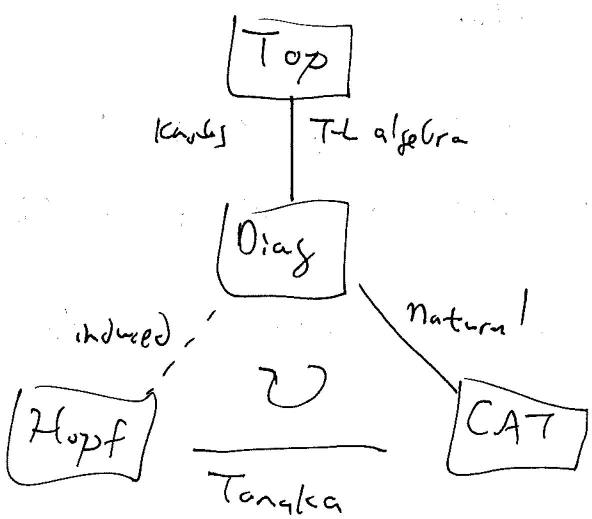
The goal of the final proj this quarter is to better formalize the rich connections between topology & algebra that underlie the structures we've worked w/ this quarter.

This is a tall task, as most of the machinery that is necessary to understand this subject w/ any coherent depth would require that I cover the content in 3 texts in 45 mins. So what I'm hoping to present instead is a broad sampling of the mathematics that pervades this area, providing evidence for <u>why</u> the diagrammatics of the string net models work, where they come from, and what is "under the hood" that leads to different theories like YM, double-semion, toric code, etc.



Primer on CAT theory:

A cat $\mathcal{C}$ is an algebraic structure similar to a group, but no closure / inverses ... so nothing like a group. (Useful to prove big general statements due to handy diagrammatic structure which takes care of algebra isomorphisms)

But has class of ~~objects~~ elements $ob(\mathcal{C})$ called "objects", & class $Mor(\mathcal{C})$ of morphisms (structure preserving maps) between objects.

functor: cat → ob & mor

If you are going from Cat to Cat, "functor", but won't need these today.

$\exists$ identity morphism $\forall$ objects $id_X : X \to X$

A "strict" "monoidal" cat has a unit object $\mathbb{1}$ & an associative tensor product of morphisms which is also bilinear.

→ comment about strict / MacLane

In the complex-linear cats we need today, $M_{or}(V,W)$ forms a vector space over $\mathbb{C}$ & morphisms also compose in the usual bilinear way that functions on vector spaces do.

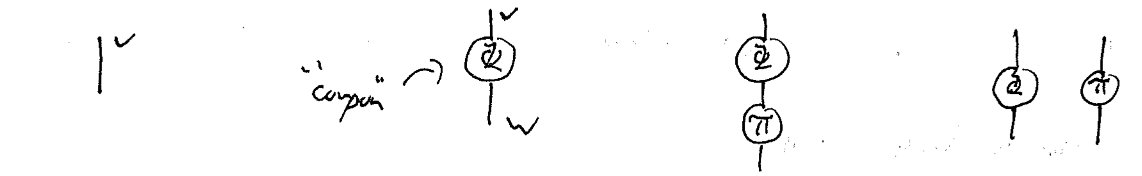
☆ We need these properties as it gives us an easy way to diagrammatically represent category theory.

☆ eg: $FinVect_{\mathbb{C}}$, finite dim vector spaces are objects, linear maps are morphisms

tensor product is usual tp of vector spaces

unit is $\mathbb{C}$

Diagrammatics: $id_V: V\to V$ $\quad \Phi: V\to W$ $\quad \pi\circ\Phi$ $\quad \Phi\otimes\pi$



why we need "strict"

first steps to string net diagrams!

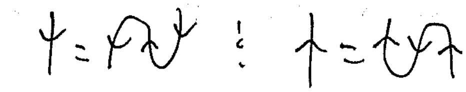
Types of Cats: Extra structure on cats gives more diagrammatics to play with
↳ won't get into detail, just show effect on diagrams

Rigid: introduce "duals" (left) ⇒ directed diagrams $\quad id_V$: ↓^{V} $\quad id_{V^*}$: ↑^{V}

Also get new morphisms: $\quad ev_V: V^*\otimes V\to 1$ $\quad coev_V: 1\to V\otimes V^*$

↳ like inner product $\quad$ ↳ like $\sum e^i\otimes e_i$ $\quad$ "drops the unit object"

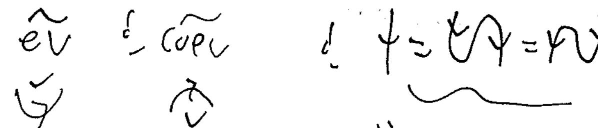
define dual s.t.



: getting closer to strings, now can curve a little

SN saw ○ a lot. Can't make one yet. Want $\overset{\text{define}}{Tr}(id_V): 1\to 1$ ○, but we can't make coherent loop yet due to arrows

Pivot: $\exists\ \tau: V\to V^{**}$ s.t. can define $\widetilde{ev}$ & $\widetilde{coev}$ &



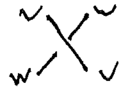
"first instance diagrammatic theorem giving rise to morphism/ISO equivalence"

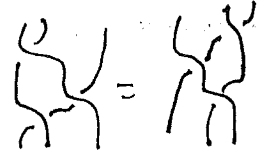
So now $Tr_+(id_V)$ = ○ $\quad Tr_-(id_V)$ = ○

But $Tr_\pm$ ...

Spherical: $Tr_+ = Tr_-$; $Tr(\overset{id_V}{○}) =$ "categorical dim" d, which shows up in our string nets

for FinVect, d = dim of vector space V

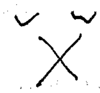
Crossing? 

<u>Braided</u>: $\exists\ c_{V,W} : V\otimes W \to W\otimes V$ gives rise to braid relation 

Start to handle crossings

~~Thickening~~

<u>Ribbon</u>: Thicken lines w/ blackboard framing: $\nu_V : V \to V$  "twist"

<u>Symmetric</u>: $\nu_V = id$, $c_{V,W} =$ 

eg: FinVect is symmetric, ~~...~~

Lastly, mention "Semisimplicity": V simple if $\dim(Mor(V,V)) = 1$

if $V \in ob(C)$ ~~...~~ can be decomposed as direct sum of simple objects, C is semi-simple

eg Rep(G) is symmetric because it inherits structure from FinVect

also semisimple for ~~finite dim~~ compact lie groups as every finite dim rep decomposed as direct sum of ~~...~~ irreps as "simple objects".

used extensively in gauge theories in this language, where lines are labeled by simple objects (irreps) & ~~...~~ ~~...~~ group intertwiners give morphisms.

like Haar measure

Trivial rep is "unit".

<u>Invariance</u>: deforming diagrams as topological entity embedded in some space gives structure isomorphisms

Cat	Deform
pivot	isotopy in $\mathbb{R}^2$
spherical	" " S^2
ribbon	" " $\mathbb{R}^3$
symmetric	combinatorial ⊔ embedding

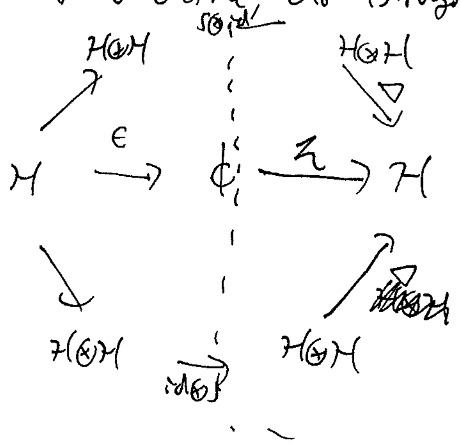
Connect to Algebra? Our theories are defined w/ algebra, so how to we make contact w/ this new diagrammatic language?

Very general ways, but going to give us a super brief peak into it.

Hopf Algebras: ~~like a group but~~ "noncommutative function algebras on groups"

↳ like a group, but not. "quantum group"

Lots to define, do through example: $C_{alg}(G)$, set of functions from G to $\mathbb{C}$ gives Hopf algebra



algebra

for $g \in G$, $f, \lambda \in H$

$\eta(g) = 1 \ \forall g$ "unit"

$\nabla(f,\lambda)(g) = f(g)\lambda(g)$ "product"

coalgebra: $\Delta f(g, g') = f(gg')$ coproduct

$\epsilon(f) = f(1)$ counit

Hopf alg is bialgebra, + "antipode" $S(f(g)) = f(g^{-1})$

↳ "convolution inverse of the identity"

Now w/o going through details, we can equip these w/ extra structure.

if equip w/ "coprotal structure" w, coprotal Hopf

if coprotal + inverse compatible $w(a) = a^{-1}(a)$ coprotal structure, "cospherical"

if Hopf + coquasitriangular structure, co... Hopf

↳ R matrix, sol to Yang Baxter eqn

if coquasi + ribbon form, coribbon Hopf

if coquasi + inverse compatible, $R(a \otimes b) = R^{-1}(b \otimes a)$ cotriangular

Based on names & the way we laid out cats, you might guess a correspondence here.
Won't prove, but yes.
Specifically, the right comodules of $\mathcal{H}$ form a rigid category $M^{\mathcal{H}}$.
finite dim

(don't worry about "right comods", just vector space V w/ $\rho: V \to V \otimes \mathcal{H}$)

if $\mathcal{H}$ ~~[illegible]~~, $M^{\mathcal{H}}$ ~~[illegible]~~	
copivot	pivot
cospherical	spherical
coquasi	braided
coribbon	ribbon
coΔ	symmetric

"Tannaka duality"

So why do we care? We are really formulating our theories not in terms of groups, but Hopf algebras.
$U(g)$ Poincaré
If $su(2)$ GT, really $\mathrm{Cat}_g(su(2))$, which is dual to symmetric cat w/ nice diagrammatics that lets us solve partition functions exactly.
But broader. If equip $\mathrm{Cat}_g(\ldots)$ w/ nontrivial coquasi Δ, ~~ribbon~~ braided cat w/ entirely different rules, different cobordisms, diff top invariants, different physics.
Key example: Drinfeld quantum group. Extra structs depend on single continuous complex ~~[illegible]~~ param $q = e^{\lambda}$: "q-deformation"
Chern-Simons has rep theory of $U_q(g)$. So if move to $C_q($~~[illegible]~~ $G)$, get back ~~[illegible]~~ $su(2)$ GT plus level k CS w/ $q = e^{\frac{2\pi i}{k+c}}$ | c = casimir in adjoint
If q not a root of unity, don't.
If $q = 1$, regular $su(2)$ GT. $\therefore$ depending on q, get different physics.

Lastly, briefly connect this to knots: look at $Cat(Su(2))$, Temperley-Lieb algebra level

Generally, associated w/ a "smoothing" of $\times$ is two options, $\underset{\frown}{\smile}\ A$ & $)A^{-1}($

Given knot K, , states are smoothings of K. Kauffman Bracket polynomial $\langle K\rangle = \sum_S \langle K|S\rangle d^{\|S\|}$

where $\langle K|S\rangle$ = product of state labels, $\|S\|$ = # disjoint Jordan curves in S, $d = -A^2 - A^{-2}$

↳ top invariant up to regular isotopy in $\mathbb{R}^3$

eg  $\langle K|S\rangle = A^3$

$\|S\| = 2$

This has prop $\langle \times \rangle = A\langle \underset{\frown}{\smile} \rangle + A^{-1}\langle)(\rangle$, $\langle O \sqcup K\rangle = d\langle K\rangle$ } rules we took as "defining" in strings, (we had $d=1$, fires) actually more general.

$A = e^{\frac{2\pi i}{r}}$

$\exists$ an entire diagrammatic calculus to evaluate diagrams that contains 6-j symbols, tet nets, "theta" diagrams, all sorts of goodies. Generally, we have Temperley-Lieb recoupling theory, where our simple objects are irreps of $U_q(su(2))$, and invariants like Turaev-Viro, RTW, etc are diagrammatically manifest & easily evaluatable.

Sidenote $U(g) = T(g)/I$ | $T(g) = K \oplus g \oplus (g\otimes g) \oplus \dots$; $I = a\otimes b - b\otimes a - [a,b]$

(K: field over which g defined)

↳ associative algebra w/ identity generated by $x_1, \dots, x_n$ subject to same relation given by commutator in Lie algebra & "nothing else"

Bilinear: left/right distributive $(x+y)\cdot z = x\cdot z + y\cdot z$, $x\cdot(y+z) = x\cdot y + x\cdot z$

& scalar compatible $ax \cdot by = ab\, x\cdot y$