OR And

Permutation & combination

n objects
- r-permutation = order r objects out of n $\quad P(n,r) = \frac{n!}{(n-r)!}$
- r-combination = select r " " " n $\quad C(n,r) = \binom{n}{r} = \frac{n!}{r!\cdot(n-r)!}$

Ex Bitstring of length 8 How many such bitstrings with exactly 3 zeros.

0001

0100

$$\binom{8}{3} = \frac{8!}{3!\cdot 5!} = \frac{8\times 7\times \cancel{6}}{\cancel{6}} = 56$$

Ex Do, but at most 3 zeros.

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3 zeros.

Ex: Do, but at most 3 zeros.

$$\binom{8}{0} + \binom{8}{1} + \binom{8}{2} + \binom{8}{3}$$

$$= \sum_{k=0}^{3} \binom{8}{k}$$

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Ex How many ways to arrange 7 books. $= 7!$

Ex 7 books
- 3 in Islamic $3!$
- 2 " Computer $2!$
- 2 " math $2!$

$$3! \times 2! \times 2! \times \underbrace{3!}_{\text{as a group}}$$

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Ex How many ways to arrange 7 books. $= 7!$

Ex 7 books
- 3 in Islamic $3!$ I
- 2 " Computer $2!$ C
- 2 " math $2!$ M

$3! \times 2! \times 2! \times \underbrace{3!}_{\text{as a group}}$

ICM
IMC
CIM
CMI
MIC
MCI

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Ex 5 people

ways to photograph in groups of 3.

7!

M. A. O

M O A $= \binom{5}{3} \times 3!$

$P(5,3)$

5	4	3

$5 \times 4 \times 3$

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Ex Do, but 1 person must be in every picture.

$\binom{4}{2} \times 3!$

$6 \times 6 = 36$

1	4	3

$\times 3$

$= 36$

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OR And

Ex Computer password. 6 - 8 characters.

lower case letter: $\underbrace{a-z}_{26}$

number: $\underbrace{0-9}_{10}$

Each password must have at least one number

Count # passwords $= \underbrace{P_6}_{\text{\# password of length} = 6} + P_7 + P_8$

$$= \sum_{k=1}^{6} 26^{6-k} \cdot 10^{k} \cdot \binom{6}{k}$$

$P_6 =$ # passwords with 1 number $\quad 26^5 \times 10 \times \binom{6}{1}$

$+$ " " " 2 " $\quad 26^4 \times 10^2 \times \binom{6}{2}$

$+$ " " " 3 "

$\vdots$

$+$ " " " 6 "

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Count # passwords $= \underbrace{P_6}_{} + P_7 + P$

password of length = 6

$P_6 =$

No constraint $= 36^6 - \underbrace{26^6}_{\text{all letters}}$

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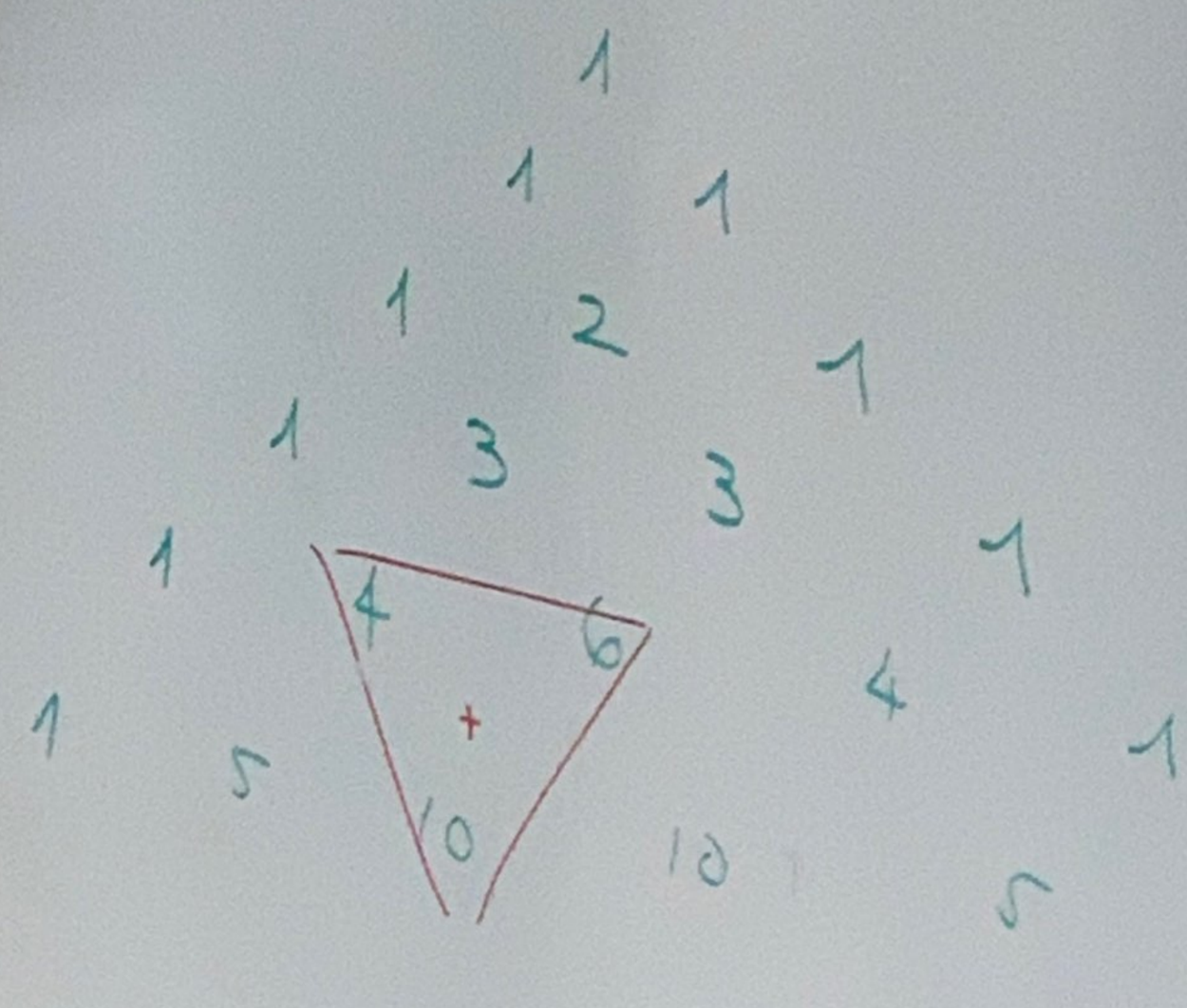
Binomial coefficient

Th Pascal's identity.

Let $n, k \in \mathbb{Z}^+$ with $n \geq k$

then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$



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OR And

Binomial coefficient

Th Pascal's identity

Let $n, k \in \mathbb{Z}^+$ with $n \geq k$

then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

$\binom{5}{4} = 1$

Binomial expansion

Th $x, y \in \mathbb{R}$

$n \in \mathbb{Z}^+$

then $$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \cdot y^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^{n-k} \cdot y^k$$

Ex $$(x+y)^2 = \sum_{k=0}^{2} \binom{2}{k} x^k \cdot y^{2-k}$$

$$= \binom{2}{0} x^0 \cdot y^2 + \binom{2}{1} x^1 \cdot y^1 + \binom{2}{2} x^2 \cdot y^0$$

$$= y^2 + 2xy + x^2$$

Ex Expand $(x+y)^5$

$$= x^5 + 5x^4 y + 10x^3 y^2 + 10x^2 y^3 + 5xy^4 + y^5$$

Ex Expand $(x+y)^{100}$

$$x^{100} + \binom{100}{1} x^{99} y + \binom{100}{2} x^{98} \cdot y^2 + \binom{100}{3} x^{97} \cdot y^3 + \dots$$

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Ex Find the coeff of $x^{90} y^{10}$ in the expansion of $(x+y)^{100}$.

$$= \binom{100}{10} = \frac{100!}{90! \times 10!}$$

$$\binom{100}{90} = \frac{100!}{10! \cdot 90!}$$

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$$\underline{\text{Th}} \quad \sum_{k=0}^{n} \binom{n}{k} = 2^n$$

$$\underline{\text{Th}} \quad \sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$$

$\cdot y^{n-k}$

y^k

1

1 1

1 2 1

1 3 3 1 $= 2^3 = 8$

1 4 6 4 1 $= 2^4 = 16$

1 5 10 10 5 1

1 6 15 20 15 6 1

⋮

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Th $\displaystyle\sum_{k=0}^{n} \binom{n}{k} = 2^n$

proof.

$x = y = 1$

Th $\displaystyle\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$

Proof.

$x = 1$

$y = -1$

$n \geq 1$

y^{n-k}

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

$= 2^3 = 8$

$= 2^4 = 16$

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Ex find the coeff of $\boxed{x^{10}}$ in the expansion of

$$\left(3x^2 - \frac{4}{x}\right)^{20}$$

$$\left(3x^2 - \frac{4}{x}\right)^{20} = \sum_{k=0}^{20} \binom{20}{k} \cdot \boxed{(3x^2)^k \cdot \left(\frac{-4}{x}\right)^{20-k}}$$

$$= 3^k \cdot x^{2k} \cdot (-4)^{20-k} \cdot x^{k-20}$$

$$= 3^k \cdot (-4)^{20-k} \cdot x^{3k-20}$$

$3k - 20 = 10$

$\boxed{k = 10}$

$$= \binom{20}{10} \cdot 3^{10} \cdot (-4)^{10}$$

Ex $(x$

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Binomial expansion

Th $x, y \in \mathbb{R}$

$n \in \mathbb{Z}^+$

then $$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \cdot y^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^{n-k} \cdot y^k$$

Multinomial Expansion

Th

$$(x_1 + x_2 + \cdots + x_k)^n$$

$$= \sum_{n_1+n_2+\cdots+n_k=n} C(n; n_1, n_2, \ldots, n_k) \cdot x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}$$

$$= \sum_{n_1+n_2+\cdots+n_k=n} \left(\frac{n!}{n_1! \cdot n_2! \cdots n_k!} \prod_{i=1}^{k} x_i^{n_i} \right)$$

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$k=2$

$$(x_1+x_2)^n = \sum_{n_1+n_2=n} \frac{n!}{n_1!\cdot n_2!} x_1^{n_1} x_2^{n_2}$$

$$= \sum_{n_1+n_2=n} \frac{n!}{n_1!\,(n-n_1)!} x_1^{n_1} \cdot x_2^{n-n_1}$$

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Ex Expand

$$(x+y+z)^2$$

$$= \sum_{\boxed{n_1+n_2+n_3=2}} \frac{2!}{n_1!\cdot n_2!\cdot n_3!} x^{n_1}\cdot y^{n_2}\cdot z^{n_3}$$

$n_1, n_2, n_3 \geq 0$

n_1	n_2	n_3	
0	0	2	z^2
0	1	1	$2yz$
0	2	0	y^2
1	0	1	$2xz$
1	1	0	$2xy$
2	0	0	x^2

$$= x^2 + 2xy + 2xz + y^2 + 2yz + z^2$$

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Ex

$\ln \quad (x^2+x+1)^3$

find coeff of x^5

$$(x^2+x+1)^3 = \sum_{n_1+n_2+n_3=3} \frac{3!}{n_1!\cdot n_2!\cdot n_3!} \underbrace{(x^2)^{n_1}\cdot(x)^{n_2}\cdot 1^{n_3}}_{x^{2n_1+n_2}}$$

want $2n_1+n_2=5$

n_1	n_2	n_3
2	1	0

Coeff of $x^5 = \dfrac{3!}{2!\cdot 1!\cdot 0!} = 3$

Ex Expand

$(x+y+z)^2$

$$= \sum_{\boxed{n_1+n_2+n_3=2}} \frac{2!}{n_1!\cdot n_2!}$$

$n_1, n_2, n_3 \geq 0$

$= x^2 + 2xy +$

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Find coeff of x^6 in

the expansion of

z^{n_3}

$$\left(2x^3 + 3x^2 + 4x + \frac{5}{x}\right)^{10}$$

$$= \sum_{n_1+n_2+n_3+n_4=10} \frac{10!}{n_1!\cdot n_2!\cdot n_3!\cdot n_4!} \underbrace{(2x^3)^{n_1}\cdot(3x^2)^{n_2}\cdot(4x)^{n_3}\cdot\left(\frac{5}{x}\right)^{n_4}}_{2^{n_1}\cdot 3^{n_2}\cdot 4^{n_3}\cdot 5^{n_4}\cdot\, x^{3n_1+2n_2+n_3-n_4}}$$

want

$$3n_1+2n_2+n_3-n_4=6$$

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OR And

Generalized Permutation & Combination

Binomial expansion

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Multinomial expansion

$$(x_1+x_2+\cdots+x_k)^n = \sum_{n_1+n_2+\cdots+n_k=n} \frac{n!}{n_1!\,n_2!\cdots n_k!} x_1^{n_1} x_2^{n_2}\cdots x_k^{n_k}$$

① Permutation

Ordering r-objects out of n objects

w/o rep: $P(n,r) = \frac{n!}{(n-r)!}$

w/ rep: n^r

② Combination

Select r objects out of n objects.

w/o rep: $\binom{n}{r} = \frac{n!}{r!\cdot(n-r)!}$

w/ rep: $\binom{n+r-1}{r}$

$r \geq 0$

letters: a, b, c, d, e

4 letter words:

w/o rep: $4!$ $\boxed{5}\boxed{4}\boxed{3}\boxed{2}$

w/rep: 5^4 $\boxed{5}\boxed{5}\boxed{5}\boxed{5}$

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Generalized Permutation & Combination

① Permutation

Ordering r-objects out of n objects.

- w/o rep. → $P(n,r) = \frac{n!}{(n-r)!}$
- w/ rep. → n^r

w/o rep. → $4!$ $\boxed{5}\boxed{4}\boxed{3}\boxed{2}$

w/rep → 5^4 $\boxed{5}\boxed{5}\boxed{5}\boxed{5}$

② Combination

Select r objects out of n objects.

- w/o rep. → $\binom{n}{r} = \frac{n!}{r!\cdot(n-r)!}$
- w/ rep. → $\binom{n+r-1}{r}$ $\quad r \geq 0$

10 apples
10 oranges
10 banana

w/o rep → Pick 2 different fruits $= \binom{3}{2} = 3$

w/rep → Pick any 2 fruits $= \binom{3+2-1}{2} = \binom{4}{2} = 6$

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Ex Find the # of solutions to the equation

$$x+y+z=10$$

$\ni x, y, z \in \mathbb{N} = \{0,1,2,\dots\}$

$$\# \text{ Solutions} = \binom{3+10-1}{10} = \binom{12}{10} = 66$$

$$\frac{12!}{10!\cdot 2!} = \frac{12\times 11}{2}$$

letters: a, b, c, d, e

4 letter words

w/o rep. — $4!$

w/rep — 5^4

Gener

& Co

① Perm

Ord

of

w/o

rep.

$P(n,r)$

$= \frac{n!}{(n-r}$

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OR And

Generalized Permutation
& Combination

Ex
find # solutions of
eq. $x+y+z=4$
$x, y, z \in \mathbb{N}$

$$\# \text{ Solutions} = \binom{3+4-1}{4}$$
$$= \binom{6}{4} = 15$$

Ex
find # Solutions of
eq. $x+y+z \leq 4$
$\ni x, y, z \in \mathbb{N}$

Solutions =
sol. to $x+y+z=4$
+ " " " $x+y+z=3$
+ " " " $x+y+z=2$
+ " " " $x+y+z=1$
+ " " " $x+y+z=0$

$$\sum_{r=0}^{4} \binom{3+r-1}{r}$$

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Ex

Find the # of solutions to the equation

$$x+y+z = 4$$

Such that $x \geq 0$

$y \geq 1$

$z \geq 2$

x	y	z
0	1	3
0	2	2
1	1	2

$$\cancel{x+y+z} = 4$$

$$x+(y+1)+(z+2) = 4$$

$$x+y+z = 4-1-2 = 1$$

$$\# \text{ solutions} = \binom{3+1-1}{1} = 3$$

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OR And

Ex Find the # of solutions to the equation

$$x+y+z = 4$$

Such that $x \geq 0$
$y \geq 1$
$z \geq 2$

x	y	z
0	1	3
0	2	2
1	1	2

$\cancel{x+y+z} = 4$

$x+(y+1)+(z+2) = 4$

$x+y+z = 4-1-2 = 1$

solutions $= \binom{3+1-1}{1} = 3$

Permutation with indistinguishable objects.

Word: JEDDAH

different words by rearranging letters of JEDDAH

$$= \frac{6!}{2!}$$

Ex ABBA # words by rearranging letters $= \frac{4!}{2! \times 2!} = \frac{24}{4} = 6$

AABB
ABBA
ABAB
BBAA
BAAB
BABA

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