Optimization

Lagrange Multiplier

Def. **Lagrange Multiplier**.

Use to convert a optimization problem with constraints to one without constraints.

$$
\begin{aligned}
&\min_\omega && f(\omega) \\
&s.t. && g_i(\omega) \le 0 && i = 1, \cdots, k \quad \Rightarrow \\
& && h_j(\omega) = 0 && j = 1, \cdots, l \\
&\min_{\omega,\alpha,\beta} \mathcal{L}(\omega, \alpha, \beta) = f(\omega) + \sum_{i=1}^{k} \alpha_i g_i(\omega) + \sum_{j=1}^{l} \beta_j h_j(\omega)
\end{aligned}
$$

α_i, β_j are so-called Lagrange multiplier, $\alpha_i \ge 0$.

Lagrange Duality

- The solution to the dual problem provides a lower bound to the solution of the primal problem.

  Def. Lagrange Dual Function

$$
\begin{aligned}
\mathcal{G}(\alpha, \beta) &= \inf_{\omega \in \mathcal{D}} \mathcal{L}(\omega, \alpha, \beta) \\
&= \inf_{\omega \in \mathcal{D}} \left(f(\omega) + \sum_{i=1}^{k} \alpha_i g_i(\omega) + \sum_{j=1}^{l} \beta_j h_j(\omega) \right)
\end{aligned}
$$

- The *Lagrange dual problem* with respect to the primal problem. The optimal value is d^*, and $d^* \le p^*$.

$$
\begin{aligned}
&\max_{\alpha,\beta} && \mathcal{G}(\alpha, \beta) \\
&s.t. && \alpha \ge 0 && \forall i = 1, \cdots, k
\end{aligned}
$$

Karush-Kuhn-Tucker (KKT) Conditions

- Let ω^* be a primal optimal point and (α^*, β^*) be a dual optimal solution.

  Def. **KKT Conditions**

  - Stationarity: $\nabla f(\omega^*) + \sum_{i=1}^{k} \alpha_i^* \nabla g_i(\omega^*) + \sum_{j=1}^{l} \beta_j^* \nabla h_j(\omega^*) = 0$
  - Primal feasibility: $g_i(\omega^*) \le 0, \forall i = 1, \cdots, k$
    $h_j(\omega^*) = 0, \forall j = 1, \cdots, l$
  - Dual feasibility: $\alpha_i^* \ge 0, \forall i = 1, \cdots, k$
  - Complementary slackness: $\alpha_i^* g_i(\omega^*) = 0, \ \forall i = 1, 2, \cdots, k$

- Proof. **Stationarity Condition**
  - ω^* is the minimizer of $\mathcal{L}(\omega, \alpha^*, \beta^*)$ over ω. Thus, $\nabla \mathcal{L} = 0$
- The primal feasibility conditions holds natrually.

- Proof. **Dual Feasibility**
  - If $\alpha \geq 0$ and $\tilde{\omega}$ is feasible, then
  - $f(\tilde{\omega}) \geq \mathcal{L}(\tilde{\omega}, \alpha, \beta) \geq \mathcal{G}(\alpha, \beta) = \inf_{\omega \in \mathcal{D}} \mathcal{L}(\omega, \alpha, \beta)$
- Proof. **Complementary Slackness**
  - If strong duality holds, then
    $f(\omega^*) = \mathcal{G}(\alpha^*, \beta^*)$

  - $$\leq f(\omega^*) + \sum_{i=1}^{k} \alpha^* g_i(\omega^*) + \sum_{j=1}^{l} \beta_j^* h_J(\omega^*)$$

    $\leq f(\omega^*)$
  - $\therefore \sum_{i=1}^{k} \alpha_i^* g_i(\omega^*) = 0$
  - Since each term is nonpositive, $\alpha_i^* g_i(\omega) = 0$.

Convex Optimization

- If objective function $f(\omega)$ and inequality constraints $g_i(\omega)$ are convex, and the equality constraints $h_j(\omega)$ are affine functions. A convex optimization problem can be represented by

$$\begin{aligned} \min_{\omega} \quad & f(w) \\ \text{s.t.} \quad & g_i(w) \leq 0, i = 1, \cdots, k \\ & Aw - b = 0 \end{aligned}$$

- where, $A \in \mathbb{R}^{l \times n}$ and $b \in \mathbb{R}^{l}$.

  Theorem. **Slarter's Condition** (one of so-called *constraint qualification*, a sufficient condition)

  Strong duality holds for a convex problem if it is strictly feasible, i.e.,
  $$\exists \omega \in \text{relint } \mathcal{D} : g_i(\omega) < 0, i = 1, \cdots, m, Aw = b$$

  relint (relative interior)