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10/20/17

PTJC Fall 2017: Lattice Field Theory

Talk 3: Lattice Gauge Theory

Outline:
1) Refresher on ctm. gauge theory
   - Local symmetries
   - Wilson lines/loops
2) Gauge fields on the lattice
3) Lattice gauge action
4) Haar measure
   - Gauge fixing
5) Confinement: ~~Static~~ $q\bar{q}$ pair
   - Strong coupling expansion
   - Phase diagrams

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1) Why do we need gauge fields? To make $\mathcal{L}(x)$ (+ $S = \int d^4x\, \mathcal{L}(x)$) invariant under local symmetry group!

e.g., $\mathcal{L}_\varphi = (D_\mu \varphi)^\dagger (D_\mu \varphi) + m^2 \varphi^\dagger \varphi + V(\varphi^\dagger \varphi)$ invariant under global symm. $\varphi(x) \to V\varphi(x)$, $\varphi^\dagger(x) \to \varphi^\dagger(x) V^\dagger$

Euclidean metric used throughout

if $V \in$ unitary gauge Lie group, is indep. of x. e.g. $V = e^{-ig\Lambda}$ ← const.

e.g., $U(1)$ (QED), $SU(2)$ (weak), $SU(3)$ (QCD)

Want local symmetry also (i.e., $V = V(x)$) so we can transform the fields at each site x independently; e.g., $V(x) = e^{-ig\Lambda(x)}$ (local $U(1)$)

- Holds for $m^2\varphi^\dagger\varphi + V(\varphi^\dagger\varphi)$, but not kinetic term.   equiv. to $\boxed{D_\mu \to V D_\mu V^\dagger}$ (*)
- Solution: introduce covariant derivative D_μ s.t. $[D_\mu \varphi](x) \to V(x)[D_\mu \varphi](x)$

Let $\boxed{D_\mu(x) \equiv \partial_\mu - igA_\mu(x)}$. Easy to show that to satisfy (*), we need $A_\mu(x)$ to transform as $\boxed{A_\mu(x) \to V(x) A_\mu(x) V^\dagger(x) + \frac{i}{g} V(x) \partial_\mu V^\dagger(x)}$

$\varphi(x) \to V(x)\varphi(x)$
$\varphi^\dagger(x) \to \varphi^\dagger(x) V^\dagger(x)$

$\overset{U(1)}{\equiv} A_\mu(x) - \partial_\mu \Lambda(x)$

$= \cancelto{0}{\partial_\mu(VV^\dagger)} - (\partial_\mu V)V^\dagger = -(\partial_\mu V)V^\dagger$

Note: $\partial_\mu V^\dagger$ is Lie algebra valued, so $A_\mu \in$ algebra of gauge group
$\Rightarrow A_\mu(x) = \sum_a A_\mu^a(x) t^a$ ← gens.

Now $\mathcal{L}'_\varphi \equiv (D_\mu\varphi)^\dagger (D_\mu \varphi) + m^2\varphi^\dagger\varphi + V(\varphi^\dagger\varphi) \to \mathcal{L}'_\varphi$ ✓
$\Rightarrow S'_\varphi \equiv \int d^4x\, \mathcal{L}_E(x) \to S'_\varphi$

Note: Also works for fermions: $\bar\psi \overset{\text{derivative}}{\not{D}}\psi \rightarrow \bar\psi \not{D}'\psi$ $\left(\psi(x) \rightarrow V(x)\psi(x),\ \bar\psi(x) \rightarrow \bar\psi(x)V^\dagger(x)\right)$

Since $D_\mu \rightarrow V D_\mu V^\dagger$, we have $\boxed{F_{\mu\nu} \equiv \frac{i}{g}[D_\mu, D_\nu] \longrightarrow V\frac{i}{g}[D_\mu, D_\nu]V^\dagger = V(x)F_{\mu\nu}(x)V^\dagger(x)}$

Thus the simplest gauge invariant (color singlet) action term we can make to describe our new "gauge field" $A_\mu(x)$ is: $\boxed{S_G^{\text{ctm.}} = \frac{1}{2}\int d^4x\, \text{tr}[F_{\mu\nu}(x)F_{\mu\nu}(x)]}$ (trace over "colors" $c=1,..,N_c$, where $SU(N_c)$ is the gauge group where $V(x)$ lives)

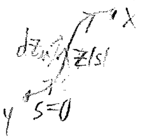
In terms of A_μ: $\boxed{F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]}$ (note: $[A_\mu, A_\nu] = 0$ is QED ($U(1)$ gauge group))

Alternate form: $F_{\mu\nu} = \sum_a F^a_{\mu\nu} t^a$, $F^a_{\mu\nu} = \partial_\mu \underset{\substack{\uparrow \\ A_\nu = \sum_a A^a_\nu}}{A^a_\nu} - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$, $\text{tr}\, t^a t^b = \frac{\delta_{ab}}{2}$ $\Rightarrow \boxed{S_G^{\text{ctm.}} = \frac{1}{4}\int d^4x\, F^a_{\mu\nu}(x)F^a_{\mu\nu}(x)}$

$[t^a, t^b] = if^{abc}t^c$ convention

Wilson lines (ctm)

Consider $\boxed{L_p(x,y) \equiv P e^{ig\int_0^1 ds \frac{dz_\mu}{ds} A_\mu(z(s))}}$, where $z(0) = y$, $z(1) = x$, + P denotes path ordering.



· Note $L_p(x,y) \in$ gauge group (since $A_\mu(z) \in$ gauge algebra)

Straightforward to show $L_p(x,y) \rightarrow V(x)L_p(x,y)V^\dagger(y)$ indep. of path

$\Rightarrow \varphi^\dagger(x)L_p(x,y)\varphi(y)$ + $\text{tr}[L_p(x,x)]$ are gauge invariant

Note: $L_p(y,x) = L_p^\dagger(x,y)$

Want lattice analogs! "Wilson loop"

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2) Moving to the lattice

Assume an isotropic hypercubic lattice $\Lambda = \{n\}$, $n = (n_1, n_2, n_3, \underset{\substack{\uparrow \\ x_4 \equiv -ix_4 = -ian_4}}{n_4})$, $\varphi_n = \varphi(\underset{\substack{\text{lattice spacing} \\ \downarrow}}{a}n)$, $\int d^4x \rightarrow a^4 \sum_n$

As Sam showed last week, kinetic terms give rise to "hopping terms" due to discretizing ∂_μ:

e.g., $\int d^4x\, \partial_\mu\varphi^\dagger(x)\partial_\mu\varphi(x) \rightarrow a^2 \sum_{n,\mu}(\varphi^\dagger_{n+\mu} - \varphi^\dagger_n)\underbrace{(\varphi_{n+\mu} - \varphi_n)}_{\equiv \partial^R_\mu \varphi_n}$, where $\varphi_{n+\mu} \equiv \varphi(a(n+\hat\mu))$.

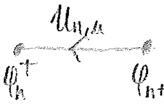
Terms like $\varphi^\dagger_n \varphi_{n+\mu}$ are not gauge invariant ($\varphi^\dagger_n$ + $\varphi_{n+\mu}$ transform differently)

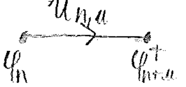
Solution: Connect with Wilson lines! (or lattice analog)

· Want $U_{n,\mu}$ s.t. $\boxed{U_{n,\mu} \xrightarrow{G.T.} V_n U_{n,\mu} V^\dagger_{n+\mu}}$, $U_{n,\mu} \in$ gauge group (e.g., $SU(N_c)$)

$\Rightarrow \sum_n \left[8\varphi^\dagger_n\varphi_n - \sum_\mu \left(\varphi^\dagger_n U_{n,\mu}\varphi_{n+\mu} + \varphi^\dagger_{n+\mu}U^\dagger_{n,\mu}\varphi_n\right)\right]$ is gauge invariant

· Note: $U_{n,\mu}$ live on links between sites, whereas φ_n live on the sites





Think of $U_{n,\mu}$ as "transporting" $\varphi_{n+\mu} \rightarrow n$, + $U^\dagger_{n,\mu}$ transporting $\varphi_n \rightarrow n+\mu$ (or $\varphi^\dagger_{n+\mu} \rightarrow n$)

Alternative Approach

Again we start with $\mathcal{L}_E \overset{\text{Euclidean}}{=} (\partial_\mu\varphi)^\dagger(\partial_\mu\varphi) + m^2\varphi^\dagger\varphi + V(\varphi^\dagger\varphi)$ + try to promote the global U(1) symmetry to a local one; i.e., $\varphi(x) \to e^{i\Lambda(x)}\varphi(x)$, $\varphi^\dagger(x) \to \overset{\text{g's but necessary}}{e^{-i\Lambda(x)}}\varphi^\dagger(x)$

As before, the ∂_μ's mess things up, but before we just replace ∂_μ with something that happens to work, let's think about why it gives us a problem.

Recall the definition of a (partial) derivative, + apply the local U(1) transf:

$$\partial_\mu\varphi(x) \equiv \lim_{\varepsilon_\mu\to 0}\frac{\varphi(x+\varepsilon_\mu)-\varphi(x)}{\varepsilon_\mu} \longrightarrow \lim_{\varepsilon_\mu\to 0}\frac{e^{i\Lambda(x+\varepsilon_\mu)}\varphi(x+\varepsilon_\mu)-e^{i\Lambda(x)}\varphi(x)}{\varepsilon_\mu} \neq e^{i\Lambda(x)}\partial_\mu\varphi(x)$$

natural reln to lattice derivative!

From this form it's clear that the issue is that the derivative is trying to relate $\overset{\varphi\text{ values at}}{\text{two}}$ different points in spacetime, but because these 2 pts. transform differently under the local U(1) + ∂_μ doesn't carry any information about the local transfs., it can't possibly preserve a local symmetry!

⇒ We need to create some object $L_P(x,y)$ that contains info. about the local transfs. s.t. it "carries" or "transports" the transf. info between 2 pts. ~~y+x~~ along some path P, i.e., a function for parallel transport on the manifold for the local U(1) group.

$$\Rightarrow L_P(x,y)\varphi(y)-\varphi(x) \xrightarrow{\text{want}} e^{i\Lambda(x)}\left[L_P(x,y)-\varphi(x)\right] \qquad \Rightarrow L_P(x,y) \to e^{i\Lambda(x)}L_P(x,y)e^{-i\Lambda(y)} \quad \overset{\text{from } y \text{ to } x}{\forall P}.$$

We then see that the replacement for $\partial_\mu\varphi(x)$ that we need is:

$$D_\mu(x)\varphi(x) \equiv \lim_{\varepsilon_\mu\to 0}\frac{L_P(x,x+\varepsilon_\mu)\varphi(x+\varepsilon_\mu)-\varphi(x)}{\varepsilon_\mu} \longrightarrow e^{i\Lambda(x)}\left[D_\mu(x)\varphi(x)\right] \Rightarrow D_\mu(x) \to e^{i\Lambda(x)}D_\mu(x)e^{-i\Lambda(x)}$$

⇒ covariant

Define $L_P(x,x) \equiv 1$ + consider an infinitesimal step ε:

$L_P(x,x+\varepsilon) = 1 - ig\varepsilon_\mu A_\mu(x) + O(\varepsilon^2)$, where $A_\mu(x)$ is just some field at this point (+ g some constant)

U(1) case only!

$$L_P(x,x+\varepsilon) \longrightarrow e^{i\Lambda(x)} L_P(x,x+\varepsilon)\, e^{-i\Lambda(x+\varepsilon)} \Rightarrow \boxed{A_\mu(x) \longrightarrow A_\mu(x) + \frac{1}{g}\partial_\mu \Lambda(x)}$$ "gauge field" ∈ gauge algebra

Plugging in $L_P(x,x+\varepsilon_\mu)$ for $D_\mu(x)\varphi(x)$, we find $\boxed{D_\mu(x) = \partial_\mu - igA_\mu(x)}$ "covariant derivative"

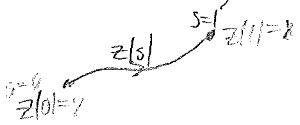
Lastly, we compound the infinitesimal form for L_P to get the general form $L_P(x,y)$:

$$L_P(x,y) = \lim_{N\to\infty} \prod_{n=1}^{N} L_P(z_{n+1}, z_n)$$, where $z_1 = y$, $z_N = x$, & $\{z_n\}_{n=2}^{N-1}$ define the intermediate path P

$$= \lim_{N\to\infty} \prod_{n=1}^{N} \left[1 + ig(z_{n+1} - z_n)A_\mu(x)\right]$$

∈ gauge group

$$\Rightarrow \boxed{L_P(x,y) = P e^{ig\int_y^x dz_\mu A_\mu(x)}}$$ "Wilson line" (technically $L_P(x,y) = P e^{ig\int_0^1 ds \frac{dz_\mu}{ds} A_\mu(z(s))}$)



path ordering (only relevant for nonabelian gauge theories)

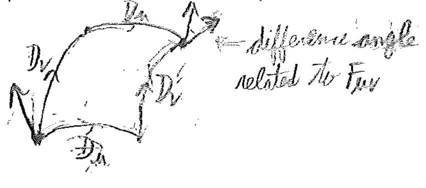
$\boxed{\varphi^\dagger(x) L_P(x,y)\varphi(y) \text{ \& } \mathrm{tr}\, L_P(x,x) \text{ are G.I.}}$

V(x) ∈ gauge group

General (nonabelian) gauge transf.: $$\boxed{\begin{aligned} &\varphi(x) \to V(x)\varphi(x), \quad \varphi^\dagger(x) \to \varphi^\dagger(x)V^{-1}(x), \\ &A_\mu(x) \to V(x)A_\mu(x)V^{-1}(x) + \frac{i}{g}V(x)\partial_\mu V^{-1}(x) \end{aligned}}$$

$$\Rightarrow \boxed{F_{\mu\nu} \equiv \frac{i}{g}[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]}$$ "field strength tensor"

$D_\mu(x)$ is related to the infinitesimal limit of the parallel transport function $L_P(x,x+\varepsilon_\mu)$, so $F_{\mu\nu} \propto [D_\mu, D_\nu] = D_\mu D_\nu - D_\nu D_\mu$ is a measure of how much a vector on the gauge group manifold that takes a "ν then μ" path differs from one that takes a "μ then ν" path; i.e., $F_{\mu\nu}$ is related to the manifold's curvature.

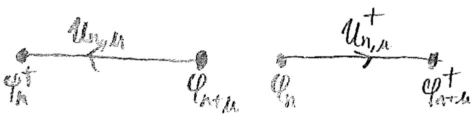


Ctm. gauge action: $\boxed{S_G^{\text{ctm.}} = \frac{1}{2}\int d^4x\, \mathrm{tr}\left[F_{\mu\nu}(x)F_{\mu\nu}(x)\right]}$

Lattice analogs: ($\Lambda = \{n\}$, $n = (n_1, n_2, n_3, n_4)$, $\overset{\text{dimensionless}}{\varphi_n} = a\varphi(an)$, $\int d^4x \to a^4 \sum_{n\in\Lambda}$, a = lattice spacing)

$\Rightarrow \int d^4x\, \partial_\mu\varphi^\dagger \partial_\mu\varphi \to \sum_{n,\mu}(\varphi^\dagger_{n+\hat{\mu}} - \varphi^\dagger_n)(\varphi_{n+\hat{\mu}} - \varphi_n)$, $\varphi_{n+\hat{\mu}} = a\varphi(a(n+\hat{\mu}))$. Local U(1) ruined by hopping terms: $\varphi_n^\dagger \varphi_{n+\hat{\mu}}$ ⇒ Define $\boxed{U_{n,\mu} \xrightarrow{\text{G.T.}} V_n U_{n,\mu} V^{-1}_{n+\hat{\mu}}}$, where $V_n \in U(1)\ \forall n \in \Lambda$ ⇒ $\varphi_n^\dagger U_{n,\mu}\varphi_{n+\hat{\mu}}$ & $\varphi^\dagger_{n+\hat{\mu}} U^\dagger_{n,\mu}\varphi_n$ are G.I.

$\boxed{U_{n,\mu} \text{ is just like } D_\mu \text{ \& } L_P\text{: a parallel transporter!}}$



Note that on the lattice, there is no A_μ, just $U_{n,\mu}$.

However, since $U_{n,\mu} \in$ gauge <u>group</u>, we're free to define $A_\mu(a(n+\frac{\hat{\mu}}{2})) \in$ gauge <u>algebra</u> s.t.

$\boxed{U_{n,\mu} \equiv e^{-iga A_\mu(a(n+\frac{\hat{\mu}}{2}))}}$ to make contact with $L_{p.t.}(an, a(n+\hat{\mu}))$.

- Easy to check gives correct ctm form $U_{n\mu}\varphi_{n+\hat{\mu}} - \varphi_n \sim a^2 D_\mu \varphi(an)$ by expanding in a

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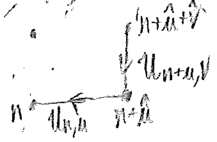
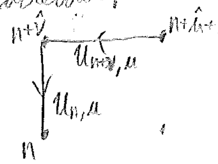
3) <u>Lattice gauge action</u>

<u>Goal</u>: Use $U_{n,\mu}$ to construct a Lagrangian that goes to $\frac{1}{2}\mathrm{tr}F_{\mu\nu}^2$ in classical ctm. limit.

Recall $F_{\mu\nu} \propto [D_\mu, D_\nu]$, + that D_μ "transports" $\varphi(x)$ in the $\hat{\mu}$-direction.

$\Rightarrow D_\mu D_\nu \longleftrightarrow$ 

, $D_\nu D_\mu \longleftrightarrow$ 

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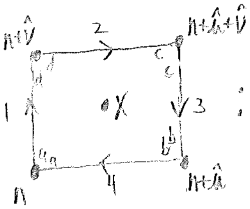
$\Rightarrow [D_\mu, D_\nu] \longleftrightarrow$ 

$:= \mathrm{tr}P_{\mu\nu,n} \equiv \mathrm{tr}[\overset{4}{U_{n,\mu}}\overset{3}{U_{n+\mu,\nu}}\overset{2}{U^\dagger_{n+\nu,\mu}}\overset{1}{U^\dagger_{n,\nu}}]$ = "plaquette"

Note: connected lines ⇒ matrix mult. ⇒ common color index

Thus $\underset{\uparrow}{\overset{\text{makes gauge invar.}}{\mathrm{tr}}} P_{\mu\nu,n}$ is a natural candidate for our lattice gauge Lagrangian term.

Note that $P_{\mu\nu,n} \overset{\mu\neq\nu}{=} P^\dagger_{\nu\mu,n}$, so at each site there are $\binom{4}{2} = 6$ indep. plaquettes ($\mu,\nu \in \{1,2,3,4\}$)

It is straightforward (yet tedious) to expand $P_{\mu\nu,n}$ about $x \equiv a(n+\frac{\hat{\mu}}{2}+\frac{\hat{\nu}}{2})$ in powers of a using
$U_{n,\mu} = e^{-iga A_\mu(a(n+\frac{\hat{\mu}}{2}))} = e^{-iga A_\mu(x-\frac{a\hat{\nu}}{2})}$ + BCH (repeatedly).

<u>Result</u>: $\boxed{\mathrm{Re}\,\mathrm{tr}\,P_{\mu\nu,n} = N_c - \frac{N_c g^2}{2} a^4 \mathrm{tr}F_{\mu\nu}^2 + O(a^6)}$, where $SU(N_c) \ni U_{n,\mu}$ is our gauge group.

$$\Rightarrow S_G^{\mathrm{ctm}} = \frac{1}{2}\int d^4x\, \mathrm{tr}F_{\mu\nu}^2 \longrightarrow \frac{1}{g^2}\sum_n \sum_{\mu\neq\nu} (N_c - \mathrm{Re}\,\mathrm{tr}P_{\mu\nu,n})$$

$$= \beta \sum_{\square} (N_c^2 - \frac{1}{N_c}\mathrm{Re}\,\mathrm{tr}\,\square)$$, where ${}_{\mu}\square^{\nu} \equiv P_{\mu\nu,n}$ + $\beta \equiv \frac{2N_c}{g^2}$ for $SU(N_c)$

($\beta = \frac{1}{g^2}$ for $U(1)$)

↖ only sums over indep. plaquettes (i.e., $\mu<\nu$)

threw away constant part '$\beta\sum_\square N_c^2$'
↓

$\therefore \boxed{S_G^{\mathrm{lattice}} = -\beta\sum_\square \frac{\mathrm{Re}\,\mathrm{tr}\,\square}{N_c}}$ "Wilson gauge action"

↖ exact! (nonperturbative result)

4) Haar measure

Physical quantity = expectation value of observable: $\langle \hat{O}(U) \rangle = \frac{1}{Z}\int \mathcal{D}U\, e^{-S_G(U)} O(U)$, (pure gauge theory)

where $Z = \int \mathcal{D}U\, e^{-S_G(U)}$ + $\int \mathcal{D}U = \prod_{n\in\Lambda}\prod_{\mu=1}^{4}\int d\underset{\nwarrow \text{matrix!}}{U_{n,\mu}}$, $U_{n,\mu}\in SU(N_c)$

· What does it mean to integrate over a group element of $SU(N_c)$ (i.e., a complex $N_c\times N_c$ matrix)?

To answer this, we first note that we want $\mathcal{D}U$ to be invariant under G.T.: $U_{n,\mu} \to U'_{n,\mu} = V_n U_{n,\mu} V^\dagger_{n+\hat\mu}$

Reason: $Z = \int \mathcal{D}U\, e^{-S_G(U)} = \int \mathcal{D}U'\, e^{-\overset{\text{gauge inv.}}{\overbrace{S_G(U')}}} = \int \mathcal{D}U'\, e^{-S_G(U)} \Rightarrow \mathcal{D}U = \mathcal{D}U' \Rightarrow dU_{n,\mu} = dU'_{n,\mu} = d(V_n U_{n,\mu} V^\dagger_{n+\hat\mu})$,

Since V_n + $V_{n+\hat\mu}$ are indep., must have $\boxed{dU = d(UV) = d(VU) \quad \forall\, V\in SU(N_c)}$ (1)

Define normalization s.t. $\boxed{\int dU = 1}$ (2)

Together (1) + (2) define the Haar measure for integration over compact Lie groups.

In general: $dU = \text{const.}\cdot\sqrt{\det g}\prod_k d\alpha_k$, where α_k are coords. on group (e.g., $U = e^{i\sum_k \alpha_k t_k}$) ← Hermitian gens. of algebra; α_k ← real #s, $k = 1,\dots,N_c^2-1$ for $SU(N_c)$

+ $g_{kl} = \mathrm{tr}\left(\frac{\partial U}{\partial\alpha^k}\frac{\partial U^\dagger}{\partial \alpha^l}\right)$ is a metric

Examples: $\underline{U(1)}$: $U = e^{i\varphi} \Rightarrow \int dU = \int_0^{2\pi}\frac{d\varphi}{2\pi}$

$\underline{SU(2)}$: $U = r^0 + \vec{r}\cdot\vec{\tau}$, $\det U = r_\mu r_\mu = 1 \Rightarrow dU = \frac{1}{\pi^2}\delta(r^2-1)\,d^4r$

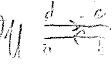
$\underline{\text{or}}$ $U = e^{i\theta\hat{n}\cdot\frac{\vec\tau}{2}} = \cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\,\hat{n}\cdot\vec\tau \Rightarrow dU = \frac{1}{4\pi^2}\sin^2\frac{\theta}{2}\,d\theta\, d\Omega(\hat{n})$

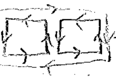
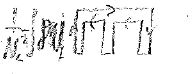
$\underline{SU(N_c\geq 3)}$: $\int dU = 1$; $\int dU\, U_{ab} = \int dU\, U_{ab}U_{cd} = 0$ (color indices ($a=1,2,3$)) (U lives in 3 of $SU(3)$, $U^\dagger$ lives in $\bar{3}$ " ")

$\int dU\, U_{ab}U^\dagger_{cd} = \frac{1}{N_c}\delta_{ad}\delta_{bc}$ (nonzero since $\overset{\text{e.g., for } N_c=3:}{3\otimes\bar{3}} = \underset{\text{singlet}}{1}\oplus 8$)

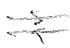
$\underset{\text{only}}{SU(3)}$: $\int dU\, U_{ab}U_{cd}U_{ef} = \frac{1}{3!}\varepsilon_{ace}\varepsilon_{bdf}$ $(3\otimes3\otimes3 = 3\otimes(\bar{3}\oplus6) = 1\oplus8\oplus8\oplus10)$

OK, now we have all the tools to do calculation.

Note: $\int \mathcal{D}U = 1$, $\int \mathcal{D}U$  $= \int \mathcal{D}U\, \mathrm{tr}[U_4U_3U_2^\dagger U_1^\dagger] = 0$, $\int \mathcal{D}U$  $= \int dU\, U_{ab}U^\dagger_{cd} = \frac{1}{N_c}\delta_{ad}\delta_{bc} = \frac{1}{N_c}$ ⇄ (color indices)

$\int \mathcal{D}U\, \mathrm{tr}(VU)\,\mathrm{tr}(U^\dagger W) = \frac{1}{N_c}\int \mathcal{D}U\, \mathrm{tr}(VW)$ ⟵ useful for "unzipping": $\int \mathcal{D}U$  $= \frac{1}{N_c^2}\int \mathcal{D}U$ 

$= \frac{1}{N_c^2}\,\mathrm{tr}\,\mathbb{1} = \frac{1}{N_c}$

Takeaway: Only get nonzero integral if integrand consists solely of (products of):  or  or ...;

SU(3) only → (or  or  or ...)

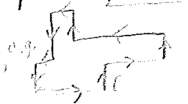
Gauge fixing

Because we're on a finite lattice, $\int \mathcal{D}U = \int \prod_{n,\mu} dU_{n,\mu}$ contains a finite # of G.T.s of each "unique" $\overset{\text{G.T.}}{\overset{\downarrow}{S_{lat}}}$

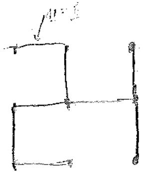
∴ We do NOT have to gauge fix in general (unlike stan. P.T., where we must gauge fix or else gluon prop. vanishes in "gauge direction")

However, it's often helpful to gauge fix to simplify matters (e.g., set $U_{n,\mu} = \mathbb{1}$ in useful places, reduce duplicates, etc.)

Note: $Z_G = \int [\mathcal{D}U]_{\substack{\text{unfixed}\\ \text{links}}} e^{-S_G[U_{\text{fixed}}]} = \int [\mathcal{D}U]_{\text{fixed}} e^{-S_G[U_{\text{fixed}}]}$; i.e., Jacobian $= 1$ ⇒ no subtleties in measure



Ex: Maximal trees: G.I. objects on lattice are closed paths (e.g., $\mathrm{tr}\left[\prod_{l \in C} U_l\right]$ where C is a closed path)



⇒ We can fix $U_{n,\mu} = \mathbb{1}$ for as many links as we want as long as we don't create any "fixed circuits".

- Note: For periodic B.C., this means we can't fix all links in 1 direction

Temporal gauge: Fix $U_{n,4} = \mathbb{1}$ for all n except for 1 time slice, which must remain unfixed.

- Useful for determining transfer matrix + showing reflection positivity, but breaks rotation inv.

Landau gauge: $\partial_\mu A_\mu(x) = 0 \Leftrightarrow$ minimize $\int d^4x \, \mathrm{tr}(A_\mu A_\mu) \overset{\text{lattice}}{\Longleftrightarrow}$ maximize $\mathrm{Re} \sum_{n,\mu} \mathrm{tr}(U_{n,\mu})$.

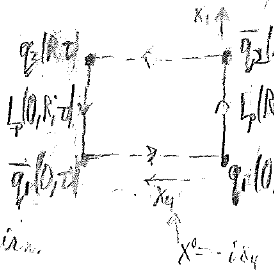
- Makes $U_{n,\mu}$ globally as close to $\mathbb{1}$ as possible.

5) Confinement: Static quark-antiquark pair

We define confinement to mean the potential energy is linearly increasing at large spatial separations

i.e., $V(R) \sim \sigma R$ for large R (expect $V(R) \sim \frac{\alpha}{R}$ for small R since $F_{\mu\nu} \xrightarrow{g \to 0} \partial_\mu A_\nu - \partial_\nu A_\mu \Rightarrow$ QED)

Consider a $\overset{\text{e.g., } \pi^+ = u\bar{d} \text{ or } \Upsilon = b\bar{b}}{\text{quark-antiquark}}$ pair $\overset{q_1, \bar{q}_2 \text{ created at } x_4 = 0}{\text{separated}}$ by some distance $\underset{\text{in } x_1\text{-dir.}}{R}$, + connected by a Wilson line to make the system gauge invariant.

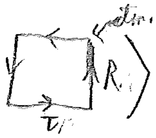


In the $m_q \to \infty$ limit, q_1 + $\bar{q}_2$ are stationary, so they take a straight path in the x_4 direction.

At some "time" $x_4 = \tau$, the $q_1 \bar{q}_2$ pair is annihilated. Thus the process forms a Wilson loop. (ignoring interactions of quarks with gauge fields since $m_q = \infty$)

Let $\hat{O}^\dagger(0)$ create the $q_1\bar{q}_2$ pair $\overset{\text{at } x_4=0}{\vee}$ & $\hat{O}(\tau)$ annihilate it at $x_4=\tau$

Then the (vacuum) expectation value is the same as that of the Wilson loop:

$\langle 0|\hat{O}(\tau)\hat{O}^\dagger(0)|0\rangle = \langle$  $\rangle$

The first form can be related to energy eigenstates of the $q_1\bar{q}_2$ system:

$$\langle 0|\hat{O}(\tau)\hat{O}^\dagger(0)|0\rangle = \sum_n \langle 0|e^{\hat{H}\tau}\hat{O}(0)e^{-\hat{H}\tau}|n\rangle\langle n|\hat{O}^\dagger(0)|0\rangle$$

(set vacuum energy $E_0=0$ & $\langle 0|0\rangle=1$)

$$= \sum_n |\langle n|\hat{O}^\dagger(0)|0\rangle|^2 e^{-E_n\tau} \longleftarrow$$

Note: can relate to stat. mech.: $e^{-E_n\tau} = e^{-\frac{E_n}{kT_{emp}}} \Rightarrow \boxed{\tau = \frac{1}{kT_{emp}}}$

$$\xrightarrow{\tau\to\infty} c_1 e^{-E_1\tau}$$

, where $E_1 = V(R)$ is the ground state energy of the $q_1,\bar{q}_2$ pair.

Setting $\tau = aT$ & $R = aL$ are the dimensions of the $\overset{\text{rectangular}}{\vee}$ Wilson loop in lattice units, we have

$W(L,T) \equiv \langle$  $\rangle \xrightarrow{T\to\infty} c_1 e^{-aV(aL)T} \Rightarrow \boxed{aV(aL) = \lim_{T\to\infty} \frac{-\ln W(L,T)}{T}}$

Thus if we have confinement, we expect $V(aL) \overset{L\text{ large}}{\sim} \sigma aL \Rightarrow W(L,T) \overset{T\text{ large}}{\sim} e^{-\sigma a^2 LT} = e^{-\sigma R\tau} = e^{-\sigma(\text{area})}$

$\therefore$ $\boxed{\text{Confinement} \Leftrightarrow \text{area law}}$

Conversely, if we <u>don't</u> have confinement, we expect a Coulomb potential form $V(aL) \overset{L\text{ large}}{\sim} \frac{\alpha}{aL}$

In this case, it can be shown that $W(L,T) \sim e^{-2c(aL+aT)} = e^{-2c(R+\tau)} = e^{-c(\text{perimeter})}$

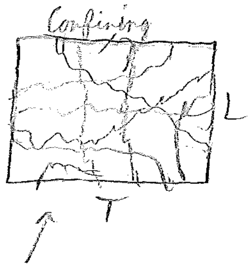
$\therefore$ $\boxed{\text{No confinement} \Leftrightarrow \text{perimeter law}}$

$\Rightarrow W(L,T)$ is an order parameter for confinement!

alternative order parameter: Polyakov loop

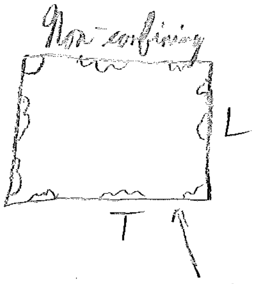
Qualitatively makes sense if you think about the strength of the interaction between a hypothetical $q_1\bar{q}_2$ pair on different parts of the Wilson loop:

Confining



Interactions over full area

Non-confining



Only interactions on the perimeter

Now let's try calculating $W(L,T)$ for different gauge groups to see if they're confining in the strong-coupling (small β) limit.

Strong coupling $\overset{\beta\ll 1}{\downarrow}$ expansion: Wilson loop

$S_G = -\beta \sum_{\square} \frac{\mathrm{Re\,tr}\,\square}{N_c}$. Note $-1 \le \frac{\mathrm{Re\,tr}\,\square}{N_c} \le 1$

$\left\langle \boxed{\circlearrowleft} \right\rangle = \left\langle \boxed{\circlearrowright} \right\rangle$, so orientation irrelevant

Consider $W(L,T) \equiv \left\langle \underset{T}{\boxed{}}\,L \right\rangle = \frac{1}{Z}\int \mathcal{D}U\, e^{\frac{\beta}{N_c}\sum_{\square}\mathrm{Re\,tr}\,\square}\; \underset{T\text{ (time dir.)}}{\boxed{}}\,L$ (spatial dir.)

trace included!

Write $\mathrm{Re\,tr}\,\square = \frac{1}{2}\left(\boxed{\circlearrowleft} + \boxed{\circlearrowright}\right)$ & expand exponential:

$$e^{\frac{\beta}{2N_c}\sum_{\square}\left(\boxed{\circlearrowleft}+\boxed{\circlearrowright}\right)} = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\beta}{2N_c}\right)^n\left[\sum_{\square}\left(\boxed{\circlearrowleft}+\boxed{\circlearrowright}\right)\right]^n$$

$$\Rightarrow W(L,T) = \frac{1}{Z}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\beta}{2N_c}\right)^n\int\mathcal{D}U\left[\sum_{\square}\left(\boxed{\circlearrowleft}+\boxed{\circlearrowright}\right)\right]^n\cdot\underset{T}{\boxed{}}\,L$$

Smallest nonzero contribution: the $(LT)!$ terms at order $n=LT$ which have the Wilson loop tiled w/ plaquettes:

(note: $Z = \int\mathcal{D}U\, e^{\frac{\beta}{2N_c}\sum_{\square}\left(\boxed{\circlearrowleft}+\boxed{\circlearrowright}\right)} = 1 + O(\beta^4)$) $\nwarrow \boxed{\boxed{\circlearrowleft}}$

$$W(L,T) = \left(\frac{\beta}{2N_c}\right)^{LT}\int\mathcal{D}U\;\underset{T}{\boxed{\begin{matrix}\square&\cdots&\square\\ \vdots&&\vdots\\ \square&\cdots&\square\end{matrix}}}\,L + \ldots$$

$$\int\mathcal{D}U\;\underset{T}{\boxed{\begin{matrix}\square&\cdots&\square\\ \vdots&&\vdots\\ \square&\cdots&\square\end{matrix}}}\,L = \frac{1}{N_c^T}\int\mathcal{D}U\;\underset{T}{\boxed{\begin{matrix}\square&\cdots&\square\\ \vdots&&\vdots\\ \square&\cdots&\square\\ \square&\cdots&\square\end{matrix}}}\,L = \frac{1}{N_c^T}\int\mathcal{D}U\;\underset{T}{\boxed{\begin{matrix}\square&\cdots&\square\\ \vdots&&\vdots\\ \square&\cdots&\square\end{matrix}}}\,L-1 = \cdots$$

$$= \frac{1}{N_c^{(L-1)T}}\int\mathcal{D}U\;\underset{T}{\boxed{\square\cdots\square}} = \frac{1}{N_c^{LT}}\cdot\overset{\mathrm{tr}\,1=N_c}{\bullet} = \frac{1}{N_c^{LT-1}}$$

$$\Rightarrow W(L,T) = N_c\left(\frac{\beta}{2N_c^2}\right)^{LT} + \ldots = e^{\overset{\text{area law!}}{-LT\ln\left(\frac{2N_c^2}{\beta}\right)}+\ln N_c} + \ldots \Rightarrow \boxed{\text{confinement } \forall SU(N_c \geq 3)} \text{ for } \beta\ll 1$$

$\Rightarrow aV(aL) \underset{L,T\text{ large}}{\sim} \frac{-\ln W(L,T)}{T} \sim \sigma L$, where $\sigma(g) \approx \ln\frac{2N_c^2}{\beta} = \ln(N_c g^2)$ is the string tension to leading order

For higher order corrections to σ, have to play "geometry games":

Leading order correction comes from cubic "bubble" out of Wilson loop:  $\Rightarrow O(\beta^4)$ correction

(Things like just adding a $\boxed{\boxed{\circlearrowleft}}$ somewhere are canceled by Z, so must get creative)

We can also check $U(1)$: $S_G = -\beta\sum_{\square}\mathrm{Re}\,\square$, $\beta = \frac{1}{g^2}$ to match $S_G^{\text{cont.}} = \frac{1}{4}\int d^4x\, F_{\mu\nu}^2$

$$U = e^{i\theta} \Rightarrow \int dU = \int_0^{2\pi}\frac{d\theta}{2\pi} \Rightarrow \int dU\, UU^\dagger = 1 \Rightarrow W(L,T) = \left(\frac{\beta}{2}\right)^{LT}\underbrace{\int\mathcal{D}U\;\boxed{\begin{matrix}\square&\cdots&\square\\ \vdots&&\vdots\\ \square&\cdots&\square\end{matrix}}}_{1} + \ldots = \left(\frac{\beta}{2}\right)^{LT} + \ldots$$

$$\Rightarrow W(L,T) = e^{-LT\ln\frac{2}{\beta}} + \ldots \Rightarrow \sigma(g) = \ln\frac{2}{\beta} = \ln 2g^2 \text{ as } \beta\to 0$$

$$\Rightarrow \boxed{\text{confinement for } U(1)}! \text{ (for } \beta\ll 1)$$

This seems wrong; we know U(1) (QED) is not confined in real life ($V(r) \sim \frac{1}{r}$).

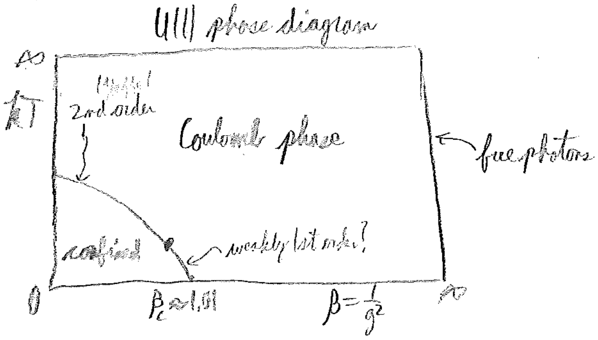
However, our result is correct <u>in the limits we considered</u>; namely, $\beta \ll 1$ & $\overset{\neq T \text{ in } W(L,T)}{N_t} \to \infty$.

- Recall $\overset{\text{from Sam's talk}}{\vee}$ that N_t is related to the temperature $\overset{\text{not the same T as in } W(L,T)}{k_B T}$ via $aN_t = \frac{1}{kT}$; thus $\underline{N_t \to \infty \Leftrightarrow kT \to 0}$.

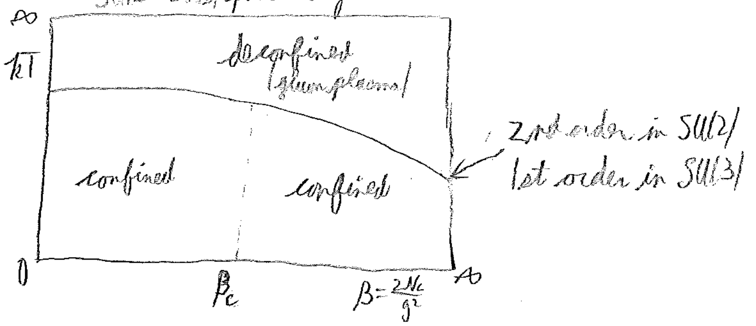
So all we've shown is that QED is confining at low temps & strong cplng.

- To get a nonconfining theory at realistic temps. & cplngs., we expect a phase transition to occur at some place in the kT vs. β diagram.
  - Lattice calculations confirm this!

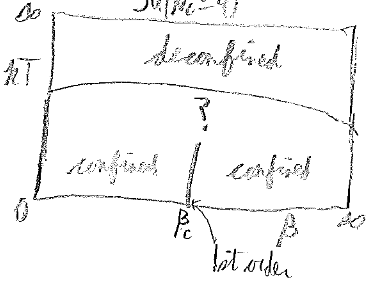
U(1) phase diagram



SU(2) & SU(3) phase diagram



SU($N_c \geq 4$)



There's a phase transition at $\beta_c \simeq 1.01$, but both sides are confining!

- Reason: large a lattice artifact

$$g^2 \sim \frac{\#}{-\ln a\Lambda} \Leftrightarrow a \sim \frac{1}{\Lambda} e^{-\#\beta}$$