PTJC - Topological phases of Matter & the Toric code.

The topic for this quarter is topological phase of matter.

Now, you may ask yourself why the particle theory journal club is attacking a subject that is usually reserved for condensed matter physicists. Well, the answer is because the ideas and formalism that underlies the theory of topological phases are cropping up in a variety of areas of theoretical physics. As such, TPM and TFT's provide a unifying thread (haha) throughout the theoretical physics community. This is true globally* as well as localy. To illustrate this, let me just talk about the theory research being conducted in this department, and perhaps it will explain why it was so easy for the members of PTJC to decide on this topic. I'll start with the obvious one:

↳ no propagating DoF

* normally reserved for strongly correlated systems.

I. Condensed Matter.

(i) Thouless & de Nijs, - TKNN in the 80's
- nobel prize in 2016

(ii) Lukasz, Sujeet & Tyler
- classifying topological phases → lots of what we describe.

(iii) Boris, Anton & Mike
- gapless topological phases - weyl & Dirac semi-metals (see also Jing-Han's lab)
- characterized by topological defects in momentum space

II Particle Theory

(i) Andreas, Andrew & Kyle

- Dualities b/w CS-matter theories
- TFT's resulting from mass deformations

(ii) Larry, Aleksey & Sri

- topological order in high density QCD
- phases of QCD - TFT's living on domain walls.

(iii) John

- combinatorial geometry & quantum gravity
- math stuff - tensor categories & invariants on 3-manifolds

III Nuclear Theory

(i) Martin, David, Natalie & Jesse

- Quantum computing for Lattice FT
- fault-tolerant/topological quantum computing
  - start thinking about these things.

So clearly the ideas we are talking about in this quarter will be relevent to a non-trivial portion of the theorists in this department. So now that I've told you why we should care about those things, lets look at the kinds of theories that will give rise to topological phases. Note, we are in 2+1 d

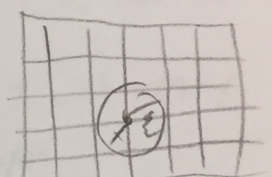
This is most easily done in the Hamiltonian formalism, where the theories involved should really be defined as living on some kind of lattice $\mathcal{L}$. We then demand the Hamiltonian satisfies the following properties:

cond-mat/0404617
String-net condensation

(i) <u>Locality</u>: $H = \sum_x H_x$ where $x \in$ "lattice" $\mathcal{L}$

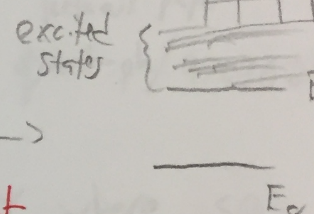
and $H_x = Y_x \otimes \mathbb{1}$ such that Y_x acts in a region of finite length ξ about the point x.

we demand $\lim_{\mathcal{L}\to\infty} \xi < 0$



(ii) <u>Gapped</u>: $\exists\ \Delta$ such that

$E_1 - E_0 = \Delta$ and $\lim_{\mathcal{L}\to\infty} \Delta > 0$ $\rightarrow$



• why demand gapped?

(iii) <u>Ground state degeneracy</u>:

$\exists$ finite # of eigenstates with energy $E < \Delta$

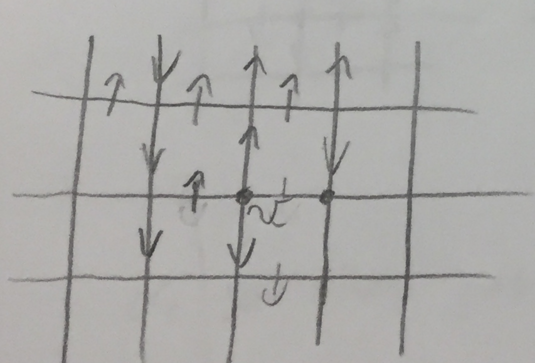
: $|1\rangle, \ldots |N_\xi\rangle$, where N_ξ depends on the topology of the 2D spatial surface on which the theory is defined. - # of handles & # of punctures.

This last point is super important. This is usually owed to the fact that in topological phases, the physical d.o.f. tend to orient themselves in string-like configurations.

and so one can obtain distinct states by seeing how these strings wrap around non-trivial cycles of the manifold. The general theory for this phenomena will be discussed in a few weeks by Kyle. But for now, I will illustrate these ideas with a simple, solvable model that has these three features. This goes by the name of the Toric code, which was proposed by Kitaev in '97 and offers an example of a topological quantum computer. I will, however, not discuss the more computational aspects of it. I will only focus on the topological properties.

Kitaev's paper
quant-ph/9707021

The toric code is a $\mathbb{Z}_2$ gauge theory where spin degrees of freedom are living on the links. The actual type of lattice doesn't matter, so for simplicity we take it to be a square. For now, let it be infinite.



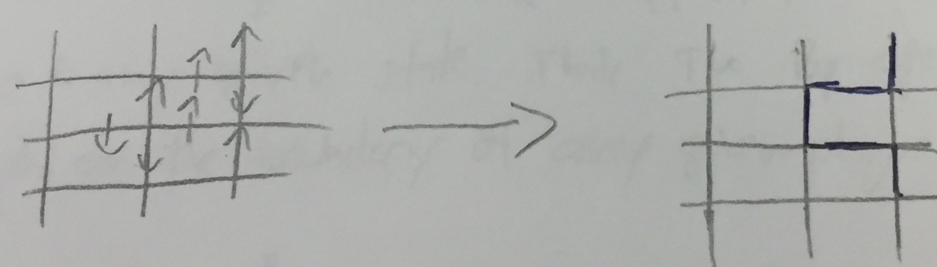
$$H = -J_1 \sum_v \underbrace{\prod_{l \in star(v)} \sigma_l^z}_{A_v} - J_2 \sum_p \underbrace{\prod_{l \in \partial p} \sigma_l^x}_{B_p}$$

The operators A_v & B_p satisfy

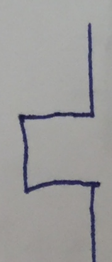
$$[A_v, A_{v'}] = [B_p, B_{p'}] = [A_v, B_p] = 0.$$

This last one is true, since if v & P are disjoint they certainly commute, but if v & P are not, they share exactly 2 sides and since σ_x & σ_z anti-commute, you pick up 2 minus signs and everything is ok.

Now, since these are mutually commuting operators, they can be simultaneously diagonalized. to find the G.S. let us examine the eigenvalues. since $\sigma_x^2 = \sigma_z^2 = 1$, we have $A_v^2 = B_p^2 = \mathbb{1}$. and so Both A_v & B_p have eigen values $\pm \mathbb{1}$. By examining the Hamiltonian, we see immediately that the eigenvalues $A_v = B_p = 1$ will minimize the Hamiltonian. what type of field configurations are expected in the ground state, then? To understand this, it's helpful to use the following visual aide. when we have a spin up particle on a link, we color it blue. if we don't its empty. So, a configuration like

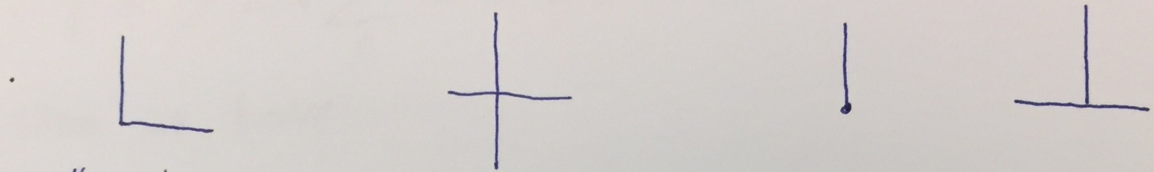


or, some times, just



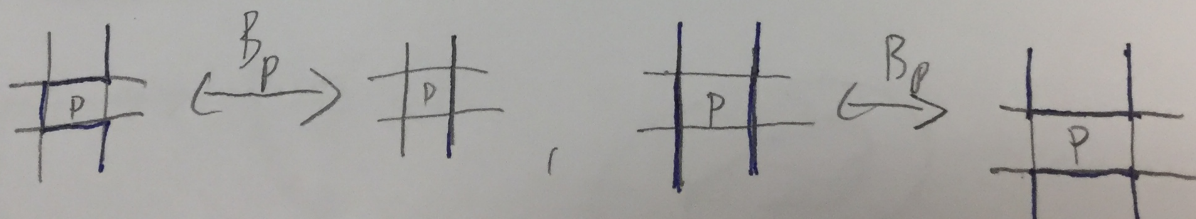
when I'm lazy

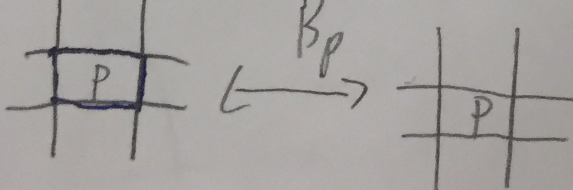
The condition $A_v = 1$, then is equivalent to saying there can only ever be an even number of Blue lines at any given vertex.



allowed allowed not allowed

verticies with an odd number of Blue lines correspond to $A_v = -1$ and so are not in the ground state. these configurations occur when the Blue lines terminate, and so we conclude that the ground state is the state in which there are only closed Blue lines. Thus, we are describing a loop gas. what about the B_p? recall that in the σ_z basis, $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and so flips the spin of any given state. Thus, The B_p operator will flip the spins on the boundary of any given plaquette p. to illustrate:





These configurations are given equal amplitude in the G.S.

By this i mean that If we expand our many-Body-G-S. wavefunction $|\Psi\rangle$ in a basis of loop configurations,

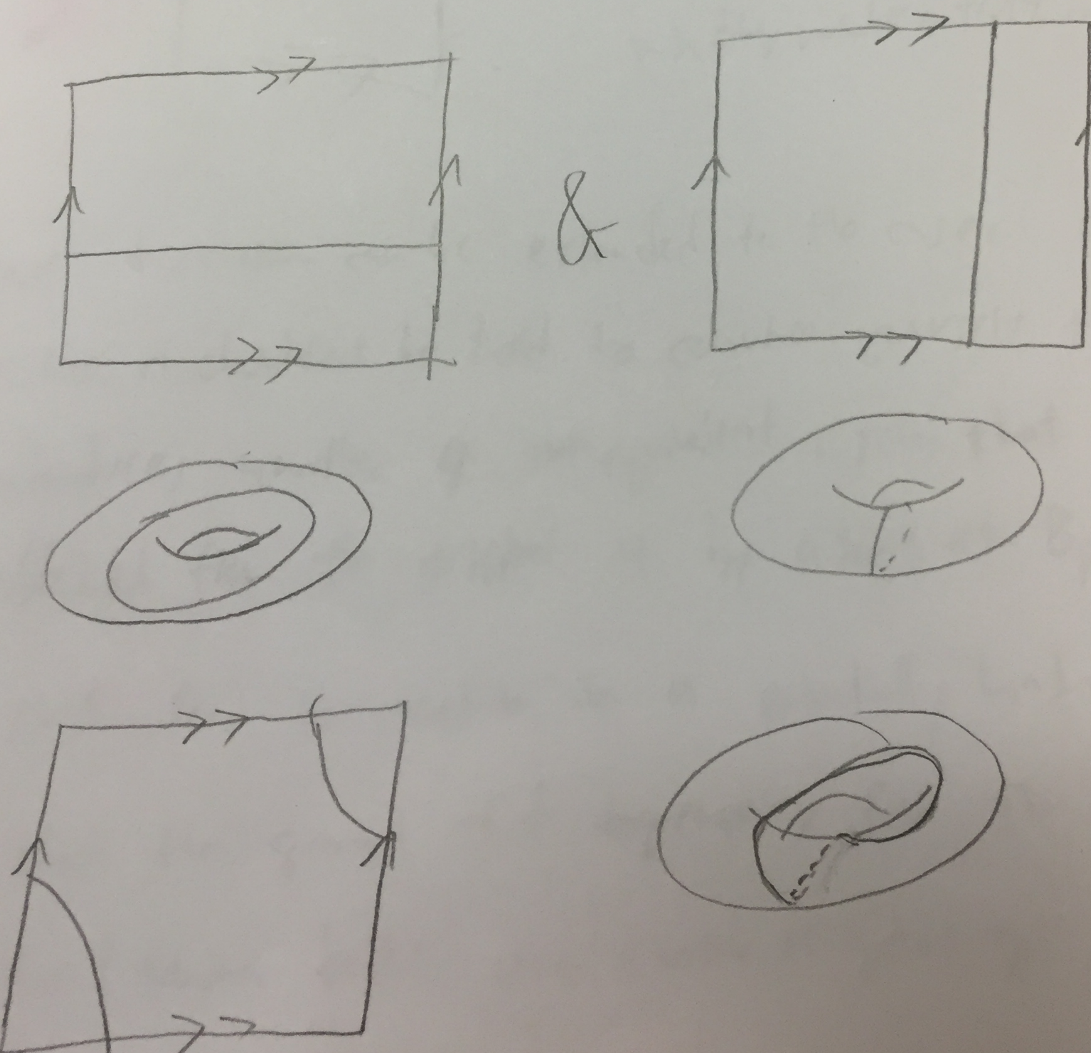
normalization → closed loop configurations.

$$|\Psi\rangle = N \sum_{C} \Psi[C]\,|C\rangle$$

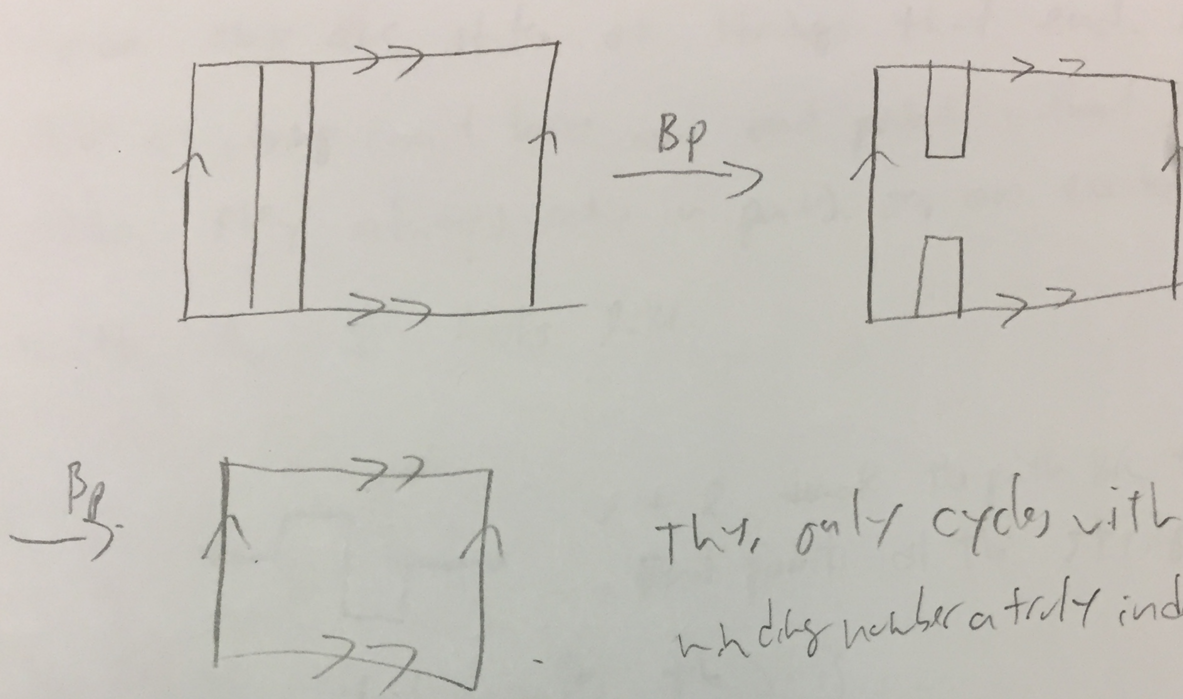
Then we have

$$\Psi[\sqsupset\!\sqsubset] = \Psi[\,||\,],\quad \Psi[\square\sqcup] = \Psi[\sqcap\!\sqcup],\quad \Psi[\square] = \Psi[\emptyset]$$

and so on. So what will happen to this model when we put it on a Torus? we have introduce 3 non-trivial cycles on our lattice. For instance



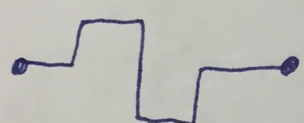
That are not obtainable via a series of B_p operations from the trivial state with no loops. However, if I had instead wrapped around my torus twice



Thus, only cycles with odd winding number are truly independent.

Our discussion can be extended to the case of a genus 2 surface. It shouldn't be hard to convince yourself that that introduces another 4 inequivalent cycles that can't be obtained from the original 4 by a series of B_p's. We are thus lead to conclude, in a potentially handwavy way, that the ground state degeneracy for the toric code Hamiltonian defined on a surface of genus g is 4^g

Let me end by quickly touching on excited states. These are states such that $A_v = -1$ or $B_p = -1$. Let me focus on the case $A_v = -1$. As I said before, these are states of strings that end. And since a string can't have one end point without the other, they always come in pairs. So, an excited state with $A_v = -1$ looks like.

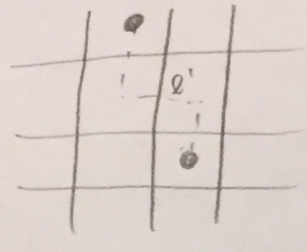


Let ℓ denote the path b/w the end points of the string. Then,

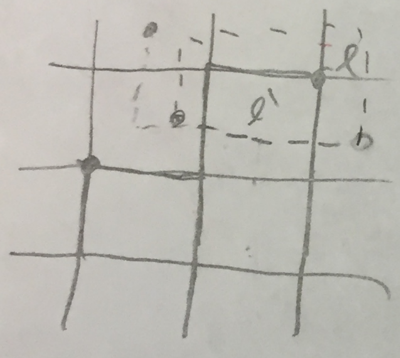
The operator that creates this is

$$\tilde{A}_\ell = \prod_\ell \sigma^z_\ell ;$$

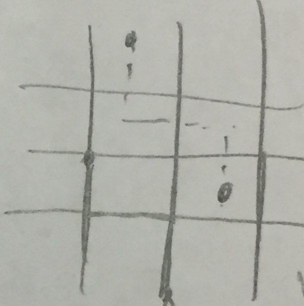
moreover, we could deform this string via any of the rules given above. A similar story happens with plaquettes. We can "frustrate" a plaquette by acting with σ_z on one of it's edges), actually it creates two such frustrated plaquettes. We can construct the operator that creates them in a similar way, except we don't consider a path ℓ in the lattice, but instead on the dual lattice



Then, $\tilde{B}_p = \prod_{l'} \sigma^x_{l'}$. Now, if l and l' share a vertex, they will anti-commute, since there is only one anti-commutation relation involved. This is equivalent to saying that the braiding of these two types of excitations is non-trivial → you pick up an overall phase.



vs.



minus sign

[illegible] sign.

These excitations then fall into representations of the Braid group, and are therefore anyons. The can be used for computation (modulo some subtleties) and hence why this is a "code"