

## Matrix Calculus

### Gradient

Def. Directional Derivative

The directional derivative of function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  in the direction  $u$  is

$$\nabla_u f(x) = \lim_{h \rightarrow 0} \frac{f(x + hu) - f(x)}{h}$$

- When  $u$  is the  $i$ -th standard unit vector  $e_i$ , then  $\nabla_u f(x) = f'_i(x) = \frac{\partial f(x)}{\partial x_i}$ .
- For any  $n$ -dimensional vector  $u$ , the directional derivative of  $f$  in the direction of  $u$  can be represented as  $\nabla_u f(x) = \sum_{i=1}^n f'_i(x) \cdot u_i$ .

let  $g(h) = f(x + hu)$

– Proof.  $\Rightarrow \nabla_u f(x) = g'(0) = \lim_{h \rightarrow 0} \frac{f(x+hu) - g(0)}{h}$

$\because g'(h) = \sum_{i=1}^n f'_i(x) \frac{d}{dh}(x_i + hu_i) = \sum_{i=1}^n f'_i(x) u_i$

let  $h = 0 \therefore \nabla_u f(x) = \sum_{i=1}^n f'_i(x) u_i$

Def. Gradient

The gradient of  $f$  is a vector function  $\nabla f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  defined by

$$\nabla f(x) = \sum_{i=1}^n \frac{\partial f}{\partial x_i} e_i$$

$$\Rightarrow \nabla f(x) = \left[ \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right]$$

- $\nabla_u f(x) = \nabla f(x) \cdot u = \|\nabla f(x)\| \cos a$  Where  $u$  is a unit vector.
- When  $u = \nabla f(x)$  such that  $a = 0$ , we have the maximum directional derivative of  $f$ .

### Matrix Derivatives

- The derivative of  $f : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$  with respect to  $A$  is defined as:

$$\nabla f(A) = \begin{bmatrix} \frac{\partial f}{\partial A_{11}} & \dots & \frac{\partial f}{\partial A_{1n}} \\ \vdots & \ddots & \vdots \\ \frac{\partial f}{\partial A_{m1}} & \dots & \frac{\partial f}{\partial A_{mn}} \end{bmatrix}$$

trace

Def.  $tr A = \sum_{i=1}^n A_{ii}$

- $tr ABCD = tr DABC = tr CDAB = tr BCDA$
- $tr A = tr A^T, tr(A + B) = tr A + tr B, tr(aA) = a \cdot tr A$
- $\nabla_A tr AB = B^T, \nabla_{A^T} f(A) = (\nabla_A f(A))^T$
- $\nabla_A tr ABA^T C = CAB + C^T AB^T, \nabla_A |A| = |A|(A^{-1})^T$
- Funky trace derivative  $\nabla_{A^T} tr ABA^T C = B^T A^T C^T + BA^T C$

Jacobian Matrix

$$J = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

Hesse Matrix

$$G(x_0) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}_{x_0}$$

$$\square H(f) = J(\nabla f)$$