Hadron Spectroscopy

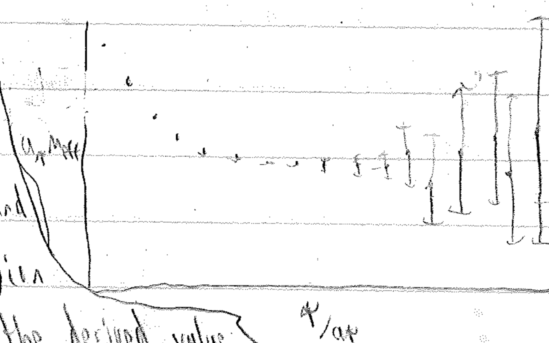
One of the simplest ways in which the lattice can be used is to measure part of the energy spectrum of a particular process.

To get ground state energies of states in QCD, we calculate a correlation function $C_{ij}(\tau) = \langle 0|\phi_i(\tau)\phi_j^\dagger(0)|0\rangle = \langle \phi_i(\tau)|\phi_j^\dagger(0)\rangle$, with ϕ_i and $\phi_j^\dagger$ operators that act on quark fields, producing a state at time $t=0$ and annihilating it at a later time t.

Inserting a complete basis of states, we get $C_{ij}(\tau) = \sum_n \langle \phi_i(0)|n\rangle\langle n|\phi_j^\dagger(0)\rangle e^{-E_n\tau}$, with a real exponential since we are in Euclidean time. For $i=j$, this is positive definite and converge monotonically from above.

At large Euclidean times τ, this goes to $C_{ij}(\tau\to\infty) = Z e^{-E_0\tau}$. Thus we can get an effective mass by taking $a_t m_{eff} = \log\left(\frac{C(\tau)}{C(\tau-1)}\right)$.

A typical plot of such a quantity might look like this. Early on, we see that the function has to settle into the ground state. Then, we have a "plateau", a region of convergence which we take to be the desired value.



For some calculations, we see that the signal-to-noise ratio plummets after a certain time has passed. This is a manifestation of the sign problem. The time at which your signal vanishes depends on the calculation, and in principle it may happen before your correlator has had time to converge, rendering the calculation useless. This is why people want to have control over the sign problem.

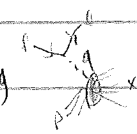
For excited and unstable states, this method is not going to make it easy to extract info.

One method people have come up with for accessing excited states is called the variational method. I won't get into much detail about this, but I just want to show that there is in principle a better method for getting excited state energies.

In this method, we take $C_{ij}(t)$ to be an element of some matrix of correlation functions $C(t)$, defined by a basis of operators ϕ_i.

Then, we solve a generalized eigenvalue problem $C(t)\vec{v} = \lambda(t,t_0)\, C(t_0)\vec{v}$, it can be shown that the eigenvalues are related to the state energies in the limit that $t-t_0 \to \infty$ by $\lim_{(t-t_0)\to\infty} \lambda_n = e^{-E_n t} + O(e^{-\Delta E_n t})$

This outlines a method for getting the energy spectra of states in QCD. Some groups have sort of reversed this in order to measure the quark mass by measuring a set of quantities for a variety of choices of quark masses and seeing which results have the best agreement with experiment. The results of these simulations generally show that the physical quark masses do indeed have the best connection to experiment.

(Hadron Structure; Deep inelastic scattering  $\Rightarrow \frac{d\sigma}{d\Omega\, dE'} = \frac{\alpha^2}{q^4}\frac{E'}{E} l_{\mu\nu} W^{\mu\nu}$

$$W^{\mu\nu} = \frac{1}{4\pi}\sum_X \langle P|J_\mu|X\rangle\langle X|J_\nu|P\rangle (2\pi)^4 \delta^4(P-q-P_X) = W_1\left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) + \frac{W_2}{M^2}\left(P^\mu - q^\mu \frac{P\cdot q}{q^2}\right)\left(P^\nu - q^\nu\frac{P\cdot q}{q^2}\right) + \text{spin dep.})$$

WIP

An Unexpected Break from χPT

One interesting result came from calculations of the pion mass dependence of the nucleon.

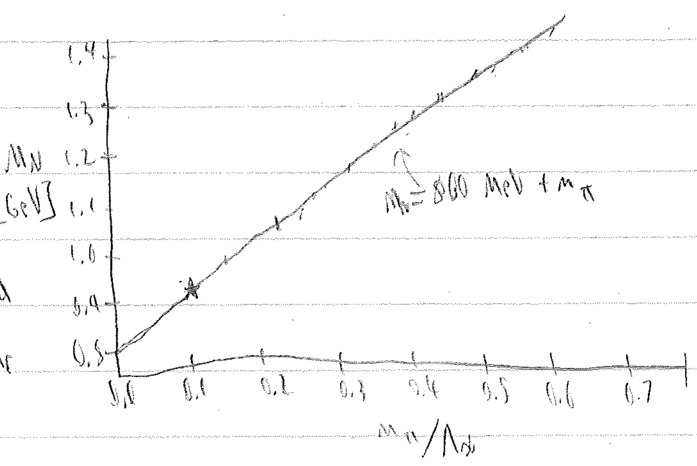
In chiral perturbation theory, the 2-flavor Lagrangian is given by (with $m_u = m_d = m_q$)
$\mathcal{L} = \bar{N}(\not{\partial} - M)N + \alpha_M v m_q \bar{N}(\Sigma + \Sigma^\dagger)N + v m_q \mathrm{Tr}[\Sigma + \Sigma^\dagger] + \cdots$, for $\Sigma = e^{i\frac{\pi_a \tau_a}{f}}$ $\quad (f \approx 93 \text{ MeV})$
This form is constrained by chiral symmetry and how it is broken by the condensate.

An expansion in the pion fields gives $\Sigma + \Sigma^\dagger \approx 2 - \frac{\vec{\pi}^2}{f^2}$. From the last term, we get a constant term which does not affect physics and a quadratic term in $\vec{\pi}^2$ given by $-\frac{2 v m_q}{f^2}\vec{\pi}^2$

This looks like a mass term for the pion $-\frac{1}{2}m_\pi^2 \vec{\pi}^2$, with $m_\pi^2 = \frac{4v}{f^2} m_q \Rightarrow v m_q = \frac{f^2 m_\pi^2}{4}$

The nucleon mass to order m_π^2 can be obtained from the other two terms to get $M_N = M - \frac{f^2 \alpha_M}{2} m_\pi^2 + \mathcal{O}(m_\pi^3)$, the remaining pieces coming from loops.

Lattice QCD calculations have been done for many values of the pion mass above the physical point, and they all have been found to match strikingly well to a linear fit of $M_N = 800 \text{ MeV} + m_\pi$.



At first glance this seems incompatible with chiral perturbation theory, but we must recall that the χPT is an expansion about $m_\pi \sim 0$. It is known to be valid for m_π ranging from 0 to the physical value, but it is not known any further.

These results would indicate that χPT breaks down just beyond the physical point, replaced with a new behavior.

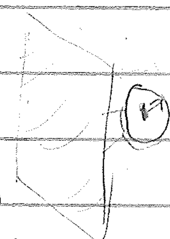
Note that this new behavior is above the threshhold of χPT but below that of perturbative QCD. This is also inaccessible to experiment as m_π is fixed in nature. This means that this feature of QCD is currently <u>Only</u> accessible to lattice methods and nothing else, illustrating that lattice QCD is not only a tool that can replicate experiments but also provide new insight into nonperturbative theories.

Lüscher's Formula in QM

One aspect of latticizing a theory that hasn't really been touched on a whole lot is the effects of putting the theory in a finite box of length L.

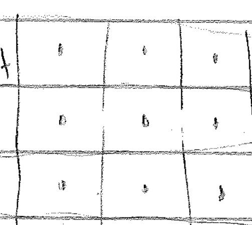
Use periodic boundary conditions so that all quantities follow $\phi(\vec{x}) = \phi(\vec{x}+\vec{n}L)$.

For low energy processes (infrared), this can be more important than the lattice spacing
- In momentum space, the finite box size has the effect of discretizing the momenta so that $\vec{p} = \frac{2\pi\vec{n}}{L}$ for some integer vector $\vec{n}$.



Let's consider QM scattering in such a box to illustrate. Scattering off of a central potential is phrased around having an incoming plane wave e^{ikrz} hitting a central, limited range, spherically symmetric potential and emanating spherically outward to infinity.

Outside the range of the potential, the wavefunction is free again, and the wavefunction is known to behave like $\psi_0(r) = N\frac{e^{-ikr}+e^{2i\delta_0}e^{ikr}}{r} = N(\cos\delta_0\, j_0(kr) + \sin\delta_0\, n_0(kr))$ for s-wave scattering, where δ_0 is the s-wave phase shift determined by the form of the potential.



In a finite volume with periodic boundary conditions, this doesn't work so well. One way to imagine this box is to think of it as infinite space but with an infinite grid of scattering potentials (hall of mirrors). There is no notion of taking the plane wave to asymptotically far away from the center, since you will always be within a certain distance from the potential.

Lüscher devised a way of solving for this wavefunction, assuming that the potential terminates before hitting the box. If this is the case, then there is a spherical shell where the particle is free, and in here the wavefunction is $\psi(r) = N(\cos\delta\, j_0(kr) + \sin\delta\, n_0(kr))$.

$\nearrow k^2 = ME$, $E =$ center mass 2-part. energy

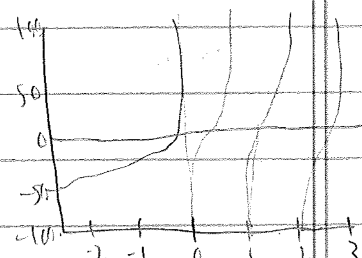
This solution accounts for the physics of the central potential, but does not include the extra boundary condition from the box, that $\Psi(\vec{x}) = \sum_{l,m} \psi_{lm}(r) Y_{lm}(\theta,\phi) = \Psi(\vec{x} + \hat{n}L)$.

Lüscher noted that if $\sin\delta_0 \neq 0$, then this partial wavefunction $\psi_0(r)$ has a pole at the origin. In other words, we have $(\nabla^2 + k^2)\, n_0(r) = \alpha\, \delta(r)$.

This resembles a Green's function equation $(\nabla^2 + k^2)\, G(\vec{x}-\vec{y}) = \delta^3(\vec{x}-\vec{y})$. The Green's function for the free Schrödinger equation is given by $G(\vec{x}) = \frac{1}{L^3} \sum_{\vec{n}} \frac{e^{i\left(\frac{2\pi \vec{n}}{L}\right)\cdot \vec{x}}}{k^2 - \left(\frac{2\pi \vec{n}}{L}\right)^2}$. Note that in the continuum, the sum $\frac{1}{L^3}\sum_{\vec{n}}$ would be replaced by an integral $\int \frac{d^3 p}{(2\pi)^3}$, but the box discretizes the momentum.

This Green's function obeys the periodic boundary conditions, but does not know about the central potential. We also know that a function $\psi_0(r)$ satisfying the Schrödinger equation and both boundary conditions must be a unique solution. Thus if we try to match the coefficients of the poles together, then the solution in terms of the spherical Bessel functions must be the same as those of the Green's function. Matching the coefficients from $\sin\delta_1$ and G gives us Lüscher's formula

$$k \cot \delta_0(k) = \frac{1}{\pi L}\left(\sum_{\vec{n}}^{\Lambda} \frac{1}{\vec{n}^2 - \left(\frac{Lk}{2\pi}\right)^2} - 4\pi\Lambda\right)$$



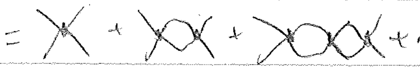
On the left hand side, we have the <u>continuum</u> phase shift.
On the right hand side, we have an artifact of the periodic boundary conditions from the box.
This tells us that the only allowed energies are the CM momenta that satisfy this equation.

Extracting the spectrum from a lattice simulation and putting it into the RHS will give us the LHS – the scattering phase shift.

Lüscher's formula in QFT.

The question remains if we can generalize this to field theories.
- Lüscher showed that you can, up to exponential corrections for large L.

In fact, we can replicate Lüscher's formula using a pionless EFT for nucleons
$\mathcal{L} = N^\dagger\left(i\partial_0 + \frac{\nabla^2}{2M}\right)N - C_0(N^\dagger N)^2$.

$A =$  $+\cdots = -iC_0 - iC_0 I_0 C_0 - iC_0 I_0 C_0 I_0 C_0 + \cdots = \frac{-iC_0}{1 - I_0 C_0} = \frac{i}{-\frac{1}{C_0} + I_0}$

Continuum: $I_0^{\overline{MS}} = -\frac{iMp}{4\pi}$, $C_0 = \frac{4\pi a}{M} \Rightarrow A = \frac{1}{-\frac{M}{4\pi a} - ip\frac{M}{4\pi}} = \frac{4\pi}{M}\frac{1}{-\frac{1}{a} - ip} = \frac{4\pi}{M}\frac{1}{p\cot\delta_0 - ip}$ for $p\cot\delta_0 \approx -\frac{1}{a}$

Finite Volume: $I_0^{\overline{MS}} = \frac{1}{L^3}\sum_{\vec{n}} \frac{M}{k^2 - \left(\frac{2\pi\vec{n}}{L}\right)^2} = \frac{-M}{4\pi^2 L}\left(\sum_{\vec{n}}^{\Lambda} \frac{1}{\vec{n}^2 - \left(\frac{Lk}{2\pi}\right)^2} - 4\pi\Lambda\right)$, $C_0 = \frac{4\pi a}{M}$, $A = \frac{4\pi}{M}\frac{1}{-\frac{1}{a} - \frac{1}{\pi L}\left(\sum_{\vec{n}}^{\Lambda}\frac{1}{\vec{n}^2 - \left(\frac{Lk}{2\pi}\right)^2} - 4\pi\Lambda\right)}$

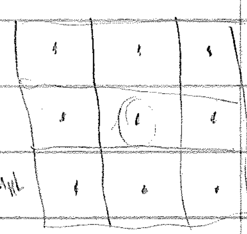
We can introduce more contact operators such as $C_2(N^\dagger(\overleftarrow{\nabla} - \overrightarrow{\nabla})^2 N)(N^\dagger N)$ to the Lagrangian. Doing this calculation again with them gives the same result but with $C_0 \to \sum_{j=0}^{\infty} C_{2j}\,p^{2j} = \frac{4\pi}{M}\frac{1}{p\cot\delta}$.
$\rightarrow A = \frac{4\pi}{M}\frac{1}{p\cot\delta - \frac{1}{\pi L}S\left(\left(\frac{Lk}{2\pi}\right)^2\right)}$

The poles of the scattering amplitude give you bound state energies, which implies that the energy levels are located at k s.t. $k\cot\delta_0(k) = \frac{1}{\pi L}S\left(\left(\frac{Lk}{2\pi}\right)^2\right)$, which is Lüscher's formula.

The benefit of recasting this as a field theory is that we can more easily incorporate pions into the theory, which will increase the range of its predicting power past the pion production threshhold.

The finite volume corrections introduced by this will be exponentially suppressed, justified by the following argument: Pion effects will arise due to a propagator of the form $\frac{1}{\vec{q}^2 + m_\pi^2}$. There is no q_0^2 term since in our nonrelativistic power counting $q_0 \sim q^2$, so its effects are lower order. The Fourier transform for this into position space is $\frac{e^{-m_\pi r}}{r}$.

If we go back to this picture of the central potential in a box, we see that the copies of the potential are separated from the NN system by a length of order L, so the potential is of order $e^{-M_\pi L}$



Thus as long as $L \gg \frac{1}{M_\pi}$, Lüscher's formula can still be used.

It has been shown that the perturbative expansion for this EFT converges slowly at NNLO for the $^1S_0$ channel and does not converge at NNLO in the $^3S_1$ channel, which is where the deuteron lives. A lattice calculation using this method could be performed using Lüscher's formula in the regime where perturbation theory struggles, so long as you remain nonrelativistic (below M_ρ).

In conclusion, we've seen how through the calculation of matrix elements on a lattice we can find a wide variety of quantities in QCD, and even some features of the theory that would not otherwise be visible to us. Note that we have restricted ourselves to QCD; there are efforts that use the lattice outside of this to calculate weak decays of quarks and even probe BSM physics such as SUSY theories.