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Optimal Regret Bounds for Stochastically Partitioning Experts

author names withheld

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Abstract

1. Introduction

In the standard decision-theoretic online learning there are N experts (or actions) at the disposal of a learner. In round t, the learner chooses a probability mass function $\boldsymbol{p}_t$ over the set of experts $\{1, 2, \ldots, N\}$. An adversary reveals the loss vector $\boldsymbol{l}_t = (l_t(1), \ldots, l_t(N)) \in [0, 1]^N$ and the learner suffers the (expected) loss $L_t = \sum_{r=1}^{t} \langle \boldsymbol{p}_r, \boldsymbol{l}_r \rangle$. Total loss for the learner after T rounds is $L_T = \sum_{r=1}^{T} l_r$, and the total loss of choosing expert i in all the rounds is $L_T(i) = \sum_{r=1}^{T} l_r(i)$. The learner aims to minimize its regret $R_T = L_T - \min_i L_T(i)$.

Decision-theoretic online learning has been extensively studied in the literature under various settings . Significant research in improving the regret bound concerning the number of rounds T. Relatively recently, Gofer et al. (2013) focused on regret bounds concerning N, when the number of experts is growing.

In this paper, we study decision-theoretic online learning under stochastically branching experts setting where the experts are convex partitions of a bounded convex set $\mathbb{B}$ that is a subset of d-dimensional Euclidean space $\mathbb{V}$. In each round t, the environment reveals new experts by partitioning the existing experts as follows. The environment draws a point i.i.d. from $\mathbb{B}$ using a fixed (unknown) distribution. The new experts are the convex partitions obtained from d orthogonal hyperplanes passing through this new point. Note that at each round the environment reveals only the losses of the existing experts and we allow the losses to be adversarial. We consider the perfect clone setting introduced in Gofer et al. (2013), where a new expert is a perfect clone of its parent expert, i.e., the cumulative loss of a new partition is equal to the cumulative loss of its parent partition. Once the new expert is revealed, its cumulative loss evolves independently from its parent expert.

1.1. Motivation

The above setting with $\mathbb{B} = [0, 1]$ arises in classification applications that use DL models with a reject/offload option Moothedath et al. (2023); Beytur et al. (2024). For each data sample, a pre-trained DL model outputs softmax values and the learner computes a confidence metric μ (value in $[0, 1]$) using the softmax values. A typical choice for the confidence metric is the maximum softmax value as the data sample is classified into the class with the maximum softmax value. The learner then uses a threshold to accept the classification if the confidence metric is above the threshold, incurring a loss of zero if the classification is correct and a loss of one, otherwise, and rejects or offloads the sample to an expert, incurring a cost $c \in [0, 1]$. In this problem, the experts are the intervals created by μ values corresponding to the data samples that arrive over time Moothedath et al. (2023). If a learner chooses an interval, then the classification of the DL model is accepted if

Algorithm 1: Hed_G

Initialize: $\mathcal{B}_0 = \{1, \ldots, n_0\}$, $w_1(i) = 1$ for all $i \in \mathcal{B}_0$, and $W_1 = 1$.
for each round $t = 1, 2, \ldots, \tau$ **do**
  if new expert revealed **then**
    $n_t = n_{t-1} + 1$ and $\mathcal{B}_t = \mathcal{B}_{t-1} \cup \{n_t\}$
    Compute new weight $w_t(n_t) = e^{-\eta L_t(n_t)}$, and $\hat{W}_t = W_t + w_t(n_t)$
  else
    $\hat{W}_t = W_t$
  end if
  Compute $p_{i,t} = \frac{w_{i,t}}{\hat{W}_t}$, for all $i \in \mathcal{B}_t$.
  Choose an expert using $\boldsymbol{p}_t$, observe $\boldsymbol{l}_t$, and incur the loss $\langle \boldsymbol{p}_t, \boldsymbol{l}_t \rangle$.
  Update the weights $w_{t+1}(i) = e^{-\eta l_i(t)} w_t(i)$
  Cumulative weight $W_{t+1} = \sum_{i=1}^{n_t} w_t(i)$.
end for

the μ value is greater than the supremum of the chosen interval, else the classification is rejected and the data sample is offloaded. For this problem, the partition of the intervals is illustrated in Fig.**??**. We also illustrate the partition of the experts for 2-dimensional plane in Fig **??**.

1.2. Related Work

2. Stochastically Partitioning Experts

The game starts with the expert $\mathbb{B}$. In round 1, the environment draws a point $X_t \in \mathbb{B}$ creating 2^d experts, which we index $1, \ldots, 2^d$. Similarly, in round t, the environment samples point X_t from $\mathbb{B}$ creating $(t+1)^d$ experts. We assign the index of each parent expert to one of its children and assign new indices $t^d + 1, \ldots, (t+1)^d$ to the remaining unindexed new experts. We use $\mathcal{B}_t = \{1, \ldots, n_t\}$, where $n_t = (t+1)^d$, to denote the set of indices at the end of round t.

Lemma 1 *Given a sequence of i.i.d. continuous random variables $X_1, X_2, \ldots, X_t$ drawn from $[a, b]$ with cumulative distribution function $F_X(x)$, for any permutation $(i_1, i_2, \ldots, i_{t-1})$ of $(1, 2, \ldots, t-1)$ and any $k \in \{2, \ldots, t\}$, we have*

$$\mathbb{P}\left(X_t \in [X_{i_{k-1}}, X_{i_k}] \,\middle|\, X_{i_1} < X_{i_2} < \cdots < X_{i_{t-1}}\right) = \frac{1}{t}.$$

Proof Given that $X_1, X_2, \cdots, X_t$ are i.i.d. continuous random variables, for any two permutations $(j_1, j_2, \ldots, j_t)$ and $(k_1, k_2, \ldots, k_t)$, we have

$$\mathbb{P}(X_{j_1} < X_{j_2} < \ldots < X_{j_t}) = \mathbb{P}(X_{k_1} < X_{k_2} < \cdots < X_{k_t}).$$

Since the events $\{X_{i_1} < X_{i_2} < \cdots < X_{i_t}\}$ are mutually exclusive and there are $t!$ permutations, we have

$$\sum_{\{i_1, i_2, \ldots, i_n\}} \mathbb{P}\left(X_{i_1} < X_{i_2} < \cdots < X_{i_t}\right) = 1 \implies \mathbb{P}\left(X_{i_1} < X_{i_2} < \cdots < X_{i_t}\right) = \frac{1}{t!}.$$

Further,

$$\mathbb{P}\left(X_t \in [X_{i_{k-1}}, X_{i_k}] \,\middle|\, X_{i_1} < X_{i_2} < \cdots < X_{i_{t-1}}\right)$$
$$= \frac{P\left(X_{i_1} < \cdots < X_{i_{k-1}} < X_t < X_{i_k} < \cdots < X_{i_{t-1}}\right)}{P\left(X_{i_1} < X_{i_2} < \cdots < X_{i_{t-1}}\right)} = \frac{\frac{1}{t!}}{\frac{1}{(t-1)!}} = \frac{1}{t}.$$

■

Theorem 2 *The regret R_T of Hed_G run with $\eta = \sqrt{\frac{8}{T}\ln T}$ for i.i.d. X_t arrivals satisfies*

$$R_T \leq \sqrt{T \ln T}.$$

Proof From line (5) of $\mathrm{Hed_G}$, we have

$$\hat{W}_t = \sum_{i \in \mathcal{B}_{t+1}} e^{-\eta L_i(t)}. \tag{1}$$

From line (12) of $\mathrm{Hed_G}$, it follows that

$$\ln \frac{W_{t+1}}{W_t} = \ln \frac{W_{t+1}}{\hat{W}_t} + \ln \frac{\hat{W}_t}{W_t}. \tag{2}$$

We now upper bound the second term in (1). Using (2) we have

$$\ln \frac{\hat{W}_t}{W_t} = \ln \frac{\sum_{i\in\mathcal{B}_t} e^{-\eta L_i(t)} + e^{\eta L_{n_t}(t)}}{\sum_{i\in\mathcal{B}_t} e^{-\eta L_i(t)}} = \ln \left(1 + \frac{e^{-\eta L_{n_t}(t)}}{\sum_{i\in\mathcal{B}_t} e^{-\eta L_i(t)}}\right) \leq \frac{e^{-\eta L_{n_t}(t)}}{\sum_{i\in\mathcal{B}_t} e^{-\eta L_i(t)}}. \tag{3}$$

Define $Y_t = \dfrac{e^{-\eta L_{n_t}(t)}}{\sum_{i\in\mathcal{B}_t} e^{-\eta L_i(t)}}$ and $H_t =$ History till time t which comprises X_τ and l_τ for $\tau \leq t$. We calculate the expectation of Y_t. For any permutation, $(i_1, i_2, \ldots, i_{t-1})$ of $(1, 2, \ldots, t-1)$, we have that

$$\begin{aligned} E[Y_t|H_{t-1}] = &\sum_{k=2}^{t} E\left[Y_t \,\middle|\, H_{t-1}, X_{i_1} < X_{i_2} < \cdots < X_{i_{t-1}}\right] \\ &\times P\left(X_{i_1} \cdots X_{i_{k-1}} < X_t < X_{i_k} \cdots < X_{i_{t-1}} | H_{t-1}\right) \qquad (4) \\ = &\sum_{k=2}^{t} \frac{e^{-\eta L_k(t)}}{\sum_{i\in\mathcal{B}_t} e^{-\eta L_i(t)}} \times \frac{1}{t} = \frac{1}{t}. \qquad (5) \end{aligned}$$

Here (5) follows from (3) and (1). Further,

$$E[Y_t] = E_{H_t}[E[Y_t|H_{t-1}]] = E\left[\frac{1}{t}\right] = \frac{1}{t} \implies \sum_{t=1}^{T} E[Y_t] = \sum_{i=1}^{T} \frac{1}{i} \leq \ln T. \tag{6}$$

Next, we upper and lower bound $\ln\left(\frac{W_T}{W_1}\right)$. By definition,

$$\ln\left(\frac{W_T}{W_1}\right) = \ln\left(\prod_{t=1}^{T-1}\frac{W_{t+1}}{W_t}\right) = \sum_{t=1}^{T-1}\ln\frac{W_{t+1}}{\hat{W}_t} + \ln\frac{\hat{W}_t}{W_t}$$
$$\leq \sum_{t=1}^{T}\left[-\eta\langle \boldsymbol{p}_t, \boldsymbol{l}_t\rangle + \frac{\eta^2}{8} + Y_t\right] \leq -\eta L_{alg}(t) + \frac{\eta^2 T}{8} + \sum Y_t. \tag{7}$$

Here (7) follows from Hoeffding's lemma. Also,

$$\ln\frac{W_T}{W_1} = \ln\sum_{i=1}^{N_T} e^{-\eta L_i(T)} \geq \ln\max_{i\in\mathcal{B}_T} e^{-\eta L_i(T)} = \max_{i\in\mathcal{B}_t}\ln e^{-\eta L_i(T)} = -\eta\min_{i\in\mathcal{B}_T} L_i(T). \tag{8}$$

From (7) and (8) it follows that

$$-\eta\min_{i\in\mathcal{B}_T} L_i(T) \leq -\eta L_{alg}(T) + \frac{\eta^2 T}{8} + \sum Y_t \implies L_{alg}(T) \leq \min_{i\in\mathcal{B}_T} L_i(T) + \frac{T\eta}{8} + \frac{\sum Y_t}{\eta}. \tag{9}$$

Substituting $\eta = \sqrt{\frac{8\ln T}{T}}$ in (9), we get

$$R_T = L_{alg}(T) - \min_{i\in\mathcal{B}_T} L_i(T) \leq \sqrt{T\ln T} + \sqrt{\frac{T}{\ln T}}\sum Y_t \leq 2\sqrt{T\ln T}. \tag{10}$$

■

Theorem 3 *The regret R_T of Hed$_G$ run for d dimensional with $\eta = \sqrt{\frac{82^d}{T}\ln T}$ for i.i.d. X_t arrivals satisfies*

$$R_T \leq \sqrt{2^d T\ln T}. \tag{11}$$

Theorem 4 *$Hedge_G$ run with $\varepsilon = \frac{m}{T}$ and $\eta = \frac{\sqrt{2m\ln(T)}}{T}$ then expected number of revealed loss rounds equals m and*

$$R_T \leq \frac{3T\ln(T)}{\sqrt{m}} \tag{12}$$

Theorem 5 *For $Hedge_G$ with Doubling Trick, we get*

$$R_T \leq 2(\sqrt{2}+1)\sqrt{T(\sum_{t=1}^{T} Y_t + 1)}$$
$$+ 2\log_2(2\sqrt{\sum_{t=1}^{T} Y_t + 1})\sqrt{T\ln T} \tag{13}$$

Theorem 6 *For Doubling Trick $Hedge_G$ With probability $\geq 1-\delta$ where $\delta = 2e^{\frac{-2\epsilon^2}{T}}$ and $\epsilon > 0$ we get*

$$R_T \leq \sqrt{T(\ln T + \epsilon)} + \ln(\ln(T) + \epsilon)\sqrt{T(\ln T)} \tag{14}$$

Theorem 7 *For Doubling Trick $Hedge_G$ With probability $\geq 1-\delta$ where $\delta = o(1)$ and $\epsilon = \omega(\sqrt{T})$ we get*

$$R_T = \omega(T^{\frac{3}{4}}) \tag{15}$$

3. Lower Bound

For the first $\frac{T}{2}$ timesteps give loss 0 to all the experts. After that the children get the same loss as their parents essentially having $\frac{T}{2}$ experts and hence the loss is lower bounded by a hedge with $N = \frac{T}{2}$ experts proving a regret lower bound of $\Omega(\sqrt{T \ln T})$.

4. Adversarial Branching Experts

We show that Hed_G has $O(\sqrt{TN_T})$ for the adversarial branching experts problem in Gofer et al. (2013).

5. Experts Drawn from Countably Finite Set

Consider a finite set $\hat{\mathcal{B}}$ containing N experts. The expert set is either unknown or $N >> 1$. At each round $t \in \{1, \ldots, T\}$, the environment draws an expert uniformly at random from the set of experts and is presented to a learner. An expert presented in a round could either be a repetition or a new expert. Let n_t denote the number of unique experts revealed till time t. The new experts are indexed in the order they are revealed, i.e., $n_t = n_{t-1} + 1$, if a new expert is revealed, and $n_t = n_{t-1}$, otherwise. Thus, the set of experts in round t is $\mathcal{B}_t = \{1, 2, \ldots, n_t\}$.

Let $\boldsymbol{p}_t$ be the probability mass function used by the learner for choosing an expert from $\mathcal{B}_t$ and $\boldsymbol{l}_t = (l_t(1), \ldots, l_t(n_t)) \in [0,1]^{n_t}$ is the loss vector revealed after choosing an expert. In round t, the cumulative loss of expert $i \in \mathcal{B}_t$ is

$$L_t(i) = \sum_{r=1}^{t} l_r(i),$$

and the expected cumulative loss of the learner is $L_t = \sum_{r=1}^{t} \langle \boldsymbol{p}_r, \boldsymbol{l}_r \rangle$. We assume that once a new expert n_t is revealed, the learner will have access to its cumulative loss $L_{t-1}(n_t)$. For $\hat{t} > t$, $L_t^{\hat{t}}(i)$ and $L_t^{\hat{t}}$ are the cumulative losss in the slots $\{t+1, \ldots, \hat{t}\}$ under expert i and the learner, respectively, given by

$$L_t^{\hat{t}}(i) = \sum_{r=t+1}^{\hat{t}} l_r(i) \text{ and } L_t^{\hat{t}} = \sum_{r=t+1}^{\hat{t}} \langle \boldsymbol{p}_r, \boldsymbol{l}_r \rangle.$$

Define

$$L_t^* = \min_{i \in \mathcal{B}_t} L_t(i).$$

Algorithm 2: Hed_G-Hedge (with parameter τ)

1: For $t = 1, \ldots, \tau$, use Hed_G

2: For $t > \tau$, use the vanilla Hedge only using the experts from $\mathcal{B}_\tau$ and resetting the weights $w_{i,\tau+1} = 1$, for all $i \in \mathcal{B}_\tau$.

The aim is to minimize the regret $R_T = L_T - L_T^*$. Let b_t denote the best expert till time slot t, i.e.,

$$b_t = \arg\min_{i \in \mathcal{B}_t} L_t(i).$$

Note that $\mathcal{B}_T \subseteq \hat{\mathcal{B}}$.

5.1. The Hed_G-Hedge Algorithm

We propose a modification to Hed_G that uses Hed_G only for the slots $t \leq \tau$, for some $\tau < T$, and uses the vanilla Hedge for $t > \tau$ using the experts from $\mathcal{B}_\tau$ and ignoring any newly revealed experts.

Let $q_\tau = \mathbb{P}(b_T \in \mathcal{B}_\tau)$ denote the probability that the best expert for T slots belongs to the set of experts revealed in the first τ slots. We have

$$q_\tau = 1 - (1 - \nu(b_T))^\tau.$$

Lemma 8 *For any $\tau \leq T$, the regret bound for Hed_G is given by*

$$L_\tau - L_\tau(b_\tau) \leq \frac{\eta\tau}{8} + \frac{e^{\eta\tau}}{\eta}(\ln n_\tau + 1).$$

Proof Step 1:

$$\begin{aligned}
\ln \frac{W_{\tau+1}}{W_1} &= \ln \sum_{i=1}^{n_\tau} e^{-\eta L_\tau(i)} \\
&\geq \ln \max_{i \in \mathcal{B}_\tau} e^{-\eta L_\tau(i)} \\
&\geq \max_{i \in \mathcal{B}_\tau} \ln e^{-\eta L_\tau(i)} \\
&= -\eta \min_{i \in \mathcal{B}_\tau} L_\tau(i) \\
&= -\eta L_\tau(b_\tau). \qquad (16)
\end{aligned}$$

Step 2: We have

$$\begin{aligned}
\frac{w_t(n_t)}{W_t} &= \frac{e^{-\eta L_t(n_t)}}{\sum_{j=1}^{n_t} e^{-\eta L_t(j)}} \\
&\leq \frac{1}{n_t}(e^{-\eta \sum_{j=1}^{n_t} L_j})^{-1/n_t} \\
&\leq \frac{1}{n_t}(e^{-\eta t n_t})^{-1/n_t} \\
&= \frac{e^{\eta t}}{n_t}. \qquad (17)
\end{aligned}$$

Let E_t denote the event of a new expert arrival in round t. For $t \leq \tau$,

$$\begin{aligned}
\ln \frac{W_{t+1}}{W_t} &= \ln \frac{W_{t+1}}{\hat{W}_t} + \ln \frac{\hat{W}_t}{W_t} \\
&\leq -\eta \langle \boldsymbol{p}_t, \boldsymbol{l}_t \rangle + \frac{\eta^2}{8} + \mathbb{1}(E_t) \ln \left(1 + \frac{w_t(n_t)}{W_t} \right) \\
&\leq -\eta \langle \boldsymbol{p}_t, \boldsymbol{l}_t \rangle + \frac{\eta^2}{8} + \mathbb{1}(E_t) \frac{w_t(n_t)}{W_t} \\
&\leq -\eta \langle \boldsymbol{p}_t, \boldsymbol{l}_t \rangle + \frac{\eta^2}{8} + \mathbb{1}(E_t) \frac{e^{\eta t}}{n_t} \qquad (18)
\end{aligned}$$

In the last equation above we have used (17). Therefore,

$$\begin{aligned}
\ln \frac{W_{\tau+1}}{W_1} &= \ln \prod_{t=1}^{\tau} \frac{W_{t+1}}{W_t} \\
&= \sum_{t=1}^{\tau} \ln \frac{W_{t+1}}{W_t} \\
&\leq -\eta \sum_{t=1}^{\tau} \langle \boldsymbol{p}_t, \boldsymbol{l}_t \rangle + \frac{\eta^2 \tau}{8} + \sum_{t=1}^{\tau} \mathbb{1}(E_t) \frac{e^{\eta t}}{n_t} \\
&\leq -\eta L_\tau + \frac{\eta^2 \tau}{8} + e^{\eta \tau} \sum_{i=n_0}^{n_\tau} \frac{1}{i} \\
&\leq -\eta L_\tau + \frac{\eta^2 \tau}{8} + e^{\eta \tau} (\ln n_t + 1) \qquad (19)
\end{aligned}$$

The result follows from further manipulation of (19) and (16). ■

For $t > \tau$, Hed$_\text{G}$-Hedge uses Hedge with the set of experts from $\mathcal{B}_\tau$. Since b_T belongs to $\mathcal{B}_\tau$ with probability q_τ, with this probability the standard regret bound of Hedge applies to the losses L_τ^T and $L_\tau^T(b_T)$ for the horizon $T - \tau$. This is stated in the following lemma and the proof is given for completeness.

Lemma 9 *For $t > \tau$, the regret bound for Hedge is given by*

$$L_\tau^T - L_\tau^T(b_T) \leq \frac{\eta(T-\tau)}{8} + \frac{\ln n_\tau}{\eta}, \text{ w.p. } q_\tau.$$

Proof Since the modified Hed$_\text{G}$ in Algorithm 3 uses vanilla Hedge using the experts from $\mathcal{B}_\tau$ for all $t > \tau$. Therefore, W_t and W_{t-1} will have the same number of weights, and using Hoeffding's inequality, we obtain

$$\ln \frac{W_t}{W_{t-1}} \leq -\eta \langle \boldsymbol{p}_t, \boldsymbol{l}_t \rangle + \frac{\eta^2}{8}, \ \forall t > \tau. \qquad (20)$$

$$\begin{aligned}\ln \frac{W_T}{W_\tau} &= \sum_{t=\tau+1}^{T} \ln \frac{W_t}{W_{t-1}} \\ &\le -\eta \sum_{t=\tau+1}^{T} \langle \boldsymbol{p}_t, \boldsymbol{l}_t \rangle + \frac{(T-\tau)\eta^2}{8} \\ &= -\eta L_\tau^T + \frac{(T-\tau)\eta^2}{8}. \end{aligned} \tag{21}$$

In the second step above, we have used (20). Again,

$$\begin{aligned}\ln \frac{W_T}{W_\tau} &= \ln \left(\frac{\sum_{i=1}^{n_t} e^{-\eta L_\tau^T(i)}}{n_\tau} \right) \\ &\ge \ln \left(\max_{i \in \mathcal{B}_\tau} e^{-\eta L_\tau^T(i)} \right) - \ln n_\tau \\ &\ge \ln e^{-\eta L_\tau^T(b_T)} - \ln n_\tau, \text{ w.p. } q_\tau \\ &= -\eta L_\tau^T(b_T) - \ln n_\tau, \text{ w.p. } q_\tau. \end{aligned} \tag{22}$$

■

Theorem 10 *The regret R_T of the modified Hed$_G$ satisfies,*

$$R_T \le \frac{\eta T}{8} + \frac{(e^{\eta\tau}+1)\ln n_\tau}{\eta} + \frac{e^{\eta\tau}}{\eta}, \text{ w.p. } q_\tau.$$

Proof Let b_t^T denote the best expert for the duration from $t+1$ to T slots, i.e.,

$$b_t^T = \arg\min_{i \in \mathcal{B}_T} L_t^T(i).$$

We have

$$\begin{aligned} L_T - L_T^* &= L_\tau - L_\tau^* + L_\tau^T - L_T^* + L_\tau^* \\ &= L_\tau - L_\tau(b_\tau) + L_\tau^T - (L_\tau(b_T) + L_\tau^T(b_T)) + L_\tau(b_\tau) \\ &\le L_\tau - L_\tau(b_\tau) + L_\tau^T - L_\tau^T(b_T), \ w.p.\ q_\tau. \end{aligned} \tag{23}$$

In the last inequality above, we have used $L_\tau(b_\tau) \le L_\tau(b_T)$, if $b_T \in \mathcal{B}_\tau$. The result follows from substituting the upper bound for $L_\tau - L_\tau(b_\tau)$ from Lemma 8 and the upper bound for $L_\tau^T - L_\tau^T(b_T)$ from Lemma 9 in (23). ■

Next Steps:

1. Extend the result to 'well-behaved' distributions under which there is a minimum probability for the best expert to be chosen.

2. Does similar bounds hold for MAB setting when the arms are revealed randomly?

3. What if all N experts are known a prior, but we randomly choose one expert at a time without replacement to reduce computation complexity? The cumulative loss of an expert is not computed and stored until the expert is chosen. This in spirit has a connection to the setting in Cohen and Mannor (2017), where the experts are sequentially added depending on whether they belong to an ϵ-covering or not.

4. Future work – Stochastic setting: Losses are drawn i.i.d. from a fixed (and unknown) distribution Amir et al. (2020). Consider pseudo regret.

6. Applications

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Appendix A. Appendix

A.1. Partial information

Define potential function $\phi_t = \frac{1}{\eta} \ln \sum_{i \in \mathcal{B}_t} \exp(-\eta L_i(T))$, hence

$$\phi_T - \phi_0 = \sum_{t=1}^{T} \phi_t - \phi_{t-1} = \sum_{t=1}^{T} \frac{1}{\eta} \ln \left(\frac{\sum_{i \in \mathcal{B}_t} \exp(-\eta L_t(i))}{\sum_{i \in \mathcal{B}_{t-1}} \exp -\eta L_{t-1}(i)} \right) \tag{24}$$

$$= \sum_{t=1}^{T} \frac{1}{\eta} \ln \frac{W_t}{\hat{W}_t} + \frac{1}{\eta} \ln \frac{\hat{W}_t}{W_{t-1}} = \sum_{t=1}^{T} \frac{1}{\eta} \ln \sum_{i \in \mathcal{B}_t} w_t(i) \exp(-\eta l_t(i)) + \frac{1}{\eta} Y_t \tag{25}$$

$$\leq \sum_{t=1}^{T} \frac{1}{\eta} \ln \left(\sum_{i \in \mathcal{B}_t} w_t(i) \left[1 - \eta l_t(i) + \frac{\eta^2 l_t(i)^2}{2} \right] \right) + \sum_{t=1}^{T} \frac{1}{\eta} Y_t \tag{26}$$

Using (2) and ($e^{-x} \leqslant 1 - x + \frac{x^2}{2}$), Further substituting $\phi_0 = 0$ and $(\ln(1+x) \leq x)$

$$\phi_T - \phi_0 \leq \sum_{t=1}^{T} \left[-w_t \cdot l_t + \eta \sum_{i \in \mathcal{B}_t} w_t(i) l_t(i)^2 \right] + \frac{1}{\eta} \sum_{t=1}^{T} Y_t \tag{27}$$

$$\phi_T + \sum_{t=1}^{T} w_t \cdot l_t \leq \frac{1}{\eta} \sum_{t=1}^{T} Y_t + \eta \sum_{t=1}^{T} \sum_{i \in \mathcal{B}_t} w_t(i) l_t(i)^2 \tag{28}$$

$$\sum_{t=1}^{T} w_t \cdot l_t - L_T^*(i) \leq \phi_T + \sum_{t=1}^{T} w_t \cdot l_t \quad (\phi_T \geq 0 \geq -L_T(i)) \tag{29}$$

Taking expectation on both the sides

$$E\left[\sum_{t=1}^{T} w_t \hat{l}_t - L_T^*(i) \right] \leq \frac{\ln(T)}{\eta} + \eta \sum_{t=1}^{T} \sum_{i \in \mathcal{B}_t} E\left[w_t(i) \cdot \hat{l}_t(i)^2 \right] \tag{30}$$

$$E\left[\sum w_t \cdot l_t - L_T^*(i) \right] \leq \frac{\ln(T)}{\eta} + \frac{\eta}{\varepsilon} \sum \sum E\left[w_t(i) . \hat{l}_t(i) \right] \tag{31}$$

$$\left(1 - \frac{T}{\varepsilon} \right) L_{alg}(T) \leq \frac{\ln(T)^2}{\eta} + L_T^*(i) \tag{32}$$

Substituting paramters $\varepsilon = \frac{m}{T} \quad \eta = \frac{\sqrt{2m \ln(T)}}{T}$ in we get (32)

$$\left(1 - \frac{\sqrt{2 \ln(T)}}{\sqrt{m}} \right) L_{alg} \leq \sqrt{\frac{\ln(T)}{2m}} T + L_T^*(i) \tag{33}$$

$$L_{alg} - L_T^*(i) \leq T \sqrt{\frac{\ln(T)}{m}} + \sqrt{\frac{2 \ln(T)}{m}} L_{alg} \leq \frac{3T \sqrt{\ln(T)}}{\sqrt{m}} \tag{34}$$

A.2. d Dimensional

Since when an expert interval is split it will yield $n_{t+1} - n_t$ new experts irrespective of which expert is split.

The expectation can be viewed as

$$E[Y_t | H_t] = \sum_{i \in \mathcal{B}_t} \frac{C_i e^{-L_i(t)}}{\sum_{j \in \mathcal{B}_t} e^{-L_j(t)}} \tag{35}$$

where C_i is the fraction of times expert i is replicated to occur in $\mathcal{B}_{t+1}$ due to some split in some other expert j at time t. Since all the splits are equally likely let $\forall i \, C_i = \frac{C}{t^d}$ as total number of experts at time t is t^d.

Algorithm 3: Doubling trick Hed_G

$l, S_y \leftarrow 0$, $L, b \leftarrow 1$
for $t = 1, \ldots, T$ **do**
  if $S_y > b$ **then**
    Start new segment
    $b \leftarrow 2b$
    $L \leftarrow L + l$
    $w = (w_1, \ldots, w_L) = \left(\frac{1}{L}, \ldots, \frac{1}{L}\right)$
    $S_y \leftarrow 0, l \leftarrow 0$
    $\eta \leftarrow \sqrt{\frac{b+\ln(L)}{T}}$
  end if
  Use Hed_G
  $S_y \leftarrow S_y + Y_t$
  $l \leftarrow l + 1$
end for

$$C = \sum \binom{d}{i}(2^i - 1)(t-1)^{d-i}1^i \tag{36}$$

$$= (t-1)^d \sum \binom{d}{i}(\frac{2}{t-1})^i - \sum \binom{d}{i}(t-1)^{d-i}1^i \tag{37}$$

$$= (t-1)^d(\frac{t+1}{t-1})^d - t^d = (t+1)^d - t^d \tag{38}$$

2 experts a, b split each other when they share some dimension. Fix a and look from its perspective. So out of the d possible dimensions let them share some i dimensions and hence there are $\binom{d}{i}$ possible ways to achieve that. There will be $(t-1)^{d-i}$ to assign b the remaining disjoint dimensions. For splits in every dimension, there will be 2 new experts getting created hence 2^i total number of experts and subtract 1 since a himself will occur in all the cases.

$$E[Y_t|H_t] = \frac{C}{t^d} = \frac{(t+1)^d - t^d}{t^d} \leq \frac{2^d}{t} \tag{39}$$

For the last inequality in the above equation, we use

$$(1+x)^r \leq 1 + (2^r - 1)x;\ x \in [0,1] \text{ and } r \in \mathbb{R} \setminus (0,1) \tag{40}$$

A.3. Doubling Trick

Using $\eta = \sqrt{\frac{8 \ln T}{T}}$ yields a suboptimal bound in a probabilistic setting. Hence we use the doubling trick to track Y_t. We assume we know the number of arrivals T of X_t beforehand and apply the doubling trick on the cumulative sum of $\sum Y_t$. Algorithm is described below Lets say we have

$m-1$ restarts of lengths $l_0, l_1, l_2, ..., l_{m-1}$ such that $\sum_{i=0}^{m-1} l_i = T$. Define the following terms

$$
\begin{aligned}
S_0 &= Y_1 + \ldots + Y_{l_0} \leq 1, \eta_0 = \sqrt{\frac{1}{T}} \\
S_1 &= Y_{l_0+1} + ... + Y_{l_0+l_1} \leq 2, \eta_1 = \sqrt{\frac{2+\ln(l_0)}{T}} \\
&\vdots \\
S_{m-1} &= Y_{(l_0+l_1+...+l_{m-2}+1)} + \ldots + Y_T \leq 2^{m-1}, \eta_{m-1} = \sqrt{\frac{2^{m-1}+\ln(l_0+l_1+...+l_{m-2})}{T}}
\end{aligned} \tag{41}
$$

Defining the regret for every restart, using (9) we can upper bound the regret as follows

$$
\begin{aligned}
R_0 &\leq \frac{\eta_0 l_0}{8} + \frac{S_0}{\eta_0} \\
R_1 &\leq \frac{\eta_1 l_1}{8} + \frac{S_1 + \ln(l_0)}{\eta_1} \\
&\vdots \\
R_{m-1} &\leq \frac{\eta_{m-1} l_{m-1}}{8} + \frac{S_{m-1} + \ln(l_0 + l_1 + \ldots l_{m-2})}{\eta_{m-1}}
\end{aligned} \tag{42}
$$

Adding the regret for every restart we have the following

$$
\begin{aligned}
R_T &\leq \sum_{i=0}^{m-1} R_i \leq \sqrt{T}(\sqrt{1} + \sqrt{2+\ln T} + \ldots + \sqrt{2^{m-1}+\ln T}) \\
&\leq \sqrt{T}\sum_{i=0}^{m-1}\sqrt{2}^i + m\sqrt{T\ln T} \leq (\sqrt{2}+1)\sqrt{T}\sqrt{2}^m + m\sqrt{T\ln T}
\end{aligned} \tag{43}
$$

We know that $\sum_{i=1}^{T} Y_t \geq \sum_{i=0}^{m-2} 2^i \geq 2^{m-1} - 1$

$$
\sqrt{2}^m \leq 2\sqrt{\sum_{t=1}^{T} Y_t + 1} \implies m \leq 2\log_2(2\sqrt{\sum_{t=1}^{T} Y_t + 1}) \tag{44}
$$

Substituting the upper bound for m in (43) we have

$$
R_T \leq 2(\sqrt{2}+1)\sqrt{T(\sum_{t=1}^{T} Y_t + 1)} + 2\log_2(2\sqrt{\sum_{t=1}^{T} Y_t + 1})\sqrt{T\ln T} \tag{45}
$$

A.4. Theorem 6

$Y_1, Y_2, ...Y_T$ are independent random variables bounded between $[0, 1]$ therefore using hoeffding's inequality we get

$$P\left(|\sum_{i=1}^{T}\frac{1}{T}(Y_i - E[Y_i])| \geq \epsilon\right) \leq 2\exp\left(-2T\epsilon^2\right) \tag{46}$$

$$P\left(|\sum_{i=1}^{T}Y_i - \ln(T)| \geq \epsilon\right) \leq 2\exp\left(\frac{-2\epsilon^2}{T}\right) \tag{47}$$

Appendix B. My Proof of Theorem 1

This is a boring technical proof.

Appendix C. My Proof of Theorem 2

This is a complete version of a proof sketched in the main text.