Problem Set 7

1. Give FO or MSO formulae for the languages of the following expressions.
   (a) a^+b^*
   (b) aab^*aa
   (c) there are at least 3 occurrences of b, and before the first b, there are at most two occurrences of a
   (d) $\{w \in \{a,b\}^* \mid w \text{ has equal number of } a \text{ and } b\}$

2. Let $L_n = \{x \mid x \text{ is a binary number that is a multiple of } n\}$. Show that for all $n \geq 1$, L_n can be captured by an MSO formula.

3. A star free expression is a regular expression built from atoms $\epsilon, a \in \Sigma, \emptyset$ using only the operations $\cdot$ for concatenation, $+$ for union, $\cap$ for intersection and $\neg$ for complementation. The language $L(r)$ defined by such an expression r is called star-free.
   As an example, Σ^* is star-free since it can be written as $\neg\emptyset$. Show that the following are star free, and in each case verify that you can write an FO formula for each.
   - a^+
   - $b(ab)^*$

4. Call $L \subseteq \Sigma^+$ non counting if

$$\exists n_0 \forall n \geq n_0 \forall u, v, w \in \Sigma^* (uv^n w \in L \Leftrightarrow uv^{n+1} w \in L)$$

   That is for all $n \geq n_0$, either all $uv^n w$ are in L, or none is.
   A language L is counting iff it is not non counting.
   - Formulate the condition for a counting language
   - Is $L = (aa)^+$ counting or not?
   - Is $L = (ab)^+$ counting or not?

5. Consider the formula
   $\varphi = \exists x \forall y (x \leq y \wedge Q_a(x)) \wedge \exists x \forall y (y \leq x \wedge Q_a(x)) \wedge \exists x (Q_b(x)) \wedge$
   $\forall x \forall y (S(x,y) \leftrightarrow \neg(Q_a(x) \wedge Q_a(y))) \wedge \forall x \forall y (S(x,y) \leftrightarrow \neg(Q_b(x) \wedge Q_b(y)))$
   Using the logic to automaton construction, construct a DFA A_φ such that $L(\varphi) = L(A_\varphi)$.