Chapter 3

Derivatives (I)

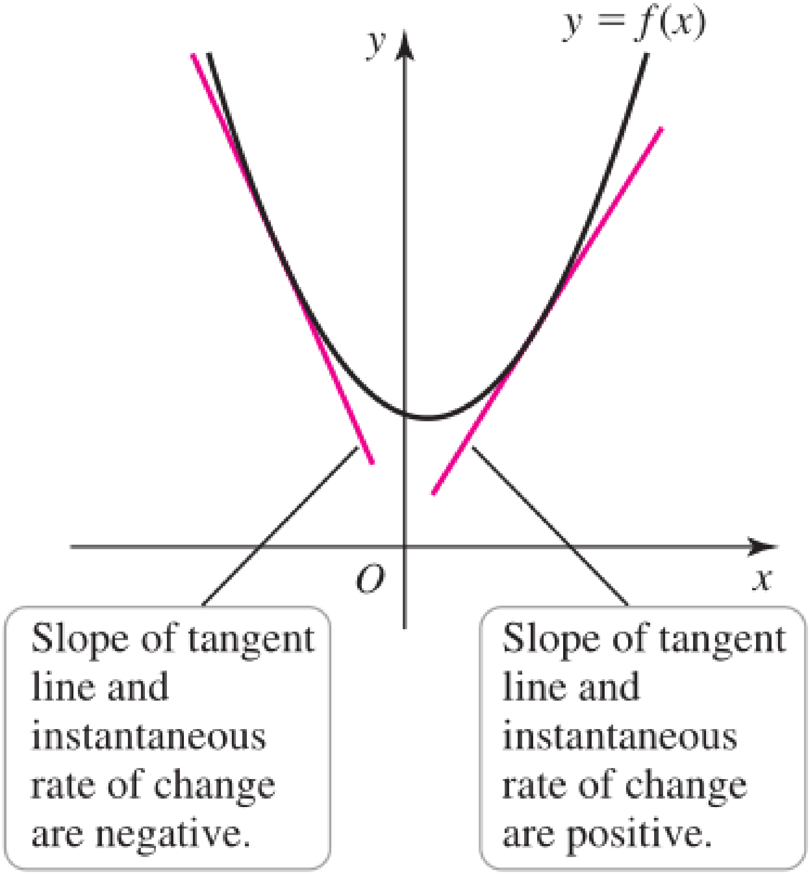
Shuwei Chen
swchen@swjtu.edu.cn

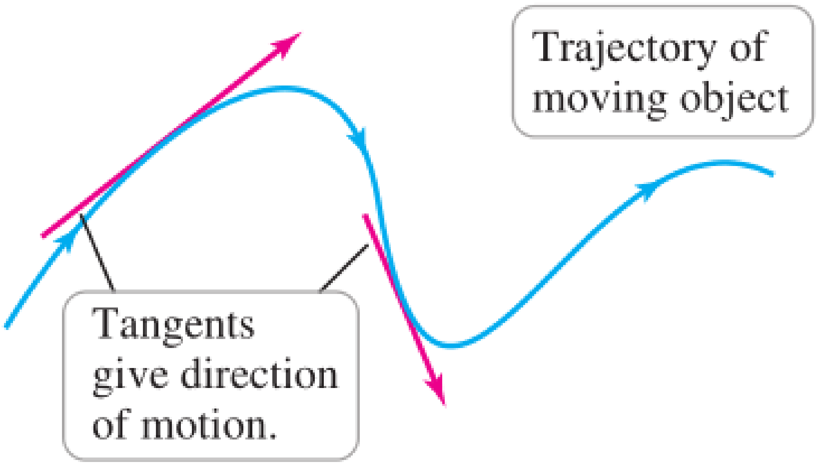
3.1

Introducing the Derivative

Slope of tangent line

Instantaneous rate of change of a function variable





Instantaneous velocity

$s(t)$: the position of an object at time t

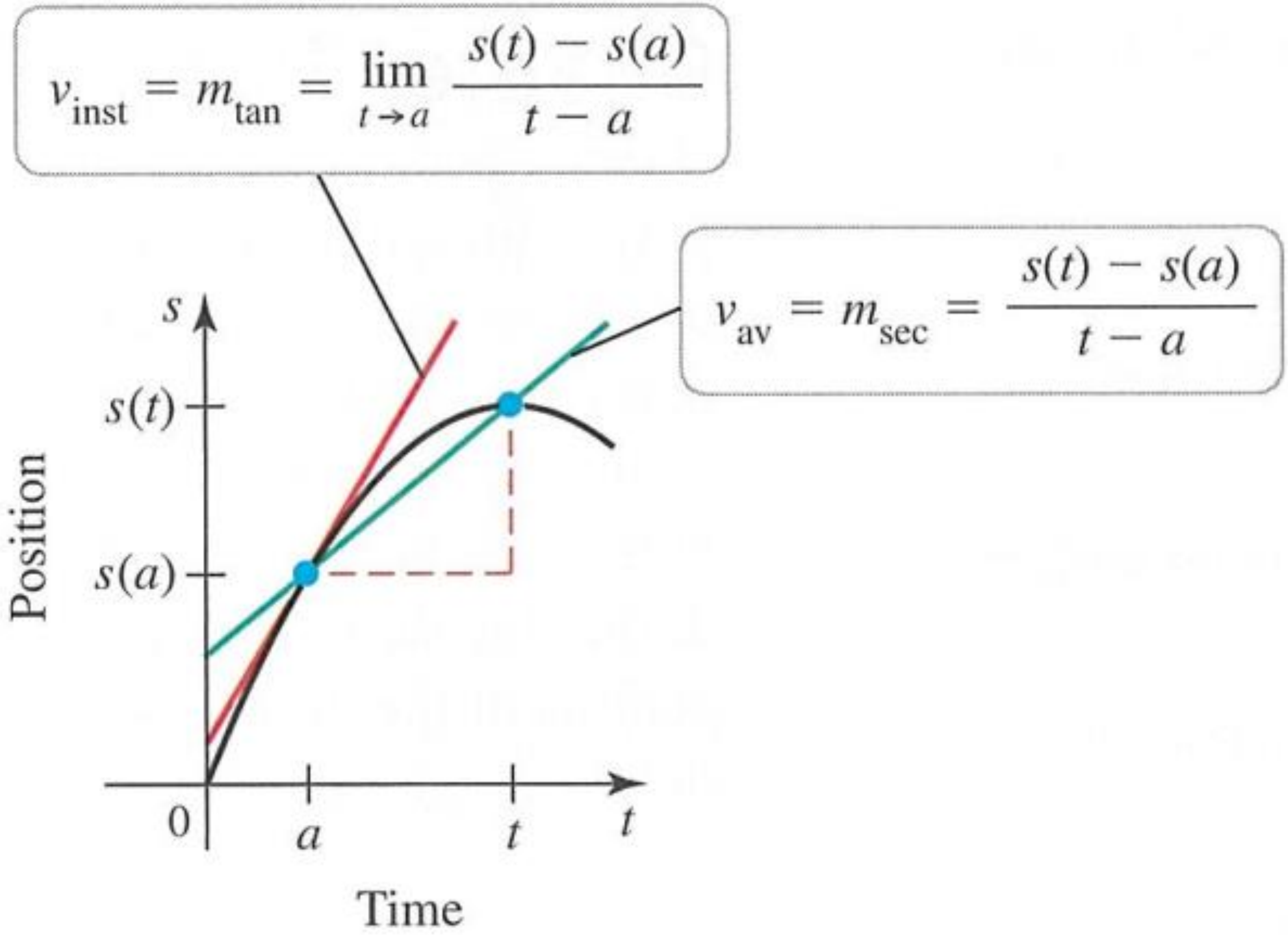
The average velocity of the object over the time interval $[a, t]$ is

$$v_{av} = \frac{s(t) - s(a)}{t - a}$$

The instantaneous velocity at time a is the limit

$$v_{instan} = \lim_{t \to a} \frac{s(t) - s(a)}{t - a}$$

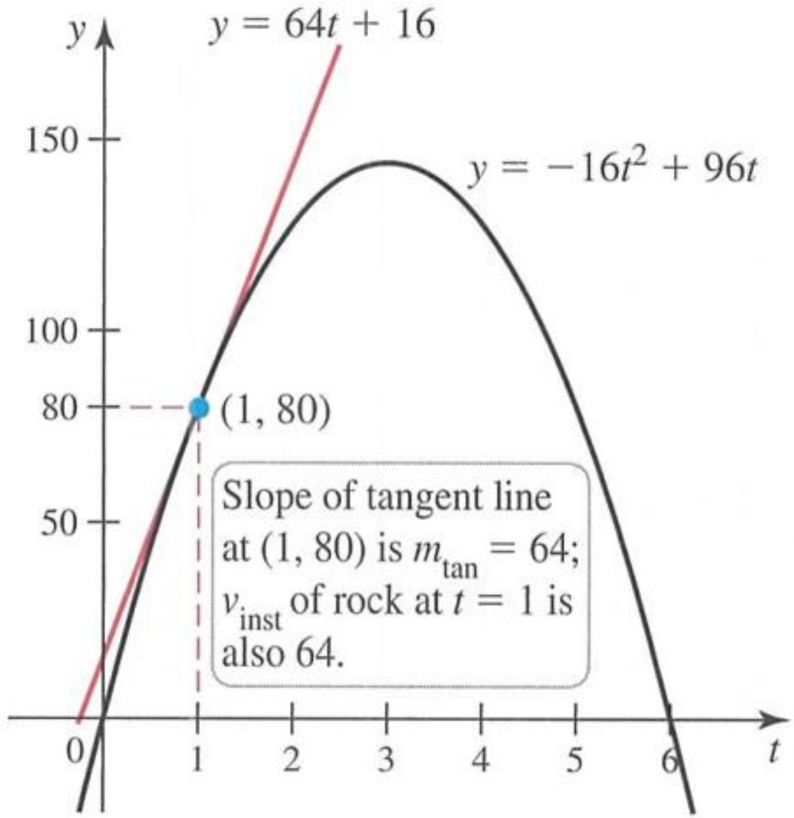
The average velocity and instantaneous velocity have important geometric interpretations



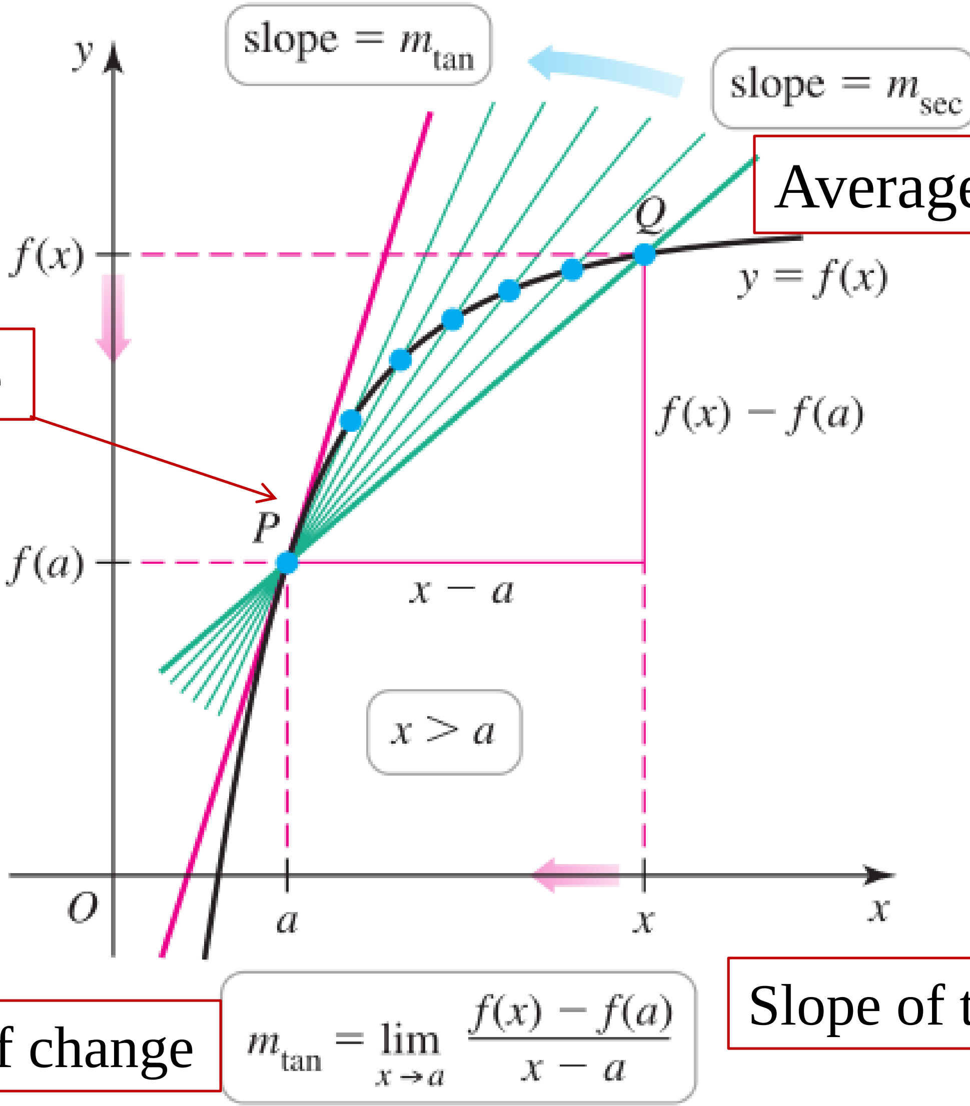
EXAMPLE 1 **Instantaneous velocity and tangent lines** A rock is launched vertically upward from the ground with an initial speed of 96 ft/s. The position of the rock in feet above the ground after t seconds is given by the function $s(t) = -16t^2 + 96t$. Consider the point $P(1, 80)$ on the curve $y = s(t)$.

a. Find the instantaneous velocity of the rock 1 second after launch and find the slope of the line tangent to the graph of s at P.

b. Find an equation of the tangent line in part (a).



Tangent Lines and Rates of Change



Average rate of change

Tangent line

Instantaneous rate of change

Slope of the tangent line

DEFINITION **Rate of Change and the Slope of the Tangent Line**

The **average rate of change** in f on the interval $[a, x]$ is the slope of the corresponding secant line:

$$m_{\text{sec}} = \frac{f(x) - f(a)}{x - a}.$$

The **instantaneous rate of change** in f at a is

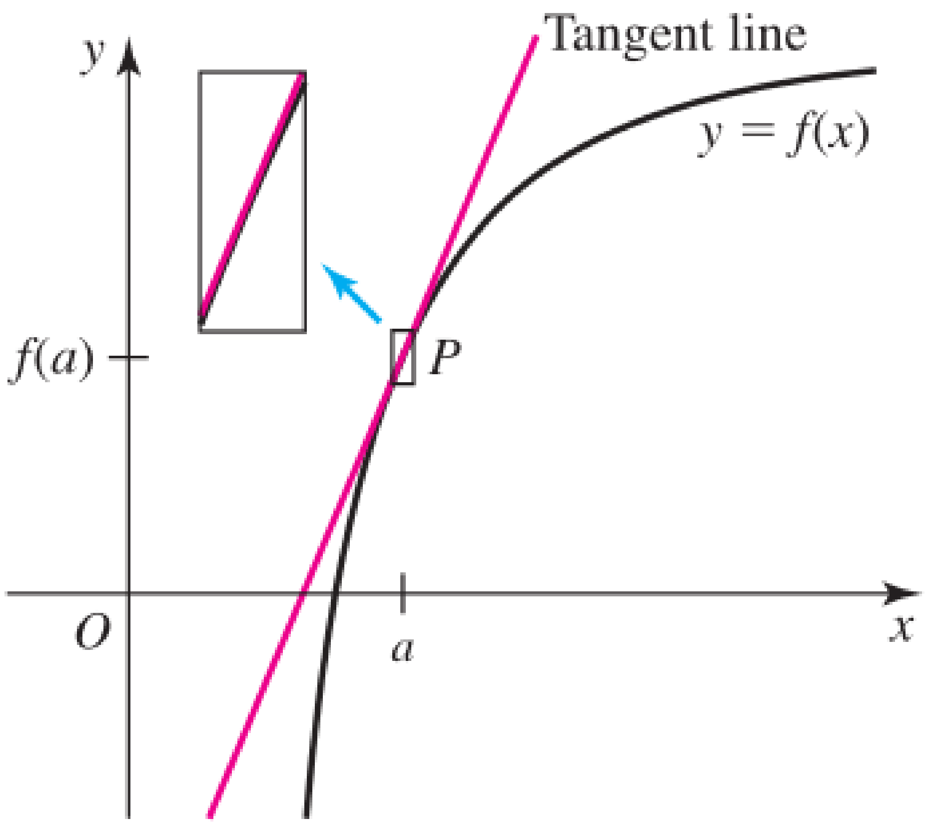
$$m_{\tan} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}, \qquad (1)$$

which is also the **slope of the tangent line** at $(a, f(a))$, provided this limit exists. The **tangent line** is the unique line through $(a, f(a))$ with slope $m_{\tan}$. Its equation is

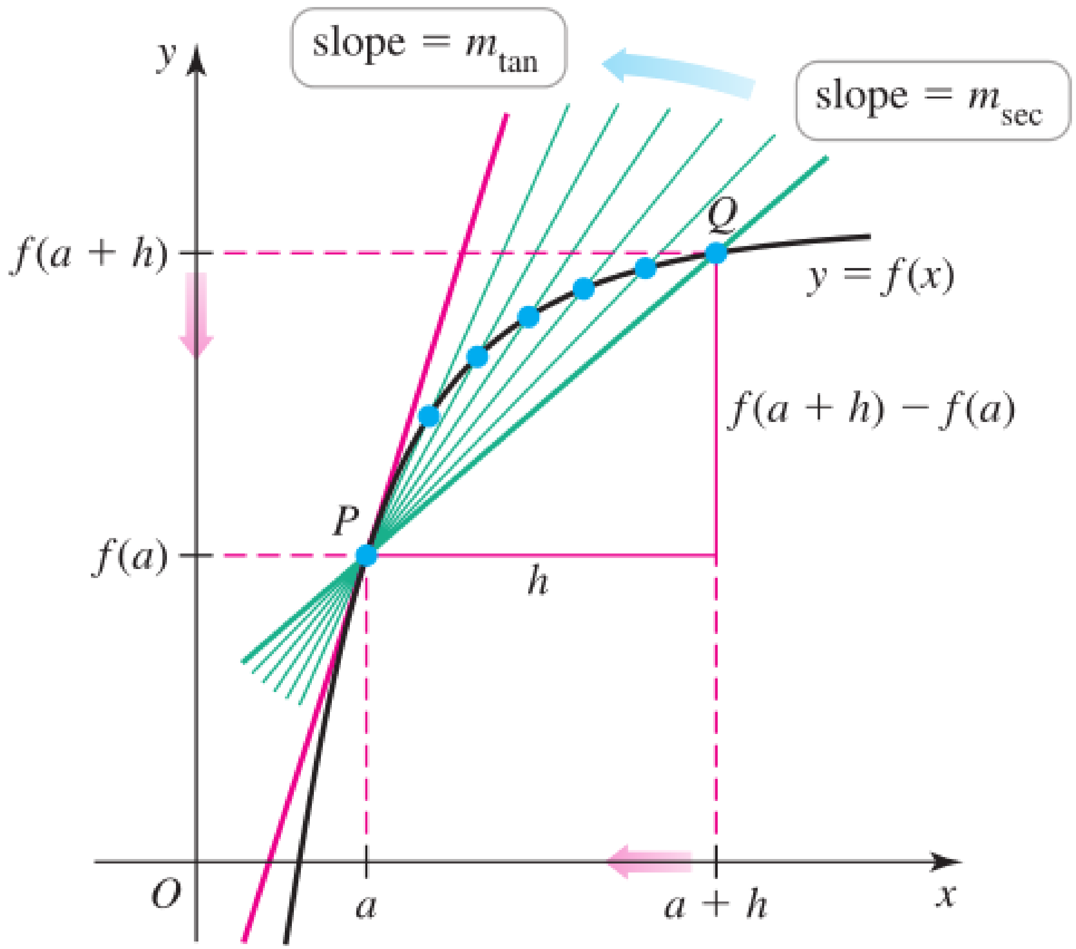
$$y - f(a) = m_{\tan}(x - a).$$

Local linearity

If we look at a point on the *smooth* curve locally (by zooming in), then the curve appears linear.



An alternative formula for the slope of the tangent line



$$m_{\tan} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

ALTERNATIVE DEFINITION **Rate of Change and the Slope of the Tangent Line**

The **average rate of change** in f on the interval $[a, a + h]$ is the slope of the corresponding secant line:

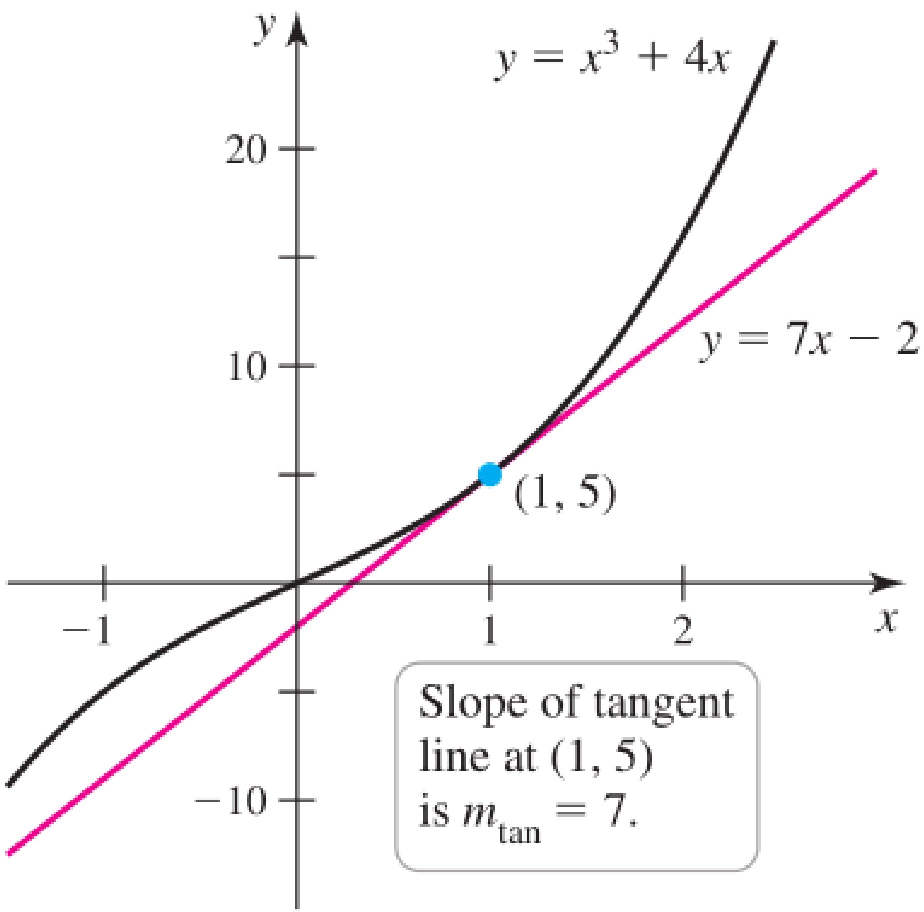
$$m_{\text{sec}} = \frac{f(a + h) - f(a)}{h}.$$

The **instantaneous rate of change** in f at a is

$$m_{\tan} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}, \qquad (2)$$

which is also the **slope of the tangent line** at $(a, f(a))$, provided this limit exists.

EXAMPLE 2 **Equation of a tangent line** Find an equation of the line tangent to the graph of $f(x) = x^3 + 4x$ at $(1, 5)$.



The Derivative

The slope of the tangent line for the graph of a function *f* at a given point *a* gives the instantaneous rate of change in *f* at *a*. Name this important behavior as ***derivative***.

DEFINITION The Derivative of a Function at a Point

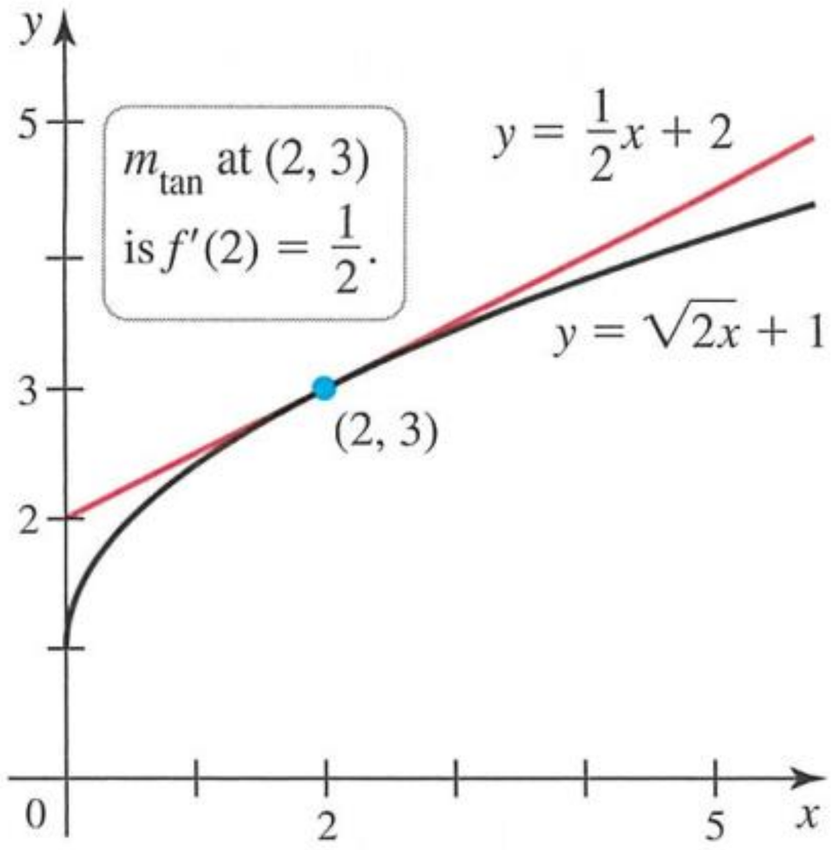
The **derivative of f at a**, denoted $f'(a)$, is given by either of the two following limits, provided the limits exist and a is in the domain of f:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \quad (1) \quad \text{or} \quad f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}. \quad (2)$$

If $f'(a)$ exists, we say that f is **differentiable** at a.

Exactly the same limits used for computing the slope of a tangent line and the instantaneous rate of change in *f* at *a*.

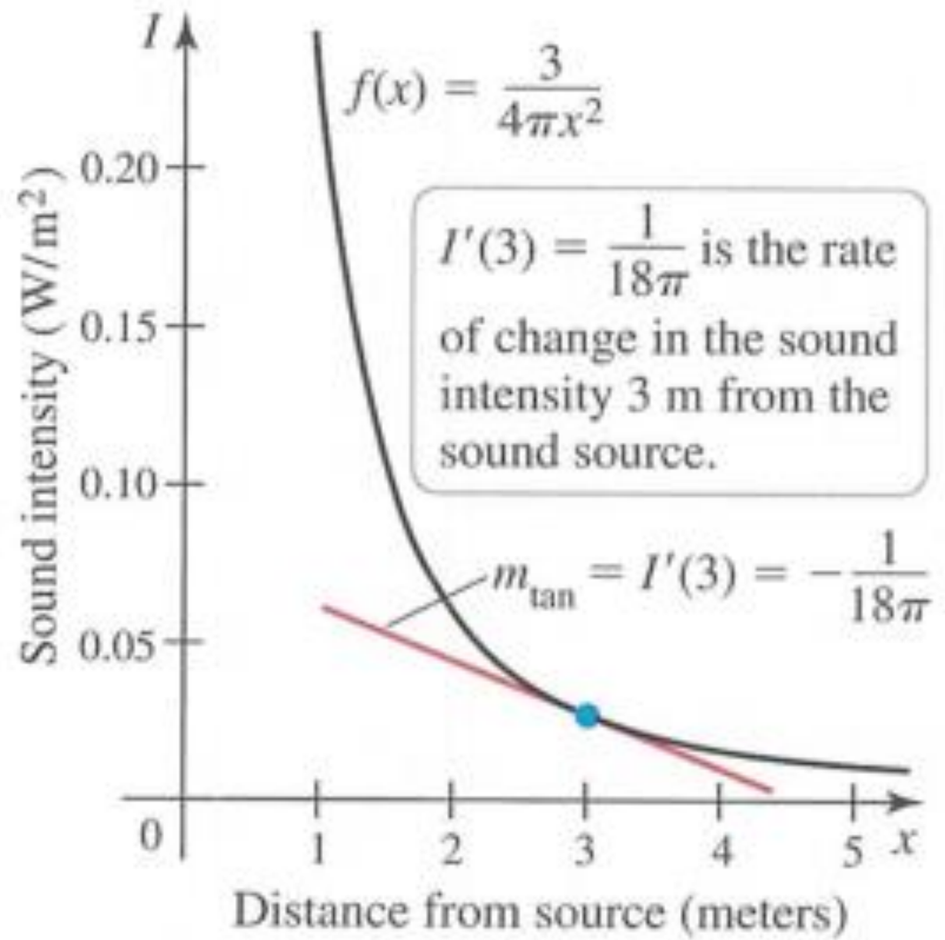
EXAMPLE 4 **Derivatives and tangent lines** Let $f(x) = \sqrt{2x} + 1$. Compute $f'(2)$, the derivative of f at $x = 2$, and use the result to find an equation of the line tangent to the graph of f at $(2, 3)$.



Interpreting the Derivative

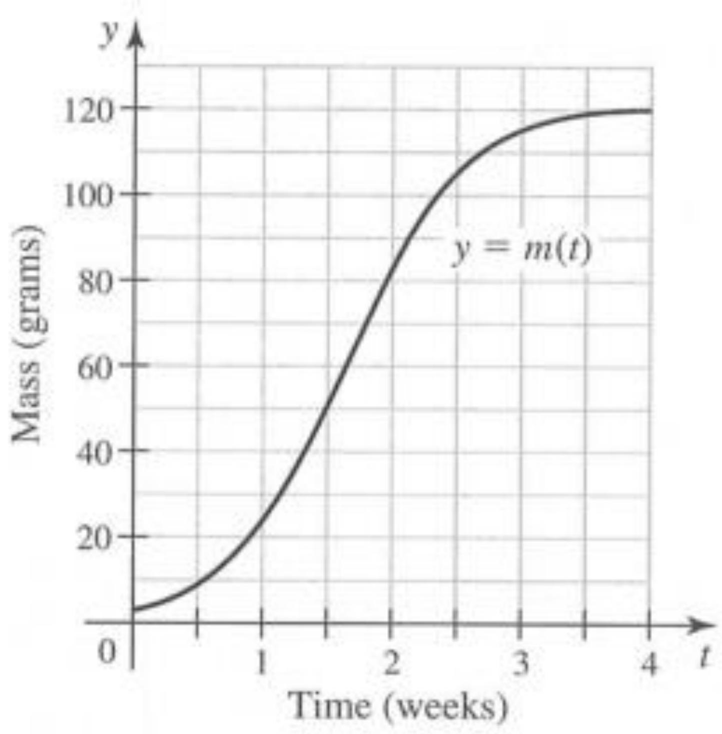
The *derivative* of a function *f* at a given point *a* measures the instantaneous *rate of change* in *f* at *a*.

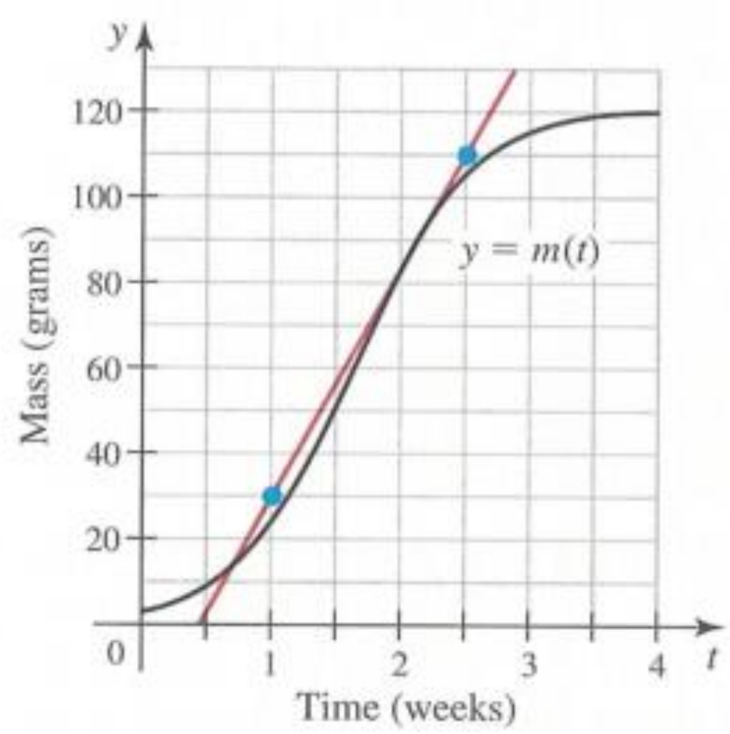
Taking specific



EXAMPLE 5 **Instai** ... ıeasured in watts per square meter (W/m^2 ... ı acoustic power P, is given by $I(x) = \dfrac{P}{4\pi x}$ ... ıroduces an acoustic power of about $P =$ ...

EXAMPLE 6 **Growth rates of Indian spotted owlets** The Indian spotted owlet is a small owl that is indigenous to Southeast Asia. The body mass $m(t)$ (in grams) of an owl at an age of t weeks is modeled by the graph in Figure 3.12. Estimate $m'(2)$ and state the physical meaning of this quantity. (*Source: ZooKeys*, 132, 2011)



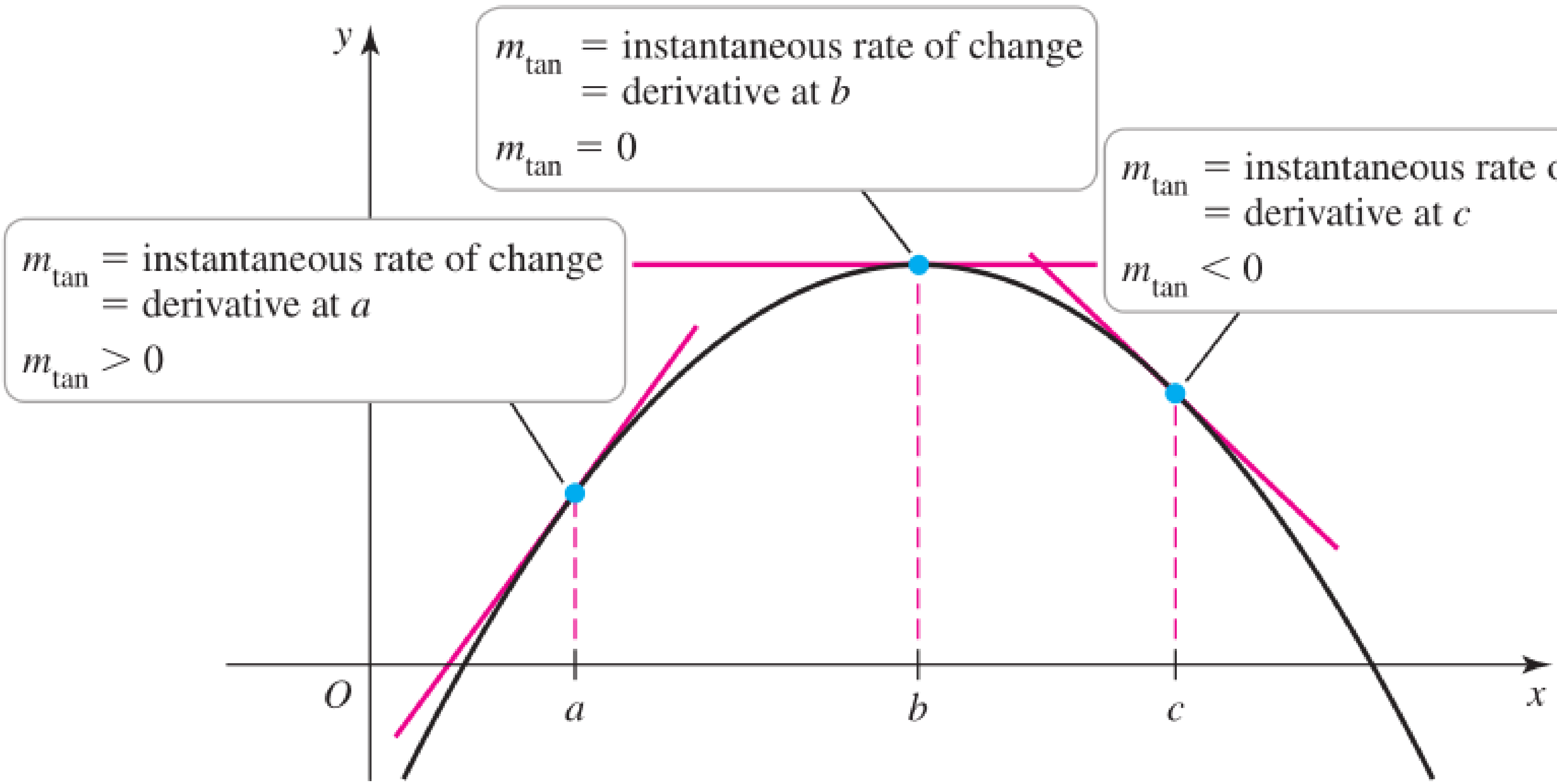


3.2

The Derivative as a Function

The Derivative Function

Let the point P move along the curve, the slope of the tangent line for function f is itself a function, called the ***derivative*** of f .



If the graph of f is smooth and 2 is in the domain of f, then $f'(2)$ is the slope of the line tangent to the graph of f at the point $(2, f(2))$. Similar case for -2 in the domain of f.

In fact, if x is *any* point in the domain of f, then $f'(x)$ is the slope of the tangent line at the point $(x, f(x))$. For a variable point x,

DEFINITION The Derivative Function

The **derivative** of f is the function

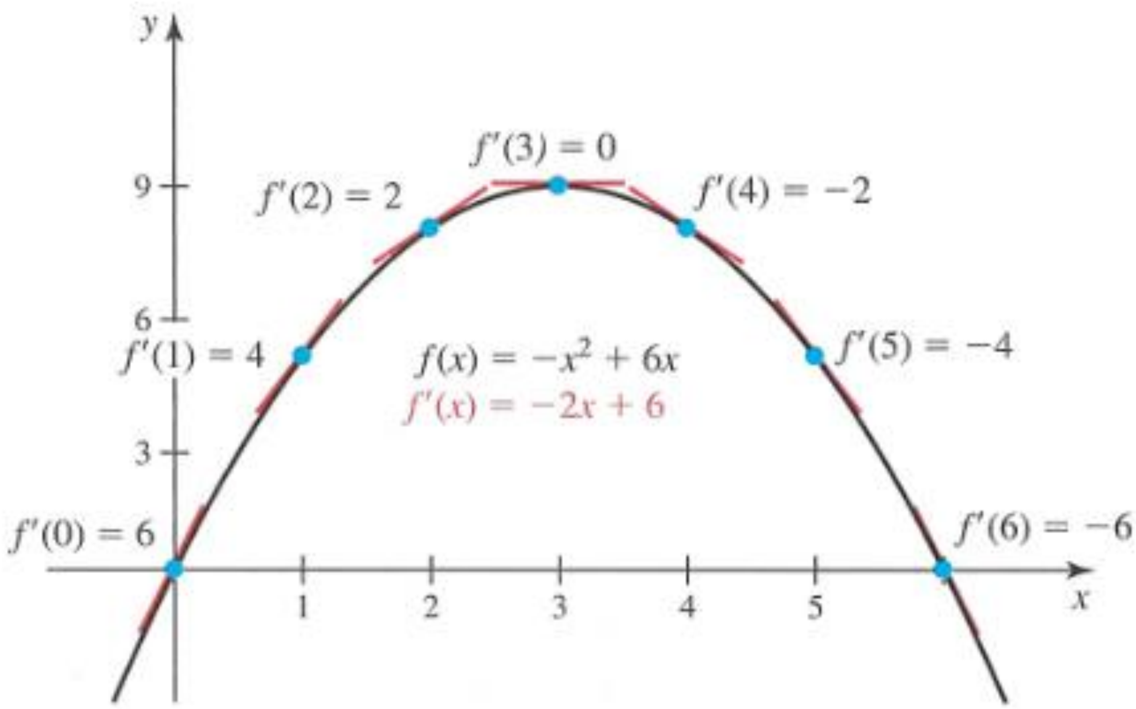
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists and x is the domain of f. If $f'(x)$ exists, we say that f is **differentiable** at x. If f is differentiable at every point of an open interval I, we say that f is differentiable on I.

The domain of f' is no larger than that of f.

If the limit in the definition of f' fails to exist at some points, then the domain of f' is a subset of the domain of f.

EXAMPLE 1 **Computing a derivative** Find the derivative of $f(x) = -x^2 + 6x$.



One-sided derivatives *The* ***right-sided*** *and* ***left-sided derivatives*** *of a function at a point a are given by*

$$f_+'(a) = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h} \quad \textit{and} \quad f_-'(a) = \lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h},$$

respectively, provided these limits exist. The derivative $f'(a)$ *exists if and only if* $f_+'(a) = f_-'(a)$.

Another way to approximate derivatives is to use the **centered difference quotient**:

$$f'(a) \approx \frac{f(a+h) - f(a-h)}{2h}.$$

Derivative Notation

Δ: a standard notation for change.

Replace h with Δx to represent the change in x.

Similarly, $f(x + \Delta x) - f(x)$ is the change in y, denoted Δy.

The slope of the secant line is

$$m_{sec} = \frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{\Delta y}{\Delta x}$$

By letting $\Delta x \to 0$,the slope of the tangent line at $(x, f(x))$ is

$$\boldsymbol{f'(x)} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \boldsymbol{\frac{dy}{dx}}$$

Other common ways of writing the derivative $\boldsymbol{f'(x)}$ $\boldsymbol{\frac{dy}{dx}}$

$$\frac{df}{dx} \qquad \frac{d}{dx}(f(x)) \qquad D_x(f(x)) \qquad y'(x)$$

Notations representing the derivative of *f* evaluated at a.

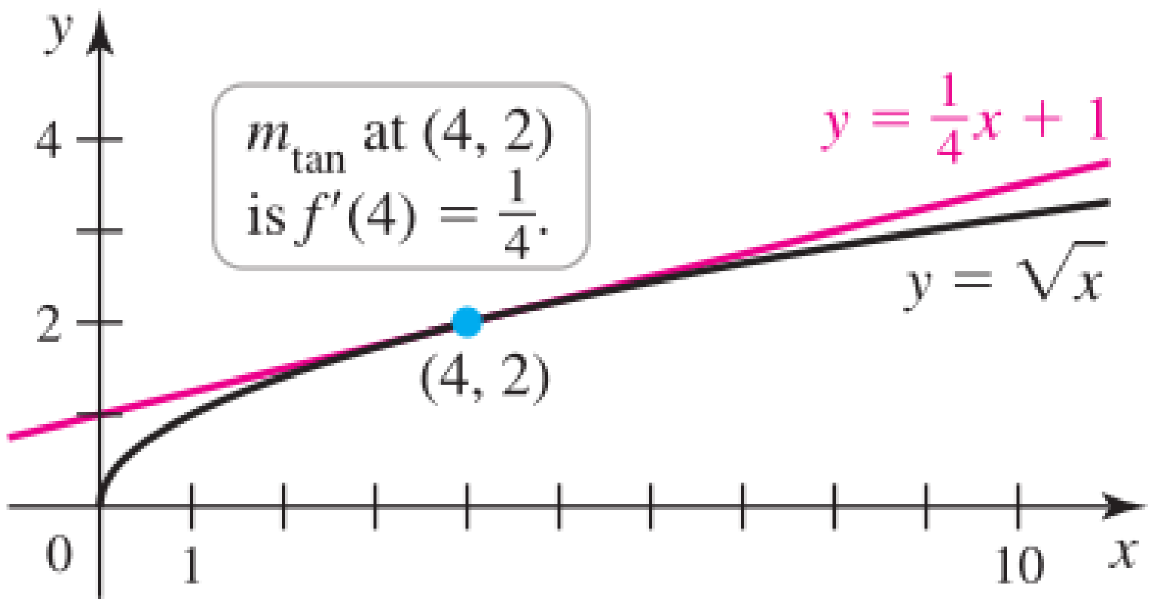
$$f'(a) \qquad y'(a) \qquad \left.\frac{df}{dx}\right|_{x=a} \qquad \left.\frac{dy}{dx}\right|_{x=a}$$

The process of finding f' is called ***differentiation***, and to ***differentiate f*** means to find f'.

EXAMPLE 4 **A derivative calculation** Let $y = f(x) = \sqrt{x}$.

a. Compute $\dfrac{dy}{dx}$.

b. Find an equation of the line tangent to the graph of f at $(4, 2)$.

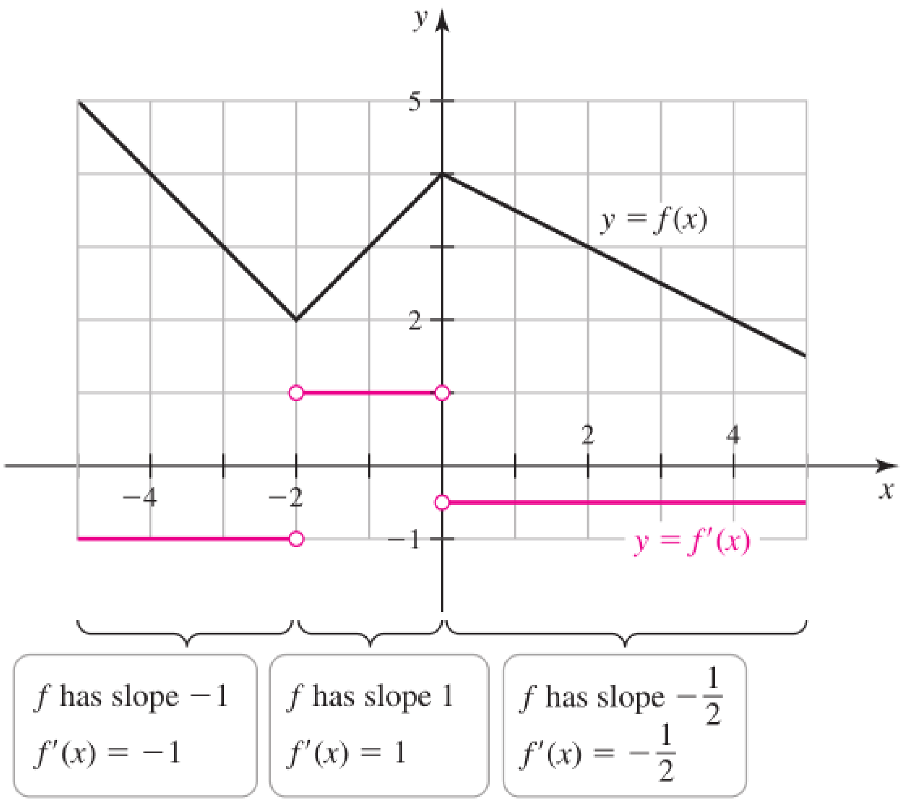


EXAMPLE 5 **Another derivative calculation** Let $g(t) = 1/t^2$ and compute $g'(t)$.

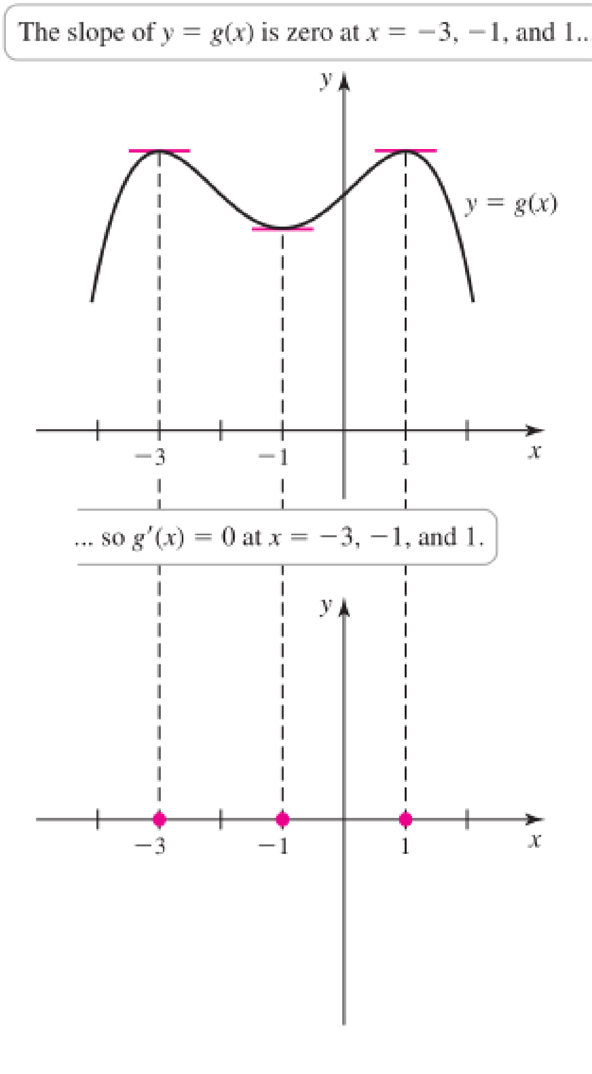
Graphs of Derivatives

The derivative f' of f is *derived* from f.
The way to *derive* the graph of from the graph of f.

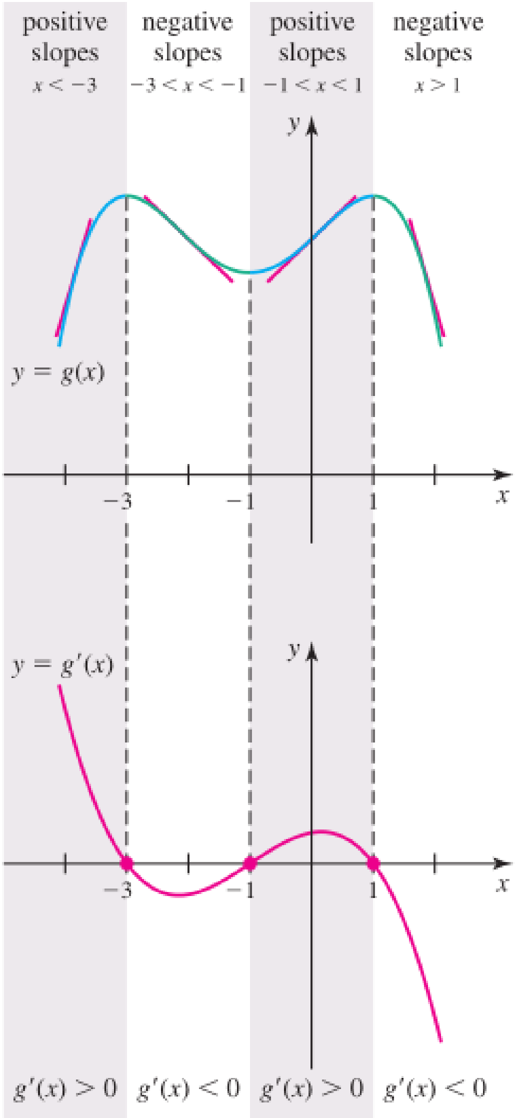
EXAMPLE 1 **Graph of the derivative** Sketch the graph of f' from the graph of f



EXAMPLE 2 **Graph of the derivative** Sketch the graph of g' using the graph of g

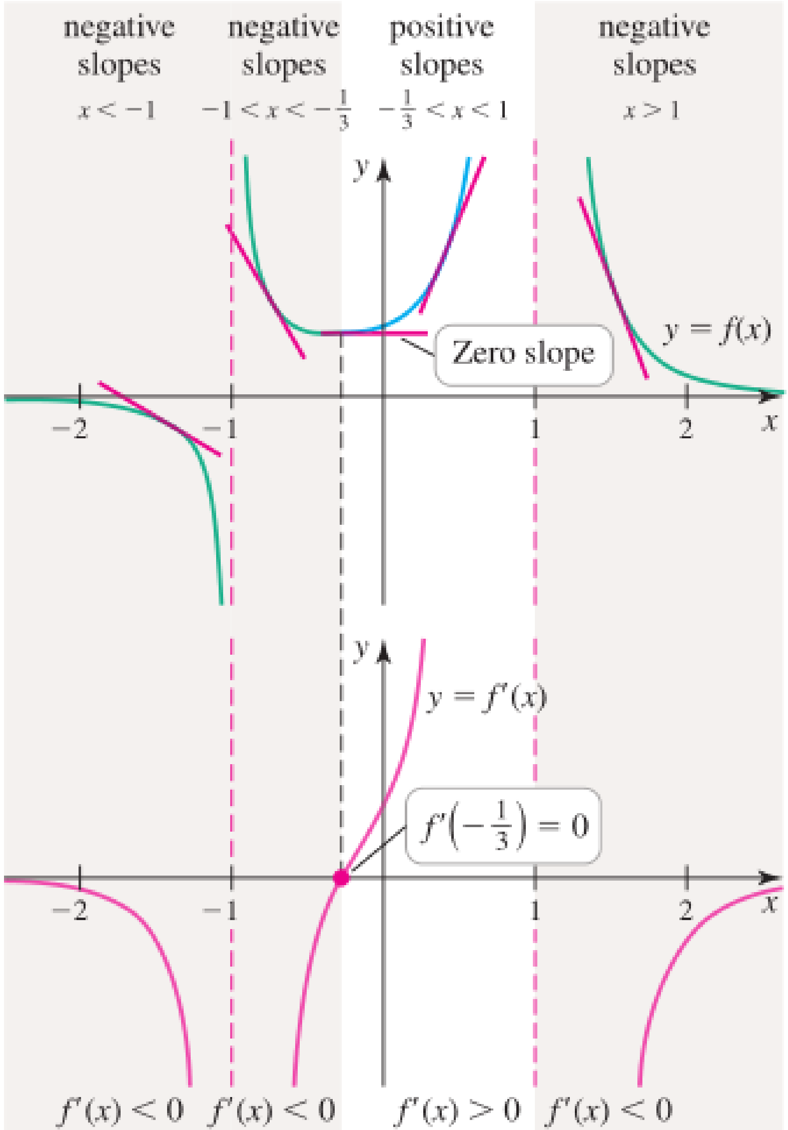


(a)



(b)

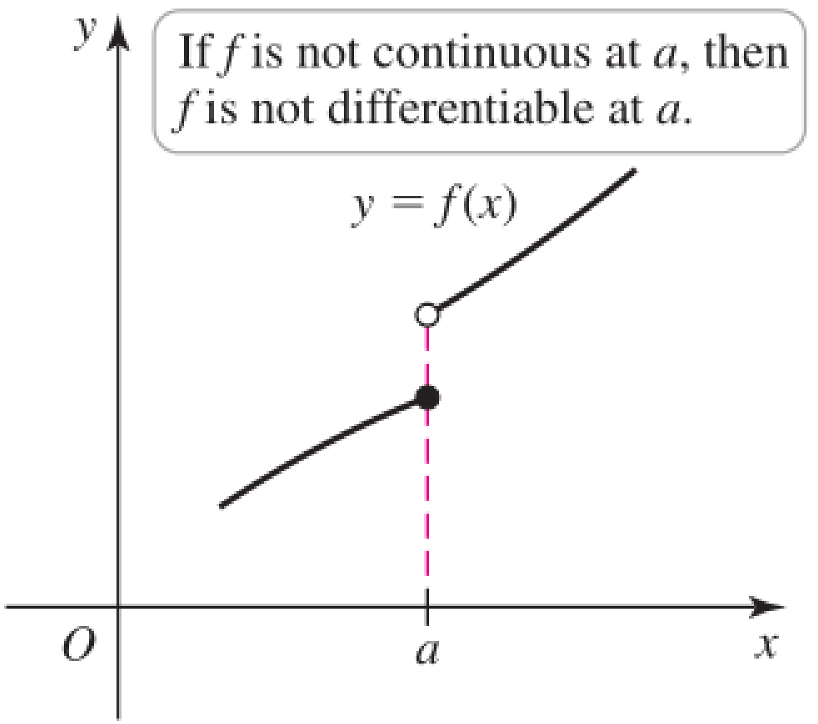
EXAMPLE 3 **Graphing the derivative with asymptotes** The graph of the function f is shown in Figure 3.16. Sketch a graph of its derivative.



Continuity

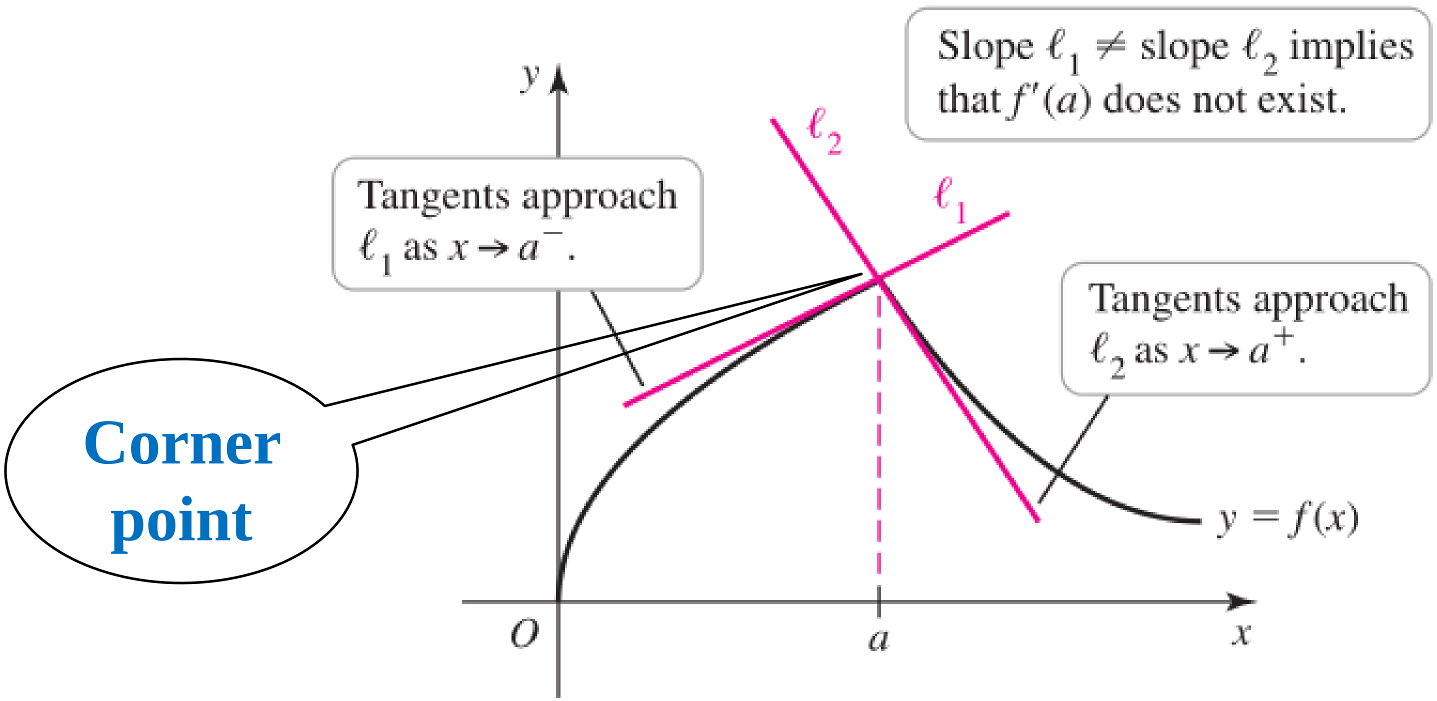
The relationship between continuity and differentiability.

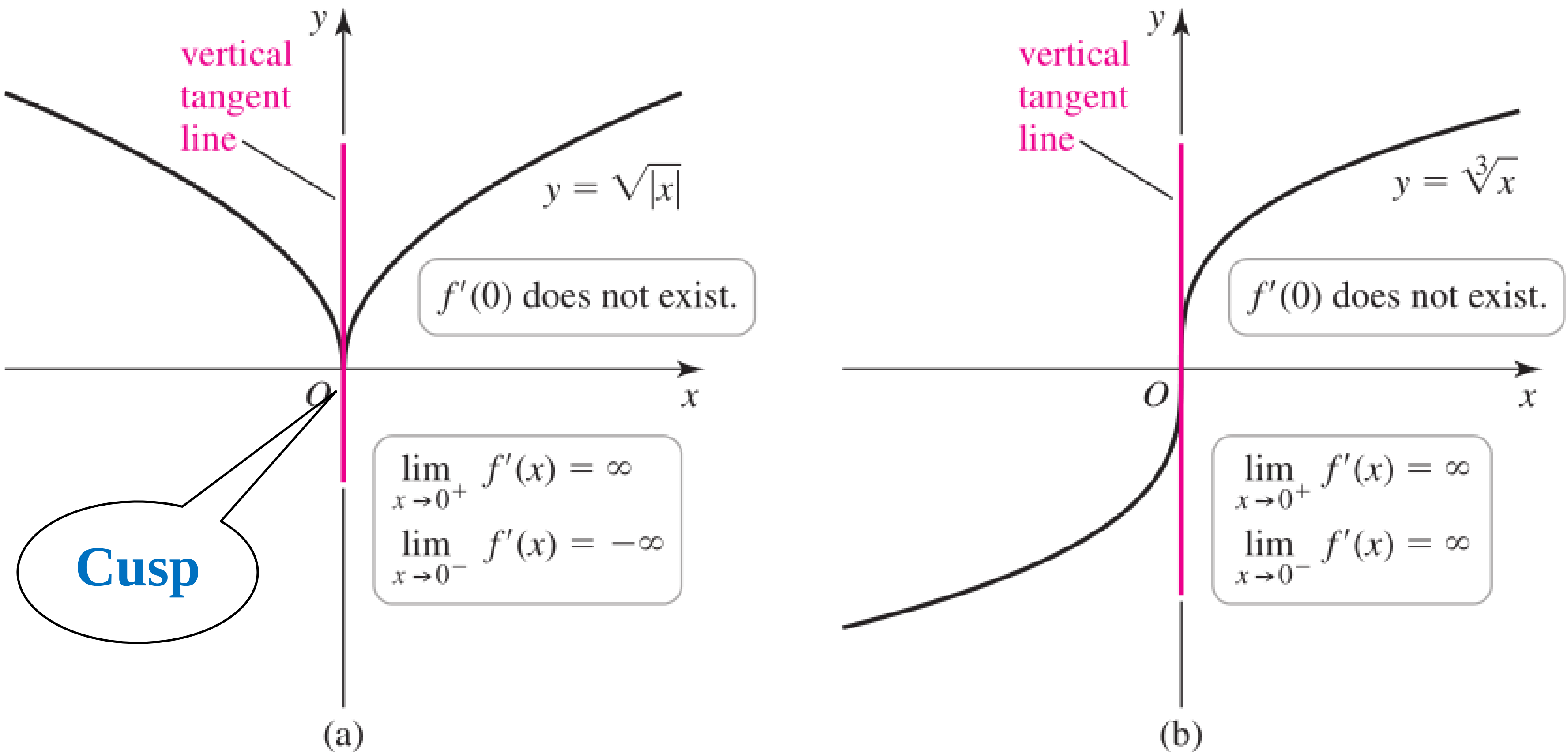
THEOREM 3.1 Differentiable Implies Continuous
If f is differentiable at a, then f is continuous at a.



THEOREM 3.1 (ALTERNATIVE VERSION) **Not Continuous Implies Not Differentiable**
If f is not continuous at a, then f is not differentiable at a.

If f is continuous at a point, f is *not* necessarily differentiable at that point.




vertical
tangent
line
$y = \sqrt{|x|}$
$f'(0)$ does not exist.
$\lim_{x \to 0^+} f'(x) = \infty$
$\lim_{x \to 0^-} f'(x) = -\infty$
Cusp
O
x
y
(a)
vertical
tangent
line
$y = \sqrt[3]{x}$
$f'(0)$ does not exist.
$\lim_{x \to 0^+} f'(x) = \infty$
$\lim_{x \to 0^-} f'(x) = \infty$
O
x
y
(b)

When Is a Function Not Differentiable at a Point?

A function f is *not* differentiable at a if at least one of the following conditions holds:

a. f is not continuous at a (Figure 3.24).

b. f has a corner at a (Figure 3.25).

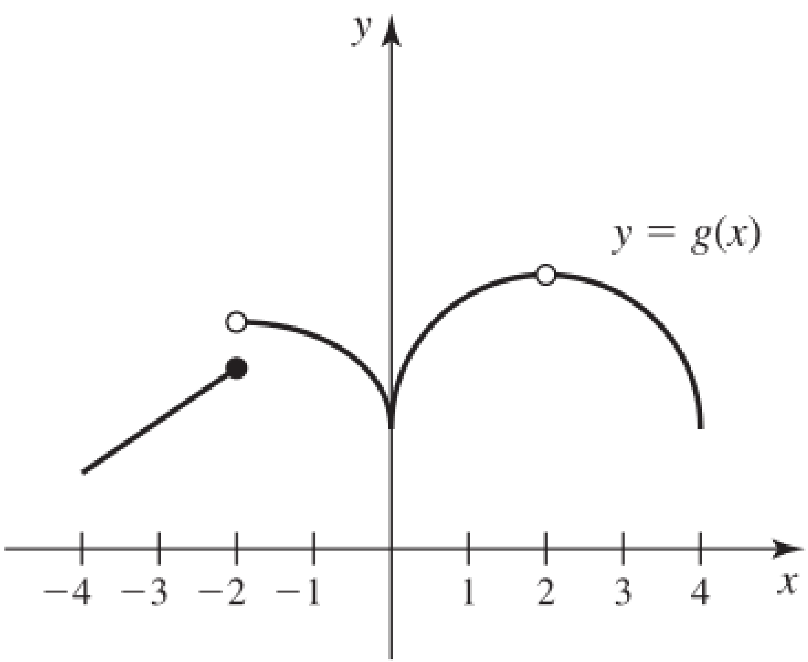
c. f has a vertical tangent at a (Figure 3.26).

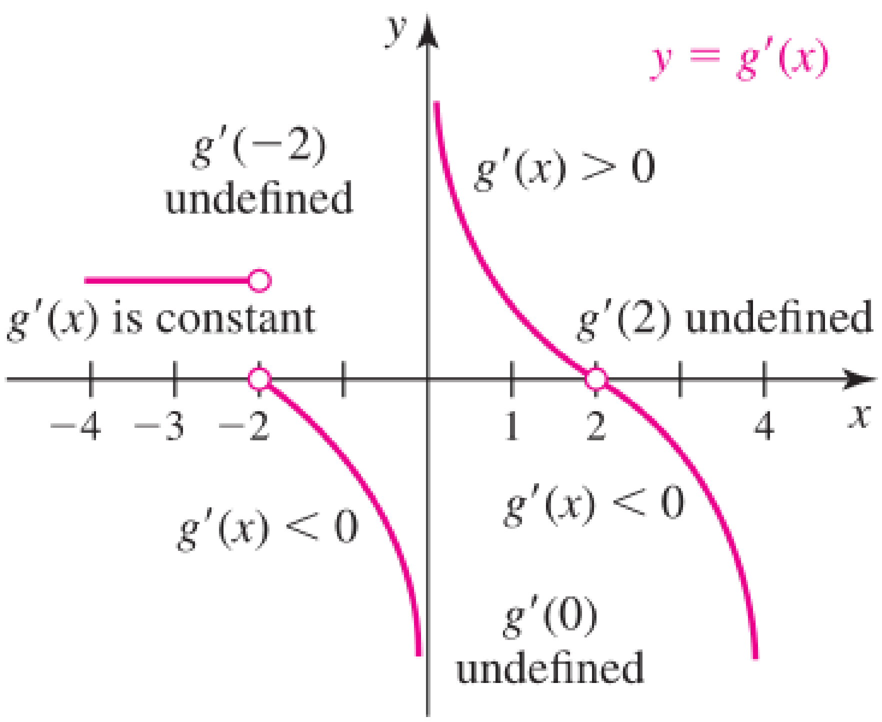
EXAMPLE 4 **Continuous and differentiable** Consider the graph of g in Figure 3.21.

a. Find the values of x in the interval $(-4, 4)$ at which g is not continuous.

b. Find the values of x in the interval $(-4, 4)$ at which g is not differentiable.

c. Sketch a graph of the derivative of g.





3.3

Rule of Differentiation

The Constant and Power Rules of Derivatives

THEOREM 3.2 Constant Rule

If c is a real number, then $\frac{d}{dx}(c) = 0$.

THEOREM 3.3 Power Rule

If n is a nonnegative integer, then $\frac{d}{dx}(x^n) = nx^{n-1}$.

EXAMPLE 1 Derivatives of power and constant functions Evaluate the following derivatives.

a. $\frac{d}{dx}(x^9)$ **b.** $\frac{d}{dx}(x)$ **c.** $\frac{d}{dx}(2^8)$

THEOREM 3.4 Constant Multiple Rule

If f is differentiable at x and c is a constant, then

$$\frac{d}{dx}(cf(x)) = cf'(x).$$

EXAMPLE 2 Derivatives of constant multiples of functions Evaluate the following derivatives.

a. $\frac{d}{dx}\left(-\frac{7x^{11}}{8}\right)$ **b.** $\frac{d}{dt}\left(\frac{3}{8}\sqrt{t}\right)$

THEOREM 3.5 Sum Rule

If f and g are differentiable at x, then

$$\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x).$$

Generalized Sum Rule

$$\frac{d}{dx}\big(f_1(x) + f_2(x) + \cdots + f_n(x)\big) = f_1'(x) + f_2'(x) + \cdots + f_n'(x)$$

Difference Rule

$$\frac{d}{dx}\big(f(x) - g(x)\big) = f'(x) - g'(x)$$

EXAMPLE 3 Derivative of a polynomial Determine $\frac{d}{dw}(2w^3 + 9w^2 - 6w + 4)$.

The Derivative of the Natural Exponential Function

Exponential function $f(x) = b^x$

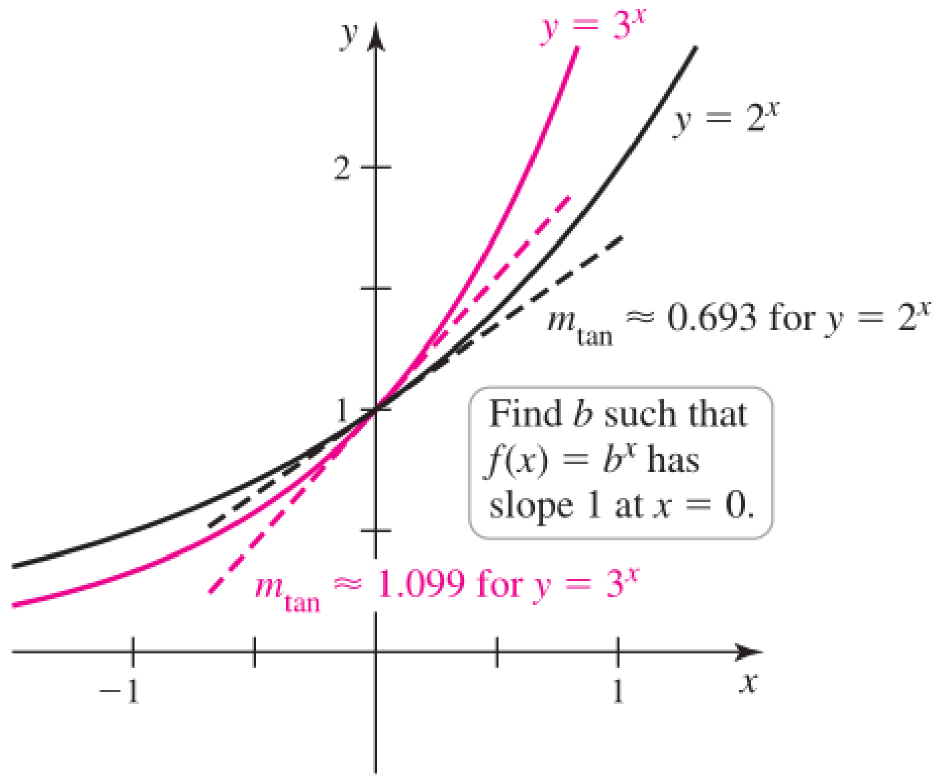
Looking at $y = 2^x$ and $y = 3^x$.

The derivative of $f(x) = b^x$ at $x = 0$

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{b^h - b^0}{h} = \lim_{h \to 0} \frac{b^h - 1}{h}.$$

Table 3.1

h	$\dfrac{2^h - 1}{h}$	$\dfrac{3^h - 1}{h}$
1.0	1.000000	2.000000
0.1	0.717735	1.161232
0.01	0.695555	1.104669
0.001	0.693387	1.099216
0.0001	0.693171	1.098673
0.00001	0.693150	1.098618

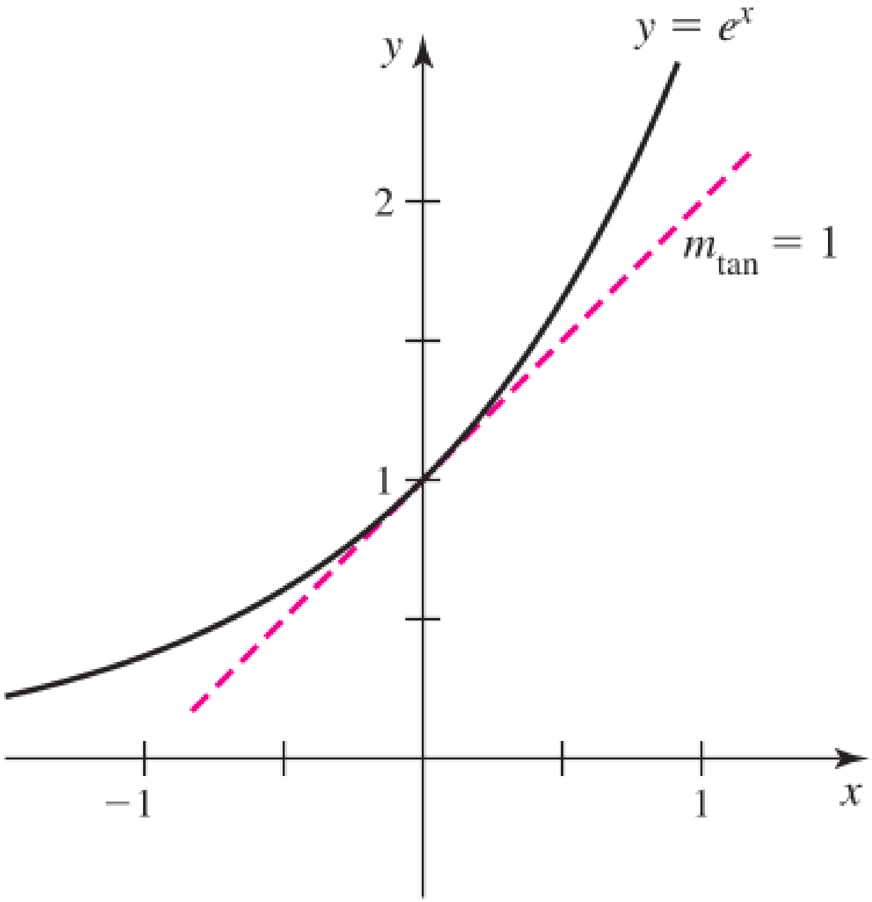


Suggest that there is a number b with $2 < b < 3$ such that the graph of $y = b^x$ has a tangent line with slope of 1 at $x = 0$.

$$\lim_{h \to 0} \frac{b^h - 1}{h} = 1$$

Such a number b exits. In fact, it is e =2.718281828459….

Therefore, the exponential function whose tangent line has slope 1 at $x = 0$ is the ***natural exponential function*** $f(x) = e^x$.



DEFINITION The Number e

The number $e = 2.718281828459\ldots$ satisfies

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1.$$

It is the base of the natural exponential function $f(x) = e^x$.

THEOREM 3.6 The Derivative of e^x

The function $f(x) = e^x$ is differentiable for all real numbers x, and

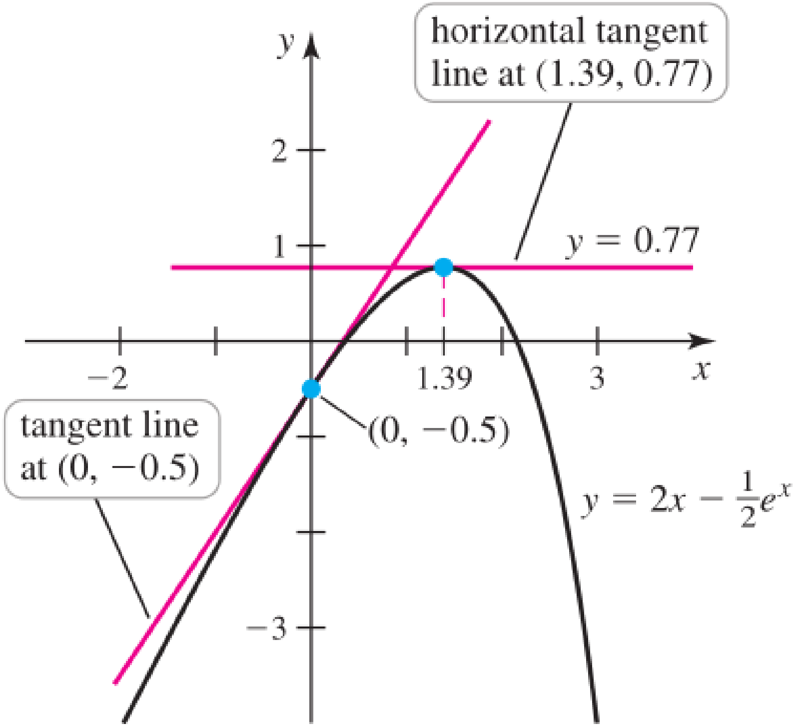
$$\frac{d}{dx}(e^x) = e^x.$$

Slopes of Tangent lines

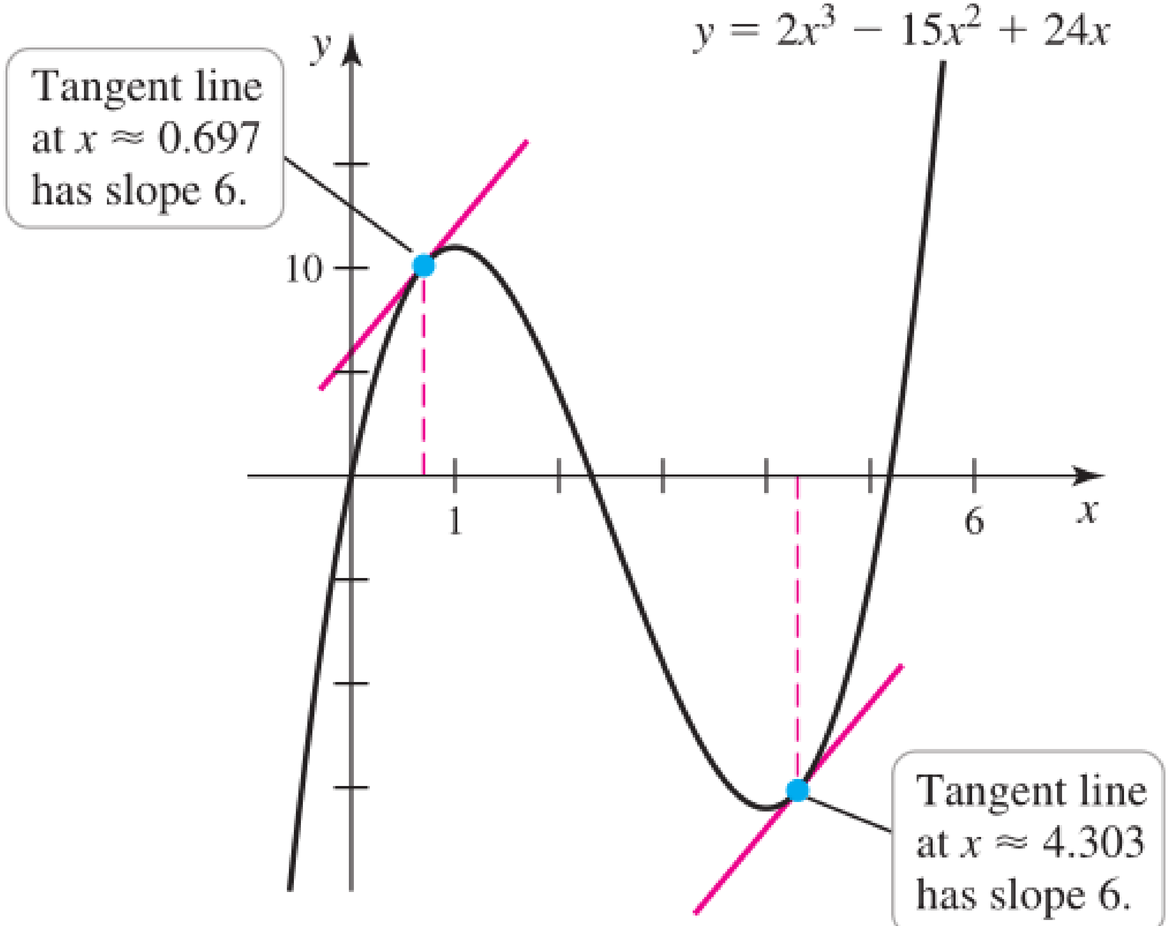
EXAMPLE 4 **Finding tangent lines**

a. Write an equation of the line tangent to the graph of $f(x) = 2x - \dfrac{e^x}{2}$ at the point $(0, -\frac{1}{2})$.

b. Find the point(s) on the graph of f at which the tangent line is horizontal.



EXAMPLE 5 **Slope of a tangent line** Let $f(x) = 2x^3 - 15x^2 + 24x$. For what values of x does the line tangent to the graph of f have a slope of 6?



Higher-Order Derivatives

The derivative of derivative function that is a function in its own

DEFINITION Higher-Order Derivatives

Assuming $y = f(x)$ can be differentiated as often as necessary, the **second derivative** of f is

$$f''(x) = \frac{d}{dx}(f'(x)).$$

For integers $n \geq 2$, the ***n*th derivative** of f is

$$f^{(n)}(x) = \frac{d}{dx}(f^{(n-1)}(x)).$$

Other common notations for the second derivative of $y = f(x)$

$$y'' \qquad f'' \qquad \frac{d^2y}{dx^2} \qquad \frac{d^2f}{dx^2}$$

The notations for the third derivative of $y = f(x)$

$$y''' \qquad f''' \qquad \frac{d^3y}{dx^3} \qquad \frac{d^3f}{dx^3}$$

The n-th derivative $\qquad y^{(n)} \qquad f^{(n)} \qquad \dfrac{d^ny}{dx^n} \qquad \dfrac{d^nf}{dx^n}$

EXAMPLE 6 **Finding higher-order derivatives** Find the third derivative of the following functions.

a. $f(x) = 3x^3 - 5x + 12$ **b.** $y = 3t + 2e^t$

3.4

The Product and Quotient Rules

Product Rule

Question: is the derivative of a product of functions is the product of their derivatives?

Counter Example. $f(x) = x^3$ and $g(x) = x^4$

Example. running speed = stride length * stride rate

change in speed = *change* in stride length * stride rate

change in speed = stride length * *change* in stride rate

change in speed = (*change* in stride length * stride rate)
+ (stride length * *change* in stride rate)

THEOREM 3.7 Product Rule

If f and g are differentiable at x, then

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x).$$

EXAMPLE 1 Using the Product Rule Find and simplify the following derivatives.

a. $\frac{d}{dv}(v^2(2\sqrt{v} + 1))$ **b.** $\frac{d}{dx}(x^2e^x)$

Quotient Rule

Consider $q(x) = \frac{f(x)}{g(x)}$ and note that $f(x) = g(x)q(x)$

Apply the Product Rule, $f'(x) = g'(x)q(x) + g(x)q'(x)$

Solving for $q'(x)$

$$q'(x) = \frac{f'(x) - g'(x)q(x)}{g(x)}$$

Substituting $q(x) = \frac{f(x)}{g(x)}$

$$q'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

THEOREM 3.8 Quotient Rule

If f and g are differentiable at x and $g(x) \neq 0$, then the derivative of f/g at x exists and

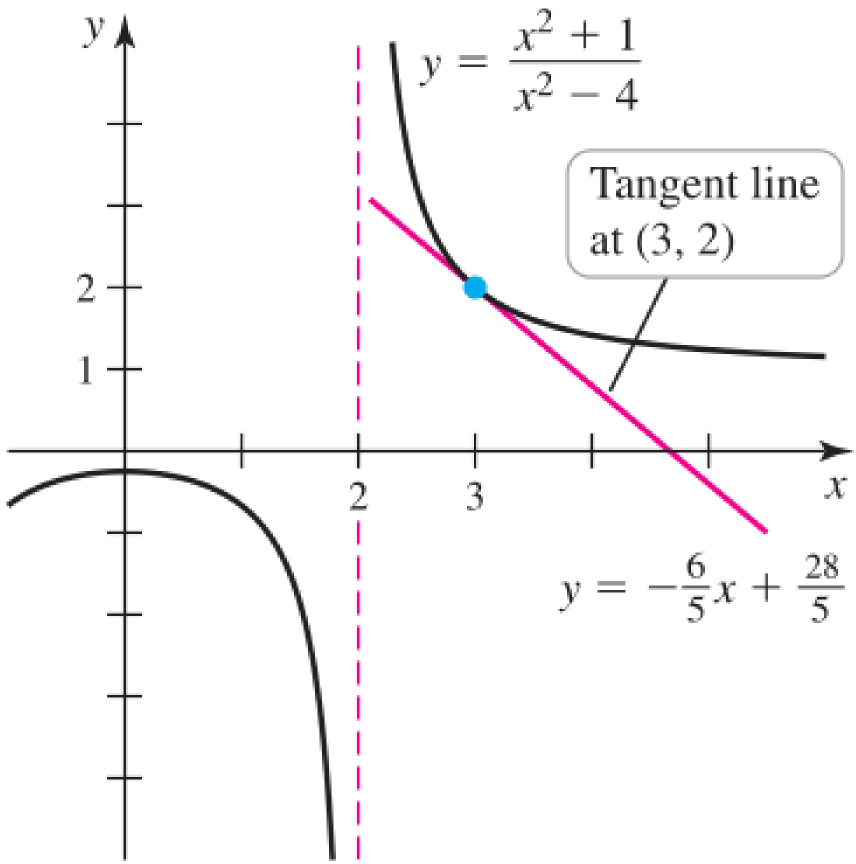
$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}.$$

Existence of the derivative of $\frac{f(x)}{g(x)}$? Refer to Ex. 96

EXAMPLE 2 **Using the Quotient Rule** Find and simplify the following derivatives.

a. $\frac{d}{dx}\left(\frac{x^2 + 3x + 4}{x^2 - 1}\right)$ **b.** $\frac{d}{dx}(e^{-x})$

EXAMPLE 3 **Finding tangent lines** Find an equation of the line tangent to the graph of $f(x) = \dfrac{x^2 + 1}{x^2 - 4}$ at the point $(3, 2)$. Plot the curve and tangent line.



Extending the Power Rule to Negative Integers

Recall Power Rule $\frac{d}{dx}(x^n) = nx^{n-1}$, for nonnegative integers

If n is a negative integer, let $m = -n$, so that $m > 0$. Then

$$\frac{d}{dx}(x^n) = \frac{d}{dx}\left(\frac{1}{x^m}\right) = \frac{x^m\left(\frac{d}{dx}(1)\right) - 1\frac{d}{dx}x^m}{(x^m)^2}$$

Quotient Rule

$$= -\frac{mx^{m-1}}{x^{2m}} = -mx^{-m-1} = nx^{n-1}$$

THEOREM 3.9 Power Rule (general form)

If n is any real number, then

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

EXAMPLE 4 **Using the Power Rule** Find the following derivatives.

a. $\dfrac{d}{dx}\left(\dfrac{9}{x^5}\right)$

b. $\dfrac{d}{dt}\left(\dfrac{3t^{16}-4}{t^6}\right)$

c. $\dfrac{d}{dz}\left(\sqrt[3]{z}\,e^z\right)$

d. $\dfrac{d}{dx}\left(\dfrac{3x^{5/2}}{2x^2+4}\right)$

Combining Derivative Rules

Multiple use of differentiation rules

EXAMPLE 7 **Combining derivative rules** Find the derivative of

$$y = \frac{4xe^x}{x^2 + 1}.$$

3.5

Derivatives of Trigonometric Functions

Two Special Limits

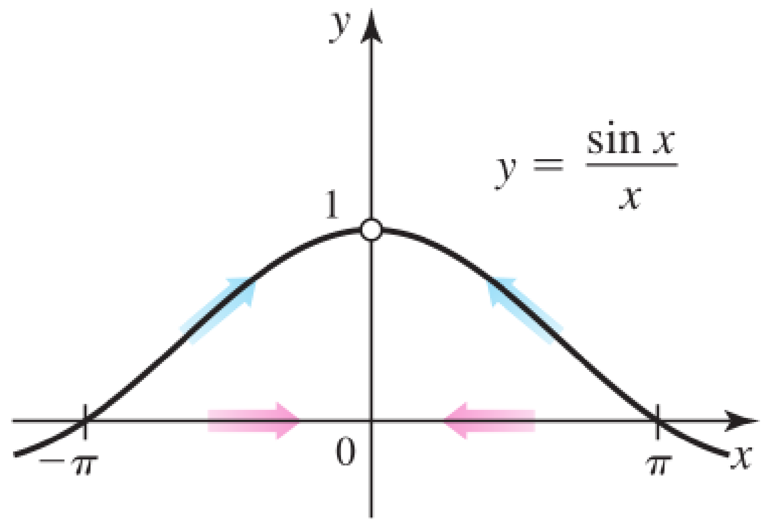
THEOREM 3.11 Trigonometric Limits

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \qquad \lim_{x \to 0} \frac{\cos x - 1}{x} = 0$$

These limits cannot be evaluated l e
the numerator and denominator ap

Table 3.2

x	$\dfrac{\sin x}{x}$
± 0.1	0.9983341665
± 0.01	0.9999833334
± 0.001	0.9999998333



The graph of $y = \dfrac{\sin x}{x}$ suggests that $\lim_{x \to 0} \dfrac{\sin x}{x} = 1$.

Proof: Consider Figure 3.31, in which $\triangle OAD$, $\triangle OBC$, and the sector OAC of the unit circle (with central angle x) are shown. Observe that with $0 < x < \pi/2$,

$$\text{area of } \triangle OAD < \text{area of sector } OAC < \text{area of } \triangle OBC. \tag{1}$$

Because the circle in Figure 3.31 is a *unit* circle, $OA = OC = 1$. It follows that $\sin x = \dfrac{AD}{OA} = AD$, $\cos x = \dfrac{OD}{OA} = OD$, and $\tan x = \dfrac{BC}{OC} = BC$. From these observations, we conclude that

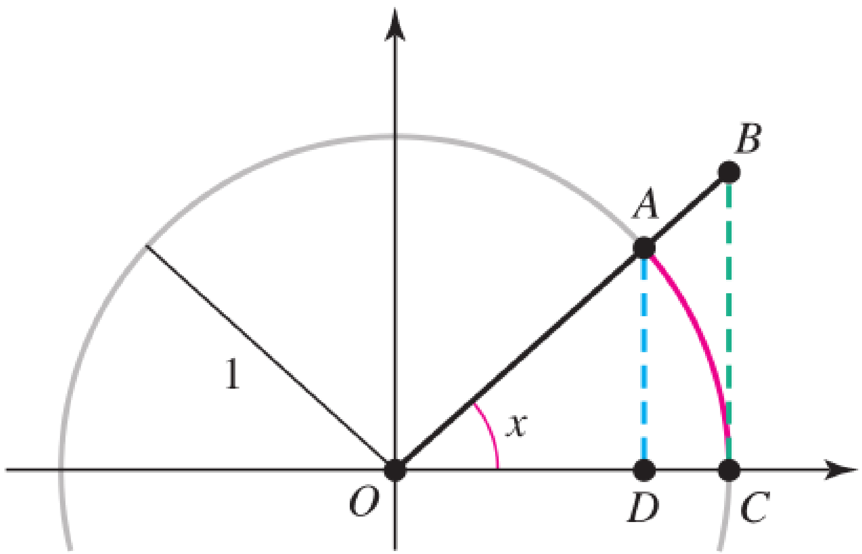


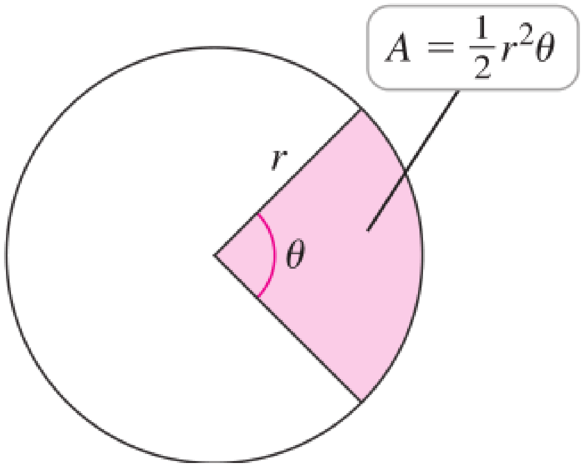
Figure 3.31

- the area of $\triangle OAD = \frac{1}{2}(OD)(AD) = \frac{1}{2}\cos x \sin x$,
- the area of sector $OAC = \frac{1}{2} \cdot 1^2 \cdot x = \frac{x}{2}$, and
- the area of $\triangle OBC = \frac{1}{2}(OC)(BC) = \frac{1}{2}\tan x$.

Substituting these results into (1), we have

$$\frac{1}{2}\cos x \sin x < \frac{x}{2} < \frac{1}{2}\tan x.$$

➤ Area of a sector of a circle of radius r formed by a central angle θ:



Replacing $\tan x$ with $\dfrac{\sin x}{\cos x}$ and multiplying the inequalities by $\dfrac{2}{\sin x}$ (which is positive) leads to the inequalities

$$\cos x < \frac{x}{\sin x} < \frac{1}{\cos x}.$$

When we take reciprocals and reverse the inequalities, we have

$$\cos x < \frac{\sin x}{x} < \frac{1}{\cos x}, \tag{2}$$

for $0 < x < \pi/2$.

A similar argument shows that the inequalities in (2) also hold for $-\pi/2 < x < 0$.

Taking the limit as $x \to 0$ in (2), we find that

$$\underbrace{\lim_{x\to 0} \cos x}_{1} < \lim_{x\to 0} \frac{\sin x}{x} < \underbrace{\lim_{x\to 0} \frac{1}{\cos x}}_{1}.$$

The Squeeze Theorem (Theorem 2.5) now implies that $\displaystyle\lim_{x\to 0} \frac{\sin x}{x} = 1$.

EXAMPLE 1 **Calculating trigonometric limits** Evaluate the following limits.

a. $\lim_{x \to 0} \frac{\sin 4x}{x}$

b. $\lim_{x \to 0} \frac{\sin 3x}{\sin 5x}$

Derivative of Sine and Cosine Functions

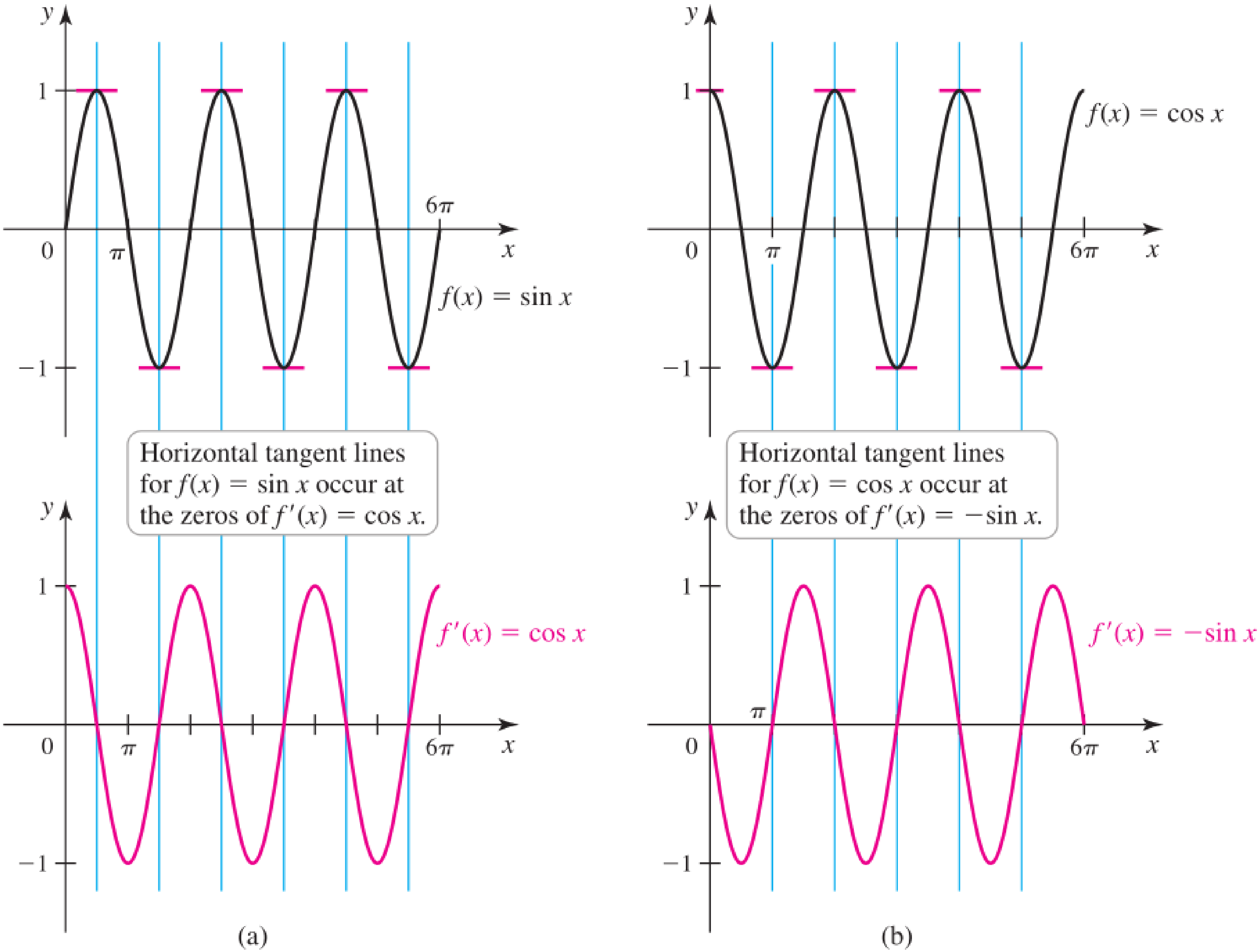
For sin$x$

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} && \text{Definition of derivative} \\
&= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} && \text{Sine addition identity} \\
&= \lim_{h\to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h} && \text{Factor } \sin x. \\
&= \lim_{h\to 0} \frac{\sin x(\cos h - 1)}{h} + \lim_{h\to 0} \frac{\cos x \sin h}{h} && \text{Theorem 2.3} \\
&= \sin x \underbrace{\left(\lim_{h\to 0} \frac{\cos h - 1}{h}\right)}_{0} + \cos x \underbrace{\left(\lim_{h\to 0} \frac{\sin h}{h}\right)}_{1} && \text{Both } \sin x \text{ and } \cos x \text{ are independent of } h. \\
&= (\sin x)(0) + \cos x(1) && \text{Theorem 3.11} \\
&= \cos x. && \text{Simplify.}
\end{aligned}
$$

The derivative of $\cos x$ can be proved using a cosine addition identity (Exercise)

THEOREM 3.12 Derivatives of Sine and Cosine

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x$$



y
1
−1
0
π
6π
x
$f(x) = \sin x$
Horizontal tangent lines for $f(x) = \sin x$ occur at the zeros of $f'(x) = \cos x$.
$f'(x) = \cos x$
(a)
$f(x) = \cos x$
Horizontal tangent lines for $f(x) = \cos x$ occur at the zeros of $f'(x) = -\sin x$.
$f'(x) = -\sin x$
(b)

EXAMPLE 2 **Derivatives involving trigonometric functions** Calculate dy/dx for the following functions.

a. $y = e^{2x} \cos x$ **b.** $y = \sin x - x \cos x$ **c.** $y = \dfrac{1 + \sin x}{1 - \sin x}$

Derivative of other Trigonometric Functions

The derivative of $\tan x$, $\cot x$, $\sec x$ and $\csc x$ can be obtained using the derivatives of $\sin x$ and $\cos x$ together with the Quotient Rule and trigonometric identities.

EXAMPLE 3 **Derivative of the tangent function** Calculate $\dfrac{d}{dx}(\tan x)$.

THEOREM 3.13 **Derivatives of the Trigonometric Functions**

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

EXAMPLE 4 **Derivatives involving sec x and csc x** Find the derivative of $y = \sec x \csc x$.

Higher-Order Trigonometric Derivatives

Important in many applications, particularly in problems that involves oscillations, vibrations or waves.

The **pattern** for $y = \sin x$

$$\frac{dy}{dx} = \cos x \qquad \frac{d^2y}{dx^2} = \frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}(-\sin x) = -\cos x \qquad \frac{d^4y}{dx^4} = \frac{d}{dx}(-\cos x) = \sin x$$

Cyclic behavior, similar for $y = cos x$,
but does not occur for other trigonometric functions

EXAMPLE 5 **Second-order derivatives** Find the second derivative of $y = \csc x$.

Chapter 3

Derivatives (I)

Shuwei Chen
swchen@swjtu.edu.cn