King Saud University

College of Sciences

Department of Mathematics

151 Math Exercises

(1)

Propositional Logic

By:

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7) $p \wedge \neg[q \rightarrow (p \vee r)]$

p	q	r	$p \vee r$	$q \rightarrow (p \vee r)$	$\neg[q \rightarrow (p \vee r)]$	$p \wedge \neg[q \rightarrow (p \vee r)]$
T	T	T	T	T	F	F
T	T	F	T	T	F	F
T	F	T	T	T	F	F
T	F	F	T	T	F	F
F	T	T	T	T	F	F
F	T	F	F	F	T	F
F	F	T	T	T	T	F
F	F	F	F	T	[illegible]	F

(By rules - without using the truth tables)

$$p \wedge \neg[q \rightarrow (p \vee r)]$$
$$\equiv p \wedge [q \wedge \neg(p \vee r)]$$
$$\equiv p \wedge [q \wedge (\neg p \wedge \neg r)]$$
$$\equiv (p \wedge q) \wedge (\neg p \wedge \neg r)$$
$$\equiv (p \wedge \neg p) \wedge (q \wedge \neg r)$$
$$\equiv F \wedge (q \wedge \neg r)$$
$$\equiv F$$

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8) $\neg u \to [(u \wedge v) \to w]$

u	v	w	$\neg u$	$u \wedge v$	$(u \wedge v) \to w$	$\neg u \to [(u \wedge v) \to w]$
T	T	T	F	T	T	T
T	T	F	F	T	F	T
T	F	T	F	F	T	T
T	F	F	F	F	T	T
F	T	T	T	F	T	T
F	T	F	T	F	T	T
F	F	T	T	F	T	T
F	F	F	T	F	T	T

(By rules " without using the truth tables)

$\neg u \to [(u \wedge v) \to w]$

$\equiv u \vee [\neg(u \wedge v) \vee w]$

$\equiv u \vee [(\neg u \vee \neg v) \vee w$

$\equiv (\neg v \vee (u \vee \neg v)) \vee w$

$\equiv (\neg v \vee T) \vee w$

$\equiv T \vee w$

$\equiv T$

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9) Show whether the following statement is a tautology or a contradiction?

$$[(p \to q) \vee (q \to r)] \to (p \to \neg r)$$

$\overset{\text{Cond}}{\equiv} \neg[\quad\quad] \vee (\quad\quad)$

$\overset{\text{Cond}}{\equiv} \neg[\neg p \vee q \vee \neg q \vee r] \vee (\quad\quad)$

$\overset{\text{Comm}}{\underset{\text{Assoc}}{\equiv}} \neg[(q \vee \neg q) \vee (\neg p \vee r)] \vee (\quad\quad)$

$\equiv \neg[T \vee (\quad\quad)] \vee (\quad\quad)$

$\equiv \neg[T] \vee (\quad\quad) \equiv F \vee (p \to \neg r)$

$\equiv p \to \neg r$ $\to T$ / $\to F$

p	$\neg r$	$p \to \neg r$
T	T	T
T	F	F

$\therefore$ contingency.

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10) *Show that the following statement is a tautology*

$$[(p \vee q) \wedge \neg p] \rightarrow q$$

$\equiv \neg[(p \vee q) \wedge \neg p] \vee q$

$\equiv \neg[(p \wedge \neg p) \vee (\neg p \wedge q)] \vee q$

$\equiv \neg[F \vee (\neg p \wedge q)] \vee q$

$\equiv \neg[\neg p \wedge q] \vee q$

$\equiv (p \vee \neg q) \vee q$

$\equiv (\neg q \vee q) \vee p$

$\equiv T \vee p$

$\equiv T$

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11) Show that the following statement is a tautology. $(p \wedge q) \rightarrow (r \rightarrow q)$

$(p \wedge q) \rightarrow (r \rightarrow q) \overset{\text{Cond}}{\equiv}$

$\neg(p \wedge q) \vee (\neg r \vee q) \overset{\text{Demorg}}{\equiv}$

$\equiv \neg p \vee \neg q \vee \neg r \vee q \equiv$

Commu / Associ $\equiv (q \vee \neg q) \vee (\neg p \vee \neg r)$

$\equiv T \vee (\quad) \equiv T$

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12) Prove the following statement is a tautology, without using the truth table.

$$(p \wedge q) \rightarrow [(q \vee r) \rightarrow p]$$

Proof:

$(p \wedge q) \rightarrow [(q \vee r) \rightarrow p] \equiv \neg(p \wedge q) \vee [\neg(q \vee r) \vee p]$

$\equiv (\neg p \vee \neg q) \vee [(\neg q \wedge \neg r) \vee p]$

$\equiv (\neg p \vee \neg q) \vee [(\neg q \vee p) \wedge (\neg r \vee p)]$

$\equiv [(\neg p \vee \neg q) \vee (\neg q \vee p)] \wedge [(\neg p \vee \neg q) \vee (\neg r \vee p)]$

$\equiv [\neg p \vee \neg q \vee \neg q \vee p] \wedge [\neg p \vee \neg q \vee \neg r \vee p]$

$\equiv [(\neg p \vee p) \vee \neg q] \wedge [(\neg p \vee p) \vee (\neg q \vee \neg r)]$

$\equiv [T \vee \neg q] \wedge [T \vee (\neg q \vee \neg r)]$

$\equiv T \wedge T \equiv T$

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13) Without using the truth table, show whether the following statement is a tautology or not.

$$(p \wedge q) \rightarrow [r \rightarrow (p \vee q)]$$

Solution:

[redacted]

$\overset{\text{cond}}{\equiv} \neg(p \wedge q) \vee [\neg r \vee (p \vee q)]$

$\overset{\text{Demorg}}{\equiv} \neg p \vee \neg q \vee \neg r \vee p \vee q$

$\overset{\text{Comm.}}{\underset{\text{Associ}}{\equiv}} (p \vee \neg p) \vee (q \vee \neg q) \vee \neg r$

$\equiv T \vee T \vee \neg r \equiv T.$

H.M

14) Show that the following statement is a tautology: $[(p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r)] \rightarrow r$

Solution:

$\overset{\text{cond}}{\equiv} \neg[(p \vee q) \wedge (\neg p \vee r) \wedge (\neg q \vee r)] \vee r$

$\overset{\text{Demorg}}{\equiv} [(\neg p \wedge \neg q) \vee (p \wedge \neg r) \vee (q \wedge \neg r)] \vee r$

$\overset{\text{Dist}}{\equiv} [(\neg p \wedge \neg q) \vee (p \vee q) \wedge \neg r] \vee r$

$\overset{\text{Associ}}{\equiv} [(\neg p \vee p) \wedge (\neg q \vee q) \wedge (\neg r \vee r)]$

$\equiv T \wedge T \wedge T \equiv T.$

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15) Show that the following statement is a tautology.

Solution:

$$[(p \to q) \wedge (q \to r)] \to (p \to r)$$

$$\equiv \neg[(p \to q) \wedge (q \to r)] \vee (p \to r)$$

$$\equiv [\neg(p \to q) \vee \neg(q \to r)] \vee (p \to r)$$

$$\equiv [\neg(\neg p \vee q) \vee \neg(\neg q \vee r)] \vee (\neg p \vee r)$$

$$\equiv [\neg(\neg p \vee r) \vee (\neg p \vee r)] \vee (q \wedge \neg r)$$

$$\equiv [T \vee (q \wedge \neg r)$$

$$\equiv T$$

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H.M

(16) Show that the following statement is a tautology : $[p \leftrightarrow (q \vee r)] \rightarrow [(\neg q \wedge \neg r) \vee p]$

Solution:

$\overset{\text{Cond}}{\equiv}\underset{\text{Bicond}}{} \neg [(p \wedge q \vee r) \vee (\neg p \wedge \neg q \wedge r] \vee (\neg q \wedge \neg r) \vee p]$

$\overset{\text{Demag}}{\equiv} [(\neg p \vee \neg q \wedge \neg r) \wedge (p \vee q \vee r)] \vee [\cancel{\neg} q \wedge \neg r) \vee p]$

$\overset{\text{Assoc}}{\equiv} [(\neg p \wedge p) \vee (\neg q \vee q) \wedge (\neg r \vee r)] \vee (\neg q \wedge \neg r) \vee p]$

$\equiv F \vee T \wedge T \vee (\neg q \wedge \neg r) \vee p]$

$\equiv T$

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17) Show that the following statement is a contradiction.

Solution:

$$[\neg(p \to q)] \wedge [q \wedge \neg r)]$$

$\equiv \neg(\neg p \vee q) \wedge (q \wedge \neg r)$

$\equiv (p \wedge \neg q) \wedge (q \wedge \neg r)$

$\equiv (\neg q \wedge q) \wedge (p \wedge \neg r)$

$\equiv F \wedge (p \wedge \neg r)$

$\equiv F$

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18) Show that the following statement is a contradiction.

$$[(p \to q) \vee (q \to r)] \to (p \to \neg r)$$

Solution:

$\equiv [(\neg p \vee q) \vee (\neg q \vee r)] \to (p \to \neg r)$

$\equiv [(q \vee \neg q) \vee (\neg p \vee r)] \to (p \to \neg r)$

$\equiv [T \vee (\neg p \vee r)] \to (p \to \neg r)$

$\equiv T \to (p \to \neg r)$

$\equiv F \vee (p \to \neg r)$

$\equiv p \to \neg r$

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3) Show that the following statements are logically equivalent.

$$(p \to q) \to q \equiv (p \vee q)$$

Solution:

$(p \to q) \to q \equiv \neg(p \to q) \vee q$

$\equiv \neg(\neg p \vee q) \vee q$

$\equiv (p \wedge \neg q) \vee q$

$\equiv (p \vee q) \wedge (\neg q \vee q)$

$\equiv (p \vee q) \wedge (T)$

$\equiv (p \vee q)$

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5) Show that the following statements are logically equivalent .

$$(p \to q) \wedge (p \to r) \equiv (\neg q \vee \neg r) \to \neg p$$

Solution:

$$(\neg q \vee \neg r) \to \neg p \overset{\text{cond}}{\equiv} \neg(\neg q \vee \neg r) \vee \neg p$$

$$\overset{\text{Demarg}}{\equiv} (q \wedge r) \vee \neg p \overset{\text{comm}}{\equiv} \neg p \vee (q \wedge r)$$

$$\overset{\text{Distri}}{\equiv} (\neg p \vee q) \wedge (\neg p \vee r)$$

$$\overset{\text{cond}}{\equiv} (p \to q) \wedge (p \to r)$$

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7) Show that the following statements are logically equivalent .

$$(\neg p \vee \neg r) \rightarrow (p \wedge q) \equiv p \wedge (q \vee r)$$

Solution:

$(\neg p \vee \neg r) \longrightarrow (p \wedge q) \equiv \neg(\neg p \vee \neg r) \vee (p \wedge q)$

$\overset{\text{Demorg}}{\equiv} (p \wedge r) \vee (p \wedge q)$

$\equiv p \wedge (r \vee q)$

$\equiv p \wedge (q \vee r)$

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10) *Show that the following statements are logically equivalent .*

$$(\neg p \vee \neg r) \rightarrow (p \wedge q) \equiv p \wedge (q \vee r)$$

Solution:

$$(\neg p \vee \neg r) \rightarrow (p \wedge q) \equiv \neg(\neg p \vee \neg r) \vee (p \wedge q)$$

$$\equiv (p \wedge r) \vee (p \wedge q)$$

$$\equiv p \wedge (r \vee q)$$

$$\equiv p \wedge (q \vee r)$$

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1) (i) Show that the following statements are logically equivalent

$$(p \to q) \wedge (q \vee \neg r) \equiv (p \vee r) \to q$$

(ii) Then use (i) to prove that

$$[(u \vee v) \to w] \wedge [w \vee \neg(x \wedge y)] \equiv [(u \vee v) \vee (x \wedge y)] \to w$$

Solution:

(i) $(p \to q) \wedge (q \vee \neg r)$

$\equiv (\neg p \vee q) \wedge (q \vee \neg r)$

$\equiv (\neg p \vee q) \wedge (\neg r \vee q)$

$\equiv q \vee (\neg p \wedge \neg r)$

$\equiv (\neg p \wedge \neg r) \vee q$

$\equiv (p \vee r) \to q$

(ii) $[(u \vee v) \to w] \wedge [w \vee \neg(x \wedge y)]$

$[(u \vee v) \to w] \wedge [w \vee \neg(x \wedge y)]$

$\equiv [\neg(u \vee v) \vee w] \wedge [w \vee \neg(x \wedge y)]$

$\equiv w \vee [\neg(u \vee v) \wedge \neg(x \wedge y)]$

$\equiv [\neg(u \vee v) \wedge \neg(x \wedge y)] \vee w$

$\equiv [(u \vee v) \vee (x \wedge y)] \to w$

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12) Show that $(p \to q) \wedge (p \to r)$ and $p \to (q \wedge r)$ are logically equivalent ?

Solution:

$$p \to (q \wedge r) \overset{\text{cond}}{\equiv} \neg p \vee (q \wedge r)$$

$$\equiv (\neg p \vee q) \wedge (\neg p \vee r)$$

$$\equiv (p \to q) \wedge (p \to r)$$

HW

13) Show that $(p \to r) \wedge (q \to r)$ and $(p \vee q) \to r$ are logically equivalent ?

Solution: I

$$\overset{\text{cond}}{\equiv} (\neg p \vee r) \wedge (\neg q \vee r)$$

$$\overset{\text{dist}}{\equiv} (\neg p \wedge \neg q) \vee r$$

$$\overset{\text{Demorg}}{\equiv} \neg [p \vee q] \vee r$$

$$\overset{\text{cond}}{\equiv} (p \vee q) \to r$$

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14) Show that $(p \to q) \vee (p \to r)$ and $p \to (q \vee r)$ are logically equivalent ?

ution:

$$(p \to q) \vee (p \to r) \equiv (\neg p \vee q) \vee (\neg p \vee r)$$
$$\equiv (\neg p \vee \neg p) \vee (q \vee r)$$
$$\equiv \neg p \vee (q \vee r)$$
$$\equiv p \to (q \vee r)$$

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15) Show that $(p \to r) \vee (q \to r)$ and $(p \wedge q) \to r$ are logically equivalent ?

Solution:

$$(p \to r) \vee (q \to r) \equiv (\neg p \vee r) \vee (\neg q \vee r)$$
$$\equiv (\neg p \vee \neg q) \vee (r \vee r)$$
$$\equiv (\neg p \vee \neg q) \vee r$$
$$\equiv \neg (p \wedge q) \to r$$

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H.M

16) Show that $\neg p \rightarrow (q \rightarrow r)$ and $q \rightarrow (p \vee r)$ are logically equivalent ?

Solution:

$$\overset{\text{cond}}{\equiv} p \vee (\neg q \vee r)$$

$$\overset{\text{Assoc}}{\equiv} \neg q \vee (p \vee r)$$

$$\overset{\text{cond}}{\equiv} q \rightarrow (p \vee r)$$

H.M T.T

17) Show that $p \leftrightarrow q$ and $(p \rightarrow q) \wedge (q \rightarrow p)$ are logically equivalent ?

Solution:

p	q	$(p \rightarrow q)$	$(q \rightarrow p)$	$(p \rightarrow q) \wedge (q \rightarrow p)$	$p \leftrightarrow q$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

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18) Show that $p \leftrightarrow q$ and $\neg p \leftrightarrow \neg q$ are logically equivalent ?

Solution:

$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$

$\equiv (\neg p \wedge \neg q) \vee (p \wedge q)$

$\equiv (\neg p \wedge \neg q) \vee (\neg(\neg p) \wedge \neg(\neg q))$

$\equiv \neg p \leftrightarrow \neg q$

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19) ***Show that the following statements are logically equivalent ?***

$$\neg p \to (q \to r) \equiv q \to (p \vee r)$$

Solution:

$$\neg p \to (q \to r) \equiv \neg(\neg p) \vee (q \to r)$$
$$\equiv p \vee (\neg q \vee r)$$
$$\equiv \neg q \vee (p \vee r)$$
$$\equiv q \to (p \vee r)$$

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H.W

20) Show that the following statements are logically equivalent ?

Solution:

$$(p \wedge \neg r) \rightarrow q \equiv (p \wedge \neg q) \rightarrow r$$

$\overset{\text{cond}}{\equiv} \neg(p \wedge \neg r) \vee q$

$\overset{\text{Demorg}}{\equiv} (\neg p \vee r) \vee q$

$\overset{\text{Assoc}}{\equiv} (\neg p \vee q) \vee r$

$\overset{\text{Demorg}}{\equiv} \neg[p \wedge \neg q] \vee r$

$\overset{\text{cond}}{\equiv} (p \wedge \neg q] \rightarrow r$

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21) Show that the following statements are logically equivalent ?

Solution:

$$\neg q \vee \neg[\neg p \vee (p \wedge q)] \equiv \neg q$$

$\neg q \vee \neg[\neg p \vee (p \wedge q)] \overset{\text{Demorg}}{\equiv}$

$\neg q \vee [p \wedge (\neg p \vee \neg q)] \overset{\text{Dist}}{\equiv}$

$\equiv \neg q \vee [(p \wedge \neg p) \vee (p \wedge \neg q)]$

$\equiv \neg q \vee [F \vee (p \wedge \neg q)] \equiv \neg q \vee (\neg q \wedge p)$

(Absor) $\equiv \neg q$

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22) Show that the following statements are logically equivalent ?

Solution:

$$\neg[p \wedge (q \vee r)] \equiv (p \to \neg q) \wedge (p \to \neg r)$$

$\neg[p \wedge (q \vee r)] \equiv [\neg p \vee \neg(q \vee r)]$

$\equiv [\neg p \vee (\neg q \wedge \neg r)]$

$\equiv (\neg p \vee \neg q) \wedge (\neg p \vee \neg r)$

$\therefore \equiv (p \to \neg q) \wedge (p \to \neg r)$

---

23) Show that the following statements are logically equivalent

Solution:

$$(p \to q) \wedge [\neg q \wedge (\neg q \vee r)] \equiv \neg(q \vee p)$$

$(p \to q) \wedge [(\underbrace{\neg q \wedge (\neg q \vee r)}_{p \wedge (p \vee q) \equiv p})]$

$\equiv (p \to q) \wedge (\neg q)$

$\equiv (\neg p \vee q) \wedge (\neg q)$

$\equiv (\neg p \wedge \neg q) \vee (\neg q \wedge q)$

$\equiv (\neg p \wedge \neg q) \vee (F)$

$\equiv (\neg q \wedge \neg p)$

$\equiv \neg(q \vee p)$

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24) Show that the following statements are logically equivalent

lution:

$$(p \wedge q) \rightarrow r \equiv (p \wedge \neg r) \rightarrow \neg q$$

$$(p \wedge q) \rightarrow r \equiv \neg(p \wedge q) \vee r$$

$$\equiv (\neg p \vee \neg q) \vee r$$

$$\equiv (\neg p \vee r) \vee \neg q$$

$$\equiv (p \wedge \neg r) \rightarrow \neg q$$

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28) Show that the following statements are logically equivalent, or not?

$$(p \rightarrow q) \vee (p \rightarrow \neg r) \equiv p \rightarrow (r \rightarrow q)$$

ution:

$$(p \rightarrow q) \vee (p \rightarrow \neg r) \equiv (\neg p \vee q) \vee (\neg p \vee \neg r)$$

$$\equiv (\neg p \vee \neg p) \vee (\neg r \vee q)$$

$$\equiv \neg p \vee (r \rightarrow q)$$

$$\equiv p \rightarrow (r \rightarrow q)$$

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HM **30)** Show that the following statements are logically equivalent , without using the truth table.

$$p \leftrightarrow (\neg q \wedge \neg r) \equiv \neg p \leftrightarrow (q \vee r)$$

Solution:

p	q	r	$\neg p$	$\neg q$	$\neg r$	$(\neg q \wedge \neg r)$	$(q \vee r)$	$p \leftrightarrow (\neg q \wedge \neg r)$	$\neg p \leftrightarrow (q \vee r)$
T	T	T	F	F	F	F	T	F	F
T	T	F	F	F	T	F	T	F	F
T	F	T	F	T	F	F	T	F	F
T	F	F	F	T	T	T	F	T	T
F	T	T	T	F	F	F	T	T	T
F	T	F	T	F	T	F	T	T	T
F	F	T	T	T	F	F	T	T	T
F	F	F	T	T	T	T	F	T	F

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32) Show that the contrapositive of $(p \land q) \to r$ is logically equivalent to $p \to (q \to r)$

Solution:

$$\neg r \to \neg(p \land q) \equiv r \lor (\neg p \lor \neg q)$$
$$\equiv \neg p \lor (\neg q \lor r)$$
$$\equiv p \to (q \to r)$$

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Antecedent مقدمة $\rightarrow$ Consquent نتيجة

3) If $m . n = l$, then $m \geq 0$ *or* $n \geq 0$ *or* $l \geq 0$, $m, n, l \in \mathbb{Z}$

Contrapositive: $\neg$ (Consquent) $\equiv$ if $m<0$ and $n<0$ and $l<0$, then $m.n \neq l$ $\equiv \neg$ (Antecedent)

Antecedent $\rightarrow$ Consquent

4) If the number $a+b-c$ is an even number, then *a is an even* or *b is an even* or *c is an even* ,where $a, b, c \in \mathbb{Z}$

If a is an odd integer and b is an odd integer also c is an odd integer then $a+b-c$ is an odd integer.

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King Saud University

College of Sciences

Department of Mathematics

$p \vee q$

$p \Rightarrow \neg p \to q$

$p \vee q$

151 Math Exercises

(2)

The Universal Quantifiers

$(p \to q) \to r$

By:

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Exercises

Q_1. Let $P(x)$ be the statement "$x = x^2$." If the domain consists of the integers, what are these truth values?

a) $P(0)$: $0 = 0^2$ True

b) $P(1)$ $1 = 1^2$. True

c) $P(2)$. $2 = 2^2$ False

d) $P(-1)$: $-1 = (-1)^2$ False

e) $\exists x\, P(x)$
True, Because If you take $x = 0$ $P(0)$: $0 = 0^2$ True
$\therefore \exists x P(x)$ True

f) $\forall x\, P(x)$
False, take $x = 2$, $P(2)$: $2 = 2^2$ False then $\forall x P(x)$ is false

Q_2. Let $Q(x)$ be the statement "$x + 1 > 2x$." If the domain consists of all integers, what are these truth values?

a) $Q(0)$: $0+1 > 2(0)$. True

b) $Q(-1)$: $-1+1 > 2(-1)$ True

c) $Q(1)$ $1+1 > 2(1)$ False

d) $\exists x\, Q(x)$
true, take $x = -1$
$Q(-1)$: $-1+1 > 2(-1)$ True
$\therefore \exists x\, Q(x)$ True

e) $\forall x\, Q(x)$

f) $\exists x\, \neg Q(x)$
$\neg Q(x)$: $x+1 \leq 2x$
True, take $x = 2$
$\neg Q(2)$: $2+1 \leq 2(2)$ True
then $\exists x \neg Q(x)$ True.

g) $\forall x\, \neg Q(x)$
$\neg Q(x)$ is not true for all integers
for example $\neg Q(0)$: $0+1 \leq 2(0)$, False
then $\forall x \neg Q(x)$ is False.

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Q_3. Determine the truth value of each of these statements if the domain consists of all integers.

a) $\forall n\,(n + 1 > n)$

True. Because n is true for all integers. then $\forall n(n+1 > n)$ True.

b) $\exists n\,(2n = 3n)$

True. take $n=0$
$2(0) = 3(0)$ True
then $\exists n(2n=3n)$ true

c) $\exists n\,(n = -n)$

True. take $n=0$
$0 = -(0)$ True
then $\exists n(n=-n)$ True

d) $\forall n\,(3n \le 4n)$

This statement is not true for all integers. for example $n=-1$. $-3 \le -4$ false
then $\forall n(3n \le 4n)$ false. $n=-1$ is counterexample of $\forall n(3n \le 4n)$

Q_4. Determine the truth value of each of these statements if the domain consists of all real numbers.

$x \in R$

a) $\exists x\,(x^3 = -1)$

True. take $x=-1$. $(-1)^3 = -1$ True
then $\exists x(x^3=-1)$ True

*

b) $\exists x\,(x^4 < x^2)$

c) $\forall x\,((-x)^2 = x^2)$

True. because $(-x)^2 = x^2$ is true for all $x \in R$
then $\forall x((-x)^2 = x^2)$ True

d) $\forall x\,(2x > x)$

False. take $x=0$. $0>0$ false
then $\forall x(2x>x)$ is false
and $x=0$ is counterexample. for $\forall x(2x>x)$.

Q_5. Determine the truth value of each of these statements if the domain consists of all integers.

a) $\forall n\,(n^2 \ge 0)$

True. because $n^2 \ge 0$ is true for all $n \in Z$
then $\forall n(n^2 \ge 0)$ True.

b) $\exists n\,(n^2 = 2)$

False. because $n^2 = 2$ is false for all $n \in Z$
then $\exists n(n^2=2)$ False.

c) $\forall n\,(n^2 \ge n)$

True. because $n^2 \ge n$ is true for all $n \in Z$
then
$\forall n(n^2 \ge n)$

d) $\exists n\,(n^2 < 0)$

False! because $n^2<0$ is false for all $n \in Z$
then
$\exists n(n^2<0)$ False

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Q6. Determine the truth value of each of these statements:

(1) $\forall x \in \mathbb{R}$, $x^2 - 4x + 4 \geq 0$

$\forall x \in \mathbb{R}$, $(x-2)^2 \geq 0$

Becauce $(x-2)^2 \geq 0$ is true for every $x \in R$

then $\forall x \in R\ (x-2)^2 \geq 0$ true

(2) $\forall x > 0$, $x \geq \frac{1}{x}$

$\frac{1}{\frac{1}{2}} = 2$

False, becauce $x \geq \frac{1}{x}$ is not true for all $x > 0$

for example take $x = \frac{1}{2}$

$\frac{1}{2} \ngeq 2$ false

then $\forall x > 0$, $x \geq \frac{1}{x}$ false

and $x = \frac{1}{2}$ is counterexample of $\forall x > 0$, $x \geq \frac{1}{x}$

(3) $\forall x \in \mathbb{Z}, ((x \geq 2) \vee (x^2 \leq 2))$

True, becauce $((x \geq 2) \vee (x^2 \leq 2))$ is true for every $x \in \mathbb{Z}$

then $\forall x \in \mathbb{Z}, ((x \geq 2) \vee (x^2 \leq 2))$ True

(4) $\exists x \in \{1,2,3,4\}$, $2^x < x!$

True becauce if we take $x = 4$

$16 < 24$ True

then

$\exists x \in \{1,2,3,4\}$, $2^x < x!$

True

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(5) $\exists x \in \mathbb{Z}^* = \mathbb{Z} - \{0\},\ \frac{x-1}{x} \in \mathbb{Z}$

True, take $x = 1$

$\frac{1-1}{1} = 0 \in \mathbb{Z}$

then $\exists x \in \mathbb{Z}^* = \mathbb{Z} - \{0\},\ \frac{x-1}{x} \in \mathbb{Z}$

True

(6) $\exists x \in \mathbb{R},\ \ x^2 = 5$

False, becauece $x^2 = 5$ is false for evers $x \in \mathbb{R}$

then $\exists x \in \mathbb{R},\ x^2 = 5$ False

Q₇ Write *the negation* of the below statements:

(*i*) Some students did not listen to the instructions.

Some Students listen to the instructions.

(*ii*) $\exists x \in D, x^2 > 3$

$\forall x \in D,\ x^2 \leq 3$

(*iii*) If you collect enough points, you will win the game.

If you don't collect enough Points ~~the~~ you will not win the game

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even = 2k
odd = 2k+1
3|x 3 = xk

3∤10 10 = 3□+1
2|10
10 = 2
3∤10 10 = 3□+1

DEFINITION 4 Let $a, b \in \mathbb{Z} : a \neq 0$. a *is a divisor to* b, $a|b$ *if there exist integer* c *such that* $b = ac$

DEFINITION 5 Let $a, b \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$. a is congruent to b modulo n, $a \equiv b (mod\ n)$ if $n|(a-b)$

$10 \equiv 6 \ mod\ 2$
$2|(10-6)$

EXAMPLE 4 : $25 \equiv 7 (\text{mod } 9)$, $3 \equiv -15 (\text{mod } 9)$

Exercises

1. Use a direct proof to show that the sum of two odd integers is even.

Solution:

let $s = 2t+1,\ t \in \mathbb{Z}$, and $n = 2k+1,\ k \in \mathbb{Z}$

$s + n = 2t+1 + 2k+1$

$s + n = 2t + 2k + 2$

$s + n = 2(\underbrace{t+k+1}_{m})$

$s + n = 2m,\ m = t+k+1 \in \mathbb{Z}$

$s+n$ is even.

2. Use a direct proof to show that the sum of two even integers is even.

Solution:

let $r = 2k,\ k \in \mathbb{Z}$, and $s = 2t,\ t \in \mathbb{Z}$

$r + s = 2k + 2t$

$r + s = 2(\underbrace{k+t}_{m})$

$r + s = 2m,\ m = k+t \in \mathbb{Z}$

$\therefore r+s$ is even.

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3. Show that the square of an even number is an even number using a direct proof.
Solution:

let a be an even number

So, $a = 2k, k \in \mathbb{Z}$

$a^2 = (2k)^2$

$a^2 = 4k^2$

$a^2 = 2(2k^2) \rightarrow h = 2k^2, h \in \mathbb{Z}$

$a^2 = 2h \rightarrow a^2$ is even

---

4. Show that the additive inverse, or negative, of an even number is an even number using a direct proof.
Solution:

---

5. Prove that if $m + n$ and $n + p$ are even integers, where m, n, and p are integers, then $m + p$ is even.

Solution: Let $m + n = 2K : K \in \mathbb{Z}$

& $n + p = 2L : L \in \mathbb{Z}$

$\rightarrow m = 2k - n \Rightarrow m + p = 2k - n + 2k - n$

$\rightarrow p = 2L - n$

$= 2\underbrace{((k - n) + (k - n))}_{M \in \mathbb{Z}}$

$= 2M$

$\therefore$ $m + p$ is even

6. Use a direct proof to show that the product of two odd numbers is odd.

Solution:

let $r = 2k+1,\ k \in \mathbb{Z}$ and $s = 2t+1,\ t \in \mathbb{Z}$

$r \cdot s = (2k+1)(2t+1)$

$rs = 4kt + 2k + 2t + 1$

$rs = 2(\underbrace{2kt + k + t}_{m}) + 1$

$rs = 2m+1,\ m = 2kt + k + t \in \mathbb{Z}$

$\therefore rs$ is odd

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11. Use a direct proof to show that if p is an odd prime number, then the number 4 is a divisor to $2p+2$.

Solution: let $p = 2k+1 : k \in \mathbb{Z}$, $p|p$ and $1|p$

$2p+2 = 2(2k+1) + 2$
$= 4k+2+2$
$= 4k+4$

$\therefore 4 | 4k+4$ where $4k+4 = 4m : m \in \mathbb{Z}$

---

12. Use a direct proof to show that if n is an odd number, then $5n+6$ is odd.

Solution: let $n = 2k+1 : k \in \mathbb{Z}$

$5n+6 = 5(2k+1)+6$
$= 10k+5+6$
$= 10k+10+1$
$= 2(\underbrace{5k+5}_{m \in \mathbb{Z}})+1$
$= 2m+1$

$\therefore$ it's odd

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13. Use a direct proof to prove that if n is an odd number, then $n^2 \equiv 1 (\bmod 4)$.

Solution:

$$n \text{ odd} \overset{?}{\longrightarrow} n^2 \equiv 1 (\bmod 4)$$

$$\overset{?}{\longrightarrow} 4 \mid (n^2 - 1)$$

let $n = 2k+1 \in \mathbb{Z} \overset{?}{\longrightarrow} n^2-1 = 4m$

$$n^2 = 4k^2 + 4k + 1$$

$$n^2 - 1 = 4k^2 + 4k + 1 - 1$$

$$n^2 - 1 = 4k^2 + 4k$$

$$n^2 - 1 = 4(\underbrace{k^2 + k}_{m})$$

$$n^2 - 1 = 4m, \quad m = k^2 + k \in \mathbb{Z}$$

$\therefore \; n^2 \equiv 1 (\bmod 4)$

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14. Use a direct proof to prove that if n is an even number, then $n^2 \equiv 0(mod4)$.

Solution:

n even $\xrightarrow{?}$ $n^2 \equiv 0 \, (mod\, 4)$

$\xrightarrow{?}$

$\xrightarrow{?}$

let $n = 2k+1 \; k \in \mathbb{Z}$

$n^2 = 4k^2$

$n^2 = 4(\underbrace{k)^2}_{s}$

$n^2 - 0 = 4s - 0 \quad s = k^2 \in \mathbb{Z}$

$4 / (n^2 - 0)$

$\therefore \; n^2 \equiv 0 \, (mod 4)$

---

15. Use a direct proof to prove that, if 3 is not a divisor of n, then $n^2 \equiv 1(mod3)$.

Solution:

$3 \nmid n \xrightarrow{?} n^2 \equiv 1 \, (mod 3)$

let $3 \nmid n$ =

then $n = 3k+1 \; . k \in \mathbb{Z}$

$n^2 = 9k^2 + 6k + 1$

$n^2 - 1 = 9k^2 + 6k + 1 - 1$

$n^2 - 1 = 3(\underbrace{3k^2 + 2k}_{m})$

$n^2 - 1 = 3m \quad , m = 3k^2 + 2k \in \mathbb{Z}$

$3 / (n^2 - 1)$

$\therefore \; n^2 \equiv 1 \, (mod\, 3)$

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Exercises

1. Use a proof by contraposition to show that if $\overset{p}{x + y \geq 2}$, where x and y are real numbers, then $\overset{q}{x \geq 1 \text{ or } y \geq 1}$.

Solution: Assume → $\neg q \equiv x < 1$
and $y < 1$
+ ———
$x + y < 2 \equiv \neg p$

$\therefore \neg q \rightarrow \neg p$

---

2. Prove that if m and n are integers and mn is even, then m is even or n is even.

Solution:

Assume → $\neg q \equiv$ (odd) $m = 2k+1 : k \in \mathbb{Z}$
and $n = 2h+1 : h \in \mathbb{Z}$

$m \cdot n = (2k+1)(2h+1)$

$m \cdot n = 4kh + 2k + 2h + 1$
$= 2\underbrace{(2kh + k + h)}_{M \in \mathbb{Z}} + 1 \rightarrow m \cdot n = 2M + 1$ (odd) $\equiv \neg p$
$\therefore \neg q \rightarrow \neg p$

---

3. Show that if $\overset{p}{n \text{ is an integer and } n^3 + 5 \text{ is odd}}$, then n $\overset{q}{\text{is even}}$ using a proof by contraposition.

Solution: Assume → $\neg q \equiv n = 2k+1$, $k \in \mathbb{Z}$

$n^3 + 5 = (2k+1)^3 + 5$

$n^3 + 5 = (8k^3 + 12k^2 + 6k + 1) + 5$

$= 8k^3 + 12k^2 + 6k + 6$

$= 2\underbrace{(4k^3 + 6k^2 + 3k + 3)}_{M \in \mathbb{Z}}$

$n^3 + 5 = 2M$ (even) $\equiv \neg p$

$\therefore \neg q \rightarrow \neg p$

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4. Prove that if n is an integer and $3n + 2$ is even, then n is even ,using a proof by contraposition.

Solution:

Assume that n is odd $\xrightarrow{?}$ $3n+2$ is odd

$n = 2k+1$, $k \in \mathbb{Z}$

$3n = 6k+3$

$3n+2 = 6k+3+2$

$3n+2 = 6k+4+1$

$= 2(\underbrace{3k+2}_{m})+1$

$3n+2 = 2m+1$, $m = 3k+2 \in \mathbb{Z}$

$3n+2$ is odd

---

5. Prove that if n is a positive integer, then n is even if and only if $7n + 4$ is even.

Solution:

n even $\overset{?}{\longleftrightarrow}$ $7n+4$ even

n even $\xrightarrow{?}$ $7n+4$ even

let $n = 2k$, $k \in \mathbb{Z}$

$7n = 14k$

$7n+4 = 14k+4$

$7n+4 = 2(\underbrace{7k+2}_{m})$

$7n+4 = 2m$, $m = 7k+2 \in \mathbb{Z}$

$\therefore 7n+4$ is even

$7n+4$ even $\xrightarrow{?}$ n is even

by contraposition

n is odd $\xrightarrow{?}$ $7n+4$ is odd

let $n = 2k+1$, $k \in \mathbb{Z}$

$7n = 14k+7$

$7n+4 = 14k+11$

$7n+4 = 14k+10+1$

$7n+4 = 2(\underbrace{7k+5}_{m})+1$

$7n+4 = 2m+1$

$\therefore 7n+4$ is odd

---

6. Prove that if a is an integer where $5 \nmid a$, then $5 \nmid (a+20)$ using a proof by contraposition.

Solution:

by contraposition :

If $5 \mid (a+20)$, then $5 \mid a$

$5 \mid (a+20) \xrightarrow{?} 5 \mid a$

let $5 \mid (a+20)$

$\therefore (a+20) = 5k$, $k \in \mathbb{Z}$

$a = 5k - 20$

$a = 5(\underbrace{k-4}_{m})$

$a = 5m$, $m = k-4 \in \mathbb{Z}$

$\therefore 5 \mid a$

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$\mathbb{Z} = \{-2, -1, 1, \ldots\}$ $\mathbb{N} = \{1, 2, 3, 4\}$

7. Use a proof by contraposition to show that if $2|mn$: $m, n \in \mathbf{N}$, then $2|m$ or $2|n$.

Solution:

by contraposition :

If $2|m$ and $2|n$, then $2|mn$

$2|m$ and $2|m \xrightarrow{?} 2|mn$

let $2|m$, $2|n$

$m = 2k$, $k \in \mathbb{Z}$, $n = 2s$, $s \in \mathbb{Z}$

$mn = 2k \times 2s$

$mn = 4ks$

$mn = 2(\underbrace{ks}_{t})$

$mn = 2t$, $t \in \mathbb{Z}$

$\therefore 2|mn$

---

8. Use a proof by contraposition to show that if xy is even number where $x, y \in \mathbb{Z}$, then x is even or y is even .

Solution:

by contraposition :

" If x is odd and y is odd , then xy is odd "

x and y odd numbers $\xrightarrow{?}$ xy is odd

let $x = 2k+1$, $k \in \mathbb{Z}$

$y = 2t+1$, $t \in \mathbb{Z}$

$xy = (2t+1)(2k+1)$

$xy = 4kt + 2t + 2k + 1$

$xy = 2(\underbrace{2kt + t + k}_{m}) + 1$

$xy = 2m+1$, $m = 2kt + t + k \in \mathbb{Z}$

$\therefore xy$ is odd

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HW

10. Use a proof by contraposition to show that if $3mn + 2$ is irrational number, then m is irrational or n is irrational .

Solution: Assume → $\neg q = m = \frac{a}{b}$: $a, b \in \mathbb{Z}$, $b \neq 0$, $\gcd(a,b) = 1$

& $n = \frac{c}{d}$: $c, d \in \mathbb{Z}$, $d \neq 0$, $\gcd(c,d) = 1$

$$\rightarrow 3mn + 2 = \frac{3}{1}\left(\frac{a}{b}\right)\left(\frac{c}{d}\right) + 2$$

$$= \frac{3ac}{bd} + \frac{2}{1} = \frac{3ac + 2bd}{bd} \equiv \neg p$$

$$\therefore \neg q \rightarrow \neg p : (T)$$

H.W

11. Use a proof by contraposition to show that if $a^2 + b^2$ is odd, then a is even or b is even .

Solution: Assume → $\neg q \equiv a = 2k + 1$, $k \in \mathbb{Z}$

& $b = 2l + 1$, $l \in \mathbb{Z}$

$$a^2 + b^2 = (2k+1)^2 + (2k+1)^2$$

$$= (4k^2 + 4k + 1) + (4k^2 + 4k + 1)$$

$$= 8k^2 + 8k + 2$$

$$= 2(\underbrace{4k^2 + 4k + 1}_{M \in \mathbb{Z}})$$

$$a^2 + b^2 = 2M$$

$$\therefore \neg q \rightarrow \neg p : (T)$$

12. Use a proof by contraposition to show that if $x + y < 15$, where x and y are real numbers, then $x < 8$ or $y < 8$.

Solution:

by contraposition

" If $x \geq 8$ and $y \geq 8$ then $x+y \geq 15$ "

let $x \geq 8$ and $y \geq 8 \xrightarrow{?} x+y \geq 15$

$x+y \geq 8+8$

$x+y \geq 16$

$\therefore x+y \geq 15$

---

13. Prove that if m is an integer where $3 \nmid m$, then $3 \nmid (m+1)^2 + 2m^2 + 5$ using a proof by contraposition.

Solution:

by contraposition

" If $3 \mid (m+1)^2 + 2m^2+5$, then $3 \mid m$

$3 \mid (m+1)^2 + 2m^2 + 5 \xrightarrow{?} 3 \mid m$

$(m+1)^2 + 2m^2+5 = 3k, k \in \mathbb{Z}$

$(m+1)^2 = 3k - 5 - 2m^2$

$m^2+2m+1 = 3k-5-2m^2$

$2m = 3k-5-2m^2-m^2-1$

$2m = 3k-6-3m^2$

$m+m = 3k-6-3m^2$

$m+m = 3(k-2-m^2)$

$m = 3(\underbrace{k-2-m^2-m}_{s})$

$m = 3s, s = k-2-m^2-m \in \mathbb{Z}$

$\therefore 3 \mid m$

---

14. Use a proof by contraposition to show that if $3n^2 + 4n + 3$ is even, then n is odd.

Solution:

by contraposition

If n even, then $3n^2+4n+3$ is odd

n even $\xrightarrow{?} 3n^2+4n+3$ is odd

$n = 2k, k \in \mathbb{Z}$

$n+3n = 6k+$

$3n+4 = 6k+4$

$n(3n+4) = n(6k+4)$

$3n^2+4n = 6kn+4n$

$3n^2+4n+3 = 6kn+4n+3$

$3n^2+4n+3 = 6kn+4n+2+1$

$= 2(\underbrace{3kn+2n+1}_{m})+1$

$= 2m+1, m = 3kn+2n+1 \in \mathbb{Z}$

$\therefore 3n^2+4n+3$ is odd

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15. Let $6 | m$: $m \in \mathbb{Z}$. Use a proof by contraposition to show that if $\underbrace{3 \nmid (m+n)}_{p}$: $n \in \mathbb{Z}$, then $\underbrace{3 \nmid n}_{q}$.

Solution: $\neg q$: $n = 3k$: $k \in \mathbb{Z}$, $m = 6L$: $L \in \mathbb{Z}$

$$m + n = 3k + 6L$$
$$= 3(\underbrace{k + 2L}_{M}) \quad : M \in \mathbb{Z}$$
$$m + n = 3M \equiv \neg p$$

$$\therefore \neg q \rightarrow \neg p$$

---

16. Prove that if n is a positive integer, then n is even <u>if and only if</u> $7n + 4$ is even .

Solution: n is even $\leftrightarrow$ $7n + 4$ is even

(i) n is even $\rightarrow$ $7n + 4$ is even

(Direct proof)

$n = 2k$: $k \in \mathbb{Z}$

$$7n + 4 = 7(2k) + 4$$
$$= 2(7k) + 4$$
$$= 2[\underbrace{7k + 2}_{h \in \mathbb{Z}}] = 2h \text{ even}$$

(ii) $7n + 4$ is even $\rightarrow$ n is even

(contrapositive)

$\neg q$: $n = 2k + 1$: $k \in \mathbb{Z}$

$$\rightarrow 7n + 4 = 7(2k + 1) + 4$$
$$= 14k + 7 + 4$$
$$= 14k + 10 + 1$$
$$= 2(\underbrace{7k + 5}_{M \in \mathbb{Z}}) + 1$$
$$= 2M + 1 \text{ odd}$$

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17. Prove that if n is a positive integer, then n is odd if and only if $5n + 6$ is odd.

Solution:

Direct

n is odd $\rightarrow 5n+6$ is odd

Let $n = 2K+1 : K \in \mathbb{Z}$

$5n+6 = 5(2K+1)+6$

$= 10K+5+6$

$= 10K+10+1$

$= 2(\underbrace{5K+5}_{M\in\mathbb{Z}})+1$

$= 2M+1$

$\therefore 5n+6$ is odd

contrapositive

$5n+6$ is even $\rightarrow n$ is even

$\neg q : n = 2K \in \mathbb{Z}$

$5n+6 = 5(2K)+6$

$= 10K+6$

$= 2(\underbrace{5K+3}_{M\in\mathbb{Z}})$

$= 2M$

$\therefore \neg q \rightarrow \neg p$

H.W

18. Prove that if n is integer, then $3n + 2$ is odd if and only if $9n + 5$ is even .

Solution:

Direct

$\overset{(odd)}{3n+2} \rightarrow \overset{(even)}{9n+5}$

$3n+2 = 2K+1 : K \in \mathbb{Z}$

$3n = 2K-1$

$3(3n)+5 = 3(2K-1)+5$

$= 6K-3+5$

$= 6K+2$

$= 2(\underbrace{3K+1}_{M\in\mathbb{Z}})$

$9n+5 = 2M$

contrapositive

$\overset{(even)}{9n+5} \rightarrow \overset{(odd)}{3n+2}$

$\neg q : 3n+2 = 2K : K \in \mathbb{Z}$

$3n = 2K-2$

$\rightarrow 3(3n)+5 = 3(2K-2)+5$

$\rightarrow 9n+5 = 6K-6+5$

$= (6K-2)+1$

$= 2(\underbrace{3K-1}_{M\in\mathbb{Z}})+1$

$= 2M+1$

$\therefore \neg q \rightarrow \neg p$

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19. Use a proof by contraposition to show that if $3x - y = z$ where $x, y, z \in \mathbb{Z}$, then x is even or y is even or z is even .

Solution:

$\neg q \equiv \quad x = 2k+1 : k \in Z$

$y = 2L+1 : L \in Z$

$z = 2b+1 : b \in Z$

$3x - y = z$

$3(2k+1) - 2L+1 = 2b+1$

$6k + 2L + 4 = 2b+1$

$2(\underbrace{2k - L + 2}_{M \in Z}) = 2b+1$

$2M = 2b+1 \equiv \neg p$

$\neg q \to \neg p . (T)$

---

20. Use a proof by contraposition to show that if $4n^2 + n - 3$ is even, then n is odd.

Solution:

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23. Use a proof by contraposition to show that if $\overset{p}{m \nmid n}$, then $\overset{q}{m \nmid l \text{ or } m \neq n + l}$ where $l, m, n \in \mathbb{Z}$.

Solution: $\neg q = n + l = ma : a \in \mathbb{Z}$

$n = ma \; , \; l = ma$

$ma + ma = ma \; : \; a \in \mathbb{Z}$

$m \mid n + l$

$\therefore \neg q \to \neg p : (T)$

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Exercises

1. Use a proof by contradiction to prove that the sum of an irrational number and a rational number is irrational.

Solution: $x \in \mathbb{Q} \wedge y \notin \mathbb{Q}$

$x + y \notin \mathbb{Q}$?

Sol. Assume the negation:

$x + y \in \mathbb{Q},\ \exists\, 0 \neq b, a \in \mathbb{Z} : x + y = \frac{a}{b}$ (*)

$\because x \in \mathbb{Q} \Rightarrow \exists\, 0 \neq b, c \in \mathbb{Z} : x = \frac{c}{d}$

By substitution (*)

$\frac{c}{d} + y = \frac{a}{b} \Rightarrow y = \frac{a}{b} - \frac{c}{d}$

$y = \frac{ad - bc}{bd} \in \mathbb{Q},\ 0 \neq bd,\ ad - bc \in \mathbb{Z}$

This is a contradiction, where $y \notin \mathbb{Q} \Rightarrow$ The statement true.

2. Show that if n is an integer and $n^3 + 5$ is odd, then n is even using a proof by contradiction.

Solution:

3. Prove that if n is an integer and $3n + 2$ is even, then n is even using a proof by contradiction.

Solution:

Assume P $p: 3n+2 \text{ even} \rightarrow n \text{ even}$

assume $\neg p: 3n+2 \text{ even} \wedge n \text{ is odd}$

$3n+2 \text{ even} \wedge n = 2k+1 \quad k \in \mathbb{Z}$

$3n+2 \text{ even} \wedge 3n = 6k+3$

$3n+2 \text{ even} \wedge 3n+2 = 6k+5$

$3n+2 \text{ even} \wedge 3n+2 = 6k+4+1$

$3n+2 = 2(\underbrace{3k+2}_{m})+1$

$3n+2 = 2m+1 \quad m = 3k+2 \in \mathbb{Z}$

$3n+2 \text{ even} \wedge \therefore 3n+2 \text{ is odd}$

contradiction

$\neg p$: False

p: True

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$\mathbb{R}$ $\mathbb{Z}$

5. Prove that $\sqrt{3}$ is irrational by giving a proof by contradiction.

Solution: Assume $\rightarrow$ the negation. $\sqrt{3} \in Q$

$\rightarrow \sqrt{3} \in Q \rightarrow \exists\ 0 \neq b, a \in Z,\ \gcd(a,b)=1$

$\rightarrow \sqrt{3} = \frac{a}{b} \xrightarrow{(\ )^2} 3 = \frac{a^2}{b^2} \longrightarrow a^2 = 3b^2$

$\rightarrow 3 \mid a^2 \rightarrow \therefore 3 \mid a \longrightarrow a = 3k \ . \ k \in Z$

$\Rightarrow 9k^2 = 3b^2 \rightarrow b^2 = 3k^2 \rightarrow 3 \mid b^2 \therefore 3 \mid b$

$\rightarrow \therefore 3 \mid b,\ 3 \mid a$

- this is a contradiction, when $\gcd(a,b)=1$

H.W **6.** Prove that $\sqrt{5}$ is irrational by giving a proof by contradiction.

Solution:

Assume $\rightarrow$ the negation. $\sqrt{5} \in Q$, $\exists\ 0 \neq b, a \in Z$, $\gcd(a,b)=1$

$\rightarrow \sqrt{5} = \frac{a}{b} \xrightarrow{(\ )^2} 5 = \frac{a^2}{b^2} \rightarrow a^2 = 5b^2$

$\rightarrow 5 \mid a^2 \therefore 5 \mid a \rightarrow a = 5k \in Z$

$\rightarrow 5b^2 = 25k^2 \rightarrow b^2 = 5k^2 \rightarrow 5 \mid b^2 \rightarrow 5 \mid b$

$\therefore 5 \mid b,\ 5 \mid b$

this is contradiction, where $\gcd(a,b)=1$

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7. Let $\sqrt{5}$ is irrational, prove that $2 - 3\sqrt{5}$ is irrational number.

Solution: Assume the negation

$2 - 3\sqrt{5} \in \mathbb{Q}$, $\exists$ $0 \neq b, a \in \mathbb{Z}$, $\gcd(a,b) = 1$

$2 - 3\sqrt{5} = \frac{a}{b} \rightarrow 3\sqrt{5} = 2 - \frac{a}{b}$

$3\sqrt{5} = \frac{2b-a}{b} \rightarrow \sqrt{5} = \frac{2b-a}{3b} \in \mathbb{Q}$

when $3b \neq 0$, $2b - a \in \mathbb{Z}$ $\rightarrow \sqrt{5}$ is rational a Contradiction

---

8. Let $\sqrt{2}$ is irrational, prove that $1 + 3\sqrt{2}$ is irrational number.

Solution:

Let negation $1 + 3\sqrt{2} \in \mathbb{Q}$, $\exists$ $0 \neq b, a \in \mathbb{Z}$, $\gcd(a,b) = 1$

$1 + 3\sqrt{2} = \frac{a}{b} \rightarrow 3\sqrt{2} = \frac{a}{b} - 1$

$\rightarrow 3\sqrt{2} = \frac{a-b}{b}$

$\rightarrow \sqrt{2} = \frac{a-b}{3b}$ where $3b \neq 0$, Contradiction

---

9. Let $\sqrt{6}$ is irrational, prove that $-3 + 2\sqrt{6}$ is irrational number.

Solution:

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10. Let $\sqrt{5}$ is irrational, prove that $\frac{\sqrt{5}-3}{2}$ is irrational number.

Solution: $\frac{\sqrt{5}-3}{2} \overset{?}{=} \frac{a}{b}$

$$\sqrt{5} = \frac{2a}{b} + 3$$

$$\sqrt{5} = \frac{2a+3b}{b} \in Q$$

contradiction

---

11. Let m, n are rational numbers : $n \neq 0$ and $\sqrt{5}$ is irrational, prove that $\frac{\sqrt{5}}{2n} - m$ is irrational number.

Solution:

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13. Let $x, y, z \in \mathbb{R}$: $x + y + z = 21$, use a proof by contradiction to show that $x \geq 8$ or $y \geq 7$ or $z \geq 6$.

Solution:

Assume the negation.

$x < 8$ and $y < 7$ and $z < 6$

$x + y + z \neq 21$

This is a contradiction

So, $x + y + z = 21$

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16. Use a proof by contradiction to prove that $\nexists\, m, n \in \mathbb{Z}$, such that $4n + 6m = 11$

Solution: #16 Assume the negation

$\exists m, n \in \mathbb{Z} : 4n + 6m = 11$

$2\underbrace{(n + 3m)}_{} = 11$

even = odd

$\Downarrow$

contradiction

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18. Prove or disprove that the sum of two irrational numbers is irrational.

Solution:

$x = 2 - \sqrt{3} \notin \mathbb{Q}$

$y = 5 + \sqrt{3} \notin \mathbb{Q}$

$x + y = 7 \in \mathbb{Q}$

19. Use a proof by contradiction to prove that $x + \frac{1}{x} \geq 2$ for $\forall\, x \in \mathbb{R}^{*}$. >0

Solution:

$x + \frac{1}{x} < 2 \quad x > 0$

$(\cdot x > 0) \quad x^2 + 1 < 2x$

$\underbrace{x^2 - 2x + 1}< 0$

$(x-1)^2 < 0$

contradiction

where

$(x+1)^2 \geq 0$

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151 Math Exercises

(3,2)

Methods of Proof

"Mathematical Induction"

(First principle)

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Sol **2.** Use mathematical induction to Show that

$$P(n): \frac{1}{1\times 3}+\frac{1}{3\times 5}+\frac{1}{5\times 7}+\cdots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1} \quad : \forall n\geq 1$$

Solution:

$$P(n): \frac{1}{1\times 3} \cdots$$

B.S. $P(1):?$ $L.H.S. = \frac{1}{1\times 3} = \frac{1}{3}$

$R.H.S = \frac{1}{2(1)+1} = \frac{1}{3}$

$\because L.H.S = R.H.S \Rightarrow \therefore P(1)$ is true

Indective step: Assume $P(k)$ is true where $k\geq 1$

$$P(k): \frac{1}{1\times 3}+\frac{1}{3\times 5}+\cdots+\frac{1}{(2k-1)(2k+1)}=\frac{k}{2k+1} \quad : k\geq 1$$

and will prove that $P(k+1)$ is also true

$$P(k+1): \frac{1}{1\times 3}+\frac{1}{3\times 5}+\cdots+\frac{1}{(2k-1)(2k+1)}$$

$$= \frac{k}{2k+1}+\frac{1}{(2k+1)(2k+3)}$$ (اخر حد رقم $n=k+1$)

$$= \frac{k(2k+3)}{(2k+1)(2k+3)}+\frac{1}{(2k+1)(2k+3)}$$

$$= \frac{2k^2+3k+1}{(2k+1)(2k+3)} = \frac{(2k+1)(k+1)}{(2k+1)(2k+3)}$$

$$= \frac{k+1}{2k+3} \Rightarrow \therefore P(k+1) \text{ is true}$$

$\therefore P(n)$ is true for $\forall n\geq 1$

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4. Use mathematical induction to Show that

$$P(n): 1.2.3 + 2.3.4 + \cdots + n(n+1)(n+2) = \frac{n(n+1)(n+2)(n+3)}{4} \ : \ \forall n \geq 1$$

Solution:

B.S. P(1)?

L.H.S $= 1.2.3 = 6$

R.HS $= \frac{1.2.3.4}{4} = 6$

$\therefore$ L.H.S = R.H.S $\Longrightarrow$ P(1) is true

Ind step: Assume P(k) is true: $k \geq 1$

$P(k): 1.2.3 + 2.3.4 + \ldots + k(k+1)(k+2) = \frac{k(k+1)(k+2)(k+3)}{4} \ : k \geq 1$

and will prove that P(k+1) is also true

$P(k+1): 1.2.3 + 2.3.4 + \ldots + k(k+1$

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H! 6. Use mathematical induction to Show that

$$p(n): \frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^n}=\frac{2^n-1}{2^n} \quad : \forall n \geq 1$$

Solution:

B.S.

$P(1):?$ $L.H.S = \boxed{\frac{1}{2}}$

$RHS = \frac{2-1}{2} = \boxed{\frac{1}{2}}$

$\therefore$ LHS = R.H.S $\Rightarrow P(1)$ is true.

Indect. step: Ass.

$$P(k): \frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{k}=\frac{2^k-1}{2^k}$$

and will prove $P(k+1)$ is also true

$$P(k+1): \frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^k}+\frac{1}{2^{k+1}+1}=\frac{2(2^k-1)}{2(2^k)}+\frac{1}{2^{k+1}}$$

$$=\frac{2\cdot 2^k-2+1}{2^{k+1}}=\frac{2^{k+1}-1}{2^{k+1}}$$

$\therefore P(k+1)$ is true

$\therefore P(n)$ is true for $\forall n \geq 1$

10. Use mathematical induction to Show that

$$1 + a + a^2 + \cdots + a^n = \frac{a^{n+1} - 1}{a-1} \quad , a \neq 1 \quad : \text{for all nonnegative integers } n.$$

$\forall n \geq 0$

Solution:

$P(n) : 1 + a + a^2 \ldots + a^n = \dfrac{a^{n+1}-1}{a-1}$

① Basis step:

Show that $P(0)$ is true

L.H.S = 1

R.H.S = $\dfrac{a^{0+1}-1}{a-1} = \dfrac{a-1}{a-1} = 1$

L.H.S = R.H.S

$P(0)$ is true

② Inductive step:

let $k \geq 1$, Assume $P(k)$ is true

$P(k): 1 + a + a^2 \ldots + a^k = \dfrac{a^{k+1}-1}{a-1} \quad a \neq 1$

and show that $P(k+1)$ is true

$P(k+1) : 1 + a + a^2 + \ldots + a^k + a^{k+1} = \dfrac{a^{k+2}-1}{a-1} \quad , a \neq 1$

L.H.S $= \dfrac{a^{k+1}-1}{a-1} + \dfrac{a^{k+1}}{1}$

$= \dfrac{a^{k+1}-1+a^{k+1}(a-1)}{a-1}$

$= \dfrac{\cancel{a^{k+1}}-1+a^{k+2}-\cancel{a^{k+1}}}{a-1}$

$= \dfrac{a^{k+2}-1}{a-1}$

= R.H.S

$P(k+1)$ is true

$\boxed{3}$ conclusion:

$\forall n \geq 0 \quad P(n)$ is true

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11. Use mathematical induction to Show that

$$2 + 4 + 6 + \cdots + 2n = n(n+1) \quad : n \geq 1$$

Solution:

$P(n): \ 2+4+6 \ldots + 2n = n(n+1)$

① Basis step

Show that $P(1)$ is true

$L.H.S = 2$

$R.H.S = 1(1+1) = 2$

$L.H.S = R.H.S$

$P(1)$ is true

---

② Inductive step – let $K \geq 2$

Assume $P(K)$ is true

$P(K): \ 2+4+6+\ldots+2K = K(K+1)$

and show that $P(K+1)$ remains true

$K^2+2K+K+1$

$P(K+1): \ 2+4+6 \ldots + 2K + 2K+1 = (K+1)(K+2)$

$L.H.S = K(K+1) + 2K+1$

$K(\underline{K+1}) + 2\underline{K+1}$

$(K+1)(K+2) = R.H.S$

$P(K+1)$ is true

---

③ conclusion

$\forall n \geq 1 \ P(n)$ is true

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21. Use mathematical induction to Show that

$$3^n < n! \quad \text{for every integer } n \text{ with } n \geq 7.$$

Solution:

---

22. Use mathematical induction to Show that

$$P(n) = n! < n^n \qquad \text{for every integer } n \text{ with } n \geq 2.$$

Solution:

Basis step: $P(n): 2! = 1 \times 2 = 2 < 2^2 = 4 \Rightarrow \therefore p(n)$ is true.

Induction step: Assume that $P(k)$ is true, where $k \geq 2$

$\Rightarrow p(k) = k! < k^k : k \geq 2$, true $(*)$

[illegible] and will prove that $P(k+1)$ is also true

$P(k+1): (k+1)! \; (k+1)^{k+1}$?

$(k+1)! = (k+1)\, \underline{k!} \leq (k+1)\, k^k$ (from $(*)$)

$< (k+1) \cdot (k+1)^k = (k+1)^{k+1}$

$\therefore (k+1)! < (k+1)^{k+1} \Rightarrow \therefore P(k+1)$ is true

$\therefore p(n)$ is true for all $n \geq 2$

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$< (k+1)^{k+1}$

?

$(k+1)(k+1)^k$

31.Use mathematical induction to Show that

$$n^2 - 7n + 12 \geq 0 \quad \text{for every integer } n \text{ with } n \geq 3.$$

Solution: $P(n)$: $\circ \geq 0$: $\forall n \geq 3$.

الطرف الأول

Basis step: $P(3)$: $3^2 - 7(3) + 12 = 21 - 21 = 0$
$= 0 \geq 0$

$\therefore P(3)$ is true.

Inductive step Assume that $P(k)$ is true where $k \geq 3 \Rightarrow$

$P(k)$: $k^2 - 2k + 12 \geq 0$: $k \geq 3$ (A)

and will prove that $P(k+1)$ is also true.

$P(k+1)$: $(k+1)^2 - 7(k+1) + 12$

$= k^2 + 2k + 1 - 7k - 7 + 12$

$= \underbrace{[k^2 - 7k + 12]}_{\text{[from (A)]} \geq 0} + \underbrace{2(k-3)}_{\geq 0 \,:\, \forall k \geq 3} \geq 0$

$\therefore (k+1)^2 - 7(k+1) + 12 \geq 0$

$\therefore P(k+1)$ is true

$\therefore P(n+1)$ is true for all $n \geq 3$

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32. Use mathematical induction to Show that

$$1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{n^2}<2-\frac{1}{n} \quad \text{for every integer } n \text{ with } n \geq 2.$$

Solution: $P(n): 1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{n^2} < 2-\frac{1}{n} : \forall n \geq 2$

$$\sum_{j=1}^{n} \frac{1}{j^2}$$

Basic step: $P(2): 1+\frac{1}{4}=\frac{5}{4} < 2-\frac{1}{2}=\frac{3}{2}=\frac{6}{4}$

$\Rightarrow \therefore P(2)$ is true

Induction step: Assume that $P(k)$ is true where

$k \geq 2 \Rightarrow P(k): 1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{k^2} < 2-\frac{1}{k}$

and will prove that $P(k+1)$ is also true.

$$P(k+1): \left[1+\frac{1}{4}+\frac{1}{9}+\cdots\frac{1}{k^2}\right]+\frac{1}{(k+1)^2} < 2-\frac{1}{k}+\frac{1}{(k+1)^2}$$

$$= 2-\frac{(k+1)^2}{k(k+1)^2}+\frac{k}{k(k+1)^2}$$

$$= 2-\frac{(k+1)^2-k}{k(k+1)^2} = 2-\frac{k^2+2k+1-k}{k(k+1)^2}$$

$$= 2-\frac{k^2+k+1}{k(k+1)^2} < 2-\frac{k^2+k}{k(k+1)^2} = 2-\frac{\cancel{k(k+1)}}{\cancel{k}(k+1)^{\cancel{2}}}$$

$= 2-\frac{1}{k+1} \Rightarrow \therefore P(k+1)$ is true $\Rightarrow P(n)$ is true for $\forall n \geq 2$

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36. Use mathematical induction to Show that

$$3\mid(n^3 + 2n) \qquad \text{for all positive integer } n$$

Solution: $P(n): \; n^3+2n = 3C : C \in \mathbb{Z} : \forall n \geq 1$

Basic step: $P(1) = 3 \mid (1^3 + 2(1)) = 3$

$\Rightarrow \therefore P(1)$ is true

Indectivestep: Assume that $P(K)$ is true: $K \geq 1 \Rightarrow P(K): K^3+2K = 3C$

$\boxed{K^3 = -2K + 3C}$ *

and will prove that $P(K+1)$ is also true

$P(K+1): (K+1)^3 + 2(K+1) =$ | $= 3M$?

$= K^3 + 3K^2 + 3K + 1 + 2K + 2$

(from *) $= -2K + 3C + 3K^2 + 3K + 3 + 2K$

$= 3(\underbrace{C + K^2 + K + 1}_{M \in \mathbb{Z}}) = 3M$

$\therefore 3 \mid (K+1)^3 + 2(K+1)$

$\Rightarrow P(K+1)$ is true

Then $P(n)$ is true for all $n \geq 1$

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37. Use mathematical induction to Show that

$$5 \mid 7^n - 2^n \qquad \forall\, n \geq 1$$

Solution: $P(n): 7^n - 2^n = 5c : c \in \mathbb{Z} : \forall n \geq 1$

B.S. $P(1): 5 \mid 7^1 - 2^1 = 5 \Rightarrow \therefore P(1)$ is true

Induct. Step: Assume that $p(k)$ is true where $k \geq 1$

$\Rightarrow P(k): 7^k - 2^k = 5c \Rightarrow \boxed{7^k = 2^k + 5c}$ (*)

and will prove that $p(k+1)$ is also true

$P(k+1): 7^{k+1} - 2^{k+1} = 7 . 7^k - 2 . 2^k$ from *

$= 7(2^k + 5c) - 2 . 2^k$

$= 7 . 2^k + 35c - 2 . 2^k$

$= 5 . 2^k + 35c = 5(2^k + 7c) = 5M$, $M \in \mathbb{Z}$

$\therefore 5 \mid 7^{k+1} - 2^{k+1} \Rightarrow P(k+1)$ is true

$\therefore P(n)$ is true for all $n \geq 1$

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King Saud University
College of Sciences
Department of Mathematics

151 Math Exercises

(4,1)

Relations and Their Properties

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EXERCISES

1. Let R be a relation defined on the set $A = \{-2,-1,0,1,2,3,4\}$

$$m,n \in A \ , \ \ m\,R\,n \Leftrightarrow n = m^2$$

(i) What are the ordered pairs in the relation R ?

(ii) Find the domain and the image of R ?

(iii) Represent R by the directed graph (diagraph) ?

Solution:

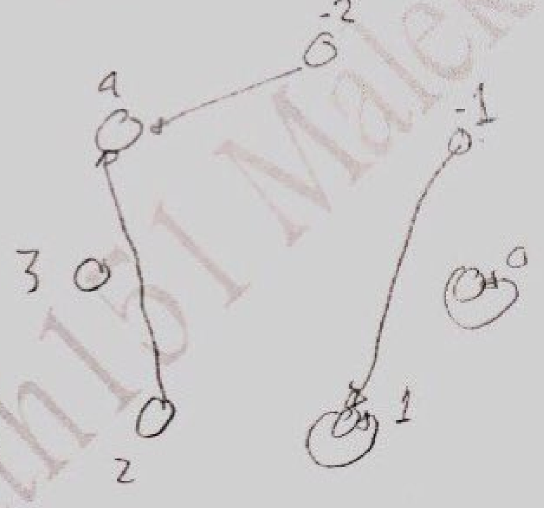
*1) $R = \{(-2,4),(-1,1),(0,0),(1,1),(2,4)\}$

(العناصر الأولى)

Domain $R = \{-2,1,0,1,2\}$

العناصر الثانية

Range $R = \{0,1,4\}$



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2. Let R be a relation defined on the set $A = \{1,2,3,4,5\}$

$$x, y \in A, \qquad x\,R\,y \Leftrightarrow xy \leq 9$$

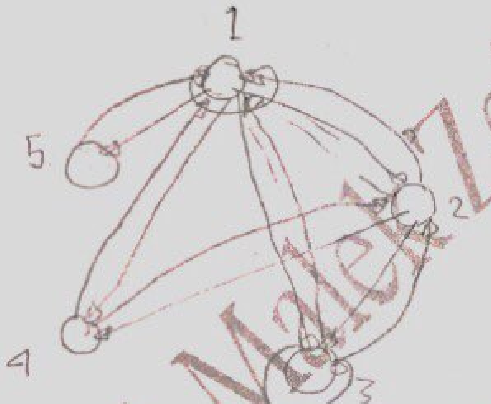
(i) What are the ordered pairs in the relation R ?
(ii) Find the domain and the image of R ?
(iii) Draw the directed graph (diagraph) that represents R ?

Solution:

$R = \{(1,1), (1,2), (1,3), (1,4), (1,5), (2,1), (2,2), (2,3), (2,4)$
$(3,1), (3,2), (3,3), (4,1), (4,2), (5,1)\}$

Domain $R = \{1,2,3,4,5\} = A$

Range $R = \{1,2,3,4,5\} = A$



$$M_R = \begin{array}{c|c} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

① R not reflexive because $(5,5) \notin R$
$5 \not R\, 5$

② R is symmetric:
$xRy \Rightarrow xy \leq 9$
الابدال
(Comm.) $\Rightarrow y \cdot x \leq 9 \Rightarrow yRx$

③ R is not antisymm. cause $(2,3) \in R$ & $(3,2) \in R$ but $2 \neq 3$

④ R is not transitive, cause:
& $(5,1) \in R$
$(1,5) \in R$ but $(5,5) \notin R$

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5. Suppose R is a relation defined on the set $A = \{-2,-1,0,1,2\}$, as

$$x, y \in A, \quad x\,R\,y \Leftrightarrow |x-y| < 2$$

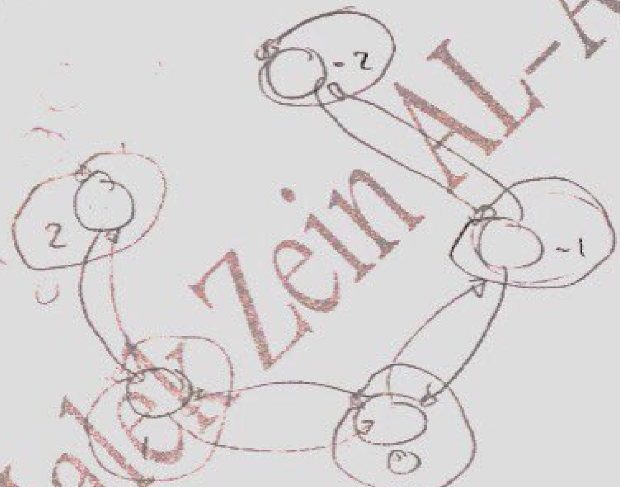
(i) What are the ordered pairs in the relation R?

(ii) Draw the directed graph (diagraph) that represents R

(iii) Determine whether the relation R is *reflexive*, *symmetric*, *antisymmetric*, and/or *transitive*.

Solution:

$R = \{(-2,-2), (-2,-1), (-1,-2)$
$(-1,-1), (-1,0), (0,-1), (0,0)$
$(0,1), (1,0), (1,1), (1,2), (2,1)$
$(2,2)\}$



(iii) ① R is reflexive

$\forall x \in A,\ |x-x| = 0 < 2 \Rightarrow$

$\therefore x R x$

اعوض y بـ x.

② R is Symmetric:

$\forall x,y \in A;\ xRy \Rightarrow |x-y| < 2$

$\Rightarrow |-(y-x)| < 2$

$\Rightarrow |-1| \cdot |y-x| < 2$

$\Rightarrow |y-x| < 2$

$\Rightarrow yRx$

③ R is not symm. Cause:

$\left.\begin{array}{l} -2R-1 \\ \& \\ -1R-2 \end{array}\right\}$ but $-1 \neq -2$

④ R is not transtive, Cause

$\left.\begin{array}{l} 1R0 \\ \& \\ 0R-1 \end{array}\right\}$ but $1 \not R -1$

(iv) $M_R = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$

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9. Let R be a relation defined on the set $A = \{-2, -1, 0, 1, 2, 3, 4\}$

$$a, b \in A, \quad a\,R\,b \Leftrightarrow a^2 = b^2$$

(i) What are the ordered pairs in the relation R?

(ii) Find the domain and the image of R?

(iii) Draw the directed graph (diagraph) that represents R ?

(iv) Determine whether the relation R is *reflexive*, *symmetric*, *antisymmetric*, and/or *transitive*.

Solution: $R = \{(-2,-2), (-2,2), (-1,-1), (-1,1), (0,0), (1,-1), (1,1), (2,-2), (2,2), (3,3), (4,4)\}$

ii Domain $R = \{-2, -1, 0, 1, 2, 3, 4\}$

iii

iv) ① R is Refl All vertices have a loop

$\forall a \in A$, $a^2 = a^2 \therefore aRa$

② R is Symmetric

$a, b \in A : aRb \Rightarrow a^2 = b^2$

$\Rightarrow b^2 = a^2 \Rightarrow bRa$

③ R is not antisymmetric, cause $(-1,1) \in R$ & $(1,-1) \in R$, but $1 = -1$

④ R is transtive:

$a, b, c : aRb \Rightarrow a^2 = b^2 \; (*)$ by comparing

&

$bRc \Rightarrow b^2 = c^2 \; (**)$ $(*)$ & $(**)$

$\Rightarrow a^2 = c^2 \Rightarrow \therefore aRC$

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12. Let $S = \{(1,1),(1,2),(1,3),(2,2),(3,1),(3,3)\}$ be a relation on the set $B = \{1, 2, 3\}$

(i) Draw the directed graph (diagraph) that represents S ?

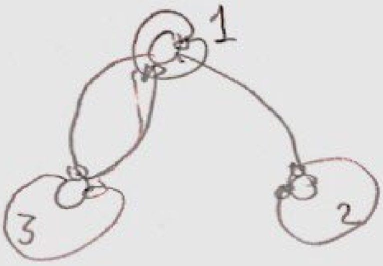
(ii) Find S^2 , $\bar{S}$, $S o \bar{S}$

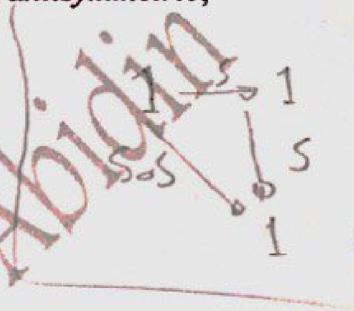
(iii) Find $\mathbf{M}_S$

(iv) Determine whether the relation S is *reflexive*, *symmetric*, *antisymmetric*, and/or *transitive*

Solution:

12)





$S = \{(1,1),(1,2),(1,3),(2,2),(3,1),(3,3)\}$

$S = \{(1,1),(1,2),(1,3),(2,2),(3,1),(3,3)\}$

$S^2 = S o S = \{(1,1),(1,2),(1,3),(2,2),(3,1),(3,2),(3,3)\}$

$S^3 = S^2 o S = \{(1,1),(1,2),(1,3) \ldots\}$

$\bar{S} = A \times A - S$

$\bar{S} = \{(2,1),(2,3),(3,2)\}$

$S o \bar{S} = \{(2,1),(2,2),(2,3),(3,2)\}$

$S^{-1} = \{(1,1),(2,1),(3,1),(3,3)\}$

$\bar{S} = \{(2,3),(3,2)\}$

$M_S = [\ ?\ ]$

(iv) ① S is Refl. every vertx is diagraph has loop.

② S is not symmetric niether antisymmetric

Cause

$(1,2) \in S$, but $(2,1) \notin S$

$(1,3) \wedge (3,1) \in S$, but $1 \neq 3$

③ S is not transtive, cause

$(3,1) \in S$, $(1,2) \in S$ but $(3,2) \notin S$

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16. Let $R = \{(a,c),(a,b),(b,b)\}$ and $S = \{(a,a),(a,c),(b,c),(c,a)\}$ are relations on the set $A = \{a, b, c\}$

(i) Find $(R \circ S) \cap R^{-1}$

(ii) Find $S^{-1} \circ R$

(iii) Find $\mathbf{M}_R$, $\mathbf{M}_S$, $\mathbf{M}_{R\cup S}$, $\mathbf{M}_{R\cap S}$, $\mathbf{M}_{R\circ S}$

Solution:

16

$$M_R = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$M_S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$M_{R\cup S} = M_R \vee M_S = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$M_{R\cap S} = M_R \wedge M_S = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$S \circ R$

$$M_{R\circ S} = M_S \odot M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

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151 Math Exercises

(4.2)

2, 6, 18

Equivalence Relations

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4.2

#2] $a S b \iff 3 \mid (a + \overset{\circ a}{2}b) \Rightarrow a+2b=3m : m \in \mathbb{Z}$

(i) $\forall a \in A,\ 3 \mid (a+2a) = 3a \Rightarrow a S a$

$\therefore S$ is refl.

(ii)* $a, b \in A : a S b \Rightarrow a+2b=3m : m \in \mathbb{Z}$

$a = 3m - 2b \xRightarrow{*(2)} 2a = 6m - 4b \xRightarrow{+b}$

$b + 2a = 6m - 3b = 3\underbrace{(2m - b)}_{H \in \mathbb{Z}} = 3H$

$\therefore b S a \Rightarrow \therefore S$ is symmetric

$\mathcal{I} = \{\{-2, 1\}, \{-1, 2\}, \{0\}\}$.

(iii) $a, b, c \in A$:

$a S b \Rightarrow a + 2b = 3m_1$

&

$b S c \Rightarrow b + 2c = 3m_2 \quad : m_1, m_2 \in \mathbb{Z}$

$(+)$ ______________

$a + 3b + 2c = 3m_1 + 3m_2$

$a + 2c = 3\underbrace{(m_1 + m_2 - b)}_{= M \in \mathbb{Z}} = 3M$

$\therefore 3 \mid (a + 2c) \Rightarrow a S c \Rightarrow S$ is transitive

S → refl., symm., trans. $\Rightarrow \therefore T$ is $\boxed{\text{Equiv. Rel.}}$

$[a] = \{x \in A : x S a\}$

$= \{x \in A : x + 2a = 3m : m \in \mathbb{Z}\}$

$= \{x \in A : x = -2a + 3m : m \in \mathbb{Z}\}$

$[-2] = \{x \in A : x = 4 + 3m : m \in \mathbb{Z}\}$

$= \{1, -2\}$

$[-1] = \{x \in A : x = 2 + 3m : m \in \mathbb{Z}\}$

$= \{2, -1\}$

$[0] = \{0\}$.

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#6 (i) $\forall a \in \mathbb{Z}^+$,

$\sqrt{a} - \sqrt{a} = 0 \in \mathbb{Z} \Rightarrow \therefore aRa$

$\therefore R$ is refl.

(ii) $a, b \in \mathbb{Z}^+$: $aRb \Rightarrow \sqrt{a} - \sqrt{b} = m \in \mathbb{Z}$

$*(-1)$

$\Rightarrow \sqrt{b} - \sqrt{a} = -m \in \mathbb{Z} \Rightarrow bRa$

$\therefore R$ is Symmetric.

(iii) $a, b, c \in \mathbb{Z}^+$:

$aRb \Rightarrow \sqrt{a} - \sqrt{b} = m_1 \in \mathbb{Z}$

&

$bRc \Rightarrow \sqrt{b} - \sqrt{c} = m_2 \in \mathbb{Z}$

$(+)$ $\sqrt{a} - \sqrt{c} = m_1 + m_2 = m \in \mathbb{Z}$

$\therefore aRc \Rightarrow \therefore R$ is transitive.

$R \to$ refl., symm., trans. $\Rightarrow R$ is Equiv. Rel.

$\because \sqrt{4} - \sqrt{9} = 2 - 3 = -1 \in \mathbb{Z}$

$\therefore 4R9 \Rightarrow 4 \in [9]$.

$a \in [b] \Longleftrightarrow aRb$

?

$aRc \Rightarrow \sqrt{a} - \sqrt{c} = M \in \mathbb{Z}$

bRa $\sqrt{b} - \sqrt{a} \in \mathbb{Z}$

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18 ① $\forall x \in \mathbb{R}$,

$|x-3| = |x-3| \Rightarrow \therefore xTx$

$\therefore$ T is refl.

② $x, y \in \mathbb{R}$:

$xTy \Rightarrow |x-3| = |y-3|$

(Rev.) $\Rightarrow |y-3| = |x-3|$

$\Rightarrow yTx$

$\therefore$ T is Symm.

③ $x, y, z \in \mathbb{R}$:

$xTy \Rightarrow |x-3| = |y-3| \quad (1)$

&

$yTz \Rightarrow |y-3| = |z-3| \quad (2)$

Comparing (1) & (2) $\Rightarrow |x-3| = |z-3|$

$\therefore xTz \Rightarrow \therefore$ T is transitive

T $\rightarrow$ refl.

$\rightarrow$ Symm. $\Rightarrow \therefore$ T is Equiv. Rel.

$\rightarrow$ trans.

$[a] = \{x \in \mathbb{R} : xTa \Rightarrow |x-3| = |a-3| \Rightarrow x-3 = \pm(a-3)\}$

$[0] = \{x \in \mathbb{R} : x-3 = \pm(0-3) = \mp 3 \Rightarrow x = 3 \mp 3\}$

$= \{0, 6\}$

$[3] = \{ \cdots$

$= \{3\}$

$[-2] = \{ \cdots$

$= \{-2, 8\}$

$|r| = |s|$

$r = \pm s$

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151 Math Exercises

(4.3)

8, 9, 10
1, 2, 21

Partial Orderings

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1440هـ

2018

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4.3

#1) (i) $aRb \Leftrightarrow a \mid b \Rightarrow b = ma$

1- $\forall a \in A,\ a \mid a \Rightarrow aRa \Rightarrow R$ is refl.

2- $a, b \in A$:

$$\begin{cases} aRb \Rightarrow a \mid b \Rightarrow b = m_1 a \\ \& \\ bRa \Rightarrow b \mid a \Rightarrow a = m_2 b \end{cases} \quad : m_1, m_2 \in \mathbb{N}$$

$\Rightarrow a = m_1 m_2 a \overset{\div a}{\Rightarrow} m_1 \cdot m_2 = 1$

$\Rightarrow m_1 = m_2 = 1$

$\Rightarrow \therefore \boxed{a = b}$

$\Rightarrow \therefore R$ is antisymmetric

③ $a, b, c \in A$:

$aRb \Rightarrow a \mid b \Rightarrow b = m_1 a$

&

$bRc \Rightarrow b \mid c \Rightarrow c = m_2 b$ $\quad : m_1, m_2 \in \mathbb{N}$

by subs. $\Rightarrow c = \underbrace{m_1 m_2}_{= m \in \mathbb{N}} a = ma$

$\Rightarrow \therefore a \mid c \Rightarrow aRc$

$\therefore R$ is transitive.

$R \begin{cases} \to \text{refl} \\ \to \text{Antisym.} \\ \to \text{trans.} \end{cases} \Rightarrow R$ is (Poset)

Partial ordering relation on A

(ii): $3, 4 \in A : 3 \nmid 4 \wedge 4 \nmid 3$

$\therefore 3, 4$ incomparable $\Rightarrow$

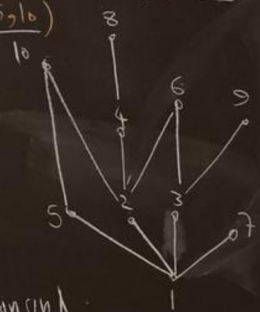
$\therefore R$ is not totally ordering rel.

(iii) $R = \{(1,1), (2,2), \ldots, (10,10)$

$(1,2), (1,3), (1,4), (1,5), (1,6), (1,7), (1,8), (1,9), (1,10)$

$(2,4), (2,6), (2,8), (2,10), (3,6), (3,9), (4,8)$

$(5,10)$



4.3

#2 $aRb \Leftrightarrow \frac{b}{a} = 2^k : k \in \{0,1,2,\ldots\}$

(i)

① $\forall a \in \mathbb{N},\ \frac{a}{a} = 1 = 2^0 \Rightarrow$

$\therefore aRa \Rightarrow \therefore R$ is refl.

② $a, b \in \mathbb{N}$:

$aRb \Rightarrow \frac{b}{a} = 2^{k_1}$

$\wedge$

$bRa \Rightarrow \frac{a}{b} = 2^{k_2}$ $: k_1, k_2 \in \{0,1,2,\ldots\}$

$(\times) \Rightarrow \frac{b}{a} \times \frac{a}{b} = 1 = 2^{k_1+k_2}$

$1 = 2^0 = 2^{k_1+k_2}$

$\therefore k_1 + k_2 = 0 \Rightarrow \therefore k_1 = k_2 = 0$

$\therefore \frac{a}{b} = 2^0 = 1 \Rightarrow \therefore \boxed{a = b}$

R is antisymmetric

③ $a, b, c \in \mathbb{N}$:

$aRb \Rightarrow \frac{b}{a} = 2^{k_1}$

&

$bRc \Rightarrow \frac{c}{b} = 2^{k_2}$ $: k_1, k_2 \in \{0,1,2,\ldots\}$

$(\times) \Rightarrow \frac{\cancel{b}}{a} \times \frac{c}{\cancel{b}} = \frac{c}{a} = 2^{k_1+k_2} = 2^k$

$k_1 + k_2 = k \in \{0,1,2,\ldots\}$

$\Downarrow$

$\therefore aRc$

$\therefore R$ is transitive.

$\Rightarrow R$ is Pratial ordering rel.

(ii) $3, 4 \in \mathbb{N}$: $k_1, k_2 \in \{0,1,2,\ldots\}$

$3 \not R 4 : \frac{4}{3} \neq 2^{k_1} \wedge 4 \not R 3 : \frac{3}{4} \neq 2^{k_2}$

$\therefore 3, 4$ incomparrable $\Rightarrow \therefore R$ is not totally oredring rel.

(iii) $R = \{(1,1), (2,2), \ldots (12,12),$
$(1,2), (1,4), (1,8), (2,4), (2,8), (3,6)$
$(3,12), (4,8), (5,10), (6,12)\}$

4.3 $x S y \Leftrightarrow \frac{x}{y}$ is odd

#8 (i)

1- $\forall x \in \mathbb{N}$, $\frac{x}{x} = 1$ (is odd)

$\therefore x S x \Rightarrow \therefore S$ is refl.

2- $x, y \in \mathbb{N}$

$x S y \Rightarrow \frac{x}{y} = m_1$

&

$y S x \Rightarrow \frac{y}{x} = m_2$ $\quad : m_1, m_2 \text{ (odd)} \in \mathbb{N}$

(X)

$\frac{x}{y} \cdot \frac{y}{x} = 1 = m_1 \cdot m_2 \Rightarrow \therefore m_1 = m_2 = 1$

$\Rightarrow \therefore \frac{x}{y} = 1 \Rightarrow \therefore x = y \Rightarrow S$ is antisymm.

3- $x, y, z \in \mathbb{N}$:

$x S y \Rightarrow \frac{x}{y} = 2k_1 + 1$

& $\quad : k_1, k_2 \in \mathbb{N}$

$y S z \Rightarrow \frac{y}{z} = 2k_2 + 1$

(X)

$\frac{x}{\cancel{y}} \cdot \frac{\cancel{y}}{z} = (2k_1+1)(2k_2+1)$

$\frac{x}{z} = 4k_1k_2 + 2k_1 + 2k_2 + 1$

$= 2(\underbrace{2k_1k_2 + k_1 + k_2}_{=k \in \mathbb{N}}) + 1$

$\frac{x}{z} = 2k + 1$ (is odd)

$\Rightarrow x S z \Rightarrow \therefore S$ is transitive

(ii) $2, 4 \in \mathbb{N}$

$\frac{4}{2} = 2$ (even), $4 \not S 2$

$\wedge$ $\quad \Rightarrow 2, 4$ incomprab.

$\frac{2}{4} = \frac{1}{2}$, $2 \not S 4$

$\therefore S$ is not totally ordering rel.

(iii) $S = \{(1,1), (2,2), (6,6), (6,2)\}$

Hasse diagram of S

#9

(i) $T = \{(p,p), (l,l), (n,n), (o,o), (m,m),$

$(p,l), (l,n), (l,o), (o,m)$

$(p,o), (p,m), (l,m), (p,n)\}$

(ii) $n, o \in C$

$(o,n) \wedge (n,o) \notin T$.

$\Rightarrow \therefore o, n$ incomparable

$\therefore T$ is not totally ordering rel. on C.

$T = \{(1,1), (2,2), (3,3), (5,5), (6,6),$

$(6,3), (6,2), (6,1), (3,1), (2,1)\}$

$= 2^{m_1} \cdot 2^{m_2} \cdot 3^{n_1} \cdot 3^{n_2}$

$= 2^{m_1+m_2} \cdot 3^{n_1+n_2} = 2^{m} \cdot 3^{n}$.

$\frac{6}{3} = 2^{1} \cdot 3^{0}$

$\frac{6}{2} = 2^{0} \cdot 3^{1}$

$\frac{6}{1} = 2^{1} \cdot 3^{1}$

$\frac{3}{1} = 2^{0} \cdot 3^{1}$

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10

$S = \{(1,1),(2,2),(3,3),(4,4),(3,1),(3,2)$

$(4,1),(4,2),(1,2)$

$A = \{a, b, c, d, e\}$

$T = \{(a,a),(b,b),(c,c),(d,d),(e,e),$

$(a,b),(a,c),(b,c),(d,c),(d,e)\}$

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#21)

① R is not refl.

Example. $6 \nmid (5)(5) = 25 \Rightarrow 5 \not R 5$

② $m, n \in \mathbb{Z}^+$: $mRn \Rightarrow 6 \mid mn \Rightarrow 6 \mid nm$

$\Rightarrow nRm \Rightarrow R$ is symm.

③ R is not antisymm.

Example: $\left.\begin{array}{l} 2R3 : 6 \mid (2)(3) \\ \wedge \\ 3R2 : 6 \mid (3)(2) \end{array}\right\}$ but $2 \neq 3$

④ R is not transitive.

Example:

$\left.\begin{array}{l} 3R2 : 6 \mid (3)(2) \\ \wedge \\ 2R3 : 6 \mid (2)(3) \end{array}\right\}$ but $6 \nmid 9 = (3)(3)$

$3 \not R 3$

R is only symmetric

$\left.\begin{array}{l} 1 R 4 \Leftrightarrow 1+4=5 \geq 2 \\ \wedge \\ 4 R (-2) \Leftrightarrow 4+(-2)=2 \geq 2 \end{array}\right\}$

but

$1 \not R (-2)$:

$1+(-2) = -1 \not\geq 2$

$\therefore R$ is not transit.

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1 2 3 4 5 6 7 8 11 15 24 26

A
B
C
D

151 Math Exercises

(5.2)

Graph Terminology and Special Types of Graphs

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Ex. 5.2.

#1

x_i	a	b	c	d	e	f
$\deg x_i$	2 even	4 even	1 odd	0 even Isolated	2 even	3 odd

odd vertices

$|\text{odd (vertice)}| = \text{even}$

$|V| = 6$, $\sum_{i=1}^{6} \deg v_i = \underbrace{2+4+1+0+2+3}_{12} = 2|E|$

$12 = 2|E| \Rightarrow \boxed{|E| = 6}$

G is a simple graph, cause has no loopes or multiple edges.

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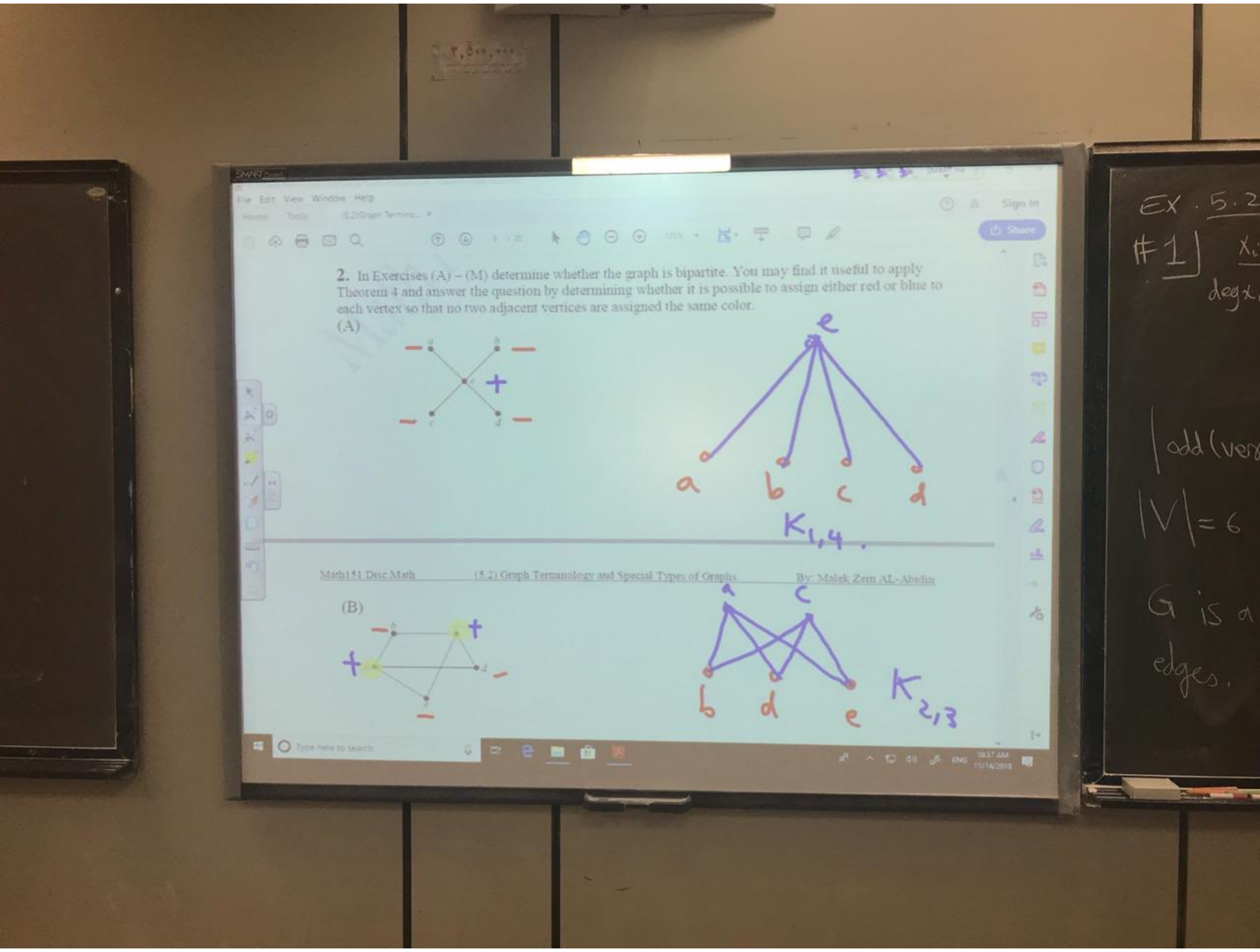

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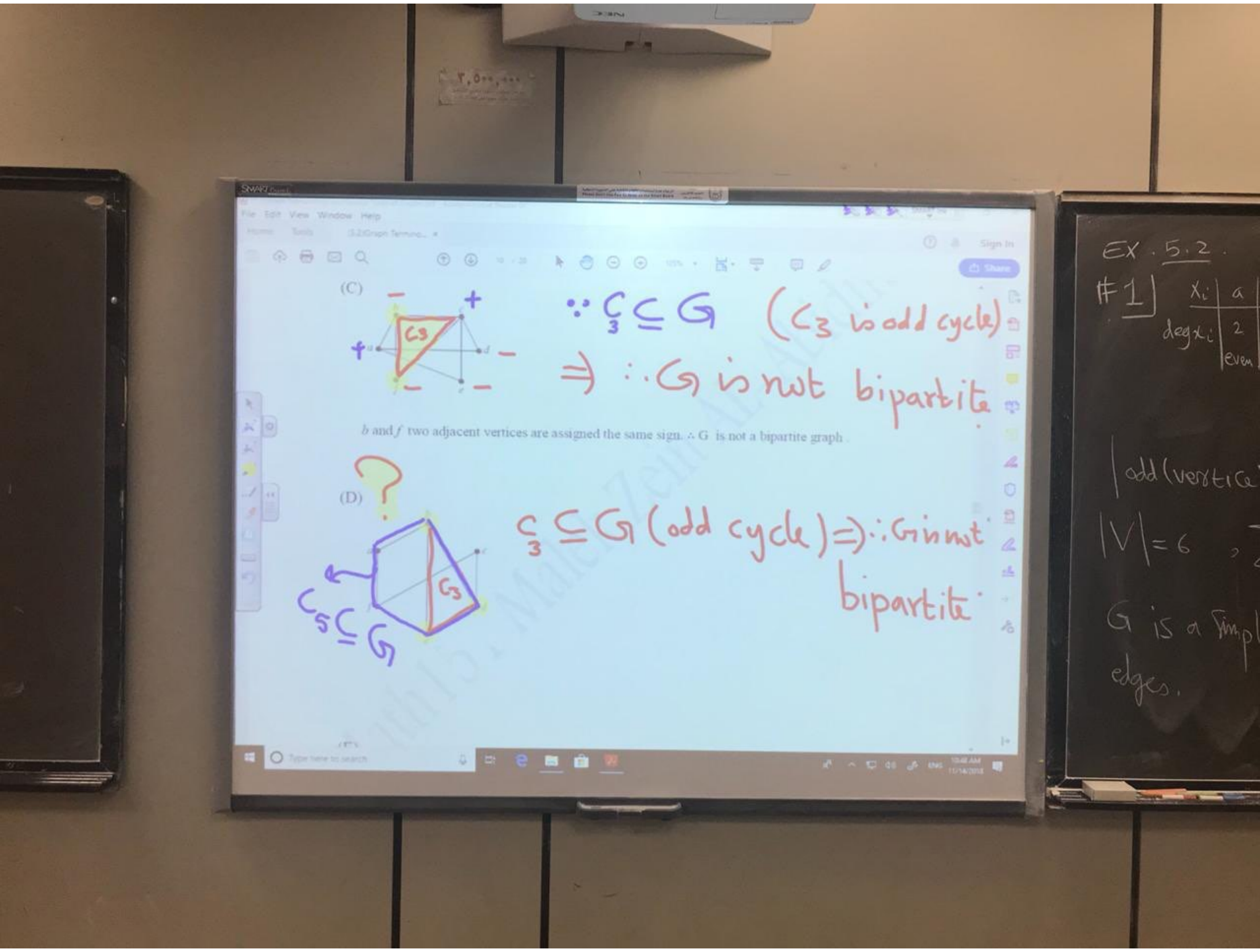

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3. Let G be a graph have 6 edges and the given degree sequence 1 , 3 , x , x . find the value of x ?

$$\sum_{i=1}^{n} \deg v_i = 2|E| \Rightarrow 1+3+x+x=12 .$$

$$2x=8 \Rightarrow \boxed{x=4}$$

4. How many vertices does a regular graph of degree four with 10 edges have?

$r=4$

$$|E| = \frac{nr}{2} \Rightarrow n = \frac{2|E|}{r} = \frac{2(10)}{4} = \boxed{5} .$$

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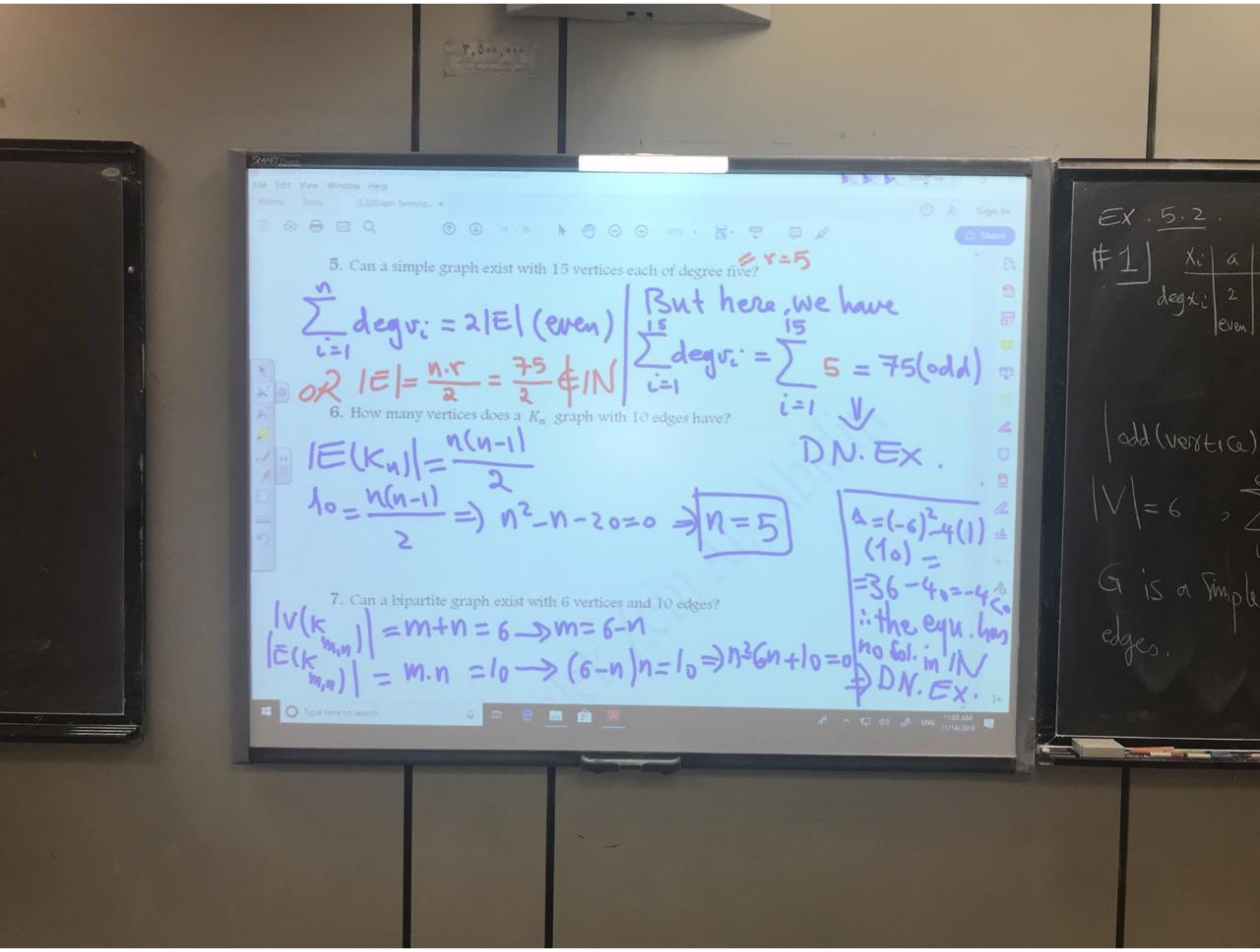

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8. Show that $K_{m,n}$ is regular if and only if $m = n$?

Sol. (i) $K_{m,n}$ is regular $\Rightarrow m = n$?

$K_{m,n} = (V_1 \cup V_2, E)$: $|V_1| = n$, $|V_2| = m$.

$\deg\limits_{x \in V_1} x = |V_2| = m$ & $\deg\limits_{y \in V_2} y = |V_1| = n$.

اكي دبي

$\because K_{m,n}$ is regular $\Rightarrow \deg\limits_{x \in V_1} x = \deg\limits_{y \in V_2} y \Rightarrow m = n$

(ii) $m = n \Rightarrow K_{m,n}$ is regular ?

اكي دبي

9. If G is a simple regular graph of degree k with n vertices, show that k is even or n is even.

Ex. 5.

#1)

deg

odd (ver

$|V| = 6$

G is a

edges.

11. If $K_{3,n}$ have the same number of edges of K_n . find the value of n ?

$$\left.\begin{array}{l} |E(K_{3,n})| = 3n \\ |E(K_n)| = \dfrac{n(n-1)}{2} \end{array}\right\} \quad 3\cancel{n} = \frac{\cancel{n}(n-1)}{2} \Rightarrow n-1=6$$

$$\boxed{n=7}$$

12. Let G be a graph have 5 edges and the given degree sequence $2,2,2,2,x$

Decide whether G is a regular ?

13. How many vertices does a regular graph of degree 3 with 10 edges have?

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15. Let G be a graph have 10 edges with two vertices of 4 degrees each, and the degree of each vertex is equal to 3. Find the number of vertices

$$\sum_{i=1}^{n} \deg v_i = 2|E| \Rightarrow 2(4) + (n-2)(3) = 20$$

$$\boxed{n=6}$$

16. Let G be a graph have 11 edges and the given degree sequence $n, n, n, n, 2n, 2n, 3n$. find the value of n ?

17. If K_{m,m^2} have 42 vertices, find the number of edges ?

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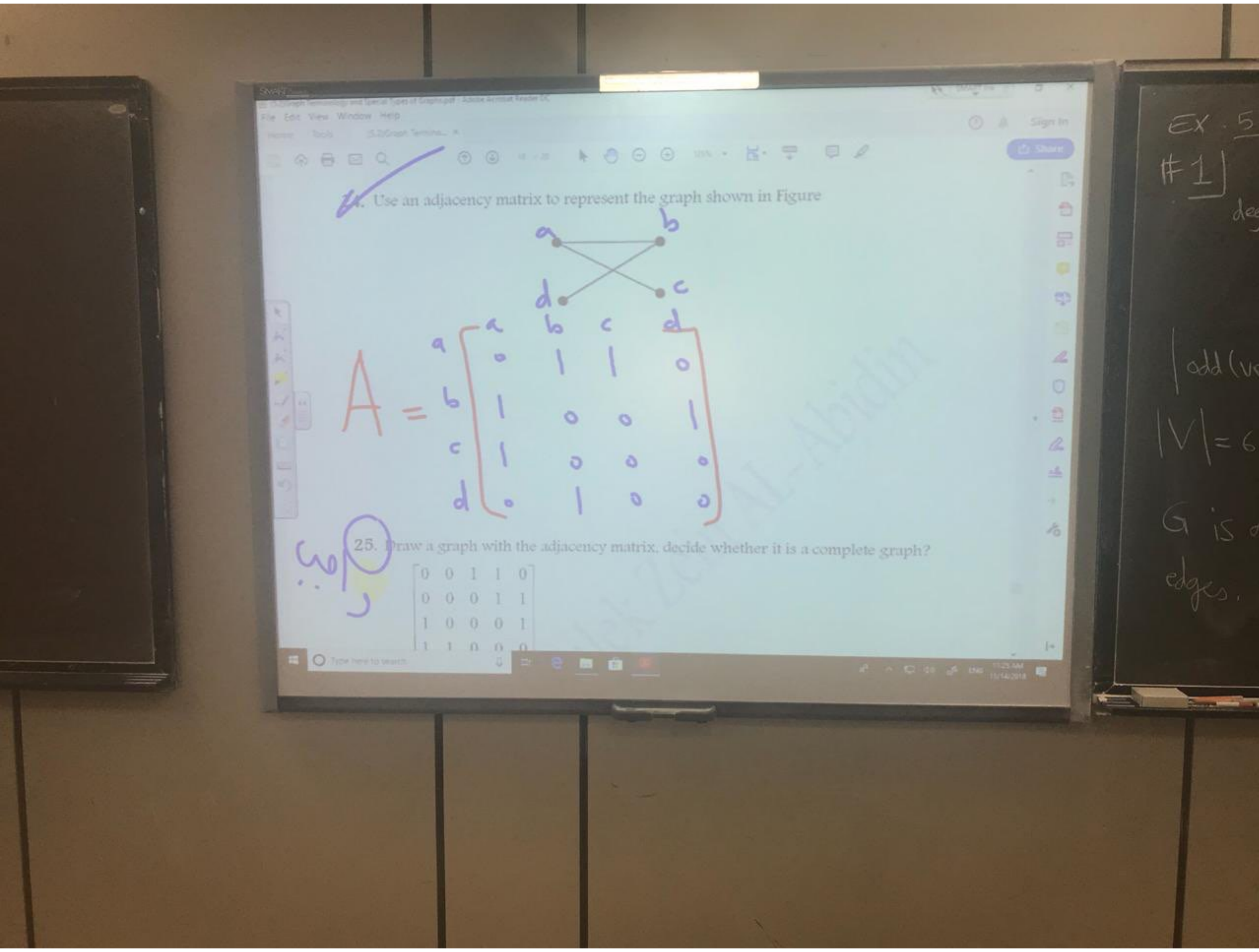

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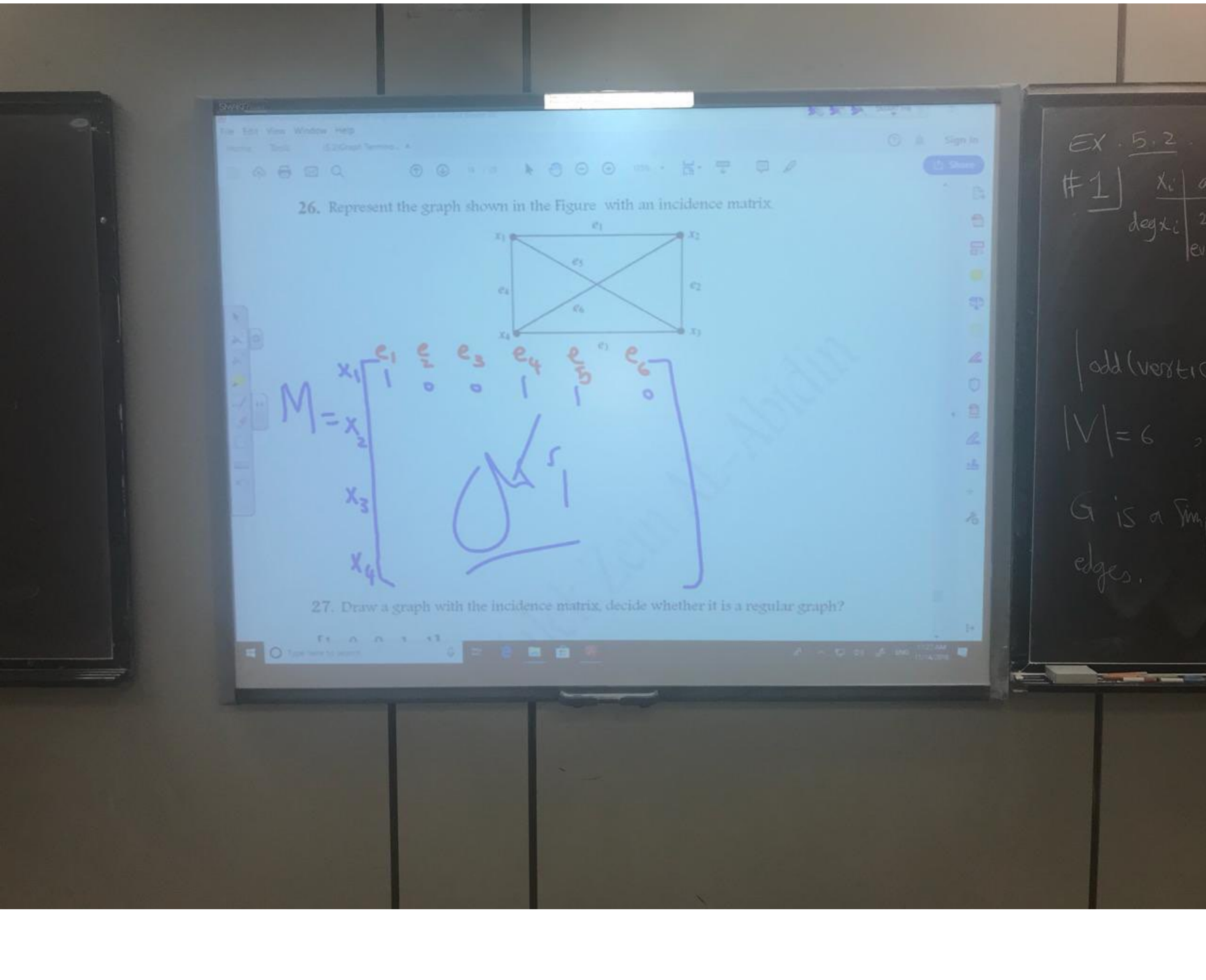

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كلية العلوم
قسم الرياضيـــــات

2 3 4

س
2 3 5 6 7 9 10

تمارين 151 ريض

نظرية الرسومات

GRAPH THEORY

~~(4-2)~~
(5, 3)

(الرسومات المتماثلة)

ISOMORPHIC GRAPHS

إعداد : مالك عبدالرحمن زين العابدين

1439 هـ
2018

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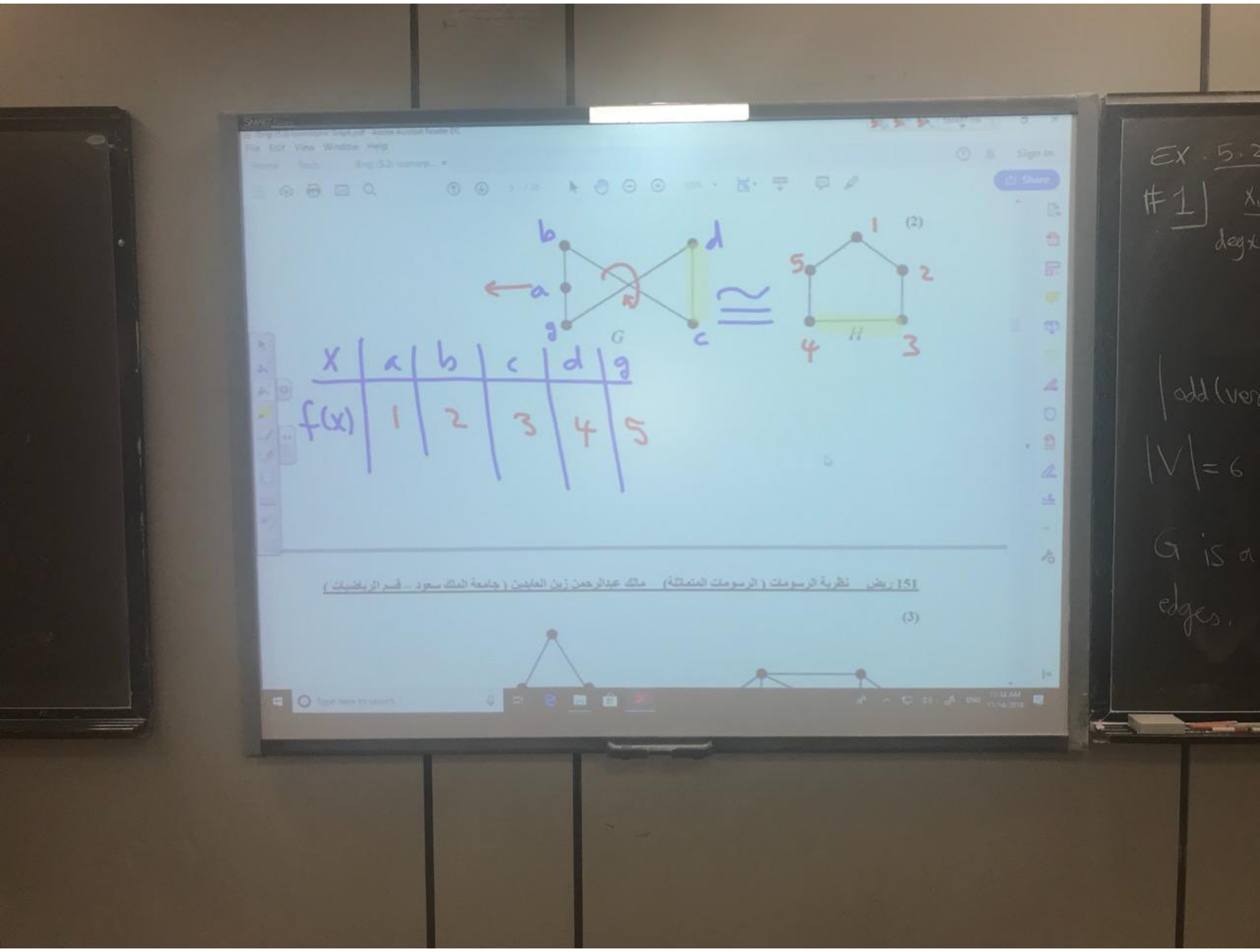

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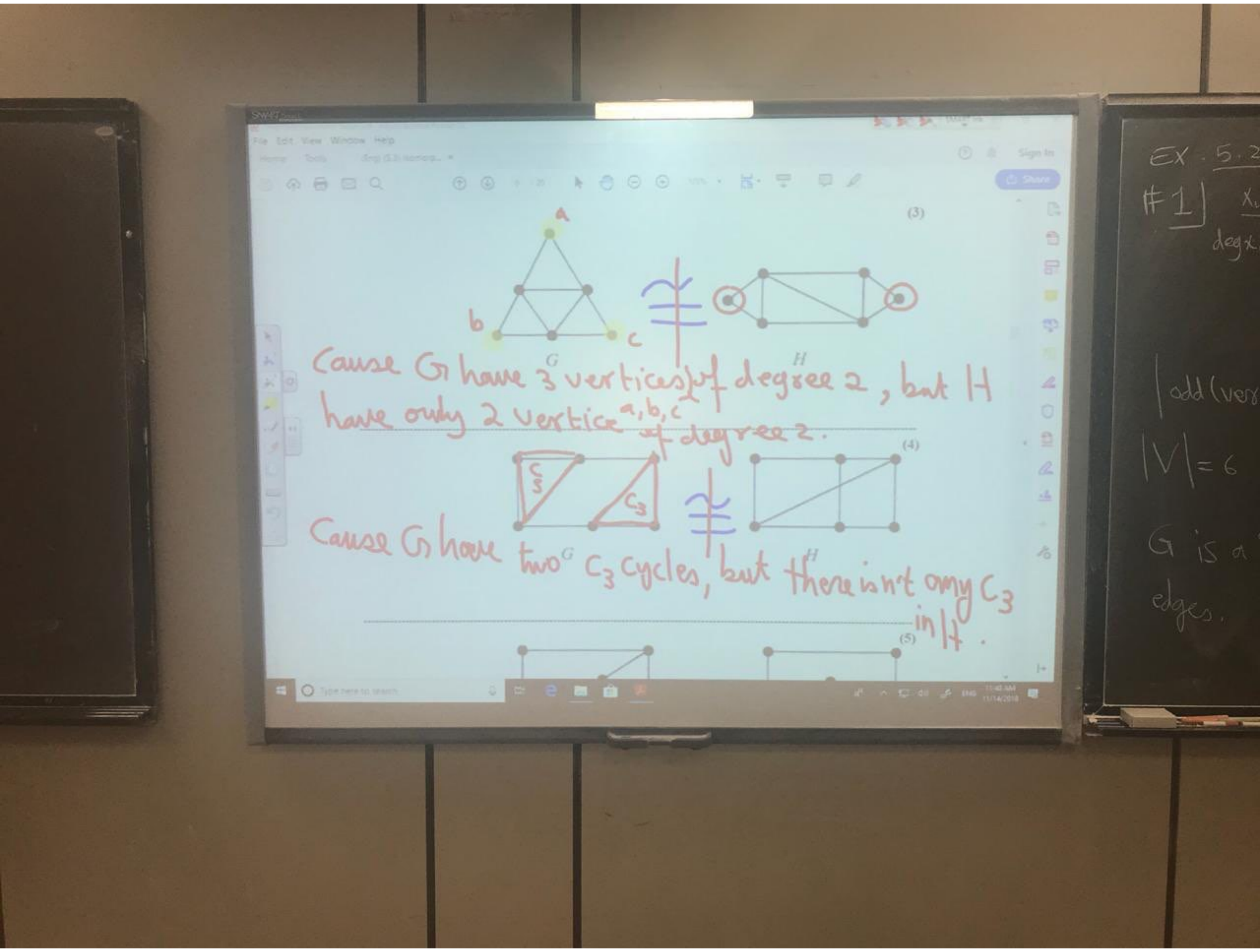

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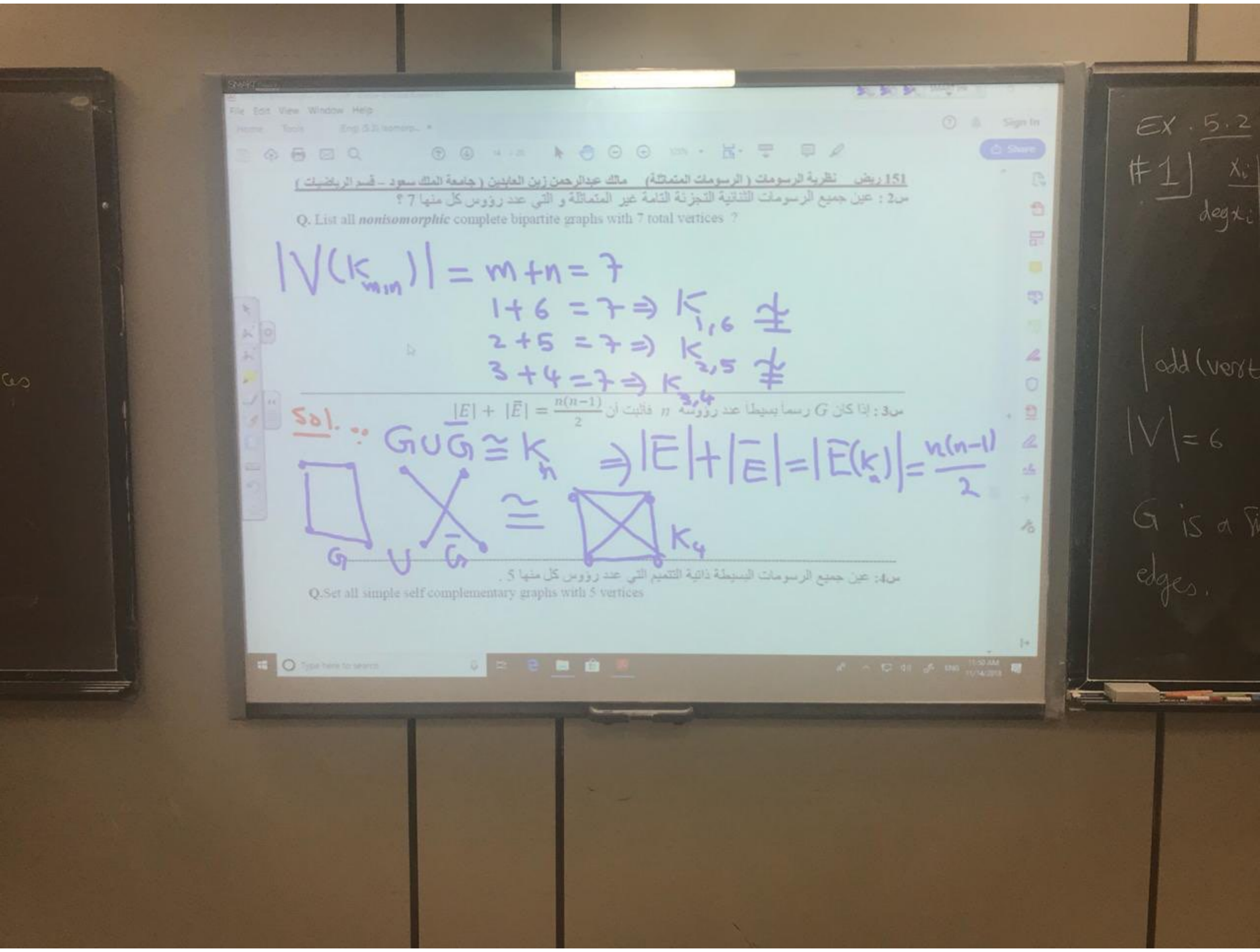

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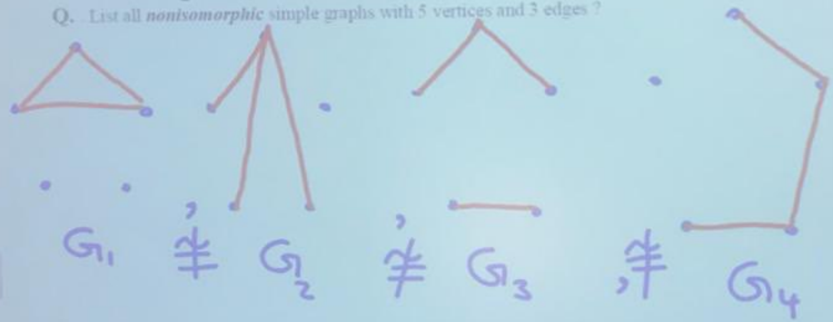
س5: عين جميع الرسومات البسيطة غير المتماثلة التي عدد رؤوس كل منها 5 و عدد أضلاع كل منها 3 .

Q. List all ***nonisomorphic*** simple graphs with 5 vertices and 3 edges ?



151 ريض نظرية الرسومات (الرسومات المتماثلة) مالك عبدالرحمن زين العابدين (جامعة الملك سعود - قسم الرياضيات)

س6: إذا كان G رسماً بسيطاً عدد رؤوسه n و عدد أضلاعه 56 و كان عدد أضلاع $\overline{G}$ هو 80 فاحسب n .

Q.Let G Be a simple graph with n vertices and 56 edges. If $\overline{G}$ have 80 edges , find the value of n ?

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س6: إذا كان G رسما بسيطا عدد رؤوسه n و عدد أضلاعه 56 و كان عدد أضلاع $\overline{G}$ هو 80 فاحسب n .

Q. Let G Be a simple graph with n vertices and 56 edges. If $\overline{G}$ have 80 edges , find the value of n ?

$$|E| + |\overline{E}| = \frac{n(n-1)}{2} \Rightarrow 56 + 80 = \frac{n(n-1)}{2}$$

$$n^2 - n - 272 = 0$$

$$(n-17)(n+16) = 0$$

$$\therefore n = 17 .$$

س7: جد مع التعليل عدد أضلاع الرسم المتمم للرسم $K_{10,14}$.

Q. Find the number of edges for the complementary graph of $K_{10,14}$. Explain the answer?

EX . 5.2

#1)

x_i	a
$\deg x_i$	2 even

odd (vertic

$|V| = 6$

G is a simp

edges.

Q. Find the number of edges for the complementary graph of $K_{10,14}$. Explain the answer?

$$|E| + |\bar{E}| = \frac{|V|(|V|-1)}{2}$$

$$|E(K_{10,14})| + |E(\overline{K}_{10,14})| = \frac{(10+14)(10+14-1)}{2}$$

$$(10)(14) + |E(\overline{K}_{10,14})| = (12)(23) \Rightarrow |E(\overline{K}_{10,14})| = 276-140 = 136$$

س8: عين جميع الرسومات البسيطة غير المتماثلة التي عدد رؤوس كل منها 4 و عدد أضلاع كل منها 3 .

Q. List all ***nonisomorphic*** simple graphs with 4 vertices and 3 edges ?

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س9: هل يوجد رسم G له 7 أضلاع و يحقق $G \cong \overline{G}$ ؟ وضح إجابتك .

Q.Is there exist a graph G with 7 edges satisfies $\overline{G} \cong G$? *explain the answer.*

Sol. $\because G \cong \overline{G}$ (self complemetary) $\Rightarrow$

$$|E| = |\overline{E}| = 7 \Rightarrow |E| + |\overline{E}| = 7+7 = \frac{n(n-1)}{2}.$$

$$n^2 - n - 28 = 0 \quad \Delta = (-1)^2 - 4(1)(-28) = 1 + 112 = 113$$

$$n_{1,2} = \frac{1 \pm \sqrt{113}}{2} \notin \mathbb{N}$$

DN.Ex.

س10: إذا كان G رسماً بسيطاً درجات رؤوسه 2,2,2,3,3,4 فأوجد عدد أضلاع $\overline{G}$ (الرسم المتمم) . *

Q.Let G be a graph with the degree sequence 2,2,2,3,3,4.Find the number of edges of $\overline{G}$?

Sol. $|E| + |\overline{E}| = \frac{n(n-1)}{2}$

$$\frac{2+2+2+3+3+4}{2} + |\overline{E}| = \frac{6(5)}{2} = 15 \Rightarrow |\overline{E}| = 7$$

س11: إذا كان G رسماً بسيطاً عدد رؤوسه n و عدد أضلاعه 5، فجد ضعف عدد أضلاع الرسم المتمم $\overline{G}$.

Q. Let G be a graph with n vertices and 5 edges. find the double edges of $\overline{G}$?

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King Saud University

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151 MATH EXERCISES

(6)

BOOLEAN ALGEBRAS

&

LOGIC GATES

&

MINIMIZATION OF CIRCUITS

Malek Zein AL-Abidin

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2018

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Q2. Let B is a Boolean Algebra and $x, y \in B$. Show that $x + y = xy + xy' + x'y$ is valid.

x	y	x'	y'	xy	xy'	$x'y$	$xy + xy' + x'y$	$x + y$
1	1	0	0					
1	0	0	1					
0	1	1	0					
0	0	1	1					

151 Math Discrete Mathematics Boolean Algebras & Logic Gates By: Malek Zein AL-Abidin

Q5. Let $f(x,y,z) = (x'+z)(x+y)$. Find $CSP(f)$ (sum-of-products expansion) and $CPS(f)$ (product-of-sums expansion) ?

$f = x'y + xz + yz$. (Sp)

$csp(f) = xyz + xy'z + x'yz + x'yz'$.

	yz	yz'	$y'z'$	$y'z$
x	1	0	0	1
x'	1	1	0	0

$f' = xyz' + xy'z' + x'y'z' + x'y'z$.

$cps(f) = (f')' = (x'+y'+z).(x'+y+z).(x+y+z).(x+y+z')$.

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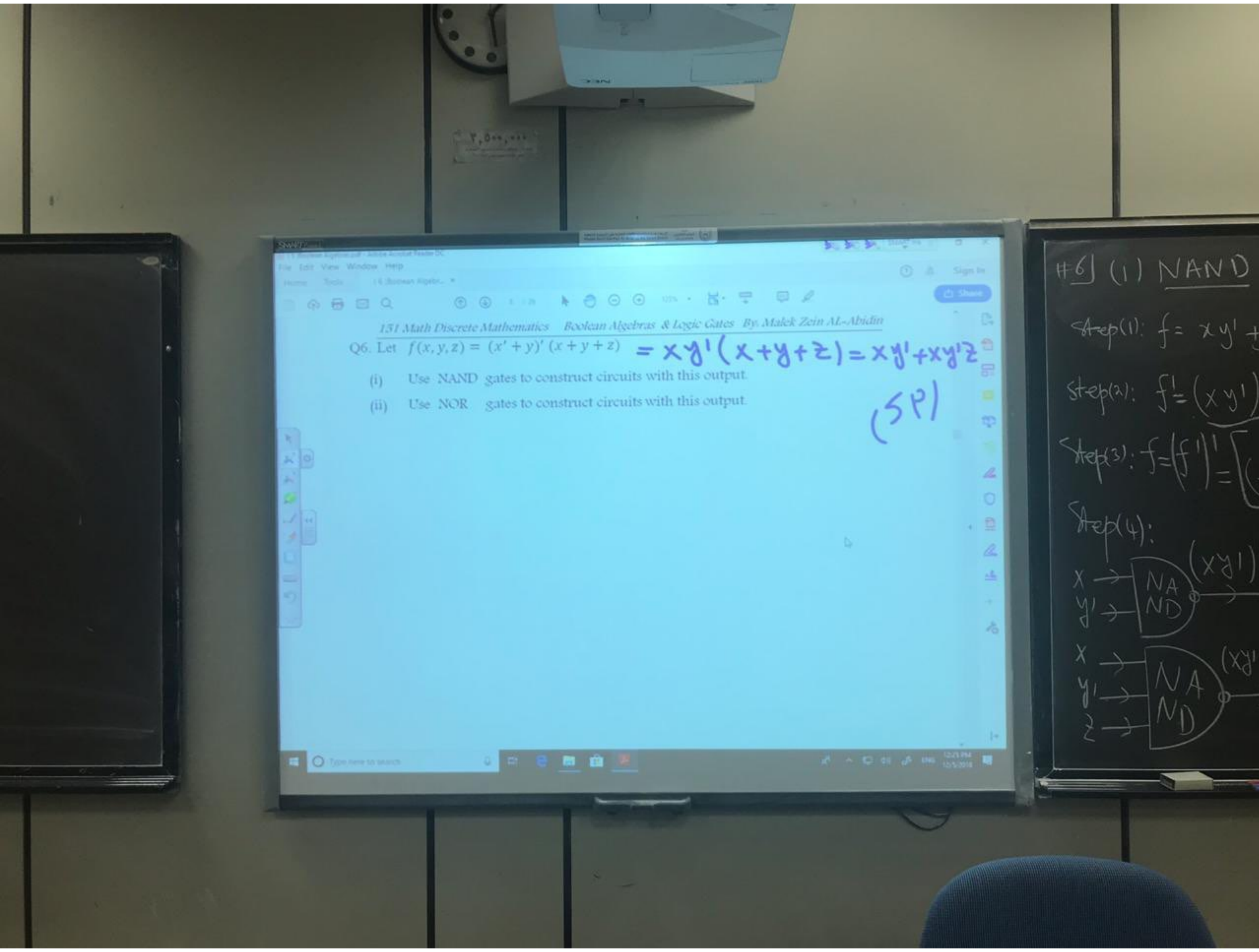

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$x y' + x y' z$

(SP)

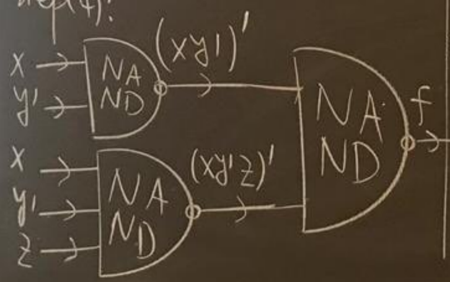
#6] (i) NAND

Step(1): $f = xy' + xy'z \quad (SP)$

Step(2): $f' = (xy')' \cdot (xy'z)' \quad (*)$

Step(3): $f = (f')' = \left[(xy')' \cdot (xy'z)'\right]'$

Step(4):



(ii) NOR

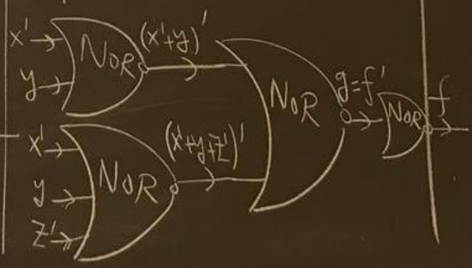
Step(1): $\underset{(*)\Rightarrow}{f'} = (x'+y)\cdot(x'+y+z') = g(x,y,z)$

(PS)

Step(2): $g' = (x'+y)' + (x'+y+z')'$

Step(3): $g = (g')' = \left[(x'+y)' + (x'+y+z')'\right]'$

Step(4):



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Q7 Let $f(x,y,z) = (x+y)(x'+yz')$ $= xyz' + x'y + yz'$ (SP)

(i) Find $CSP(f)$ and $CPS(f)$.

(ii) Find $MSP(f)$ and $MPS(f)$.

(iii) Construct a minimal circuit using (AND–OR) gates, with $f(x,y',z)$ output.

<u>Sol.</u> $CSP(f) = xyz' + x'yz + x'yz'$.

$f' = xyz + xy'z' + xy'z + x'y'z' + x'y'z$.

$CPS(f) = (f')' = (x'+y'+z')(x'+y+z).(x'+y+z').(x+y+z).(x+y+z')$.

	yz	yz'	$y'z'$	$y'z$
x	0	1	0	0
x'	1	1	0	0

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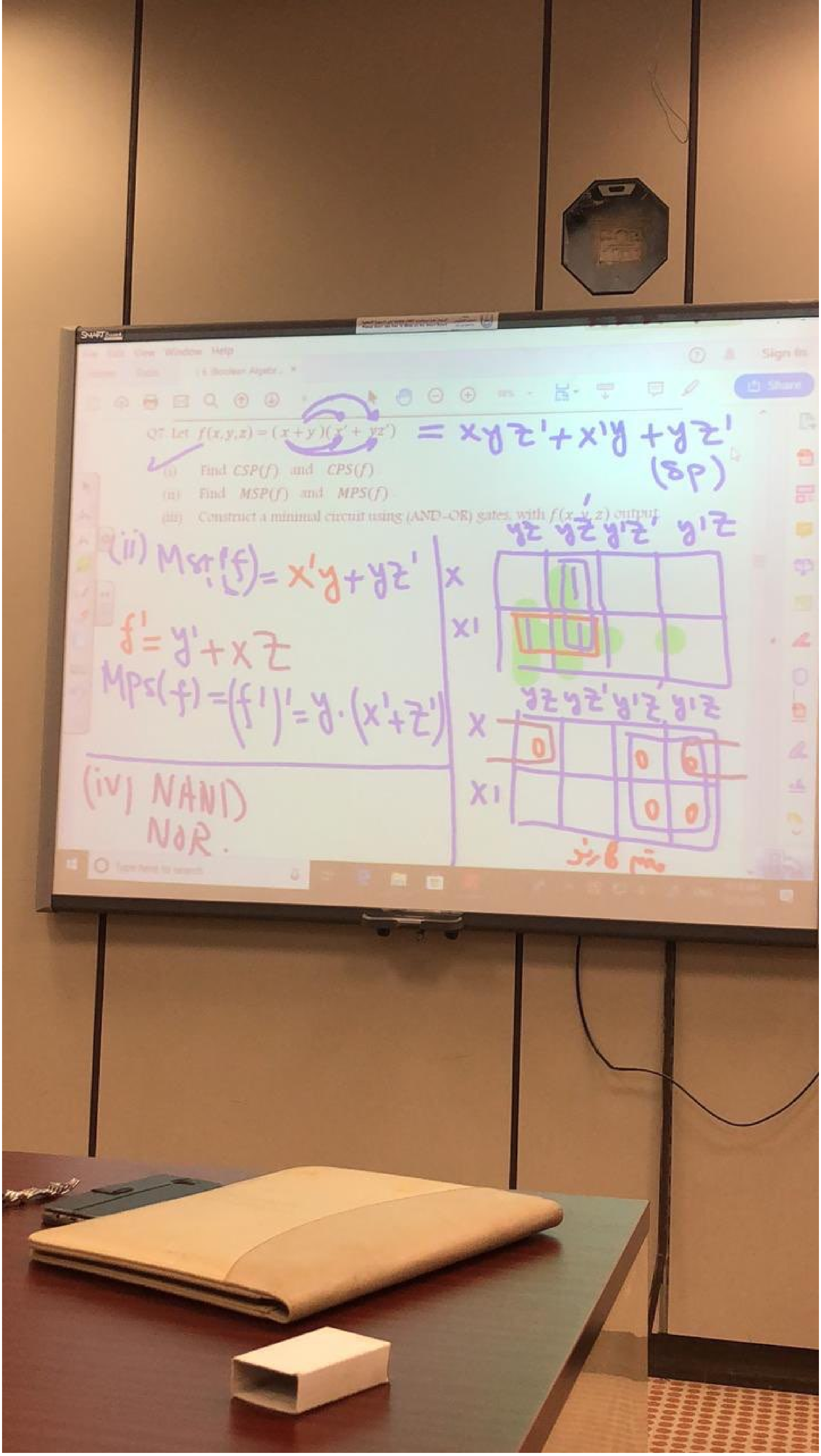

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#7

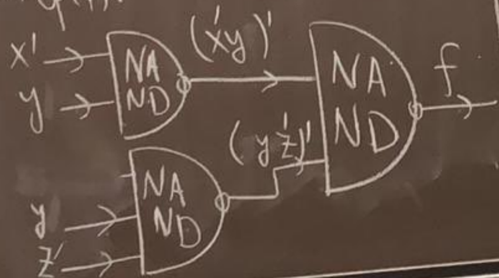
(ii) (a) NAND

Step(1): $Msp(f) = x'y + yz'$.

Step(2): $f' = (x'y)' \cdot (yz')'$.

Step(3): $f = (f')' = [(x'y)'(yz')']'$

Step(4):



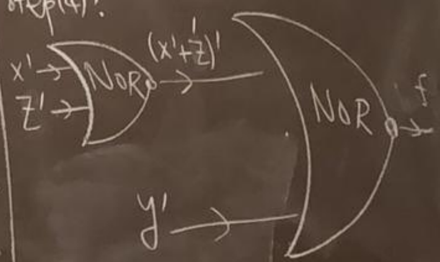
(b) NOR.

Step(1): $Mps(f) = y \cdot (x' + z')$.

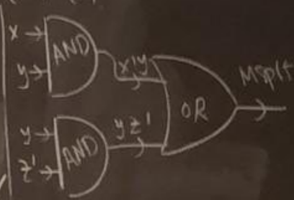
Step(2): $f' = y' + (x' + z')'$

Step(3): $f = (f')' = [y' + (x' + z')']'$

Step(4):

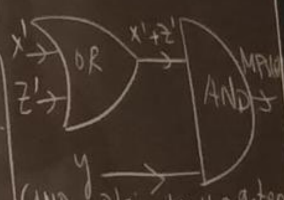


(iii) (a)



(AND-OR) circuit with 3 gates

(b)



(AND-OR) circuit with 2 gates (is the minimal).

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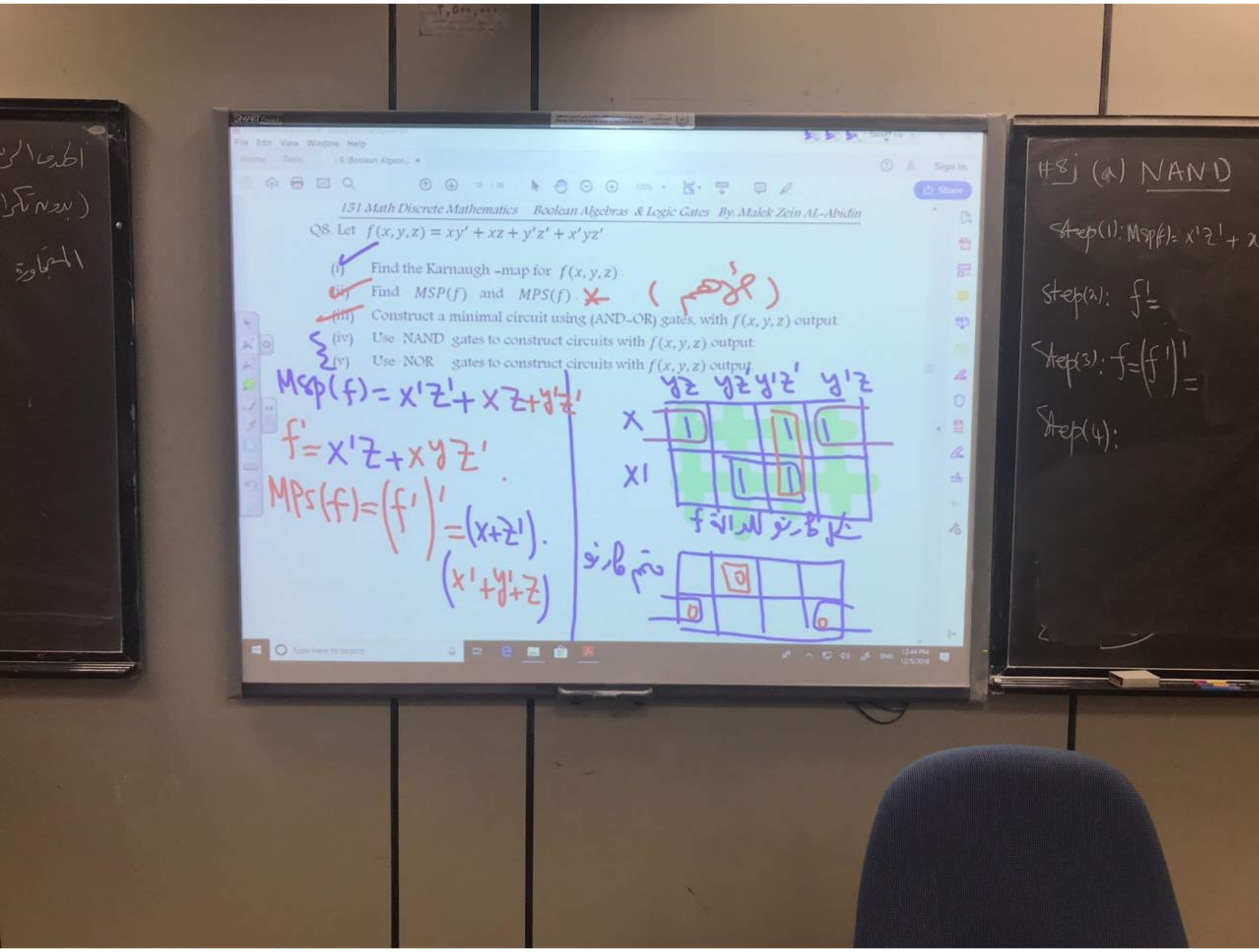
151 Math Discrete Mathematics Boolean Algebras & Logic Gates By: Malek Zein AL-Abidin
Q8. Let f(x,y,z) = xy' + xz + y'z' + x'yz'
(i) Find the Karnaugh -map for f(x,y,z).
(ii) Find MSP(f) and MPS(f).
(iii) Construct a minimal circuit using (AND-OR) gates, with f(x,y,z) output.
(iv) Use NAND gates to construct circuits with f(x,y,z) output.
(v) Use NOR gates to construct circuits with f(x,y,z) output.
Msp(f) = x'z' + xz + y'z'
f' = x'z + xyz'
MPs(f) = (f')' = (x+z')·(x'+y'+z)
yz yz' y'z' y'z
x
x'
#8 j (a) NAND
Step(1): Msp(f) = x'z' + x
Step(2): f' =
Step(3): f = (f')' =
Step(4):
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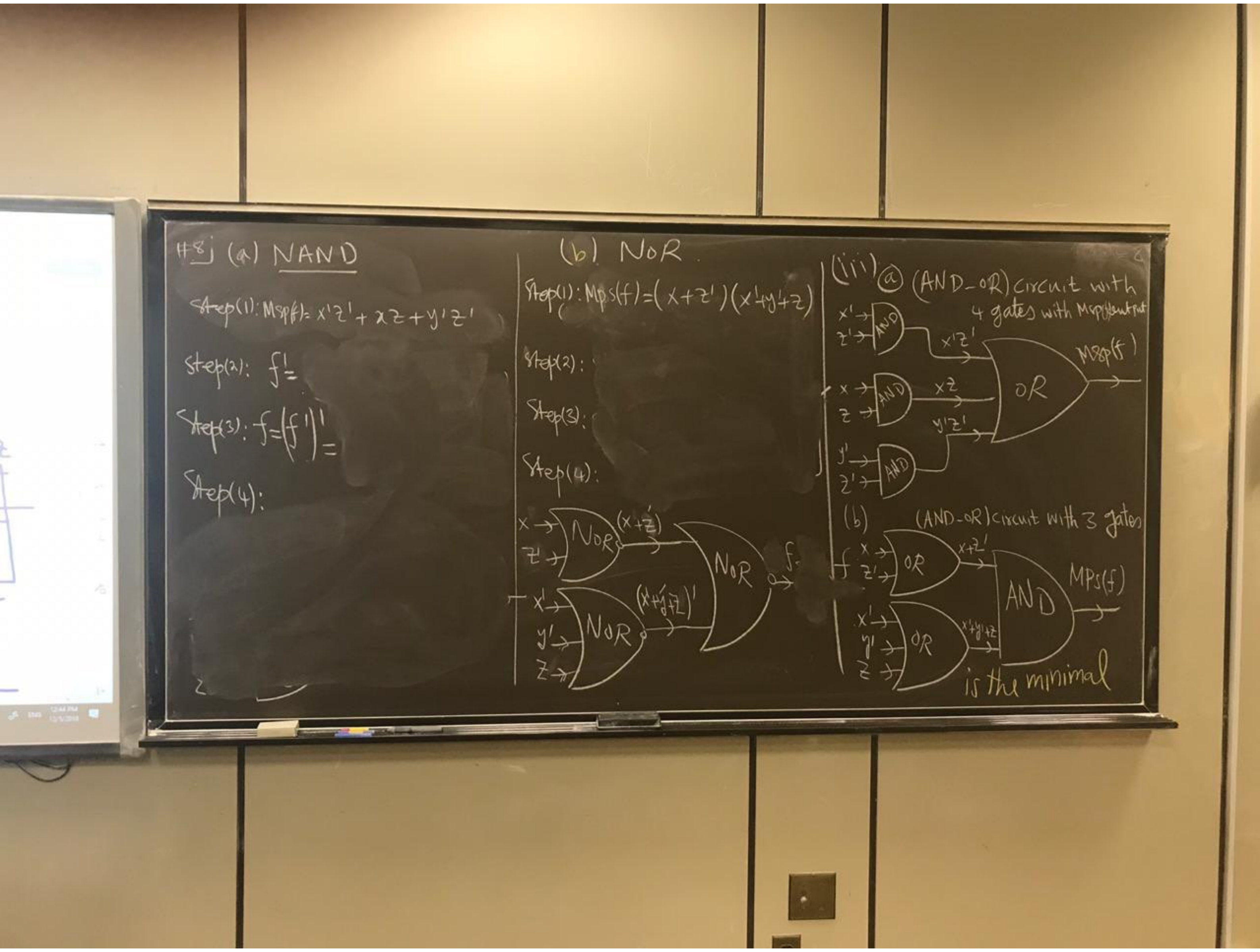

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الهدف الرئيسي هو ايجاد أقل عدد ممكن من المستطيلات الأساسية (بدون تكرار) والتي تحتوي على أكبر عدد ممكن من الخلايا المتجاورة أفقياً أو عمودياً والتي عددها:

عدد الخلايا

$2^3 = 8$

$2^2 = 4$

$2^1 = 2$

$2^0 = 1$ (خلية منفردة)

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Q12. Let

#

	zw	zw'	$z'w'$	$z'w$
xy	0	1	0	1
xy'	1	1	1	1
$x'y'$	1	0	1	0
$x'y$	0	0	0	0

[illegible] f'

$$f' = x'z'w + yz'w' + yzw + x'zw'.$$

$$Mps(f) = (f')' = (x+z+w')\cdot(y'+z+w)(y'+z'+w')(x+z'+w)$$

✓ Be the Karnaugh -map for $f(x,y,z)$.

(i)Find MSP(f) and MPS(f)

(ii)Construct a minimal circuit using (AND-OR) gates, with $f(x,y,z)$ output.

(iii)Use NAND gates to construct circuits with $f(x,y,z)$ output.

(iv)Use NOR gates to construct circuits with $f(x,y,z)$ output.

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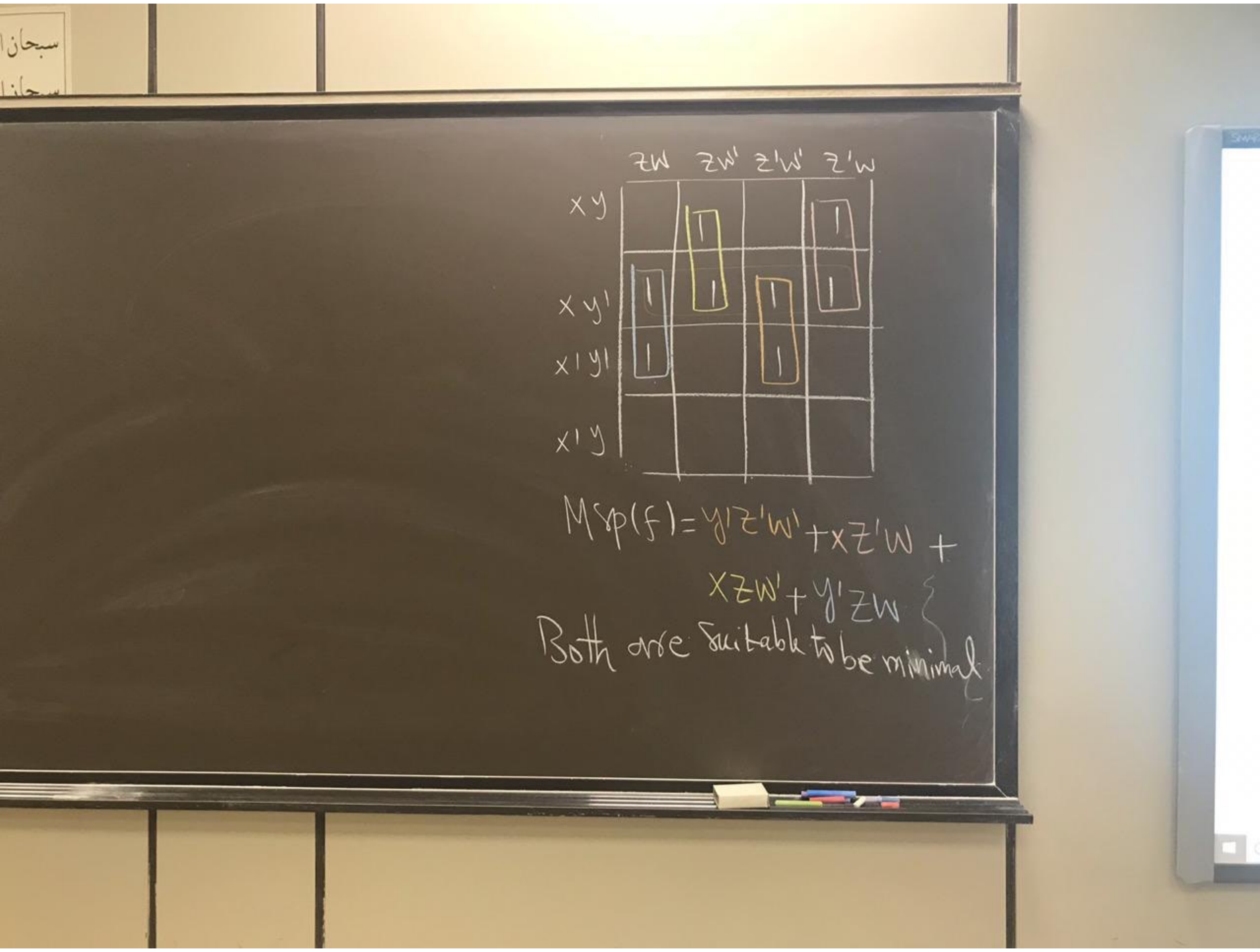

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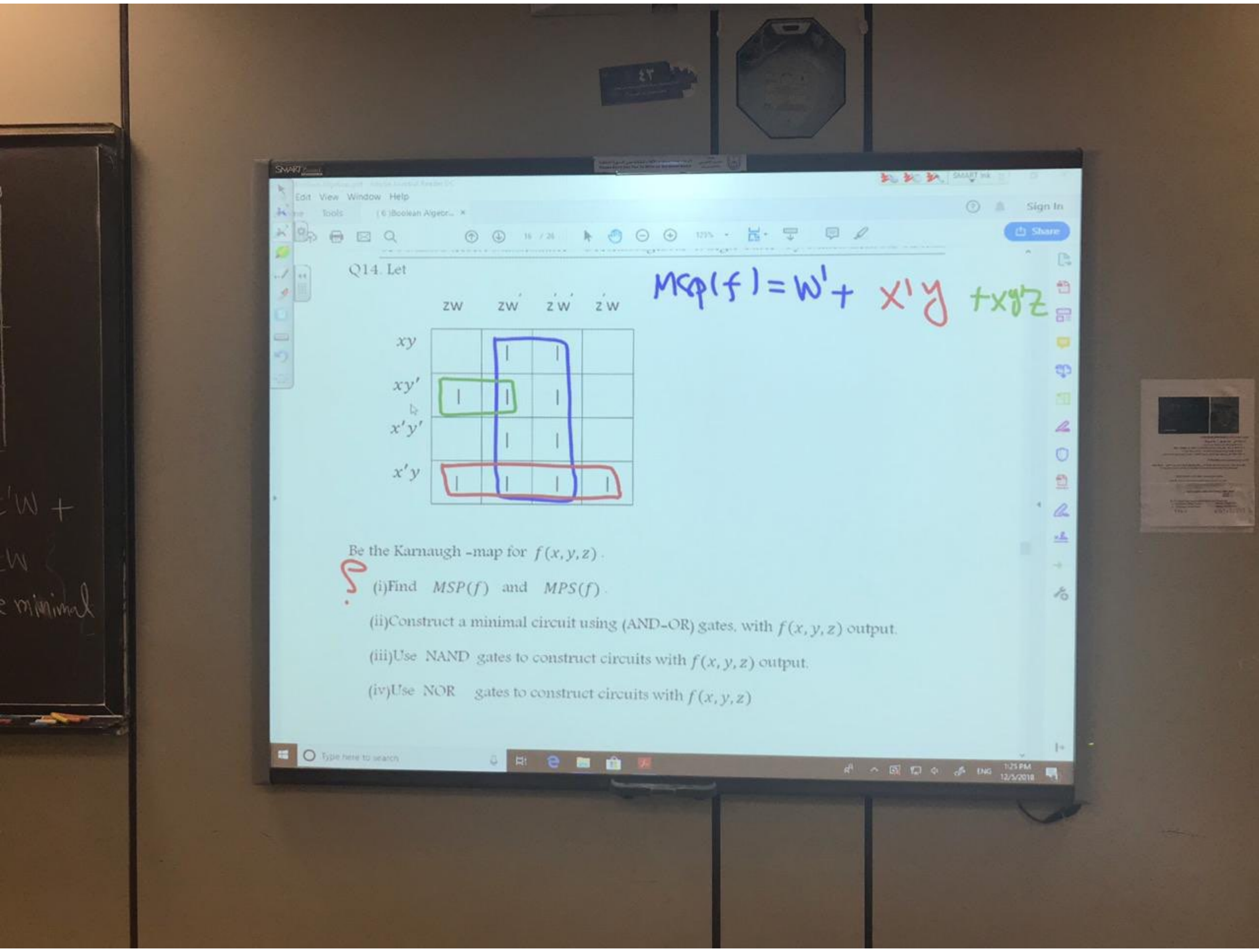

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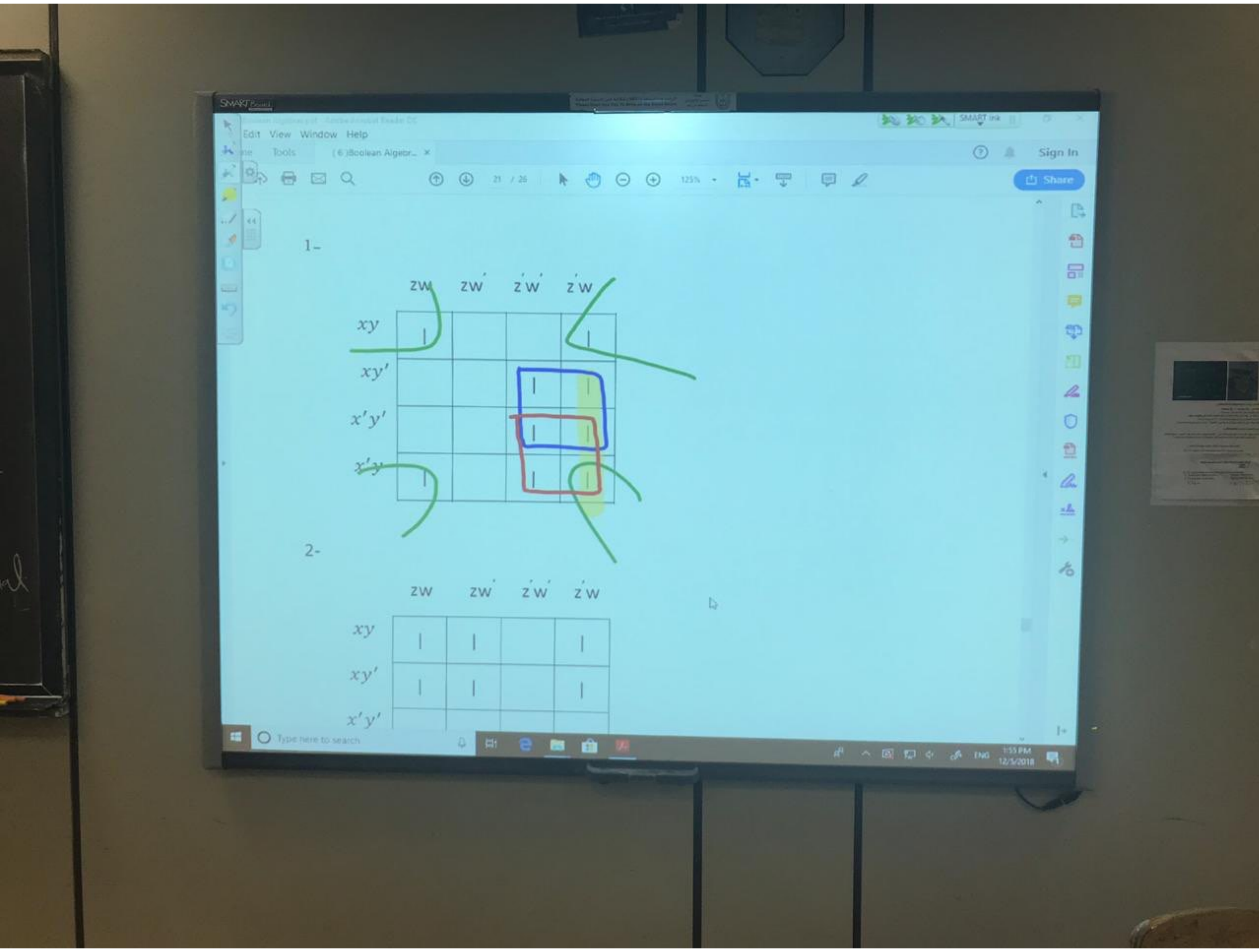

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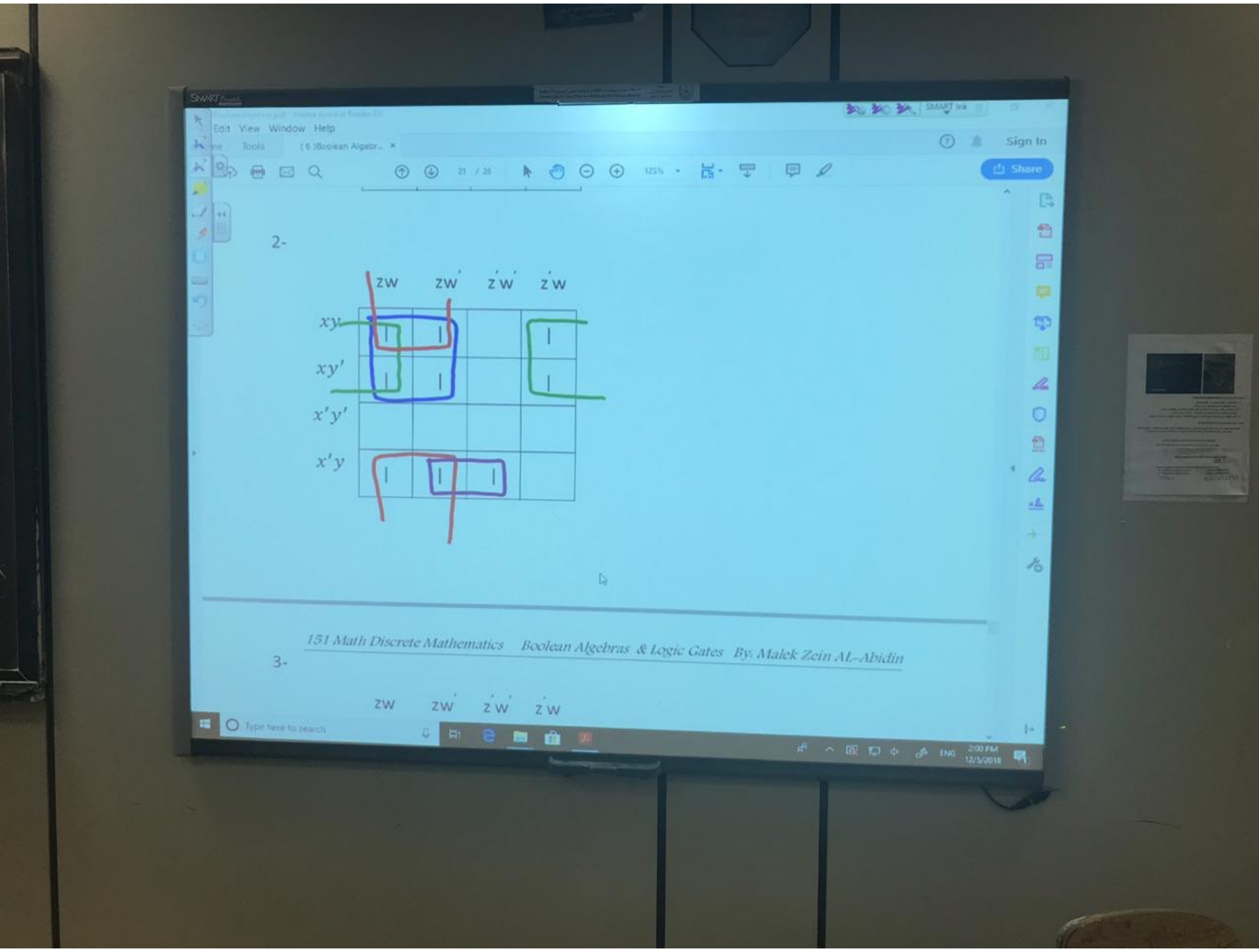

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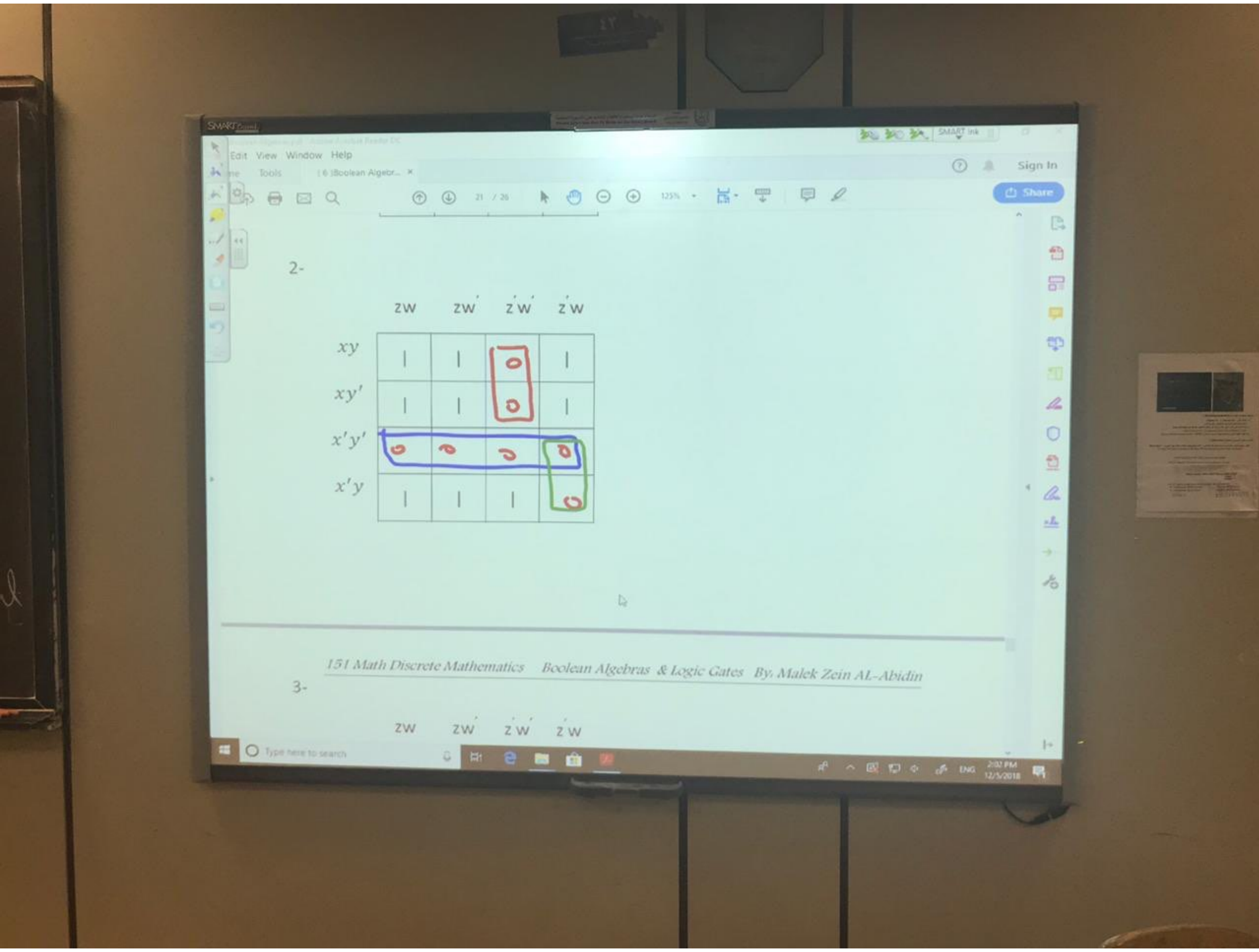

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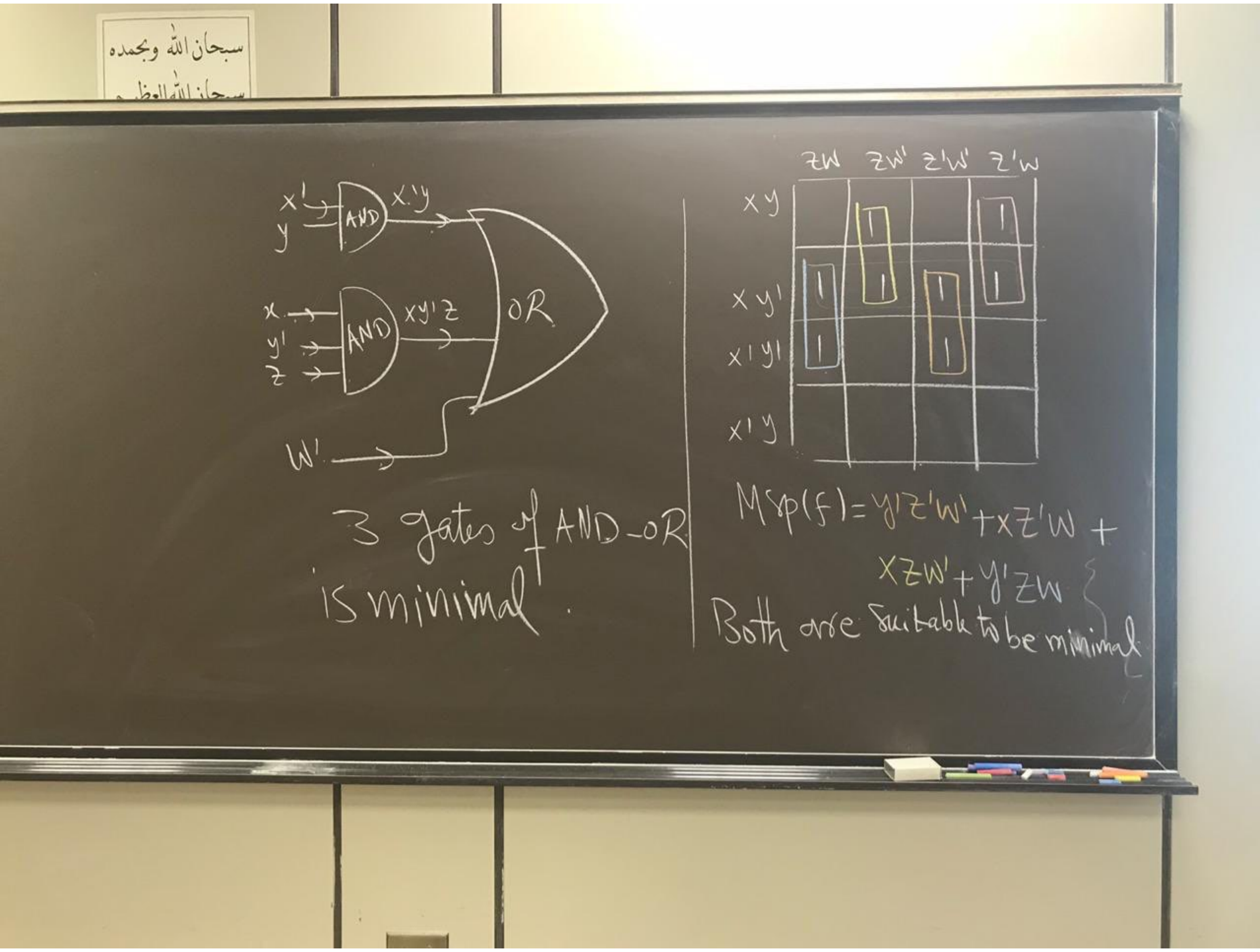

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Q14. Let

	zw	zw'	$z'w'$	$z'w$
xy	0	1	1	0
xy'	1	1	1	0
$x'y'$	0	1	1	0
$x'y$	1	1	1	1

$MSP(f) = w' + x'y + xy'z$

$f' = x'y'w + y'z'w + xyw$

$MPS(f) = (f') = (x+y+w') \cdot (y+z+w') \cdot (x'+y'+w')$

Be the Karnaugh -map for $f(x,y,z)$.

(i)Find $MSP(f)$ and $MPS(f)$.

(ii)Construct a minimal circuit using (AND-OR) gates, with $f(x,y,z)$ output.

(iii)Use NAND gates to construct circuits with $f(x,y,z)$ output.

(iv)Use NOR gates to construct circuits with $f(x,y,z)$

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