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King Saud University

College of Science

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151 Math Exercises

(4.3)

Partial Order

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Def.(1) R is a relation defined on the set $A \neq \phi$

R is Partial order

① R **reflexive** ✓

② R is **antisymmetric**

$a, b \in A$:

$$\left.\begin{array}{l} aRb \\ \& \wedge \\ bRa \end{array}\right\} \Rightarrow a = b$$

$$\text{or } R \left.\begin{array}{l} aRb \\ \text{But} \\ b \not R a \end{array}\right\} \Rightarrow R \text{ is antisymm}$$

Ex. $2 < 3$ but $3 \not< 2$

$\Downarrow$

$<$ is antisymm.

③ R is **Transitive** ✓

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(2) R is **total** order:

(i) R is Partial order & (ii) $\forall a, b \in A$: $a \neq b$

(1) aRb or $bRa \Rightarrow$ a, b are comparable.

Partial Orderings

DEFINITION 1 A relation R on a non empty set S is called *a partial ordering* or *partial order* if it is reflexive, antisymmetric, and transitive. A set S together with a partial ordering R is called a *partially ordered set*, or *poset*, and is denoted by (S,R). Members of S are called *elements* of the poset.

DEFINITION 2 Let a and b *be in the set S. Assume* (S,R) is a poset. We say that a and b are *comparable* if either $a\ R\ b$ or $b\ R\ a$. When a and b are elements of S such that neither $a\ R\ b$ nor $b\ R\ a$, a and b are called *incomparable*.

DEFINITION 3 If (S,R) is a poset and every two elements of S are comparable, S is called a ***totally ordered*** or ***linearly ordered set***, and R is called a ***total order*** or a ***linear order***. A totally ordered set is also called a ***chain***.

Hasse Diagrams

If (S,R) is a partially ordered set, we can represent this group in a schematic format called (Hasse diagram), where S is a finite set, as follows:

We represent each element of S in a small circle and if a < b we place b above a and connect between them by a straight line segment, ignoring the cut of the lines we automatically get by means of a transitive property.

For example, if $(a < b) \wedge (b < c)$ and there is no x such that $a < x < b$,also there is no y such that $b < y < c$, then we get a straight line segment between a and b and between b and c but we do not get between a and c.

Example 1. Let R be a relation defined on the set $\mathbb{Z}^+ : a,b \in \mathbb{Z}^+$, $a\ R\ b \Leftrightarrow a \mid b$

(i) Show that R is a partial ordering relation (poset) on $\mathbb{Z}^+$.

(ii) In case R is defined on $\mathbb{Z}$, is R still a partial ordering relation on $\mathbb{Z}$, why?

(iii) Draw the Hasse diagram representing the partial ordering relation R on the set $A = \{1,2,3,4,6,8,12,24\}$

(iv) Decide whether R is totally ordering relation on $\mathbb{Z}^+$, why?

(v) In case R is defined on $B = \{1,3,9,27,81\}$, is R a total ordering relation on B, why?

(vi) Draw the Hasse diagram representing the totally ordering relation R on the set $B = \{1,3,9,27,81\}$

Solution: (i)

1- $\forall a \in \mathbb{Z}^+ \;,\; a|a \;\Rightarrow \therefore a\,R\,a \;\Rightarrow\; R$ is *reflexive* .

2- $a, b \in \mathbb{Z}^+ \;,\; a\,R\,b \Rightarrow a|b \Rightarrow b = m_1 a : m_1 \in \mathbb{Z}^+$

$b\,R\,a \Rightarrow b|a \Rightarrow a = m_2 b : m_2 \in \mathbb{Z}^+$

$(\times) \quad \Rightarrow \overline{\qquad\qquad\qquad}$

$ab = m_1 m_2\, ab$

$\div ab \Rightarrow 1 = m_1 m_2 \;\Rightarrow \therefore\; m_1 = m_2 = 1 \;\Rightarrow \therefore\; a = b \;\Rightarrow \therefore\;$ R is *antisymmetric*

3- $a, b, c \in \mathbb{Z}^+ \;,\; a\,R\,b \Rightarrow a|b \Rightarrow b = m_1 a : m_1 \in \mathbb{Z}^+$

$b\,R\,c \Rightarrow b|c \Rightarrow c = m_2 b : m_2 \in \mathbb{Z}^+$

$(\times) \quad \Rightarrow \overline{\qquad\qquad\qquad}$

$bc = m_1 m_2\, ab$

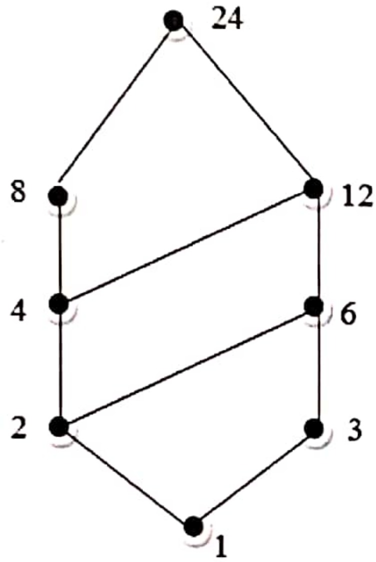
$\Rightarrow \therefore\;$ R *is transitive*

$\because$ R is reflexive , *antisymmetric* and *transitive* $\therefore$ R is partial ordering .

(ii) $\because 3, -3 \in \mathbb{Z} \;,\; : 3|(-3) \;\wedge\; -3|3$ *but* $3 \neq -3 \;\Rightarrow\; R$ is not *antisymmetric*

$\therefore$ R is not a *partial ordering* on $\mathbb{Z}$.

(iii)
$$R = \left\{ \begin{array}{c} (1,1),(2,2),(3,3),(4,4),(6,6),(8,8),(12,12),(24,24), \\ (1,2),(1,3),(1,4),(1,6),(1,8),(1,12),(1,24), \\ (2,4),(2,6),(2,8),(2,12),(2,24) \\ (3,6),(3,12),(3,24),(4,8),(4,12), \\ (6,12),(6,24),(8,24),(12,24) \end{array} \right\}$$



Hasse diagram of R

(iv) $3, 5 \in \mathbb{Z}^+$, $\because 3 \nmid 5 \wedge 5 \nmid 3 \Rightarrow$ $3, 5$ *incomparable* $\Rightarrow \therefore$ R is not a total ordering

(v) $\because$ *all elements in* B *are powers of* $3 \Rightarrow \forall a, b \in B$, $a|b$ or $b|a$

(B, R) *is comparable* $\Rightarrow \therefore (B, R)$ is *totally ordered.*

(iv) $$R = \left\{ \begin{matrix} (1,1), (3,3), (9,9), (27,27), (81,81), (1,3), (1,9), (1,27), (1,81) \\ (3,9), (3,27), (3,81), (9,27), (9,81), (27,81) \end{matrix} \right\}$$

Hasse diagram of R (chain)

Example 2. Let R be a relation defined on the set $\mathbb{Q}^+$:

$$a,b \in \mathbb{Q}^+ \quad , \; a\,R\,b \;\Leftrightarrow\; \frac{a}{b} \in \mathbb{Z}^+$$

(i) Show that R is a partial ordering relation on $\mathbb{Q}^+$.

(ii) Decide whether R is totally ordering relation on $\mathbb{Q}^+$, why?

(iii) Draw the Hasse diagram representing the partial ordering relation R on the set

$$A = \{\tfrac{1}{6}, \tfrac{1}{3}, \tfrac{1}{2}, 1,2,3,6\}$$

Solution: (*i*)

(1) $\forall a \in \mathbb{Q}^+ , \frac{a}{a} = 1 \in \mathbb{Z}^+ \Rightarrow \therefore\; a\,R\,a \;\Rightarrow\; \therefore$ R is reflexive

(2) $a,b \in \mathbb{Q}^+ \;,\; a\,R\,b \Leftrightarrow \frac{a}{b} = m_1 \in \mathbb{Z}^+ \;\&\; b\,R\,a \Leftrightarrow \frac{b}{a} = m_2 \in \mathbb{Z}^+$

$(\times) \Rightarrow \frac{a}{b} \times \frac{b}{a} = m_1 m_2 \Rightarrow 1 = m_1 m_2 \Rightarrow m_1 = m_2 = 1 \Rightarrow \therefore a = b$

$\therefore$ *R is antisymmetric*

(3) $a,b,c \in \mathbb{Q}^+, a\,R\,b \Leftrightarrow \frac{a}{b} = m_1 \in \mathbb{Z}^+ \;\&\; b\,R\,c \Leftrightarrow \frac{b}{c} = m_2 \in \mathbb{Z}^+$

$(\times) \Rightarrow \frac{a}{b} \times \frac{b}{c} = m_1 m_2 \Rightarrow \frac{a}{c} = m_1 m_2 = m \Rightarrow \therefore aRc : m = m_1 m_2 \in \mathbb{Z}^+$

$\therefore$ *R is transitive*

$\because$ *R is reflexive , antisymmetric* and *transitive*

$\therefore$ R is partial ordering relation .

(*ii*) $3\,,5 \in \mathbb{Q}^+ , \frac{3}{5} \notin \mathbb{Z}^+ \wedge \frac{5}{3} \notin \mathbb{Z}^+ \Rightarrow 3\,,5$ *incomparable* $\Rightarrow \therefore$ *R is not totally ordering relation .*

(*iii*)

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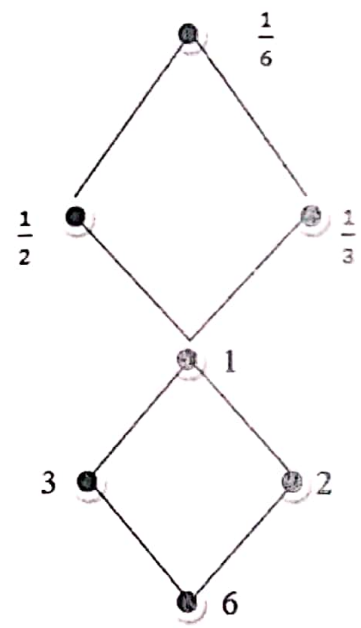
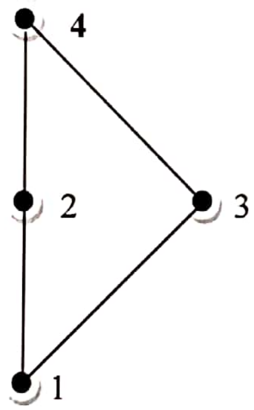


Hasse diagram of R

Example 3. Draw the Hasse diagram representing the partial ordering relation

$$R = \{(1,1),(2,2),(3,3),(4,4),(1,2),(1,3),(1,4),(2,4),(3,4)\}$$

on the set $A = \{1,2,3,4\}$



1. Let R be a relation defined on the set $\mathbb{Z}^+ = \mathbb{N} = \{1,2,3\ldots\} : a,b \in \mathbb{Z}^+ = \mathbb{N}$, $a\,R\,b \Leftrightarrow a \mid b$

✓(i) Show that R is a partial order relation on $\mathbb{N}$.

(ii) In case R is defined on $\mathbb{Z}^*$, is R still a partial order relation on $\mathbb{Z}$, why?

(iii) Draw the Hasse diagram representing the partial order relation R on the set $A=\{1,2,3,4,6,8,12,24\}$

(iv) Decide whether R is total order relation on $\mathbb{Z}^+ = \mathbb{N}$, why?

(v) In case R is defined on $B = \{1,3,9,27,81\}$, is R a total order relation on B, why?

(vi) Draw the Hasse diagram representing the total order relation R on the set $B = \{1,3,9,27,81\}$

Sol. (i) ① $\forall a \in \mathbb{Z}^+,\ a \mid a \Rightarrow \therefore aRa \Rightarrow R$ is refl.

② $a,b \in \mathbb{Z}^+$: $aRb \Rightarrow a \mid b \Rightarrow b = k_1 a$

$\wedge$ $\quad : k_1, k_2 \in \mathbb{N} = \mathbb{Z}^+$

$bRa \Rightarrow b \mid a \Rightarrow a = k_2 b$

$\Rightarrow \cancel{a} = k_2 k_1 \cancel{a} \Rightarrow k_1 . k_2 = 1 \Rightarrow \therefore k_1 = k_2 = 1$

$\Rightarrow \therefore \boxed{a=b} \Rightarrow R$ is antisymm.

③ $a,b,c \in \mathbb{Z}^+ = \mathbb{N}$: $aRb \Rightarrow a \mid b \Rightarrow b = h_1 a$

$\wedge$ $\quad : h_1, h_2 \in \mathbb{N}$

$bRc \Rightarrow b \mid c \Rightarrow c = h_2 b$

$\Rightarrow c = \underbrace{h_2 h_1}_{h \in \mathbb{N}} a = ha \Rightarrow \therefore a \mid c \Rightarrow aRc$

$\therefore R$ is transitive.

① & ② & ③ $\Rightarrow \therefore R$ is Partial Order.

(iii) $-3, 3 \in \mathbb{Z}^*$

$$\left.\begin{array}{l} -3R3: -3 \mid 3 \\ \wedge \\ 3R-3: 3 \mid -3 \end{array}\right\} \text{but } -3 \neq 3.$$

$\Downarrow$

$\therefore$ R is not antisymm.

$\Downarrow$

$\therefore$ R is not Partial order.

(iv) R is Partial order on $\mathbb{Z}^+ = \mathbb{N}$.

$3, 4 \in \mathbb{Z}^+ = \mathbb{N}$, $3 \nmid 4 \wedge 4 \nmid 3$

$3 \neq 4$

$$\Rightarrow \underbrace{3 \not R 4 \wedge 4 \not R 3}$$

$\Downarrow$

$3, 4$ incomparable.

$\Downarrow$

$\therefore$ R is not total order on $\mathbb{Z}^+$.

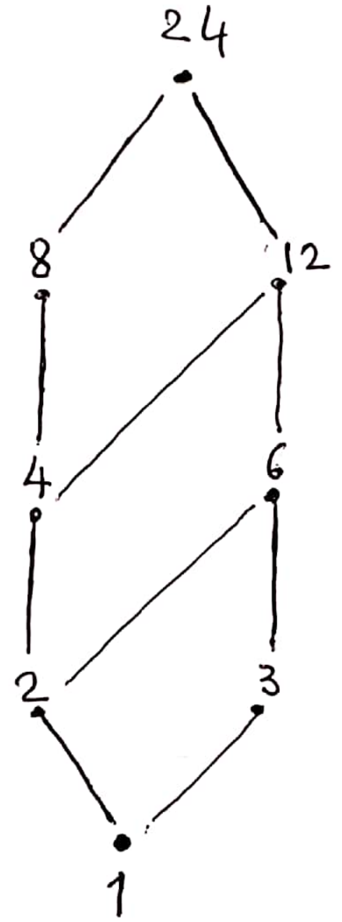
$A = \{1, 2, 3, 4, 6, 8, 12, 24\}$ divisors of 24.

(iii) $R = \{(1,1), (2,2), (3,3), (4,4), (6,6), (8,8), (12,12), (24,24)$

$(1,2), (1,3), (1,4), (1,6), (1,8), (1,12), (1,24), (2,4), (2,6)$

$(2,8), (2,12), (2,24), (3,6), (3,12), (3,24), (6,12)$

$(6,24), (8,24), (12,24), (4,8), (4,12), (4,24)\}$.



Hasse Diagram for R on A.

(v) $B = \{\overset{3^0}{1}, \overset{3^1}{3}, \overset{3^2}{9}, \overset{3^3}{27}, \overset{3^4}{81}\}$ = divisiors of 81.

R is Partial order on B and every element in B is a power of 3

$\Rightarrow \therefore \forall a, b \in B \Rightarrow a \mid b \lor b \mid a$.

(B, R) Comparable. $\Rightarrow \therefore R$ is total order on B.

(vi) $R = \{(1,1), (3,3), (9,9), (27,27), (81,81), (1,3), (1,9), (1,27), (1,81), (3,9), (3,27), (3,81), (9,27), (9,81), (27,81)\}$

(chain)
Total order

Hasse Diagram

Exercises

1. Let R be a relation defined on the set $A = \{1,2,3,4,5,6,7,8,9,10\}$

$$a, b \in A \quad , \quad a\,R\,b \Leftrightarrow a \mid b$$

(i) Show that R is a partial ordering relation on A .

(ii) Decide whether R is totally ordering relation on A, why?

(iii) Draw the Hasse diagram representing the partial ordering relation R on the set A

Solution (i) ① $\forall a \in A,\ a \mid a \Rightarrow \therefore aRa \Rightarrow R$ is refl.

② $a, b \in A$: $aRb \Rightarrow a \mid b \Rightarrow b = K_1 a$

& $bRa \Rightarrow b \mid a \Rightarrow a = K_2 b$: $K_1, K_2 \in \mathbb{N}$

$\Rightarrow \cancel{a} = K_2 . K_1 \cancel{a} \Rightarrow K_1 K_2 = 1 \Rightarrow \therefore K_1 = K_2 = \boxed{1}$

$\Rightarrow \therefore \boxed{a = b} \Rightarrow R$ is antisymm.

③ $a, b, c \in A$: $aRb \Rightarrow a \mid b \Rightarrow b = h_1 a$

& $bRc \Rightarrow b \mid c \Rightarrow c = h_2 b$: $h_1, h_2 \in \mathbb{N}$

$\Rightarrow c = \underbrace{h_2 . h_1}_{h \in \mathbb{N}} a \Rightarrow c = ha \Rightarrow a \mid c \Rightarrow aRc$

$\therefore R$ is transitive.

R is: Refl, antisymm, and transitive

$\therefore R$ is partial order.

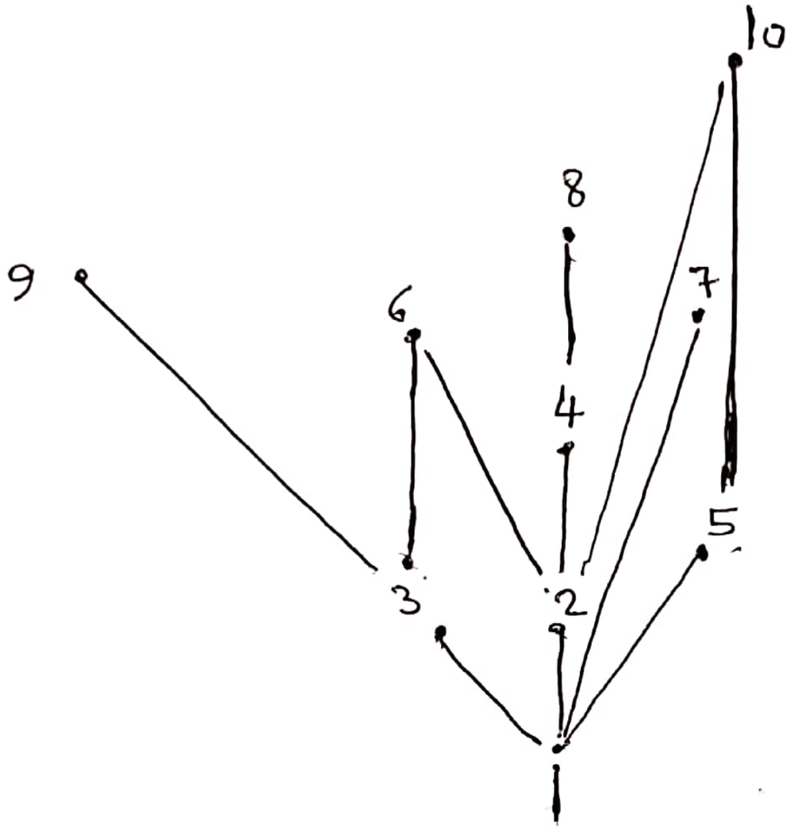
(ii) $5, 7 \in A$, $(5 \neq 7)$, $5 \nmid 7 \Rightarrow 5 \not R 7$ & $7 \nmid 5 \Rightarrow 7 \not R 5$ $\Big\}$ $\therefore 5, 7$ are incomparable.

$\Downarrow$

$\therefore R$ is not total order.

iiii)

$R = \{(1,1),(2,2),(3,3),\ldots,(10,10),(1,2),(1,3),(1,4),(1,5),$
$(1,6),(1,7),(1,8),(1,9),(1,10),(2,4),(2,6),(2,8),(2,10),$
$(3,6),(3,9),(4,8),(5,10)\}$



Hasse diagram.

2. Let R be a relation defined on the set $\mathbb{Q}^+$:

$$a,b \in \mathbb{Q}^+ \ , a\,R\,b \Leftrightarrow \frac{a}{b} \in \mathbb{Z}^+ , \quad \frac{a}{b} = m \in \mathbb{Z}^+ = \mathbb{N}$$

(i) Show that R is a partial order relation on $\mathbb{Q}^+$.

(ii) Decide whether R is total order relation on $\mathbb{Q}^+$, why?

(iii) Draw the Hasse diagram representing the partial order relation R on the set

$$A = \{\frac{1}{6}, \frac{1}{3}, \frac{1}{2}, 1,2,3,6\}$$

Sol. (i) ① $\forall a \in \mathbb{Q}^+, \ \frac{a}{a} = 1 \in \mathbb{Z}^+ = \mathbb{N} \Rightarrow \therefore aRa \Rightarrow$ R is refl.

② $a,b \in \mathbb{Q}^+ : aRb \Rightarrow \frac{a}{b} = m_1 \in \mathbb{Z}^+$

$\wedge$

$bRa \Rightarrow \frac{b}{a} = m_2 \in \mathbb{Z}^+$.

(*)

$$\frac{a}{b} \cdot \frac{b}{a} = 1 = m_1 \cdot m_2 \Rightarrow \therefore m_1 = m_2 = 1.$$

$$\Rightarrow \frac{a}{b} = 1 \Rightarrow \therefore a = b \Rightarrow R \text{ is antisymm.}$$

③ $a,b,c \in \mathbb{Q}^+ : aRb \Rightarrow \frac{a}{b} = k_1 \in \mathbb{Z}^+$

$\wedge$

$bRc \Rightarrow \frac{b}{c} = k_2 \in \mathbb{Z}^+$.

(*)

$$\frac{a}{\cancel{b}} \cdot \frac{\cancel{b}}{c} = \frac{a}{c} = k_1 \cdot k_2 = k \in \mathbb{Z}^+ \Rightarrow \therefore aRc$$

$\Rightarrow \therefore R$ is transitive.

① & ② & ③ $\Rightarrow R$ is Partial order on $\mathbb{Q}^+$

(ii) We have R is Partial order on $\mathbb{Q}^+$.

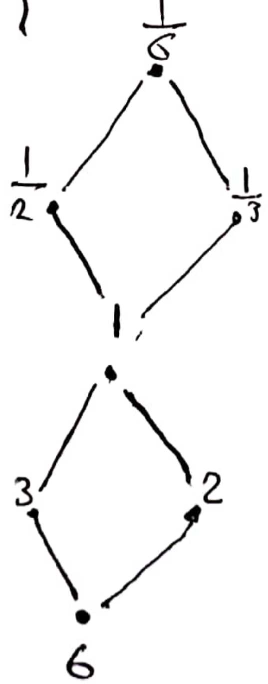
$3, 5 \in \mathbb{Q}^+$: $\frac{3}{5} \notin \mathbb{Z}^+ \Rightarrow 3 \not R 5$
$3 \neq 5$ $\quad \wedge$
$\frac{5}{3} \notin \mathbb{Z}^+ \Rightarrow 5 \not R 3$

$\Big\}$ 3, 5 incomparable $\Downarrow$

$\therefore R$ is not total order on $\mathbb{Q}^+$.

(iii) $A = \{\frac{1}{6}, \frac{1}{3}, \frac{1}{2}, 1, 2, 3, 6\}$

$R = \{(\frac{1}{6}, \frac{1}{6}), (\frac{1}{3}, \frac{1}{3}), (\frac{1}{2}, \frac{1}{2}), (1,1), (2,2), (3,3), (6,6),$
$(6,3), (6,2), (6,1), (6, \frac{1}{2}), (6, \frac{1}{3}), (6, \frac{1}{6}), (3,1), (3, \frac{1}{2})$
$(3, \frac{1}{3}), (3, \frac{1}{6}), (2,1), (2, \frac{1}{2}), (2, \frac{1}{3}), (2, \frac{1}{6}), (1, \frac{1}{2}), (1, \frac{1}{3})$
$(1, \frac{1}{6}), (\frac{1}{2}, \frac{1}{6}), (\frac{1}{3}, \frac{1}{6})\}$



Hasse Diagram

3. Let R be a relation defined on the set $\mathbb{N} = \{1,2,3,\dots\}$:

$$a, b \in \mathbb{N}\,,\quad a\,R\,b \Leftrightarrow \frac{b}{a} = 2^k \;:\; k \in \{0,1,2,\dots\}$$

(i) Show that R is a partial ordering relation on $\mathbb{N}$.

(ii) Decide whether R is totally ordering relation on $\mathbb{N}$, why?

(iii) Draw the Hasse diagram representing the partial ordering relation R on the set $A = \{1,2,3,\dots,12\}$

Sol. (i) ① $\forall a \in \mathbb{N},\ \frac{a}{a} = 1 = 2^0 \Rightarrow aRa \Rightarrow R$ is refl.

② $a, b \in \mathbb{N}:\ aRb \Rightarrow \frac{b}{a} = 2^{k_1}$ $: k_1, k_2 \in \{0,1,2,\dots\}$

$\wedge$

$bRa \Rightarrow \frac{a}{b} = 2^{k_2}$

(*) -- ?

$\frac{b}{a} \cdot \frac{a}{b} = 1 = 2^{k_1} \cdot 2^{k_2} \Rightarrow 2^0 = 2^{k_1+k_2} \Rightarrow$

$k_1 + k_2 = 0 \Rightarrow \therefore k_1 = k_2 = 0 \Rightarrow \frac{a}{b} = 2^0 = 1$

$\Rightarrow \therefore a = b \Rightarrow \therefore R$ is antisymm.

③ $a, b, c \in \mathbb{N}:\ aRb \Rightarrow \frac{b}{a} = 2^{h_1}$ $: h_1, h_2 \in \{0,1,2,\dots\}$

$\wedge$

$bRc \Rightarrow \frac{c}{b} = 2^{h_2}$

(*) --

$\frac{\cancel{b}}{a} \cdot \frac{c}{\cancel{b}} = 2^{h_1} \cdot 2^{h_2} \Rightarrow \frac{c}{a} = 2^{h_1+h_2} = 2^{h}$ $: h_1 + h_2 = h \in \{0,1,2,\dots\}$

$\Rightarrow \therefore aRc \Rightarrow R$ is transitive.

①&② & ③ $\Rightarrow R$ is partial order on $\mathbb{N}$.

(ii) We have R is Partial order on $\mathbb{N}$.

$$3,5 \in \mathbb{N}, \quad \frac{5}{3} \neq 2^K \wedge \frac{3}{5} \neq 2^K : K \in \{0,1,2,\dots\}$$

$$\Rightarrow 3 \not R 5 \wedge 5 \not R 3$$

$$\Downarrow$$

3,5 incomparable.

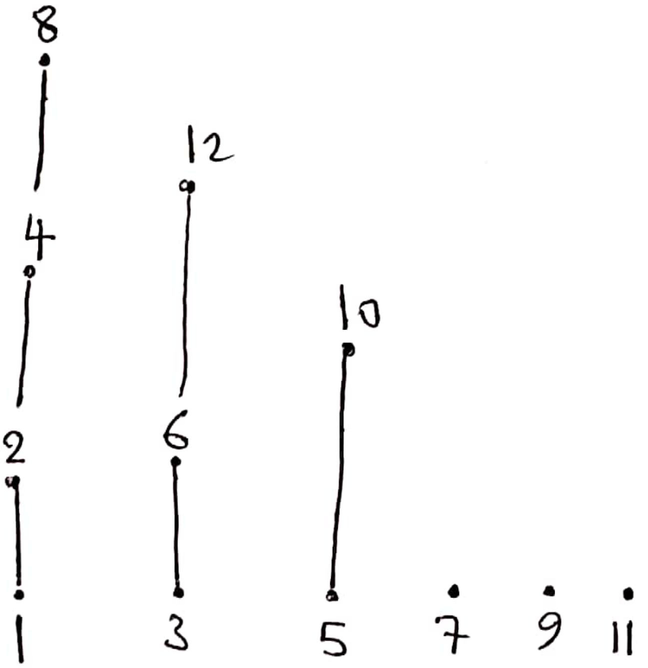
$$\Downarrow$$

$\therefore R$ is not total order.

(iii) $R = \{(1,1),(2,2),(3,3),\dots,(12,12),$

$(1,2),(1,4),(1,8),(2,4),(2,8),(3,6),(3,12),$

$(4,8),(5,10),(6,12)\}$



H.W **5.** Let T be a relation defined on the set $\mathbb{Z}^* = \{\dots, -2, -1, 1, 2, \dots\}$:

$$a, b \in \mathbb{Z}^*, a\ T\ b \Leftrightarrow \frac{a}{b} = 3^k \quad : k \in \{0,1,2,\dots\}$$

(i) Show that T is a partial order relation on $\mathbb{Z}^*$.

(ii) Decide whether T is total order relation on $\mathbb{Z}^*$, why?

(iii) Draw the Hasse diagram representing the partial order relation T on the set $A = \{-27, -18, -9, -6, -3, 1, 2, 3, 6, 9\}$.

Solution :

4. Let T be a relation defined on the set $\mathbb{Z}$:

$$x, y \in \mathbb{Z}\,, \quad x\,T\,y \Leftrightarrow x - y = 2k \qquad : k \in \{0,1,2,\ldots\}$$

(i) Show that T is a partial ordering relation on $\mathbb{Z}$.

(ii) Decide whether T is totally ordering relation on $\mathbb{Z}$, why?

(iii) Draw the Hasse diagram representing the partial ordering relation T on the set $A = \{0,1,2,3\}$.

Solution: (i) ① $\forall x \in \mathbb{Z}\,,\ x - x = 0 = 2(0) \Rightarrow \therefore xTx \Rightarrow T$ is refl.

② $x, y \in \mathbb{Z} : xTy \Rightarrow x - y = 2K_1$

$: K_1, K_2 \in \{0,1,2,\ldots\}$

$\wedge$

$yTx \Rightarrow y - x = 2K_2$?

(+) ———————————

$0 = 2(K_1 + K_2) \Rightarrow K_1 + K_2 = 0 \Rightarrow K_1 = K_2 = 0$

$\Rightarrow x - y = 0 \Rightarrow \therefore \boxed{x = y} \Rightarrow \therefore T$ is antisymm.

③ $x, y, z \in \mathbb{Z} : xTy \Rightarrow x - \cancel{y} = 2h_1$

$: h_1, h_2 \in \{0,1,2,\ldots\}$

$\wedge$

$yTz \Rightarrow \cancel{y} - z = 2h_2$

(+) ———————————

$x - z = 2(h_1 + h_2) = 2h \ :\ h_1 + h_2 = h \in \{0,1,2,\ldots\}$

$\Rightarrow \therefore xTz \Rightarrow \therefore T$ is transitive.

① & ② & ③ $\Rightarrow \therefore T$ is Partial order.

(ii) $4, 5 \in \mathbb{Z}\,,\ 4 - 5 = -1$ (odd) $,\ 4 \not T 5$

$\wedge$

$5 - 4 = 1$ (odd) $,\ 5 \not T 4$

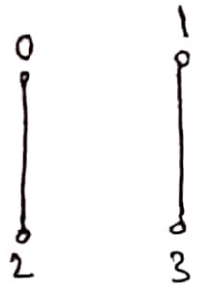
$\Big\}$ 4, 5 incomparable

$\Downarrow$

$\therefore T$ is not total order.

(iii)

$T = \{(0,0), (1,1), (2,2), (3,3), (2,0), (3,1)\}$.



Hasse Diagram.

6. Let T be a relation defined on the set $\mathbb{N} = \{1,2,3,\dots\}$:

$$x, y \in \mathbb{N}, \quad x\,T\,y \Leftrightarrow x = y^k \quad : k \in \{0,1,2,\dots\}$$

(i) Show that T is a partial ordering relation on $\mathbb{N}$.

(ii) Decide whether T is totally ordering relation on $\mathbb{N}$, why?

(iii) Draw the Hasse diagram representing the partial ordering relation T on the set $A = \{1,2,3,4\}$.

Solution : (i) ① $\forall x \in \mathbb{N},\ x = x^1 \Rightarrow \therefore x T x \Rightarrow T$ is refl.

② $x, y \in \mathbb{N} : xTy \Rightarrow x = y^{k_1}$ $\quad : k_1, k_2 \in \{0,1,2,\dots\}$

$\wedge$

$yTx \Rightarrow y = x^{k_2}$

$\Rightarrow y^1 = (y^{k_1})^{k_2} = y^{k_1 k_2} \Rightarrow \therefore k_1 k_2 = 1$

$\Rightarrow \therefore k_1 = k_2 = 1 \Rightarrow x = y \Rightarrow T$ is antisymm.

③ $x, y, z \in \mathbb{N} : xTy \Rightarrow x = y^{h_1}$

$\wedge$ $\quad \Big\} : h_1, h_2 \in \{0,1,2,\dots\}$

$yTz \Rightarrow y = z^{h_2}$

By subst. $\Rightarrow x = (z^{h_2})^{h_1} = z^{h_1 h_2} = z^h \; ; h_1 h_2 = h \in \{0,1,2,\dots\}$

$\Rightarrow \therefore xTz \Rightarrow T$ is transitive.

①&②&③ $\Rightarrow \therefore T$ is Partial order.

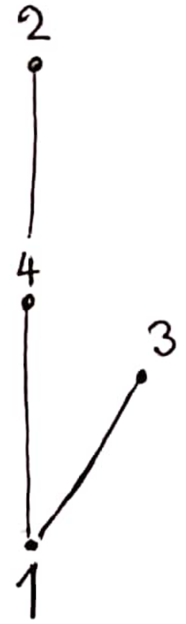
(ii) $2, 3 \in \mathbb{N}$, $\quad 2 \not{T} 3 : 2 \neq 3^k \quad : k \in \{0,1,2,\dots\}$

$\wedge \qquad\qquad \wedge$

$3 \not{T} 2 : 3 \neq 2^k$

$\therefore 2, 3$ incomparable $\Rightarrow \therefore T$ is not total order.

(iii) $T = \{(1,1),(2,2),(3,3),(4,4),(1,2),(1,3),(1,4),(4,2)\}$



Hasse Diagram.

13. Let T be a relation defined on the set $\mathbb{Z}^* = \{\dots, -2, -1, 1, 2, \dots\} = \mathbb{Z} - \{0\}$

H.W.

$$x, y \in \mathbb{Z}^*, \quad x\,R\,y \Leftrightarrow x = y^{2k+1} \quad : k \in \{0,1,2,\dots\}$$

(i) Show that T is a partial ordering relation on $\mathbb{Z}^*$.

(ii) Decide whether T is totally ordering relation on $\mathbb{Z}^*$, why?

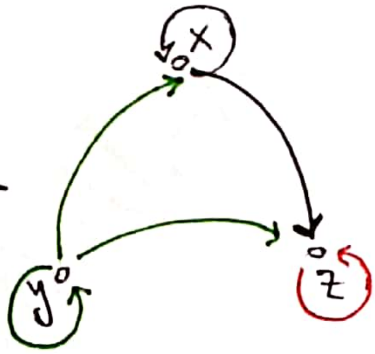
(iii) Draw the Hasse diagram representing the partial ordering relation T on the set $A = \{1,2,3,4,8,27\}$.

Solution : (i)

Exam

7. Let $T = \{(x,x), (x,z), (y,x), (y,y), (y,z), (z,z)\}$ be a relation defined on the set $B = \{x, y, z\}$
   - (i) Represent the relation T by diagram .
   - (ii) Show that T is a partial order relation on B .
   - (iii) Decide whether T is total order relation on A. why?
   - (iv) Draw the Hasse diagram representing the partial order relation T on the set B .

(i)



(ii) ① $(x,x), (y,y)$ and $(z,z) \in T$

$\therefore T$ is refl.

② $\because (x,z) T$, but $(z,x) \notin T$
$\wedge$ $(y,z) \in T$, but $(z,y) \notin T$
and $(y,x) \in T$, but $(x,y) \notin T$ $\Bigg\} \Rightarrow \therefore T$ is antisymm.

③ $\because \left.\begin{matrix}(y,x) \in T \\ \wedge (x,z) \in T\end{matrix}\right\}$ and $(y,z) \in T \Rightarrow T$ is transitive.

① & ② & ③ $\Rightarrow \therefore T$ is Partial order.

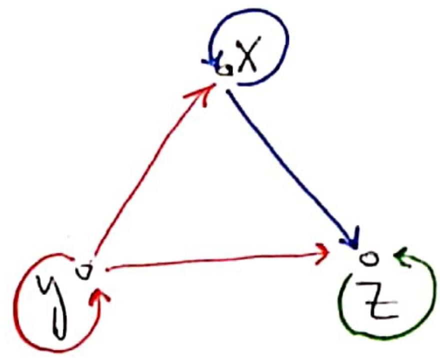
(ii) $\because (y,x) \wedge (x,z)$ and $(y,z) \in T \Rightarrow \therefore B = \{x, y, z\}$

(iii) (B, T) is Comparable $\Rightarrow \therefore T$ is total order.

Hasse diagram

#7]

(i)



(ii) ① $\because (x,x), (y,y)$ and $(z,z) \in T$

$\therefore T$ is refl.

② $\left.\begin{array}{l} \because (x,z) \in T, \text{ but } (z,x) \notin T \\ \wedge\ (y,x) \in T, \text{ but } (x,y) \notin T \\ \wedge\ (y,z) \in T, \text{ but } (z,y) \notin T \end{array}\right\} \therefore T$ is antisymm.

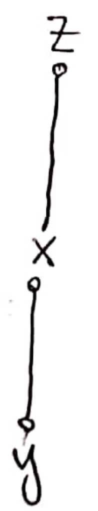
③ $\because (y,x) \wedge (x,z)$ and $(y,z) \in T \Rightarrow T$ is transitive.

① & ② & ③ $\Rightarrow \therefore T$ is Partial order.

(iii) $\left.\begin{array}{l} \because (x,z) \in T \Rightarrow xTz \\ (y,x) \in T \Rightarrow yTx \\ (y,z) \in T \Rightarrow yTz \end{array}\right\} (B,T)$ Comparable.

$\Downarrow$

T is total order.



Hasse diagram (chain).

8. Let S be a relation defined on the set $\mathbb{N} = \{1,2,3,\dots\}$:

$$x, y \in \mathbb{N}\ ,\ x\,S\,y \Leftrightarrow \frac{x}{y} \text{ is odd } \text{integer}^{+ve}.$$

(i) Show that S is a partial order relation on $\mathbb{N}$.

(ii) Decide whether S is total order relation on $\mathbb{N}$, why?

(iii) Draw the Hasse diagram representing the partial order relation S on the set $A = \{1,2,6\}$.

Solution: (i) ① $\forall x \in \mathbb{N},\ \frac{x}{x} = 1$ (odd) $\Rightarrow \therefore xSx \Rightarrow S$ is refl.

② $x, y \in \mathbb{N}$: $xSy \Rightarrow \frac{x}{y} = m_1$ (odd) $\in \mathbb{N}$

$\wedge$

$ySx \Rightarrow \frac{y}{x} = m_2$ (odd) $\in \mathbb{N}$

(*) ――――――――

$\frac{x}{y} \cdot \frac{y}{x} = m_1 \cdot m_2 \Rightarrow m_1 \cdot m_2 = 1 \Rightarrow \therefore m_1 = m_2 = 1$

$\Rightarrow \therefore \frac{x}{y} = 1 \Rightarrow \boxed{x = y} \Rightarrow S$ is antisymm.

③ $x, y, z \in \mathbb{N}$

fi

$xSy \Rightarrow \frac{x}{y} = 2K_1 + 1$: $K_1, K_2 \in \mathbb{N}$.

$\wedge$

$ySz \Rightarrow \frac{y}{z} = 2K_2 + 1$

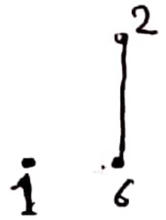
(*) ――――――――

$\frac{x}{\cancel{y}} \cdot \frac{\cancel{y}}{z} = (2K_1+1)(2K_2+1) \Rightarrow \frac{x}{z} = 4K_1K_2 + 2K_1 + 2K_2 + 1$

$\Rightarrow \frac{x}{z} = 2(\underbrace{2K_1K_2 + K_1 + K_2}_{\| K \in \mathbb{N}}) + 1 = 2K + 1$ (odd) $\Rightarrow \therefore xSz$

$\therefore S$ is transitive.

(iii) $S = \{(1,1), (2,2), (6,6), (6,2)\}$



Hasse diagram.

#8] (i) ① $\forall x \in \mathbb{N}$, $\frac{x}{x} = 1$ (odd) $\Rightarrow \therefore xTx \Rightarrow T$ is refl.

② $x, y \in \mathbb{N}$: $xTy \Rightarrow \frac{x}{y} = m_1$ (odd) $\in \mathbb{N}$.

$\wedge$

$yTx \Rightarrow \frac{y}{x} = m_2$ (odd) $\in \mathbb{N}$.

(*) ─────────────

$\frac{x}{y} \cdot \frac{y}{x} = m_1 \cdot m_2 \Rightarrow 1 = m_1 \cdot m_2 \Rightarrow \therefore m_1 = m_2 = 1$

$\Rightarrow \frac{x}{y} = 1 \Rightarrow \boxed{x = y} \Rightarrow T$ is antisymm.

③ $x, y, z \in \mathbb{N}$: $xTy \Rightarrow \frac{x}{y} = 2k+1 \in \mathbb{N}$

; $k, h \in \mathbb{N}$ ~~$\{0,1,2,\dots\}$~~.

$\wedge$

$yTz \Rightarrow \frac{y}{z} = 2h+1 \in \mathbb{N}$

(*) ─────────────

$\frac{x}{\cancel{y}} \cdot \frac{\cancel{y}}{z} = (2k+1)(2h+1)$.

$\Rightarrow \frac{x}{z} = \underbrace{4kh + 2k + 2h}{} + 1 = 2(\underbrace{2kh + k + h}_{H \in \mathbb{N}}) + 1$

$\frac{x}{z} = 2H+1$ (odd) $\in \mathbb{N} \Rightarrow xTz \Rightarrow$

T is transitive.

① & ② & ③ $\Rightarrow T$ is Partial order on $\mathbb{N}$.

(ii) $3, 6 \in \mathbb{N}$: $3 \not T 6$, $\frac{3}{6} = \frac{1}{2}$ (not odd) $\}$ 3, 6 incomparable

$\wedge$ $\Downarrow$

$6 \not T 3$, $\frac{6}{3} = 2$ (even) $\}$ T is not total order

8. Let S be a relation defined on the set $\mathbb{N} = \{1,2,3,\dots\}$:

$$x, y \in \mathbb{N}\ ,\ x\,S\,y \Leftrightarrow \frac{x}{y} \text{ is odd integer.}$$

(i) Show that S is a partial order relation on $\mathbb{N}$.

(ii) Decide whether S is total order relation on $\mathbb{N}$, why?

(iii) Draw the Hasse diagram representing the partial order relation S on the set $A = \{1,2,6\}$.

Solution: (i)

① $\forall x \in \mathbb{N}\ ,\ \frac{x}{x} = 1 \text{ (odd)} \Rightarrow \therefore xSx \Rightarrow S$ is refl.

② $x, y \in \mathbb{N}:\ xSy \Rightarrow \frac{x}{y} = m_1 \text{ (odd)} \in \mathbb{N}$

$\wedge$

$ySx \Rightarrow \frac{y}{x} = m_2 \text{ (odd)} \in \mathbb{N}$

$(*)$

$\frac{x}{y} \cdot \frac{y}{x} = 1 = m_1 \cdot m_2 \Rightarrow \therefore m_1 = m_2 = \boxed{1}$

$\Rightarrow \frac{x}{y} = 1 \Rightarrow \boxed{x = y} \Rightarrow \therefore S$ is antisymmetric.

③ $x, y, z \in \mathbb{N}:\ x\,Sy \Rightarrow \frac{x}{y} = h_1 \text{ (odd)} \in \mathbb{N}.$

$\wedge$

$ySz \Rightarrow \frac{y}{z} = h_2 \text{ (odd)} \in \mathbb{N}$

$(*)$

$\frac{x}{\cancel{y}} \cdot \frac{\cancel{y}}{z} = \frac{x}{z} = h_1 \cdot h_2 \stackrel{\checkmark}{=} h \text{ (odd)} \in \mathbb{N}$

$\therefore \frac{x}{z} = h \text{ (odd)} \in \mathbb{N} \Rightarrow \therefore xRz \Rightarrow R$ is transitive

① & ② & ③ $\Rightarrow$ S is Partial order.

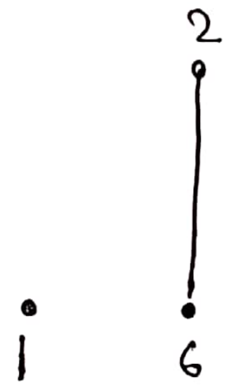
(ii) we have S is Partial order on $\mathbb{N}$

$3, 6 \in \mathbb{N}$, $\frac{6}{3} = 2$ (even) $\Rightarrow 6 \not{S} 3$

$3 \neq 6$ and $\frac{3}{6} = \frac{1}{2}$ (not odd integer) $\Rightarrow 3 \not{S} 6$

$\therefore$ 3, 6 incomparable $\Rightarrow \therefore S$ is not total order on $\mathbb{N}$.

(iii) $S = \{(1,1), (2,2), (6,6), \underline{(6,2)}\}$



Hasse diagram.

H.W.

6. Let T be a relation defined on the set $\mathbb{N} = \{1,2,3,\dots\}$:

$$x, y \in \mathbb{N}\,,\quad x\,T\,y \Leftrightarrow x = y^k \quad : k \in \{0,1,2,\dots\}$$

(i) Show that T is a partial ordering relation on $\mathbb{N}$.

(ii) Decide whether T is totally ordering relation on $\mathbb{N}$, why?

(iii) Draw the Hasse diagram representing the partial ordering relation T on the set $A = \{1,2,3,4\}$.

Solution : (i)

1- $\forall x \in \mathbb{N}\,,\ x = x^1 \Rightarrow \therefore\ x\,T\,x \Rightarrow \therefore\quad T$ is *reflexive*

2- $x, y \in \mathbb{N}\,,\ x\,T\,y \Leftrightarrow x = y^{k_1} \quad : k_1 \in \{0,1,2,\dots\}$

$\wedge$

$y\,T\,x \Leftrightarrow y = x^{k_2} \quad : k_2 \in \{0,1,2,\dots\}$

(*by substitution*) $\Rightarrow\ x^1 = x^{k_1 k_2} \Rightarrow k_1 k_2 = 1 \Rightarrow k_1 = k_2 = 1$

$\Rightarrow x = y \Rightarrow \therefore\quad T$ *is antisymmetric*

3- $x, y, z \in \mathbb{N}\,,\ x\,T\,y \Leftrightarrow x = y^{k_1} \quad : k_1 \in \{0,1,2,\dots\}$

$\wedge$

$y\,T\,z \Leftrightarrow y = z^{k_2} \quad : k_2 \in \{0,1,2,\dots\}$

(*by substitution*) $\Rightarrow\ x = z^{k_1 k_2} \Rightarrow x = z^k \quad : k_1 k_2 = k \in \{0,1,2,\dots\}$

$\Rightarrow x\,T\,z \Rightarrow \therefore\quad T$ *is transitive*

$\because\ T$ is *reflexive*, *antisymmetric* and *transitive*

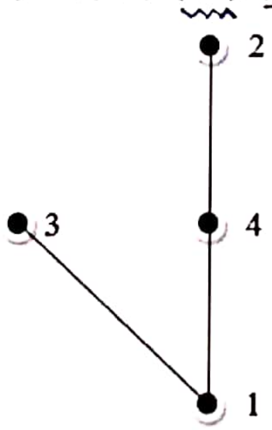
$\therefore\ T$ *is partial ordering relation on* $\mathbb{N}$.

(ii) $2\,,\,5 \in \mathbb{N}\,,\ 2 \neq 5^k \ \wedge\ 5 \neq 2^k \ : K \in \{0,1,2,\dots\}$

$2 \not{T} 5 \ \wedge\ 5 \not{T} 2$

$\therefore 2\,,5$ *incomparable* $\Rightarrow \therefore\ T$ *is not totally ordering relation .*

(iii) $T = \{(1,1), \dots, (4,4), (1,2), (1,3), (1,4)(4,2)\}$



Hasse diagram of T

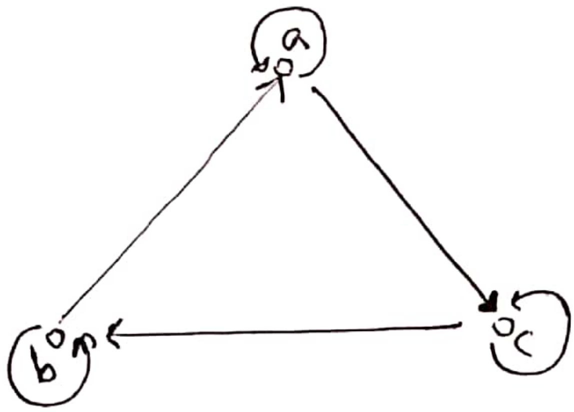
$1\,T\,2 : 1 = 2^0$

$1\,T\,3 : 1 = 3^0$

$1\,T\,4 : 1 = 4^0$

$4\,T\,2 : 4 = 2^2$

Exam



Q. Decide whether R is Partial order?

Sol.

$R = \{(a,a), (b,b), (c,c), (b,a), (a,c), (c,b)\}$.

R is not transitive: $(a,c) \wedge (c,b) \in R$ But $(a,b) \notin R$.

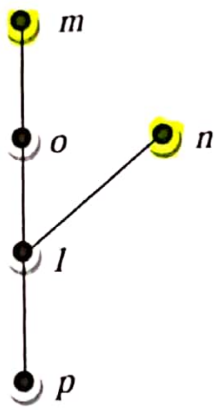
$\Downarrow$

R is not Partial order.

9. Let T be a partial order relation defined on the set $C = \{l, m, n, o, p\}$ shown in the given Hasse diagram

(i) List all ordered pairs of T.

(ii) Decide whether T is totally order relation on C, why?

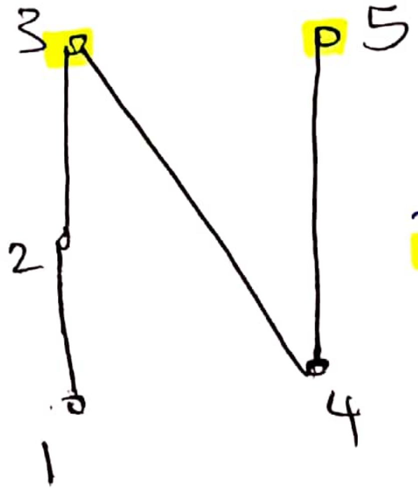


Sol. (i) $T = \{(l,l), (m,m), (n,n), (o,o), (p,p)$
$(p,l), (p,o), (p,m), (p,n), (l,n), (l,o), (l,m)$
$(o,m)\}$

(ii) $(m,n) \wedge (n,m) \notin T \Rightarrow$
$\therefore$ m, n incomparable $\Rightarrow \therefore T$ is not total order.

Exam.

Let T be a partial order on $B = \{1, 2, 3, 4, 5\}$ shown in the given Hasse diagram.



ملاحظة:

"مخطط هاسي: هو مخطط متسلسل صاعداً من الأسفل إلى الأعلى دوماً"

(i)

$$T = \{(1,1), (2,2), (3,3), (4,4), (5,5), (1,2), (1,3), (2,3), (4,3), (4,5)\}.$$

(ii) $(3,5) \wedge (5,3) \notin T \Rightarrow 3, 5$ incomparable

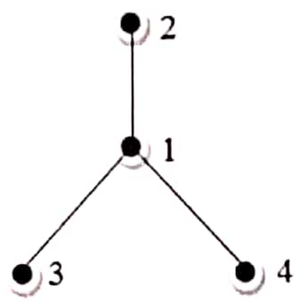
$\Downarrow$

$\therefore T$ is not a total order.

10. Let S be a partial ordering relation defined on the set $A = \{1,2,3,4\}$ shown in the given Hasse diagram

(i) List all ordered pairs of S.

(ii) Decide whether S is totally ordering relation on A. why?



Solution:

$$S = \{(1,1),(2,2),(3,3),(4,4),(3,1),(4,1),(1,2),(3,2),(4,2)\}$$

$\because\ (3,4) \wedge (4,3) \notin S \quad \therefore 3\ ,4\ \text{incomparable} \Rightarrow \therefore\ S\ \text{is not totally ordering relation}.$

? Exam

#: Let $P = \{(1,1),(1,3),(2,1),(2,2),(2,3),(3,3),(4,1),(4,3),(4,4)\}$

H.W.

be a partial order on the set $A = \{1,2,3,4\}$.

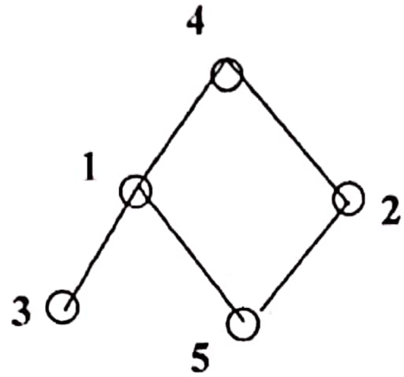
(i) Draw the Hasse diagram of P.

(ii) Decide whether P is a total order.

23. Let T be a partial ordering relation defined on the set $A = \{1,2,3,4,5\}$ shown in the given Hasse diagram

(i) List all ordered pairs of T.

(ii) Decide whether T is totally ordering relation on A, why?



24. Let R be a relation defined on the set $\mathbb{Z}^+ = \{1,2,3,\dots\}$:

$$m,n \in \mathbb{Z}^+ , \ m\,R\,n \Leftrightarrow m = n^a \quad : a \in \{0,1,2,\dots\}$$

(i) Show that R is a partial ordering relation on $\mathbb{Z}^+$.

(ii) Decide whether R is totally ordering relation on $\mathbb{Z}^+$, why?

(iii) Draw the Hasse diagram representing the partial ordering relation R on the set $A = \{2,3,4,5,6,7,8,9\}$