Chapter 11

Power Series

Shuwei Chen
swchen@swjtu.edu.cn

11.1

Approximating Functions with Polynomials

What Is a Power Series?

Power series provide a way to represent familiar functions and to define new functions. Infinite series taking the form

$$\sum_{k=0}^{\infty} c_k x^k = \underbrace{c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n}_{n\text{th-degree polynomial}} + \underbrace{c_{n+1} x^{n+1} + \cdots}_{\text{terms continue}},$$

or, more generally,

$$\sum_{k=0}^{\infty} c_k (x-a)^k = \underbrace{c_0 + c_1 (x-a) + \cdots + c_n (x-a)^n}_{n\text{th-degree polynomial}} + \underbrace{c_{n+1}(x-a)^{n+1} + \cdots}_{\text{terms continue}},$$

where the *center* of the series a and the coefficients c_k are constants.

From another point of view, a power series is built up from polynomials of increasing degree, *super-polynomials*

$$\left.\begin{array}{ll}
\text{Degree } 0: & c_0 \\
\text{Degree } 1: & c_0 + c_1 x \\
\text{Degree } 2: & c_0 + c_1 x + c_2 x^2 \\
\quad\vdots & \quad\vdots \qquad \vdots \\
\text{Degree } n: & c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n = \displaystyle\sum_{k=0}^{n} c_k x^k
\end{array}\right\} \text{Polynomials}$$

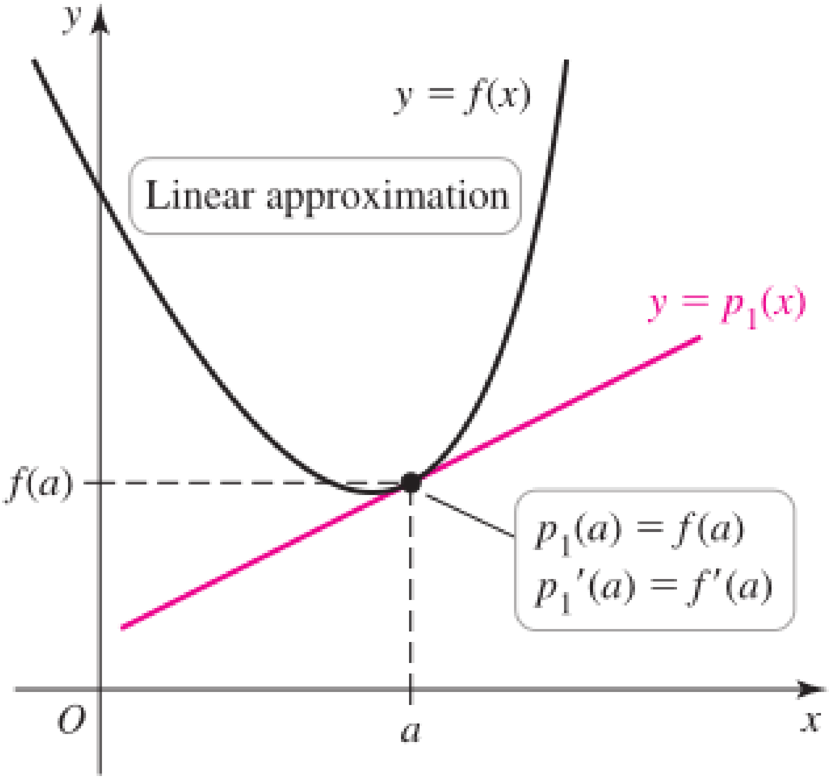
$$\vdots \qquad \vdots \qquad \vdots$$

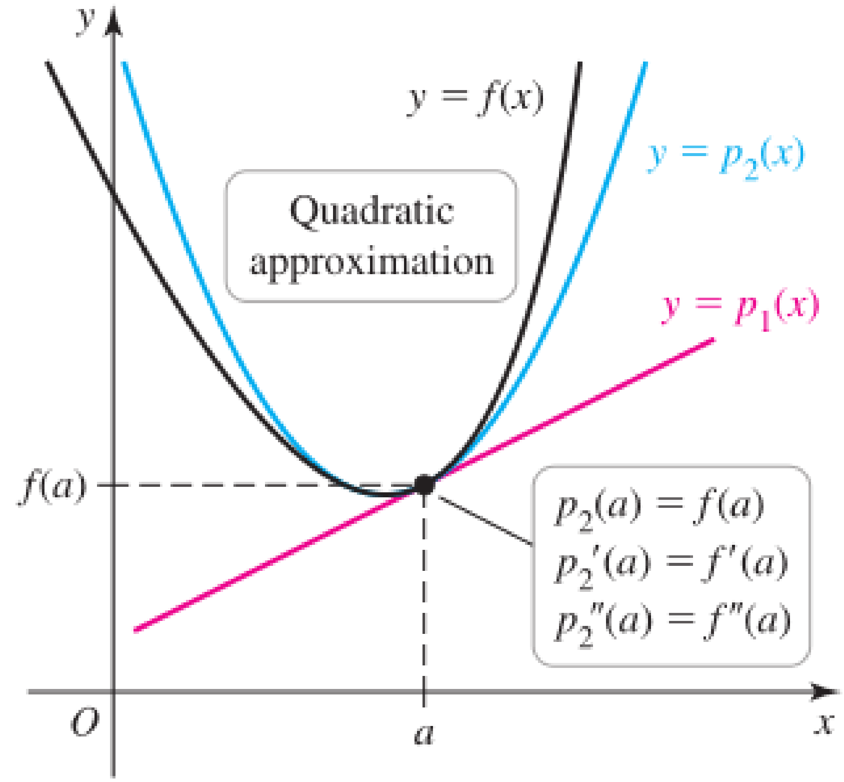
$$\left. c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots = \sum_{k=0}^{\infty} c_k x^k \right\} \text{Power series}$$

Polynomial Approximation

Linear Approximation: $p_1(x) = f(a) + f'(a)(x - a)$

Quadratic Approximation: $p_2(x) = \underbrace{f(a) + f'(a)(x - a)}_{p_1(x)} + \underbrace{c_2(x - a)^2}_{\text{quadratic term}}.$





To ensure that p_2 is a good approximation to f near the point a, we require that p_2 agree with f in *value, slope,* and *concavity* at a
That is,

$$p_2(a) = f(a), p_2'(a) = f'(a), p_2''(a) = f''(a)$$

$$p_2''(a) = 2c_2 = f''(a)$$

It follows that

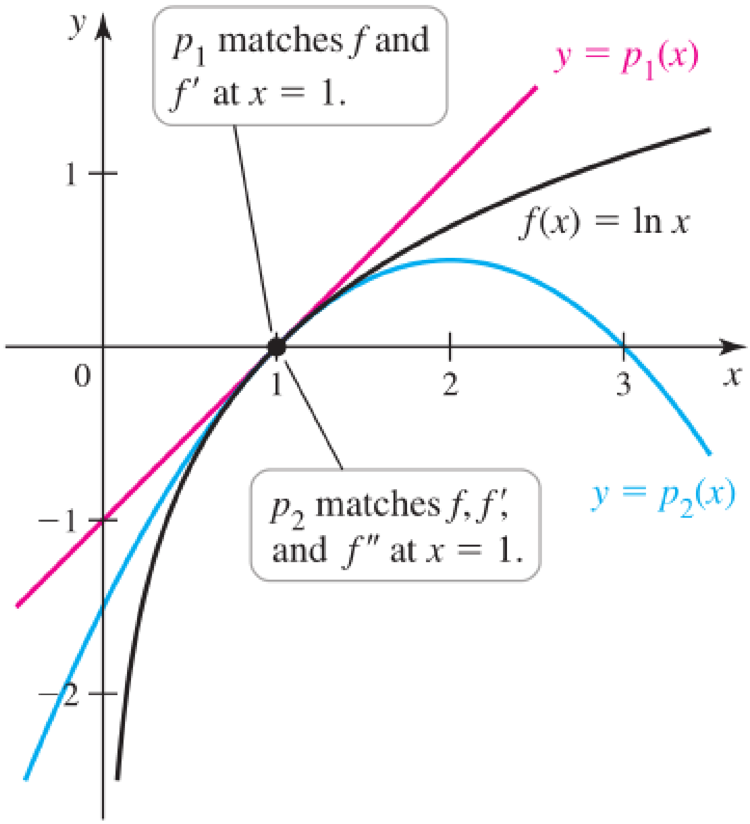
$$p_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

EXAMPLE 1 Linear and quadratic approximations for $\ln x$

a. Find the linear approximation to $f(x) = \ln x$ at $x = 1$.

b. Find the quadratic approximation to $f(x) = \ln x$ at $x = 1$.

c. Use these approximations to estimate $\ln 1.05$.



Taylor Polynomials

To find an nth-degree polynomial that approximates the values of f near a.

DEFINITION Taylor Polynomials

Let f be a function with f', f'', . . . , and $f^{(n)}$ defined at a. The ***n*th-order Taylor polynomial** for f with its **center** at a, denoted p_n, has the property that it matches f in value, slope, and all derivatives up to the nth derivative at a; that is,

$$p_n(a) = f(a), p_n'(a) = f'(a), \ldots, \text{ and } p_n^{(n)}(a) = f^{(n)}(a).$$

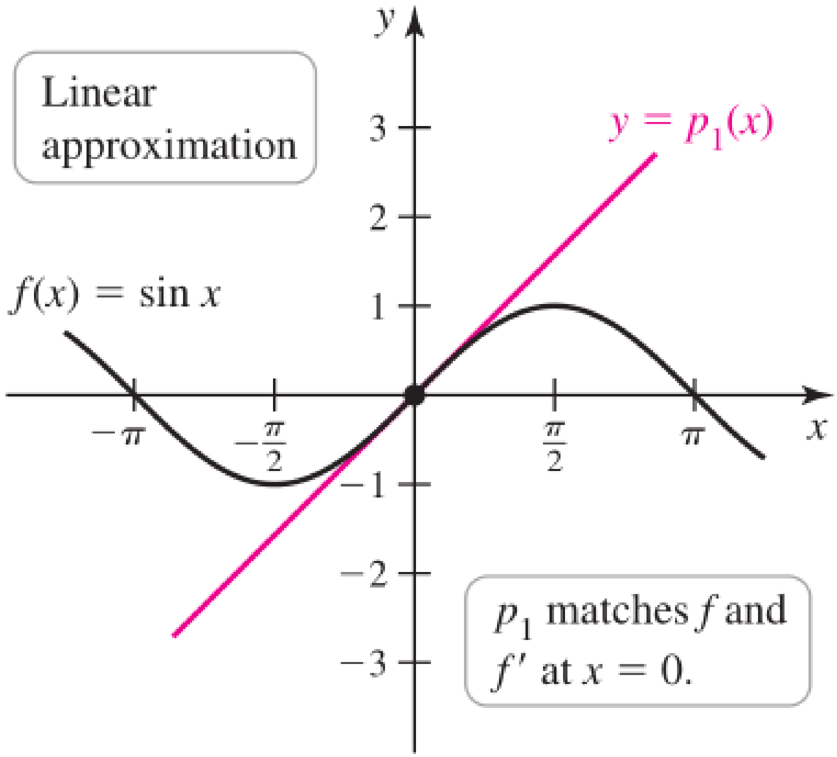
The nth-order Taylor polynomial centered at a is

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

More compactly, $p_n(x) = \sum_{k=0}^{n} c_k(x-a)^k$, where the **coefficients** are

$$c_k = \frac{f^{(k)}(a)}{k!}, \qquad \text{for } k = 0, 1, 2, \ldots, n.$$

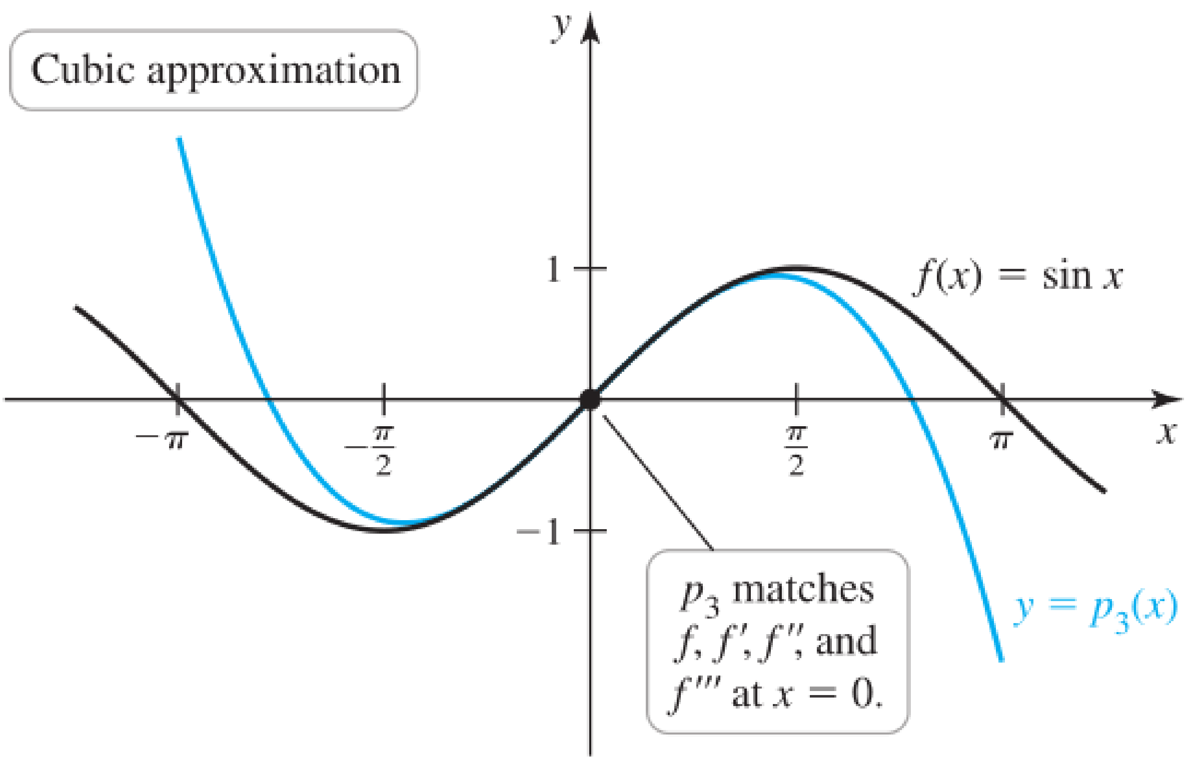
EXAMPLE 2 **Taylor polynomials for sin x** Find the Taylor polynomials $p_1, \ldots, p_7$ centered at $x = 0$ for $f(x) = \sin x$.



$$p_1(x) = f(0) + f'(0)(x - 0) = x,$$

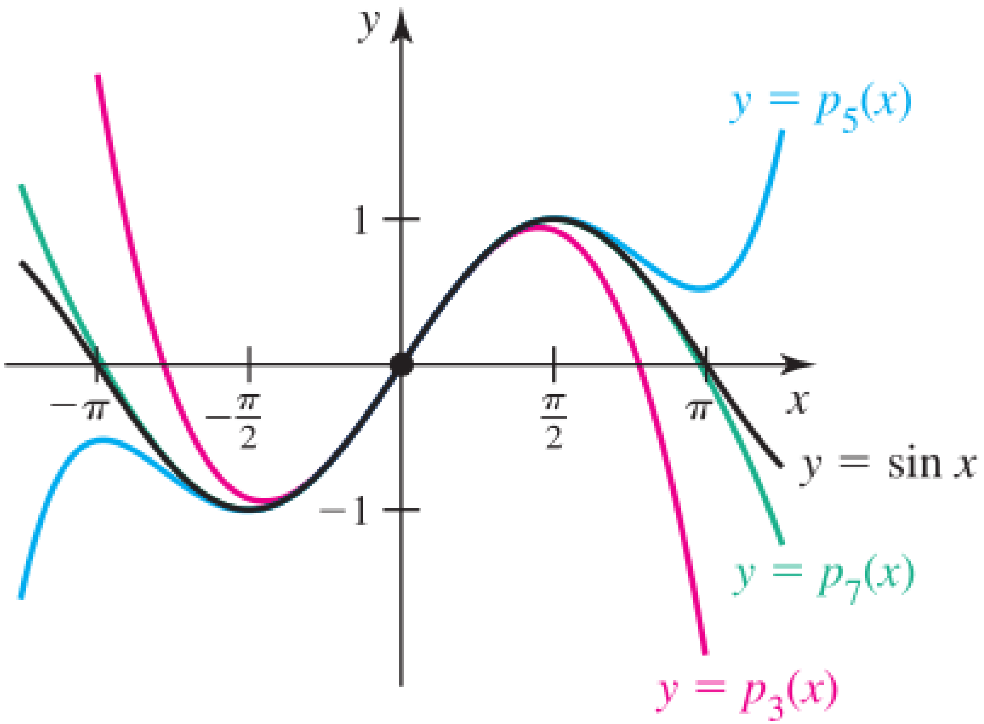
$$p_2(x) = \underbrace{f(0)}_{0} + \underbrace{f'(0)}_{1}x + \underbrace{\frac{f''(0)}{2!}}_{0}x^2 = x,$$

$$p_3(x) = \underbrace{f(0) + f'(0)x + \frac{f''(0)}{2!}x^2}_{p_2(x) = x} + \underbrace{\frac{f'''(0)}{3!}}_{-1/3!}x^3 = x - \frac{x^3}{6}.$$



$$p_6(x) = p_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}. \quad c_5 = \frac{f^{(5)}(0)}{5!} = \frac{1}{5!}$$

$$p_7(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}. \quad c_7 = \frac{f^{(7)}(0)}{7!} = -\frac{1}{7!}$$



Approximations with Taylor Polynomials

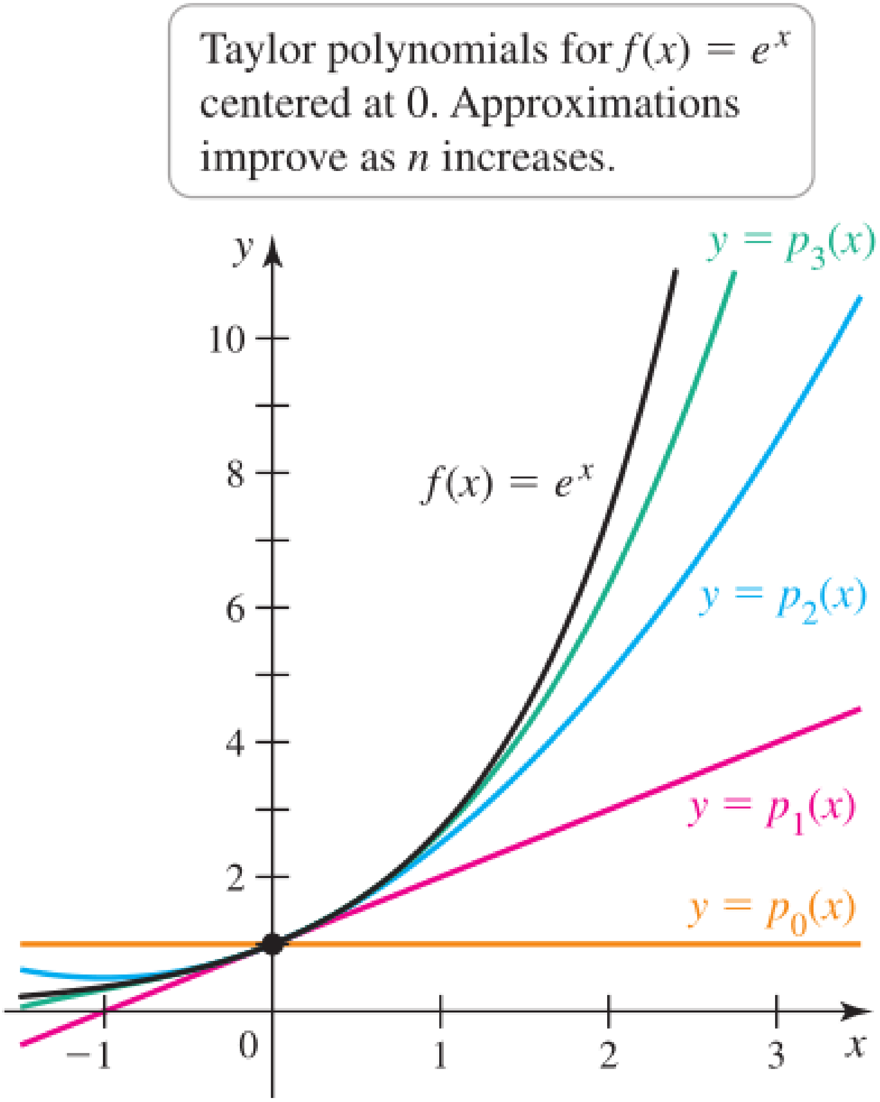
EXAMPLE 3 **Taylor polynomials for e^x**

a. Find the Taylor polynomials of order $n = 0, 1, 2,$ and 3 for $f(x) = e^x$ centered at 0. Graph f and the polynomials.

b. Use the polynomials in part (a) to approximate $e^{0.1}$ and $e^{-0.25}$. Find the absolute errors, $|f(x) - p_n(x)|$, in the approximations. Use calculator values for the exact values of f.

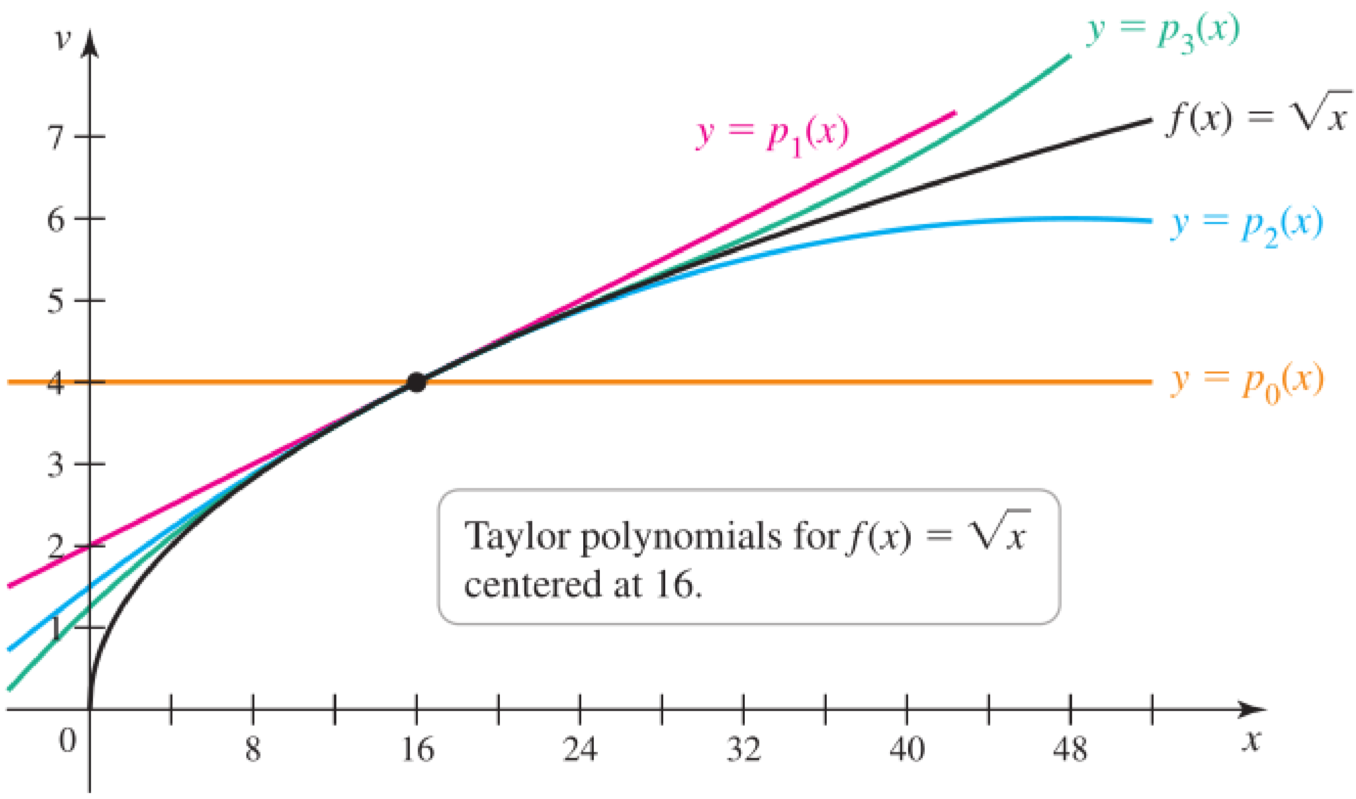
Table 1

n	Approximation $p_n(0.1)$	Absolute Error $\lvert e^{0.1} - p_n(0.1)\rvert$	Approximation $p_n(-0.25)$	Absolute Error $\lvert e^{-0.25} - p_n(-0.25)\rvert$
0	1	1.1×10^{-1}	1	2.2×10^{-1}
1	1.1	5.2×10^{-3}	0.75	2.9×10^{-2}
2	1.105	1.7×10^{-4}	0.78125	2.4×10^{-3}
3	1.105167	4.3×10^{-6}	0.778646	1.5×10^{-4}


Taylor polynomials for $f(x) = e^x$ centered at 0. Approximations improve as n increases.
y
10
8
6
4
2
0
-1
1
2
3
x
$f(x) = e^x$
$y = p_3(x)$
$y = p_2(x)$
$y = p_1(x)$
$y = p_0(x)$

EXAMPLE 4 **Approximating a real number using Taylor polynomials** Use polynomials of order $n = 0, 1, 2$, and 3 to approximate $\sqrt{18}$.

$$p_n(x) = f(16) + f'(16)(x - 16) + \frac{f''(16)}{2!}(x - 16)^2 + \cdots + \frac{f^{(n)}(16)}{n!}(x - 16)^n.$$



$$p_3(x) = \underbrace{\underbrace{\underbrace{4}_{p_0(x)} + \frac{1}{8}(x-16)}_{p_1(x)} - \frac{1}{512}(x-16)^2}_{p_2(x)} + \frac{1}{16{,}384}(x-16)^3.$$

Table 2

n	Approximation $p_n(18)$	Absolute Error $\lvert\sqrt{18} - p_n(18)\rvert$
0	4	2.4×10^{-1}
1	4.25	7.4×10^{-3}
2	4.242188	4.5×10^{-4}
3	4.242676	3.5×10^{-5}

Remainder in a Taylor Polynomial

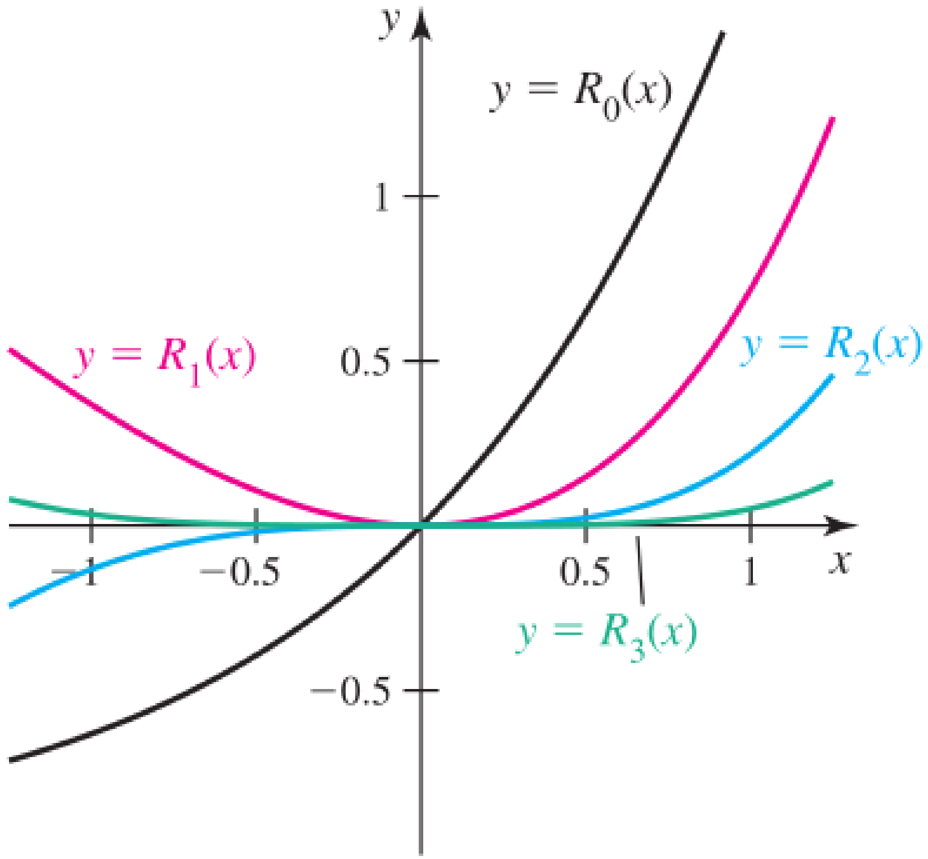
How accurate are the approximations?

Remainder: $R_n(x) = f(x) - p_n(x)$

DEFINITION Remainder in a Taylor Polynomial

Let p_n be the Taylor polynomial of order n for f. The **remainder** in using p_n to approximate f at the point x is

$$R_n(x) = f(x) - p_n(x).$$



Remainders increase in magnitude as $|x|$ increases. Remainders decrease in magnitude to zero as n increases.

THEOREM 1 Taylor's Theorem (Remainder Theorem)
Let f have continuous derivatives up to $f^{(n+1)}$ on an open interval I containing a. For all x in I,

$$f(x) = p_n(x) + R_n(x),$$

where p_n is the nth-order Taylor polynomial for f centered at a and the remainder is

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1},$$

for some point c between x and a.

Discussion:

First, the case $n = 0$ is the Mean Value Theorem, $c \in (x, a)$

$$\frac{f(x) - f(a)}{x - a} = f'(c)$$

Rearrange it to obtain

$$f(x) = f(a) + f'(c)(x - a) = p_0(x) + R_0(x)$$

Second observation

Write the $(n + 1)$st Taylor polynomial p_{n+1} , the highest-degree term is $\frac{f^{(n+1)}(a)}{(n+1)!}(x - a)^{(n+1)}$.

Replacing $f^{(n+1)}(a)$ with $f^{(n+1)}(c)$ results in the remainder for p_n.

Estimating the Remainder

The remainder has both *practical* and *theoretical* importance.

THEOREM 2 Estimate of the Remainder
Let n be a fixed positive integer. Suppose there exists a number M such that $|f^{(n+1)}(c)| \leq M$, for all c between a and x inclusive. The remainder in the nth-order Taylor polynomial for f centered at a satisfies

$$|R_n(x)| = |f(x) - p_n(x)| \leq M \frac{|x - a|^{n+1}}{(n + 1)!}.$$

EXAMPLE 5 **Estimating the remainder for cos x** Find a bound for the magnitude of the remainder for the Taylor polynomials of $f(x) = \cos x$ centered at 0.

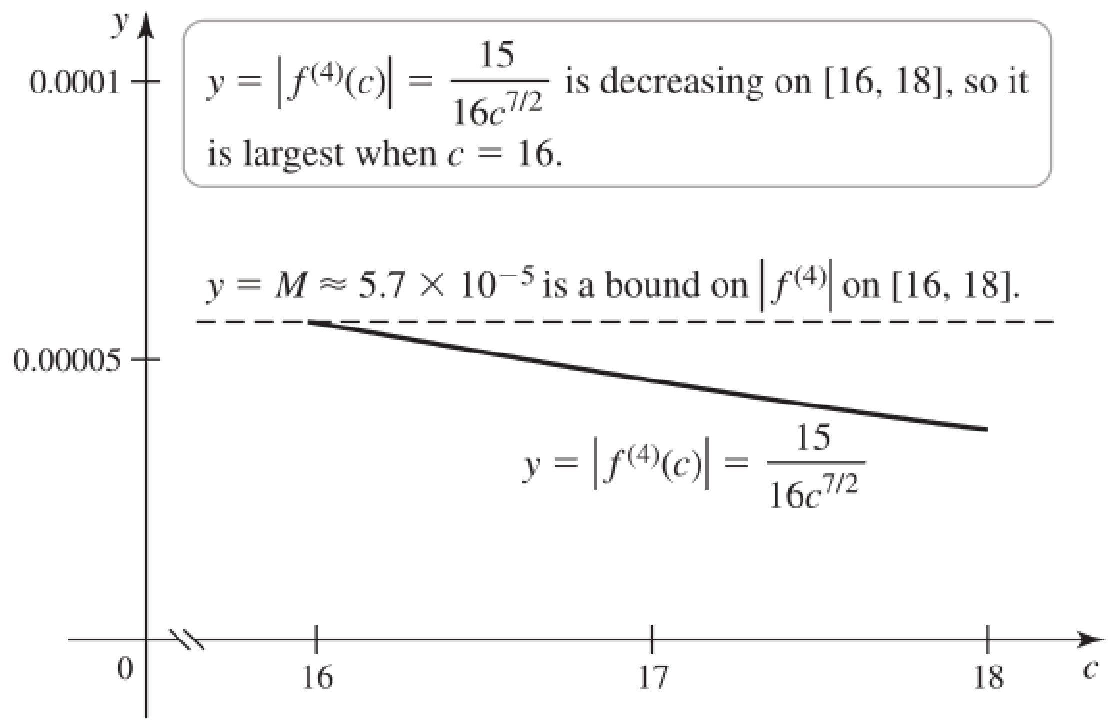
$$|R_n(x)| = \left|\frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}\right| \le \frac{|x|^{n+1}}{(n+1)!}.$$

EXAMPLE 6 **Estimating a remainder** Consider again Example 4 in which we approximated $\sqrt{18}$ using the Taylor polynomial

$$p_3(x) = 4 + \frac{1}{8}(x - 16) - \frac{1}{512}(x - 16)^2 + \frac{1}{16{,}384}(x - 16)^3.$$

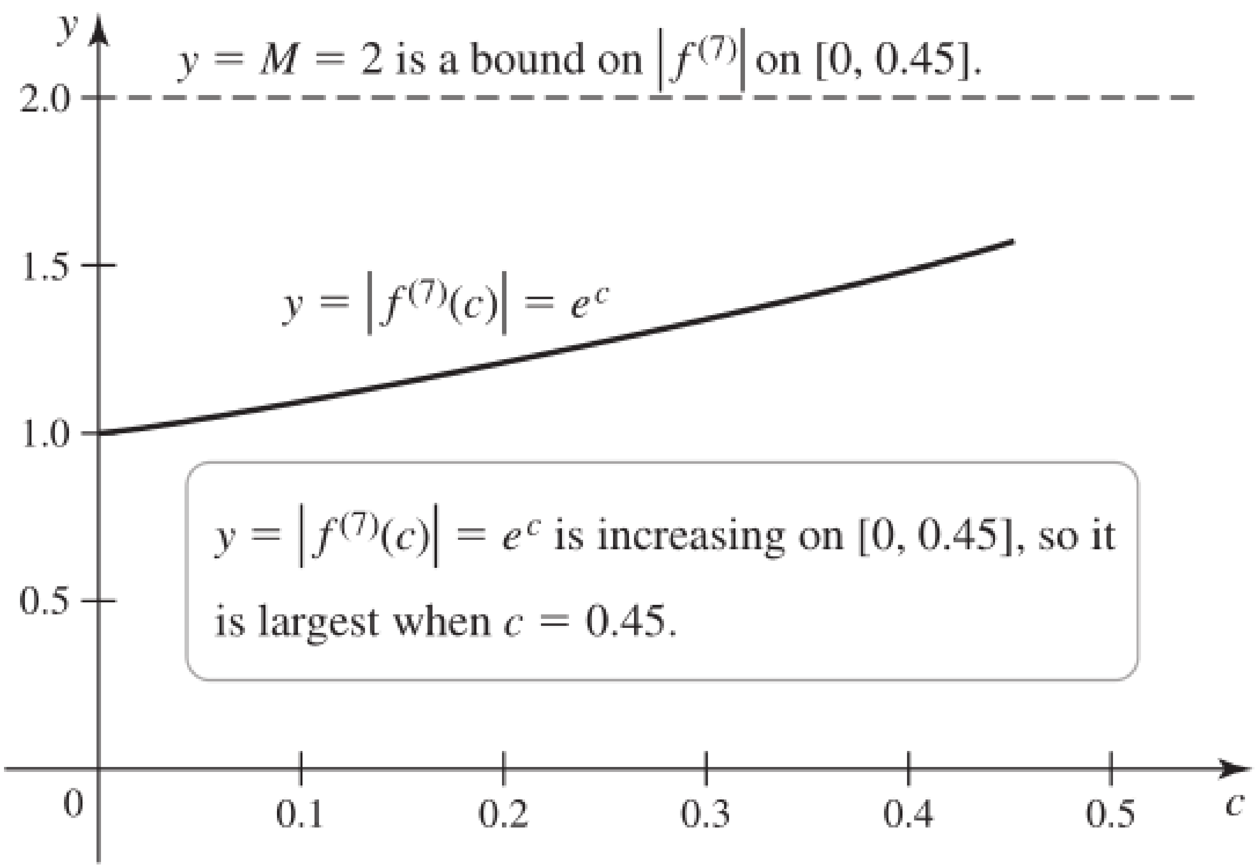
Find an upper bound on the magnitude of the remainder when using $p_3(x)$ to approximate $\sqrt{18}$.

$$|R_3(18)| \leq M\frac{(18 - 16)^4}{4!} = \frac{2}{3}M,$$



EXAMPLE 7 **Estimating the remainder for e^x** Find a bound on the remainder in approximating $e^{0.45}$ using the Taylor polynomial of order $n = 6$ for $f(x) = e^x$ centered at 0.

$$|R_6(0.45)| \le M\frac{|0.45 - 0|^7}{7!} \approx 7.4 \times 10^{-7} M, \approx 1.5 \times 10^{-6}.$$



EXAMPLE 8 **Working with the remainder** The nth-order Taylor polynomial for $f(x) = \ln(1 - x)$ centered at 0 is

$$p_n(x) = -\sum_{k=1}^{n} \frac{x^k}{k} = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots - \frac{x^n}{n}.$$

a. Find a bound on the error in approximating $\ln(1 - x)$ by $p_3(x)$ for values of x in the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$.

b. How many terms of the Taylor polynomial are needed to approximate values of $f(x) = \ln(1 - x)$ with an error less than 10^{-3} on the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$?

$$\circ R_n(x) \circ = \frac{1}{(n+1)!} \cdot \underbrace{\circ f^{(n+1)}(c) \circ}_{\le n!2^{n+1}} \cdot \underbrace{\circ x \circ^{n+1}}_{\le \left(\frac{1}{2}\right)^{n+1}}$$

$$\le \frac{1}{(n+1)!} \cdot n!2^{n+1} \cdot \frac{1}{2^{n+1}}$$

$$= \frac{1}{n+1}.$$

11.2

Properties of Power Series

Geometric Series as Power Series

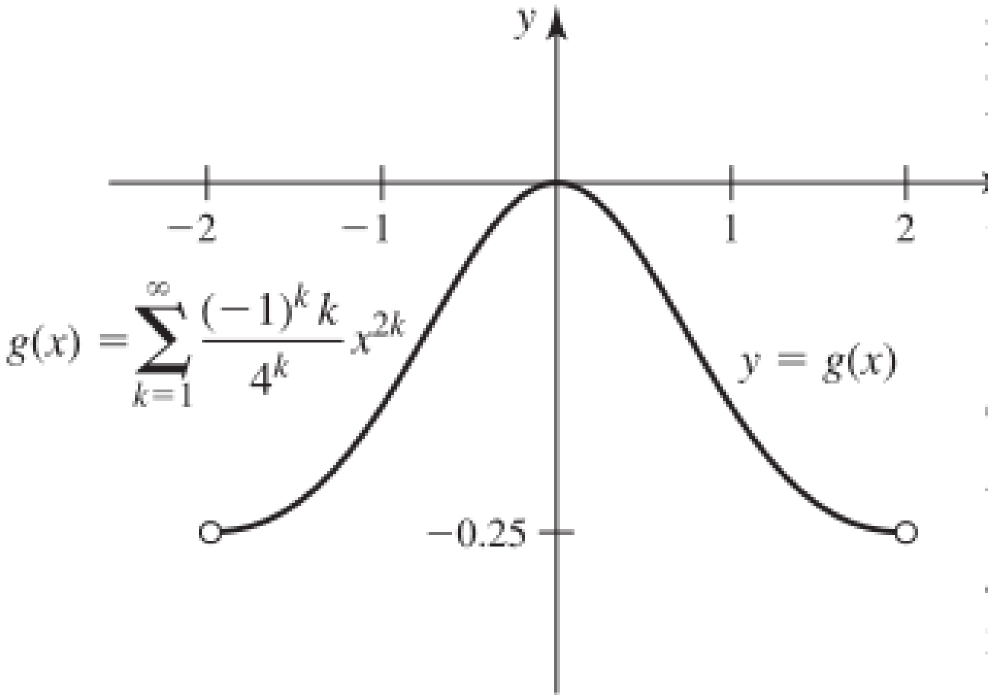
$$\sum_{k=0}^{\infty} r^k = 1 + r + r^2 + \cdots = \frac{1}{1-r}, \qquad \text{provided } |r| < 1.$$

A small change to replace the real number r with the variable x

$$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots = \frac{1}{1-x}, \qquad \text{provided } |x| < 1.$$

Power series are generally used to represent functions such as trigonometric, exponential, and logarithmic functions, e.g.,

$$g(x) = \sum_{k=1}^{\infty} \frac{(-1)^k k}{4^k} x^{2k}$$



Problems:

Is g really a function?

If so, is it continuous? Does it have a derivative?

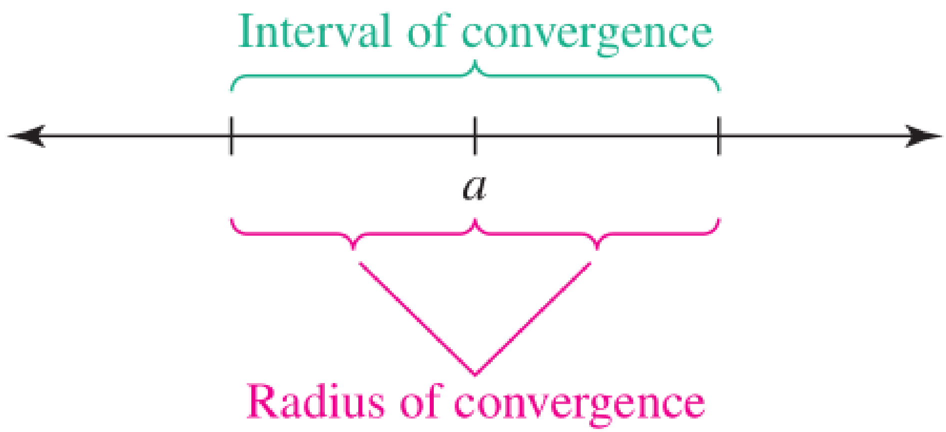
Convergence of Power Series

DEFINITION Power Series

A **power series** has the general form

$$\sum_{k=0}^{\infty} c_k (x - a)^k,$$

where a and c_k are real numbers, and x is a variable. The c_k's are the **coefficients** of the power series and a is the **center** of the power series. The set of values of x for which the series converges is its **interval of convergence**. The **radius of convergence** of the power series, denoted R, is the distance from the center of the series to the boundary of the interval of convergence



Problems: How to determine the interval of convergence for a given power series?

Ratio Test or the Root Test to test a power series for absolute convergence

$$r = \lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right| < 1$$

$$\rho = \lim_{k \to \infty} \sqrt[k]{|a_k|} < 1$$

EXAMPLE 1 **Interval and radius of convergence** Find the interval and radius of convergence for each power series.

a. $\displaystyle\sum_{k=0}^{\infty} \frac{x^k}{k!}$ **b.** $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k(x-2)^k}{4^k}$ **c.** $\displaystyle\sum_{k=1}^{\infty} k!\, x^k$

b. Need to test the convergence at the endpoints separately where the Root Test results is $\rho = 1$.

Example 1 illustrates the ***three common types* of intervals of convergence**, which are summarized in the following theorem

THEOREM 3 Convergence of Power Series

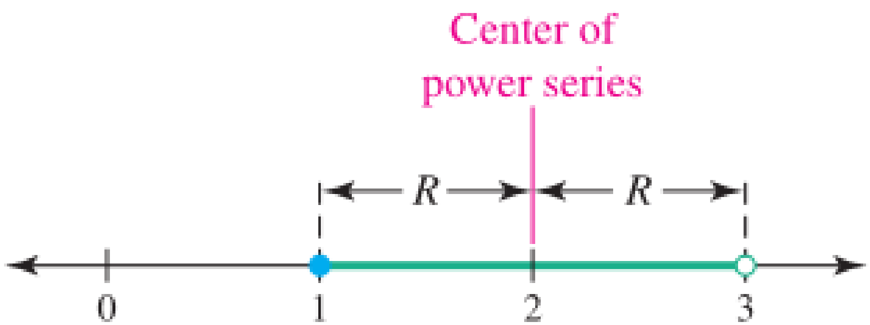
A power series $\sum_{k=0}^{\infty} c_k(x-a)^k$ centered at a converges in one of three ways:

1. The series converges for all x, in which case the interval of convergence is $(-\infty, \infty)$ and the radius of convergence is $R = \infty$.
2. There is a real number $R > 0$ such that the series converges for $|x - a| < R$ and diverges for $|x - a| > R$, in which case the radius of convergence is R.
3. The series converges only at a, in which case the radius of convergence is $R = 0$.

Interval of convergence is *symmetric* about the center of the series;
Radius of convergence R is determined by analyzing r from the Ratio Test (Root Test)

EXAMPLE 2 **Interval and radius of convergence** Use the Ratio Test to find the radius and interval of convergence of $\sum_{k=1}^{\infty} \frac{(x-2)^k}{\sqrt{k}}$.

Geometric series



Interval of convergence $[1, 3)$
Radius of convergence $R = 1$

Need to test the convergence at the endpoints separately.

Combining Power Series

THEOREM 4 Combining Power Series

Suppose the power series $\sum c_k x^k$ and $\sum d_k x^k$ converge to $f(x)$ and $g(x)$, respectively, on an interval I.

1. **Sum and difference:** The power series $\sum (c_k \pm d_k)x^k$ converges to $f(x) \pm g(x)$ on I.
2. **Multiplication by a power:** Suppose m is an integer such that $k + m \geq 0$ for all terms of the power series $x^m \sum c_k x^k = \sum c_k x^{k+m}$. This series converges to $x^m f(x)$ for all $x \neq 0$ in I. When $x = 0$, the series converges to $\lim_{x \to 0} x^m f(x)$.
3. **Composition:** If $h(x) = bx^m$, where m is a positive integer and b is a nonzero real number, the power series $\sum c_k (h(x))^k$ converges to the composite function $f(h(x))$, for all x such that $h(x)$ is in I.

EXAMPLE 3 **Combining power series** Given the geometric series

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots, \qquad \text{for } |x| < 1,$$

find the power series and interval of convergence for the following functions.

a. $\dfrac{x^5}{1-x}$ **b.** $\dfrac{1}{1-2x}$ **c.** $\dfrac{1}{1+x^2}$

All are geometric series

Differentiating and Integrating Power Series

THEOREM 5 Differentiating and Integrating Power Series
Suppose the power series $\sum c_k(x-a)^k$ converges for $|x-a| < R$ and defines a function f on that interval.

1. Then f is differentiable (which implies continuous) for $|x-a| < R$, and f' is found by differentiating the power series for f term by term; that is,
$$f'(x) = \sum k c_k (x-a)^{k-1},$$
for $|x-a| < R$.
2. The indefinite integral of f is found by integrating the power series for f term by term; that is,
$$\int f(x)dx = \sum c_k \frac{(x-a)^{k+1}}{k+1} + C,$$
for $|x-a| < R$, where C is an arbitrary constant.

Term-by-term differentiation and integration say two things:

- The differentiated and integrated power series converge, provided x belongs to the interior of interval of convergence
- The differentiated and integrated power series converge to the derivative and indefinite integral of f, respectively, on the interior of the interval of convergence

EXAMPLE 4 **Differentiating and integrating power series** Consider the geometric series

$$f(x) = \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots, \quad \text{for } |x| < 1.$$

a. Differentiate this series term by term to find the power series for f' and identify the function it represents.

b. Integrate this series term by term and identify the function it represents.

Obtain a series representation for $\ln(1-x)$

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots = -\sum_{k=1}^{\infty} \frac{x^k}{k}$$

Interesting to test the endpoints of the interval $|x| < 1$

EXAMPLE 5 **Functions to power series** Find power series representations centered at 0 for the following functions and give their intervals of convergence.

a. $\tan^{-1}x$ **b.** $\ln\left(\frac{1+x}{1-x}\right)$

Work with known power series and use differentiation, integration, and other combinations

Convergence at the endpoints?

Every example in this section is ultimately based on the geometric series

Taylor Series for a Function

Question:

Given a function, what is its power series representation?

From *Taylor polynomials* to *Taylor series*

$$\underbrace{c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n}_{\text{Taylor polynomial of order } n} + \underbrace{\cdots}_{n \to \infty} = \sum_{k=0}^{\infty} c_k(x-a)^k.$$

The coefficients of the Taylor polynomial, also the power series

$$c_k = \frac{f^{(k)}(a)}{k!}$$

11.3

Taylor Series

DEFINITION Taylor/Maclaurin Series for a Function

Suppose the function f has derivatives of all orders on an interval centered at the point a. The **Taylor series for f centered at a** is

$$f(a) + f'(a)(x-a) + \frac{f'(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

A Taylor series centered at 0 is called a **Maclaurin series**.

Two key points:

- the values of x for which the Taylor series converges
- the values of x for which the Taylor series for f equals f

First step: Verify that if the Taylor series for f centered at a is evaluated at $x = a$, then the Taylor series equals $f(a)$.

EXAMPLE 1 **Maclaurin series and convergence** Find the Maclaurin series (which is the Taylor series centered at 0) for the following functions. Find the interval of convergence.

a. $f(x) = \cos x$ **b.** $f(x) = \dfrac{1}{1 - x}$

Important lesson:

- There is only one power series representation for a given function about a given point.
- However, there may be several ways to find it.

EXAMPLE 2 **Center other than 0** Find the first four nonzero terms of the Taylor series for $f(x) = \sqrt[3]{x}$ centered at 8.

EXAMPLE 3 **Manipulating Maclaurin series** Let $f(x) = e^x$.

a. Find the Maclaurin series for f.

b. Find its interval of convergence.

c. Use the Maclaurin series for e^x to find the Maclaurin series for the functions $x^4 e^x$, e^{-2x}, and e^{-x^2}.

The Binomial Series

$(1+x)^p$ is a polynomial of degree p, i.e.,

$$(1+x)^p = \binom{p}{0} + \binom{p}{1}x + \binom{p}{2}x^2 + \cdots + \binom{p}{p}x^p$$

where the binomial coefficients $\binom{p}{k}$ are defined as follows.

DEFINITION Binomial Coefficients

For real numbers p and integers $k \geq 1$,

$$\binom{p}{k} = \frac{p(p-1)(p-2)\cdots(p-k+1)}{k!}, \qquad \binom{p}{0} = 1.$$

Goal: extend to $f(x) = (1+x)^p$ where $p \neq 0$ is a real number.

THEOREM 6 Binomial Series

For real numbers $p \neq 0$, the Taylor series for $f(x) = (1 + x)^p$ centered at 0 is the **binomial series**

$$\sum_{k=0}^{\infty} \binom{p}{k} x^k = 1 + \sum_{k=1}^{\infty} \frac{p(p-1)(p-2)\cdots(p-k+1)}{k!} x^k$$

$$= 1 + px + \frac{p(p-1)}{2!} x^2 + \frac{p(p-1)(p-2)}{3!} x^3 + \cdots.$$

The series converges for $|x| < 1$ (and possibly at the endpoints, depending on p). If p is a nonnegative integer, the series terminates and results in a polynomial of degree p.

- $(-1, 1)$ if $p \leq -1$,
- $(-1, 1]$ if $-1 < p < 0$, and
- $[-1, 1]$ if $p > 0$ and not an integer.

The interval of convergence for the binomial series is determined by the Ratio Test.

$$r = \lim_{k\to\infty}\left|\frac{x^{k+1}p(p-1)\cdots(p-k+1)(p-k)/(k+1)!}{x^k p(p-1)\cdots(p-k+1)/k!}\right|$$

Ratio of $(k+1)$st to kth term

$$= |x|\lim_{k\to\infty}\underbrace{\left|\frac{p-k}{k+1}\right|}_{\text{approaches } 1}$$

Cancel factors and simplify.

$$= |x|.$$

With p fixed, $\lim_{k\to\infty}\left|\frac{(p-k)}{k+1}\right| = 1.$

Therefore, the series converges for $|x| < 1$.

EXAMPLE 4 **Binomial series** Consider the function $f(x) = \sqrt{1 + x}$.

a. Find the first four terms of the binomial series for f centered at 0.

b. Approximate $\sqrt{1.15}$ to three decimal places. Assume the series for f converges to f on its interval of convergence, which is $[-1, 1]$.

Table 3

n	Approximation $p_n(0.15)$
0	1.0
1	1.075
2	1.0721875
3	1.072398438

EXAMPLE 5 **Working with binomial series** Consider the functions

$$f(x) = \sqrt[3]{1 + x} \quad \text{and} \quad g(x) = \sqrt[3]{c + x}, \text{ where } c > 0 \text{ is a constant.}$$

a. Find the first four terms of the binomial series for f centered at 0.

b. Use part (a) to find the first four terms of the binomial series for g centered at 0.

c. Use part (b) to approximate $\sqrt[3]{23}, \sqrt[3]{24}, \ldots, \sqrt[3]{31}$. Assume the series for g converges to g on its interval of convergence.

Table 4

	Approximation	Absolute Error
$\sqrt[3]{23}$	2.8439	6.7×10^{-5}
$\sqrt[3]{24}$	2.8845	2.0×10^{-5}
$\sqrt[3]{25}$	2.9240	3.9×10^{-6}
$\sqrt[3]{26}$	2.9625	2.4×10^{-7}
$\sqrt[3]{27}$	3	0

Convergence of Taylor Series

Question:
When the Taylor series for f actually converges to f on its interval of convergence?

Assume f has derivatives of all orders on an open interval containing the point a.
Taylor's Theorem (Remainder Theorem) tells

$$f(x) = p_n(x) + R_n(x)$$

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

where c is a point between x and a.
The question is then: when $\lim_{n\to\infty} p_n(x) = f(x)$?

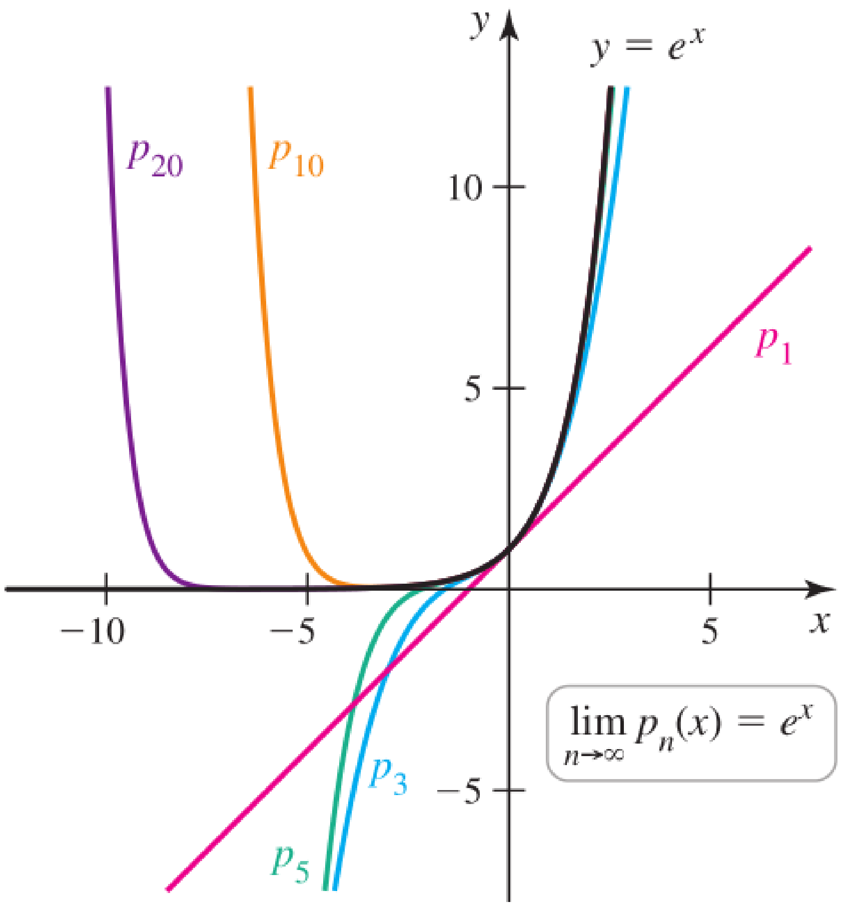
THEOREM 7 Convergence of Taylor Series

Let f have derivatives of all orders on an open interval I containing a. The Taylor series for f centered at a converges to f, for all x in I, if and only if $\lim_{n\to\infty} R_n(x) = 0$, for all x in I, where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

is the remainder at x (with c between x and a).

EXAMPLE 6 Remainder in the Maclaurin series for e^x Show that the Maclaurin series for $f(x) = e^x$ converges to $f(x)$, for $-\infty < x < \infty$.



EXAMPLE 7 **Maclaurin series convergence for cos x** Show that the Maclaurin series for cos x,

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!},$$

converges to $f(x) = \cos x$, for $-\infty < x < \infty$.

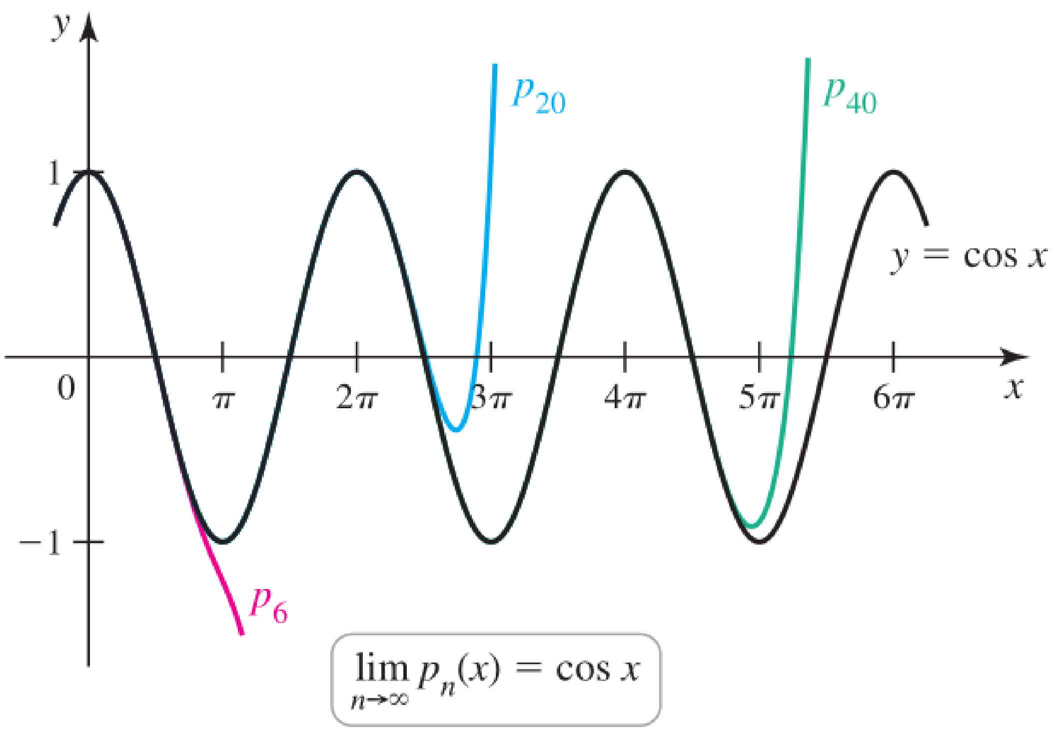


Table 5

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^k + \cdots = \sum_{k=0}^{\infty} x^k, \quad \text{for } |x| < 1$$

$$\frac{1}{1+x} = 1 - x + x^2 - \cdots + (-1)^k x^k + \cdots = \sum_{k=0}^{\infty} (-1)^k x^k, \quad \text{for } |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \quad \text{for } |x| < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^k x^{2k+1}}{(2k+1)!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}, \quad \text{for } |x| < \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^k x^{2k}}{(2k)!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}, \quad \text{for } |x| < \infty$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{k+1}x^k}{k} + \cdots = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}x^k}{k}, \quad \text{for } -1 < x \le 1$$

$$-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^k}{k} + \cdots = \sum_{k=1}^{\infty} \frac{x^k}{k}, \quad \text{for } -1 \le x < 1$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + \frac{(-1)^k x^{2k+1}}{2k+1} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}, \quad \text{for } |x| \le 1$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{x^{2k+1}}{(2k+1)!} + \cdots = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}, \quad \text{for } |x| < \infty$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{2k}}{(2k)!} + \cdots = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}, \quad \text{for } |x| < \infty$$

$$(1+x)^p = \sum_{k=0}^{\infty} \binom{p}{k} x^k, \text{ for } |x| < 1 \text{ and } \binom{p}{k} = \frac{p(p-1)(p-2)\cdots(p-k+1)}{k!}, \binom{p}{0} = 1$$

11.4

Working with Taylor Series

From limits and derivatives to integrals and approximation

Limits by Taylor Series

EXAMPLE 1 **A limit by Taylor series** Evaluate $\lim_{x \to 0} \frac{x^2 + 2\cos x - 2}{3x^4}$.

L'Hôpital's Rule can be used, but requires four applications of the rule.

Substitute the Maclaurin series for $\cos x$

EXAMPLE 2 **A limit by Taylor series** Evaluate

$$\lim_{x \to \infty} \left(6x^5 \sin \frac{1}{x} - 6x^4 + x^2 \right).$$

Technique: replace x with $\frac{1}{t}$ and note that as $x \to \infty$, $t \to 0^+$

Differentiating Power Series

EXAMPLE 3 **Power series for derivatives** Differentiate the Maclaurin series for $f(x) = \sin x$ to verify that $\frac{d}{dx}(\sin x) = \cos x$.

EXAMPLE 4 **A differential equation** Find a power series solution of the differential equation $y'(t) = y + 2$, subject to the initial condition $y(0) = 6$. Identify the function represented by the power series.

Integrating Power Series

EXAMPLE 5 **Approximating a definite integral** Approximate the value of the integral $\int_0^1 e^{-x^2}\,dx$ with an error no greater than 5×10^{-4}.

$$\int_0^1 e^{-x^2}\,dx \approx 1 - \frac{1}{3} + \frac{1}{5\cdot 2!} - \frac{1}{7\cdot 3!} + \frac{1}{9\cdot 4!} - \frac{1}{11\cdot 5!} \approx 0.747.$$

Representing Real Numbers

Substitute values of x into a convergent power series, the result may be a series representation of a familiar real number.

EXAMPLE 6 **Evaluating infinite series**

a. Use the Maclaurin series for $f(x) = \tan^{-1} x$ to evaluate

$$1 - \frac{1}{3} + \frac{1}{5} - \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k + 1}.$$

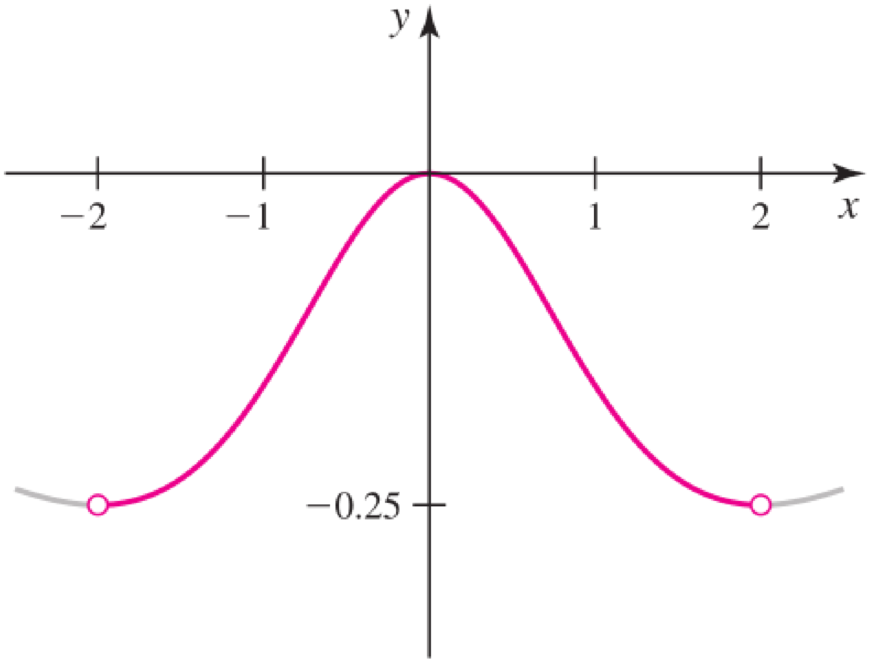
b. Let $f(x) = (e^x - 1)/x$, for $x \neq 0$, and $f(0) = 1$. Use the Maclaurin series for f to evaluate $f'(1)$ and $\displaystyle\sum_{k=1}^{\infty} \frac{k}{(k + 1)!}$.

Representing Functions as Power Series

Providing alternative representations of familiar functions

EXAMPLE 7 **Identify the series** Identify the function represented by the power series $\sum_{k=0}^{\infty} \frac{(1-2x)^k}{k!}$ and give its interval of convergence.

EXAMPLE 8 **Mystery series** The power series $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k k}{4^k} x^{2k}$ appeared in the opening of Section 9.2. Determine the interval of convergence of the power series and find the function it represents on this interval.



$$\sum_{k=1}^{\infty} \frac{(-1)^k k}{4^k} x^{2k} = -\frac{4x^2}{(4 + x^2)^2} \text{ on } (-2, 2)$$

Chapter 11

Power Series

Shuwei Chen
swchen@swjtu.edu.cn