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An approximation algorithm for interval data minmax regret combinatorial optimization problems

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Abstract

The general problem of minimizing the maximal regret in combinatorial optimization problems with interval data is considered. In many cases, the minmax regret versions of the classical, polynomially solvable, combinatorial optimization problems become NP-hard and no approximation algorithms for them have been known. Our main result is a polynomial time approximation algorithm with a performance ratio of 2 for this class of problems.

Keywords: Robust optimization; Maximal regret; Approximation algorithms; Combinatorial optimization

1. Introduction

In many combinatorial optimization problems, data (parameters) are not precisely known. One of the simplest form of uncertainty representation is modeling the parameters as intervals. It is then assumed that the value of each parameter may fall within a given range, independently of the values taken by the other parameters. Various frameworks for determining the optimal solution of such stated problems have been proposed. Among them are *robust criteria* described in [6]. One of the robust criteria, studied in [6], is the *maximal regret*. The maximal regret criterion (called also the *robust deviation criterion*) is a worst-case measure of performance. It expresses the maximal possible deviation of a given solution from optimum. The aim is to find a solution for which the maximal regret is minimal.

Our study is focused on the maximal regret criterion applied to combinatorial optimization problems with interval data. This approach has attracted considerable attention recently [1–11]. For instance, it has been applied to many interval versions of the basic combinatorial optimization problems like: the minimum spanning tree problem [1,2,8,10], the shortest path problem [9,7,11] and the assignment problem [5]. Unfortunately, all of them turned out to be NP-hard [2,4,5,11]. Thus, one of the most important subjects of research is the development of efficient approximation algorithms for them. Up to now, no approximation results have been known.

In this paper, we present an approximation algorithm with a performance ratio of 2, which can be applied to a wide class of the interval data minmax regret combi-

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natorial optimization problems, in particular to the minimum spanning tree problem, the shortest path problem and the assignment problem.

2. Problem formulation

Let $E = \{e_1, e_2, \dots, e_n\}$ be a finite set, $|E| = n$, and let $\Phi \subseteq 2^E$ be a set of subsets of E. Set Φ is called the set of *the feasible solutions*. For every element $e \in E$, there is given an interval $\tilde{c}_e = [\underline{c}_e, \bar{c}_e]$, which expresses a range of possible values of the cost (the weights). A vector $S = (c_e^S)_{e\in E}$ that represents a particular assignment of costs c_e^S to elements $e \in E$ is called *scenario*. We will denote by Γ the set of all the scenarios, i.e., Γ is the Cartesian product of the corresponding intervals $\tilde{c}_e$, $e \in E$. For a given solution $X \in \Phi$, we define its cost under a fixed scenario $S \in \Gamma$ as follows:

$$F(X, S) = \sum_{e\in X} c_e^S.$$

We will denote by $F^*(S)$ the value of the cost of the optimal solution under scenario $S \in \Gamma$,

$$F^*(S) = \min_{X\in\Phi} F(X, S). \tag{1}$$

If scenario S is fixed, then problem (1) is *the classical combinatorial optimization problem*. In this paper, we assume that there exists a polynomial time algorithm which outputs an optimal solution for problem (1) for a fixed scenario S. Let us define:

$$Z(X) = \max_{S\in\Gamma}\bigl\{F(X, S) - F^*(S)\bigr\}. \tag{2}$$

The value of $Z(X)$ is called *the maximal regret* for X. Scenario S which maximizes the right-hand side of (2) is called *the worst case scenario* for X.

In this paper we focus on the following *minmax regret* (*robust*) *combinatorial optimization problem* that corresponds to problem (1):

$$\text{ROB:}\quad \min_{X\in\Phi} Z(X).$$

So, we seek a feasible solution, for which the maximal regret is minimal. Note that problem (1) is a special case of problem ROB if set Γ consists of a single scenario.

The following proposition is well known (see [3]) and it characterizes the worst case scenario for a given solution $X \in \Phi$.

Proposition 1. *Given a solution $X \in \Phi$, the worst case scenario S^X for X is the one where elements $e \in X$ have costs $\bar{c}_e$ and all the other elements have costs $\underline{c}_e$, i.e., $c_e^{S^X} = \bar{c}_e$, if $e \in X$; and $c_e^{S^X} = \underline{c}_e$, if $e \in E \setminus X$.*

Making use of Proposition 1, we can express the maximal regret of a given solution $X \in \Phi$ as follows:

$$Z(X) = F\bigl(X, S^X\bigr) - F^*\bigl(S^X\bigr).$$

3. A 2-approximation algorithm for problem ROB

In this section, we propose an algorithm with performance ratio of 2 for solving problem ROB. Let us start with proving the following propositions.

Proposition 2. *Let $X \in \Phi$ and $Y \in \Phi$ be two feasible solutions. Then, the following inequality holds*:

$$Z(X) \geqslant \sum_{e\in X\setminus Y} \bar{c}_e - \sum_{e\in Y\setminus X} \underline{c}_e. \tag{3}$$

Proof. From the definition of the maximal regret we obtain:

$$\begin{aligned} Z(X) &= F\bigl(X, S^X\bigr) - F^*\bigl(S^X\bigr) \\ &\geqslant F\bigl(X, S^X\bigr) - F\bigl(Y, S^X\bigr). \end{aligned} \tag{4}$$

It is easily seen that $F(X, S^X) - F(Y, S^X) = \sum_{e\in X\setminus Y} \bar{c}_e - \sum_{e\in Y\setminus X} \underline{c}_e$, which, together with (4), imply (3). □

Proposition 3. *Let $X \in \Phi$ and $Y \in \Phi$ be two feasible solutions. Then, the following inequality holds*:

$$Z(Y) \leqslant Z(X) + \sum_{e\in Y\setminus X} \bar{c}_e - \sum_{e\in X\setminus Y} \underline{c}_e. \tag{5}$$

Proof. The following equality is easy to verify:

$$F\bigl(Y, S^Y\bigr) = F\bigl(X, S^X\bigr) + \sum_{e\in Y\setminus X} \bar{c}_e - \sum_{e\in X\setminus Y} \bar{c}_e. \tag{6}$$

We first show that:

$$F^*\bigl(S^Y\bigr) \geqslant F^*\bigl(S^X\bigr) - \sum_{e\in X\setminus Y} (\bar{c}_e - \underline{c}_e). \tag{7}$$

Suppose that inequality (7) is false. Let Y^* be a feasible solution with $F(Y^*, S^Y) = F^*(S^Y)$. We thus get

$$\begin{aligned} F^*\bigl(S^X\bigr) &> F\bigl(Y^*, S^Y\bigr) + \sum_{e\in X\setminus Y} (\bar{c}_e - \underline{c}_e) \\ &\geqslant F\bigl(Y^*, S^{X\cup Y}\bigr) \geqslant F\bigl(Y^*, S^X\bigr), \end{aligned}$$

which contradicts the definition of $F^*(S^X)$. Now subtracting (7) from (6) yields (5). □

We are now in a position to give a polynomial time approximation algorithm for the minmax regret combinatorial optimization problem (ROB). Let $\mathtt{AOpt}(S)$

denote a polynomial algorithm that outputs an optimal solution for the underlying classical combinatorial optimization problem for a fixed scenario S (see problem (1)). The idea of the algorithm for problem ROB (Algorithm AM) is as follows: we first determine scenario S in which the costs of the elements are the midpoints of their corresponding cost intervals; then we apply algorithm $\mathtt{AOpt}(S)$ for determined scenario S.

Algorithm AM
(Algorithm for Midpoint interval scenario)
for all $e \in E$ **do**
$c_e^S \leftarrow \frac{1}{2}(\underline{c}_e + \bar{c}_e)$;
end for
$M \leftarrow \mathtt{AOpt}(S)$;
return M;

The next proposition characterizes the solution computed by Algorithm AM.

Proposition 4. *Let M be the solution constructed by Algorithm* AM. *Then for every $X \in \Phi$ it holds $Z(M) \leqslant 2Z(X)$.*

Proof. Since M is the solution constructed by Algorithm AM, it fulfills inequality

$$\frac{1}{2}\sum_{e\in M}(\underline{c}_e + \bar{c}_e) \leqslant \frac{1}{2}\sum_{e\in X}(\underline{c}_e + \bar{c}_e).$$

Hence

$$\sum_{e\in M\setminus X}(\underline{c}_e + \bar{c}_e) \leqslant \sum_{e\in X\setminus M}(\underline{c}_e + \bar{c}_e),$$

which implies:

$$\sum_{e\in X\setminus M}\bar{c}_e - \sum_{e\in M\setminus X}\underline{c}_e \geqslant \sum_{e\in M\setminus X}\bar{c}_e - \sum_{e\in X\setminus M}\underline{c}_e. \tag{8}$$

Using inequality (5) we get:

$$Z(M) \leqslant Z(X) + \sum_{e\in M\setminus X}\bar{c}_e - \sum_{e\in X\setminus M}\underline{c}_e. \tag{9}$$

Inequality (3) together with (8) yield:

$$\begin{aligned} Z(X) &\geqslant \sum_{e\in X\setminus M}\bar{c}_e - \sum_{e\in M\setminus X}\underline{c}_e \\ &\geqslant \sum_{e\in M\setminus X}\bar{c}_e - \sum_{e\in X\setminus M}\underline{c}_e. \end{aligned} \tag{10}$$

Now, conditions (9) and (10) imply $Z(M) \leqslant 2Z(X)$ which completes the proof. □

We are now ready to formulate the main result of this paper.

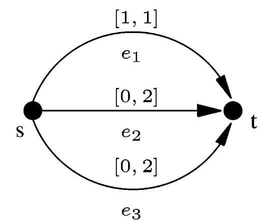

Fig. 1. An example of the robust shortest path problem for which Algorithm AM achieves a ratio of 2.

Theorem 1. *The performance ratio of Algorithm* AM *is at most* 2.

Proof. It is a direct consequence of Proposition 4. If X^* is the optimal solution to problem ROB, then Algorithm AM returns solution M, such that $Z(M) \leqslant 2Z(X^*)$. □

It easy to construct an example of problem ROB showing that 2 is the worst case performance ratio of Algorithm AM.

Consider the minmax regret (robust) version of the *shortest path problem*. That is, for a given directed graph with arc costs specified as intervals, we seek a path from s to t for which the maximal regret is minimal. A sample problem is presented in Fig. 1 (sets: $E = \{e_1, e_2, e_3\}$, $\Phi = \{\{e_1\}, \{e_2\}, \{e_3\}\}$ and the arc costs: $\tilde{c}_{e_1} = [1, 1]$, $\tilde{c}_{e_2} = [0, 2]$, $\tilde{c}_{e_3} = [0, 2]$). The maximal regrets of the paths (the solutions from Φ) are as follows: $Z(\{e_1\}) = 1$, $Z(\{e_2\}) = Z(\{e_3\}) = 2$. It is easy to check that Algorithm AM may return each of these paths. The worst alternatives are $\{e_2\}$ or $\{e_3\}$, which give the bound of 2.

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