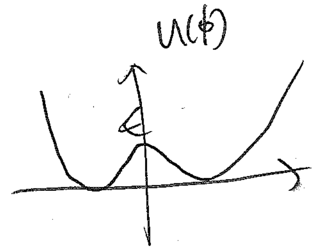
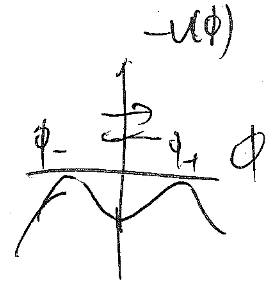
~~Brief~~ Start with QM instanton. Translate into scalar field language.

Tunneling in QM.

Euclidean action

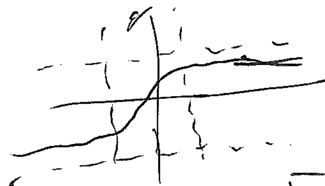
$$S_E = \int d^4x \left[\frac{1}{2} \partial_\mu \phi \partial_\mu \phi + U(\phi) \right]$$





EOM $\quad \partial_\mu \partial_\mu \phi - U'(\phi) = 0$

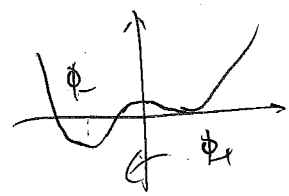
Solution is classical path that minimize S_E. Dominate PI.

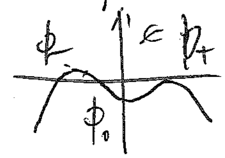


$\sim \frac{1}{m}$ $\quad m = \sqrt{U''(\phi_\pm)}$.

Asymmetric Double well

"Bounce"

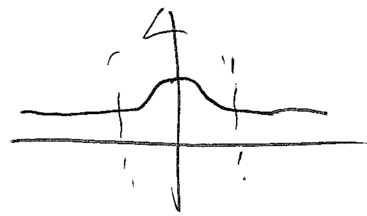




$$\frac{\Gamma}{V} \sim A e^{-S_{instanton}}$$

V: 4 translational invariance.

$\int$ all config $= V$.



$\frac{1}{m} \sim \sqrt{U''(\phi_+)}$

EOM invariant under 4D Euclidean rot + reflections.

Expect ~~bounce sols~~ there exist $O(4)$ invariant Bounces.

Assume $O(4)$ invariant bounces are global minimum for S_E. Coleman's erratum.

Find $\phi(r)$ with $r = \sqrt{\tau^2 + \vec{x}\cdot\vec{x}}$.

"Bounce" need to satisfy $\lim_{\tau \to \pm\infty} \phi \to \phi_+$. Finite action: $\lim_{|\vec{x}| \to \infty} \phi \to \phi_+$

initial velocity $\left.\frac{d\phi}{d\tau}\right|_{\tau=0} = 0$.

In terms of $\phi(r)$:

~~No a $O(4)$ invariant~~ No angular dep. 4D laplacian gives

$$\frac{d^2\phi}{dr^2} + \frac{3}{r}\frac{d\phi}{dr} \mp U'(\phi) = 0.$$

$$\lim_{r\to\infty} \phi \to \phi_+ ,$$

Regular at $r=0$ $\Rightarrow$ $\left.\frac{d\phi}{dr}\right|_{r=0} = 0.$

$$S_E = 2\pi^2 \int r^3 dr \left[\frac{1}{2}\left(\frac{d\phi}{dr}\right)^2 + U(\phi)\right]$$ angular dep integrated over.

Existence for those BCs? Guaranteed to have sln with $\phi_{\infty} \to \phi_+$ and $\phi'(0) = 0$.

ϕ' term analogous to damping in classical mechanics.

① Start at some points right to ϕ_0,

$$\frac{d}{dr}\left[\frac{1}{2}(\phi')^2 \mp U(\phi)\right] = -\frac{3}{r}\left(\frac{d\phi}{dr}\right)^2 < 0$$ (Euclidean) Energy decreased.

Undershoot guaranteed.

② Start close to ϕ_-. Expanding $U(\phi)$ near ϕ_-. E-9M term, ($\mu^2 = U''(\phi_-)$).

$$\left[\frac{d^2}{dr^2} + \frac{3}{r}\frac{d}{dr} - \mu^2\right](\phi - \phi_-) = 0 \Rightarrow \phi(r) - \phi_- = \frac{2I_1(\mu r)}{\mu r}\left[\phi(0) - \phi_-\right] \xrightarrow{r\to\infty} 0$$

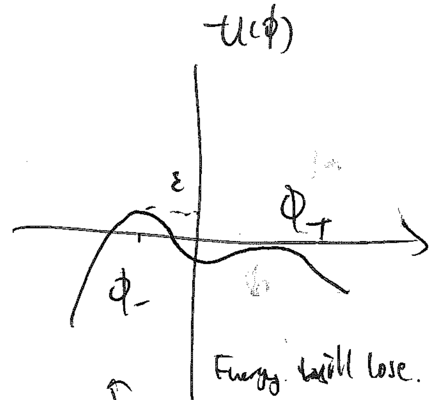
~~Close~~ If initially very close to ϕ_-, for larger r still close to ϕ_-. ~~Overshoot~~.

However at large r damping term unimportant. Overshoot guaranteed.

Then by smooth interpolation ~~exist~~ if initial ϕ properly chosen will find $\phi \xrightarrow{r\to\infty} \phi_+$.

Assuming $U(\phi_+) - U(\phi_-) = \varepsilon$ ~~≪~~ small

Thin wall approximation



Can construct such potential from symmetric one $U_0(\phi)$.

$U(\phi) = U_0(\phi) + O(\varepsilon)$.

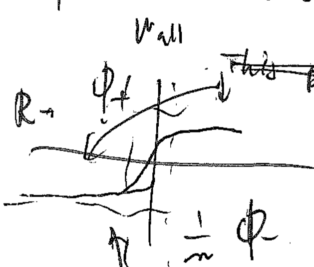
$\delta U = \varepsilon$

Energy will lose. ε is small.

We want to start near ϕ_- to reach ϕ_+.

In this limit, ϕ' term only important when ϕ traverse the valley.

Also ~~Bounce ⟷ instantons~~. Take long "Time" R to reach valley. At $r=R$, $\frac{\phi'}{r}$ also negligible.



In euclidean space this looks like a wall localized at $r=R$.

$\Rightarrow$ CAN drop $\frac{\phi'}{r}$ term all together. $\quad \frac{\phi'^2}{2} = E + V = V$

$\phi'' = U_0'(\phi)$. $\quad S_1 = \int [\frac{\phi'^2}{2} + U_0(\phi)]\, dr$

$= \int \frac{d\phi}{dr}\frac{d\phi}{dr}\, dr$

$= \int \frac{d\phi}{dr}\, d\phi = \int_{\phi_-}^{\phi_+} \sqrt{2U_0(\phi)}\, d\phi$

$S_E = 2\pi^2 \int r^3 dr \, [\frac{\phi'^2}{2} + U] \quad = S_E^{in} + S_E^{out} + S_E^{\text{"in"}}$

Inside & outside $\phi' \approx 0$. $\quad$ Inside $\phi \sim \phi_-$, $U(\phi) = U(\phi_-) = -\varepsilon$.

Outside $\phi \sim \phi_+$, $U(\phi) = 0$. $\quad \Rightarrow S_E^{out} = 0$.

Drop $O(\varepsilon)$, $\quad \to S_1$ 1d soliton action.

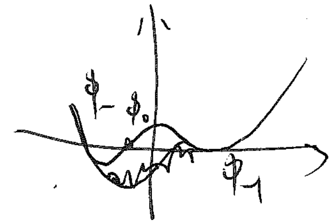
$S_E = -\frac{\pi^2}{2} R^4 \varepsilon + 2\pi^2 R^3 \int dr \, [\frac{\phi'^2}{2} + U_0] + O(\varepsilon)$

Minimize S_E: $\frac{\partial S_E}{\partial R} = 0 \Rightarrow R = \frac{3S_1}{\varepsilon} + O(1)$ $\quad$ Can throw away $O(\varepsilon)$ piece in the line above.

$\frac{\text{size}}{\text{wall thickness}} = \frac{R}{1/m} = mR = \frac{3mS_1}{\varepsilon} \gg 1$. $\quad$ condition for thin wall

$\Rightarrow \frac{P}{V} \propto e^{-S_E}$, $\quad S_E = \frac{27\pi^2 S_1^4}{2\varepsilon^3}$

How to interpret? Go back to Minkowski



Analytical continue to real time

$$\phi(r) = \phi\left(\sqrt{|\vec{x}|^2 - t^2}\right)$$

Boiling water analogy. Thermodynamic fluctuation creates superheated vapor in ~~small~~ some regions.

~~If bubble size~~ Due to surface/volume competition not all bubbles will grow. Some will collapse.

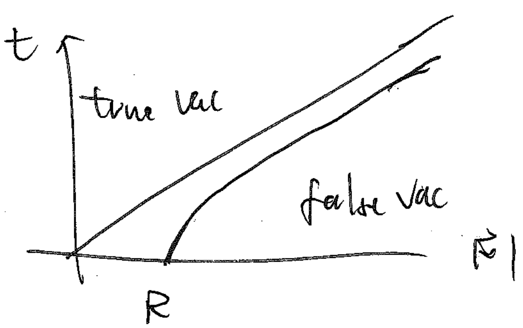
In similar fashion the tunneling has certain probability. Once it reaches escape point, zero kinetic energy, will ~~fall~~ rolling down along the curve for sure. $\phi(t=0,\vec{x}) = \phi(\tau=0,\vec{x})$. By the translation $\frac{d\phi}{dt}\Big|_{t=0} = 0$. chooses $\phi(r=0) = \phi_0$.

After the exit point the field evolves classically. Given by

$$(-\partial_t^2 + \nabla^2)\phi = U'(\phi)$$ Just the continuation of Euclidean EOM we've studied.

The bubble $\tau^2 + |\vec{x}|^2 = R^2 \Rightarrow -t^2 + |\vec{x}|^2 = R^2$ is Minkowski spacetime.

Hyperbolic



$v \to c$.

E_{wall} per area $= \frac{S_1}{\sqrt{1-v^2}}$ $\Rightarrow E_{wall} = 4\pi |\vec{x}|^2 \frac{S_1}{\sqrt{1-v^2}}$

$$v = \frac{d|\vec{x}|}{dt} = \frac{(|\vec{x}|^2 - R^2)^{1/2}}{|\vec{x}|}, \quad E_{wall} = \frac{4\pi |\vec{x}|^3}{3}\varepsilon.$$ Carries all energy.

Hawking, Moss, Stewart PRD 26 2681 (1981), Guth & Weinberg Nucl Phys B 212 (1983)

Gravitational effect. (Finite temperature / presence of other fields)

Importance?

$$R_{sc} = \frac{2GM}{c^2} = \frac{8\pi G}{3}\frac{R_{sc}^3}{c^2}\varepsilon \Rightarrow R_{sc} = \left(\sqrt{\frac{8\pi G\varepsilon}{3}}\right)^{-1} = \left(\frac{\sqrt{3\varepsilon}}{M_{pl}}\right)^{-1}$$

at $(10^{0}\,\mathrm{GeV})^4$, $R_{sc} \approx$ meters. >> GUT scale.

Effects on Formation of bubbles negligible. During the growth will reach this scale eventually.

Shortcut for

$$S_E = \int d^4x \sqrt{g}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - U(\phi) - \frac{R}{16\pi G}\right] \quad \text{Ignoring } R\phi^2 \text{ term.}$$

change $U(\phi)$ by const $\Rightarrow$ $+\Lambda$ term. formally modify gravitational theory.

$O(4)$ Invariance
$$S_E = 2\pi^2 \int d\tau \left[\rho^3\left(\frac{1}{2}\phi'^2 + U\right)\right] + \frac{3\rho^3}{8\pi G}\left(\frac{\rho''}{\rho} + \frac{\rho'^2}{\rho^2} - \frac{1}{\rho^2}\right)$$

$$ds^2 = d\tau^2 + \rho^2(\tau)\, d\Omega_3^2 .$$

obtain EOM from S_E : $\phi'' + \frac{3\rho'}{\rho}\phi' = U'(\phi)$

→ As usual throw away $\frac{\rho'}{\rho}\phi'$ term

Einstein equation $\rho'^2 = 1 + \frac{8\pi G}{3}\rho^2\left(\frac{1}{2}\phi'^2 - U\right)$

$$\downarrow$$

$$S_E = 4\pi^2 \int d\tau \left[\rho^3\left(\frac{\phi'^2}{2} + U\right) - \frac{3}{8\pi G}\left(\rho\rho'^2 + \rho\right)\right] = 4\pi^2 \int d\tau \left[\rho^3 U - \frac{3\rho}{8\pi G}\right]$$