

# Graphtheory Theorems

## 1 Important Graphs

### Complete Graph, Clique:

the *complete graph*  $K_n$  on  $n$  vertices is isomorphic to  $([n], \binom{[n]}{2})$

### Cycle

$C_n$  on  $n$  vertices with  $n \geq 3$  is isomorphic to  $([n], \{\{i, i+1\} : i = 1, \dots, n-1\} \cup \{n, 1\})$ , the *length of a cycle* is its number of edges.

### Empty Graph

$E_n$  on  $n$  vertices is isomorphic to  $([n], \emptyset)$ .

Empty graphs correspond to *independent sets*.

### Complete Bipartite Graph

$K_{m,n}$  on  $n+m$  vertices is isomorphic to  $(A \cup B, \{xy : x \in A, y \in B\})$  where  $|A| = m$  and  $|B| = n$  and  $A \cap B = \emptyset$ .

**Complete r-partite graph** with  $r \geq 2$  is isomorphic to

$$(A_1 \cup \dots \cup A_r, \{xy : x \in A_i, y \in A_j, i \neq j\})$$

where  $A_1, \dots, A_r$  are disjoint non-empty finite sets.

**n-dimensional hypercube**  $Q_n$  is the graph isomorphic to

$$(2^{[n]}, \{\{S, T\} : S, T \in 2^{[n]}, |S \Delta T| = 1\})$$

Vertices are labeled either by corresponding sets or binar indicator vectors. For example the vertex  $\{1, 3, 4\}$  in  $Q_6$  is coded by  $(1, 0, 1, 1, 0, 0)$ . A 1 is indicating we take this Element and a 0 if not.

**k-uniform hypergraph** is a hypergraph where all hyperedges have the same cardinality

## 2 Basics

### Definitions

- A graph  $G$  is *non-trivial* if it contains at least one edge, equivalently if  $G$  is not an empty graph
- The *order* of  $G$  written  $|G|$ , is the number of vertices of  $G$ , i.e.  $|G| = |V|$
- The *size* of  $G$  written  $\|G\|$ , is the number of edges of  $G$ , i. e.  $\|G\| = |E|$ , if order of  $G$  is  $n$  then the size of  $G$  is between 0 and  $\binom{n}{2}$
- $N(S)$  the *neighbourhood* of  $S \subseteq V$  is the set of vertices in  $V$ . that have and adjacent vertex in  $S$ . Instead of  $N(\{v\})$  for  $v \in V$  we write  $N(v)$

- vertex of degree 1 is called *leaf*
- vertex of degree 0 is called *isolated vertex*
- *minimum degree of  $G$* , denoted by  $\delta(G)$  is the smallest vertex degree in  $G$
- *maximum degree of  $G$* , denoted by  $\Delta(G)$  is the highest vertex degree in  $G$
- graph  $G$  is called  *$k$ -regular*, with  $k \in \mathbb{N}$ , if all vertices have degree  $k$ .
- *average degree of  $G$*  is defined as  $d(G) = \frac{\sum_{v \in V} \deg(v)}{|V|}$

We have

$$\delta(G) \leq d(G) \leq \Delta(G)$$

with equality if and only if  $G$  is  $k$ -regular

**Handshake Lemma** For ever graph  $G = (V, E)$  we have

$$2|E| = \sum_{v \in V} d(v)$$

*Proof.* By double counting the set  $X = \{(e, x) : e \in E(G), x \in V(G), x \in e\}$  then

$$|X| = \sum_{v \in V(G)} d(v)$$

and

$$|X| = \sum_{e \in E(G)} 2 = 2|E(G)|$$

by the principle of double counting the terms are equal.  $\square$

**Corollary** From this follows that the sum of all vertex degrees is even and therefore the number of vertices with odd degree is even.

**Proposition 3** If a graph  $G$  has minimum degree  $\delta(G) \geq 2$ , then  $G$  has a path of length  $\delta(G)$  and a cycle with at least  $\delta(G) + 1$  vertices.

*Proof.* Let  $P = (x_0, \dots, x_k)$  be a longest path in  $G$ . Then  $N(x_0) \subseteq V(P)$ , otherwise for  $x \in N(x_0) \setminus V(P)$  the path  $(x, x_0, x_1, \dots, x_k)$  would be a longer path.

Let  $i$  be the largest index such that  $x_i \in N(x_0)$ , then  $i \geq |N(x_0)| \geq \delta$ . So  $(x_0, x_1, \dots, x_i, x_0)$  is a cycle of length at least  $\delta(G) + 1$ .  $\square$

**Proposition 4** If for distinct vertices  $u$  and  $v$  a graph has a  $u$ - $v$ -walk, then it has a  $u$ - $v$ -path.

*Proof.* Consider a  $u$ - $v$ -walk  $W$  with the smallest number of edges. Assume that  $W$  does not form a path, then there is a repeated vertex,  $w$ , i.e.

$$W = u, e, v_1, e_1, \dots, e_k, w, e_{k+1}, \dots, e_l, w, e_{l+1}, \dots, v$$

Then  $W_1 = u, e, v_1, \dots, e_k, w, e_{k+1}, \dots, v$  is a shorter  $u$ - $v$ -walk, a contradiction.  $\square$

**Proposition 5** If a graph has a closed walk of odd length, then it contains an odd cycle.

*Proof.* Let  $W$  be the shortest closed odd walk. If  $W$  is a cycle the Proposition holds. Otherwise there is a repeated vertex, so  $W$  is an edge-disjoint union of two closed walks. The sum of the lengths of these walks is odd, therefore one of them is an odd closed walk shorter than  $W$  a contradiction to the minimality of  $W$ .  $\square$

**Proposition 6** If a graph has a closed walk with a non-repeated edge, then the graph contains a cycle.

*Proof.* Let  $W$  be a shortest walk with a non-repeated edge  $e$ . If  $W$  is a cycle, we are done. Otherwise, there is a repeated vertex and  $W$  is a union of two closed walks  $W_1$  and  $W_2$  that are shorter than  $W$ . One of them say  $W_1$ , contains  $e$ , a non-repeated edge. This contradicts the minimality of  $W$ .  $\square$

**Definition bipartite** A graph  $G = (V, E)$  is called *bipartite* if there exists natural numbers  $m, n$  such that  $G$  is isomorphic to a subgraph of  $K_{m,n}$ . Then the vertex set can be written as  $V = A \cup B$  such that  $E \subseteq \{ab : a \in A, b \in B\}$ . The sets  $A$  and  $B$  are called the *partite sets* of  $G$

**Proposition 1.5** A graph is bipartite if and only if it has no cycles of odd length.

*Proof.* skript  
“ $\Rightarrow$ ”

Assume that  $G$  is a bipartite graph with parts  $A$  and  $B$ . Then any cycle has a form  $a_1, b_1, a_2, b_2, \dots, a_k, b_k, a_1$  where  $a_i \in A, b_i \in B, i \in [k]$ . Thus every cycle has even length.

“ $\Leftarrow$ ”

Assume  $G$  does not have cycles of odd length. We can assume that  $G$  is connected, otherwise we can treat the connected components separately. Let  $v \in V(G)$ . Let  $A = \{u \in V(G) : \text{dist}(u, v) \equiv 0 \pmod{2}\}$  and let  $B = \{u \in V(G) : \text{dist}(u, v) \equiv 1 \pmod{2}\}$ . We claim that  $G$  is bipartite with parts  $A$  and  $B$ . To verify this it is sufficient to prove that  $A$  and  $B$  are independent sets. Let  $u_1 u_2 \in E(G)$  and let  $P_1$  be a shortest  $u_1$ - $v$ -path and  $P_2$  a shortest  $u_2$ - $v$ -path. Then the union of  $P_1, P_2$  and  $u_1 u_2$  forms a closed walk  $W$ . If  $u_1, u_2 \in A$  or  $u_1, u_2 \in B$  then  $W$  is a closed odd walk, because  $\text{dist}(v, u_1)$  and  $\text{dist}(v, u_2)$  are both even or odd. Thus by Prop. 5  $G$  contains an odd cycle, a contradiction. Thus for any edge  $u_1 u_2$  the adjacent vertices  $u_1$  and  $u_2$  are in different parts  $A$  or  $B$ . Therefore  $A$  and  $B$  are independent sets.  $\square$

*Proof.* Diestel  
“ $\Leftarrow$ ”

Let  $T$  be a spanning tree in  $G$ , pick a root  $r \in T$  and denote the associated tree-order on  $V$  by  $\leq_T$  (this order expressing height if  $x < y$  then  $x$  lies below  $y$  in  $T$ ). For each  $v \in V(G)$  the unique path  $r$ - $v$ - $T$  has odd or even length. This defines a bipartition of  $V(G)$ , we show that  $G$  is bipartite with this partition. Let  $e = xy$  be an edge of  $G$ . If  $e \in T$  with  $x <_T y$  say, then  $r$ - $y$ - $T = r$ - $xy$ - $T$

and so  $x$  and  $y$  lie in different partition classes. If  $e \notin T$  then  $C_e := x-y-T+e$  is a cycle, and by the case treated already the vertices along  $x-y-T$  alternate between the two classes. Since  $C_e$  is even by assumption,  $x$  and  $y$  again lie in different classes.  $\square$

**Euler tour** A closed walk that traverses every edge of the graph exactly once is called an *Euler tour*.

**Theorem 1.6 (Eulerian Tour Condition)** A connected graph has an Eulerian Tour if and only if every vertex has even degree.

*Proof.* “ $\Rightarrow$ ”

The degree condition is necessary for an Euler tour, because a vertex appearing  $k$  times in an Euler tour (or  $k+1$  times if it is the starting and finishing vertex) must have degree  $2k$ .

“ $\Leftarrow$ ”

Show by induction on  $\|G\|$  that every connected Graph  $G$  with all degrees even has an Euler tour.  $\|G\| = 0$  is trivial.

Now let  $\|G\| \geq 1$ , since all degrees are even, we can find in  $G$  a non-trivial closed walk that contains no edge more than once. To find this walk we consider  $W$  a walk of maximal length and write  $F$  for the set of its edges. If  $F = E(G)$ , then  $W$  is an Euler tour.

Suppose, therefore  $G' := G - F$  has an edge.

For every vertex  $v \in G$ , an even number of edges of  $G$  at  $v$  lies in  $F$ , so the degrees of  $G'$  are again all even. Since  $G$  is connected,  $G'$  has an edge  $e$  incident with a vertex on  $W$ . By I.H. the component  $C$  of  $G'$  containing  $e$  has an Euler tour. Concatenating this with  $W$  (suitably re-indexed), we obtain a closed walk in  $G$  that contradicts the maximal length of  $W$ .  $\square$

## Definitions

- graph  $G$  is *connected* if any two vertices are linked by a path.
- a maximal connected subgraph of  $G$  is called a *connected component* of  $G$ .
- acyclic graphs are called *forests*
- a graph  $G$  is called a *tree* if  $G$  is connected and acyclic.

**Lemma 7** Every tree on at least two vertices has a leaf.

*Proof.* If a tree  $T$  on at least two vertices does not have leaves then every vertex has degree  $> 2$ , so we have a cycle in  $T$  with length  $\geq 3$ , a contradiction.  $\square$

**Lemma 8** A tree of order  $n \geq 1$  has exactly  $n - 1$  edges.

*Proof.* We prove the statement by induction on  $n$ . When  $n = 1$ , there are no edges.

**I.H.:** Assume that each tree on  $n = k$  vertices has  $k - 1$  edges, with  $k \geq 1$ .

**Step:** Lets prove that each tree on  $k + 1$  vertices has  $k$  edges. Consider a tree  $T$  on  $k + 1$  vertices. Since  $k + 1 \geq 2$ ,  $T$  has a leaf  $v$ . Let  $T' = T - \{v\}$ . We see that  $T'$  is connected because any  $u$ - $w$ -path in  $T$ , for  $u \neq v$  and  $w \neq v$ , does not contain  $v$ . We see also that  $T'$  is acyclic, because deleting vertices from an acyclic graph does not create new cycles. Thus  $T'$  is a tree on  $k$  vertices. By I.H.  $|E(T')| = k - 1$ . Thus  $|E(T)| = |E(T')| + 1 = (k - 1) + 1 = k$ . □

**Lemma 9** Every connected graph contains a spanning tree.

*Proof.* Let  $G$  be a connected graph. Consider  $T$ , an acyclic spanning subgraph of  $G$  with largest number of edges. If it is a tree we are done.

Otherwise,  $T$  has more than one component. Consider vertices  $u$  and  $v$  from different components of  $G$ . Consider a shortest  $u$ - $v$ -path  $P$  in  $G$ . Then  $P$  has an edge  $e = xy$  with exactly one vertex  $x$  in one of the components of  $T$ . Then  $P$  has an edge  $e = xy$  with exactly one vertex  $x$  in one of the components of  $T$ . Then  $T \cup \{e\}$  is acyclic. If there would be a cycle, it would contain  $e$ , however  $e$  connects to components, therefore cannot be part of a cycle ( $e$  would be a repeated edge). Thus  $T \cup \{e\}$  is a bigger spanning acyclic subgraph of  $G$  contradicting the maximality of  $T$ . □

**Lemma 10** A connected graph on  $n \geq 1$  vertices and  $n - 1$  edges is tree.

*Proof.* Let  $G$  be a connected graph on  $n$  vertices with  $n - 1$  edges. Assume  $G$  is not a tree, i.e. contains a cycle. We therefore can remove a edge so that  $G$  is still connected. This is a contradiction because a graph on  $n$  vertices with  $n - 2$  edges cannot be connected. Because a walk from vertex 1 to vertex  $n$  has to have at least  $n - 1$  edges. □

**Lemma 11** The vertices of every connected graph on  $n \geq 2$  vertices can be ordered  $(v_1, \dots, v_n)$  so that for every  $i \in \{1, \dots, n\}$  the Graph  $G[\{v_1, \dots, v_i\}]$  is connected.

*Proof.* skript

Let  $G$  be a connected graph on  $n$  vertices. It contains a spanning tree  $T$ . Let  $v_n$  be a leaf of  $T$ , let  $v_{n-1}$  be a leaf of  $T - \{v_n\}$ , let  $v_{n-2}$  be a leaf of  $T - \{v_n, v_{n-1}\}$  and so on. Let  $v_k$  be a leaf in  $T - \{v_n, v_{n-1}, \dots, v_{k+1}\}$ ,  $k = 2, \dots, n$ . Since deleting a leaf does not disconnect a tree, all resulting graphs form a spanning trees of  $G[v_1, \dots, v_i]$ ,  $i = 1, \dots, n$ . A graph  $H$  having a spanning tree or any connected spanning subgraph  $H'$  is connected because a  $u$ - $v$ -path in  $H'$  is a  $u$ - $v$ -path in  $H$ . This observation completes the proof. □

*Proof.* diestel

Pick any vertex as  $v_1$ , and assume inductively that  $v_1, \dots, v_i$  have been chosen for some  $i < |G|$ . Now pick a vertex  $v \in G - G_i$ . As  $G$  is connected, it contains

a  $v-v_1$  path  $P$ . Choose  $v_{i+1}$  as the last vertex of  $P$  in  $G - G_i$ , then  $v_{i+1}$  has a neighbour in  $G_i$ . If we consider  $i + 1$  then we simply add  $v_{i+1}$  to our  $G_i$ . Thus  $G_{i+1} := G_i \cup \{v_{i+1}\}$  which is also connected.  $\square$

**Tree equivalences** For any graph  $G = (V, E)$  the following are equivalent:

1.  $G$  is a tree, i.e.  $G$  is connected and acyclic.
2.  $G$  is connected, but for any  $e \in E$  the graph  $G - e$  is not connected (minimally connected)
3.  $G$  is acyclic, but for any  $x, y \in V(G), xy \notin E(G)$  the graph  $G + xy$  has a cycle. (maximally acyclic)
4.  $G$  is connected and 1-degenerate
5.  $G$  is connected and  $|E| = |V| - 1$
6.  $G$  is acyclic and  $|E| = |V| - 1$
7.  $G$  is connected and every non-trivial subgraph of  $G$  has a vertex of degree at most 1.
8. Any two vertices are joined by a unique path in  $G$ .

*Proof.*

(i)  $\Rightarrow$  (ii):

$G$  is connected and acyclic, now assume for any edge  $e = xy$  the graph  $G' = G - e$  would still be connected. Then  $G'$  has a  $x$ - $y$ -path  $P$ . But  $P \cup e$  is a cycle in  $G$  which contradicts that  $G$  is acyclic.

(ii)  $\Rightarrow$  (i):

$G$  is connected and for any edge  $e$  the graph  $G - e$  is not connected. We want to show that  $G$  is acyclic. If  $G$  would have a cycle we could simply remove an edge from that and the resulting graph would still be connected, a contradiction.  $\square$

*Proof.*

TODO complete all equivalences for exam

(i)  $\Rightarrow$  (iv):

(vi)  $\Rightarrow$  (i):

(i)  $\Rightarrow$  (vii):

(vii)  $\Rightarrow$  (i):

(i)  $\Rightarrow$  (viii):

(viii)  $\Rightarrow$  (i):

$\square$

**Definition  $d$ -degenerate** If there is a vertex ordering  $v_1, \dots, v_n$  of  $G$  and a  $d \in \mathbb{N}$  such that

$$|N(v_i) \cap \{v_{i+1}, \dots, v_n\}| \leq d$$

for all  $i \in [n-1]$  then  $G$  is called  $d$ -degenerate. The minimum  $d$  for which  $G$  is  $d$ -degenerate is called the *degeneracy* of  $G$ .

Every finite planar graph has a vertex of degree five or less, therefore every planar graph is 5-degenerate.

**Definition arboricity** The least number of trees that can cover the edges of a graph is its arboricity.

It is a measure for the graphs maximum local density: it is small if and only if the graph is nowhere dense, in the sense that there is no subgraph  $H$  with large  $\epsilon(H) = \frac{E(H)}{V(H)}$ .

**Definition Contract** For an edge  $e = xy$  in  $G$  we define  $G \circ e$  as the graph obtained from  $G$  by identifying  $x$  and  $y$  and removing (if necessary) loops and multiple edges. We say that  $G \circ e$  arises from  $G$  by *contracting the edge  $e$* .

**Definition Complement** The *complement* of  $G$ , denoted by  $\overline{G}$  is defined as the graph  $(V, \binom{V}{2} \setminus E)$ . In particular  $G + \overline{G}$  is a complete graph and  $\overline{G} = (G + \overline{G}) - E$ .

## Definitions

- *girth* of  $G$ , denoted by  $g(G)$  is the length of the shortest cycle in  $G$ , if  $G$  is acyclic, its girth is said to be  $\infty$
- *circumference* of  $G$ , is the length of the longest cycle if  $G$  is acyclic the circumference is said to be 0
- $G$  is called *Hamiltonian* if  $G$  has a spanning cycle, i.e. a cycle that contains every vertex of  $G$ . In other words the circumference is  $|V|$
- $G$  is called *traceable* if  $G$  has a spanning path
- For two vertices  $v$  and  $u$  in  $G$ , the *distance between  $u$  and  $v$* , denoted by  $d(v, u)$  is the length of a shortest  $u$ - $v$ -path in  $G$ . If no such path exists  $d(u, v) = \infty$
- The *diameter* of  $G$ , denoted by  $\text{diam}(G)$ , is the maximum distance among all pairs of vertices in  $G$ , i.e.

$$\text{diam}(G) = \max_{u, v \in V} d(u, v)$$

- *eccentricity*,  $\text{ecc}(v)$  is the greatest distance of  $v$  to any other vertex.
- The *radius* of  $G$ , denoted by  $\text{rad}(G)$  is defined as

$$\text{rad}(G) = \min_{u \in V} \max_{v \in V} d(u, v)$$

its the vertex that has the smallest eccentricity

### 3 Matchings

#### Definitions

- *matching* is a 1-regular graph, i.e. a matching is a graph  $M$  so that  $E(M)$  is a union of pairwise non-adjacent edges and  $2|E(M)| = |V(M)|$
- a matching in  $G$  is a subgraph of  $G$  isomorphic to a matching. We denote the size of the largest matching in  $G$  by  $\nu(G)$
- a *vertex cover* in  $G$  is a set of vertices  $U \subseteq V$  such that each edge in  $E$  is incident to at least one vertex in  $U$ . We denote the size of the smallest vertex cover in  $G$  by  $\tau(G)$
- a *k-factor* of  $G$  is a  $k$ -regular spanning subgraph of  $G$ .
- A *1-factor* of  $G$  is also called a *perfect matching* since it is a matching of largest possible size of order  $|V|$ . Clearly  $G$  can only contain a perfect matching if  $|V|$  is even.

#### Theorem 2.2 (Hall's Marriage Theorem)

Let  $G$  be a bipartite graph with partite sets  $A$  and  $B$ . Then  $G$  has a matching containing all vertices of  $A$  if and only if  $|N(S)| \geq |S|$  for all  $S \subseteq A$

*Proof.*

$\Rightarrow$ :

If  $G$  has a matching  $M$  containing all vertices of  $A$ , then for any  $S \subseteq A$ ,  $N(S)$  in  $G$  is at least as large as  $N(S)$  in  $M$ , thus  $|N(S)| \geq |S|$ .

$\Leftarrow$ :

We shall prove by induction on  $|A|$  that any bipartite graph with parts  $A$  and  $B$  satisfying Hall's condition has a matching containing all vertices of  $A$ , in other words, saturating  $A$ .

When  $|A| = 1$ , there is at least one edge in  $G$  and thus a matching saturating  $A$ . Assume that the statement is true for all graphs  $G$  satisfying Hall's condition and with  $|A| = k \geq 1$ .

Now consider a bipartite graph  $G$  with  $|A| = k + 1$  and satisfying Hall's condition.

Case 1:  $|N(S)| \geq |S| + 1$  for any  $S \subsetneq A$ .

Let  $G' = G - \{x, y\}$ , for some edge  $xy$ .  $G'$  has parts  $A' = A - \{x\}$  and  $B' = B - \{y\}$ . For any  $S \subseteq A'$ ,  $|N_{G'}(S)| \geq |N_G(S)| - 1 \geq |S| + 1 - 1 = |S|$ . Thus  $G'$  satisfies Hall's condition and by induction has a matching  $M'$  saturating  $A'$ . Then  $M = M' \cup \{xy\}$  is a matching in  $G$  saturating  $A$ .

Case 2:  $|N(S_1)| = |S_1|$  for some  $S_1 \subsetneq A$ .

Let  $A' = S_1, B' = N(A'), G' = G[A' \cup B']$ . Since  $|A'| < |A|$ , and  $G'$  satisfies Hall's condition,  $G'$  has a matching  $M'$  saturating  $A'$  by induction. Now consider  $A'' = A - A', B'' = B - B', G'' = G[A'' \cup B'']$ . We claim  $G''$  also satisfies Hall's condition. Assume not, and there is  $S \subseteq A''$  such that  $|N_{G''}(S)| < |S|$ . Then  $|N_G(S \cup A')| = |B' \cup N_{G''}(S)| = |B'| + |N_{G''}(S)| < |A'| + |S| = |A' \cup S|$ , a contradiction to Hall's condition. Thus there is a Matching  $M''$  in  $G''$  and  $M' \cup M''$  is a matching saturating  $A$  in  $G$ .  $\square$



**Corollary 12** Let  $G$  be a bipartite graph with partite sets  $A$  and  $B$  such that  $|N(S)| \geq |S| - d$  holds for all  $S \subseteq A$ , and for a fixed positive integer  $d$ . Then  $G$  contains a matching of size at least  $|A| - d$ .

*Proof.* TODO □

**Corollary 13** If  $G$  is a regular bipartite graph, it has a perfect matching.

*Proof.* Let  $k \in \mathbb{N}$  and let  $G$  be a  $k$ -regular bipartite graph with parts  $A$  and  $B$ . Then  $|E(G)| = k|A| = k|B|$ , and thus  $|A| = |B|$ . Consider  $S \subseteq A$ , let  $r$  be the number of edges between  $S$  and  $N(S)$ . On one hand,  $r = |S|k$ , on the other hand  $r \leq |N(S)|k$ . Thus  $|N(S)| \geq |S|$  and by Hall's theorem there is a matching saturating  $A$ . Since  $|A| = |B|$ , it is a perfect matching. □

**Corollary 14** A  $k$ -regular bipartite graph has a proper  $k$ -edge-coloring.

*Proof.* TODO □

**Theorem 2.3** (König's Theorem)

Let  $G$  be bipartite. Then the size of a largest matching is the same as the size of a smallest vertex cover.

*Proof.* Let  $c$  be the vertex-cover number of  $G$  and  $m$  be the size of the largest matching of  $G$ . Since a vertex cover should contain at least one vertex from each matching edge,  $c \geq m$ .

Now, we shall prove that  $c \leq m$ . Let  $M$  be a largest matching in  $G$ , we need to show that  $c \leq |M|$ . Let  $A$  and  $B$  be the partite sets of  $G$ . An *alternating path* is a path that starts with a vertex in  $A$  not incident to an edge of  $M$ , and alternates between edges in  $M$  and edges in  $M$ . Note that an alternating path must end in a vertex saturated by  $M$ , otherwise one can find a larger matching. Let

$$U' = \{b : ab \in E(M) \text{ for some } a \in A \text{ and some alternating path ends in } b\}$$

$$U = U' \cup \{a : ab \in E(M), b \notin U'\}$$

We see that  $|U| = m$ . We shall show that  $U$  is a vertex cover, i.e. that every edge of  $G$  contains a vertex from  $U$ . Indeed, if  $ab \in E(M)$ , then either  $a$  or  $b$  is in  $U$ . If  $ab \notin E(M)$ , we consider the following cases:

**Case 0:**  $a \in U$ . We are done.

**Case 1:**  $a$  is not incident to  $M$ . Then  $ab$  is an alternating path. If  $b$  is also not incident to  $M$  then  $M \cup \{ab\}$  is a larger matching, a contradiction. Thus  $b$  is incident to  $M$  and then  $b \in U$ .

**Case 2:**  $a$  is incident to  $M$ . Then  $ab' \in E(M)$  for some  $b'$ . Since  $a \notin U$ , we have that  $b' \in U$ , thus there is an alternating path  $P$  ending in  $b'$ . If  $P$  contains  $b$ , then  $b \in U$ , otherwise  $Pb'ab$  is an alternating path ending in  $b$ , so  $b \in U$ . □

**Theorem 2.4** (Tutte's Theorem)

A graph  $G$  has a perfect matching if and only if  $q(G - S) \leq |S|$  for all  $S \subseteq V$ . We define  $q(H)$  to be the number of odd components of  $H$ , i.e. the number of connected components of  $H$  consisting of an odd number of vertices.

*Proof.*

$\Rightarrow$ :

Assume first that  $G$  has a perfect matching  $M$ . Consider a set  $S$  of vertices and an odd component  $G'$  of  $G - S$ . We see that there is at least one vertex in  $G'$  that is incident to an edge of  $M$  that has another endpoint not in  $G'$ . This endpoint must be in  $S$ . Thus  $|S|$  is at least as large as the number of odd components.

$\Leftarrow$ :

Now, assume that  $q(G - S) \leq |S|$  for all  $S \subseteq V$ . Assume that  $G$  has no perfect matching and  $|V(G)| = n$ . Note that  $|V(G)|$  is even (it follows from the assumption  $q(G - S) \leq |S|$  applied to  $S = \emptyset$ ). Let  $G'$  be constructed from  $G$  by adding missing edges as long as no perfect matching appears. Let  $S$  be a set of vertices of degree  $n - 1$ . Note that it could be empty.

**Claim 1:** Each component of  $G' - S$  is complete. Assume not, there is a component  $\square$

## 4 Connectivity

### Definitions

- for a natural number  $k \geq 1$ , a graph  $G$  is called *k-connected* if  $|V(G)| \geq k + 1$  and for any set  $U$  of  $k - 1$  vertices in  $G$  the graph  $G - U$  is connected. In particular  $K_n$  is  $(n - 1)$ -connected.

this implies: a graph is *k-connected* if any two of its vertices can be joined by  $k$  independent paths

- a graph  $G$  is called *k-linked* if  $|G| \geq 2k$  and for any  $2k$  distinct vertices  $s_1, \dots, s_k, t_1, \dots, t_k$  there are vertex disjoint  $s_i$ - $t_i$ -paths,  $i = 1, \dots, k$

**Lemma 3.2** For any connected, non-trivial graph  $G$  we have

$$\kappa(G) \leq \kappa'(G) \leq \delta(G)$$

*Proof.* If  $G$  is complete  $\kappa(G) = \kappa'(G) = \delta(G) = n - 1$

Assume  $G$  is not complete:

$\kappa'(G) \leq \delta(G)$ :

Simply remove all the edges of a vertex with minimum degree.

$\kappa(G) \leq \kappa'(G)$ :

Consider smallest separating set of edges  $F$ :

case1:

there is a vertex  $v$  not incident to  $F$ , then the vertices incident to  $F$  separate this vertex from any vertex in the other component.

case2:

Every vertex is incident to  $F$ , consider  $v$  of degree  $< |G| - 1$ , exists because  $G$  is not complete. Show the neighbourhood of  $v$  is less than  $|F|$  so  $N(v)$  is a separating set.  $\square$

**Definition** A subset  $X$  of vertices and edges of  $G$  separates two vertex sets  $A, B$  if each  $A$ - $B$ -path (starts in  $A$  ends in  $B$ ) contains an element of  $X$ .

**Theorem 3.3 (Menger's Theorem)** For any graph  $G$  and any two vertex sets  $A, B \subseteq V(G)$ , the smallest number of vertices separating  $A$  and  $B$  is equal to the largest number of disjoint  $A$ - $B$ -paths.

**Theorem 3.4 (Global Version of Menger's Theorem)** A graph  $G$  is  $k$ -connected if and only if for any two vertices  $a, b$  in  $G$  there exist  $k$  independent  $a$ - $b$ -paths.

Note that Menger's Theorem implies that if  $G$  is  $k$ -linked, then  $G$  is  $k$ -connected.

**Definition: Line Graph** For a graph  $G = (V, E)$  the line graph  $L(G)$  is the graph  $L(G) = (E, E')$  where

$$E' = \{\{e_1, e_2\} \in \binom{E}{2} : e_1 \text{ adjacent to } e_2 \text{ in } G\}$$

**Corollary 24** If  $a, b$  are vertices of  $G$ , then

$$\min \# \text{edges separating } a \text{ and } b = \max \# \text{edge-disjoint } a\text{-}b\text{-paths}$$

**Definition**

- *H-path*: Given a graph  $H$ , we call a path  $P$  an  $H$ -path if  $P$  is non-trivial and meets  $H$  exactly in its ends. Such a path is also called an *ear* of the graph  $H \cup P$ .
- An *ear-decomposition* of a graph  $G$  is a sequence  $G_0 \subseteq G_1 \subseteq \dots \subseteq G_k$  of graphs, such that
  - $G_0$  is a cycle
  - for each  $i = 1, \dots, k$  the graph  $G_i$  arises from  $G_{i-1}$  by adding a  $G_{i-1}$ -path  $P_i$ , i.e.  $P_i$  is an ear of  $G_i$
  - $G_k = G$

**Theorem 25 (Ear-decomposition)** A graph  $G$  is 2-connected if and only if it has an ear decomposition starting from any cycle of  $G$ .

*Proof.*  $\Leftarrow$ :

Assume first that  $G$  has a ear-decomposition starting from a cycle  $C$ , i.e.  $C = G_0 \subseteq G_1 \subseteq \dots \subseteq G_k = G$  Use induction on  $i$  to show that  $G_i$  is 2-connected.  $G_0$  is a cycle and therefore 2-connected.  $G_{i+1}$  is obtained from  $G_i$  by adding an ear. We now by IH that  $G_i$  is 2-connected, therefore a cut vertex has to be on the

ear. The ear is contained in a cycle therefore deleting a vertex from it does not disconnect  $G_{i+1}$  therefore  $G_{i+1}$  is 2-connected.

$\Rightarrow$ :

Assume  $G$  is 2-connected and  $C$  is a cycle in  $G$ . Let  $H$  be the largest subgraph of  $G$  obtained by ear decomposition starting with  $C$ .  $H$  has to be an induced subgraph of  $G$  otherwise an edge incident to two vertices of  $H$  would be an ear that could be added to  $H$ .

Assume  $H \neq G$ . Since  $G$  is connected there is an edge from  $v \in H$  to  $u \in G \setminus H$ . Since  $G - u$  is connected consider a  $v$ - $w$ -path  $P$  in  $G - u$  for some vertex  $w \in (V(H) - u)$ . Let  $w'$  be the first vertex from  $V(H) - u$  on this path. Then  $Pw' \cup uw$  is an ear of  $H$ , a contradiction to minimality of  $H$ .  $\square$

**Lemma 26** If  $G$  is 3-connected with  $G \neq K_4$ , then there exists an edge  $e$  of  $G$  such that  $G \circ e$  is also 3-connected.

**Theorem 3.6 (Tutte)** A graph  $G$  is 3-connected if and only if there exists a sequence of graphs  $G_0, G_1, \dots, G_k$ , such that

- $G_0 = K_4$
- for each  $i = 1, \dots, k$  the graph  $G_i$  has two adjacent vertices  $x', x''$  of degree at least 3, so that  $G_{i-1} = G_i \circ x'x''$
- $G_k = G$

*Proof.*

$\Rightarrow$ :

If  $G$  is 3-connected, such a sequence exists by Lemma 26. To see the degree condition is satisfied, recall that  $\delta(H) \geq 3$  for any 3-connected graph  $H$ . Note that with each contraction, the number of vertices decrease by 1 and until we have at least 5 vertices, we can apply Lemma 26 and contract one more edge. Thus we stop at a graph  $G_0$  which has 4 vertices and  $\delta(G_0) \geq 3$  from which  $G_0 \cong K_4$  follows.

$\Leftarrow$

We consider a sequence of graphs satisfying the given conditions and show that each graph in the sequence is 3-connected. Assume that  $G_i$  is 3-connected,  $G_{i+1}$  is not, and  $G_i = G_{i+1} \circ xy$ , for an edge  $xy$  of  $G_{i+1}$  such that  $d(x), d(y) \geq 3$ . Then  $G_{i+1}$  has a cut-set  $S$  with at most two vertices.

Case 1:  $x, y \in S$ .

Then  $G_i$  has a cut vertex, a contradiction.

Case 2:  $x \in S, y \notin S, y$  is not the only vertex of its component in  $G_{i+1} - S$ .

Then  $G_i$  has a cut set of size at most 2, a contradiction.

Case 3:  $x \in S, y \notin S, y$  is the only vertex of its component in  $G_{i+1} - S$ .

Then  $d(y) \leq 2$ , a contradiction to the fact that  $d(y) \geq 3$

Case 4:  $x, y \notin S$

Then  $x$  and  $y$  are in the same component of  $G_{i+1} - S$ . So  $S$  is a cut set of  $G_i$ , a contradiction.  $\square$

Note that this theorem gives a way to generate all 3-connected graphs by starting with  $K_4$  and creating a sequence of graphs by “uncontracting” a vertex such that the degrees of new vertices are at least 3 each.

**Theorem 27 (Mader)** Every graph  $G = (V, E)$  of average degree at least  $4k$  has a  $k$ -connected subgraph.

### Definition

- let  $G$  be a graph, a maximal connected subgraph of  $G$  without a cut vertex is called a *block* of  $G$ . In particular, the blocks of  $G$  are exactly the bridges and the maximal 2-connected subgraphs of  $G$ .
- *block-cut-vertex graph* or *block graph* of  $G$  is a bipartite graph  $H$  whose partite sets are the *blocks* of  $G$  and the cut vertices of  $G$  respectively. There is an edge between a block  $B$  and a cut vertex  $a$  if and only if  $a \in B$ , i.e. the block contains the cut vertex. The leaves of this graph are called *leaf blocks*.

**Theorem 28** The block-cut-vertex graph of a connected graph is a tree.

## 5 Planar graphs

**planar graph vs. plane graph** Plane Graph is topological object  $(V, E)$ ,  $V \subseteq \mathbb{R}^2$ ,  $e \in E$  are arcs in  $\mathbb{R}^2$ . Planar graph is combinatorial object  $(V, E)$ ,  $E \subseteq \binom{V}{2}$  s.t. it has a plane graph realization (planar embedding)

### Definitions

- For any plane graph  $G$ , the set  $\mathbb{R}^2 \setminus G$  is open, its regions are the *faces* of  $G$ .
- the face of  $G$  corresponding to the unbounded region is the *outer face* of  $G$ , the other faces are its *inner faces*. The set of all Faces is denoted by  $F(G)$
- The *frontier* of a set  $X \subseteq \mathbb{R}^2$  is the set  $Y$  of all points  $y \in \mathbb{R}^2$  such that every neighbourhood of  $y$  meets both  $X$  and  $\mathbb{R}^2 \setminus X$ . Note that if  $X$  is closed, its frontier lies in  $X$ , while if  $X$  is open, its frontier lies in  $\mathbb{R}^2 \setminus X$ .
- The subgraph of  $G$  whose point set is the frontier of a face  $f$  is said to *bound*  $f$  and is called its *boundary*, we denote it by  $G[f]$
- Let  $G$  be a plane graph. If one cannot add an edge to form plane  $G' \supset G$  with  $V(G') = V(G)$ , then  $G$  is called *maximally plane*.
- If every face in  $F(G)$  (including the outer face) is bounded by a triangle in  $G$ , then  $G$  is called a *plane triangulation*
- A *planar embedding* of an abstract graph  $G = (V, E)$  is a bijective mapping  $f : V \rightarrow V'$ , where  $G' = (V', E')$  is a plane graph and  $uv \in E(G)$ , then there is an edge in  $E'$  with endpoints  $f(u)$  and  $f(v)$ . We say that  $G'$  is a *drawing* of  $G$ . A graph  $G = (V, E)$  is *planar* if it has a planar embedding.

- A graph  $G$  is *outerplanar* if it has a plane embedding such that the boundary of the outer face contains all vertices  $V$ .

**Theorem 32 (Plane triangulation)** A graph of order at least 3 is maximally plane if and only if it is a plane triangulation.

*Proof.* If  $G$  is a plane triangulation, then each face is bounded by a triangle. If an edge is added to  $G$  so that the resulting graph is plane, the interior of the edge must be in some face  $f$  of  $G$ . The endpoints of the added edge must be two of the three vertices on the frontier of  $f$ . However, these vertices already are endpoints of an edge of  $G$ , a contradiction. Thus no edge could be added to  $G$  so that the graph remains plane.

Now assume that  $G$  is maximally plane. Let  $f$  be a face and  $H = G[f]$ . Then we see that  $H$  is a complete graph, otherwise we could add a new edge with interior in  $f$ . If  $H$  has at least 4 vertices  $v_1, v_2, v_3, v_4, \dots$  then we see that  $v_i-v_j$ -paths,  $i, j \in [4]$  can not all be pairwise disjoint. If  $H$  has at most 2 vertices, then  $f$  is a face having at most one edge on its boundary, thus  $f = \mathbb{R}^2 - G$  and one can add another edge to  $G$ . Therefore, we see that  $H$  is a complete graph on 3 vertices.  $\square$

**Theorem 4.2 (Euler's Formula)** Let  $G$  be a connected plane graph with  $n$  vertices,  $m$  edges and  $l$  faces. Then

$$n - m + l = 2$$

*Proof.* Apply induction on  $m$ . A connected graph has at least  $n - 1$  edge. If  $m = n - 1$ ,  $G$  is a tree. Then  $l = 1$  and  $n - (n - 1) + 1 = 2$

Let  $m \geq n$  and assume the claim holds for smaller values of  $m$ . Then there is an edge  $e$  on a cycle. Then  $e$  is on the boundary of exactly two faces  $f_1$  and  $f_2$ , these make a become one face in  $G' := G - e$ , by induction the claim holds.  $\square$

**Corollary 33** A plane graph with  $n \geq 3$  vertices has at most  $3n - 6$  edges. Every plane triangulation has exactly  $3n - 6$  edges.

*Proof.* Count the number of edges for every face and plug into Euler's formula.  $\square$

**Corollary 34** A triangle-free plane graph with  $n \geq 3$  vertices has at most  $2n - 4$  edges

*Proof.* Double count the set of pairs of edges with faces and use eulers formula.  $\square$

**Theorem 4.4 (Kuratowski's Theorem)** The following statements are equivalent for graphs  $G$ :

1.  $G$  is planar
2.  $G$  does not have  $K_5$  or  $K_{3,3}$  as minors
3.  $G$  does not have  $K_5$  or  $K_{3,3}$  as topological minors

### Definitions: Poset

- Let  $X$  be a set and  $\leq \subseteq X^2$  be a relation on  $X$ . Then  $\leq$  is a *partial order* if it is reflexive, antisymmetric and transitive. A partial order is *total* if  $x \leq y$  or  $y \leq x$  for every  $x, y \in X$
- The *poset dimension* of  $(X, \leq)$  is the smallest number  $d$  such that there are total orders  $R_1, \dots, R_d$  on  $X$  with  $\leq = R_1 \cap \dots \cap R_d$ .
- The *incidence poset*  $(V \cup E, \leq)$  on a graph  $G = (V, E)$  is given by  $v \leq e$  iff  $e$  is incident to  $v$  for all  $v \in V$  and  $e \in E$

**Theorem (Schnyder)** Let  $G$  be a graph and  $P$  its incidence poset. Then  $G$  is planar iff  $\dim(P) \leq 3$ .

**Theorem 4.7 (5-Color Theorem)** Every planar graph is 5-colorable

*Proof.* We shall apply induction on  $|V(G)|$  with a trivial basis when  $|V(G)| \leq 5$ . Assume  $|V(G)| > 5$ , assume further that  $G$  is maximally planar (plane triangulation). By Euler's Formula there is a vertex  $v$  of degree at most 5. By induction, there is a proper coloring  $c$  of  $G - v$  in at most 5 colors. If  $c$  assigns at most 4 colors to  $N(v)$ , we can assign  $v$  a color such that we use only 5. Otherwise, assume  $v_1, \dots, v_5$ ,  $c(v_i) = i$  and  $v_i$ 's are cyclically arranged on the face of  $G - v$ . We make a color switch from 1 to 3 at vertex  $v_1$ , if this is valid and now both  $v_1$  and  $v_3$  have the same color 3, then  $v$  can get color 1. Otherwise there has to be a  $v_1$ - $v_3$ -path colored in 1 and 3. Similarly there is a  $v_2$ - $v_4$ -path colored 2 and 4, but these paths have to intersect at a single vertex, so that graph is still planar. But this vertex cannot lie on both path since the sets of available colors are disjoint.  $\square$

**Wrong proof of Four Color Theorem by Kempe** Uses contradiction from 2 colored path which intersect, but there is an example which shows that we have two vertices of the same color.

**Definition** List coloring

- Let  $L(v) \subseteq \mathbb{N}$  be a list of colors for each vertex  $v \in V$ . We say that  $G$  is *L-list-colorable* if there is a coloring  $c : V \rightarrow \mathbb{N}$  such that  $c(v) \in L(v)$  for each  $v \in V$  and adjacent vertices receive different colors.
- Let  $k \in \mathbb{N}$ . We say that  $G$  is *k-list-colorable* or *k-choosable* if  $G$  is *L-list-colorable* for each list  $L$  with  $|L(v)| = k$  for all  $v \in V$
- the *choosability*, denoted by  $\text{ch}(G)$  is the smallest  $k$  such that  $G$  is *k-choosable*.
- the *edge choosability*, denoted by  $\text{ch}'(G)$  is the smallest  $k$  such that  $G$  is *L-edge-list-colorable* for each list  $L$  with  $|L(e)| = k$  for  $e \in E$

**Theorem 4.10 (5-List-Color Theorem)** Let  $G$  be a planar graph. Then the list chromatic number of  $G$  is at most 5.

*Proof.* We prove a stronger statement  $(\star)$ :

Let  $G$  be an outer triangulation (triangular inner faces and an outer face forming a cycle) Suppose two adjacent vertices  $x, y$  on the boundary of the outer face have already been colored. For all other vertices on the cycle have list of length 3 and all other vertices on bounded faces have list of length 5. Then the coloring of  $x, y$  can be extended to a coloring of  $G$  from the given lists.

We prove  $(\star)$  by induction on  $|V(G)|$ , with trivial basis  $|V(G)| = 3$  Consider an outer triangulation on more than 3 vertices.

Case 1: There is a chord

Then  $G = G_1 \cup G_2$ , such that  $\{u, v\} = V(G_1) \cap V(G_2)$ ,  $|G| > |G_i| \geq 3$ ,  $G_i$  is an outer triangulation,  $i = 1, 2$ . Without loss of generality  $x, y$  are on the outer face of  $G_1$ . Apply induction to  $G_1$  to obtain a proper  $L$ -coloring  $c'$  of  $G_1$ . Next apply induction to  $G_2$  with  $u$  and  $v$  playing the role of  $x, y$  and list assignments  $L'$  such that they have the same color as in  $c'$ . Then there is a proper  $L'$ -coloring  $c''$  of  $G_2$ . Since these colorings have the same colors for  $u$  and  $v$  they form together a proper coloring of  $G$ .

Case 2: There is no chord

Let  $z$  be a neighbor of  $x$  on the boundary of the outer face,  $z \neq y$ . Let  $Z$  be the set of neighbors of  $z$  not on the outer face. Let  $L(x) = \{a\}$ ,  $L(y) = \{b\}$ . Let  $c, d \in L(z)$  such that  $c \neq a$  and  $d \neq a$ . Remove  $z$  from  $G$ ,  $G' := G - z$ . Let  $L'$  be the list assignment for  $V(G')$  such that  $L(v) = \{c, d\}$  for  $v \in Z$  and  $L'(v) = L(v)$  for  $v \notin Z$ . Remove the colors  $c, d$  from the vertices in the neighbourhood of  $z$ . and leave the colors for all other vertices the same. By induction  $G'$  has a proper  $L'$ -coloring  $c'$ . We shall extend a coloring  $c'$  to a coloring  $c$  of  $G$ . Simply by giving  $z$  the color  $c, d$  depending on the color of his neighbor on the outer face.

□

## 6 Colorings

Note that a  $k$ -coloring is nothing but a vertex partition into  $k$  independent sets, now called *colorclasses*; the non-trivial 2-colourable graphs, are precisely the bipartite graphs. The chromatic number is a key parameter in many extremal question, therefore it is studied a lot.

### Definitions

- *vertex coloring* of a Graph  $G$  is a map  $c : V \rightarrow S$  such that  $c(v) \neq c(w)$  whenever  $v$  and  $w$  are adjacent. The elements of the set  $S$  are called available colors. The minimal  $k = |S|$  such that  $G$  has a  $k$ -coloring is the *chromatic number* of  $G$ , denoted by  $\chi(G)$ . A graph with  $\chi(G) = k$  is called  $k$ -chromatic.
- *edge coloring* is a map  $c : E \rightarrow S$  with  $c(e) \neq c(f)$  for any adjacent edges  $e, f$ . We say the *edge-chromatic number*, or *chromatic index* of  $G$  is the minimal  $k$  for which a  $k$ -edge-coloring exists, it is denoted by  $\chi'(G)$ .



**Relationship of  $\chi(G)$  and  $\chi'(G)$**  Every edge coloring of  $G$  is a vertex coloring of its line graph  $L(G)$ , and vice versa, in particular  $\chi'(G) = \chi(L(G))$ . The problem of finding good edge colorings may thus be viewed as a restriction of the more general vertex coloring problem to this special class of graphs. There are only very rough estimates for  $\chi$  but  $\chi'$  always takes one of two values, either  $\Delta$  or  $\Delta + 1$

### Definitions

- *clique number*  $\omega(G)$  of  $G$  is the largest order of a clique in  $G$
- *co-clique number*  $\alpha(G)$  of  $G$  is the largest order of an independent set in  $G$
- Graph is *perfect* if  $\chi(H) = \omega(H)$  for each induced subgraph  $H$  of  $G$ . For example bipartite graphs are perfect with  $\chi = \omega = 2$

**Lemma 43** For any connected graph  $G$  and for any vertex  $v$  there is an ordering of the vertices of  $G: v_1, \dots, v_n$  such that  $v = v_n$  and for each  $i, 1 \leq i < n$ ,  $v_i$  has a higher indexed neighbor.

*Proof.* Consider a spanning tree  $T$  of  $G$  and create a sequence of sets  $X_1, \dots, X_{n-1}$  with  $X_1 = V(G_1)$ ,  $X_i = X_{i-1} - \{v_{i-1}\}$ , where  $v_i$  is a leaf of  $T[X_i]$  not equal to  $v$ , for  $i = 1, \dots, n-1$ . Then  $v_1, \dots, v_n$  is a desired ordering with  $v_n = v$   $\square$

**Lemma 45** Let  $G$  be a 2-connected non-complete graph of minimum degree at least 3. Then there are vertices  $x, y$  and  $v$  such that  $xy \notin E(G)$ ,  $xv, yv \in E(G)$  and  $G - \{x, y\}$  is connected.

*Proof.* Consider a vertex  $w$  of degree at most  $|G| - 2$

Case 1:  $G - w$  has no cutvertices.

Let  $x = w$  and  $y$  be a vertex at distance 2 from  $x$  and let  $v$  be a common neighbor of  $x$  and  $y$ . Since  $y$  is not a cut-vertex in  $G - x$ ,  $G - \{x, y\}$  is connected.

Case 2:  $G - w$  has a cutvertex.

In this case, let  $v = w$ . Then  $v$  must be adjacent to non-cutvertex members of each leaf-block of  $G - v$ . Let  $x$  and  $y$  be such neighbors in distinct leaf-blocks. Since  $v$  has another neighbor besides  $x$  and  $y$ ,  $G - \{x, y\}$  is connected.  $\square$

### Lemma 46 (Simple Coloring Results)

For any graph  $G$  the following hold:

- $\chi(G) \geq \max\{\omega(G), \frac{|G|}{\alpha(G)}\}$
- $\|G\| \geq \binom{\chi(G)}{2}$  and  $\chi(G) \leq 1/2 + \sqrt{2\|G\| + 1/4}$

### Greedy coloring algorithm

starting from a fixed vertex enumeration  $v_1, \dots, v_n$  of  $G$ , we consider the vertices in turn and color each  $v_i$  with the first available color, meaning with the smallest positive integer not already used to color any neighbour of  $v_i$  among  $v_1, \dots, v_{i-1}$ . In this way we never use more than  $\Delta(G) + 1$  colors, even for unfavourable choices of the enumeration. If  $G$  is complete or an odd cycle, then this is even best possible.

**Lovasz Perfect Graph Theorem** A graph  $G$  is perfect if and only if its complement  $\overline{G}$  is perfect.

**Strong Perfect Graph Theorem** A graph  $G$  is perfect if and only if it does not contain an odd cycle on at least 5 vertices (an *odd hole*) or the complement of an odd hole as an induced subgraph.

**Graphs with  $\omega(G) \leq \chi(G)$  exist** Meaning  $\exists G : \omega < \chi(G)$ , we have 3 proofs

- Mycielski's construction
- Tutte's construction
- Erdős-Hajnal theorem: girth greater than  $k$  and chromatic number greater than  $k$

**Theorem 5.1 (Brook's Theorem)** Let  $G$  be a connected graph. Then  $\chi(G) \leq \Delta(G)$  unless  $G$  is a complete graph or an odd cycle.

*Proof.* Induction on  $|V(G)| = n$ . The theorem holds for any graph on at most 3 vertices. Assume  $|V(G)| > 3$ , we have two cases:

case 1:  $G$  has a cut-vertex  $v$

If  $G$  has a cut-vertex  $v$ , we can apply induction to the graphs  $G_1$  and  $G_2$  such that  $G_1 \cup G_2 = G$  and  $V(G_1) \cap V(G_2) = \{v\}$  and  $|G_1| < |G|$  and  $|G_2| < |G|$ . By I.H. we get that even if  $G_1$  and  $G_2$  are complete or an odd cycle  $\chi(G_i) \leq \Delta(G_i) \leq \Delta(G)$  we can make sure that the color of  $v$  is the same in both colorings and still get  $\chi(G) \leq \Delta(G)$ .

If  $\Delta(G) \leq 2$  then  $G$  is a path or a cycle and the theorem holds. Assume  $\Delta(G) \geq 3$

case 2:  $G$  is 2-connected.

case 2.1: There is a vertex  $v$  of degree at most  $\Delta - 1$

We shall order the vertices of  $G$   $v_1, \dots, v_n$  such that  $v = v_n$  and each  $v_i, i < n$  has a neighbor with larger index, such an ordering exists by Lemma 43. Color  $G$  greedily with respect to this ordering. We see at step  $i$ , there are at most  $\Delta - 1$  neighbors of  $v_i$  that has been colored, so there is an available color for  $v_i$

case 2.2: All vertices of  $G$  have degree  $\Delta$

Consider vertices  $x, y, v$  guaranteed by Lemma 45, such that  $xy \notin E(G)$  and  $xv, yv \in E(G)$  and  $G - \{x, y\}$  is connected. Order the vertices of  $G$  as  $v_1, \dots, v_n$  such that  $v_1 = x, v_2 = y, v_n = v$  for each  $v_i, 3 \leq i < n$  there is a neighbor of  $v_i$  with a higher index, such an ordering exists by Lemma 43. Color  $G$  greedily according to this ordering. We see that  $v_1$  and  $v_2$  get the same color as in the previous case, at step  $i, 3 \leq i < n$ ,  $v_i$  has at most  $\Delta - 1$  colored neighbors so it could be colored with a remaining color. At the last step, we see that  $v_n$  has  $\Delta$  colored neighbors, but two of them  $v_1$  and  $v_2$  have the same color so there are at most  $\Delta - 1$  colors used by the neighbors of  $v_n$ . Thus  $v_n$  can be colored with a remaining color.  $\square$

**Graphs with arbitrarily high chromatic number**

**Mycielski's Construction** We can construct a family  $(G_k = (V_k, E_k))_{k \in \mathbb{N}}$  of triangle-free graphs with  $\chi(G_k) = k$  as follows:

- $G_1$  is the single-vertex graph,  $G_2$  is the single-edge  $K_2$
- $V_{k+1} := V_k \cup U \cup \{w\}$  where  $V_k \cap (U \cup \{w\}) = \emptyset$ ,  $V_k = \{v_1, \dots, v_n\}$  and  $U = \{u_1, \dots, u_n\}$
- $E_{k+1} := E_k \cup \{wu_i : i = 1, \dots, n\} \cup \bigcup_{i=1}^n \{u_i v : v \in N_{G_k}(v_i)\}$

**Lemma** For any  $k \geq 1$ , Mycielski's graph  $G_k$  has chromatic number  $k$ . Moreover,  $G_k$  is triangle-free.

*Proof.* We shall prove this statement by induction on  $k$  with trivial basis  $k = 1$ . Assume  $k \geq 2$  and  $\chi(G_{k-1}) = k - 1$  and  $G_{k-1}$  is triangle-free.

First we show that  $\chi(G_k) = k$ :

We see that  $\chi(G) \leq k$  by considering a proper coloring  $c$  of  $G_{k-1}$  with colors from  $[k - 1]$  and letting  $c' : V_k \rightarrow [k]$  such that every  $u_i$  gets the same color as  $v_i$  and the vertex  $w$  gets the new color  $k$ .

Now assume that  $\chi(G_k) < k$ . Let  $c$  be a proper coloring of  $G_k$  with colors from  $[k - 1]$ . Derive contradiction if  $\chi(G_k) = k - 1$  by coloring  $w$  with color  $k - 1$  and the vertices from  $U_{k-1}$  are colored using  $k - 2$  colors, but then the vertices in  $V_{k-1}$  could also be colored from  $[k - 2]$ .

To see that  $G_k$  has no triangle, observe that a triangle could only have one vertex in  $u_i \in U_{k-1}$  and two vertices in  $v_j, v_m \in V_{k-1}$ . Then  $v_i, v_j, v_m$  form a triangle in  $G_{k-1}$  a contradiction.  $\square$

**Tutte's Construction** We can construct a family  $(G_k)_{k \in \mathbb{N}}$  of triangle-free graphs with  $\chi(G_k) = k$  as follows:  $G_1$  is the single-vertex graph. To get from  $G_k$  to  $G_{k+1}$ , take an independent set  $U$  of size  $k(|G_k| - 1) + 1$  and  $\binom{|U|}{|G_k|}$  vertex-disjoint copies of  $G_k$ . For each subset of size  $|G_k|$  in  $U$  then introduce a perfect matching to exactly one of the copies of  $G_k$ .

**Lemma** For any  $k$ , Tutte's graph  $G_k$  has chromatic number  $k$  and it is triangle-free.

*Proof.* We argue by induction on  $k$  with trivial basis  $k = 1$ . We see that  $\chi(G_k) \leq \chi(G_{k-1}) + 1$  because we can assign the same set of  $\chi(G_{k-1})$  colors to each copy of  $G_{k-1}$  and a new color to  $U$ . Assume that  $\chi(G_k) \leq \chi(G_{k-1})$ . Consider a coloring of  $G_k$  with  $\chi(G_{k-1})$  colors. By pigeonhole principle there is a set  $U'$  of  $|G_{k-1}|$  vertices in  $U$  of the same color, say 1. The vertices of  $U'$  are matched to a copy  $G'$  of  $G_{k-1}$ . Then  $G'$  does not use color 1 on its vertices and thus colored with less than  $\chi(G_{k-1})$  colors. Therefore there are two adjacent vertices of the same color. So, any proper coloring of  $G_k$  uses more than  $\chi(G_{k-1})$  colors.

To see that  $G_k$  has no triangles, observe that any two adjacent edges incident to  $U$  have endpoints in distinct copies of  $G_{k-1}$ , thus are not part of any triangle.  $\square$

**Theorem 54 (Kőnig, 1916)** If  $G$  is a bipartite graph with maximum degree  $\Delta$  then  $\chi'(G) = \Delta$ .

*Proof.* We see, that  $\chi'(G) \geq \Delta$  because the edges incident to a vertex of maximum degree require distinct colors in a proper edge-coloring.

To prove that  $\chi'(G) \leq \Delta$  we use induction on  $\|G\|$  with a basis  $\|G\| = 1$ . Let  $G$  be given,  $\|G\| \geq 2$  and assume that the statement is true for any graph on at most  $\|G\| - 1$  edges. Let  $e = xy \in E(G)$ . By induction, there is a proper edge coloring  $c$  of  $G' = G - e$  using  $\Delta$  colors.

In  $G'$  both  $x$  and  $y$  are incident to at most  $\Delta - 1$  edges. Thus there are non-empty color sets  $Mis(x), Mis(y) \subseteq [\Delta]$ , which are the colors that are not used on edges incident to  $x$  or  $y$ . If  $Mis(x) \cap Mis(y) \neq \emptyset$ , color  $e$  with  $\alpha \in Mis(x) \cap Mis(y)$ . This gives  $\chi'(G) \leq \Delta$ .

If  $Mis(x) \cap Mis(y) = \emptyset$ , let  $a \in Mis(x)$  and  $b \in Mis(y)$ , consider the longest path  $P$  colored  $a$  and  $b$  starting at  $x$ . Because of parity,  $P$  does not end in  $y$ , and because  $y$  is not incident to  $b$ ,  $y$  is not a vertex on  $P$ . Switch colors  $a$  and  $b$  on  $P$ . Then we obtain a proper edge-coloring in which  $b \in Mis(x) \cap Mis(y)$ , which allows  $e$  to be colored  $b$ . Thus  $\chi'(G) \leq \Delta$ .  $\square$

Proof sketch: induction: remove an edge, apply IH if the vertices incident to the edge have same color available then put back the edge and color it with it. If they have different ones then consider longest path (alternating between the available colors) starting in one of the vts, and switch colors of that one. So the vertices have the same color available, now put back the edge and color it with that color.

**Theorem 5.4 (Vizing's Theorem)** For any graph  $G$  with maximum degree  $\Delta$

$$\Delta \leq \chi'(G) \leq \Delta + 1$$

*Proof.* proof sketch:

- lower bound: because of vertex of maximum degree
- upper bound: induction on number of edges
- **Assume that  $G$  has no proper edge-coloring with max degree + 1 colors**
- introduce claim that when removing an edge we have a path between the incident vertices in the two colors that are available at them
- proof the claim with contradiction (we could switch colors and obtain coloring with less colors)
- get contradiction using claim and creating multiple Paths, this means  $G$  has a proper edge-coloring with  $\Delta + 1$

The lower bound holds because the edges incident to a vertex of maximum degree require distinct colors in a proper edge-coloring. For the upper bound use induction on  $\|G\|$  with the trivial basis  $\|G\| = 1$ . Let  $G$  be a graph,  $\|G\| > 1$ , assume that the assertion holds for all graphs with smaller number of edges. For any edge-coloring  $c$  of a subgraph  $H$  of  $G$  with colors  $[\Delta + 1]$ . Assume now that  $G$  has no proper edge-coloring with  $\Delta + 1$  colors.

**Claim** For any  $e = xy \in E(G)$ , for any proper coloring  $c$  of  $G - e$  from  $[\Delta + 1]$ , for any  $\alpha \in \text{Mis}_c(x)$  and any  $\beta \in \text{Mis}_c(y)$ , there is an  $x$ - $y$ -path colored  $\alpha$  and  $\beta$ .

We see that  $\text{Mis}_c \neq \emptyset$  for any  $v$ . If  $\text{Mis}_c(x) \cap \text{Mis}_c(y) \neq \emptyset$ , let  $\alpha \in \text{Mis}_c(x) \cap \text{Mis}_c(y)$ . Color  $xy$  with  $\alpha$ , this gives a proper coloring of  $G$  with at most  $\Delta + 1$  colors, a contradiction.

If  $\text{Mis}_c(x) \cap \text{Mis}_c(y) = \emptyset$ , let  $\alpha \in \text{Mis}_c(x), \beta \in \text{Mis}_c(y), \alpha \neq \beta$ . If there is a maximal path  $P$  colored  $\alpha$  and  $\beta$  that contains  $x$  and does not contain  $y$ , switch the colors  $\alpha$  and  $\beta$  in  $P$  and color  $xy$  with  $\beta$ . This gives a proper coloring of  $G$  with at most  $\Delta + 1$  colors, a contradiction, this proves the claim.

Let  $xy_0 \in E(G)$ . Let  $c_0$  be a proper coloring of  $G_0 := G - xy_0$  from  $[\Delta + 1]$ . Let  $\alpha \in \text{Mis}_{c_0}(x)$ . Let  $y_0, y_1, \dots, y_k$  be a maximal sequence of distinct neighbors of  $x$  such that  $c_0(xy_{i+1}) \in \text{Mis}_{c_0}(y_i), 0 \leq i < k$ .

Let  $c_i$  be a coloring of  $G_i = G - xy_i$  such that

$$c_i(xy_j) = \begin{cases} c_0(xy_{j+1}), & \text{for } j \in \{0, \dots, i-1\} \\ c_0(e), & \text{otherwise} \end{cases}$$

Note that  $\text{Mis}_{c_i}(x) = \text{Mis}_{c_j}(x)$  for all  $i, j \in \{0, \dots, k\}$ .

Let  $\beta \in \text{Mis}_{c_0}(y_k)$ . Let  $y = y_i$  be a vertex so that  $c_0(yx) = \beta$ . Such a vertex exists, otherwise either  $\beta \in \text{Mis}_{c_k}(y_k) \cap \text{Mis}_{c_k}(x)$  contradicting Claim, or the sequence  $y_0, \dots, y_k$  can be extended, contradicting its maximality.

Then  $G_k$  has an  $\alpha$ - $\beta$  path  $P$  with endpoints  $y_{i-1}, y_k$  in  $G_k - x$ . On the other hand  $G_i$  has an  $\alpha$ - $\beta$ -path  $P'$  with endpoints  $y_{i-1}, y_i$  in  $G_i - x$ . Since  $G_x$  is colored identically in  $c_k$  and  $c_i$ , we have that  $P \cup P'$  is a two-colored graph, connected since both paths contain  $y_{i-1}$  and having three vertices of degree 1. This is impossible.  $\square$

**Lemma 55** The list chromatic number of  $G = K_{n,n}$ , with  $n = \binom{2k}{k}$  is at least  $k + 1$ .

This means there exists a graph which has much greater list-chromatic number than chromatic number

## 7 Extremal graph theory

In this chapter we study how global parameters of a graph, such as its edge density or chromatic number, can influence its local substructures. Two categories of questions:

- global assumptions that might imply that  $H$  as a *minor* (or topological minor) it will suffice to raise  $\|G\|$  above the value of some linear function of  $|G|$ .
- global assumptions that might imply the existence of some given graph  $H$  as a *subgraph*

### Definition

- The *extremal number*  $\text{ex}(n, H)$  denotes the maximum size (amount of edges) of a graph of order  $n$  that does not contain  $H$  as a subgraph and  $\text{EX}(n, H)$  is the set of  $H$ -free graphs on  $n$  vertices with  $\text{ex}(n, H)$  edges.
- Example:  $\text{ex}(n, P_3) = \lfloor \frac{n}{2} \rfloor$ ,  $\text{EX}(n, P_3) = \{\lfloor n \rfloor \cdot K_2 + (n \bmod 2) \cdot E_1\}$
- Let  $n$  and  $r$  be integers with  $1 \leq r \leq n$ . The *Turan graph*  $T_r(n)$  is the unique complete  $r$ -partite graph of order  $n$  whose partite sets differ by at most 1 in size. It does not contain  $K_{r+1}$ , we denote  $\|T_r(n)\|$  by  $t_r(n)$

### Calculating Turan Number

The Turan number  $t_r(n)$  is calculated as follows:

Let  $n = pr + s$  where  $p$  and  $s$  are integers, and  $0 \leq s < r$ , then  $T_r(n)$  has  $s$  parts of size  $p + 1$  and  $r - s$  parts of size  $p$ . And then simply count edges by hand.

**Lemma 58** For any  $r, n \geq 1$ ,  $t_r(n + r) = t_r(n) + n(r - 1) + \binom{r}{2}$

*Proof.* Consider  $G = T_r(n + r)$  graph with parts  $V_1, \dots, V_r$ . Let  $v_i \in V_i, i = 1, \dots, r$ . Then  $G' = G - \{v_1, \dots, v_r\}$  is isomorphic to  $T_r(n)$ . We have that  $\|G\| - \|G'\|$  is equal to the number of edges incident to  $v_i$ 's for some  $i = 1, \dots, r$ . This number is

$$\underbrace{n(r - 1)}_{\text{every vertex in } T_r(n) \text{ gets } (r-1) \text{ new}} + \underbrace{\binom{r}{2}}_{\text{edges between new } r \text{ vts}}$$

□

**Lemma 59** Among all  $n$ -vertex  $r$ -partite graphs,  $T_r(n)$  has the largest number of edges.

*Proof.* Let first  $r = 2$ :

Let  $G$  be an  $n$ -vertex bipartite graph with largest possible number of edges. Then clearly  $G$  is complete bipartite. Assume that two parts  $V$  and  $U$  of  $G$  differ in size by at least 2, so  $|V| > |U| + 1$ . Put one vertex from  $V$  to  $U$  to obtain new parts  $V'$  and  $U'$  and let  $G'$  be complete bipartite graph with parts  $V'$  and  $U'$ . Then  $\|G'\| = |V'||U'| = (|V| - 1)(|U| + 1) = |V||U| - |U| + |V| - 1 > |V||U| - |U| + |U| + 1 - 1 = |V||U| = \|G\|$ , a contradiction to maximality of  $G$ .

If  $r > 2$ :

Consider any two parts  $U, V$  of an  $r$ -partite  $G$ . Assume that  $U$  differs from  $V$  by at least 2 in size. Let  $X$  be the remaining set of vertices. Then  $\|G\| = \|G[X]\| + |X|(n - |X|) + \|G[U \cup V]\|$ . Let  $G'$  be a graph on the same set of vertices as  $G$  that differs from  $G$  only on edges induced by  $U \cup V$  and so that  $G'[U \cup V]$

is a balanced complete bipartite graph. Then from the previous paragraph with  $r = 2$ , we see that  $\|G'[U \cup V]\| > \|G[U \cup V]\|$ . Thus  $\|G'\| > \|G\|$ , a contradiction. Thus any two parts of  $G$  differ in size by at most 1. In addition we see as before that  $G$  is complete  $r$ -partite. Thus  $G$  is isomorphic to  $T_r(n)$ .  $\square$

**Lemma 60** For a fixed  $r$ ,

$$\lim_{n \rightarrow \infty} \frac{t_r(n)}{\binom{n}{2}} = 1 - \frac{1}{r}$$

*Proof.* Since each part in  $T_r(n)$  has size either  $\lfloor \frac{n}{r} \rfloor$  or  $\lceil \frac{n}{r} \rceil$ , we see that each part has size between  $\frac{n-r}{r}$  and  $\frac{n+r}{r}$ . We have that

$$\underbrace{\binom{n}{2}}_{\text{all edges}} - \underbrace{r \binom{(n+r)/r}{2}}_{r \text{ independent sets}} \leq t_r(n) \leq \binom{n}{2} - r \binom{(n-r)/r}{2}$$

Thus

$$\binom{n}{2} - r \frac{1}{2} \frac{(n+r)}{r} \frac{n}{r} \leq t_r(n) \leq \binom{n}{2} - r \frac{1}{2} \frac{(n-r)}{r} \frac{(n-2r)}{r}$$

More manipulation and dividing by  $\binom{n}{2}$  gives the result.  $\square$

**Theorem 62 (Mantel's theorem)** If a graph  $G$  on  $n$  vertices contains no triangle then it contains at most  $\frac{n^2}{4}$  edges.

*Proof.* We proceed by induction on  $n$ . For  $n = 1$  and  $n = 2$ . The result is trivial, so assume that  $n > 2$  and we know it to be true for  $n - 1$ . Let  $G$  be a graph on  $n$  vertices. Let  $x$  and  $y$  be two adjacent vertices in  $G$ . Since every vertex in  $G$  is connected to at most one of  $x$  and  $y$ , there are at most  $n - 2$  edges between  $\{x, y\}$  and  $V(G) - \{x, y\}$ . Let  $H = G - \{x, y\}$ . Then  $H$  contains no triangles and thus, by induction,  $H$  has at most  $\frac{(n-2)^2}{4}$  edges. Therefore the total number of edges in  $G$  is at most  $\frac{n^2}{4}$ .  $\square$

**Theorem 6.2 (Turan's Theorem)** For all integers  $r > 1$  and  $n \geq 1$ , any graph  $G$  with  $n$  vertices,  $\text{ex}(n, K_r)$  edges and  $K_r \not\subseteq G$  is a  $T_{r-1}(n)$ . In other words  $\text{EX}(n, K_r) = \{T_{r-1}(n)\}$ .

*Proof.* **TODO**  $\square$

**Conjecture Erdős-Sos** If  $|G| = n$  and  $\|G\| > \frac{(k-1)n}{2}$ , then  $G$  contains all  $k$ -edge trees as subgraphs, i.e. for any tree  $T$  on  $k$  edges  $\text{ex}(n, T) \leq \frac{(k-1)n}{2}$

**Theorem 66 (Erdős-Stone-Simonovits)** For any graph  $H$  and for any fixed  $\epsilon > 0$  there is  $n_0$  such that for any  $n \geq n_0$ :

$$\left(1 - \frac{1}{\chi(H) - 1} - \epsilon\right) \binom{n}{2} \leq \text{ex}(n, H) \leq \left(1 - \frac{1}{\chi(H) - 1} + \epsilon\right) \binom{n}{2}$$

### Corollary

- $\chi(H) = 3$ ,  $\text{ex}(n, H) \approx \frac{1}{2} \binom{n}{2}$
- $\chi(H) \geq 3$ ,  $\text{ex}(n, H) \approx \text{ex}(n, K_{\chi(H)})$

For exam, find extremal number for an ugly big graph,  
by applying Erdős-Stone

*Proof outline:* Let  $r = \chi(H) - 1$

For the upper bound, let  $G$  be a graph on  $n$  vertices that has  $(1 - \frac{1}{\chi(H)-1} + \epsilon) \binom{n}{2}$  edges. We shall show that  $G$  has a subgraph isomorphic to  $H$ . Let  $G'$  be a large subgraph of  $G$  that has minimum degree at least  $(1 - \frac{1}{r} + \frac{\epsilon}{2})|V(G')|$ , we can find such  $G'$  by greedily deleting vertices of smaller degrees. Then show, by induction on  $r$  that  $G'$  contains a complete  $(r+1)$ -partite graph  $H'$  with sufficiently large parts. Finally observe that  $H \subseteq H'$

**Definition Zarankiewicz function**  $z(m, n; s, t)$  denotes the maximum number of edges that a bipartite graph with parts  $X, Y$  of sizes  $m, n$  respectively, can have without containing  $K_{s,t}$  respecting sides (i.e. there is no copy of  $K_{s,t}$  with partition sets  $S, T$  of sizes  $s, t$  respectively, such that  $S \subseteq X$  and  $T \subseteq Y$ )

**Theorem 67 (Kovari-Sos-Turan)** We have the upper bound

$$z(m, n; s, t) \leq (s-1)^{\frac{1}{t}}(n-t+1)m^{1-\frac{1}{t}} + (t-1)m$$

In particular for  $m = n$  and  $t = s$

$$z(m, n; s, t) \leq c_1 \cdot n \cdot n^{1-\frac{1}{t}} + c_2 \cdot n = \mathcal{O}(n^{2-\frac{1}{t}})$$

*Proof.* Let  $G$  be a bipartite graph with parts  $A, B$ ,  $|A| = m$  and  $|B| = n$  such that it does not contain a copy of  $K_{s,t}$  with part of size  $s$  in  $A$  and part of size  $t$  in  $B$ . Let  $T$  be the number of stars of size  $t$  with a center in  $A$ . Then

$$T = \sum_{v \in A} \binom{\deg(v)}{t}$$

On the other hand

$$T \leq (s-1) \binom{n}{t}$$

Since for each subset  $Q$  of  $t$  vertices in  $B$  there are at most  $s-1$  stars counted by  $T$  with a leaf-set  $Q$ .

TODO

□

### Hadwiger's Conjecture

$$\chi(G) \geq r \Rightarrow G \text{ has a } K_r \text{ minor}$$

*Proof.* proof cases  $r = 3$  and  $r = 4$  for exam

□

**Lemma 68** For any positive integers  $n, t$  with  $t < n$ ,

$$\text{ex}(n, K_{t,t}) \leq \frac{z(n, n; t, t)}{2}$$



**Theorem 69** For any positive  $t$ , and  $n > t$ , there are positive constants  $c$  and  $c'$  such that

$$c' \cdot n^{2 - \frac{2}{t+1}} \leq ex(n, K_{t,t}) \leq c \cdot n^{2 - \frac{1}{t}}$$

**Theorem 71**

$$ex(n, C_4) = \frac{1}{2}n^{3/2} + (n^{3/2})$$

$$ex(n, C_6) = \Theta(n^{4/3})$$

$$ex(n, C_{10}) = \Theta(n^{6/5})$$

$$c' \cdot n^{1 - \frac{2}{3k-2-\epsilon}} \leq ex(n, C_{2k}) \leq c \cdot n^{1 + \frac{1}{k}}$$

**Definition 6.4** Let  $X, Y \subseteq V(G)$  be disjoint vertex sets and  $\epsilon > 0$ .

- the density  $d(X, Y)$  of  $(X, Y)$  is

$$d(X, Y) := \frac{\|X, Y\|}{|X| \cdot |Y|}$$

- For  $\epsilon > 0$  the pair  $(X, Y)$  is an  $\epsilon$ -regular pair if we have

$$|d(X, Y) - d(A, B)| \leq \epsilon$$

for all  $A \subseteq X, B \subseteq Y$  with  $|A| \geq \epsilon|X|$  and  $|B| \geq \epsilon|Y|$

In other words, the edges in an  $\epsilon$ -regular pair are distributed very uniformly, with the density between any pair of reasonably large subsets of vertices being very close to the overall density of the pair. This uniform distribution of edges is typical in a random bipartite graph, and captures what we mean when we say a (bipartite) graph 'looks random'. It remains to define what kinds of partitions of the vertices we will be concerned with.

- An  $\epsilon$ -regular partition of the graph  $G = (V, E)$  is a partition of the vertex set  $V = V_0 \cup \dots \cup V_k$  with the following properties:

1.  $|V_0| \leq \epsilon|V|$
2.  $|V_1| = |V_2| = \dots = |V_k|$
3. all but at most  $\epsilon k^2$  of the pairs  $(V_i, V_j)$  for  $1 \leq i < j \leq k$  are  $\epsilon$ -regular.

Note the parameter  $\epsilon$  play three roles here: bounding the size of the exceptional set  $V_0$ , bounding the number of irregular pairs, and controlling the regularity of the regular pairs

In an  $\epsilon$ -regular partition we have control over the distribution of edges between the  $\epsilon$ -regular pairs, but not over the edges within any of the parts, involving the exceptional set, or in irregular pairs. Thus, in light of the three roles described above, the smaller  $\epsilon$  is, the greater our control over the distribution of edges in an  $\epsilon$ -regular partition.

Informally, the Regularity Lemma tells us that the vertices of *any* large graph can be partitioned into a bounded number of parts, with the subgraph between most pairs of parts looking random.

**Szemerédi's Regularity Lemma** For any  $\epsilon > 0$  and any integer  $m \geq 1$  there is an  $M \in \mathbb{N}$  such that every graph of order at least  $m$  has an  $\epsilon$ -regular partition  $V_0 \cup \dots \cup V_k$  with  $m \leq k \leq M$ .

### Analyzing large Graphs with regularity Lemma

1. start with  $\epsilon$ -partition
2. create auxillary graph with an edge between parts if  $(V_i, V_j)$   $\epsilon$ -regular with density  $d > 0$
3. use blowup graph to complete partite, use blowup lemma:

$$H \subseteq R_\epsilon \Rightarrow H \subseteq G$$

**Erdős-Stone Theorem** For all integers  $r > s \geq 1$  and any  $\epsilon > 0$  there exists an integer  $n_0$  such that every graph with  $n \geq n_0$  vertices and at least

$$t_{r-1}(n) + \epsilon n^2$$

edges contains  $K_r^s$  (is the complete  $r$ -partite graph where each part contains exactly  $s$  vertices) as a subgraph.

**Corollary 73** Erdős-Stone together with  $\lim_{n \rightarrow \infty} \frac{t_{r-1}(n)}{\binom{n}{2}} = 1 - 1/r$  yields an asymptotic formula for the extremal number of any graph  $H$  on at least one edge:

$$\lim_{n \rightarrow \infty} \frac{ex(n, H)}{\binom{n}{2}} = \frac{\chi(H) - 2}{\chi(H) - 1}$$

**Substructures in sparse Graphs** **Key message:** extremal numbers for minors are linear in  $|E|$

- $\|G\| \geq cr^2|G| \Rightarrow G \supseteq \text{TK}_r$
- $\|G\| \geq c\sqrt{\log(r)} \Rightarrow G \supseteq \text{MK}_r$

## 8 Ramsey Theory

### Definitions

- In an edge-coloring of a graph, a set of edges is
  - *monochromatic* if all edges have the same color
  - *rainbow* if no two edges have the same color
  - *lexical* if two edges have the same color iff they have the same lower endpoint in some ordering of the vertices
- Let  $k$  be a natural number. Then the *Ramsey number*  $R(k) \in \mathbb{N}$  is the smallest  $n$  such that every 2-edge-coloring of  $K_n$  contains a monochromatic  $K_k$ .

- **asymmetric Ramsey number**  $R(k, l)$ : is the smallest  $n \in \mathbb{N}$  such that every 2-edge-coloring of a  $K_n$  contains a red  $K_k$  or a blue  $K_l$ .
- **graph Ramsey number**  $R(G, H)$ : is the smallest  $n \in \mathbb{N}$  such that every red-blue edge-coloring of  $K_n$  contains a red  $G$  or a blue  $H$ .
- **hypergraph Ramsey number**  $R_r(l_1, \dots, l_k)$ : is the smallest  $n \in \mathbb{N}$  such that every  $k$ -coloring of the edges of the complete hypergraph on  $n$  vertices and edges of size  $r$  contains a clique of size  $l_i$  whose edges all have color  $i$ , for some  $i \in \{1, \dots, k\}$ .
- **induced Ramsey number**  $IR(G, H)$ : is the smallest  $n \in \mathbb{N}$  for which there is a graph  $F$  on  $n$  vertices such that in any red-blue coloring of  $E(F)$ , there is an induced subgraph of  $F$  isomorphic to  $G$  with all its edges colored red or there is an induced subgraph of  $F$  isomorphic to  $H$  with all its edges colored blue.
- **anti-Ramsey number**  $AR(n, H)$ : is the maximum number of colors that an edge-coloring of  $K_n$  can have without containing a rainbow copy of  $H$ .

**Ramsey Theorem** For any  $k \in \mathbb{N}$  we have

$$\sqrt{2}^k \leq R(k) \leq 4^k$$

In particular the Ramsey numbers, the asymmetric Ramsey numbers and the graph Ramsey numbers are finite.

*Proof.* **TODO** □

**Remark**  $R(2) = 2, R(3) = 6, R(4) = 18$  and  $43 \leq R(5) \leq 48$ .

### Applications of Ramsey theory

**Theorem (Erdős, Szekeres)** Any list of more than  $n^2$  numbers contains a nondecreasing or non-increasing sublist of more than  $n$  numbers.

*Proof.* **TODO** □

**Theorem (Erdős, Szekeres)** For any integer  $m \geq 3$  there is an integer  $N = N(m)$  such that if  $X$  is a set of  $N$  points on the plane such that no three points are on a line, then  $X$  contains a vertex set of a convex  $m$ -gon.

*Proof.* **TODO** □

**Definition** Let  $R(p, q; r)$  be the hypergraph Ramsey number for  $r$ -uniform hypergraphs. The following Theorem show the existence of hypergraph Ramsey number.

**Theorem 83** For any parametres  $p, q, r \geq 2$

$$R(p, q; r) \leq R(R(p-1, q; r), R(p, q-1; r); r-1) + 1$$

## 9 Random Graphs

### Definitions

- Erdos-Renyi model of random graphs  $\mathcal{G}(n, p)$  is the probability space on all  $n$ -vertex graphs that results from independently deciding whether to include each of the  $\binom{n}{2}$  possible edges with fixed probability  $p \in [0, 1]$
- A property  $\mathcal{P}$  is a set of graphs, for example  $\mathcal{P} = \{G : G \text{ is } k\text{-connected}\}$
- Let  $(p_n) \in [0, 1]^{\mathbb{N}}$  be a sequence. We say that  $G \in \mathcal{G}(n, p_n)$  *almost always* has a property  $\mathcal{P}$  if  $\text{Prob}(G \in \mathcal{G}(n, p_n) \cap \mathcal{P}) \rightarrow 1$  for  $n \rightarrow \infty$ . If  $(p_n)$  is constant  $p$ , we also say in this case that *almost all* graphs in  $\mathcal{G}(n, p)$  have property  $\mathcal{P}$
- A function  $f(n) : \mathbb{N} \rightarrow [0, 1]$  is a *threshold function* for property  $\mathcal{P}$  if:
  - for all  $(p_n) \in [0, 1]^{\mathbb{N}}$  with

$$\frac{p_n}{f(n)} \xrightarrow{n \rightarrow \infty} 0$$

the graph  $G \in \mathcal{G}(n, p_n)$  almost always does not have property  $\mathcal{P}$

- for all  $(p_n) \in [0, 1]^{\mathbb{N}}$  with

$$\frac{p_n}{f(n)} \xrightarrow{n \rightarrow \infty} \infty$$

the graph  $G \in \mathcal{G}(n, p_n)$  almost always has property  $\mathcal{P}$

- threshold function for containing a cycle is  $f(n) = 1/n$  meaning if  $p > 1/n$  we almost always have a cycle

### Usage of probabilistic Method

- $\text{ex}(n, K_{t,t})$
- $\sqrt{2}^k \leq R(k)$
- Erdős-Hajnal

**Lemma 107** Erdős-Renyi model is universal **TODO** Lemma 107 tells us that Erdos-Renyi model is universal, it gives us any graph.

**Lemma 108** **TODO**

**Lemma 110** Expected number of cycles  
**TODO**

**Theorem 9.2 (Erdős)** For any  $k \geq 2$  there is a Graph  $G$  on  $\sqrt{2}^k$  vertices such that  $\alpha(G) < k$  and  $\omega(G) < k$ . This implies  $R(k, k) \geq 2^{k/2}$ .

*Proof.* **TODO** □

**Theorem 9.3 (Erdos-Hajnal)** sketch: take random graph, number of short cycles (length at most  $k$ ) is  $< \frac{n}{2}$  delete a vertex from each of these cycles and get a graph  $G'$  so the girth of  $G' > k$ . We know chromatic number of  $G$  prime is  $\geq \frac{|V(G')|}{\alpha(G')}$  with lemma 108 we can show  $\alpha$  is not so big and therefore get a lower bound

**two tricks:**

- random graph and kill short cycle
- bound the independence number (co-clique number) size of the largest independent set

## 10 Hamiltonian cycles

**Definition**

- A cycle  $C$  in a graph  $G$  is *Hamiltonian* if it contains all vertices.
- A graph that has a Hamiltonian cycle is called Hamiltonian graph.

**Lemma 10.1 (Necessary condition for existence of Ham. cycle)**

If  $G$  has a Hamiltonian cycle, then for every non-empty  $S \subseteq V$  the graph  $G - S$  cannot have more than  $|S|$  components.

*Proof.* Let  $C$  be a Hamiltonian cycle of  $G$ . Let  $S \subseteq V(G)$ ,  $S \neq \emptyset$ ,  $t := \#$  components of  $G - S$ . There are at least 2 edges of  $C$  between each component of  $G - S$  and  $S$ . If  $e = \#$  edges of  $C$  between  $S$  and  $V - S$ , we have

$$e \geq t \cdot 2 \text{ and } e \underset{C \text{ is 2-regular}}{\leq} |S| \cdot 2$$

□

**Theorem 10.2 Dirac** Every graph with  $n \geq 3$  vertices and minimum degree at least  $\frac{n}{2}$  has a Hamiltonian cycle.

*Proof.* **TODO** □

**Ore's Thm.** A graph  $G$  on  $n \geq 3$  vertices is Hamiltonian if  $\forall u, v \in V(G), uv \notin E(G), d(u) + d(v) \geq n$

## 11 Networkflows

**Definitions**

- circulation
- network, source, sink, capacity

### Definition Network Flow

(F1)

(F2)

(F3)

- cut
- capacity of a cut

**Lemma 100** For any cut  $(S, \bar{S})$  and a network flow  $f$  in a network  $N$ ,  $f(S, \bar{S}) = f(s, V(G))$

*Proof.*

$$f(S, \bar{S}) = f(S, V) - f(S, S) = f(s, V) + \underbrace{\sum_{v \in S \setminus \{s\}} f(v, V)}_{0 \text{ by } F2} - \underbrace{f(S, S)}_{0 \text{ by } F1} = f(s, V) + 0 - 0$$

□

**Ford-Fulkerson Theorem** Let  $N = (G, s, t, c)$  be a network. Then

$$\max\{|f| : f \text{ is an } N\text{-flow}\} = \min\{c(S, \bar{S}) : (S, \bar{S}) \text{ is a cut}\}$$

and there is an integral flow  $f : T \rightarrow \mathbb{Z}_{\geq 0}$  with this maximum flow value.

*Proof.* **TODO**

□

**No Group-valued flows!!**

## 12 problems

**problem 1** Determine the number of edges, average degree, diameter and girth of the  $d$ -dimensional hypercube.

*Proof.* For number of edges consider vertex degree, for average degree observe that  $Q_d$  is regular.

claim:  $Q_d$  has diameter  $d$ .

First prove that for any  $x, y \in V$  the distance  $d(x, y)$  in  $Q_d$  is the number of positions where they differ. Suppose they differ in  $l \geq 1$  positions then we get a path of length  $l$  by inverting each entry where they differ sequentially. There can not be a shorter Path since each step can only change one position.

Therefore the diameter is  $d$ , because that's the maximum amount of positions where two sequences can differ, consider 00...0 and 11...1.

claim: girth of  $Q_d$  is  $\infty$  if  $d = 1$  and otherwise 4.

If  $d = 1$ , the resulting graph  $Q_1$  is acyclic. If  $d \geq 2$  we first observe that then  $Q_d$  is triangle-free. Suppose otherwise than there is a triangle  $xyz$  in  $Q_d$ , now consider the amount of 1's in  $|v|, v \in V$ . Wlog  $|x|$  is even then  $|y|$  must be odd

and  $|z|$  must be even. But  $zx$  is an edge, a contradiction.  
On the other hand  $Q_d$  does contain 4-cycles:

$$(00\dots 0, 010\dots 0, 100\dots 0, 00\dots 0)$$

□

**problem 2** Show that any tree  $T$  has at least  $\Delta(T)$  leaves.

*Proof.* Induction on  $n = |T|$ .

**Base:** tree on two vertices has 2 leaves and maximum degree is 1.

**Step:** remove a leaf  $v$  from  $T$  and let  $u$  be its neighbour, so that  $T' := T - v$  we then have two cases:

case 1:  $\Delta(T) = \Delta(T')$

We know by I.H. that  $T'$  has at least  $\Delta(T')$  leaves, thus if the maximum degree of  $T$  is the same the claim holds.

case 2:  $\Delta(T) = \Delta(T') + 1$  This means  $u$  is the vertex with maximal degree in  $T$ , but we know that the leaves of  $T$  can only be the leaves of  $T'$  plus the leaf  $v$  we removed therefore  $T$  has  $\Delta(T') + 1$  leaves.

□

*Proof.* By Counting

Let  $T$  be any tree. Let  $L \subseteq V$  be the set of leaves and  $N = V \setminus L$  the set of non-leaves in  $T$ . Let  $u$  be a vertex with maximum degree  $\Delta(T)$ . We know that a tree has  $|V| + 1$  edges.

$$2 \cdot (|V| - 1) = \sum_{v \in V(T)} d(v) \tag{1}$$

$$= d(u) + \sum_{v \in L} d(v) + \sum_{v \in N \setminus \{u\}} d(v) \tag{2}$$

$$\geq \Delta(T) + \sum_{v \in L} 1 + \sum_{v \in N \setminus \{u\}} 2 \tag{3}$$

$$= \Delta(T) + |L| \cdot 1 + (|V| - |L| - 1) \cdot 2 \tag{4}$$

$$= \Delta(T) + 2 \cdot (|V| - 1) - |L| \tag{5}$$

From this follows  $|L| \geq \Delta(T)$ , as desired.

□

**problem 3** Prove that either a graph or its complement is connected.

*Proof.* Let  $G = (V, E)$  be any non-empty graph. We assume  $G$  is not connected and shall argue that  $\overline{G}$  is connected.

Since  $G$  is disconnect, we find two vertices  $u, w \in V$  and a connected component  $C$  of  $G$  such that  $u \in C$  and  $w \notin C$ . Now in  $\overline{G}$  all vertices in  $C$  are adjacent to  $w$ . And in particular  $uw \in E(\overline{G})$ , so all vertices lie in a single connected component of  $\overline{G}$ , which is therefore connected.

□

**problem 4** Prove that the vertex set of any graph can be partitioned into two sets such that for each vertex, at least half of its neighbors belong to the other set.

*Proof.* Consider partition of  $V(G)$  into disjoint sets  $A, B$ , maximizing the number of edges between  $A$  and  $B$ . We will show that moving any vertex to the other sets results in a partition that has less edges between the sets than the original one.

Pick wlog a vertex  $v \in A$ . Let  $d_B = N(v) \cap B$  and  $d_A = d - d_B$ . Now consider  $A' = A - v$  and  $B' = B + v$

$$|\{uw \in E : u \in A', w \in B'\}| = |\{uw \in E : u \in A, w \in B\}| - d_B + (d - d_B)$$

but by the maximality of edges between  $A$  and  $B$  we have:

$$|\{uw \in E : u \in A', w \in B'\}| \leq |\{uw \in E : u \in A, w \in B\}|$$

which gives us

$$-d_B + (d - d_B) \leq 0 \iff d_B \geq d/2$$

□

**problem 5** Show that the following statements are equivalent

1.  $G$  is connected, but  $G - e$  is disconnected for every edge  $e$
2. Any two vertices in  $G$  are linked by a unique path

*Proof.*

“(ii)  $\Rightarrow$  (i)”: trivial

“(i)  $\Rightarrow$  (ii)”:

As  $G$  is connected, there is at least one path between any two vertices of  $G$ . Lets assume for the sake of contradiction that there are some vertices  $x$  and  $y$  that are joined by at least two paths  $P_1, P_2$ . As  $P_1 \neq P_2$  there is an edge  $e_0$  that lies in  $P_1$  but not in  $P_2$ . We shall show that the graph  $G - e_0$  is still connected, which will be a contradiction.

Idea is to consider the path between any two vertices  $u, v$  and remove the edge  $e_0$  and still show we can reach  $v$  from  $u$  by using the vertices  $x$  and  $y$  which have two paths between them and simply put  $e_0$  in one of them, then we can use the other one by going from one endpoint of  $e_0$  to  $x$ , possible because  $G$  is connected. Then to  $y$  using the other path and then from  $y$  to the other endpoint of  $e_0$  and finally to  $v$ . Thus  $u$  and  $v$  are still connected after removing  $e_0$  a contradiction.

□

**problem 6** Let  $T_1, \dots, T_k$  be subtrees of a tree  $T$ , any two of which have at least one vertex in common. Prove that there is a vertex common to all  $T_i$ .

*Proof.* Apply induction on  $|T|$ . If  $|T| = 1$  then  $T$  consists of a single vertex, say  $v$ . If  $T_1, \dots, T_k$  are subtrees of  $T$  with pairwise intersecting vertex sets, then we must have  $T_i = T$  for each  $i = 1, \dots, k$ . It follows that  $v$  belongs to each  $T_i$ .



So assume  $n \geq$  is an integer, and let us suppose this result holds for all trees of order  $n-1$ . Suppose  $T$  is a tree with  $|T| = n$  and  $T_i, \dots, T_k$  are subtrees of  $T$ . We assume  $k \geq 2$ . Let  $v$  be a leaf in  $T$  and let  $T' = T - v$  be the tree resulting from removing this leaf.  $\square$

**problem 7**