Optimization

Lagrange Multiplier

Def. Lagrange Multiplier.

Use to convert a optimization problem with constraints to one without constraints.

$$\begin{array}{lll} \min_\omega & f(\omega) & \\ s.t. & g_i(\omega) \le 0 & i = 1, \cdots, k \quad \Rightarrow \\ & h_j(\omega) = 0 & j = 1, \cdots, l \end{array}$$
$$\min_{\omega,\alpha,\beta} \mathcal{L}(\omega, \alpha, \beta) = f(\omega) + \sum_{i=1}^k \alpha_i g_i(\omega) + \sum_{j=1}^l \beta_j h_j(\omega)$$

α_i, β_j are so-called Lagrange multiplier, $\alpha_i \ge 0$.

Lagrange Duality

- The solution to the dual problem provides a lower bound to the solution of the primal problem.

  Def. Lagrange Dual Function

$$\begin{aligned} \mathcal{G}(\alpha, \beta) &= \inf_{\omega \in \mathcal{D}} \mathcal{L}(\omega, \alpha, \beta) \\ &= \inf_{\omega \in \mathcal{D}} \left(f(\omega) + \sum_{i=1}^k \alpha_i g_i(\omega) + \sum_{j=1}^l \beta_j h_j(\omega) \right) \end{aligned}$$

- The Lagrange dual problem with respect to the primal problem. The optimal value is d^*, and $d^* \le p^*$.

$$\begin{array}{lll} \max_{\alpha,\beta} & \mathcal{G}(\alpha, \beta) & \\ s.t. & \alpha \ge 0 & \forall i = 1, \cdots, k \end{array}$$

Karush-Kuhn-Tucker (KKT) Conditions

- Let ω^* be a primal optimal point and (α^*, β^*) be a dual optimal solution.

  Def. KKT Conditions

  - Stationarity: $\nabla f(\omega^*) + \sum_{i=1}^k \alpha_i^* \nabla g_i(\omega^*) + \sum_{j=1}^l \beta_j^* \nabla h_j(\omega^*) = 0$
  - Primal feasibility: $g_i(\omega^*) \le 0, \forall i = 1, \cdots, k$
    $h_j(\omega^*) = 0, \forall j = 1, \cdots, l$
  - Dual feasibility: $\alpha_i^* \ge 0, \forall i = 1, \cdots, k$
  - Complementary slackness: $\alpha_i^* g_i(\omega^*) = 0,\ \forall i = 1, 2, \cdots, k$

- Proof. Stationarity Condition
  - ω^* is the minimizer of $\mathcal{L}(\omega, \alpha^*, \beta^*)$ over ω. Thus, $\nabla \mathcal{L} = 0$
- The primal feasibility conditions holds natrually.
- Proof. Dual Feasibility

- If $\alpha \geq 0$ and $\tilde{\omega}$ is feasible, then
- $f(\tilde{\omega}) \geq \mathcal{L}(\tilde{\omega}, \alpha, \beta) \geq \mathcal{G}(\alpha, \beta) = \inf_{\omega \in \mathcal{D}} \mathcal{L}(\omega, \alpha, \beta)$

- Proof. Complementary Slackness
  - If strong duality holds, then
    $f(\omega^*) = \mathcal{G}(\alpha^*, \beta^*)$
  - $$\leq f(\omega^*) + \sum_{i=1}^{k} \alpha^* g_i(\omega^*) + \sum_{j=1}^{l} \beta_j^* h_J(\omega^*)$$
    $$\leq f(\omega^*)$$
  - $\therefore \sum_{i=1}^{k} \alpha_i^* g_i(\omega^*) = 0$
  - Since each term is nonpositive, $\alpha_i^* g_i(\omega) = 0$.

Convex Optimization

- If objective function $f(\omega)$ and inequality constraints $g_i(\omega)$ are convex, and the equality constraints $h_j(\omega)$ are affine functions. A convex optimization problem can be represented by

$$\begin{array}{ll} \text{min}_\omega & f(w) \\ \text{s.t.} & g_i(w) \leq 0, i = 1, \cdots, k \\ & Aw - b = 0 \end{array}$$

- where, $A \in \mathbb{R}^{l \times n}$ and $b \in \mathbb{R}^l$.

  Theorem. Slarter's Condition (one of so-called constraint qualification, a sufficient condition)

  Strong duality holds for a convex problem if it is strictly feasible, i.e.,

$$\exists \omega \in \text{relint } \mathcal{D} : g_i(\omega) < 0, i = 1, \cdots, m, Aw = b$$

  relint (relative interior)