1.]

(a) $a^+ b^*$

$$\left[\exists x \left(First(x) \wedge Q_a(x)\right)\right] \wedge \left[\forall x \left(Q_b(x) \rightarrow \forall y \left((x<y) \rightarrow Q_b(y)\right)\right)\right]$$

OR

$$\forall x \forall y \left[\left(Q_a(x) \wedge Q_b(y)\right) \rightarrow (x<y)\right]$$

(b) $a a b^* a a$

$$\exists x_1 x_2 x_3 x_4 \left(First(x_1) \wedge Second(x_2) \wedge Last(x_4) \wedge SecondLast(x_3) \wedge \forall y \left\{\left((x_2<y) \wedge (y<x_3)\right) \rightarrow Q_b(y)\right\} \right)$$

(c) $$\left[\exists x_1 x_2 x_3 \left((x_1<x_2) \wedge (x_2<x_3) \wedge Q_b(x_1) \wedge Q_b(x_2) \wedge Q_b(x_3)\right)\right]$$

$$\wedge \left[\exists x \; \underbrace{Q_b(x) \wedge \forall y \left((y<x) \rightarrow \neg Q_b(x)\right)}_{\text{first 'b'}} \wedge \neg\left[\exists x_1 x_2 x_3 \left((x_1<x_2) \wedge (x_2<x_3) \wedge (x_3<x) \wedge Q_a(x_1) \wedge Q_a(x_2) \wedge Q_a(x_3)\right)\right]\right]$$

3 a's do not occur before the first 'b'.

2] MSO

every x belongs to atleast one part

$$\exists R_0 R_1 \cdots R_{n-1} \left[\forall x \left(R_0(x) \vee R_1(x) \cdots \vee R_{n-1}(x) \right) \wedge \bigwedge_{0 \le i < j < n} \neg(R_i(x) \wedge R_j(x)) \right]$$

{ no 'x' belongs to 2 parts }

$$\bigwedge_{0 \le i < n} \left[R_i(x) \rightarrow \forall y \left(S(x,y) \rightarrow \left\{ (Q_0(y) \rightarrow R_{2i \% n}(y)) \wedge (Q_1(y) \rightarrow R_{(2i+1) \% n}(y)) \right\} \right) \right]$$

$w = \underline{\quad - - - \quad}$ - - ; $R_i(x)$ is 1 iff remainder of the prefix of w upto x = i
($\uparrow x$)

$$\wedge \left[\exists x \; First(x) \wedge (Q_0(x) \rightarrow R_0(x)) \wedge (Q_1(x) \rightarrow R_1(x)) \right]$$

(base condition)

$$\wedge \left[\exists x (Last(x) \wedge R_0(x)) \right]$$

final condition that remainder of entire string is 0

This is for $n \geq 2$.

For $n=1$ we will just have all non-empty words

3 i) $\neg\epsilon + \neg((\neg\emptyset).(\Sigma \cap \neg a).(\neg\emptyset))$

$\Sigma^{+} \cap \neg(\Sigma^{*}(\Sigma - \{a\})\Sigma^{*})$

Not empty, and does not contain anything except a.

ii) $b.[[(a\Sigma^{*} \cap \Sigma^{*}b) \cap \neg(\Sigma^{*}aa\Sigma^{*} \cup \Sigma^{*}bb\Sigma^{*})]$
$\cup\ \epsilon]$

$b.[[(a\Sigma^{*} \cap \Sigma^{*}b) - \Sigma^{*}aa\Sigma^{*} \cup \Sigma^{*}bb\Sigma^{*}]$

$\cup\ \epsilon]$ Also empty word

Starts with a and ends with b

No consecutive a's or b's

4.
i) $\forall n_0\ \exists n \geq n_0\ \exists u, v, w \in \Sigma^{*}\ ((\neg(uv^{n}w \in L) \wedge uv^{n+1}w \in L)$
$\vee (uv^{n}w \in L \wedge \neg uv^{n+1}w \in L))$

For any n_0 there is some $n \geq n_0$ such that either $uv^{n}w$ is in L, and $uv^{n+1}w$ is not, or $uv^{n}w$ is not in L, $uv^{n+1}w$ is in L.

ii) $L = (aa)^+$

Take $u = \epsilon$, $v = a$, $w = \epsilon$

$uv^n w \in L$ if n is even,
$\notin L$ if n is odd

For any n_0

$uv^{2n_0} w \in L$

$uv^{2n_0+1} w \notin L$

$\therefore$ We can't have any n_0 such that $\forall n > n_0$

$uv^n w$ are all in L, or none are

$\therefore$ L is counting

iii) $L = (ab)^+$

Take $n_0 = 2$

a) For any v, w, if v is not of the form $(ab)^k$ or $(ba)^k$,

then $uv^n w$ will have two consecutive a's or two consecutive b's and cannot be in L $\forall n \geq 2$

b) Suppose $v = (ab)^k$ or $(ba)^k$ and $uv^n w = \cancel{(ab)^{k+d}}\ (ab)^{nk+d} \in L$,

then $uv^{n+1} w = (ab)^{nk+d+k} \in L$

$\therefore$ L is non counting