" and ν counts instantons − anti instantons.

generically of +8 − zero modes at separately cancel "accidental" zero modes ... $=0$ or $N_-=0$ ... so that N_++N_- is minimized

Deep result called Atiyah-Singer index thm says (see Kaplan 09)

$$\nu = N_- - N_+ \qquad Z_{QCD} = \left\langle \prod_f m_f^{|\nu|} \prod_n{}' (\lambda_n^2 + m_f^2) \right\rangle_{YM}$$

N_-: # RH modes; N_+: # LH modes

turns out calculating these in arbitrary background & averaging is hard

We are lead to define a generic χRMT $\prod_{i=1}^{N_-}\prod_{j=1}^{N_+} d\mathrm{Re}\, W_{ij}\, d\mathrm{Im}\, W_{ij}\; P_\nu(W_{ij})$

$$Z_{\chi RMT}(\nu) = \left\langle \prod_f \det\begin{pmatrix} m & iW \\ iW^\dagger & m \end{pmatrix} \right\rangle_P = \int \mathcal{D}W\, P_\nu(W)\, \prod_f \det\begin{pmatrix} m & iW \\ iW^\dagger & m \end{pmatrix}$$

W $N_- \times N_+$

$N_+ + N_- = V$

$N_- - N_+ = \nu$

W includes zero modes

First thing to explore, what's the chiral condensate?
(is χral symm spontaneously broken in the thermodynamic limit of the χRMT?)

$$\frac{1}{N_f}\sum_f \langle \bar{q}_f q_f \rangle_{QCD} = \frac{-1}{N_f V}\frac{\partial}{\partial m}\ln Z_{QCD} = \int \prod_f \mathcal{D}q_f \mathcal{D}\bar{q}_f\, \Sigma_f\left(\frac{1}{N_f V}\int d^4x\, q_f(x)\bar{q}_f(x)\right) e^{-\int d^4x\, \bar{q}_f(x)(\not{D}+m)q_f(x)}$$

$$\frac{1}{N_f}\sum_f \langle \bar{q}_f q_f \rangle_{\chi RMT} \equiv \frac{-1}{N_f V}\frac{\partial}{\partial m}\ln Z_{\chi RMT}$$

in either case, we have same mass dep in Z, so

Banks-Casher

$$\Sigma \equiv \lim_{m\to 0}\lim_{V\to\infty} \frac{1}{N_f}\sum_f \langle \bar{q}_f q_f \rangle\Big|_{\chi RMT} = \lim_{m\to 0}\lim_{V\to\infty}\left\langle \frac{|\nu|}{mV} + \sum_n^{V}{}' \frac{1}{i\lambda_n + m}\right\rangle_P \qquad |\nu| \sim \sqrt{V} \quad (\langle \nu^2 \rangle \sim V)$$

$$= \left\langle \frac{\pi}{V}\sum_n{}' \delta(\lambda_n)\right\rangle_P \equiv \frac{\pi\rho(0)}{V}$$

where we intro spectral density $\qquad \rho_s(x) = \lim_{\substack{V\to\infty \\ Vm_f\ \text{fixed}}} \frac{1}{V}\rho\left(\frac{\lambda}{V}\right)$ density near $\lambda = 0$ enlarged

$$\rho(\lambda) \equiv \left\langle \frac{1}{V}\sum_n{}' \delta(\lambda_n - \lambda)\right\rangle_P \qquad \text{Eval spacing } \Delta\lambda \sim \frac{1}{V\Sigma}$$

Large N (large V) χRMT breaks χsym spontaneously provided there is a dense accumulation of (non-zero-mode) eigenvalues around $\lambda = 0$.

Can be physically understood in context on Anderson localization, need long-range correlations in our eigenvalues to introduce competing eval forces w/ the repulsion away from $\lambda = 0$ present in the $\det(\not{D}+m)$ measure in χral limit.

To determine the quark condensate and symmetry breaking pattern of χRMT, it is helpful to make simplest anstaz for the probability dist $P(W)$ – Gaussian

$$Z_{\chi GUE}(\nu) = \int \prod_{i=1}^{N_-} \prod_{j=1}^{N_+} d\mathrm{Re}\, W_{ij}\, d\mathrm{Im}\, W_{ij}\; e^{-\frac{N}{2\lambda^2} \mathrm{Tr}(WW^\dagger)} \prod_f \det \begin{pmatrix} m & iW \\ iW^\dagger & m \end{pmatrix}$$

It can be shown that the properties we are about to derive are universal – any $O(W^4)$ or other non-Gaussian terms in $P(W)$ do not contribute in large V limit

To concretely study the theory, want to evaluate partition fn

$$Z_{\chi GUE}(\nu) = \int \prod_f \mathcal{D}\psi_f \mathcal{D}\bar\psi_f \prod_{i=1}^{N_-} \prod_{j=1}^{N_+} d\mathrm{Re}\, W_{ij}\, d\mathrm{Im}\, W_{ij} \exp\Big[\frac{-N}{2\lambda^2} W_{ij} W_{ij}^* + \sum_f i\psi_{fi}^* W_{ij} \varphi_{fj} + i\varphi_j^* W_{ij}^* \psi_i + m(\psi_{fi}^* \psi_{fi} + \varphi_{fi}^* \varphi_{fi})\Big]$$

$$\propto \int \prod_f \mathcal{D}\psi_f \mathcal{D}\bar\psi_f \exp\Big[+\frac{2\lambda^2}{N} \sum_{f,g} \psi_f^* \psi_g \varphi_g^* \varphi_f + m \sum_f \psi_f^* \psi_f + \varphi_f^* \varphi_f\Big]$$

Splitting 4-Fermi into $\psi\psi + \varphi\varphi$ & $\psi\psi - \varphi\varphi$ pieces and peforming a pair of Hubbard-Stratonovich transformations gives

$$\propto \int \prod_{f,g} \mathcal{D}\sigma_{fg} \mathcal{D}\bar\sigma_{fg} \det{}^{N_+}(\sigma + i\bar\sigma + m) \det{}^{N_-}(\sigma - i\bar\sigma + m^*) \exp\Big[\frac{-N}{2\lambda^2} \mathrm{Tr}(\sigma + i\bar\sigma)(\sigma - i\bar\sigma)\Big]$$

What have we gained? Gone from $V \times V$ matrix to $N_f \times N_f$!

$\sigma + i\bar\sigma$ generic complex matrix, might not be diagonalizable. But can always separate transform to left & right eigenbasis, that is find a $U, V \in U(N_f)$ s.t.

$\sigma + i\bar\sigma = U \Lambda V^{-1}$, Λ real diagonal positive def

$U \in U(N_f)$, $V \in U(N_f)/U(1)^{N_f}$ ← degree of freedom counting

$(\sigma + i\bar\sigma) \to L(\sigma + i\bar\sigma)R^\dagger$

$$Z_{\chi GUE}(\nu, m) = \int \mathcal{D}\Lambda \mathcal{D}U \mathcal{D}V\, J(\Lambda) \det{}^{N_+}(U\Lambda V^{-1} + m) \det{}^{N_-}(V\Lambda U^{-1} + m)\, e^{-\frac{N}{2\lambda^2} \mathrm{Tr}\Lambda^2}$$

$$Z_{\chi GUE} = \int \mathcal{D}\Lambda \mathcal{D}U \mathcal{D}V \, J(\Lambda) \det{}^{N_f}(U\Lambda V^{-1}+m) \det{}^{N_f}(V\Lambda U^{-1}+m^*) \, e^{-\frac{N}{2\lambda^2}\mathrm{Tr}\Lambda^2}$$

~~our theory now is explicitly a theory of N_f massive modes Λ~~

~~and~~ if we set $m=0$, this becomes

$$Z_{\chi GUE}(\nu, m=0) = \int \mathcal{D}V \mathcal{D}U \mathcal{D}V \, J(\Lambda) \, e^{N \mathrm{Tr} \ln(\Lambda) - \frac{N}{2\lambda^2}\mathrm{Tr}\Lambda^2} \det{}^{\nu}(VU^{-1})$$

in the large N limit we can evaluate this w/ saddle point approximation. relevant saddle point $\frac{N}{\Lambda} = \frac{-N\Lambda}{\lambda}$

$\Lambda = \lambda$ - N_f condensates $SU(N_f)\times SU(N_f) \to SU(N_f)$ N_f^2-1 GBs! extra zero mode in $\nu \neq 0$

$\mathrm{Tr}\,\sigma\tau\bar{\sigma} \rangle \neq 0$

$\Rightarrow \sigma\tau\bar{\sigma} \to L(\sigma\tau\bar{\sigma})R^t$

not symmetry of vacuum, but ~~this~~ $L=R$ is!!

to determine the χral condensate we need to keep $m \neq 0$, but small. the $m=0$ saddle point solution is justified if $N \to \infty$ w/ $mN \ll 1$ fixed, at which point we ~~expand~~

$e^{N_f \mathrm{Tr}\ln(\Lambda U + m) + N_f \mathrm{Tr}(\Lambda U^{-1} + m^*)}$

$\simeq e^{\frac{N}{2\lambda} \mathrm{Tr}(mU^{-1}) + \frac{N}{2\lambda}\mathrm{Tr}(mU)}$

$$Z_{\chi GUE}(\nu, m) = \int \mathcal{D}U \det{}^{\nu} U \, e^{\frac{N}{2\lambda}\mathrm{Tr}(mU + m^* U^{-1})}$$

From this, we can determine our χral condensate, $m\nu \to 0$

$$\Sigma = \lim_{m\to 0} \lim_{N\to\infty} \frac{\partial}{\partial m} Z_{\chi GUE} \quad (N\Sigma N)$$

$$= \lim_{m\to 0} \lim_{N\to\infty} \frac{1}{2N_f \lambda} \int \mathcal{D}U \det{}^{\nu} U \, \mathrm{Tr}(U + U^{-1}) \, e^{\frac{1}{2\lambda} N \mathrm{Tr}(mU + mU^{-1})}$$

finite in large N limit

the large N limit allows us to do the U integral w/ a saddle-point approx, exponential dominated by $U=1$

$$\Sigma = \frac{1}{2N_f\lambda}(2N_f) = \lambda$$

we now see large N χRMT has spontaneous χSB, partition fn depends on $N\Sigma m$

diverges in large N limit

$$Z_{\chi}(\nu, mN\Sigma) = \int_{U \in U(N_f)} \mathcal{D}U \det{}^{\nu} U \exp\left[\mathrm{Tr}\,\mathrm{Re}(mN\Sigma U)\right]$$

$$Z_{\chi GUE}(\nu,m) = \int_{U\in U(N_f)} \mathcal{D}U \det{}^{\nu} U \; e^{N\Sigma \frac{1}{2}\mathrm{Tr}(mU + mU^{-1})}$$

We can now separate U into a phase and an $SU(N_f)$ piece

$$Z_{\chi GUE}(\nu,m) = \int \frac{d\theta}{2\pi} \int_{U\in SU(N_f)} \mathcal{D}U \; e^{i\nu\theta} \; e^{N\Sigma \frac{1}{2}\mathrm{Tr}(me^{i\theta/N_f}U + m e^{-i\theta/N_f}U^{-1})}$$

$$\equiv \int \frac{d\theta}{2\pi} e^{i\nu\theta} Z_{\chi GUE}(\theta,m)$$

~~For $\nu \neq 0$, θ is an additional zero mode the the partition fn.~~

Identifying θ as vacuum angle,

$$Z_{\chi GUE}(\theta,m) = \int_{U\in SU(N_f)} \mathcal{D}U \; e^{N\Sigma \frac{1}{2}\mathrm{Tr}(me^{i\theta/N_f}U + me^{-i\theta/N_f}U^{-1})}$$

We can now finally check consistency of our assumption of neglecting zero-modes in Banks-Casher. Using

$$Z(\nu,m) = \sum_\nu e^{-i\nu\theta} Z(\theta,m) \Rightarrow \langle \nu^2 \rangle = -\frac{\partial^2}{\partial\theta^2} Z(\theta)$$

We have

$$\langle \nu^2 \rangle_{\theta=0} = \frac{mN\Sigma}{N_f}, \qquad \frac{\sqrt{\langle \nu^2 \rangle_{\theta=0}}}{N} \to 0 \text{ for large } N$$

shows physics of topological quenching by chiral quarks captured by our χRMT!
(not surprising, consequence of axial Ward identities ensured by $me^{i\theta/N_f}$ combination).

Now, this partition fn should look familiar to many of you.

~~$Z_{\chi GUE}(\theta,m) = \int_{U\in SU(N_f)} \mathcal{D}U \ldots (mU + mU^{-1})$~~

$$Z_{\chi GUE}(\theta,m) = \int_{U\in SU(N_f)} \mathcal{D}U \; e^{N\Sigma \frac{1}{2}\mathrm{Tr}\left(m e^{i\theta/N_f} U + m e^{-i\theta/N_f} U^{-1}\right)}$$

χPT says that we should be able to parametrize the dynamics of Goldstone bosons by $U = e^{i\pi/f}$ w/ $U \to LUR^{+}$

$$\mathcal{L}_{\chi PT} = \frac{f^2}{4}\mathrm{Tr}\left(\partial_\mu U \partial_\mu U^{-1}\right) - \frac{\Sigma}{2}\mathrm{Tr}\left(m e^{i\theta/N_f} U + m e^{-i\theta/N_f} U^{-1}\right)$$

Our χRMT partition fn will reproduce the χPT Lagrangian exactly provided kinetic term can be neglected

$\partial_\mu U \sim 1/L$

so condition for $\chi RMT \sim \chi PT$ is

$$\frac{m\Sigma}{f^2} \ll \frac{1}{L^2}, \qquad L \ll \sqrt{\frac{f^2}{m\Sigma}} = \frac{1}{m_\pi}$$

Can you use χRMT to say interesting things about static pion physics?

Make predictions for the low energy Dirac spectrum in lattice QCD, these have by and large been tested and verified.

Give a model for static properties where lattice QCD cannot eg finite density χRMT been under active investigation

XSB, χral phase transition at finite T & μ universal phenomena, critical pt

Value of χral condensate not universal. FV described by same EFT w/ same LEC. (UV physics only?) but values of these LECs depend on dynamics of the theory.

Investigate universality of χral physics