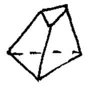
$\mathcal{A} =$ 

1) Motivation Overview
2) Motivation Depths
3) Intro to Shapes
4) NMHV
5) ∂
6) Bonus Comments

1) • Considered talking about relation between $Gr(n,k)$ & Amps $\Rightarrow$ relies on Yangian symmetry
   → but would just result in a punchline of nasty integrals over weird spaces, no grand insight unless you are already really familiar with Grassmannians.

• Instead, I want to expand on Isaac's talk, and push the geometric interpretation to its limit

• We'll motivate how we can treat not just our variables but our whole Amplitudes geometrically, connect it with very intuitive pictures to give us new perspective on the analytical relations we've already encountered, and even ~~clearing up~~ work towards our discussion of spurious poles

2) • We'll frame our investigation in the context of 6-pt NMHV amp $A_6[1^+2^-3^+4^-5^+6^-]$

• Andrew taught us that $A_n^{NMHV} = A_n^{MHV} \sum\sum R_{ijk}$ & easy $\hookrightarrow \equiv [i,j-1,j,k-1,k]$ | $[i,j,k,l,m] = \dfrac{\delta^4(\chi_{iA}\langle jklm\rangle + \text{cyc})}{\underbrace{\langle ijkl\rangle\langle jklm\rangle\ldots\langle mijk\rangle}_{5\text{ terms}}}$

$\hookrightarrow$ want to understand better, since they also show up in our loops as products w/ Y invariants etc, as Isaac mentioned.

• What do we know about them? a) Cyclic invariance b) reflection symmetry c) 6-term identity

e) Hinted at, but we didn't really write yet: $[2,3,4,6,1] + [2,3,4,5,6] + [2,4,5,6,1] = [3,1,6,5,4] + [3,2,1,6,5] + [3,2,1,5,4]$

$\hookrightarrow$ comes from invariance of $\mathcal{A}$ under different BCFW super-shifts, $[2,3\rangle = [3,2\rangle$ in our case

$\Rightarrow$ But this bracket relation is more fundamental than a shift in a particular amp since they are invariants that have to hold $\forall$ 6 momentum twistors, as long as we want conservation of momentum.

• Is it Schouten id? $\Rightarrow$ any antisymmetric 5-bracket defined by $\langle ijklm\rangle = \epsilon_{IJKLM} Z_1^I Z_2^J Z_3^K Z_4^L Z_5^M$ would have a Schouten that looks like our relation above, But in our case, we have 4-component twistors, not 5.

Can we massage our representation a bit? Let $Z_i^{Z_5} = \begin{pmatrix} Z_i^{Z_4} \\ \underbrace{\chi_i \cdot \eta} \end{pmatrix}$

SU(4) invariant, η_A an auxiliary Grassmann $\forall$ external particles

Turns out this gives the right Schouten, but it's not our 5-bracket since ours is built out of purely Bosonic contractions, with fermionic weights, not intertwined within the angle brackets, So integrate them out!

~~OA~~ If you stare hard, can convince yourself that $[i,j,k,l,m] = \frac{1}{4!}\int d^4 Y \frac{\langle i,j,k,l,m\rangle^4}{\langle 0,i,j,k,l\rangle \dots \langle 0,m,i,j,k\rangle}\Big|_{Z_0^Z = \begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix}}$

ref vector.

~~also~~ so related to Schouten, but not. So what is it?

Note that $Z_i^Z \to t_i Z_i^Z$ invariance at all pts except for our reference vector $\therefore Z_i^Z \in \mathbb{CP}^4 \setminus \{Z_0^Z\}$

↳ we've seen this before! Issue: $Y_{ij}^2 = \frac{\langle i-1, i, j-1, j\rangle}{\langle i-1, i\rangle\langle j-1, j\rangle} = \frac{\langle i-1, i, j-1, j\rangle}{\langle Z_0^{Zj}, i-1, i\rangle\langle Z_0^{Zj}, j-1, j\rangle}\Big|$ Z_0^{Zj} = reference bi-twistor,

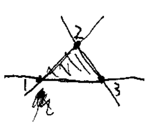
gives us a way to break $\mathbb{CP}^3$ & measure distance between Y_i & Y_j

$\Rightarrow$ both projective except for reference, both built out of only ~~(with~~ angle brackets,

each Z_0^{Zj} fixes each pt

so extrapolate: if Y_{ij} gives dist between pts Y_i & Y_j, the ~~thdd~~ ours gives the volume of some object defined by 5 pts in $\mathbb{CP}^4$. Let's make it precise & see what it buys us!

3) ~~Recall Vol~~ ~~\~~ ~~..~~ who remembers how to calc area of triangle? $\frac{1}{2}bh \Rightarrow$ let's ~~do~~ it a little more sophisticated by using determinants.

$A^{\text{Tri}} =$  $= \frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1\end{vmatrix}$; now this last row is redundant, since we could just write ~~the~~ the area as a sum over $|2\times 2|$. When faced w/ redundancy in physics, we either get rid of it, or promote it as a feature. Let's do the latter, as will be clear why in a moment,

Let $W_{iZ} = \begin{pmatrix} x_i \\ y_i \\ 1\end{pmatrix}$, $Z_0^Z = \begin{pmatrix}0\\0\\1\end{pmatrix}$, $Z = \{1,2,3\}$; Now $A^{\text{Tri}} = \frac{1}{2}\frac{\langle 1,2,3\rangle}{(Z_0 \cdot W_1)(Z_0\cdot W_2)(Z_0 \cdot W_3)}$ ← usual Levi-Civita form

Now it's clear that denom is just 1 & all content is in $\langle\ \rangle$, but if we introduce this structure, now ~~our~~ A^{Tri} is invariant under $\mathbb{CP}^2$, $W_i \to t_i W_i$ modulo the origin. We're close!

Can we get the dot products into bracket form too? Yes, with dual variables!

Let Z_a^Z be lines in W-space. A line is set of pts constrained s.t. $Z^Z W_Z = 0$

vector in 2D ↑ ; 1 constraint i.e. 1 dof, line

Each W_{iZ} lies at intersection of 2 lines, eg W_{1Z} needs to satisfy $Z_a^Z W_{1Z} = Z_c^Z W_{1Z} = 0$

inverting, we find $W_{1Z} = \langle *, Z_c, Z_a\rangle = \epsilon_{ZJK} Z_c^J Z_a^K$

$\therefore A^{\text{Tri}} = \frac{1}{2}\frac{\langle a b c\rangle^2}{\langle 0,b,c\rangle\langle 0,a,b\rangle\langle 0,c,a\rangle} \cong [a,b,c]$

We can generalize easily to the volume of an n-simplex in $\mathbb{CP}^n$

$$[Z_{i_1},\dots,Z_{i_{n+1}}] = \frac{1}{n!}\frac{\langle i_1,\dots,i_{n+1}\rangle^n}{\langle 0,i_1,\dots,i_n\rangle\cdots\langle 0,i_{n+1},i_1,\dots,i_{n-1}\rangle}$$

; n+1 Z's correspond to the n+1 boundaries of the simplex since set of W's satisfying incidence relation $Z_i^Z W_Z = 0$ spans (n-1) dim subspace of $\mathbb{CP}^n$

looking back at our 5-bracket, it's identical to integrand. ∴ NMHV tree amps are literally just sums of ~~[illegible]~~ brackets ∴ just some volume in $\mathbb{CP}^4$, (integrated)

4) $A_5^{NMHV}[1,2,3,4,5] = A_5^{MHV} \times \underbrace{[1,2,3,4,5]}_{\text{vol of 4 simplex in } \mathbb{CP}^4}$, simplest case

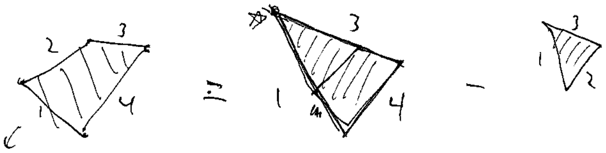
Next nontrivial? $A_6^{NMHV}[1,2,3,4,5,6] \propto \underbrace{[2,3,4,6,1]}_{\langle 4,6,1,2\rangle\langle 2,3,4,6\rangle} + \underbrace{[2,3,4,5,6]}_{\langle 6,2,3,4\rangle\langle 4,5,6,2\rangle} + \underbrace{[2,4,5,6,1]}_{\langle 2,4,5,6\rangle\langle 6,1,2,4\rangle}$

leave space

∴ sum of 3 vol of 4 simps. But are they 3 ⊥ shapes? ~~[illegible]~~ do they fit together somehow?

Recall that each ~~[illegible]~~ ⟨i-1, i, j-1, j⟩ in denom of [5] gives a local pole, while something like ⟨1,2,4,6⟩ is non-local & spurious. Refer to above for spurious poles hidden in each term. Since A_6 ~~[illegible]~~ must be local, these poles must cancel, & they do as shown in pairs above. But each denom ~~[illegible]~~ bracket is associated w/ vertex in polytope, ∴ somehow ~~[illegible]~~ ∃ extra vertices that must disappear in our simplex ~~[illegible]~~ gluing. We attack this by studying the geometric equivalent of the partial fraction decomposition that gave rise to our spurious pole discussion in 4d case in Notebook talk ⇒ look at geometric decomposition as triangulations of our ~~[illegible]~~ polytope!

2-D for simplicity again!,

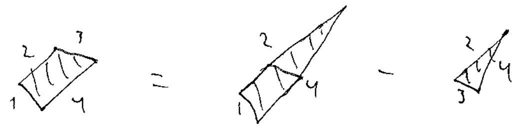


label is subspace label Z_i

$= [4,1,3] - [2,1,3]$

* introduce new non-local pt as intersection of lines 1 & 3 which cancels out between the diagrams. / also pick orientation for pos ~~...~~

But can also pick



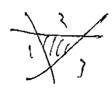
$= [1,2,4] - [2,4,3]$

since triangulation invariant, must be same amplitude ∴ $[4,1,3]-[2,1,3] = [1,2,4]-[2,4,3]$

↳ just like our bracket relation!

∴ our 6 pt amp is just triangulation of associated polytope by adding 3 auxillary/spurious vertices. Different BCFW terms are just different tesselations into 4 simplices with different spurious poles/vertices that cancel since they are not present in original local polytope determined by Z_i boundaries.

5) Recall some boundary geometry:



length of line segment 1 is just projection to subspace Z_1 of intersection of lines 2 & 3. ∴ perimeter of triangle can be denoted $[12]+[23]+[31]$ where we pick an edge orientation to point into volume of polytope. (RHR)

We can generalize this & define boundary operator $\partial[1,\dots,n] = \sum_{i=1}^{n} (-1)^{i+1} [1,\dots,\hat{i},\dots,n]$ ← removed index ! trust that this handles signs such that orientation of each piece is correct.

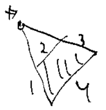
How is this useful to us?

Well, can't build a tetrahedron in 2-D ∴ boundary op acting on a 2-D tet must vanish.

$\Rightarrow 0 = \partial[1,2,3,4] = [2,3,4] - [1,3,4] + [1,2,4] - [1,2,3]$ } same BCFW identity!

∴ our original motivation for understanding 6-term [5] identity is finally manifest ⇒ it's the statement that you can't build a 5 simplex in $\mathbb{CP}^4$!

Lastly, can we see our spurious poles cancel in this boundary language?

 =  −  $= [4,1,3] - [2,1,3]$

$\Rightarrow \partial[\qquad] = [1,3] - [4,3] + [4,1] - ([1,3] - [2,3] + [2,1])$

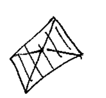
cancel?, length of side 1, length of side 3

Note, different subspaces. If we label them by subspace, get $= ([4,3]-[2,3])_{z_1} + [3,1]_{z_2} + ([4,1]-[2,1])_{z_3} + [1,3]_{z_4}$ ← flipping orientation

= circumference of original polytope

Now, in simplex, each boundary uniquely defines a vertex opposite it. So we can relabel this operation by vertices instead. In a vertex rep, $[3,1]_{z_2} \to * \leftarrow [1,3]_{z_4}$ ∴ in original ∂ op, can cancel them. A way to see actual cancelation of spurious poles in boundary op.

$\cancel{\Rightarrow \partial = V[3,4] + V[4,1] + V[2,3] + V[1,2]}$

6) What about triangulating internally?
a)  ? works, but now have 4 terms instead of 2
∴ BCFW is more efficient.

b) Does ∃ a triangulation w/ no spurious poles? Yes for tree NMHV! Still generalizing for more

c) Does every polytope give an amplitude? ⇒ No. ~~Ex~~ eg, take only first 2 terms of A_6 ⇒ won't cancel all spurious poles ∴ nonlocal even though a fine polytope.

d) we needed dual superconformal invariance of planar N=4 SYM. But this also works for pure YM ~~also~~ at tree level since they are related as we discussed ~~in~~ earlier in the quarter.
Generally valid for NMHV n-pt tree superamps as well as 1-loop n-pt MHV integrands. ← integrate out pairs of "hidden" pts
If want N^kMHV, see Amplituhedron, an extension of our construction dual to our polytopes w/ manifest locality but emergent unitarity