

## Logistic Regression

- Classification problem is similar to predict only a small number of discrete values instead of continuous values.

### 1. Logistic Function (Sigmoid Function)

$$g(z) = \frac{1}{1 + e^{-z}}$$

#### Properties

- Bound:  $g(z) \in (0, 1)$
- Symmetric:  $1 - g(z) = g(-z)$
- Gradient:  $g'(z) = g(z)(1 - g(z))$

### 2. Logistic Regression

$$h_{\theta}(x) = g(\theta^T x) = \frac{1}{(1 + e^{-\theta^T x})}$$

- $\theta^T x$  is called score.  $h_{\theta}$  is called logistic regression.
- $Pr(Y = 1|X = x; \theta) = h_{\theta}(x) = \frac{1}{1 + e^{-\theta^T x}}$
- $Pr(Y = 0|X = x; \theta) = 1 - h_{\theta}(x) = \frac{1}{1 + e^{\theta^T x}}$

#### Decision Boundary

- $Pr(Y = 1|X = x; \theta) = Pr(Y = 0|X = x; \theta) \Rightarrow \theta^T x = 0$
- $\therefore$  The decision boundary is a linear hyperplane.
- The score  $\theta^T x$  is also a measure of distance  $x$  from the hyperplane.

#### Probability Mass Function

$$p(y|x; \theta) = Pr(Y = y|X = x; \theta) = (h_{\theta}(x))^y (1 - h_{\theta}(x))^{1-y}, \text{ where } y \in \{0, 1\}$$

$$p(y|x; \theta) = \frac{1}{1 + \exp(-y\theta^T x)}, \text{ where } y \in \{-1, 1\} \text{ instead of } y \in \{0, 1\},$$

- Maximize the log likelihood  $\ell(\theta) = \log L(\theta) = \prod_{i=1}^m p(y|x; \theta)$
- $\ell(\theta) = \log L(\theta) = \sum_{i=1}^m (y^{(i)} \log h_{\theta}(x^{(i)}) + (1 - y^{(i)}) \log (1 - h_{\theta}(x^{(i)})))$ , still assume  $y \in \{-1, 1\}$

$$\frac{\partial}{\partial \theta_j} \ell(\theta) = \sum_{i=1}^m \frac{y^{(i)} - h_{\theta}(x^{(i)})}{h_{\theta}(x^{(i)})(1 - h_{\theta}(x^{(i)}))} \cdot \frac{\partial h_{\theta}(x^{(i)})}{\partial \theta_j} = \sum_{i=1}^m (y^{(i)} - h_{\theta}(x^{(i)})) x_j^{(i)}$$

### 3. Newton's Method

#### Properties

- Highly dependent on initial guess
- Quadratic convergence once it is sufficiently close to  $x^*$
- If  $f' = 0$ , only has linear convergence
- Is not guaranteed to converge at all, depending on function or initial guess

#### Update

$$x \leftarrow x - \frac{f'(x)}{f''(x)}$$

- For  $l : \mathbb{R}^n \rightarrow \mathbb{R}$ ,

$$\theta \leftarrow \theta - H^{-1} \nabla_{\theta} l(\theta), \text{ where } H_{i,j} = \frac{\partial^2 l(\theta)}{\partial \theta_i \partial \theta_j}$$

### 4. Multiclass Classification

- Transformation to binary
  - One-vs.-rest (OvR, train a single classifier per class, with the samples of that class as positive samples and all other samples as negative ones)
    - \*  $y^* = \arg \max_k f_k(x)$
    - \*  $f_k(x)$  implies high probability that  $x$  is in class  $k$ .
  - One-vs.-one (OvO, to train  $K(K-1)/2$  binary classifiers)
    - \*  $y^* = \arg \max_s (\sum_t f_{s,t}(x))$
    - \*  $f_{s,t}(x)$  implies that label  $s$  has higher probability than label  $t$ .
- Extension from binary
- Hierarchical classification

#### Softmax Regression

$$\begin{aligned} \ell(\theta) &= \sum_{i=1}^m \log p(y^{(i)} | x^{(i)}; \theta) \\ &= \sum_{i=1}^m \log \prod_{k=1}^K \left( \frac{\exp(\theta^{(k)T} x^{(i)})}{\sum_{k'=1}^K \exp(\theta^{(k')T} x^{(i)})} \right)^{\mathbb{I}(y^{(i)}=k)} \end{aligned}$$

- where  $\mathbb{I} : \{True, False\} \rightarrow \{0, 1\}$  is an indicator function.