Introduction

In the previous talks, we've been discussing solitons and instantons in 2d Euclidean space to build up some background knowledge of instantons.

We've so far avoided discussing 4-d Euclidean space because of Derrick's Thm; we need to know how instantons work with gauge fields to go to 4-d. This is what we will do today.

It will turn out that our $SU(N)$ gauge theories have important subtleties that give rise to a continuous range of degenerate vacuum states, which flies in the face of conventional knowledge (of Coleman's time, at least).

To start heading down this direction, let us consider our gauge theory in a semiclassical limit. We care about finite energy solutions here, so since our action is $S = -\frac{1}{2}\int d^4x \, Tr[F_{\mu\nu}F^{\mu\nu}]$, we need $F_{\mu\nu}$ to scale faster than $\frac{1}{r^2}$ at infinity.

Let's say it has a nice Taylor expansion here so that it is $F_{\mu\nu} \sim O(\frac{1}{r^3})$. This would naively imply that $A_\mu \sim O(\frac{1}{r^2})$ (∂_μ counts as $\frac{1}{r}$), but due to gauge invariance we are allowed to have $A_\mu = \frac{i}{g} f \partial_\mu f^\dagger + O(\frac{1}{r^2})$, where f is a member of $SU(N)$ and a function only of angles (not r).

This looks like it should be removable by a gauge transformation, but as it turns out there are properties of f that are gauge invariant, namely the winding number.

We've already seen some of this in 2-d space. Briefly turning to $U(1)$ in 2d for a moment, we can see that the functions $f = e^{i\nu\theta}$ are valid choices as long as ν is an integer (because θ is the 2d angle and f must be periodic).

The winding number ν can be expressed as $\nu = \frac{1}{2\pi i}\int_0^{2\pi} f \frac{df^\dagger}{d\theta}\, d\theta$, clearly true for $e^{i\nu\theta}$.
Under an infinitesimal gauge transformation $f \to (1+i\delta\lambda)f$ and so $\delta\nu = \frac{1}{2\pi i}\int_0^{2\pi} \underbrace{f f^\dagger}_{1} \frac{d(\delta\lambda)}{d\theta} d\theta = \frac{1}{2\pi}\left(\delta\lambda(2\pi) - \delta\lambda(0)\right) = 0$
Thus the winding number ν cannot be changed by gauge transformation. It represents physical information that does not affect the vacuum state action of our theory.

Winding number in SU(2) in 4-d Euclidean space.

Now we will take a closer look at homotopy classes of SU(2) in 4-d.

In the previous example, we were essentially looking for mappings between the S^1 of 2d space at fixed r to the S^1 of the U(1) gauge theory. Here, we want to go from the spatial S^3 at fixed r to the topology of $SU(2) \simeq S^3$ (because $U = a + i\vec{b}\cdot\vec{\sigma}$ with $a^2+|\vec{b}|^2=1$)

The identity mapping here will be $f^{(1)}(x) = \frac{x_4 + i\vec{x}\cdot\vec{\sigma}}{r}$. Higher mappings are given by $f^{(\nu)}(x) = (f^{(1)}(x))^\nu$, just as with U(1). The number ν represents the number of times we wrap around the hypersphere, and all choices of $f(x)$ can be deformed into one of the above cases.

Let's define $\tilde{\nu} = \frac{-1}{24\pi^2}\int d^3\theta\, \epsilon^{ijk}\, \mathrm{Tr}[(f\partial_i f^\dagger)(f\partial_j f^\dagger)(f\partial_k f^\dagger)]$, which will serve as our formal definition of the winding number. Under an infinitesimal gauge transformation, we can show that by rearranging the derivatives using unitarity of f and integration by parts, the shift $\delta\tilde{\nu} = 0$.

Finally, consider the object $G_\mu = 2\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}[A_\nu \partial_\lambda A_\sigma - ig\frac{2}{3}A_\nu A_\lambda A_\sigma] = 2\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}[A_\nu F_{\rho\sigma} + ig\frac{2}{3}A_\nu A_\lambda A_\sigma]$
Its divergence can be shown to be $\partial_\mu G_\mu = \frac{1}{2}\epsilon_{\mu\nu\lambda\sigma}\mathrm{Tr}[F_{\mu\nu}F_{\rho\sigma}] = \mathrm{Tr}[\tilde{F}F]$.
This can be used to show that $\int d^4x\, \mathrm{Tr}[\tilde{F}F] = \int d^3S\, \hat{e}_\mu G_\mu$, integrating at infinity.

Thus for our minimal action solutions near infinity, we then have

$$\int d^4x\, \mathrm{Tr}[\tilde{F}F] = \int d^3S\, \hat{r}_\mu \epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}[A_\nu F_{\rho\sigma} + ig\tfrac{2}{3}A_\nu A_\rho A_\sigma] \approx \int d^3S\, \epsilon_{ijk}\mathrm{Tr}[-\tfrac{2}{3g^2}(f\partial_i f^\dagger)(f\partial_j f^\dagger)(f\partial_k f^\dagger) + \mathcal{O}(\tfrac{1}{r^4})]$$

$$= \frac{16\pi^2\nu}{g^2} \qquad \left(\frac{g^2}{16\pi^2}\int d^4x\, \mathrm{Tr}[\tilde{F}F] = \nu\right)$$

Finally, there exists a theorem from Raoul Bott states that any continuous mapping from S^3 to any simple Lie group G can be continuously deformed into a mapping into an SU(2) subgroup of G. Thus the work here tells you how to map to any Lie group namely all SU(N) theories.

Quantizing the Theory

We have looked at minimal action solutions asymptotically for a classical gauge theory, so now we must quantize the theory.

To have a well-defined path integral, we need to choose a gauge to work in. We will choose axial gauge, $A_3=0$. The reasons for this is that non-singular gauge configuration can be put into axial gauge with non-singular transformations, there is no need for ghost fields or subsidiary conditions, other works use $A_0=0$ gauge, and it can be shifted into other gauges for specific calculations.

We will also work in a finite but large 3-volume V and a finite but large Euclidean time T. Doing this may actually gain us info as theories with many vacua may be made evident by showing that certain properties always depend on boundary cond. Plus, the condition used to eliminate A_0 in the canonical quantization is made unique.

The surface term for a gauge field theory is given by $\delta S = \frac{1}{g^2}\int d^3 s\, n^\mu F_{\mu\nu} \delta A^\nu + \dots$
We need to choose a boundary condition consistent with $A_3=0$ and our semiclassical calculation from before. This means that only field configurations of definite winding number are allowed in the box, and this turns out to be the only aspect of finite volume that survives in the continuum limit.

Thus for large enough boxes we can forget about boundary conditions and integrate only over all configurations of definite winding number n. This is denoted by the integral $F[V,T,n] = N \int [dA]\, e^{-S}\, \delta_{\nu n}$.

$F[V,T,n]$ is a transition matrix element from some initial state to some final state. For large times T_1, T_2 this should be $F[V,T_1+T_2,n] = \sum_{n_1+n_2=n} F[V,T_1,n_1]\, F[V,T_2,n_2]$, as the sub-boxes should be big enough that their boundary terms don't matter either.

However, we want to find a solution with definite energy, which would have a purely multiplicative composition rule. Luckily, we can make this happen with a Fourier transform. $F(V,T,\theta) = \sum_{\nu} e^{i\nu\theta} F(V,T,\nu) = N \int [dA] e^{-S} e^{i\nu\theta}$
$\longrightarrow F(V, T_1 + T_2, \theta) = F(V, T_1, \theta) F(V, T_2, \theta)$

Thus we identify $F(V,T,\theta)$ as proportional to $\langle \theta | e^{-HT} | \theta \rangle = N' \int [dA] e^{-S} e^{i\nu\theta}$, where the θ state represents a distinct vacuum state. So we have shown that there is an infinite continuum of vacua, each with nearly identical action save for an $F\tilde{F}$ term.

The instanton solutions

Now we can finally proceed with constructing the $\nu=1$ solutions for our gauge theory. Just as in previous talks, we can take this state and the $\nu=-1$ anti-instantons and get approximate solutions for other winding numbers out of $\nu=n-\bar{n}$ (anti)instantons.

We will approximate $F(V,T,\theta)$ by summing over these configurations to get $\langle\theta|e^{-HT}|\theta\rangle \propto \sum_{n,\bar{n}} (Ke^{-S_0})^{n+\bar{n}} (VT)^{n+\bar{n}} e^{i(n-\bar{n})\theta} \frac{1}{n!\,\bar{n}!} = \exp(2KVTe^{-S_0}\cos\theta)$. Thus the energy density can be read off as $\frac{E(\theta)}{V} = -2K\cos\theta\, e^{-S_0}$.

We can also calculate $\langle\theta|\mathrm{Tr}[\tilde{F}(x)F(x)]|\theta\rangle = \frac{1}{VT}\int d^4x\,\langle\theta|\mathrm{Tr}[\tilde{F}F]|\theta\rangle$ by translational invariance

$$= \frac{16\pi^2 \int[dA]\,\nu\, e^{-S} e^{i\nu\theta}}{g^2 VT\int[dA]\,e^{-S}e^{i\nu\theta}} = \frac{16\pi^2 i}{g^2 VT}\frac{d}{d\theta}\ln\Big(\int[dA]\,e^{-S}e^{i\nu\theta}\Big) \qquad \frac{d}{d\theta}\langle\theta|e^{-HT}|\theta\rangle. = +\frac{32\pi^2 i}{g^2}Ke^{-S_0}\sin\theta$$

Note that this is independent of V and T, and is imaginary so that the Minkowski space quantity is guaranteed to be real.

Also, note the dependence on θ, differentiating the vacua.

From the Schwartz inequality, $\int d^4x\,\mathrm{Tr}[FF] = \sqrt{\int d^4x\,\mathrm{Tr}[FF]\int d^4x\,\mathrm{Tr}[\tilde{F}\tilde{F}]} \geq \int d^4x\,\mathrm{Tr}[F\tilde{F}]$

Thus for any winding number we have $S \geq \frac{8\pi^2}{g^2}|\nu|$, where the equality holds when $F=\pm\tilde{F}$. $\pm = \mathrm{sign}(\nu)$.

If solutions to this exist, then they must be minima for the action, and are the only solutions we need to consider here.

For $\nu=1$, we know that the field can be transformed to be $A_\mu = \frac{i}{g}f^{(1)}\partial_\mu f^{(1)\dagger} + \mathcal{O}(\frac{1}{r^2})$ for $f^{(1)} = \frac{x_4 + i\vec{x}\cdot\vec{\sigma}}{r}$. This is rotationally invariant in the sense that it may be reverted back with a gauge transformation, since $SO(4)\simeq SU(2)\times SU(2)$. Thus our solution should have the same property.

We make the ansatz $A_\mu = R(r^2) f^{(1)}\partial_\mu f^{(1)\dagger}$, with $R(0)=0$. Doing the algebra gives $R(r^2) = \frac{r^2}{r^2+\rho^2}$ for arbitrary ρ.

We can then get more solutions by applying symmetries of the Lagrangian to this one. The symmetries we have are scale transformations, rotations, spatial translations, special conformal transformations, and gauge transformations.

Scale transformations only change ρ, the size of the instanton.
Rotations & gauge transformations are equivalent, give more general f's
Spatial translations will give us new solutions, shifting the center of the instanton (center of this one is 0)
Conformal turns out to also be equivalent to gauge + translations.
Fixed gauge still counts because we can still make global transformations.
Total of 8 parameters.
Approximate solutions for higher ν have 8ν parameters.

Evaluation of $K = \left(\frac{S_0}{2\pi\hbar}\right)^{1/2} \left|\frac{\det(-\partial_t^2+w^2)}{\det'(-\partial_t^2+V''(\bar{x}))}\right|^{1/2}$ for dd

Finally, one last thing we might want to accomplish is evaluating K for this instanton. In this case, we know that $S_0 = \frac{8\pi^2}{g^2}$, there are 8 parameters $\rightarrow$ 8 eigenmodes $\rightarrow$ factor of $\frac{1}{g^8\hbar^4}$. Instanton location integral is taken care of, and the integral over gauge transformations gives a constant factor, thus only an integral over scale remains.

Thus $\frac{E(\theta)}{V} = -\cos\theta \cdot e^{-\frac{8\pi^2}{g^2}} g^{-8} \int_0^\infty \frac{d\rho}{\rho^5} X(\rho M)$, where X is an unknown function and M is an arbitrary mass needed to define a renormalization condition. The $\frac{1}{\rho^5}$ is required by dimensional analysis.

Renormalization group analysis says that the combination $\frac{1}{g^2} - \beta_1 \ln M + O(g^2)$ is observable. $\beta_1 = \frac{11}{12\pi^2}$. From the e^{-S_0} factor, this fixes X to be $Ae^{8\pi^2 \beta_1 \ln(\rho M)} = A(\rho M)^{\frac{22}{3}}$. A is a difficult constant to obtain, so our final answer will be $\frac{E(\theta)}{V} = -A\cos\theta\, e^{-\frac{8\pi^2}{g^2}} g^{-8} \int_0^\infty \frac{d\rho}{\rho^5} (\rho M)^{\frac{22}{3}} [1+O(g^2)]$

This integral is divergent for large ρ, which is a consequence of the $\beta_1 \ln(\rho M)$ term getting large for large ρ, meaning our small g^2 expansion fails in the large ρ limit. It is feasible that if we had exact instanton solutions that worked in high g^2 regimes that this integral (and therefore K) would be finite.

For SU(3), the procedure for instantons follows very similarly. Our SU(2) instanton can be used as a starting point by doing the 5 other gauge transformations. Thus we would have not 3 but 7 parameters for gauge (T^8 commutes with $T^{1,2,3}$), the g^{-8} is replaced with a g^{-12}, and $\beta_1 = \frac{11}{8\pi^2}$. It is not known whether this would exhaust all instanton solutions for SU(3).

Conclusion