King Saud University

College of Science

Department of Mathematics

151 Math Exercises

(3,3)

Methods of Proof

"Mathematical Induction"

(STRONG INDUCTION)

Malek Zein AL-Abidin

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2021

Exercises

1. Assume $\{a_n\}_{n=1}^{\infty}$ is a sequence defined as:

$$a_1 = 3\,,\quad a_2 = 6\quad ,\qquad a_n = a_{n-1} + a_{n-2}\ (*)\qquad :\forall n \geq 3$$

Prove that $3|\,a_n$ for all positive integers n , $\forall n \geq 1$

Solution: Let $P(n)$ be the proposition , $P(n):\ 3|\,a_n\ ,\ \Rightarrow a_n = 3c : c \in \mathbb{N}$

BASIS STEP: *When* $n = 1 \Rightarrow 3|\ a_1\ :\ a_1 = 3 = 3(1)\ \Rightarrow\ \therefore\ P(1)$ *is true* .

When $n = (2) \Rightarrow\ 3|\ a_2\ :\ a_2 = 6 = 3(2)\ \Rightarrow\ \therefore\ P(2)$ *is true* .

INDUCTIVE STEP: Let $k \geq 2$ and assume that

$P(1), P(2), \ldots, P(k-2), P(k-1)\,,\,P(k)$ All are true . (**)

Our goal is to show that $P(\,k+1\,)$ *is also true* ?

$$a_{k+1} = 3c\ \Rightarrow\ 3|\,a_{k+1}\qquad (\text{ our goal })\ ??$$

from $(*)\ \Rightarrow\qquad a_{k+1} = a_k + a_{k-1}\qquad (***)$

$\because\ P(k)\ \&\ P(k-1)$ both are true, (from inductive hypothesis **) $\Rightarrow$

from $P(k) \Rightarrow\ 3|\,a_k\ \Rightarrow a_k = 3c_1 : c_1 \in \mathbb{N}$

from $P(k-1) \Rightarrow\ 3|\,a_{k-1}\ \Rightarrow a_{k-1} = 3c_2\ :\ c_2 \in \mathbb{N}$

$\xRightarrow{\text{by subist.into}(***)}$

$$a_{k+1} = a_k + a_{k-1} = 3c_1 + 3c_2 = 3(c_1 + c_2) = 3c$$

$$: c = (c_1 + c_2) \in \mathbb{N}$$

$\therefore\ a_{k+1} = 3c\ \Rightarrow\ 3|\,a_{k+1} \Rightarrow \therefore\ P(\,k+1\,)$ is true . #

2. Assume $\{a_n\}_{n=1}^{\infty}$ is a sequence defined as: let $P(n)$: a_n is even

$$a_1 = 8\,,\ a_2 = 4\,,\ a_n = a_{n-1} + a_{n-2} \ :\forall n \geq 3 \quad (*)$$

BS: when $n=1, n=2$

Prove that a_n is even for $\forall n \geq 1$.

Solution: $a_1 = 8$, $a_2 = 4$ so that $P(1) \wedge P(2)$ are both true

Ind step: let $k \geq 2$ ∧ assume that $P(1), P(2), \dots, P(k-1), P(k)$ all are true (**)

our goal to show that $P(k+1)$ is also true

a_{k+1} is even

$(*) \underset{(k+1)}{\Rightarrow} a_{k+1} = a_k + a_{k-1}$ (***)

∵ $P(k) \wedge P(k-1)$ both are true from (**)

from $P(k) \wedge P(k-1) \Rightarrow a_k = 2c_1$ (even), $a_{k-1} = 2c_2$ (even) ∴ $c_1, c_2 \in \mathbb{N}$

$$(***) \Rightarrow a_{k+1} = 2c_1 + 2c_2 = 2\underbrace{(c_1 + c_2)}_{c \in \mathbb{N}} = 2c$$

is even ∴ $P(k+1)$ is true

done #

11. Assume $\{a_n\}_{n=0}^{\infty}$ is a sequence defined as:

$$a_0 = 0 \;,\quad a_1 = 2 \;,\quad \boxed{a_n = 4a_{n-1} - 3a_{n-2} \quad (*)} \qquad : \forall n \geq 2$$

Prove that $a_n = 3^n - 1$ for all integers n. $\forall n \geq 0$

Solution:

Let $P(n)$: "$a_n = 3^n - 1$"

BASIS STEP: *When* $n = 0 \Rightarrow$ $3^0 - 1 = 1 - 1 = 0$ then P(0), P(1) are true

When $n = (1) \Rightarrow$ $3^1 - 1 = 3 - 1 = 2$

INDUCTIVE STEP: Let $k \geq 1$ and assume that

$P(1), P(2), \dots, P(k-2), P(k-1)\,, P(k)$ All are true . (**)

Our goal is to show that $P(k+1)$ *is also true* ? $3^{k+1} - 1$??

from (*) $\Rightarrow$ $a_{k+1} = 4a_k - 3a_{k-1}$

$\because$ $P(k)$ & $P(k-1)$ both are true, (from inductive hypothesis **) $\Rightarrow$

from $P(k) \Rightarrow$ $a_k = 3^k - 1 \xRightarrow{4\times} 4a_k = 4\cdot 3^k - 4$ ①

from $P(k-1) \Rightarrow$ $a_{k-1} = 3^{k-1} - 1 \xRightarrow{\times 3} -3a_{k-1} = -3\cdot 3^{k-1} + 3 = -3^k + 3$ ②

by subist.into(***) [illegible] [illegible]

① + ② $\Rightarrow$ $a_{k+1} = 4\cdot 3^k - 4 - 3^k + 3$

$= \boxed{3\cdot 3^k} - 1 = 3^{k+1} - 1$

so P(k+1) is true

done #

12. Assume $\{a_n\}_{n=0}^{\infty}$ is a sequence defined as:

$$a_0 = 9\,,\ a_1 = 15\ ,\ a_2 = 3 \quad \boxed{a_n = \frac{a_{n-1}\,a_{n-2}\,a_{n-3}}{9} + 6} : \forall n \geq 3\ \ (*)$$

Prove that $3|a_n$ for all integers n. $\forall n \geq 0$

Solution:

Let $P(n)$: " $3 \mid a_n$ "

<u>*BASIS STEP*</u>: *When* $n = 0 \Rightarrow 3 \mid a_0 : a_0 = 9 = 3(3) \Rightarrow \therefore P(0)$ is true

When $n = 1 \Rightarrow 3 \mid a_1 : a_1 = 15 = 3(5) \Rightarrow \therefore P(1)$ is true

When $n = (2) \Rightarrow 3 \mid a_2 : a_2 = 3 = 3(1) \Rightarrow P(2)$ is true

<u>*INDUCTIVE STEP*</u>: Let $k \geq 2$ and assume that

$P(1), P(2), \dots, P(k-2), P(k-1)\ , P(k)$ All are true . (**)

Our goal is to show that $P(\,k+1\,)$ *is also true* ? $\Rightarrow 3 \mid a_{k+1}$

from (*) $\Rightarrow a_{k+1} = \dfrac{(a_k)(a_{k-1})(a_{k-2})}{9} + 6$

$\because$ $P(k)\ , P(k-1)$ & $P(k-2)$ all are true, (from inductive hypothesis **) $\Rightarrow$

from $P(k) \Rightarrow 3 \mid a_k \Rightarrow 3c_1 = a_k\ ;\ c_1 \in \mathbb{Z}$

from $P(k-1) \Rightarrow 3 \mid a_{k-1} \Rightarrow 3c_2 = a_{k-1}\ ;\ c_2 \in \mathbb{Z}$

from $P(k-2) \Rightarrow 3 \mid a_{k-2} \Rightarrow 3c_3 = a_{k-2}\ ;\ c_3 \in \mathbb{Z}$

by subist.into(***) $\Longrightarrow$

$$\underline{a_{k+1}} = \frac{3c_1 \cdot \cancel{3}c_2 \cdot \cancel{3}c_3}{\cancel{9}} + 6$$

$$= 3c_1c_2c_3 + 6 = 3(\underbrace{c_1c_2c_3 + 2}_{c \in \mathbb{N}}) = 3c$$

$\Rightarrow 3 \mid a_{k+1}$ So $P(k+1)$ is true done #

16. Assume $\{u_n\}_{n=1}^{\infty}$ is a sequence defined as:

$$u_1 = 2\,,\ u_2 = 4 \quad , \quad u_n = \frac{2u_{n-1} + u_{n-2} + 8}{3} \quad : \forall n \geq 3 \quad (*)$$

Prove that $u_n = 2n$ for all positive integers n. $\forall n \geq 1$

Solution:

Let $P(n)$: " $u_n = 2n$ "

BASIS STEP: *When* $n = 1 \Rightarrow u_1 = 2(1) = 2$

When $n = (2) \Rightarrow u_2 = 2(2) = 4$ $\Big] \to$ so $P(1), P(2)$ all are true

INDUCTIVE STEP: Let $k \geq 2$ and assume that

$P(1), P(2), \dots, P(k-2), P(k-1)\,, P(k)$ All are true (**)

Our goal is to show that $P(k+1)$ *is also true* ? $u_{k+1} = 2(k+1)$

from (*) $\Rightarrow u_{k+1} = \dfrac{2u_k + u_{k-1} + 8}{3}$

$\because$ $P(k)$ & $P(k-1)$ both are true, (from inductive hypothesis **) $\Rightarrow$

from $P(k) \Rightarrow u_k = 2k \overset{\times 2}{\Longrightarrow} 2u_k = 4k$

from $P(k-1) \Rightarrow u_{k-1} = 2(k-1) = 2k - 2$

by subist.into(***) $\Longrightarrow$ $u_{k+1} = \dfrac{4k + 2k - 2 + 8}{3}$

$= \dfrac{6k + 6}{3} = \dfrac{6}{3}(k+1) = 2(k+1)$

$\Rightarrow u_{k+1} = 2(k+1)$. so $P(k+1)$ is true done #

17. Assume $\{a_n\}_{n=0}^{\infty}$ is a sequence defined as:

$$a_0 = 1 \,, \quad a_1 = 2 \,, \; a_2 = 3 \,, \; a_n = a_{n-1} + a_{n-2} + 2a_{n-3} \; : \forall n \geq 3 \; (*)$$

Prove that $a_n \leq 3^n$ for all integers $n \geq 0$.

Solution:

Let $P(n)$:

<u>*BASIS STEP*</u>: *When* $n = 0 \;\Rightarrow$

$When\, n = 1 \;\Rightarrow$

$When\, n = (2) \Rightarrow$

<u>*INDUCTIVE STEP*</u>: Let $k \geq 2$ and assume that

$P(1), P(2), \ldots, P(k-2), P(k-1) \,, P(k)$ All are true . (**)

$Our\ goal\ is\ to\ show\ that\ P(\,k+1\,)\ is\ also\ true$?

from (*) $\Rightarrow$

$\because$ $P(k) \,, P(k-1)$ & $P(k-2)$ all are true, (from inductive hypothesis **) $\Rightarrow$

from $P(k) \Rightarrow$

from $P(k-1) \Rightarrow$

from $P(k-2) \Rightarrow$

$\xRightarrow{by\ subist.into (***)}$

18. Assume $\{a_n\}_{n=0}^{\infty}$ is a sequence defined as:

$$a_0 = 1\,, \quad a_1 = 2 \quad , \quad a_n = 4a_{n-1} - 4a_{n-2} \quad : \forall n \geq 2 \quad (*)$$

Prove that $a_n = 2^n$ for all integers $n \geq 0$.

Solution:

Let $P(n)$: "$a_n = 2^n$"

<u>*BASIS STEP*</u>*: When* $n = 0 \;\Rightarrow$

$When\; n = (1) \Rightarrow$

<u>*INDUCTIVE STEP*</u>*:* Let $k \geq 1$ and assume that

$P(1), P(2), \ldots, P(k-2), P(k-1)\,, P(k)$ All are true . (**)

Our goal is to show that $P(\,k+1\,)$ *is also true* ?

from (*) $\Rightarrow$

$\because$ $P(k)\; \&\; P(k-1)$ both are true, (from inductive hypothesis **) $\Rightarrow$

from $P(k) \Rightarrow$

from $P(k-1) \Rightarrow$

$\xRightarrow{by\ subist.into(***)}$

6. Assume $\{a_n\}_{n=1}^{\infty}$ is a sequence defined as:

$$a_1 = 1\,,\ a_2 = 5\ ,\quad a_{n+1} = 2a_n + 3\,a_{n-1}\quad : \forall n \geq 2 \qquad (*)$$

Prove that $3^n \leq a_{n+1} \leq 2\,.\,3^n$ for all positive integers n.

Solution:

Let $P(n)$: " $3^n \leq a_{n+1} \leq 2\cdot 3^n$ "

<u>*BASIS STEP*</u>: *When* $n = 1$ $\Rightarrow$ $3^1 \leq a_{1+1} \leq 2\cdot 3^1 \Rightarrow 3 \leq a_2 = 5 \leq 6 \Rightarrow P(1)$ is true

~~*When* $n = (2) \Rightarrow$~~ a_3 [illegible]

<u>*INDUCTIVE STEP*</u>: Let $k \geq \cancel{2}\ 1$ and assume that

$P(1), P(2), \ldots, P(k-2), P(k-1)\ , P(k)$ All are true . (**)

Our goal is to show that $P(\,k+1\,)$ *is also true* ?

$$3^{k+1} \leq a_{k+2} \leq 2\cdot 3^{k+1}$$

from (*) $\Rightarrow$

$$a_{k+2} = 2a_{k+1} + 3a_k \qquad (***)$$

$\because$ $P(k)$ & $P(k-1)$ both are true, (from inductive hypothesis **) $\Rightarrow$

from $P(k) \Rightarrow$ $3^k \leq a_{k+1} \leq 2\cdot 3^k \overset{\times 2}{\Rightarrow} 2\cdot 3 \leq 2a_{k+1} \leq 4\cdot 3^k$

from $P(k-1) \Rightarrow$ $3^{k-1} \leq a_k \leq 2\cdot 3^{k-1} \overset{\times 3}{\Rightarrow} 3^1\cdot 3^{k-1} \leq 3a_k \leq 2\cdot 3\cdot 3^{k-1}$

$\Rightarrow 3^k \leq 3a_k \leq 2\cdot 3^k$

$\xRightarrow{\textit{by subist.into}(***)}$

$① + ② \Rightarrow 2\cdot 3^k + 3^k \leq 2a_{k+1} + 3a_k \leq 4\cdot 3^k + 2\cdot 3^k$

$3\cdot 3^k \leq a_{k+2} \leq 6\cdot 3^k = 2\cdot 3\cdot 3^k$

$$3^{k+1} \leq a_{k+2} \leq 2\cdot 3^{k+1}$$

so $P(k+1)$ is true done #

9. Assume $\{a_n\}_{n=0}^{\infty}$ is a sequence defined as:

$$a_0 = 1\,,\ a_1 = 1\ ,\ \ a_n = 2a_{n-1} + a_{n-2}\ :\forall n \geq 2 \quad (*)$$

Prove that a_n is odd for all integers $n \geq 0$

Solution:

Let $P(n)$: " a_n is odd "

<u>*BASIS STEP*</u>: *When* $n = 0 \Rightarrow a_0 = 1$ is odd so $P(0)$ is true

When $n = (1) \Rightarrow a_1 = 1$ is odd so $P(1)$ is true

<u>*INDUCTIVE STEP*</u>: Let $k \geq 1$ and assume that

$P(1), P(2), \dots, P(k-2), P(k-1)\,, P(k)$ All are true (**)

Our goal is to show that $P(\,k+1\,)$ *is also true* : a_{k+1} is odd

from (*) $\Rightarrow a_{k+1} = 2a_k + a_{k-1}\ (***)$

$\because$ $P(k)\ \&\ P(k-1)$ both are true, (from inductive hypothesis **) $\Rightarrow$ (**)

from $P(k) \Rightarrow a_k = 2h+1 : h \in \mathbb{Z} \Rightarrow 2a_k = 4h+2$

from $P(k-1) \Rightarrow a_{k-1} = 2L+1 : L \in \mathbb{Z}$

$\xRightarrow{\text{by subist.into}(***)}$ $a_{k+1} = 4h+2L+2+1$

$= 2\underbrace{(2h+L+1)}_{M \in \mathbb{Z}}+1 = 2M+1$ is odd

So $P(k+1)$ is true done#