Propositional Logic: Logical Entailment ⇒ Resolution

Source: Computational Logic Lecture Notes
Stanford University

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Contents

- Review
- Propositional Logic ⇒ Logical Entailment (CONTINUE)
  - Proof Method: Rule of inference, axiom schema, Propositional Resolution

Review

- Reasoning: information ⇒ conclusion
  - Deduction, Induction, Abduction, Analogy
  - Which one is truth preserving?
- Formal Logic
  - Formal language ⇒ syntax, semantics, proof systems
  - Encode information, legal transformation
- Computational Logic
  - **Propositional Logic:**
    - Sintax ⇒ Simple sentence, Compound Sentence
    - Semantics ⇒ interpretation, evaluation, reverse evaluation, types of compund sentence
    - Logical Entailment ⇒ Semantic Reasoning, **Proof Method**
  - Relational Logic

Review Semantic Reasoning

- $\Delta \models \phi$
  - Set of premises Δ logically entails a conclusion ϕ iff every interpretation that satisfies the premises also satisfies the conclusion
- Example:
  - $\{p\} \models (p \lor q)$
  - $\{p\} \not\models (p \land q)$
  - $\{p,q\} \models (p \land q)$
- Semantic reasoning:
  - Truth table
  - Validity checking
  - Unsatisfiability checking

Proof Method

- Proof of a conclusion from set of premises:
  - Sequence of sentences terminating in conclusion in which each item is either a premise, an instance of axiom schema, or the result of applying a rule of inference to earlier items in sequence.
  - Base: Applied Rule of Inference to premises

- A rule of inference (if we have premises to apply rules of inference):
  - Rule of Replacement

- Axiom Schemata

Axiom Schemata

II: $A \to (B \to A)$

ID: $A \to (B \to C) \to ((A \to B) \to (A \to C))$

CR: $(A \to \sim B) \to ((A \to B) \to \sim A)$
$(\sim A \to \sim B) \to ((\sim A \to B) \to A)$

EQ: $(A \leftrightarrow B) \to (A \to B)$
$(A \leftrightarrow B) \to (B \to A)$
$(A \to B) \to ((B \to A) \to (A \leftrightarrow B))$

O: $(A \leftarrow B) \leftrightarrow (B \to A)$
$(A \lor B) \leftrightarrow (\sim A \to B)$
$(A \land B) \leftrightarrow \sim(\sim A \lor \sim B)$

Propositional Resolution

- Rule of inference, only for expression in clausal form
- Clause: set of literals in a clause expression.
- Clause expression:
  - Literal: atomic proposition / negation
    Example: p, ~p
  - Disjunction of literals
    Example: p ∨ q

- Clause examples: {p}, {~p}, {p,q}, {}
- {}: empty disjunctions; unsatisfiable

Resolution Provability

- Prove $\Delta \models P$ by proving that $\Delta \cup \{\sim P\}$ unsatisfiable
- Steps:
  - Rewrite $\Delta \cup \{\sim P\}$ in clausal forms
  - Derive empty clause with Resolution Principle

Conversion to Clausal Form

1. Implications (I):
   $P_1 \rightarrow P_2$: $\sim P_1 \vee P_2$
   $P_1 \leftarrow P_2$: $P_1 \vee \sim P_2$
   $P_1 \leftrightarrow P_2$: $(\sim P_1 \vee P_2) \wedge (P_1 \vee \sim P_2)$

2. Negations (N):
   $\sim\sim P$: P
   $\sim(P_1 \wedge P_2)$: $\sim P_1 \vee \sim P_2$
   $\sim(P_1 \vee P_2)$: $\sim P_1 \wedge \sim P_2$

Konversi ke Clausal Form (2)

3. Distribution (D):

   $P_1 \lor (P_2 \land P_3)$: $(P_1 \lor P_2) \land (P_1 \lor P_3)$
   $(P_1 \land P_2) \lor P_3$: $(P_1 \lor P_3) \land (P_2 \lor P_3)$
   $(P_1 \lor P_2) \lor P_3$: $P_1 \lor (P_2 \lor P_3)$
   $(P_1 \land P_2) \land P_3$: $P_1 \land (P_2 \land P_3)$

4. Operators (O):

   $P_1 \lor \ldots \lor P_n$: $\{P_1, \ldots, P_n\}$
   $P_1 \land \ldots \land P_n$: $\{P_1\} \ldots \{P_n\}$

Conversion Examples

1. $g \land (r \rightarrow f)$

I $g \land (\sim r \lor f)$

N $g \land (\sim r \lor f)$

D $g \land (\sim r \lor f)$

O $\{g\}$

$\{\sim r, f\}$

2. $\sim(g \land (r \rightarrow f))$

I $\sim(g \land (\sim r \lor f))$

N $\sim g \lor \sim(\sim r \lor f)$

$\sim g \lor (\sim\sim r \land \sim f)$

$\sim g \lor (r \land \sim f)$

D $(\sim g \lor r) \land (\sim g \lor \sim f)$

O $\{\sim g, r\}$

$\{\sim g, \sim f\}$

Resolution Principle

$$\frac{\begin{array}{l}\{\varphi_1,\ldots,\chi,\ldots,\varphi_m\}\\ \{\psi_1,\ldots,\neg\chi,\ldots,\psi_n\}\end{array}}{\{\varphi_1,\ldots,\varphi_m,\psi_1,\ldots,\psi_n\}}$$

$$\frac{\begin{array}{l}\{p,q\}\\ \{\neg p,r\}\end{array}}{\{q,r\}}$$

- {~p,q}, {p} ⇒ {q}
- {~p},{p} ⇒ {}

Resolution Notes

- In a clause: No literal repetition
  {~p,q}, {p,q} ⇒ {q} **NOT {q,q}**
- Only a pair of literals that can be resolved (even though there are some possibilities)
  - {p,q},{~p,~q} ⇒ {p,~p} / {q,~q} **NOT {}**

Resolution Provability

1. Prove $\Delta \models P$ by proving that $\Delta \cup \{\sim P\}$ unsatisfiable
2. Steps:
   a. Rewrite $\Delta \cup \{\sim P\}$ in clausal forms
   b. Derive empty clause with Resolution Principle

Example

If Mary loves Pat, then Mary loves Quincy. If it is Monday, Mary loves Pat or Quincy. Prove that, if it is Monday, then Mary loves Quincy.

1.	$\{\neg p, q\}$	Premise
2.	$\{\neg m, p, q\}$	Premise
3.	$\{m\}$	Negated Goal
4.	$\{\neg q\}$	Negated Goal
5.	$\{p, q\}$	3,2
6.	$\{q\}$	5,1
7.	$\{\}$	6,4

Another Example

Premises: $p \rightarrow q$, $q \rightarrow r$

Conclusion: $p \rightarrow r$

1. {~p,q} premise $p \rightarrow q$
2. {~q,r} premise $q \rightarrow r$
3. {p} Negated Goal $p \rightarrow r$
4. {~r} Negated Goal $p \rightarrow r$
5. {q} 1,3
6. {r} 2,5
7. {} 4,6

Exercise

- Prove:

  $$p\rightarrow q,\ q\rightarrow r\}\ |= (q\rightarrow r)\rightarrow((p\rightarrow \sim r)\rightarrow \sim p)$$

  - using:
    - Propositional Resolution

Exercise

1. Use propositional resolution to show that the following sets of clauses are unsatisfiable.

a) {p,q}, {~p,r}, {~p,~r}, {p,~q}

b) {p,q,~r,s},{~p,r,s},{~q,~r},{p,~s},{~p,~r},{r}

2. Buktikan dengan menggunakan prinsip resolusi bahwa ekspresi logika di bawah ini adalah valid/tautologi

$(\sim r \Rightarrow ((\sim q \vee r) \Rightarrow (p \wedge \sim q \wedge \sim r))) \vee (\sim p \wedge \sim q \wedge \sim r)$

Exercise (con't)

3. Terdapat premis sebagai berikut.

1. Seseorang yang pergi belajar, selalu menyisir rambut.
2. Seorang yang tidak pergi belajar tidak memiliki kontrol diri.
3. Seseorang tidak tampak menarik jika orang tersebut tidak rapi.
4. Seseorang yang menyisir rambut, tampak menarik.

Buktikan bahwa kesimpulan: **Jika seseorang memiliki kontrol diri maka orang tersebut rapi**, dapat ditarik dari kumpulan premis tersebut, dengan menggunakan *propositional resolution*.

Exercise 4 (con’t)

Gunakan proposisi:

- *p*: seseorang pergi belajar;
- *q*: seseorang menyisir rambut;
- *r*: seseorang tampak menarik;
- *s*: seseorang rapi;
- *t*: seseorang memiliki kontrol diri.

Exercise: Syntax

5 Translasikan kalimat alami berikut ke dalam representasi propositional logic, dengan menggunakan proposisi sebagai berikut:

- p: Anda mendapat nilai A pada UAS
- q: Anda mengerjakan semua latihan pada buku
- r: Anda mendapat nilai akhir A

a) Anda mendapat nilai akhir A tetapi anda tidak mengerjakan semua latihan pada buku.

b) Anda mendapatkan nilai A pada UAS, anda mengerjakan semua latihan pada buku, dan anda mendapat nilai akhir A.

c) Untuk mendapatkan nilai akhir A, maka anda harus mendapatkan nilai A pada UAS.

d) Anda mendapatkan nilai A pada UAS tapi anda tidak mengerjakan semua latihan pada buku; meski demikian anda mendapatkan nilai akhir A.

e) Anda mendapat nilai akhir A jika dan hanya jika anda mengerjakan semua latihan pada buku atau mendapat nilai A pada UAS.

Review

- Propositional Logic:
  - Syntax: Simple Sentence, Compound Sentence
  - Semantics: Interpretation, Evaluation, Reverse Evaluation, Type of Sentences (valid, satisfiable, unsatisfiable)
  - Logical Entailments $\Rightarrow$
    - Semantic Reasoning: Truth Table, Validity Checking, Unsatisfiability Checking
    - Proof Method: Rule of inference, axiom schema, resolution

THANK YOU