CS 228 : Logic in Computer Science

S. Krishna

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- Let ψ be the formula $\exists y \exists w \{Q_a(w) \wedge Q_b(y) \wedge \forall x(Q_a(x) \rightarrow x > y) \wedge \exists z[Q_b(z) \wedge \forall t(z \geqslant t)]\}$. What is $L(\psi)$?

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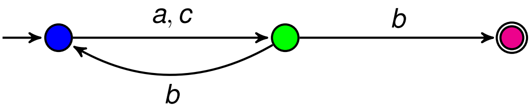
- Formula φ is satisfiable iff $L(\varphi) \neq \emptyset$.
- Formula φ is valid iff $L(\varphi) = \Sigma^*$.
- Question : How to check satisfiability of FO over words?

Idea for SAT checking

- Given FO formula φ over an alphabet Σ, construct an edge labeled graph G_φ : a graph whose edges are labeled by Σ.



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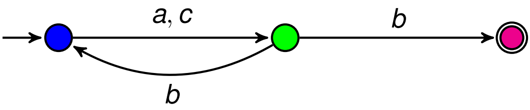

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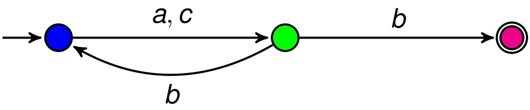

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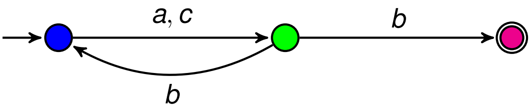

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- G_φ has some special kinds of vertices
  - There is a unique vertex called the start vertex (blue vertex)
  - There are some vertices called good vertices (magenta vertex)
- Read off words on paths from the start vertex to any final vertex and call this set of words $L(G_\varphi)$
- Ensure that G_φ is constructed such that $L(\varphi) = L(G_\varphi)$.

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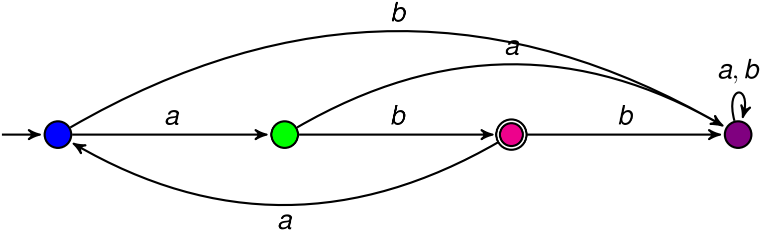
- How to construct G_φ?

A First Labeled Graph A



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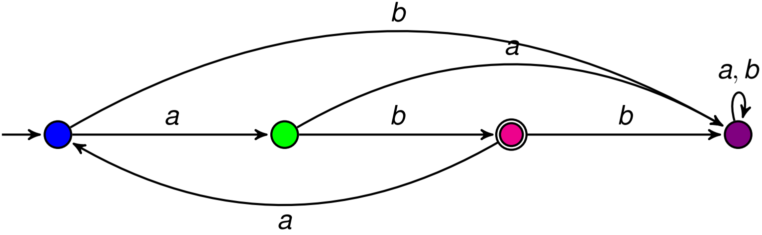

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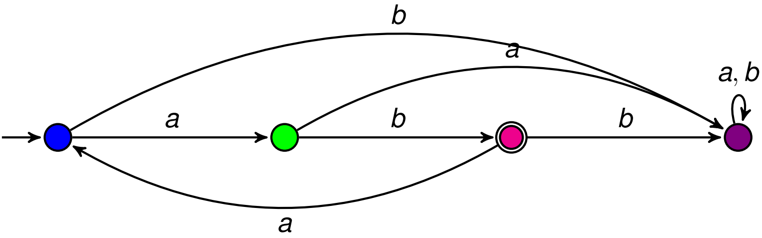

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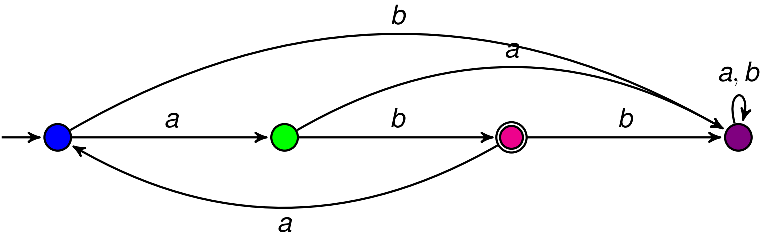

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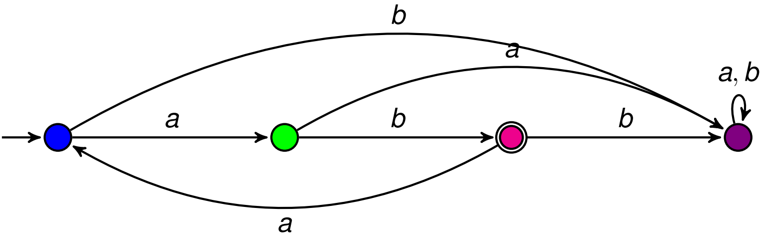

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- The set of words accepted by the graph is called the language of the graph

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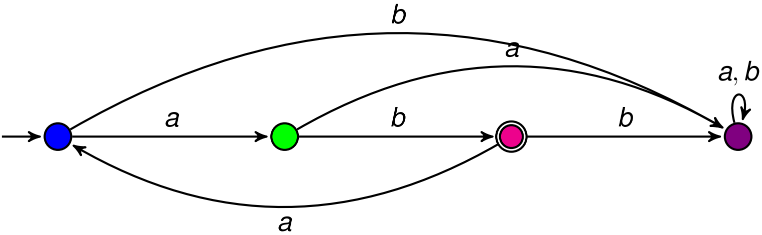

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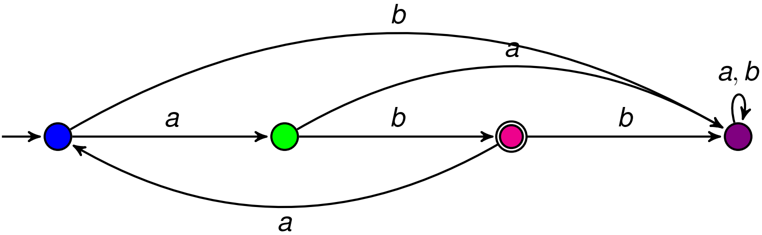
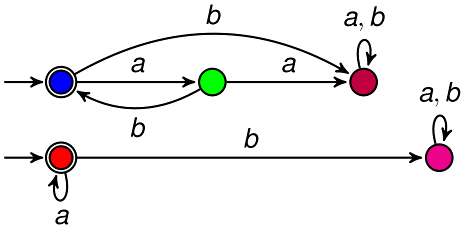


- What is the language L accepted by this graph, $L(A)$?
- Write an FO formula φ such that $L(\varphi) = L(A)$

A Second and a Third Graph B, C



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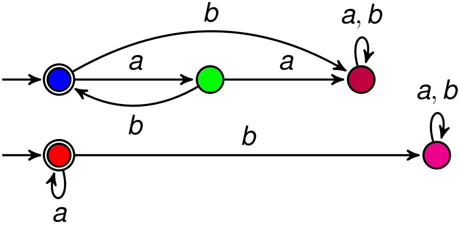

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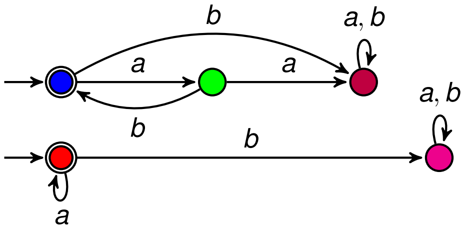

- What are $L(B), L(C)$?
- Give an FO formula φ such that $L(\varphi) = L(B) \cup L(C)$

$\neg\exists x(x = x) \vee \exists x(Q_a(x) \wedge first(x)) \wedge \exists y(Q_b(y) \wedge last(y)) \wedge$
$\forall x \forall y[(S(x,y) \wedge Q_a(x) \to Q_b(y)) \wedge (S(x,y) \wedge Q_b(x) \to Q_a(y))]$

$$\vee$$

$\forall x(Q_a(x))$

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- $q_0 \in Q$ is the initial state
- $F \subseteq Q$ is the set of final states
- $L(A)$=all words leading from q_0 to some $f \in F$

Languages, Machines and Logic

A language $L \subseteq \Sigma^*$ is called regular iff there exists some DFA A such that $L = L(A)$.

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A language $L \subseteq \Sigma^*$ is called regular iff there exists some DFA A such that $L = L(A)$.
A language $L \subseteq \Sigma^*$ is called FO-definable iff there exists an FO formula φ such that $L = L(\varphi)$.

Is it Regular? Is it FO-definable?

$\Sigma = \{a, b\}$. Consider the following languages $L \subseteq \Sigma^*$:

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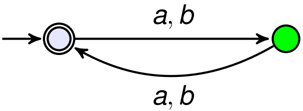

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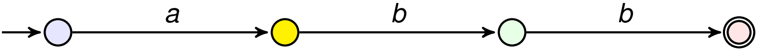

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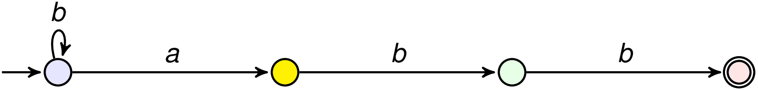

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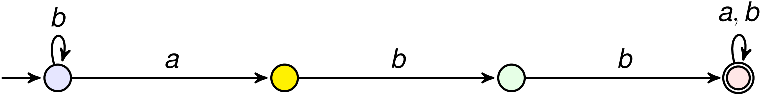

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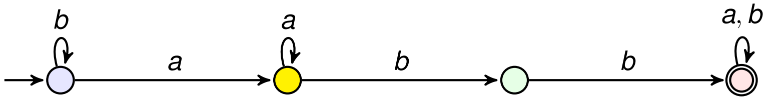

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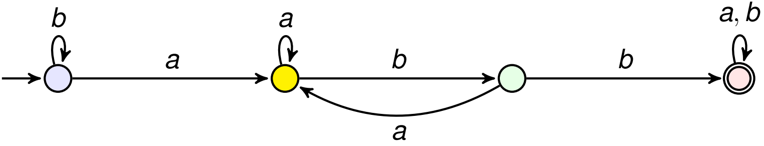

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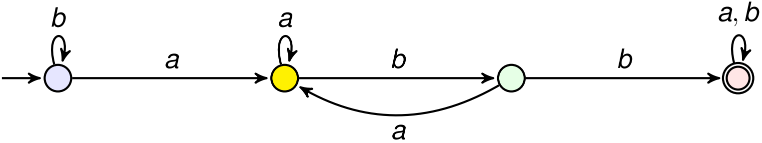

$$\exists x \exists y \exists z (Q_a(x) \wedge Q_b(y) \wedge Q_b(z) \wedge S(x, y) \wedge S(y, z))$$

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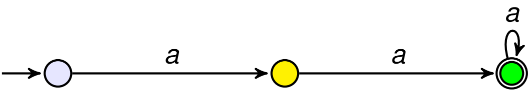

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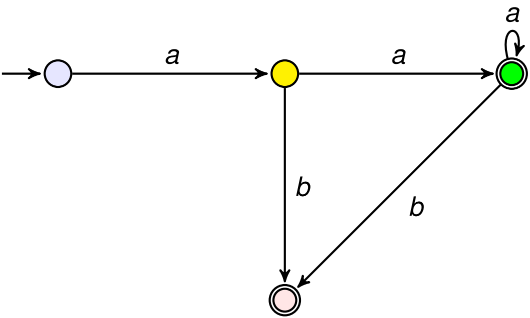

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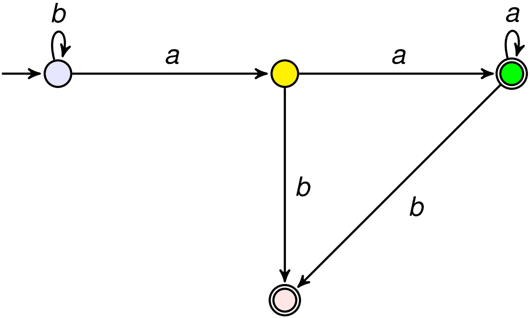

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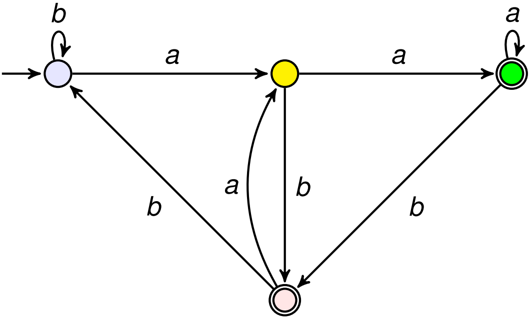

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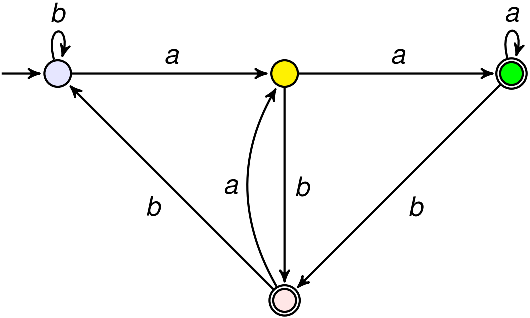

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