Before start

Left	Right
Talk sections	Full $\mathcal{L}$
	Eff $\mathcal{L}$

↑ leave room for Mass term.

Hello, my name is Bryce and I will be giving todays talk on effective field theories.

So far we have seen examples of effective field theories and seen parts of calculations but I am going to try to put all these concepts together and show the essentials of how to create an effective field theory from a full Lagrangian up to one loop order. Describing how to do calculations while hopefully not losing sight of the overall gaol. (probably assuming knowledge of qft in some places)

-~~Power counting and~~ tree level matching (quickly, similar to Issac and Andrews talks about matching)
-Renormalization group running
-one loop matching

Power counting and tree level matching

We start with a theory of a massless fermion interacting with 2 real scalar fields: one light and one heavy.

$$\mathcal{L}_{full} = i\bar{\psi}\partial\!\!\!/\psi + \frac{1}{2}(\partial_\mu\Phi)^2 - \frac{M^2}{2}\Phi^2 + \frac{1}{2}(\partial_\mu\phi)^2 - \frac{m^2}{2}\phi^2 - \lambda\bar{\psi}\psi\Phi - \eta\bar{\psi}\psi\phi \quad (1)$$

$$\mathcal{L}_{eff} = i\bar{\psi}\partial\!\!\!/\psi + \frac{1}{2}(\partial_\mu\phi)^2 - \frac{m^2}{2}\phi^2 - \eta\bar{\psi}\psi\phi + \frac{c}{\underset{\nwarrow \text{symmetry factor}}{2}}\bar{\psi}\psi\bar{\psi}\psi + d\partial_\mu\bar{\psi}\partial^\mu\psi\bar{\psi}\psi + ... \quad (2)$$

assume $M >> m$

We want an effective theory with only the light fields ψ and ϕ.

Formally we are finding an effective Lagrangian by integrating out the heavy fields in the full Lagrangian

$$\int \mathcal{D}\Phi e^{i\int\mathcal{L}(\psi,\phi,\Phi)} = e^{i\int\mathcal{L}_{eff}(\psi,\phi)} \quad (3)$$

This has been mentioned before, last week Natalie told us that this is an impractical integral in general and instead rewrote our Lagrangian into a loop expansion.

(((Space to write up the recap + explanation of momentum expansion (includes diagrams))))

Want to match amplitudes

Split up full theory + effective theory

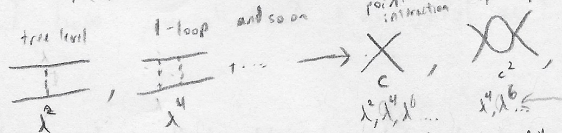
$$\mathcal{L}_F(\Phi,\psi)+\mathcal{L}_F(\psi,\phi)\longrightarrow \mathcal{L}_E(\psi,\phi)+\delta\mathcal{L}_E(\psi)$$

point like interactions

(errase)

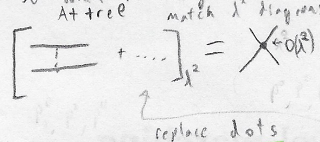
Removing terms that are the same

$$\mathcal{L}_F(\Phi,\psi)\longrightarrow\delta\mathcal{L}_E(\psi)$$



So when matching we are looking for an expansion of c in terms of λ^2

At tree match λ^2 diagrams in full theory to all λ^2 contributions it Eff. Theory



In addition, each amplitude has an expansion in powers of the external momentum. On the left this comes from the Propogator while in the effective theory this comes from adding more operators with more derivatives.

The matching that I will do will be for $\psi\psi\to\psi\psi$ scattering.

At tree level our full theory has 2 diagrams for psi psi scattering which involve the heavy scaler and thus must be matched in our effective theory.

So our first step is to calculate the amplitude in the full theory.

$$A_F=\bar{u}(p_3)u(p_1)\bar{u}(p_4)u(p_2)(-i\lambda)^2\frac{i}{(p_3-p_1)^2-M^2}-\{3\leftrightarrow 4\} \tag{4}$$

Where $\{3\leftrightarrow 4\}$ means the same term with p_3 and p_4 swapped.

In our effective theory (and for the rest of the talk) the Dirac spinors will have the same structure so I am going to write them as U_S from now on. Since the dirac structure is the same lets expand the propagator in p/M because, as I said, our effective theory comes with both an expansion in number of loops and in powers of p over M

$$(-i\lambda)^2\frac{i}{(p_3-p_1)^2-M^2}=i\frac{\lambda^2}{M^2}\frac{1}{1-\frac{(p_3-p_1)^2}{M^2}}=i\frac{\lambda^2}{M^2}(1+\frac{(p_3-p_1)^2}{M^2}+O(\frac{p^4}{M^4})) \tag{5}$$

~~Now we will need an effective Lagrangian to match to this amplitude. From the Feynman diagrams we can see that there will need to be a four~~

From our Feynman diagrams in the full theory we can see why we needed the 4 point interaction in Eff theory. But

~~point function in our effective Lagrangian.~~ We can also have higher order momentum operators by including more derivatives. These operators will correspond to terms in the momentum expansion that I just wrote down.

~~(the one half in the first 4 point interaction is a symmetry factor)~~

The more derivatives (higher order in p) the more singular and the more terms we can match in our previous expansion in p/M. Since we have assumed $p << M$ we can ignore terms further in the expansion once we have reached our desired accuracy. I will just work to order p^2 for now.
(((It happens to be that each of the operators I wrote can exactly account for one term in the momentum expansion. In a more complex example it might not work or it might not work at a higher order in momentum)))

so in our effective theory we can easily calculate the corresponding amplitude to tree level.

$$A_E = \bar{u}(p_3)u(p_1)\bar{u}(p_4)u(p_2)(ic + id(p_1 \cdot p_3 + p_2 \cdot p_4)) - \{3 \leftrightarrow 4\} \qquad (6)$$

using momentum conservation we can rewrite this as

$$A_E = \bar{u}(p_3)u(p_1)\bar{u}(p_4)u(p_2)(ic + 2idp_1 \cdot p_3) - \{3 \leftrightarrow 4\} \qquad (7)$$

(since ψ is massless $(p_3 - p_1)^2$ nicely works out to $= -2p_1 \cdot p_3$)

Now we can compare the UV and IR amplitudes and easily see that $c = \frac{\lambda^2}{M^2}$ and $d = -\frac{\lambda^2}{M^4}$. And we have now matched our effective theory to the full theory at tree level

(((There are other two derivative operators we could have included. It happens that the one we used is sufficient at tree level. (could mention what the other operators and why they don't have to show up There is actually only 2 independent operators and the second one wasnt necessary))))

Renormalization Group Running

Matching is the process of comparing theories with different field contents at some renormalization scale in both theories, usually the mass of the lightest particle being integrated out. Since there is only one scale there is no reason large logs would show up.

At tree level it is not obvious, but the coupling constants we have calculated through matching are dependent on the mass scale. That is $c(\mu = M) = \frac{\lambda^2}{M^2}$ and $d(\mu = M) = -\frac{\lambda^2}{M^4}$. However the calculations we want to do with our effective theory are nearer to the mass of the light boson. So it is important to know how the couplings change at different mass scales. What we need are called Beta functions, they tell us how the couplings change with our mass scale. They are defined to be the logarithmic derivative of a coupling with respect to our mass scale μ. (write example for c or something).

We will see that our original Lagrangian will give us divergent answers for things we know shouldn't be divergent. To fix this we need to think of our original lagrangian as containing bare couplings, which are independent of the mass scale $\psi_0 = \sqrt{Z_\psi}\psi$, $c_0 = c\mu^{2\epsilon}Z_c$, this will allow us a way of writing our counter terms to cancel our divergences.

We are going to be using dim reg and MS bar for all of our regularization. Because we are using dim reg the dimensions of all our couplings will change since we take space to have $4-2\epsilon$ dimensions But we really want the couplings to keep the same dimensions, so we need the mass scale mu here to make that happen

To find our beta functions we will solve equations like $0 = \mu\frac{dc_0}{d\mu} = \mu\frac{d}{d\mu}(c\mu^{2\epsilon}Z_c)$ since we know that the bare couplings are independent of the scale. The beta functions will come from the derivatives of our couplings, so the only thing we don't know about in this equation is how the Z factor depends on mu.

if we look at the ψ terms in our effective Lagrangian. can separate out counter terms

$$\mathcal{L}_{p^0,\lambda^2\eta^2} = i\bar{\psi}_0\partial\!\!\!/\psi_0 + \frac{c_0}{2}\bar{\psi}_0\psi_0\bar{\psi}_0\psi_0 \tag{8}$$

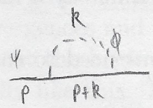
$$= i\bar{\psi}\partial\!\!\!/\psi + \mu^{2\epsilon}\frac{c}{2}\bar{\psi}\psi\bar{\psi}\psi + i(Z_\psi - 1)\bar{\psi}\partial\!\!\!/\psi + \mu^{2\epsilon}\frac{c}{2}(Z_cZ_\psi^2 - 1)\bar{\psi}\psi\bar{\psi}\psi \tag{9}$$

Since we are using ms bar the counter terms will exactly cancel the divergences we encounter, allowing us to find the values of Z_ψ and Z_c

recap

Finally, We have the concept down so I will work through our example to show how this works. I will work to lowest order in momentum and order $\lambda^2\eta^2$.

First of all we have the fermion self energy diagram



$$= (-i\eta)^2 \int \frac{d^dk}{(2\pi)^d} \frac{i(\not k + \not p)}{(k+p)^2} \frac{i}{k^2 - m^2} \tag{10}$$

Since all we are looking for is the counter terms, and in MS bar the counter terms will just cancel the divergent terms, we can exclusively work out the $1/\epsilon$ poles in these diagrams. Here we find

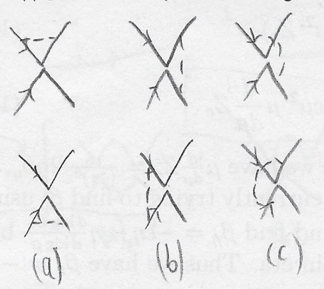
$$\frac{i\eta^2 \not p}{2(4\pi)^2\epsilon} \tag{11}$$

where $d = 4 - 2\epsilon$

Now we need the loop corrections to our 4 fermion vertex. We have a total of 6 different diagrams but there are only 3 different types.

((space for feynman diagrams))

leaving space at verticies to make it clear how the fermions travel through the lines



Since all of the diagrams are logarithmically divergent in the UV we can neglect the external momenta and the masses in order to find the divergent parts. Diagrams (a) are given by

$$2(-i\eta)^2 ic \int \frac{d^dk}{(2\pi)^d} \frac{i\not k}{k^2} \frac{i\not k}{k^2} \frac{i}{k^2} \tag{12}$$

For which the divergent term goes like

$$\frac{-2ic\eta^2}{(4\pi)^2\epsilon} \tag{13}$$

(((((Cross diagrams $\{3 \leftrightarrow 4\}$ not mentioned but they are there and have been accounted for)))))

Now the (b) and (c) diagrams are nice enough to only differ by a minus sign so their divergent parts end up canceling.

We now have the divergent parts that our counter terms need to cancel and therefore can calculate our z factors.

So in order to get finite quantities our counter terms need to be $Z_\psi - 1 = \frac{-\eta^2}{2(4\pi)^2\epsilon}$ and $c(Z_c Z_\psi^2 - 1) = \frac{2c\eta^2}{(4\pi)^2\epsilon}$

Using some algebra and keeping in mind that we are working to order lambda squared and eta squared and we can find $Z_\psi = 1 - \frac{\eta^2}{2(4\pi)^2\epsilon}$ and $Z_c = 1 + \frac{3\eta^2}{(4\pi)^2\epsilon}$

Now going back this (point) equation which we can actually solve now

hopefully still written down

$$0 = \mu\frac{dc_0}{d\mu} = \mu\frac{d}{d\mu}(c\mu^{2\epsilon}Z_c) \tag{14}$$

$$= \boxed{\beta_c\mu^{2\epsilon}Z_c + 2\epsilon c\mu^{2\epsilon}Z_c + c\mu^{2\epsilon}\mu\frac{d}{d\mu}Z_c} \tag{15}$$

we can see from the Z_c we calculated that we have $\mu\frac{d}{d\mu}Z_c = \frac{3}{(4\pi)^2\epsilon}2\eta\beta_\eta$. If we were to try to find β_η the same way we are currently trying to find β_c using an equation exactly like this (point). we wound find $\beta_\eta = -\epsilon\eta - \eta\frac{d\log Z_\eta}{d\log\mu}$ but the second term is higher than second order in eta. Thus we have $\beta_\eta = -\epsilon\eta$ and we can find that $\beta_c = \frac{6\eta^2}{(4\pi)^2}c$ if we take epsilon to zero. Great! So now we have the beta function in terms of things we know and that is just a differential equation for c. solving it gives us

gonna get rid of that really quick

off by 2? easy fix if necessary...

to leading log order

$$c(m) = c(M) - \frac{6\eta^2}{(4\pi)^2}c(M)\log\frac{M}{m} = \frac{\lambda^2}{M^2}(1 - \frac{6\eta^2}{(4\pi)^2}\log(\frac{M}{m})) \tag{16}$$

Where in the last step we have substituted in the value we found for c at tree level at the scale of the large mass.

One Loop Matching

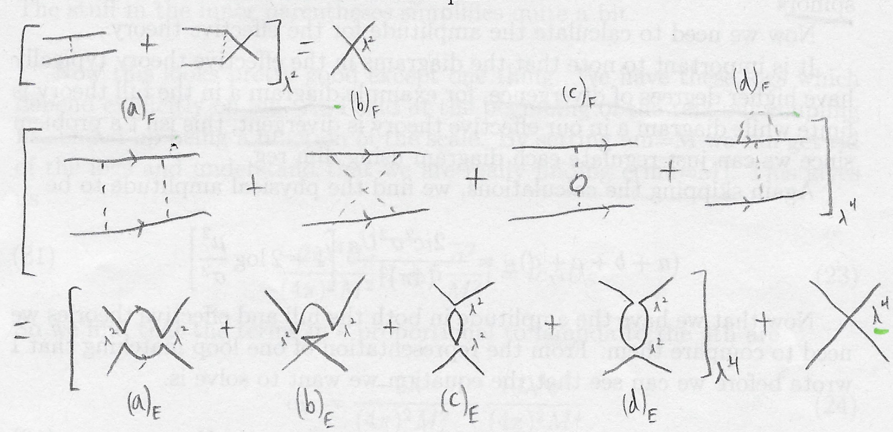
or just a reminder

As we saw in the tree level matching, the terms in our Lagrangian which are not directly coupled to the field we are integrating out will not affect the matching calculation. We will start from the same Lagrangian but this time add a small mass term for the ψ particle so that we avoid possible IR divergences and also so that we can see terms proportional to $1/M^4$. (at $O(p^0)$ no other scale to cancel the extra M^2))

$-\sigma\bar{\psi}\psi$

At the beginning of the talk we saw tree level matching presented pictorially like this. And in the same way we can express matching at one loop like so.

——pictorial representation of tree level matching from beginning. then the one for one loop matching will also include all the Feynman diagrams! Label them abcd still!! but use subscripts.



The vertices for the effective loop diagrams are order λ^2

In the full and effective theories diagram (d) stands for 2 diagrams which are reflections of each other.

Since we have already found the coupling c to order lambda squared, we can calculate both the full theory's amplitude at one loop and the effective theories amplitude at one loop with veracities of order lambda squared.

When we subtract these two we can enforce that the remainder is the 4 point interaction at order lambda to the 4th.

leading order in momentum

Unlike at tree level I will only match terms to order p^0 since the next term is analogous to the tree level case. I am also going to skip the evaluation of these diagrams and just write the physically renormalized amplitudes since calculating them is standard qft calculations not closely related to our subject.

——————mention subscript F, mu, and U sub s

$$(a+b+c+d)_F = \frac{2i\lambda^4 U_S}{(4\pi)^2 M^2}\left[-1-\log\frac{\mu^2}{M^2}+\frac{\sigma^2}{M^2}(-6\log\frac{\mu^2}{\sigma^2}-6+4\log\frac{M^2}{\sigma^2})\right] \tag{17}$$

The mu factors are our regularization mass scale and U_S is our Dirac spinors

Now we need to calculate the amplitude for the effective theory.

It is important to note that the diagrams in the effective theory typically have higher degrees of divergence, for example diagram a in the full theory is finite while diagram a in our effective theory is divergent, this isn't a problem since we can just regulate each diagram using dim reg.

Again skipping the calculations, we find the physical amplitude to be

$$(a+b+c+d)_E = \frac{-2ic^2\sigma^2 U_S}{(4\pi)^2}\left[1+2\log\frac{\mu^2}{\sigma^2}\right] \tag{18}$$

Now that we have the amplitude in both the full and effective theories we need to compare them. From the representation of one loop matching that I wrote before we can see that the equation we want to solve is.

$$(a+b+c+d)_F = (a+b+c+d)_E + ic_{\lambda^4}U_S \tag{19}$$

Since we have the amplitude in the full theory to one loop, and we have the amplitude in our effective theory to one loop AND we know the terms in c proportional to lambda squared (remember that we found $c = \lambda^2/M^2$ at tree level) then we can subtract these two and demand that the remainder is exactly the amplitude of the 4 point function which is proportional to lambda to the 4th.

Let's run through the algebra really quick to illustrate a couple points.

$$(a+b+c+d)_F-(a+b+c+d)_E=\frac{-2i\lambda^4U_S}{(4\pi)^2M^2}[1+\log\frac{\mu^2}{M^2} \tag{20}$$

$$+\frac{\sigma^2}{M^2}\left(6\log\frac{\mu^2}{\sigma^2}+6-4\log\frac{M^2}{\sigma^2}-\underbrace{2log\frac{M^2}{\sigma^2}-1}\right)] \tag{21}$$

Where these two terms are those from our effective theory.

Note that when we do this subtraction all of the $\log\sigma^2$ end up canceling which is exactly what we expect since sigma is a small mass and we have already demanded that our theories agree in the IR limit.

$$\frac{-2i\lambda^4U_S}{(4\pi)^2M^2}[1+\log\frac{\mu^2}{M^2}+\frac{\sigma^2}{M^2}\left(6\log\frac{\mu^2}{M^2}+5\right)] \tag{22}$$

The stuff in the inner parentheses simplifies quite a bit.

Now this looks pretty good except one thing. We have these logs which depend explicitly on mu. As I said at the beginning of the talk our coupling has ended up being a function of the scale. By setting mu=M we can get rid of the logs and understand that we are really finding c(mu=M). This gives us

$$\frac{-2i\lambda^4U_S}{(4\pi)^2M^2}[1+5\frac{\sigma^2}{M^2}]=ic_{\lambda^4}U_S \tag{23}$$

So we find that the terms in c proportional to lambda to the 4th are

$$c_{\lambda^4}=\frac{-2\lambda^4}{(4\pi)^2M^2}-\frac{10\lambda^4\sigma^2}{(4\pi)^2M^4} \tag{24}$$

and finally by combining this with our original matching at tree level we find

$$c(\mu=M)=\frac{\lambda^2}{M^2}-\frac{2\lambda^4}{(4\pi)^2M^2}-\frac{10\lambda^4\sigma^2}{(4\pi)^2M^4} \tag{25}$$

This is matching at one loop.

Recap/conclusion

So starting from a full Lagrangian we have integrated out the heavy scaler and our calculations in the effective theory will be accurate to order lambda to the 4th since we matched up to one loop. We have also seen that our couplings depend on the mass scale and we therefore need to run them to a lower mass scale.

I hope you all found it as useful as I did