An introduction to continuum mechanics

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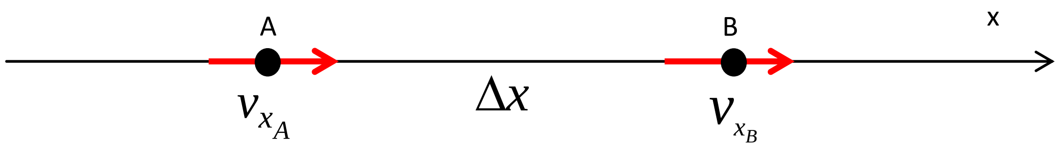
1. Conservation of mass (continuity equation)

1.1 The 1D continuity equation

The continuity equation (conservation of mass) in 1D can be derived by analyzing material fluxes along a small, immobile 1D volume (distance) with a constant length Δx. The initial mass in this volume is m_0. We can calculate the initial average density ρ_0 within this 1D volume:

$$\rho_0 = \frac{m_0}{\Delta x}$$

A density in 1D has the units $\frac{kg}{m}$ because the volume is not a cube but only a line.



Mass enters the volume through the boundary point A and leaves it through the opposite boundary point B. After a small period of time Δt, this mass becomes m_1 and the average density ρ_1 changes to:

$$\rho_1 = \frac{m_1}{\Delta x}$$

But we can also calculate the new mass m_1 using the mass fluxes through the boundary points.

$$\begin{aligned} m_1 &= m_0 + \mathrm{m_{in}} - m_{out} \\ \mathrm{m_{in}} &= m_A \\ m_{out} &= m_B \\ m_A &= \rho_A v_{x_A} \Delta t \\ m_B &= \rho_B v_{x_B} \Delta t \end{aligned}$$

where $\mathrm{m_{in}}$ and m_{out} are the incoming and outgoing mass, respectively. m_A and m_B are the masses that pass through the boundary points A and B during the time Δt. ρ_A and ρ_B are the densities in the respective boundary points and v_{x_A} and v_{x_B} are the velocities responsible for the material fluxes through the boundary points. ρv isa mass flux and has the units $\frac{kg}{m}\frac{m}{s}=\frac{kg}{s}$.If Δt is small we can write an approximate expression for the time derivative of the average density in the volume:

$$\frac{\partial \rho}{\partial t} \approx \frac{\Delta \rho}{\Delta t} = \frac{\rho_1 - \rho_2}{\Delta t} = \frac{m_1 - m_0}{\Delta x \Delta t}$$

By using the equations for m_A and m_B the following expression can be obtained:

$$\frac{\Delta \rho}{\Delta t} = -\frac{\rho_B v_{x_B} \Delta t - \rho_A v_{x_A} \Delta t}{\Delta t \Delta x} = -\frac{\rho_B v_{x_B} - \rho_A v_{x_A}}{\Delta x}$$

or

$$\frac{\Delta \rho}{\Delta t} = -\frac{\Delta(\rho v_x)}{\Delta x}$$

or

$$\frac{\Delta \rho}{\Delta t} + \frac{\Delta(\rho v_x)}{\Delta x} = 0$$

$$\Delta(\rho v_x) = \rho_B v_{x_B} \Delta t - \rho_A v_{x_A} \Delta t$$

where $\Delta(\rho v_x)$ is the difference in the mass flux through the two boundary points. If Δt and Δx both tend to zero, the differences can be replaced by derivatives and we end up with the 1D continuity equation

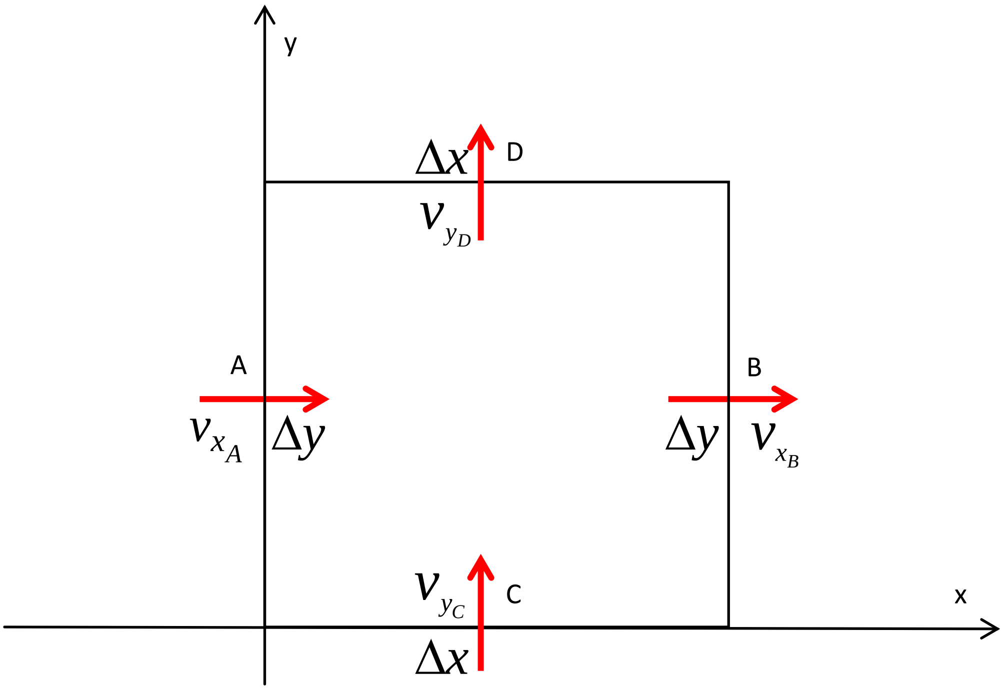
$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_x)}{\partial x} = 0$$

1.2 The 2D continuity equation

The continuity equation in 2D is derived analogue to the 1D equation by analyzing the material fluxes through a small, immobile 2D volume (square) with constant lengths Δx and Δy. Again the initial average density ρ_0 in this volume can be calculated out of the initial mass m_0:

$$\rho_0 = \frac{m_0}{\Delta x \Delta y}$$

In 2D the density has the units $\frac{kg}{m^2}$ because the volume in 2D is a square.



In 2D mass enters through the boundaries A and C and leaves it through the boundaries B and D and after a small period of time (incremental time) the mass within the volume becomes m_1 and the new density ρ_1 is:

$$\rho_1 = \frac{m_1}{\Delta x \Delta y}$$

m_1 is calculated as follows:

$$\begin{aligned}
m_1 &= m_0 + m_{in} - m_{out} \\
m_{in} &= m_A + m_C \\
m_{out} &= m_B + m_D \\
m_A &= \rho_A v_{x_A} \Delta y \Delta t \\
m_B &= \rho_B v_{x_B} \Delta y \Delta t \\
m_C &= \rho_C v_{x_C} \Delta x \Delta t \\
m_D &= \rho_D v_{x_D} \Delta x \Delta t
\end{aligned}$$

where m_{in} and m_{out} are the incoming and outgoing mass, respectively. m_A - m_D are the masses that passed through the respective boundaries during the time Δt. ρ_A - ρ_D are the densities at the respective boundaries and v_{x_A} - v_{x_D} are the velocities responsible for the mass fluxes through the boundaries A-D. If Δt is small we can write an approximate expression for the time derivative of the average density in our 2D volume as:

$$\frac{\partial\rho}{\partial t} \approx \frac{\Delta\rho}{\Delta t} = \frac{\rho_1 - \rho_0}{\Delta t} = \frac{m_1 - m_0}{\Delta x \Delta y \Delta t}$$

By using the equations for m_A - m_D we can reform this to:

$$\frac{\Delta\rho}{\Delta t} = -\frac{\rho_B v_{x_B} - \rho_A v_{x_A}}{\Delta x} - \frac{\rho_D v_{x_D} - \rho_C v_{xC}}{\Delta y}$$

$$\frac{\Delta\rho}{\Delta t} = -\frac{\Delta(\rho v_x)}{\Delta x} - \frac{\Delta(\rho v_y)}{\Delta y}$$

$$\frac{\Delta\rho}{\Delta t} + \frac{\Delta(\rho v_x)}{\Delta x} + \frac{\Delta(\rho v_y)}{\Delta y} = 0$$

$$\Delta(\rho v_x) = \rho_B v_{x_B} - \rho_A v_{x_A}$$

$$\Delta(\rho v_y) = \rho_D v_{x_D} - \rho_C v_{x_C}$$

If Δx , Δy and Δt all tend to zero, the differences can be replaced by derivatives and we obtain the 2D continuity equation (conservation of mass).

$$\frac{\partial\rho}{\partial t} + \frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} = 0$$

1.3 The 3D continuity equation

The continuity equation in 3D is derived analogue to the 1D and 2D equation by analyzing the material fluxes through a small, immobile 3D volume (cube) with constant lengths. As an exercise you can verify the 3D equation by deriving it step by step.

$$\frac{\partial\rho}{\partial t} + \frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} + \frac{\partial(\rho v_z)}{\partial z} = 0$$

1.4 The divergence div()

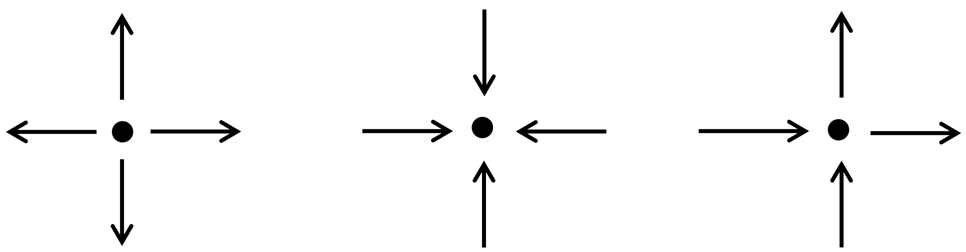
The divergence of a vector field is a scalar function and is defined as follows:

In 1D: $$div(\vec{v}) = \frac{\partial v_x}{\partial x}$$

In 2D: $$div(\vec{v}) = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \nabla \cdot \vec{v}$$

In 3D: $$div(\vec{v}) = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{pmatrix} \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \nabla \cdot \vec{v}$$

where x, y, and z are the Cartesian coordinates and v_x, v_y and v_z are the velocities parallel to the respective coordinate axes. In principle the divergence div() of a vector field is the scalar product of the vector field with the vector $\nabla = \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{pmatrix}$. ∇ is called the divergence operator. In other words the divergence operator maps vector fields to scalar fields and matrices to vectors. The divergence at a given point is positive when the surrounding vector field is pointing away from the point (divergent flow) and is negative when this flow field is pointing towards the point (convergent flow). In other words the divergence measures the magnitude of a source or sink at a given point in a vector field.



$div(\vec{v}) > 0$ $\qquad$ $div(\vec{v}) < 0$ $\qquad$ $div(\vec{v}) = 0$

This means that we can write the continuity equation in a shorter form:

$$\frac{\partial \rho}{\partial t} + div(\rho \vec{v}) = 0$$

This is equation is valid for 1D,2D and 3D.

1.5 Incompressible continuity equation

For many geological media we assume no density change of the material, i.e. we assume the density to be constant with time. With this assumption we end up with the so-called incompressible continuity equation.

$$\frac{\partial \rho}{\partial t} = 0$$

$$div\left(\rho \vec{v}\right) = 0$$

$$div\left(\vec{v}\right) = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$$

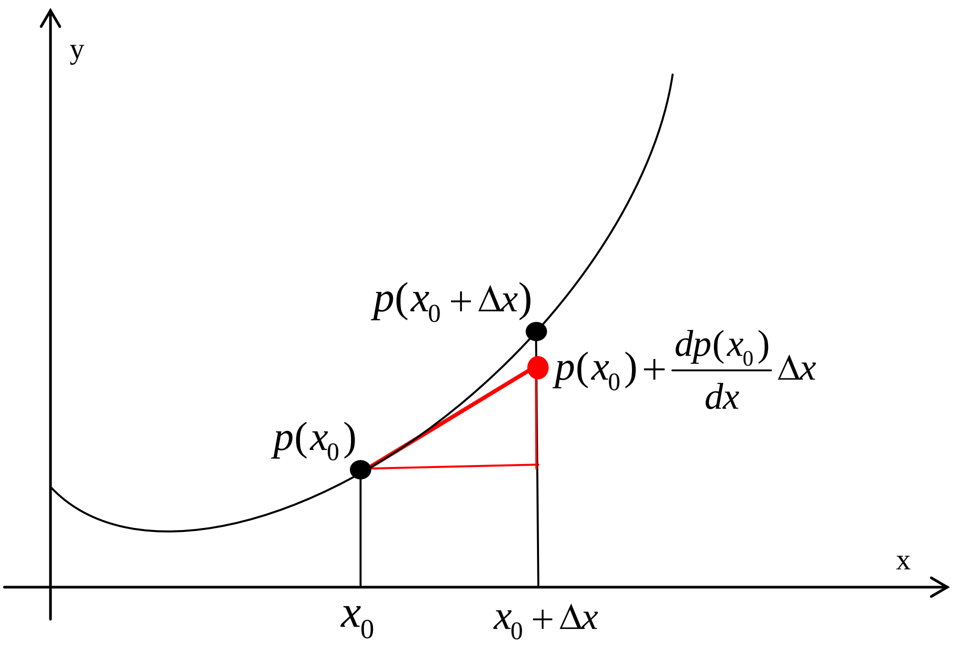
This is valid when no phase transitions lead to volume changes and pressure and temperature changes are not very large. In geodynamic modeling it is very common to use the incompressible continuity equation, although it is in many cases a rather big simplification.

1.6 Another way to Rome

There is always more than one way to Rome. By now we did the derivation of the continuity equation in a very intuitive way by analyzing mass fluxes through the boundaries of a small, immobile volume. But we can also derive it by using Taylor series or integro-differential equations.

1.6.1 Using Taylor series

Using Taylor series one can approximate the value of a function $p(x_0 + \Delta x)$ by the value $p(x_0)$ and the derivative of the function $\frac{dp(x_0)}{dx}$ at x_0.



$$p(x_0+\Delta x)\approx p(x_0)+\frac{dp(x_0)}{dx}\Delta x+O\left(\Delta x^2\right)$$

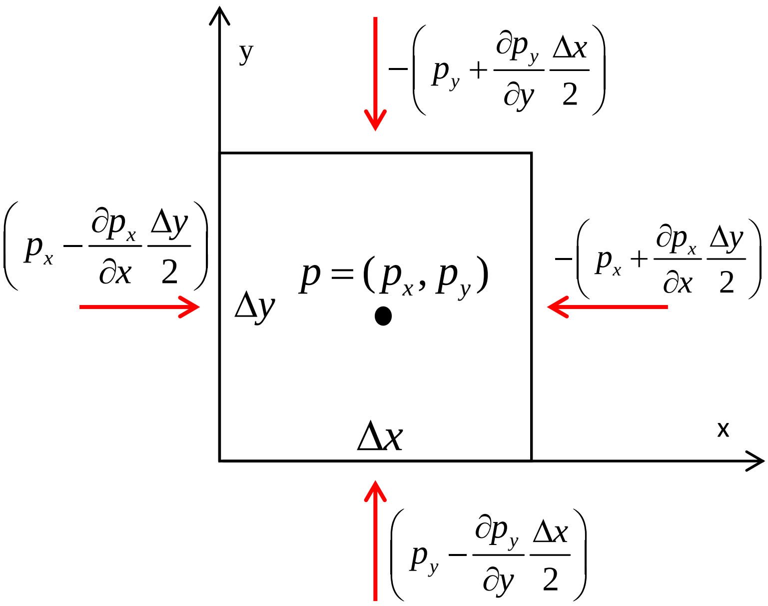
These are the first two terms of the infinite Taylor series. From this we get:

$$\frac{p(x_0+\Delta x)-p(x_0)}{\Delta x}\approx\frac{dp(x_0)}{dx}$$

In 2D we can write the net rate of mass change within a 2D volumeas:

$$\rho=\frac{m}{V}\Rightarrow m=\rho V=\rho\Delta x\Delta y$$
$$\frac{\partial m}{\partial t}=\frac{\partial}{\partial t}(\rho\Delta x\Delta y)=\frac{\partial\rho}{\partial t}(\Delta x\Delta y)=\left[\frac{1}{s}\frac{kg}{m^2}mm\right]=\left[\frac{kg}{s}\right]$$

The net rate of mass change is balanced by the net rate of mass flow into and out of the volume we are looking at. The mass fluxes are defined as $p_x=\rho v_x\Delta y$ and $p_y=\rho v_y\Delta x$ in x- and y-direction, respectively and have the units $\frac{kg}{m^2}\frac{m}{s}m=\frac{kg}{s}$. Using the Taylor series the mass flux at the boundaries can be approximated.



Summing up the mass fluxes through the boundaries we then get:

$$\left(\rho v_x - \frac{\partial \rho v_x}{\partial x}\frac{\Delta x}{2}\right)\Delta y - \left(\rho v_x + \frac{\partial \rho v_x}{\partial x}\frac{\Delta x}{2}\right)\Delta y + \left(\rho v_y - \frac{\partial \rho v_y}{\partial y}\frac{\Delta y}{2}\right)\Delta x - \left(\rho v_y + \frac{\partial \rho v_y}{\partial y}\frac{\Delta y}{2}\right)\Delta x = -\frac{\partial \rho v_x}{\partial x}\Delta x\Delta y - \frac{\partial \rho v_y}{\partial y}\Delta y\Delta x$$

This should be balanced with the net rate of mass change which gives us the 2D continuity equation.

$$\frac{\partial \rho}{\partial t}\cancel{\Delta x\Delta y} = -\frac{\partial \rho v_x}{\partial x}\cancel{\Delta x\Delta y} - \frac{\partial \rho v_y}{\partial y}\cancel{\Delta y\Delta x}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v_x}{\partial x} + \frac{\partial \rho v_y}{\partial y} = 0$$

$$\frac{\partial \rho}{\partial t} + div\left(\rho \vec{v}\right) = 0$$

This way looks much shorter than to analyze all the mass fluxes as we did it before but you have to know much more mathematical rules, as the concept of Taylor-series, to understand it and it is not so intuitive anymore.

1.6.2 Using integro-differential equations

A third method to derive the conservation of mass is the so called integro-differential method. To understand this you have to now even more mathematical rules, laws and standard notation. Although it is still the same physical law, the same mechanism.

The density ρ is a function of the position x_j (j=1, 2 in 2D and j=1, 2, 3 in 3D) and v_j are the components of the velocity. Further we have to use the notation of Cartesian tensors with the usual summation convention. This means that we can write a tensor, for example the stress tensor as:

$$\sigma_{ij}$$
$$i = 1,2,3$$
$$j = 1,2,3$$

This is much shorter than to write the whole stress tensor in every step of the derivation. The usual summation convention (Einstein summation convention) says: When an index variable (i,j,k,l) appears twice in a single term it implies that we are summing over all of its possible values, for example:

$$v_j \frac{\partial \rho}{\partial x_j} = \sum_{j=1}^{3} v_j \frac{\partial \rho}{\partial x_j} = v_1 \frac{\partial \rho}{\partial x_1} + v_2 \frac{\partial \rho}{\partial x_2} + v_3 \frac{\partial \rho}{\partial x_3}$$

This is again much shorter then to write out all the summand.
According to the physical law of the conservation of mass the rate of change of the mass contained in a fixed volume V is given by the rate at which the material flows in and out of it across the boundary S of the volume. Mathematically this is expressed as:

$$\frac{\partial}{\partial t} \int_V \rho d\tau = -\int_S \rho v_j dS_j$$

where τ is the volume element. Now we use the so called Gauss theorem (divergence theorem) and get:

$$\frac{\partial}{\partial t} \int_V \rho d\tau = -\int_V \frac{\partial}{\partial x_j} (\rho v_j) d\tau = -\int_V div\left(\rho \vec{v}\right) d\tau$$

If we assume ρ to be constant within the volume V we end up with the same equation as before:

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x_j} (\rho v_j) = -\frac{\partial (\rho v_x)}{\partial x} - \frac{\partial (\rho v_y)}{\partial y} - \frac{\partial (\rho v_z)}{\partial z} = -div\left(\rho \vec{v}\right)$$

$$\frac{\partial \rho}{\partial t} + div\left(\rho \vec{v}\right) = 0$$

2. Conservation of linear momentum (force balance)

2.1 Repetition: Stress and Strain

2.2 The 2D momentum equation

The deformation of a continuous material always results from a net force which acts on the material. The net force is derived from the balance of various internal and external forces which act on the material. The momentum equation describes the conservation of linear momentum for a continuous material. The momentum equation is furthermore an equivalent to Newton's second law of motion.

$$\vec{f} = m\vec{a}$$

where f is a net force acting on the object and a is the acceleration of the object. In 2D this gives:

$$f_x = ma_x$$
$$f_y = ma_y$$
$$a_x = \frac{\partial v_x}{\partial t}$$
$$a_y = \frac{\partial v_y}{\partial t}$$

The 2D momentum equation is derived by considering forces that act on a small 2D volume with dimensions Δx and Δy. Let us start with the x-momentum equation. The net force f_x acting in the x-direction is defined as follows:

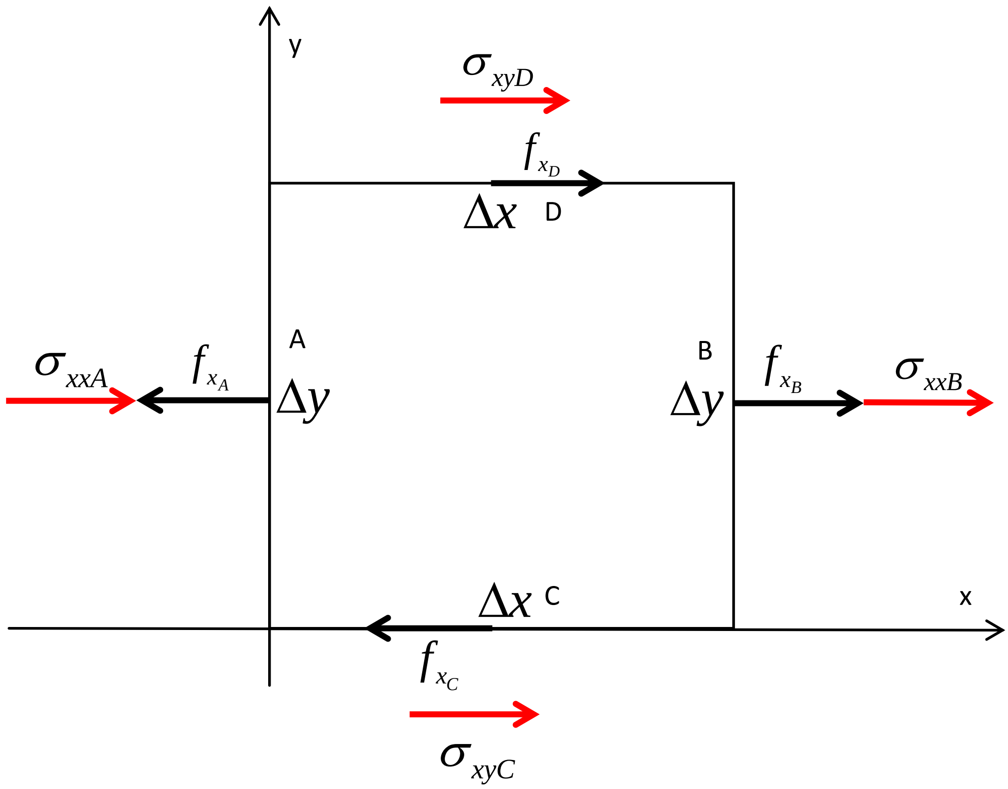
$$f_x = f_{x_A} + f_{x_B} + f_{x_C} + f_{x_D} + F_x$$

where f_{x_A} - f_{x_D} are stress-related forces acting from the outside of the volume on the respective boundaries A-D and F_x is the x-component of an additional external body force $\vec{F}$ acting on the object.This external body force can for example be the gravitational force.

The stress-related forces can be defined using the stresses which are acting on the respective boundaries:

$$
\begin{aligned}
f_{x_A} &= -\sigma_{xxA}\Delta y \\
f_{x_B} &= +\sigma_{xxB}\Delta y \\
f_{x_C} &= -\sigma_{xyC}\Delta x \\
f_{x_D} &= +\sigma_{xyD}\Delta x
\end{aligned}
$$

where σ_{xxA}, σ_{xxB} and σ_{xyC}, σ_{xyD} are the normal and shear stress components defined at the respective boundaries.



One can now write Newton's second law using the expressions for f_{x_A} - f_{x_D}.

$$(\sigma_{xxB} - \sigma_{xxA})\Delta y + (\sigma_{xyD} - \sigma_{xyC})\Delta x + F_x = ma_x$$

Now we divide the whole equation by the volume ($V = \Delta x \Delta y$):

$$\frac{(\sigma_{xxB}-\sigma_{xxA})\Delta y}{V}+\frac{(\sigma_{xyD}-\sigma_{xyC})\Delta x}{V}+\frac{F_x}{V}=\frac{m}{V}a_x$$

$$\frac{(\sigma_{xxB}-\sigma_{xxA})}{\Delta x}+\frac{(\sigma_{xyD}-\sigma_{xyC})}{\Delta y}+\frac{F_x}{\Delta x\Delta y}=\rho a_x$$

$$\frac{\Delta\sigma_{xx}}{\Delta x}+\frac{\Delta\sigma_{xy}}{\Delta y}+\frac{F_x}{\Delta x\Delta y}=\rho a_x$$

$$V=\Delta x\Delta y$$

$$\rho=\frac{m}{V}$$

$$\Delta\sigma_{xx}=(\sigma_{xxB}-\sigma_{xxA})$$

$$\Delta\sigma_{xy}=(\sigma_{xyD}-\sigma_{xyC})$$

where ρ is the average material density and $\Delta\sigma_{xx}$ and $\Delta\sigma_{xy}$ are the differences of respective stress components in x- and y-direction, respectively.When all the differences tend to zero they can be replaced by derivatives and we end up with the x-momentum equation:

$$\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{xy}}{\partial y}+\bar{F}_x=\rho a_x=\rho\frac{\partial v_x}{\partial t}$$

$$\bar{F}_x=\frac{F_x}{\Delta x\Delta y}$$

The same procedure we can do for the y-momentum equation and we will end up with:

$$\frac{\partial\sigma_{xy}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}+\bar{F}_y=\rho a_y=\rho\frac{\partial v_y}{\partial t}$$

$$\bar{F}_y=\frac{F_y}{\Delta x\Delta y}=\frac{F_y}{V}$$

As an exercise you can derive the y-momentum equation. If the external body force is the gravitational force acting in y-direction the equation is:

$$F_y=mg=\rho Vg$$

$$\bar{F}_y=\frac{F_y}{V}=\frac{\rho Vg}{V}=\rho g$$

$$\frac{\partial\sigma_{xy}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}+\rho g=\rho\frac{\partial v_y}{\partial t}$$

Since the stress-related forces have to be conserved they sum up to zero and when we assume the external Force F to be zero the linear momentum equation is:

$$\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \sigma_{xy}}{\partial y}=0$$
$$\frac{\partial \sigma_{xy}}{\partial x}+\frac{\partial \sigma_{yy}}{\partial y}=0$$

or

$$div(\boldsymbol{\sigma})=0$$

where the divergence of the matrix σ is in 2D:

$$div(\boldsymbol{\sigma})=\begin{pmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y}\end{pmatrix}\begin{pmatrix}\sigma_{xx} & \sigma_{xy}\\ \sigma_{xy} & \sigma_{yy}\end{pmatrix}=\begin{pmatrix}\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \sigma_{xy}}{\partial y}\\ \frac{\partial \sigma_{xy}}{\partial x}+\frac{\partial \sigma_{yy}}{\partial y}\end{pmatrix}$$

2.3 The 3D momentum equation

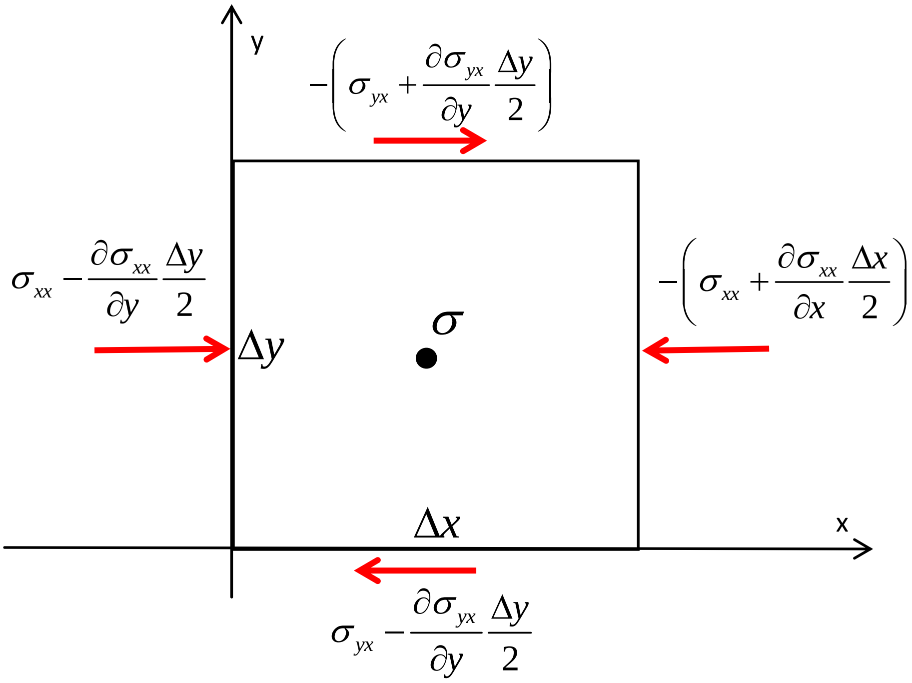
Analogue to the derivation of the 2D linear momentum equation we can derive the 3D momentum equation. You can verify this as an exercise.

$$\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \sigma_{xy}}{\partial y}+\frac{\partial \sigma_{xz}}{\partial z}=0$$
$$\frac{\partial \sigma_{xy}}{\partial x}+\frac{\partial \sigma_{yy}}{\partial y}+\frac{\partial \sigma_{yz}}{\partial z}=0$$
$$\frac{\partial \sigma_{xz}}{\partial x}+\frac{\partial \sigma_{yz}}{\partial y}+\frac{\partial \sigma_{zz}}{\partial z}=0$$
$$div(\boldsymbol{\sigma})=0$$

2.4 Another way to Rome

2.4.1 Using Taylor series

To derive the x-momentum equation we first approximate the stresses in x-direction at the boundaries of the 2D volumeusing the concept of Taylor-series.



Now we can balance the forces using the fact that the stress is force per area:

$$\left[\sigma_{xx}-\frac{\partial\sigma_{xx}}{\partial x}\frac{\Delta x}{2}-\left(\sigma_{xx}+\frac{\partial\sigma_{xx}}{\partial x}\frac{\Delta x}{2}\right)\right]\Delta y+\left[\sigma_{yx}-\frac{\partial\sigma_{yx}}{\partial y}\frac{\Delta y}{2}-\left(\sigma_{yx}+\frac{\partial\sigma_{yx}}{\partial y}\frac{\Delta y}{2}\right)\right]\Delta x=0$$

By rearranging this we get:

$$\left[\cancel{\sigma_{xx}}-\frac{\partial\sigma_{xx}}{\partial x}\frac{\cancel{\Delta x}}{2}-\left(\cancel{\sigma_{xx}}+\frac{\partial\sigma_{xx}}{\partial x}\frac{\cancel{\Delta x}}{2}\right)\right]\cancel{\Delta y}+\left[\cancel{\sigma_{yx}}-\frac{\partial\sigma_{yx}}{\partial y}\frac{\cancel{\Delta y}}{2}-\left(\cancel{\sigma_{yx}}+\frac{\partial\sigma_{yx}}{\partial y}\frac{\cancel{\Delta y}}{2}\right)\right]\cancel{\Delta x}=0$$

$$\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{yx}}{\partial y}=0$$

As an exercise you can do this for the y-momentum equation.

2.4.2 Using integro-differential equations

The total stress-related forces acting on an arbitrary volume V is:

$$\int_S T dS = 0$$

Where T is the internal traction acting on the surface S of the volume V. We can express T in terms of stresses and use the Gauss theorem:

$$\int_S \mathbf{T} dS = \int_S \sigma_{ij} n_j dS = \int_V div(\boldsymbol{\sigma}) dV = \int_V \frac{\partial \sigma_{ji}}{\partial x_j} dV = 0$$

Since an integral is zero, when all integrands are zero we can write:

$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0$$

Using the Einstein summation convention we get in 3D:

$$\begin{aligned}
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} &= 0 \\
\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} &= 0 \\
\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} &= 0 \\
div(\boldsymbol{\sigma}) &= 0
\end{aligned}$$

3. Conservation of angular momentum (angular momentum equation)

If there are no internal sources of angular momentum the stress tensor is symmetric.

In 2D: $\partial \sigma_{xy} = \partial \sigma_{yx}$

In 3D: $\begin{aligned} \partial \sigma_{xy} &= \partial \sigma_{yx} \\ \partial \sigma_{xz} &= \partial \sigma_{zx} \\ \partial \sigma_{yz} &= \partial \sigma_{zy} \end{aligned}$ or $\sigma_{ij} = \sigma_{ji}$

This is the simplest version of the conservation of momentum and most common.

4. Rheology

The conservation equations are independent of material properties and can be applied to all types of materials, such as solids, fluids and gases. In 2D we have now four conservation equations for six unknowns.

Conservation of linear momentum

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} = 0$$
$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0$$

Conservation of angular momentum

$$\sigma_{yx} = \sigma_{xy}$$

Conservation of mass

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

Unknowns

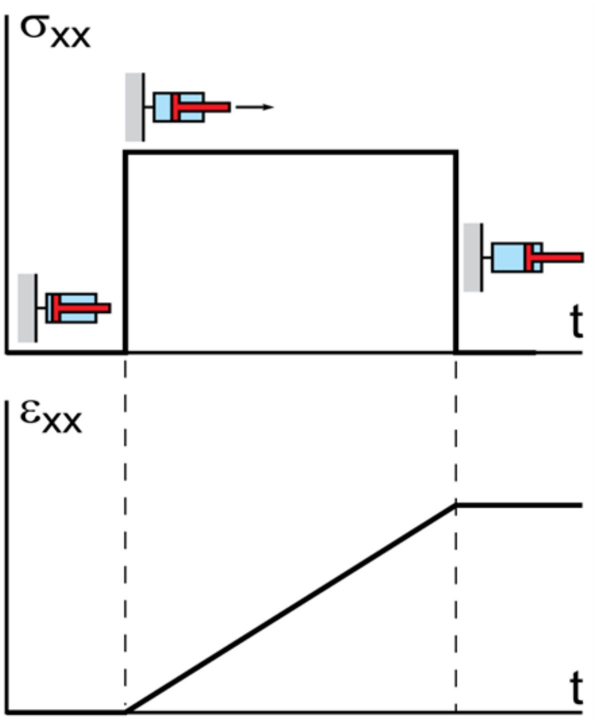
$$\sigma_{xx} / \sigma_{yx} / \sigma_{xy} / \sigma_{yy} / v_x / v_y$$

Therefore, we need more equations, the so-called constitutive equations which specify the material behavior. For fluids, the constitutive equations are often termed rheological equations.

4.1 Ductile rheology

The viscous behavior has the following properties:

- The viscous deformation is time dependent



- The Energy which is used to deform the material is dissipated (for example transformed to heat)
- The deformation of viscous materials is unrecoverable
- The stress is a function of the strain rate

The stress can be related to the strain rate in two different ways: Linear or non-linear.

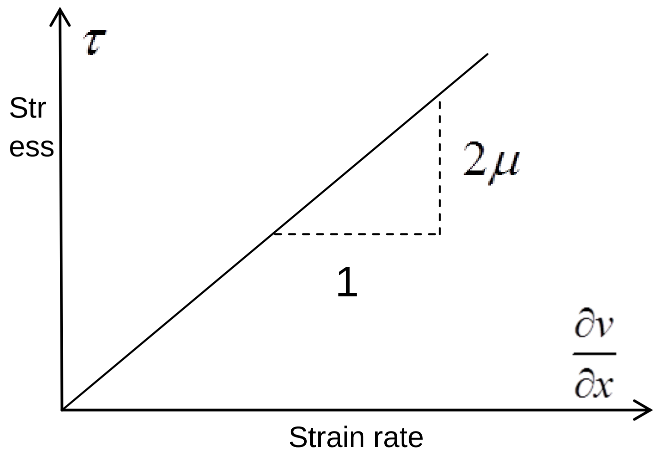
4.1.1 Linear viscous rheology (Newtonian)

1D linear viscous rheology (Newtonian rheology)

The Newtonian rheology is linear, and the deviatoric stress is related to the deviatoric strain rate:

$$\tau = 2\mu \frac{\partial v}{\partial x}$$

where τ is the deviatoric stress and μ is the viscosity.



2D linear viscous rheology (Newtonian rheology)

In 2D the linear viscous rheology equation is:

$$\sigma_{xx} = -p + 2\mu \frac{\partial v_x}{\partial x}$$

$$\sigma_{yy} = -p + 2\mu \frac{\partial v_y}{\partial y}$$

$$\sigma_{yx} = \mu \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right)$$

where σ_{xx}, σ_{yy} and σ_{yx} are the components of the stress tensor, p is the pressure and μ is the viscosity.

4.1.2 power-law viscous rheology

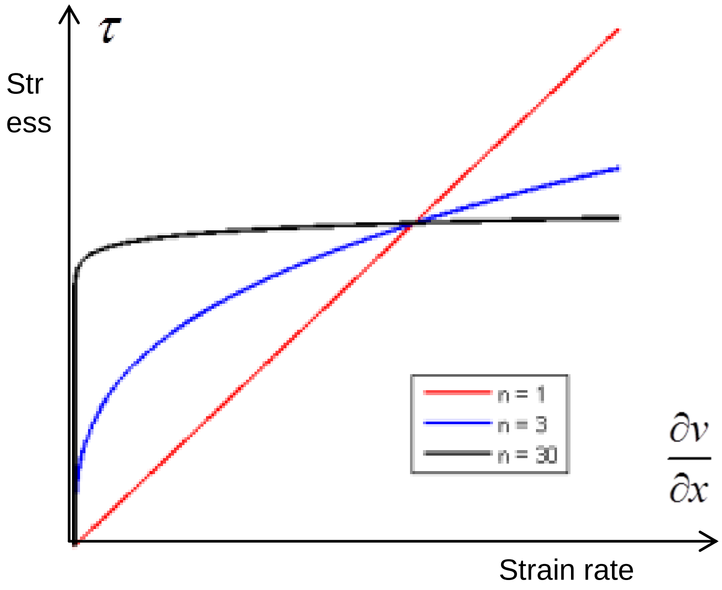
1D power-law viscous rheology

For the power-law viscous rheology the stress is non-linear proportional to the strain rate:

$$\tau^n = 2\mu \frac{\partial v}{\partial x}$$

$$\tau = \left(2\mu \frac{\partial v}{\partial x}\right)^{\frac{1}{n}} =$$

$$= \underbrace{(2\mu)^{\frac{1}{n}} \left(\frac{\partial v}{\partial x}\right)^{\frac{1}{n}-1}}_{\mu_{eff}} \frac{\partial v}{\partial x} = \mu_{eff} \frac{\partial v}{\partial x}$$



where τ is the stress, n is the power-law stress exponent, μ is the viscosity and μ_{eff} is the effective viscosity and a function of the strain rate. To solve this numerically it is necessary to iterate within a loop. The typical structure of rock rheology is:

$$\mu = \varepsilon^{\frac{1}{n}-1} A^{\frac{1}{n}} \exp\left(\frac{E+V}{nRT}\right)$$

where ε is the strain rate, A is a material parameter, the so-called pre-exponential parameter, E is the activation energy, R is the gas constant, T is the temperature and n is the power-law stress exponent.

2D power-law viscous rheology

In 2D this is:

$$\sigma_{xx} = -p + 2\eta \dot{\varepsilon}_{II}^{\frac{1}{n}-1} \frac{\partial v_x}{\partial x}$$

$$\sigma_{yy} = -p + 2\eta \dot{\varepsilon}_{II}^{\frac{1}{n}-1} \frac{\partial v_y}{\partial y}$$

$$\sigma_{xy} = 2\eta \dot{\varepsilon}_{II}^{\frac{1}{n}-1} \frac{1}{2}\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right)$$

$$\dot{\varepsilon}_{II} = \sqrt{\frac{1}{4}\left(\frac{\partial v_x}{\partial x} - \frac{\partial v_y}{\partial y} \right)^2 + \frac{1}{4}\left(\frac{\partial v_x}{\partial y} - \frac{\partial v_y}{\partial x} \right)^2}$$

where $\dot{\varepsilon}_{II}$ is the second invariant of the strain rate tensor.

With these three equations in addition we end up with a closed system of 7 equations of 7 unknowns.

Conservation of linear momentum

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} = 0$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0$$

Conservation of angular momentum

$$\sigma_{yx} = \sigma_{xy}$$

Conservation of mass

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

Rheological equations

$$\sigma_{xx} = -p + 2\eta \dot{\varepsilon}_{II}^{\frac{1}{n}-1} \frac{\partial v_x}{\partial x}$$

$$\sigma_{yy} = -p + 2\eta \dot{\varepsilon}_{II}^{\frac{1}{n}-1} \frac{\partial v_y}{\partial y}$$

$$\sigma_{xy} = 2\eta \dot{\varepsilon}_{II}^{\frac{1}{n}-1} \frac{1}{2}\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right)$$

$$\dot{\varepsilon}_{II} = \sqrt{\frac{1}{4}\left(\frac{\partial v_x}{\partial x} - \frac{\partial v_y}{\partial y} \right)^2 + \frac{1}{4}\left(\frac{\partial v_x}{\partial y} - \frac{\partial v_y}{\partial x} \right)^2}$$

Unknowns

$$\sigma_{xx} \,/\, \sigma_{yx} \,/\, \sigma_{xy} \,/\, \sigma_{yy} \,/\, v_x \,/\, v_y \,/\, p$$

4.2 Elastic rheology

Other than the ductile rheology the elastic behavior is time independent and the energy is conserved during the deformation, stored within the material. This means that there is no dissipation and no shear heating. In 2D there are different forms of the equations for plane strain and plane stress.

Plane strain

$$\sigma_{xx} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\left(\frac{\partial u_x}{\partial x} + \frac{\nu}{(1-\nu)}\frac{\partial u_y}{\partial y}\right)$$
$$\sigma_{yy} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\left(\frac{\nu}{(1-\nu)}\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right)$$
$$\sigma_{yx} = \frac{E}{2(1+\nu)}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)$$

where E is the Young's modulus and ν is the Poisson ratio, u is the displacement.

Plane stress

$$\sigma_{xx} = (L+2G)\frac{\partial u_x}{\partial x} + L\frac{\partial u_y}{\partial y}$$
$$\sigma_{yy} = L\frac{\partial u_x}{\partial x} + (L+2G)\frac{\partial u_y}{\partial y}$$
$$\sigma_{yx} = G\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)$$

where L is the Lame parameter and G is the shear modulus.

5. Energy conservation

The 2D Energy conservation is described by the Heat equation with the Heat conduction advection, the Heat source and the Heat production due to shear heating:

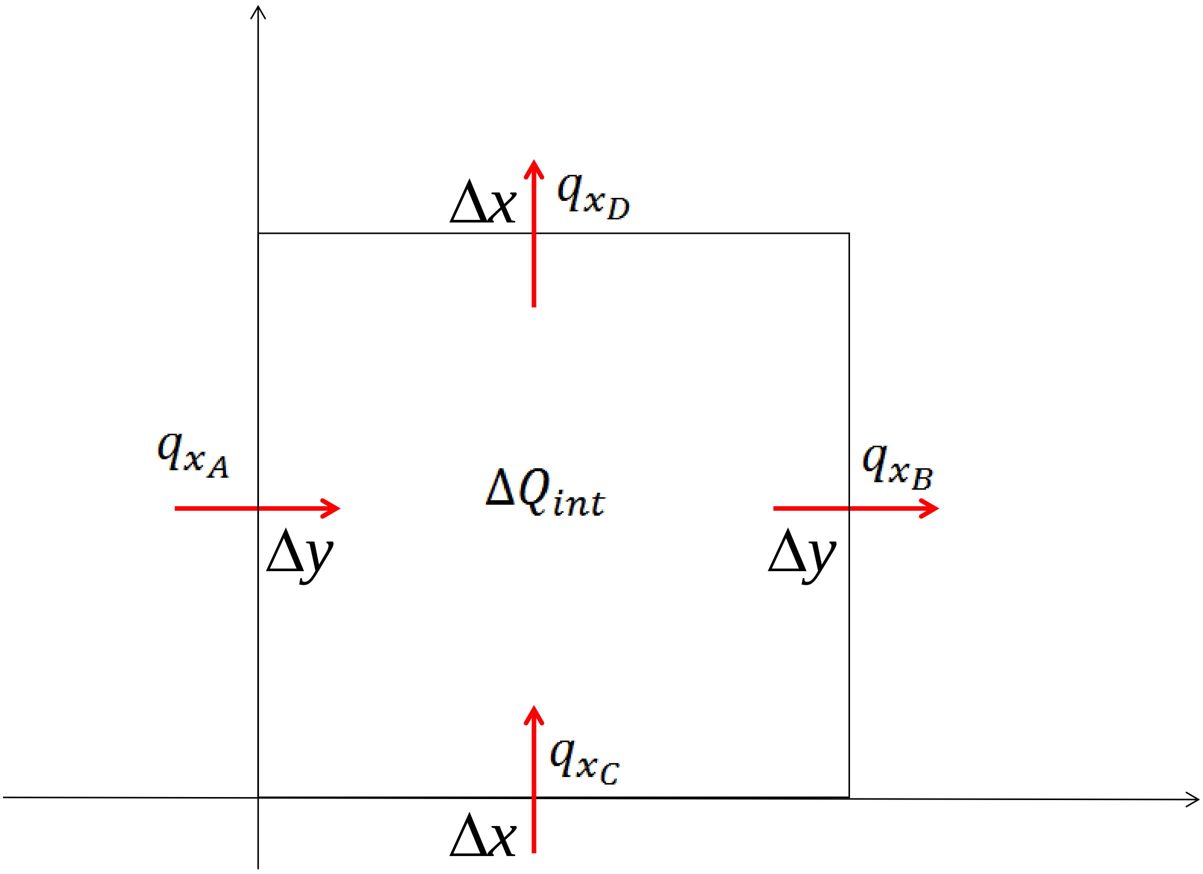
$$\underbrace{\rho c\frac{DT}{Dt} = \frac{\partial}{\partial x}\left(k_x\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y\frac{\partial T}{\partial y}\right)}_{\text{Heat conduction-advection}} + \underbrace{Q}_{\text{Heat source}} + \underbrace{\sigma_{xx}\frac{\partial v_x}{\partial x} + \sigma_{yx}\frac{\partial v_x}{\partial y} + \sigma_{xy}\frac{\partial v_y}{\partial x} + \sigma_{yy}\frac{\partial v_y}{\partial y}}_{\text{Heat production due to shear heating}}$$

where ρ is the density, c is the heat capacity, T is the temperature, k_i are constants, Q is the Heat source, σ_{ij} are the components of the stress tensor and v_i are the velocities. The total time derivative (Lagrangian) can be written using only partial time derivatives (Eulerian):

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y}$$

Therefore, we end up with one equation for one unknown which is the Temperature,T.

The derivation of the Temperature equation is very similar to the derivation of the momentum equation. In a first step we analyze the heat fluxes in a Lagrangian 2D volume:



Where $q_{x_A} - q_{y_D}$ are the heat flux components which are responsible for the heat flux through the respective boundaries. The heat flux is defined as:

$$q = -k\frac{\partial T}{\partial x}$$

where k is the thermal conductivity of the material (in W/m/K). The total heat fluxes through the different boundaries are:

$$\Delta Q_A = q_{x_A} \Delta y \Delta t$$

$$\Delta Q_B = q_{x_B} \Delta y \Delta t$$

$$\Delta Q_C = q_{y_c} \Delta x \Delta t$$

$$\Delta Q_D = q_{y_D} \Delta x \Delta t$$

Therefore:

$$\Delta Q = \Delta Q_A - \Delta Q_B + \Delta Q_C - \Delta Q_D + \Delta Q_{int}$$

The heat comes into the Lagrangian volume through the boundaries A and C and leaves it through the boundaries B and D. In addition there is an internal Heat source ΔQ_{int} within the Lagrangian volume. The heat which comes in and leaves the volume will change the Temperature. This is described by the following thermodynamic equation:

$$\Delta Q = m C_p \Delta T$$

Where m is the mass, C_p is the heat capacity at constant pressure (in J/kg/K) and ΔT is the change in Temperature. Therefore we come up with:

$$m C_p \Delta T = \Delta Q = \Delta Q_A - \Delta Q_B + \Delta Q_C - \Delta Q_D + \Delta Q_{int}$$

$$m C_p \Delta T = q_{x_A} \Delta y \Delta t - q_{x_B} \Delta y \Delta t + q_{y_c} \Delta x \Delta t - q_{y_D} \Delta x \Delta t$$

$$m C_p \Delta T = -\left(q_{x_B} - q_{x_A}\right) \Delta y \Delta t - (q_{y_D} - q_{y_c}) \Delta x \Delta t$$

We divide this equation by Δt and $V = \Delta x \Delta y$:

$$\frac{m}{V} C_p \frac{\Delta T}{\Delta t} = -\frac{\left(q_{x_B} - q_{x_A}\right) \Delta y \Delta t}{\Delta x \Delta y \Delta t} - \frac{(q_{y_D} - q_{y_c}) \Delta x \Delta t}{\Delta x \Delta y \Delta t} + \frac{\Delta Q_{int}}{V \Delta t}$$

$$\frac{m}{V} C_p \frac{\Delta T}{\Delta t} = -\frac{\left(q_{x_B} - q_{x_A}\right)}{\Delta x} - \frac{(q_{y_D} - q_{y_c})}{\Delta y} + \frac{\Delta Q_{int}}{V \Delta t}$$

With $\rho = \frac{m}{V}$ and $Q = \frac{1}{V}\frac{DQ_{int}}{Dt}$ and assuming that $\Delta x, \Delta y\ and\ \Delta t$ all tend towards zero we come up with:

$$\rho C_p \frac{DT}{Dt} = -\frac{\partial q_x}{\partial x} - \frac{\partial q_y}{\partial y} + Q$$

With

$$q_i = -k_i \frac{\partial T}{\partial x_i}$$

we get:

$$\rho C_p \frac{DT}{Dt} = \frac{\partial}{\partial x}\left(k_x \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y \frac{\partial T}{\partial y}\right) + Q$$

Therefore we derived the conduction advection part of the heat equation. The heat production due to shear heating is defined as:

$$Q_{shear} = \sigma_{xx} \frac{\partial v_x}{\partial x} + \sigma_{yx} \frac{\partial v_x}{\partial y} + \sigma_{xy} \frac{\partial v_y}{\partial x} + \sigma_{yy} \frac{\partial v_y}{\partial y}$$