Relational Logic: Semantic & Logical Entailment

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Review

- Reasoning: information ⇒ conclusion
- Computational Logic
  - **Propositional Logic:**
    - Syntax ⇒ Simple sentence, Compound Sentence
    - Semantics ⇒ interpretation, evaluation, reverse evaluation, types of compound sentence
    - Logical Entailment :
      - Semantic Reasoning ⇒ Two tables, Validity Checking, Unsatisfiability Checking
      - Proof Method ⇒ Rules of Inference, Axiom Schemata, Propositional Resolution
  - **Relational Logic:**
    - Syntax ⇒ Objects, Variable, Functions, Relations, Terms, Quantified Sentence
    - Semantic ⇒ Objects, Functions, Relations, Data, Model, Atomic sentence, Logical Sentence, Quantified Sentence

Objects

- An object is an entity presumed or hypothesized to exist in the world we are discussing.
  - Primitive: *a quark*
  - Composite: *an engine, this class*

  - Real: *Sun, Mike*
  - Imaginary: *a unicorn, Sherlock Holmes*

  - Physical: *Earth, Moon, Sun*
  - Abstract: *Justice*

Relations

- An *n-ary* relation is a property that holds of various combinations of *n objects.*

- *clear - true of a block iff it has no blocks above it.*
- *table - true of a block iff it is resting on the table.*

- *on - true of 2 blocks if f one is immediately on the other.*
- *above - true of 2 blocks if f one is anywhere above the other.*

- *below - true of 2 blocks if f one is anywhere below the other.*
- *stack - true of 3 blocks if f they form a stack of 3 blocks.*

Function

- An *n-ary* function is a relation associating each combination of n objects in a universe of discourse (called the arguments) with a single object (called the *value).*
- Numerical Examples:
  - Unary: *sqrt, log*
  - Binary: +, -, *, /
- People Examples:
  - Unary: *mother, father*

Function (2)

- Functions are *total and single-valued* - one and exactly one value for each combination of arguments.
- Partial - not defined for some combination of arguments
- Multivalued - more than one value for some argument combination
- *NB: We ignore partial and multi-valued functions.*

Ingredients for a Model

- *Universe of Discourse* - a set of object constants, one for each object under discussion.
- *Functional Basis Set* - a set of function constants, one for each function under discussion.
- *Relational Basis Set* - a set of relation constants, one for each relation under discussion.
- Note that, in some cases, we need more object constants than we can form from 26 letters and 10 digits. We can solve this problem by extending our alphabet. However, this won't be necessary in this course.

Data

- A *datum* is a ground, atomic sentence in which all arguments are object constants.

  on(a,b)

- Intuitively, a datum is true if and only if the relation holds of the arguments. It can also be viewed as an instance of a relation.

Models

- A *model is an arbitrary set of data.*

  {clear(a),clear(d),table(c),table(e),
  on(a,b),on(b,c),on(d,e),stack(a, b,c)}

- Intuitively, a model is the set of *all* data that are true in the world being considered. If a datum is *not* included in a model, it is assumed to be false in that model.

Atomic Sentences

- A *ground atomic* sentence ϕ is true in a model Γ (written $\models_\Gamma \phi$) if and only if ϕ is a member of Γ.

- Model:

  {*clear(a),clear(d),table(c),table(e),on(a,b), on(b,c),on(d,e),stack(a, b,c)*}

- True: *clear(a); clear(d)*
- Not True: *clear(b); clear(c); clear(e)*

Logical Sentences

- A negation is true if and only if its target is false.
- A conjunction is true if and only if every conjunct is true.
- A disjunction is true if and only if some disjunct is true.
- An implication is false if and only if the antecedent is true or the consequent is false.
- A reduction is false if and only if the antecedent is true or the consequent is false.
- An equivalence is true if and only if the arguments are either both false or both true.

Instances

- An *instance* of a sentence relative to a model is a sentence obtained by consistently substituting an object constant from the model's universe of discourse for each free variable in the sentence.

  $$p(x,y) \land q(x,b,z) \Rightarrow p(a,a) \land q(a,b,b)$$

- Note that we do not substitute for bound variables (until later).

  $$\exists y. \forall z.p(x,y,z) \Rightarrow \exists y. \forall z.p(a,y,z)$$

Quantified Sentences

- A *universally quantified sentence is true if and only if* every instance of the scope is true. An *existentially quantified sentence is true if and only if some instance of* the scope is true.
- Model:

  $\{clear(a), clear(d), table(c), table(e), on(a,b), on(b,c), on(d,e), stack(a, b,c)\}$
- True: Not True:
- $\forall x.(on(x,y) \Rightarrow \neg on(y,x))$ $\quad \forall x.on(x,y)$
- $\exists x.clear(x)$ $\quad \exists x.(table(x) \land clear(x))$

Open Sentences

- An *open sentence* is true in a model if and only if it satisfies every instance of the sentence relative to the model.
- True: Not True:

$on(x,y) \Rightarrow \neg on(y,x)$ $on(x,y)$

- This just formalizes the notion that free variables are universally quantified

Instances Again

- An instance of a sentence relative to a model is a sentence obtained by:

  (1) consistently substituting an object constant from the model for each *free variable in the* sentence

  (2) replacing all ground functional terms by their values in the model.
- Model:

  {boss(art)=art,boss(joe)=art}
- Example:

  p(x,boss(x)) menjadi p(joe,boss(joe)) menjadi p(joe,art)

Definitions

- A sentence is *valid* if and only if every model satisfies it.
- *A* sentence is *unsatisfiable* if and only if no model satisfies it.
- *A* sentence is *contingent* if and only there is some model that makes it true and some model that makes it false.
- A set of premises Δ *logically entails* a conclusion ϕ (written $\Delta \models \phi$) if and only if every model that satisfies the premises also satisfies the conclusion (i.e. $\models_\Gamma \Delta$ implies $\models_\Gamma \phi$, f or all Γ).

Review

- Modeling the World
  - Objects, Functions, Relations
  - Data
  - Models
- Semantics of Relational Logic
  - Atomic/ Relational Sentences
  - Logical Sentences
  - Quantified Sentences

Logical Entailment

- A sentence is *valid* if and only if every model satisfies it.
- *A* sentence is *unsatisfiable* if and only if no model satisfies it.
- *A* sentence is *contingent* if and only there is some model that makes it true and some model that makes it false.
- A set of premises Δ *logically entails* a conclusion ϕ (written $\Delta \models \phi$) if and only if every model that satisfies the premises also satisfies the conclusion (i.e. $\models_\Gamma \Delta$ implies $\models_\Gamma \phi$, f or all Γ).

Logical Entailment Checking

- Metode pemeriksaan langsung (interpretasi Herbrand) ⇒
  - Finite Relational Language (FRL): finite object constant, no function constant
  - Omega Relational Language: infinite object constant, no function constant
- Metode pembuktian (Proof Method)
  - Kaidah inferensi
  - Axiom schemata
  - Resolution

Kaidah Inferensi

- Modus ponen (MP): $(p \rightarrow q, p) \rightarrow q$
- Modus tollen (MT): $(p \rightarrow q, \sim q) \rightarrow \sim p$
- Simplifikasi/And Elimination (AE):
  $(p \wedge q) \rightarrow p, (p \wedge q) \rightarrow q$
- Konjungsi/And Introduction (AI):
  $(p, q) \rightarrow (p \wedge q)$
- Silogisme hipotetis (SH): $(p \rightarrow q, q \rightarrow r) \rightarrow (p \rightarrow r)$
- Silogisme disjungtif (SD): $(p \vee q, \sim p) \rightarrow q$
- Penjumlahan: $p \rightarrow (p \vee q)$
- Equivalence Elimination (EE):
  $(p \leftrightarrow q) \rightarrow (p \rightarrow q, q \rightarrow p)$

Kaidah Inferensi (2)

- Universal Instantiation (UI):

  $\forall x.\, \phi \quad \rightarrow \phi\, [x \square \tau]$

  $\forall x.\, p(x) \quad \rightarrow p(a)$

  - ϕ tidak mengandung free variabel
  - τ adalah konstanta objek baru atau fungsi dengan argumen yang bebas

- Contoh:

  Semua manusia dapat mati.

  $\forall x.\, (manusia(x) \rightarrow mati(x))$

  UI: manusia(Adam) → mati(Adam)

Kaidah Inferensi (3)

- UI dari $\forall x.\ sunny(x)$:
  $sunny(today)$
  $sunny(a)$
  …
- UI dari $\forall x.\ \exists y.kenal(x,y)$:
  $\exists y.kenal(Budi,y)$
  $\exists y.kenal(a,y)$
  $\exists y.kenal(f(a),y)$

  …
  $\exists y.kenal(y,y)$ salah!!
  $\exists y.kenal(f(y),y)$ salah!!

Kaidah Inferensi (4)

- Existential Instantiation (EI):
  $\exists x.\ \phi \quad \rightarrow \phi\ [x \square \tau]$
  $\exists x.\ p(x) \quad \rightarrow p(a)$
  $\exists x.\ p(x) \quad \rightarrow p(f(a))$
  $\exists x.\ p(x) \quad \rightarrow p(f(y))$
- ϕ tidak mengandung free variabel
- τ adalah konstanta objek baru atau fungsi dengan argumen yang bebas.

Kaidah Inferensi (5)

- Universal Generalization (UG):
  $\phi \rightarrow \forall x.\phi$, x bukan variabel bebas pada ϕ
  $A \rightarrow \forall x.A$

- Existential Generalization (EG):
  $\phi \rightarrow \exists x.\phi$, x bukan variabel bebas pada ϕ
  $A \rightarrow \exists x.A$

Contoh 1

- Semua manusia dapat mati. Adam adalah manusia. Buktikan bahwa Adam pun dapat mati.

1. $\forall x. (\text{manusia}(x) \rightarrow \text{mati}(x))$ premis
2. manusia(Adam) premis
3. manusia(Adam) → mati(Adam) UI 1
4. mati(Adam) MP 2,3

Contoh 2: Deduksi yang salah

- Ada kucing yang mencuri daging. Ada tikus yang mencuri daging.

1. $\exists x.(kucing(x) \land pencuri(x))$ premis
2. $\exists x.(tikus(x) \land pencuri(x))$ premis
3. $kucing(a) \land pencuri(a)$ EI 1
4. $tikus(a) \land pencuri(a)$ EI 2
5. $kucing(a)$ AE 3
6. $pencuri(a)$ AE 3
7. $tikus(a)$ AE 4
8. $pencuri(a)$ AE 4
9. $Kucing(a) \land tikus(a)$ AI 5,7

Contoh 3

- It is not sunny this afternoon and it is colder than yesterday. We will go swimming only if it is sunny. If we do not go swimming then we will take a canoe trip. If we take a canoe trip, then we will be home by sunset. Prove that we will be home by sunset.

Representasi: Logika Proposisi

- p: It is sunny this afternoon
- q: It is colder than yesterday.
- r: We will go swimming.
- s: We will take a canoe trip.
- t: we will be home by sunset.
- Premis: $\sim p \wedge q$, $r \rightarrow p$, $\sim r \rightarrow s$, $s \rightarrow t$
- Konklusi: t

Pembuktian: Logika Proposisi

- Premis: $\sim p \wedge q$, $r \rightarrow p$, $\sim r \rightarrow s$, $s \rightarrow t$
- Konklusi: t
- Bukti:
  - $\sim p \wedge q$ premis
  - $r \rightarrow p$ premis
  - $\sim r \rightarrow s$ premis
  - $s \rightarrow t$ premis
  - $\sim p$ simplifikasi 1
  - $\sim r$ Modus tollen 2,5
  - s Modus ponen 3,6
  - t Modus ponen 4,7

Jadi kesimpulan dapat ditarik dari premis yang ada.

Representasi: Logika Predikat

It is not sunny this afternoon and it is colder than yesterday.

~sunny(this_afternoon) $\wedge$ colder(this_afternoon,yesterday)

We will go swimming only if it is sunny.

swimming(we) $\rightarrow$ $\forall$x. sunny(x)

If we do not go swimming then we will take a canoe trip.

~swimming(we) $\rightarrow$ canoe_trip(we)

If we take a canoe trip, then we will be home by sunset.

canoe_trip(we) $\rightarrow$ homebysunset(we)

Prove that we will be home by sunset.

homebysunset(we)

Pembuktian: Logika Predikat

1. ~sunny(this_afternoon) $\wedge$ colder(this_afternoon,yesterday) premis
2. swimming(we) → ∀x. sunny(x) premis
3. ~swimming(we) → canoe_trip(we) premis
4. canoe_trip(we) → homebysunset(we) premis
5. swimming(we) → sunny(this_afternoon) UI 2
6. ~sunny(this_afternoon) AE 1
7. colder(this_afternoon,yesterday) AE 1
8. ~swimming(we) MT 5,6
9. canoe_trip(we) MP 3,8
10. homebysunset(we) MP 4,9

Jadi terbukti bahwa "we will be home by sunset"

Contoh 4

- We know that horses are faster than dogs and that there is a greyhound that is faster than every rabbit. We know that Harry is a horse and that Ralph is a rabbit. Derive the fact that Harry is faster than Ralph.

- Use relation constant:
  - horse(x), dog(x), faster(x,y), greyhound(x), rabbit(x)

- Additional Information:
  - All greyhounds are dogs
  - If x faster than y and y faster than z, then x is faster then z

Contoh 4: Representasi

- Horses are faster than dogs
  $\forall x,y.(horse(x) \land dog(y) \rightarrow faster(x,y))$
- There is a greyhound that is faster than every rabbit.
  $\exists x.(greyhound(x) \land \forall y.(rabbit(y) \rightarrow faster(x,y)))$
- Harry is a horse
  horse(Harry)
- Ralph is a rabbit.
  rabbit(Ralph)
- Harry is faster than Ralph.
  faster(Harry,Ralph)

- Greyhounds are dogs.
  $\forall x.(greyhound(x) \rightarrow dog(x))$
- $\forall x,y,z.(faster(x,y) \land faster(y,z) \rightarrow faster(x,z))$

Contoh 4: Kaidah Inferensi

1. $\forall x,y.(\text{horse}(x) \land \text{dog}(y) \rightarrow \text{faster}(x,y))$
2. $\exists x.(\text{greyhound}(x) \land \forall y.(\text{rabbit}(y) \rightarrow \text{faster}(x,y)))$
3. $\forall x.(\text{greyhound}(x) \rightarrow \text{dog}(x))$
4. $\forall x,y,z.(\text{faster}(x,y) \land \text{faster}(y,z) \rightarrow \text{faster}(x,z))$
5. horse(Harry)
6. rabbit(Ralph)
7. $\text{greyhound}(a) \land \forall y.(\text{rabbit}(y) \rightarrow \text{faster}(a,y)))$ EI 2
8. greyhound(a) AE 7
9. $\forall y.(\text{rabbit}(y) \rightarrow \text{faster}(a,y))$ AE 7
10. $\text{rabbit(Ralph)} \rightarrow \text{faster(a,Ralph)}$ UI 9
11. faster(a,Ralph) MP 6,10
12. $\text{greyhound}(a) \rightarrow \text{dog}(a)$ UI 3
13. dog(a) MP 8,12
14. $\forall y.(\text{horse(Harry)} \land \text{dog}(y) \rightarrow \text{faster(Harry},y))$ UI 1
15. $\text{horse(Harry)} \land \text{dog}(a) \rightarrow \text{faster(Harry},a)$ UI 14
16. $\text{Horse(Harry)} \land \text{dog}(a)$ AI 5,13
17. faster(Harry,a) MP 15,16
18. $\forall y,z.(\text{faster(Harry},y) \land \text{faster}(y,z) \rightarrow \text{faster(Harry},z))$ UI 4
19. $\forall y.(\text{faster(Harry},y) \land \text{faster}(y,\text{Ralph}) \rightarrow \text{faster(Harry,Ralph)})$ UI 18
20. $\text{faster(Harry},a) \land \text{faster}(a,\text{Ralph}) \rightarrow \text{faster(Harry,Ralph)}$ UI 19
21. $\text{faster(Harry},a) \land \text{faster}(a,\text{Ralph})$ AI 17,11
22. faster(Harry,Ralph) MP 20,21

Jadi terbukti bahwa “Harry is faster than Ralph”

Standard Axiom Schemata

- Implication Introduction (II):
  $\phi \rightarrow (\psi \rightarrow \phi)$
  $A \rightarrow (B \rightarrow A)$

- Implication Distribution (ID):
  $\phi \rightarrow (\psi \rightarrow \chi) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \chi))$
  $A \rightarrow (B \rightarrow C) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$

Standard Axiom Schemata (2)

- Contradiction Realization (CR):
  $(\psi\rightarrow\sim\phi)\rightarrow((\psi\rightarrow\phi)\rightarrow\sim\psi)$
  $(A\rightarrow\sim B)\rightarrow((A\rightarrow B)\rightarrow\sim A)$

  $(\sim\psi\rightarrow\sim\phi)\rightarrow((\sim\psi\rightarrow\phi)\rightarrow\psi)$
  $(\sim A\rightarrow\sim B)\rightarrow((\sim A\rightarrow B)\rightarrow A)$

Standard Axiom Schemata (3)

- Equivalence (EQ):
  $(\phi\leftrightarrow\psi)\rightarrow(\phi\rightarrow\psi)$
  $(A\leftrightarrow B)\rightarrow(A\rightarrow B)$

  $(\phi\leftrightarrow\psi)\rightarrow(\psi\rightarrow\phi)$
  $(A\leftrightarrow B)\rightarrow(B\rightarrow A)$

  $(\phi\rightarrow\psi)\rightarrow((\psi\rightarrow\phi)\rightarrow(\phi\leftrightarrow\psi))$
  $(A\rightarrow B)\rightarrow((B\rightarrow A)\rightarrow(A\leftrightarrow B))$

Standard Axiom Schemata (4)

- Other operators:
  $(\phi \leftarrow \psi) \leftrightarrow (\psi \rightarrow \phi)$
  $(A \leftarrow B) \leftrightarrow (B \rightarrow A)$

  $(\phi \vee \psi) \leftrightarrow (\sim\phi \rightarrow \psi)$
  $(A \vee B) \leftrightarrow (\sim A \rightarrow B)$

  $(\phi \wedge \psi) \leftrightarrow \sim(\sim\phi \vee \sim\psi)$
  $(A \wedge B) \leftrightarrow \sim(\sim A \vee \sim B)$

Standard Axiom Schemata (5)

- Universal Distribution (UD):
  $\forall x.(\phi \rightarrow \psi) \rightarrow (\forall x.\phi \rightarrow \forall x.\psi)$
  $\forall x.(A \rightarrow B) \rightarrow (\forall x.A \rightarrow \forall x.B)$

- Universal Generalization (UG):
  $\phi \rightarrow \forall x.\phi$, x bukan variabel bebas pada ϕ
  $A \rightarrow \forall x.A$

- Existential Generalization (EG):
  $\phi \rightarrow \exists x.\phi$, x bukan variabel bebas pada ϕ
  $A \rightarrow \exists x.A$

Standard Axiom Schemata (6)

- Universal Instantiation (UI):

  $\forall x.\phi \rightarrow \phi\ [x \square \tau]$, τ bebas untuk x pada ϕ

  $\forall x.A \rightarrow A\ [x \square \tau]$

- Existential Instantiation (EI):

  $\exists x.\ \phi \quad \rightarrow \phi\ [x \square \tau]$

  $\exists x.\ p(x) \quad \rightarrow p(a)$

  $\exists x.\ p(x) \quad \rightarrow p(f(a))$

  $\exists x.\ p(x) \quad \rightarrow p(f(y))$

- Existential Definition (ED):

  $\exists x.\phi \leftrightarrow \sim\forall x.\sim\phi$

  $\exists x.A \leftrightarrow \sim\forall x.\sim A$

Exercise 5

Diketahui fakta sebagai berikut :

1. Penjual software bajakan adalah seorang kriminal.
2. Ketua kelas mempunyai beberapa software bajakan dan semuanya dibeli dari Gayus.

Buktikan bahwa Gayus adalah seorang kriminal dengan menggunakan *relational proof* tanpa memakai standard axioms.

Exercise 6

Terdapat premis sebagai berikut:

- John owns a dog.
- Anyone who owns a dog is a lover-of-animals.
- Lovers-of-animals do not kill animals.
- Either John killed Tuna or curiosity killed Tuna.
- Tuna is a cat.
- All cats are animals.

Kesimpulan:

- Did curiosity kill Tuna?

Representasikan premis dan kesimpulan tersebut dalam relational logic. Buatlah pembuktian apakah kesimpulan tersebut dapat ditarik dari kumpulan premis yang ada, dengan metode kaidah inferensi (*rules of inference*) dan/ atau axiom schemata. Gunakanlah:

```
dog(x): x is a dog
own(x,y): x owns y
lover(x): x is a
lover-of-animals
animal(x): x is an animal
kill(x,y): x kills y
cat(x): x is a cat
```

Exercise 7

Diketahui fakta sebagai berikut

$$\forall x.(p(x) \rightarrow q(x)) \rightarrow \exists x.(r(x) \wedge s(x))$$

$$\forall x.(p(x) \rightarrow s(x)) \wedge \forall x.(s(x) \rightarrow q(x))$$

Buktikan bahwa kesimpulan: $\exists x.s(x)$

dapat ditarik dari kumpulan fakta tersebut dengan memanfaatkan kaidah inferensi dan/ atau *axiom schema*

Exercise 8

Diberikan kalimat dalam representasi logika relasional.

$\exists x. (\textbf{makanan}(x) \wedge \forall y. (\textbf{mahasiswa}(y) \rightarrow \textbf{suka}(y,x)))$

$\forall y.(\textbf{mahasiswa}(y) \rightarrow \exists x. (\textbf{makanan}(x) \wedge \textbf{suka}(y,x)))$

Terjemahkanlah kedua kalimat tersebut ke dalam bahasa alami, dengan aturan relasi sbb.

makanan(x) : x adalah makanan

mahasiswa(x) : x adalah mahasiswa

suka(x,y) : x suka y

Exercise 9

Diberikan pernyataan berikut:

All hungry animals are caterpillars. All caterpillars have 42 legs. Edward is a hungry animal. Therefore, Edward has 42 legs.

Buktikanlah dengan kaidah inferensi dan/atau standard axiom schemata. Gunakanlah *unary relation* hungry, caterpillar, dan 42legs

Exercise 10

- Jeki, seorang murid di kelas ini berumur 19 tahun. Setiap orang yang berumur 19 tahun boleh mendapatkan SIM. Buktikan bahwa **seseorang di kelas ini boleh mendapatkan SIM**, dengan menggunakan relational proof (kaidah inferensi saja untuk *relational logic*). Gunakan relasi kelas(x) untuk x ada di kelas ini, umur19(x) untuk x berumur 19 tahun, dan sim(x) untuk x boleh mendapatkan SIM.

THANK YOU