Graphtheory Theorems + Exercises

1 Basics

Definitions

- A graph G is *non-trivial* if it contains at least one edge, equivalently if G is not an empty graph
- The *order* of G writen $|G|$, is the number of vertices of G, i.e. $|G| = |V|$
- The *size* of G wirten $||G||$, is the number of edges of G, i. e. $||G|| = |E|$, if order of G is n then the size of G is between 0 and $\binom{n}{2}$
- $N(S)$ the *neighbourhood* of $S \subseteq V$ is the set of vertices in V. that have and adjacent vertex in S. Instead of $N(\{v\})$ for $v \in V$ we write $N(v)$
- vertex of degree 1 is called *leaf*
- vertex of degree 0 is called *isolated vertex*
- *minimum degree of* G, denoted by $\delta(G)$ is the smallest vertex degree in G
- *maximum degree of* G, denoted by $\Delta(G)$ is the highest vertex degree in G
- graph G is called *k-regular*, with $k \in \mathbb{N}$, if all vertices have degree k.
- *average degree of* G is defined as $d(G) = \frac{\sum_{v \in V} deg(v)}{|V|}$
  We have
  $$\delta(G) \leq d(G) \leq \Delta(G)$$
  with equality if and only if G is k-regular

Handshake Lemma For ever graph $G = (V, E)$ we have

$$2|E| = \sum_{v \in V} d(v)$$

Proof. By double counting the set $X = \{(e, x) : e \in E(G), x \in V(G), x \in e\}$ then

$$|X| = \sum_{v \in V(G)} d(x)$$

and

$$|X| = \sum_{e \in E(G)} 2 = 2|E(G)|$$

by the principle of double counting the terms are equal. □

Corollary From this follows that the sum of all vertex degrees is even and therefore the number of vertices with odd degree is even.

Proposition 3 If a graph G has minimum degree $\delta(G) \geq 2$, then G has a path of length $\delta(G)$ and a cycle with at least $\delta(G) + 1$ vertices.

Proof. Let $P = (x_0, ..., x_k)$ be a longest path in G. Then $N(x_0) \subseteq V(P)$, otherwise for $x \in N(x_0) \setminus V(P)$ the path $(x, x_0, x_1, ..., x_k)$ would be a longer path.

Let i be the largest index such that $x_i \in N(x_0)$, then $i \geq |N(x_0)| \geq \delta$. So $(x_0, x_1, ..., x_i, x_0)$ is a cycle of length at least $\delta(G) + 1$. □

Proposition 4 If for distinct vertices u and v a graph has a u-v-walk, then it has a u-v-path.

Proof. Consider a u-v-walk W with the smallest number of edges. Assume that W does not form a path, then there is a repeated vertex, w, i.e.

$$W = u, e, v_1, e_1, ..., e_k, w, e_{k+1}, ..., e_l, w, e_{l+1}, ..., v$$

Then $W_1 = u, e, v_1, ..., e_k, w, e_{k+1}, ..., v$ is a shorter u-v-walk, a contradiction. □

Propostion 5 If a graph has a closed walk of odd length, then it contains an odd cycle.

Proof. Let W be the shortest closed odd walk. If W is a cycle the Proposition holds. Otherwise there is a repeated vertex, so W is an edge-disjoint union of two closed walks. The sum of the lengths of these walks is odd, therefore one of them is an odd closed walk shorter than W a contradiction to the minimality of W. □

Proposition 6 If a graph has a closed walk with a non-repeated edge, then the graph contains a cycle.

Proof. Let W be a shortest walk with a non-repeated edge e. If W is a cycle, we are done. Otherwise, there is a repeated vertex and W is a union of two closed walks W_1 and W_2 that are shorter than W. One of them say W_1, contains e, a non-repeated edge. This contradicts the minimality of W. □

Definition bipartite A graph $G = (V, E)$ is called *bipartite* if there exists natural numbers m, n such that G is isomorphic to a subgraph of $K_{m,n}$. Then the vertex set can be written as $V = A \cup B$ such that $E \subseteq \{ab : a \in A, b \in B\}$. The sets A and B are called the *partite sets* of G

Proposition 1.5 A graph is bipartite if and only if it has no cycles of odd length.

Proof. skript
"$\Rightarrow$"
Assume that G is a bipartite graph with parts A and B. Then any cycle has a form $a_1, b_1, a_2, b_2, ..., a_k, b_k, a_1$ where $a_i \in A, b_i \in B, i \in [k]$. Thus every cycle has even length.
"$\Leftarrow$"
Assume G does not have cycles of odd length. We can assume that G is

connected, otherwise we can treat the connected components separately. Let $v \in V(G)$. Let $A = \{u \in V(G) : dist(u,v) \equiv 0 \pmod 2\}$ and let $B = \{u \in V(G) : dist(u,v) \equiv 1 \pmod 2\}$ We claim that G is bipartite with parts A and B. To verify this it is sufficient to prove that A and B are independent sets. Let $u_1u_2 \in E(G)$ and let P_1 be a shortest u_1-v-path and P_2 a shortest u_2-v-path. Then the union of P_1, P_2 and u_1u_2 forms a closed walk W. If $u_1, u_2 \in A$ or $u_1, u_2 \in B$ then W is a closed odd walk, because $dist(v,u_1)$ and $dist(v,u_2)$ are both even or odd. Thus by Prop. 5 G contains an odd cycle, a contradiction. Thus for any edge u_1u_2 the adjacent vertices u_1 and u_2 are in different parts A or B. Therefore A and B are independent sets. ☐

Proof. Diestel
"$\Leftarrow$"
Let T be a spanning tree in G, pick a root $r \in T$ and denote the associated tree-order on V by $\leq_T$ (this order expressing height if $x < y$ then x lies *below* y in T). For each $v \in V(G)$ the unique path r-v-T has odd or even length. This defines a bipartition of $V(G)$, we show that G is bipartite with this partition. Let $e = xy$ be an edge of G. If $e \in T$ with $x <_T y$ say, then r-y-T $=$ r-xy-T and so x and y lie in different partition classes. If $e \notin T$ then $C_e := x$-y-T+ e is a cycle, and by the case treated already the vertices along x-y-T alternate between the two classes. Since C_e is even by assumption, x and y again lie in different classes. ☐

Euler tour A closed walk that traverses every edge of the graph exactly once is called an *Euler tour*.

Theorem 1.6 (Eulerian Tour Condition) A connected graph has an Eulerian Tour if and only if every vertex has even degree.

Proof. "$\Rightarrow$"
The degree condition is necessary for an euler tour, because a vertex appearing k times in an Euler tour (or $k+1$ times if it is the starting and finishing vertex) must have degree $2k$.
"$\Leftarrow$"
Show by induction on $||G||$ that every connected Graph G with all degrees even has an Euler tour. $||G|| = 0$ is trivial.

Now let $||G|| \geq 1$, since all degrees are even, we can find in G a non-trivial closed walk that contains no edge more than once. To find this walk we consider W a walk of maximal length and write F for the set of its edges. If $F = E(G)$, then W is an Euler tour.

Suppose, therefore $G' := G - F$ has an edge.
For every vertex $v \in G$, an even number of edges of G at v lies in F, so the degrees of G' are again all even. Since G is connected, G' has an edge e incident with a vertex on W. By I.H. the component C of G' containing e has an Euler tour. Concatenating this with W (suitably re-indexed), we obtain a closed walk in G that contradicts the maximal length of W.

☐

Definitions

- graph G is *connected* if any two vertices are linked by a path.
- a maximal connected subgraph of G is called a *connected component* of G.
- acyclic graphs are called $forests$
- a graph G is called a *tree* if G is connected and acyclic.

Lemma 7 Every tree on at least two vertices has a leaf.

Proof. If a tree T on at least two vertices does not have leaves then every vertex has degree $>$ than 2, so we have a cycle in T with length ≥ 3, a contradiction. □

Lemma 8 A tree of order $n \geq 1$ has exactly $n - 1$ edges.

Proof. We prove the statement by induction on n. When $n = 1$, there are no edges.

I.H.: Assume that each tree on $n = k$ vertices has $k - 1$ edges, with $k \geq 1$.

Step: Lets prove that each tree on $k + 1$ vertices has k edges. Consider a tree T on $k + 1$ vertices. Since $k + 1 \geq 2$, T has a leaf v. Let $T' = T - \{v\}$. We see that T' is connected because any u-w-path in T, for $u \neq v$ and $w \neq v$, does not contain v. We see also that T' is acyclic, because deleting vertices from an acyclic graph does not create new cycles. Thus T' is a tree on k vertices. By I.H. $|E(T')| = k - 1$. Thus $|E(T)| = |E(T')| + 1 = (k - 1) + 1 = k$.

□

Lemma 9 Every connected graph contains a spanning tree.

Proof. Let G be a connected graph. Consider T, an acyclic spanning subgraph of G with largest number of edges. If it is a tree we are done.

Otherwise, T has more than one component. Consider vertices u and v from different components of G. Consider a shortest u-v-path P in G. Then P has an edge $e = xy$ with exactly one vertex x in one of the components of T. Then P has an edge $e = xy$ with exactly one vertex x in one of the components of T. Then $T \cup \{e\}$ is acyclic. If there would be a cycle, it would contain e, however e connects to components, therefore cannot be part of a cylce (e would be a repeated edge). Thus $T \cup \{e\}$ is a bigger spanning acyclic subgraph of G contradicting the maximality of T. □

Lemma 10 A connected graph on $n \geq 1$ vertices and $n - 1$ edges is tree.

Proof. Let G be a connected graph on n vertices with $n - 1$ edges. Assume G is not a tree, i.e. contains a cycle. We therefore can remove a edge so that G is still connected. This is a contradiction because a graph on n vertices with $n - 2$ edges cannot be connected. Because a walk from vertex 1 to vertex n has to have at least $n - 1$ edges. □

Lemma 11 The vertices of every connected graph on $n \geq 2$ vertices can be ordered $(v_1, ..., v_n)$ so that for every $i \in \{1, ..., n\}$ the Graph $G[\{v_1, ..., v_i\}]$ is connected.

Proof. skript
Let G be a connected graph on n vertices. It contains a spanning tree T. Let v_n be a leaf of T, let v_{n-1} be a leaf of $T - \{v_n\}$, let v_{n-2} be a leaf of $T - \{v_n, v_{n-1}\}$ and so on. Let v_k be a leaf in $T - \{v_n, v_{n-1}, ..., v_{k+1}\}$, $k = 2, ..., n$. Since deleting a leaf does not disconnect a tree, all resulting graphs form a spanning trees of $G[v_1, ..., v_i]$, $i = 1, ..., n$. A graph H having a spanning tree or any connected spanning subgraph H' is connected because a u-v-path in H' is a u-v-path in H. This observation completes the proof. □

Proof. diestel
Pick any vertex as v_1, and assume inductively that $v_1, ..., v_i$ have been chosen for some $i < |G|$. Now pick a vertex $v \in G - G_i$. As G is connected, it contains a v-v_1 path P. Choose v_{i+1} as the last vertex of P in $G - G_i$, then v_{i+1} has a neighbour in G_i. If we consider $i + 1$ then we simply add v_{i+1} to our G_i, Thus $G_{i+1} := G_i \cup \{v_{i+1}\}$ which is also connected. □

Tree equivalences For any graph $G = (V, E)$ the following are equivalent:

(i) G is a tree, i.e. G is connected and acyclic.

(ii) G is connected, but for any $e \in E$ the graph $G - e$ is not connected (minimally connected)

(iii) G is acyclic, but for any $x, y \in V(G), xy \notin E(G)$ the graph $G + xy$ has a cycle. (maximaly acyclic)

(iv) G is connected and 1-degenerate

(v) G is connected and $|E| = |V| - 1$

(vi) G is acyclic and $|E| = |V| - 1$

(vii) G is connected and every non-trivial subgraph of G has a vertex of degree at most 1.

(viii) Any two vertices are joined by a unique path in G.

Proof.
(i) ⇒ (ii):
G is connected and acyclic, now assume for any edge $e = xy$ the graph $G' = G - e$ would still be connected. Then G' has a x-y-path P. But $P \cup e$ is a cycle in G which contradicts that G is acyclic.
(ii) ⇒ (i):
G is connected and for any edge e the graph $G - e$ is not connected. We want to show that G is acyclic. If G would have a cycle we could simply remove an edge from that and the resulting graph would still be connected, a contradiction. □

Proof. (i) $\Rightarrow$ (iv):

(vi) $\Rightarrow$ (i):

$\square$

Proof. (i) $\Rightarrow$ (vii):

(vii) $\Rightarrow$ (i):

$\square$

Proof. (i) $\Rightarrow$ (viii):

(viii) $\Rightarrow$ (i):

$\square$

Definition d-degenerate If there is a vertex ordering $v_1, ..., v_n$ of G and a $d \in \mathbb{N}$ such that

$$|N(v_i) \cap \{v_{i+1}, ..., v_n\}| \leq d$$

for all $i \in [n-1]$ then G is called *d-degenerate*. The minimum d for which G is *d-degenerate* is called the *degeneracy* of G.

Every finite planar graph has a vertex of degree five or less, therefore every planar graph is 5-degenerate.

Definition arboricity The least number of trees that can cover the edges of a graph is its arborictiy.

It is a measure for the graphs maximum local density: it is small if and only if the graph is nowhere dense, in the sense that there is no subgraph H with large $\epsilon(H) = \frac{E(H)}{V(H)}$.

Definition Contract For an edge $e = xy$ in G we define $G \circ e$ as the graph obtained from G by identifying x and y and removing (if necessary) loops and multiple edges. We say that $G \circ e$ arises from G by *contracting the edge e*.

Definition Complement The *complement* of G, denoted by $\overline{G}$ is defined as the graph $(V, \binom{V}{2} \setminus E)$. In particular $G + \overline{G}$ is a complete graph and $\overline{G} = (G + \overline{G}) - E$.

Definitions

- *girth of* G, denoted by $g(G)$ is the length of the shortest cycle in G, if G is acyclic, its girth is said to be ∞
- *circumference of* G, is the length of the longest cycle if G is acyclic the circumference is said to be 0
- G is called *Hamiltonian* if G has a spanning cycle, i.e. a cycle that contains every vertex of G. In other words the circumference is $|V|$
- G is called *traceable* if G has a spanning path

- For two vertices v and u in G, the *distance between u and v* , denoted by $d(v,u)$ is the length of a shortest u-v-path in G. If no such path exists $d(u,v) = \infty$
- The *diameter* of G, denoted by $\operatorname{diam}(G)$, is the maximum distance among all pairs of vertices in G, i.e.

$$\operatorname{diam}(G) = \max_{u,v \in V} d(u,v)$$

- *eccentricity*, $\operatorname{ecc}(v)$ is the greatest distance of v to any other vertex.
- The *radius of G* , denoted by $\operatorname{rad}(G)$ is defined as

$$\operatorname{rad}(G) = \min_{u \in V} \max_{n \in V} d(u,v)$$

  its the vertex that has the smallest eccentricity

problem sheets 1 and 2

problem 1

problem 2

problem 3

problem 4

problem 5

problem 6

problem 7

2 Important Graphs

Complete Graph, Clique:
the *complete graph K_n* on n vertices is isomorphic to $([n], \binom{[n]}{2})$

Cycle
C_n on n vertices with $n \geq 3$ is isomorphic to $([n], \{\{i, i+1\} : i = 1, ..., n-1\} \cup \{n, 1\})$, the *length of a cycle* is its number of edges.

Empty Graph
E_n on n vertices is isomorphic to $([n], \varnothing)$.
Empty graphs correspond to *independent sets*.

Complete Bipartite Graph
$K_{m,n}$ on $n+m$ vertices is isomorphic to $(A \cup B, \{xy : x \in A, y \in B\})$ where $|A| = m$ and $|B| = n$ and $A \cap B = \varnothing$.

Complete r-partite graph with $r \geq 2$ is isomorphic to

$$(A_1 \cup ... \cup A_r, \{xy : x \in A_i, y \in A_j, i \neq j\})$$

where $A_1, ..., A_r$ are disjoint non-empty finite sets.