PTJC Talk #1: Intro to Solitons

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(1) Series overview

This term we are working through sections of *Aspects of Symmetry.* A theme of several talks is the quantum descendants of interesting classical solutions to systems: we begin with two talks on solitons (or lumps, per Coleman), then move to ∼4 talks on instantons, rounding out the term with a handful of grab bag topics TBD. Today, I'm going to introduce solitons in a classical context and illustrate their relationship to the topology of a theory's vacuum space, from which will emerge a concept of conserved topological charges.

(2) What's a soliton and why do we care?

So first off, what are these things we're looking for? Given a classical field theory, we look for solutions which are **spatially localized** and **have constant shape**. Physically, we want a finite energy with a localized density that doesn't spread out over time, which is not the typical behavior of a solution to a nonlinear system. Most perturbations will eventually have their energy spread out over the whole domain, and only soliton solutions avoid this through the interaction of dispersion and non-superpositive effects. It seems like we may be able to look at this as a kind of particle, but that's left to next week. Let's start with a classic classical example.

(3) Korteweg–de Vries equation

$$\frac{\partial \phi}{\partial t} + \frac{\partial^3 \phi}{\partial x^3} - 6\phi(t,x)\frac{\partial \phi}{\partial x} = 0 \tag{3.1}$$

This equation originated to model gravity waves in shallow water, which is indeed the system in which solitons were first described. The equation is of mathematical interest as it is an integrable nonlinear system. To see the soliton solutions in particular, let us make an ansatz

$$\phi(t,x) = f(x - ct - a), \tag{3.2}$$

which describes a wave of constant shape translating with speed c. We can recast KdV as an ODE in a single parameter τ (dimensionally a length, but what's a dimension to a mathematician?):

$$-cf'(\tau) + f'''(\tau) - 6f(\tau)f'(\tau) = 0 \tag{3.3}$$

$$\text{integrate}$$

$$-cf(\tau) + f''(\tau) - 6f^2(\tau) = C. \tag{3.4}$$

Rearranging a little we can get

$$f''(\tau) = -\frac{\mathrm{d}}{\mathrm{d}\tau}\left(-2f^3(\tau) - \frac{c}{2}f^2(\tau) - Cf(\tau)\right), \tag{3.5}$$

which looks like a Newtonian equation of motion for a trajectory f in the potential

$$U(f) = -f(2f^2 + \frac{c}{2}f + C), \tag{3.6}$$

which is particularly nice when $C = 0$. We can see that there is an initial condition where $f(\tau)$ starts on the hill as $\tau \to -\infty$, rolls to the left, then rolls back to the top of the hill as $\tau \to \infty$. Considering a fixed position, this will look like a localized pulse passing by before the height returns to zero. Specifically,

$$f(\tau) = -\frac{c}{2}\text{sech}^2\left(\frac{\sqrt{c}}{2}(\tau - a)\right). \tag{3.7}$$

We can see that this does everything we want. Moreover, we can actually construct n-soliton solutions with different speeds and directions. When they meet, they pass through each other unchanged up to a phase shift, which is an almost-superpositive behavior that we cannot assume in a nonlinear system, so its appearance in this special solution is interesting.

(4) 1+1 dimensional scalars

Let's now consider a classical theory of a single real scalar field in $1 + 1$ dimensions. Here we will think of solitons as having localized *energy density* of constant shape, rather than the field itself, and also require finite total energy. We will look at first for time-independent solutions, knowing that we can boost them. Let our theory have the Lagrangian density

$$\mathcal{L} = -\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{8}\lambda(\phi^2 - v^2)^2, \tag{4.1}$$

which gives the ϕ particle a mass $m = \sqrt{\lambda}v$. This corresponds to two classical vacuum solutions $\phi(x) = \pm v$: uniformity kills kinetic term, $\pm v$ are minima of potential. For energy density to be localized, we want the potential density to fall in either direction, so the solution must tend toward a vacuum value at either end. If we assign both ends the same vacuum value, we will get a vacuum out, so we are interested in a solution that goes (let's say) from $-v$ at $-\infty$ to $+v$ at $+\infty$. This is our first example of a choice of map from the spatial boundary set to the space of vacuum values: this theme will recur, and will have a more interesting array of choices, in higher dimensions.

Let's minimize the total energy subject to these boundary conditions. We know it will be nonzero, because the solution has to switch between vacua at some point, and that costs energy.

$$E = \int \mathrm{d}x \left[\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\phi'^2 + \frac{1}{8}\lambda(\phi^2 - v^2)^2\right] = \int \mathrm{d}x \left[\frac{1}{2}\left(\phi' - \sqrt{\frac{\lambda}{4}(\phi^2 - v^2)^2}\right)^2 + \phi'\sqrt{\frac{\lambda}{4}(\phi^2 - v^2)^2}\right] \tag{4.2}$$

We have set $\dot{\phi}$ to zero to get a time-independent solution, then completed the square in a way that lets us replace $\phi'\mathrm{d}x \to \mathrm{d}\phi$ in the second term. Our chosen boundary conditions appear as the limits of integration:

$$E = \int_{-\infty}^{\infty}\mathrm{d}x \left[\frac{1}{2}\left(\phi' - \sqrt{\frac{\lambda}{4}(\phi^2 - v^2)^2}\right)^2\right] + \int_{-v}^{v}\mathrm{d}\phi\sqrt{\frac{\lambda}{4}(\phi^2 - v^2)^2}, \tag{4.3}$$

and

$$\int_{-v}^{v} \mathrm{d}\phi \sqrt{\frac{\lambda}{4}(\phi^2 - v^2)^2} = \frac{2}{3}\frac{m^2}{\lambda}m. \tag{4.4}$$

Thus, we have reduced the energy to a constant term plus a non-negative integral. This gives us what is known as a Bogomolny or BPS bound (depending who you ask) on the energy, and we can find the special state that saturates this bound by setting the remaining integrand to zero, which gives an ODE for $\phi(x)$ whose solution is

$$\phi(x) = v \tanh\left[\frac{m}{2}(x - x_0)\right]. \tag{4.5}$$

We can see that all terms of the Hamiltonian density tend to zero at either end, giving a localized lump and finite total energy. If we had imposed a trivial map, we would have gotten out a constant vacuum solution. Boosting this solution by a velocity β gives

$$\phi(t, x) = v \tanh\left(\frac{\gamma m}{2}(x - x_0 - \beta t)\right) \tag{4.6}$$

$$E = \gamma \frac{2}{3}\frac{m^2}{\lambda}m \tag{4.7}$$

$$p = \gamma\beta \frac{2}{3}\frac{m^2}{\lambda}m, \tag{4.8}$$

where we see how we can associate this solution with an on-shell particle.

(5) 2+1 dimensional scalars — vortices

We have pretty much exhausted one space dimension, so it's time to step up to 2+1. The easiest way we can do this is to just take our soliton and stretch it into the new x^2 direction without alteration. This is no longer localized in all available space directions, and no longer has finite energy, but is an interesting physical structure: the domain wall, whose surface tension is related to our previous rest energy. However, we've lost most of what interested us about the 1-D soliton, including the topological correspondence between vacua and the the spatial boundary, so let's step up our Lagrangian to that of a complex scalar field

$$\mathcal{L} = -\partial^\mu \phi^\dagger \partial_\mu \phi - \frac{\lambda}{4}\left(\phi^\dagger \phi - v^2\right)^2. \tag{5.1}$$

The classical vacuum values here now live on a circle $\phi(x) = ve^{i\alpha}$ with a real parameter α, and our spatial boundary is also topologically a circle. The argument still goes through that we want the field to take a vacuum value as we go to ∞ in any direction, and as well that we are not interested in trivial maps. Our choice of map is now much richer, however! We're not barbarians, so the map should still be continuous, and mapping one circle to another continuously suggests the family of maps

$$e^{i\alpha(\theta)} = e^{in\theta} \tag{5.2}$$

where $n \in \mathbb{Z}$ and θ is the polar coordinate giving position around the boundary. For obvious reasons, we call n the winding number. There are of course other choices, but we will argue that these are the important ones. For now, let's make an ansatz (still time-independent)

$$\phi(r, \phi) = vf(r)e^{in\theta}, \tag{5.3}$$

where $f(r \to \infty) = 1$ imposes our boundary map and $f(r \to 0) = 0$ deals with the gradient singularity at the origin. Consider doing a similar calculation to before:

$$E = \int \mathrm{d}^2x \left(|\vec{\nabla}\phi|^2 + \frac{\lambda}{4}(\phi^\dagger\phi - v^2)^2 \right), \tag{5.4}$$

$$|\vec{\nabla}\phi|^2 = v^2 f'(r)^2 + n^2 v^2 \frac{f(r)^2}{r^2}. \tag{5.5}$$

As we integrate the second term against $r\mathrm{d}r$ and approach infinity, the limiting behavior of $f(r)$ makes the integral approach something logarithmically divergent, while the first term is finite. We have actually run up against a mathematical constraint: Derrick's theorem states that with scalar fields only in $D + 1$ dimensions, the only smooth, time-independent, finite-energy solutions of our desired form are vacuum solutions if $D \geqslant 2$.

To get around Derrick we need more fields. It is possible to use fermion fields, but Coleman considers this a mess, so we'll shy away. It turns out that if instead we keep our current Lagrangian and promote its global $U(1)$ symmetry to a local symmetry by introducing a gauge field A^μ, we can find an interesting solution. The Lagrangian becomes

$$\mathcal{L} = -\left[\partial^\mu\phi - ieA^\mu\phi\right]^\dagger \left[\partial_\mu\phi - ieA_\mu\phi\right] - \frac{\lambda}{4}(\phi^\dagger\phi - v^2)^2 - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}, \tag{5.6}$$

where we can see that the covariant derivative will furnish the term we need to cancel out the log divergence in the gradient if we pick the right gauge. The gauge symmetry is spontaneously broken, giving the vector particle a mass $m_v = ev$ while the scalar particle has mass $m_s = \sqrt{\lambda}v$; these quantities will pop up later. Let us pick A^μ so that

$$\lim_{r\to\infty} \vec{A}(r,\theta) = \frac{i}{e} U \vec{\nabla} U^\dagger, \tag{5.7}$$

where $U(\theta)$ is the map that assigns vacuum values to the boundary. Note that since all vacuum values are v times a phase, this potential is a $U(1)$ gauge transformation of a single vacuum state around the boundary. If U has the form $e^{in\theta}$ as above, then $\lim \vec{A} = \frac{n}{er}\hat{\theta}$. It turns out, however, that as we probe inward from the boundary we cannot everywhere write the solution as a well-defined gauge transformation of a vacuum (happily, since that would rob us of our solitons). There must exist some number of defects, which cost energy, and these are the solitons.

Let us make an ansatz:

$$\phi(r,\theta) = v f(r) U(\theta); \qquad \vec{A}(r,\theta) = \frac{i}{e} a(r) U(\theta) \vec{\nabla} U^\dagger(\theta) \tag{5.8}$$

where $f(r)$ and $a(r)$ are real functions with the same b.c. as $f(r)$ earlier, and we have left the Us general for reasons that will become clear. We can derive a magnetic flux from the vector potential via Stokes' theorem:

$$\Phi = \int_\partial \mathrm{d}\vec{\ell} \cdot \vec{A} = \frac{i}{e} \lim_{r\to\infty} a(r) \int_0^{2\pi} \mathrm{d}\theta U \frac{\mathrm{d}U^\dagger}{\mathrm{d}\theta}. \tag{5.9}$$

We must now define

$$n = \frac{i}{2\pi} \int_0^{2\pi} \mathrm{d}\theta U \frac{\mathrm{d}U^\dagger}{\mathrm{d}\theta}, \tag{5.10}$$

which is the winding number of *any* smooth map $S^1 \to S^1$. It is easy to check for our earlier example $e^{in\theta}$, but it turns out that n is invariant under any smooth deformation (homotopy) of U, so those $e^{in\theta}$ are representatives of homotopy classes which partition the space of valid maps. This observation gives us

$$\Phi = \frac{2\pi n}{e}, \tag{5.11}$$

so we have astoundingly produced a *quantized* magnetic flux depending on a topological charge which is the same for any map continuously deformable to $e^{in\theta}$. These solutions are Nielsen–Olesen vortices defined by a conserved quantum number which can merge and decay when energetically favorable.

We can play the same BPS energy minimization game as before and get a strict bound $E > 2\pi v^2|n|$ whenever $m_v^2/m_s^2 > 1$ (permitting breakup of high-$|n|$ solitons into $n = 1$ solitons), as well as coupled ODEs for the saturating solution:

$$f'' + \frac{f'}{\rho} - \frac{n^2 f}{\rho^2}(1-a)^2 + \frac{1}{2}\beta^2(1-f^2)f = 0 \tag{5.12}$$

$$a'' - \frac{a'}{\rho} + (1-a)f^2 = 0. \tag{5.13}$$

There is no closed-form solution, but $f \sim \rho^n$ and $a \sim \rho^2$ near the origin and both fall exponentially to 1 near the boundary. As before, we can translate and boost these to make whatever moving configurations we want.

(6) A glimpse of 3+1 and monopoles

As when we went from 1+1 to 2+1, one path is to extend the vortices into Nielsen–Olesen strings, which can be bent and joined to themselves to make cosmic strings, which occur in the early universe in some theories. Also as before, though, this is not the most exciting solution. There is not much enlightening in the detail of the calculation, but the outline follows.

We extend to three real scalar fields with a 2-sphere of vacuum values and consider maps between 2-spheres (for which we can also define a winding number, maybe better called the degree in this case). To cancel divergences, we now need an $SU(2)$ gauge field. The symmetry is spontaneously broken to $U(1)$, producing two massive W fields and a massless photon field. Picking a gauge appropriate to our space–vacuum map, we can again compute a magnetic flux, but now out of the volume of the universe rather than just the plane:

$$\Phi = -\frac{4\pi n}{e}, \tag{6.1}$$

where n is the winding number. We have again pulled a quantization condition out of a classical theory, but now it describes a quantized magnetic charge: a monopole! (This close to the Dirac quantization condition.) There is an ansatz for the form of these monopoles due to 't Hooft and Polyakov, which may be discussed next week, and a corresponding Bogomolny bound which makes the monopole extremely heavy compared to the Ws. The bound can again be saturated by a BPS solution

$$a(\rho) = 1 - \frac{\rho}{\sinh \rho} \tag{6.2}$$

$$f(\rho) = \coth \rho - \frac{1}{\rho}. \tag{6.3}$$

Next week will explore Coleman's treatment of the quantum correspondents to these soliton solutions, maybe in the context of the Sine–Gordon equation.