

## Optimization

### Lagrange Multiplier

Def. Lagrange Multiplier.

Use to convert a optimization problem with constraints to one without constraints.

$$\begin{aligned} \min_{\omega} \quad & f(\omega) \\ \text{s.t.} \quad & g_i(\omega) \leq 0 \quad i = 1, \dots, k \Rightarrow \\ & h_j(\omega) = 0 \quad j = 1, \dots, l \\ \min_{\omega, \alpha, \beta} \quad & \mathcal{L}(\omega, \alpha, \beta) = f(\omega) + \sum_{i=1}^k \alpha_i g_i(\omega) + \sum_{j=1}^l \beta_j h_j(\omega) \end{aligned}$$

$\alpha_i, \beta_j$  are so-called Lagrange multiplier,  $\alpha_i \geq 0$ .

### Lagrange Duality

- The solution to the dual problem provides a lower bound to the solution of the primal problem.

Def. Lagrange Dual Function

$$\begin{aligned} \mathcal{G}(\alpha, \beta) &= \inf_{\omega \in \mathcal{D}} \mathcal{L}(\omega, \alpha, \beta) \\ &= \inf_{\omega \in \mathcal{D}} \left( f(\omega) + \sum_{i=1}^k \alpha_i g_i(\omega) + \sum_{j=1}^l \beta_j h_j(\omega) \right) \end{aligned}$$

- The Lagrange dual problem with respect to the primal problem. The optimal value is  $d^*$ , and  $d^* \leq p^*$ .

$$\begin{aligned} \max_{\alpha, \beta} \quad & \mathcal{G}(\alpha, \beta) \\ \text{s.t.} \quad & \alpha \geq 0 \quad \forall i = 1, \dots, k \end{aligned}$$

### Karush-Kuhn-Tucker (KKT) Conditions

- Let  $\omega^*$  be a primal optimal point and  $(\alpha^*, \beta^*)$  be a dual optimal solution.

Def. KKT Conditions

- Stationarity:  $\nabla f(\omega^*) + \sum_{i=1}^k \alpha_i^* \nabla g_i(\omega^*) + \sum_{j=1}^l \beta_j^* \nabla h_j(\omega^*) = 0$
- Primal feasibility:  $g_i(\omega^*) \leq 0, \forall i = 1, \dots, k$   
 $h_j(\omega^*) = 0, \forall j = 1, \dots, l$
- Dual feasibility:  $\alpha_i^* \geq 0, \forall i = 1, \dots, k$
- Complementary slackness:  $\alpha_i^* g_i(\omega^*) = 0, \forall i = 1, 2, \dots, k$
- Proof. Stationarity Condition
  - $\omega^*$  is the minimizer of  $\mathcal{L}(\omega, \alpha^*, \beta^*)$  over  $\omega$ . Thus,  $\nabla \mathcal{L} = 0$
- The primal feasibility conditions holds naturally.
- Proof. Dual Feasibility

- If  $\alpha \geq 0$  and  $\tilde{\omega}$  is feasible, then
- $f(\tilde{\omega}) \geq \mathcal{L}(\tilde{\omega}, \alpha, \beta) \geq \mathcal{G}(\alpha, \beta) = \inf_{\omega \in \mathcal{D}} \mathcal{L}(\omega, \alpha, \beta)$
- Proof. Complementary Slackness
  - If strong duality holds, then
  - $f(\omega^*) = \mathcal{G}(\alpha^*, \beta^*)$
  - $$\leq f(\omega^*) + \sum_{i=1}^k \alpha_i^* g_i(\omega^*) + \sum_{j=1}^l \beta_j^* h_j(\omega^*)$$
  - $$\leq f(\omega^*)$$
  - $\therefore \sum_{i=1}^k \alpha_i^* g_i(\omega^*) = 0$
  - Since each term is nonpositive,  $\alpha_i^* g_i(\omega) = 0$ .

#### Convex Optimization

- If objective function  $f(\omega)$  and inequality constraints  $g_i(\omega)$  are convex, and the equality constraints  $h_j(\omega)$  are affine functions. A convex optimization problem can be represented by

$$\begin{aligned} \min_{\omega} \quad & f(w) \\ \text{s.t.} \quad & g_i(w) \leq 0, i = 1, \dots, k \\ & Aw - b = 0 \end{aligned}$$

- where,  $A \in \mathbb{R}^{l \times n}$  and  $b \in \mathbb{R}^l$ .

Theorem. Slater's Condition (one of so-called constraint qualification, a sufficient condition)

Strong duality holds for a convex problem if it is strictly feasible, i.e.,

$$\exists \omega \in \text{relint } \mathcal{D} : g_i(\omega) < 0, i = 1, \dots, m, Aw = b$$

relint (relative interior)