PHASE TRANSITIONS

Hermetian 1-matrix model

Gauge Fields on a lattice

2-Dimensions

Large-N

Gross-Witten Transition

Cayley Map

Remarks

Review from Brandon's Talk
but shows
simple phase transition
to connect G-W
to later

$$Z_{1h} = \int d\varphi \, e^{-N \mathrm{Tr}\, V(\varphi)}$$

⋮ Skip steps

Log repulsive potential

$$= \int \prod_{i=1}^{N} dp_i \underbrace{\prod_{i<j} (p_i - p_j)^2}_{\text{Vandermonde det} \rightarrow} \exp\left[-N \sum_{i=1}^{N} V(p_i)\right]$$

Introduce $\rho(p) = \frac{1}{N} \sum_{i=1}^{N} \delta^{(1)}(p - p_i)$

satisfies $\int dp \, \rho(p) = 1$ $\quad \rho(p) \geq 0$

Saddle pt:

Extrema at

$$V'(p_i) = \frac{2}{N} \sum_j \frac{1}{p_i - p_j}$$

large N limit:

$$V'(p) = 2 \fint d\lambda \, \frac{\rho(\lambda)}{p - \lambda}$$

Idea: intro resolvent

$$W(\omega) = \int_a^b d\lambda \, \frac{\rho(\lambda)}{\omega - \lambda} \xrightarrow{\omega \to \infty} \frac{1}{\omega}$$

Equate $\mathrm{Re}\, W = \frac{V'}{2}$ at support of ρ

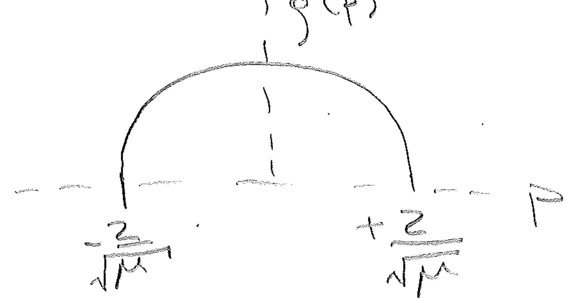
⋮ Skip steps

$\rightarrow$ Set $W = \frac{V'}{2} - \frac{1}{2}\sqrt{(V')^2 - 4Q}$

$$W(p \pm i\varepsilon) = \frac{V'(p)}{2} \mp i\pi \rho(p)$$

Gaussian Example:

$$\rho(p) = \frac{\mu}{2\pi}\sqrt{\frac{4}{\mu} - p^2}$$



Far right:

General potential

$$W(\omega) = \frac{V'(\omega)}{2} - \frac{M(\omega)}{2}\sqrt{(\omega-a)(\omega-b)} \qquad \text{Ansatz}$$

$V(\omega)$ degree K

$M(\omega)$ " $K-2$

Asymptotic behavior:

$$\frac{V'(\omega)}{\sqrt{(\omega-a)(\omega-b)}} - M(\omega) \longrightarrow \frac{2}{\omega^2}$$

$$\rho(p) = \frac{M(p)}{2\pi}\sqrt{(p-a)(b-p)}$$

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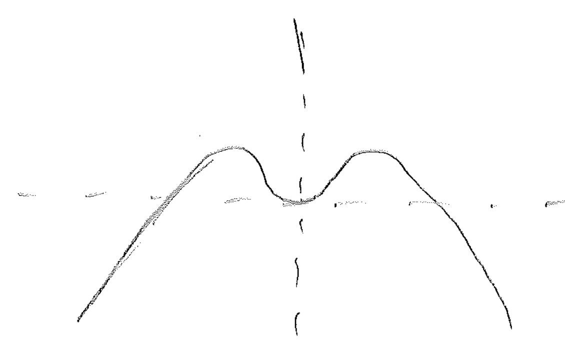
e.g. $V(p) = \frac{1}{2}\mu p^2 + \frac{1}{4}gp^4$



For small enough $(-g)$ the p^2 term can contain the repulsive log effective potential. After they spill out, $\Rightarrow$ non-compact support.

In particular for $\mu > 0$ $g < 0$

$$V'(p) = \mu p + g p^3$$

$$M(\omega) = \alpha + \beta\omega + \gamma\omega^2$$

Expand $\dfrac{V'(\omega)}{\sqrt{(\omega-a)(\omega-b)}} - M(\omega) \xrightarrow{\omega\to\infty} \dfrac{2}{\omega^2}$

5 powers $\omega^2, \omega, 1, \frac{1}{\omega}, \frac{1}{\omega^2}$

5 unknowns $\alpha, \beta, \gamma, a, b$

Soln:

$$\rho(p) = \frac{1}{2\pi}\underbrace{\left[\mu + g p^2 + g\,\eta^2/2\right]}_{M(p)}\sqrt{\eta^2 - p^2}$$

$$\eta^2 \equiv \frac{2\mu}{3g}\left(-1 + \sqrt{1 + \frac{12g}{\mu^2}}\right)$$

- as $g \to 0$ $\eta^2 \to \frac{4}{\mu}$

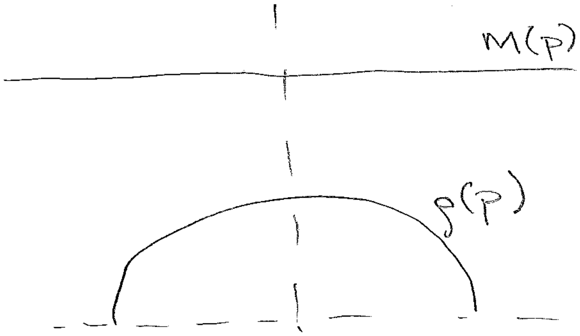
  Semi-circle dist.

- Critical value:

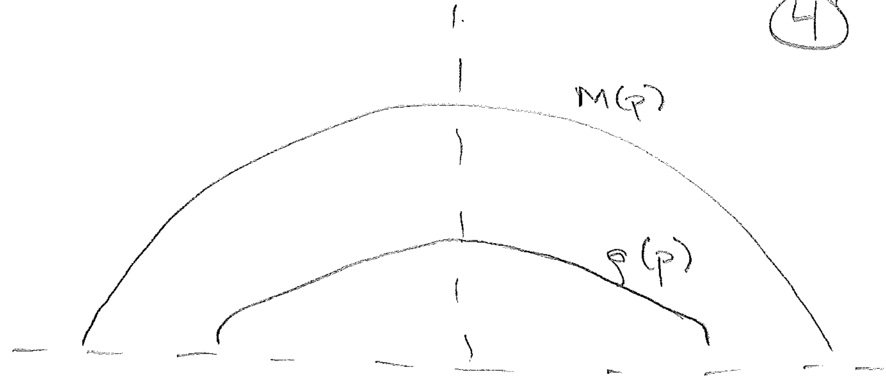
$$-g \leq \frac{\mu^2}{12}$$

For larger neg g, resolvent ansatz failed

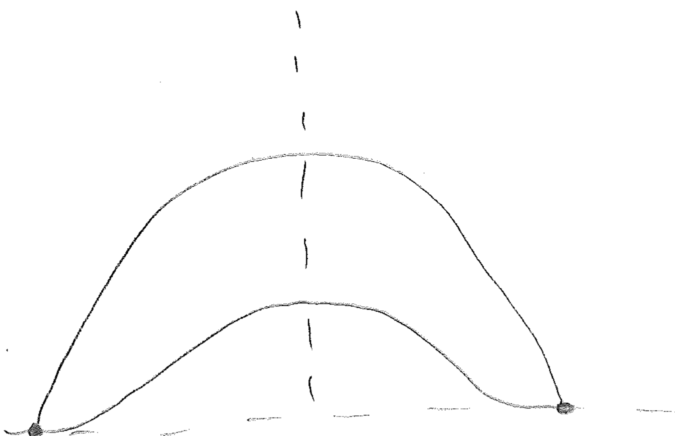
<u>Planar diagrams diverge</u>



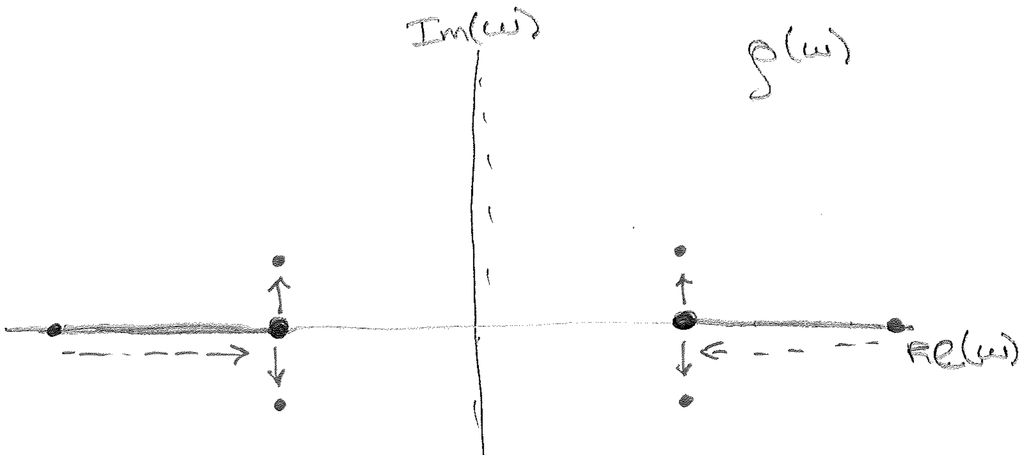
$g=0$



$-\frac{\mu^2}{12} < g \leq 0$

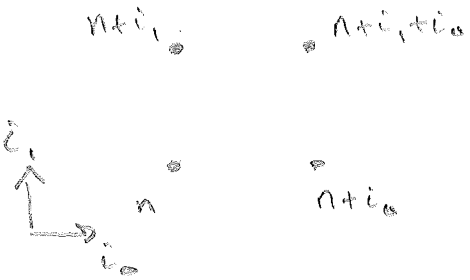


$g = g_* = -\frac{\mu^2}{12}$



zeros of $M(\omega)$
"Resolvent ansatz polynomial"
coalesces w/ endpoint
changing edge's singular behavior

Lattice gauge fields



$U_{n,i}$ unitary matrix
adjoint rep of $U(N)$

transports a matter field
from $n \longrightarrow n+i$

Aside: $U_{x\mu} = P \exp\left[i \int_x^{x+a\mu} dz^\mu A_\mu(z) \right]$

As $a \to 0$ $\quad U_{n,i} \longrightarrow e^{iaA_i(\bar{n})}$

exponential of i^{th} component of vector potential
at "center" of link.

~~Aside: Adjoint:~~

~~Want covariant deriv~~

$$\cancel{D_\mu \psi \to U D_\mu \psi = (U D_\mu U^{-1})(U\psi)}$$

$$\cancel{D_\mu \to U D_\mu U^{-1}}$$

$$\cancel{D_\mu = \partial_\mu - igA_\mu}$$

$n = \dim(G)$

WILSON ACTION:

$$S(U) = \sum_{P} \frac{1}{g^2} \mathrm{Tr}\left(\square + \square^{\dagger}\right)$$

$$\square = U_{n,i_0}\, U_{n+i_0,i_1}\, U^{\dagger}_{n+i_1,i_0}\, U^{\dagger}_{n,i_1}$$

$$Z_{2D} = \int \prod_{\vec{n},i} dU_{\vec{n},i}\, e^{-S(U)}$$

↑ Haar measure on $U(N)$

Vacuum energy density:

$$E_0 = \frac{F(g^2,N)}{\lambda} \equiv -\frac{\log Z}{V} \qquad \lambda = g^2 N$$

Analogy $\lambda = kT$

$d=2$ Why?

Exploit Gauge Invariance

$$U_{n,i} \longrightarrow V_n U_{n,i} V_{n+i}^{\dagger}$$ arbitrary V_n unitary

Gauge choice:

$$A_0 = 0 \Rightarrow U_{n,i_0} = 1 \quad \forall \vec{n}$$

$$S = \frac{1}{g^2} \sum_n{}' \operatorname{Tr}\left(\square_n + \square_n^{\dagger} \right)$$

Let $W = \square$

So

$$Z_{2D} = \int \prod_n dW_n \exp\left[- \sum_n{}' \frac{1}{g^2} \operatorname{Tr}\left(W_n + W_n^{\dagger} \right) \right]$$

$$= \left[\int dW \exp\left[-\frac{1}{g^2} \operatorname{Tr}\left(W + W^{\dagger} \right) \right] \right]^{V/a^2}$$

$$= \left(Z_{1p} \right)^{V/a^2}$$

$$E_0 = \frac{F}{\lambda} = -\frac{1}{a^2} \log Z$$

So 2-D gauge theory reduced to single integral, the one plaquette world.

Large-N [Circular unitary ensemble]

Integrand depends only on eigen values

so let

$$W = TDT^{+} \qquad D = \mathrm{diag}(e^{i\alpha_1}, \ldots, e^{i\alpha_N})$$

$$TT^{+} = \mathbb{1}$$

$$dW = \text{const } dT \prod_{i=1}^{N} d\alpha_i \, \Delta^2(\alpha_i)$$

$$\Delta^2(\alpha_i) = \prod_{i<j} \sin^2\left|\frac{\alpha_i - \alpha_j}{2}\right|$$

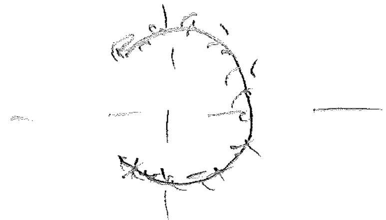
$$\boxed{Z = \int_{-\pi}^{\pi} d\alpha_1 \ldots d\alpha_n \exp\left[\frac{2}{g^2}\sum \cos\alpha_i + \sum_{i\neq j}' \log\left|\sin\frac{\alpha_i - \alpha_j}{2}\right|\right]}$$

Leave Room

stationarity condition:

$$\frac{2}{g^2}\sin\alpha_i = \sum_{j\neq i}' \cot\left(\frac{\alpha_i - \alpha_j}{2}\right)$$

Intro: $\rho(\alpha) \geq 0$ s.t. $\int_{-\alpha_c}^{\alpha_c} d\alpha \, \rho(\alpha) = 1$



$$\frac{2}{\lambda}\sin\alpha_i = \int_{-\alpha_c}^{\alpha_c} d\beta \, \rho(\beta) \cot\left(\frac{\alpha - \beta}{2}\right)$$

Two Solutions

(1) Assume $\alpha_c = \pi$

$$\Longrightarrow \quad \rho(\beta) = \frac{1}{2\pi}\left[1 + \frac{2}{\lambda}\cos\alpha\right]$$

But $\rho(\beta) \geq 0 \Longrightarrow \lambda \geq 2$

Strong coupling soln

eigenvalues fill unit circle

As $\lambda \to \infty$, uniform distribution.

(2) For other soln,

$\lambda \leq 2$ allow $\alpha_c < \pi$

Intro

$$F(z) = \int_{-\alpha_c}^{\alpha_c} d\beta \; \rho(\beta) \cot\left(\frac{z-\beta}{2}\right)$$

- Periodic $F(z) = F(z+2\pi)$
- Analytic for $z \notin (-\alpha_c + 2\pi n, \alpha_c + 2\pi n)$
- $F(z) \in \mathbb{R}$ for $z \in \mathbb{R} \notin (-\alpha_c + 2\pi n, \alpha_c + 2\pi n)$
- $F(\alpha \pm i\epsilon) = \frac{2}{\lambda}\sin\alpha \mp 2\pi i \rho(\beta)$

  since simple poles of $\cot\left(\frac{z-\beta}{2}\right)$

  at $z = \beta + 2n\pi$
- $F(z) \to \pm 1$ as $|z| \to \infty$ except along real axis

  From $\cot z \sim \dfrac{e^{2iz}+1}{e^{2iz}-1}$ & $\int \rho(\beta)d\beta = 1$

Should all be familiar if we remember constraints on resolvent $W(\omega)$ from Brandon's talk & from Hermetian 1-matrix model.

Unique soln:

$$\rho(\alpha) = \begin{cases} \frac{2}{\pi\lambda}\cos\frac{\alpha}{2}\left(\frac{\lambda}{2} - \sin^2\frac{\alpha}{2}\right)^{1/2} & \lambda \le 2 \\ & |\alpha| < 2\sin^{-1}\left(\frac{\lambda}{2}\right)^{1/2} \\ \frac{1}{2\pi}\left(1 + \frac{2}{\lambda}\cos\alpha\right) & \lambda \ge 2 \\ & |\alpha| \le \pi \end{cases}$$

$\lambda \to 0 \qquad \alpha \approx \mathcal{O}(\sqrt{\lambda})$

$$\rho(\alpha) \to \sim \frac{1}{\pi}\left(1 - \frac{\alpha^2}{2\lambda}\right)^{1/2} \qquad |\alpha| \le \sqrt{2\lambda}$$

$$S_{eff} \sim \frac{2}{\lambda}\sum \cos\alpha_i + \frac{1}{N}\sum_{i \ne j} \log\left|\sin\frac{\alpha_i - \alpha_j}{2}\right|$$

strong coupling: Wilson action neglected
Δ^2 term dominates

weak " : Wilson action dominates

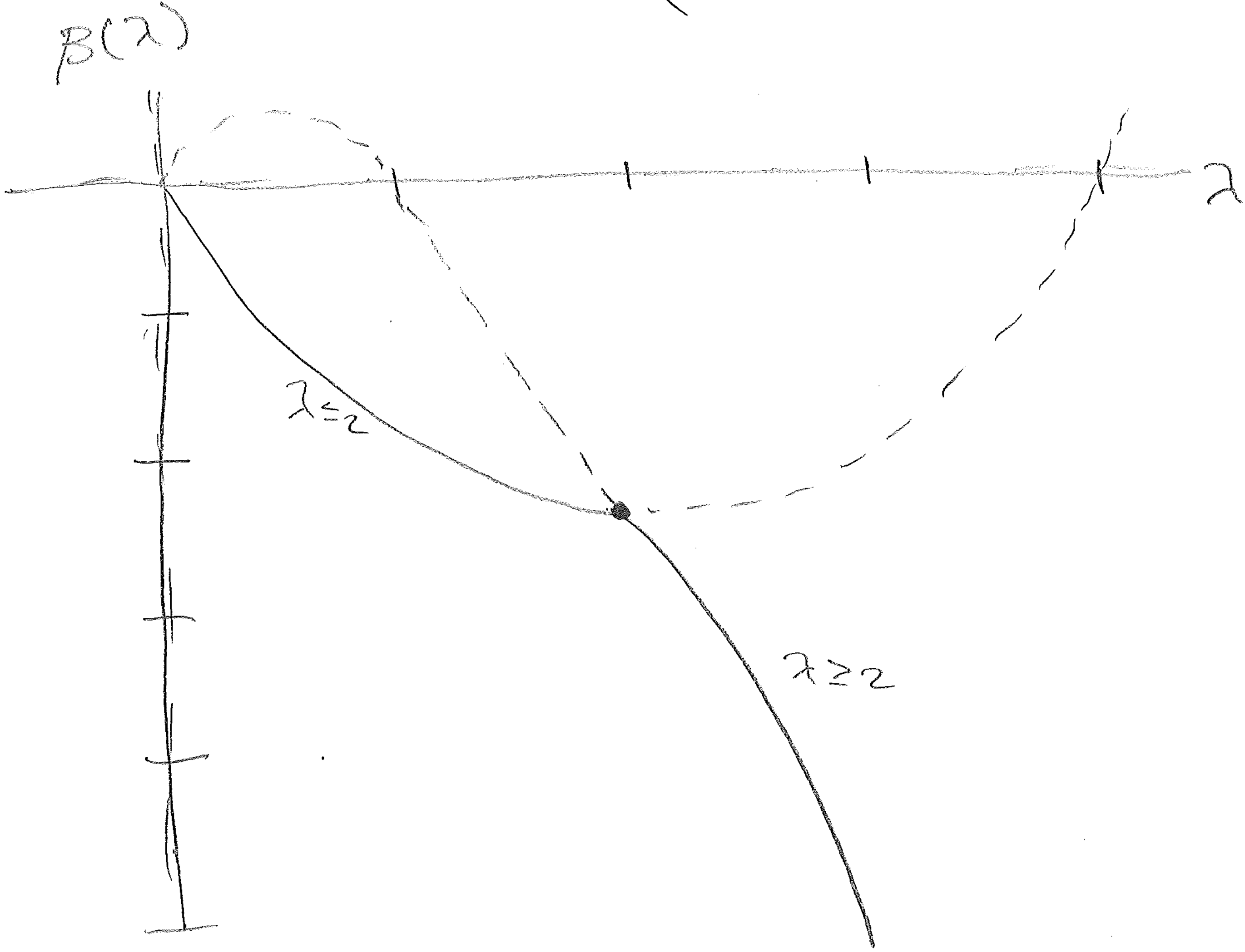
Phase transition when eigenvalues fill unit circle

3rd order:

$$-E_0(\lambda) = -\frac{Fa^2}{\lambda N^2} = \begin{cases} 1/\lambda^2, & \lambda \geq 2 \\ \frac{2}{\lambda} + \frac{1}{2}\log\frac{\lambda}{2} - \frac{3}{4}, & \lambda \leq 2 \end{cases}$$

<u>β - Function</u>

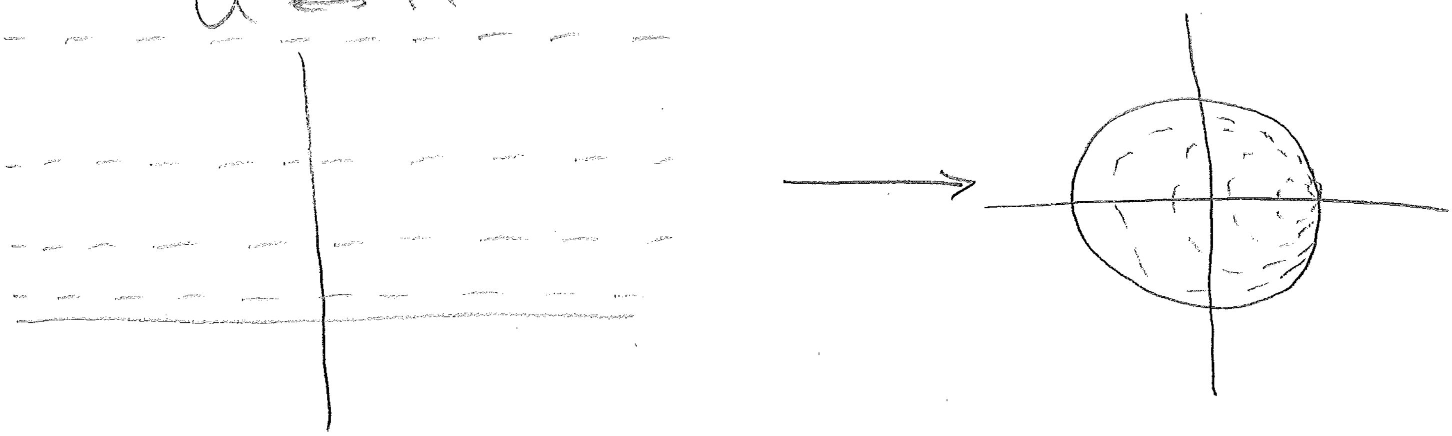
$$-\beta(\lambda) \equiv -\frac{\partial\lambda(a)}{\partial \log a} = \begin{cases} 2\lambda\log\lambda & \lambda \geq 2 \\ 2(4-\lambda)\log\frac{4}{4-\lambda} & \lambda \leq 2 \end{cases}$$



Cayley Map

$$U = \frac{i-H}{i+H} \qquad dU = \frac{dH}{\det(1+H^2)}$$

$$U \Leftrightarrow H$$



Cayley Map takes G-W phase transition to a Hermetian matrix model phase transition.

- compact support (confining potential)
- infinite support (log external potential)

Remarks

- Phase transtion in finite volume
- Thermodynamic limit from $N \to \infty$
  infinite degrees of freedom.
  even in finite volume
- Typical 2nd-order phase transition
  at $\beta = 0$ from naive extrapolation
  of strong coupling
- So large N is crucial for actual physics here

  finite N $\to$ F, Z, etc. all analytic
  functions of λ
  $0 < \lambda \leq \infty$

  As $N \to \infty$ zeros of Z coalesce
  to form boundary $\lambda > 2$ to $\lambda < 2$

  Finite $N \to$ none of these zeros lie on
  real axis and will not be dense.

If In 4D theory phase transion would exist
in $N \to \infty$ limit. But for finite N
large enough, one would expect sharp transition
at $\lambda \simeq \lambda_c$ from weak-to-strong behavior