Approximating the Embedded *m* Out Of *n* Day Soft-Call Option of a Convertible bond: An Auxiliary Reversed Binomial Tree Method

Qiang Liu
School of Finance
Southwestern University of Finance and Economics
Chengdu, Sichuan 610074, P. R. China

First version: June 19, 2006; this version: July 2, 2008

Abstract: A convertible bond (CB) with a right of *m* out of *n* day provisional call or soft-call becomes callable given that the underlying stock closes above a pre-set trigger price for any *m* or more days over the *n* consecutive trading days up to the current day. It is computationally challenge to value the contribution of this embedded option to the price of CB. This paper proposes an approximation based on the idea of an auxiliary reversed binomial (ARB) tree, and shows that the approach can be efficiently implemented under the Cox-Ross-Rubinstein parameterization. Two important insights emerge from ARB. First, the convertible bond is unconditionally callable at higher stock prices, but uniformly not callable at lower prices. Second, the effect of the soft-call is rather localized around the trigger price. Surprisingly, the simple One-Touch, or *one* out of *one*, approximation is found to yield CB prices that are very close to those from ARB, even though ARB is found to a better approximation in almost every aspect. Further numerical results suggest that in order to generate well-behaved CB prices, cautions need to be taken while designing the terms of soft-call. Being independent of the finite difference pricing grid, the proposed ARB tree can also be used in association with the tree method or Monte Carlo simulation, and could in principle be applicable to exotic derivatives with similar embedded options.
Keywords: *m* out of *n* day provisional call; soft-call; call protection; embedded option; path-dependence; auxiliary reversed binomial tree; convertible bond
JEL Classification: G13, G12, C63

1. Introduction

Convertible bond (CB), a corporate debt, provides its investors with the right to convert itself into the common shares of its issuing company. Since the early 19^{th} century, companies have been issuing CB's to attract investors (Calamos, 1988). Around 1837, Jacob Little surprised his trading enemies by delivering CB when short-squeezed on the Erie Railroad stocks and made money (Geisst, 1997). As an important asset class, convertible bonds are still popular among hedge fund managers today. For example, Highbridge Capital Management,[1] a hedge fund where the author worked for many years, started its business in convertible arbitrage. Even on the young Chinese capital markets on which options are still prohibited, convertible bonds have surprisingly been traded for a relatively long time (Liu, 2007).

With a number of embedded options possible, such as put, call, and/or convert, convertible bonds are complex hybrid financial instruments. Take the call option as an example. Issuers of CB's usually give themselves an option to retire a CB or "call" it back earlier, which would benefit the issuer but may affect investors adversely. To protect the interests of investors (and thus make a CB attractive), various restrictions (or protections) are placed on calls. As a standard practice nowadays a convertible bond is not allowed to be called for a number of years right after issuing. This restriction is commonly referred to as hard-call protection. A second type of protection is the soft-call or provisional call. In this case, a convertible bond could only be called back when the underlying stock closes above[2] a pre-set trigger price for any m or more days out of the n consecutive trading days up to the current day.

Apparently as options to the issuer, both hard-call and soft-call reduce the value of a CB. Therefore, a reasonably accurate pricing method of CB should take both calls into consideration. Typically in backwards induction[3], calls, along with all those other embedded options, are handled exactly as the early exercise option of American puts. At each time step, the called value of the CB is kept if the convertible bond is within hard-call period and the called value is lower than the discounted cash-flow from a later time step. Thus, the contribution of a hard-call option to the value of CB is very easy to handle.

Soft-call is completely a different story, since it is path-dependent. At each time step, it is necessary to look into the past to see whether the soft-call condition is satisfied. This is challenging and so far no easy solution is available. For path-dependent options, Kwok and Lau (2001) proposed to use an auxiliary state variable to keep track of the conditions along all the paths in a trinomial tree. Their approach is in principle accurate and might be applicable to the soft-call of CB as well, but seems not computationally feasible, since the number of variables for tracking the conditions grows exponentially as 3^n, where n is the total number of tree steps. Monte Carlo simulation could potentially be an alternative for handling soft-calls (Landau & Binder, 2000; Longstaff & Schwartz, 2001; Hull, 2003), but again it is infeasible to keep track of the gigantic number of conditions.

Industry practitioners have been using various kinds of approximations for the soft-call modeling, such as a One-Touch with increased trigger price and the Navin (1999) approach. The so-called One-Touch[4] model simply treats a CB as callable when the current stock price is above the trigger by ignoring the m out of n condition altogether. It can be regarded as *one* out of *one*, is in a sense a limiting case of the provisional call, and can be handled exactly as hard-call when *one* out of *one* is satisfied. In principle, as is argued later in this paper, One-Touch should

[1] Highbridge is now a part of JPMorgan Chase.
[2] Here "above" usually means "not less than" and this meaning is used throughout the paper.
[3] For either tree or finite difference method.
[4] Following Navin (1999).

produce correct results when the stock prices are either quite low or very high (given that the stock price is not allowed to jump). One might believe that One-Touch overestimates the call probability. To compensate for this supposedly overestimation, the trigger price is arbitrarily set higher, which is questionable to say the least. The Navin algorithm[5] generates a daily indicator series for the entire valuation period, so that some of the days are set to be callable, while the rest of the days are not. This is computationally quite efficient, but the proposed series generating algorithm appears not to be based on any sound theoretical ground. Further, Navin may lead to divergence from the One-Touch model at very high stock prices. Therefore, both those approximations seem to be unsatisfactory.

This paper proposes a more reasonable yet computationally efficient approximation for estimating the contribution of the embedded soft-call option to the value of CB. This approach will be called the auxiliary reversed binomial (ARB) tree approximation. In the rest of the paper, the idea of ARB is first introduced by looking at an illustrative case of *three* out of *five* and then a realistic case of *20* out of *30*. Next, extensive numerical results with various parameters under both ARB and One-Touch are presented and analyzed. Finally, the paper concluded with summaries and comments.

2. Auxiliary Reversed Binomial Tree Approximation

2.1. Finite Difference and the Auxiliary Reversed Binomial Tree

Let's begin by emphasizing that the paper is solely concerned about how to approximate the contribution of the embedded soft-call option to the value of CB in a backwards induction setting, and is not proposing a new model for pricing the whole CB. In backwards induction, either the finite difference or tree method could be utilized. Throughout this paper however, finite difference is chosen for the discussions and for obtaining numerical results, because it can easily handle the CB model with credit risk of Ayache et al (2003), while it is difficult to do so using a tree approach.

To price a CB, finite difference goes backwards from maturity to the valuation date by solving repeatedly a PDE for a chosen CB model. At each grid point, the embedded options are checked against the discounted, expected cash-flows from a later grid time to see whether it is optimal to "exercise" those options. Convert, put, and hard-call all can be treated this way. Only soft-call presents problems however, since it is necessary to know whether the provisional condition is satisfied. This requires closing prices for the underlying $n - 1$ days before the grid time, which is not available from the finite difference grid. One might be tempted to think that a tree would provide such information, but unfortunately it does not either due to the following reasons. First, the step size of a tree for valuing CB with a long maturity may be longer than one day, while soft-call needs daily closing prices. Second, numerous possible stock paths lead to each tree node, whose implication will be discussed in the following paragraph.

Still, one is not completely out of luck here, since it is possible to settle for an approximation. Imagine one is sitting on a particular grid point (t, S_j) and wants to find out whether a convertible bond is provisionally callable. Imagine further that one can look back in time and connect earlier stock prices to S_j. If there were only one such path, one could simply count the number of prices that is above the trigger and determine whether the condition of call is satisfied. Unfortunately, the grid time t is in the "future" (relative to the valuation date), and numerous paths from times before t may end at S_j. This observation suggests that it may not be possible to know for sure whether it is callable for a CB at the grid point (t, S_j); only a probability of call can be associated

[5] Courtesy of Dr. Robert Navin. Note that the algorithm is not stated explicitly in Navin (1999).

with S_j if all paths end at S_j could be enumerated. As a result, the contribution of soft-call to the value of CB can only be approximated by some kind of average.

A model for the underlying price process is needed in order to enumerate the paths. Many alternatives are available; for simplicity, a binomial tree model is used here. Sitting on S_j and looking back in time, one sees a binomial tree grows from S_j into the past (and thus clearly only price nodes on this tree could possibly be connected to S_j). Such a tree is attached in imagination to every S_j in the finite difference grid and grows backwards; hence it is called the auxiliary reversed binomial tree.

2.2. A Simple Case: *Three* Out Of *Five*

Let's begin with a simple case to illustrate the idea. For *three* out of *five*, only sixteen (2^4) possible paths lead to S_j (from now on, the subscript j will be dropped for simplicity). It is then easy to count the number of paths with three or more stock prices above the trigger price, S_{Tr} (see Figure 1 below):

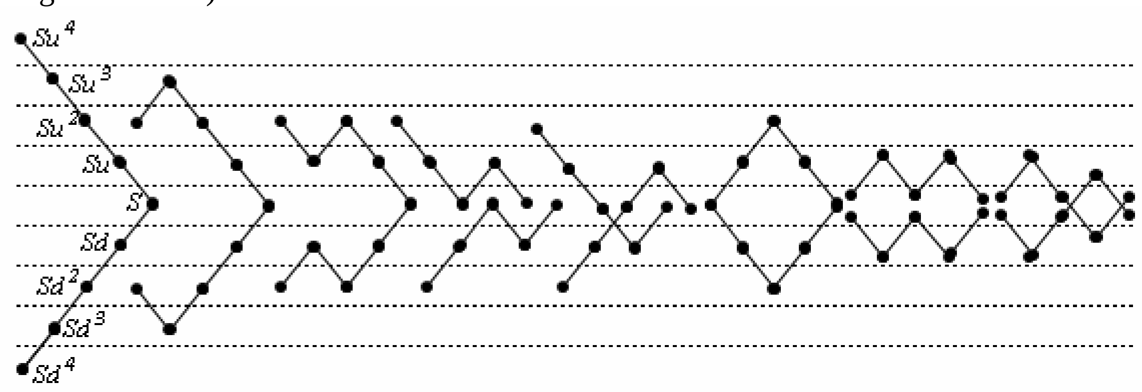

Figure 1. The sixteen four-step paths that end at some stock price S.
Possible trigger positions are indicated by dotted lines. Symmetrical pairs of paths that may result in ambiguity are dislocated slightly from each other.

The Cox-Ross-Rubinstein (CRR) parameterization (Cox et al, 1979), namely $ud = 1$ and $u = e^{\sigma\sqrt{\Delta t}}$, in risk-neutral is used here. A total of nine distinct prices (see Figure 1) are possible in the four-step tree. The dotted horizontal lines show possible locations of the trigger price; the trigger could of course be above the highest or below the lowest stock price shown. Let's count the callable paths. For example, Path No. 1 and 2 (numbered from left to right and then from top to bottom) have three prices above the trigger indicated by the third dotted line. In another word, the possibility of call in this case is two out of sixteen, assuming that each path is equally likely to occur.[6] Table 1 shows the numbers of callable paths for various trigger levels.

Observing from Table 1 that when the trigger price is above Su^2, the convertible bond will not be callable at all, since no matter from where it is originated, no path that ends at S will have three or more nodes with prices above the trigger. On the other hand, the convertible bond will be definitely callable given that the trigger is below Sd^2, for then all paths will satisfy *three* out of *five*. Therefore, at both relatively low and high stock prices the provisional call behaves just as One-Touch assumes, and in exactly that sense the effect of soft-call is quite localized and could

[6] The assumption is not unreasonable, since the steps are independent and the probability of any single step is $P(up) \approx \frac{1}{2} + \frac{1}{2\sigma}(r - \frac{1}{2}\sigma^2)\sqrt{\Delta t} \rightarrow \frac{1}{2}$ when $\Delta t \rightarrow 0$.

be significant only around the trigger price.

Table 1. Number of callable paths and call probability
in *three* out of *five* for all possible locations of the trigger price.

$S_{Tr} \in$	Number of Callable Paths	Call Probability
(Su^4, ∞)	0	0
$(Su^3, Su^4]$	0	0
$(Su^2, Su^3]$	0	0
$(Su, Su^2]$	2	2/16
$(S, Su]$	5	5/16
$(Sd, S]$	11	11/16
$(Sd^2, Sd]$	14	14/16
$(Sd^3, Sd^2]$	16	16/16
$(Sd^4, Sd^3]$	16	16/16
$[0, Sd^4]$	16	16/16

For any grid point (t, S) between the valuation date and maturity, it is assumed that each one of the sixteen paths ends at S is equally likely to occur. Further, if S is close to the trigger, some of the paths will make the CB callable, but the other paths will not (Table 1). Thus, it is impossible to decide outright whether or not to call the CB at (t, S). The best one can do in this situation is once again to forecast the "future" by averaging over all the possible paths, as has always been done in derivatives pricing. One can think of this as Monte Carlo reversed.

Now for each grid point (t, S) while S is close to the trigger price, two values for the CB are calculated. First, assuming that it is not soft-callable, a value for the CB at (t, S) is obtained by checking all the other embedded options against the discounted, expected cash-flows from grid time $t+1$ to see whether it is optimal to "exercise" those options. The result is denoted as $V_{nc}(t, S)$, where *nc* stands for not soft-callable. Second, another value for the CB is computed in the same way but by assuming the CB is soft-callable. This result will be denoted as $V_{sc}(t, S)$, where *sc* indicates soft-callable. Clearly, the first would overestimate, while the second underestimate, the value of the CB. With the call probability P_{sc} at (t, S) given (Table 1), the value of the CB at (t, S) could then be approximated as $V(t, S)$ by the following average:

$$V(t, S) = P_{sc} \times V_{sc}(t, S) + [1 - P_c] \times V_{nc}(t, S) \tag{1}$$

Note that $V(t, S)$ will be the desired CB value for point (t, S) in finite difference, and is going to be discounted further back to grid time $t - 1$.

2.3. The Realistic Case: *20* Out Of *30*

For the *three* out of *five* provisional call discussed above, paths can be enumerated by simply drawing diagrams. The same approach does not work, however, for the more realistic *20* out of *30* case.[7] For 30 prices, there are 536870912 (2^{29}) paths. Further imagine what one has to do with, say, *30* out of *40*! Simply speaking, it is impossible to draw out all the possible paths by

[7] In the United States *20* out of *30* is most common, even though other cases are possible as well.

Table 2. Number of callable paths and call probability
in *20* out of *30* for all various trigger levels calculated with *S=100* and *u=1.1*.

$S_{Tr} \in$	Number of Callable Paths	Probability of Call (%)
(Su^{10}, ∞)	0	0.00
$(Su^{9}, Su^{10}]$	92378	0.02
$(Su^{8}, Su^{9}]$	277134	0.05
$(Su^{7}, Su^{8}]$	1368874	0.25
$(Su^{6}, Su^{7}]$	3367598	0.63
$(Su^{5}, Su^{6}]$	9330178	1.74
$(Su^{4}, Su^{5}]$	19256614	3.59
$(Su^{3}, Su^{4}]$	39237394	7.31
$(Su^{2}, Su^{3}]$	69272518	12.90
$(Su, Su^{2}]$	115309708	21.48
$(S, Su]$	177348964	33.03
$(Sd, S]$	254907724	47.48
$(Sd^{2}, Sd]$	324706732	60.48
$(Sd^{3}, Sd^{2}]$	381413581	71.04
$(Sd^{4}, Sd^{3}]$	429260863	79.96
$(Sd^{5}, Sd^{4}]$	463636309	86.36
$(Sd^{6}, Sd^{5}]$	490888807	91.44
$(Sd^{7}, Sd^{6}]$	507955766	94.61
$(Sd^{8}, Sd^{7}]$	520697698	96.99
$(Sd^{9}, Sd^{8}]$	527518414	98.26
$(Sd^{10}, Sd^{9}]$	532324922	99.15
$(Sd^{11}, Sd^{10}]$	534464675	99.55
$(Sd^{12}, Sd^{11}]$	535891177	99.82
$(Sd^{13}, Sd^{12}]$	536398475	99.91
$(Sd^{14}, Sd^{13}]$	536719133	99.97
$(Sd^{15}, Sd^{14}]$	536804570	99.99
$(Sd^{16}, Sd^{15}]$	536855882	100.00
$(Sd^{17}, Sd^{16}]$	536864990	100.00
$(Sd^{18}, Sd^{17}]$	536870198	100.00
$(Sd^{19}, Sd^{18}]$	536870660	100.00
$[0, Sd^{19}]$	536870912	100.00

hand and one needs to resort to computers.

The computation turns out not to be trivial at all, if one considers the potentially gigantic memory usage and the complexity of the possible paths. The graph method, well studied by mathematicians and computer scientists (Sedgewick, 2001), was chosen out of many possible approaches to handle the binomial tree. Table 2 is generated by utilizing the graph method as follows. 1. A binomial tree with 29 steps is created as an adjacency list by using the freely available Boost Graph Library (Lee et al, 2002). 2. Each node on the binomial tree is associated with a stock price computed according to the CRR parameterization with *u=1.1* and *S=100*. 3. Given a trigger price (say 99) and a terminal node N_T, all the paths between N_T and the initial node are traversed while the result of comparison between a node price and 99 is counted and accumulated. As a result, the number of callable paths for N_T is obtained. 4. Step 3 is repeated for all the other terminal nodes to obtain one entry (the row indicated by $(Sd, S]$, to be exact) for Table 2. 5. Step 4 is repeated for different trigger prices to arrive at Table 2. Note that a few computational tricks are used as well to make the code more efficient, which are considered unnecessary to be described here in details however.

Note that in Table 2 actual trigger prices used in the numerical computation are replaced by $(Sd^{i+1}, Sd^i], i = 0,1,...,18$ or $(Su^j, Su^{j+1}], j = 0,1,...,9$. The rationale for doing so will be explained below shortly. Clearly, the convertible bond becomes not callable at all at a trigger above Su^{10} and definitely callable at a trigger below Sd^{19}, even though the binomial tree extends well beyond these two points on both sides. Once again, the provisional call behaves just as One-Touch does in regions far away from the trigger, and the value of CB under ARB and One-Touch should converge for both very low and high stock prices. This is of course assuring since One-Touch can be viewed as a limiting case for an *m* out of *n* provisional call, as was mentioned earlier in the introduction. Further note that the effect of soft-call is actually limited to even a smaller region (from Sd^8 to Su^4 roughly).

With call probabilities as given in Table 2, the value of the CB at any grid point (t, S) could again be approximated as $V(t, S)$ by using Equation (1).

2.4. Making it Practical

It took more than ninety minutes to compute Table 2 on an IBM Thinkpad X32 (512 MB memory and 1.70 GHz Pentium Processor) running Windows XP. Apparently, that is too long for the ARB approximation to be useful in practice.

Fortunately, it is actually unnecessary to re-calculate this table repeatedly. The binomial stock process under the CRR parameterization has a very nice structure, with only *2n – 1* distinct prices for a tree with *n – 1* steps and *n(n+1)/2* nodes. In another word, *2n – 1* horizontal lines (across the time dimension) will cover all the tree nodes.[8] As a result, the tree might become wider or narrower with the change of the stock price *S* or the parameter *u*, but the number of nodes above a given relative level (say for example somewhere within $(Sd^3, Sd^2]$) for a particular path does not change at all. In another word, as long as the trigger is within $(Sd^3, Sd^2]$, the call probability is always 71.04%, no matter what the actual values of *S*, *u*, or S_{Tr} are. This

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[8] Another well-known parameterization, the Jarrow-Rudd parameterization with $P(up) = \frac{1}{2}$, $u = e^{(r-\frac{1}{2}\sigma^2)\delta t + \sigma\sqrt{\delta t}}$, and $d = e^{(r-\frac{1}{2}\sigma^2)\delta t - \sigma\sqrt{\delta t}}$ (Jarrow & Rudd, 1983), could be used instead. The result would be more complicated unfortunately, since in general there will be as many distinct prices as the number of nodes in the tree.

explains why entries such as $(Sd^3, Sd^2]$ are put into Table 2, and is consistent with the fact that Table 1 is generated without specifying any values for *S*, *u*, or S_{Tr}. Therefore, only one set of probabilities, such as Column 3 of Table 2 for *20* out of *30*, needs to be computed for any specific *m* out of *n*. Further, to make the approximation efficient this probability data can be pre-computed and stored.

Now again for any grid point (*t*, *S*), an imaginary binomial tree can grow out of *S* and the call probability for a given trigger price can be obtained directly from Table 2. Note that in this approximation, the table created depends on *S* and *u* (and thus on volatility) only, but has nothing to do with the grid time *t* (for exceptions, see the Appendix).

The procedure suggested in the previous paragraph is not efficient, however. Note that soft-call usually has one single fixed trigger[9], and thus it is better to generate only once a table based on the trigger, and then read call probabilities for various stock prices off this table. This is not as hard to figure out as it appears, however. Due to the nature of the binomial tree as is discussed above, it is obvious to see that the trigger level and the stock price are anti-symmetrical in some sense. For example, a stock price *S* within $[S_{Tr}, S_{Tr}u)$ is equivalent to a trigger S_{Tr} within $(Sd, S]$; *S* within $[S_{Tr}u, S_{Tr}u^2)$ is equivalent to S_{Tr} within $(Sd^2, Sd]$; and so on. As a result, a trigger price based table (Table 3) can easily be generated from Table 2:

Table 3. Call probability in *20* out of *30* for various stock Prices.
Some of the rows are omitted to avoid repeating the contents of Table 2.

$S \in$	Probability of Call (%)
$[0, S_{Tr}d^{10})$	0.00
…	…
$[S_{Tr}d^2, S_{Tr}d)$	21.48
$[S_{Tr}d, S_{Tr})$	33.03
$[S_{Tr}, S_{Tr}u)$	47.48
$[S_{Tr}u, S_{Tr}u^2)$	60.48
…	…
$[S_{Tr}u^{19}, \infty)$	100.00

As was argued earlier, the set of call probabilities remain the same for *20* out of *30*, no matter what the actual S_{Tr} and *u* are for a particular CB!

The probabilities in Table 3 are correct for grid points that are well inside the finite difference grid, but not so for boundary grid points located either at the upper or lower edge of the grid or close to the valuation date. In most cases, however, the pricing grid is built around the trigger, so the upper or lower grid edge does not pose any problem at all. The situation is more complicated, however, for grid points that are very close to (or within *n* days of) the valuation date and within the soft-call period. Here the path might be shorter (for there could be a historical price series as input), and the total number of paths could be fewer (for some of the starting price nodes in the imaginary tree that lead to *S* are simply not available). As a result, path enumeration

[9] One trigger price that can be reset and multiple triggers for different periods are possible, but a single fixed trigger is most common and is thus discussed in this paper.

(and thus call probability) has to be dealt with on a case-by-case basis (please refer to the Appendix for more discussion). Fortunately, one does not have to worry about this complexity when the soft-call period is far away from the valuation date. Furthermore, for a long-dated convertible bond, the effect of this complication is probably insignificant and could be neglected. Therefore, those complications will not be addressed further in this paper.

For simplicity, the ARB approximation does not consider the effect of any dividend payout over the previous $n - 1$ days, as is done with One-Touch and the Navin Approach. One might think that the trigger were dividend-protected or lowered accordingly with dividend payments, which is believed to be the majority of cases. Further, as is usually done in the finite difference grid, weekends or holidays or both are not treated explicitly as well.

Since the proposed ARB tree is independent of the finite difference grid, it can be used together with other pricing approaches, such as the tree method or Monte Carlo simulation. Even though the step size of ARB tree has to be one day, the step size of the main pricing tree or Monte Carlo could be longer or shorter. Finally, it is not hard to see that the ARB approximation could be applied to similar embedded option in derivatives other than convertible bonds.

3. Numerical Results

3.1. Convertible Bond Model and an Example

The Ayache-Forsyth-Vetzal (AFV) model, or more precisely the equation (4.6) of Ayache et al (2003),

$$V_t + (r + p\eta)SV_S + \frac{1}{2}\sigma^2 S^2 V_{SS} + p\max[\kappa S(1-\eta), RX] = (r+p)V \ ,$$

implemented with the Crank-Nicolson finite difference method (Clewlow & Strickland, 1998; Hull, 2003) is applied here in all calculations. Note that σ denotes the volatility as usual, and X

Table 4. Terms and provisions of a convertible bond.

Item	Value
Valuation date	01/01/2008
Maturity	01/01/2013
Conversion ratio, κ	1
Convertible period	01/01/2008 to 01/01/2013
Call price	110
Callable period	01/01/2010 to 01/01/2013
Call notice period	0
Put price	105
Putable time	On 01/01/2011 (one day only)
Coupon rate	8%
Coupon frequency	Semiannual
First coupon date	07/01/2008
Par	100
Hazard rate, p	0.02
Recovery rate, R	0.0
Partial default, η	0.0
Risk-free interest rate, r	0.05

the notional amount for computing the payment *RX* upon default; the other parameters are explained in Table 4 below. For easy comparison, the CB data are taken almost directly from Ayache et al (2003) and summarized in Table 4. Note that explicit dates are used in Table 4 to match the terms such as maturity given by Ayache et al (2003), which gives a maturity in years. Further, calculations are limited to the partial default case with zero recovery. The computations use as default a grid time step of one day, or a total 1827 steps, and a stock price spacing of $\Delta(\ln S) = \sigma\sqrt{3\Delta t}$.[10]

In the US, when a convertible bond is called, the CB holders usually have the option to convert the CB in a certain period, called the notice period, if doing so is in their interest. For simplicity, this notice period is ignored outright, as is done in Ayache et al (2003). Thus, investors have to convert immediately if called, or get the call price.

At maturity, a CB holder gets $max(\kappa S, Par+Coupon)$.

One last thing to specify is accrued interest. It is calculated on an actual-actual basis. The accruing starts on the valuation date. When the CB is called or an investor puts the CB back to the issuer, she gets accrued in either case, in addition to the (clean) call or put price, which is once again the same as in Ayache et al (2003).

When the hazard rate is zero, the AFV model reduces to the well-known Black-Scholes equation. For calibration, the CB prices in this case from three different implementations are obtained. Given a stock price of 100, the CB price is 125.953 from Ayache et al (2003), 124.730 from a NY hedge fund,[11] and 124.414 from this implementation. The difference between prices from Ayache et al (2003) and this paper is noticeable, yet within an acceptable error range, so the cause of it, which is irrelevant here, will not be addressed further.[12] It is then assumed that the current implementation is good enough for the soft-call results done in the following.

3.2. ARB Approximation vs. One-Touch

As was pointed out earlier, no benchmark for the *m* out of *n* soft-call is available yet. The results from ARB approximation are then compared to those of One-Touch, which could be regarded as a limiting case of the soft-call. In the following, the ARB for *20* out of *30* is investigated.

Isolated Soft-Call Days. It is first assumed that the convertible bond will be hypothetically soft-callable every 31 days, starting on November 1, 2008 and ending on December 26, 2009 (before the hard call period begins) for a total of 15 callable days. With such a soft-call period, no two consecutive callable days interfere with each other.

In order to see the provisional call effect more clearly, a high volatility of 90% is first assumed. Three comparisons are carried out by varying the trigger price, the call price, or the step size of the finite difference grid. The CB prices corresponding to a stock price of 100 are

Table 5. CB prices for three different trigger prices.
The call price is 110.

Trigger Price	ARB	OT
95	141.180	140.821
100	141.647	140.821
105	142.421	141.768

[10] This is the convergence condition for the explicit finite difference method (Clewlow & Strickland, 1998). For simplicity of programming however, this price spacing is used for all finite difference methods, including Crank-Nicolson, for which the condition is of course unnecessary.

[11] Courtesy of Mr. Zhi-Guo Huang.

[12] In this paper a discrete form of the equation (4.6) of Ayache et al (2003) is applied directly, while Ayache et al (2003) used more complicated approaches.

shown in Tables 5 and 6. Note that OT stands for One-Touch.

Table 5 shows that prices under ARB are higher than these under OT in all three cases. Consistent with intuition, the CB price (under ARB) increases with the trigger price, while OT treats the triggers 95 and 100 as the same, because the stock price in the finite difference grid "jumps" directly from 92.165 to 100. Therefore, ARB does a better job in this case.

With a trigger price of 100, the CB price under ARB, as well as OT, increases when the call price goes from 105 to 110, and then to 115. Again prices under ARB are higher than these under OT in all three cases, as is expected.

Table 6. CB prices with three different step sizes for finite difference.
The trigger price is 100, and call price 110.

Days per Step	ARB	OT
3	141.307	140.451
2	141.709	141.868
1	141.647	140.821

The convergences of ARB and OT could only be tested indirectly, since the step size of ARB tree is fixed at one day. In addition to a default grid size of one day, finite difference valuations with two and three days per step are run as well (Table 6). The results under ARB are reasonably close, while under OT the result with two days deviates dramatically from either one day or three days, and is even higher than that under ARB. Apparently, ARB converges better while OT is not so stable with regard to the step size of finite difference.

Finally, a summary graph shows various results given a trigger price of 100, a call price of 110, and a grid step size of one day for the finite difference (Figure 2). Note that AFV indicates the CB given in Table 4, Navin stands for the results from the Navin Algorithm, and HC (in the figure caption) is the result of replacing the 15 isolated soft-call days by hard-call days.

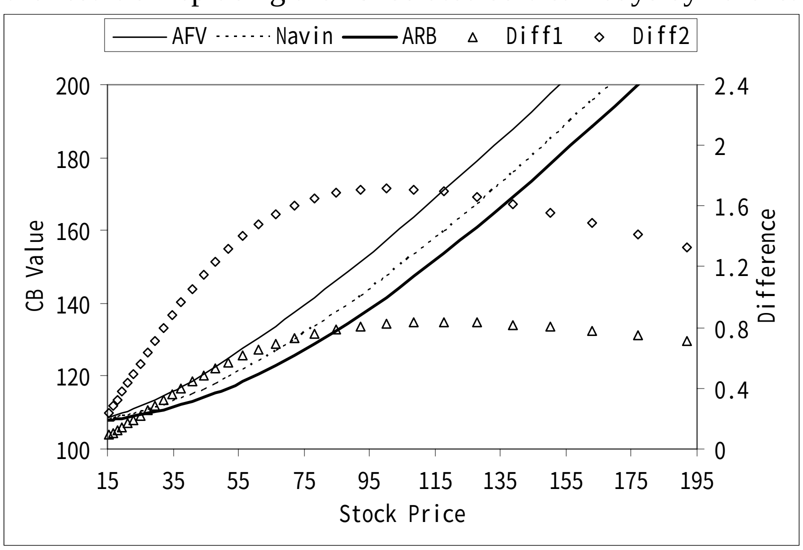

Figure 2. CB prices (left y-axis) and price differences (right y-axis) vs. stock prices.
Diff1 is ARB minus OT. Diff2 is ARB minus HC.

As is expected, all the CB prices converge at very low stock prices (Figure 2); in all range, AFV is the highest, ARB is above OT, while HC is the lowest.[13] The maximum difference between ARB and OT is 0.837 at a stock price of 117.726. Deep in the money, ARB, OT, and HC converge. It is somewhat surprising however, that the additional soft-call of only 15 days depresses the prices so dramatically at higher stock prices.

Note that Navin appears to diverge noticeably from that of the high stock price limit (Figure 2) and yield a different result from OT or ARB. This behavior seems unsatisfactory.[14] Since at very high stock prices, a convertible bond becomes unconditionally callable, it is reasonable to require any approximations of soft-call, including Navin, to do so. Therefore, Navin Algorithm will not be discussed further from now on.

Before closing this section, let's look at a set of results given a more reasonable volatility of 20%. Table 7 presents the CB price again corresponding to a stock price of 100:

Table 7. CB prices for different trigger prices.
The call price is 110.

Trigger Price	ARB	OT	ARB minus OT
100	116.301	116.313	-0.012
101	116.338	116.313	0.025
102	116.389	116.396	-0.007
103	116.453	116.396	0.057
104	116.528	116.508	0.020
105	116.608	116.508	0.100

Apparently, the CB price under ARB increases smoothly with the trigger price, while under OT six trigger prices yield only three CB prices. Over all, the differences between ARB and OT are quite small, but somewhat contrary to intuition, ARB is even below OT for trigger prices 100 and 102. It might be simply argued that since OT yields the same CB price for triggers 100 and 101 (102 and 103 as well), the results under OT are not so dependable, so one should not read too much into those small differences.

On the other hand, it might be argued that compared with *20* out of *30*, it may not be so apparent that One-Touch definitely overestimates the call probability (and hence yields a lower CB price). In terms of pricing CB using finite difference, tree methods, or Monte Carlo simulation, one looks to the future and has to consider many likely paths or possible realizations of the underlying process. Imagine paths pass through (t, S). If S is above the trigger, One-Touch gives a higher probability of call; otherwise, ARB will have a higher call probability, since a stock price below the trigger could still be callable under ARB, but not under OT.[15] Any price of CB at the valuation date is determined not by any single path, but by all those possible future paths together, the contributions of which to the CB value commingle in a very complicated fashion. As a result, the effect of OT or ARB on the CB value is not so clear-cut, as one might

[13] Shown in Figure 2 are not OT and HC directly, but their differences with ARB instead.
[14] For the case at hand, the Navin series has the following repeating pattern of 42 days: callable for 5 days, not callable for 2 days, callable for 5 days, not callable for 2 days, callable for 1 day, and not callable for 27 days. The six-week series begins on November 3, 2008. To any callable days, One-Touch is applied. Out of the 15 isolated callable days, only four are in the Navin series (30 January 2009, 30 April 2009, 28 August 2009, and 26 November 2009). It is not surprising then Navin may underestimate the call probability and thus yield a higher CB price.
[15] The true limiting case for *20* out of *30* is actually *one* out of *30*, not *one* out of *one*. Since *one* out of *30* is definitely callable when the trigger is above the stock price, it has a higher call probability than *20* out of *30*. Further, because *one* out of *30* may also be called with certain probability when the trigger is below the stock price, it has a higher call probability than One-Touch.

tend to believe superficially.[16]

In sum, ARB is above OT when the volatility is high enough, but the difference between ARB and OT becomes smaller when the volatility gets lower. ARB yields more smooth results in every respect, and thus is believed to be a better approximation than OT.

Continuous Soft-Call Days. Now let's look at what happens in practice when the period of soft-call is often continuous. To do this, the convertible bond is set to be callable from November 1, 2008 to December 31, 2009 (again before the hard call period begins). The trigger price is 100 and the call price 110.

First, assume the volatility is 90%. ARB is still above OT, but the differences between ARB and OT become much smaller, with a maximum difference of 0.047 at a stock price of 117.726. OT is now indistinguishable from hard-call (the case in which the whole soft-call period is assumed to be hard-callable). It is concluded that for any practical purpose, ARB and OT are basically the same.

Second, set the volatility to 20%. As is in the case of isolated soft-call days, ARB is still below OT, but surprisingly the differences become even larger. As is argued previously, this may have caused by the defect of OT in approximating the soft-call. Note further that even the maximum difference of 0.136 at a stock price of 86.497 is still quite small and negligible. Therefore, ARB and OT can still be regarded as the same.

Interestingly as simple as it is, One-Touch appears to be a reasonable approximation to the provisional call of the CB specified in Table 4, if ARB is believed to be a reasonable approximation in the first place. This is probably not so surprising after all. Remember that it is argued and shown earlier that ARB and One-Touch converge at prices both below and above the trigger price, and thus due to the complicated commingling effect of CB pricing in backwards induction and the requirement of CB prices being a monotonic function of the stock price, the CB values from ARB and OT could not be too different even around the trigger price.

3.3. Issues of CB Design. With so many parameters affecting the value of CB in complicated ways, it is no easy a task to design a suitable set of terms so that the value of CB increases monotonically with the underlying stock price. One abnormal example is shown in Figure 3.

Clearly the curve in Figure 3 is not monotonic.[17] A local maximum of 110.100 can be seen at a stock price of 63.553, and a local minimum of 109.886 at a stock price of 77.581. Such prices of CB are not acceptable, since they decrease when the stock price increases, and further imply zero as well as negative hedge ratios. Fortunately, such abnormality is not observed for call prices 110 and 115 or when the volatility is 90%. It is concluded that 105 is not an appropriate soft-call price in this case.

[16] This paragraph is a result of in-depth discussions with Dr. Peter Forsyth.

[17] One-Touch shows this kind of abnormality as well.

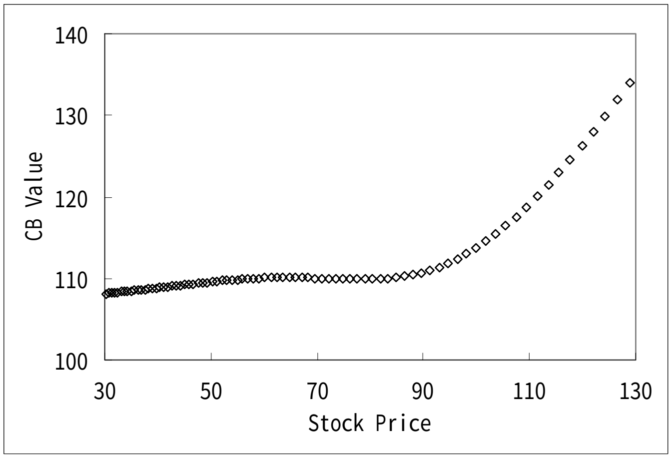

Figure 3. CB prices with a volatility of 20%, a trigger price of 100, and a call price of 105 while the full soft-call period between 1 November 2008 and 31 December 2009 is approximated by ARB.

What more surprising perhaps is that abnormal CB prices may be caused by merely removing the soft-call protection[18] (Figure 4). At stock prices below 100, the CB prices form a concave curve; from 100 to 110, the CB prices fall on a horizontal line. Compared with the abnormality reported in Figure 3, these are even worse. Note that the two noticeable kinks in the curve under OT is probably a result of applying the call on the valuation date, which get no chance to be smoothed out by "diffusion."

Note that neither OT nor ARB yields the correct CB prices (Figure 4). Nevertheless, it appears that the error under ARB is less pronounced. As a result, one might argue that it is even better in some sense for ARB to be below OT in this case!

To summarize, in order to get the terms or parameters right, careful modeling of CB values should be carried out when designing CB.

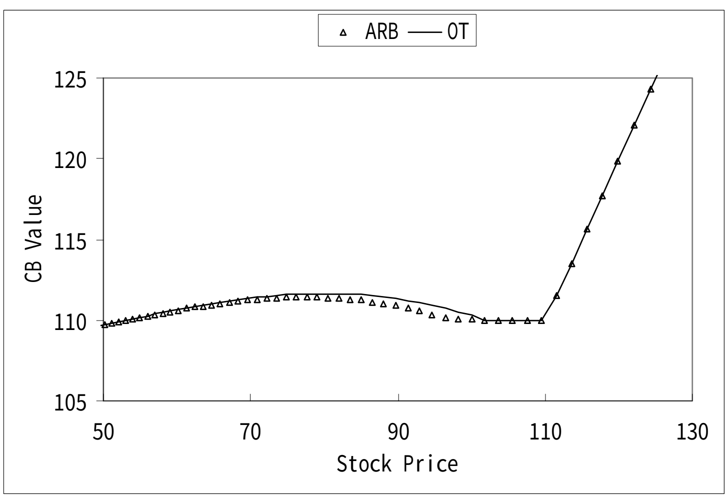

Figure 4. CB prices with a volatility of 20%, a trigger price of 100, and a call price of 110 while the convertible bond is soft-callable from the valuation date (1 January 2008) to 31 December 2009.

[18] This test is suggested by Dr. Peter Forsyth.

4. Conclusions

The auxiliary reversed binomial tree is proposed and seems to be a better approximation to the computationally difficult problem of valuing the contribution of the *m* out of *n* provisional call to the CB value than either One-Touch or the Navin algorithm. The approach can be efficiently implemented using the Cox-Ross-Rubinstein parameterization, since the probabilities of call under ARB are in some sense independent of the trigger price and the volatility.

Several insights emerge from ARB. Given certain combination of the trigger price and soft-call price, the CB Value can be reduced dramatically, as the numerical example shows. Seen from the perspective of the ARB tree, the convertible bond is unconditionally callable at prices much higher than the trigger price, but uniformly not callable at prices well below the trigger. In both these cases, the contributions of soft-call under ARB approach the correct limits, which are treated correctly by the simple One-Touch method as well. As a result, the effect of the provisional call is found to be quite localized around the trigger price.

What surprising perhaps is that compared with the ARB approach, the simple method of One-Touch seems to approximate the provisional call rather well (at least) for the numerical example used in the paper. On the other hand, the Navin algorithm appears to diverge from the correct upper limit while deep in-the-money, so its validity is at best questionable.

Finally, the ARB approach appears to be quite general in treating the *m* out of *n* provision, because it is independent of the main pricing grid (or tree as well). Exotic options other than CB but with similar embedded options could be handled similarly. Further, it is natural to speculate that the idea of the ARB tree could be employed to treat other complex embedded options.

Acknowledgement

The author thanks Zhi-Guo Huang for providing one price calculation, Robert Navin for comments, Yingzi Zhu for inspiration, and two anonymous reviewers for detailed comments. Special thanks are due to the reviewer whose suggestions helped improve the presentation of the paper greatly and to Peter Forsyth for taking a lot of time to discuss the problem with me and for insightful suggestions. The work is supported by a National Natural Science Foundation of China grant (No. 70571012).

Appendix

A.1. Upper or Lower Edges

In a typical implementation of the finite difference method, the stock grid covers a significantly wide range of prices from deep out-of-the-money to deep in-the-money with hundreds of points in between. Therefore, it is too rare for the trigger price to be close to the upper or lower edges of the grid to warrant special attention.

On the other hand, however, it is not difficult to count the paths when the trigger is indeed close to the edges. Let look at Figure 1 again with the trigger indicated by the third dotted line. Say the upper edge is shown by the top dotted line. Now Path No. 1 is then not available, while only Path No. 2 has three nodes above the trigger. This results in a callable probability of one-fifteenth, instead of two-sixteenth as originally.

Probabilities for other locations of trigger as well as for the case where the trigger is near the lower edge can be counted in a similar fashion.

A.2. Close to the Valuation Date

The situation is much hard to deal with when the convertible is provisionally callable on or very close to the valuation date. Historical close prices up to *n-1* days before the valuation date have to be considered. For the valuation date, it is trivial to count whether *m* out of *n* is satisfied by looking at the *n-1* closes as well as the stock price on the valuation date. But for days after the valuation date, the algorithm becomes more complex.

Again use *20* out of *30* for easy exposure. Say it is nineteen days after the valuation date. Conditional on the ten historical closes, eleven tables, namely *20* out of *20* (no historical close is above the trigger), *19* out of *20* (one historical close is above the trigger), …, and *10* out of *20* (all ten historical closes are above the trigger), have to be pre-computed. As an example, it takes about three seconds to compute *10* out of *20*. Therefore, tables for times before nineteen days can be calculated on-the-fly, while a total of (11+2)*10/2=65 tables for times after nineteen days need to be pre-computed for efficiency.

Note that the above conditional *m* out of *n* tables are only an approximation to the problem. To be more precise, the level of the historical closes needs to be taken into consideration as well. As it is argued before, however, the added complication may not lead to significantly enhanced accuracy, since the maturity of a typical convertible bond is much longer than 30 days.

Of course, to make matters simpler, the twenty-eight days after the valuation date could be approximated roughly by either One-Touch or ARB as well.

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