LAGRANGIAN RELAXATION AND DUALITY

A very useful tool for integer, mixed integer, large scale, and non-linear programming is **lagrangian relaxation**. The general situation is as follows. Consider the problem:

$$(P)\quad \begin{aligned} &\text{Min } f(x) \\ &\text{s.t.} \quad g(x) \geq 0 \\ &\qquad\quad x \in \mathcal{D} \end{aligned}$$

where $x \in R^n$, and g is an vector of m constraints. We suppose that the problem without the inequality constraints is relatively simple to solve. The inequality constraints complicate things. To get around the problem it is sometime useful to put the offending constraints into the objective:

$$(P_\pi)\quad \begin{aligned} &\text{Min} \quad L(x,\pi) = f(x) - \pi g(x) \\ &\text{s.t.} \qquad\qquad\qquad x \in \mathcal{D} \end{aligned}$$

where π is a non-negative vector. (πg is nothing but a non-negatively weighted sum of the constraints.) (P_π) is called a lagrangian relaxation.

Let $\mathcal{D}^0 = \{x \mid g(x) \geq 0, x \in \mathcal{D}\}$ then the domain, $\mathcal{D}^0$, of (P) is a subset of, $\mathcal{D}$, the domain of (P_π). Moreover for any point in $\mathcal{D}^0$ we have $L(x;\pi) = f(x) - \pi g(x) \leq f(x)$ since $\pi \geq 0, \text{ and } g(x) \geq 0$. Thus the conditions for the relaxation lemma are satisfied for any non-negative π.

Therefore any optimal solution of (P_π) is a lower bound on the optimal solution of (P); moreover, if we have a solution, x^*, of (P_π), so that $x^* \in \mathcal{D}^0$, and $f(x^*) = L(x^*; \pi)$, then x^* optimizes (P). Note that for this to happen πg must be zero, which is a complementary slackness condition. Also for any $x \in \mathcal{D} - \mathcal{D}^0$ πg becomes a penalty function. That is by increasing the weight in π for the constraints which are violated, these constraints can be penalized, and, in general, will decrease when (P_π) is reoptimized.

The **lagrangian dual** of (P) is:

$$(LD)\qquad v_d = \max_{\pi} \min_{x \in \mathcal{D}} L(x,\pi)$$

Since the optimal value of $L(x; \pi)$ in (P_π) is a lower bound for the objective function in (P) for any non-negative π, the maximum of these is the best lower bound of this sort we can get. Of course $v_d \leq v_P$. The difference is the "duality gap." In general, it is positive; however, for linear programs it is 0. For linear programs we have something of the form:

$$(PLP)\quad \begin{aligned} &\text{Min } cx \\ &\text{s.t.} \quad Ax \geq b \\ &\qquad\quad Dx \geq d \\ &\qquad\qquad x \geq 0 \end{aligned}$$

where $Ax \geq b$ is the set of difficult constraints, and $\mathcal{D} = \{x \mid Dx \geq d, x \geq 0\}$.

Assuming that (PLP) has an optimal solution x*; then it also has an optimal dual solution in the k-vector π^*, and (m-k)-vector σ^*. Then use the same π^* in (P_π). Then use the same x*, and σ^* as primal and dual solutions for (P_π). They satisfy the complementary slackness conditions for (P_π), and the objectives have the same value, so in this case v_d, = v_P. In other applications such as to integer programming and other non-convex programs, the gap will generally be positive.

Historical Notes:

The lagrangian relaxation is strongly related to the earlier decomposition method of Dantzig and Wolfe [1960]; Magnanti, Shapiro, and Wagner connected the dots [1976]. Relaxation/decomposition can also, all be done in the dual. That is, in situations where there are easy and difficult variables (say integer) to deal with. This was first developed by Benders [1962].

References:

Every branch of optimization seems to have its own development of lagrangian relaxation and duality:

Ahuja, Ravindra K., Thomas L. Magnanti, and James B. Orlin, Network Flows, Prentice-Hall, 1993
Has a excellent network oriented treatment in Chapter 16. Orlin has also posted power point slides for his courses based on the book:
http://web.mit.edu/jorlin/www/15.082/lectures/19_lagrangian_Relaxation_1.ppt and
http://web.mit.edu/jorlin/www/15.082/lectures/20_lagrangian_Relaxation_2.ppt.

Benders, J. F., "Partitioning Procedures for solving mixed-variables programming problems," Numerische Mathematik, v. 4, pp. 238-252.

Dantzig, George, and Philip Wolfe, "Decomposition principle for linear programs," Operations Research, v. 8, pp. 101-111, 1960.

Magnanti, T. L., J. Shapiro, and M. Wagner, "Generalized linear programming solves the dual," Management Science, v. 22, pp.1195-1203, 1976.

Nemhauser, George, and Laurence A. Wolsey, Integer and Combinatorial Optimization, John Wiley, 1988
Give an excellent treatment of the uses of lagrangian relaxation and duality in solving integer and mixed integer linear programs in pp. 323-337 for the theory, and in pp. 409-417 for computation.

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