Notes: Quantum lumps (Aspects of Symmetry)

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Abstract

These are notes I compiled for my University of Washington Particle Theory Journal Club talk, winter quarter 2018. I review some fun material selected from section 4 of the "Classical lumps and their quantum descendents" chapter in *Aspects of Symmetry*. I do little in the way of derivation, focusing primarily on the results and interpretation of the counterparts to lumps in the quantization of sine Gordon theory.

Contents

1 Lumps

In this talk I will carry on the discussion of topological defects started by Sam Kowash in the first lecture of this series. Let's start with a quick recap of some points that will be built on today:

1.1 Old friends

In Sam's talk we considered solutions to classical field equations having particular properties

- Finite energy
- Time-independent (or boosted)
- Non-dissipative

E.g., there was the Korteweg-de Vries equation describing waves on the surface of shallow water,

$$\partial_t \phi + \partial_x^2 \phi - 6\phi \partial_x \phi = 0 \tag{1}$$

with the right-mover solution

$$\phi(x,t) = \frac{-c}{2} \mathrm{sech}^2 \left[\frac{\sqrt{c}}{2}(x - ct - a) \right] \tag{2}$$

Of more relevance to us, there was also the (spontaneously broken) ϕ^4 theory

$$\boxed{\mathscr{L} = \frac{\lambda}{2}(\phi^2 - a^2)^2 \qquad a^2 = \mu^2/\lambda} \tag{3}$$

which has a time-independent lump solutions given by

$$\phi = a \tanh(\mu x), \tag{4}$$

and boosts and shifts of this solution. These are sometimes referred to as *kinks*.

1.2 A new friend

Today we'll make a new friend, the *sine-Gordon theory*, specified by the lagrangian

$$\boxed{\mathscr{L} = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{\alpha}{\beta^2}(\cos(\beta\phi) - 1)} \tag{5}$$

where α, β are real parameters. It is easy to convince oneself we can take $\alpha > 0$ and $\beta > 0$ without loss of generality (an appropriate field redefinition can always make this true, possibly introducing an unimportant shift of the vacuum energy). By the way, today we will live exclusively in $d = 1+1$ and concern ourselves with real scalar fields.

If the sine Gordon theory is new to you, you may be uncomfortable by the appearance of a cosine. The Maclaurin series of this interaction density, which begins with

$$U(\phi) = \frac{\alpha}{2}\phi^2 - \frac{\alpha}{4!}\beta^2\phi^4 - \frac{\alpha}{6!}\beta^4\phi^6 + \cdots, \tag{6}$$

implies the theory has an infinite number of interaction terms. We won't be concerned with computing Feynman diagrams so this need not bother us. Actually the sine-Gordon model has an abundance of interesting properties, of which I will only be able to scrape the surface during a one-hour talk. We should nonetheless begin to love the sine Gordon model and view it as a useful addition to our theoretical toolbox.

1.3 Elementary lump properties of s-G

Let me note a few more properties of s-G obvious at the lagrangian level: The interaction density has degenerate global minima:

$$\phi = \frac{2\pi}{\beta}\mathbb{Z} \qquad \text{(s-G ground states)} \tag{7}$$

And, since we have $\alpha > 0$, we can identify the two lowest order terms with ϕ^4 theory via

$$\alpha \leftrightarrow \mu^2, \qquad \alpha\beta^2 \leftrightarrow \lambda \tag{8}$$

If α had been negative, the apparently negative mass-squared would only be a reflection of the fact that the series (6) was expanded about a maximum rather than a minimum.

As before, we will concern ourselves with the question

Question 1. *Are there finite energy, time-independent solutions to the field equations?*

Obviously, the answer is yes. Sine Gordon theory admits lump solutions of the form

$$\boxed{\phi(x) = \frac{4}{\beta}\arctan\left(e^{\sqrt{\alpha}x}\right)} \tag{9}$$

For later use, let f designate solitons centered at points b not necessarily the origin:

$$f(x-b) = \phi(x) \tag{10}$$

One can compute the total energy of a sine Gordon soliton:

$$\boxed{M = \frac{8\sqrt{\alpha}}{\beta^2}} \tag{11}$$

And, as before, moving solitons can be obtained by boosting the time-independent solutions. They obey the familiar relativistic energy-momentum relation.

2 Quantum corrections to the classical picture

2.1 Lumps as "particles"

I would like to emphasize some of the remarks made at the end of the last section: **classical lumps are in many ways like particles (rather, extensive rigid bodies)** since

- The lumps are localized in space
- They have finite energy greater than the vacuum state, M^2
- Propagating solutions are related by boosts, with $E^2 = P^2 + M^2$.
- New solutions to the field equations are obtained by patching together distant lumps: **many-particle states**

Obviously, at the end of the day, what pays our bills is not classical mechanics but quantum mechanics. We are thus lead to inquire

Question 2. *In what way—if any–do the classical lumps carry over into the quantum theory? And how is the classical picture modified?*

I think it's quite unreasonable to seek an answer in such generality. Therefore, to narrow our scope, we will do the obvious thing:

1. Quantize the field theory
2. Examine the semi-classical limit of the field theory by sending $\hbar \to 0$

We will **use the sine Gordon model as a testing grounds**.

2.2 Departure at the lagrangian level; the classical limit

To begin, let us recall the lagrangian of the classical theory we seek to quantize:

$$\mathscr{L} = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{\alpha}{\beta^2}(\cos(\beta\phi) - 1) \tag{12}$$

If we had been only interested in classical physics, I would have done you an injustice by not pointing out sooner that β **is a extraneous parameter**: By this, I mean we can perform a field redefinition,

$$\phi' = \beta\phi \tag{13}$$

to arrive at the equivalent lagrangian

$$\boxed{\mathscr{L} = \frac{1}{\beta^2}\left[\frac{1}{2}(\partial_\mu \phi')^2 + \alpha(\cos\phi' - 1)\right]} \tag{14}$$

From now on, we will mostly concern ourselves with this rescaled field and lagrangian. We see all the β-dependence has been extracted as an overall multiplicative constant, which is entirely invisible to the field equations. Thus, any classical solution involving the ϕ' fields is instantly translated into one for ϕ. α is effectively the only parameter for the classical theory.

The reason we cannot discard β in the quantum theory is most easily seen from the partition function:

$$Z \propto \int [d\phi']_{\text{PBC}} e^{-S[\phi']/\hbar} \tag{15}$$

A rescaling of the action leads to different relative weights of the field configurations being summed over in the functional integral. E.g., if we consider two field configurations $\phi'_{1,2}$, rescaling the lagrangian by $\mathscr{L} \to \kappa\mathscr{L}$ changes the relative weights of the configurations according to

$$\frac{e^{-\kappa S_2}}{e^{-\kappa S_1}} = e^{(\kappa-1)(S_1 - S_2)}\left(\frac{e^{-S_2}}{e^{-S_1}}\right) \tag{16}$$

In this setting, rescaling the lagrangian is no different than dialing the temperature knob. The path integral is also useful since it shows how we can examine the classical limit: since the important quantity is

$$\frac{S[\phi']}{\hbar} = \frac{1}{\hbar\beta^2}\int dx\, dt\, \left[\frac{1}{2}(\partial_\mu \phi')^2 + \alpha(\cos\phi' - 1)\right] \tag{17}$$

the classical limit is **the same as the weak coupling limit**, $\beta \to 0$. Therefore, instead of fussing with explicit $\hbar$'s, we will simply do perturbation theory in β^2. So, before diving in let me clearly state our goal: **We will use the results of perturbative quantum corrections and identify them as corresponding to the topological defects of the classical theory.**

Order	Lowest eigenstates	Energy
0	$\|r_0, \Omega\rangle$	E_0
1	$\|n, \Omega\rangle$	$+(n+\frac{1}{2})\sqrt{\frac{V''(r_0)}{M}}$
2	$\|n, \ell, m\rangle$	$+\frac{\ell(\ell+1)}{2Mr_0^2} + \cdots$

Table 1: Perturbative results for diatomic molecule

2.3 Toward perturbation theory

We will specifically consider perturbative corrections to the lowest-lying energy states of the theory and their energies. For this task, we are in need of a Hamiltonian. Performing the Legendre transform is a straightforward task, with the result

$$\boxed{\begin{aligned} \beta^2 H &= V + \beta^4 \int dx\, \frac{(\pi')^2}{2}, \\ V &= \int dx\, \left[\frac{1}{2}(\partial_1\phi)^2 + U(\phi)\right] \end{aligned}} \tag{18}$$

We will work with $\beta^2 H$ as "the hamiltonian" and just divide β^2 out of the spectrum later. Looking at this hamiltonian, we note peculiarity stands out: the zeroth order hamiltonian is the potential, whereas the kinetic energy is, evidently, the perturbative correction. Obviously this departs from the usual most familiar scenario, in which the interaction is the perturbation. *From where, then, can we draw intuition?*

2.4 P.T. of diatomic molecule

One answer is a diatomic molecule of heavy atoms. The hamiltonian for such a system is

$$H = \frac{\vec{P}^2}{2M} + V(|\vec{r}|) \tag{19}$$

which uses center-of-mass coordinates. By assumption, the reduced mass M is large relative to the scale of the potential. I have also taken the interaction to be rotationally invariant.

What does perturbation theory look like in such a system? The first few orders of perturbation theory are sketched in Table 1. I will speak of the system as if it were just a particle in a central potential.

- At zeroth order, where H is nothing but the potential alone, the ground state is just a position eigenstate corresponding to the classical ground state. Because of the rotational symmetry, there is a degeneracy of the ground state in the angular coordinates, ω. One such state can be obtained from another by an appropriate rotation.
- At first order we pick up small radial oscillations corresponding to fluctuations in the harmonic approximation. Correspondingly, the radial eigenvalue r_0 no longer characterizes the ground state, so this quantum number is replaced with the level number, n.
- Finally, at second order, the angular degeneracy is lifted by decomposing into orbital angular momentum modes: states within a given irrep ℓ transform among each other under rotations.

Further corrections depend on the detailed form of $V(r)$, which we won't concern ourselves with.

I'd like to point out the organization of the perturbation theory: we assigned the vibrational excitations to the first order, and rotational excitations to the second order. This is not a general feature, but a consequence of our assuming M is very large. Because the vibrational corrections must come in at $\mathcal{O}(M^{-1/2})$, and the rotational at order $\mathcal{O}(I^{-1}) \sim \mathcal{O}(M^{-1})$ (with $I = Mr_0^2$ the zeroth order moment of inertia), the corrections had to enter in this way. Note that the proper perturbative parameter here is $M^{-1/2}$.

2.5 Generalizing to multiple degrees of freedom

Our field theory problem is more akin to the quantum mechanics of a polyatomic molecule. The generalization of the previous discussion to many nuclei is straightforward. Instead of a rigid rotator, the zeroth order picture is now that of a rigid body with the (heavy) nuclei all pinned to position eigenstates in normal coordinates. The first order corrections are now level numbers corresponding to the classical normal modes:

$$\sqrt{V''/M}\left(n + \frac{1}{2}\right) \to \sum_i \omega_i \left(n_i + \frac{1}{2}\right) \tag{20}$$

with the ω_i being frequencies of the normal modes: these still scale as the inverse square root of the mass scale. The rotational degeneracy consists of the three Euler angles of the rigid body.

2.6 Aside: Field normal modes

We nearly have enough in place to make the leap to field theory. But first, we need to make a quick detour to give context for the generalization of the oscillator modes. If $f(x)$ describes a time-independent classical lump, consider a slightly perturbed solution to the equations of motion:

$$\phi(x,t) = f(x) + \delta(x,t) \tag{21}$$

Inserting this into the equations of motion, we find the first order requirement

$$\Box^2 \delta + U''(f(x)) = 0 \tag{22}$$

The general solution for such a perturbation can therefore be expressed in normal modes,

$$\delta(x,t) = \mathrm{Re} \sum_n a_n e^{i\omega_n t} \psi_n(x) \tag{23}$$

where the eigenfunctions satisfy

$$-\frac{d^2\psi_n}{dx^2} + U''(f(x))\psi_n = \omega_n^2 \psi_n \tag{24}$$

What all this is saying is that if we give small kicks to a classical lump, we will excite normal modes which are deviations from the lump that oscillate periodically in time. (I refer you to Coleman for the short argument that the lump is in fact stable against such kicks.) These fluctuations about a stable equilibrium are what become oscillator levels in the quantum theory.

Order	Lowest eigenstates	Energy
0	$\|b\rangle$	E_0/β^2
1	$\|n_1, n_2, \ldots; b\rangle$	$+\sum_i (n_i + \frac{1}{2})\omega_i$
2	$\|n_1, n_2, \ldots; p\rangle$	$+\frac{\beta^2 p^2}{2E_0} + \cdots$

Table 2: Perturbative results for $d = 1 + 1$ lumps

2.7 Sine-Gordon P.T.

We can now understand the lowest-lying states in perturbation theory, which are summarized in Table 2. Note that since the kinetic term carried a factor β^4 in the field theory and M^{-1} in quantum mechanics, we have the correspondence $\beta^2 \leftrightarrow M^{-1/2}$. An additional β^2 has been divided out as promised earlier.

- At zeroth order we have eigenstates corresponding to time-independent lumps, with degeneracy in the center of the lump. These are eigenstates of the field operator in the same way our lowest order states in quantum mechanics were coordinate eigenstates.
- At first order, we have independent oscillator modes corresponding to the motions I just described. In the quantum setting, these are viewed as scalars interacting with the lump.
- Finally at second order the translational degeneracy is broken by the states acquiring momentum:

$$|n_1, \ldots; p\rangle = \int \frac{db}{\sqrt{2\pi}} e^{ipb} |n_1, \ldots; b\rangle \tag{25}$$

  Whereas all the states $|b\rangle$ at zeroth order were related to each other by the translation group, a momentum eigenstate at second order goes into itself under translations. (A parity degeneracy clearly remains.) These states are viewed as boosted lumps.

3 Uniquely quantum properties

I think all the claims I have made thus far have been very plausible. You might even say that we have hardly seen anything special about passing to the quantum theory: all I really did was classify low-energy states and identify their classical meaning.

3.1 A more interesting classical lump

To get at some uniquely quantum behavior, let's again take fruit from the sine Gordon tree. Allowing for time dependence, the classical equation of motion for the rescaled s-G field takes the form

$$-\Box^2 \phi' - \alpha \sin \phi' = 0 \tag{26}$$

A one-parameter family of solutions (which aren't minimal in energy, but are nonetheless non-dissipative) is given by

$$\phi'_{\text{doublet}} = 4 \arctan \left[\frac{\sqrt{\alpha - \omega^2} \sin \omega t}{\omega \cosh\left(\sqrt{\alpha - \omega^2} x\right)} \right] \qquad (0 < \omega < \sqrt{\alpha}) \tag{27}$$

I have designated this as a "doublet" solution because, in the parameter regime $\omega \ll \sqrt{\alpha}$, there is a very natural interpretation of this solution as a solition-antisoliton bound state, with the lumps oscillating back and forth about their center of mass. In these notes I omit the mathematical arguments for this, because in my talk I simply showed this as an movie on my laptop.

The energy of the doublet solution can be computed without too much trouble with the result

$$E_{\text{doublet}} = 2M\sqrt{1 - \frac{\omega^2}{\alpha}} \tag{28}$$

where $M = 8\sqrt{\alpha}/\beta^2$ is the soliton rest energy computed earlier. This supports the interpretation of the bound state, since the total energy is manifestly less than the sum of two isolated solitons.

3.2 Bohr-Summerfeld quantization

Going over to the quantum theory, it is reasonable to expect these solutions to show up as energy eigenstates. A quick and dirty way to examine the leading order dynamics in the (old) quantum theory is to appeal to the Bohr-Summerfeld quantization condition. According to this principle, if there is a one-parameter family of periodic solutions labeled by the period $T = 2\pi\omega$, then an energy eigenstate shows up in the spectrum whenever

$$\int_0^T dt\, p\dot{q} = 2\pi n \tag{29}$$

with n an integer. For us, in the field theory language, this quantization condition becomes

$$2\pi n = \int_0^T dt \int dx\, \pi(t,x)\partial_t\phi(t,x) \tag{30}$$

An equivalent, and much easier, way to solve this condition is by using the equivalent statement that *the frequency of a quantum emitted when the system transitions from one state to the next is equal to the classical frequency of motion*:

$$dE = \omega\, dn \tag{31}$$

Solving (28) for the classical frequency of the periodic doublet solutions, we have the differential equation

$$\frac{dn}{dE} = \frac{1}{\omega} = \frac{1}{\sqrt{\alpha}\sqrt{1 - \left(\frac{E}{2M}\right)^2}} \tag{32}$$

Then, integrating this with the initial condition $dn/dE = 0$ at $E = 0$, our quantization becomes

$$n = \int_0^{E_n} dE \frac{dn}{dE} = \frac{2M}{\sqrt{\alpha}} \arcsin\left(\frac{E_n}{2M}\right) \tag{33}$$

From this relation, because of the maximum of arcsin, it can be inferred that $n_{\max} \leq 8\pi/\beta^2$, so that

$$E_n = 2M \sin(\beta^2 n/16) = \frac{16\sqrt{\alpha}}{\beta^2} \sin(\beta^2 n/16), \qquad (n = 0, 1, \ldots) \tag{34}$$

There is a nice interpretation of this quantum mechanical calculation in the classical limit. Expanding this in powers of β,

$$E_n = \sqrt{\alpha} n + \mathcal{O}(\beta^4) \tag{35}$$

When we first met the sine Gordon theory, we identified $\sqrt{\alpha}$ with μ of ϕ^4 theory (valid for weak coupling). Thus, in the quantum picture, we have the soliton-antisoliton doublet bound state as well as a cloud of weakly interacting scalar excitations. This result was not apparent from the classical field solution alone.

3.3 Extra: Seeing the doublet

For the purposes of creating an animation of the doublet, it's worth pointing out that a vast simplification can be made. I non-dimensionalized using

$$y = \sqrt{\alpha}x, \qquad \tau = \sqrt{\alpha}t \tag{36}$$

and I parametrized the solutions by the ratio of the doublet energy to the sum of rest energies:

$$f = \sqrt{1 - \frac{\omega^2}{\alpha}} \tag{37}$$

When f is very close to one, the binding energy is small. Hence, the solitons are very weakly bound, and it is possible to really see the almost independent behavior of the two lumps, exchange for brief moments when they run into each other. The form of the solution is expressed as

$$\phi' = 4 \arctan\left(\frac{f}{\sqrt{1-f^2}} \frac{\sin\left(\sqrt{1-f^2}\tau\right)}{\cosh(fy)}\right) \tag{38}$$

4 Summary

In this talk my aim was to tell a small bit of the story of lumps in the quantum domain, and to tell it in my own words based on what I learned from *Aspects of Symmetry*. I'd strongly recommend anyone wanting to learn this topic to start with the book or perhaps a more modern reference, but these notes can serve as an even briefer summary of some of the interesting results. I swept many details under the rug and didn't even touch on the equivalence between the sine Gordon theory and the massive Thirring model, which is another reason why sine Gordon is such a useful laboratory.