King Saud University

College of Science

Department of Mathematics

① $A(n) = B(n)$

② $A(n) \geq B(n)$

③ $c \mid A(n)$

151 Math Exercises

$P(n) : A(n)$ is odd (even)

(3,2)

Methods of Proof

"Mathematical Induction"

(First principle)

Malek Zein AL-Abidin

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Mathematical Induction

In general, mathematical induction ∗ can be used to prove statements that assert that *P(n)* is true for all positive integers *n*, where *P(n)* is a propositional function. A proof by mathematical induction has two parts, a **basis step**, where we show that *P(1)* is true, and an **inductive step**, where we show that for all positive integers *k*, if *P(k)* is true, then *P(k + 1)* is true.

PRINCIPLE OF MATHEMATICAL INDUCTION To prove that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function, we complete two steps:
BASIS STEP: We verify that $P(1)$ is true.
INDUCTIVE STEP: We show that the conditional statement $P(k) \rightarrow P(k+1)$ is true for all positive integers k.

Exercises

1. Use mathematical induction to Show that

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2} \quad , \text{ where } n \text{ is a positive integer} : \forall n \geq 1$$

Solution: Let $P(n)$ be the proposition , $P(n)$: " $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ "

BASIS STEP: $P(1)$? when $n = 1$, L H S = 1 , R H S = $\frac{1(1+1)}{2} = 1$

$\because$ L H S = R H S $\Rightarrow \therefore P(1)$ is true .

INDUCTIVE STEP: Let k is integer where $k \geq 1$ and assume that $p(k)$ is true , $\Rightarrow$

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2} \qquad (*)$$

Our goal is to show that $p(k+1)$ is also true

$$1 + 2 + \cdots + k + (k+1) = \frac{(k+1)(k+2)}{2} \qquad \text{(Our Goal) ?}$$

From (*) add the term $(k+1)$ (the term # $n = k+1$) to both sides $\Rightarrow$

$$1 + 2 + \cdots + k + (k+1) = \frac{k(k+1)}{2} + (k+1)$$
$$= \frac{k(k+1) + 2(k+1)}{2}$$
$$= \frac{k^2 + 3k + 2}{2}$$
$$= \frac{(k+1)(k+2)}{2}$$

So $p(k+1)$ is true . #

2. Use mathematical induction to Show that

$$\frac{1}{1\times 3}+\frac{1}{3\times 5}+\frac{1}{5\times 7}+\cdots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1} \quad : \forall n\geq 1$$

Solution:

Let $p(n)$: " $\frac{1}{1\times3}+\frac{1}{3\times5}+\frac{1}{5\times7}+\cdots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$ "

BASIS STEP: $P(1)$?? When $=1$, L H S $=\frac{1}{(2(1)-1)(2(1)+1)}=\frac{1}{(1)(3)}=\frac{1}{3}$

R H S $=\frac{1}{2(1)+1}=\frac{1}{3}$

$\because$ L H S = R H S $\Rightarrow$ $\therefore$ $P(1)$ is true .

INDUCTIVE STEP: Let k is integer where $k\geq 1$ and assume that $p(k)$ is true , $\Rightarrow$

$$\frac{1}{1\times3}+\frac{1}{3\times5}+\frac{1}{5\times7}+\cdots+\frac{1}{(2k-1)(2k+1)}=\frac{k}{2k+1} \qquad (*)$$

Our goal is to show that $p(k+1)$ is also true

$$\frac{1}{1\times3}+\frac{1}{3\times5}+\frac{1}{5\times7}+\cdots+\frac{1}{(2k-1)(2k+1)}+\frac{1}{(2k+1)(2k+3)}=\frac{k+1}{2k+3} \quad (\text{ Our Goal })\ ?$$

add the term $\frac{1}{[2(k+1)-1][2(k+1)+1]}$ (the term # $n=k+1$) to the both sides in $(*)$ $\Rightarrow$

$$\frac{1}{1\times 3}+\frac{1}{3\times 5}+\frac{1}{5\times 7}+\cdots+\frac{1}{(2k-1)(2k+1)}+\frac{1}{(2k+1)(2k+3)}=\frac{k}{2k+1}+\frac{1}{(2k+1)(2k+3)}$$

$$=\frac{k(2k+3)+1}{(2k+1)(2k+3)}$$

$$=\frac{2k^2+3k+1}{(2k+1)(2k+3)}$$

$$=\frac{\cancel{(2k+1)}(k+1)}{\cancel{(2k+1)}(2k+3)}$$

$$=\frac{k+1}{2k+3}$$

So $p(k+1)$ is true . #

3. Use mathematical induction to Show that

$$1.2 + 2.3 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3} : \forall n \geq 1$$

Solution: let $P(n)$ $1.2+2.3+\dots+n(n+1)=\frac{n(n+1)(n+2)}{3}$

BS: $P(1)$? When $n=1 \Rightarrow$ R.H.S $= \frac{1(2)(3)}{3} = 2$

L.H.S $= 1(1+1) = 2$ $\therefore$ L.H.S $=$ R.H.S $\Rightarrow P(1)$ is true

Induc. Step: let $K \geq 1$ and assume that $P(K)$ is true. $1.2+2.3+\dots+K(K+1) = \frac{K(K+1)(K+2)}{3}$ $(*)$

our goal to Show that $P(K+1)$ is also true.

$1.2+2.3+\dots+(K+1)(K+2) = \frac{(K+1)(K+2)(K+3)}{3}$?

$\overset{(*)}{\Rightarrow}$ $1.2+2.3+\dots+K(K+1)+\underline{(K+1)(K+2)}$ $\Leftarrow$ الحد رقم $n=K+1$

$= \underbrace{\frac{K(K+1)(K+2)}{3} + (K+1)(K+2)} =$

$= \frac{K(K+1)(K+2)}{3} + \frac{3(K+1)(K+2)}{3} = \frac{(K+1)(K+2)}{3}(K+3)$

$= \frac{(K+1)(K+2)(K+3)}{3} \Rightarrow$ so $P(K+1)$ is true $\Rightarrow P(n)$ is true for $\forall n \geq 1$

done #

4. Use mathematical induction to Show that

$$1.2.3 + 2.3.4 + \cdots + n(n+1)(n+2) = \frac{n(n+1)(n+2)(n+3)}{4} : \ \forall n \geq 1$$

Solution:

5. Use mathematical induction to Show that

if n is a positive integer, then $1 + 3 + 5 + \cdots + (2n - 1) = n^2$

Solution: Let $P(n)$ be the proposition ,

$$P(n):\ \text{“}\ 1 + 3 + 5 + \cdots + (2n - 1) = n^2\ \text{“}$$

BASIS STEP: $P(1)$? when $n = 1 \Rightarrow$ L H S = $2(1) - 1 = 1$, R H S = $1^2 = 1$

$\because$ L H S = R H S $\Rightarrow \therefore P(1)$ is true .

INDUCTIVE STEP: Let k is integer where $k \geq 1$ and assume that $p(k)$ is true ,

$$1 + 3 + 5 + \cdots + (2k - 1) = k^2 \quad (*)$$

Our goal is to show that $p(k+1)$ is also true

$$1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) = (k+1)^2 \quad \text{(Our Goal) ?}$$

add the term $[2(k+1) - 1]$ (the term # $n = k + 1$) to both sides in $(*)$ $\Rightarrow$

$$1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) = k^2 + (2k + 1)$$

(where $P(k)$ is true *) [the term # n= k+1]

$$= k^2 + 2k + 1$$

$$= (k+1)^2$$

So $p(k+1)$ is true . #

6. Use mathematical induction to Show that

$$\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^n}=\frac{2^n-1}{2^n} \quad : \forall n \geq 1$$

$\boxed{2^1-1}$

Solution: let $P(n): \frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^n}=\frac{2^n-1}{2^n}$

BS: $P(1)$? when $n=1 \Rightarrow$ LHS $= \frac{1}{2}$ = R.H.S $= \frac{2-1}{2} = \frac{1}{2}$

$\Rightarrow \therefore P(1)$ is true.

Induc step: let $k \geq 1$ and assume that $P(k)$ is true.

$\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^k}=\frac{2^k-1}{2^k}$ (*). our goal to show that $P(k+1)$ is also true.

$\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^{k+1}}=\frac{2^{k+1}-1}{2^{k+1}}$??

$\overset{(*)}{\Longrightarrow} \frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^k}+\frac{1}{2^{k+1}}=\frac{2^k-1}{2^k}+\frac{1}{2^{k+1}}$

$= \frac{2(2^k-1)}{2 \cdot 2^k}+\frac{1}{2^{k+1}}=\frac{2^{k+1}-2+1}{2^{k+1}}=\frac{2^{k+1}-1}{2^{k+1}}$

so $P(k+1)$ is true $\Rightarrow \therefore P(n)$ is true for $\forall n \geq 1$ done #

7. Use mathematical induction to Show that

$$1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$$

for all nonnegative integers n.

Solution: Let $P(n)$ be the proposition ,

$$P(n):\ \text{“ } 1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1 \text{ “}$$

BASIS STEP: $P(0)$? when $n = 0 \Rightarrow$ L H S = $2^0 = \boxed{1}$, R H S = $2^{0+1} - 1 = 2 - 1 = \boxed{1}$

$\because$ L H S = R H S $\Rightarrow \therefore P(0)$ is true .

INDUCTIVE STEP Let k is integer where $k \geq 1$ and assume that $p(k)$ is true ,

$$1 + 2 + 2^2 + \cdots + 2^k = 2^{k+1} - 1 \qquad (*)$$

Our <u>goal</u> is to show that $p(k+1)$ is also true

$$1 + 2 + 2^2 + \cdots + 2^n + 2^{n+1} = 2^{n+2} - 1 \qquad (\text{ Our Goal }) ?$$

add the term $[2^{k+1}]$ (the term # $n = k + 1$) to both sides in $(*)$ $\Rightarrow$

$$1 + 2 + 2^2 + \cdots + 2^k + (2^{k+1}) = 2^{k+1} - 1 + 2^{k+1}$$

(where $P(k)$ is true *) [the term # n= k+1]

$$= 2\, 2^{k+1} - 1$$

$$= 2^{(k+1)+1} - 1$$

$$= 2^{n+2} - 1$$

So $p(k+1)$ is true . #

8. Use mathematical induction to Show that

$$2+2(-7)+2(-7)^2+2(-7)^3+\cdots+2(-7)^n=\frac{1-(-7)^{n+1}}{4} : n\geq 0$$

Solution:

9. Use mathematical induction to Show that

$$1 \times 2 + 2 \times 2^2 + 3 \times 2^3 + \cdots + n \times 2^n = 2 + (n-1)2^{n+1} \ : n \geq 1$$

Solution:

10. Use mathematical induction to Show that

$$\sum_{j=0}^{n} ar^j = a + ar + ar^2 + \cdots + ar^n = \frac{ar^{n+1} - a}{r - 1}$$

where n is a nonnegative integer. , when $r \neq 1$

Solution: Let $P(n)$ be the proposition ,

$P(n)$: "

$$\sum_{j=0}^{n} ar^j = a + ar + ar^2 + \cdots + ar^n = \frac{ar^{n+1} - a}{r - 1}$$

Where $r \neq 1$, "

BASIS STEP: $P(0)$? when n=0 L H S = $ar^0 = a.1 = a$,

R H S = $\frac{ar^{0+1}-a}{r-1} = \frac{a(r-1)}{r-1} = a$

$\because$ L H S = R H S $\Rightarrow$ $\therefore$ $P(0)$ is true .

INDUCTIVE STEP: Let k is integer where $k \geq 0$ and assume that $p(k)$ is true ,

$$a + ar + ar^2 + \cdots + ar^k = \frac{a\,r^{k+1}-a}{r-1} \quad (*)$$

Our goal is to show $p(k+1)$ is also true

$$a + ar + ar^2 + \cdots + ar^k + ar^{k+1} = \frac{a\,r^{k+2}-a}{r-1} \quad \text{(Our Goal)?}$$

Add ar^{k+1} to both sides of the equation in (*) we obtain

$$a + ar + ar^2 + \cdots + ar^k + (ar^{k+1}) = \frac{a\,r^{k+1}-a}{r-1} + ar^{k+1}$$

(where $P(k)$ is true *) [the term # $n = k+1$]

$\Rightarrow$

$$\frac{ar^{k+1} - a}{r - 1} + ar^{k+1} = \frac{\cancel{ar^{k+1}} - a}{r - 1} + \frac{ar^{k+2} - \cancel{ar^{k+1}}}{r - 1}$$

$$= \frac{ar^{k+2} - a}{r - 1}.$$

Combining these last two equations gives

$$a + ar + ar^2 + \cdots + ar^k + ar^{k+1} = \frac{ar^{k+2} - a}{r - 1}.$$

So $P(k+1)$ is true . #

11. Use mathematical induction to Show that

$$1 + a + a^2 + \cdots + a^n = \frac{a^{n+1} - 1}{a-1} \quad , a \neq 1$$: for all nonnegative integers n.

Solution:

12. Use mathematical induction to Show that

$$2 + 4 + 6 + \cdots + 2n = n(n+1) \quad : n \geq 1$$

Solution:

13. Use mathematical induction to Show that

$$4+8+12+\cdots+4n=2n(n+1) \quad : n \geq 1$$

3k+3

14. Use mathematical induction to Show that

$$1+4+7+\cdots+(3n-2)=\frac{n(3n-1)}{2} \quad : n \geq 1$$

Solution: let $P(n)$: $1+4+7+\dots+(3n-2) \frac{n(3n-1)}{2}$

BS: $P(1)$?? when $n=1$ $(3)1-2)=1 \Rightarrow$ L.H.S, $\frac{1(3(1)-1)}{2}=1 \Rightarrow$ R.H.S

$\therefore$ L.H.S = R.H.S $\Rightarrow P(1)$ is true. Induc step: let $k \geq 1$ and assume that $P(k)$ is true

$1+4+7+\dots+(3k-2)=\frac{k(3k-1)}{2}$ (*). our goal is to show that $P(k+1)$ is also true.

$1+4+7+\dots+(3k-2)=\frac{(k+1)(3k+2)}{2}$

(*) $\Rightarrow 1+4+7+\dots(3k-2)(3k-2)=\frac{k(3k-1)}{2}+(3k-2)$

$=\frac{k(3k-1)}{2}+\frac{2(3k-2)}{2}=\frac{3k^2-k+6k-2}{2}$

$=\frac{3k^2+5k+2}{2}=\frac{(k+1)(3k+2)}{2}$

so $P(k+1)$ is true

done #

15. Use mathematical induction to Show that

$$3 + 3^2 + 3^3 + \cdots + 3^n = \frac{3}{2}(3^n - 1) : \forall n \geq 1$$

Solution:

16. Use mathematical induction to Show that

$$3+\frac{3}{4}+\frac{3}{4^2}+\cdots+\frac{3}{4^n}=\frac{4^{n+1}-1}{4^n}\ :\ \forall n\geq 0$$

17. Use mathematical induction to Show that

$$\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+\cdots+\frac{n}{2^n}=2-\frac{n+2}{2^n} \quad : n \geq 1$$

Solution:

18. Use mathematical induction to Show that

$$2 + 2 \times 3 + 2 \times 3^2 + 2 \times 3^3 + \cdots + 2 \times 3^{n-1} = 3^n - 1 : n \geq 1$$

Solution:

19. Use mathematical induction to Show that

$$1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3} : \ \forall n \geq 1$$

Solution:

20. Use mathematical induction to Show that

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = [\frac{n(n+1)}{2}]^2 \quad : \qquad \forall n \geq 1$$

Solution:

21. Use mathematical induction to Show that

$$1 + 5 + 9 + \cdots + (4n - 3) = n(2n - 1) \ : \ \forall n \geq 1$$

22. Use mathematical induction to Show that

$$n < 2^n \quad : \forall n \geq 1$$

Solution: Let $P(n)$ be the proposition ,

$P(n)$: " $n < 2^n$ "

BASIS STEP: $P(1)$? when $n = 1$, $1 < 2^1 = 2$, $\therefore$ $P(1)$ is true

INDUCTIVE STEP: Let k is integer where $k \geq 1$ and assume that $p(k)$ is true ,

$$k < 2^k \qquad (*)$$

Our <u>goal</u> is to show that $p(k+1)$ is also true.

$$k + 1 < 2^{k+1} \qquad (\text{ Our Goal }) \ ?$$

From (*) $\Rightarrow k + 1 < 2^k + 1 < 2^k + 2^k = 2\ 2^k = 2^{k+1}$ where $1 < 2^k$

$\therefore$ $k + 1 < 2^{k+1}$ $\Rightarrow$ So $P(k+1)$ is true .

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23. Use mathematical induction to Show that

$2^n < n!$ for every integer n with $n \geq 4$.

Solution: Let $P(n)$ be the proposition ,

$P(n)$: $2^n < n!$

BASIS STEP: $P(4)$? when $n = 4$, $2^4 = 16 < 4! = 24$, $\Rightarrow 2^4 < 4!$ $\therefore$ $P(4)$ is true

INDUCTIVE STEP: Let k is integer where $k \geq 4$ and assume that $p(k)$ is true ,

$$2^k < k! \qquad (*)$$

Our <u>goal</u> is to show that $p(k+1)$ is also true

$$2^{k+!} < (k+1)! \qquad (\text{ Our Goal }) \ ?$$

$2^{k+1} = 2\ 2^k < 2\ k!$ (from inductive hypothesis *)

$< (k+1)\ k!$ (because $2 < k + 1$)

$= (k+1)!$ (by definition of factorial function)

$\therefore$ $2^{k+1} < (k+1)!$ $\Rightarrow$ So $P(k+1)$ is true . #

24. Use mathematical induction to Show that

$$3^n < n! \quad \text{for every integer } n \text{ with } n \geq 7.$$

Solution:

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25. Use mathematical induction to Show that let $P(n)$ $n! < n^n$

$$n! < n^n \quad \text{for every integer } n \text{ with } n \geq 2.$$

BS: $P(2)$ when $n=2$ $2! < 2^2 \Rightarrow 2 < 4$

Solution:

$P(2)$ is true.

Induc Step: let $k \geq 2$ and assume that $P(k)$ is the

$k! < k^k$ $(*)$. our goal to Show that $P(k+1)$ is also true

$(k+1)(k+1)^k$

from $(*)$

$(k+1)! < (k+1)^{(k+1)}$. $(k+1)! = (k+1)k! < (k+1)k^k$

$< (k+1)(k+1)^k = (k+1)^k$

$\therefore (k+1)! < (k+1)^{k+1}$ $\therefore P(k)$ is true done #

26. Use mathematical induction to Show that

$$2^n \geq n + 12 \quad \text{for every integer } n \text{ with } n \geq 4.$$

Solution: Let $P(n)$ be the proposition ,

$P(n)$: " $2^n \geq n + 12$ "

BASIS STEP: $P(4)$? $when\ n = 4 \Rightarrow 2^4 = 16 \geq 16 = 4 + 12 \Rightarrow \therefore P(4)\ is\ true$.

INDUCTIVE STEP: Let k is integer where $k \geq 4$ and assume that $p(k)$ is true ,

$$2^k \geq k + 12 \qquad (*)$$

Our goal is to show that $p(k+1)$ is also true

$$[\,2^{k+1} \geq k + 13\,] \qquad (\text{ Our Goal })\ ?$$

$2^{k+1} = 2\ 2^k \geq 2\ (k + 12)$ (from inductive hypothesis *)

$= 2k + 24 = k + 13 + (k + 11)$

$\geq k + 13$ (because $k + 11 > 1$ ∵ $k \geq 4$)

$\therefore \quad 2^{k+1} \geq k + 13 \quad \Rightarrow \quad \therefore$ *P(k+1) is true* .

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27. Use mathematical induction to Show that

$$2^n > n^2 \quad \text{for every integer } n \text{ with } n \geq 5.$$

Solution:

28. Use mathematical induction to Show that

$$n^2 > 4 + n \quad \text{for every integer } n \text{ with } n \geq 3.$$

Solution: Let $P(n)$ be the proposition ,

$$P(n): \quad " \ n^2 > 4 + n \ "$$

BASIS STEP: $P(3)$? $when\ n = 3 \Rightarrow 3^2 = 9 > 7 = 4 + 3$

$$\Rightarrow 3^2 > 4 + 3 \qquad \therefore\ p(3)\ is\ true$$

INDUCTIVE STEP: Let k is integer where $k \geq 3$ and assume that $p(k)$ is true ,

$$k^2 > 4 + k \qquad (*)$$

Our goal is to show $p(k+1)$ is also true

$$(k+1)^2 > 4 + (k+1) \qquad \text{(Our Goal) ?}$$

$$\begin{aligned}(k+1)^2 &= k^2 + 2k + 1 \\ &> 4 + k + 2k + 1 && \text{(from inductive hypothesis *)} \\ &> 4 + k + 1 && \text{(because } 2k > 1 : k \geq 3 \text{)}\end{aligned}$$

$$\therefore\ (k+1)^2 > 4 + (k+1) \Rightarrow \quad \therefore\ P(k+1) \text{ is true .}$$

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29. Use mathematical induction to Show that let $P(n): 2^n > n^2+19$

$$2^n > n^2 + 19 \qquad \text{for every integer } n \text{ with } n \geq 6.$$

B.S: let P(6) ?? when $n = 6$

Solution:

$2^6 > 6^2 + 19 \Rightarrow 64 > 55 \Rightarrow P(6)$ is true

Induc step: let $k \geq 6$ and assume that $P(k)$ is true

$2^k > k^2 + 19$ (*). our goal to show that $P(k+1)$ is also true

$2^{k+1} > (k+1)^2 + 19 = k^2 + 2k + 20$ ($1+19$)

$2^{k+1} = 2^1 \, 2^k > 2(k^2+19) = k^2+19+k^2+19$

$> k^2 (2k+1) + 19 > (k+1)^2 + 19$

$\therefore$ so $P(k+1)$ is true

done #

30. Use mathematical induction to Show that

$$n^3 > 2n + 1 \quad \text{for every integer } n \text{ with } n \geq 2.$$

Solution: Let $P(n)$ be the proposition ,

$$P(n): \quad " \ n^3 > 2n + 1 \ "$$

BASIS STEP: $P(2)?$ $when\ n = 2 \Rightarrow$

$$2^3 = 8 > 5 = 4 + 1 = 2(2) + 1 \Rightarrow 2^3 > 2(2) + 1 \ \therefore\ P(2) \ is\ true$$

INDUCTIVE STEP: Let k is integer where $k \geq 2$ and assume that $p(k)$ is true ,

$$k^3 > 2k + 1 \qquad (*)$$

Our goal is to show that $p(k+1)$ is also true

$$(k+1)^3 > 2k + 3 \qquad (\text{ Our Goal }) ?$$

$$\begin{aligned}(k+1)^3 &= k^3 + 3k^2 + 3k + 1 \\ &> 2k + 1 + 3k^2 + 3k + 1 \qquad (\text{ from inductive hypothesis } *) \\ &= 2k + 2 + [3k(k+1)] \\ &> 2k + 2 + 1 = 2k + 3 \qquad (\text{ because } 3k(k+1) > 1 \ \because\ k \geq 2)\end{aligned}$$

$$\therefore\ (k+1)^3 > 2(k+1) + 1 = 2k + 3$$

$$(k+1)^3 > 2k + 3$$

$$\Rightarrow \ \therefore\ P(k+1) \ \text{ is true .}$$

31. Use mathematical induction to Show that

$$\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}} \geq \sqrt{n} \qquad \text{for every integer } n \text{ with } n \geq 1.$$

Solution: Let $P(n)$ be the proposition ,

$$p(n): \ " \frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}} \geq \sqrt{n} \ "$$

BASIS STEP: $p(1)?\ when\ n=1 \Rightarrow \frac{1}{\sqrt{1}}=1 \geq \sqrt{1}=1 \Rightarrow \therefore \quad p(1)\ is\ true$.

INDUCTIVE STEP: Let k is integer where $k \geq 1$ and assume that $p(k)$ is true

$$\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{k}} \geq \sqrt{k} \qquad (*)$$

Our goal is to show $p(k+1)$ is also true

$$\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{k+1}} \geq \sqrt{k+1} \qquad (\text{ Our Goal })\ ?$$

$$\left[\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{k}}\right]+\frac{1}{\sqrt{k+1}} \geq \left[\sqrt{k}\,\right]+\frac{1}{\sqrt{k+1}} \qquad (\text{ from inductive hypothesis } *)$$

$$=\frac{\sqrt{k}\,\sqrt{k+1}+1}{\sqrt{k+1}}$$

$$=\frac{\sqrt{k(k+1)}+1}{\sqrt{k+1}}$$

$$\geq \frac{\sqrt{k(k)}+1}{\sqrt{k+1}}=\frac{\sqrt{k^2}+1}{\sqrt{k+1}} \qquad (\text{because } k+1>k)$$

$$=\frac{k+1}{\sqrt{k+1}}=\sqrt{k+1}$$

$$\therefore \quad \frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{k+1}} \geq \sqrt{k+1}$$

$\therefore \quad p(k+1)$ is true .

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32. Use mathematical induction to prove that let $P(n): 3^{n-1} \geq 2^n + 1$

$$3^{n-1} \geq 2^n + 1 \qquad \text{for all integers } n \geq 3.$$

Solution:

BS: let P(3) when $n=3$. $3^{3-1} \geq 2^3+1 \Rightarrow 9 \geq 9$ so P(3) is true. Induc Step: let $k \geq 3$ and assume that P(k) is true

$3^{k-1} \geq 2^k + 1$ (*) our goal to show that P(k+1) is also true.

$3^{k+1-1} \geq 2^{k+1}+1$. $3^1 \cdot 3^{k-1} \geq 3^1(2^k+1)$ $\xrightarrow{(*)}$

$= 3^1 \cdot 2^k + 3 \geq 2 \cdot 2^k + 1 = 2^{k+1}+1$

so P(k+1) is true done #

33. Use mathematical induction to Show that

$$\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^n} < 1 \qquad \text{for every integer } n \text{ with } n \geq 1.$$

Solution:

34. Use mathematical induction to Show that

$$n^2 - 7n + 12 \geq 0 \qquad \text{for every integer } n \text{ with } n \geq 3.$$

Solution: $P(n)$ be the proposition ,

$$P(n): \quad " \; n^2 - 7n + 12 \geq 0 \; "$$

BASIS STEP: $P(3)$? $when\, n = 3 \Rightarrow$

$$3^2 - 7(3) + 12 = 9 - 21 + 12 = 21 - 21 = 0 \geq 0 \quad \Rightarrow \therefore P(3) \;\; is\ true$$

INDUCTIVE STEP: Let k is integer where $k \geq 3$ and assume that $p(k)$ is true

$$k^2 - 7k + 12 \geq 0 \qquad (*)$$

Our goal is to show $p(k+1)$ is also true

$$(k+1)^2 - 7(k+1) + 12 \geq 0 \qquad \text{(Our Goal) ?}$$

$$(k+1)^2 - 7(k+1) + 12 = k^2 + 2k + 1 - 7k - 7 + 12$$

$$= [\, k^2 - 7k + 12] + 2(k - 3)$$

$$\geq 0 \quad + \quad 0 \qquad \geq 0$$

(from inductive hypothesis *) and

(because $2(k-3) \geq 0 \;:\; k \geq 3$)

$$\therefore \; (k+1)^2 - 7(k+1) + 12 \geq 0$$

$$\Rightarrow \quad \therefore \; P(k+1) \quad \text{is true .}$$

#

35. Use mathematical induction to Show that

$$1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{n^2}<2-\frac{1}{n}$$ for every integer n with $n \geq 2$.

Solution: $P(n)$ be the proposition ,

$$P(n): \quad " \ 1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{n^2}<2-\frac{1}{n} \ "$$

BASIS STEP: $P(2)?$ *when* $n = 2 \Rightarrow 1+\frac{1}{2^2}=1+\frac{1}{4}=\frac{5}{4}<2-\frac{1}{2}=\frac{3}{2}=\frac{6}{4}$

$$\Rightarrow \frac{5}{4}<\frac{6}{4} \qquad \therefore\ P(2) \text{ is true}$$

INDUCTIVE STEP: Let k is integer where $k \geq 2$ and assume that $p(k)$ is true

$$1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{k^2}<2-\frac{1}{k} \qquad (*)$$

Our <u>goal</u> is to show $p(k+1)$ is also true

$$1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{k^2}+\frac{1}{(k+1)^2}<2-\frac{1}{k+1} \qquad \text{(Our Goal) ?}$$

Add the term $\frac{1}{(k+1)^2}$ (the term # n= k+1) to the both sides in (*)

$$\left[1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{k^2}\right]+\frac{1}{(k+1)^2}<\left[2-\frac{1}{k}\right]+\frac{1}{(k+1)^2} \qquad \text{(from inductive hypothesis *)}$$

$$=2-\frac{(k+1)^2}{k(k+1)^2}+\frac{k}{k(k+1)^2}=2-\frac{(k+1)^2-k}{k(k+1)^2}$$

$$=2-\frac{k^2+2k+1-k}{k(k+1)^2}=2-\frac{k^2+k+\cancel{1}}{k(k+1)^2}$$

$$<\ 2-\frac{k^2+k}{k(k+1)^2}=2-\frac{k(K+1)}{k(k+1)^2}=2-\frac{1}{k+1}$$

$$\therefore\ 1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{k^2}+\frac{1}{(k+1)^2}<2-\frac{1}{k+1} \Rightarrow \therefore\ P(k+1) \text{ is true . \#}$$

36. Use mathematical induction to Show that

$$3^n \geq 2^{n+2} \qquad \text{for every integer } n \text{ with } n \geq 4.$$

Solution: Let $P(n)$ be the proposition ,

$$P(n): \quad " \quad 3^n \geq 2^{n+2} \quad "$$

BASIS STEP: $P(4)?\, when\ n = 4 \;\Rightarrow\; 3^4 = 81 \geq 64 = 2^6 = 2^{4+2}$

$$3^4 \geq 2^{4+2} \Rightarrow \therefore\ P(4)\ \ is\ true$$

INDUCTIVE STEP: Let k is integer where $k \geq 4$ and assume that $p(k)$ is true

$$3^k \geq 2^{k+2} \qquad . \quad (*)$$

Our goal is to show $p(k+1)$ is also true

$$3^{k+1} \geq 2^{k+3} \qquad \text{(Our Goal) ?}$$

$$3^{k+1} = 3\ 3^k \geq 3\ 2^{k+2} \qquad \text{(from inductive hypothesis *)}$$

$$\geq 2\ 2^{k+2} = 2^{k+3}$$

$$\therefore \quad 3^{k+1} \geq 2^{k+3} \qquad \Rightarrow \quad \therefore\ P(k+1) \text{ is true .}$$

#

37. Use mathematical induction to Show that

$$n^2 - 3n + 5 \quad \text{is odd} \quad \text{for all nonnegative integers } n.$$

Solution: let $P(n)$: n^2-3n+5 is odd. BS: $P(0)$??

When $n=0$ $0^2-3(0)+5=5$ is odd

$\therefore P(0)$ is true. let $k\geq 0$ and assume $P(k)$ is true. $k^2-3k+5=2h+1 : h\in\mathbb{Z}$ $(*)$

our goal to show that $P(k+1)$ is also true. $(k+1)^2-3(k+1)+5=2c+1$

$\Rightarrow k^2+2k+1-3k-3+5$

$\overset{(*)}{\Rightarrow} (2h+1)+2k-2 = 2h+2k-2+1$

$=2(h+k-1)+1 = 2c+1 : c\in\mathbb{N}$

so $P(k+1)$ is true. odd

done #

38. Use mathematical induction to Show that

$$3|4^n - 1 \qquad \text{for all nonnegative integers } n.$$

Solution: Let $P(n)$ be the proposition ,

$$p(n): \quad "\ 3|4^n - 1\ " \quad \Rightarrow \ \exists c \in \mathbb{Z} : \quad 4^n - 1 = 3c$$

BASIS STEP: $p(0)?\ when\ n = 0 \quad \Rightarrow 4^0 - 1 = 1 - 1 = 0 = 3\,(0) : 0 \in \mathbb{Z}$

$$\Rightarrow 3|4^0 - 1 \ \Rightarrow \ \therefore \quad p(0)\ \ is\ true\ .$$

INDUCTIVE STEP: Let k is integer where $k \geq 0$ and assume that $p(k)$ is true $\Rightarrow$

$$3|4^k - 1$$

$$\Rightarrow \ 4^k - 1 = 3c \ \Rightarrow \ 4^k = 3c + 1 \quad : c \in \mathbb{Z} \quad (*)$$

Our <u>goal</u> is to show that $p(k+1)$ is also true

$$3|4^{k+1} - 1 \qquad (\text{ Our Goal })\ ?$$

$$4^{k+1} - 1 = 4\ 4^k - 1$$

$$= 4\,(\,3c + 1) - 1 \qquad (\text{ from inductive hypothesis } *\ )$$

$$= 12c + 4 - 1 = 12c + 3$$

$$= 3(4c + 1) = 3h \qquad : h = (4c + 1) \in \mathbb{Z}$$

$$\Rightarrow \ 3|4^{k+1} - 1$$

$\therefore \qquad p(k+1)$ is true

#

39. Use mathematical induction to Show that

$$3|(n^3 + 2n) \qquad \text{for all positive integer } n$$

Solution: let $P(n): 3|(n^3+2n)$ BS:

let $P(1)$?? when $n=1$ $3|1^3+2(1)=3|3$

$\therefore P(1)$ is true.

Induc step: let $k\geq 1$ and assume that $P(k)$ is true

$3|k^3+2k \Rightarrow 3c = k^3+2k$ (*) $c\in\mathbb{Z}$. Our goal to show that

$P(k+1)$ is also true $3|(k+1)^3+2(k+1) \Rightarrow 3h = (k+1)^3+2(k+1)$ $h\in\mathbb{Z}$

$(k+1)^3+2(k+1) = k^3+3k^2+3k+1+2k+2$

$\overset{(*)}{\Rightarrow} 3c+3k^2+3k+3 = 3\underbrace{(c+k^2+k+1)}_{h\in\mathbb{Z}}$

$=3h$ $\therefore P(k+1)$ is true

done #

40. Use mathematical induction to Show that

$$5|7^{n} - 2^{n} \qquad \forall \ n \geq 1$$

Solution:

41. Use mathematical induction to Show that

$3|5^n - 2^{n+2}$ for all nonnegative integers n.

Solution: let $P(n)$: $3 \mid 5^n - 2^{n+2}$ BS: $P(0)$?? When $n=0$

$3|5^0 - 2^{0+2} = 3|1-4 = 3|-3 \Rightarrow P(0)$ is true.

Induc Step: let $k \geq 0$ and assume that $P(k)$ is true

$3|5^k - 2^{k+2} \Rightarrow 3c = 5^k - 2^{k+2} \Rightarrow 5^k = 3c + 2^{k+2}$: $c \in \mathbb{Z}$ (*)

Our goal to show that $P(k+1)$ is also

$3|5^{k+1} - 2^{k+3} \Rightarrow 3h = 5^{k+1} - 2^{k+3}$: $h \in \mathbb{Z}$

$= 5^1 \cdot 5^k - 2^1 \cdot 2^{k+2} = 5(3c + 2^{k+2}) - 2 \cdot 2^{k+2}$

$= 3(5c) + 5(2^{k+2}) - 2 \cdot 2^{k+2} = 3(5c) + 3 \cdot 2^{k+2} =$

$= 3\underbrace{(5c + 2^{k+2})}_{h \in \mathbb{Z}} = 3h$

$\therefore P(k+1)$ is also true

done #

42. Use mathematical induction to Show that

$$7 | 9^{2n} - 5^{2n} \qquad \forall \ n \geq 1$$

Solution:

43. Use mathematical induction to prove that $n^3 - n$ is divisible by 3 whenever n is a positive integer .

$$3|n^3 - n \qquad \forall\ n \geq 1$$

Solution:

44. Use mathematical induction to prove that $n(n^2+5)$ is divisible by 6 whenever n is a positive integer .

$$6 \mid n(n^2+5) \qquad \forall\ n \geq 1$$

Solution:

45. Use mathematical induction to prove that $n^3 - n + 3$ is divisible by 3 whenever n is a nonnegative integer .

$$3 | n^3 - n + 3 \qquad \forall \ n \geq 0$$

Solution:

46. Use mathematical induction to Show that

$$5 | 2^{2n-1} + 3^{2n-1} \qquad \forall\ n \geq 1$$

Solution:

47. Use mathematical induction to Show that

$$2|n^2 + n \qquad \forall \ n \geq 0$$

Solution:

48. Use mathematical induction to prove that $n^5 - n$ is divisible by 5 whenever n is a nonnegative integer .

$$5 | n^5 - n \qquad \forall \; n \geq 0$$

Solution:

49. Prove that 21 divides $4^{n+1} + 5^{2n-1}$ whenever n is a positive integer .

Solution:

50. Prove that if n is a positive integer, then 133 divides $11^{n+1} + 12^{2n-1}$

Solution:

51. Use mathematical induction to prove that $7^{n+2} + 8^{2n+1}$ is divisible by 57 for every nonnegative integer n.

Solution: