Graphtheory Theorems

1 Important Graphs

Complete Graph, Clique:
the *complete graph* K_n on n vertices is isomorphic to $([n], \binom{[n]}{2})$

Cycle
C_n on n vertices with $n \geq 3$ is isomorphic to $([n], \{\{i, i+1\} : i = 1, ..., n-1\} \cup \{n, 1\})$, the *length of a cycle* is its number of edges.

Empty Graph
E_n on n vertices is isomorphic to $([n], \varnothing)$.
Empty graphs correspond to *independent sets.*

Complete Bipartite Graph
$K_{m,n}$ on $n+m$ vertices is isomorphic to $(A \cup B, \{xy : x \in A, y \in B\})$ where $|A| = m$ and $|B| = n$ and $A \cap B = \varnothing$.

Complete r-partite graph with $r \geq 2$ is isomorphic to

$$(A_1 \uplus ... \uplus A_r, \{xy : x \in A_i, y \in A_j, i \neq j\})$$

where $A_1, ..., A_r$ are disjoint non-empty finite sets.

n-dimensional hypercube Q_n is the graph isomorphic to

$$(2^{[n]}, \{\{S, T\} : S, T \in 2^{[n]}, |S \Delta T| = 1\})$$

Vertices are labeled either by corresponding sets or binar indicator vectors. For example the vertex $\{1, 3, 4\}$ in Q_6 is coded by $(1, 0, 1, 1, 0, 0)$. A 1 is indicating we take this Element and a 0 if not.

k-uniform hypergraph is a hypergraph where all hyperedges have the same cardinality

2 Basics

Definitions

- A graph G is *non-trivial* if it contains at least one edge, equivalently if G is not an empty graph
- The *order* of G writen $|G|$, is the number of vertices of G, i.e. $|G| = |V|$
- The *size* of G wirten $||G||$, is the number of edges of G, i. e. $||G|| = |E|$, if order of G is n then the size of G is between 0 and $\binom{n}{2}$
- $N(S)$ the *neighbourhood* of $S \subseteq V$ is the set of vertices in V. that have and adjacent vertex in S. Instead of $N(\{v\})$ for $v \in V$ we write $N(v)$

- vertex of degree 1 is called *leaf*
- vertex of degree 0 is called *isolated vertex*
- *minimum degree of* G, denoted by $\delta(G)$ is the smallest vertex degree in G
- *maximum degree of* G, denoted by $\Delta(G)$ is the highest vertex degree in G
- graph G is called *k-regular*, with $k \in \mathbb{N}$, if all vertices have degree k.
- *average degree of* G is defined as $d(G) = \frac{\sum_{v \in V} deg(v)}{|V|}$
  We have
  $$\delta(G) \leq d(G) \leq \Delta(G)$$
  with equality if and only if G is k-regular

Handshake Lemma For ever graph $G = (V, E)$ we have

$$2|E| = \sum_{v \in V} d(v)$$

Proof. By double counting the set $X = \{(e, x) : e \in E(G), x \in V(G), x \in e\}$ then

$$|X| = \sum_{v \in V(G)} d(x)$$

and

$$|X| = \sum_{e \in E(G)} 2 = 2|E(G)|$$

by the principle of double counting the terms are equal. □

Corollary From this follows that the sum of all vertex degrees is even and therefore the number of vertices with odd degree is even.

Proposition 3 If a graph G has minimum degree $\delta(G) \geq 2$, then G has a path of length $\delta(G)$ and a cycle with at least $\delta(G) + 1$ vertices.

Proof. Let $P = (x_0, ..., x_k)$ be a longest path in G. Then $N(x_0) \subseteq V(P)$, otherwise for $x \in N(x_0) \setminus V(P)$ the path $(x, x_0, x_1, ..., x_k)$ would be a longer path.

Let i be the largest index such that $x_i \in N(x_0)$, then $i \geq |N(x_0)| \geq \delta$. So $(x_0, x_1, ..., x_i, x_0)$ is a cycle of length at least $\delta(G) + 1$. □

Proposition 4 If for distinct vertices u and v a graph has a u-v-walk, then it has a u-v-path.

Proof. Consider a u-v-walk W with the smallest number of edges. Assume that W does not form a path, then there is a repeated vertex, w, i.e.

$$W = u, e, v_1, e_1, ..., e_k, w, e_{k+1}, ..., e_l, w, e_{l+1}, ..., v$$

Then $W_1 = u, e, v_1, ..., e_k, w, e_{k+1}, ..., v$ is a shorter u-v-walk, a contradiction. □

Propostion 5 If a graph has a closed walk of odd length, then it contains an odd cycle.

Proof. Let W be the shortest closed odd walk. If W is a cycle the Proposition holds. Otherwise there is a repeated vertex, so W is an edge-disjoint union of two closed walks. The sum of the lengths of these walks is odd, therefore one of them is an odd closed walk shorter than W a contradiction to the minimality of W. □

Proposition 6 If a graph has a closed walk with a non-repeated edge, then the graph contains a cycle.

Proof. Let W be a shortest walk with a non-repeated edge e. If W is a cycle, we are done. Otherwise, there is a repeated vertex and W is a union of two closed walks W_1 and W_2 that are shorter than W. One of them say W_1, contains e, a non-repeated edge. This contradicts the minimality of W. □

Definition bipartite A graph $G = (V, E)$ is called *bipartite* if there exists natural numbers m, n such that G is isomorphic to a subgraph of $K_{m,n}$. Then the vertex set can be written as $V = A \cup B$ such that $E \subseteq \{ab : a \in A, b \in B\}$. The sets A and B are called the *partite sets* of G

Proposition 1.5 A graph is bipartite if and only if it has no cycles of odd length.

Proof. skript
"$\Rightarrow$"
Assume that G is a bipartite graph with parts A and B. Then any cycle has a form $a_1, b_1, a_2, b_2, ..., a_k, b_k, a_1$ where $a_i \in A, b_i \in B, i \in [k]$. Thus every cycle has even length.
"$\Leftarrow$"
Assume G does not have cycles of odd length. We can assume that G is connected, otherwise we can treat the connected components separately. Let $v \in V(G)$. Let $A = \{u \in V(G) : dist(u, v) \equiv 0 \pmod 2\}$ and let $B = \{u \in V(G) : dist(u, v) \equiv 1 \pmod 2\}$ We claim that G is bipartite with parts A and B. To verify this it is sufficient to prove that A and B are independent sets. Let $u_1u_2 \in E(G)$ and let P_1 be a shortest u_1-v-path and P_2 a shortest u_2-v-path. Then the union of P_1, P_2 and u_1u_2 forms a closed walk W. If $u_1, u_2 \in A$ or $u_1, u_2 \in B$ then W is a closed odd walk, because $dist(v, u_1)$ and $dist(v, u_2)$ are both even or odd. Thus by Prop. 5 G contains an odd cycle, a contradiction. Thus for any edge u_1u_2 the adjacent vertices u_1 and u_2 are in different parts A or B. Therefore A and B are independent sets. □

Proof. Diestel
"$\Leftarrow$"
Let T be a spanning tree in G, pick a root $r \in T$ and denote the associated tree-order on V by $\leq_T$ (this order expressing height if $x < y$ then x lies *below* y in T). For each $v \in V(G)$ the unique path r-v-T has odd or even length. This defines a bipartition of $V(G)$, we show that G is bipartite with this partition. Let $e = xy$ be an edge of G. If $e \in T$ with $x <_T y$ say, then r-y-T = r-xy-T

and so x and y lie in different partition classes. If $e \notin T$ then $C_e := x\text{-}y\text{-}T + e$ is a cycle, and by the case treated already the vertices along $x\text{-}y\text{-}T$ alternate between the two classes. Since C_e is even by assumption, x and y again lie in different classes. □

Euler tour A closed walk that traverses every edge of the graph exactly once is called an *Euler tour*.

Theorem 1.6 (Eulerian Tour Condition) A connected graph has an Eulerian Tour if and only if every vertex has even degree.

Proof. "$\Rightarrow$"
The degree condition is necessary for an euler tour, because a vertex appearing k times in an Euler tour (or $k+1$ times if it is the starting and finishing vertex) must have degree $2k$.
"$\Leftarrow$"
Show by induction on $\|G\|$ that every connected Graph G with all degrees even has an Euler tour. $\|G\| = 0$ is trivial.

Now let $\|G\| \geq 1$, since all degrees are even, we can find in G a non-trivial closed walk that contains no edge more than once. To find this walk we consider W a walk of maximal length and write F for the set of its edges. If $F = E(G)$, then W is an Euler tour.

Suppose, therefore $G' := G - F$ has an edge.
For every vertex $v \in G$, an even number of edges of G at v lies in F, so the degrees of G' are again all even. Since G is connected, G' has an edge e incident with a vertex on W. By I.H. the component C of G' containing e has an Euler tour. Concatenating this with W (suitably re-indexed), we obtain a closed walk in G that contradicts the maximal length of W.

□

Definitions

- graph G is *connected* if any two vertices are linked by a path.
- a maximal connected subgraph of G is called a *connected component* of G.
- acyclic graphs are called *forests*
- a graph G is called a *tree* if G is connected and acyclic.

Lemma 7 Every tree on at least two vertices has a leaf.

Proof. If a tree T on at least two vertices does not have leaves then every vertex has degree $>$ than 2, so we have a cycle in T with length ≥ 3, a contradiction. □

Lemma 8 A tree of order $n \geq 1$ has exactly $n - 1$ edges.

Proof. We prove the statement by induction on n. When $n = 1$, there are no edges.

I.H.: Assume that each tree on $n = k$ vertices has $k - 1$ edges, with $k \geq 1$.

Step: Lets prove that each tree on $k + 1$ vertices has k edges. Consider a tree T on $k + 1$ vertices. Since $k + 1 \geq 2$, T has a leaf v. Let $T' = T - \{v\}$. We see that T' is connected because any u-w-path in T, for $u \neq v$ and $w \neq v$, does not contain v. We see also that T' is acyclic, because deleting vertices from an acyclic graph does not create new cycles. Thus T' is a tree on k vertices. By I.H. $|E(T')| = k - 1$. Thus $|E(T)| = |E(T')| + 1 = (k - 1) + 1 = k$.

□

Lemma 9 Every connected graph contains a spanning tree.

Proof. Let G be a connected graph. Consider T, an acyclic spanning subgraph of G with largest number of edges. If it is a tree we are done.

Otherwise, T has more than one component. Consider vertices u and v from different components of G. Consider a shortest u-v-path P in G. Then P has an edge $e = xy$ with exactly one vertex x in one of the components of T. Then P has an edge $e = xy$ with exactly one vertex x in one of the components of T. Then $T \cup \{e\}$ is acyclic. If there would be a cycle, it would contain e, however e connects to components, therefore cannot be part of a cylce (e would be a repeated edge). Thus $T \cup \{e\}$ is a bigger spanning acyclic subgraph of G contradicting the maximality of T. □

Lemma 10 A connected graph on $n \geq 1$ vertices and $n - 1$ edges is tree.

Proof. Let G be a connected graph on n vertices with $n - 1$ edges. Assume G is not a tree, i.e. contains a cycle. We therefore can remove a edge so that G is still connected. This is a contradiction because a graph on n vertices with $n - 2$ edges cannot be connected. Because a walk from vertex 1 to vertex n has to have at least $n - 1$ edges. □

Lemma 11 The vertices of every connected graph on $n \geq 2$ vertices can be ordered $(v_1, ..., v_n)$ so that for every $i \in \{1, ..., n\}$ the Graph $G[\{v_1, ..., v_i\}]$ is connected.

Proof. skript
Let G be a connected graph on n vertices. It contains a spanning tree T. Let v_n be a leaf of T, let v_{n-1} be a leaf of $T - \{v_n\}$, let v_{n-2} be a leaf of $T - \{v_n, v_{n-1}\}$ and so on. Let v_k be a leaf in $T - \{v_n, v_{n-1}, ..., v_{k+1}\}$, $k = 2, ..., n$. Since deleting a leaf does not disconnect a tree, all resulting graphs form a spanning trees of $G[v_1, ..., v_i]$, $i = 1, ..., n$. A graph H having a spanning tree or any connected spanning subgraph H' is connected because a u-v-path in H' is a u-v-path in H. This observation completes the proof. □

Proof. diestel
Pick any vertex as v_1, and assume inductively that $v_1, ..., v_i$ have been chosen for some $i < |G|$. Now pick a vertex $v \in G - G_i$. As G is connected, it contains

a v-v_1 path P. Choose v_{i+1} as the last vertex of P in $G - G_i$, then v_{i+1} has a neighbour in G_i. If we consider $i + 1$ then we simply add v_{i+1} to our G_i, Thus $G_{i+1} := G_i \cup \{v_{i+1}\}$ which is also connected. □

Tree equivalences For any graph $G = (V, E)$ the following are equivalent:

1. G is a tree, i.e. G is connected and acyclic.
2. G is connected, but for any $e \in E$ the graph $G - e$ is not connected (minimally connected)
3. G is acyclic, but for any $x, y \in V(G), xy \notin E(G)$ the graph $G + xy$ has a cycle. (maximaly acyclic)
4. G is connected and 1-degenerate
5. G is connected and $|E| = |V| - 1$
6. G is acyclic and $|E| = |V| - 1$
7. G is connected and every non-trivial subgraph of G has a vertex of degree at most 1.
8. Any two vertices are joined by a unique path in G.

Proof.
(i) ⇒ (ii):
G is connected and acyclic, now assume for any edge $e = xy$ the graph $G' = G - e$ would still be connected. Then G' has a x-y-path P. But $P \cup e$ is a cycle in G which contradicts that G is acyclic.
(ii) ⇒ (i):
G is connected and for any edge e the graph $G - e$ is not connected. We want to show that G is acyclic. If G would have a cycle we could simply remove an edge from that and the resulting graph would still be connected, a contradiction. □

Proof.
TODO complete all equivalences for exam
(i) ⇒ (iv):

(vi) ⇒ (i):

(i) ⇒ (vii):

(vii) ⇒ (i):

(i) ⇒ (viii):

(viii) ⇒ (i):

□

Definition d-degenerate If there is a vertex ordering $v_1, ..., v_n$ of G and a $d \in \mathbb{N}$ such that

$$|N(v_i) \cap \{v_{i+1}, ..., v_n\}| \leq d$$

for all $i \in [n-1]$ then G is called *d-degenerate*. The minimum d for which G is *d-degenerate* is called the *degeneracy* of G.

Every finite planar graph has a vertex of degree five or less, therefore every planar graph is 5-degenerate.

Definition arboricity The least number of trees that can cover the edges of a graph is its arborictiy.

It is a measure for the graphs maximum local density: it is small if and only if the graph is nowhere dense, in the sense that there is no subgraph H with large $\epsilon(H) = \frac{E(H)}{V(H)}$.

Definition Contract For an edge $e = xy$ in G we define $G \circ e$ as the graph obtained from G by identifying x and y and removing (if necessary) loops and multiple edges. We say that $G \circ e$ arises from G by *contracting the edge e*.

Definition Complement The *complement* of G, denoted by $\overline{G}$ is defined as the graph $(V, \binom{V}{2} \setminus E)$. In particular $G + \overline{G}$ is a complete graph and $\overline{G} = (G + \overline{G}) - E$.

Definitions

- *girth of* G, denoted by $g(G)$ is the length of the shortest cycle in G, if G is acyclic, its girth is said to be ∞
- *circumference of* G, is the length of the longest cycle if G is acyclic the circumference is said to be 0
- G is called *Hamiltonian* if G has a spanning cycle, i.e. a cycle that contains every vertex of G. In other words the circumference is $|V|$
- G is called *traceable* if G has a spanning path
- For two vertices v and u in G, the *distance between u and v*, denoted by $d(v, u)$ is the length of a shortest u-v-path in G. If no such path exists $d(u, v) = \infty$
- The *diameter* of G, denoted by $\text{diam}(G)$, is the maximum distance among all pairs of vertices in G, i.e.

$$\text{diam}(G) = \max_{u,v \in V} d(u, v)$$

- *eccentricity*, $\text{ecc}(v)$ is the greatest distance of v to any other vertex.
- The *radius of G*, denoted by $\text{rad}(G)$ is defined as

$$\text{rad}(G) = \min_{u \in V} \max_{n \in V} d(u, v)$$

  its the vertex that has the smallest eccentricity

3 Matchings

Definitions

- *matching* is a 1-regular graph, i.e. a matching is a graph M so that $E(M)$ is a union of pairwise non-adjacent edges and $2|E(M)| = |V(M)|$
- a matching in G is a subgraph of G isomorphic to a matching. We denote the size of the largest matching in G by $\upsilon(G)$
- a *vertex cover* in G is a set of vertices $U \subseteq V$ such that each edge in E is incident to at least one vertex in U. We denote the size of the smallest vertex cover in G by $\tau(G)$
- a *k-factor of G* is a k-regular spanning subgraph of G.
- A *1-factor of G* is also called a *perfect matching* since it is a matching of largest possible size of order $|V|$. Clearly G can only contain a perfect matching if $|V|$ is even.

Theorem 2.2 (Hall's Marriage Theorem)
Let G be a bipartite graph with partite sets A and B. Then G has a matching containing all vertices of A if and only if $|N(S)| \geq |S|$ for all $S \subseteq A$

Proof.
$\Rightarrow$:
If G has a matching M containing all vertices of A, then for any $S \subseteq A, N(S)$ in G is at least as large as $N(S)$ in M, thus $|N(S)| \geq |S|$.

$\Leftarrow$:
We shall prove by induction on $|A|$ that any bipartite graph with parts A and B satisfying Hall's condition has a matching containing all vertices of A, in other words, saturating A.

When $|A| = 1$, there is at least one edge in G and thus a matching saturating A. Assume that the statement is true for all graphs G satisfying Hall's condition and with $|A| = k \geq 1$.

Now consider a bipartite graph G with $|A| = k+1$ and satisfying Hall's condition.

Case 1: $|N(S)| \geq |S| + 1$ for any $S \subsetneq A$.
Let $G' = G\{x, y\}$, for some edge xy. G' has parts $A' = A - \{x\}$ and $B' = B - \{y\}$. For any $S \subseteq A'$, $|N_{G'}(S')| \geq |N_G(S)| - 1 \geq |S| + 1 - 1 = |S|$. Thus G' satisfies Hall's condition and by induction has a matching M' saturating A'. Then $M = M' \cup \{xy\}$ is a matching in G saturating A.

Case 2: $|N(S_1)| = |S_1|$ for some $S_1 \subsetneq A$.
Let $A' = S_1, B' = N(A'), G' = G[A' \cup B']$. Since $|A'| < |A|$, and G' satisfies Hall's condition, G' has a matching M' saturating A' by induction. Now consider $A'' = A - A'$, $B'' = B - B'$, $G'' = G[A'' \cup B'']$. We claim G'' also satisfies Hall's condition. Assume not, and there is $S \subseteq A''$ such that $|N_{G''}(S)| < |S|$. Then $|N_G(S \cup A') = |B' \cup N_{G''}(S)| = |B'| + |N_{G''}(S)| < |A'| + |S| = |A' \cup S|$, a contradiction to Hall's condition. Thus there is a Matching M'' in G'' and $M' \cup M''$ is a matching saturating A in G. $\square$

Corollary 12 Let G be a bipartite graph with partite sets A and B such that $|N(S)| \geq |S| - d$ holds for all $S \subseteq A$, and for a fixed positive integer d. Then G contains a matching of size at least $|A| - d$.

Proof. TODO □

Corollary 13 If G is a regular bipartite graph, it has a perfect matching.

Proof. Let $k \in \mathbb{N}$ and let G be a k-regulat bipartite graph with parts A and B. Then $|E(G)| = k|A| = k|B|$, and thus $|A| = |B|$. Consider $S \subseteq A$, let r be the number of edges between S and $N(S)$. On one hand, $r = |S|k$, on the other hand $r \leq |N(S)|k$. Thus $|N(S)| \geq |S|$ and by Hall's theorem there is a matching saturating A. Since $|A| = |B|$, it is a perfect matching. □

Corollary 14 A k-regular bipartite graph has a proper k-edge-coloring.

Proof. TODO □

Theorem 2.3 (Kőnig's Theorem)
Let G be bipartite. Then the size of a largset matching is the same as the size of a smallest vertex cover.

Proof. Let c be the vertex-cover number of G and m be the size of the largest matching of G. Since a vertex cover should contain at least one vertex from each matching edge, $c \geq m$.

Now, we shall prove that $c \leq m$. Let M be a largest matching in G, we need to show that $c \leq |M|$. Let A and B be the partite sets of G. An *alternating path* is a path that starts with a vertex in A not incident to an edge of M, and alternates between edges in M and edges in M. Note that an alternating path must end in a vertex saturated by M, otherwise one can find a larger matching. Let

$$U' = \{b : ab \in E(M) \text{ for some } a \in A \text{ and some alternating path ends in } b\}$$

$$U = U' \cup \{a : ab \in E(M), b \notin U'\}$$

We see that $|U| = m$. We shall show that U is a vertex cover, i.e. that every edge of G contains a vertex from U. Indeed, if $ab \in E(M)$, then either a or b is in U. If $ab \notin E(M)$, we consider the following cases:

Case 0: $a \in U$. We are done.

Case 1: a in not incident to M. Then ab is an alternating path. If b is also not incident to M then $M \cup \{ab\}$ is a larger matching, a contradiction. Thus b is incident to M and then $b \in U$.

Case 2: a is incident to M. Then $ab' \in E(M)$ for some b'. Since $a \notin U$, we have that $b' \in U$, thus there is an alternating path P ending in b'. If P contains b, then $b \in U$, otherwise $Pb'ab$ is an alternating path ending in b, so $b \in U$.

□

Theorem 2.4 (Tutte's Theorem)
A graph G has a perfect matching if and only if $q(G-S) \leq |S|$ for all $S \subseteq V$. We define $q(H)$ to be the number of odd components of H, i.e. the number of connected components of H consisting of an odd number of vertices.

Proof.
$\Rightarrow$:
Assume first that G has a perfect matching M. Consider a set S of vertices and an odd component G' of $G-S$. We see that there is at least one vertex in G' that is incident to an edge of M that has another endpoint not in G'. This endpoint must be in S. Thus $|S|$ is at least as large as the number of odd components.

$\Leftarrow$:
Now, assume that $q(G-S) \leq |S|$ for all $S \subseteq V$. Assume that G has no perfect matching and $|V(G)| = n$. Note that $|V(G)|$ is even (it follows from the assumption $q(G-S) \leq |S|$) applied to $S = \varnothing$). Let G' be constructed from G by adding missing edges as long as no perfect matching appears. Let S be a set of vertices of degree $n-1$. Note that it could be empty.

Claim 1: Each component of $G' - S$ is complete. Assume not, there is a component □

4 Connectivity

Definitions

- for a natural number $k \geq 1$, a graph G is called *k-connected* if $|V(G)| \geq k+1$ and for any set U of $k-1$ vertices in G the graph $G-U$ is connected. In particular K_n is $(n-1)$-connected.

  this implies: a graph is k-connected if any two of its vertices can be joined by k independent paths

- a graph G is called *k-linked* if $|G| \geq 2k$ and for any $2k$ distinct vertices $s_1, ..., s_k, t_1, ..., t_k$ there are vertex disjoint s_i-t_i-paths, $i = 1, ..., k$

Lemma 3.2 For any connected, non-trivial graph G we have

$$\kappa(G) \leq \kappa'(G) \leq \delta(G)$$

Proof. If G is complete $\kappa(G) = \kappa'(G) = \delta(G) = n-1$

Assume G is not complete:
$\kappa'(G) \leq \delta(G)$:
Simply remove all the edges of a vertex with minimum degree.

$\kappa(G) \leq \kappa'(G)$:
Consider smallest separating set of edges F:

case1:
there is a vertex v not incident to F, then the vertices incident to F separate this vertex from any vertex in the other component.

case2:
Every vertex is incident to F, consider v of degree $< |G| - 1$, exists because G is not complete. Show the neighbourhood of v is less than $|F|$ so $N(v)$ is a separating set. □

Definition A subset X of vertices and edges of G seperates two vertex sets A, B if each A-B-path (starts in A ends in B) contains an element of X.

Theorem 3.3 (Menger's Theorem) For any graph G and any two vertex sets $A, B \subseteq V(G)$, the smallest number of vertices separating A and B is equal to the largest number of disjoint A-B-paths.

Theorem 3.4 (Global Version of Menger's Theorem) A graph G is k-connected if and only if for any two vertices a, b in G there exist k independent a-b-paths.

Note that Menger's Theorem implies that if G is k-linked, then G is k-connected.

Definition: Line Graph For a graph $G = (V, E)$ the line graph $L(G)$ is the graph $L(G) = (E, E')$ where

$$E' = \{\{e_1, e_2\} \in \binom{E}{2} : e_1 \text{ adjacent to } e_2 \text{ in } G\}$$

Corollary 24 If a, b are vertices of G, then

$$\min \#\text{edges separating } a \text{ and } b = \max \#\text{edge-disjoint } a\text{-}b\text{-paths}$$

Definition

- *H-path*: Given a graph H, we call a path P an $H-path$ if P is non-trivial and meets H exactly in its ends. Such a path is also called an *ear* of the graph $H \cup P$.
- An *ear-decomposition* of a graph G is a sequence $G_0 \subseteq G_1 \subseteq \ldots \subseteq G_k$ of graphs, such that
  - G_0 is a cycle
  - for each $i = 1, \ldots, k$ the graph G_i arises from G_{i-1} by adding a G_{i-1}-path P_i, i.e. P_i is an ear of G_i
  - $G_k = G$

Theorem 25 (Ear-decomposition) A graph G is 2-connected if and only if it has an ear decomposition starting from any cycle of G.

Proof. TODO □

Lemma 26 If G is 3-connected with $G \neq K_4$, then there exists an edge e of G such that $G \circ e$ is also 3-connected.

Proof. TODO □

Theorem 3.6 (Tutte) A graph G is 3-connected if and only if there exists a sequence of graphs $G_0, G_1, ..., G_k$, such that

- $G_0 = K_4$
- for each $i = 1, ..., k$ the graph G_i has two adjacent vertices x', x'' of degree at least 3, so that $G_{i-1} = G_i \circ x^{prime} x''$
- $G_k = G$

Proof. TODO □

Theorem 27 (Mader) Every graph $G = (V, E)$ of average degree at least $4k$ has a k-connected subgraph.

Proof. □

Definition

- let G be a graph, a maximal connected subgraph of G without a cut vertex is called a *block* of G. In particular, the blocks of G are exactly the bridges and the maximal 2-connected subgraphs of G.
- *block-cut-vertex graph or block graph* of G is a bipartite graph H whose partite sets are the *blocks* of G and the cute vertices of G respectively. There is an edge between a block B and a cut vertex a if and only if $a \in B$, i.e. the block contains the cut vertex. The leaves of this graph are called $leaf\ blocks$.

Theorem 28 The block-cut-vertex graph of a connected graph is a tree.

5 Planar graphs

planar graph vs. plane graph Plane Graph is topological object $(V, E), V \subseteq \mathbb{R}^2$, $e \in E$ are arcs in $\mathbb{R}^2$. Planar graph is combinatorial object $(V, E), E \subseteq \binom{n}{2}$ s.t. it has a plane graph realization (planar embedding)

Definition outerplanar A graph G is *outerplanar* if it has a plane embedding such that the boundary of the outer face contains all vertices V.

Theorem 32 (Plane triangulation) A graph of order at least 3 is maximally plane if and only if it is a plane triangulation.

Proof. TODO □

Theorem 4.2 (Euler's Formula) Let G be a connected plane graph with n vertices, m edges and l faces. Then

$$n - m + l = 2$$

Proof. TODO □

Corollary 33 A plane graph with $n \geq 3$ vertices has at most $3n - 6$ edges. Every plane triangulation has exactly $3n - 6$ edges.

Proof. TODO □

Theorem 4.4 (Kuratowski's Theorem) The following statements are equivalent for graphs G:

1. G is planar
2. G does not have K_5 or $K_{3,3}$ as minors
3. G does not have K_5 or $K_{3,3}$ as topological minors

Theorem 4.7 (5-Color Theorem) Every planar graph is 5-colorabel

Proof. TODO □

Definition List coloring

- Let $L(v) \subseteq \mathbb{N}$ be a list of colors for each vertex $v \in V$. We say that G is *L-list-colorable* if there is a coloring $c : V \to \mathbb{N}$ such that $c(v) \in L(v)$ for each $v \in V$ and adjacent vertices receive different colors.
- Let $k \in \mathbb{N}$. We say that G is *k-list-colorable* or *k-choosable* if G is L-list-colorable for each list L with $|L(v)| = k$ for all $v \in V$
- the *choosability*, denoted by $\mathrm{ch}(G)$ is the smallest k such that G is k-choosable.
- the *edge choosability*, denoted by $\mathrm{ch}'(G)$ is the smallest k such that G is L-edge-list-colorable for each list L with $|L(e)| = k$ for $e \in E$

Theorem 4.10 (5-List-Color Theorem) Let G be a planar graph. Then the list chromatic number of G is at most 5.

Proof. TODO □

6 Colorings

Note that a k-coloring is nothing but a vertex partition into k independent sets, now called *colorclasses*; the non-trivial 2-colourable graphs, are precisely the bipartite graphs. The chromatic number is a key parameter in many extremal question, therefore it is studied a lot.

Definitions

- *vertex coloring* of a Graph G is a map $c : V \to S$ such that $c(v) \neq c(w)$ whenever v and w are adjacent. The elements of the set S are called available colors. The minimal $k = |S|$ such that G has a k-coloring is the *chromatic number* of G, denoted by $\chi(G)$. A graph with $\chi(G) = k$ is called k-chromatic.
- edge coloring is a map $c : E \to S$ with $c(e) \neq c(f)$ for any adjacent edges e, f. We say the *edge-chromatic number*, or *chromatic index* of G is the minimal k for which a k-edge-coloring exists, it is denoted by .

Relationship of $\chi(G)$ and $\chi'(G)$ Every edge coloring of G is a vertex coloring of its line graph $L(G)$, and vice versa, in particular $\chi'(G) = \chi(L(G))$. The problem of finding good edge colorings may thus be viewed as a restriction of the more general vertex coloring problem to this special class of graphs. There are only very rough estimates for χ but χ' always takes one ot two values, either Δ or $\Delta + 1$

Definitions

- *clique number* $\omega(G)$ of G is the largest order of a clique in G
- *co-clique number* $\alpha(G)$ of G is the largest order of an independent set in G
- Graph is *perfect* if $\chi(H) = \omega(H)$ for each induced subgraph H of G. For example bipartite graphs are perfect with $\chi = \omega = 2$

Lemma 46 (Simple Coloring Results)
For any graph G the following hold:

- $\chi(G) \geq max\{\omega(G), \frac{|G|}{\alpha(G)}\}$
- $||G|| \geq \binom{\chi(G)}{2}$ and $\chi(G) \leq 1/2 + \sqrt{2||G|| + 1/4}$

Greedy coloring algorithm

Proof. TODO □

Lovasz Perfect Graph Theorem A graph G is perfect if and only if its complement $\overline{G}$ is perfect.

Strong Perfect Graph Theorem A graph G is pefect if and only if it does not contain an odd cycle on at least 5 vertics (an *odd hole*) or the complement of an odd hole as an induced subgraph.

Graphs with $\omega(G) \leq \chi(G)$ exist Meaning $\exists G : \omega << \chi(G)$, we have 3 proofs

- Mycielski's construction: TODO
- Tutte's construction: TODO
- Erdős-Hajnal theorem: girth greater than k and chromatic number greater than k

Theorem 5.1 (Brook's Theorem) Let G be a connected graph. Then $\chi(G) \leq \Delta(G)$ unless G is a complete graph or and odd cycle.

Proof. TODO □

Theorem 54 (Kőnig, 1916) If G is a bipartite graph with maximum degree Δ then $\chi'(G) = \Delta$.

Proof. TODO □

Theorem 5.4 (Vizing's Theorem) For any graph G with maximum degree Δ

$$\Delta \leq \chi'(G) \leq \Delta + 1$$

Proof. TODO □

Lemma 55 The list chromatic number of $G = K_{n,n}$, with $n = \binom{2k}{k}$ is at least $k + 1$.
This means there exists a graph which has much greater list-chromatikc number than chromatic number

7 Extremal graph theory

In this chapter we study how global parameters of a graph, such as its edge density or chromatic number, can influence its local substructures. Two categories of questions:

- global assumptions that might imply that H as a *minor* (or topological minor) it will suffice to raise $\|G\|$ above the value of some linear function of $|G|$.
- global assumptions that might imply the existence of some given graph H as a *subgraph*

Definition

- The *extremal number* $\text{ex}(n, H)$ denotes the maximum size (amount of edges) of a graph of order n that does not contain H as a subgraph and $\text{EX}(n, H)$ is the set of H-free graphs on n vetices with $\text{ex}(n, H)$ edges.
- Example: $\text{ex}(n, P_3) = \lfloor \frac{n}{2} \rfloor$, $\text{EX}(n, P_3) = \{\lfloor n \rfloor \cdot K_2 + (n \bmod 2) \cdot E_1\}$
- Let n and r be integers with $1 \leq r \leq n$ The *Turan graph* $T_r(n)$ is the unique complete r-partite graph of order n whose partite sets differ by at most 1 in size. It does not contain K_{r+1}, we denote $\|T_r(n)\|$ by $t_r(n)$

Calculating Turan Number
The Turan number $t_r(n)$ is calculated as follows:
Let $n = pr + s$ where p and s are integers, and $0 \leq s < r$, then $T_r(n)$ has s parts of size $p + 1$ and $r - s$ parts of size p. And then simply count edges by hand.

Lemma 58 For any $r, n \geq 1, t_r(n+r) = t_r(n) + n(r-1) + \binom{r}{2}$

Proof. Consider $G = T_r(n+r)$ graph with parts $V_1, ..., V_r$. Let $v_i \in V_i, i = 1, ..., r$. Then $G' = G - \{v_1, ..., v_r\}$ is isomporphic to $T_r(n)$. We have that $||G|| - ||G'||$ is equal to the number of edges incident to v_i's for some $i = 1, ..., r$. This number is

$$\underbrace{n(r-1)}_{\text{every vertex in } T_r(n) \text{ gets (r-1) new}} + \underbrace{\binom{r}{2}}_{\text{edges between new r vts}}$$

□

Lemma 59 Among all $n-$vertex r-partite graphs, $T_r(n)$ has the largest number of edges.

Proof. Let first $r = 2$:
Let G be an n-vertex bipartite graph with largest possible number of edges. Then clearly G is complete bipartite. Assume that two parts V and U of G differ in size by at least 2, so $|V| > |U| + 1$. Put one vertex from V to U to obtain new parts V' and U' and let G' be complete bipartite graph with parts V' and U'. Then $||G'|| = |V'||U'| = (|V| - 1)(|U| + 1) = |V||U| - |U| + |V| - 1 > |V||U| - |U| + |U| + 1 - 1 = |V||U| = ||G||$, a contradiction to maximality of G.

If $r > 2$:
Consider any two parts U, V of an r-partite G. Assume that U differs from V by at least 2 in size. Let X be the remaining set of vertices. Then $||G|| = ||G[X]|| + |X|(n - |X|) + ||G[U \cup V]||$. Let G' be a graph on the same set of vertices as G that differs from G only on edges induced by $U \cup V$ and so that $G'[U \cup V]$ is a balanced complete bipartite graph. Then from the previous paragraph with $r = 2$, we see that $||G'[U \cup V]|| > ||G[U \cup V]||$. Thus $||G'|| > ||G||$, a contradiction. Thus any two parts of G differ in size by at most 1. In addition we see as before that G is complete r-partite. Thus G is isomporphic to $T_r(n)$. □

Lemma 60 For a fixed r,

$$\lim_{n\to\infty} \frac{t_r(n)}{\binom{n}{2}} = 1 - \frac{1}{r}$$

Proof. Since each part in $T_r(n)$ has size either $\lfloor \frac{n}{r} \rfloor$ or $\lceil \frac{n}{r} \rceil$, we see that each part has size between $\frac{n-r}{r}$ and $\frac{n+r}{r}$. We have that

$$\underbrace{\binom{n}{2}}_{\text{all edges}} - \underbrace{r\binom{(n+r)/r}{2}}_{r \text{ independent sets}} \leq t_r(n) \leq \binom{n}{2} - r\binom{(n-r)/r}{2}$$

Thus

$$\binom{n}{2} - r\frac{1}{2}\frac{(n+r)}{r}\frac{n}{r} \leq t_r(n) \leq \binom{n}{2} - r\frac{1}{2}\frac{(n-r)}{r}\frac{(n-2r)}{r}$$

More manipulation and dividing by $\binom{n}{2}$ gives the result. □

Theorem 62 (Mantel's theorem) If a graph G on n vertices contains no triangle then it contains at most $\frac{n^2}{4}$ edges.

Proof. We proceed by induction on n. For $n = 1$ and $n = 2$. The result is trivial, so assume that $n > 2$ and we know it to be true for $n - 1$. Let G be a graph on n vertices. Let x and y be two adjacent vertices in G. Since every vertex in G is connected to at most one of x and y, there are at most $n - 2$ edges between $\{x, y\}$ and $V(G) - \{x, y\}$. Let $H = G - \{x, y\}$. Then H contains no triangles and thus, by induction, H has at most $\frac{(n-2)^2}{4}$ edges. Therefore the total number of edges in G is at most

□

Theorem 6.2 (Turan's Theorem) For all integers $r > 1$ and $n \geq 1$, any graph G with n vertices, $\text{ex}(n, K_r)$ edges and $K_r \nsubseteq G$ is a $T_{r-1}(n)$. In other words $\text{EX}(n, K_r) = \{T_{r-1}(n)\}$.

Proof. TODO □

Conjecture Erdős-Sos If $|G| = n$ and $||G|| > \frac{(k-1)n}{2}$, then G contains all k-edge trees as subgraphs, i.e. for any tree T on k edges $\text{ex}(n, T) \leq \frac{(k-1)n}{2}$

Theorem 66 (Erdős-Stone-Simonovits) For any graph H and for any fixed $\epsilon > 0$ there is n_0 such that for any $n \geq n_0$:

$$(1 - \frac{1}{\chi(H) - 1} - \epsilon)\binom{n}{2} \leq ex(n, H) \leq (1 - \frac{1}{\chi(H) - 1} + \epsilon)\binom{n}{2}$$

Corollary

- $\chi(H) = 3$, $\text{ex}(n, H) \approx \frac{1}{2}\binom{n}{2}$
- $\chi(H) \geq 3$, $\text{ex}(n, H) \approx \text{ex}(n, K_{\chi(H)})$

For exam, find extremal number for an ugly big graph,
by applying Erdős-Stone

Proof outline: Let $r = \chi(H) - 1$
For the upper bound, let G be a graph on n vertices that has $(1 - \frac{1}{\chi(H)-1} + \epsilon)\binom{n}{2}$ edges. We shall show that G has a subgraph isomorphic to H. Let G' be a large subgraph of G that has minimum degree at least $(1 - \frac{1}{r} + \frac{\epsilon}{2})|V(G')|$, we can finde such G' by greedily deleting vertices of smaller degrees. Then show, by induction on r that G' contains a complete $(r + 1)$-partite graph H' with sufficiently large parts. Finally observe that $H \subseteq H'$

Definition Zarankiewicz function $z(m, n; s, t)$ denotes the maximum number of edges that a bipartite graph with parts x, Y of sizes m, n respectively, can have without containing $K_{s,t}$ respecting sides (i.e. there is no copy of $K_{s,t}$ with partition sets S, T of sizes s, t respectively, such that $S \subseteq X$ and $T \subseteq Y$)

Theorem 67 (Kovari-Sos-Turan) We have the upper bound

$$z(m,n;s,t) \leq (s-1)^{\frac{1}{t}}(n-t+1)m^{1-\frac{1}{t}} + (t-1)m$$

In particular for $m = n$ and $t = s$

$$z(m,n;s,t) \leq c_1 \cdot n \cdot n^{1-\frac{1}{t}} + c_2 \cdot n = \mathcal{O}(n^{2-\frac{1}{t}})$$

Proof. Let G be a bipartite graph with parts A, $|A| = m$ and B, $|B| = n$ such that it does not contain a copy of $K_{s,t}$ with part of size s in A and part of size t in B. Let T be the number of stars of size t with a center in A Then

$$T = \sum_{v \in A} \binom{deg(v)}{t}$$

On the other hand

$$T \leq (s-1)\binom{n}{t}$$

Since for each subset $\mathcal{Q}$ of t vertices in B there are at most $s-1$ stars counted by T with a leaf-set $\mathcal{Q}$.

TODO □

Hadwiger's Conjecture

$$\chi(G) \geq r \Rightarrow G \text{ has a } K_r \text{ minor}$$

Proof. proof cases $r = 3$ and $r = 4$ for exam □

Lemma 68 For any positive integers n, t with $t < n$,

$$ex(n, K_{t,t}) \leq \frac{z(n,n;t,t)}{2}$$

Theorem 69 For any positive t, and $n > t$, there are positive constants c and c' such that

$$c' \cdot n^{2-\frac{2}{t+1}} \leq ex(n, K_{t,t}) \leq c \cdot n^{2-\frac{1}{t}}$$

Theorem 71

$$ex(n, C_4) = \frac{1}{2}n^{3/2} + (n^{3/2})$$

$$ex(n, C_6) = \Theta(n^{4/3})$$

$$ex(n, C_{10}) = \Theta(n^{6/5})$$

$$c' \cdot n^{1-\frac{2}{3k-2-\epsilon}} \leq ex(n, C_{2k}) \leq c \cdot n^{1+\frac{1}{k}}$$

Definition 6.4 Let $X, Y \subseteq V(G)$ be disjoint vertex sets and $\epsilon > 0$.

- *the density* $d(X,Y)$ of (X,Y) is

$$d(X,Y) := \frac{\|X,Y\|}{|X|,|Y|}$$

- For $\epsilon > 0$ the pair (X,Y) is an *ϵ-regular pair* if we have

$$|d(X,Y) - d(A,B)| \leq \epsilon$$

  for all $A \subseteq X, B \subseteq Y$ with $|A| \geq \epsilon|X|$ and $|B| \geq \epsilon|Y|$
  In other words, the edges in an ϵ-regular pair are distributed very uniformly, with the density between any pair of reasonably large subsets of vertices being very close to the overall density of the pair. This uniform distribution of edges is typical in a random bipartite graph, and captures what we mean when we say a (bipartite) graph 'looks random'. It remains to define what kinds of partitions of the vertices we will be concerned with.

- An *ϵ-regular partition* of the graph $G = (V,E)$ is a partition of the vertex set $V = V_0 \uplus ... \uplus V_k$ with the following properties:

  1. $|V_0| \leq \epsilon|V|$
  2. $|V_1| = |V_2| = ... = |V_k|$
  3. all but at most ϵk^2 of the pairs (V_i, V_j) for $1 \leq i < j \leq k$ are ϵ-regular.

  Note the parameter ϵ play three roles here: bounding the size of the exceptional set V_0, bounding the number of irregular pairs, and controlling the regularity of the regular pairs

In an ϵ-regular parition we have control over the distribution of edges between the ϵ-regular pairs, but not over the edges within any of the parts, involving the exceptional set, or in irregular pairs. Thus, in light of the three roles described above, the smaller ϵ is, the greater our control over the distribution of edges in an ϵ-regular partition.

Informally, the Regularity Lemma tells us that the vertices of *any* large graph can be partitioned into a bounded number of parts, with the subgraph between most pairs of parts looking random.

Szemeredi's Regularity Lemma For any $\epsilon > 0$ and any integer $m \geq 1$ there is an $M \in \mathbb{N}$ such that every graph of order at least m has an ϵ-regular partition $V_0 \uplus ... \uplus V_k$ with $m \leq k \leq M$.

Analyzing large Graphs with regularity Lemma

1. start with ϵ-partition
2. create auxillary graph with an edge between parts if $(V_i, V_j)\epsilon$-regular with density $d > 0$
3. use blowup graph to complete partite, use blowup lemma:

$$H \subseteq R_s \Rightarrow H \subseteq G$$

Erdos-Stone Theorem For all integers $r > s \geq 1$ and any $\epsilon > 0$ there exists an integer n_0 such that every graph with $n \geq n_0$ vertices and at least

$$t_{r-1}(n) + \epsilon n^2$$

edges contains K_r^s (is the complete r-partite graph where each part contains exactly s vertices) as a subgraph.

Corollary 73 Erdős-Stone together with $\lim_{n\to\infty} / \binom{n}{2} = 1 - 1/r$ yields an asymptotic formula for the extremal number of any graph H on at least one edge:

$$lim_{n\to\infty} \frac{ex(n,H)}{\binom{n}{2}} = \frac{\chi(H) - 2}{\chi(H) - 1}$$

Substructures in sparse Graphs **Key message:** extremal numbers for minors are linear in $|E|$

- $||G|| \geq cr^2|G| \Rightarrow G \supseteq \mathrm{TK}_r$
- $||G|| \geq c\sqrt{log(r)} \Rightarrow G \supseteq \mathrm{MK}_r$

8 Ramsey Theory

Definitions

- In an edge-coloring of a graph, a set of edges is
  - *monochromatic* if all edges have the same color
  - *rainbow* if no two edges have the same color
  - *lexical* if two edges have the same color iff they have the same lower endpoint in some ordering of the vertices
- Let k be a natural number. Then the *Ramsey number* $R(k) \in \mathbb{N}$ is the smallest n such that every 2-edge-coloring of K_n contains a monochromatic K_k.
- **asymmetric Ramsey number** $R(k,l)$**:** is the smallest $n \in \mathbb{N}$ such that every 2-edge-coloring of a K_n contains a red K_k or a blue K_l.
- **graph Ramsey number** $R(G,H)$**:** is the smallest $n \in \mathbb{N}$ such that every red-blue edge-coloring of K_n contains a red G or a blue H.
- **hypergraph Ramsey number** $R_r(l_1,...,l_k)$**:** is the smallest $n \in \mathbb{N}$ such that every k-coloring of the edges of the complete hypergraph on n vertices and edges of size r contains a clique of size l_i whose edges all have color i, for some $i \in \{1,...,k\}$.
- **induced Ramsey number** $IR(G,H)$**:** is the smallest $n \in \mathbb{N}$ for which there is a graph F on n vertices such that in any red-blue coloring of $E(F)$, there is an induced subgraph of F isomorphic to G with all its edgs colored red or there is an induced subgraph of F isomorphic to H with all its edges colored blue.

- **anti-Ramsey number $AR(n,H)$:** is the maximum number of colors that an edge-coloring of K_n can have without containing a rainbow copy of H.

Ramsey Theorem For any $k \in \mathbb{N}$ we have

$$\sqrt{2}^k \leq R(k) \leq 4^k$$

In particular the Ramsey numbers, the asymmetric Ramsey numbers an the graph Ramsey numbers are finite.

Proof. TODO □

Remark $R(2) = 2, R(3) = 6, R(4) = 18$ and $43 \leq R(5) \leq 48$.

Applications of Ramsey theory

Theorem (Erdős, Szekeres) Any list of more than n^2 numbers contains a nondecreasing or non-increasing sublist of more than n numbers.

Proof. TODO □

Theorem (Erdős, Szekeres) For any integer $m \geq 3$ there is an integer $N = N(m)$ such that if X is a set of N points on the plane such that no three points are on a line, then X contains a vertex set of a convex m-gon.

Proof. TODO □

Definition Let $R(p,q;r)$ be the hypergraph Ramsey number for r-uniform hypergraphs. The following Theorem show the existence of hypergraph Ramsey number.

Theorem 83 For any parametres $p,q,r \geq 2$

$$R(p,q;r) \leq R(R(p-1,q;r), R(p,q-1;r); r-1) + 1$$

9 Random Graphs

Definitions

- Erdos-Renyi model of random graphs $\mathcal{G}(n,p)$ is the probability space on all n-vertex graphs that results from independently deciding whether to include each of the $\binom{n}{2}$ possible edges with fixed probability $p \in [0,1]$
- A property $\mathcal{P}$ is a set of graphs, for example $\mathcal{P} = \{G : G \text{ is } k\text{-connected}\}$
- Let $(p_n) \in [0,1]^{\mathbb{N}}$ be a sequence. We say that $G \in \mathcal{G}(n,p_n)$ *almost always* has a property $\mathcal{P}$ if $\text{Prob}(G \in \mathcal{G}(n,p_n) \cap \mathcal{P}) \to 1$ for $n \to \infty$. If (p_n) is constant p, we also say in this case that *almost all* graphs in $\mathcal{G}(n,p)$ have property $\mathcal{P}$

- A function $f(n) : \mathbb{N} \to [0,1]$ is a *threshold function* for property $\mathcal{P}$ if:
  - for all $(p_n) \in [0,1]^{\mathbb{N}}$ with

$$\frac{p_n}{f(n)} \underset{n\to\infty}{\to} 0$$

    the graph $G \in \mathcal{G}(n, p_n)$ almost always does not have property $\mathcal{P}$
  - for all $(p_n) \in [0,1]^{\mathbb{N}}$ with

$$\frac{p_n}{f(n)} \underset{n\to\infty}{\to} \infty$$

    the graph $G \in \mathcal{G}(n, p_n)$ almost always has property $\mathcal{P}$
  - threshold function for containing a cycle is $f(n) = 1/n$ meaning if $p > 1/n$ we almost always have a cycle

Usage of probabilistic Method

- $\mathrm{ex}(n, K_{t,t})$
- $\sqrt{2}^k \leq R(k)$
- Erdős-Hajnal

Lemma 107 Edős-Renyi model is universal TODO Lemma 107 tells us that Erdos-Renyi model is universal, it gives us any graph.

Lemma 108 TODO

Lemma 110 Expected number of cycles
TODO

Theorem 9.2 (Erdős) For any $k \geq 2$ there is a Graph G on $\sqrt{2}^k$ vertices such that $\alpha(G) < k$ and $\omega(G) < k$. This implies $R(k,k) \geq 2^{k/2}$.

Proof. TODO □

Theorem 9.3 (Erdos-Hajnal) sketch: take random graph, number of short cycles (length at most k) is $< \frac{n}{2}$ delete a vertex from each of these cycles and get a graph G' so the girth of $G' > k$. We know chromatic number of G prime is $\geq \frac{|V(G')|}{\alpha(G')}$ with lemma 108 we can show α is not so big and therefore get a lower bound

two tricks:

- random graph and kill short cycle
- bound the independence number (co-clique number) size of the largest independent set

10 Hamiltonian cycles

Definition

- A cycle C in a graph G is *Hamiltonian* if it contains all vertices.
- A graph that has a Hamiltonian cycle is called Hamiltonian graph.

Lemma 10.1 (Neccessary condition for existence of Ham. cycle)
If G has a Hamiltonian cycle, the for every non-empty $S \subseteq V$ the graph $G - S$ cannot have more than $|S|$ components.

Proof. Let C be a Hamiltonian cycle of G. Let $S \subseteq V(G)$, $S \neq \emptyset$, $t := \#$ components of $G - S$ There are at least 2 edges of C between each component of $G - S$ and S. If $e = \#$ edges of C between S and $V - S$, we have

$$e \geq t \cdot 2 \text{ and } e \underset{Cis2\text{–regular}}{\leq} |S| \cdot 2$$

□

Theorem 10.2 Dirac Every graph with $n \geq 3$ vertices and minimum degree at least $\frac{n}{2}$ has a Hamiltonian cycle.

Proof. TODO □

Ore's Thm. A graph G on $n \geq 3$ vertices is Hamiltonian if $\forall u, v \in V(G), uv \notin E(G), d(u) + d(v) \geq n$

11 Networkflows

Definitions

- circulation
- network, source, sink, capacity

Definition Network Flow
(F1)
(F2)
(F3)

- cut
- capacity of a cut

Lemma 100 For any cut $(S, \overline{S})$ and a network flow f in a network N, $f(S, \overline{S}) = f(s, V(G))$

Proof.

$$f(S, \overline{S}) = f(S, V) - f(S, S) = f(s, V) + \underbrace{\sum_{v \in S \setminus \{s\}} f(v, V)}_{0 \text{ by } F2} - \underbrace{f(S, S)}_{0 \text{ by } F1} = f(s, V) + 0 - 0$$

□

Ford-Fulkerson Theorem Let $N = (G, s, t, c)$ be a network. Then

$$max\{|f| : f \text{ is an } N\text{-flow}\} = \min\{c(S, \overline{S}) : (S, \overline{S}) \text{ is a cut}\}$$

and there is an integral flow $f : T \to \mathbb{Z}_{\geq 0}$ with this maximum flow value.

Proof. TODO □

No Group-valued flows!!

12 problems

problem 1 Determine the number of edges, average degree, diameter and girth of the *d-dimensional hypercube.*

Proof. For number of edges consider vertex degree, for average degree observe that Q_d is regular.

claim: Q_d has diameter d.

First prove that for any $x, y \in V$ the distance $d(x, y)$ in Q_d is the number of postions where they differ. Suppose they differ in $l \geq 1$ positions then we get a path of length l by inverting each entry where they differ sequentially. There can not be a shorter Path since each step can only change one position.

Therefore the diameter is d, because thats the maximum amount of positions where two sequences can differ, consider 00...0 and 11...1.

claim: girth of Q_d is ∞ if $d = 1$ and otherwise 4.

If $d = 1$, the resulting graph Q_1 is acyclic. If $d \geq 2$ we first observe that then Q_d is triangel-free. Suppose otherwise than there is a triangle xyz in Q_d, now consider the amount of 1's in $|v|, v \in V$. Wlog $|x|$ is even then $|y|$ must be odd and $|z|$ must be even. But zx is an edge, a contradiction.

On the other hand Q_d does contain 4-cycles:

$$(00...0, 010...0, 100...0, 00...0)$$

□

problem 2 Show that any tree T has at least $\Delta(T)$ leaves.

Proof. Induction on $n = |T|$.

Base: tree on two vertices has 2 leaves and maximum degree is 1.

Step: remove a leaf v from T and let u be its neighbour, so that $T' := T - v$ we than have two cases:

case 1: $\Delta(T) = \Delta(T')$

We know by I.H. that T' has at least $\Delta(T')$ leaves, thus if the maximum degree of T is the same the claim holds.

case 2: $\Delta(T) = \Delta(T') + 1$ This means u is the vertex with maximal degree in T, but we know that the leafs of T can only be the leafs of T' plus the leaf v we removed therefore T has $\Delta(T') + 1$ leafs.

□

Proof. By Counting
Let T be any tree. Let $L \subseteq V$ be the set of leaves and $N = V \setminus L$ the set of non-leaves in T. Let u be a vertex with maximum degree $\Delta(T)$. We know that a tree has $|V| + 1$ edges.

$$\begin{aligned}
2 \cdot (|V| - 1) &= \sum_{v \in V(T)} d(v) && (1)\\
&= d(u) + \sum_{v \in L} d(v) + \sum_{v \in N \setminus \{u\}} d(v) && (2)\\
&\geq \Delta(T) + \sum_{v \in L} 1 + \sum_{v \in N \setminus \{u\}} 2 && (3)\\
&= \Delta(T) + |L| \cdot 1 + (|V| - |L| - 1) \cdot 2 && (4)\\
&= \Delta(T) + 2 \cdot (|V| - 1) - |L| && (5)
\end{aligned}$$

From this follows $|L| \geq \Delta(T)$, as desired. □

problem 3 Prove that either a graph or its complement is connected.

Proof. Let $G = (V, E)$ be any non-empty graph. We assume G is not connected and shall argue that $\overline{G}$ is connected.
Since G is disconnect, we find two vertices $u, w \in V$ and a connected component C of G such that $u \in C$ and $w \notin C$. Now in $\overline{G}$ all vertices in C are adjacent to w. And in particular $uw \in E(\overline{G})$, so all vertices lie in a single connected component of $\overline{G}$, which is therefore connected. □

problem 4 Prove that the vertex set of any graph can be partitioned into two sets such that for each vertex, at least half of its neighbors belong to the other set.

Proof. Consider partition of $V(G)$ into disjoint sets A, B, maximizing the number of edges between A and B. We will show that moving any vertex to the other sets results in a partition that has less edges between the sets than the original one.
Pick wlog a vertex $v \in A$. Let $d_B = N(v) \cap B$ and $d_A = d - d_B$. Now consider $A' = A - v$ and $B' = B + v$

$$|\{uw \in E : u \in A', w \in B'\}| = |\{uw \in E : u \in A, w \in B\}| - d_B + (d - d_B)$$

but by the maximality of edges between A and B we have:

$$|\{uw \in E : u \in A', w \in B'\}| \leq |\{uw \in E : u \in A, w \in B\}|$$

which gives us

$$-d_B + (d - d_B) \leq 0 \iff d_B \geq d/2$$

□

problem 5 Show that the following statements are equivalent

1. G is connected, but $G - e$ is disconnected for every edge e
2. Any two vertices in G are linked by a unique path

Proof.
"(ii) $\Rightarrow$ (i)": trivial
"(i) $\Rightarrow$ (ii)":
As G is connected, there is at least one path between any two vertices of G. Lets assume for the sake of contradiction that there are some vertices x and y that are joind by at least two paths P_1, P_2. As $P_1 \neq P_2$ there is and edge e_0 that lies in P_1 but not in P_2. We shall show that the graph $G - e_0$ is still connected, which will be a contradiction.

Idea is to consider the path between any two vertices u, v and remove remove the edge e_0 and still show we can reach v from v by using the vertices x and y which have two paths between them and simply put e_0 in one of them, then we can user the other one by going from one endpoint from e_0 to x, possible because G is connected. Then to y using the other path and then from y to the other endpoint of e_0 and finally to v. Thus u and v are still connected after removing e_0 a contradiction.

□

problem 6 Let $T_1, ..., T_k$ be subtrees of a tree T, any two of which have at least one vertex in common. Prove that there is a vertex common to all T_i.

Proof. Apply induction on $|T|$. If $|T| = 1$ then T consists of a single vertex, say v. If $T_1, ..., T_k$ are subtrees of T with pairwise intersecting vertex sets, then we must have $T_i = T$ for each $i = 1, ..., k$. It follows that v belongs to each T_i.

So assume $n \geq$ is an integer, and let us suppose this result holds for all trees of order $n - 1$. Suppose T is a tree with $|T| = n$ and $T_i, ..., T_k$ are subtrees of T. We assume $k \geq 2$. Let v be a leaf in T and let $T' = T - v$ be the tree resulting from removing this leaf. □

problem 7