Back To the Future
An Approximate Solution for n out of m Soft-call Option

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Abstract: In convertible bond market, it is very common to protect the conversion privilege from being called away too early by using soft-call constraint, or to protect the bond being converted too early by using provision convert constraint. Both constrains have the common feature that the option can be exercised only when the underlying stock closes above a pre-set barrier for any *n* or more days over *m* consecutive trading. This feature brings challenge for pricing. This paper will propose an efficient approximation solution to this problem. The idea is using an auxiliary state variable to keep track of the number of days stock is above the barrier. Once the transition probability of this new state viable can be calculated, then we can use rollback procedure to calculate the bond price. The results are compared with the exact solution given by Zhang (2008), the maximum error is about 30bp for trinomial tree and 50bp for binomial tree implementation based on all test cases in this article.

Keywords: *n* out of *m* day provisional call; soft-call; provision convert; coco; call protection; convertible bond; binomial tree, path dependent option;

1. Introduction

Convertible bond is a very popular hybrid financial instrument for companies with difficult in issuing regular bonds. A convertible bond has the embedded option to convert the bond into a fixed number of stock shares. This feature helps to attract investors with a long dated stock option.

But usually these companies do not want the bonds to be converted too early, and impose restrictions so that during the first few years they can be converted only when *n out of m* day the stock price is above a pre-set barrier. This type of bonds is usually referred to as *provisional convertible* or *conditional convertible.* (CoCo)

At the same time, a regular convertible bond can also be called when the market is in favor of the issuer. In order to protect the bonds from being called too early, and to keep the convertibles more attractive, soft and hard call protection features are introduced for the first a few years of the bond. The hard call constraint restrains the issuer from initiating the call; the soft call constraint requires the stock price to be above some pre-set barrier for *n out of m* day in order to *knock-in* the call option. The most popular case is a *20 out of 30* soft-call.

In both CoCo and soft-call, the *n out of m* embedded options are path dependent, and they remains as one of the most difficult path dependent problem in derivative pricing. Currently around 20% of total notional of the convertible bonds have soft-call feature. The need for soft-call or conditional convertible feature becomes especially pressing as the stock market and convertible bond market recovers from the financial crisis. Even though many banks or venders provide price for the soft-callalbe convertible bonds, either use approximations or Monte Carlo, but there is very few literatures discussing the methods.

In practice, most people use approximation methods. For example, approximate the *20 out of 30* as *2 out of 3*; the later can be handled with PDE. Even further, people could approximate the soft call with the one-touch barrier with the barrier slightly above the started barrier in the contract. Navin (1999) approach gives a perturbation method, the order of perturbation is based on how many times the stock price crosses the barrier. The leading order will be just the one-touch barrier option with a daily indicator for the option to exercise or not for the entire valuation period. Except an example, how to generate the daily indicator is not explicitly described in that paper. Liu (2008) proposes another approximation method (ARB). The idea is that at any state of the stock, if the stock goes backward *m* steps on a tree, all historical *m*-step stock paths can be generated, based on counting the number of paths which satisfy the soft-call trigger condition, the probability of trigger can then be calculated. The value of the bond at any stock state is then approximated as an average of callable and non-callable bond based on the trigger probability. The problem of this method is that there is no mathematic prove that using trigger probability is a good approximation method. It has been proved that this method could be worse than the simple one touch method in some cases in Zhang (2008).

Recently Zhang (2008) proposed an exact solution to this problem on binomial tree. The idea is to use an auxiliary state variable to represent the historical *m* day stock path, and then the path dependent problem becomes a Markovian problem and can be numerically solved. This exact solution can be a benchmark for all approximation methods. But it is slow to compute and it has limitation in extending to trinomial tree or PDE due to current computation power. Fast and good approximation method is still useful in real practice.

This paper will propose an approximation solution to this problem. The idea is using an auxiliary state variable to keep track of the historical path, but instead of tracking the exactly path, here we will only track the number of days stock is above the barrier (It will be called *level* in this article). The idea is similar to the ARB tree proposed by Liu (2008), instead of counting the trigger probably of stock at every stock grid point, here we will generate the conditional trigger probability based on the new auxiliary state variable. If walk backward on the tree, this probability can be calculated out.

In the following, I will first define the American soft-call option and describe the new approximation method. A step by step detail calculation will be given based on Lognormal stock dynamic on Binomial tree. Then I will give numerical results of an example convertible bond with 20 out of 30 soft-call. The results will be compared with exact solution, one-touch approximation. The idea will then be generalized to more general stock dynamic with PDE solver. For this case I will only lay out the steps, since the generalization is very simple. In the Appendix, I will give the detail calculation on the transition probability for both binomial and trinomial trees.

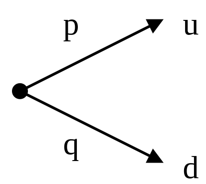
2. Looking Backward Method On Binomial Tree

Since we are focusing on how to solve the *n out of m* problem, we will not discuss the general convertible bond model, the method discussed here can be applied to any convertible bond model with a simply generalization. We will start with a simple *n out of m* soft American call option, and describe how to price it step by step. In practice this type of option does not exist yet. When there is no dividend, in general it is not optimal to exercise American call option before expiring. But in Convertible bond, because the bond can be converted into stock if it's in favor of bond holder, the pricing of convertible bond becomes a game problem and the American call option can be exercised before expiring due to the competition with the convertible option. So the pricing of a pure *n out of m* soft American call option is the key component here. The method described here can be directly applied to any other type of soft-call or soft-put or provision-convert (conditional convert) option etc.

***The n out of m American soft call option*.**

Consider a T year expiring (with total number of N days) American soft call option on a non-dividend paying stock. The soft condition is that the call option can be exercised only when the pass m trading days the stock price is above barrier H with n or more days (In general there could be a delay between checking the soft condition and the exercise date, to simplify the problem I will assume there is no delay, which means the option can be exercise on a date that the soft condition is met. The general delayed exercise can also be derived out easily with the current idea). Assume on today the stock is S_0, the interest rate is r, the call strike is K, and volatility is σ.

Then with the standard CRR method, we can build the daily binomial tree from today to *T* with *N* steps



With $\Delta t = 1day, u = e^{\sigma\sqrt{\Delta t}}, d = \frac{1}{u} = e^{-\sigma\sqrt{\Delta t}}, R = e^{r\Delta t}, p = \frac{R-d}{u-d}, q = 1 - p.$
So on any time step *i* which is *i* days ahead of today, the stock will be on one of the *i+1* nodes with stock price

$$S_{i,j} = S_0 u^j d^{i-j} \qquad j = 0,1, \dots, i \tag{1}$$

And the probability of one path starting from S_0 ends at $S_{i,j}$,

$$P\left(S_{i,j} \middle| S_0\right) = p^j q^{i-j} \tag{2}$$

If it was a standard American call option without soft constrain, we can just rollback from *i+1* to *i* until today to get the option value. This is in any option pricing text book.

$$O_{i,j} = max\left(I_{i,j}, \left(pO_{i+1,j+1} + qO_{i+1,j}\right)/R\right) \tag{3}$$

Here $O_{i,j}$ is the option value on time *i* at node with stock price $S_{i,j}$,
$I_{i,j} = max(0, S_{i,j} - k)$ is the intrinsic value of the call option on time *i* at node *j*, which is the value of the option if it is exercised immediately.

Now since we have the soft constrain, the algorithm above does not work directly any more. Let's define *L* be the integrated random number which counts the number of days the stock price is above trigger *H* within the last *m* days. Then we can expand the above tree into two dimensions: on each notes$S_{i,j}$, both the option value and the intrinsic value and stock prices are expanded into vectors

$$S_{i,j,l}, O_{i,j,l}, I_{i,j,l} \quad l = 0,1, \dots, m$$

With $S_{i,j,l} \equiv S_{i,j},\ I_{i,j,l} = max\left(0, S_{i,j} - k\right) \mathbb{I}(l \geq n)$

Here *l* is the possible value of *L*. From now on I will call it *level*.

The option value can be approximated with

$$O_{i,j,l} = max\left[I_{i,j,l}, (p\pi_{i+1,j+1,l} + q\pi_{i+1,j,l})/R\right] \tag{4a}$$

$$\pi_{i+1,j+1,l} = h_{i+1,j+1,l+1}O_{i+1,j+1,l+1} + h_{i+1,j+1,l}O_{i+1,j+1,l} + h_{i+1,j+1,l-1}O_{i+1,j+1,l-1} \tag{4b}$$

$$\pi_{i+1,j,l} = h_{i+1,j,l+1}O_{i+1,j,l+1} + h_{i+1,j,l}O_{i+1,j,l} + h_{i+1,j,l-1}O_{i+1,j,l-1} \tag{4c}$$

Here I define the general conditional transition probability h which is path dependent.

$$h_{i+1,j+1,l+1} \equiv Prob\left\{L_{i+1,j+1} = L_{i,j} + 1 | S_{i+1,j+1} = uS_{i,j}\right\} \tag{5a}$$

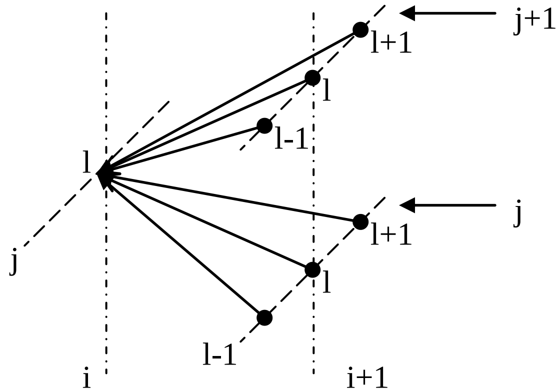
is the probability of level increased by 1 conditioned on that the stock $S_{i,j}$ moving up to notes $S_{i+1,j+1}$.Same way the other probabilities are defined as:

$$\begin{aligned}
h_{i+1,j+1,l} &\equiv Prob\left\{L_{i+1,j+1} = L_{i,j} | S_{i+1,j+1} = uS_{i,j}\right\} \\
h_{i+1,j+1,l-1} &\equiv Prob\left\{L_{i+1,j+1} = L_{i,j} - 1 | S_{i+1,j+1} = uS_{i,j}\right\} \\
h_{i+1,j,l+1} &\equiv Prob\left\{L_{i+1,j} = L_{i,j} + 1 | S_{i+1,j} = dS_{i,j}\right\} \\
h_{i+1,j,l} &\equiv Prob\left\{L_{i+1,j} = L_{i,j} | S_{i+1,j} = dS_{i,j}\right\} \\
h_{i+1,j,l-1} &\equiv Prob\left\{L_{i+1,j} = L_{i,j} - 1 | S_{i+1,j} = dS_{i,j}\right\}
\end{aligned}$$

Since the time step is one day, the level of the stock path can be only increase by one or keep the same or decrease by one. The conservation of probability tells us

$$\begin{aligned}
h_{i+1,j+1,l+1} + h_{i+1,j+1,j} + h_{i+1,j+1,l-1} &= 1 \\
h_{i+1,j,l+1} + h_{i+1,j,j} + h_{i+1,j,l-1} &= 1
\end{aligned}$$

This algorithm can be demonstrated in the following diagram



Here $S_{i,j}$ and $L_{i,j}$are not enough to define a Markovian dynamic for the soft call problem, because with same $L_{i,j}$, the option value can be different based on different historical path. Here I ignore these difference and only get an averaged value.

So, if we can solve these probabilities, then we have an approximating solution to our problem. This task looks difficult at the first glance, but if we really look into the detail, it is not that scaring and we can include some important features in:

1) if $S_{i,j} > Hu^{m-1}$, $L_{i,j}$ has only one state $L_{i,j} = m$ and $h_{i+1,j+1,m} = 1$, all other probabilities are 0.
2) if $S_{i,j} \leq Hd^{m-1}$, $L_{i,j}$ has only one state $L_{i,j} = 0$ and $h_{i+1,j,0} = 1$, all the other probabilities are 0.

So we only need to calculate the conditional probability on the whole tree within nodes with $Hd^{m-1} < S_{i,j} \leq Hu^{m-1}$, outside of this range the level is fixed. Within the range we have following properties:

$$h_{i+1,j+1,l-1} = Prob\{L_{i+1,j+1} = L_{i,j} - 1 | S_{i+1,j+1} = uS_{i,j} > H\} = 0$$
$$h_{i+1,j,l-1} = Prob\{L_{i+1,j} = L_{i,j} - 1 | S_{i+1,j} = dS_{i,j} > H\} = 0$$

Which means when the stock is above the trigger *H* in the next step, then the total level cannot decrease.
Same idea, when the stock price is not above the trigger *H* in the next step, the total level cannot increase. This is described in the following equation.

$$h_{i+1,j+1,l+1} = Prob\{L_{i+1,j+1} = L_{i,j} + 1 | S_{i+1,j+1} = uS_{i,j} \leq H\} = 0$$
$$h_{i+1,j,l+1} = Prob\{L_{i+1,j} = L_{i,j} + 1 | S_{i+1,j} = dS_{i,j} \leq H\} = 0$$

So we only need to solve two probabilities instead of there at any node.

Looking backward on the tree.
By examine all paths starting from time$S_{0,0}$, end at $S_{i,j}$for $i \in [0, n], j = 0, \dots, i$, we only consider the paths with level equal *l* for the final *m* day period. For those paths, if we look at the stock price at time *i-m+1*, they will be distributed either above barrier *H* or below. If stock price at step *i+1* with $S_{i+1,(*)} > H$, then for those paths with stock price at time *i-m+1* below the trigger, the level *l* will increase by 1 for step $i \to i+1$, for those above the trigger, the level *l* will remain the same. If stock price at step *i+1* with$S_{i+1,(*)} \leq H$, then for those paths with stock price at time *i-m+1* below the trigger, the level *l* will remain the same for step $i \to i+1$, for those above the trigger, the level *l* will decrease by 1.

So starting from $S_{i,j}$, on the binomial tree, we can look backward *m-1* day to check the history. So the probability of $h_{i+1,(*),l+1} = Prob\{L_{i+1} = L_i + 1 | S_{i+1,(*)} > H\}$ is equivalent to get the probability of $\rho\left(S_{i-m+1} \leq H \middle| S_{i,j}, S_{0,0}, l\right)$ which is the conditional probability of the stock price S_{i-m+1} is below the barrier *H* with process starting from time 0 with $S_{0,0}$, end at $S_{i,j}$ with fixed number of level *l*. With same idea we get

$$h_{i+1,(*),l} = Prob\{L_{i+1} = L_i | S_{i+1,(*)} > H\} = \rho\left(S_{i-m+1} > H \middle| S_{i,j}, S_{0,0}, l\right)$$
$$h_{i+1,(*),l} = Prob\{L_{i+1} = L_i | S_{i+1,(*)} \leq H\} = \rho\left(S_{i-m+1} \leq H \middle| S_{i,j}, S_{0,0}, l\right)$$

$$h_{i+1,(*),l-1} = Prob\{L_{i+1} = L_i - 1 | S_{i+1,(*)} \le H\} = \rho(S_{i-m+1} > H | S_{i,j}, S_{0,0}, l)$$

This can be done by just counting the path with fixed level l within the 2^{m-1} *total paths within the m-1 steps, the number of path above the trigger, and also the number of path below the trigger separately, weighted by the probability that the stock coming from time 0 to that states.*

This sounds like a huge numerical task, but actually it is not since there is a backward induction numerical scheme, and for Black model we can pre-calculate a table $M(k,b,l;m)$, which gives the number of path for an *m-1* step tree, starting from $\xi_{0,0} = S_{i,j}$ end at notes $\xi_{m-1,k}$ with trigger at $\xi_{0,0} u^b d^{m-1-b} \le H < \xi_{0,0} u^{b+1} d^{m-2-b}$ and the level equal *l*. So this table would not depend on $S_{i,j}$ and can be pre-calculated.

$$\rho\left(S_{i-m+1,(*)} \le H \middle| S_{i,j}, S_{0,0}, l\right) = \rho\left(\xi_{m-1,(*)} \le H \middle| \xi_{0,0} = S_{i,j}, S_{0,0}, l\right)$$
$$= \begin{cases} \dfrac{\sum_{k=0}^{b} M(k,b,l;m)\binom{j+k-m+1}{i-m+1}}{\sum_{k=0}^{m-1} M(k,b,l;m)\binom{j+k-m+1}{i-m+1}} \ldots\ldots\ldots\ldots if \displaystyle\sum_{k=0}^{m-1} M(k,b,l;m)\binom{j+k-m+1}{i-m+1} > 0 \\ 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. if \displaystyle\sum_{k=0}^{m-1} M(k,b,l;m)\binom{j+k-m+1}{i-m+1} = 0 \end{cases}$$

…....…………………………………………………………………………… (6a)

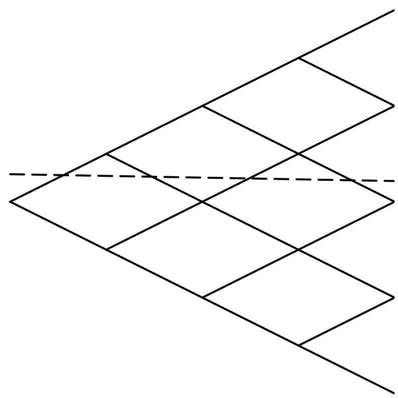
With same logic we get:

$$\rho\left(S_{i-m+1,(*)} > H \middle| S_{i,j}, S_{0,0}, l\right) = \rho\left(\xi_{m-1,(*)} > H \middle| \xi_{0,0} = S_{i,j}, S_{0,0}, l\right)$$
$$= \begin{cases} 1 - \rho\left(S_{i-m+1,(*)} \le H \middle| S_{i,j}, S_{0,0}, l\right) \ldots if \displaystyle\sum_{k=0}^{m-1} M(k,b,l;m)\binom{j+k-m+1}{i-m+1} > 0 \\ 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots... if \displaystyle\sum_{k=0}^{m-1} M(k,b,l;m)\binom{j+k-m+1}{i-m+1} = 0 \end{cases}$$

………………………………………………………………………...…… (6b)

Here I define $\binom{j+k-m+1}{i-m+1}$ is the binomial coefficient, which is the number of paths end at note $\xi_{m-1,k}$ starting from $S_{0,0}$ (remember *k* is the notes on the reverse tree, the corresponding notes on the original mother tree is *j+k-m+1*), which gives the weight from $S_{0,0}$ to $S_{i-m+1,k}$; $M(k,b,l;m)$gives the weight from $S_{i-m+1,k}$ to $S_{i,j}$ with fixed level *l*. So equations (6a) (6b) give the probability by considering all possible scenarios.

To illustrate how to count the table, Lets look at a 5 step tree. With trigger $b = 0$:
By counting the entire path we can get the table as:



How to calculate a more general Table for *m* steps will be discussed in Appendix.

The boundary effect.
The probability equation can be applied for most of the notes on the original binomial tree, but actually there is boundary effect when the notes are very close to the up boundary or lower boundary or close to today.

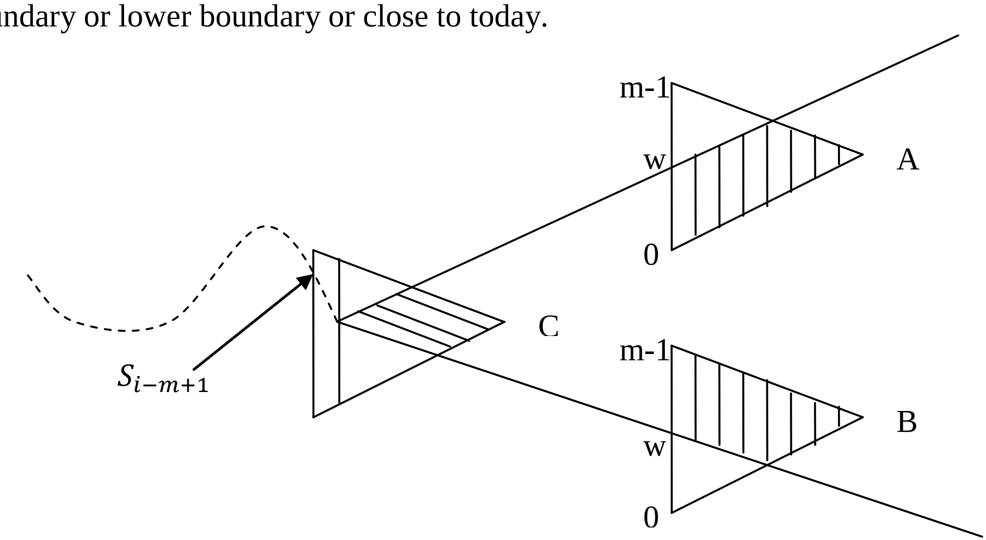


Figure 1. The boundary effects.

As show in picture, the backward *m-1* step binomial trees are showed as a triangle. For Case A, when the notes is very close to the up boundary, To calculate the probability we only need to count the table $M(k,b,l;m)$ with $0 \leq k \leq w$ where *w* is the crossing point. For case B when the notes is very close to the lower boundary, we only need to count table with $w \leq k \leq m-1$

For case C when the time step with $i \leq m-1$, since we know the historical path of the stock price S_{i-m+1}, so

$$\rho\left(S_{i-m+1,(*)} \leq H \middle| S_{i,j}, S_{0,0}, l\right) = \begin{cases} 0 \ldots\ldots\ldots\ldots S_{i-m+1} > H \\ 1 \ldots\ldots\ldots\ldots S_{i-m+1} \leq H \end{cases}$$

Since we solved the conditional transition probability, then we can apply into the rollback scheme to get the option price rollback equations

$$O_{i,j,l} = \max\left[I_{i,j,l}, (p\pi_{i+1,j+1,l} + q\pi_{i+1,j,l})/R\right] \quad (7a)$$

$$\pi_{i+1,j+1,l} = \begin{cases} \left[\rho\left(S_{i-m+1,(*)} \leq H \middle| S_{i,j}, S_{0,0}, l\right) O_{i+1,j+1,l+1} + \rho\left(S_{i-m+1,(*)} > H \middle| S_{i,j}, S_{0,0}, l\right) O_{i+1,j+1,l}\right] \mathbb{I}(S_{i+1,j+1} > H) + \\ \left[\rho\left(S_{i-m+1,(*)} \leq H \middle| S_{i,j}, S_{0,0}, l\right) O_{i+1,j+1,l} + \rho\left(S_{i-m+1,(*)} > H \middle| S_{i,j}, S_{0,0}, l\right) O_{i+1,j+1,l-1}\right] \mathbb{I}(S_{i+1,j+1} \leq H) \end{cases}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (7b)$$

$$\pi_{i+1,j,l} = \begin{cases} \left[\rho\left(S_{i-m+1,(*)} \leq H \middle| S_{i,j}, S_{0,0}, l\right) O_{i+1,j,l+1} + \rho\left(S_{i-m+1,(*)} > H \middle| S_{i,j}, S_{0,0}, l\right) O_{i+1,j,l}\right] \mathbb{I}(S_{i+1,j} > H) + \\ \left[\rho\left(S_{i-m+1,(*)} \leq H \middle| S_{i,j}, S_{0,0}, l\right) O_{i+1,j,l} + \rho\left(S_{i-m+1,(*)} > H \middle| S_{i,j}, S_{0,0}, l\right) O_{i+1,j,l-1}\right] \mathbb{I}(S_{i+1,j} \leq H) \end{cases}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (7c)$$

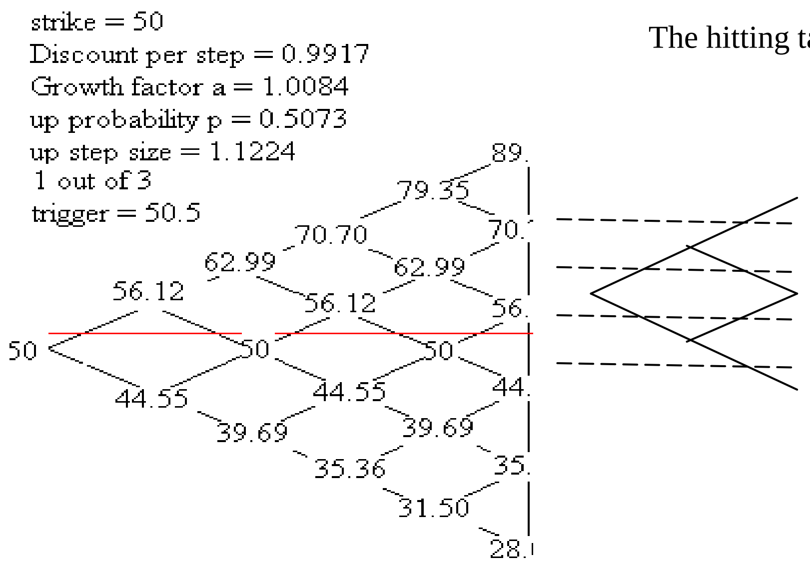
The value today will be $O_{0,0,l_0}$, here l_0 is the level of the stock path for the passed *m* day including today.

3. Results of Convertible Bond with 20 out of 30 Soft Call Option

In the section, I will give some true calculation result. Before try to price convertible bond, I will first illustrate how to price an American option with soft call constrain, even though such kind of derivative does not exist in the market. Consider a 5 step binomial tree with parameters given in the picture. The soft constrain is 1 out of 3. Assuming the historical stock price is always above the barrier. Then the state at today is level=2.

The roll back prices are given in the table 1. The intrinsic value calculation is not optimized, the other steps have done the optimization (you did not need to calculate all the level states, instead only calculate the level states could be reached based on the tree notes and the trigger condition. This will speed up the calculation a lot especially when the total time steps are very large).

Then let’s consider a true convertible bond. To simplify the problem and focus on the soft call feature of the convertible bond, we will assume a no default lognormal stock dynamic. The term of the convertible bond is given in table 2.



The hitting table for 3 step tree:

M(2,-2,3;3)=1
M(2,-1,3;3)=1
M(2,0,2;3) =1
M(2,1,1;3) =1

M(1,-2,3;3)=2;
M(1,-1,2;3)=M(1,-1,3;3)=1;
M(1, 0,0;3)=M(1, 0,1;3)=1;
M(1, 1,0;3)=2;

M(0,-2,2;3)=1;
M(0,-1,1;3)=1;
M(0, 0,0;3)=1;
M(0, 1,0;3)=1

					0,39.07,39.07,39.07
				0,0,0,29.77	
			0,0,0,21.53		0,20.7,20.7,20.7
		0,0,14.86,0		0,0,13.4,13.4	
	0,0,9.87,0		0,8.25,8.25,8.25		0,6.12,6.12,6.12
0,0,6.36,0		4.91,4.91,0,0		3.08,3.08,3.08,0	
	0,2.85,0,0		1.55,1.55,0,0		0,0,0,0
		0.78,0,0,0		0,0,0,0	
			0,0,0,0		0,0,0,0
				0,0,0,0	
					0,0,0,0

Table 1 Option price of a 5 step tree. At each node, the option price for level is also shown.

T	5 year
Clean hard call price	110 in years 2-5
Clean soft call price	110 in year 0-2
Soft call barrier	100-160
Soft trigger condition	20 out of 30
Clean put price	105 at 3 years
r	0.05
σ	0.20 or 0.40
Conversion ratio	1.0
Face value of bond	100
Coupon dates	0.1,1.0,1.5…,5.0
Coupon payments	4.0

Table 2: Data for numerical example of a convertible bond with soft call

Here we consider a 5 year convertible bond with callable and puttable features. The bond is hard callable from year 2 until maturity. During first 2 years it is soft callable when the stock is above barrier for 20 out of 30 trade days. The clean call price is 110. In order to see the impact of the barrier, we will vary the barrier from 100 to 160. There is a one time put option at 3 years with price 105. risk-free interest rate is 5% and volatility is 20% or 40%. The bond pays out 8% annual coupon with semiannual pay frequency. Assume at today the stock price is 100, and the historical price is always below soft call barrier. The conversion ratio is 1. When the convertible bond is called, the bond holder still has the privilege to convert the bond into stocks.

Similar to standard barrier option, there is a convergence problem if we used discrete price method to approximate the true barriers. To solve this problem, assume the true barrier is in between of two nodes on the tree: $H_{in} \leq H \leq H_{out}$, then we will calculate two times the price of the convertible bond, one assume the barrier is at the inner barrier H_{in}, there other use the outer barrier H_{out}, and the true price can be calculated using linear interpretation method.

The results are show in the Table 3 & 4 and Figure 2 & 3, for different volatilities. For comparison, I also give the results with different barrier *H*. By comparing with the exact results derived from method in Zhang (2008). We can see for all cases the LB approximating have less than 0.5% relative errors.

The method is also extended to trinomial tree. The parameters for the trinomial tree are chosen as

$$u = e^{\sigma\sqrt{1.5\Delta t}}$$

$$d = \frac{1}{u}$$

$$p_d = -\sqrt{\frac{\Delta t}{6\sigma^2}}\left(r - \frac{\sigma^2}{2}\right) + \frac{1}{3}$$

$$p_m = \frac{1}{3}$$

$$p_u = \sqrt{\frac{\Delta t}{6\sigma^2}}\left(r - \frac{\sigma^2}{2}\right) + \frac{1}{3}$$

The results are also shown in the tables 3&4. The errors are always smaller than the binomial results, this is probably because the trinomial tree handles the barrier pricing better. As a known advantage in barrier option pricing, the trinomial tree can be adjusted such that the barrier is right on the notes to reduce the pricing error, we believe same technique can be used here.

Some people used 1 out of 1 (one touch OT) to simulate the 20 out of 30 soft constrains, for comparison purpose, I also show the result. We can conclude that the error can be very big in some cases if you do not change the barrier. This is especially true in the callable bond world, where the barrier is usually in the money of the option either for soft call or provisional conversion.

In figure 4, I show the convertible bond price depending on the underline stock price, when the soft call has fixed barrier at 100. Both the Look Back result and one touch result are shown, we can see when the stock price is very low or very high, these two methods converge to each other, but when the stock is close to barrier, the difference is significant.

4. Extension to General Stock Dynamic with PDE

In the real convertible bond market, the stock dynamic people use is usually more complicated than the Black model, and PDE is used to solve the problem (for example Ayache, *et al* (2003)). In this case, the idea above can still be generalized and make it works. Consider PDE is very similar to trinomial tree, all above idea can be automatically adopted on trinomial tree. The only difference is that, now the stock move probability need to be calculated with PDE. Here are the steps we can do to solve the problem:

a) Solve the green function of the original stock dynamic process to get the transition probability of the stock movement using the PDE method.
b) at each notes of the PDE grid, calculate the transition probability of the hitting level based on the pass 30 day trinomial paths. Including the stock probability calculated in step a), we can first build the table $M(k, b, j; m)$ (Here the table already consider the weight of each path starting from time 0 to time *i-m+1* at a note corresponding to the reverse tree note k and then from k to current notes.), and then sum the paths up to get the level move probability as above.
c) Then we can use the rollback idea to get the option value as in the tree, since we know both the stock move probability and the level move probability.

In this way, we can solve the problem. But practically there is a difficult that, since each step the stock move probability is different, we do not have a simple general table M can be reused, and we have to recalculate on each note. This will increase the computation time. Fortunately, there is two short-cuts: first we only need to calculate with the range of the barrier plus/minus a *m* day maximum price range; second the soft constrain is usually within 30 day period, which is very short period, so we can assume the stock dynamic can be simplified with simply dynamic again. How good the simplification is can be tested since we have the true value.

5. Discussion

A Looking Backward (LB) method is proposed to approximate solving the soft-call/provision-convert problem for convertible bond. This method gives very good result for all testing cases by comparing with the exact solution with much fast algorithm. People may think the increase of an extra dimension will increase the computation burden, but actually most soft constrains in the real market are only in the first a couple years, so we only need to calculate the soft option in these period.

The similar idea can be employed to solve other path dependent problem as long as the path dependent problem can be parameterized as a few extra parameters, and the transition probability on those extra dimensions can be calculated based on the reverse steps. This idea expended the limitation of the current path dependent problem solving method on PDE or tree, which usually requires the Markov property of the dynamics on the expended dimensions.

6. Acknowledgement

The author would like to thank Gheis Hamati for introduce the soft-call problem and inspiring discussion and also thank for Peter Carr and Bruno Dupire to point out the limitation of this method.

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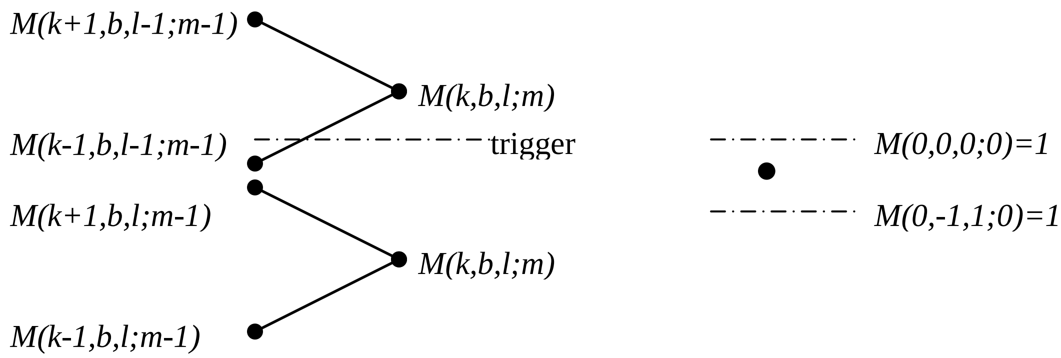
7. Appendix

Calculation of the conditional probability on tree

There is a very simple recursion relationship in calculating the $M(k, b, j; m)$, which gives the number of path for an *m* step tree, starting from $\xi_{0,0}$ end at notes $\xi(m, k) = \xi_{0,0}u^k$ with trigger at $\xi_{0,0}u^b \leq H < \xi_{0,0}u^{b+1}$ and the level is *l*.

$$M(k, b, l; m) = \begin{cases} [M(k-1, b, l; m-1) + M(k+1, b, l; m-1)]\mathbb{I}(\xi(m, k) \leq H) + \\ [M(k-1, b, l-1; m-1) + M(k+1, b, l-1; m-1)]\mathbb{I}(\xi(m, k) > H) \end{cases}$$

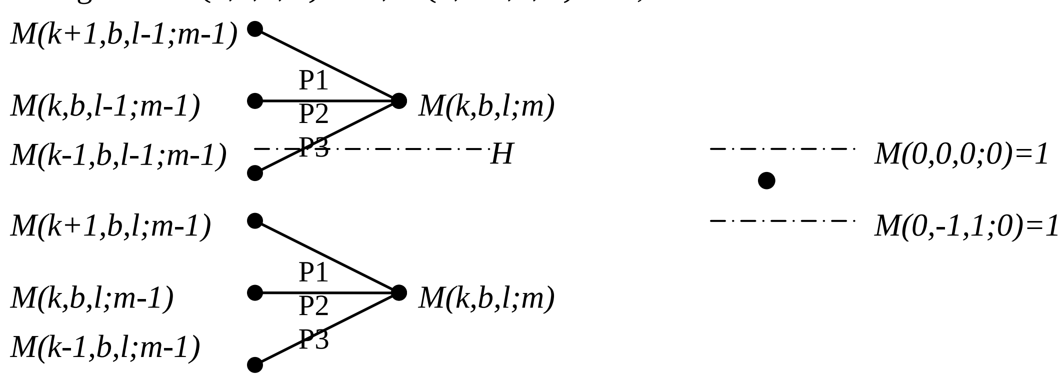
Starting from $M(0,0,0; 0) = 1; M(0, -1,1; 0) = 1;$



Same idea for calculating the table for trinomial tree or PDE case, with the probability on each step as p1,p2,p3

$$M(k, b, l; m) = \begin{cases} [M(k-1, b, l; m-1)p3 + M(k, b, l; m-1)p2 + M(k+1, b, l; m-1)p1]\mathbb{I}(\xi(m, k) \leq H) + \\ [M(k-1, b, l-1; m-1)p3 + M(k, b, l-1; m-1)p2 + M(k+1, b, l-1; m-1)p1]\mathbb{I}(\xi(m, k) > H) \end{cases}$$

Starting from $M(0,0,0; 0) = 1; M(0, -1,1; 0) = 1;$



σ =40%, T = 5 year.				
Barrier	***One touch***	***LB(Tri)***	***LB(Bi)***	***Exact***
H=100	112.8314 (-0.9%)	113.9127 (6bp)	113.9662 (11bp)	113.8435
H=105	112.9339 (-2.1%)	115.4980 (17bp)	115.6147 (27bp)	115.3002
H=110	113.0399 (-3.9%)	117.9622 (30bp)	118.1226 (44bp)	117.6029
H=115	114.4056 (-5.0%)	120.8641 (32bp)	121.0576 (48bp)	120.4842
H=120	117.8162 (-4.4%)	123.5566 (28bp)	123.7141 (41bp)	123.2079
H=125	120.7615 (-3.7%)	125.7482 (23bp)	125.8903 (35bp)	125.4563
*H=*130	123.2487 (-3.2%)	127.5458 (20bp)	127.6622 (29bp)	127.295
H=135	125.3202 (-2.7%)	129.0008 (16bp)	129.1022 (24bp)	128.7945
H=140	127.0530 (-2.3%)	130.1878 (13bp)	130.2725 (20bp)	130.0165
H=145	128.4705 (-1.9%)	131.1345 (10bp)	131.2084 (16bp)	130.996
H=150	129.6561 (-1.6%)	131.9106 (9bp)	131.9672 (13bp)	131.7913
H=155	130.6320 (-1.4%)	132.5370 (7bp)	132.5853 (11bp)	132.4401
H=160	131.4252 (-1.2%)	133.0367 (5bp)	133.0774 (9bp)	132.9572

Table 3 Convertible bond price with 20 out of 30 soft call option. Stock price at time 0 is 100. H is the soft call barrier, σ is the volatility. One Touch is simple one touch approximation methods. LB is the current Look Backward method. Bi is on binomial tree, Tri is on trinomial tree. Exact result is using unique words on binomial tree method described in [4].

σ =*20%*, T = 5 year.				
Barrier	***One touch***	***LB(Tri)***	***LB(Bi)***	***Exact***
H=100	112.8314 (-0.14%)	112.9965 (0.9bp)	113.0019 (1.4bp)	112.9863
H=105	113.0145 (-0.32%)	113.4297 (4.2bp)	113.4453 (5.6bp)	113.3819
H=110	113.2091 (-1.30%)	114.8780 (15bp)	114.9297 (20bp)	114.7044
H=115	114.4649 (-2.39%)	117.4608 (16bp)	117.5412 (23bp)	117.2686
H=120	117.3987 (-1.80%)	119.6876 (11bp)	119.7492 (16bp)	119.5541
H=125	119.5569 (-1.30%)	121.2206 (7.3bp)	121.2667 (11bp)	121.1327
*H=*130	121.0717 (-0.92%)	122.2525 (4.6bp)	122.2873 (7.4bp)	122.1967
H=135	122.1072 (-0.65%)	122.9400 (3.1bp)	122.9627 (5.0bp)	122.9017
H=140	122.8149 (-0.45%)	123.3989 (2.1bp)	123.4148 (3.4bp)	123.3734
H=145	123.2923 (-0.32%)	123.7082 (1.5bp)	123.7182 (2.3bp)	123.6897
H=150	123.6204 (-0.24%)	123.9171 (0.1bp)	123.9257 (0.7bp)	123.9172
H=155	123.8493 (-0.17%)	124.0627 (0.2bp)	124.0689 (0.7bp)	124.0598
H=160	112.8314 (-0.14%)	124.1641 (0.2bp)	124.1693 (0.6bp)	124.1622

Table 4. Same as Table 2. with different volatility of 20%.

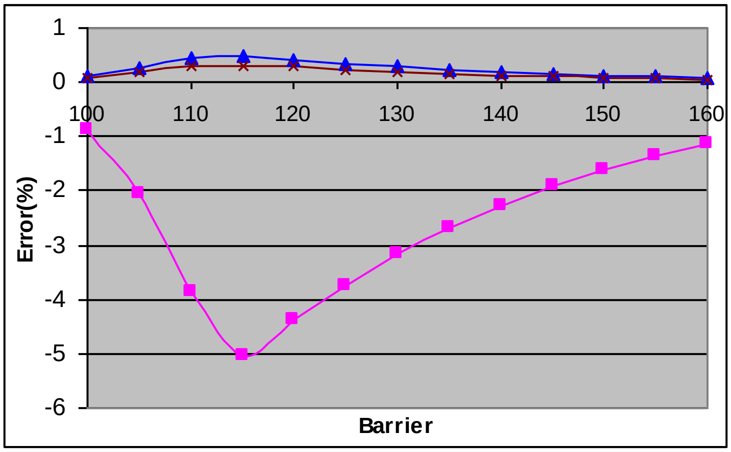


Figure 2. The relative errors in Table 3 when volatility is 40%. **From top to bottom curve are Binomial Look Back result, Trinomial Look Back method, and One Touch method.**

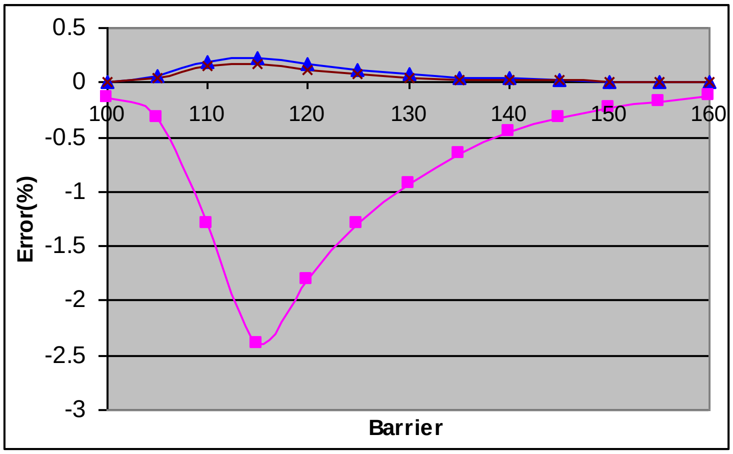


Figure 3. The relative errors in Table 4 when volatility is 20%. **From top to bottom curve are Binomial Look Back result, Trinomial Look Back method, and One Touch method.**

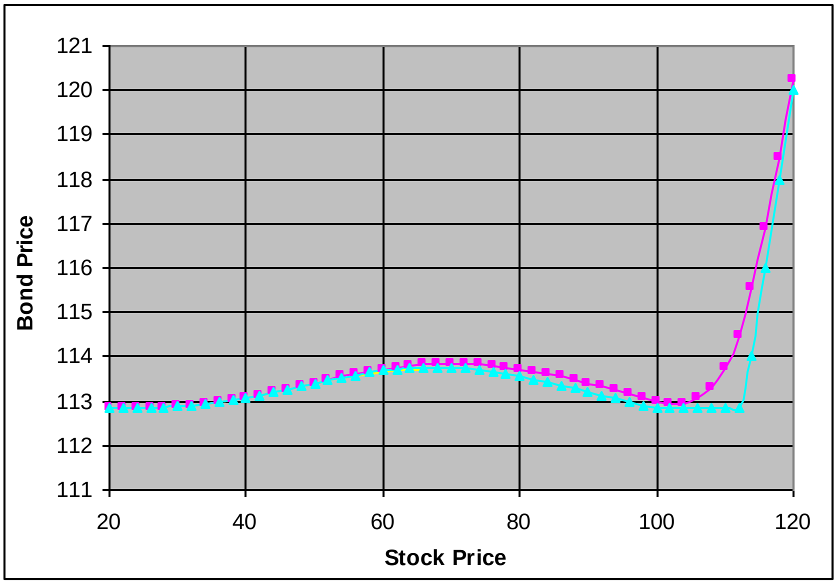


Figure 4. The price of the convertible bond depends on the stock price, the soft call barrier is fixed at 100. Top curve is the Binomial tree Look Back result, the bottom is One touch result.