Vacuum Topology

θ-Vacua

$\mathcal{L} = -\frac{1}{4}G^2 + \frac{\theta g_s^2}{32\pi^2} G\tilde{G}$, 2 Puzzles

1 Hamiltonian is θ independent

All physics comes from evolution of states $e^{-iHt}|x_i\rangle$

2 θ-term is a total derivative

$G\tilde{G} = \partial_\mu K^\mu$, $K^\mu = \varepsilon^{\mu\nu\rho\sigma} A^a_\nu \left(F^a_{\rho\sigma} - \frac{g}{3} f^{abc} A^b_\rho A^c_\sigma\right)$

$\hookrightarrow$ divergence thm

$S = \int d^4x\, \mathcal{L} \supset \int d^4x \frac{\theta g_s^2}{32\pi^2} G\tilde{G} = \int d^3x \frac{\theta g_s^2}{32\pi^2} K^r \Big|_{r\to\infty}$

If $K \to 0$ faster than $1/r^3$, this vanishes & θ has no physical effect

System w/ finite energy $\sim \int d^3x\, E^2 + B^2$, $E < 1/r^2$

$\partial K \sim K/r \sim G\tilde{G} \sim G^2 < 1/r^4 \to$ no physical effect

Puzzle 1: θ is in the initial state

Puzzle 2: instatons where $K \sim 1/r^3$, but surface integral doesn't vanish

~~Want~~ Boundary conditions for the theory: want finite energy so

$E, B \to 0$ as $r \to \infty \Rightarrow A \to U \partial_\mu U^\dagger \to$ i.e. all related by gauge trans.

are all gauge config. equivalent? No, can look at the homotopy group

$\pi_3(SU(3)) = \mathbb{Z} \to$ maps from S_3 ~~(G)~~ to S_3 that can be continuesly deformed into eachother, $\to$ characterized by an integer

Simple example $\pi_1(U(1))$ mapping circles to circles

$\phi_1 \to n\phi_2$ winding number

For $SU(3)$, $SU(2) \subset SU(3)$ symmetry group of S_3, this is just a higher dim analog

The winding number is $\int d^4x \frac{1}{32\pi^2} G\tilde{G} = \underset{r=\infty}{n_1} - \underset{r=0}{n_2}$

$\hookrightarrow$ Look at Srednicki for details

#11

Vacuum is eigenstate w/ lowest energy $\Rightarrow$ b/c states have same E & B fields they have the same energy. Do they mix?

Physical quantities come from $\int_{A_i}^{A_f} d[A]\, e^{iS}$

start w/ $n=0$, do we tunnel to $n=1$?

Instaton solution $A_\mu = \frac{r^2}{r^2+\rho^2}\, g\,\partial_\mu g^{-1}$, $g = \frac{x_4 1 + i\vec{x}\cdot\vec{\tau}}{r}$ (ρ constant)

Can connect any two n's by superimposing these solutions.

$\Rightarrow$ different n states do mix

truncating n at D

$$H = \begin{pmatrix} E & \varepsilon_1 & \varepsilon_2 & \cdots & \varepsilon_{D-1} \\ \varepsilon_{D-1} & E & \varepsilon_1 & \varepsilon_2 & \\ & & \ddots & E & \\ & & & \ddots & \\ \varepsilon_1 & \cdots & & \varepsilon_{D-1} & E \end{pmatrix}$$

eigenvectors are $\sum_n e^{i\frac{2\pi k}{D} n} |n\rangle$, $D \to \infty$, $|\theta\rangle = \sum_n e^{i\theta n} |n\rangle$

specify different sectors of the theory

$\langle\theta|\mathcal{O}|\theta\rangle = \sum_{m,n} e^{i\theta(m-n)} \langle m|\mathcal{O}|n\rangle = \sum_{n\Delta} e^{i\theta\Delta} \langle n+\Delta|\mathcal{O}|n\rangle$

$= \int dA\, e^{i\int d^4x\, \mathcal{L}_n + \frac{\theta}{32\pi^2} G\tilde{G}}$

Topology in cosmic structures real fields ϕ w/ $O(N)$ gauge field

$\mathcal{L}_n = \frac{1}{2}(D_\mu\phi)^2 - \frac{1}{8} g^2 (\phi^2 - \eta^2)^2 + \frac{1}{6}\, tr(B_{\mu\nu}B^{\mu\nu})$

$D_\mu\phi = \partial_\mu\phi - e B_\mu \phi$

$B_{\mu\nu} = \partial_\nu B_\mu - \partial_\mu B_\nu + e[B_\mu, B_\nu]$

$T \to 0$ $O(N)$ sym. breaks to $O(N-1)$, $\langle\phi\rangle^2 = \eta^2$ degenerate vacua $= S^{N-1}$

More generally sym group G, H subgroup that leaves $\langle\phi\rangle$ unaltered then $\langle\phi\rangle$ lies on $M = G/H$ re- unbroken symmetries

Finite T $\langle\phi\rangle$ comes from minimizing the free energy

at one-loop $V(\phi) = \frac{1}{8} g^2 (\phi^2 - \eta^2)^2 + \frac{1}{48}\left[(N+2) g^2 + 6(N-1) e^2\right] T^2 \phi^2$

$T_c = \eta\left(\frac{N+2}{12} + \frac{N-1}{2}\frac{e^2}{g^2}\right)^{-1/2}$ $\langle\phi\rangle^2 = \eta^2\left(1 - \left(\frac{T}{T_c}\right)^2\right)$

$M = O(N)/O(N-1) = S^{N-1}$

Start in hot universe & cool below $T_c \sim G_F^{-1/2} \sim 100$s GeV, time $\sim 10^{-11}$–10^{-10} s

$T \to T_c$ large fluctuations, diff regions can have diff $\langle \phi \rangle$

$(\nabla \phi)^2$ term → energy penalty for sharp trans.

What structure remains?

$T < T_c$ free energy diff between $\langle \phi \rangle = 0$ & eq. value $\Delta f = \frac{1}{8} g^2 (\langle \phi \rangle^2)^2$

corr. length $\xi^{-1} = g\, |\langle \phi \rangle|$

~~Prob~~ free energy associated w/ fluc to zero w/ scale ξ, $(2\xi)^3 \Delta f \simeq \frac{|\langle \phi \rangle|}{g}$

likely w/ $T \sim T_c$, as universe cools, becomes less likely

~~Consider~~ What are the domain structures?

divide space into cells w/ constant $\langle \phi \rangle$

look at 2 cells first

$\phi_1 \mid \phi_2$

sharp boundary energetically unfavorable b/c of $(\nabla \phi)^2$ term

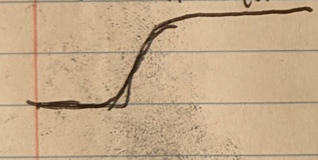
if M is connected & $\langle \phi_1 \rangle$ can be deformed to $\langle \phi_2 \rangle$ (smooth)

dis " no trans. possible, get a wall of normal phase

i.e. $\pi_0(M)$ is non-trivial

ex. $N=1$, only a reflection symmetry, $T=0$, $\langle \phi \rangle = \pm \eta$

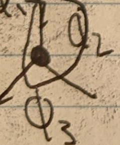
a domain wall forms between the regions due to balance between Δf & $(\nabla \phi)^2$



evolution determined by surface tension $\sigma \simeq \xi \Delta f \simeq g |\langle \phi \rangle|^3$

↳ will lead to area of wall shrinking w/ time, so one phase will dominate

3 cells. (assume M connected)

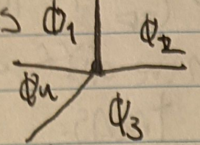


$\phi_1 \to \phi_2 \to \phi_3 \to \phi_1$ smooth path around the edge

If the path can be deformed to a point, then ϕ's can be deformed into each other & fuse. If not, then a region of normal phase is trapped along the edge, gauge field makes energy/length finite

like a flux tube in superconductors. i.e. strings form if $\pi_1(M)$ is nontrivial)

If $\pi_1(M)$ has multiple generators, there can be multiple types of strings & can have vertices where 3 strings join

4 cells Φ_1 

meet at a vertex

assume all closed circuits $\Phi_1 - \Phi_2 - \Phi_3 - \Phi_1$ is null homo.

If closed surface around the vertex can be shrunk to a point in M $\langle\Phi\rangle$ can be continuously extended, otherwise a normal region is trapped at the vertex (monopole), exists if $\pi_2(M)$ is nontrivial

size of elementary particle, not significant on cosmic scale

In general to trap a k-dim normal phase in the nontrivial phase $\pi_{2-k}(M)$ must be nontrivial

$\pi_0(M)$ & $\pi_1(M)$ give rise to possible cosmic structures