String-Net Models

- Review & Leaps of Faith
- General String-Net Picture
- Examples

Thus far this quarter we have been discussing topologically protected phases which are presumed to be some low-energy description of certain system in which only topological DoF are relevant.

- Andrew: showed us the toric code / or a square lattice, which had an equivalent description in terms of closed strings (aka $\mathbb{Z}_2$ gauge theory)
- Tyler: showed us a continuum TQFT and the double-semion / $U(1)_2 \times \overline{U(1)}_2$ model, which also had a closed string description & was closely related to the toric code
- Sam: showed us a method to generalize CS theories and deduce properties of certain TQFT's from the K-matrix.

This talk will be in a similar vein as Sam's but it'll be even more general! * And reversed: I will try to tie these more general picture back to the $\mathbb{Z}_2$ GT & $U(1)_2 \times \overline{U(1)}_2$ examples to keep us grounded

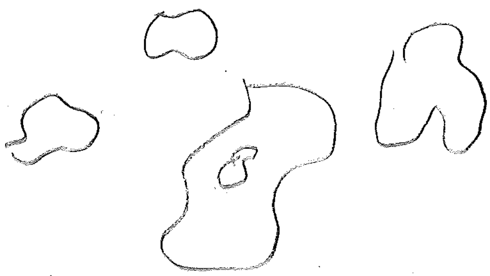
Today: Only working on $g=1$ surfaces

Review & Leaps of Faith: Reformulate as theory of flux lines

$H = -\sum_v A_v - \sum_p B_p$

- $\mathbb{Z}_2$ gauge theory:

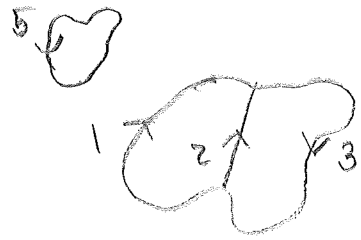
  ground state consisted of configs where:
  - Only closed loops of "electric flux"
  - One type of flux
  - Gauss's law-like const: no branching



- U(1) gauge theory: theory of flux lines
  - More than one type of flux, links $\in \mathbb{Z}$ — reps are labelled by ints
  - Need to specify orientation.
  - Gauss' law: $E_1 + E_2 + E_3 = 0$

    for three flux lines $E_{1,2,3}$

    So branching allowed "string-network"



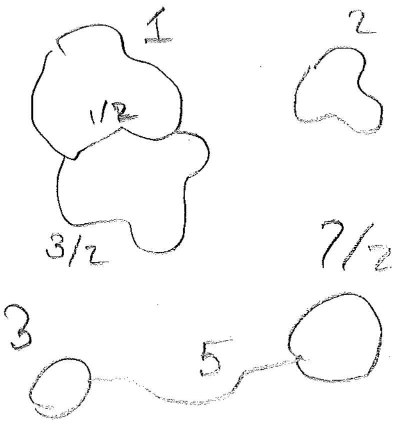
- General gauge theory, G
  - Flux lines fall in representations
  - Gauss' law: $E_1 \otimes E_2 \otimes E_3 \supset \underline{1}$ — trivial rep
- SU(2): reps in $E = \frac{1}{2}, 1, \frac{3}{2}, \ldots$

  $\{E_1, E_2, E_3\}$ can meet at point if: $E_1 \leq E_2 + E_3$

  $E_2 \leq E_1 + E_3$

  $E_3 \leq E_1 + E_2$

  & $E_1 + E_2 + E_3 \in \mathbb{Z}$



General String-Net Picture

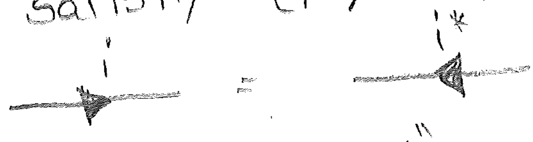
- Generalize string nets beyond the scope of gauge theories.
- Not the <u>most</u> general case, but focus on this case for simplicity.
  - Most general has "spin" DoF at each node, the dimension of which depends on incoming strings. We focus on cases with $\delta_{ijk} = 0, 1$ (boo…)
- Focus on trivalent networks. – <u>Claim</u>: All "doubled" top. Phas… can be described by this pict… Parity & time-rev invariant.
- Three pieces of data to define the network (i.e. pre-Hilbert space):

I nclude… $CS \times CS$ … discrete gauge t…

1. <u>String types</u>, label by $i = 1, \ldots, N$. ($N = 1$ in Z_2 case, $N = \infty$ in $U(1)$ & $SU(2)$)

2. <u>Branching rules</u>, $\{\{i, j, k\}, \ldots\}$

3. <u>String orientations</u>, dual of string type i^*, must satisfy $(i^*)^* = i$. Z_2 & $SU(2)$: $i = i^*$ (no orientations)

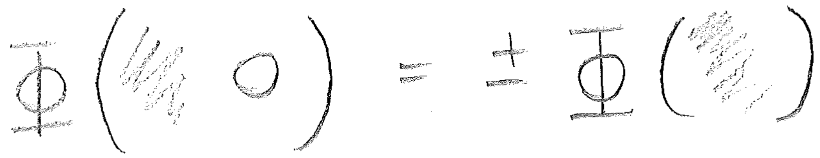


- This is the "pre-Hilbert" space, we need to pro… additional information to specify the ground state.
  - This could come in the form of a Hamilt… or local constraints. Seen <u>one</u> example o… this already in Z_2 & $U(1)_2 \times U(1)_2$ models. Same <u>pre-Hilbert</u> but diff. g… states, etc

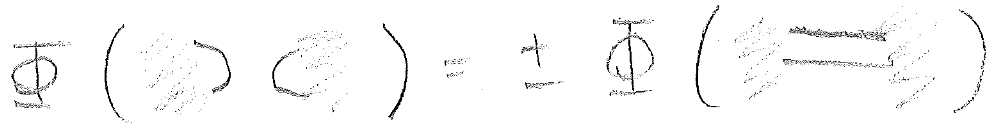
- Instead of starting from Hamiltonians, let us start with constraints.

Again, we've already seen this: ($= \sum_X \Phi(X)|X\rangle$)

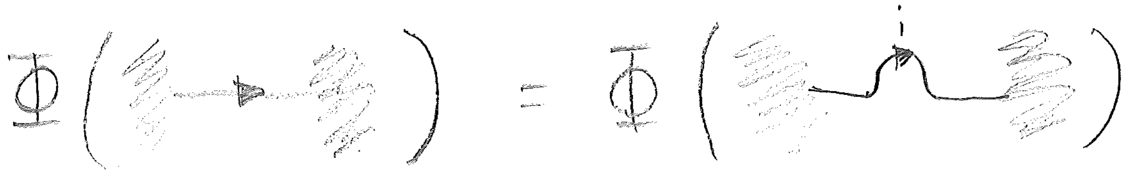
For $\Phi(X) = \langle X | \Phi \rangle$ with $|\Phi\rangle$ my grd state and X some arb. string net, we saw



$\pm$ corresp to $\mathbb{Z}_2$ & $u(1)_2 \times \overline{u(1)_2}$ resp.

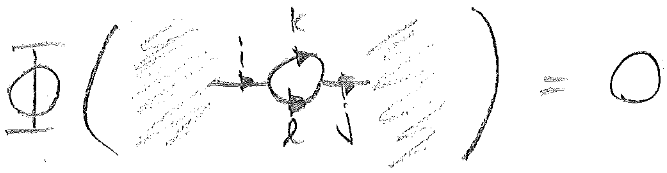


- For a general string-net, we write down constraints motivated by certain properties we'd like the wfn to obey, namely properties of a topological phase

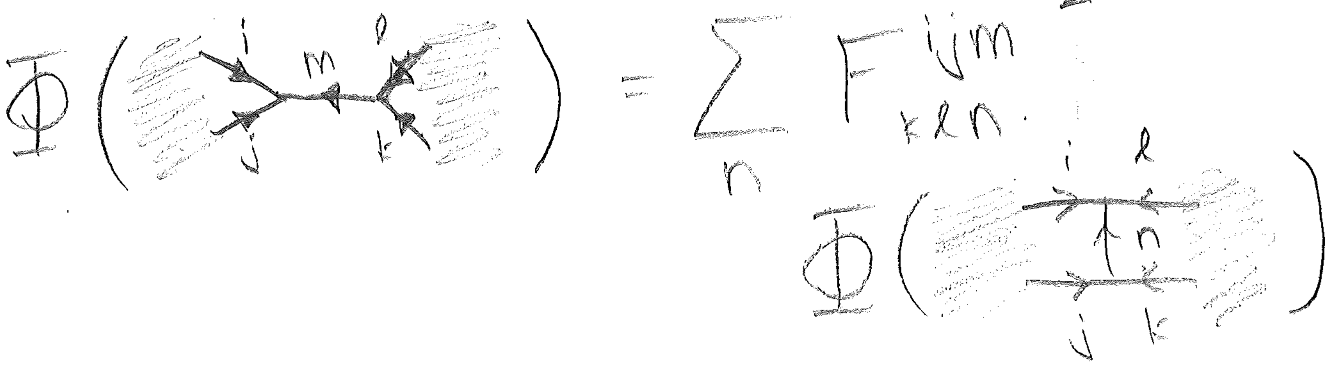
1. 

2. 

$d_i \in \mathbb{C}$

3. 

if $i \neq j \ \forall \, l, k$

4. 

$F^{ijm}_{kln} \in \mathbb{C}$

New notation:

- If any branching is not allowed, $F^{ijm}_{\ell k n} = 0$
- $i=0$ string is "empty string", so now $i = 0, \dots, N$. Hence $\{i, j, 0\}$ is an allowed branching iff $i = j^*$.

Motivation for rules:

1 ⇒ Topological invariance, two configs should have same amp if they can be topologically deformed from one to another

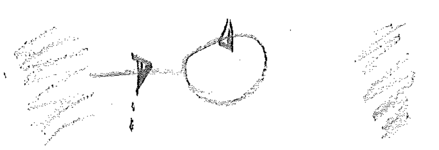
2 ⇒ Scale invariance, Φ should look the same at all distances. Closed string disappears at some scale, so amp without it should be proportional

3. ⇒ Also scale invariance,

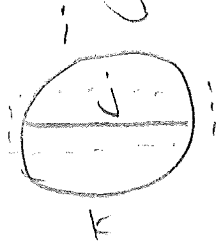
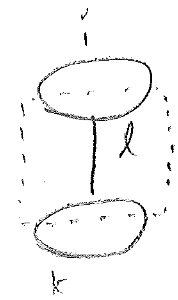
if $i \neq j$, the latter is not allowed

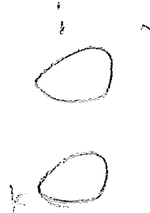
4. ⇒ Needed for completeness, cannot specify gnd state uniquely without it. Also motivated by fusion algebra in CFT.

- These rules uniquely specify the ground state Φ; Equivalently, the grd state is expressible via $(F^{ijm}_{\ell k n}, d_i)$.
- Example of how to use these rules:
  - Rule 3: $\Rightarrow$ No tadpoles

$$\Phi\Big($$  $$\Big) = 0 \text{ unless } i=0$$

  - Find a given amplitude:

$$\Phi\Big($$  $$\Big) = \sum_{\ell} F^{ikj}_{ki\ell}\, \Phi\Big($$  $$\Big)$$

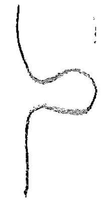
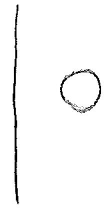
$$= F^{ikj}_{ki0}\, \Phi\Big($$  $$\Big) = F^{ikj}_{ki0}\, d_i d_k$$

where I have defined $\Phi(\emptyset) = 1$.

This can be done with all amplitudes!

$\Rightarrow$ Uniquely specify the wavefunction

- However, not every choice of $(F^{ijm}_{k\ell n}, d_i)$ corresponds to a string-net phase. We must demand self-consistency. Ex:

 $\Rightarrow$  $\Rightarrow -1 \times$  $\Rightarrow$ 

Inconsistent if two rules had diff. signs!

- Can show: with $v_i = v_{i^*} = \sqrt{d_i}$, $v_0 = 1$

$$F^{ijk}_{j^*i^*0} = \frac{v_k}{v_i v_j}\delta_{ijk}$$

$$F^{ijm}_{kln} = F^{lkm^*}_{jin} = F^{jim}_{lkn^*} = F^{imj}_{k^*nl}\frac{v_m v_n}{v_j v_l}$$

$$\sum_{n=0}^{N} F^{mlq}_{kp^*n} F^{jip}_{mns^*} F^{js^*n}_{lkr^*} = F^{jip}_{q^*kr^*} F^{riq^*}_{mls^*}$$

$$\delta_{ijk} = \begin{cases} 1 & \text{if } \{i,j,k\} \text{ is allowed} \\ 0 & \text{otherwise} \end{cases}$$

Claim:

(2+1)D string-net phases (T & P inv.) $\Longleftrightarrow$ $\underbrace{\text{Solutions of above}}_{\text{"tensor categories"}}$ (with allowed stri

- Each group G provides a solution: i runs over irreps, d_i is dimension of irreps, F^{ijm}_{kln} is "6j symbol" of the group
- Doubled CS theories also solutions
- This list is not exhaustive

Summary: Three pieces of data $\Rightarrow$ Look for solutions $\Rightarrow$ (2+1)D string-net phase (s
string type, branches, conj (not nec. uniqe).

How can we use this?

- Construct (exactly solvable) Hamiltonian whose gnd state is string-net:

$$H = -\underbrace{\sum_{I \text{ vert}} Q_I}_{\text{constraint on allowed vert.}} - \sum_P B_P$$

B_P ← Fn of $(F^{ijm}_{k\ell n}, d_i)$

- Allow us to build intuition
- Derive univ. properties

Hermitian if $F^{i^* j^* m^*}_{k^* \ell^* n^*} = (F^{ijm}_{k\ell n})^*$ ← Believed that only these are physically realizable

- Provides framework for deriving physical properties of quasiparticles, "twists" & S-matrices
- Generalizable to higher dimensions

Z_2 & $U(1)_2 \times \overline{U(1)}_2$:

1. $N = 1$
2. No branching: $\emptyset$
3. $1^* = 1$

$\Rightarrow$

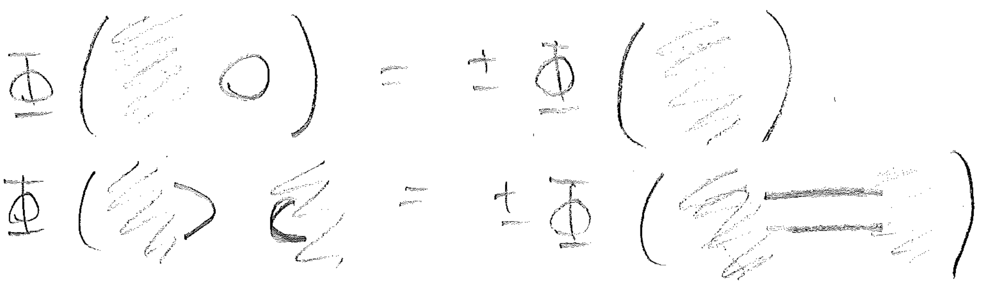
$d_0 = +1$

$d_1 = F^{110}_{110} = \pm 1$

$F^{000}_{000} = F^{101}_{101} = F^{011}_{011} = +1$

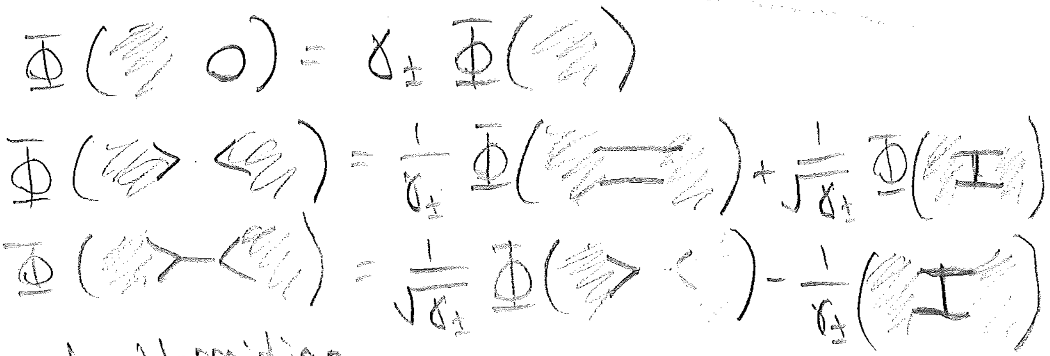
$F^{000}_{111} = F^{110}_{001} = F^{101}_{010} = F^{011}_{100} = 1$

Rules



---

1. $N = 1$
2. $\{\{1,1,1\}\}$
3. $1^* = 1$



$\gamma_\pm = \frac{1 \pm \sqrt{5}}{2}$

γ_- not Hermitian

γ_+: $SO(3)_3 \times \overline{SO(3)}_3$ CS theory "Yang-Lee"

Non-Abelian anyons

→ univ. quantum comp.

1. $N=2$
2. $\{\{1,1,1\},\{2,2,2\}\}$
3. $1^* = 2,\ 2^* = 1$

Two solutions
- $\mathbb{Z}_3$ gauge theory
- $U(1)\times U(1)$ with $k = \begin{pmatrix} 3 & 0 \\ 0 & -3 \end{pmatrix}$

9 quasiparticles

---

1. $N=2$
2. $\{\{1,2,2\},\{2,2,2\}\}$
3. $1^* = 1,\ 2^* = 2$

S^3 gauge theory

8 quasiparticles (reps of S^3)