Graphtheory Proof Techniques

General Problem Solving, Testprep

1. consider smallest, largest, maximal, minimal Set with some porperty (very often and in different forms)
2. small information are important for example that a set has to have more than two vertices, this will be used later etc.
3. draw alot of **figures** to stimulate thinking and discovering patterns, but keep in mind that they could be misleading
4. Look at small examples
5. what similar problems theorems do I know what is different
6. take a step back and dont get fixiated on the first idea and try a completly different way of looking at the problem
7. start with the most difficult problems
8. work on a problem as long as I can make progress, if I do not know how to continue switch to another problem
9. see solution in big picture, "does this make sense?"
10. relax while taking test, focus on breathing
11. see test as sports event to show of all the hard work in the past

1 Graph invariants, properties and their relations

- bipartite
- density, amount of edges
- forest, tree
- girth (length of shortest cycle), circumference (longest cycle)
- diameter (longest shortest path), radius (shortest longest shortest path, considering all vertices)
- d-degenerate, degeneracy
- vertex degrees (min, max, average)
- k-factor
- structural implication (subgraphs), cycles, triangles
- smallest vertex cover
- largest matching
- clique number $\omega(G)$

- co-clique number $\alpha(G)$
- perfect graph
- independent sets
- connectivity $\kappa(G)$
- edge-connectivity $\kappa'(G)$
- planarity
- outerplanar
- minor, topologiacl minor
- chroamtic number $\chi(X)$
- critical chroamtic number
- item edge-chroamtic number $\chi'(G)$
- list-coloring number
- extremal number
- zarankiewicz function
- Ramsey number
- property of graphs: almost always, threshold functions
- hamiltonian cycles

2 Proof Techniques

1. Induction

2. Extremal principle, with Contradiction

- Consider a longest path ...

3. Counting Arguments

- Double Counting, Example:
  - $ex(n, K_{t,t}) \leq c \cdot n^{2-\frac{1}{t}}$ (skript Theorem 67)
- Pigeonhole Principle
- Parity Arguments (even vs. odd)

4. Algorithmic, Iterative ("Just do it")

5. "Dichotomy", Ramsey Either a red coloring has a structure we want or if not then this implies some structural information about the blue coloring.

6. Probabilistic Method

- $\mathbb{P}(\bigcup$"Bad Event"$) < 1$, therefore the Probablity that none of these Bad Events happen is greater than 0, this simple fact often allows to show that some object with desired "good" properties exists.
- Conputing $\mathbb{E}X$ (using linearity of $\mathbb{E}$), Example:
  - Computing the expected number of k-cycles in $G \in \mathcal{G}(n,p)$
- Alterations (random Object has some unwanted structure, simply destroy it by removing edge etc.), Example:
  - $ex(n, K_{t,t}) \geq c \cdot n^{2-\frac{1}{t+1}}$ (skript Theorem 69)

7. Apply a Theorem!
Most used:

- Regularity Lemma, graph removal, triangle removal
- Fit Ramsey number onto a Problem
- Eulers Formula (plane graphs)
- Menger's Theorem
- Hall's Condition
- Kőnigs Theorem

8. Define Auxiliary Graph
Examples:

- For Hall's condition (Problem 8)

Exam topics

1. Tree problem, argue only with basic properties, degree sequence, sequences
2. plane graphs: determine edges, triangles, faces bounden by cycle etc.
3. list chroamtic number
4. computing Turan graph
5. prove for almost all graphs (asymptotic behaviour)
6. True, False Statements: know important theorems