The anomaly inflow on axion strings

Assumption: 1° Aion has the following coupling to photons

$$\mathcal{L} \supset -\frac{e^2}{16\pi^2 f_a} a F_{\mu\nu} \tilde{F}_{\mu\nu}$$

2° No domain walls.

Axion remain massless (Can't be our DM).

critical time $m_a \sim H$ (Can't be our QCD axion).

$$S = \frac{-e^2}{16\pi^2 f_a} \int d^4x \, a F \tilde{F}$$

$$S_{3+1} = \frac{e^2}{16\pi^2 f_a} \int d^4x \, \underbrace{(1+\rho)}_{\rho(r) = \text{"bump function"}} \epsilon^{\mu\nu\rho\sigma} \partial_\mu a A_\nu F_{\rho\sigma}$$

Parametrization of the string in 3+1 D.

$$\frac{1}{2\pi f_a} \partial_r \rho(r) \partial_\phi a \equiv \delta^{(2)}(\vec{x}_\perp) \quad \text{(winding number 1)}$$

Boundary Condit

$$\rho(0) = -1 \qquad \rho(r \to \infty) = 0$$

$$\rho'(0) = \rho'(\infty) = 0$$

Now we are ready to compute the current

$$j^\mu_{3+1} = j^\mu_{G-W} + j^\mu_A = \frac{\delta S_{3+1}}{\delta A_\mu}$$

$$j^\mu_{G-W} = \frac{e^2}{8\pi^2 f_a} (1+\rho) \epsilon^{\mu\nu\alpha\beta} \partial_\nu a F_{\alpha\beta}$$

$$j^\mu_A = \frac{e^2}{8\pi^2 f_a} \epsilon^{\mu\nu\rho\beta} \partial_\alpha \rho \, \partial_\nu a A_\beta \simeq -\frac{e^2}{4\pi} \delta^{(2)}(\vec{x}_\perp) \epsilon^{\mu\nu} A_\nu$$

$$\partial_\mu j^\mu_A = -\delta^{(2)}(\vec{x}_\perp) \frac{e^2}{8\pi} \epsilon_{ab} F^{ab}$$

$$\partial_\mu j^\mu_{G-W} = \frac{e^2}{8\pi^2 f_a} \partial_\mu \rho \, \epsilon^{\mu\nu\alpha\beta} \partial_\nu a F_{\alpha\beta} = \delta^{(2)}(\vec{x}_\perp) \frac{e^2}{4\pi} \epsilon_{ab} F^{ab}$$

$$\underbrace{\partial_\mu j^\mu_A}_{\text{bulk current}} + \underbrace{\partial_\mu j^\mu_{G-W}}_{\text{axion-gradient}} \neq 0 \qquad \text{what's going on ???}$$

The bulk theory is anomalous. String core is doing something.

On the UV Side, axion may couple to charged fermions or Bosons

Index theorem guarantees the existence of fermion zero-mode solution.

* Atiyah-Singer Index Theorem.

Define the index of the Dirac operator.

$$\mathrm{Index}(i\not{D}) = n_+ - n_- \qquad \begin{cases} i\not{D}\psi_n = \lambda_n \psi_n \\ i\not{D}(\gamma^5\psi_n) = -i\gamma^5 \not{D}\psi_n = -\lambda_n \gamma^5 \psi_n \end{cases}$$

$\lambda_n = 0 \Rightarrow$ zero modes. $\Rightarrow$ both $i\not{D}$, γ^5 simultaneously diagonalised

$$\mathrm{Index}(i\not{D}) = \sum_n \psi_n^\dagger(x)\gamma_5\psi_n(x) = \lim_{\epsilon\to 0} e^{i\lambda_n^2\epsilon}\psi_n^\dagger(x)\gamma_5\psi(x)$$

$$= \begin{cases} \frac{e}{4\pi}\epsilon_{\mu\nu}F^{\mu\nu} & \text{for } U(1) \text{ in } d=2 \\ \frac{e^2}{16\pi^2}\mathrm{tr}(F^{\mu\nu}\tilde{F}_{\mu\nu}) & \text{for } SU(N) \text{ in } d=4. \end{cases}$$

$$Z = \int \mathcal{D}A \det(i\not{D})\, e^{in\theta}$$

Z should be independent of θ. $\Rightarrow$ if $n \neq 0$. $\det(i\not{D}) = 0$

$\Rightarrow$ there must be fermion zero modes

1+1 D Dirac Eq. ($A_0 = 0$ gauge)

$$i\partial_0\psi_R = (-i\partial_1 - A^1)\psi_R \qquad i\partial_0\psi_L = (i\partial_1 - A^1)\psi_L(x)$$

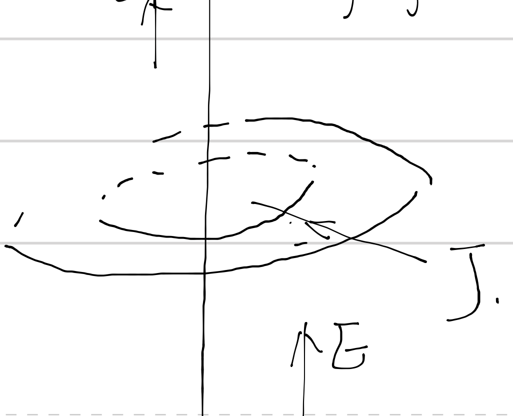
Fermi momentum changes with A^1 ($P_F = \epsilon_F$)

$$\frac{dP_F}{dt} = eE$$

$$\frac{dN_R}{dt} = \frac{1}{2}\frac{1}{2\pi}\frac{d\epsilon_F}{dt} = \frac{e}{4\pi}E = \frac{e}{8\pi}\epsilon^{ab}F_{ab}.$$

only left-moving

phase density



$$\partial_\mu J^\mu = -\partial_t I = -\frac{e^2}{2\pi} E_z$$

$\Longrightarrow$ I increases linearly with electric field.

$\lambda_{Q,zm} = I_{zm}$ Superconducting because charge

$\lambda_{Q,a} = I_a$ carriers are massless.

If there's only left mover

otherwise $\lambda_{Q,zm} = -I_{zm}$

charge conservation

$$\Delta Q = \int dt\, \dot{Q} = \int dx \frac{d\lambda_Q}{dt} = \int d^2x\, \partial_a j^a_{string} = -\frac{e^2}{4\pi}\int d^2x\, \epsilon^{ab} F_{ab}$$

$$= -\frac{e^2}{2\pi}\int d^2x\, E = \frac{e^2}{2\pi}\int dt \frac{d\Phi}{dt} = \frac{e^2 \Delta\Phi}{2\pi}$$

$$\Rightarrow Q_{string} = \frac{e^2\Phi}{2\pi} = en. \quad \Rightarrow \Phi = \frac{2\pi n}{e}$$

$$Q_{total} = Q_{axion} + Q_{string}$$

axion clound. $Q_{axion} = Q_{C-W} = -\dfrac{e^2\Phi}{2\pi}$