Problem Set 8

1. Let Σ be a finite alphabet. The atomic formulae in MSO defined over Σ^* are $x = y, x < y, S(x, y), X(x)$ and $Q_a(x)$, $a \in \Sigma$. Consider the following logic called MSO_0 having atomic formulae of the following forms:

$$Sing(X), X \subseteq Y, X < Y, S(X, Y), Q_a(X)$$

where
- $Sing(X)$ means that X is a SO variable of cardinality 1;
- $X \subseteq Y$ means that every element of the SO variable X is contained in the SO variable Y;
- $X < Y$ means that SO variables X, Y have cardinality 1, and that the element in Y is greater than the element in X;
- $S(X, Y)$ means that SO variables X, Y have cardinality 1, and Y contains the successor of the element in X; and,
- $Q_a(X)$ means that all positions in X are decorated by $a \in \Sigma$.

If φ is an atomic formula in MSO, then $\varphi \wedge \varphi, \neg\varphi, \varphi \vee \varphi$, $\forall x\ \varphi$ and $\forall X\ \varphi$ are formulae in MSO. Similarly, if φ is an atomic formula in MSO_0, then, $\varphi \wedge \varphi, \neg\varphi, \varphi \vee \varphi$ and $\forall X\ \varphi$ are formulae in MSO_0.
Compare the expressiveness of MSO and MSO_0.

2. For the formula $\exists x \forall y (x < y \rightarrow Q_a(y))$ give an equivalent MSO_0 formula.

3. Consider the following NFA $N = (\{0, 1, 2, 3\}, \{a, b\}, \Delta, \{0\}, \{1\})$ with $\Delta(0, b) = \{1\}$, $\Delta(1, a) = \{2\}$, $\Delta(2, a) = \{2\}$, $\Delta(2, b) = \{3\}$ and $\Delta(3, b) = \{0\}$. Write an MSO formula with two SO variables that characterizes $L(N)$.

4. Prove or disprove : Every MSO formula $\varphi(X_1, \ldots, X_n)$ over words is equivalent to an EMSO formula, that is a formula of the form

$$\exists Y_1 \ldots \exists Y_m \psi(X_1, \ldots, X_n, Y_1, \ldots, Y_m)$$

where ψ is an MSO formula.

5. Formulas in Presburger Arithmetic (PA) use first-order variables $x, y, \ldots$ that are evaluated over the structure $\mathcal{A} = (N, +)$, where $N = \{0, 1, \ldots\}$ is the set of natural numbers, and, $+$ is the ternary addition predicate. Here is a valid sentence in this logic:

$$\forall\ x\ \forall\ y\ \exists\ z\ [(x + z = y) \vee (y + z = x)]$$

Show that the relations $x < y, S(x, y)$ can be defined in PA.

As future work, think about the satisfiability problem of PA : given a formula φ in PA, can you decide if φ is satisfiable?