Basic Defs

ODE:

$$\begin{cases} f : D \subset \mathbb{R}^{n+1} \mapsto \mathbb{R} \\ y^{(n)} = f(t, y(t), y'(t), \cdots, y^{(n-1)}(t)) \end{cases}$$

Solution of a Diff Eq:

$\phi(t)$ solves an ODE on $I = (t_1, t_2)$ if

1. $\phi(t), \phi'(t), \cdots, \phi^{(n-1)}(t), \phi^{(n)}(t)$ exists for $t \in I$
2. $(\phi(t), \phi'(t), \cdots, \phi^{(n-1)}(t)) \in D$ for $t \in I$
3. $\phi^{(n)}(t) = f(t, \phi(t), \phi'(t), \cdots, \phi^{(n-1)}(t))$

Solution Techniques

Integrating Factors

$$\dot{y}(t) + a(t)y(t) = b(t)$$

Giving $m(t) \triangleq e^{\int a(t)\,\mathrm{d}t}$

$$y(t) = \frac{1}{m(t)}\left[\int m(t)b(t)\,\mathrm{d}t + C\right]$$

Bernoulli Eq.

$$\frac{dy}{dt} + a(t)y = b(t)y^n \qquad n \geq 0$$

Substitute $z = y^{1-n} \implies \frac{1}{1-n}z' = y^{-n}\frac{dy}{dt}$

2nd Order ORDE (linear homo)

$$\ddot{y} + a(t)\dot{y} + b(t)y = 0$$

If y_1 and y_2 are solns, then so is any linear combination.

Theorem: Two solns $y_1(t)$ and $y_2(t)$ are linearly dependent iff $W(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = 0$.

Existence

Thm: Picard's Existence Theorem

Suppose f defined on a rectangle R of size $2a \times 2b$ is bounded, i.e. $|f(t,y)| \leq M \quad \forall (t,y) \in R. \quad M > 0.$ and is a cts function satisfying Lipschitz condition

$$|f(t,y_1) - f(t,y_2)| \leq L|y_1 - y_2|$$

for some constant $L > 0$.

Then the IVP has a soln on the interval $\{t : |t - t_0| \leq \alpha\}$ for some constant $\alpha > 0$, $\alpha = \min\{a, \frac{b}{M}\}$.

Picard's Iteration:

$$y_0(s) \triangleq y_0$$

$$y_n(t) \triangleq y_0 + \int_{t_0}^{t} f(s, y_{n-1}(s))\,\mathrm{d}s$$

$y(t) = \lim_{n\to\infty} y_n(t)$ exists and solves IVP.

Uniform Convergence (allows interchange of limits and integrals) (Note N before t in the qualifiers)

$$\forall \epsilon > 0, \exists N : \forall n > N, \forall t \in I : |f_n(t) - f(t)| < \epsilon$$

If $|f_n(t)| \leq M_n$ for all $t \in I$ and $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} f_n(t)$ converges uniformly.

Peano's Existence Theorem

Suppose f is CTS on rectangle R. Then there exists a soln of IVP on the interval $|t - t_0| < \alpha$ for some $\alpha > 0$

Uniqueness

Thm (Gronwall's Ineq.)

Let $K \geq 0$ constant, f and g are cts *non-negative* functions defined on $t \in [a, b]$ satisfying

$$\forall t \in [a, b] : f(t) \leq k + \int_a^t f(s)g(s)\,\mathrm{d}s$$

$$f(t) \leq k \exp\left(\int_a^t g(s)\,\mathrm{d}s\right)$$

Uniqueness Theorem

Suppose f is CTS satisfying Lip. condition, i.e.

$$|f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2|$$

such that $L > 0$ constant, on the "box" $R = \{(t, y) : |t - t_0| \leq a, |y - y_0| \leq b\}$ then the soln (defined by local existence thm) is unique.

Sufficient condition for Lip

$$\left|\frac{\partial f}{\partial y}\right| \leq L.$$

Lemma

Suppose f is CTS in a domain D, $|f| \leq M$ in D. Let ϕ be a soln of $\begin{cases} \frac{\mathrm{d}y}{\mathrm{d}t} = f(t, y) \\ y(t_0) = y_0 \end{cases}$ that exists a finite interval (a, b). Then $\lim_{t \to a^+} \phi(t)$ and $\lim_{t \to b^-} \phi(t)$ exists.

Suppose f is CTS in a given region D satisfying Lip condition.

f is bounded in D. Let $(t_0, y_0) \in D$. Then the unique soln of $\frac{\mathrm{d}y}{\mathrm{d}t} = f(t, y)$, passing through the point (t_0, y_0) can be extended until its graph meets the boundary of D.

Corrollary: If D is (t, y) space, and if f is CTS and Lip on D, then the soln of IVP can be extended uniquely in both directions as long as $|\phi(t)|$ remain finite.

Apriori estimate: $|\phi(t)| \leq M$

Thm: f CTS in (t, y), bdd, lip in y. in D Lip. const: L.

Let ϕ be the soln of the IVP with $y(t_0) = y_0$, and ψ be the soln of IVP with $y(t_0) = \tilde{y}_0$

Suppose ϕ, ψ exist on some interval $a < t < b$.

Then $\forall \epsilon > 0, \exists \delta > 0 : |y_0 - \tilde{y}_0| < \delta \implies (\forall t \in (a, b) : |\phi(t) - \psi(t)| < \epsilon)$

Thm: Let f and g def on D. CTS in (t, y), bdd $\begin{cases} |f| \leq M \\ |g| \leq M \end{cases}$

Lip cts y. w/ same Lip constant L.

Let ϕ be $\begin{cases} y' = f(t, y) \\ y(t_0) = y_0 \end{cases}$ and ψ be $\begin{cases} y' = g(t, y) \\ y(t_0) = y_0 \end{cases}$ exists a common interval $a < t < b$. Suppose $|f(t, y) - g(t, y)| \leq \epsilon \quad \forall (t, y) \in D$. Then solns ϕ and ψ satisfy the estimate $|\phi(t) - \psi(t)| \leq \epsilon(b - a)\exp(L|t - t_0|)$.

Thm: $\frac{dy}{dt} = A(t)y + g(t)$ with $y(t_0) = y_0$. If $A(t)$, $g(t)$ are CTS on some interval $[a, b]$ and $t_0 \in [a, b], y_0 < \infty$ then the system has a unique soln $\phi(t)$ satisfying $\phi(t_0) = y_0$ and existing on $[a, b]$.

Thm $\frac{dy}{dt} = A(t)y$ with $y \in \mathbb{R}^n$ (W5B)

If $n \times n$ complex $A(t)$ is CTS on an interval I, then the soln of the system on I form a vector space of dimension n over complex numbers.

Def. Linearliy indep. solns $\phi_1, \cdots, \phi_n$ are called fundamental set of solns.

$$\Phi = \begin{bmatrix} \phi_1 & \cdots & \phi_n \end{bmatrix}$$

1. Satisfies $\frac{d\Phi}{dt} = A(t)\Phi$
2. $\forall \vec{c} \in \mathbb{C}^n : \Phi(t)\vec{c}$ solves IVP.
3. $\forall \psi(t) \in S : \exists \vec{c} : \psi(t) = \Phi(t)\vec{c}$
4. $\forall t : \det(\Phi(t)) \neq 0$

Lemma: $\Phi(t)$ satisfies IVP on an interval I, it is a fund. matrix of IVP on I iff $\forall t \in I : \det(\Phi(t)) \neq 0$

Thm: Abel's Formula

If Φ is a fund. matrix of IVP on I, and $t_0 \in I$, then

$$\det \Phi(t) = \det \Phi(t_0) \exp\left(\int_{t_0}^{t} \sum_{k=1}^{n} A_{kk}(s)\, ds\right)$$

A soln. matrix $\Phi(t)$ of IVP is a fund. matrix iff $\det(\Phi(t)) \neq 0$ for some $t = t_0$.

Cor: $\Phi(t)$ is a fund. matrix of IVP on I and C is a non-singular const matrix, then $\Phi(t)$C is a fund. matrix of IVP on I.

Variation of const formula

$$y(t) = \Phi(t)\Phi^{-1}(t_0)y_0 + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)g(s)\mathrm{d}s$$

Matrix exponential:

$$e^M \triangleq \sum_{n=0}^{\infty} \frac{M^n}{n!}$$

Properties:

1. $e^{\mathbf{0}} = I$
2. If $AB = BA$ then $e^{A+B} = e^A e^B$ and $Ae^B = e^B A$
3. e^A is always invertible.
4. If T is nonsingular $n \times n$ mmatrix, then $e^{TAT^{-1}} = Te^A T^{-1}$

The Matrix $\Phi(t) = e^{At}$ is a fund. matrix of $\frac{\mathrm{d}\Phi}{\mathrm{d}t} = A\Phi(t)$ w/ $\Phi(0) = I$

Thm: λ is a complex e-val of real matrix A w/ e-vec v then $\bar{\lambda}$ is also an e-val w/ e-vec $\bar{\vec{v}}$