Topological entanglement entropy

* Entropy in QM

; Density matrix ρ , $\rho^\dagger = \rho$. $\rho = \sum_i p_i |i\rangle\langle i|$

· Neumann entropy $S = -\mathrm{tr}\, \rho \ln \rho$

pure state $\rho^2 = \rho \Rightarrow S = 0$.

Equal mixture / random $\rho = \frac{1}{N} \sum_{i=1}^{N} |i\rangle\langle i| \Rightarrow S_{max} = \ln N. \sim \ln \dim H$

~~Entanglement~~
Joint system $H_{AB} = H_A \otimes H_B$. ρ_{AB} on AB.   (Bipartite)

only access to subsys A : $\rho_A = \mathrm{tr}_B\, \rho_{AB}$ (marginal distribution).
reduced density matrix.

(Entanglement entropy) $S(A) = \mathrm{tr}\, \rho_A \ln \rho_A$.

· Entanglement

generally vectors in H_{AB} can be represented by

$$|\psi_{AB}\rangle = \sum_{a,b} c_{a,b} |\psi_a\rangle \otimes |\psi_b\rangle.$$

NOT entangled state can factorize:

$$|\psi_{AB}\rangle_{unentangled} = \left(\sum_a c_a |\psi_a\rangle\right) \otimes \left(\sum_b \tilde{c}_b |\psi_b\rangle\right)$$

Ex 1. $|\Psi\rangle$ on AB not entangled. 2 spin $\frac{1}{2}$ system

$$|\Psi\rangle = \frac{1}{\sqrt{2}} \left(|\uparrow\uparrow\rangle + |\downarrow\uparrow\rangle \right).$$

$$\Rightarrow \rho_A = \frac{1}{2} \left(|\uparrow\rangle + |\downarrow\rangle \right)\left(\langle\uparrow| + \langle\downarrow| \right)$$

$$\Rightarrow S(A) = 0$$

Ex 2. $|\Psi\rangle$ on AB entangled.

$$|\Psi\rangle = \frac{1}{\sqrt{2}} \left(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle \right)$$

$\Rightarrow \rho_A = \frac{1}{2} \left(|\downarrow\rangle\langle\downarrow| + |\uparrow\rangle\langle\uparrow| \right)$ * mixed subsystem, even AB is pure!

$\Rightarrow S(A) = \ln 2$ (Entanglement entropy). EE

More generally, start with PURE state $|\Psi\rangle$ on AB,

$$S(A) > 0 \iff |\Psi\rangle_{AB} \text{ is entangled.}$$

Can use $S(A)$ to identify entanglement in $|\Psi\rangle_{AB}$.

Caveat. If ρ_{AB} not pure, $\rho_{AB} = \sum_i p_i \rho_A^i \otimes \rho_B^i$, $S(A) > 0 \not\Leftrightarrow$ AB entangled.

Can't distinguish entanglement ~~and~~ classical mixing.

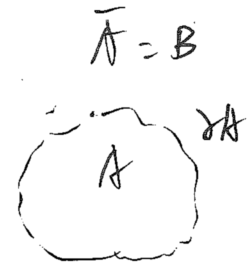
Application. Use in reverse direction, start with mixed state $|\Psi_A\rangle$. can find system B, such that $|\Psi_{AB}\rangle$ in $H_A \otimes H_B$ is ~~of~~ a pure state

$\Rightarrow$ purification

* EE in field theory.

Space of QFT in ground state.

How a region A entangled with its environment.



(subalgebra commute)

$\mathcal{H} \overset{?}{=} \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$.   Focus on GS.

not exact in continuum, but close, at least in GS (correl length small)

Should be able to calculate $S(A)$ as in QM, which tells me about entanglement.

$S(A)$ depends on
- shape
- size
- state (GS)
- region
- UV.

Understand behavior of $S(A)$ first.

(Srednicki, '93)

Express $|0\rangle = \sum_{ii'} \psi_{ii'} |\psi_i\rangle_A |\psi_{i'}\rangle_{\bar{A}}$, $\rho = |0\rangle\langle 0|$

$\rho_A = \mathrm{tr}_B \rho = \sum_{ij} |i\rangle\langle j| (\psi\psi^\dagger)_{ij}$ Only depends on DOF of region A

$\rho_B = \ - - - = \sum_{i'j'} |i'\rangle\langle j'| (\psi^T \psi^*)_{i'j'}$ — — — $\bar{A}$.

For any integer k, $\mathrm{tr}\, \rho_A^k = \mathrm{tr}\, \rho_B^k \Rightarrow \rho_A, \rho_B$ same e'values, up to zeros.

$\Rightarrow S(A) = \mathrm{tr}\, \rho_A \ln \rho_A = S(B)$.

Furthermore, $S(A)$ only depends on DOF in A

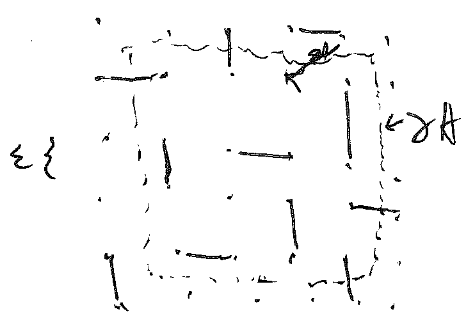
$S(B)$ ... ... B

If $S(A)=S(B)$ $\Rightarrow$ they depend on A & B share in common.

$\Rightarrow \partial A$. $\Rightarrow$ $f(L) = L + L^2 + L^3 + \dots$?

For G.S. of local QFT, ~~correl~~ and assuming correlation length dies off exp,
(not proven)

local short range correlations.



$S(A) \propto$ # of bonds broken

$\propto \frac{L}{\varepsilon}$ $\qquad D = 2+1$

leading order size dep of GS (Area law)

Other convincing evidence, but not proven in $D > 2$ (?)

+ Topology

natural guess: const piece $\Rightarrow$ topo invariant

$$S(A) = \alpha L - \gamma + O(L^{-\beta}) \qquad (2+1)$$

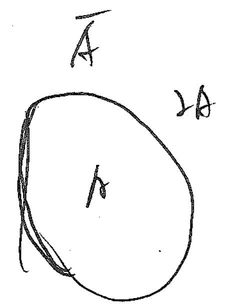
(Kitaev & Preskill, Levin-Wen, 06.)

$\gamma = 0 \Rightarrow$ no topological order

$\gamma > 0 \Rightarrow$ topological phase

Can we understand this in a handwavy way? Yes..

Local QFT, short range correlation $\Rightarrow S(A)$ depends on local properties of ∂A



$$\Rightarrow S(A) = \int_{\partial A} f(\overset{\text{curvature}}{k}, \partial k)$$

Smooth bd, no short-distance phys

Justify EFT.

$$= \int_{\partial A} \#\text{const} + \#k + \#k^2 + (\#\underbrace{\partial k}_{=0 \text{ total div.}}) + \cdots$$

$$= L + \text{const} + \frac{1}{L} + \cdots$$

Similarly for $S(\bar{A})$. However symmetry constraints $S(A) = S(\bar{A})$ exclude const.

k term (in $f(k, \partial k)$) switch sign going from $A \to \bar{A}$ locally.

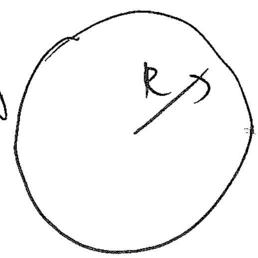
$\Rightarrow$ No const term in $S(A)$ in $D = 2+1$ ~~state~~ if ~~short-ed~~ corre falls off exp.

sort of indicate

$\Rightarrow \gamma \neq 0$ implies long-distance entanglement, global property

How to extract γ? size-independent piece

$S(A)$ for a range $R \gg \xi$

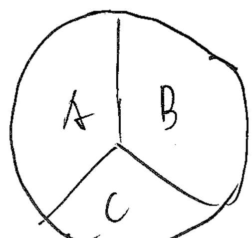
∂A 

$$+\gamma = R\frac{dS(A)}{dR} - S(A)$$

$\Rightarrow$ in practice (lattice simulation) can't choose ∂A smooth

$\Rightarrow$ may be ~~sensitive~~ short distance physics many, not justifying effective TQFT.

Divide regions in a more contrived way

D 

$$S_{topo} = S_A + S_B + S_C - S_{AB} - S_{AC} - S_{BC} + \underset{\overset{\|}{S_D}}{S_{ABC}} = -\gamma$$

Only dep on topology not geometry

① Obviously L dep. vanish: C in $+S_C - S_{AC} - S_{BC} + S_{ABC} = 0$ in S_{topo}

Can do same for other bds. Corner's cancelling.

② Deform bds, say AC. Only change S_{ijk} where index involve A or C, other regions far away

$$\Delta S_{topo} = (\Delta S_A - \Delta S_{AB}) + (\Delta S_C - \Delta S_{BC})$$

B is far away from A & C, ~~deforming~~ appending B to A, C have negligible effect

$\Rightarrow$ all brackets $= 0$. $\Delta S_{topo} = 0$.

③

③ Deform triple points, say ACD.

$$\Delta S_{topo} = (\Delta S_A - \Delta S_{AB}) + (\Delta S_C - \Delta S_{BC}) + (\Delta S_D - \Delta S_{BD})$$

ΔS_{ABC} ∥, ΔS_{AC} ∥

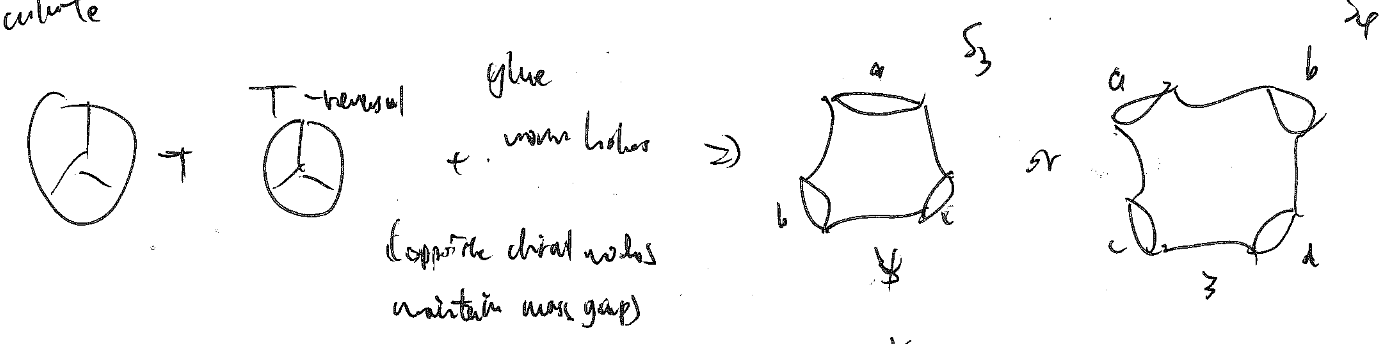
Again append B to regions A, C, D should not affect on change of entropy, since B is far away. $\Delta S_{topo} = 0$.

④ Deform Hamiltonian smoothly? assume local, ξ remain finite during deformation, $\ll$ size of A.

→ If deformation far away from bds, negligible effect on bds (vicinity of GS), expect $\Delta S_{topo} = 0$

* If deformation is far away from bds, can deform bds far away, deform Hamiltonian, then remove bd back.

Calculate



→ trivial anyons ⇒ TQFT summary $P_a = |S_1^a|^2 = \frac{d_a^2}{D^2}$ ...

$P_{ab\to c} = \frac{N_{ab}^c d_c}{d_a d_b}$ (fusion $a \times b = \sum_c N_{ab}^c c$)

⇒ $S_3 = 4\log D - 3\sum_a P_a \ln d_a$, $S_4 = 6\ln D - 4\sum_a P_a \ln d_a$.

The end of the day $\gamma = \ln D$, $D \to$ total quantum dimension

$D = \sqrt{\sum_a d_a^2}$ $\quad d_a \to$ quantum dimension of anyon of type a

~~e.g.~~ ∝ Abelian anyons has $d_a = 1$. $\Rightarrow D^2 =$ # of type of anyons.

o Fractional quantum Hall $\nu = 1/q$, $D = \sqrt{q}$.

② Toric code (or Z_2 with Gauss's law)

$\dim H = 2^2 = 4 \Rightarrow D = 2 \Rightarrow \gamma = \ln 2$.

∝ Nonabelian anyons $d_a \geq 1$ and $d_a \in R$.

$SU(2)_k \quad D^{-1} = \sqrt{\frac{2}{k+2}} \sin\left(\frac{\pi}{k+2}\right)$

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* Concise, simple, clear ex. Toric Code. $H = \sum_v H_v + \sum_p H_p$ (vertex, plaquette)

$\Rightarrow$ string like configurations $\Rightarrow$ closed loops in GS.

$\bar{A}$ fix flux on vertices on ∂A. (local continuous strings). (mod 2)

Equal mixture in ρ_A.

$S(A) = \ln(\#$ of flux configurations on $\partial A)$

$= (\#$ of vertices on $\partial A - 1) \ln 2$

$= \alpha \frac{L}{\varepsilon} - \ln 2$ $\quad$ global Gauss's law constraint.