Tyler Blanton

PTJC Talk #3:

Tunneling + Instantons in QM

Outline:
0| Motivation
1| Transition
2| Double well
3| Periodic potential

Cheat sheet:

$$\langle x_f | e^{-HT/\hbar} | x_i \rangle = e^{-E_0 T/\hbar} \langle x_f | 0 \rangle \langle 0 | x_i \rangle \left[1 + O\left(e^{-(E_1 - E_0)T/\hbar}\right) \right]$$

$$= N e^{-S_0/\hbar} \left[\det\left(-\partial_\tau^2 + V''(\bar{x})\right) \right]^{-1/2} \left[1 + O(\hbar) \right]$$

$$N \left[\det\left(-\partial_\tau^2 + \omega^2\right) \right]^{-1/2} \xrightarrow[T \to \infty]{} \sqrt{\frac{\omega}{\pi\hbar}}\, e^{-\omega T/\hbar}$$

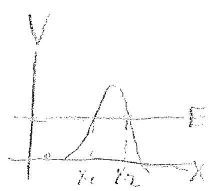
Motivation: PT not always good enough

$$\mathcal{L} = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \frac{1}{2} m^2 \varphi^2 - g^2 \varphi^4$$

$$\varphi \to \varphi' = g\varphi \;:\; \mathcal{L} = \frac{1}{g^2} \left[\partial_\mu \varphi' \partial^\mu \varphi' - \frac{1}{2} m^2 \varphi'^2 - \varphi'^4 \right]$$

- Classically: g is irrelevant (not in EoM)
- QM: $g^2\hbar$ is relevant, e.g., $Z \sim e^{-S/\hbar} \Rightarrow \frac{\mathcal{L}}{\hbar} \sim \frac{1}{g^2\hbar}$ relevant
  - $\hbar \to 0 \Leftrightarrow g \to 0$
  - $\Rightarrow$ PT? not always!

Tunneling / WKB:



$$|T(E)| = e^{-\frac{1}{\hbar} \int_{x_1}^{x_2} dx \sqrt{2(V-E)}} \left[1 + O(\hbar) \right]$$

- $e^{-1/\hbar}$ decays faster than any $\hbar^n$ (Taylor exp. has RoC $= 0$)

  $\Rightarrow$ PT useless for tunneling!

- Analogous situations in QFT where PT fails
- Solution: study propagator
  - relate to QFT $\Rightarrow$ PI
  - compute PI $\Rightarrow$ Euclidean spacetime (1+1 for this talk)

Formalism

Euclidean time

$$\langle x_f | e^{-HT/\hbar} | x_i \rangle = N \int \mathcal{D}x(\tau)\big|_{BC}\, e^{-S[x(\tau)]/\hbar}$$

- $S[x(\tau)] = \int_{-T/2}^{T/2} d\tau \left[\frac{\dot{x}^2}{2} + V(x) \right]$ = Eucl. action, $\dot{x} = \frac{dx}{d\tau}$, $m=1$
- B.C: $x(-\frac{T}{2}) = x_i$, $x(\frac{T}{2}) = x_f$
- $\int \mathcal{D}x(\tau)\big|_{BC}$ = integral over all paths satisfying BC
- N = normalization (won't worry about)
- T = Euclidean time it takes for pt. to go from $x_i \to x_f$

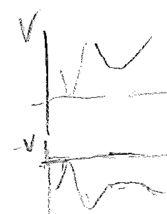
$\boxed{+V \text{ in action is super important!}}$

denotes "classical" soln to EoM

$$S = \int_{-T/2}^{T/2} d\tau\, L, \quad \delta S = 0 \Rightarrow \boxed{\ddot{\bar{x}} = +V'(\bar{x})} \quad (\text{minimizes } S[x])$$

$\Rightarrow$ constant of motion is $\boxed{E = \frac{\dot{\bar{x}}^2}{2} - V(\bar{x})}$

Normally (Minkowski): $\ddot{x}_M = -V'(x_M)$



$\therefore$ $\boxed{\text{Particle sees inverted potential in Euclidean spacetime}}$

Compute P.I.

Saddle pt. approx: $\hbar$ small $\Rightarrow$ only smallest $S[x]$ contribute much

$\Rightarrow$ only paths close to $\bar{x}(\tau)$ important

small!

Let $x(\tau) = \bar{x}(\tau) + y(\tau)$, $y(\pm\frac{T}{2}) = 0$

don't write this!

$$\Rightarrow S[x] = \underbrace{S[\bar{x}]}_{S_0} + \frac{1}{2}\int_{-T/2}^{T/2} d\tau\, y\left[-\partial_\tau^2 + V''(\bar{x})\right] y + O(y^3)$$

don't write!

linear term vanishes due to EoM

Tyler Bleh

$$\Rightarrow \int \mathcal{D}x\Big|_{BC} e^{-S/\hbar} = e^{-S_0/\hbar} \underbrace{\int \mathcal{D}y\Big|_{BC} e^{-\frac{1}{2\hbar}\int_{-T/2}^{T/2} d\tau\, y\left[-\partial_\tau^2 + V''[\bar{x}]\right]y}}_{\equiv I} \left[1+O(\hbar)\right]$$

- I looks Gaussian, but ∂_τ^2 makes it weird
- Solution: Decompose $y = \sum_n c_n y_n$, where $\left[-\partial_\tau^2 + V''[\bar{x}]\right]y_n = \lambda_n y_n$ (just like usual QM)

  $\int_{-T/2}^{T/2} d\tau\, y_m y_n = \delta_{mn}$ (orthonormal)

$$\Rightarrow \int \mathcal{D}y = \prod_n \int \frac{dc_n}{\sqrt{2\pi\hbar}}, \qquad S = S_0 + \frac{1}{2}\sum_n \lambda_n c_n^2$$

just point to

$$\Rightarrow \boxed{\langle x_f | e^{-HT/\hbar} | x_i \rangle = N e^{-S_0/\hbar} \overbrace{\left[\det\left(-\partial_\tau^2 + V''(\bar{x})\right)\right]^{-1/2}}^{=\prod_n \lambda_n^{-1/2}} \left[1+O(\hbar)\right]} \quad (1)$$

$$\text{Also: } \langle x_f | e^{-HT/\hbar} | x_i \rangle = \langle x_f | 0 \rangle \langle 0 | x_i \rangle\, e^{-E_0 T/\hbar}\left[1 + O\left(e^{-(E_1-E_0)T/\hbar}\right)\right] \quad (2)$$

Plan:
- Use (1) to compute prop.
- Take $T \to \infty$ to make (2) valid
- Read off E_0 + $\langle x_f|0\rangle\langle 0|x_i\rangle$ = overlap of pos. + energy eigenstates
- If there are multiple stationary pt. sol'ns $\bar{x}_n$ w/ action S_n, then (1) must include a $\sum_n$

Simple case: SHO

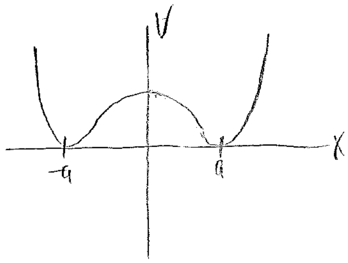
$V''(\bar{x}) = \omega^2 \rightarrow$ can evaluate $N\left[\det\left(-\partial_\tau^2 + \omega^2\right)\right]^{-1/2}$ exactly

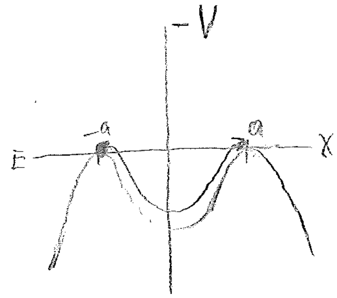
Result: $\boxed{N\left[\det\left(-\partial_\tau^2 + \omega^2\right)\right]^{-1/2} = \sqrt{\frac{\omega}{\pi\hbar}}\left[2\sinh \omega T\right]^{-1/2} \xrightarrow{T\to\infty} \sqrt{\frac{\omega}{\pi\hbar}}\, e^{-\omega T/2}}$

$S_0 = 0 \Rightarrow e^{-S_0/\hbar} = 1 \Rightarrow \langle x_f | e^{-HT/\hbar} | x_i \rangle =$

$\Rightarrow E_0 = \frac{\hbar\omega}{2}$ ✓, $\left|\langle x=0 | n=0\rangle\right|^2 = \sqrt{\frac{\omega}{\pi\hbar}}$ ✓

Real test: Double well





$$V(x) = \frac{\omega^2}{8a^2}(x^2 - a^2)^2$$

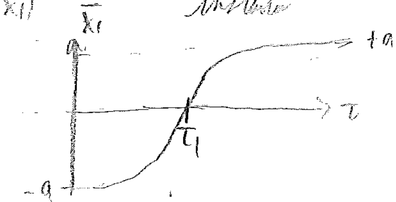
$$\Rightarrow \boxed{V''(\pm a) = \omega^2}$$

- $S = \frac{\dot{x}^2}{2} + V$ minimized when $x = \pm a$, $\dot{x} = 0$ $(S = 0)$
- However, this also means $E = \frac{\dot{x}^2}{2} - V = 0$

$\Rightarrow$ particle is sitting on top of a hill!

- If particle at $-a$ gets nudged to the right at all (≈ 0 energy cost), it will roll all the way (infinitesimally slow) to $+a$
- EoM for this process: $E = \frac{\dot{\bar{x}}_1^2}{2} - V(\bar{x}_1) = 0 \Rightarrow \dot{\bar{x}}_1 = \sqrt{2V(\bar{x}_1)}$

$$\Rightarrow \boxed{\bar{x}_1(\tau) = a \tanh \frac{\omega(\tau - \tau_1)}{2}} \quad \bar{x}_1(\tau_1) = 0$$

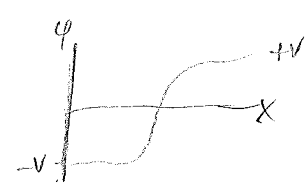


This solution is localized to an "instant" τ_1 in Eucl. time

- width $\sim \frac{1}{\omega}$ (property of the potential)

We call it an instanton!

- Corresponds to tunneling from $-a \to +a$ in original (Minkowski) picture w/ $+V$
- Graph looks like soliton, but completely different situation



Solitons:
- Minkowski
- finite E soln
- local in space (like real particle)

Instantons:
- Euclidean
- finite S
- local in "time" (not like real pt.)
- describes tunneling

Tyler Blanton

Check: Is this result consistent w/ WKB?

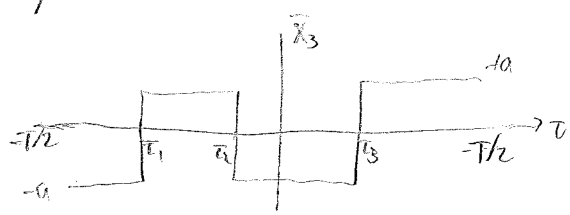
$$S_0 = \int_{-T/2}^{T/2} d\tau \left[\frac{\dot{\bar{x}}_1^2}{2} + V(\bar{x}_1) \right] \overset{\text{use } \dot{\bar{x}} = \sqrt{2V(\bar{x})}}{=} \cdots = \underline{\int_{-a}^{a} dx \sqrt{2V(x)}} \Rightarrow e^{-S_0/\hbar} = e^{-\frac{1}{\hbar}\int_{-a}^{a} dx \sqrt{2V(x)}} \checkmark$$

So, we can roll from $-a \to +a$ for no energy using 1 instanton...

$\Rightarrow$ It can roll back & forth n times & still satisfy $E = \frac{\dot{\bar{x}}_n^2}{2} - V(\bar{x}_n) = 0$!

$\therefore$ To compute the true propagator $\langle +a | e^{-HT/\hbar} | -a \rangle$, we must sum over all odd n, where $n = \#$ insts. $+ \#$ anti-instans.

Since each instn. is localized to $\Delta\tau \sim \frac{1}{w}$, if we take $T \gg \frac{1}{w}$ (where $x(\mp \frac{T}{2}) = \mp a$), then the instns/anti-instns. will be widely separated (can address at end if necessary)



$\Rightarrow \underline{S_n = n S_0}$

Need to integrate over $\tau_1, \dots, \tau_n$:

$$\int_{-T/2}^{T/2} d\tau_1 \int_{\tau_1}^{T/2} d\tau_2 \cdots \int_{\tau_{n-1}}^{T/2} d\tau_n = \frac{T^n}{n!}$$

The vast majority of the time, $\bar{x}_n(\tau) \approx \pm a$ & $V''(\bar{x}_n) \approx w^2$

$$\Rightarrow \text{expect } N[\det(-\partial_\tau^2 + V''(\bar{x}_n))]^{-1/2} = N[\det(-\partial_\tau^2 + w^2)]^{-1/2} K^n \xrightarrow[T\to\infty]{} \sqrt{\frac{w}{\pi\hbar}}\, e^{-wT/\hbar} K^n,$$

where K is the correction due to a single instanton

- Consistent w/ naive degenerate G.S. energy $\frac{\hbar w}{2}$ at bottom of each well

$\Rightarrow$ $\boxed{\text{expect } K \text{ to split degeneracy}}$

$$\Rightarrow \langle +a|e^{-HT/\hbar}|-a\rangle = \sqrt{\frac{\omega}{\pi\hbar}}\, e^{-\frac{\omega T}{2}} \sum_{n\,\text{odd}} \frac{(KTe^{-S_0/\hbar})^n}{n!}\left[1+O(\hbar)\right]$$

$$\langle \pm a|e^{-HT/\hbar}|-a\rangle = \frac{1}{2}\sqrt{\frac{\omega}{\pi\hbar}}\left\{\exp\left[-\underbrace{\left(\frac{\omega}{2}-Ke^{-S_0/\hbar}\right)}_{=E_-/\hbar}T\right] \mp \exp\left[-\underbrace{\left(\frac{\omega}{2}+Ke^{-S_0/\hbar}\right)}_{=E_+/\hbar}T\right]\right\}$$

$$\Rightarrow \boxed{E_\mp = \frac{\hbar\omega}{2}\mp\hbar K e^{-S_0/\hbar}}$$ $\Rightarrow$ degeneracy lifted ✓

- For $\langle -a|e^{-HT/\hbar}|-a\rangle$, same procedure except $n=$ even
- We can even check the parity of our sol'ns $|E_\mp\rangle$:

$$\langle +a|E_-\rangle\langle E_-|-a\rangle = +\frac{1}{2}\sqrt{\frac{\omega}{\pi\hbar}}$$ $\Rightarrow$ spatially even ✓ (true GS)

$$\langle +a|E_+\rangle\langle E_+|-a\rangle = -\frac{1}{2}\sqrt{\frac{\omega}{\pi\hbar}}$$ $\Rightarrow$ spatially odd ✓ (1st excited state)

- Haven't even needed K!

Computing K:

- Until now, we've implicitly assumed $-\partial_\tau^2 + V''(\bar{x})$ had only positive eival's λ_n
- Double well has zero mode $n=0$, $\lambda_0 = 0$ due to time-transln. invar.

$$\Rightarrow \left[\det(-\partial_\tau^2+V''(\bar{x}))\right]^{-1/2} = \prod_n \lambda_n^{-1/2} = \infty \ ?$$ no!

- P.I. $\longrightarrow \prod_n \int \frac{dc_n}{\sqrt{2\pi\hbar}}\, e^{-\frac{1}{2\hbar}\lambda_n c_n^2} = \left[\int \frac{dc_0}{\sqrt{2\pi\hbar}}\right]\left[\overset{\text{only } \lambda_n>0}{\det'}(-\partial_\tau^2+V''(\bar{x}))\right]^{-1/2}$

$$\left[-\partial_\tau^2+V''(\bar{x}_1)\right]y_n = \lambda_n y_n$$

- Make sense of $\int\frac{dc_0}{\sqrt{2\pi\hbar}}$: $y_0 = S_0^{-1/2}\dot{\bar{x}}_1$, $x = \bar{x}_1 + y = \bar{x}_1 + c_0 y_0 + \sum_{n=1}^{\infty} c_n y_n$

under $\tau_1 \to \tau_1 + d\tau_1$

$$\Rightarrow dx = \dot{\bar{x}}_1\, d\tau_1 = y_0\, dc_0 \Rightarrow dc_0 = \frac{\dot{\bar{x}}_1\, d\tau_1}{y_0} = S_0^{1/2}\, d\tau_1 \Rightarrow \boxed{\int\frac{dc_0}{\sqrt{2\pi\hbar}} = \sqrt{\frac{S_0}{2\pi\hbar}}\int d\tau_1}$$

$$\Rightarrow \det(-\partial_\tau^2+V''(\bar{x}_1))^{-1/2} \underset{\text{int. over } c_0}{=} \sqrt{\frac{S_0}{2\pi\hbar}}\,\det'(-\partial_\tau^2+V''(\bar{x}_1)) \Rightarrow \boxed{K = \sqrt{\frac{S_0}{2\pi\hbar}}\left|\frac{\det(-\partial_\tau^2+\omega^2)}{\det'(-\partial_\tau^2+V''(\bar{x}_1))}\right|^{1/2}}$$