Graphtheory Theorems

1 Important Graphs

Complete Graph, Clique:
the *complete graph* K_n on n vertices is isomorphic to $([n], \binom{[n]}{2})$

Cycle
C_n on n vertices with $n \geq 3$ is isomorphic to $([n], \{\{i, i+1\} : i = 1, ..., n-1\} \cup \{n, 1\})$, the *length of a cycle* is its number of edges.

Empty Graph
E_n on n vertices is isomorphic to $([n], \varnothing)$.
Empty graphs correspond to *independent sets*.

Complete Bipartite Graph
$K_{m,n}$ on $n+m$ vertices is isomorphic to $(A \cup B, \{xy : x \in A, y \in B\})$ where $|A| = m$ and $|B| = n$ and $A \cap B = \varnothing$.

Complete r-partite graph with $r \geq 2$ is isomorphic to

$$(A_1 \uplus ... \uplus A_r, \{xy : x \in A_i, y \in A_j, i \neq j\})$$

where $A_1, ..., A_r$ are disjoint non-empty finite sets.

n-dimensional hypercube Q_n is the graph isomorphic to

$$(2^{[n]}, \{\{S, T\} : S, T \in 2^{[n]}, |S \Delta T| = 1\})$$

Vertices are labeled either by corresponding sets or binar indicator vectors. For example the vertex $\{1, 3, 4\}$ in Q_6 is coded by $(1, 0, 1, 1, 0, 0)$. A 1 is indicating we take this Element and a 0 if not.

2 Basics

Definitions

- A graph G is *non-trivial* if it contains at least one edge, equivalently if G is not an empty graph
- The *order* of G writen $|G|$, is the number of vertices of G, i.e. $|G| = |V|$
- The *size* of G wirten $||G||$, is the number of edges of G, i. e. $||G|| = |E|$, if order of G is n then the size of G is between 0 and $\binom{n}{2}$
- $N(S)$ the *neighbourhood* of $S \subseteq V$ is the set of vertices in V. that have and adjacent vertex in S. Instead of $N(\{v\})$ for $v \in V$ we write $N(v)$
- vertex of degree 1 is called *leaf*
- vertex of degree 0 is called *isolated vertex*

- *minimum degree of* G, denoted by $\delta(G)$ is the smallest vertex degree in G
- *maximum degree of* G, denoted by $\Delta(G)$ is the highest vertex degree in G
- graph G is called *k-regular*, with $k \in \mathbb{N}$, if all vertices have degree k.
- *average degree of* G is defined as $d(G) = \frac{\sum_{v \in V} deg(v)}{|V|}$
  We have
  $$\delta(G) \leq d(G) \leq \Delta(G)$$
  with equality if and only if G is k-regular

Handshake Lemma For ever graph $G = (V, E)$ we have

$$2|E| = \sum_{v \in V} d(v)$$

Proof. By double counting the set $X = \{(e, x) : e \in E(G), x \in V(G), x \in e\}$ then

$$|X| = \sum_{v \in V(G)} d(x)$$

and

$$|X| = \sum_{e \in E(G)} 2 = 2|E(G)|$$

by the principle of double counting the terms are equal. □

Corollary From this follows that the sum of all vertex degrees is even and therefore the number of vertices with odd degree is even.

Proposition 3 If a graph G has minimum degree $\delta(G) \geq 2$, then G has a path of length $\delta(G)$ and a cycle with at least $\delta(G) + 1$ vertices.

Proof. Let $P = (x_0, ..., x_k)$ be a longest path in G. Then $N(x_0) \subseteq V(P)$, otherwise for $x \in N(x_0) \setminus V(P)$ the path $(x, x_0, x_1, ..., x_k)$ would be a longer path.

Let i be the largest index such that $x_i \in N(x_0)$, then $i \geq |N(x_0)| \geq \delta$. So $(x_0, x_1, ..., x_i, x_0)$ is a cycle of length at least $\delta(G) + 1$. □

Proposition 4 If for distinct vertices u and v a graph has a u-v-walk, then it has a u-v-path.

Proof. Consider a u-v-walk W with the smallest number of edges. Assume that W does not form a path, then there is a repeated vertex, w, i.e.

$$W = u, e, v_1, e_1, ..., e_k, w, e_{k+1}, ..., e_l, w, e_{l+1}, ..., v$$

Then $W_1 = u, e, v_1, ..., e_k, w, e_{k+1}, ..., v$ is a shorter u-v-walk, a contradiction. □

Propostion 5 If a graph has a closed walk of odd length, then it contains an odd cycle.

Proof. Let W be the shortest closed odd walk. If W is a cycle the Proposition holds. Otherwise there is a repeated vertex, so W is an edge-disjoint union of two closed walks. The sum of the lengths of these walks is odd, therefore one of them is an odd closed walk shorter than W a contradiction to the minimality of W. □

Proposition 6 If a graph has a closed walk with a non-repeated edge, then the graph contains a cycle.

Proof. Let W be a shortest walk with a non-repeated edge e. If W is a cycle, we are done. Otherwise, there is a repeated vertex and W is a union of two closed walks W_1 and W_2 that are shorter than W. One of them say W_1, contains e, a non-repeated edge. This contradicts the minimality of W. □

Definition bipartite A graph $G = (V, E)$ is called *bipartite* if there exists natural numbers m, n such that G is isomorphic to a subgraph of $K_{m,n}$. Then the vertex set can be written as $V = A \cup B$ such that $E \subseteq \{ab : a \in A, b \in B\}$. The sets A and B are called the *partite sets* of G

Proposition 1.5 A graph is bipartite if and only if it has no cycles of odd length.

Proof. skript
"$\Rightarrow$"
Assume that G is a bipartite graph with parts A and B. Then any cycle has a form $a_1, b_1, a_2, b_2, ..., a_k, b_k, a_1$ where $a_i \in A, b_i \in B, i \in [k]$. Thus every cycle has even length.
"$\Leftarrow$"
Assume G does not have cycles of odd length. We can assume that G is connected, otherwise we can treat the connected components separately. Let $v \in V(G)$. Let $A = \{u \in V(G) : dist(u, v) \equiv 0 \pmod 2\}$ and let $B = \{u \in V(G) : dist(u, v) \equiv 1 \pmod 2\}$ We claim that G is bipartite with parts A and B. To verify this it is sufficient to prove that A and B are independent sets. Let $u_1u_2 \in E(G)$ and let P_1 be a shortest u_1-v-path and P_2 a shortest u_2-v-path. Then the union of P_1, P_2 and u_1u_2 forms a closed walk W. If $u_1, u_2 \in A$ or $u_1, u_2 \in B$ then W is a closed odd walk, because $dist(v, u_1)$ and $dist(v, u_2)$ are both even or odd. Thus by Prop. 5 G contains an odd cycle, a contradiction. Thus for any edge u_1u_2 the adjacent vertices u_1 and u_2 are in different parts A or B. Therefore A and B are independent sets. □

Proof. Diestel
"$\Leftarrow$"
Let T be a spanning tree in G, pick a root $r \in T$ and denote the associated tree-order on V by $\leq_T$ (this order expressing height if $x < y$ then x lies *below* y in T). For each $v \in V(G)$ the unique path r-v-T has odd or even length. This defines a bipartition of $V(G)$, we show that G is bipartite with this partition. Let $e = xy$ be an edge of G. If $e \in T$ with $x <_T y$ say, then r-y-T = r-xy-T and so x and y lie in different partition classes. If $e \notin T$ then $C_e := x$-y-T+ e is a cycle, and by the case treated already the vertices along x-y-T alternate between the two classes. Since C_e is even by assumption, x and y again lie in different classes. □

Euler tour A closed walk that traverses every edge of the graph exactly once is called an *Euler tour*.

Theorem 1.6 (Eulerian Tour Condition) A connected graph has an Eulerian Tour if and only if every vertex has even degree.

Proof. "⇒"
The degree condition is necessary for an euler tour, because a vertex appearing k times in an Euler tour (or $k+1$ times if it is the starting and finishing vertex) must have degree $2k$.
"⇐"
Show by induction on $||G||$ that every connected Graph G with all degrees even has an Euler tour. $||G|| = 0$ is trivial.

Now let $||G|| \geq 1$, since all degrees are even, we can find in G a non-trivial closed walk that contains no edge more than once. To find this walk we consider W a walk of maximal length and write F for the set of its edges. If $F = E(G)$, then W is an Euler tour.

Suppose, therefore $G' := G - F$ has an edge.
For every vertex $v \in G$, an even number of edges of G at v lies in F, so the degrees of G' are again all even. Since G is connected, G' has an edge e incident with a vertex on W. By I.H. the component C of G' containing e has an Euler tour. Concatenating this with W (suitably re-indexed), we obtain a closed walk in G that contradicts the maximal length of W.

□

Definitions

- graph G is *connected* if any two vertices are linked by a path.
- a maximal connected subgraph of G is called a *connected component* of G.
- acyclic graphs are called $forests$
- a graph G is called a *tree* if G is connected and acyclic.

Lemma 7 Every tree on at least two vertices has a leaf.

Proof. If a tree T on at least two vertices does not have leaves then every vertex has degree $>$ than 2, so we have a cycle in T with length ≥ 3, a contradiction. □

Lemma 8 A tree of order $n \geq 1$ has exactly $n-1$ edges.

Proof. We prove the statement by induction on n. When $n = 1$, there are no edges.

I.H.: Assume that each tree on $n = k$ vertices has $k-1$ edges, with $k \geq 1$.

Step: Lets prove that each tree on $k+1$ vertices has k edges. Consider a tree T on $k+1$ vertices. Since $k+1 \geq 2$, T has a leaf v. Let $T' = T - \{v\}$. We see that T' is connected because any u-w-path in T, for $u \neq v$ and $w \neq v$, does not contain v. We see also that T' is acyclic, because deleting vertices from an acyclic graph does not create new cycles. Thus T' is a tree on k vertices. By I.H. $|E(T')| = k-1$. Thus $|E(T)| = |E(T')| + 1 = (k-1) + 1 = k$.

□

Lemma 9 Every connected graph contains a spanning tree.

Proof. Let G be a connected graph. Consider T, an acyclic spanning subgraph of G with largest number of edges. If it is a tree we are done.

Otherwise, T has more than one component. Consider vertices u and v from different components of G. Consider a shortest u-v-path P in G. Then P has an edge $e = xy$ with exactly one vertex x in one of the components of T. Then P has an edge $e = xy$ with exactly one vertex x in one of the components of T. Then $T \cup \{e\}$ is acyclic. If there would be a cycle, it would contain e, however e connects to components, therefore cannot be part of a cylce (e would be a repeated edge). Thus $T \cup \{e\}$ is a bigger spanning acyclic subgraph of G contradicting the maximality of T. □

Lemma 10 A connected graph on $n \geq 1$ vertices and $n - 1$ edges is tree.

Proof. Let G be a connected graph on n vertices with $n - 1$ edges. Assume G is not a tree, i.e. contains a cycle. We therefore can remove a edge so that G is still connected. This is a contradiction because a graph on n vertices with $n - 2$ edges cannot be connected. Because a walk from vertex 1 to vertex n has to have at least $n - 1$ edges. □

Lemma 11 The vertices of every connected graph on $n \geq 2$ vertices can be ordered $(v_1, ..., v_n)$ so that for every $i \in \{1, ..., n\}$ the Graph $G[\{v_1, ..., v_i\}]$ is connected.

Proof. skript
Let G be a connected graph on n vertices. It contains a spanning tree T. Let v_n be a leaf of T, let v_{n-1} be a leaf of $T - \{v_n\}$, let v_{n-2} be a leaf of $T - \{v_n, v_{n-1}\}$ and so on. Let v_k be a leaf in $T - \{v_n, v_{n-1}, ..., v_{k+1}\}$, $k = 2, ..., n$. Since deleting a leaf does not disconnect a tree, all resulting graphs form a spanning trees of $G[v_1, ..., v_i]$, $i = 1, ..., n$. A graph H having a spanning tree or any connected spanning subgraph H' is connected because a u-v-path in H' is a u-v-path in H. This observation completes the proof. □

Proof. diestel
Pick any vertex as v_1, and assume inductively that $v_1, ..., v_i$ have been chosen for some $i < |G|$. Now pick a vertex $v \in G - G_i$. As G is connected, it contains a v-v_1 path P. Choose v_{i+1} as the last vertex of P in $G - G_i$, then v_{i+1} has a neighbour in G_i. If we consider $i + 1$ then we simply add v_{i+1} to our G_i, Thus $G_{i+1} := G_i \cup \{v_{i+1}\}$ which is also connected. □

Tree equivalences For any graph $G = (V, E)$ the following are equivalent:

(i) G is a tree, i.e. G is connected and acyclic.

(ii) G is connected, but for any $e \in E$ the graph $G - e$ is not connected (minimally connected)

(iii) G is acyclic, but for any $x, y \in V(G), xy \notin E(G)$ the graph $G + xy$ has a cycle. (maximaly acyclic)

(iv) G is connected and 1-degenerate

(v) G is connected and $|E| = |V| - 1$

(vi) G is acyclic and $|E| = |V| - 1$

(vii) G is connected and every non-trivial subgraph of G has a vertex of degree at most 1.

(viii) Any two vertices are joined by a unique path in G.

Proof.
(i) $\Rightarrow$ (ii):
G is connected and acyclic, now assume for any edge $e = xy$ the graph $G' = G - e$ would still be connected. Then G' has a x-y-path P. But $P \cup e$ is a cycle in G which contradicts that G is acyclic.
(ii) $\Rightarrow$ (i):
G is connected and for any edge e the graph $G - e$ is not connected. We want to show that G is acyclic. If G would have a cycle we could simply remove an edge from that and the resulting graph would still be connected, a contradiction. □

Proof. (i) $\Rightarrow$ (iv):

(vi) $\Rightarrow$ (i):

□

Proof. (i) $\Rightarrow$ (vii):

(vii) $\Rightarrow$ (i):

□

Proof. (i) $\Rightarrow$ (viii):

(viii) $\Rightarrow$ (i):

□

Definition d-degenerate If there is a vertex ordering $v_1, ..., v_n$ of G and a $d \in \mathbb{N}$ such that

$$|N(v_i) \cap \{v_{i+1}, ..., v_n\}| \leq d$$

for all $i \in [n-1]$ then G is called *d-degenerate*. The minimum d for which G is *d-degenerate* is called the *degeneracy* of G.

Every finite planar graph has a vertex of degree five or less, therefore every planar graph is 5-degenerate.

Definition arboricity The least number of trees that can cover the edges of a graph is its arborictiy.

It is a measure for the graphs maximum local density: it is small if and only if the graph is nowhere dense, in the sense that there is no subgraph H with large $\epsilon(H) = \frac{E(H)}{V(H)}$.

Definition Contract For an edge $e = xy$ in G we define $G \circ e$ as the graph obtained from G by identifying x and y and removing (if necessary) loops and multiple edges. We say that $G \circ e$ arises from G by *contracting the edge e.*

Definition Complement The *complement* of G, denoted by $\overline{G}$ is defined as the graph $(V, \binom{V}{2} \setminus E)$. In particular $G + \overline{G}$ is a complete graph and $\overline{G} = (G + \overline{G}) - E$.

Definitions

- *girth of* G, denoted by $g(G)$ is the length of the shortest cycle in G, if G is acyclic, its girth is said to be ∞
- *circumference of* G, is the length of the longest cycle if G is acyclic the circumference is said to be 0
- G is called *Hamiltonian* if G has a spanning cycle, i.e. a cycle that contains every vertex of G. In other words the circumference is $|V|$
- G is called *traceable* if G has a spanning path
- For two vertices v and u in G, the *distance between u and v* , denoted by $d(v, u)$ is the length of a shortest u-v-path in G. If no such path exists $d(u, v) = \infty$
- The *diameter* of G, denoted by $\operatorname{diam}(G)$, is the maximum distance among all pairs of vertices in G, i.e.

$$\operatorname{diam}(G) = \max_{u,v \in V} d(u, v)$$

- *eccentricity*, $\operatorname{ecc}(v)$ is the greatest distance of v to any other vertex.
- The *radius of G* , denoted by $\operatorname{rad}(G)$ is defined as

$$\operatorname{rad}(G) = \min_{u \in V} \max_{n \in V} d(u, v)$$

its the vertex that has the smallest eccentricity

problem sheets

problem 1 Determine the number of edges, average degree, diameter and girth of the *d-dimensional hypercube.*

Proof. For number of edges consider vertex degree, for average degree observe that Q_d is regular.

claim: Q_d has diameter d.
First prove that for any $x, y \in V$ the distance $d(x, y)$ in Q_d is the number of postions where they differ. Suppose they differ in $l \geq 1$ positions then we get a path of length l by inverting each entry where they differ sequentially. There can not be a shorter Path since each step can only change one position.

Therefore the diameter is d, because thats the maximum amount of positions where two sequences can differ, consider 00...0 and 11...1.

claim: girth of Q_d is ∞ if $d = 1$ and otherwise 4.

If $d = 1$, the resulting graph Q_1 is acyclic. If $d \geq 2$ we first observe that then Q_d is triangel-free. Suppose otherwise than there is a triangle xyz in Q_d, now consider the amount of 1's in $|v|, v \in V$. Wlog $|x|$ is even then $|y|$ must be odd and $|z|$ must be even. But zx is an edge, a contradiction.
On the other hand Q_d does contain 4-cycles:

$$(00...0, 010...0, 100...0, 00...0)$$

□

problem 2 Show that any tree T has at least $\Delta(T)$ leaves.

Proof. Induction on $n = |T|$.
Base: tree on two vertices has 2 leaves and maximum degree is 1.

Step: remove a leaf v from T and let u be its neighbour, so that $T' := T - v$ we than have two cases:

case 1: $\Delta(T) = \Delta(T')$
We know by I.H. that T' has at least $\Delta(T')$ leaves, thus if the maximum degree of T is the same the claim holds.

case 2: $\Delta(T) = \Delta(T') + 1$ This means u is the vertex with maximal degree in T, but we know that the leafs of T can only be the leafs of T' plus the leaf v we removed therefore T has $\Delta(T^prime) + 1$ leafs.

□

Proof. By Counting
Let T be any tree. Let $L \subseteq V$ be the set of leaves and $N = V \setminus L$ the set of non-leaves in T. Let u be a vertex with maximum degree $\Delta(T)$. We know that a tree has $|V| + 1$ edges.

$$\begin{aligned}
2 \cdot (|V| - 1) &= \sum_{v \in V(T)} d(v) && (1)\\
&= d(u) + \sum_{v \in L} d(v) + \sum_{v \in N \setminus \{u\}} d(v) && (2)\\
&\geq \Delta(T) + \sum_{v \in L} 1 + \sum_{v \in N \setminus \{u\}} 2 && (3)\\
&= \Delta(T) + |L| \cdot 1 + (|V| - |L| - 1) \cdot 2 && (4)\\
&= \Delta(T) + 2 \cdot (|V| - 1) - |L| && (5)
\end{aligned}$$

From this follows $|L| \geq \Delta(T)$, as desired. □

problem 3 Prove that either a graph or its complement is connected.

Proof. Let $G = (V, E)$ be any non-empty graph. We assume G is not connected and shall argue that $\overline{G}$ is connected.
Since G is disconnect, we find two vertices $u, w \in V$ and a connected component C of G such that $u \in C$ and $w \notin C$. Now in $\overline{G}$ all vertices in C are adjacent to w. And in particular $uw \in E(\overline{G})$, so all vertices lie in a single connected component of $\overline{G}$, which is therefore connected. □

problem 4 Prove that the vertex set of any graph can be partitioned into two sets such that for each vertex, at least half of its neighbors belong to the other set.

Proof. Consider partition of $V(G)$ into disjoint sets A, B, maximizing the number of edges between A and B. We will show that moving any vertex to the other sets results in a partition that has less edges between the sets than the original one.
Pick wlog a vertex $v \in A$. Let $d_B = N(v) \cap B$ and $d_A = d - d_B$. Now consider $A' = A - v$ and $B' = B + v$

$$|\{uw \in E : u \in A', w \in B'\}| = |\{uw \in E : u \in A, w \in B\}| - d_B + (d - d_B)$$

but by the maximality of edges between A and B we have:

$$|\{uw \in E : u \in A', w \in B'\}| \leq |\{uw \in E : u \in A, w \in B\}|$$

which gives us

$$-d_B + (d - d_B) \leq 0 \iff d_B \geq d/2$$

□

problem 5 Show that the following statements are equivalent

(i) G is connected, but $G - e$ is disconnected for every edge e

(ii) Any two vertices in G are linked by a unique path

Proof.
"(ii) ⇒ (i)": trivial
"(i) ⇒ (ii)":
As G is connected, there is at least one path between any two vertices of G. Lets assume for the sake of contradiction that there are some vertices x and y that are joind by at least two paths P_1, P_2. As $P_1 \neq P_2$ there is and edge e_0 that lies in P_1 but not in P_2. We shall show that the graph $G - e_0$ is still connected, which will be a contradiction.

Idea is to consider the path between any two vertices u, v and remove remove the edge e_0 and still show we can reach v from v by using the vertices x and y which have two paths between them and simply put e_0 in one of them, then we can user the other one by going from one endpoint from e_0 to x, possible because G is connected. Then to y using the other path and then from y to the other endpoint of e_0 and finally to v. Thus u and v are still connected after removing e_0 a contradiction.

□

problem 6 Let $T_1, ..., T_k$ be subtrees of a tree T, any two of which have at least one vertex in common. Prove that there is a vertex common to all T_i.

Proof. Apply induction on $|T|$. If $|T| = 1$ then T consists of a single vertex, say v. If $T_1, ..., T_k$ are subtrees of T with pairwise intersecting vertex sets, then we must have $T_i = T$ for each $i = 1, ..., k$. It follows that v belongs to each T_i.

So assume $n \geq$ is an integer, and let us suppose this result holds for all trees of order $n-1$. Suppose T is a tree with $|T| = n$ and $T_i, ..., T_k$ are subtrees of T. We assume $k \geq 2$. Let v be a leaf in T and let $T' = T - v$ be the tree resulting from removing this leaf. □

problem 7

Diestel

Lovasz

3 Matchings

Definitions

- *matching* is a 1-regular graph, i.e. a matching is a graph M so that $E(M)$ is a union of pairwise non-adjacent edges and $2|E(M)| = |V(M)|$
- a matching in G is a subgraph of G isomorphic to a matching. We denote the size of the largest matching in G by $v(G)$
- a *vertex cover* in G is a set of vertices $U \subseteq V$ such that each edge in E is incident to at least one vertex in U. We denote the size of the smallest vertex cover in G by $\tau(G)$
- a *k-factor of G* is a k-regular spanning subgraph of G.
- A *1-factor of G* is also called a *perfect matching* since it is a matching of largest possible size of order $|V|$. Clearly G can only contain a perfect matching if $|V|$ is even.

Theorem 2.2 (Hall's Marriage Theorem)
Let G be a bipartite graph with partite sets A and B. Then G has a matching containing all vertices of A if and only if $|N(S)| \geq |S|$ for all $S \subseteq A$

Proof.
$\Rightarrow$:
If G has a matching M containing all vertices of A, then for any $S \subseteq A, N(S)$ in G is at least as large as $N(S)$ in M, thus $|N(S)| \geq |S|$.

$\Leftarrow$:
We shall prove by induction on $|A|$ that any bipartite graph with parts A and B satisfying Hall's condition has a matching containing all vertices of A, in other words, saturating A.

When $|A| = 1$, there is at least one edge in G and thus a matching saturating A. Assume that the statement is true for all graphs G satisfying Hall's condition and with $|A| = k \geq 1$.

Now consider a bipartite graph G with $|A| = k+1$ and satisfying Hall's condition.

Case 1: $|N(S)| \geq |S| + 1$ for any $S \subsetneq A$.
Let $G' = G\{x, y\}$, for some edge xy. G' has parts $A' = A - \{x\}$ and $B' = B - \{y\}$. For any $S \subseteq A'$, $|N_{G'}(S')| \geq |N_G(S)| - 1 \geq |S| + 1 - 1 = |S|$. Thus G' satisfies Hall's condition and by induction has a matching M' saturating A'. Then $M = M' \cup \{xy\}$ is a matching in G saturating A.

Case 2: $|N(S_1)| = |S_1|$ for some $S_1 \subsetneq A$.
Let $A' = S_1, B' = N(A'), G' = G[A' \cup B']$. Since $|A'| < |A|$, and G' satisfies Hall's condition, G' has a matching M' saturating A' by induction. Now consider $A'' = A - A'$, $B'' = B - B'$, $G'' = G[A'' \cup B'']$. We claim G'' also satisfies Hall's condition. Assume not, and there is $S \subseteq A''$ such that $|N_{G''}(S)| < |S|$. Then $|N_G(S \cup A') = |B' \cup N_{G''}(S)| = |B'| + |N_{G''}(S)| < |A'| + |S| = |A' \cup S|$, a contradiction to Hall's condition. Thus there is a Matching M'' in G'' and $M' \cup M''$ is a matching saturating A in G. □

Corollary 12 Let G be a bipartite graph with partite sets A and B such that $|N(S)| \geq |S| - d$ holds for all $S \subseteq A$, and for a fixed positive integer d. Then G contains a matching of size at least $|A| - d$.

Proof. TODO □

Corollary 13 If G is a regular bipartite graph, it has a perfect matching.

Proof. Let $k \in \mathbb{N}$ and let G be a k-regulat bipartite graph with parts A and B. Then $|E(G)| = k|A| = k|B|$, and thus $|A| = |B|$. Consider $S \subseteq A$, let r be the number of edges between S and $N(S)$. On one hand, $r = |S|k$, on the other hand $r \leq |N(S)|k$. Thus $|N(S)| \geq |S|$ and by Hall's theorem there is a matching saturating A. Since $|A| = |B|$, it is a perfect matching. □

Corollary 14 A k-regular bipartite graph has a proper k-edge-coloring.

Proof. TODO □

Theorem 2.3 (Kőnig's Theorem)
Let G be bipartite. Then the size of a largset matching is the same as the size of a smallest vertex cover.

Proof. Let c be the vertex-cover number of G and m be the size of the largest matching of G. Since a vertex cover should contain at least one vertex from each matching edge, $c \geq m$.

Now, we shall prove that $c \leq m$. Let M be a largest matching in G, we need to show that $c \leq |M|$. Let A and B be the partite sets of G. An *alternating path* is a path that starts with a vertex in A not incident to an edge of M, and alternates between edges in M and edges in M. Note that an alternating path must end in a vertex saturated by M, otherwise one can find a larger matching. Let

$$U' = \{b : ab \in E(M) \text{ for some } a \in A \text{ and some alternating path ends in } b\}$$

$$U = U' \cup \{a : ab \in E(M), b \notin U'\}$$

We see that $|U| = m$. We shall show that U is a vertex cover, i.e. that every edge of G contains a vertex from U. Indeed, if $ab \in E(M)$, then either a or b is in U. If $ab \notin E(M)$, we consider the following cases:

Case 0: $a \in U$. We are done.

Case 1: a in not incident to M. Then ab is an alternating path. If b is also not incident to M then $M \cup \{ab\}$ is a larger matching, a contradiction. Thus b is incident to M and then $b \in U$.

Case 2: a is incident to M. Then $ab' \in E(M)$ for some b'. Since $a \notin U$, we have that $b' \in U$, thus there is an alternating path P ending in b'. If P contains b, then $b \in U$, otherwise $Pb'ab$ is an alternating path ending in b, so $b \in U$.

□

Theorem 2.4 (Tutte's Theorem)
A graph G has a perfect matching if and only if $q(G - S) \leq |S|$ for all $S \subseteq V$. We define $q(H)$ to be the number of odd components of H, i.e. the number of connected components of H consisting of an odd number of vertices.

4 Connectivity

Definitions

- for a natural number $k \geq 1$, a graph G is called *k-connected* if $|V(G)| \geq k+1$ and for any set U of $k-1$ vertices in G the graph $G - U$ is connected. In particular K_n is $(n-1)$-connected.

Lemma 3.2 For any connected, non-trivial graph G we have

$$\kappa(G) \leq \kappa'(G) \leq \delta(G)$$

Proof. □

Theorem 3.3 (Menger's Theorem) For any graph G and any two vertex sets $A, B \subseteq V(G)$, the smallest number of vertices separating A and B is equal to the largest number of disjoint A-B-paths.

Theorem 3.4 (Global Version of Menger's Theorem) A graph G is k-connected if and only if for any two vertices a, b in G there exist k independent a-b-paths.

Definition

- *H-path*: Given a graph H, we call a path P an $H-path$ if P is non-trivial and meets H exactly in its ends. Such a path is also called an *ear* of the graph $H \cup P$.

- An *ear-decomposition* of a graph G is a sequence $G_0 \subseteq G_1 \subseteq ... \subseteq G_k$ of graphs, such that
  - G_0 is a cycle
  - for each $i = 1, ..., k$ the graph G_i arises from G_{i-1} by adding a G_{i-1}-path P_i, i.e. P_i is an ear of G_i
  - $G_k = G$

Theorem 25 (Ear-decomposition) A graph G is 2-connected if and only if it has an ear decomposition starting from any cycle of G.

Lemma 26 If G is 3-connected with $G \neq K_4$, then there exists an edge e of G such that $G \circ e$ is also 3-connected.

Theorem 3.6 (Tutte) A graph G is 3-connected if and only if there exists a sequence of graphs $G_0, G_1, ..., G_k$, such that

- $G_0 = K_4$
- for each $i = 1, ..., k$ the graph G_i has two adjacent vertices x', x'' of degree at least 3, so that $G_{i-1} = G_i \circ x^{prime} x''$
- $G_k = G$

Theorem 27 (Mader) Every graph $G = (V, E)$ of average degree at least $4k$ has a k-connected subgraph.

Proof. □

5 Planar graphs

Theorem 32 (Plane triangulation) A graph of order at least 3 is maximally plane if and only if it is a plane triangulation.

Proof. □

Theorem 4.2 (Euler's Formula) Let G be a connected plane graph with n vertices, m edges and l faces. Then

$$n - m + l = 2$$

Intuition: Eulers Thm for counting faces, edges and vertices of a polyhedron, and then projecting them into the plane gives us a plane graph (see Weitz video)

Proof. □

Corollary 33 A plane graph with $n \geq 3$ vertices has at most $3n - 6$ edges. Every plane triangulation has exactly $3n - 6$ edges.

Proof. □

Lemma 37 A graph G contains K_5 or $K_{3,3}$ iff G contains K_5 or $K_{3,3}$, as topological minor.

Proof. □

Lemma 38 Let G be a 3-connected graph, $MK_5 \nsubseteq G$ and $MK_{3,3} \nsubseteq G$. Then G is planar.

Proof. □

Theorem 4.4 (Kuratowski's Theorem) The following statements are equivalent for graphs G:

(i) G is planar

(ii) G does not have K_5 or $K_{3,3}$ as minors

(iii) G does not have K_5 or $K_{3,3}$ as topological minors

Proof. TODO □

Theorem 41 (Schnyder) Let G be a graph and P be its incidence poset. Then G is planar if and only if $\dim(P) \leq 3$.

Theorem 4.7 (5-Color Theorem) Every planar graph is 5-colorabel

Proof. □

6 Ramseytheory

Lecture 27.1.20

Theorem 7.4 For any positive integer Δ there exists a $c \in \mathbb{N}$ such that

Choonbu Lee 2015 For every positive d $\exists c = c(d) \forall H$ H-d-degenerate $R(H) \leq c|V(H)|$

7 Random Graphs

Example: $n = 4$ and $p = \frac{1}{3}$
empty graph has prob = $\frac{2}{3}^6$

graph with 2 edges has prob $\frac{1}{3}^2 \cdot \frac{2}{3}^4$

Note let m be number of edges in G

$$\sum_{G graphon[n]} Prob(G) = \sum_{m=0}^{\binom{n}{2}} \binom{\binom{n}{2}}{m} \cdot p^m (1-p)^{\binom{n}{2}-m} = (p + (1-p)^{\binom{n}{2}}) = 1$$

Lemma 107 tells us that Erdos-Renyi model is universal, it gives us any graph.

The type of estimats of lemma 108 are used in many different results.

Lemma 109: count cycle of k vertices overcounting for k in also in different order therefore divide by $2k$ and times p^k because we have k edges in our cycle

purpose: for this lemma, to prove ramsey result, but the main purpose is the result of erdos-Hajnal (thm 9.3) that graph with arbitrarily high girth (no cycle of length smaller) and abitrarily high chromatic number.

Theorem 9.3 (Erdos-Hajnal) sketch: take random graph, number of short cycles (length at most k) is $< \frac{n}{2}$ delete a vertex from each of these cycles and get a graph G' so the girth of $G' > k$. We know chromatic number of G prime is $\geq \frac{|V(G')|}{\alpha(G')}$ with lemma 108 we can show α is not so big and therefore get a lower bound

two tricks:

- random graph and kill short cycle
- bound the independence number (co-clique number) size of the largest independent set

papers on probablistic method are not easy to read, because it involves choosing the parameters correctly and asymptotic analysis. But the results obtained by probablistic mehtod cannot be obtained by any other method.

Graph Properties and threshold functions *graphproperty* is a set of graphs. Example:

$$P = \{G : \text{ G is connected}\}$$

8 Hamiltionian cycles

Definition A cycle C in a graph G is Hamiltionian if it is spanning, if it contains all vertices.
A graph that has a Ham. cycle is called Hamiltionian graph.

Example Every complete graph is Hamiltionian.

History Was introduced by Sir WIlliam Rovar Hamilton in 1857. Introduced it via a game "Icosian game".

related is Traveling Salesperson problem

Properties for Ham. Cycles

- extremal number not fit (to big??)
- vertices of low degree
- connectivity

Lemma 10.1 Neccessary cond. Ham. cycle)

Proof. Let C be a Hamiltionian cycle of G. Let $S \subseteq V(G)$, $S \neq \emptyset$, $t := \#$ components of $G - S$ There are at least 2 edges of C between each component of $G - S$ and S. If $e = \#$ edges of C between S and $V - S$, we have

$$e \geq t \cdot 2 \text{ and } e \underset{Cis2\text{-regular}}{\leq} |S| \cdot 2$$

□

Ore's Thm. A graph G on $n \geq 3$ vertices is Hamiltionian $\forall u, v \in V(G), uv \notin E(G), d(u) + d(v) \geq n$

Komős-Sovkozy,Szemeredi (generalization of Dirac)
$\delta(G) \geq \frac{k}{k+1}n$, then G has a k^{th} power of a Hamiltionian cycle, that is a subgraph obtained from a Ham. cycle by joining all vts at distance $\leq k$ on the cycle by an edge.

Csaba, Kühn, Osthus, Lu, Treglown 2014 (usage of regularity lemma)
For sufficiently large n, each d-regular graph with $d \geq \lfloor \frac{n}{2} \rfloor$ has an edge-decomposition into Ham. cycles of at most one matching.

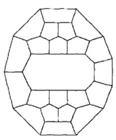
Theorem 114 Every graph on $n \geq 3$ vts with $\alpha(G) \leq \kappa(G)$ is Hamiltionian.

Proof. (from Diestel)
3 Let C be a longest cycle in G, $C = (v_0, v_1, ..., v_{m-1}, v_0)$. If C is not Hamiltionian, $\exists v \in V(G) \setminus V(C)$. Let F be a C-v-fan, i.e. $F = \{P_i : P_i isav_i{-}v{-}path, i \in I\}$ P_i's share only v pairwise. Moreover let F be of max. cadinality. By Menger's thm $|F| \leq min\{k, |C|\}$ We have $\forall i \in I, i + 1(\mod m) \notin I$. (otherwise C is not longest) $\forall i, j \in I, i \neq j : v_{i+1}v_{j+1} \notin E(G)$

□

Planarity and Hamiltonicity Example: G is planar, cubic $\kappa(G) = 3$ and not Hamiltionian.

Example with 42 vertices



9 Networkflows