SF - YM Duality, S'16: 1) Motivation 2) Review Lattice YM

3) Introduce SF 4) Derive NAH dual a) Partition b) graphical calculus c) Final form

4) 2D SU(2) exact sol example?

1) Outside of string theory & AdS/CFT, ∃ other neat dualities that can be used to gain insight into physical theories;

Strong-weak dualities are especially useful if one side has an expansion in 1 regime (eg, strong coupling expansion in Lattice YM) that is perturbatively hopeless in the other ⇒ dual model may be able to shed light on the ~~[illegible]~~ regime inaccessible by the usual techniques.

Here, we will derive the SF-L~~[illegible]~~ duality, which is a perturbative - non-perturbative SW duality.

(we'll see SF are a generalizations of Wilson Loops, colorings of spin networks, which are) : Non PT; Traditional proof started w/ BF, gravitational constants, ! draw relation to YM uniqueness; ~~[illegible]~~ [illegible] many theories, like gravity; and finally, started for [illegible] ground / [illegible] backgrounds,

2) Restrict ourselves to a hypercubic lattice K in $\mathbb{R}^d$, of spacing a ~~[illegible]~~, with bnd ∂K

orient the edges, & label the edges by e; faces by f

Pick a compact Lie group of interest that you want to study, G; Our gauge connections are represented by fctns $g: E_K \to G$

$e \mapsto g_e$

~~[illegible]~~ As usual, we are interested in partition fctn observables:

Let $\Omega(\Phi) = \int \Big(\underbrace{\prod_{e\in K} dg_e}_{\text{Haar}}\Big) \exp(i S(g))\, \Phi(g)$

↳ if $\Phi = 1$, partition fctn ; (for convenience many formed, restrict to
if $\Phi = \phi$, $w = \langle\phi\rangle$ $\Phi(g) = \phi^*(\partial g)$ ↳ restrict conn. to boundary, bnd-bnd observables)

Let $S = \sum_{f\in K^\circ} S_f$, face centered real gauge invariant action on interior Lattice

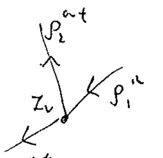
eg: $\underset{\text{for } SU(N)}{S_{\text{Wilson}}} = \frac{2N}{a^{4-d}\gamma^2}\Big[1 - \frac{1}{2N}\mathrm{tr}\big(U_f(g) + U_f^\dagger(g)\big)\Big]$ ↳ coupling ; tr over defining rep $\Big|$ $U_f(g) = g_4^{-1} g_3^{-1} g_2 g_1$ ↳ holonomy

In the appropriate scaling limit, recover continuum YM;

3) To transform to dual, we use a new basis for our functionals of connections

⇒ spin network basis! ~~span space of~~ $L^2(\mathcal{A}_K)$, gauge invariant functionals of connections
"moduli space of flat forms"

spin network = directed compact graph, valence ≥ 2, edges labeled by irreps ρ_e of group of interest,
vertices get intertwining ops I_v in tensor product $V_{\rho_1^{out}} \otimes \cdots \otimes V_{\rho_m^{out}} \otimes V^*_{\rho_1^{in}} \otimes \cdots \otimes V^*_{\rho_n^{in}}$



invariant tensor/equivariant map that commutes w/ group action,
SU(2), related to CG coeffs, "how to add spins such that their sum is 0"

[more args only if covers]

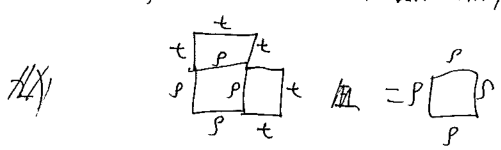
Every network S corresponds to a state functional, $\Psi_S(g) = \prod_{v \in K} I_v \cdot \prod_{e \in K} (\dim V_{\rho_e})^{1/2} \rho_e(g_e)$

2 S are = if they have same Ψ_S

contract all rep maps w/ intertwiners in the way indicated by the diagram.

(when S orientations opposite orientations, get $\rho(g_e^{-1})$ of K,)

3 properties: 1) we can let trivial irrep tensors just contribute factors of 1 ∴ prune from graph



2) (b) $I_2{}^{\cdots}{}_{\cdots a}\, \rho_e(g_e^{-1})^a{}_b\, I_1{}^{b \cdots}{}_{\cdots} = I_1{}^{b \cdots}{}_{\cdots}\, \rho_e^*(g_e)_b{}^a\, I_2{}^{\cdots}{}_{\cdots a}$, or $\xrightarrow{\rho} = \xleftarrow{\rho^*}$

2(a)

3) Can pick ONB of intertwiners for tensor product space, gives ONB of spin network states B_K

↳ these three result in spin networks being able to span $L^2_0(\mathcal{A}_K)$, space of gauge invariant functionals of ~~flat~~ connection! (Pf by Peter-Weyl thm on each edge of K)

↳ $L^2(G) = \widehat{\bigoplus}_{\rho \in \Sigma} V_\rho^{\oplus \dim V_\rho}$ → multiplicity equal to dim of underlying space of ρ

space of irrep ρ

Σ isomorphism classes of irreps of G

closure of direct sum

pts plus limit pts

" matrix coeff of irreps of G are dense in space C(G) of continuous complex-valued fctns on G ∴ thus also in $L^2(G)$

and, if suitably normalized, are an ONB since the space $L^2(G)$ decomposes into orthogonal direct sum of irreps

For an ultralocal LGT (H has no grad terms, diagonal in $\mathcal{G}$ space)

4) a) Dual transformation maps functional integral to discrete sum over 2-D surfaces w/ boundaries, called spin foams, weighted by an associated amplitude. Foams are graphs congruent to underlying lattice.

$$\Omega(\Psi) = \int \prod_{e \in K} dg_e \, \exp(iS(g))\, \Psi^*(\partial g)$$; leave aside bnd vars for now & focus on interior, integrating them out

$$\Omega(g) = \int_{\partial g' = g} \prod_{e \in K^\circ} dg'_e \, \underbrace{\exp(iS(g'))}$$, $\overbrace{\text{functional}}$, $\therefore \Omega(\Psi) = \int \prod_{e \in \partial K} dg_e \, \Omega(g)\, \Psi^*(g)$

leaves common bnd as free

~~$\int_{\partial g'=g} \left(\prod_{e\in K^\circ} dg'_e\right) \prod_{f \in K^\circ} \exp(iS_f(g'))$~~

~~what are C_{S_f}? motivates why character expansion!~~

~~$\sum_{S_f \in B_f} C_{S_f} \mathcal{I}_{S_f}$, expand in spin network basis B_f "sum over all networks whose states lie in B_f"~~

~~↳ loop basis, smallest graph of edges around a face~~

~~↳ w/o spt, $\mathcal{I}_{S_f}(g) = \mathrm{tr}_f [\rho(U_f(g))]$~~

~~well known eg $U(1)$: ... $e^{iS_f} = \sum_\rho C_{S_\rho} \dim V_\rho \chi_\rho(U_f)$~~

~~$\therefore$ conjecture w/ the usual character expansion, ∃ group traces, invariant maps~~

~~$C_{S_f} = \sum \dim V_\rho C_{f\rho}$~~

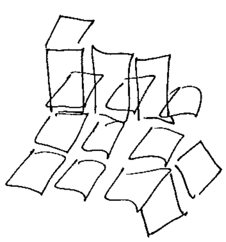
~~$\sum_\rho C_{f\rho} (\dim V_\rho) \chi_\rho(U_f)$, peter-weyl ... Assuming trivial ... $C_{S_f} = 1$,~~

$\underbrace{\sum_{S_f \in B_f} C_{S_f} \mathcal{I}_{S_f}}$

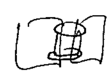
$$\Omega(g) = \sum_{\{f\} \to \{S_f\}} \underbrace{\int_{\partial g' = g} \left(\prod_{e \in K^\circ} dg'_e\right) \prod_{f \in K^\circ} C_{S_f} \mathcal{I}_{S_f}}$$, switching $\int$ & $\sum$ ⟹ Fubini needs $\int \sum |f| < \infty$ or $\sum \int |f| < \infty$

sum over all ways to assign basis spin networks to faces

Lets look at this summand, ; organize the non-trivial networks into surfaces F_i → two loops are members of the same surface if they share an edge with each other & none-other,



$\bigcup_{ij} F_i \cap F_j = \Gamma_F$, Boundary graph on F, $\bigcup F_i$, "Branched surface"

Each group integration on a lattice edge will be denoted graphically by cabling together the wires of the network  like so.

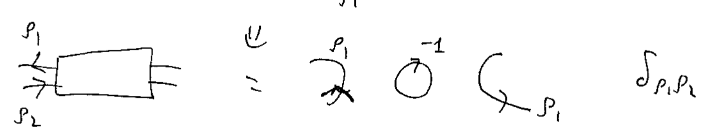
start marking categories! ⟹ this notation is actually not just notation of convenience, but has rich roots in the theory of modular functors! I won't dive into that here, but know that all the framework I've developed and am about to develop can be done in the absence of any physics at all → worth comment that this connects to knot invariants.

∃ a graphical calculus used to evaluate the representation theoretic objects we've defined here, called the Temperley Lieb algebra; It is a remarkable achievement of mathematics, and allows us to compactly evaluate these spin networks contractions and the group integrals.

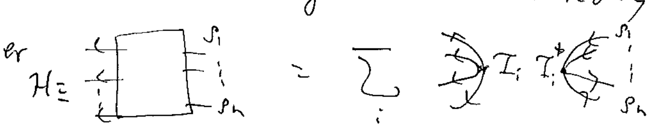
Let lines  $= \delta^{i}_{j}$,  $= \langle i|\rho(g)|j\rangle$,  $=$ ~~ρ~~ $\mathrm{tr}(\delta^\rho) = \dim V_\rho$; $\int dg$  $= \int dg\, \rho(g)$

Now formulas like Schur's lemma look like this: $\int dg\, \rho_1(g)^{a_1}{}_{b_1}\, \rho_2(g^{-1})^{b_2}{}_{a_2} = \frac{1}{\dim V_{\rho_1}} \delta^{a_1}{}_{a_2} \delta_{b_1}{}^{b_2} \delta_{\rho_1 \rho_2}$

↳ If 2 have 2 finite dim reps of a Group & ϕ is an equivariant map, then either ϕ is invertible or $\phi = 0$



More generally, Schur's lemma becomes a "Haar splitting": when e shared by more than 2 non-trivial loops, get Haar intertwiner



, a projector;

for orientably lattices, we can pick all face orientations & hence coherently coloring.

$\therefore$ every F_i, as a 2-D surface, must be of a single rep coloring (spin) (Schur), else would be 0.

~~Now have Oriented colored surfaces~~

get $\dim V_\rho$ for each vertex, $\dim^{-1}$ for each edge

~~each face contributes $c_{s_f} = \dim V_\rho c_{f\rho}$,~~ → if pick basis as trace of face holonomy, smallest spin network admissible loop elements

Return back to $\sum_f c_{s_f} Z_f$; Z_f for a basis loop is just  $= \mathrm{Tr}(U_f)$ $\therefore$ can decompose as character expansion of class fctn (Tr) just like LGT: $e^{(iS_f)} = \sum_\rho c_{f\rho} \dim V_\rho\, \chi_\rho(U_f)$

$\therefore$ ~~$\sum_f c$~~

$= \sum_f c_{s_f} Z_f = \sum_f c_{f\rho}\chi_\rho(U_f)$

$\therefore c_{s_\rho} = c_{f\rho} \dim V_\rho$

$\therefore$ each face contributes extra $\dim V_\rho$ $\therefore$ surface Amplitude:

$A_{F_i} = (\dim V_\rho)^{\chi(F_i)} \prod_{f \in F_i} c_{f\rho}$

(boundary edges are on S^1 & $\chi(S^1) = 0$ $\therefore$ no contribution of extra dims)

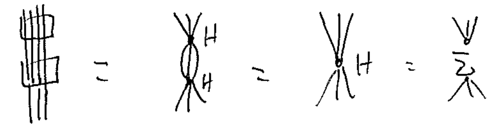
e.g. $U(1)$ $c_\rho = \frac{I_\rho(\beta)}{I_0(\beta)}$ for Wilson Action (β: inverse Temp)

$\sim e^{\left(-\frac{\rho^2}{2\beta}\right)}$

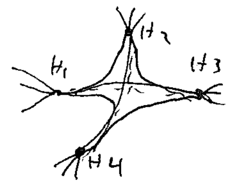
↝ inverse β, strong-weak

Now we have ~~Haar~~ surface Amplitude, we can get Amplitude of foam Γ_F

On any branching line, all Haar intertwiners are contracted by the color property of surfaces, and Haar is a projector ∴

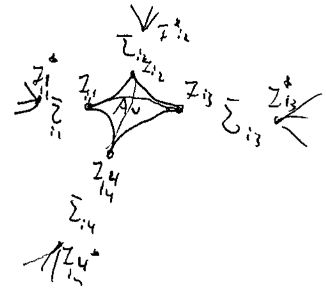


Now foam is left in picture like this:

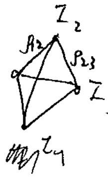


for each former vertex on a branching graph intersection

split! =



$A_v =$



$=$ 3nj symbol (for SU(2)); like higher dim Clebsch-Gordan coeff, known, computable.

At boundary, left with unintegrated spin network functional after Haar splitting pushes all the basis intertwiners outward. ~~Boundary Stuff~~ (Bnd-Bnd contributions are null, $z \bullet\!\!\ominus\!\!\bullet z^* = 1$ since intertwiners are normalized, $\mathcal{I}(\alpha\, \mathcal{I}\mathcal{I}^* = 1$

$$\therefore \Omega(g) = \sum_{F\subset K}\left(\prod_{v\in\Gamma_F} A_v\right)\left(\prod_i A_{F_i}\right) \mathcal{I}_{s_F}(g)$$

Lastly, contract integral w/ boundary functional $\Phi(\partial g)$: expression spin network basis, $\Phi^*(\partial g) = \sum_{s\in B(\partial K)} \Phi_s \mathcal{I}_s(g)$

inner product gives us only contributions when $s = s_F$ due to ONB

$$\therefore \Omega(\Phi) = \sum_{F\subset K}\left(\prod_{v\in\Gamma_F} A_v\right)\left(\prod_i A_{F_i}\right) \Phi^*_{s_F} \rightarrow \text{boundary functional weighting, non-pert, exact sol!}$$

sum over all foams congruent to lattice; measure piece from dimensionality & topology; action piece from decomposition into new basis

See gains! If ~~...~~ 2D, all networks are only ~~...~~ surfaces ∴ $A_v = 1$ & ~~...~~

~~...~~, If we topological theory like B-F theory where all action coeffs are also 1, get

$$Z = \sum_{F\subset K} \prod_i (\dim V_\rho)^{\chi(F_i)}, \text{ done!}$$

If $SU(N)$ in $d \geq 3$ dims; for convenience, use heat kernel action instead of Wilson, (same universality class, ok for low energy physics)

can look up, character expansion of this action has coeffs $c_{\rho} = \exp(-a^{d-4}\gamma^2(a)\, C_\rho)$

eval of Casimir in the ρ rep

$$\therefore \prod_{f \in F_i} c_{\rho_i} = \exp(-N_{F_i}\, a^{d-4}\gamma^2(a)\, C_{\rho_i})$$

↳ # faces on a surface ; $N_{F_i} = \dfrac{\tilde{A}_{F_i}}{a^2}$ ← area

$\Rightarrow$ for $U(1)$, irreps $\in \mathbb{Z}$, & $C_\rho = \rho^2$

for $SU(2)$, $C_\rho = 2\rho(\rho+1)$, $\rho \in \frac{\mathbb{N}_0}{2}$

write suggestive constant $T_\rho(a) \equiv a^{d-4}\gamma^2(a)\, C_{\rho_i}$

$$\Omega(\Phi) = \sum_{F \subset K} \left(\prod_{v \in F_0} A_v\right)\left(\prod_i (\dim V_{\rho_i})^{\chi(F_i)}\right) \exp\left(-T_{\rho_i}\tilde{A}_{F_i}\right) \Phi^d_{SF}\,,$$ exact sol!

Looks like ~~string quantization~~ lattice quantization of Nambu Goto string!

Transform from gauge to SF is non-pert, no restrictions on analysis; ~~...~~

Spin foam ≡ world sheet for spin network; exponent is Action for each colored sheet, proportional to area of sheet; T_{ρ_i} is color-dependent tension on edges!

Running of original gauge coupling $\gamma(a)$ mapped into running of Tension w/ inverse power :: S.W.

new dual DoF are rep labels; in many cases can be re-encoded as product of local constraints; ~~Expansion of original model in strong coupling~~

Conc: Many cool connections to math & physics; gives new insight into exact quantization

Background: models like this are used to quantize gravity.