PTJC Talk #3: General Multi-Chern–Simons Theories

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So far this quarter we've seen two major examples of topological phases: the toric code and the doubled semion system. Recall that Andrew laid down a definition of a topological phase demanding a gapped spectrum, something like local correlations, and a finite exact ground state degeneracy dependent only on topological properties of the surface on which the theory is defined. In both cases we've seen it was possible to devise a graphical scheme for describing the degeneracy of the ground state and classifying excited states.

Today, we will focus less on solutions to particular systems, but rather on a general description of a class of theories which exhibit these sorts of phases. Tyler introduced the concept of a Chern–Simons gauge theory, and showed a particular example where a combination of two Chern–Simons terms produced a P, T-invariant theory with topological behavior seen in its ground state degeneracy and excitation statistics. Here we will worry less about constructing solutions, but rather pulling back and seeing how both theories addressed so far are particular cases of a more general formalism.

(1) K matrices

The interesting behavior in Tyler's theory was produced by the Chern–Simons action

$$S_{CS}[a_\mu] = \frac{k}{4\pi} \int \mathrm{d}^3x \, \epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho, \tag{1.1}$$

where a_μ is a gauge field which here will lie in $U(1)$ (non-abelian also possible but not considered here), and k is referred to as the *level* of the C–S term. It has been argued handwavily that k must be integer in order to ensure gauge invariance of the path integral, and we will continue waving that hand. These terms can arise in several ways. The semion model more or less started with the CS term for the purpose of simplicity, but they often arise in attempting to write down a low-energy effective theory for something complicated (e.g. QH systems). In such cases, we may want more terms, either governing the a-field for coupling it to external fields, but for now we leave those be. The semion theory ended up needing a second term to recover P, T and give the nice soluble system we ended up with:

$$S_{\text{semion}}[a_\mu, c_\mu] = \frac{2}{4\pi} \int \mathrm{d}^3x \, e^{\mu\nu\rho} a_\mu \partial_\nu a_\rho - \frac{2}{4\pi} \int \mathrm{d}^3x \, e^{\mu\nu\rho} c_\mu \partial_\nu c_\rho, \tag{1.2}$$

where c_μ was an additional gauge field. But once we do this, why not think about adding more fields? Why not permit mixed CS terms in which two gauge fields appear? Are these situations so different that we should consider them case-by-case? No! Let us consider the more general action

$$S_K[a^i_\mu] = \int \mathrm{d}^3x \, \left[\frac{1}{4\pi} K_{ij} \epsilon^{\mu\nu\rho} a^i_\mu \partial_\nu a^j_\rho + j^\mu_i a^i_\mu \right] \tag{1.3}$$

where we have introduced the *K-matrix* containing the levels of the different terms as well as a term coupling the fields to currents of their quasiparticles. By the same gauge-invariance arguments as before, we can agree that K had better be an integer matrix. Further, we are going to take it symmetric. (This seems to be convention and also to describe physically interesting situations, esp FQH states. If the boundary is trivial, then K^T represents the same theory as K with relabeled fields. If not, harder. Symmetry needed to ensure real eigenvalues, which are nice.) We can see that the doubled semion theory had

$$K = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}, \tag{1.4}$$

and it will turn out that the toric code can be described by $K = 2\sigma_x$.

(a) Eigensystem/Lattice

We have an RSM, know we have real eigenvectors $\vec{e}_a$ s.t.

$$K_{ij}(e_a)_j = \lambda_a (e_a)_i \tag{1.5}$$

with λ_a real. We choose the normalization

$$(e_a)_i (e_b)_i = |\lambda_a| \delta_{ab} \tag{1.6}$$

so that

$$K_{ij} = \eta^{ab}(e_a)_i (e_b)_j, \tag{1.7}$$

where

$$\eta^{ab} = \operatorname{sgn}(\lambda^a)\delta^{ab} \text{ (no sum on a)} \tag{1.8}$$

defines a pseudo-Riemannian metric for these eigenvectors. In a sense, this moves the information carried by K into this eigenbasis and its inner product. We'll further repackage this data into a *lattice* defined by

$$\Lambda = \left\{ m^i \vec{e}_i \mid m^i \in \mathbb{Z} \right\}, \tag{1.9}$$

where the lattice vectors are defined with respect to a basis for $\mathbb{R}^{N_+, N_-}$ where N^+ (N^-) is the number of positive (negative) eigenvalues of K.

To get topological data out of this formalism we'll also need K^{-1} or, equivalently, the dual lattice Λ^* with generators $\vec{f^i}$ where $\vec{f^i} \cdot \vec{e}^j = \delta^{ij}$

(b) Statistics

A characteristic result in topological phases is the presence of nontrivial braiding behavior for the particles of the theory: phases as we take particles around each other. The K-matrix formalism can get us this information, and the reason for it makes sense if we view the Chern–Simons term as representing an attachment of flux to particles. Consider the equation of motion in the presences of sources corresponding to a_0^i:

$$\epsilon^{IJ}\partial_I a_J^i = 2\pi (K^{-1})^{ij} j_j^0. \tag{1.10}$$

The left-hand side the magnetic flux of the ith gauge field, and we are relating it to some combination of source charges.

[draw a picture of braiding]

Say we want to braid two particles together, and they are characterized by integer (possibly binary?) charge vectors $\vec{n}$ and $\vec{m}$, where the ith entry of the charge vector gives that particle's charge under the ith gauge field. The equation of motion tells us to attach flux to charges, and the Aharanov–Bohm effect tells us to accrue phase as a charge encircles flux, so we can see that the total phase in this braid is

$$\theta = 2\pi m_i \left(K^{-1}\right)^{ij} n_j. \tag{1.11}$$

In the lattice language, we can attach these charges to dual lattice vectors and interpret the above as

$$\theta = 2\pi(m_i \vec{f^i}) \cdot (n_j \vec{f^j}), \tag{1.12}$$

where the dot is with respect to η. Since everybody in the direct lattice has integer inner product by construction, $\Lambda \leqslant \Lambda^*$. We can see that the phase θ is invariant under shifts of either vector by a direct lattice vector, so to see distinct topological behavior we really want to think about equivalence classes mod Λ. It turns out that this group's structure is exactly the anyonic fusion structure.To figure out how many of these distinct particles there are, compare the unit cell volume of Λ and Λ^*, finding

$$|D| = \frac{\sqrt{|\det K|}}{\sqrt{|\det K^{-1}|}} = |\det K|. \tag{1.13}$$

For higher genus surface, raise to the g. This is related to the ground-state degeneracy as $|\det K|^g$ for reasons that aren't obvious to me. Possibly can draw analogy to GSD shown before for single and doubled Chern–Simons theories to all diagonal K-matrix theories, or to fact that excitations came from frustrations of ground state modes.

(2) Example: the toric code

The toric code has K matrix

$$K = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}. \tag{2.1}$$

Let's take a second to write out the action explicitly:

$$S = \int \mathrm{d}^3x \left[\frac{2}{4\pi} \epsilon^{\mu\nu\rho} a^1_\mu \partial_\nu a^2_\rho + \frac{2}{4\pi} \epsilon^{\mu\nu\rho} a^2_\mu \partial_\nu a^1_\rho + j^\mu_1 a^1_\mu + j^\mu_2 a^2_\mu \right]. \tag{2.2}$$

This couples each gauge field to its own source and to the flux of the *other* field, rather than its own. That is, the a_0 equations of motion are

$$\epsilon^{IJ} \partial_I a^1_J = \frac{2\pi}{2} (j^2)^0 \tag{2.3}$$

$$\epsilon^{IJ} \partial_I a^2_J = \frac{2\pi}{2} (j^1)^0, \tag{2.4}$$

so it's clear that 1-type particles (e-excitations) see only m flux and vice versa. This leads to the phases that we saw in Andrew's talk:

$$\frac{\theta_{ij}}{2\pi} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}. \tag{2.5}$$

Braiding of es with ms gave us a relative sign, but interchange of identical particles was strictly bosonic (note that exchange of identical particles is "half" of a braiding). The fusion ψ of e and m, however, has fermionic statistics.

(3) Example: doubled semions

We saw this theory constructed explicitly as a Chern–Simons theory, but we'll now interpret it in K-matrix form. As mentioned before, it is clear that the K-matrix is

$$K = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}, \tag{3.1}$$

so we expect ground state degeneracy of 4^g, which was realized in Tyler's Wilson line construction as $(2 \times 2)^g$ via the combination of Hilbert spaces as a tensor product. Likewise, there should be four topologically distinct excitations on the torus.

[draw lattice/dual lattice]

The phase matrix for this system is again just the inverse of K:

$$\frac{\theta_{ij}}{2\pi} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \tag{3.2}$$

so we see that as advertised last week, there are particles which gain a $\pi/2$ phase under interchange.

(4) Equivalence of K-matrices and extension

Back at the introduction of the K-matrix, it would have been natural to wonder whether we are free to just redefine the a^i in a way that amounts to a basis change of K. At the level of the C–S term by itself, of course we are. We can always take

$$K \to SKS^T \tag{4.1}$$
$$S \in SL(N, \mathbb{Z}) \tag{4.2}$$

and we will have an equivalent C–S theory with the same particles and same statistics. However, this is not the entire content of a topological phase! There are a few places that this limit becomes apparent.

One conventional example is the calculation of Landau levels on the sphere, where the degeneracy of a given level is not exactly proportional to the flux out of the sphere but rather has a *shift* resulting from the different angular momentum configurations of the Landau levels. This shift is *not* evident in the K-matrix formalism and we must construct it from additional information.

In systems with non-trivial boundary, it is also found that the K-matrix cannot specify the edge modes completely, essentially because they are sensitive to short-distance physics not captured by the C–S terms. We can have many equivalent K-matrices describing the same bulk topological phase, but we may have less freedom once we fix data about the edge modes.

There are, of course, non-abelian C–S theories not treated here, but they also support K-matrix descriptions and lead to parallel, though somewhat more complicated, results.

(5) Coda: Fractional quantum Hall effect

Much of this formalism was developed in pursuit of descriptions of fractional quantum Hall states, so it seems fitting to give them some airtime. From a very broad perspective, the story is that in the presence of magnetic flux, conductors with impurities develop a transverse (Hall) conductivity σ_{xy}. This is exploited to great advantage in cheap magnetic field sensing.

What is fascinating is that this conductivity turns out to be quantized to *extreme* precision in experiment, which screams topology. This quantization occurs at integer and some set of rational values of the Landau level filling fraction ν. The states with fractional ν have been pretty successfully described by effective Chern–Simons theories with levels equal or related to $\frac{1}{\nu}$, where the statistical fields describe the long-range behavior of the electron currents. The reason for this is that if we couple our statistical gauge fields to a background field A_μ by

$$S = \int \mathrm{d}^3x \, \frac{1}{2\pi} t_i \epsilon^{\mu\nu\rho} A_\mu \partial_\nu a^i_\rho, \tag{5.1}$$

where t_i is a vector of charges coupling the statistical fields to the Maxwell field (and introducing an outside scale), it turns out that we can compute the response to the external field (particularly an $\vec{E}$-field) by integrating out the statistical fields and get

$$\sigma_{xy} = t_i \left(K^{-1}\right)^{ij} t_j. \tag{5.2}$$

This reproduces a lot of well-known states, but there's a particularly pretty one with

$$K = \begin{pmatrix} k_1 & -1 & 0 & 0 & \cdots \\ -1 & k_2 & -1 & 0 & \\ 0 & -1 & k_3 & -1 & \\ 0 & 0 & -1 & k_4 & \\ \vdots & & & & \ddots \end{pmatrix}, \qquad \vec{t} = \begin{pmatrix} 1 & 0 & \cdots \end{pmatrix}. \tag{5.3}$$

We should interpret this as coupling only one gauge field directly to the background field, and then coupling "neighboring" statistical fields in a constant way through the off-diagonal terms. This indirectly gives combinations of statistical fields various fractional charges under the background field.

On a physical level, we might think of this as starting with an effective theory for the electron currents, identifying quasiparticles in this theory, then defining an effective theory for the FQH states of those quasiparticles, which will have quasiquasiparticles, then defining. . . We can see why it is reasonable to call these the *hierarchy* states. Sparing the details, we can perhaps also trust that this gives the elegant result

$$\nu = \frac{1}{k_1 \pm \frac{1}{k_2 \pm \cdots}}. \tag{5.4}$$

This explains many observed FQH states, though not all. There is a generalized form of this hierarchy construction due to Wen & Zee which is claimed to describe all possible abelian FQH states.

Next week, we're going to move forward to the string-net condensation picture of topological phases with Kyle.