Finite Temperature Yang-Mills and Confinement

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So far this quarter we've been dealing with various topologically protected field configurations in a variety of toy models. We've studied kinks, vorticies, monopoles and instantons, examined their dynamics and found the effect they have on physical observables. We would now like to turn to the question of confinement: can these tools help us understand the nature of confinement, in particular the confinement/deconfinment transition? What type of field configurations lead us to confinement? Moreover, what do we even mean by confinement? Today, I will try to answer some of these questions.

Let me start with a brief introduction to finite temperature Yang-Mills. This is a process most of you have seen before, so I will be sparse on the exact details. The spirit is as follows: (i) Start with a theory defined on $\mathbb{R}^{3,1}$, (ii) wick rotate the time direction so that the theory is defined on $\mathbb{R}^4$, and (iii) compactify the time direction with period $\beta = \frac{1}{T}$ so that the manifold is $\mathbb{R}^3 \times S^1$. By doing this, we are introducing one non-trivial cycle in our theory (more on this later), which will play an incredibly important role in what follows. For completness, the action in our theory is

$$S = -\frac{1}{4g^2} \int_{\mathbb{R}^3 \times S^1} \mathrm{tr}\left(F^{\mu\nu} F_{\mu\nu}\right)$$

where we specify periodic boundary conditions

$$A_\mu(t+\beta, \mathbf{x}) = A_\mu(t, \mathbf{x})$$

to ensure that we are summing over all physical states, i.e. states that satisfy Gauss' law.

Last week Kyle showed us how confiment arises in the Polyakov model–an $SU(2)$ gauge theory in 3 Euclidean dimensions. He was able to demonstrate for us that the contribution of instantons to the effective potential gives rise to a linear force between probe quarks, and that the chromo-electric flux is loalized to a small region of space. This is a feature that is very pervasive in confining theories, and can be used as a necessary criterion for confiment to occur. If there is alinear potential, the theory is likely confining. I say likely here because there are other conditions that the theory must satisfy, but I will not go into that today.

In addition to a linear potential, we should expect our theory to contain a physical observable that will act as some sort of order parameter for the confinment/deconfinment phase transition. For us, this is known as the Polyakov loop, or holonomy. It is defined as

$$L(\vec{x}) = \mathcal{P}e^{i\int_{S^1} A}$$

an the corresponding physical observable of interest is

$$\langle \mathrm{tr} L(\vec{x})\rangle = \left\langle \mathrm{tr}\mathcal{P}e^{i\int_{S^1} A}\right\rangle$$

which we would expcet to be 0 when we are in a deconfined phase and non-zero when we are in a confined phase.

The concept of a holonomy is very important in what follows, so it is worthwhile to expand on it further. Moreover, holonomies are an important part of differential geometry in general and so it is worth considering an easy-to-visualize example. Let us consider a closed cycle anchored at a point p on an S^2 (DRAW IT). If I parallell transport my tangent vector about any closed cycle, it does not return to its original orientation. In fact, it is rotated by some angle in the tangent plane at p. Different cycles will return me at different angles. The group whose elements generates the transformations of these tangent vectors is known as the holonomy group. For the case of tangent vectors on S^2 the corresponding holonomy group is $SO(2) \sim U(1)$.

The situation with YM on $\mathbb{R}^3 \times S^1$ is more complicated, but qualitatively similar. Except now, instead of transporting tangent vectors around closed cycles, we are transporting gauge fields (particularly the spatial components of the gauge field, since they enter in the action with time derivatives and are therefore dynamical). We expect that when A_i returns from its journey around the thermal circle, it yields an identical physical state. But we are in a gauge theory, so different gauge field configurations yield identical physical states if they are gauge equivalent. Thus, we find that the A_i's can be transformed by an element of $SU(N)$ as it completes this cycle, similar to how the tangent vectors can be transformed by an element of $SO(2)$. We conclude that the holonomy group of $SU(N)$ fibered over $\mathbb{R}^3 \times S^1$ is $SU(N)$. For cycles wrapping the S^1, the group element that characterizes this gauge transformation is exactly the Polyakov line:

$$L(\vec{x}) = \mathcal{P}e^{i\int_{S^1} A} \in SU(N).$$

which is why we call it the holonomy. There is some pretty abstract mathematics I am sweeping under the rug. If you are at all interested, I highly reccomend Nakahara's book on the subject.

We not quite interested in this operator itself, since its a pretty complicated object, but rather the eigenvalues of this operator. Being an element of $SU(N)$, we know these eigenvalues will be complex number of unit modulous–i.e. they all sit on the unit circle. We can perform a unitary transformation to diagonalize $L(\mathbf{x})$ with the eigenvalues sitting on the diagonal. We can parameterize it as

$$L(\mathbf{x}) = diag\left(e^{2\pi i\mu_1}, e^{2\pi i\mu_2}, ..., e^{2\pi i\mu_N}\right)$$

where we impose the following conditions

$$\sum_m \mu_m = 1, \quad \mu_1 \leq \mu_2 \leq ... \leq \mu_N \equiv \mu_1 + 1$$

this last requirment is just a convinient specification–we are free to order them however we want. Often times these eigenphases $\{\mu_m\}$ are called the holonomy as well. Somewhere, a mathematican is throwing up. An important thing to note here is that these eigenphases are specified only modulo an integer–this should be obvious since they are all sitting up in the complex exponentials.

We call the holonomy "trivial" if all the eigenphases are equal modulo an integer:

$$\mu_m - \mu_n \in \mathbb{Z}$$

in which case, the Polyakov loop is just the identity up to a phase

$$L = (\text{phase}) \times \mathbb{I}_N$$

which, of course, gives us a non-zero value of our order parameter

$$\langle \text{tr} L \rangle \underbrace{=}_{normalize} 1.$$

This is what we expect to happen in a deconfined phase. All the holonomies are sitting on the same point on the unit circle and nothing interesting happens to spatial components of the gauge field as they traverse the unit circle.

We will call the holonomy confining if all the $\mu_m's$ are equidistant about the unit circle. Specifically, they will be given by

$$\mu_m = -\frac{1}{2} - \frac{1}{2N} + \frac{m}{N}$$

In this case, each eigenvalue of L corresponds to an N^{th} root of unity–the eigenvalues spread out as much as they can along the unit circle (DRAW THIS). This gives

$$\langle \mathrm{tr} L \rangle = 0$$

which is what we expect in a deconfined phase.

Now heres a question: what holonomy does thermal YM prefer? Does it prefer a confining, a trivial, or some other holonomy? We can answer this question by integrating out all the degrees of freedom in our theory except for the μ_m's. By doing this, we can generate an effective potential for the $\mu' s$ and find where the minimum is. From there, we can compute $\langle \mathrm{tr} L \rangle$ and determine if we are in a confined or a deconfined phase. This, of course, is easier said than done, since YM is strongly coupled at low energies. But fear not, we can still perform a perturbative analysis so long as we are working at high enough temperatures. This was done long ago by our very own Larry Yaffe and his colleagues. They found tthat for high tempertures, the effective potential for the $\mu' s$ is

$$U_{eff} = \frac{(2\pi)^2 T^3}{3} \sum_{n<m}^{N} (\mu_m - \mu_n)^2 [1 - (\mu_m - \mu_n)]^2$$

which is minimized when

$$\mu_m = \mu_n \mod 1$$

which is precisely the trivial holonomy. This should be unsuprising–we know that YM is deconfined at high temperatures, and so effective potential in this regime should be . For $SU(2)$, we can plot this potential as a function of $\mathrm{tr} L$ (DRAW IT).

However, we are not done, obviously, because we know that there will be contibutions to the effective potential coming from topologically non-trivial field configurations. If there is any justice in this world, then these contributions should look something like this (DRAW THE DYON CONTRIBUTION), which will prefer a confining holonomy. This is in fact the case. The name of the game will then be the balance between these two potentials–one will dominate at high tempertures, and one at low, with a phase transition at some critical temperature. But the real question is what are the relevant field configurations? How would we even go about finding them?

Fortunately for us, this task is not as forboding as it seems. This is because we know how to classify all finite energy solutions to the YM equations using three sets of numbers.

- Topological charge: $\nu = \frac{1}{8\pi^2}\int_{\mathcal{M}} F \wedge F$–this should be familiar to us by now. The standard instanton solutions we studied with Peter a number of weeks ago have $\nu = 1$. We then obtain states with larger charge by considering any number of instantons.
- Holonomy–rather, eigenvalues of the holonomy at spatial infinity.
- Magnetic charges–these can be obtained by considering the mapping of the S^2 at spatial infinity into the manifold formed by the eigenvalues of the holonomy. These will also be very important to us as we will see shortly.

Now, with these three sets of numbers, we can attempt to find an field configurations thats lead to an effective potential that prefers a confining holonomy.

The most straightforward thing to do is to find a generalization of the standard four dimensional instanton that satisfies the required periodicity condition. This has been done, and is dubbed the periodic instanton. It has

- $\nu_{PI} = 1$
- $\{\mu_m\}$=trivial at spatial infinity
- zero magnetic charge

Moreover, these things are genuinly perioidic. We can plot them in two different directions. Along any spatial direction they will look like this (DRAW IT). In the time direction, however, it looks like this (DRAW IT). To check to see if this can contribute to the effective potential, one must compute the measure associated with integration of the zero modes. This is the equivalent to Kyle's μ^3 last week, which played an important role in supressing the mass of the dual photon in the Polyakov model. Such a measure will always show up when dealing with these types of field configurations, and is obtained by the so called moduli space metric–an overlap of the zero modes in the problem. The moduli space metric for the periodic instanton can be computed in high temperature pertubation theory. I won't write down the whole thing. because it is quite messy. But I would like to highlight the dependence on the zero mode associated to the size. For $\pi\rho T >> 1$, we have

$$d\mu_{PI} \sim \rho^5 e^{-\frac{4}{3}(\pi\rho T)^2}$$

which means that for high temperature, the size of the instantons are highly suppressed. They prefer to be arbitrarily small, and so do not really contribute anything interesting to the partition function or the effective potential. This is the "dilute gas" approximation–an ensemble of non-interacting, tiny instantons going about their merry way.

The fact that this configuration does not contribute to the parititon function at high temperatures should not be suprising. We know that YM is deconfined at high temperatures, and so the perturbative potential I wrote on the board before should be the dominant contribution since the effective potential prefers the trivial holonomy. This is really the behavior we should be seeing with ANY possible canidates for confining field configuration.

The real meat of the story is what is going on at low temperatures. This is a more difficult task, since pertubation theory does not work at this regime. Still, there is work to be done, particularly using variational techniques to see what the prefered holonomy is. This was done by Dimitri Diakonov and Vitor Petrov in the 90's for $SU(2)$, although they alledgedly never published (except in this review I'm reading lol). They find the average behavior of $\langle \mathrm{tr} L\rangle$ to be something like this (DRAW IT). This is concerning, since it is not zero at low temperatures and there is no sharp fall off. Lattice data shows the actual behavior is something like (DRAW IT). Now, these do have similar qualitative behavior for high temperatures, but behave wildly differently at low temperatures.

So our dilute gas of instantons is not quite the field configuration we are hoping for. What can we change? Well, heuristically, we could imagine modifying our "gas" of instantons to possibly include interactions. But that would require the introduction of some type of charge, be it electric of magnetic. But including non-zero magnetic charge will force us to have a non trivial holonomy. Lucky for us, again, we already know of solutions to the YM equations that possess both of these requirements: the BPS monopoles that Kyle told us about last week. These solutions posses both magnetic AND electric charge (owed to the fact that the field configurations are self dual), and have nontrivial holonomy at spatial infinity. For the $SU(2)$, this is because we have $A_4 = \pi T \frac{\tau_3}{2}$ at spatial infinity, which leads to a confining holonomy. For higher rank gauge groups, we will find that we can replace the τ_3 above with any of the $N-1$ Cartan generators $C_m = diag(0, ..1, -1, ...0)$ together with $C_N = diag(-1, 0, ...0, 1)$ and obtain equivalent results. These configurations may not seem like they will help us, since they are localized in three dimensions, but a remarkable thing will happen when we consider bound states of these dyons. But before that, let us take a closer look at the BPS monopole.

Consider $SU(2)$. The monopole solutions Kyle introduced to us last week have the explict form of

$$B_i^a = (\delta_{ai} - n_a n_i)\frac{v}{\sinh vr}\left(\frac{1}{r} - v\coth vr\right) + n_a n_i\left(-\frac{1}{r^2} + \frac{v^2}{\sinh^2 vr}\right)$$
$$E_i^a = \pm B_i^a$$

where n is a unit vector and the upper and lower signs here are for the self and anti-self dual solutions to the EoM. By taking the large r-limit, we see

$$B_i^a \underbrace{\rightarrow}_{r\rightarrow\infty} \frac{n_a n_i}{r^2}, \quad E_i^a = \pm\frac{n_a n_i}{r^2}$$

which shows that these solutions carry electric and magnetic charges $(q_e, q_m) = (1,1), (1,-1)$ for the dual and self dual solutions respectively. This begs the question, can we find solutions that have charges $(-1,-1)$ and $(-1,1)$? In fact, this can be done. First, one must transform to a simpler gauge, an then make the simple replacement $v \rightarrow 2\pi T - v$ in the above equations. The result of this procedure is that the for small values of r the solutions are time dependent, but for $r \rightarrow \infty$ we have the same asymptotic behavior with opposite charges. So, we have the following conclusion: for $SU(2)$, we have 2 different types of dyons and 2 types of anti-dyons (the anti-self dual solutions).

Passing over to $SU(N)$, we find that we have N types of dyons and anti-dyons. This is because, as I alluded to before, we can always pass to the Cartan basis and make the replacement $\tau_3 \rightarrow C_m$ in the analysis that was performed above. We then obtain a dyon for each value of m. To see this, we define the quantity

$$\nu_m = \mu_{m+1} - \mu_m$$

and demand that the holonomy approach an arbitrary value at spatial infiniy

$$A_4(\infty) = 2\pi T diag(\mu_1, \mu_2, ...\mu_N)$$

One then finds that far away from the center of the dyon, the field is Abelian and the it has the form

$$\mathbf{E}^{(m)} = \pm\mathbf{B}^{(m)} = C_m \frac{\mathbf{x}}{2\left|\mathbf{x}\right|^3}$$

and since m ranges from 1 to N, we see that we have N different types of dyons in our system. Each one has topological charge ν_m and action

$$S^{(m)} = \frac{8\pi}{g^2}\nu_m$$

However, something remarkable happens. When we consider a system of all N types of dyons, the total topological charge is

$$\sum_m \nu_m = 1$$

and so the total action is

$$S = \sum_m S^{(m)} = \frac{8\pi}{g^2}$$

which is exactly the same charge and action as a single instanton! This is a highly non-trivial fact: a system of N interaction dyons is essentially indistinguisable from an instanton. Also, we never had to specify a holonomy! It is completely arbitrary, and so it is possible to obtain an effective potential from it. Full solutions for this have been found by Kraal and van Baal and Lee and Lu. We call the monopole instantons, for obvious reaosns.

The resulting bound state is localized in 4 dimensions despite the constituient dyons being localized in 3 dimensions. This is weird, but not completely crazy. For instance, consider the size modulous of the instanton. This is, heuristically speaking, related to the seperation of the constituent dyons. As this size approaches the length of the compact direction, we find that it is forced to wrap around. From a purely temporal point of view, this will look as if the instanton is getting more delocalized.

Now, this does not address on of the stranger, and more interesting points of the solution, which is that the instanton will fractionalize into many constituent dyons. For $SU(3)$, where we have 3 different types of instantons, it will look something like this (DRAW IT!!). This type of behavior is really a difficult thing to have a heuristic/intuitive understanding of. It is a weird quirk of the full monopole-instanton soluton.

Now we are still a long way off from obtaining a full picture of confinement. The next task at hand would be to find the moduli space metric so that we may integrate over the zero modes. To motivate this, I will move to the semi-classical picture where we will treat this ensemble of dyons as fixed in space, interacting via Coulomb potentials, and integrate over all the possible positions and sizes of the dyons. This is essentially equivalent to actually going in and computing the moduli space of the metric as I will now motivate.

Let K_m be the number of dyons of charge m. The semi-classical partition function for a system of this type will be

$$Z = \sum_{K_1 \dots K_N} \frac{1}{K_1! \dots K_N!} \prod_{m=1}^{N} \prod_{i=1}^{K_m} \int d\mathbf{x}_{mi} f e^{-U_{col}}$$

where $f = \frac{4\pi}{g^4} \frac{\Lambda^4}{T}$ is the fugacity and U_{col} contains all the different coulomb interactions between every different dyon. If we use the identity

$$\exp \operatorname{tr} \log G = \det G$$

and identify $-U_{col} = \operatorname{tr} \log G$, then we can rewrite the boltzman factor as the determinant of a matrix. Doing so, one finds that this is exactly the moduli space metric you would obtain by doing a straightforward overlap of all the zero modes in the problem. Now a remarkable thing happens. When we write it in the form of the determinant, and rather than the boltzman fator, and use the gross, formal definition of the determinant, you find that the contribution from all the Coulomb terms cancel! This is crazy, since it now renders the problem solvable. Going through and evaluating the partition function yields

$$Z = \exp\left(4\pi f V N (\prod_m \nu_m)^{\frac{1}{N}}\right)$$

Evidently, it has a minimum at

$$\nu_1 = ... = \nu_N = \frac{1}{N}$$

which corresponds precisely to the confining holonomy! Hoorah! We have found a non-perturbative field configuration that prefers a confining holonomy!

However, this is still semi-classical in nature. Moreover, is this cancellation a semi-classical artifact, or does it still occur in the full quantum picture? To answer this, we would like to somehow like to intepret the semi-classical paritition function as stemming from a quantum field theory. To do so, we would have to employ the follow tricks

- Fermionization: Formally, we can rewrite the determinant of any matrix as an integral over grassmann variables. Essntially, we use the identity

$$\det G_{AB} = \int \Pi_A d\psi_A^\dagger d\psi_A \exp\left(\psi_A^\dagger G_{AB} \psi_B\right)$$

- Bosoinzation: We can rewrite the coulomb interactions using

$$\exp\left(\sum_{m,n} \frac{Q_m Q_n}{|\mathbf{x}_m - \mathbf{x}_n|}\right) = \int \mathcal{D}\psi \exp\left[-d\mathbf{x}\left(\frac{1}{16\pi}\partial_i\phi\partial_i\phi + \rho\phi\right)\right]$$

In doing the bosonization step, in order to obtain off diagonal elements, we need to introduce 2 anti-commuting, bosonic ghost fields. The combine action of the bosonic fields from the bosonization procedure, and the ghost fields defines a QFT with a paritition function equivalent to the semi-classical analysis of the dyon ensemble. From there, we can integrate out the fields and obtain a fully quantum result.

Miraculously, the cancellation of the coulomb terms still happens in the quantum picture! This is because the bosonic fields and the ghost fields give equal contributions to the path integral, but the results differ due to the statistics. Since one contribution is from Grassmann fields and the other is from regular fields, they cancel each other's contribution! This can be traced back even further to properties of the moduli space metric. This particular metric is "hyper-Kahler" which basically means that the metric is hemritian and closed and satisfies some other interesting properties I am not going in to.

The work is not done, but my time is. With this full QFT describing the contrubituon of monopole instantons in YM theory you can calculate a whole bunch of things. For instance, the 2 point function of the polyakov loops, expectation values of wilson loops, spectrum and mass gaps and all sorts of good things. In doing so, you can further show that this theory is confining by checking that, for instance, the quark-antiquark potential is linear, as in the polyakov model last week.

Also being neglected is the contribution from the anti-instantons. Those are super important for the story too and can be included in a natural way. I will not go over that. Just know that the story I've presented to you is far from complete, but still contains a nice picture of the physics that is relevent for confinement. Thank you