Chapter 15

Functions of Several Variables (I)

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15.1

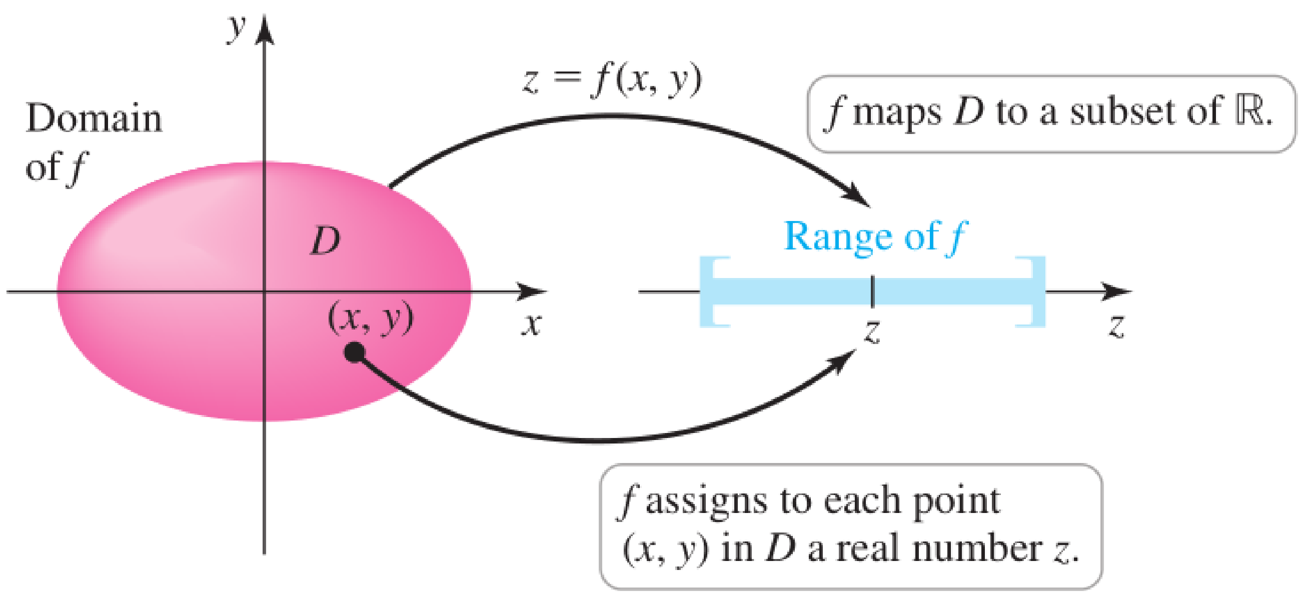
Graphs and Level Curves

Functions of Two Variables

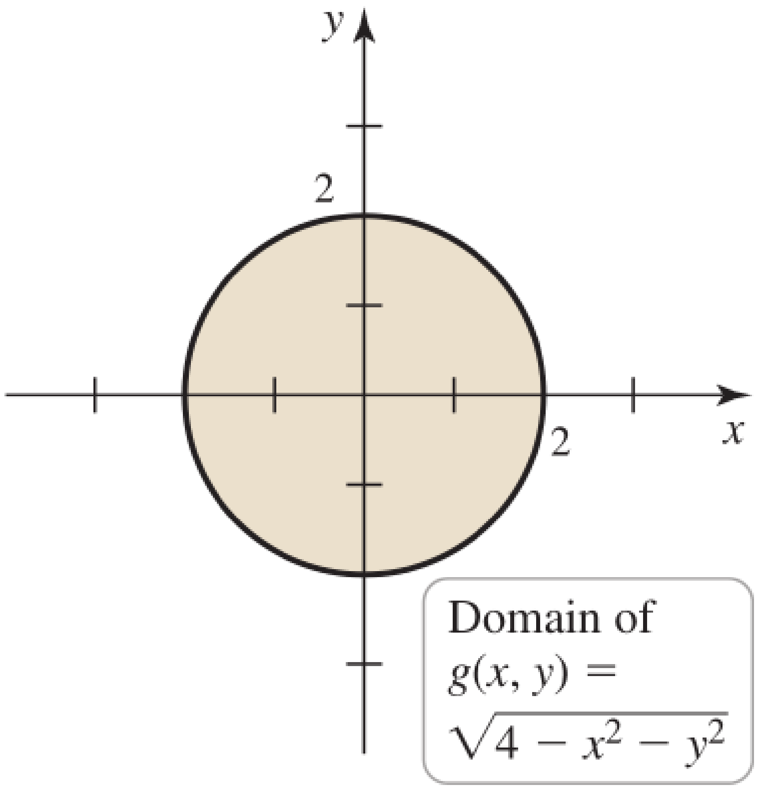
Explicitly: $z = f(x, y)$, or implicitly: $F(x, y, z) = 0$

DEFINITION Function, Domain, and Range with Two Independent Variables

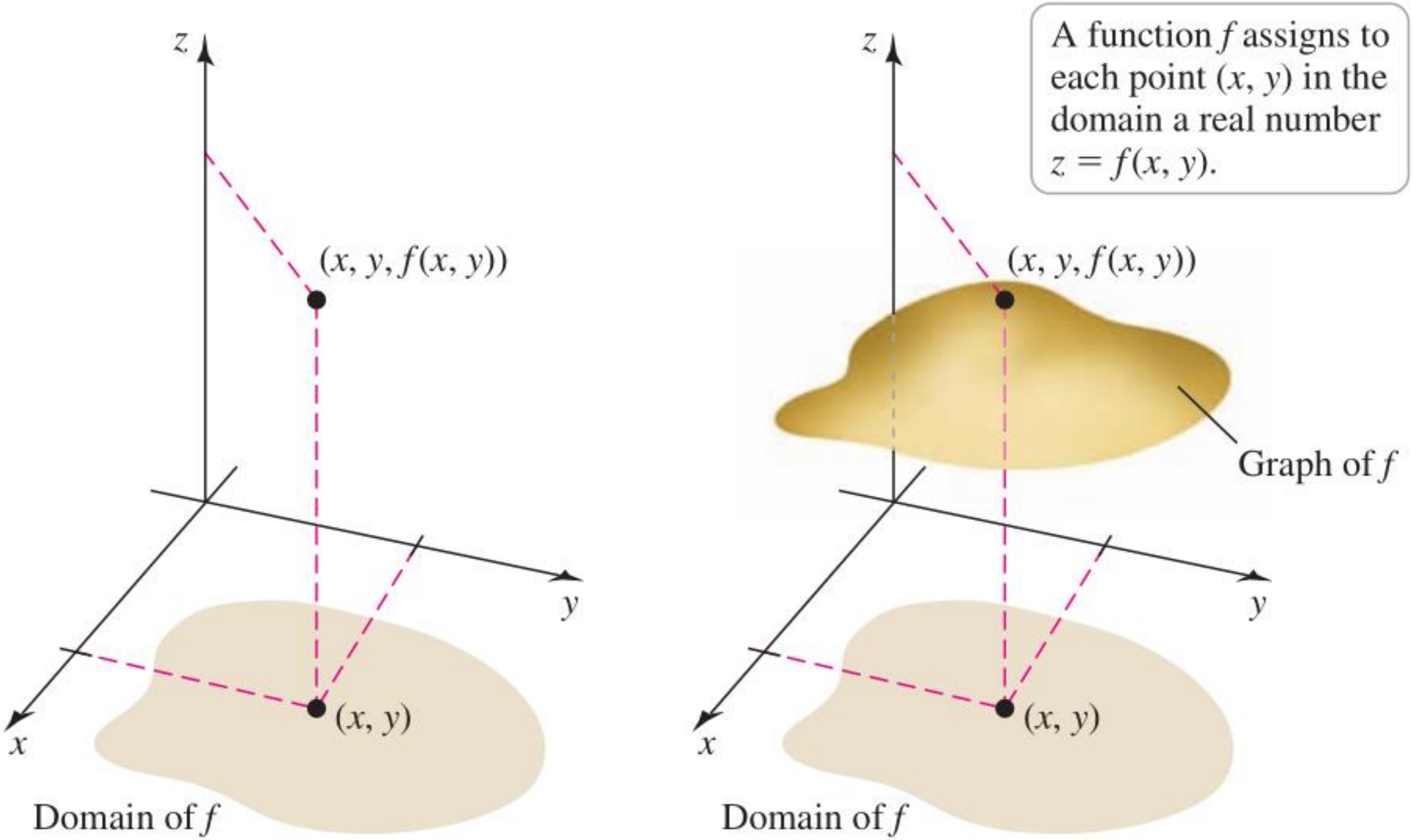
A **function** $z = f(x, y)$ assigns to each point (x, y) in a set D in $\mathbb{R}^2$ a unique real number z in a subset of $\mathbb{R}$. The set D is the **domain** of f. The **range** of f is the set of real numbers z that are assumed as the points (x, y) vary over the domain (Figure 19).



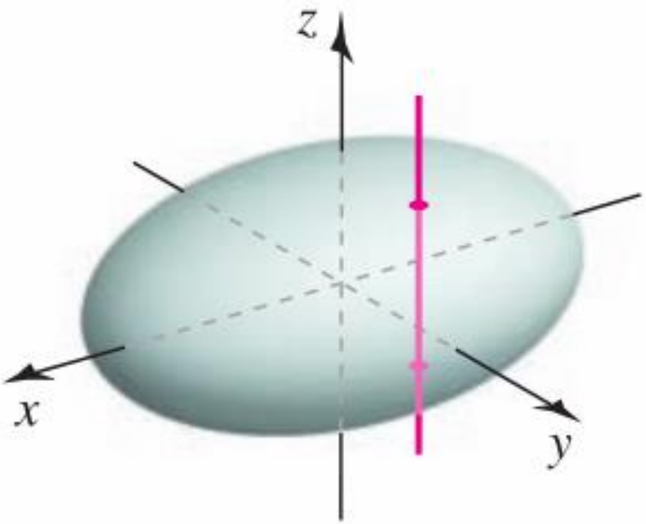
EXAMPLE 1 **Finding domains** Find the domain of the function $g(x, y) = \sqrt{4 - x^2 - y^2}$.



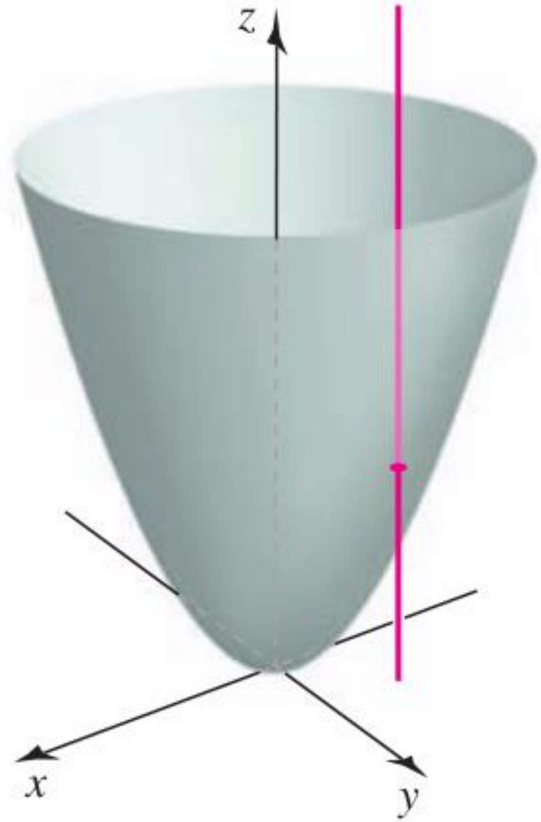
Graphs of Functions of Two Variables



Vertical line test



An ellipsoid does not pass the vertical line test: not the graph of a function.

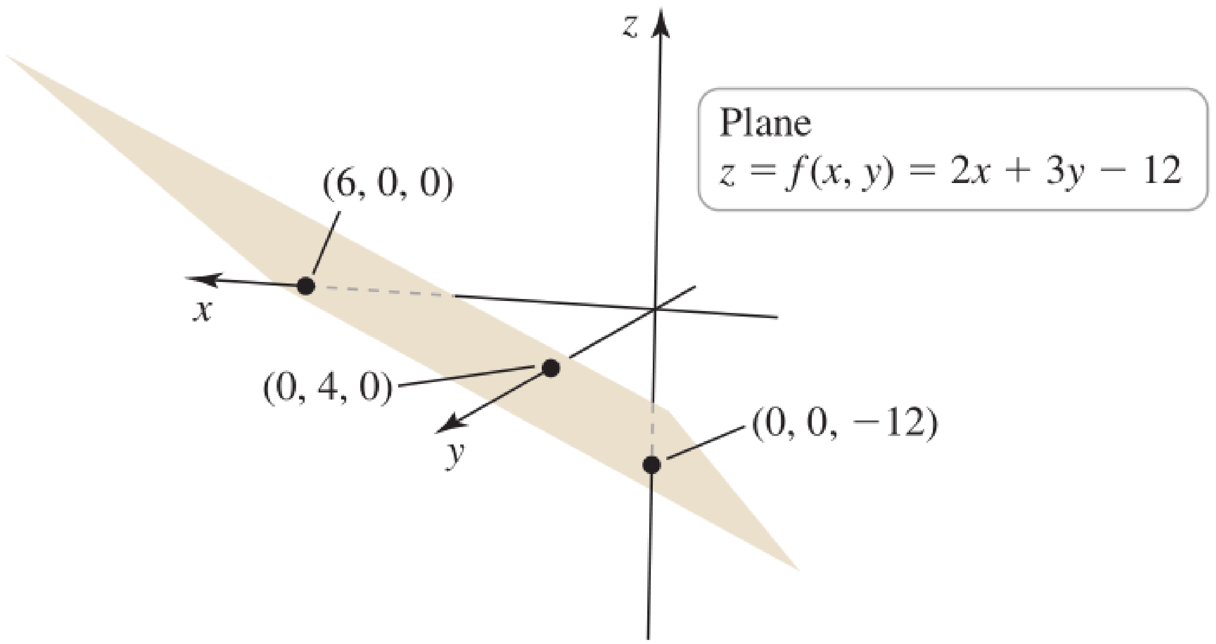


This elliptic paraboloid passes the vertical line test: graph of a function.

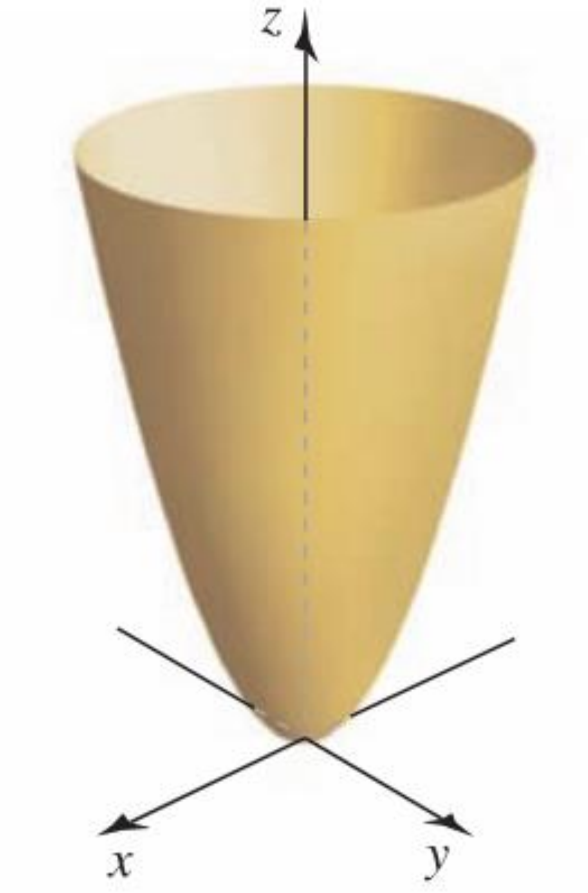
EXAMPLE 2 **Graphing two-variable functions** Find the domain and range of the following functions. Then sketch a graph.

a. $f(x, y) = 2x + 3y - 12$ **b.** $g(x, y) = x^2 + y^2$

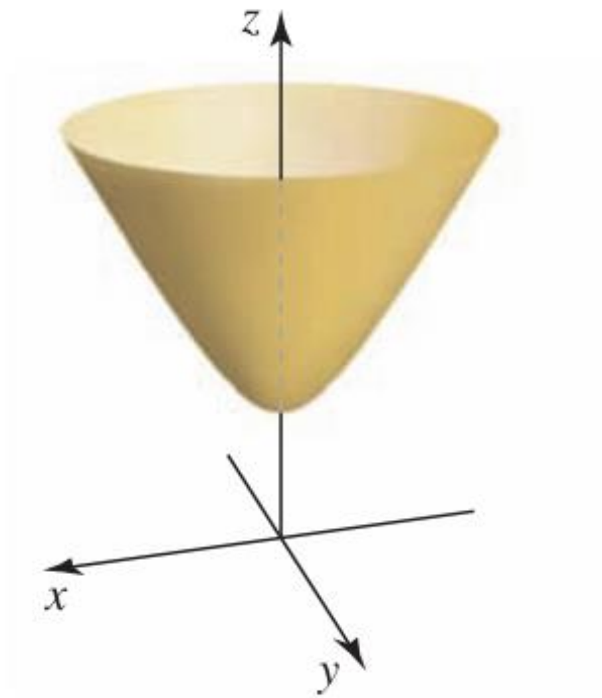
c. $h(x, y) = \sqrt{1 + x^2 + y^2}$



Paraboloid
$z = f(x, y) = x^2 + y^2$

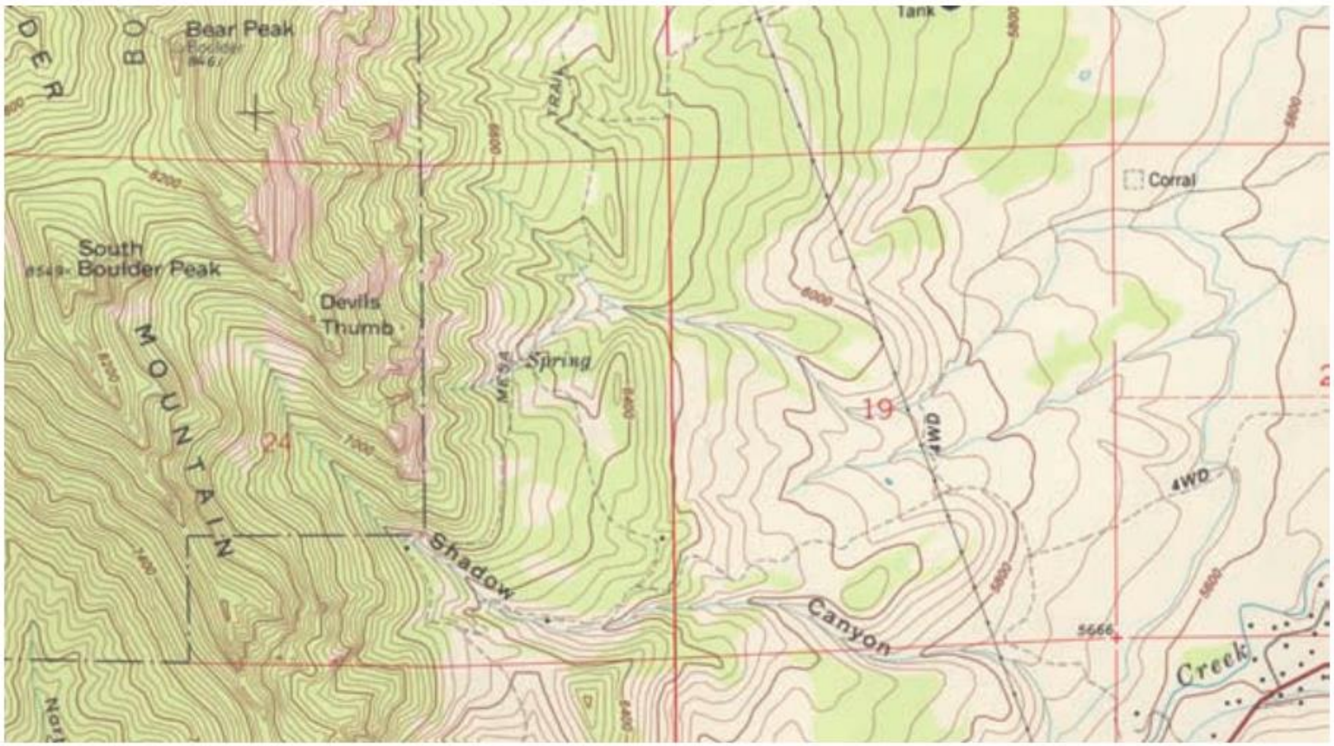


Upper sheet of hyperboloid of two sheets
$z = \sqrt{1 + x^2 + y^2}$



Level curves

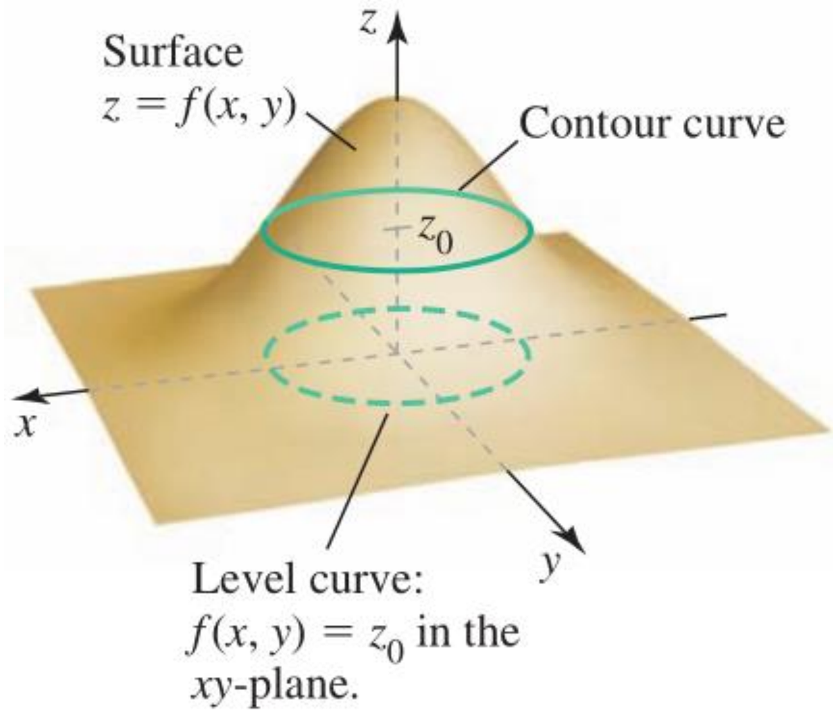
Closely spaced contours: rapid changes in elevation

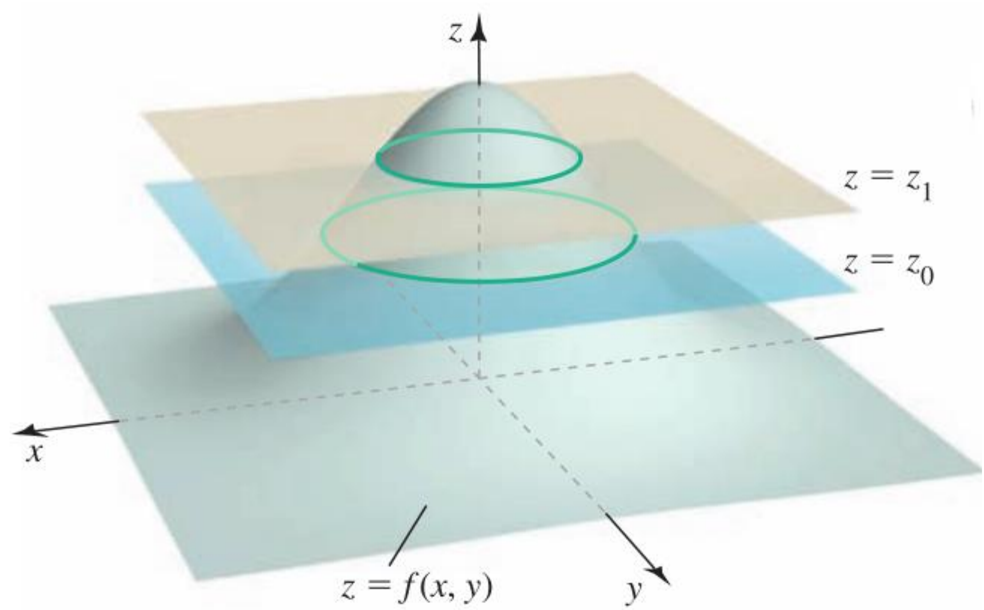


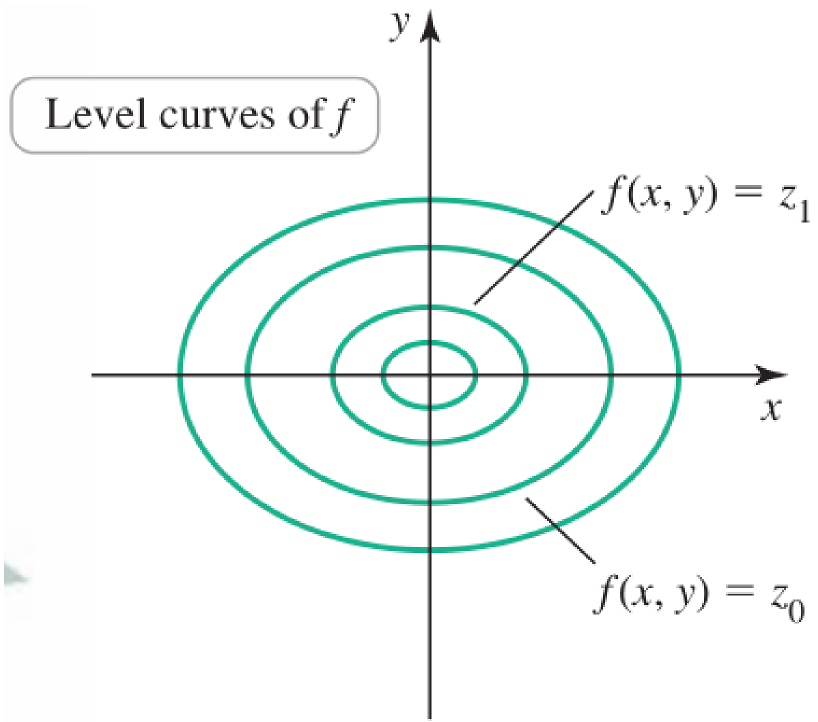
Widely spaced contours: slow changes in elevation

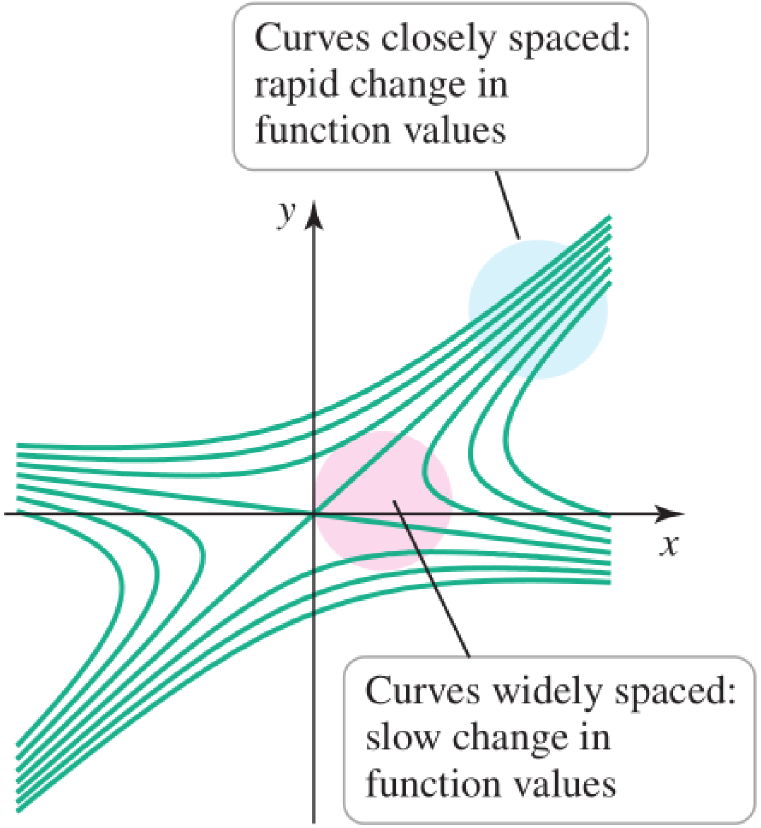
contour curve: the intersection of the surface and the horizontal plane $z = z_0$

level curve: the projection of contour curve onto the xy-plane



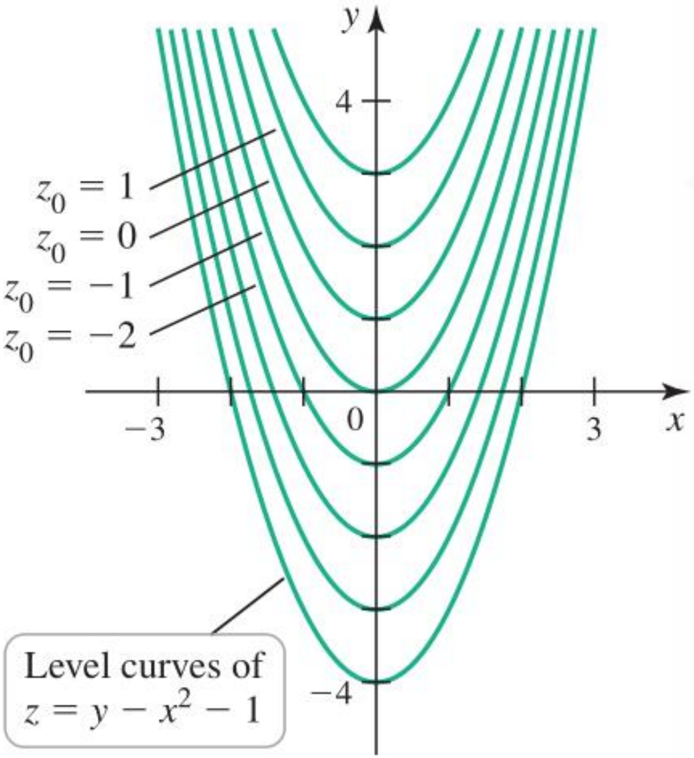


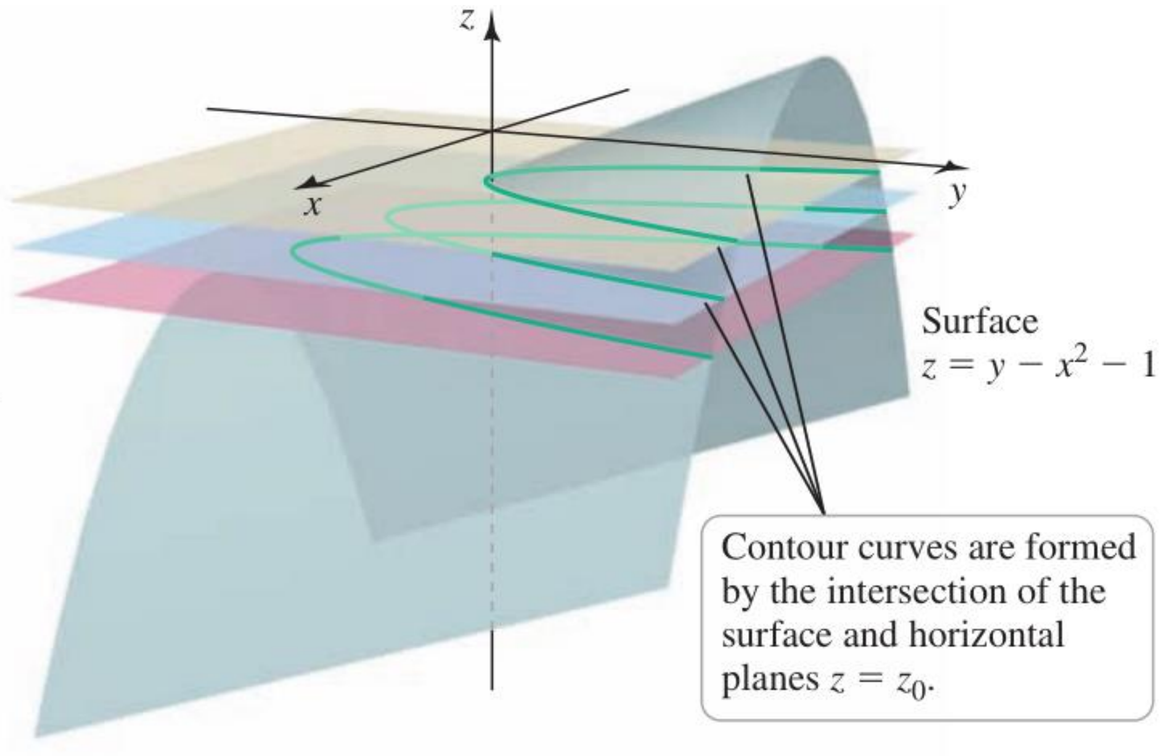

Level curves of f
y
x
$f(x, y) = z_1$
$f(x, y) = z_0$

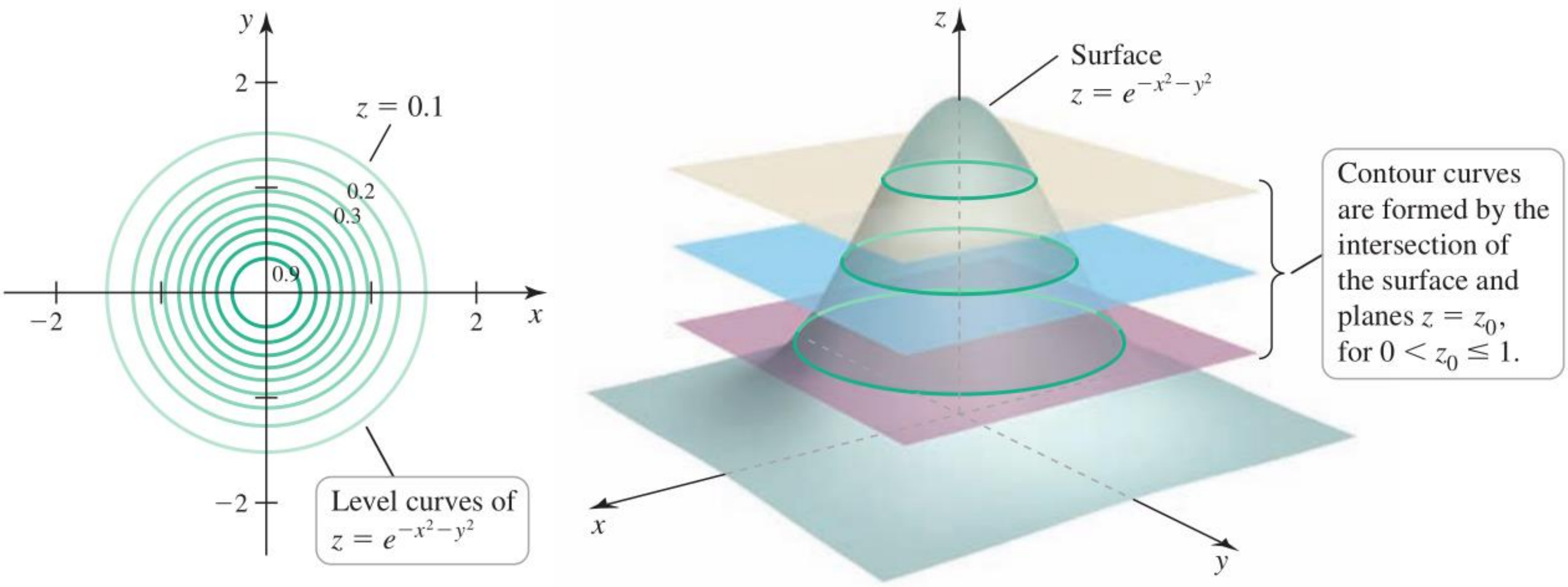

Curves closely spaced: rapid change in function values
y
x
Curves widely spaced: slow change in function values

EXAMPLE 3 **Level curves** Find and sketch the level curves of the following surfaces.

a. $f(x, y) = y - x^2 - 1$ **b.** $f(x, y) = e^{-x^2 - y^2}$



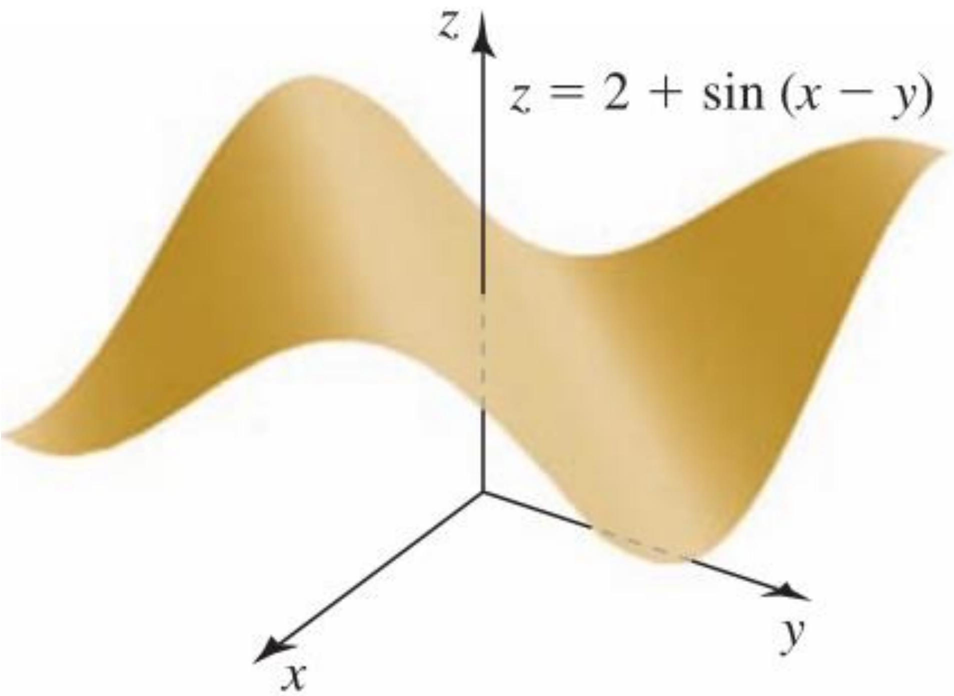



y
2
$z = 0.1$
0.2
0.3
0.9
−2
2
x
−2
Level curves of
$z = e^{-x^2-y^2}$
z
Surface
$z = e^{-x^2-y^2}$
Contour curves are formed by the intersection of the surface and planes $z = z_0$, for $0 < z_0 \le 1$.
x
y

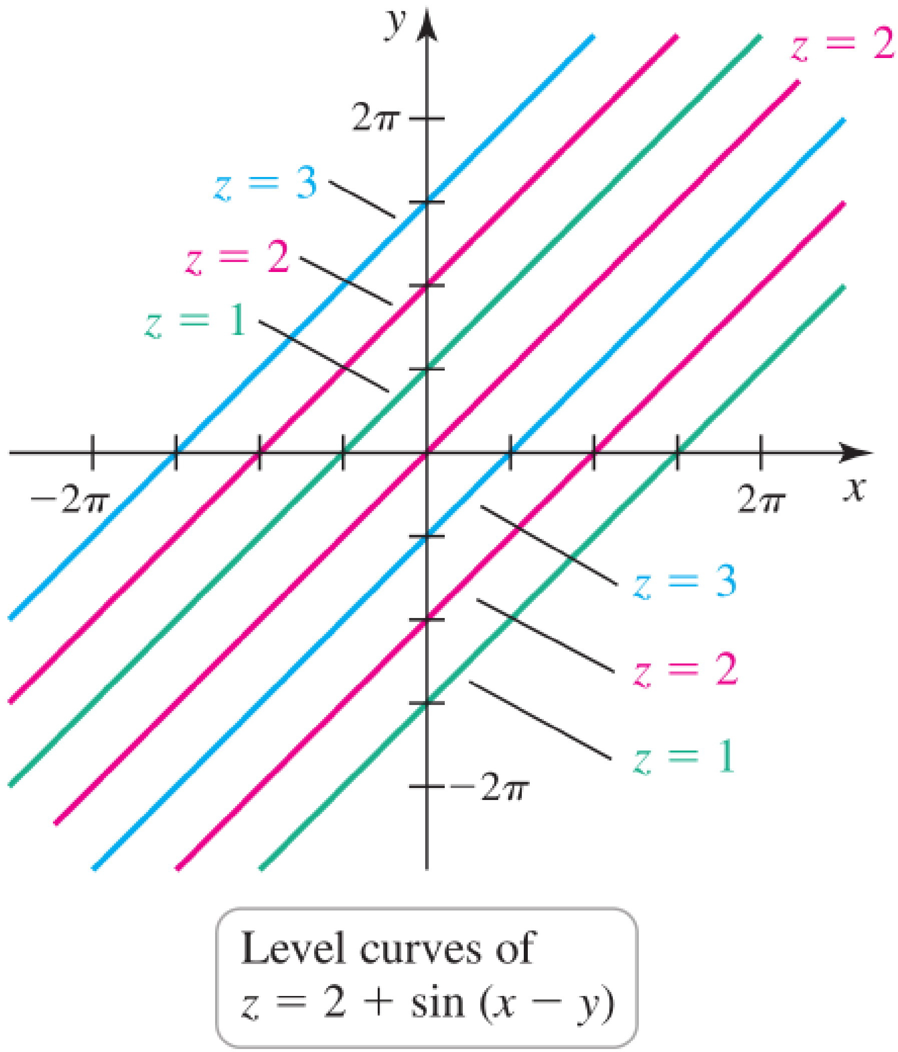
EXAMPLE 4 **Level curves** The graph of the function

$$f(x, y) = 2 + \sin(x - y)$$

is shown in Figure 32a. Sketch several level curves of the function.



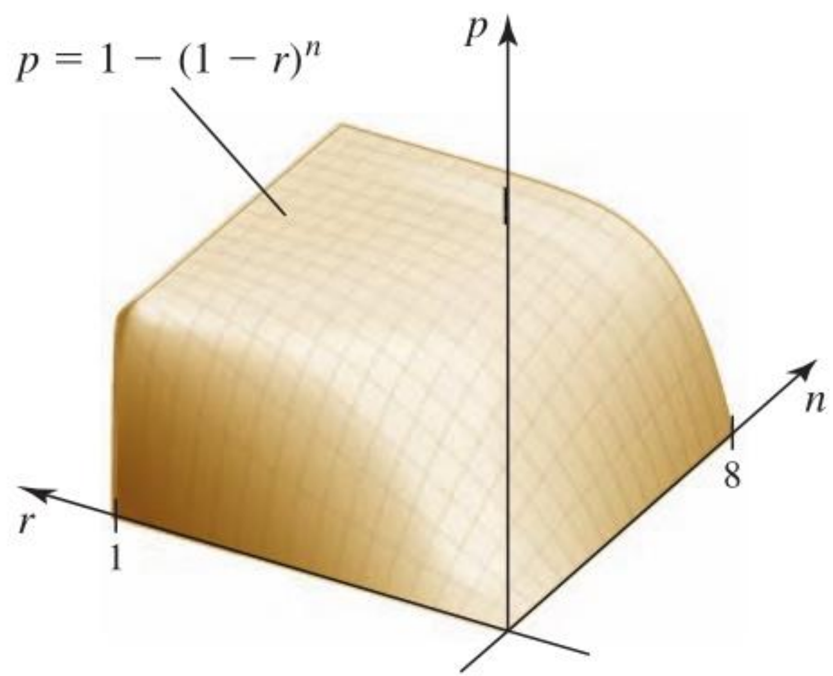
(a)



y
2π
$z = 2$
$z = 3$
$z = 2$
$z = 1$
-2π
2π
x
$z = 3$
$z = 2$
$z = 1$
-2π
Level curves of
$z = 2 + \sin(x - y)$

Applications of Functions of Two Variables

EXAMPLE 5 **A probability function of two variables** Suppose that on a particular day, the fraction of students on campus infected with flu is r, where $0 \le r \le 1$. If you have n random (possibly repeated) encounters with students during the day, the probability of meeting *at least* one infected person is $p(n, r) = 1 - (1 - r)^n$ (Figure 33a). Discuss this probability function.



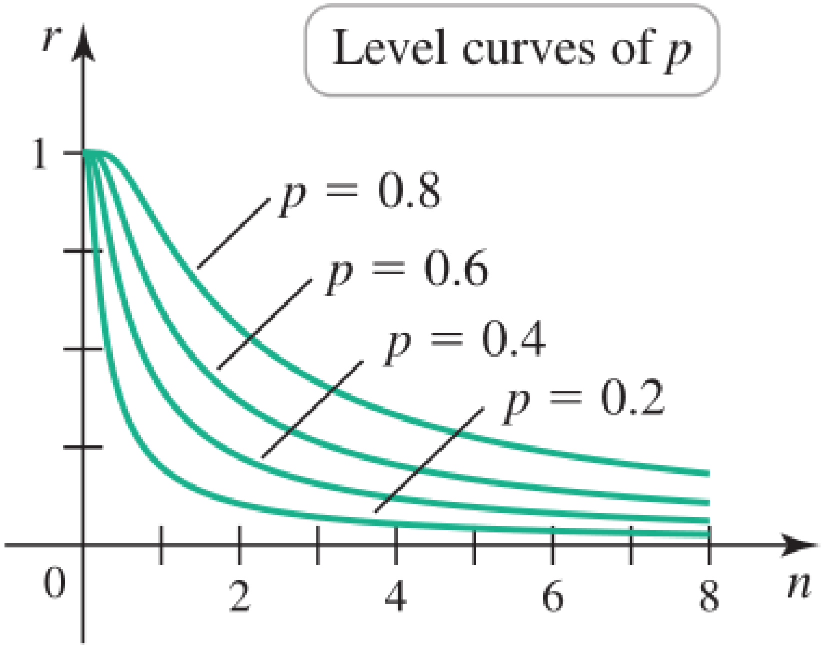


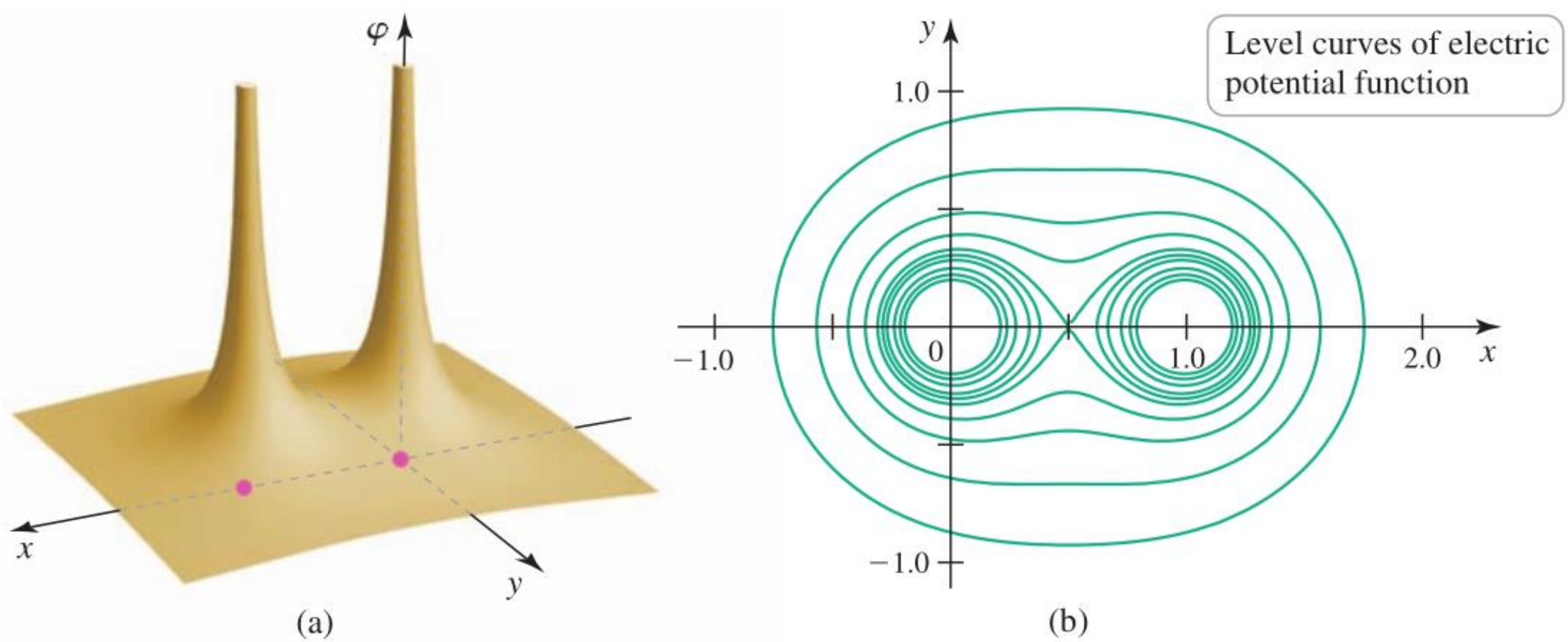
Table 2

	n: 2	5	10	15	20
r: **0.05**	0.10	0.23	0.40	0.54	0.64
0.1	0.19	0.41	0.65	0.79	0.88
0.3	0.51	0.83	0.97	1	1
0.5	0.75	0.97	1	1	1
0.7	0.91	1	1	1	1

EXAMPLE 6 **Electric potential function in two variables** The electric field at points in the xy-plane due to two point charges located at $(0, 0)$ and $(1, 0)$ is related to the electric potential function

$$\varphi(x, y) = \frac{2}{\sqrt{x^2 + y^2}} + \frac{2}{\sqrt{(x - 1)^2 + y^2}}.$$

Discuss the electric potential function.



Functions of More Than Two Variables

Table 3

Number of Independent Variables	Explicit Form	Implicit Form	Graph Resides In...
1	$y = f(x)$	$F(x, y) = 0$	$\mathbb{R}^2$ (xy-plane)
2	$z = f(x, y)$	$F(x, y, z) = 0$	$\mathbb{R}^3$ (xyz-space)
3	$w = f(x, y, z)$	$F(x, y, z, w) = 0$	$\mathbb{R}^4$
n	$x_{n+1} = f(x_1, x_2, \ldots, x_n)$	$F(x_1, x_2, \ldots, x_n, x_{n+1}) = 0$	$\mathbb{R}^{n+1}$

DEFINITION Function, Domain, and Range with *n* Independent Variables

The **function** $x_{n+1} = f(x_1, x_2, \ldots, x_n)$ assigns a unique real number x_{n+1} to each point $(x_1, x_2, \ldots, x_n)$ in a set D in $\mathbb{R}^n$. The set D is the **domain** of f. The **range** is the set of real numbers x_{n+1} that are assumed as the points $(x_1, x_2, \ldots, x_n)$ vary over the domain.

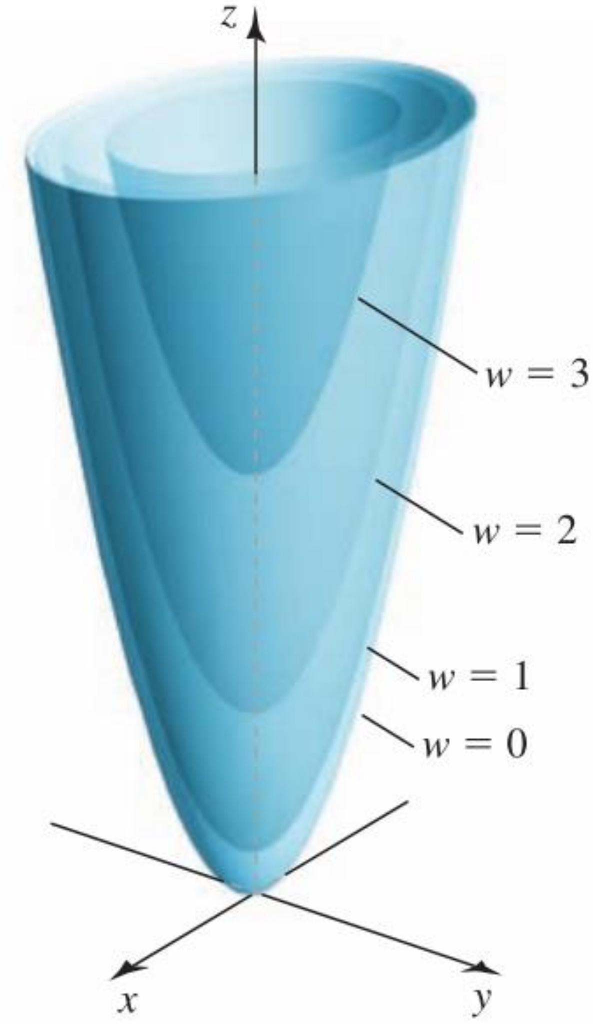
EXAMPLE 7 **Finding domains** Find the domain of the following functions.

a. $g(x, y, z) = \sqrt{16 - x^2 - y^2 - z^2}$ **b.** $h(x, y, z) = \dfrac{12y^2}{z - y}$

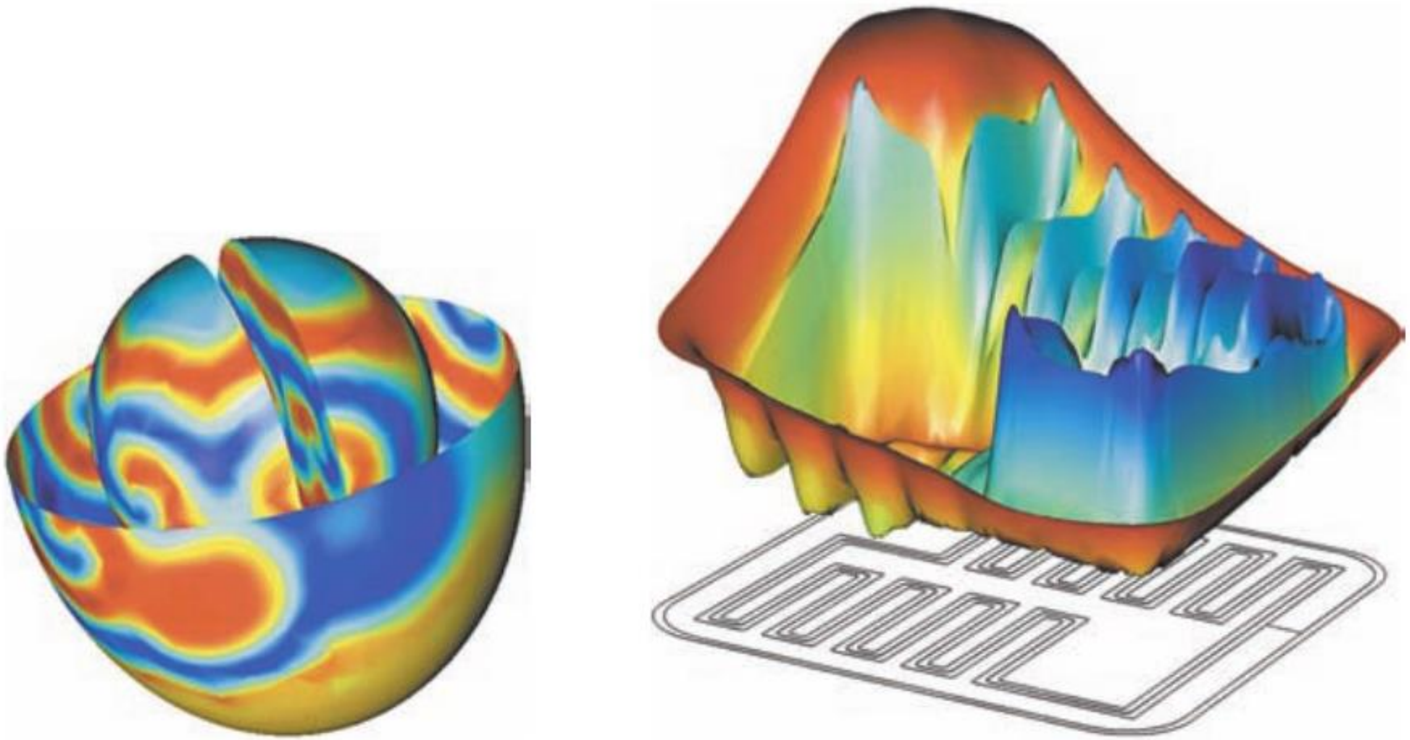
Graphs of Functions of More Than Two Variables

With the function $w = f(x, y, z)$, level curves become **level surfaces**, where w is constant

Example, $w = \sqrt{z - x^2 - 2y^2}$



Another approach to displaying functions of three variables is to **use colors** to represent the fourth dimension.



(a) (b)

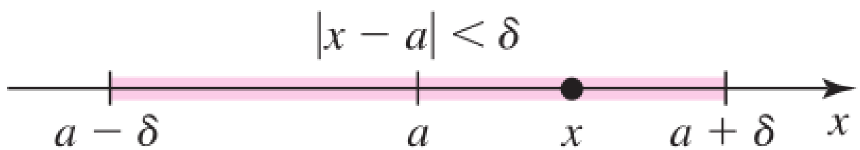
15.2

Limits and Continuity

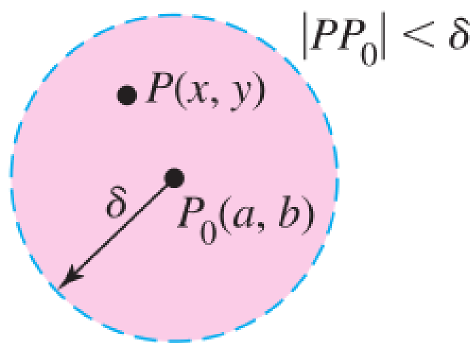
Limit of a Function of Two Variables

$|f(x,y) - L|$ can be made arbitrarily small for all P in the domain that are sufficiently close to P_0

$$\lim_{(x,y)\to(a,b)} f(x,y) = \lim_{P\to P_0} f(x,y) = L$$



(a)



(b)

DEFINITION Limit of a Function of Two Variables

The function f has the **limit** L as $P(x, y)$ approaches $P_0(a, b)$, written

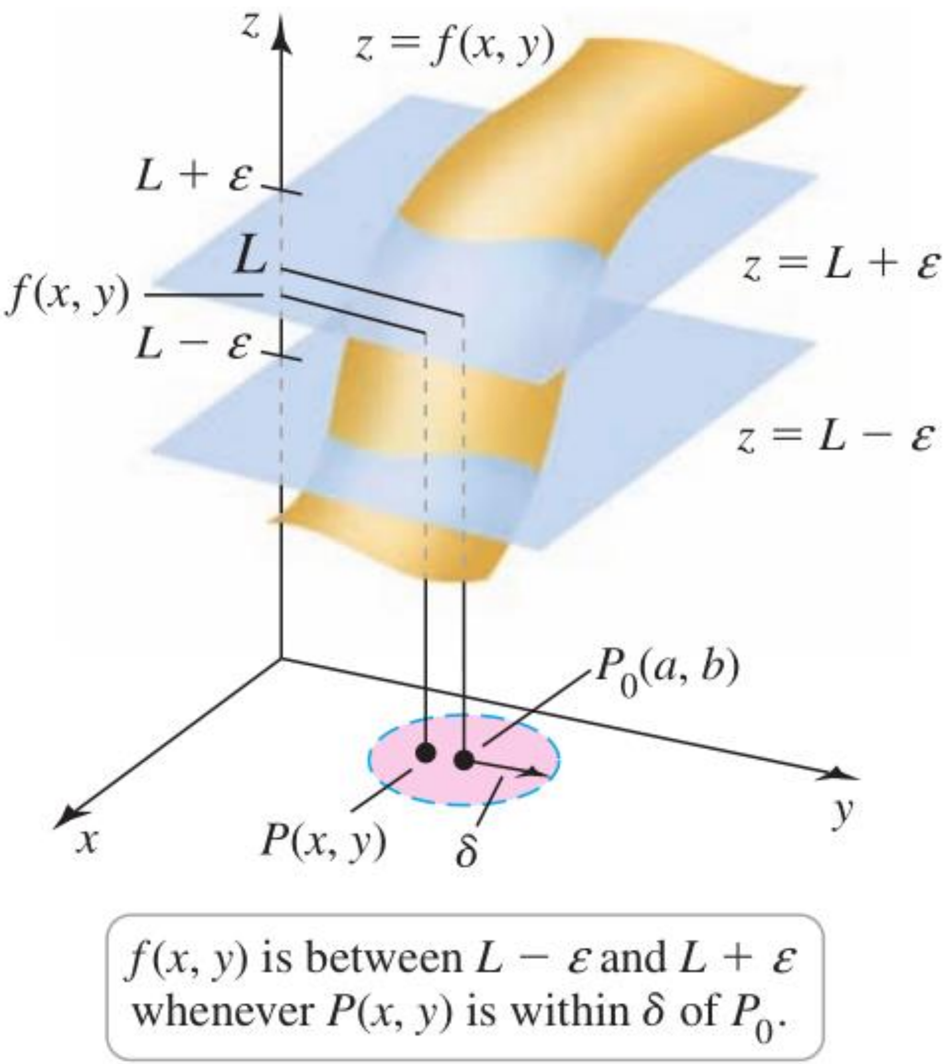
$$\lim_{(x,y)\to(a,b)} f(x, y) = \lim_{P\to P_0} f(x, y) = L,$$

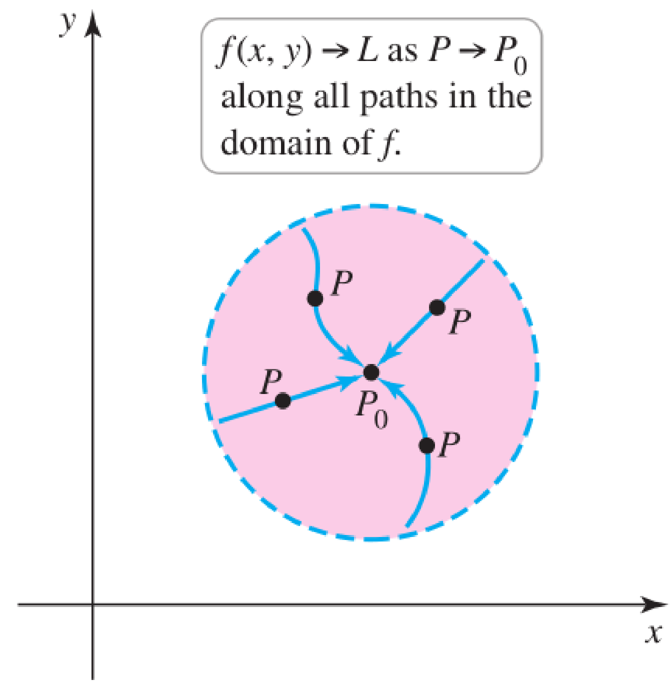
if, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x, y) - L| < \varepsilon$$

whenever (x, y) is in the domain of f and

$$0 < |PP_0| = \sqrt{(x - a)^2 + (y - b)^2} < \delta.$$


z
z = f(x, y)
L + ε
L
f(x, y)
L − ε
z = L + ε
z = L − ε
P0(a, b)
x
P(x, y)
δ
y
f(x, y) is between L − ε and L + ε whenever P(x, y) is within δ of P0.


y
f(x, y) → L as P → P0 along all paths in the domain of f.
P
P
P
P0
P
x

THEOREM 1 Limits of Constant and Linear Functions

Let a, b, and c be real numbers.

1. Constant function $f(x, y) = c$: $\lim_{(x,y) \to (a,b)} c = c$
2. Linear function $f(x, y) = x$: $\lim_{(x,y) \to (a,b)} x = a$
3. Linear function $f(x, y) = y$: $\lim_{(x,y) \to (a,b)} y = b$

THEOREM 2 Limit Laws for Functions of Two Variables

Let L and M be real numbers and suppose that $\lim_{(x,y)\to(a,b)} f(x, y) = L$ and $\lim_{(x,y)\to(a,b)} g(x, y) = M$. Assume c is a constant, and m and n are integers.

1. **Sum** $\lim_{(x,y)\to(a,b)} (f(x, y) + g(x, y)) = L + M$
2. **Difference** $\lim_{(x,y)\to(a,b)} (f(x, y) - g(x, y)) = L - M$
3. **Constant multiple** $\lim_{(x,y)\to(a,b)} cf(x, y) = cL$
4. **Product** $\lim_{(x,y)\to(a,b)} f(x, y)g(x, y) = LM$
5. **Quotient** $\lim_{(x,y)\to(a,b)} \frac{f(x, y)}{g(x, y)} = \frac{L}{M}$, provided $M \neq 0$
6. **Power** $\lim_{(x,y)\to(a,b)} (f(x, y))^n = L^n$
7. **Root** $\lim_{(x, y)\to(a, b)} (f(x, y))^{1/n} = L^{1/n}$, where we assume $L > 0$ if n is even.

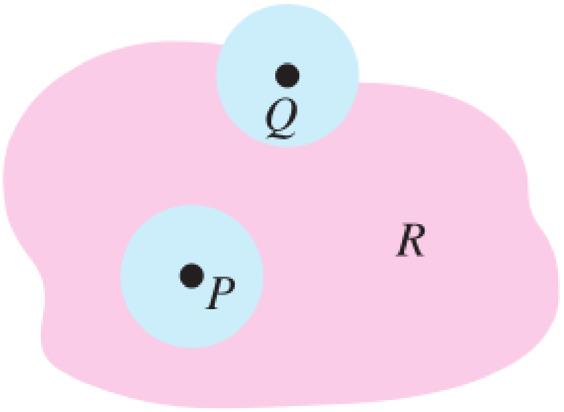
EXAMPLE 1 **Limits of two-variable functions** Evaluate $\lim_{(x,y)\to(2,8)} (3x^2y + \sqrt{xy})$.

Limits at Boundary Points

DEFINITION Interior and Boundary Points

Let R be a region in $\mathbb{R}^2$. An **interior point** P of R lies entirely within R, which means it is possible to find a disk centered at P that contains only points of R (Figure 40).

A **boundary point** Q of R lies on the edge of R in the sense that *every* disk centered at Q contains at least one point in R and at least one point not in R.



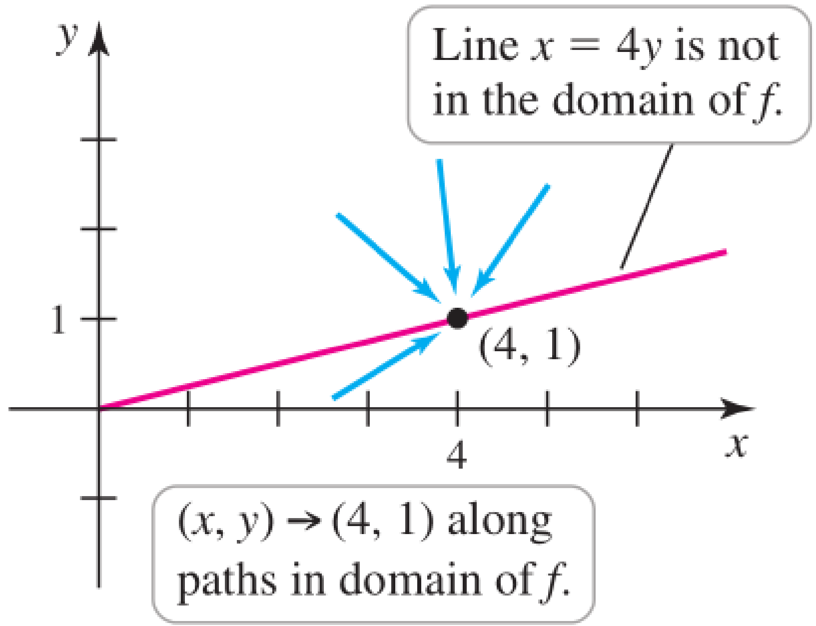
Q is a boundary point: Every disk centered at Q contains points in R and points not in R.

P is an interior point: There is a disk centered at P that lies entirely in R.

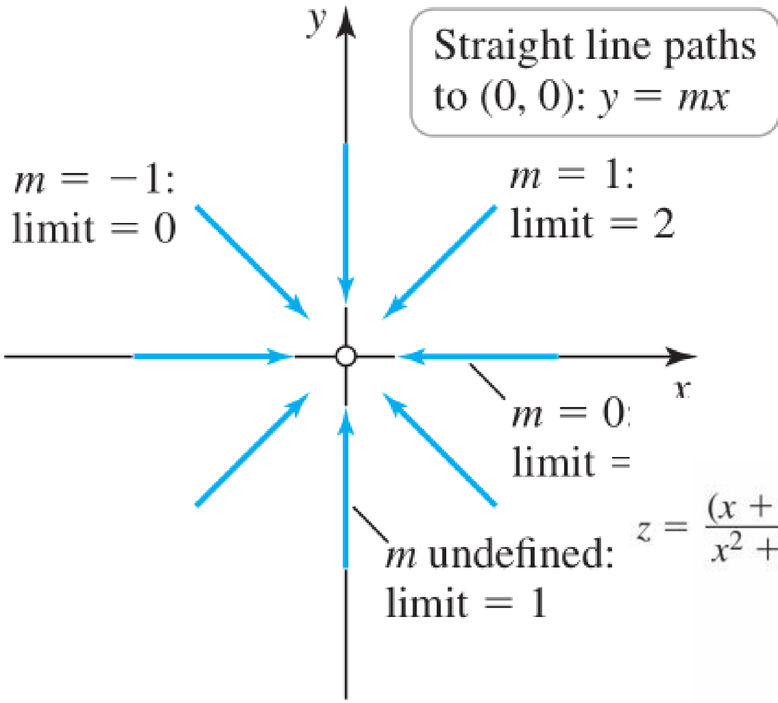
DEFINITION **Open and Closed Sets**

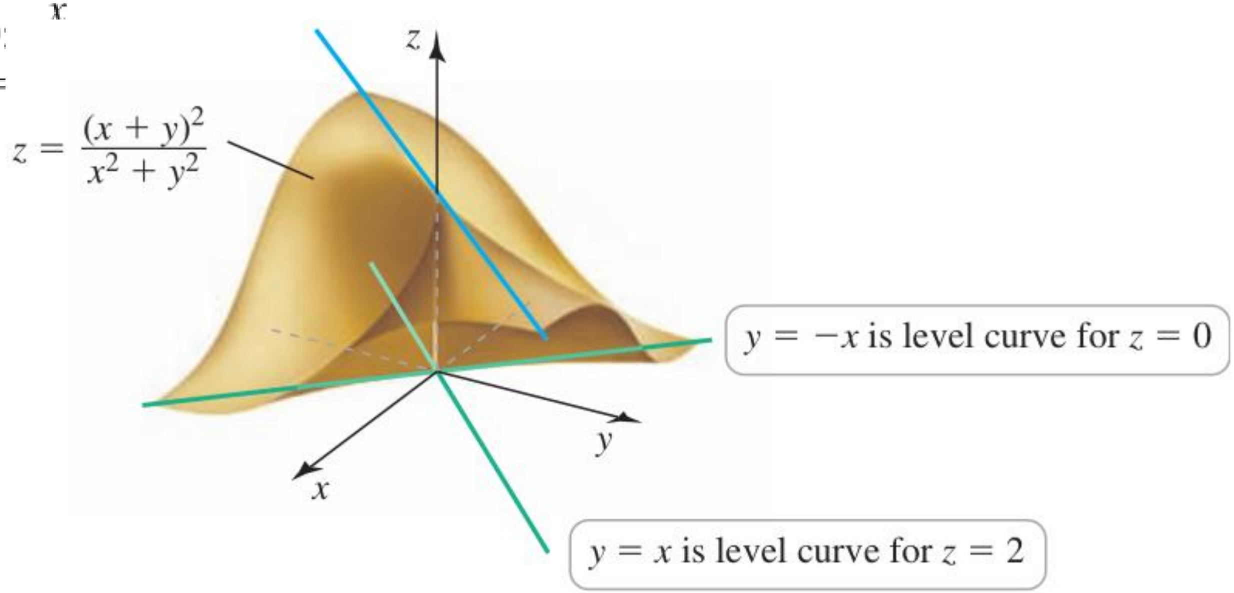
A region is **open** if it consists entirely of interior points. A region is **closed** if it contains all its boundary points.

EXAMPLE 2 **Limits at boundary points** Evaluate $\lim_{(x,y)\to(4,1)} \dfrac{xy - 4y^2}{\sqrt{x} - 2\sqrt{y}}$.



EXAMPLE 3 **Nonexistence of a limit** Investigate the limit $\lim_{(x,y)\to(0,0)} \frac{(x+y)^2}{x^2+y^2}$.





PROCEDURE **Two-Path Test for Nonexistence of Limits**

If $f(x, y)$ approaches two different values as (x, y) approaches (a, b) along two different paths in the domain of f, then $\lim_{(x,y)\to(a,b)} f(x, y)$ does not exist.

Continuity of Functions of Two Variables

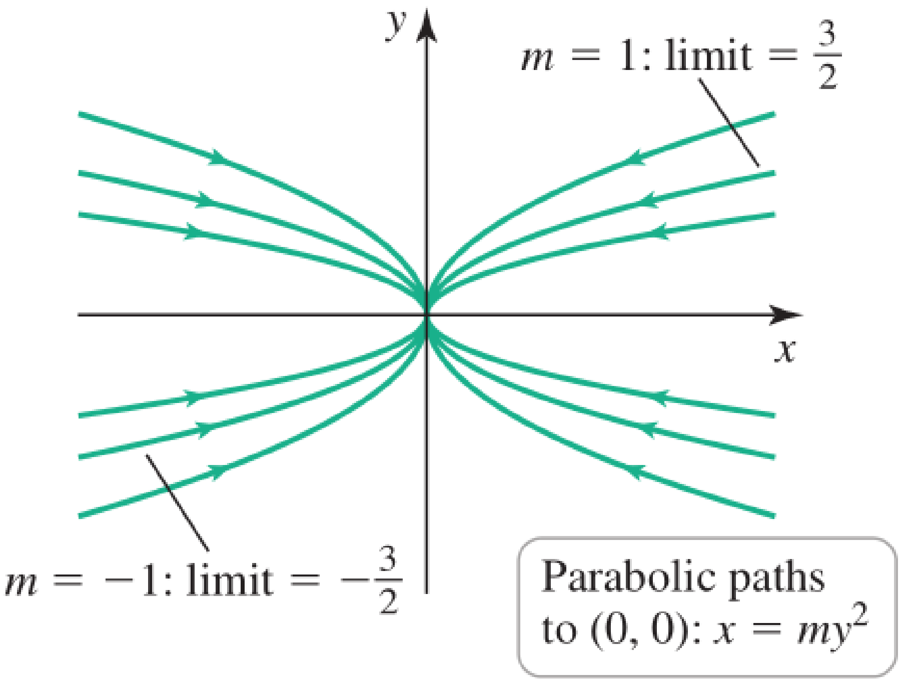
DEFINITION **Continuity**

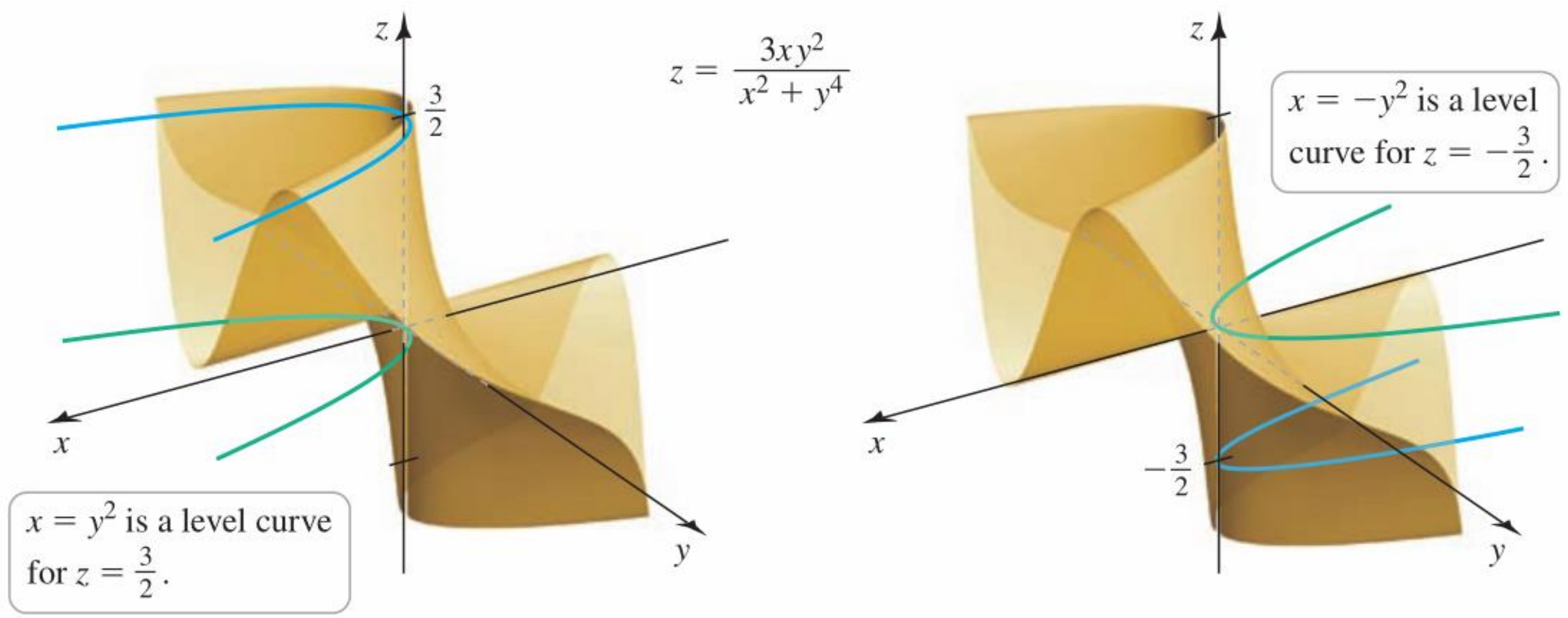
The function f is continuous at the point (a, b) provided

1. f is defined at (a, b).
2. $\lim_{(x,y)\to(a,b)} f(x, y)$ exists.
3. $\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$.

EXAMPLE 4 **Checking continuity** Determine the points at which the following function is continuous.

$$f(x, y) = \begin{cases} \dfrac{3xy^2}{x^2 + y^4} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$





$z = \frac{3xy^2}{x^2 + y^4}$
$x = y^2$ is a level curve for $z = \frac{3}{2}$.
$x = -y^2$ is a level curve for $z = -\frac{3}{2}$.
z
x
y
$\frac{3}{2}$
$-\frac{3}{2}$

Composite Functions

THEOREM 3 Continuity of Composite Functions

If $u = g(x, y)$ is continuous at (a, b) and $z = f(u)$ is continuous at $g(a, b)$, then the composite function $z = f(g(x, y))$ is continuous at (a, b).

EXAMPLE 5 **Continuity of composite functions.** Determine the points at which the following functions are continuous.

a. $h(x, y) = \ln (x^2 + y^2 + 4)$ **b.** $h(x, y) = e^{x/y}$

Functions of Three Variables

EXAMPLE 6 **Functions of three variables**

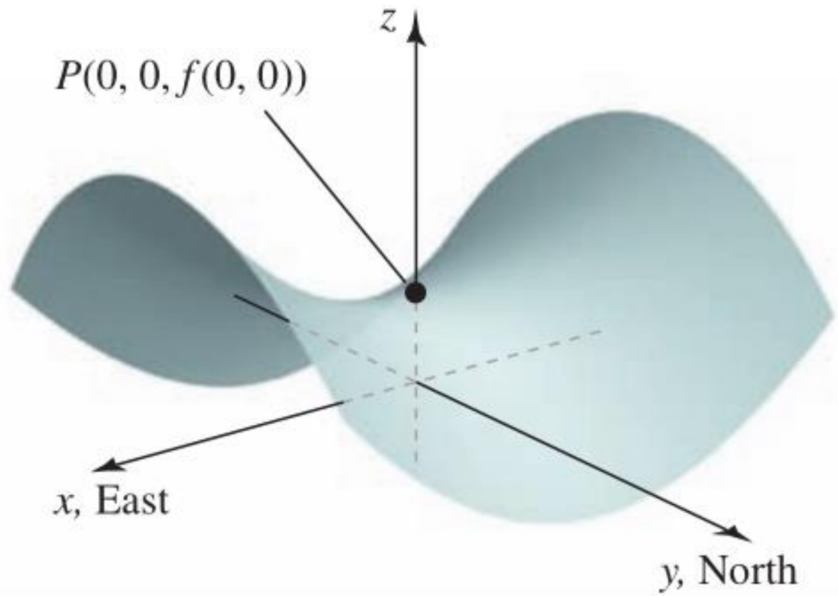
a. Evaluate $\lim_{(x,y,z)\to(2,\pi/2,0)} \dfrac{x^2 \sin y}{z^2 + 4}$.

b. Find the points at which $h(x, y, z) = \sqrt{x^2 + y^2 + z^2 - 1}$ is continuous.

15.3

Partial Derivatives

Derivatives with Two Variables



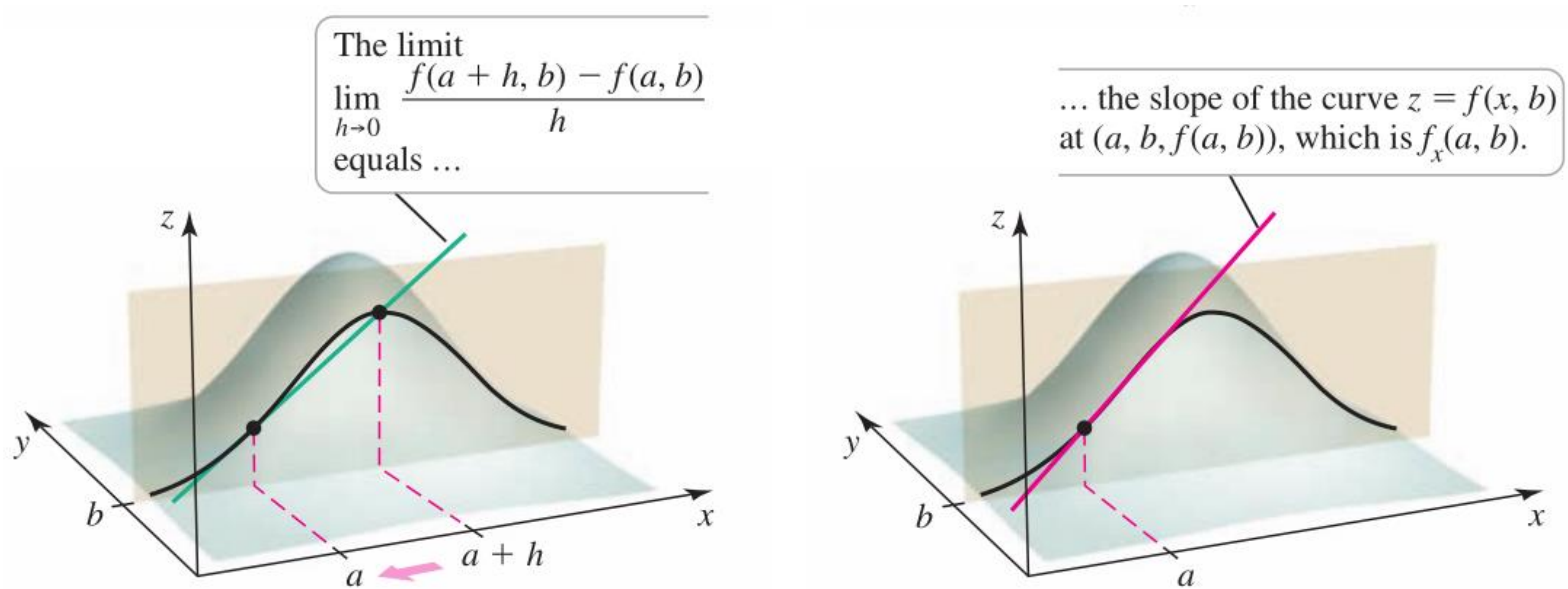
(a)

(b)

Partial derivatives: An ordinary derivative with respect to the remaining variable when one independent variable fixed.

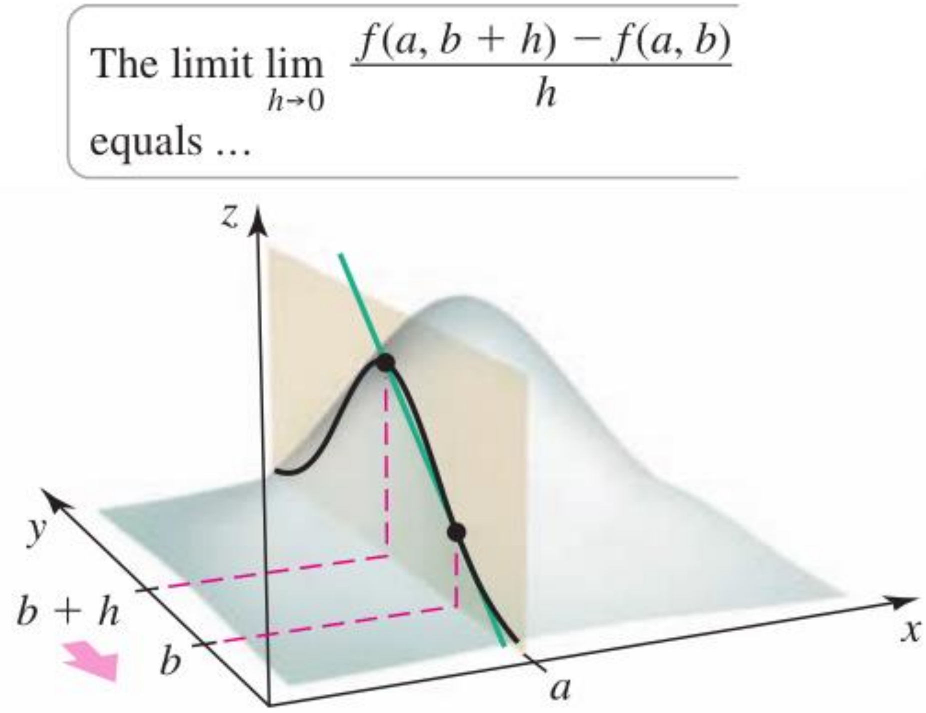
For example, let *y*-coordinate be fixed at $y = b$

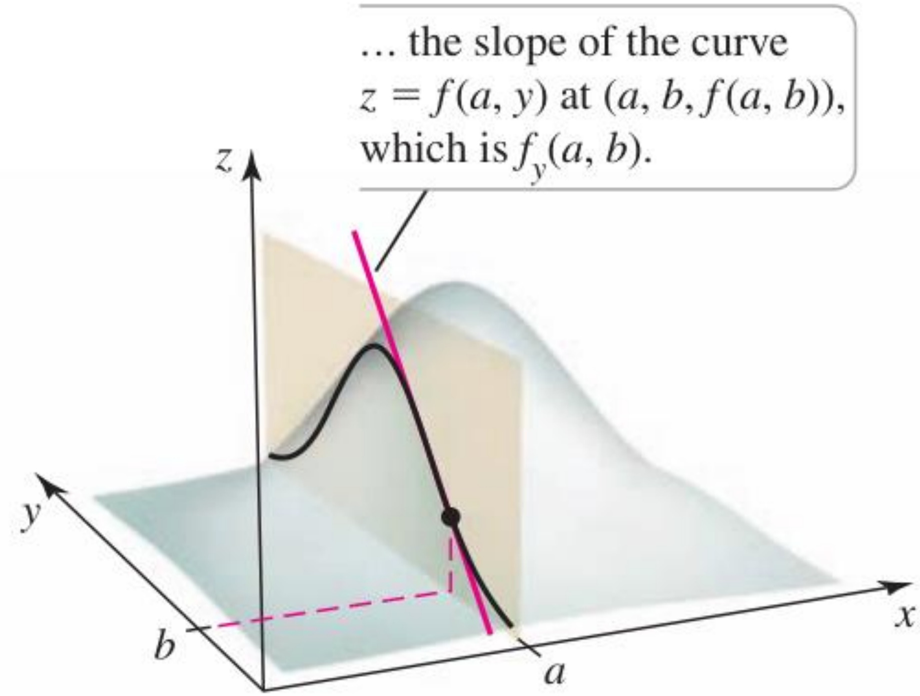
$$f_x(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}$$



Similarly, let *x*-coordinate be fixed at $x = a$

$$f_y(a,b) = \lim_{h\to 0} \frac{f(a,b+h) - f(a,b)}{h}$$





DEFINITION **Partial Derivatives**

The **partial derivative of f with respect to x at the point (a, b)** is

$$f_x(a, b) = \lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h}.$$

The **partial derivative of f with respect to y at the point (a, b)** is

$$f_y(a, b) = \lim_{h \to 0} \frac{f(a, b + h) - f(a, b)}{h},$$

provided these limits exist.

Notations

$$\frac{\partial f}{\partial x}(a,b) = \left.\frac{\partial f}{\partial x}\right|_{(a,b)} = f_x(a,b), \frac{\partial f}{\partial y}(a,b) = \left.\frac{\partial f}{\partial y}\right|_{(a,b)} = f_y(a,b)$$

Calculating Partial Derivatives

EXAMPLE 1 **Partial derivatives from the definition** Suppose $f(x, y) = x^2y$. Use the limit definition of partial derivatives to compute $f_x(x, y)$ and $f_y(x, y)$.

EXAMPLE 2 **Partial derivatives** Let $f(x, y) = x^3 - y^2 + 4$.

a. Compute $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$.

b. Evaluate each derivative at $(2, -4)$.

EXAMPLE 3 **Partial derivatives** Compute the partial derivatives of the following functions.

a. $f(x, y) = \sin xy$

b. $g(x, y) = x^2 e^{xy}$

Higher-Order Partial Derivatives

Table 4

Notation 1	Notation 2	What we say . . .
$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$	$(f_x)_x = f_{xx}$	*d squared f dx squared* or *f-x-x*
$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$	$(f_y)_y = f_{yy}$	*d squared f dy squared* or *f-y-y*
$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$	$(f_y)_x = f_{yx}$	*f-y-x*
$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$	$(f_x)_y = f_{xy}$	*f-x-y*

The order of differentiation can make a difference in the **mixed partial derivatives** f_{xy} and f_{yx}

EXAMPLE 4 **Second partial derivatives** Find the four second partial derivatives of $f(x, y) = 3x^4y - 2xy + 5xy^3$.

Equality of Mixed Partial Derivatives

THEOREM 4 (Clairaut) Equality of Mixed Partial Derivatives
Assume that f is defined on an open set D of $\mathbb{R}^2$, and that f_{xy} and f_{yx} are continuous throughout D. Then $f_{xy} = f_{yx}$ at all points of D.

Functions of Three Variables

EXAMPLE 5 **Partial derivatives with more than two variables** Find f_x, f_y, and f_z when $f(x, y, z) = e^{-xy} \cos z$.

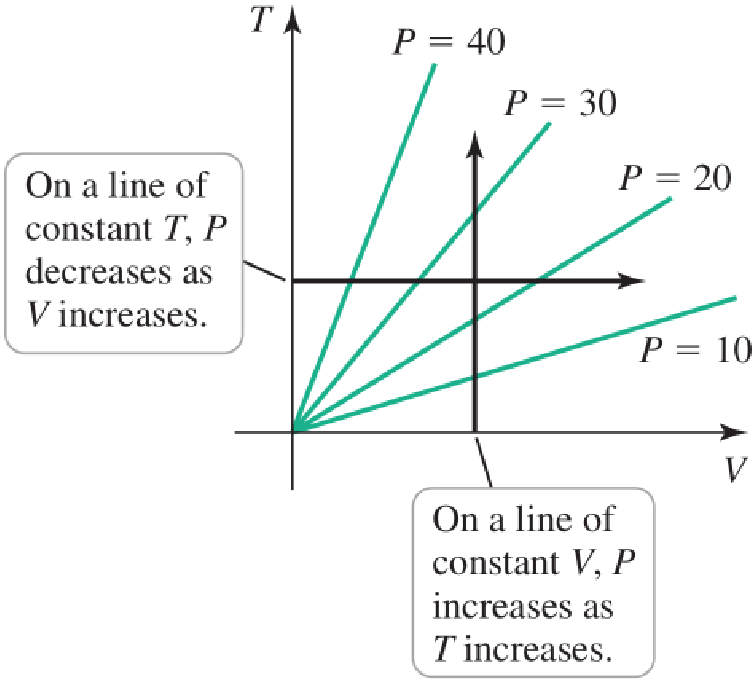
Applications of Partial Derivatives

EXAMPLE 6 **Ideal Gas Law** The pressure P, volume V, and temperature T of an ideal gas are related by the equation $PV = kT$, where $k > 0$ is a constant depending on the amount of gas.

a. Determine the rate of change of the pressure with respect to the volume at constant temperature. Interpret the result.

b. Determine the rate of change of the pressure with respect to the temperature at constant volume. Interpret the result.

c. Explain these results using level curves.



Differentiability

Is f *differentiable* there if the partial derivatives f_x and f_y exist at a point? Not that simple!

Recall that a function f of one variable is differentiable at $x = a$

$$f'(a) = \lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x}$$

$$\varepsilon = \underbrace{\frac{f(a + \Delta x) - f(a)}{\Delta x}}_{\text{slope of secant line}} - \underbrace{f'(a)}_{\substack{\text{slope of} \\ \text{tangent line}}},$$

$$\Delta y = f(a + \Delta x) - f(a) = f'(a)\,\Delta x + \underbrace{\varepsilon}_{\varepsilon \to 0 \text{ as } \Delta x \to 0}\,\Delta x.$$

DEFINITION Differentiability

The function $z = f(x, y)$ is **differentiable at** $(\boldsymbol{a}, \boldsymbol{b})$ provided $f_x(a, b)$ and $f_y(a, b)$ exist and the change $\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b)$ equals

$$\Delta z = f_x(a, b)\,\Delta x + f_y(a, b)\,\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y,$$

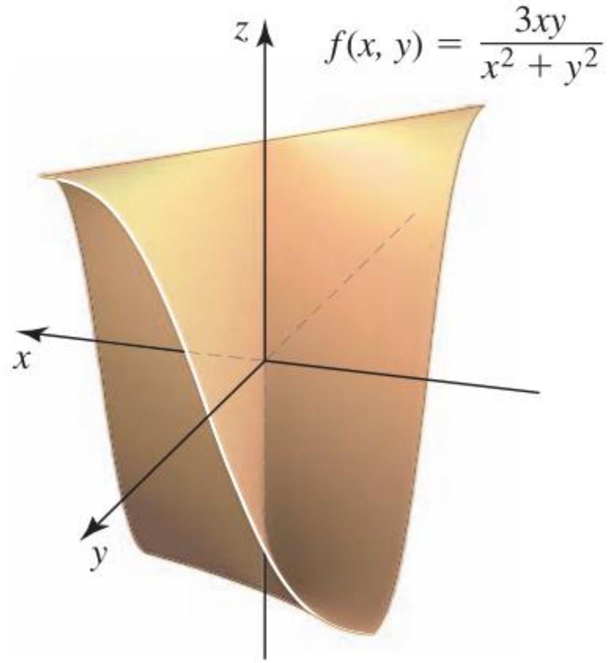
where for fixed a and b, ε_1 and ε_2 are functions that depend only on Δx and Δy, with $(\varepsilon_1, \varepsilon_2) \rightarrow (0, 0)$ as $(\Delta x, \Delta y) \rightarrow (0, 0)$. A function is **differentiable** on an open set R if it is differentiable at every point of R.

THEOREM 5 Conditions for Differentiability
Suppose the function f has partial derivatives f_x and f_y defined on an open set containing (a, b), with f_x and f_y continuous at (a, b). Then f is differentiable at (a, b).

THEOREM 6 Differentiable Implies Continuous
If a function f is differentiable at (a, b), then it is continuous at (a, b).

EXAMPLE 7 **A nondifferentiable function** Discuss the differentiability and continuity of the function

$$f(x, y) = \begin{cases} \dfrac{3xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$



f is not continuous at (0, 0), even though $f_x(0, 0) = f_y(0, 0) = 0$.

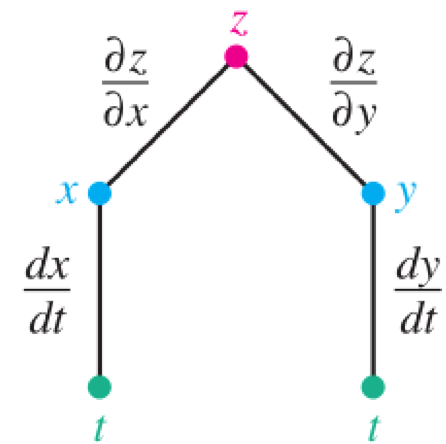
15.4

The Chain Rule

The Chain Rule with One Independent Variable

Recall the basic Chain Rule: $\dfrac{dy}{dt} = \dfrac{dy}{du}\dfrac{du}{dt}$

For $z = f(x, y)$,
where x and y are functions of t.
What is $\dfrac{dz}{dt}$?



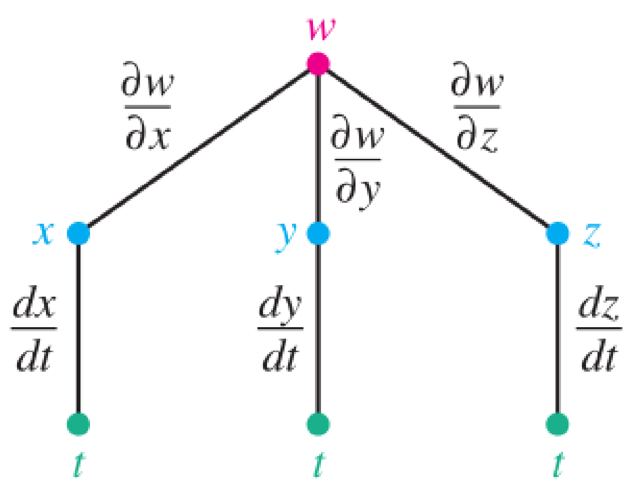
$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

THEOREM 7 Chain Rule (One Independent Variable)
Let z be a differentiable function of x and y on its domain, where x and y are differentiable functions of t on an interval I. Then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

Several comments

1. For $z = f(x(t), y(t))$, the sole independent variable is t, and x and y are **intermediate variables**.
2. The choice of notation for partial and ordinary derivatives in the Chain Rule is important.
3. Theorem 7 can be generalized directly to functions of more than two intermediate variables.

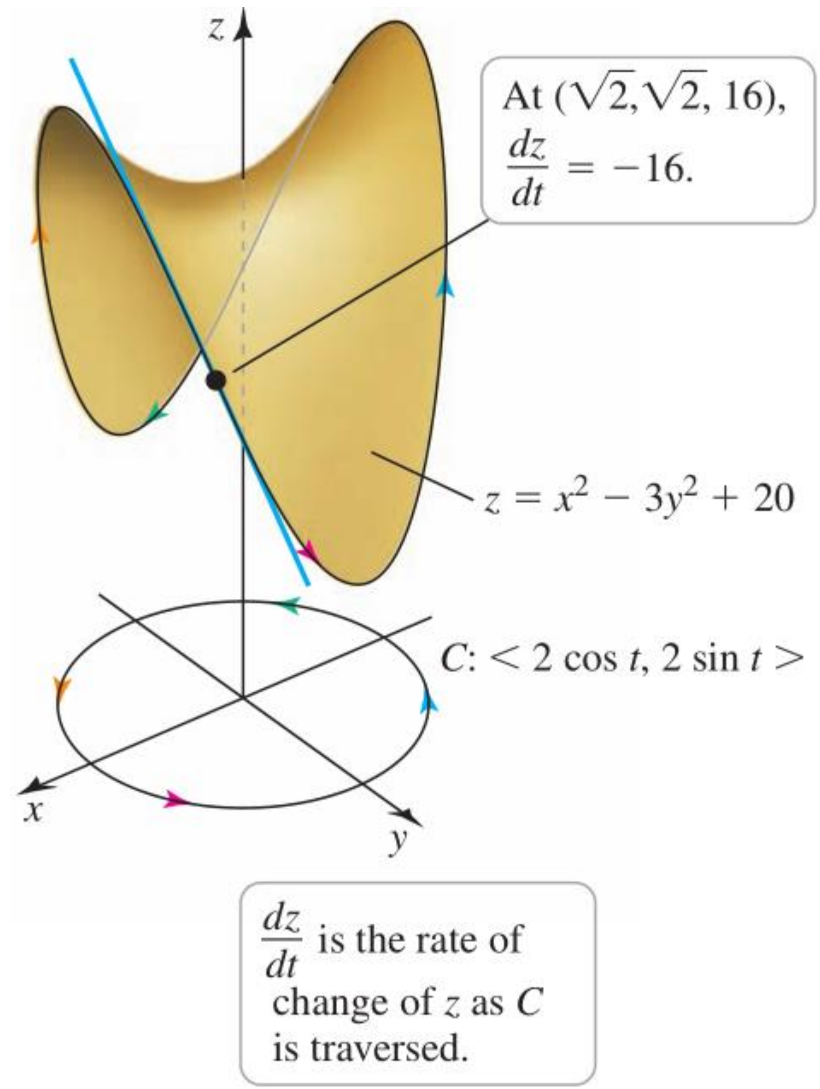


$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

EXAMPLE 1 **Chain Rule with one independent variable** Let $z = x^2 - 3y^2 + 20$, where $x = 2\cos t$ and $y = 2\sin t$.

a. Find $\dfrac{dz}{dt}$ and evaluate it at $t = \pi/4$.

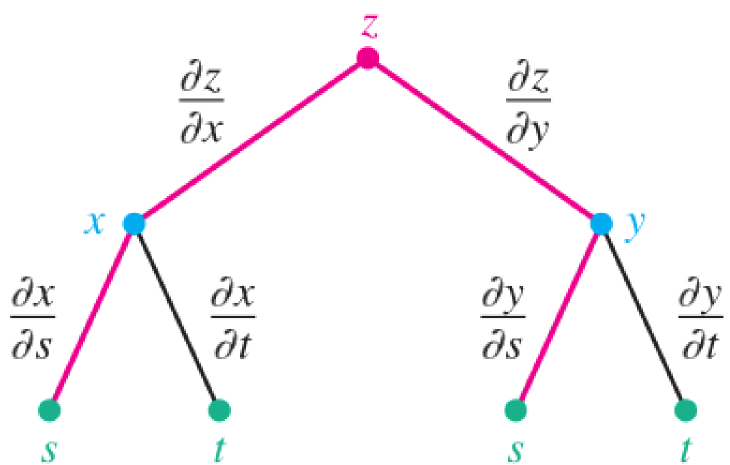
b. Interpret the result geometrically.



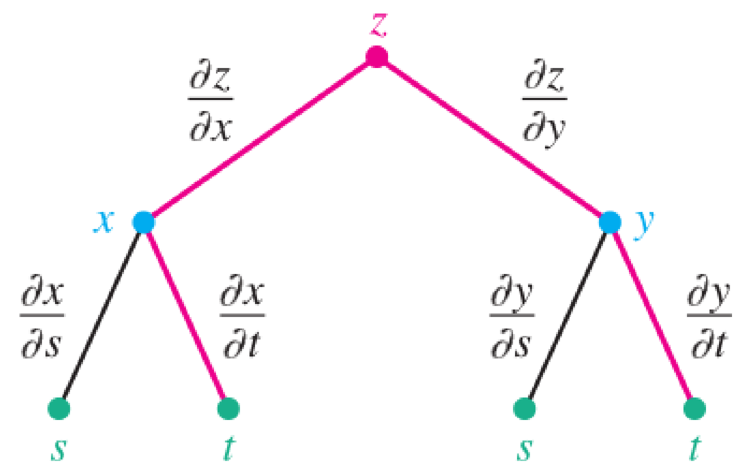
The Chain Rule with Several Independent Variables

THEOREM 8 Chain Rule (Two Independent Variables)
Let z be a differentiable function of x and y, where x and y are differentiable functions of s and t. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \quad \text{and} \quad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}.$$



$$\boxed{\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}}$$



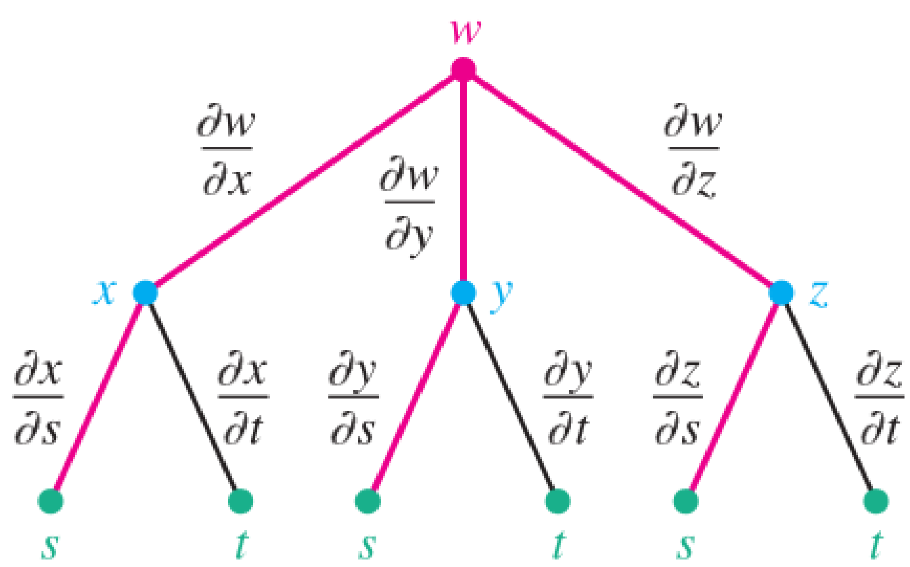
$$\boxed{\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}}$$

EXAMPLE 2 **Chain Rule with two independent variables** Let $z = \sin 2x \cos 3y$, where $x = s + t$ and $y = s - t$. Evaluate $\partial z/\partial s$ and $\partial z/\partial t$.

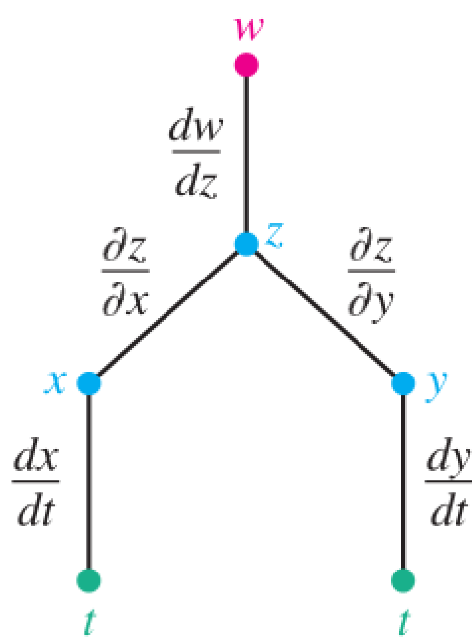
EXAMPLE 3 **More variables** Let w be a function of x, y, and z, each of which is a function of s and t.

a. Draw a labeled tree diagram showing the relationships among the variables.

b. Write the Chain Rule formula for $\dfrac{\partial w}{\partial s}$.



EXAMPLE 4 **A different kind of tree** Let w be a function of z, where z is a function of x and y, and each of x and y is a function of t. Draw a labeled tree diagram and write the Chain Rule formula for dw/dt.



Implicit Differentiation

Recall: Given $F(x,y)=0$, to find $\frac{dy}{dx}$. Treat x as the independent variable and differentiate both sides of $F(x,y)=0$. Apply the Chain Rule of partial derivatives now,

$$\frac{\partial F}{\partial x}\frac{dx}{dx}+\frac{\partial F}{\partial y}\frac{dy}{dx}=0$$

THEOREM 9 Implicit Differentiation
Let F be differentiable on its domain and suppose that $F(x, y) = 0$ defines y as a differentiable function of x. Provided $F_y \neq 0$,

$$\frac{dy}{dx} = -\frac{F_x}{F_y}.$$

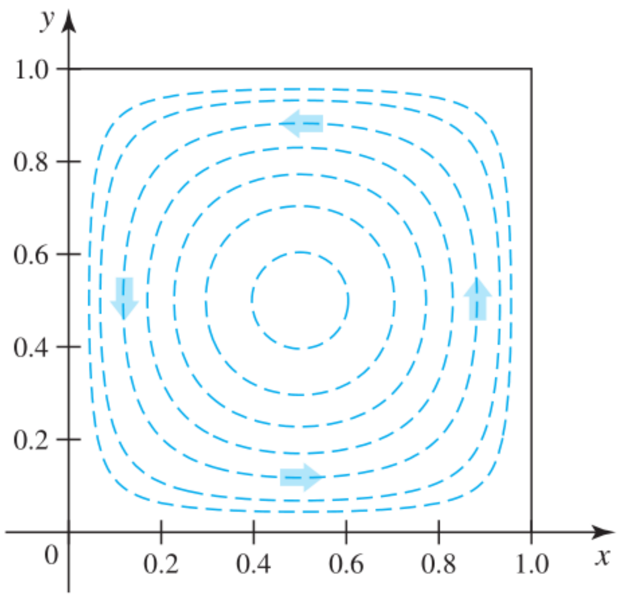
To compute $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ with functions of the form $F(x,y,z)=0$

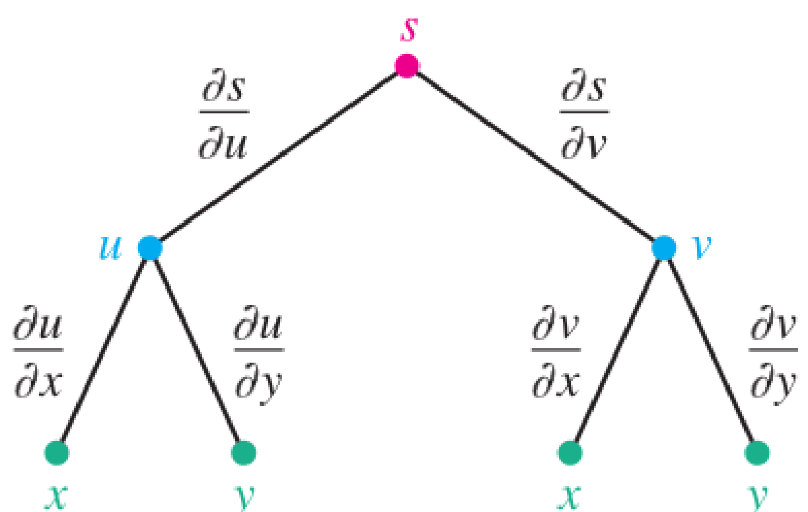
EXAMPLE 5 **Implicit differentiation** Find dy/dx when $F(x, y) = \sin xy + \pi y^2 - x = 0$.

EXAMPLE 6 **Fluid flow** A basin of circulating water is represented by the square region $\{(x, y): 0 \leq x \leq 1, 0 \leq y \leq 1\}$, where x is positive in the eastward direction and y is positive in the northward direction. The velocity components of the water are

$$\text{the east-west velocity } u(x, y) = 2 \sin \pi x \cos \pi y \text{ and}$$
$$\text{the north-south velocity } v(x, y) = -2 \cos \pi x \sin \pi y;$$

these velocity components produce the flow pattern shown in Figure 61. The *streamlines* shown in the figure are the paths followed by small parcels of water. The speed of the water at a point (x, y) is given by the function $s(x, y) = \sqrt{u(x, y)^2 + v(x, y)^2}$. Find $\partial s/\partial x$ and $\partial s/\partial y$, the rates of change of the water speed in the x- and y-directions, respectively.





EXAMPLE 7 **Second derivatives** Let $z = f(x, y) = \frac{x}{y}$, where $x = s + t^2$ and $y = s^2 - t$. Compute $\frac{\partial^2 z}{\partial s^2} = z_{ss}$, $\frac{\partial^2 z}{\partial t \partial s} = z_{st}$, and $\frac{\partial^2 z}{\partial t^2} = z_{tt}$, and express the results in terms of s and t. We use subscripts for partial derivatives in this example to simplify the notation.

Chapter 15

Functions of Several Variables (I)

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