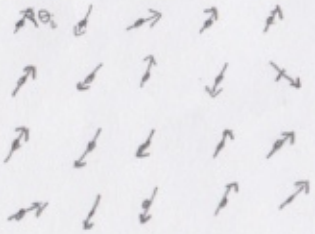
My Talk

- XY Model
- MWHB/C's argument
- High/Low T correlation
- vortices/2d Coulomb gas
- KT transition

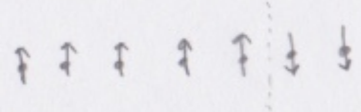
XY Model



$$H_{XY} = -J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j = -J \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j)$$

MWHB/C's argument

1d discrete



"Ising-like"

$$H = -J \sum_{\langle ij \rangle} s_i^z s_j^z$$

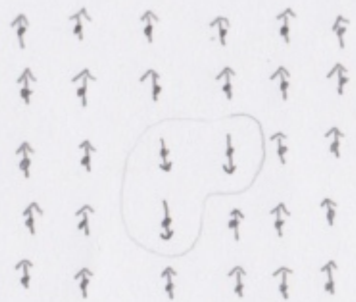
$$Z = \sum_n e^{-\beta E_n} = \sum_{E_n} \Omega(E_n) e^{-\beta E_n}$$

$$= \sum_{E_n} e^{S(E_n)/k_B} e^{-\beta E_n} = \sum_{E_n} e^{-\beta(E_n - TS(E_n))}$$

$E_1 \sim J$

$S(E_1) \sim \ln L$

2d discrete



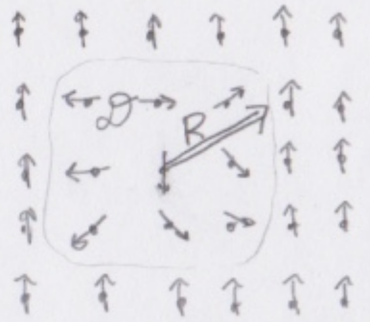
"Ising-like"

$$H = -J \sum_{\langle ij \rangle} s_i^z s_j^z$$

$E_n \sim J\ell$

$S(E_n) \sim \ln\left(\frac{L^2}{A}\right) + \ell \ln \mu$

2d continuous



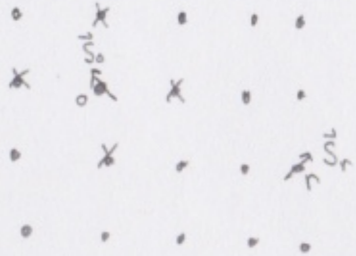
XY Model

In $\mathcal{D}$, $(\theta_i - \theta_j) \sim \frac{1}{R}$

so, $\cos(\theta_i - \theta_j) \sim \frac{1}{R^2}$

$E_n \sim \frac{1}{R^2} \quad R^2 \sim 1$

High/Low T correlations



$$\langle \vec{S}_0 \cdot \vec{S}_r \rangle = \langle \cos(\theta_0 - \theta_r) \rangle$$

$$Z = \int_0^{2\pi} \prod_{i=0}^{N} \frac{d\theta_i}{2\pi} \, e^{K \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j)}, \quad (K := \beta J)$$

High T

$$\langle \cos(\theta_0 - \theta_r) \rangle = \frac{1}{Z} \int_0^{2\pi} \prod_{i=0}^{N} \frac{d\theta_i}{2\pi} \, e^{K \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j)} \cos(\theta_0 - \theta_r)$$

$$\underset{(\text{small } K)}{\overset{(\text{Hi T})}{\approx}} \frac{1}{Z} \int_0^{2\pi} \prod_{i=0}^{N} \frac{d\theta_i}{2\pi} \underbrace{\left[\prod_{\langle ij \rangle} \left(1 + K \cos(\theta_i - \theta_j) \right) \right]}_{\left(1 + K \sum \cos + K^2 \sum \cos\cos + \cdots \right)} \cos(\theta_0 - \theta_r)$$

$$\int_0^{2\pi} \frac{d\theta_0}{2\pi} \cos(\theta_0 - \theta_r) = 0$$

$$K \int_0^{2\pi} \frac{d\theta_0}{2\pi} \cos(\theta_0 - \theta_r) \cos(\theta_0 - \theta_1) = \frac{K}{2} \cos(\theta_1 - \theta_r)$$

$$\langle \vec{S}_0 \cdot \vec{S}_r \rangle \sim \left(\frac{K}{2} \right)^{\|r\|} = e^{-\|r\|/\xi}$$

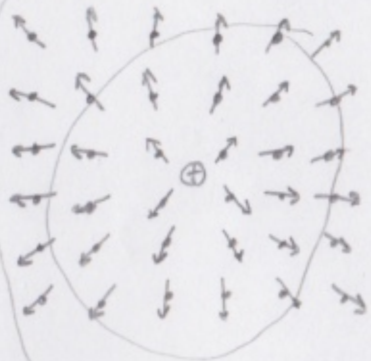
$\boxed{\text{Low T}}$

$$H_{XY} \underset{((\theta_i-\theta_j)\ \text{small})}{\overset{(\text{Low T})}{\approx}} -J \sum_{\langle ij \rangle} \frac{1}{2}(\theta_i - \theta_j)^2$$

$$\langle \vec{S}_0 \cdot \vec{S}_r \rangle \sim \left(\frac{a}{\|r\|}\right)^{\frac{1}{2\pi K}}$$

$\boxed{\text{vortices / 2d Coulomb gas}}$

$$H_{XY} \overset{(\text{Low T})}{\approx} -\frac{J}{2} \sum_{\langle ij \rangle} (\theta_i - \theta_j)^2$$



$$\approx -\frac{J}{2} \int d^2x \, |\nabla\theta(x)|^2$$

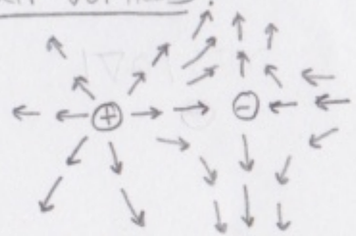
Single vortex:

$$|\nabla\theta| 2\pi r = 2\pi m$$

$$\Rightarrow |\nabla\theta| = \frac{m}{r}$$

$$E_m \sim J\pi \int_a^L dr\, r\left(\frac{m}{r}\right)^2 + E_{core} \sim J\pi m^2 \ln\left|\frac{L}{a}\right|$$

pair vortices:



$$|\nabla\theta| \sim \frac{d}{r^2}$$

$$E_{dip} = \text{finite}$$

$\boxed{\text{KT transition}}$

$$Z = \sum_{E_n} e^{-\beta(E_n - TS(E_n))}$$

$$E_m \sim J\pi m^2 \ln\left|\frac{L}{a}\right|$$

$$\Omega(E_m) \sim \left(\frac{L}{a}\right)^2$$

$$S(E_m) \sim 2k_B \ln\left(\frac{L}{a}\right)$$

$$E_m - TS(E_m) = J\pi m^2 \ln\left|\frac{L}{a}\right| - 2k_B T \ln\left(\frac{L}{a}\right)$$

$$= \left(J\pi m^2 - 2k_B T\right) \ln\frac{L}{a}$$

$$T_c \approx \frac{J\pi}{2k_B}$$

Outline

- 1d, discrete "KT-like" transition
- 2d, continuous BKT transition
- 2d Coulomb gas

1d discrete "KT-like" transition

$\underbrace{\uparrow \quad \uparrow}_{a} \quad \uparrow \quad \downarrow \quad \downarrow$ "Ising-like"

$$H = -J \sum_{i<j} \frac{1}{(j-i)^2} s_i s_j$$

$$Z = \sum_n e^{-\beta E_n} = \sum_E \underbrace{\Omega(E)}_{e^{S(E)/k_B}} e^{-\beta E} = \sum_E e^{-\beta(\overbrace{E - TS(E)}^{F})}$$

$$\Delta F = \Delta E - T\Delta S$$

1 dw

$\uparrow \quad \uparrow \quad \uparrow \quad \uparrow \;\vdots\; \downarrow \quad \downarrow$

$$\Delta E_1 \sim J \int_{x_j = x_{dw}+a/2}^{L/2} \int_{x_i = -\frac{L}{2}}^{x_{dw}-\frac{a}{2}} dx_i \, dx_j \frac{1}{(x_j - x_i)^2} \sim J \ln \frac{L}{a}$$

$$\Delta S(E_1) \sim k_B \ln \frac{L}{a}$$

$$T_c \approx J/k_B$$

pair dw

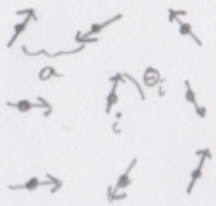
$\uparrow \quad \uparrow \quad \downarrow \quad \downarrow \quad \uparrow \quad \uparrow$

$$\Delta E_{pair} \sim J \ln \frac{x_{dw_2} - x_{dw_1}}{a}$$

$$\Delta S(E_{pair}) \sim k_B \ln \frac{L}{a}$$

2d Continuous BKT transition

XY Model

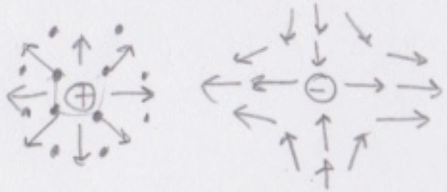


$$H_{XY} = -J\sum_{\langle ij\rangle} \vec{S}_i \cdot \vec{S}_j = -J\sum_{\langle ij\rangle} \cos(\theta_i - \theta_j)$$

$$H_{XY} \overset{(\text{low } T)}{\approx} -\frac{J}{2}\sum_{\langle ij\rangle} (\theta_i - \theta_j)^2$$

$$\approx -\frac{J}{2}\int d^2x\, |\nabla\theta|^2$$

Vortices



$$H'_{XY} \approx -\frac{J}{2}\int d^2x\, \vec{J}^2 = -\frac{J}{2}\int d^2x\, \vec{J}_{\parallel}^2 - \frac{J}{2}\int d^2x\, \vec{J}_{\perp}^2$$

$$\oint d\vec{l}\, \vec{J}_\perp = 2\pi \sum_i m_i$$

1 vortex

$$|\vec{J}_\perp| 2\pi r = 2\pi$$

$$|\vec{J}_\perp| \sim \frac{1}{r}$$

$$\Delta E_1 \sim J\pi \int_a^L dr\, r \frac{1}{r^2} \sim J\pi \ln\frac{L}{a}$$

$$\Delta S(E_1) \sim 2k_B \ln\frac{L}{a}$$

Pair of vortices

$$|\vec{J}_\perp| \sim \frac{1}{r^2}$$

$$\Delta E_{pair} \sim J$$

$$\Delta S(E_{pair}) \sim 2k_B \ln\frac{L}{a}$$

$$Z = \sum_E e^{-\beta(E - TS(E))}$$

$$\Delta F = \Delta E - T\Delta S(E)$$

$$T_{BKT} \approx \frac{J\pi}{2k_B}$$

$\boxed{\text{2d Coulomb Gas}}$

$$\vec{J}_\perp = \nabla \times (\hat{z} W) = (\partial_y W, -\partial_x W, 0)$$

$$\nabla \times \vec{J}_\perp = (0, 0, -\nabla^2 W)$$

$$-\nabla^2 W = 2\pi \sum_i m_i \delta(\vec{x} - \vec{x}_i), \quad W = \sum_i m_i \ln \frac{|\vec{x} - \vec{x}_i|}{a}$$

$$H'_{XY_\perp} \simeq -\frac{J}{2} \int d^2x \, |\vec{J}_\perp|^2 = -\frac{J}{2} \int d^2x \, |\nabla \times (\hat{z} W)|^2 = -\frac{J}{2} \int d^2x \, |\nabla W|^2$$

$$= \frac{J}{2} \int d^2x \, W \nabla^2 W + \cancel{\text{bdry}} \quad \sum m_i = 0$$

$$= \frac{J}{2} \sum_{i,j} (2\pi m_i)(2\pi m_j) \frac{1}{2\pi} \ln \frac{|\vec{x}_j - \vec{x}_i|}{a}$$