Hagedorn / Deconfinement phase transition in Large N Weakly coupled gauge Theories.
PTJC Lecture Notes.
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We would like to understand deconfinement in yang-mills gauge theories. These are believed to be confining for small N, however it is still unproven in general.

This however is not possible to do ~~(at least at~~ ~~locally~~ analytically; however if we take the large N limit we arrive at a theory which is analytically tractable, and is believed to share many common features with finite N gauge theories.

Furthermore in this talk we will be specifically focusing on Large N gauge theories which are weakly coupled on a compact manifold. ~~Wll~~ We will rotate to euclidean time which gives us the manifold of $M \times U(1)$ where M is a $d-1$ dim manifold and the $U(1)$ in the time direction.

— Deconfinement phase transition. ~~its~~ believed that at low temperatures $SU(N)$ gauge theories are confining and that the ~~fp~~ particles of the theory at 0T are interacting glueballs.

At large T there is a different story. We can ~~have got a go~~ explore the high temperature limit via perturbative calculations ~~since the~~ due to asymptotic freedom.

In the high temperature state, the theory consists of a weakly coupled gas of gluons, which can be thought of as a different phase.

To make this idea of a phase transition more clear, lets consider an order parameter which can clearly show the nature of the phase transition. At zero T we have the free energy of one quark is $F_q \to e^{-\beta F_q} = \langle P \rangle$ where P is the polyakov loop. $P = \frac{1}{N} \mathrm{Tr}\, P e^{-\oint A}$. At zero temperature $\langle P \rangle = 0$, and at large temperatures we have that $\langle P \rangle \neq 0$, so this can be used as an order parameter.

There is also another order parameter we can use which is the free energy. At low T, we have $F(T)$ scales as N^0 at large N. However at large T $F(T)$ goes as N^2 as the gluons (In the adjoint, # gluon goes as N^2) increase with N. So the quantity $\lim_{N\to\infty} \frac{F(T)}{N^2}$ also can serve as an order parameter.

A Note on $N=4$ SYM. If you stopped by NTC last semester you will be slightly familiar with ~~the~~ $N=4$ SYM. This theory has conformal symmetry. Like the previous lectures we will take the limit as $g_{YM} \to 0$, $N \to \infty$ with $\lambda = g_{YM}^2 N$ held fixed. The conformal symmetry is then given by $F = -\frac{\pi^2}{6} N^2 T^4 f(\lambda)$

at small λ perturbative calculations give $F(\lambda) = 1 - \frac{3\lambda}{2\pi^2}$, while at large λ the ADS/CFT correspondance can be used to show $F(\lambda) = \frac{3}{4} + \frac{45}{32}\frac{\zeta(3)}{(2\lambda)^{3/2}}$

— Effects of compact space.

The coupling constant can no longer be run to arbitrarily small energies, instead, the scale $\frac{1}{R}$ sets a new scale of the problem and $\lambda(\mu = \frac{1}{R})$ ~~sets a new scale~~ is a new dimensionless coupling in the problem.

If we have $R \gg \frac{1}{\Lambda_{QCD}}$, ~~then~~ we expect the theory to resemble the theory in flat space, not perfectly though, ~~due to~~ There cannot be a phase transition for a system with a finite # of DOF, so there cannot be a phase transition for finite N, However as $N \to \infty$, the smooth transition at finite N grows more jagged until it becomes less and less ~~smooth~~. Then the deconfinement transition mimics its flat space counterpart more closely (which has a phase transition for finite N).

The polyakov loop which was useful before is no longer a good order parameter. If we apply gauss law to a compact manifold, we find that since there is no surface, then $\int (D_\mu F^{\mu\nu})^a = 0$, which means $\int (J^\nu)^a = 0$. So by construction of the manifold we cannot have a free quark.

However, the free energy will be seen to have the same behavior as it does in a non-compact manifold, notably it changes from scaling as N^0 to scaling as N^2.

In this talk I will focus on the case with $R\Lambda_{QCD} << 1$. The theory is weakly coupled in this regime as the coupling (stronger at large momenta) stops ~~running~~ running at the scale $\frac{1}{R} >> \Lambda_{QCD}$, so the weak, high energy coupling is frozen in at the scale. At this scale the low temperature phase has hagedorn growth in the density of states.

— Partition function

$$Z(\beta) = \sum_{\text{states}} e^{-\beta E} = \int \rho(E) x^E \, dE.$$

Exact partition function for free Yang-Mills theory

The free particles will behave as modes in the space for example spherical harmonics. Each of these modes will be in some representation of the gauge group. Let the energy of one of these modes be E_i, then the partition function is given by

$$Z(\beta) = \sum_{n_1 \ldots n_k} x^{n_1 E_1} x^{n_2 E_2} \ldots \left\{ \#\text{ of singlets in } \cancel{\text{sym}^{n_i} R_i} \prod_i \begin{Bmatrix} \text{sym}^{n_i}(R_i) & \text{bosonic} \\ \text{anti}_i^{n_i}(R_i) & \text{fermionic} \end{Bmatrix} \right\}$$

where we take the symmetric wavefunction for bosonic modes and the antisymmetric wavefunction for fermionic modes. We only take the singlets since they are the only physical states that can exist.

This symmetry factor is where our matrix models come in. Recall for finite groups ~~have~~ the character is defined as a map $\chi_{R_i}: G \to \mathbb{C}$ where for a representation R, and group element $U \in R$, $\chi(U \in R) = Tr(U)$, which then can be shown to satisfy $\chi_{R_1 \oplus R_2} = \chi_{R_1} + \chi_{R_2}$; $\chi_{R_1 \otimes R_2} = \chi_{R_1} \cdot \chi_{R_2}$. furthermore

$\sum_{g \in G} \chi^*_{R_1^I}(U(g)) \chi_{R_2^I}(U(g)) = \delta_{R_1^I R_2^I}$. For lie groups this

becomes $\int [dU] \chi^*_{R_1^I}(U) \chi^*_{R_2^I}(U) = \delta_{R_1^I R_2^I}$, where $\int [dU]$ is the haar measure normalized such that $\int [dU] = 1$

We can then use this to find the number of singlet states. For example consider the tensor product of representations $R_1 \ldots R_k$. $R_k^T = R_1 \otimes R_2 \otimes \ldots \otimes R_k$. This will decompose into

$$R_k^T = \left(R_1^I \oplus R_1^I \oplus \ldots \oplus R_1^I\right) \oplus \left(R_2^I \oplus \ldots \oplus R_2^I\right) \oplus \ldots$$

to figure out how many copies of irreducible representation ℓ there are we can note

$$\chi_{R_k^T}(U) = n_1 \chi_{R_1^I} + \ldots + n_k \chi_{R_k^I},$$

however $\chi_{R_k^T}(U) = \chi_{R_1} \cdot \chi_{R_2} \cdots \chi_{R_k}$, so

$$\int [dU]\, \chi^*_{R_\ell^I}(U) \left(n_1 \chi_{R_1^I} + \ldots + n_k \chi_{R_k^I}(U)\right) = \int [dU]\, \chi^*_{R_\ell^I}(U) \prod_{i=1}^{k} \chi_{R_i}(U)$$

$$n_{I^\ell} = \int [dU]\, \chi^*_{R_\ell^I}(U) \prod_{i=1}^{k} \chi_{R_i}(U)$$

The character of the singlet state is easy to find, its just one, so the number of singlet representations in the tensor product is

$$n_s = \int [dU] \prod_{i=1}^{k} \chi_{R_i}(U).$$

For the applications of this talk we require the number of singlet representations of the tensor product $(U_R)^{a_1}_{[a_1} \cdots (U_R)^{a_k}_{a_k]}$ where $[\ ]$ indicates symmetrization or antisymmetrization

We can evaluate this exactly using

$G_\pm(U,t) = \frac{1}{\pi^{\dim(R)}}\int[d\varphi_\pm]\, e^{-\bar\varphi_\pm\varphi_\pm + t\bar\varphi_\pm U_R \varphi_\pm}$, where $\varphi_\pm$ are complex bosonic or fermionic variables respectively. If we expand this ~~and take $t=1$~~, the n^{th} term has the coefficient we are interested in

$$G_\pm = \det(1 \mp tU_R)^{\mp 1}$$

$$G_+(U,t) = \sum_{n=0}^{\infty} t^n \chi_{sym^n(R)}(U) = \exp \sum_{l=1}^{\infty} t^l \chi_R(U^l)/l$$

$$G_-(U,t) = \sum_{n=0}^{\infty} t^n \chi_{anti^n(R)}(U) = \exp \sum_{l=1}^{\infty} (-1)^{l+1} t^l \chi_R(U^l)/l$$

Back to the main topic.

$$Z(\beta) = \prod_{i=1}^{h}\left(\sum_{n_i=0}^{\infty} x^{n_i E_i}\right)\cdot\left\{\#\text{ singlets in } sym^{n_i}(R_i)\otimes\ldots \times anti^{n_j}(R_j)\ldots\right\}$$

$$= \int[dU]\left(\prod_i\left\{\sum_{n_i=0}^{\infty} \underbrace{x^{n_i E_i}}_{x_i^{n}} \chi_{sym^{n_i}(R_i)}(U)\right\}\right)\left(\prod_j\left\{\sum_{n_j=0}^{\infty} \underbrace{x^{n_j E_j}}_{x_j^{n_j}} \chi_{anti^{n_j}(R_j)}(U)\right\}\right)$$

$$= \int[dU]\, e^{\sum_{i,n=1}^{\infty}\frac{1}{n} x^{nE_i}\chi_{R_i}(U^n)}\, e^{\sum_{j,n=1}^{\infty}\frac{(-1)^{n+1}}{n} x^{nE_j}\chi_{R_j}(U^n)}$$

If we define $z_B^R(\beta) = \sum_{R_j = R} x^{E_i}$ $\quad z_F^R(\beta) = \sum_{R_j = R} x^{E_j}$,

then we may write

$$Z(\beta) = \int[dU]\sum_R\sum_{n=1}^{\infty}\frac{1}{n}\left(z_B^R(x^n) + (-1)^{n+1} z_F^R(x^n)\right)\chi_R(U^n)$$

Single particle partition functions on $S^3 \times \mathbb{R}$.

To calculate the single particle partition functions let us use a conformal transformation from $S^3 \times \mathbb{R}$ to $\mathbb{R}$. This will result in an immense simplification of the path integral to $Z \to \sum_{\substack{\text{local} \\ \text{operators}}} x^{\Delta}$ where Δ is the scaling dimension of the operator.

- A map $x \to x'$ is a conformal transformation if $g_{\rho\sigma} \frac{dx'^{\rho} dx'^{\sigma}}{dx^{\mu} dx^{\nu}} = \Lambda(x) g_{\mu\nu}$.

$g'_{\rho\sigma} \frac{dx'^{\rho} dx'^{\sigma}}{dx^{\mu} dx^{\nu}} = \Lambda(x) g_{\mu\nu}$. For $\mathbb{R}^d$ we can write our metric as

$ds^2 = d\rho^2 + \rho^{d-1} d\Omega_{d-1}'^{2}$, For $S_{d-1} \times \mathbb{R}$, the metric is ~~$ds^2 = dz^2 = r^{-(d-1)} dr^2$~~ $ds^2 = dz^2 + d\Omega_{d-1}^2$. There is a conformal transformation that maps $\Omega \Rightarrow \Omega'$, $z = \frac{-2}{\sqrt{r}}$, where $z = -\infty$ corresponds to the origin. Furthermore, the z direction which corresponded to euclidian time now corresponds to radial distance. ~~This means that~~ This leads to the generator of scale transformations in the flat euclidian theory is mapped to the hamiltonian on the sphere. These operators are related by an isomorphism of the conformal group, which means that there spectra is identicle. Thus instead of finding the spectra of the hamiltonian on a 3 sphere, we can equivalently find the spectra of scaling dimensions of operators in flat space.

The ~~scaling~~ dimensions of operators ~~are~~

Field	Scaling dim
φ	1
fermion $\xi, \bar{\xi}$	3/2
vector A	2
∂	1

~~so our local operators are $\sum_{n=0}^{\infty} \partial^n \varphi, = \frac{x}{1-x}$, however, we do have to account that $\partial^2\varphi = 0$ so $\partial(\partial^2\varphi) = 0$, so we have to subtract this to get $z = \frac{x - x^3}{1-x}$~~

The local operators are φ, $\partial_i \varphi$, $\partial_i \partial_j \varphi$ where i and j are different indices. Summing over all of these gives us $z' = \frac{x}{(1-x)^d}$, where the factor of d comes from there being d different spacial derivatives.

However, this overcounted the number of operators since $\partial^2 \varphi = 0$, so we need to subtract the operators which are zero ie $\partial_i \partial^2 \varphi$, $\partial_i \partial_j \partial^2 \varphi$ which gives us $\frac{x^3}{(1-x)^d}$, this gives us

$$Z_s = \frac{x^2 + x}{(1-x)^3}$$

— For a vector ~~particle~~ field we have to only sum gauge invariant states. To do this let us choose the gauge $A_0 = 0$ on $\mathbb{R} \times S^3$ which becomes ~~$\partial_\mu A^\mu = 0$ in $\mathbb{R}^d$~~, $x^\mu A_\mu = 0$ on $\mathbb{R}^d$ (recall the operators are analyzed at $x=0$)

W/o the gauge condition there are 4 dof for $A_1 \ldots A_4$ and the operators are of the form $\partial_i \partial_j \ldots A_k$, so the partition function is $4 \frac{1}{(1-x)^4}$, The e.o.m. in the gauge are just

$\partial^2 A_\mu = 0$, which gives us $z' = \frac{4x^2(1+x)}{(1-x)^3}$, However we still have to account for gauge invariance. Differentiating our gauge condition gives us $\partial_i(x^\mu A_\mu) = \partial_i \partial_j (x A_\mu) = \ldots = 0$, which in turn gives $A_\mu = 0$, ~~∂~~ $\partial_\mu A_\nu + \partial_\nu A_\mu = 0, \ldots, \partial_{(\mu_1} \partial_{\mu_2} \ldots A_{\nu)} = 0$. These operators which are set to zero are tensors of rank Δ, where Δ is their dimension. After doing some algebra this gives us

$$Z_v = 1 - \frac{(1+x)(1+x^2-4x)}{(1-x)^3}$$

Finally, (not doing the math out), the partition function for dirac spinors is $Z_f = \dfrac{2^3 x^{3/2}}{(1-x)^3}$.

Now going back to our matrix model.

The character in the adjoint of $U(N)$ is given by $\chi_{adj} = Tr(U)Tr(U^T)$. for $SU(N)$ this changes to $\chi = Tr(U)Tr(U^T) - 1$. Now that we have the full partition function we can analyze this via the standard techniques. ~~which have been~~

$$Z_S = \frac{x + x^2}{(1-x)^3}; \; Z_V = \frac{6x^2 - 2x^3}{(1-x)^3}; \; Z_{F_4} = \frac{4x^{3/2}}{(1-x)^3}$$

The corresponding bosonic & fermionic single particle partition functions are

$$Z_B = n_S Z_{S_4} + n_V Z_{V_4}; \quad Z_F = n_f Z_{F_4}.$$

~~The~~ $$Z = \int [dU] \exp \frac{1}{n}\left(Z_B(x^n) + (-1)^{n+1} Z_F(x^n)\right) \chi_R(U^n)$$

Recall the eigenvalues will lie on a circle so we ~~have~~ can express the eigenvalues as ~~...~~ $e^{i\alpha}$, where $\alpha \in (-\pi, \pi)$. Changing variables to α we obtain. $\int [dU] \to \prod_i \int_{-\pi}^{\pi} (d\alpha_i) \prod_{i<j} \sin^2\left(\frac{\alpha_i - \alpha_j}{2}\right)$; $Tr(U_a^n) \to \sum_i e^{in\alpha_i}$

With only adjoint matter, ~~this is further~~ the partition function is reduced to

$$Z(x) = \int [d\alpha_i] \, e^{-\sum_{i \neq j} V(\alpha_i - \alpha_j)}$$

where $$V(\theta) = -\log\left|\sin\left(\frac{\theta}{2}\right)\right| - \sum \frac{1}{n}\left(Z_B(x^n) + (-1)^{n+1} Z_F(x^n)\right)$$

$$= \log(2) + \sum_{n=1}^{\infty} \frac{1}{n}\left(1 - Z_B(x^n) - (-1)^{n+1} Z_F(x^n)\right) \cos(n\theta).$$

Let us introduce the eigenvalue density $\rho(\theta)$ s.t. $\int_{-\pi}^{\pi} d\theta \, \rho(\theta) = 1$,

~~with~~ with this variable our action becomes
$$S(\rho) = N^2 \int d\theta_1\, d\theta_2\, \rho(\theta_1)\, \rho(\theta_2)\, V(\theta_1 - \theta_2)$$

We can further write ~~this~~ as

$$S = \frac{N^2}{2\pi} \sum_{n=1}^{\infty} |\rho_n|^2 V_n(T)$$

Where $\rho_n = \int d\theta\, \rho(\theta) \cos(n\theta)$
$V_n = \int d\theta\, V(\theta) \cos(n\theta)$

From this definition it is clear that $\rho_n = 0$ will be the minimum of the potential as long as all of the V_n's are positive. ~~From 5.3 we~~
From our expression for V, we can see that
The uniform dist. is an abs minimum if and only if $Z_B(x^n) + (-1)^{n+1} Z_F(x^n) < 1 \quad \forall n.$

the Z_n's are monotonically increasing with x, so the uniform distribution ~~is uniform~~ an abs minimum iff. $Z_B(x) + Z_F(x) < 1.$

~~Let x_H~~

Let x_H be defined as the x s.t.
$$Z(x_H) \equiv Z_B(x_H) + Z_F(x_H) = 1.$$

This always has one unique solution and past this we will have the uniform distribution no ~~longer~~ longer being the absolute minimum (the ρ_n space, the exact solution is no longer at the origin) ~~(recall ρ_n space is an ∞ dim product of $[0,$~~

Let us consider the behavior of Z below the critical temperature (recall that classically all of the ρ_n's vanish, so $Tr(U^n)$ vanishes for any $n \geq 1$ for the uniform dist., and the classical contribution to the action (and thus the leading $O(N^2)$ contribution to the free energy vanishes).

The first non-zero contribution to the free energy ~~any~~ arises from the gaussian integral around this configuration. each n yields a factor of $\frac{2\pi^2}{N^2}\frac{1}{V_n}$

(~~its~~ $\frac{1}{V}$ instead of $\frac{1}{\sqrt{V}}$ due to the integral being over ~~$|\rho|^2 = \rho_r(Re\rho)^2 + (Im\rho)^2$~~ the real and imaginary parts of ρ

We want this to evaluate to 1 in the limit as $x \to 0$ (where z_b and $z_f \to 0$) ~~which kills~~ so that the free energy vanishes. Fixing this gives us to leading order

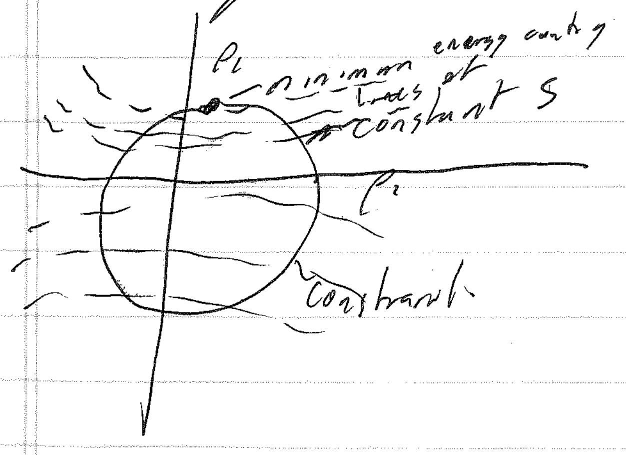
$$Z(x) = \prod_{n=1}^{\infty} \frac{1}{1 - z_b(x^n) - (-1)^n z_f(x^n)}$$

Notice that this has no N dependence which is what we expect, so it reaffirms that what were dealing with slightly resembles reality.

Furthermore it might be of interest that the ~~free~~ energy diverges as

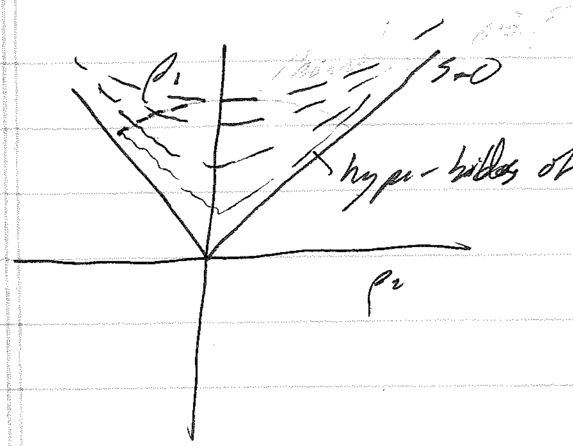
$$F \to T_H \log(T_H - T)$$

At the Hagedorn temperature, V_1 vanishes so the lowest mode becomes massless. Since the action is quadratic this corresponds to a flat direction in the potential, and the minimum action configurations are $\rho(\theta) = \frac{1}{2\pi} 1 + t\cos\theta$ where, $t \in [0,1]$ (note $t \in (0,-1)$ is simply changing the origin of θ to π and that values of $t > 1$ are disallowed under the positivity condition for ρ. Since S has negative coefficients for $T > T_H$, so all minimum action coefficients must lie on the boundary of configuration space. The boundary on acceptable values for all of the ρ's is provided by the positivity condition $\rho(\theta) \geq 0$, at the point where a hyperboloid of constant S lies tangent to this boundary is the minimum action coefficient.



Since this boundary is $\rho(\theta) \geq 0$, then it is necessary that $\rho(\theta)$ vanishes for some θ.

In the limit of small positive $\Delta T = T - T_H$, the action countour in ρ_2 space is a a cone with its opening angle going to zero, the countours of smaller S are hyperboloids within the cone and the minimum action configuration is within the cone



The leading coefficient in the action then comes from the ~~TH~~ $t=1$ configuration. ~~Evaluating the path integral at this point.~~ at this point.

$$S_{T>T_H} = \frac{N^2}{8\pi}\left(T - T_H\right) V_1'\left(T_H\right)$$

~~Constant~~ taylor expansion around T_H.

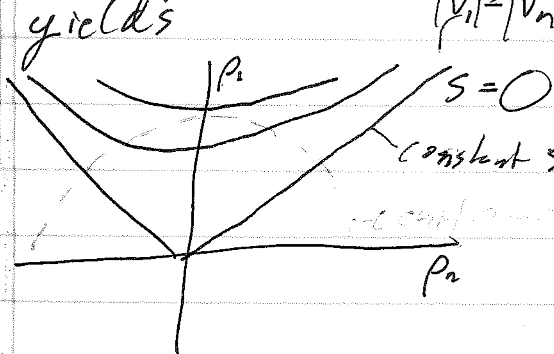
So above the transition the free energy goes as N^2!! This is exactly what we wanted.

$$S = \frac{N^2}{2\pi} \sum_{n=1}^{\infty} |\rho_n|^2 V_n(T)$$

only $V_1 < 0$, so we have

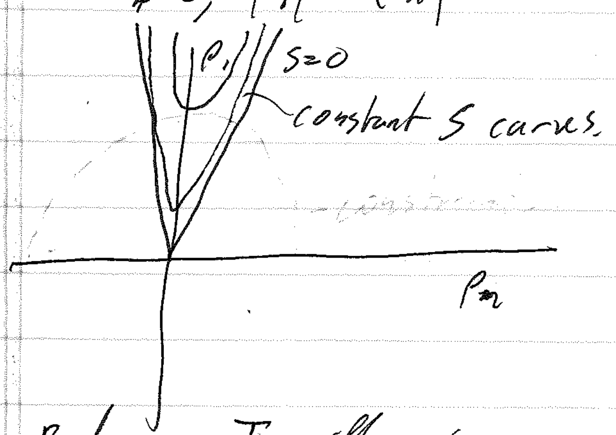
$$S = \frac{N^2}{2\pi} \; V_1(T)\rho_1^2 + \sum_{n=2}^{\infty} |\rho_n|^2 V_n T,$$

Plotting the action countours in ρ_n space yields



as we lower the value of V_1 this becomes narrower

$T = T_H + \varepsilon, \; |V_1| << |V_n|$



$$Z = \int d\rho_1 d\rho_2 \ldots e^{-\frac{W^2}{2\pi}}$$

Below T_H the ρ_n's are all zero, so the classical solution to the action ~~is the solution~~ is simply $S=0$. However above the transition we have $\rho_1 \neq 0$ so the classical action does vary with q.