Graphtheory Theorems

1 Important Graphs

Complete Graph, Clique:
the *complete graph* K_n on n vertices is isomorphic to $([n], \binom{[n]}{2})$

Cycle
C_n on n vertices with $n \geq 3$ is isomorphic to $([n], \{\{i, i+1\} : i = 1, ..., n-1\} \cup \{n, 1\})$, the *length of a cycle* is its number of edges.

Empty Graph
E_n on n vertices is isomorphic to $([n], \varnothing)$.
Empty graphs correspond to *independent sets*.

Complete Bipartite Graph
$K_{m,n}$ on $n+m$ vertices is isomorphic to $(A \cup B, \{xy : x \in A, y \in B\})$ where $|A| = m$ and $|B| = n$ and $A \cap B = \varnothing$.

Complete r-partite graph with $r \geq 2$ is isomorphic to

$$(A_1 \uplus ... \uplus A_r, \{xy : x \in A_i, y \in A_j, i \neq j\})$$

where $A_1, ..., A_r$ are disjoint non-empty finite sets.

n-dimensional hypercube Q_n is the graph isomorphic to

$$(2^{[n]}, \{\{S, T\} : S, T \in 2^{[n]}, |S \Delta T| = 1\})$$

Vertices are labeled either by corresponding sets or binar indicator vectors. For example the vertex $\{1, 3, 4\}$ in Q_6 is coded by $(1, 0, 1, 1, 0, 0)$. A 1 is indicating we take this Element and a 0 if not.

k-uniform hypergraph is a hypergraph where all hyperedges have the same cardinality

2 Basics

Definitions

- A graph G is *non-trivial* if it contains at least one edge, equivalently if G is not an empty graph
- The *order* of G writen $|G|$, is the number of vertices of G, i.e. $|G| = |V|$
- The *size* of G wirten $||G||$, is the number of edges of G, i. e. $||G|| = |E|$, if order of G is n then the size of G is between 0 and $\binom{n}{2}$
- $N(S)$ the *neighbourhood* of $S \subseteq V$ is the set of vertices in V. that have and adjacent vertex in S. Instead of $N(\{v\})$ for $v \in V$ we write $N(v)$

- vertex of degree 1 is called *leaf*
- vertex of degree 0 is called *isolated vertex*
- *minimum degree of* G, denoted by $\delta(G)$ is the smallest vertex degree in G
- *maximum degree of* G, denoted by $\Delta(G)$ is the highest vertex degree in G
- graph G is called *k-regular*, with $k \in \mathbb{N}$, if all vertices have degree k.
- *average degree of* G is defined as $d(G) = \frac{\sum_{v \in V} deg(v)}{|V|}$
  We have
  $$\delta(G) \leq d(G) \leq \Delta(G)$$
  with equality if and only if G is k-regular

Handshake Lemma For ever graph $G = (V, E)$ we have

$$2|E| = \sum_{v \in V} d(v)$$

Proof. By double counting the set $X = \{(e, x) : e \in E(G), x \in V(G), x \in e\}$ then

$$|X| = \sum_{v \in V(G)} d(x)$$

and

$$|X| = \sum_{e \in E(G)} 2 = 2|E(G)|$$

by the principle of double counting the terms are equal. □

Corollary From this follows that the sum of all vertex degrees is even and therefore the number of vertices with odd degree is even.

Proposition 3 If a graph G has minimum degree $\delta(G) \geq 2$, then G has a path of length $\delta(G)$ and a cycle with at least $\delta(G) + 1$ vertices.

Proof. Let $P = (x_0, ..., x_k)$ be a longest path in G. Then $N(x_0) \subseteq V(P)$, otherwise for $x \in N(x_0) \setminus V(P)$ the path $(x, x_0, x_1, ..., x_k)$ would be a longer path.

Let i be the largest index such that $x_i \in N(x_0)$, then $i \geq |N(x_0)| \geq \delta$. So $(x_0, x_1, ..., x_i, x_0)$ is a cycle of length at least $\delta(G) + 1$. □

Proposition 4 If for distinct vertices u and v a graph has a u-v-walk, then it has a u-v-path.

Proof. Consider a u-v-walk W with the smallest number of edges. Assume that W does not form a path, then there is a repeated vertex, w, i.e.

$$W = u, e, v_1, e_1, ..., e_k, w, e_{k+1}, ..., e_l, w, e_{l+1}, ..., v$$

Then $W_1 = u, e, v_1, ..., e_k, w, e_{k+1}, ..., v$ is a shorter u-v-walk, a contradiction. □

Propostion 5 If a graph has a closed walk of odd length, then it contains an odd cycle.

Proof. Let W be the shortest closed odd walk. If W is a cycle the Proposition holds. Otherwise there is a repeated vertex, so W is an edge-disjoint union of two closed walks. The sum of the lengths of these walks is odd, therefore one of them is an odd closed walk shorter than W a contradiction to the minimality of W. □

Proposition 6 If a graph has a closed walk with a non-repeated edge, then the graph contains a cycle.

Proof. Let W be a shortest walk with a non-repeated edge e. If W is a cycle, we are done. Otherwise, there is a repeated vertex and W is a union of two closed walks W_1 and W_2 that are shorter than W. One of them say W_1, contains e, a non-repeated edge. This contradicts the minimality of W. □

Definition bipartite A graph $G = (V, E)$ is called *bipartite* if there exists natural numbers m, n such that G is isomorphic to a subgraph of $K_{m,n}$. Then the vertex set can be written as $V = A \cup B$ such that $E \subseteq \{ab : a \in A, b \in B\}$. The sets A and B are called the *partite sets* of G

Proposition 1.5 A graph is bipartite if and only if it has no cycles of odd length.

Proof. skript
"$\Rightarrow$"
Assume that G is a bipartite graph with parts A and B. Then any cycle has a form $a_1, b_1, a_2, b_2, ..., a_k, b_k, a_1$ where $a_i \in A, b_i \in B, i \in [k]$. Thus every cycle has even length.
"$\Leftarrow$"
Assume G does not have cycles of odd length. We can assume that G is connected, otherwise we can treat the connected components separately. Let $v \in V(G)$. Let $A = \{u \in V(G) : dist(u, v) \equiv 0 \pmod 2\}$ and let $B = \{u \in V(G) : dist(u, v) \equiv 1 \pmod 2\}$ We claim that G is bipartite with parts A and B. To verify this it is sufficient to prove that A and B are independent sets. Let $u_1u_2 \in E(G)$ and let P_1 be a shortest u_1-v-path and P_2 a shortest u_2-v-path. Then the union of P_1, P_2 and u_1u_2 forms a closed walk W. If $u_1, u_2 \in A$ or $u_1, u_2 \in B$ then W is a closed odd walk, because $dist(v, u_1)$ and $dist(v, u_2)$ are both even or odd. Thus by Prop. 5 G contains an odd cycle, a contradiction. Thus for any edge u_1u_2 the adjacent vertices u_1 and u_2 are in different parts A or B. Therefore A and B are independent sets. □

Proof. Diestel
"$\Leftarrow$"
Let T be a spanning tree in G, pick a root $r \in T$ and denote the associated tree-order on V by $\leq_T$ (this order expressing height if $x < y$ then x lies *below* y in T). For each $v \in V(G)$ the unique path r-v-T has odd or even length. This defines a bipartition of $V(G)$, we show that G is bipartite with this partition. Let $e = xy$ be an edge of G. If $e \in T$ with $x <_T y$ say, then r-y-T = r-xy-T

and so x and y lie in different partition classes. If $e \notin T$ then $C_e := x\text{-}y\text{-}T + e$ is a cycle, and by the case treated already the vertices along $x\text{-}y\text{-}T$ alternate between the two classes. Since C_e is even by assumption, x and y again lie in different classes. □

Euler tour A closed walk that traverses every edge of the graph exactly once is called an *Euler tour*.

Theorem 1.6 (Eulerian Tour Condition) A connected graph has an Eulerian Tour if and only if every vertex has even degree.

Proof. "⇒"
The degree condition is necessary for an euler tour, because a vertex appearing k times in an Euler tour (or $k+1$ times if it is the starting and finishing vertex) must have degree $2k$.
"⇐"
Show by induction on $||G||$ that every connected Graph G with all degrees even has an Euler tour. $||G|| = 0$ is trivial.

Now let $||G|| \geq 1$, since all degrees are even, we can find in G a non-trivial closed walk that contains no edge more than once. To find this walk we consider W a walk of maximal length and write F for the set of its edges. If $F = E(G)$, then W is an Euler tour.

Suppose, therefore $G' := G - F$ has an edge.
For every vertex $v \in G$, an even number of edges of G at v lies in F, so the degrees of G' are again all even. Since G is connected, G' has an edge e incident with a vertex on W. By I.H. the component C of G' containing e has an Euler tour. Concatenating this with W (suitably re-indexed), we obtain a closed walk in G that contradicts the maximal length of W.

□

Definitions

- graph G is *connected* if any two vertices are linked by a path.
- a maximal connected subgraph of G is called a *connected component* of G.
- acyclic graphs are called *forests*
- a graph G is called a *tree* if G is connected and acyclic.

Lemma 7 Every tree on at least two vertices has a leaf.

Proof. If a tree T on at least two vertices does not have leaves then every vertex has degree $>$ than 2, so we have a cycle in T with length ≥ 3, a contradiction. □

Lemma 8 A tree of order $n \geq 1$ has exactly $n - 1$ edges.

Proof. We prove the statement by induction on n. When $n = 1$, there are no edges.

I.H.: Assume that each tree on $n = k$ vertices has $k - 1$ edges, with $k \geq 1$.

Step: Lets prove that each tree on $k + 1$ vertices has k edges. Consider a tree T on $k + 1$ vertices. Since $k + 1 \geq 2$, T has a leaf v. Let $T' = T - \{v\}$. We see that T' is connected because any u-w-path in T, for $u \neq v$ and $w \neq v$, does not contain v. We see also that T' is acyclic, because deleting vertices from an acyclic graph does not create new cycles. Thus T' is a tree on k vertices. By I.H. $|E(T')| = k - 1$. Thus $|E(T)| = |E(T')| + 1 = (k - 1) + 1 = k$.

□

Lemma 9 Every connected graph contains a spanning tree.

Proof. Let G be a connected graph. Consider T, an acyclic spanning subgraph of G with largest number of edges. If it is a tree we are done.

Otherwise, T has more than one component. Consider vertices u and v from different components of G. Consider a shortest u-v-path P in G. Then P has an edge $e = xy$ with exactly one vertex x in one of the components of T. Then P has an edge $e = xy$ with exactly one vertex x in one of the components of T. Then $T \cup \{e\}$ is acyclic. If there would be a cycle, it would contain e, however e connects to components, therefore cannot be part of a cylce (e would be a repeated edge). Thus $T \cup \{e\}$ is a bigger spanning acyclic subgraph of G contradicting the maximality of T. □

Lemma 10 A connected graph on $n \geq 1$ vertices and $n - 1$ edges is tree.

Proof. Let G be a connected graph on n vertices with $n - 1$ edges. Assume G is not a tree, i.e. contains a cycle. We therefore can remove a edge so that G is still connected. This is a contradiction because a graph on n vertices with $n - 2$ edges cannot be connected. Because a walk from vertex 1 to vertex n has to have at least $n - 1$ edges. □

Lemma 11 The vertices of every connected graph on $n \geq 2$ vertices can be ordered $(v_1, ..., v_n)$ so that for every $i \in \{1, ..., n\}$ the Graph $G[\{v_1, ..., v_i\}]$ is connected.

Proof. skript
Let G be a connected graph on n vertices. It contains a spanning tree T. Let v_n be a leaf of T, let v_{n-1} be a leaf of $T - \{v_n\}$, let v_{n-2} be a leaf of $T - \{v_n, v_{n-1}\}$ and so on. Let v_k be a leaf in $T - \{v_n, v_{n-1}, ..., v_{k+1}\}$, $k = 2, ..., n$. Since deleting a leaf does not disconnect a tree, all resulting graphs form a spanning trees of $G[v_1, ..., v_i]$, $i = 1, ..., n$. A graph H having a spanning tree or any connected spanning subgraph H' is connected because a u-v-path in H' is a u-v-path in H. This observation completes the proof. □

Proof. diestel
Pick any vertex as v_1, and assume inductively that $v_1, ..., v_i$ have been chosen for some $i < |G|$. Now pick a vertex $v \in G - G_i$. As G is connected, it contains

a v-v_1 path P. Choose v_{i+1} as the last vertex of P in $G - G_i$, then v_{i+1} has a neighbour in G_i. If we consider $i+1$ then we simply add v_{i+1} to our G_i, Thus $G_{i+1} := G_i \cup \{v_{i+1}\}$ which is also connected. $\square$

Tree equivalences For any graph $G = (V, E)$ the following are equivalent:

1. G is a tree, i.e. G is connected and acyclic.
2. G is connected, but for any $e \in E$ the graph $G - e$ is not connected (minimally connected)
3. G is acyclic, but for any $x, y \in V(G), xy \notin E(G)$ the graph $G + xy$ has a cycle. (maximaly acyclic)
4. G is connected and 1-degenerate
5. G is connected and $|E| = |V| - 1$
6. G is acyclic and $|E| = |V| - 1$
7. G is connected and every non-trivial subgraph of G has a vertex of degree at most 1.
8. Any two vertices are joined by a unique path in G.

Proof.
(i) $\Rightarrow$ (ii):
G is connected and acyclic, now assume for any edge $e = xy$ the graph $G' = G - e$ would still be connected. Then G' has a x-y-path P. But $P \cup e$ is a cycle in G which contradicts that G is acyclic.
(ii) $\Rightarrow$ (i):
G is connected and for any edge e the graph $G - e$ is not connected. We want to show that G is acyclic. If G would have a cycle we could simply remove an edge from that and the resulting graph would still be connected, a contradiction. $\square$

Proof.
TODO complete all equivalences for exam
(i) $\Rightarrow$ (iv):

(vi) $\Rightarrow$ (i):

(i) $\Rightarrow$ (vii):

(vii) $\Rightarrow$ (i):

(i) $\Rightarrow$ (viii):

(viii) $\Rightarrow$ (i):

$\square$

Definition d-degenerate If there is a vertex ordering $v_1, ..., v_n$ of G and a $d \in \mathbb{N}$ such that

$$|N(v_i) \cap \{v_{i+1}, ..., v_n\}| \leq d$$

for all $i \in [n-1]$ then G is called *d-degenerate*. The minimum d for which G is *d-degenerate* is called the *degeneracy* of G.

Every finite planar graph has a vertex of degree five or less, therefore every planar graph is 5-degenerate.

Definition arboricity The least number of trees that can cover the edges of a graph is its arborictiy.

It is a measure for the graphs maximum local density: it is small if and only if the graph is nowhere dense, in the sense that there is no subgraph H with large $\epsilon(H) = \frac{E(H)}{V(H)}$.

Definition Contract For an edge $e = xy$ in G we define $G \circ e$ as the graph obtained from G by identifying x and y and removing (if necessary) loops and multiple edges. We say that $G \circ e$ arises from G by *contracting the edge e*.

Definition Complement The *complement* of G, denoted by $\overline{G}$ is defined as the graph $(V, \binom{V}{2} \setminus E)$. In particular $G + \overline{G}$ is a complete graph and $\overline{G} = (G + \overline{G}) - E$.

Definitions

- *girth of* G, denoted by $g(G)$ is the length of the shortest cycle in G, if G is acyclic, its girth is said to be ∞
- *circumference of* G, is the length of the longest cycle if G is acyclic the circumference is said to be 0
- G is called *Hamiltonian* if G has a spanning cycle, i.e. a cycle that contains every vertex of G. In other words the circumference is $|V|$
- G is called *traceable* if G has a spanning path
- For two vertices v and u in G, the *distance between u and v* , denoted by $d(v, u)$ is the length of a shortest u-v-path in G. If no such path exists $d(u, v) = \infty$
- The *diameter* of G, denoted by $\text{diam}(G)$, is the maximum distance among all pairs of vertices in G, i.e.

  $$\text{diam}(G) = \max_{u,v \in V} d(u, v)$$

- *eccentricity*, $\text{ecc}(v)$ is the greatest distance of v to any other vertex.
- The *radius of G* , denoted by $\text{rad}(G)$ is defined as

  $$\text{rad}(G) = \min_{u \in V} \max_{n \in V} d(u, v)$$

  its the vertex that has the smallest eccentricity

3 Matchings

Definitions

- *matching* is a 1-regular graph, i.e. a matching is a graph M so that $E(M)$ is a union of pairwise non-adjacent edges and $2|E(M)| = |V(M)|$
- a matching in G is a subgraph of G isomorphic to a matching. We denote the size of the largest matching in G by $\upsilon(G)$
- a *vertex cover* in G is a set of vertices $U \subseteq V$ such that each edge in E is incident to at least one vertex in U. We denote the size of the smallest vertex cover in G by $\tau(G)$
- a *k-factor of G* is a k-regular spanning subgraph of G.
- A *1-factor of G* is also called a *perfect matching* since it is a matching of largest possible size of order $|V|$. Clearly G can only contain a perfect matching if $|V|$ is even.

Theorem 2.2 (Hall's Marriage Theorem)
Let G be a bipartite graph with partite sets A and B. Then G has a matching containing all vertices of A if and only if $|N(S)| \geq |S|$ for all $S \subseteq A$

Proof.
$\Rightarrow$:
If G has a matching M containing all vertices of A, then for any $S \subseteq A, N(S)$ in G is at least as large as $N(S)$ in M, thus $|N(S)| \geq |S|$.

$\Leftarrow$:
We shall prove by induction on $|A|$ that any bipartite graph with parts A and B satisfying Hall's condition has a matching containing all vertices of A, in other words, saturating A.

When $|A| = 1$, there is at least one edge in G and thus a matching saturating A. Assume that the statement is true for all graphs G satisfying Hall's condition and with $|A| = k \geq 1$.

Now consider a bipartite graph G with $|A| = k+1$ and satisfying Hall's condition.

Case 1: $|N(S)| \geq |S| + 1$ for any $S \subsetneq A$.
Let $G' = G\{x, y\}$, for some edge xy. G' has parts $A' = A - \{x\}$ and $B' = B - \{y\}$. For any $S \subseteq A'$, $|N_{G'}(S')| \geq |N_G(S)| - 1 \geq |S| + 1 - 1 = |S|$. Thus G' satisfies Hall's condition and by induction has a matching M' saturating A'. Then $M = M' \cup \{xy\}$ is a matching in G saturating A.

Case 2: $|N(S_1)| = |S_1|$ for some $S_1 \subsetneq A$.
Let $A' = S_1, B' = N(A'), G' = G[A' \cup B']$. Since $|A'| < |A|$, and G' satisfies Hall's condition, G' has a matching M' saturating A' by induction. Now consider $A'' = A - A'$, $B'' = B - B'$, $G'' = G[A'' \cup B'']$. We claim G'' also satisfies Hall's condition. Assume not, and there is $S \subseteq A''$ such that $|N_{G''}(S)| < |S|$. Then $|N_G(S \cup A') = |B' \cup N_{G''}(S)| = |B'| + |N_{G''}(S)| < |A'| + |S| = |A' \cup S|$, a contradiction to Hall's condition. Thus there is a Matching M'' in G'' and $M' \cup M''$ is a matching saturating A in G. $\square$

Corollary 12 Let G be a bipartite graph with partite sets A and B such that $|N(S)| \geq |S| - d$ holds for all $S \subseteq A$, and for a fixed positive integer d. Then G contains a matching of size at least $|A| - d$.

Proof. TODO □

Corollary 13 If G is a regular bipartite graph, it has a perfect matching.

Proof. Let $k \in \mathbb{N}$ and let G be a k-regulat bipartite graph with parts A and B. Then $|E(G)| = k|A| = k|B|$, and thus $|A| = |B|$. Consider $S \subseteq A$, let r be the number of edges between S and $N(S)$. On one hand, $r = |S|k$, on the other hand $r \leq |N(S)|k$. Thus $|N(S)| \geq |S|$ and by Hall's theorem there is a matching saturating A. Since $|A| = |B|$, it is a perfect matching. □

Corollary 14 A k-regular bipartite graph has a proper k-edge-coloring.

Proof. TODO □

Theorem 2.3 (Kőnig's Theorem)
Let G be bipartite. Then the size of a largset matching is the same as the size of a smallest vertex cover.

Proof. Let c be the vertex-cover number of G and m be the size of the largest matching of G. Since a vertex cover should contain at least one vertex from each matching edge, $c \geq m$.

Now, we shall prove that $c \leq m$. Let M be a largest matching in G, we need to show that $c \leq |M|$. Let A and B be the partite sets of G. An *alternating path* is a path that starts with a vertex in A not incident to an edge of M, and alternates between edges in M and edges in M. Note that an alternating path must end in a vertex saturated by M, otherwise one can find a larger matching. Let

$$U' = \{b : ab \in E(M) \text{ for some } a \in A \text{ and some alternating path ends in } b\}$$

$$U = U' \cup \{a : ab \in E(M), b \notin U'\}$$

We see that $|U| = m$. We shall show that U is a vertex cover, i.e. that every edge of G contains a vertex from U. Indeed, if $ab \in E(M)$, then either a or b is in U. If $ab \notin E(M)$, we consider the following cases:

Case 0: $a \in U$. We are done.

Case 1: a in not incident to M. Then ab is an alternating path. If b is also not incident to M then $M \cup \{ab\}$ is a larger matching, a contradiction. Thus b is incident to M and then $b \in U$.

Case 2: a is incident to M. Then $ab' \in E(M)$ for some b'. Since $a \notin U$, we have that $b' \in U$, thus there is an alternating path P ending in b'. If P contains b, then $b \in U$, otherwise $Pb'ab$ is an alternating path ending in b, so $b \in U$.

□

Theorem 2.4 (Tutte's Theorem)
A graph G has a perfect matching if and only if $q(G-S) \le |S|$ for all $S \subseteq V$. We define $q(H)$ to be the number of odd components of H, i.e. the number of connected components of H consisting of an odd number of vertices.

Proof.
$\Rightarrow$:
Assume first that G has a perfect matching M. Consider a set S of vertices and an odd component G' of $G-S$. We see that there is at least one vertex in G' that is incident to an edge of M that has another endpoint not in G'. This endpoint must be in S. Thus $|S|$ is at least as large as the number of odd components.

$\Leftarrow$:
Now, assume that $q(G-S) \le |S|$ for all $S \subseteq V$. Assume that G has no perfect matching and $|V(G)| = n$. Note that $|V(G)|$ is even (it follows from the assumption $q(G-S) \le |S|$) applied to $S = \varnothing$). Let G' be constructed from G by adding missing edges as long as no perfect matching appears. Let S be a set of vertices of degree $n-1$. Note that it could be empty.

Claim 1: Each component of $G' - S$ is complete. Assume not, there is a component □

4 Connectivity

Definitions

- for a natural number $k \ge 1$, a graph G is called *k-connected* if $|V(G)| \ge k+1$ and for any set U of $k-1$ vertices in G the graph $G-U$ is connected. In particular K_n is $(n-1)$-connected.

  this implies: a graph is k-connected if any two of its vertices can be joined by k independent paths

- a graph G is called *k-linked* if $|G| \ge 2k$ and for any $2k$ distinct vertices $s_1, ..., s_k, t_1, ..., t_k$ there are vertex disjoint s_i-t_i-paths, $i = 1, ..., k$

Lemma 3.2 For any connected, non-trivial graph G we have

$$\kappa(G) \le \kappa'(G) \le \delta(G)$$

Proof. If G is complete $\kappa(G) = \kappa'(G) = \delta(G) = n-1$

Assume G is not complete:
$\kappa'(G) \le \delta(G)$:
Simply remove all the edges of a vertex with minimum degree.

$\kappa(G) \le \kappa'(G)$:
Consider smallest separating set of edges F:

case1:
there is a vertex v not incident to F, then the vertices incident to F separate this vertex from any vertex in the other component.

case2:
Every vertex is incident to F, consider v of degree $< |G| - 1$, exists because G is not complete. Show the neighbourhood of v is less than $|F|$ so $N(v)$ is a separating set. □

Definition A subset X of vertices and edges of G seperates two vertex sets A, B if each A-B-path (starts in A ends in B) contains an element of X.

Theorem 3.3 (Menger's Theorem) For any graph G and any two vertex sets $A, B \subseteq V(G)$, the smallest number of vertices separating A and B is equal to the largest number of disjoint A-B-paths.

Theorem 3.4 (Global Version of Menger's Theorem) A graph G is k-connected if and only if for any two vertices a, b in G there exist k independent a-b-paths.

Note that Menger's Theorem implies that if G is k-linked, then G is k-connected.

Definition: Line Graph For a graph $G = (V, E)$ the line graph $L(G)$ is the graph $L(G) = (E, E')$ where

$$E' = \{\{e_1, e_2\} \in \binom{E}{2} : e_1 \text{ adjacent to } e_2 \text{ in } G\}$$

Corollary 24 If a, b are vertices of G, then

$$\min \#\text{edges separating } a \text{ and } b = \max \#\text{edge-disjoint } a\text{-}b\text{-paths}$$

Definition

- *H-path*: Given a graph H, we call a path P an $H-path$ if P is non-trivial and meets H exactly in its ends. Such a path is also called an *ear* of the graph $H \cup P$.
- An *ear-decomposition* of a graph G is a sequence $G_0 \subseteq G_1 \subseteq ... \subseteq G_k$ of graphs, such that
  - G_0 is a cycle
  - for each $i = 1, ..., k$ the graph G_i arises from G_{i-1} by adding a G_{i-1}-path P_i, i.e. P_i is an ear of G_i
  - $G_k = G$

Theorem 25 (Ear-decomposition) A graph G is 2-connected if and only if it has an ear decomposition starting from any cycle of G.

Proof. ⇐:
Assume first that G has a ear-decomposition starting from a cycle C, i.e. $C = G_0 \subseteq G_1 \subseteq ... \subseteq G_k = G$ Use induction on i to show that G_i is 2-connected. G_0 is a cylce and therefore 2-connected. G_{i+1} is obtained from G_i by adding an ear. We now by IH that G_i is 2-connected, therefore a cut vertex has to be on the

ear. The ear is contained in a cycle therefore deleting a vertex from it does not disconnect G_{i+1} therefore G_{i+1} is 2-connected.

$\Rightarrow$:
Assume G is 2-connected and C is a cycle in G. Let H be the largest subgraph of G obtained by ear decomposition starting with C. H has to be an induced subgraph of G otherwise an edge incident to two vertice of H would be and ear that could be added to H.

Assume $H \neq G$. Since G is connected there is an edge from $v \in H$ to $u \in G \setminus H$. Since $G - u$ is connected consider a v-w-path P in $G - u$ for some vertex $w \in (V(H) - u)$. Let w' be the first vertex from $V(H) - u$ on this path. Then $Pw' \cup uv$ is an ear of H, a contradiction to minimality of H. $\square$

Lemma 26 If G is 3-connected with $G \neq K_4$, then there exists an edge e of G such that $G \circ e$ is also 3-connected.

Theorem 3.6 (Tutte) A graph G is 3-connected if and only if there exists a sequence of graphs $G_0, G_1, ..., G_k$, such that

- $G_0 = K_4$
- for each $i = 1, ..., k$ the graph G_i has two adjacent vertices x', x'' of degree at least 3, so that $G_{i-1} = G_i \circ x'x''$
- $G_k = G$

Proof.
$\Rightarrow$:
If G is 3-connected, such a sequence exists by Lemma 26. To see the degree condition is satisfied, recall that $\delta(H) \geq 3$ for any 3-connected graph H. Note that with each contraction, the number of vertices decrease by 1 and until we have at least 5 vertices, we can apply Lemma 26 and contract one more edge. Thus we stop at a graph G_0 which has 4 vertices and $\delta(G_0) \geq 3$ from which $G_0 \cong K_4$ follows.

$\Leftarrow$
We consider a sequence of graphs satisfying the given conditions and show that each graph in the sequence is 3-connected. Assume that G_i is 3-connected, G_{i+1} is not, and $G_i = G_{i+1} \circ xy$, for an edge xy of G_{i+1} such that $d(x), d(y) \geq 3$. Then G_{i+1} has a cut-set S with at most two vertices.

Case 1: $x, y \in S$.
Then G_i has a cut vertex, a contradiction.

Case 2: $x \in S, y \notin S, y$ is not the only vertex of its component in $G_{i+1} - S$.
Then G_i has a cut set of size at most 2, a contradiction.

Case 3: $x \in S, y \notin S, y$ is the only vertex of its component in $G_{i+1} - S$.
Then $d(y) \leq 2$, a contradiction to the fact that $d(y) \geq 3$

Case 4: $x, y \notin S$
Then x and y are in the same component of $G_{i+1} - S$. So S is a cut set of G_i, a contradiction. $\square$

Note that this theorem gives a way to generate all 3-connected graphs by starting with K_4 and creating a sequence of graphs by "uncontracting" a vertex such that the degrees of new vertices are at least 3 each.

Theorem 27 (Mader) Every graph $G = (V, E)$ of average degree at least $4k$ has a k-connected subgraph.

Definition

- let G be a graph, a maximal connected subgraph of G without a cut vertex is called a *block* of G. In particular, the blocks of G are exactly the bridges and the maximal 2-connected subgraphs of G.
- *block-cut-vertex graph or block graph* of G is a bipartite graph H whose partite sets are the *blocks* of G and the cute vertices of G respectively. There is an edge between a block B and a cut vertex a if and only if $a \in B$, i.e. the block contains the cut vertex. The leaves of this graph are called $leaf\ blocks$.

Theorem 28 The block-cut-vertex graph of a connected graph is a tree.

5 Planar graphs

planar graph vs. plane graph Plane Graph is topological object $(V, E), V \subseteq \mathbb{R}^2$, $e \in E$ are arcs in $\mathbb{R}^2$. Planar graph is combinatorial object $(V, E), E \subseteq \binom{n}{2}$ s.t. it has a plane graph realization (planar embedding)

Definitions

- For any plane graph G, the set $\mathbb{R}^2 \setminus G$ is open, its regions are the *faces* of G.
- the face of G corresponding to the unbounded region is the *outer face* of G, the other faces are its *inner faces*. The set of all Faces is denoted by $F(G)$
- The *frontier* of a set $X \subseteq \mathbb{R}^2$ is the set Y of all points $y \in \mathbb{R}^2$ such that every neighbourhood of y meets both X and $\mathbb{R}^2 \setminus X$. Note that if X is closed, its frontier lies in X, while if X is open, its frontier lies in $\mathbb{R}^2 \setminus X$.
- The subgraph of G whose point set is the frontier of a face f is said to *bound* f and is called its *boundary*, we denote it by $G[f]$
- Let G be a plane graph. If one cannot add an edge to form plane $G' \supset G$ with $V(G') = V(G)$, then G is called *maximally plane*.
- If every face in $F(G)$ (including the outer face) is bounded by a triangle in G, then G is called a *plane triangulation*
- A *planar embedding* of an abstract graph $G = (V, E)$ is a bijective mapping $f : V \to V'$, where $G' = (V', E')$ is a plane graph and $uv \in E(G)$, then there is an edge in E' with endpoints $f(u)$ and $f(v)$. We say that G' is a *drawing* of G. A graph $G = (V, E)$ is *planar* if it has a planar embedding.

- A graph G is *outerplanar* if it has a plane embedding such that the boundary of the outer face contains all vertices V.

Theorem 32 (Plane triangulation) A graph of order at least 3 is maximally plane if and only if it is a plane triangulation.

Proof. If G is a plane triangulation, then each face is bounded by a triangle. If an edge is added to G so that the resulting graph is plane, the interior of the edge must be in some face f of G. The endpoints of the added edge must be two of the three vertices on the frontier of f. However, these vertices already are endpoints of an edge of G, a contradiction. Thus no edge could be added to G so that the graph remains plane.

Now assume that G is maximally plane. Let f be a face and $H = G[f]$. Then we see that H is a complete graph, otherwise we could add a new edge with interior in f. If H has at least 4 vertices $v_1, v_2, v_3, v_4, ...$ then we see that v_i-v_j-paths, $i, j \in [4]$ can not all be pairwise disjoint. If H has at most 2 vertices, then f is a face having at most one edge on its boundary, thus $f = \mathbb{R}^2 - G$ and one can add another edge to G. Therefore, we see that H is a complete graph on 3 vertices. □

Theorem 4.2 (Euler's Formula) Let G be a connected plane graph with n vertices, m edges and l faces. Then

$$n - m + l = 2$$

Proof. Apply induction on m. A connected graph has at least $n - 1$ edge. If $m = n - 1$, G is a tree. Then $l = 1$ and $n - (n - 1) + 1 = 2$

Let $m \geq n$ and assume the claim holds for smaller values of m. Then there is an edge e on a cycle. Then e is on the boundary of exactly two faces f_1 and f_2, these make a become one face in $G' := G - e$, by induction the claim holds. □

Corollary 33 A plane graph with $n \geq 3$ vertices has at most $3n - 6$ edges. Every plane triangulation has exactly $3n - 6$ edges.

Proof. Count the number of edges for every face and plug into Euler's formula. □

Corollary 34 A triangle-free plane graph with $n \geq 3$ vertices has at most $2n - 4$ edges

Proof. Double count the set of pairs of edges with faces and use eulers formula. □

Theorem 4.4 (Kuratowski's Theorem) The following statements are equivalent for graphs G:

1. G is planar
2. G does not have K_5 or $K_{3,3}$ as minors
3. G does not have K_5 or $K_{3,3}$ as topological minors

Definitions: Poset

- Let X be a set and $\leq \subseteq X^2$ be a relation on X. Then $\leq$ is a *partial order* if it is reflexive, antisymmetric and transitive. A partial order is *total* if $x \leq y$ or $y \leq x$ for every $x, y \in X$
- The *poset dimension* of $(X, \leq)$ is the smallest number d such that there are total orders $R_1, ..., R_d$ on X with $\leq = R_1 \cap ... \cap R_d$.
- The *incidence poset* $(V \cup E, \leq)$ on a graph $G = (V, E)$ is given by $v \leq e$ iff e is incident to v for all $v \in V$ and $e \in E$

Theorem (Schnyder) Let G be a graph and P its incidence poset. Then G is planar iff $\dim(P) \leq 3$.

Theorem 4.7 (5-Color Theorem) Every planar graph is 5-colorable

Proof. We shall apply induction on $|V(G)|$ with a trivial basis when $|V(G)| \leq 5$. Assume $|V(G)| > 5$, assume further that G is maximally planar (plane triangulation). By Euler's Formula there is a vertex v of degree at most 5. By induction, there is a proper coloring c of $G - v$ in at most 5 colors. If c assigns at most 4 colors to $N(v)$, we can assign v a color such that we use only 5. Otherwise, assume $v_1, .., v_5$, $c(v_i) = i$ and v_i's are cyclically arranged on the face of $G - v$. We make a color switch from 1 to 3 at vertex v_1, if this is valid and now both v_1 and v_3 have the same color 3, then v can get color 1. Otherwise there has to be a v_1-v_3-path colored in 1 and 3. Similary there is a v_2-v_4-path colored 2 and 4, but these paths have to intersect at a single vertex, so that graph is still planar. But this vertex cannot lie on both path since the sets of available colors are disjoint. □

Wrong proof of Four Color Theorem by Kempe Uses contradiction from 2 colored path which intersect, but there is an example which shows that we have two vertices of the same color.

Definition List coloring

- Let $L(v) \subseteq \mathbb{N}$ be a list of colors for each vertex $v \in V$. We say that G is *L-list-colorable* if there is a coloring $c : V \to \mathbb{N}$ such that $c(v) \in L(v)$ for each $v \in V$ and adjacent vertices receive different colors.
- Let $k \in \mathbb{N}$. We say that G is *k-list-colorable* or *k-choosable* if G is L-list-colorable for each list L with $|L(v)| = k$ for all $v \in V$
- the *choosability*, denoted by $\text{ch}(G)$ is the smallest k such that G is k-choosable.
- the *edge choosability*, denoted by $\text{ch}'(G)$ is the smallest k such that G is L-edge-list-colorable for each list L with $|L(e)| = k$ for $e \in E$

Theorem 4.10 (5-List-Color Theorem) Let G be a planar graph. Then the list chromatic number of G is at most 5.

Proof. We prove a stronger statement ($\star$):
Let G be an outer triangulation (triangular inner faces and an outer face forming a cycle) Suppose two adjacent vertices x, y on the boundary of the outer face have already been colored. For all other vertices on the cycle have list of length 3 and all other vertices on bounded faces have list of length 5. Then the coloring of x, y can be extended to a coloring of G from the given lists.

We prove ($\star$) by induction on $|V(G)|$, with trivial basis $|V(G)| = 3$ Consider an outer triangulation on more than 3 vertices.
Case 1: There is a chord
Then $G = G_1 \cup G_2$, such that $\{u, v\} = V(G_1) \cap V(G_2)$, $|G| > |G_i| \geq 3$, G_i is an outer triangulation, $i = 1, 2$. Without loss of generality x, y are on the outer face of G_1. Apply induction to G_1 to obtain a proper L-coloring c' of G_1. Next apply induction to G_2 with u and v playing the role of x, y and list assignments L' such that they have the same color as in c'. Then there is a proper L'-coloring c'' of G_2. Since these colorings have the same colors for u and v they form together a proper coloring of G.
Case 2: There is no chord
Let z be a neighbor of x on the boundary of the outer face, $z \neq y$. Let Z be the set of neighbors of z not on the outer face. Let $L(x) = \{a\}$, $L(y) = \{b\}$. Let $c, d \in L(z)$ such that $c \neq a$ and $d \neq a$. Remove z from G, $G' := G - z$. Let L' be the list assignment for $V(G')$ such that $L(v) = \{c, d\}$ for $v \in Z$ and $L'(v) = L(v)$ for $v \notin Z$. Remove the colors c, d from the vertices in the neighbourhood of z. and leave the colors for all other vertices the same. By induction G' has a proper L'-coloring c'. We shall extend a coloring c' to a coloring c of G. Simply by giving z the color c, d depending on the color of his neighbor on the outer face.

□

6 Colorings

Note that a k-coloring is nothing but a vertex partition into k independent sets, now called *colorclasses*; the non-trivial 2-colourable graphs, are precisely the bipartite graphs. The chromatic number is a key parameter in many extremal question, therefore it is studied a lot.

Definitions

- *vertex coloring* of a Graph G is a map $c : V \to S$ such that $c(v) \neq c(w)$ whenever v and w are adjacent. The elements of the set S are called available colors. The minimal $k = |S|$ such that G has a k-coloring is the *chromatic number* of G, denoted by $\chi(G)$. A graph with $\chi(G) = k$ is called k-chromatic.
- edge coloring is a map $c : E \to S$ with $c(e) \neq c(f)$ for any adjacent edges e, f. We say the *edge-chromatic number*, or *chromatic index* of G is the minimal k for which a k-edge-coloring exists, it is denoted by .

Relationship of $\chi(G)$ and $\chi'(G)$ Every edge coloring of G is a vertex coloring of its line graph $L(G)$, and vice versa, in particular $\chi'(G) = \chi(L(G))$. The problem of finding good edge colorings may thus be viewed as a restriction of the more general vertex coloring problem to this special class of graphs. There are only very rough estimates for χ but χ' always takes one ot two values, either Δ or $\Delta + 1$

Definitions

- *clique number* $\omega(G)$ of G is the largest order of a clique in G
- *co-clique number* $\alpha(G)$ of G is the largest order of an independent set in G
- Graph is *perfect* if $\chi(H) = \omega(H)$ for each induced subgraph H of G. For example bipartite graphs are perfect with $\chi = \omega = 2$

Lemma 43 For any connected graph G and for any vertex v there is an ordering of the vertices of $G : v_1, ..., v_n$ such that $v = v_n$ and for each $i, 1 \leq i < n$, v_i has a higher indexed neighbor.

Proof. Consider a spanning tree T of G and create a sequence of sets $X_1, ..., X_{n-1}$ with $X_1 = V(G_1), X_i = X_{i-1} - \{v_{i-1}\}$, where v_i is a leaf of $T[X_i]$ not equal to v, for $i = 1, ..., n-1$. Then $v_1, ..., v_n$ is a desired ordering with $v_n = v$ □

Lemma 45 Let G be a 2-connected non-complete graph of minimum degree at least 3. Then there are vertices x, y and v such that $xy \notin E(G), xv, yv \in E(G)$ and $G - \{x, y\}$ is connected.

Proof. Consider a vertex w of degree at most $|G| - 2$
Case 1: $G - w$ has no cutvertices.
Let $x = w$ and y be a vertex at distance 2 from x and let v be a common neighbor of x and y. Since y is not a cut-vertex in $G - x$, $G - \{x, y\}$ is connected.
Case 2: $G - w$ has a cutvertex.
In this case, let $v = w$. Then v must be adjacent to non-cutvertex members of each leaf-block of $G - v$. Let x and y be such neighbors in distinct leaf-blocks. Since v has another neighbor besides x and y, $G - \{x, y\}$ is connected. □

Lemma 46 (Simple Coloring Results)
For any graph G the following hold:

- $\chi(G) \geq max\{\omega(G), \frac{|G|}{\alpha(G)}\}$
- $||G|| \geq \binom{\chi(G)}{2}$ and $\chi(G) \leq 1/2 + \sqrt{2||G|| + 1/4}$

Greedy coloring algorithm
starting from a fixed vertex enumeration $v_1, ..., v_n$ of G, we consider the vertices in turn and color each v_i with the first available color, meaning with the smallest positive integer not already used to color any neighbour of v_i among $v_1, ..., v_{i-1}$. In this way we never use more than $\Delta(G) + 1$ colors, even for unfavourable choices of the enumeration. If G is complete or an odd cycle, then this is even best possible.

Lovasz Perfect Graph Theorem A graph G is perfect if and only if its complement $\overline{G}$ is perfect.

Strong Perfect Graph Theorem A graph G is pefect if and only if it does not contain an odd cycle on at least 5 vertics (an *odd hole*) or the complement of an odd hole as an induced subgraph.

Graphs with $\omega(G) \leq \chi(G)$ exist Meaning $\exists G : \omega << \chi(G)$, we have 3 proofs

- Mycielski's construction
- Tutte's construction
- Erdős-Hajnal theorem: girth greater than k and chromatic number greater than k

Theorem 5.1 (Brook's Theorem) Let G be a connected graph. Then $\chi(G) \leq \Delta(G)$ unless G is a complete graph or and odd cycle.

Proof. Induction on $|V(G)| = n$. The theorem holds for any graph on at most 3 vertices. Assume $|V(G)| > 3$, we have two cases:

case 1: G has a cut-vertex v
If G has a cut-vertex v, we can apply induction to the graphs G_1 and G_2 such that $G_1 \cup G_2 = G$ and $V(G_1) \cap V(G_2) = \{v\}$ and $|G_1| < |G|$ and $|G_2| < G$. By I.H. we get that even if G_1 and G_2 are complete or an odd cycle $\chi G_i \leq \Delta(G_i) \leq \Delta(G)$ we can make sure that the color of v is the same in both colorings and still get $\chi(G) \leq \Delta(G)$.
If $\Delta(G) \leq 2$ then G is a path or a cycle and the theorem holds. Assume $\Delta(G) \geq 3$

case 2: G is 2-connected.
case 2.1: There is a vertex v of degree at most $\Delta - 1$
We shall order the vertices of G $v_1, ..., v_n$ such that $v = v_n$ and each $v_i, i < n$ has a neighbor with larger index, such an ordering exists by Lemma 43. Color G greedily with respect to this ordering. We see at step i, there are at most $\Delta - 1$ neighbors of v_i that has been colored, so there is an available color for v_i
case 2.2: All vertices of G have degree Δ
Consider vertices x, y, v guaranteed by Lemma 45, such that $xy \notin E(G)$ and $xv, yv \in E(G)$ and $G - \{x, y\}$ is connected. Order the vertices of G as $v_1, ..., v_n$ such that $v_1 = x, v_2 = y, v_n = v$ for each $v_i, 3 \leq i < n$ there is a neighbor of v_i with a higher index, such an ordering exists by Lemma 43. Color G greedily according to this ordering. We see that v_1 and v_2 get the same color as in the previous case, at step $i, 3 \leq i < n$, v_i has at most $\Delta - 1$ colored neighbors so it could be colored with a remaining color. At the last step, we see that v_n has Δ colored neighbors, but two of them v_1 and v_2 have the same color so there are at most $\Delta - 1$ colors used by the neighbors of v_n. Thus v_n can be colored with a remaining color. □

Graphs with arbitrarily high chromatic number

Mycielski's Construction We can construct a family $(G_k = (V_k, E_k))_{k \in \mathbb{N}}$ of triangle-free graphs with $\chi(G_k) = k$ as follows:

- G_1 is the single-vertex graph, G_2 is the single-edge K_2
- $V_{k+1} \coloneqq V_k \cup U \cup \{w\}$ where $V_k \cap (U \cup \{w\}) = \varnothing, V_k = \{v_1, ..., v_n\}$ and $U = \{u_1, ..., u_n\}$
- $E_{k+1} \coloneqq E_k \cup \{wu_i : i = 1, ..., k\} \cup \bigcup_{i=1}^{n} \{u_i v : v \in N_{G_k}(v_i)\}$

Lemma For any $k \geq 1$, Miycielsk's graph G_k has chromatic number k. Moreover, G_k is triangle-free.

Proof. We shall prove this statement by induction on k with trivial basis $k = 1$. Assume $k \geq 2$ and $\chi(G_{k-1} = k-1$ and G_{k-1} is triangle-free.

First we show that $\chi(G_k) = k$:
We see that $\chi(G) \leq k$ by considering a proper coloring c of G_{k-1} with colors from $[k-1]$ and letting $c' : V_k \to [k]$ such that every u_i gets the same color as v_i and the vertex w gets the new color k.
Now assume that $\chi(G_k) < k$. Let c be a proper coloring of G_k with colors from $[k-1]$. Derive contradiction if $\chi(G_k) = k-1$ by coloring w with color $k-1$ and the vertices from U_{k-1} are colored using $k-2$ colors, but then the vertices in V_{k-1} could also be colored from $[k-2]$.

To see that G_k has no triangle, observe that a triangle could only have one vertex in $u_i \in U_{k-1}$ and two vertices in $v_j, v_m \in V_{k-1}$. Then v_i, v_j, v_m form a triangle in G_{k-1} a contradiction. □

Tutte's Construction We can construct a family $(G_k)_{k \in \mathbb{N}}$ of triangle-free graphs with $\chi(G_k) = k$ as follows: G_1 is the single-vertex graph. To get from G_k to G_{k+1}, take an independent set U of size $k(|G_k| - 1) + 1$ and $\binom{|U|}{|G_k|}$ vertex-disjoint copies of G_k. For each subset of size $|G_k|$ in U then introduce a perfect matching to exactly one of the copies of G_k.

Lemma For any k, Tutte's graph G_k has chromatic number k and it is triangle-free.

Proof. We argue by induction on k with trivial basis $k = 1$. We see that $\chi(G_k) \leq \chi(G_{k_1}) + 1$ because we can assign the same set of $\chi(G_{k-1})$ colors to each copy of G_{k-1} and a new color to U. Assume that $\chi(G_k) \leq \chi(G_{k-1})$. Consider a coloring of G_k with $\chi(G_{k-1})$ colors. By pigeonhole principle there is a set U' of $|G_{k-1}|$ vertices in U of the same color, say 1. The vertices of U' are matched to a copy G' of G_{k-1}. Then G' does not use color 1 on its vertices and thus colored with less than $\chi(G_{k-1})$ colors. Therefore there are two adjacent vertices of the same color. So, any proper coloring of G_k uses more than $\chi(G_{k-1})$ colors.

To see that G_k has no triangles, observe that any two adjacent edges incident to U have endpoints in distinct copies of G_{k-1}, thus are not part of any triangle. □

Theorem 54 (Kőnig, 1916) If G is a bipartite graph with maximum degree Δ then $\chi'(G) = \Delta$.

Proof. We see, that $\chi'(G) \geq \Delta$ because the edges incident to a vertex of maximum degree require distinct colors in a proper edge-coloring.

To prove that $\chi'(G) \leq \Delta$ we use induction on $||G||$ with a basis $||G|| = 1$. Let G be given, $||G|| \geq 2$ and assume that the statement is true for any graph on at most $||G|| - 1$ edges. Let $e = xy \in E(G)$. By induction, there is a proper edge coloring c of $G' = G - e$ using Δ colors.

In G' both x and y are incident to at most $\Delta - 1$ edges. Thus there are non-empty color sets $Mis(y), Mis(y) \subseteq [\Delta]$, which are the colors that are not used on edges incident to x or y. If $Mis(x) \cap Mis(y) \neq \emptyset$, color e with $\alpha \in Mis(x) \cap Mis(y)$. This gives $\chi'(G) \leq \Delta$.

If $Mis(x) \cap Mis(y) = \emptyset$, let $a \in Mis(x)$ and $b \in Mis(y)$, consider the longest path P colored a and b starting at x. Because of parity, P does not end in y, and because y is not incident to b, y is not a vertex on P. Switch colors a and b on P. Then we obtain a proper edge-coloring in which $b \in Mis(x) \cap Mis(y)$, which allows e to be colored b. Thus $\chi'(G) \leq \Delta$. □

Proof sketch: induction: remove an edge, apply IH if the vertices incident to the edge have same color available then put back the edge and color it with it. If they have different ones then consider longest path (alternating between the available colors) starting in one of the vts, and switch colors of that one. So the vertices have the same color available, now put back the edge and color it with that color.

Theorem 5.4 (Vizing's Theorem) For any graph G with maximum degree Δ

$$\Delta \leq \chi'(G) \leq \Delta + 1$$

Proof. proof sketch:

- lower bound: because of vertex of maximum degree
- upper bound: induction on number of edges
- **Assume that G has no proper edge-coloring with max degree + 1 colors**
- introduce claim that when removing an edge we have a path between the incident vertices in the two colors that are available at them
- proof the claim with contradiction (we could switch colors and obtain coloring with less colors)
- get contradiction using claim and creating multiple Paths, this means G has a proper edge-coloring with $\Delta + 1$

The lower bound holds because the edges incident to a vertex of maximum degree require distinct colors in a proper edge-coloring. For the upper bound use induction on $||G||$ with the trivial basis $||G|| = 1$. Let G be a graph, $||G|| > 1$, assume that the assertion holds for all graphs with smaller number of edges. For any edge-coloring c of a subgraph H of G with colors $[\Delta + 1]$. Assume now that G has no proper edge-coloring with $\Delta + 1$ colors.

Claim For any $e = xy \in E(G)$, for any proper coloring c of $G - e$ from $[\Delta + 1]$, for any $\alpha \in Mis_c(x)$ and any $\beta \in Mis_c(y)$, there is an x-y-path colored α and β.

We see that $Mis_c \neq \varnothing$ for any v. If $Mis_c(x) \cap Mis_c(y) \neq \varnothing$, let $\alpha \in Mis_c(x) \cap Mis_c(y)$. Color xy with α, this gives a proper coloring of G with at most $\Delta + 1$ colors, a contradiction.

If $Mis_c(x) \cap Mis_c(y) = \varnothing$, let $\alpha \in Mis_c(x), \beta \in Mis_c(y), \alpha \neq \beta$. If there is a maximal path P colored α and β that contains x and does not contain y, switch the colors α and β in P and color xy with β. This gives a proper coloring of G with at most $\Delta + 1$ colors, a contradiction, this proves the claim.

Let $xy_0 \in E(G)$. Let c_0 be a proper coloring of $G_0 := G - xy_0$ from $[\Delta + 1]$. Let $\alpha \in Mis_{c_0}(x)$. Let $y_0, y_1, ..., y_k$ be a maximal sequence of distinct neighbors of x such that $c_0(xy_{i+1}) \in Mis_{c_0}(y_i), 0 \leq i < k$.

Let c_i be a coloring of $G_i = G - xy_i$ such that

$$c_i(xy_j) = \begin{cases} c_0(xy_{j+1}), & \text{for } j \in \{0, ..., i-1\} \\ c_0(e), & \text{otherwise} \end{cases}$$

Note that $Mis_{c_i}(x) = Mis_{c_j}(x)$ for all $i, j \in \{0, ..., k\}$.

Let $\beta \in Mis_{c_0}(y_k)$. Let $y = y_i$ be a vertex so that $c_0(yx) = \beta$. Such a vertex exists, otherwise either $\beta \in Mis_{c_k}(y_k) \cap Mis_{c_k}(x)$ contradicting Claim, or the sequence $y_0, ..., y_k$ can be extenden, contradicting its maximality.

Then G_k has an α-β path P with endpoints y_{i-1}, y_k in $G_k - x$. On the other hand G_i has an α-β-path P' with endpoints y_{i-1}, y_i in $G_i - x$. Since G_x is colored identically in c_k and c_i, we have that $P \cup P'$ is a two-colored graph, connected since both paths contain y_{i-1} and having three vertices of degree 1. This is impossible. $\square$

Lemma 55 The list chromatic number of $G = K_{n,n}$, with $n = \binom{2k}{k}$ is at least $k + 1$.
This means there exists a graph which has much greater list-chromatic number than chromatic number

7 Extremal graph theory

In this chapter we study how global parameters of a graph, such as its edge density or chromatic number, can influence its local substructures. Two categories of questions:

- global assumptions that might imply that H as a *minor* (or topological minor)it will suffice to raise $\|G\|$ above the value of some linear function of $|G|$.
- global assumptions that might imply the existence of some given graph H as a *subgraph*

Definition

- The *extremal number* $\text{ex}(n, H)$ denotes the maximum size (amount of edges) of a graph of order n that does not contain H as a subgraph and $\text{EX}(n, H)$ is the set of H-free graphs on n vetices with $\text{ex}(n, H)$ edges.
- Example: $\text{ex}(n, P_3) = \lfloor \frac{n}{2} \rfloor$, $\text{EX}(n, P_3) = \{\lfloor n \rfloor \cdot K_2 + (n \bmod 2) \cdot E_1\}$
- Let n and r be integers with $1 \leq r \leq n$ The *Turan graph* $T_r(n)$ is the unique complete r-partite graph of order n whose partite sets differ by at most 1 in size. It does not contain K_{r+1}, we denote $\|T_r(n)\|$ by $t_r(n)$

Calculating Turan Number
The Turan number $t_r(n)$ is calculated as follows:
Let $n = pr + s$ where p and s are integers, and $0 \leq s < r$, then $T_r(n)$ has s parts of size $p+1$ and $r-s$ parts of size p. And then simply count edges by hand.

Lemma 58 For any $r, n \geq 1, t_r(n+r) = t_r(n) + n(r-1) + \binom{r}{2}$

Proof. Consider $G = T_r(n+r)$ graph with parts $V_1, ..., V_r$. Let $v_i \in V_i, i = 1, ..., r$. Then $G' = G - \{v_1, ..., v_r\}$ is isomporphic to $T_r(n)$. We have that $\|G\| - \|G'\|$ is equal to the number of edges incident to v_i's for some $i = 1, ..., r$. This number is

$$\underbrace{n(r-1)}_{\text{every vertex in } T_r(n) \text{ gets (r-1) new}} \qquad + \qquad \underbrace{\binom{r}{2}}_{\text{edges between new r vts}}$$

□

Lemma 59 Among all $n-$vertex r-partite graphs, $T_r(n)$ has the largest number of edges.

Proof. Let first $r = 2$:
Let G be an n-vertex bipartite graph with largest possible number of edges. Then clearly G is complete bipartite. Assume that two parts V and U of G differ in size by at least 2, so $|V| > |U| + 1$. Put one vertex from V to U to obtain new parts V' and U' and let G' be complete bipartite graph with parts V' and U'. Then $\|G'\| = |V'||U'| = (|V|-1)(|U|+1) = |V||U| - |U| + |V| - 1 > |V||U| - |U| + |U| + 1 - 1 = |V||U| = \|G\|$, a contradiction to maximality of G.

If $r > 2$:
Consider any two parts U, V of an r-partite G. Assume that U differs from V by at least 2 in size. Let X be the remaining set of vertices. Then $\|G\| = \|G[X]\| + |X|(n - |X|) + \|G[U \cup V]\|$. Let G' be a graph on the same set of vertices as G that differs from G only on edges induced by $U \cup V$ and so that $G'[U \cup V]$

is a balanced complete bipartite graph. Then from the previous paragraph with $r = 2$, we see that $\|G'[U \cup V]\| > \|G[U \cup V]\|$. Thus $\|G'\| > \|G\|$, a contradiction. Thus any two parts of G differ in size by at most 1. In addition we see as before that G is complete r-partite. Thus G is isomporphic to $T_r(n)$. □

Lemma 60 For a fixed r,

$$\lim_{n\to\infty} \frac{t_r(n)}{\binom{n}{2}} = 1 - \frac{1}{r}$$

Proof. Since each part in $T_r(n)$ has size either $\lfloor \frac{n}{r} \rfloor$ or $\lceil \frac{n}{r} \rceil$, we see that each part has size between $\frac{n-r}{r}$ and $\frac{n+r}{r}$. We have that

$$\underbrace{\binom{n}{2}}_{\text{all edges}} - \underbrace{r\binom{(n+r)/r}{2}}_{r \text{ independent sets}} \le t_r(n) \le \binom{n}{2} - r\binom{(n-r)/r}{2}$$

Thus

$$\binom{n}{2} - r\frac{1}{2}\frac{(n+r)}{r}\frac{n}{r} \le t_r(n) \le \binom{n}{2} - r\frac{1}{2}\frac{(n-r)}{r}\frac{(n-2r)}{r}$$

More manipulation and dividing by $\binom{n}{2}$ gives the result. □

Theorem 62 (Mantel's theorem) If a graph G on n vertices contains no triangle then it contains at most $\frac{n^2}{4}$ edges.

Proof. We proceed by induction on n. For $n = 1$ and $n = 2$. The result is trivial, so assume that $n > 2$ and we know it to be true for $n - 1$. Let G be a graph on n vertices. Let x and y be two adjacent vertices in G. Since every vertex in G is connected to at most one of x and y, there are at most $n - 2$ edges between $\{x, y\}$ and $V(G) - \{x, y\}$. Let $H = G - \{x, y\}$. Then H contains no triangles and thus, by induction, H has at most $\frac{(n-2)^2}{4}$ edges. Therefore the total number of edges in G is at most □

Theorem 6.2 (Turan's Theorem) For all integers $r > 1$ and $n \ge 1$, any graph G with n vertices, $\text{ex}(n, K_r)$ edges and $K_r \notin G$ is a $T_{r-1}(n)$. In other words $\text{EX}(n, K_r) = \{T_{r-1}(n)\}$.

Proof. We use induction on n for a fixed r. If $n \le r - 1$, then K_n is the graph with largest number of edges, n vertices and no copy of K_r. Since the Turan graph is K_n if $n \le r - 1$ the basis case is complete.

Assume that $n > r - 1$. Let $G \in EX(n, K_r)$. Then G contains a copy K of K_{r-1} otherwise we could add an edge to G without creating a copy K_r, thus violating maximality of G. Let $G' = G - V(K)$. By I.H.

$$\|G'\| \le t_{r-1}(n - (r - 1))$$

Thus

$$\|G\| = \|G'\| + \|K\| + \|G[V(K), V - V(K)]\| \le t_{r-1}(n - r + 1) + \binom{r-1}{2} + (n - r + 1)(r - 2) \tag{1}$$

Indeed, the last term holds since any vertex of $V - V(K)$ is adjacent to at most $|V(K)| - 1 = r - 2$ vertices of K (otherwise we would have a copy of K_r in G). By Lemma 58 $t_{r-1}(n) = t_{r-1}(n-r+1) + \binom{r-1}{2} + (n-r+1)(r-2)$ and thus

$$||G|| \leq t_{r-1}(n)$$

On the other hand we know that $T_{r-1}(n)$ does not have K_r as a subgraph, so the densest K_r-free graph G should have at least as many edges as $T_{r-1}(n)$. Thus $||G|| \geq t_{r-1}(n)$, therefore we get $||G|| = t_{r-1}(n)$ and all inequalities in (1) are equalities.

So, in particular $||G'|| = t_{r-1}(n-r+1)$, by induction $G' = T_{r-1}(n-r+1)$, and each vertex of $V - V(K)$ sends exactly $r-2$ edges to K. Let $V_1, ..., V_{r-1}$ be the parts of G'. For all $v \in V_1 \cup ... \cup V_{r-1}$, let $f(v) \in V(K)$ so that v is not adjacent to $f(v)$. If there are indices $i, j \in [r-1]$, $i \neq j$ so that there are vertices $v \in V_i, v' \in V_j$ for which $f(v) = f(v')$, then $V(K) \cup \{v, v'\}$ induces an r-clique in G, a contradiction.

Therefore we can suppose that for all $i, j \in [r-1], i \neq j$ and for any $v \in V_i$ and $v' \in V_j, f(v) \neq f(v')$. It implies that for any $i \in [r-1]$ and any $u, u' \in V_i, f(u) = f(u')$. Denote the vertices of K by $v_1, ..., v_{r-1}$ where $v_i f(u_i)$ with $u_i \in V_i$.

Then $G = T_{r-1}(n)$ with parts $V_i \cup \{v_i\}$ □

Conjecture Erdős-Sos If $|G| = n$ and $||G|| > \frac{(k-1)n}{2}$, then G contains all k-edge trees as subgraphs, i.e. for any tree T on k edges $\text{ex}(n, T) \leq \frac{(k-1)n}{2}$

Theorem 66 (Erdős-Stone-Simonovits) For any graph H and for any fixed $\epsilon > 0$ there is n_0 such that for any $n \geq n_0$:

$$(1 - \frac{1}{\chi(H) - 1} - \epsilon)\binom{n}{2} \leq ex(n, H) \leq (1 - \frac{1}{\chi(H) - 1} + \epsilon)\binom{n}{2}$$

Corollary

- $\chi(H) = 3$, $\text{ex}(n, H) \approx \frac{1}{2}\binom{n}{2}$
- $\chi(H) \geq 3$, $\text{ex}(n, H) \approx \text{ex}(n, K_{\chi(H)})$

For exam, find extremal number for an ugly big graph,
by applying Erdős-Stone

Proof outline: Let $r = \chi(H) - 1$

For the upper bound, let G be a graph on n vertices that has $(1 - \frac{1}{\chi(H)-1} + \epsilon)\binom{n}{2}$ edges. We shall show that G has a subgraph isomorphic to H. Let G' be a large subgraph of G that has minimum degree at least $(1 - \frac{1}{r} + \frac{\epsilon}{2})|V(G')|$, we can finde such G' by greedily deleting vertices of smaller degrees. Then show, by induction on r that G' contains a complete $(r+1)$-partite graph H' with sufficiently large parts. Finally observe that $H \subseteq H'$

Definition Zarankiewicz function $z(m, n; s, t)$ denotes the maximum number of edges that a bipartite graph with parts x, Y of sizes m, n respectively, can have without containing $K_{s,t}$ respecting sides (i.e. there is no copy of $K_{s,t}$ with partition sets S, T of sizes s, t respectively, such that $S \subseteq X$ and $T \subseteq Y$)

Theorem 67 (Kovari-Sos-Turan) We have the upper bound

$$z(m,n;s,t) \leq (s-1)^{\frac{1}{t}}(n-t+1)m^{1-\frac{1}{t}} + (t-1)m$$

In particular for $m = n$ and $t = s$

$$z(m,n;s,t) \leq c_1 \cdot n \cdot n^{1-\frac{1}{t}} + c_2 \cdot n = \mathcal{O}(n^{2-\frac{1}{t}})$$

Proof. Let G be a bipartite graph with parts A, $|A| = m$ and B, $|B| = n$ such that it does not contain a copy of $K_{s,t}$ with part of size s in A and part of size t in B. Let T be the number of stars of size t with a center in A Then

$$T = \sum_{v \in A} \binom{deg(v)}{t}$$

On the other hand

$$T \leq (s-1)\binom{n}{t}$$

Since for each subset $\mathcal{Q}$ of t vertices in B there are at most $s-1$ stars counted by T with a leaf-set $\mathcal{Q}$.

TODO □

Hadwiger's Conjecture

$$\chi(G) \geq r \Rightarrow G \text{ has a } K_r \text{ minor}$$

Proof. proof cases $r = 3$ and $r = 4$ for exam □

Lemma 68 For any positive integers n, t with $t < n$,

$$ex(n, K_{t,t}) \leq \frac{z(n,n;t,t)}{2}$$

Theorem 69 For any positive t, and $n > t$, there are positive constants c and c' such that

$$c' \cdot n^{2-\frac{2}{t+1}} \leq ex(n, K_{t,t}) \leq c \cdot n^{2-\frac{1}{t}}$$

Theorem 71

$$ex(n, C_4) = \frac{1}{2}n^{3/2} + (n^{3/2})$$

$$ex(n, C_6) = \Theta(n^{4/3})$$

$$ex(n, C_{10}) = \Theta(n^{6/5})$$

$$c' \cdot n^{1-\frac{2}{3k-2-\epsilon}} \leq ex(n, C_{2k}) \leq c \cdot n^{1+\frac{1}{k}}$$

Definition 6.4 Let $X, Y \subseteq V(G)$ be disjoint vertex sets and $\epsilon > 0$.

- *the density* $d(X,Y)$ of (X,Y) is

$$d(X,Y) := \frac{\|X,Y\|}{|X|,|Y|}$$

- For $\epsilon > 0$ the pair (X,Y) is an ϵ*-regular pair* if we have

$$|d(X,Y) - d(A,B)| \leq \epsilon$$

  for all $A \subseteq X, B \subseteq Y$ with $|A| \geq \epsilon|X|$ and $|B| \geq \epsilon|Y|$
  In other words, the edges in an ϵ-regular pair are distributed very uniformly, with the density between any pair of reasonably large subsets of vertices being very close to the overall density of the pair. This uniform distribution of edges is typical in a random bipartite graph, and captures what we mean when we say a (bipartite) graph 'looks random'. It remains to define what kinds of partitions of the vertices we will be concerned with.

- An ϵ*-regular partition* of the graph $G = (V,E)$ is a partition of the vertex set $V = V_0 \uplus ... \uplus V_k$ with the following properties:

  1. $|V_0| \leq \epsilon|V|$
  2. $|V_1| = |V_2| = ... = |V_k|$
  3. all but at most ϵk^2 of the pairs (V_i, V_j) for $1 \leq i < j \leq k$ are ϵ-regular.

  Note the parameter ϵ play three roles here: bounding the size of the exceptional set V_0, bounding the number of irregular pairs, and controlling the regularity of the regular pairs

In an ϵ-regular parition we have control over the distribution of edges between the ϵ-regular pairs, but not over the edges within any of the parts, involving the exceptional set, or in irregular pairs. Thus, in light of the three roles described above, the smaller ϵ is, the greater our control over the distribution of edges in an ϵ-regular partition.

Informally, the Regularity Lemma tells us that the vertices of *any* large graph can be partitioned into a bounded number of parts, with the subgraph between most pairs of parts looking random.

Szemeredi's Regularity Lemma For any $\epsilon > 0$ and any integer $m \geq 1$ there is an $M \in \mathbb{N}$ such that every graph of order at least m has an ϵ-regular partition $V_0 \uplus ... \uplus V_k$ with $m \leq k \leq M$.

Analyzing large Graphs with regularity Lemma

1. start with ϵ-partition
2. create auxillary graph with an edge between parts if $(V_i, V_j)\epsilon$-regular with density $d > 0$
3. use blowup graph to complete partite, use blowup lemma:

$$H \subseteq R_s \Rightarrow H \subseteq G$$

Erdos-Stone Theorem For all integers $r > s \geq 1$ and any $\epsilon > 0$ there exists an integer n_0 such that every graph with $n \geq n_0$ vertices and at least

$$t_{r-1}(n) + \epsilon n^2$$

edges contains K_r^s (is the complete r-partite graph where each part contains exactly s vertices) as a subgraph.

Corollary 73 Erdős-Stone together with $\lim_{n\to\infty} / \binom{n}{2} = 1 - 1/r$ yields an asymptotic formula for the extremal number of any graph H on at least one edge:

$$lim_{n\to\infty} \frac{ex(n,H)}{\binom{n}{2}} = \frac{\chi(H) - 2}{\chi(H) - 1}$$

Substructures in sparse Graphs **Key message:** extremal numbers for minors are linear in $|E|$

- $||G|| \geq cr^2|G| \Rightarrow G \supseteq \mathrm{TK}_r$
- $||G|| \geq c\sqrt{log(r)} \Rightarrow G \supseteq \mathrm{MK}_r$

8 Ramsey Theory

Definitions

- In an edge-coloring of a graph, a set of edges is
  - *monochromatic* if all edges have the same color
  - *rainbow* if no two edges have the same color
  - *lexical* if two edges have the same color iff they have the same lower endpoint in some ordering of the vertices
- Let k be a natural number. Then the *Ramsey number* $R(k) \in \mathbb{N}$ is the smallest n such that every 2-edge-coloring of K_n contains a monochromatic K_k.
- **asymmetric Ramsey number** $R(k,l)$**:** is the smallest $n \in \mathbb{N}$ such that every 2-edge-coloring of a K_n contains a red K_k or a blue K_l.
- **graph Ramsey number** $R(G,H)$**:** is the smallest $n \in \mathbb{N}$ such that every red-blue edge-coloring of K_n contains a red G or a blue H.
- **hypergraph Ramsey number** $R_r(l_1, ..., l_k)$**:** is the smallest $n \in \mathbb{N}$ such that every k-coloring of the edges of the complete hypergraph on n vertices and edges of size r contains a clique of size l_i whose edges all have color i, for some $i \in \{1, ..., k\}$.
- **induced Ramsey number** $IR(G,H)$**:** is the smallest $n \in \mathbb{N}$ for which there is a graph F on n vertices such that in any red-blue coloring of $E(F)$, there is an induced subgraph of F isomorphic to G with all its edgs colored red or there is an induced subgraph of F isomorphic to H with all its edges colored blue.

- **anti-Ramsey number** $AR(n, H)$**:** is the maximum number of colors that an edge-coloring of K_n can have without containing a rainbow copy of H.

Ramsey Theorem For any $k \in \mathbb{N}$ we have

$$\sqrt{2}^k \leq R(k) \leq 4^k$$

In particular the Ramsey numbers, the asymmetric Ramsey numbers and the graph Ramsey numbers are finite.

Proof. Proofsketch:
Upperbound: consider sequence of vts and their monochromatic nbhs, the number is divided by two at each step

Lowerbound: Consider $K_{\sqrt{2}^k}$ we will show that there exists a coloring for which there is no monochromatic coloring of a clique with k vertices. Color edges blue and red with prob. 1/2, show that Prob(there is a monochromatic clique on k vertices) is less than one

For the upper bound, consider a red blue edge-coloring of $G = K_{4^k}$. Construct a sequence of vertices $x_1, ..., x_{2k}$, a sequence of vertex sets $X_1, ..., X_{2k}$, and a sequence of colors $c_1, ..., c_{2k_1}$ as follows. Let x_1 be any arbitrary vertex, $X_1 = V(G)$. Let X_2 be the largest monochromatic neighborhood of x_1 in X_1, i.e. the largest subset of vertices from X_1, such that all edges from this subset to x_1 have the same color. Call this color c_1. We see that $|X_2| \geq \lceil \frac{|X_1|-1}{2} \rceil \geq 4^k/2$. Let x_2 be an arbitrary vertex in X_2. Let X_3 be the largest monochromatic neighborhood of x_2 in X_2 with respective edges of color c_2, and so on let X_m be the largest monochromatic neighborhood of x_{m-1} with respective color c_{m-1} in $X_{m-1}, x_m \in X_m$. Wee see that $|X_m| \leq 4^k/2^{m-1}$. Thus $|X_m| > 0$ as long as $2k > (m-1)$, i.e. as long as $m \leq 2k$. Consider vertices $x_1, ..., x_{2k}$ and colors $c_1, ..., c_{2k-1}$. At least k of the colors say $c_{i1}, c_{i2}, ..., c_{ik}$ are the same by pigeonhole principle, say wlog red. Then $c_{i1}, c_{i2}, ..., c_{ik}$ induce a k-vertex clique all of whose edges are red.

For the lower bound, we shall construct a coloring of $K_n, n = 2^{k/2}$ with no monochromatic cliques on k vertices. Let's color each edge red with probability $\frac{1}{2}$ and blue with probability $\frac{1}{2}$. Let S be a fixed set of k vertices. Then

$$\text{Prob}(\ S \text{ induces a red clique}) = 2^{-\binom{k}{2}}$$

Thus Prob(S induces monochromatic clique) $= 2^{-\binom{k}{2}+1}$. Therefore

$$\begin{aligned}
\text{Prob(monochr. clique on k vertices)} &\leq \binom{n}{k} 2^{-\binom{k}{2}+1} \\
&\leq \frac{n^k}{k!} 2^{-k^2/2+k/2+1} \\
&\leq \frac{2^{k/2k+1}}{k!} \\
&\leq 1
\end{aligned}$$

□

Remark $R(2) = 2, R(3) = 6, R(4) = 18$ and $43 \le R(5) \le 48$.

Applications of Ramsey theory

Theorem (Erdős, Szekeres) Any list of more than n^2 numbers contains a nondecreasing or non-increasing sublist of more than n numbers.

Proof. Let $a_1, ..., a_{n^2+1}$ be a list of numbers. Let u_i be the length of a longest non-decreasing sublist ending with a_i. Let d_i be the length of a longest non-increasing sublist ending with a_i. Assume that the statement of the theorem is false. Then $u_i, d_i \le n$ and there are at most n^2 distinct pairs (u_i, d_i). Since there are more than n^2 numbers there are indices $i < j$ such that $(u_i, d_i) = (u_j, d_j)$. If $a_i \le a_j$, we have $u_i < u_j$. If $a_i \ge a_j$, we have $d_i < d_j$, a contradiction. $\square$

Definition Let $R(p, q; r)$ be the hypergraph Ramsey number for r-uniform hypergraphs. The following Theorem show the existence of hypergraph Ramsey number.

$$\begin{gathered} R(p,q;r) = \min\{N : \forall c : \tbinom{[N]}{r} \to \{0,1\} \\ \exists A \subseteq [N], |A| = p, \forall A' \in \tbinom{A}{r}\ c(A') = 0 \text{ or} \\ \exists B \subseteq [N], |B| = q, \forall B' \in \tbinom{B}{r}\ c(B') = 1 \end{gathered}$$

Theorem 83 For any parametres $p, q, r \ge 2$

$$R(p,q;r) \le R(R(p-1,q;r), R(p,q-1;r); r-1) + 1$$

Proof. Let $c : \binom{X}{r} \to \{red, blue\}$, where $|X| = R(R(p-1,q;r), R(p,q-1;r); r-1) + 1$. We shall show that there is a red r-clique on p vertices or a blue r-clique on q vertices. Let $x \in X$. Let $c' : \binom{X-x}{r-1} \to \{red, blue\}$ be defined as follows: for any $A \subseteq X - x$, let $c'(A) = c(A \cup x)$. Let $p_1 = R(p-1,q;r)$ and $q_1 = R(p,q-1;r)$. Since $|X - x| = R(p_1, q_1; r-1)$, there is a red $(r-1)$-clique on vertex set $X', |X'| = p_1$, or a blue $(r-1)$-clique on vertex set $X'', |X''| = q_1$. Assume the former. The latter is treated similary. Then in c, all sets $A \cup x$ are red where $A \subseteq X'$. Since $|X'| = p_1 = R(p-1,q;r)$, then in X' under c there is either a blue r-clique of size q and we are done, or there is a red r-clique on vertex set $X^* \subseteq X', |X^*| = p-1$. But then $X^* \cup x$ forms a red r-clique under c on p vertices and we are done. $\square$

Theorem (Erdős, Szekeres) For any integer $m \ge 3$ there is an integer $N = N(m)$ such that if X is a set of N points on the plane such that no three points are on a line, then X contains a vertex set of a convex m-gon.

Proof. Let $N = R(m, 5; 4)$. For each 4-element subset X' of X color it red if the convex hull X' is a 4-gon, it blue if the convex hull of X' is a triangle. By definition of R, there is either a set A of m points, such that $\binom{A}{4}$ is red, or a set B of 5 points such that $\binom{B}{4}$ is blue. Assume the latter. Then we see in particular that the convex hull of B is a triangle T and there are two vertices u, v of B inside this triangle. Consider a line through u, v it splits the plane in two parts, one containing one vertex of T, another two vertices of T, call them x, y. Then the convex hull of $\{u, v, x, y\}$ is a 4-gon, so that $\binom{A}{4}$ is red. We claim that A forms a vertex set of a convex m-gon. Assume not, and there is a point

x of A inside the convex hull A' of A. Triangulate A'. Then x will be inside one of the triangles, say with vertex set $\{y, z, w\}$. Then $\{x, y, z, w\}$ must be colored blue, a contradiction. $\square$

9 Random Graphs

Definitions

- Erdos-Renyi model of random graphs $\mathcal{G}(n,p)$ is the probability space on all n-vertex graphs that results from independently deciding whether to include each of the $\binom{n}{2}$ possible edges with fixed probability $p \in [0,1]$
- A property $\mathcal{P}$ is a set of graphs, for example $\mathcal{P} = \{G : G \text{ is } k\text{-connected}\}$
- Let $(p_n) \in [0,1]^{\mathbb{N}}$ be a sequence. We say that $G \in \mathcal{G}(n,p_n)$ *almost always* has a property $\mathcal{P}$ if $\text{Prob}(G \in \mathcal{G}(n,p_n) \cap \mathcal{P}) \to 1$ for $n \to \infty$. If (p_n) is constant p, we also say in this case that *almost all* graphs in $\mathcal{G}(n,p)$ have property $\mathcal{P}$
- A function $f(n) : \mathbb{N} \to [0,1]$ is a *threshold function* for property $\mathcal{P}$ if:
  - for all $(p_n) \in [0,1]^{\mathbb{N}}$ with
$$\frac{p_n}{f(n)} \underset{n\to\infty}{\to} 0$$
    the graph $G \in \mathcal{G}(n,p_n)$ almost always does not have property $\mathcal{P}$
  - for all $(p_n) \in [0,1]^{\mathbb{N}}$ with
$$\frac{p_n}{f(n)} \underset{n\to\infty}{\to} \infty$$
    the graph $G \in \mathcal{G}(n,p_n)$ almost always has property $\mathcal{P}$
  - threshold function for containing a cycle is $f(n) = 1/n$ meaning if $p > 1/n$ we almost always have a cycle

Usage of probabilistic Method

- $\text{ex}(n, K_{t,t})$
- $\sqrt{2}^k \leq R(k)$
- Erdős-Hajnal

Theorem 9.3 (Erdos-Hajnal) sketch: take random graph, number of short cycles (length at most k) is $< \frac{n}{2}$ delete a vertex from each of these cycles and get a graph G' so the girth of $G' > k$. We know chromatic number of G prime is $\geq \frac{|V(G')|}{\alpha(G')}$ with lemma 108 we can show α is not so big and therefore get a lower bound

two tricks:

- random graph and kill short cycle
- bound the independence number (co-clique number) size of the largest independent set

10 Hamiltonian cycles

Definition

- A cycle C in a graph G is *Hamiltonian* if it contains all vertices.
- A graph that has a Hamiltonian cycle is called Hamiltonian graph.

Lemma 10.1 (Neccessary condition for existence of Ham. cycle)
If G has a Hamiltonian cycle, the for every non-empty $S \subseteq V$ the graph $G - S$ cannot have more than $|S|$ components.

Proof. Let C be a Hamiltonian cycle of G. Let $S \subseteq V(G)$, $S \neq \emptyset$, $t :=$ # components of $G - S$ There are at least 2 edges of C between each component of $G - S$ and S. If $e =$ # edges of C between S and $V - S$, we have

$$e \geq t \cdot 2 \text{ and } e \underset{Cis2\text{-regular}}{\leq} |S| \cdot 2$$

□

Theorem 10.2 Dirac Every graph with $n \geq 3$ vertices and minimum degree at least $\frac{n}{2}$ has a Hamiltonian cycle.

Proof. First we note that G is connected, otherwise a smaller component has all vertices of degree at most $n/2 - 1$. Consider a longest path $P = (v_0, ..., v_k)$. Then $N(v_0), N(v_k) \subseteq V(P)$. Since $|N(_0)|, |N(v_k)| \geq n/2$, and $k \leq n - 1$, we have by pigeonhole principle that $v_0 v_k \in E(G)$ or there is $i, 0 < i < k - 1$ such that $v_0 v_{i+1} \in E(G)$ and $v_i v_k \in E(G)$. In any case there is a cycle C on $k + 1$ vertices in G. If $k + 1 = n$ we are done. If $k + 1 < n$, since G is connected there is a vertex v not in C that is adjacent to a vertex with C. Then v and C induce a graph that contains a spanning path, i.e. a path on $k + 2$ vertices, a contradiction to maximality of P. □

Ore's Thm. A graph G on $n \geq 3$ vertices is Hamiltonian if $\forall u, v \in V(G), uv \notin E(G), d(u) + d(v) \geq n$

11 Networkflows

Definitions

- A *multigraph* is a triple, (V, E, T^*), where T^* is a set of tuples $(\{x, y\}, e)$ with $x, y \in V(G)$ and $e \in E$, such that for each $e \in E$ there is a unique tuple $(\{x, y\}, e)$ in T^*. We say x and y are endpoints of e, if $x = y$ we say that e is a loop. If $(\{x, y\}, e), (\{x, y\}, e') \in T^*$, $e \neq e', x \neq y$ we say that e and e' are parallel or multiple edges.
- we assign a value to an ordered triple (x, e, y). Let $T(G) = \{(x, e, y) : (\{x, y\}, e) \in T^*(G)\}$

- Let $f : T(G) \to H$ and let $X, Y \subseteq V(G)$, we define
$$f(X,Y) := \sum_{x\in X, y\in Y, (x,e,y)\in T(G), x\neq y} f(x,e,y)$$
and write $f(x,Y)$ for $f(\{x\},Y)$
- Let G be a graph $s,t \in V(G)$, be distinct vertices and $c : T(G) \to \mathbb{N}\cup\{0\}$. We call a quadruple $N = (G,s,t,c)$ a *network* with *source* s, *sink* t and *capacity function* c

Definition Network Flow A function $f : T(G) \to \mathbb{R}$ is a *network flow* or *N-flow* if

(F1) $f(x,e,y) = -f(y,e,x)$ for any edge e with endpoints x,y and $x \neq y$

(F2) $f(x,V(G)) = 0, x \in V(G) - \{s,t\}$

(F3) $f(x,e,y) \le c(x,e,y)$ for any edge e with endpoints x,y and $x \neq y$

- A *cut* in a network N is a pair $(S,\overline{S})$, where S is a subset of vertices of G such that $s \in S, t \notin S$ and $\overline{S} = V(G) - S$
- *capacity of a cut*:
$$c(S,\overline{S}) = \sum_{x\in S, y\in\overline{S}, (x,e,y)\in T(G)} c(x,y)$$

Lemma 100 For any cut $(S,\overline{S})$ and a network flow f in a network N, $f(S,\overline{S})$ $= f(s,V(G))$

Proof.
$$f(S,\overline{S}) = f(S,V) - f(S,S) = f(s,V) + \underbrace{\sum_{v\in S\setminus\{s\}} f(v,V)}_{0 \text{ by } F2} - \underbrace{f(S,S)}_{0 \text{ by } F1} = f(s,V) + 0 - 0$$

□

Ford-Fulkerson Theorem Let $N = (G,s,t,c)$ be a network. Then
$$max\{|f| : f \text{ is an } N\text{-flow}\} = \min\{c(S,\overline{S}) : (S,\overline{S}) \text{ is a cut}\}$$
and there is an integral flow $f : T \to \mathbb{Z}_{\geq 0}$ with this maximum flow value.

Proof. Since $|f| = f(s,V) = f(S,\overline{S}) \le c(S,\overline{S})$, for any cut $(S,\overline{S})$, we have that
$$\max\{|f| : f \text{ is an } N\text{-flow}\} \le \min\{c(S,\overline{S} : (S,\overline{S}) \text{ is a cut}\}$$
Next, we shall construct a flow f such that $|f| = \min\{c(S,\overline{S}) : (S,\overline{S}) \text{ is a cut}\}$

We shall define $f_0, f_1, ...$ - a sequence of N-flows such that $f_0(x,e,y) = 0$ for all $(x,e,y) \in T(G), f_i$ assigns integer values and $|f_i| \ge |f_{i-1}| + 1$ for $i \ge 1$. Note that since $|f_i| \le \min\{c(S,\overline{S} : (S,\overline{S} \text{ is a cut}\}$ for all $i = 0,1,...$, the sequence $f_0, f_1, ...$ is finite. Let f_n be defined. We shall either let $f = f_n$ or define $f_n + 1$.

Case 1 There is a sequence of vertices $x_0 = s, x_1, ..., x_m = t$ and edges $e_0, ..., e_{m-1}$ such that $x_i x_{i+1} = e_i \in E(G)$ and $f(x_i, e_i, x_{i+1}) < c(x_i, e_i, x_{i+1}, i = 0, ..., m-1$

Let $\epsilon =$ $\min\{c(x_i, e_i, x_{i+1} - f(x_i, e_i, x_{i+1}) : i = 0, ..., m-1\}$. Note that $\epsilon \in \mathbb{N}$. Let

$$f_{n+1}(x,e,y) = \begin{cases} f_n(x,e,y), (x,e,y) \neq (x_i,e_i,x_{i+1}), = i = 0,...,m-1 \\ f_n(x,e,y) + \epsilon, (x,e,y) = (x_i,e_i,x_{i+1}), = i = 0,...,m-1 \\ f_n(x,e,y) - \epsilon, (x,e,y) = (x_{i+1},e_i,x_i), = i = 0,...,m-1 \end{cases}$$

Note that f_{n+1} is an N-flow, it takes integer values, and $|f_{n+1}| = |f_n| + \epsilon \geq |f_n| + 1$.

Case 2 Case 1 does not hold. Let

$$S = \{v \in V : \exists \text{path } s = x_0, e_0, x_1, ..., e_q, x_{q+1} = v, \\ f(x_i,e_i,x_{i+1}) < c(x_i,e_i,x_{i+1}), i = 0,...,q\}$$

Note that since we are not in Case 1, $t \notin S$. Also $s \in S$. Thus $(S, \overline{S})$. From the definition of S, we see that $f_n(x,e,y) = c(x,e,y)$ for all $x \in S, y \in \overline{S}, (x,e,y) \in T(G)$. Thus $f_n(S,\overline{S}) = c(S,\overline{S})$ and so $|f_n| \geq \min\{c(S,\overline{S} : (S,\overline{S})$ is a cut$\}$. Let $f = f_n$. Since the sequence $f_0, f_1, ...$ is finite Case 2 must occur. □

No Group-valued flows!!

12 problems

problem 1 Determine the number of edges, average degree, diameter and girth of the *d-dimensional hypercube.*

Proof. For number of edges consider vertex degree, for average degree observe that Q_d is regular.

claim: Q_d has diameter d.

First prove that for any $x, y \in V$ the distance $d(x,y)$ in Q_d is the number of postions where they differ. Suppose they differ in $l \geq 1$ positions then we get a path of length l by inverting each entry where they differ sequentially. There can not be a shorter Path since each step can only change one position.
Therefore the diameter is d, because thats the maximum amount of positions where two sequences can differ, consider 00...0 and 11...1.

claim: girth of Q_d is ∞ if $d = 1$ and otherwise 4.

If $d = 1$, the resulting graph Q_1 is acyclic. If $d \geq 2$ we first observe that then Q_d is triangel-free. Suppose otherwise than there is a triangle xyz in Q_d, now consider the amount of 1's in $|v|, v \in V$. Wlog $|x|$ is even then $|y|$ must be odd and $|z|$ must be even. But zx is an edge, a contradiction.
On the other hand Q_d does contain 4-cycles:

$$(00...0, 010...0, 100...0, 00...0)$$

□

problem 2 Show that any tree T has at least $\Delta(T)$ leaves.

Proof. Induction on $n = |T|$.
Base: tree on two vertices has 2 leaves and maximum degree is 1.

Step: remove a leaf v from T and let u be its neighbour, so that $T' := T - v$ we than have two cases:

case 1: $\Delta(T) = \Delta(T')$
We know by I.H. that T' has at least $\Delta(T')$ leaves, thus if the maximum degree of T is the same the claim holds.

case 2: $\Delta(T) = \Delta(T') + 1$ This means u is the vertex with maximal degree in T, but we know that the leafs of T can only be the leafs of T' plus the leaf v we removed therefore T has $\Delta(T') + 1$ leafs.

□

Proof. By Counting
Let T be any tree. Let $L \subseteq V$ be the set of leaves and $N = V \setminus L$ the set of non-leaves in T. Let u be a vertex with maximum degree $\Delta(T)$. We know that a tree has $|V| + 1$ edges.

$$
\begin{aligned}
2 \cdot (|V| - 1) &= \sum_{v \in V(T)} d(v) && (2)\\
&= d(u) + \sum_{v \in L} d(v) + \sum_{v \in N \setminus \{u\}} d(v) && (3)\\
&\geq \Delta(T) + \sum_{v \in L} 1 + \sum_{v \in N \setminus \{u\}} 2 && (4)\\
&= \Delta(T) + |L| \cdot 1 + (|V| - |L| - 1) \cdot 2 && (5)\\
&= \Delta(T) + 2 \cdot (|V| - 1) - |L| && (6)
\end{aligned}
$$

From this follows $|L| \geq \Delta(T)$, as desired. □

problem 3 Prove that either a graph or its complement is connected.

Proof. Let $G = (V, E)$ be any non-empty graph. We assume G is not connected and shall argue that $\overline{G}$ is connected.
Since G is disconnect, we find two vertices $u, w \in V$ and a connected component C of G such that $u \in C$ and $w \notin C$. Now in $\overline{G}$ all vertices in C are adjacent to w. And in particular $uw \in E(\overline{G})$, so all vertices lie in a single connected component of $\overline{G}$, which is therefore connected. □

problem 4 Prove that the vertex set of any graph can be partitioned into two sets such that for each vertex, at least half of its neighbors belong to the other set.

Proof. Consider partition of $V(G)$ into disjoint sets A, B, maximizing the number of edges between A and B. We will show that moving any vertex to the other sets results in a partition that has less edges between the sets than the original one.

Pick wlog a vertex $v \in A$. Let $d_B = N(v) \cap B$ and $d_A = d - d_B$. Now consider $A' = A - v$ and $B' = B + v$

$$|\{uw \in E : u \in A', w \in B'\}| = |\{uw \in E : u \in A, w \in B\}| - d_B + (d - d_B)$$

but by the maximality of edges between A and B we have:

$$|\{uw \in E : u \in A', w \in B'\}| \leq |\{uw \in E : u \in A, w \in B\}|$$

which gives us

$$-d_B + (d - d_B) \leq 0 \iff d_B \geq d/2$$

□

problem 5 Show that the following statements are equivalent

1. G is connected, but $G - e$ is disconnected for every edge e
2. Any two vertices in G are linked by a unique path

Proof.
"(ii) ⇒ (i)": trivial
"(i) ⇒ (ii)":
As G is connected, there is at least one path between any two vertices of G. Lets assume for the sake of contradiction that there are some vertices x and y that are joind by at least two paths P_1, P_2. As $P_1 \neq P_2$ there is and edge e_0 that lies in P_1 but not in P_2. We shall show that the graph $G - e_0$ is still connected, which will be a contradiction.

Idea is to consider the path between any two vertices u, v and remove the edge e_0 and still show we can reach v from v by using the vertices x and y which have two paths between them and simply put e_0 in one of them, then we can user the other one by going from one endpoint from e_0 to x, possible because G is connected. Then to y using the other path and then from y to the other endpoint of e_0 and finally to v. Thus u and v are still connected after removing e_0 a contradiction.

□

problem 6 Let $T_1, ..., T_k$ be subtrees of a tree T, any two of which have at least one vertex in common. Prove that there is a vertex common to all T_i.

Proof. Apply induction on $|T|$. If $|T| = 1$ then T consists of a single vertex, say v. If $T_1, ..., T_k$ are subtrees of T with pairwise intersecting vertex sets, then we must have $T_i = T$ for each $i = 1, ..., k$. It follows that v belongs to each T_i.

So assume $n \geq$ is an integer, and let us suppose this result holds for all trees of order $n - 1$. Suppose T is a tree with $|T| = n$ and $T_i, ..., T_k$ are subtrees of T. We assume $k \geq 2$. Let v be a leaf in T and let $T' = T - v$ be the tree resulting from removing this leaf. □

Problem 15 Let G be a nonempty connected graph and let $x \in V(G)$. For each $r \geq 0$ let $B_r = B_r(x)$ be the set of vertices in G distance exactly r from x. Prove that for some r:

$$\chi(G) \leq \chi(G[B_r]) + \chi(G[B_{r+1}])$$

Proof.
Proof sketch: color layer by layer use induction create proper coloring from from one with one less layer. Take new colors from $\{1, ..., \chi(B_r) + \chi(B_{r+1})\}$ these exists otherwise choice of r not maximal.

Proof:
Let $x \in V(G)$ and note that since G is connected, for som $t \geq 0$ the sets $B_0 = \{x\}, ..., B_t$ partition the vertex set. By the definition of each B_r, there are no edges in G between B_i and B_j whenever $|i-j| > 1$. In other words, the only edges in G are possibly inside each B_i and between consecutive $B_i's$. For simplicity of notation we write B_r for $G[B_r]$. Now let $r \geq 0$ be such that $\chi(B_r) + \chi(B_{r+1})$ is maximized and write

$$C = \{1, ..., \chi(B_r) + \chi(B_{r+1})\}$$

Our aim is to color G 'layer by layer'. Assign an arbitrary color from C to $B_0 = \{x\}$. Now assume that $B_0, ..., B_{k-1}$ have been colored for some $k \geq 1$ with $\chi(B_0), ..., \chi(B_{k-1})$ colors from C, respectiviely, such that the graph $G[B_0 \cup ... \cup B_{k-1}]$ is properly colored. Let us show how to color B_k with $\chi(B_k)$ colors from C such that $G[B_0 \cup ... \cup B_k]$ is properly colored. By assumption, there is some subset $C' \subseteq C$ of $\chi(B_{k-1})$ colors used to color B_{k-1} in the coloring constructed so far. We claim that $\chi(B_k) \leq |C \setminus C'|$. Indeed, if otherwise then

$$\chi(B_{k-1} + \chi(B_k) > |C'| + (|C| - |C'|) = |C| = \chi(B_r) + \chi(B_{r+1})$$

contradicting the maximal choice of r. Therefore we max choose a set of $\chi(B_k)$ colors from $C \setminus C'$. These colors do not appear in the coloring of B_{k-1}, so this is a legal coloring of the graph induced on $B_{k-1} \cup B_k$. As we observed earlier, there are no edges between non-consecutive B_i's so this defines a proper coloring of G with $|C| = \chi(B_r) + \chi(B_{r+1})$ colors. □

Problem 16 Let G be a connected graph with minimum degree $\delta(G) = k \geq 1$. Prove that G contains a path $x_1, ..., x_k$ such that $G - \{x_1, ..., x_k\}$ is also connected.

Proof sketch: consider longest path, suppose removing path of length k is disconnecting G, consider the two components. Consider a new longest path in one of the components starting at a vertex u with maximum number of nbhs, and we use that to build a longer path in G, from this we get a inequality which has to hold to not get contradiction, but if the inequality holds, then get contradiction of choice of u.

Proof. Consider a longest path $P = x_1, .., x_l$. As every neighbor of x_l lies on P, we must have $l \geq k + 1$ Suppose,by way of contradiction, that $G' = G - \{x_1, ..., x_k\}$ is disconnected. Therefore, G' has at least two components; let C be the component not containing x_{k+1} Let $u \in C$ have the maximum number, say

d, of neighbors in $\{x_1, ..., x_k\}$. Note that u cannot be joined to x_1 as otherwise we obtain a path longer than P. Also u has at least one neighbor in $\{x_1, ..., x_k\}$ (otherwise G is disconnected). It follows that $\delta(G[C]) \geq k - d > 0$ since vertices in C only have neighbors in $\{x_1, ..., x_k\} \cup C$ and $\delta(G) = k$ and $\delta(G) = k$. Let x_i be a neighbor of u for some $i \in [k]$ and consider a longest path Q in $G[C]$ that starts with u. This path has length at least $k - d$. Indeed, the last vertex of Q has all its neighbors in the path Q, and $\delta(G[Q]) \geq k - d$. Lets try to finde a longer path in G using Q. If u' denotes the last vertex of Q, consider the path $u'Qux_i...x_l$. It has length at least

$$(k - d) + 1 + (l - i)$$

As P was a longest path in G, this is a contradiction unless

$$(k - d) + 1 + (l - i) \leq l - 1$$

which is equivalent to $i \geq k - d + 2$. But then the neighborhood of u in $\{x_1, ..., x_k\}$ is contained in $\{x_{k-d+2}, ..., x_k\}$, which has size $k - (k - d + 2) + 1 = d - 1 < d$, contradictin our choice of u. Thus $G - \{x_1, ..., x_k\}$ must be connected. □

13 Exam Problems

13.1 summer 18

13.2 winter 17/18

13.3 summer 16

13.4 winter 15/16

13.5 winter 12/13

problem 1 Let $d = (d_1, ..., d_n)$ be a sequence of $n \geq 2$ positive integers with $\sum_{i=1}^{n} d_i = 2n - 2$. Show that there exists an n-vertex tree with degree sequence d.

Solution: Proof by induction on n : W.l.o.g. d is sorted a descending order. $n = 2$: because each element is positive and their sum is 2, we have $d = (1, 1)$. K_2 is a tree with this degree sequence. $n \geq 3$: because of the sum condition, there is at least one element of the sequence $d_j \geq 2$ and the last element d_n is 1. The sequence $d' = (d_1, ..., d_j - 1, ..., d_{n-1}$ fulfills the I.H., so there is a tree T' with vertices $v_1, ..., v_{n-1}$ of degrees $d'_1, ..., d'_{n-1}$. Then $T := T' + v_j v_n$ is a tree with n vertices and degree sequence d.

problem 2 Show that every 2-connected graph, which is not an odd cycle, contains an even cycle.

Solution: Let G be a 2-connected graph. Then G has an ear-decomposition, so there are graphs $G_1, ..., G_k = G$ such that G_1 is a cycle and G_{i+1} is constructed from G_i by adding a path with distinct endpoints in G_i, but not intersecting G_i otherwise.

If G_1 is an even cycle, we are done. Otherwise k is at least 2, since G is not an odd cycle. Let u, v be the endpoints of the path P used to construct G_2. Then there are two paths P_1, P_2 connecting u and v in G_1. Since G_1 is and odd cycle, w.l.o.g. P_1 is of even length, and P_2 is odd. If P is even, then $P_1 \cup P$ is an even cycle, otherwise $P_2 \cup P$ is an even cycle.

problem 3 Let G be a bipartite graph with parts A and B and let S be the set of vertices of maximum degree in G. Prove that there exists a matching M of G covering every vertex in $A \cap S$.

Solution: Let $A' \subseteq A \cap S$. Since all vertices in S are of maximum degree, in $N(A')$ there are $k|A'|$ edges arriving from A'. If $|N(A')| < |A'|$, then some vertex in $N(A')$ would need to have degree $> k$, a contradiction to the maximum degree. So for all $A' \subseteq A \cap S$, we have $|N(A')| \geq |A'|$. Thus the subgraph induced by $(A \cap S) \cup B$ has a perfect matching by Hall's thm.

problem 4 Show that the maximum number of triangles in an n-vertex outerplanar graph equals $n - 2$, provided $n \geq 3$.

Solution: Proof by induction on n, any graph on 3 vts contains at most one triangle, so let $n \geq 4$ and G be a outerplanar graph on n vts.

Claim: G contains a vertex v of degree ≤ 2

If G is not maximally outerplanar, add edges until this is the case. Then the outer face is a cycle and every inner face is a triangle. We thus can find an edge on the inside whose vertices are of distance 2 on the cycle (otherwise we could add another edge), the middle vertex on this path is of degree 2. Now $G - v$ is outerplanar with at most $n - 3$ triangles (I.H.). By removing v, we removed at most one triangle, so G has at most $n - 2$ triangles, thus proving the upper bound.

We give a construction for the lower bound:

Let G be outerplanar and uv an edge on the outer face. Add a new vertex w on the outside of G and add the edges uw, vw to G, thus creating a new triangle, but not removing any vertices from the outer face.

problem 7 Prove that almost all graphs G in $\mathcal{G}(n, \frac{1}{2})$ fulfill $\chi(G) \geq \sqrt{n}$

Solution: Since each color class in a graph froms an independent set, the chromatic number of an n-vertex graph G can be bounded by $\chi(G) \geq \frac{n}{\alpha(G)}$.

For a random graph $G \in \mathcal{G}(n, \frac{1}{2})$ we thus have

$$\mathrm{P}(\chi(G) \leq \sqrt{n}) \leq \mathrm{P}(\alpha(G) \geq \sqrt{n}) \leq \binom{n}{\sqrt{n}} 2^{-\binom{\sqrt{n}}{2}} \leq n^{\sqrt{n}} 2^{\sqrt{n}-n} = \frac{(2n)^{\sqrt{n}}}{2^n} \underset{n\to\infty}{\to} 0$$

problem 8 Prove that for $n \geq 5$, every triangle-free non-bipartite graph on n vertices contains at most $\lfloor \frac{(n-1)^2}{4} \rfloor + 1$ edges.

Solution: PRoof by contradiction: Assume we have a edge-maximal triangle-free non-bipartite graph G on n vertices and at least $\lfloor \frac{(n-1)^2}{4} \rfloor + 2$ edges. Since G is non-bipartite, it contains an odd cycle. Consider a shortest odd cycle C.

Claim 1: C is of length 5.

Assume C was of length ≥ 7, then we could split the cycle into an even path P_1 of length 4 and an odd path P_2 of length ≥ 3. Denote the endpoints of the paths by u and v. Then uv cannot be an edge of G, because otherwise $P_1 + uv$ would be a shorter odd cycle. If u and v have a common neighbor w, then $P_2 + uw + wv$ would be a shorter odd cycle. Adding uv could thous not induce a triangle, so $G + uv$ would still be triangle-free and non-bipartite, so G could not have been edge-maximal, contradicting our assumption.

Claim 2: Removing C from G removes at most $2n - 5$ edges.
Any vertex in $G - C$ can be connected to at most 2 vts of C, because otherwise, two neighboring vertices in C would share a neighbor in $G - C$, thus inducing a triangle. We thus lose at most $2(n-5)$ edges outside C and 5 edges in C, which gives us the claim.

From these claims, we can derive that we can always remove a C_5 from G without losing more than $2n - 5$ edges. We have $(n-1)^2/4 + 2 - (2n-5) = (n-5)^2/4 + 1$ For odd n, the number of edges in $G - C$ is greater than $t_2(n-5) = (n-5)^2/4$, so by Turan's theorem, $G - C$ contains a triangle, a contradiction. For even n the number of edges in G is at least $\lceil \frac{(n-1)^2}{4} \rceil + 1$, so $G - C$ contains at least $\lceil (n-5)^2/4 \rceil > \lceil (n-5)/2 \rceil \lfloor (n-5)/2 \rfloor = t_2(n-5)$ edges. By Turans theorem G cannot be triangle free, a contradiction.

We thus have shown that such a graph can have at most $\lfloor (n-1)^2/4 \rfloor + 1$ edges, otherwise it would not be triangle-free.

problem 9 Prove that for every k there exists an $n = n(k)$ such that any set of n points in the plane contains a subset of k points that can be covered with a disc of radius 1, or a subset of k points which cannot be covered with less than k discs of radius 1 each.

Solution:

problem 10
For every k there exists an $m = m(k)$ such that every graph on m edges contains a vertex of degree at least k or a matching on at least k edges.

There is a 3-connected graph on n vertices with at most $2n$ edges.

For every $k \geq 3$ there exists a $\chi = \chi(k)$ such that every graph G

Any n-vertex graph with at least $\frac{n^2}{4}$ eges is Hamiltionian.

13.6 winter 11/12