Polyakov's Model of Confinement in 2+1-Dimensions 

- Linear confinement
- Monopoles
- Dual photon
- Instanton-induced interactions

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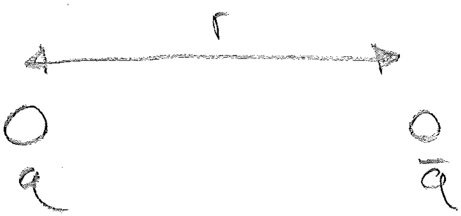
<u>Linear confinement</u>: In pure YM, if one takes a heavy probe quark and antiquark separated by a large distance, the force does not fall off with distance!

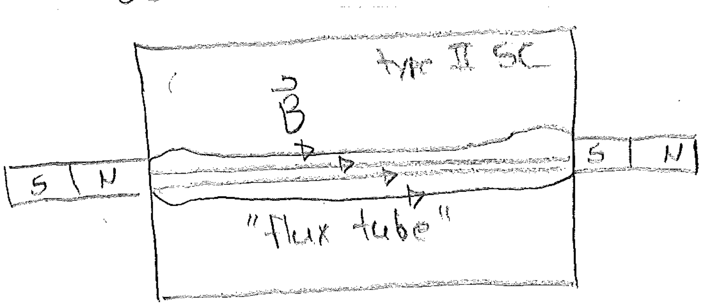



In fact the potential grows linearly with distance: $\underline{V(r) \sim r}$.

- Explanation of the emperical fact quarks & gluons are not seen as asymptotic states.
- Unfortunately, QCD and YM theories at strong coupling in general are not yet analytically solved.
- One should then ask: Are there physical phenomena in which $V(r) \sim r$? And do we understand the underlying mechanism?

Answer: Yes! Example: <u>type II superconductors</u>

superconducting medium does not tolerate a B-field, but flux must be conserved $\Rightarrow$ <u>flux tube</u>




<u>has fixed tension</u>

<u>imply constant force</u> $\Rightarrow$

<u>linear potential</u>

<u>"Meissner effect"</u>

Two problems:

- Occurs in an Abelian theory, QED
- Flux tube is magnetic. Quark conf. "chromoelectric" rather than chromomagnetic.

## Polyakov's model of color confinement (1977)

- Historically, first gauge model where confinement was analytically established in 2+1 dimensions.

"Compact electrodynamics confines electric charges in 2+1 dimensions"

- Cannot be generalized to four dimensions
- Color confinement is essentially Abelian.

## Aside: Monopoles

So far we have seen:

| Real Space | → | Field Space | |
|---|---|---|---|
| $S^0$ | → | $S^0$ | "kinks" |
| $S^1$ | → | $S^1$ | "vortices" |
| Today ⟹ $S^2$ | → | $S^2$ | "monopoles" |
| $S^3$ | → | $S^3$ | "YM Inst" |

## Georgi-Glashow Model: (3+1 dim)

$$S_{GG} = \int d^4x \left[ -\frac{1}{4g^2} \underbrace{G^a_{\mu\nu} G^{\mu\nu,a}}_{SU(2)\text{ field strength}} + \frac{1}{2} (D_\mu \phi^a)(D^\mu \phi^a) - \lambda(\phi^a\phi^a - v^2)^2 \right]$$

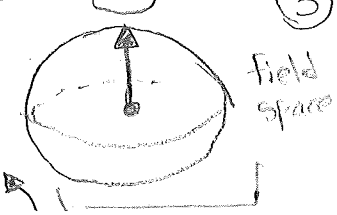
$a = 1,2,3$

$\phi^a$: real scalar field (vector rep), 3-DoF

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \epsilon^{abc} A^b_\mu A^c_\nu \qquad D_\mu \phi^a = \partial_\mu \phi^a + \epsilon^{abc} A^b_\mu \phi^c$$

- Note: $V(\phi^a)$ shows space of vevs is $S^2$. Boundary of space is $S^2$ as well. ⟹ topologically stable solutions
- Work in BPS limit (minimize energy): $\lambda \to 0$ only role of $V(\phi^a)$ is to enforce BC on field $\phi^a$

Vev causes breaking: $SU(2) \rightarrow U(1)$ [!]
(still do color rotations about $\langle\phi^a\rangle = v\delta^{3a}$)




Three components of gauge field $\Longrightarrow$ $\begin{cases} 2 \text{ "W-bosons" } m_W = gv \\ 1 \text{ "Photon"} \end{cases}$ Higgs mech

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Monopole solutions:

Same reason vortex gets a field

$$M_M = \frac{4\pi \, m_W}{g^2} \quad \left(\begin{array}{c}\text{Localized} \\ \text{in 3 spatial} \\ \text{dir!}\end{array}\right)$$

$$B_i \equiv \frac{1}{v} B_i^a \phi^a \xrightarrow[r\to\infty]{} n^i \frac{1}{r^2} \Longrightarrow Q_M = \frac{4\pi}{g}$$

Four zero modes: Three translations
One rotation (redundant with gauge trans)

(other two change vacuum)

Assumed to be chosen in a particular (and unique) way.

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Back to the Polyakov Model: 2+1 dimensions but Wick rotate to 3-dim Euclidean (to look for inst.)

$$S_{GG}^E = \int d^{3}x \left[ + \frac{1}{4g^2} G_{\mu\nu}^a G^{\mu\nu,a} + \frac{1}{2}(D_\mu \phi^a)(D^\mu \phi^a) + \lambda(\phi^a\phi^a - v^2)^2 \right]$$

| | | |
|---|---|---|
| Monopole solutions localized in 3 spatial dir | $\Longrightarrow$ | Monopole-instanton sol. localized in 3 spacetime dir |
| Mass $M_M = \frac{4\pi m_W}{g^2}$ | $\longrightarrow$ | Action $S_{inst} = \frac{4\pi m_W}{g^2}$ |
| Field config. & zero modes | $\longrightarrow$ | Unchanged |

# Dual Photon: of the Polyakov model

Let us focus in on low-energy limit) $E \ll m_W = gv$

- W bosons in spectrum don't matter
- Two scalars eaten by Higgs mech, last is also massive $m_s =$
- Only relevant DoF unbroken $U(1)$ photon field, which for the moment, appears to be massless. $\mathcal{L}^{E}_{GG} \Rightarrow \mathcal{L}_{GG} = +\frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} \quad (E \ll m_W)$
- Note that in 2+1 dimensions, a photon has only one physical (transverse) polarization. We should then be able to describe it with a scalar field, since this also has only one physical degree of freedom! "Dual photon"

$$F_{\mu\nu} = \underbrace{\frac{g^2}{4\pi}}_{\text{Constants fixed by matching}} \epsilon_{\mu\nu\rho} \partial^\rho \varphi$$

- $\varphi$ is compact, that is $\varphi \sim \varphi + 2\pi$ (gauge field itself is compact).
- Non-local in the sense $\frac{F}{\partial} \sim \varphi \sim \frac{F}{p}$ so momentum series never terminates
- However, if I know $\partial^\rho \varphi$ somewhere, $F_{\mu\nu}$ is determined.

$$\mathcal{L} = -\frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} \quad \Longleftrightarrow \quad \mathcal{L}_{dual} = \frac{1}{2}\left(\frac{g}{4\pi}\right)^2 (\partial_\mu \varphi)(\partial^\mu \varphi)$$

- At this level $\varphi$ remains massless, as it should.

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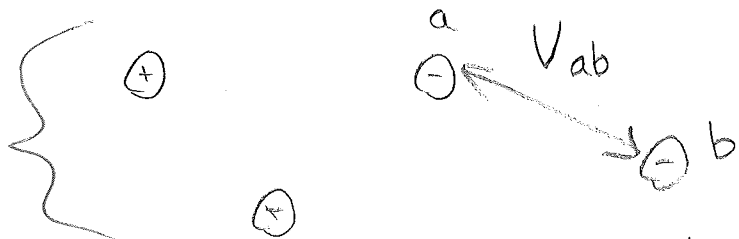
<u>Instanton induced interaction</u>: What are inst. effects on interactions?

- <u>Could proceed as usual</u> (this is what Polyakov does to an extent):
  - <u>Sum over instanton configurations in the path integral</u>
  - Integrate over the four zero modes (translation & rotations) $\underline{d\mu_{inst} = \mu^3 d^3x_0}$

$$\mu^3 \sim \frac{m_W^5}{g^4} e^{-S_{inst}}$$




  - Must account for interactions between monopole istantons as well (Partition f'n for coulomb gas)
  - This looks very ugly, but Polyakov realized the sum could be simplified by moving to the dual photon description. We're going to use this fact to slightly simplify the calculation by working from the dual description from the beginning.

Interactions are now long range compared to, say, 1 d QM

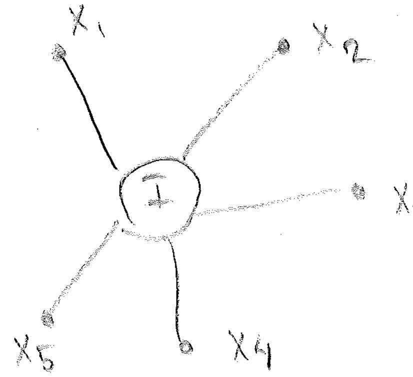
- Assume instanton contribution enters into the Lagrangian by adding a term $\mathcal{L}_{dual} \to \mathcal{L}_{dual} + \mathcal{L}_{inst}$
  - Follow usual EFT methods: do calculation in two distinct ways and compare

1. Directly from instanton field configuration

2. Using effective Lagrangian

(Eucl. space)

$\langle O(x) \rangle_{inst} = \langle O(x) \mathcal{L}_{inst} \rangle$




The correlation function we'd like to consider is motivated by the dual photon identification:

$$\langle 0 | T\{B_{\gamma_1}(x_1) B_{\gamma_2}(x_2), \dots B_{\gamma_n}(x_n)\} | 0 \rangle_{\text{one-inst (at origin)}}$$

where $B^\gamma(x) \equiv -\frac{1}{2} \epsilon^{\mu\nu\gamma} F_{\mu\nu}(x)$ — $B^\gamma$ is 3+1 dim magnetic field

- Instanton is at the origin
- All $|x_k| \gg m_W^{-1}$ (pass to Eucl. space)
- To leading approximation, one can just substitute the classical value of each operator in the monopole-instanton field

$$B_{\gamma_k}(x_k) \to \frac{n_{\gamma k}}{(x_k)^2} = \frac{(x_k)_{\gamma_k}}{(x_k)^2 \sqrt{(x_k)^2}}$$

$$\langle n\text{-point} \rangle_{1\text{-inst}} = \frac{1}{2} \mu^3 \sum_{k=1}^{n} \frac{(x_k)_{\gamma_k}}{(x_k)^2 \sqrt{(x_k)^2}}$$

since only one location of inst;

$$\mu^3 \sim \frac{m_W^5}{g^4} e^{-S_{inst}}$$

comes from int. over rotation zero modes

Now compare to second way:

using $B_\gamma(x) = -i\frac{g^2}{4\pi}[\partial_\gamma \varphi(x)]$ find:

$$\left(-i\frac{g^2}{4\pi}\right)^n \langle 0| T\{\partial_{\gamma_1}\varphi(x_1)\,\partial_{\gamma_2}\varphi(x_2)\cdots\partial_{\gamma_n}\varphi(x_n)\,\mathcal{L}_{inst}(0)\}|0\rangle$$

Leading order pert. exp.

If we choose $\mathcal{L}_{inst} = \frac{1}{2}\mu^3 e^{i\varphi}$,

then we get the exact same leading order result for the correlation function.
(Note this is periodic in $\varphi$, as it must be).

- Same calculation for anti-instanton, find a $\mathcal{L}_{\overline{inst}} = \frac{1}{2}\mu^3 e^{-i\varphi}$

- Combine the two:

$$\mathcal{L}_{dual} = \frac{1}{2}\left(\frac{g^2}{4\pi}\right)^2(\partial_\mu\varphi)(\partial^\mu\varphi) + \mu^3\cos(\varphi)$$

$m_\varphi = \frac{4\pi}{g}\mu^{3/2}$

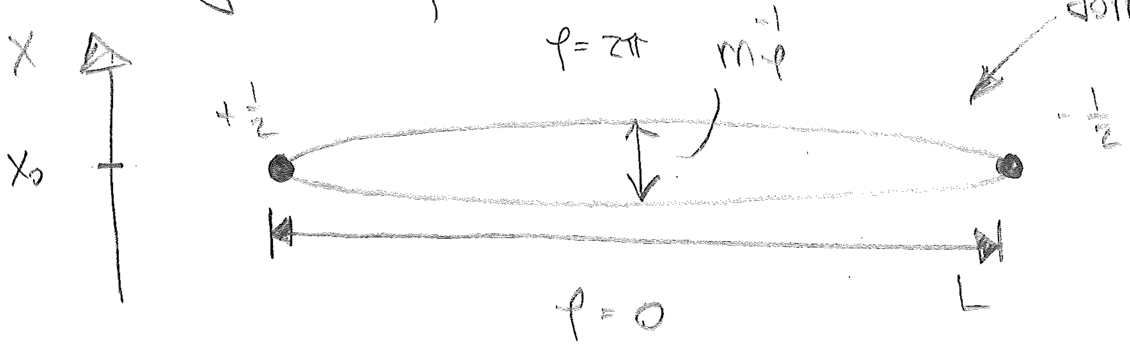
- Expand $\cos(\varphi) \sim 1 - \frac{1}{2}\varphi^2 + \frac{1}{24}\varphi^4 + \dots$

dual photon has acquired an exponentially small mass!
(actual photon gets a mass too, harder to describe)

- Note this is same Lagrangian as sine-Gordon, with one extra spatial direction, Except now $\varphi$ is periodic!

- Why does photon mass $\Rightarrow$ Linear confinement?

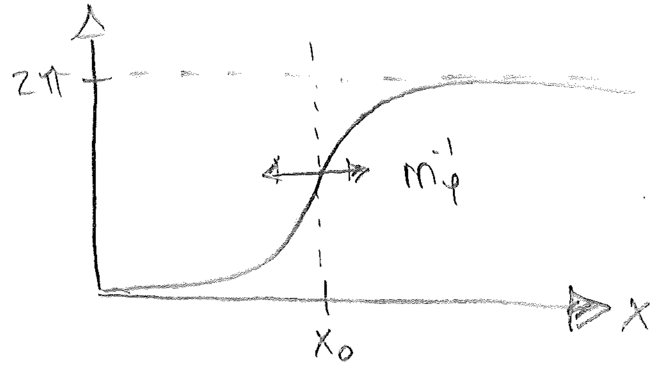
Consider the insertion of probe charges $\varphi \to \varphi + 2\pi$




As one goes around a •, $\phi$ must change by $2\pi$.

$$\varphi = 2\left[\arcsin \tanh (m_\varphi x) + \frac{\pi}{2}\right]$$

(same solution Jesse gave, but in different form)

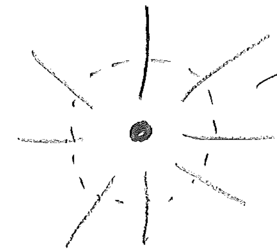



Remember, $\partial_x \varphi$ is related to $F_{yt}$, so this implies nonzero E-field between charges which is in a flux line

- Can only have domain lines because $\varphi$ is periodic. since can travel around the string & still be in the same vac.
- Can find $T = \frac{2g}{\pi}\mu^{3/2}$. Energy grows as $TL$
- At distances $\lesssim m_\varphi^{-1}$ each charge has a two-dim Coulomb field with lines spreading out homogeneously.

Again, consider the insertion of probe charges

Single charge:

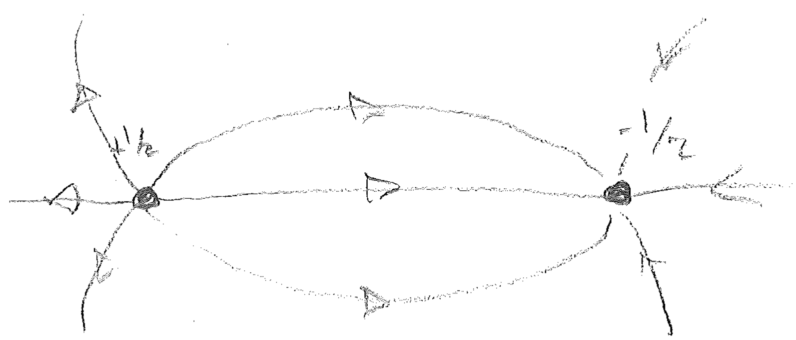


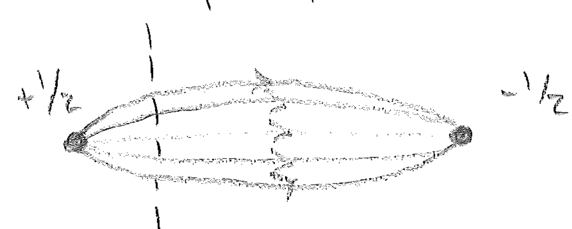
$\varphi \to \varphi + 2\pi$

$F_{\mu\nu} \sim \varepsilon_{\mu\nu\rho}\,\partial^{\rho}\varphi$

$E_j \sim F_{jt} \sim \partial_i \varphi$ $\Leftarrow$ Electric field $\Leftrightarrow$ Charge in $\varphi$

Consider two cases

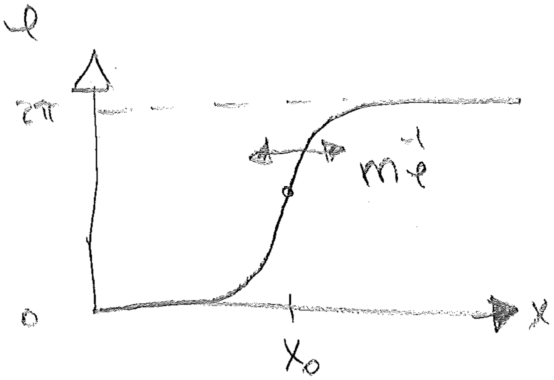



$$\mathcal{E}_{dual} = \underbrace{\frac{1}{2}\left(\frac{g^2}{4\pi}\right)(\partial_i\varphi)(\partial^i\varphi)}_{\text{KE term}} \underbrace{- \mu^3\cos(\varphi)}_{\text{PE term}}$$

KE term favors slow variation

PE term favors $\varphi = 0, 2\pi, \ldots$




$\varphi = 2\left[\arcsin\tanh(m_\varphi x) + \frac{\pi}{2}\right]$

- Can find $T = \frac{2g}{\pi}\mu^{3/2}$

$\Rightarrow$ Linear confinement!