Chapter 4

Applications of the Derivative (II)

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4.5

Optimization Problems

Optimization Problems

Assume $x + y = 20$ for a pair of nonnegative real numbers x and y between 0 and 20.
Of all possible pairs, which has the greatest product?

Table 4.3

x	y	$x + y$	$P = xy$
1	19	20	19
5.5	14.5	20	79.75
9	11	20	99
13	7	20	91
18	2	20	36

The condition that $x + y = 20$ is called a **constraint**
The quantity that we wish to maximize (or minimize) is called the **objective function**, $P = xy$ here.

General form of optimization problems

What is the maximum (minimum) value of an objective function subject to the given constraint(s)?

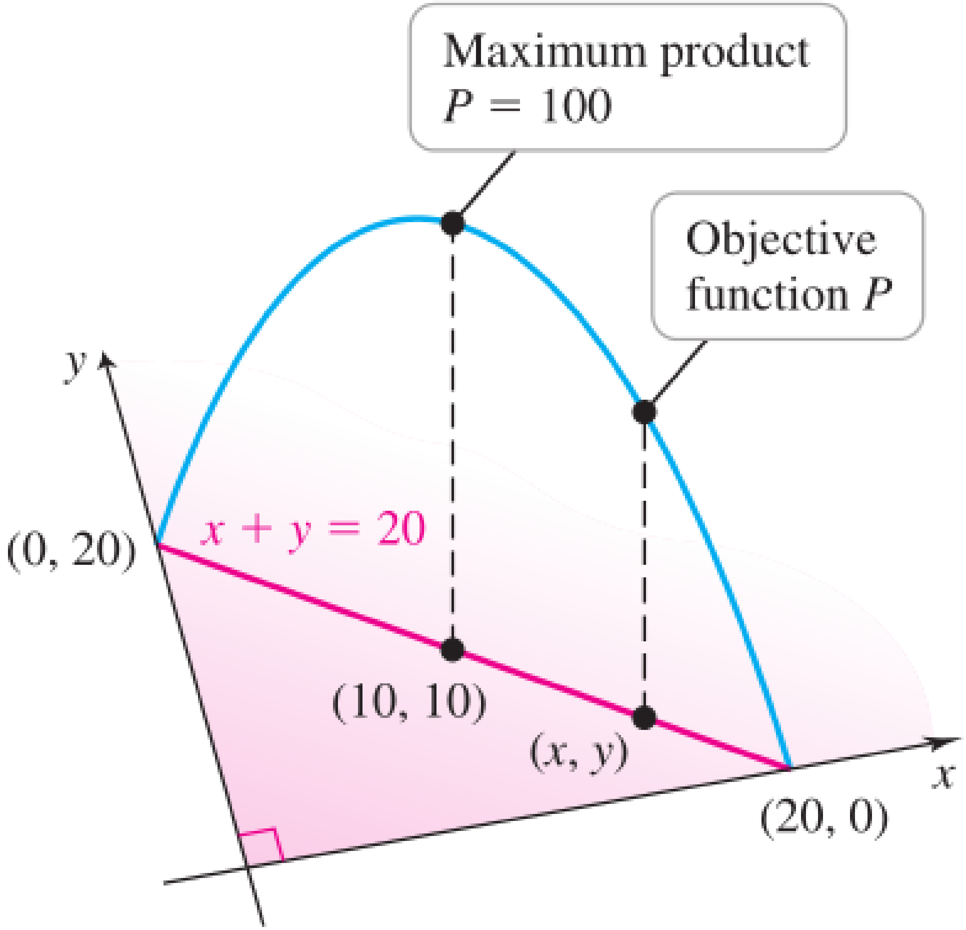
First step: express the objective function in terms of a single variable using the constraint

$$P = xy = x(20 - x) = 20x - x^2$$

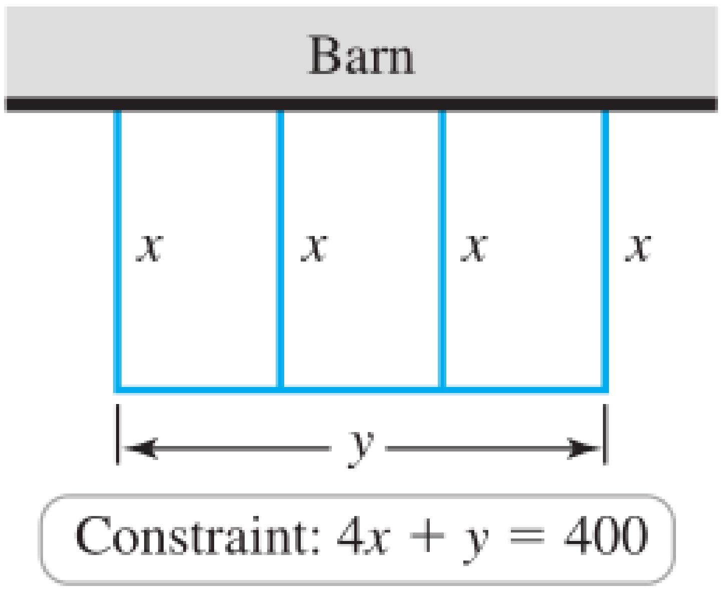
Find the critical points: Solving $P'(x) = 20 - 2x = 0$ to obtain the solution $x = 10$.

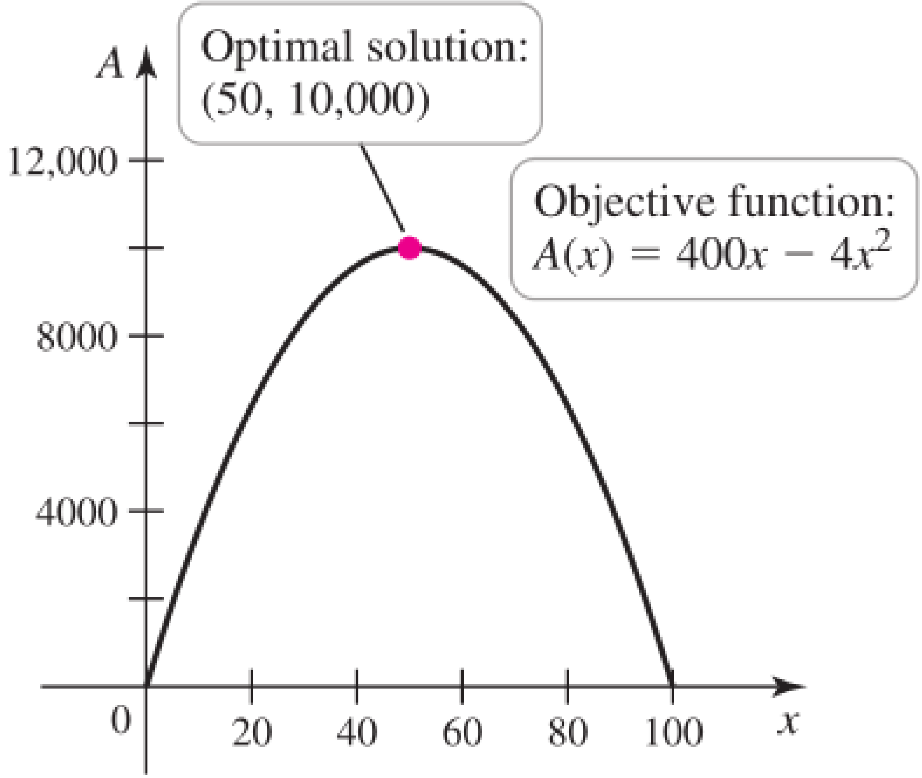
Check the values of endpoints and critical points:
$P(0) = P(20) = 0$ and $P(10) = 100$, then P has its absolute maximum value at $x = 10$, and the product is $P = 100$.

Most optimization problems have the same basic structure.

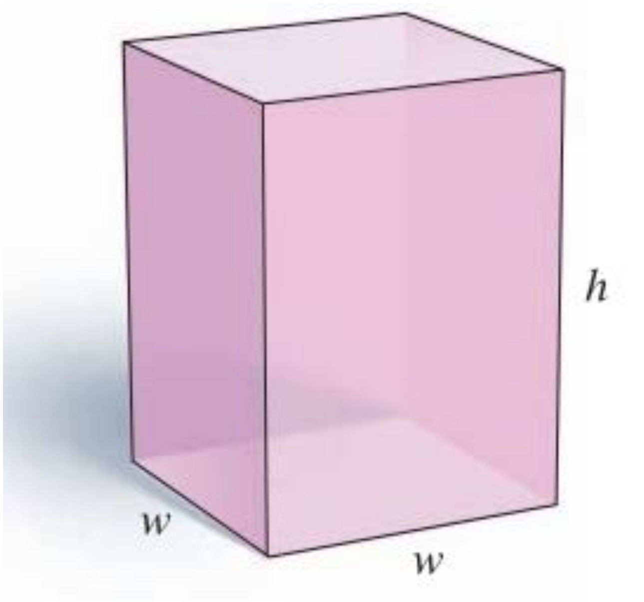

Maximum product
$P = 100$
Objective
function P
y
(0, 20)
$x + y = 20$
(10, 10)
(x, y)
x
(20, 0)

EXAMPLE 1 **Rancher's dilemma** A rancher has 400 ft of fence for constructing a rectangular corral. One side of the corral will be formed by a barn and requires no fence. Three exterior fences and two interior fences partition the corral into three rectangular regions as shown in Figure 4.51. What are the dimensions of the corral that maximize the enclosed area? What is the area of that corral?

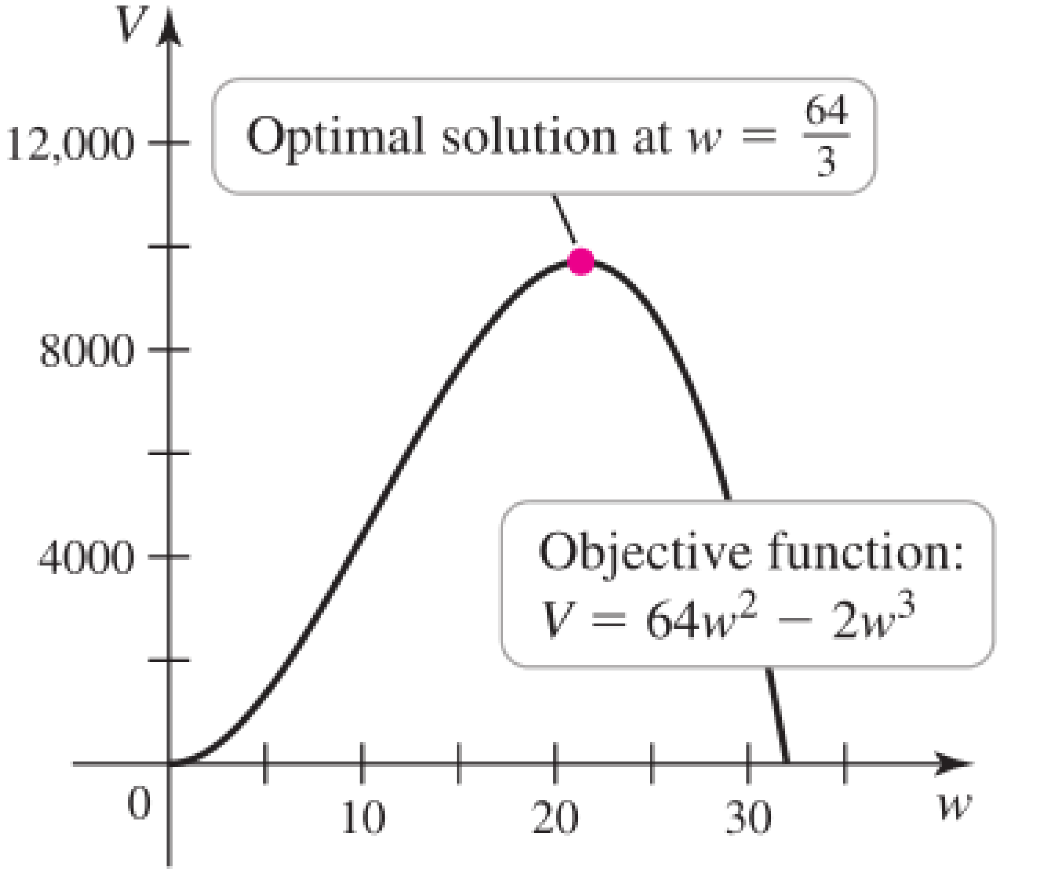




EXAMPLE 2 **Airline regulations** Suppose an airline policy states that all baggage must be box-shaped with a sum of length, width, and height not exceeding 64 in. What are the dimensions and volume of a square-based box with the greatest volume under these conditions?



Objective function: $V = w^2h$
Constraint: $2w + h = 64$

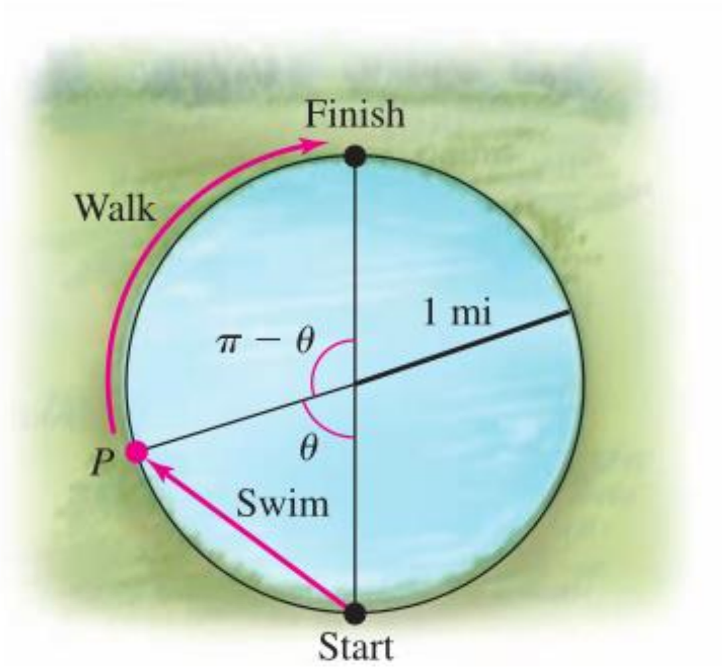


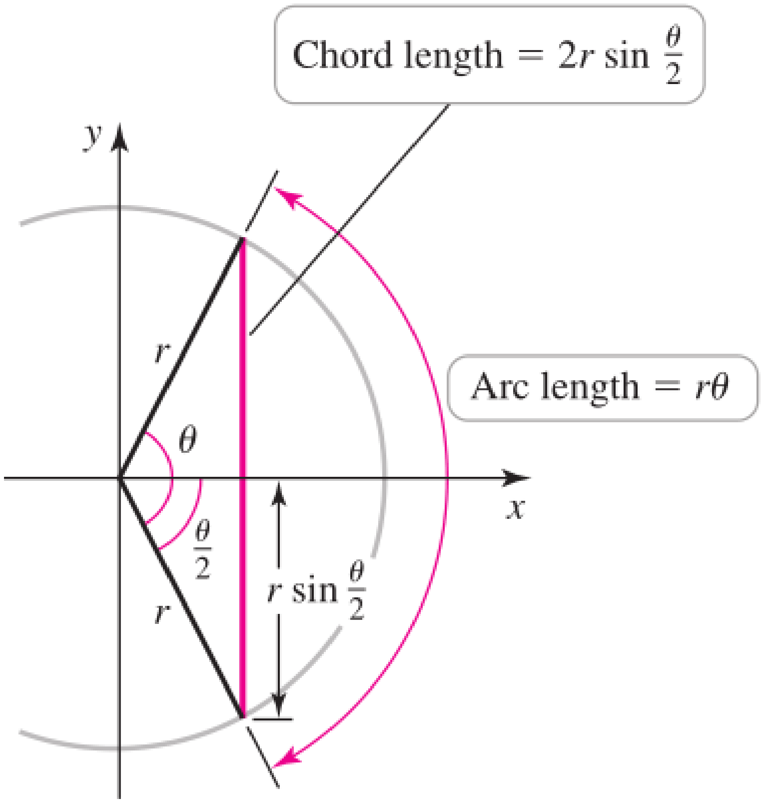
Optimization Guidelines

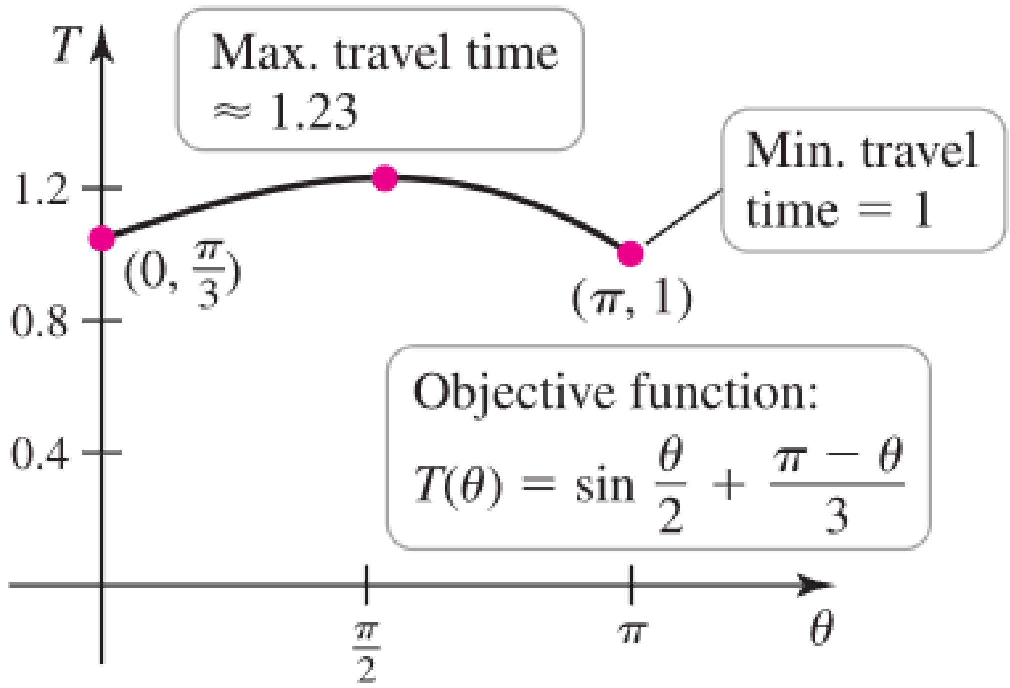
Guidelines for Optimization Problems

1. Read the problem carefully, identify the variables, and organize the given information with a picture.
2. Identify the objective function (the function to be optimized). Write it in terms of the variables of the problem.
3. Identify the constraint(s). Write them in terms of the variables of the problem.
4. Use the constraint(s) to eliminate all but one independent variable of the objective function.
5. With the objective function expressed in terms of a single variable, find the interval of interest for that variable.
6. Use methods of calculus to find the absolute maximum or minimum value of the objective function on the interval of interest. If necessary, check the endpoints.

EXAMPLE 3 **Walking and swimming** Suppose you are standing on the shore of a circular pond with a radius of 1 mile and you want to get to a point on the shore directly opposite your position (on the other end of a diameter). You plan to swim at 2 mi/hr from your current position to another point P on the shore and then walk at 3 mi/hr along the shore to the terminal point (Figure 4.55). How should you choose P to minimize the total time for the trip?

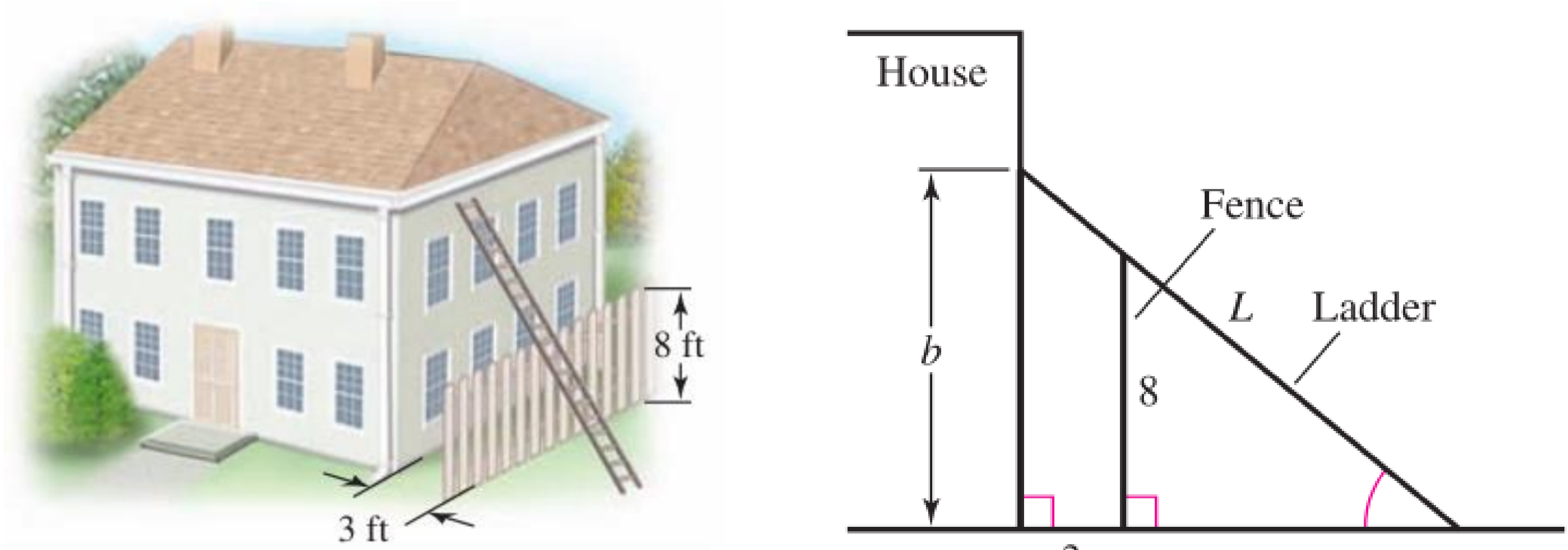






T
Max. travel time ≈ 1.23
Min. travel time $= 1$
1.2
0.8
0.4
$(0, \frac{\pi}{3})$
$(\pi, 1)$
Objective function:
$T(\theta) = \sin\frac{\theta}{2} + \frac{\pi - \theta}{3}$
$\frac{\pi}{2}$
π
θ

EXAMPLE 4 **Ladder over the fence** An 8-foot-tall fence runs parallel to the side of a house 3 feet away (Figure 4.58a). What is the length of the shortest ladder that clears the fence and reaches the house? Assume that the vertical wall of the house and the horizontal ground have infinite extent (see Exercise 23 for more realistic assumptions).

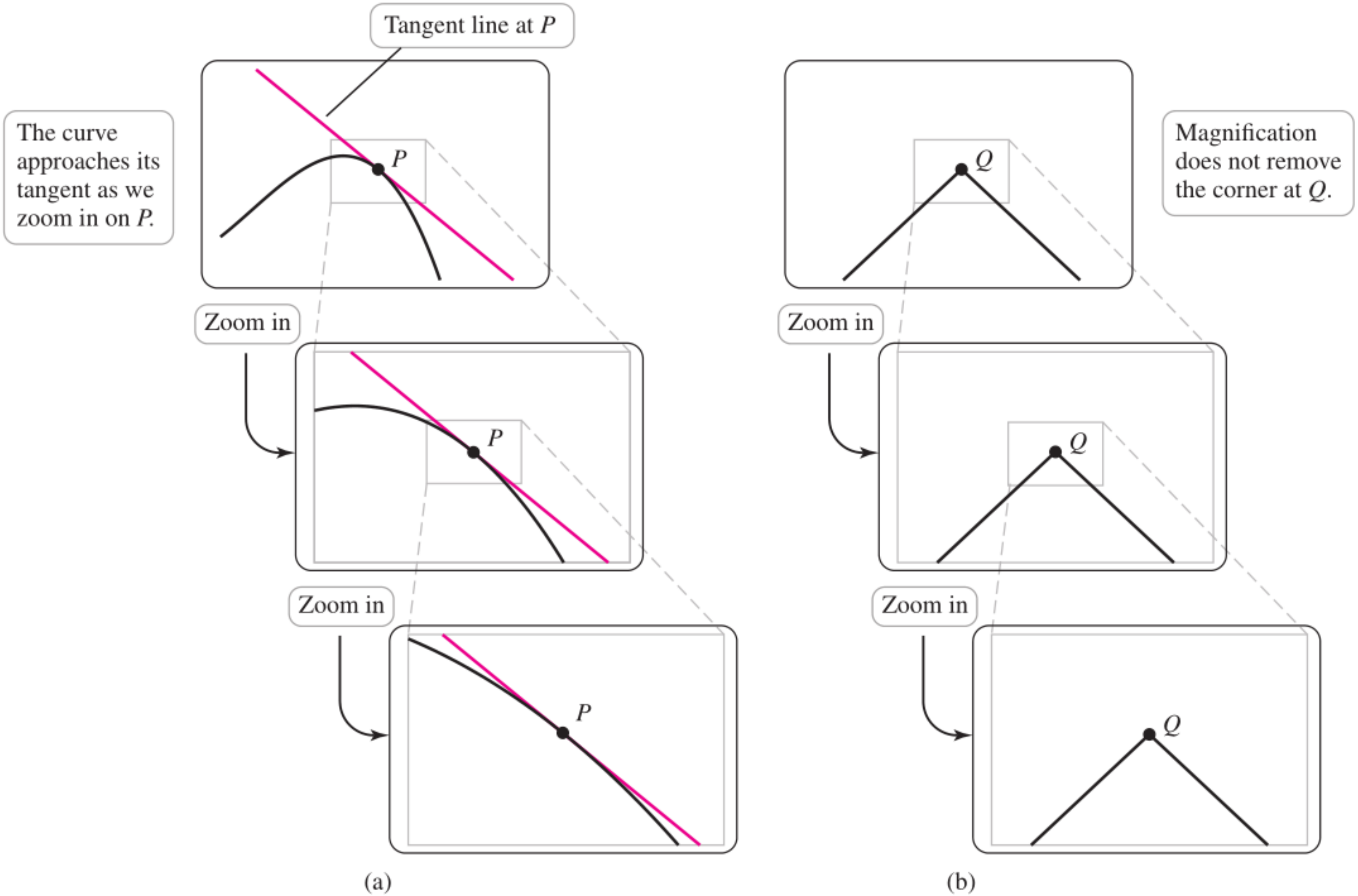


(a) (b)

$$L^2 = (x+3)^2 + \underbrace{\left(\frac{8(x+3)}{x}\right)}_{b}{}^2 = (x+3)^2\left(1 + \frac{64}{x^2}\right).$$

4.6

Linear Approximation and Differentials

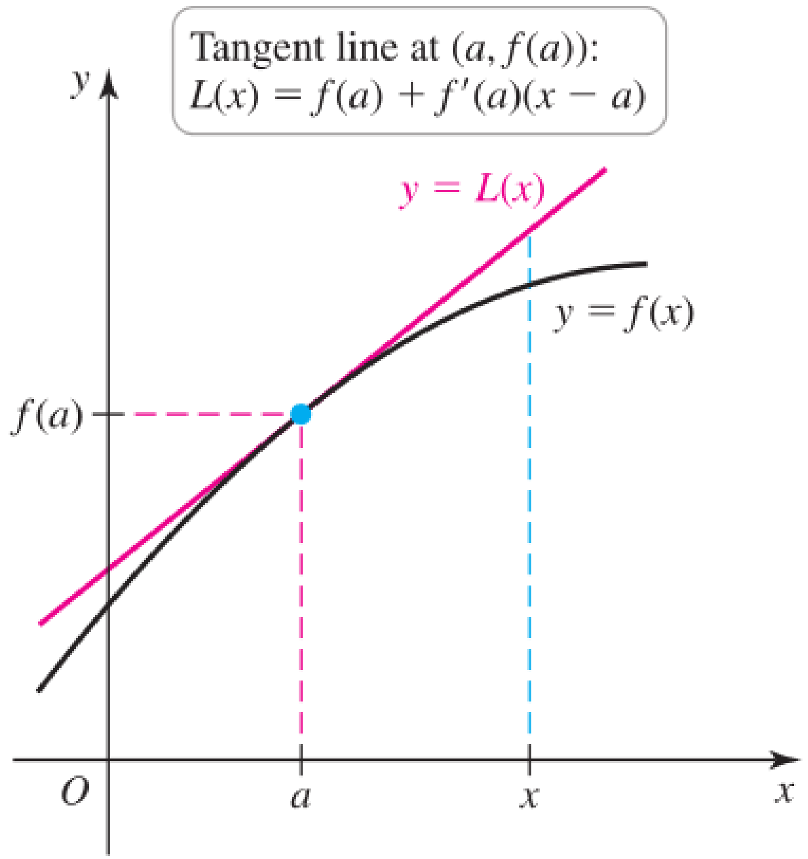


The fundamental observation—that smooth curves appear straighter on smaller scales—is called *local linearity*.

The basis of many important mathematical ideas, one of which is *linear approximation*.

Linear Approximation

Figure suggests that when we zoom in on the graph of a smooth function at a point P, the curve approaches its tangent line at P.



An equation of the tangent line is

$$y - f(a) = f'(a)(x - a) \quad \text{or} \quad y = \underbrace{f(a) + f'(a)(x - a)}_{L(x)}.$$

This tangent line represents a new function L that we call the *linear approximation* to f at the point a. That is,

$$f(x) \approx L(x) = f(a) + f'(a)(x - a)$$

DEFINITION Linear Approximation to f at a

Suppose f is differentiable on an interval I containing the point a. The **linear approximation** to f at a is the linear function

$$L(x) = f(a) + f'(a)(x - a), \quad \text{for } x \text{ in } I.$$

EXAMPLE 1 **Useful driving math** Suppose you are driving along a highway at a nearly constant speed and you record the number of seconds it takes to travel between two consecutive mile markers. If it takes 60 seconds, to travel one mile, then your average speed is 1 mi/60 s or 60 mi/hr. Now suppose that you travel one mile in $60 + x$ seconds; for example, if it takes 62 seconds, then $x = 2$, and if it takes 57 seconds, then $x = -3$. In this case, your average speed over one mile is 1 mi/$(60 + x)$s. Because there are 3600 s in 1 hr, the function

$$s(x) = \frac{3600}{60 + x} = 3600(60 + x)^{-1}$$

gives your average speed in mi/hr if you travel one mile in x seconds more or less than 60 seconds. For example, if you travel one mile in 62 seconds, then $x = 2$ and your average speed is $s(2) \approx 58.06$ mi/hr. If you travel one mile in 57 seconds, then $x = -3$ and your average speed is $s(-3) \approx 63.16$ mi/hr. Because you don't want to use a calculator while driving, you need an easy approximation to this function. Use linear approximation to derive such a formula.

EXAMPLE 2 Linear approximations and errors

a. Find the linear approximation to $f(x) = \sqrt{x}$ at $x = 1$ and use it to approximate $\sqrt{1.1}$.

b. Use linear approximation to estimate the value of $\sqrt{0.1}$.

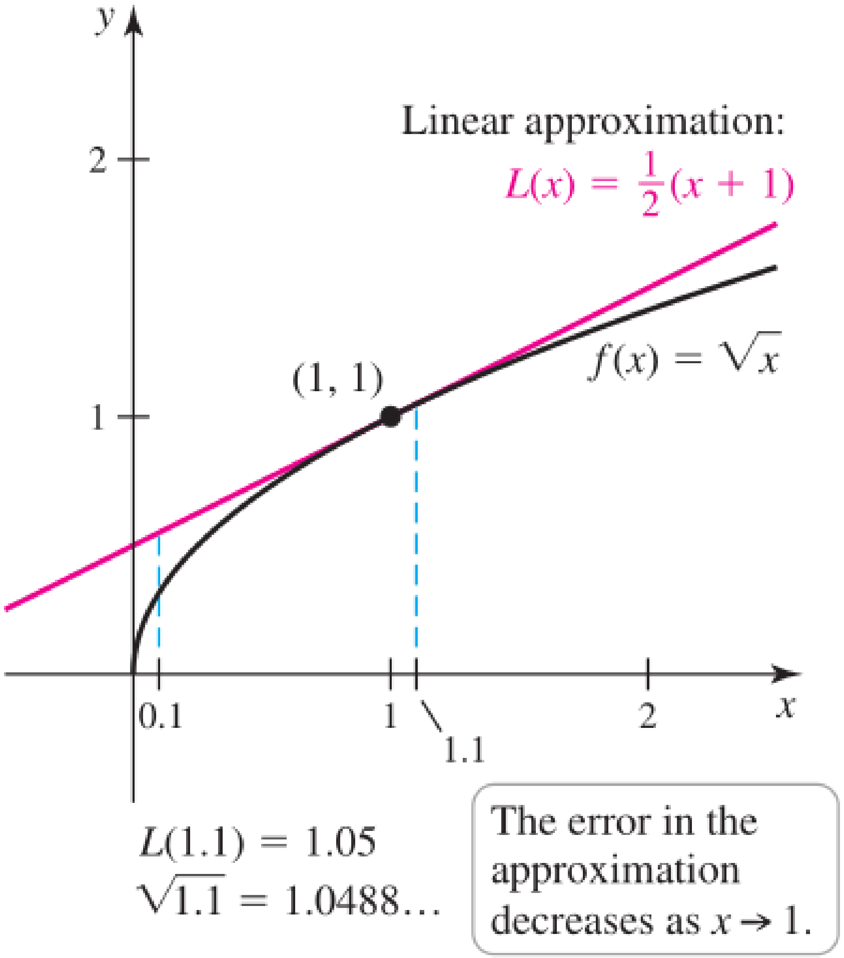
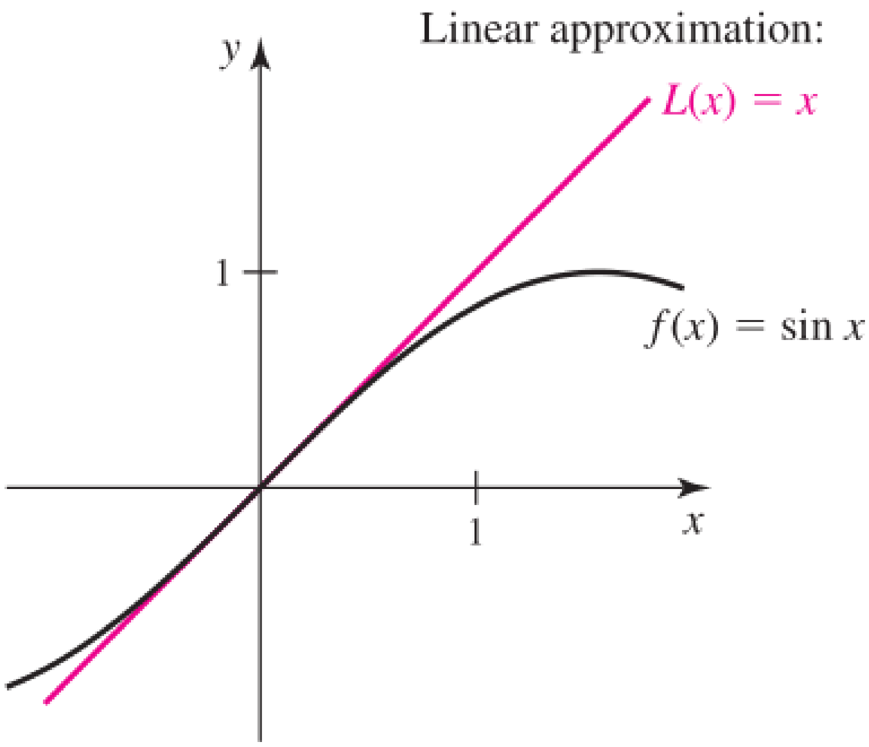


Table 4.4

x	$L(x)$	Exact $\sqrt{x}$	Error
1.2	1.1	1.0954 . . .	4.6×10^{-3}
1.1	1.05	1.0488 . . .	1.2×10^{-3}
1.01	1.005	1.0049 . . .	1.2×10^{-5}
1.001	1.0005	1.0005 . . .	1.2×10^{-7}

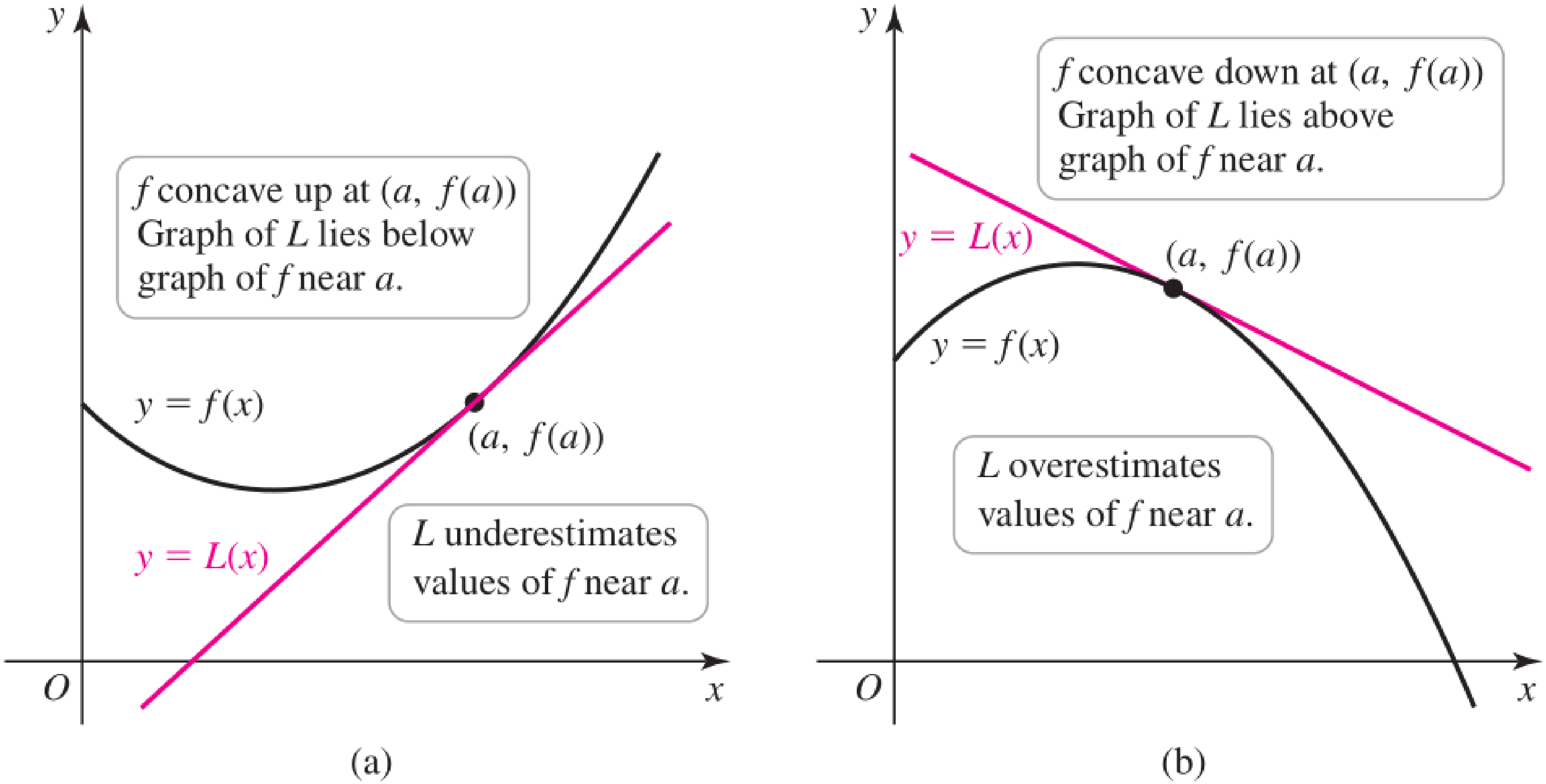
EXAMPLE 3 Linear approximation for the sine function Find the linear approximation to $f(x) = \sin x$ at $x = 0$ and use it to approximate $\sin 2.5°$.



Why bother with linear approximation when a calculator does a better job?

- Linear approximation is actually just the first step in the process of *polynomial approximation*.
- Linear approximation also allows us to discover simple approximations to complicated functions.

Linear Approximation and Concavity



If f is concave up, the linear approximation *underestimates* values of f near a.
If f is concave down, the linear approximation *overestimates* values of f near a.

Curvature

Another observation related to the *degree of concavity* (also called *curvature*).

- A large value of $|f''(a)|$ (large curvature) means that near $(a, f(a))$, the slope of the curve changes rapidly and the graph of f separates quickly from the tangent line.
- A small value of $|f''(a)|$ (small curvature) means the slope of the curve changes slowly and the curve is relatively flat near $(a, f(a))$; therefore, the curve remains close to the tangent line.
- As a result, absolute errors in linear approximation are larger when $|f''(a)|$ is large.

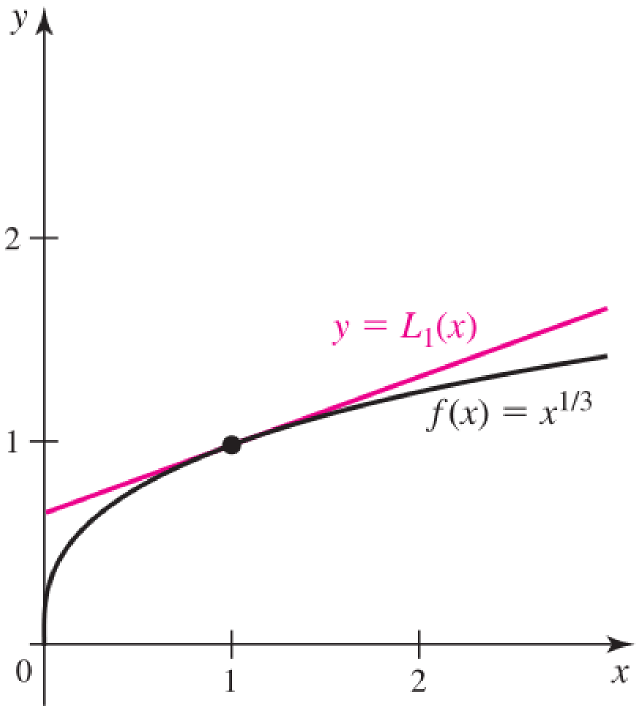
EXAMPLE 4 **Linear approximation and concavity**

a. Find the linear approximation to $f(x) = x^{1/3}$ at $x = 1$ and $x = 27$.

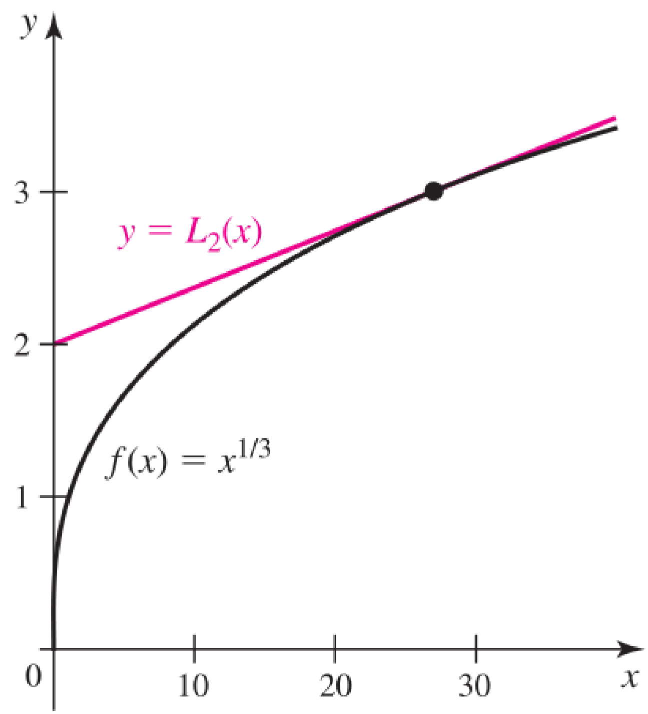
b. Use the linear approximations of part (a) to approximate $\sqrt[3]{2}$ and $\sqrt[3]{26}$

c. Are the approximations in part (b) overestimates or underestimates?

d. Compute the error in the approximations of part (b). Which error is greater? Explain.



(a)



(b)

A Variation on Linear Approximation

Linear approximation

$$f(x) \approx f(a) + f'(a)(x - a)$$

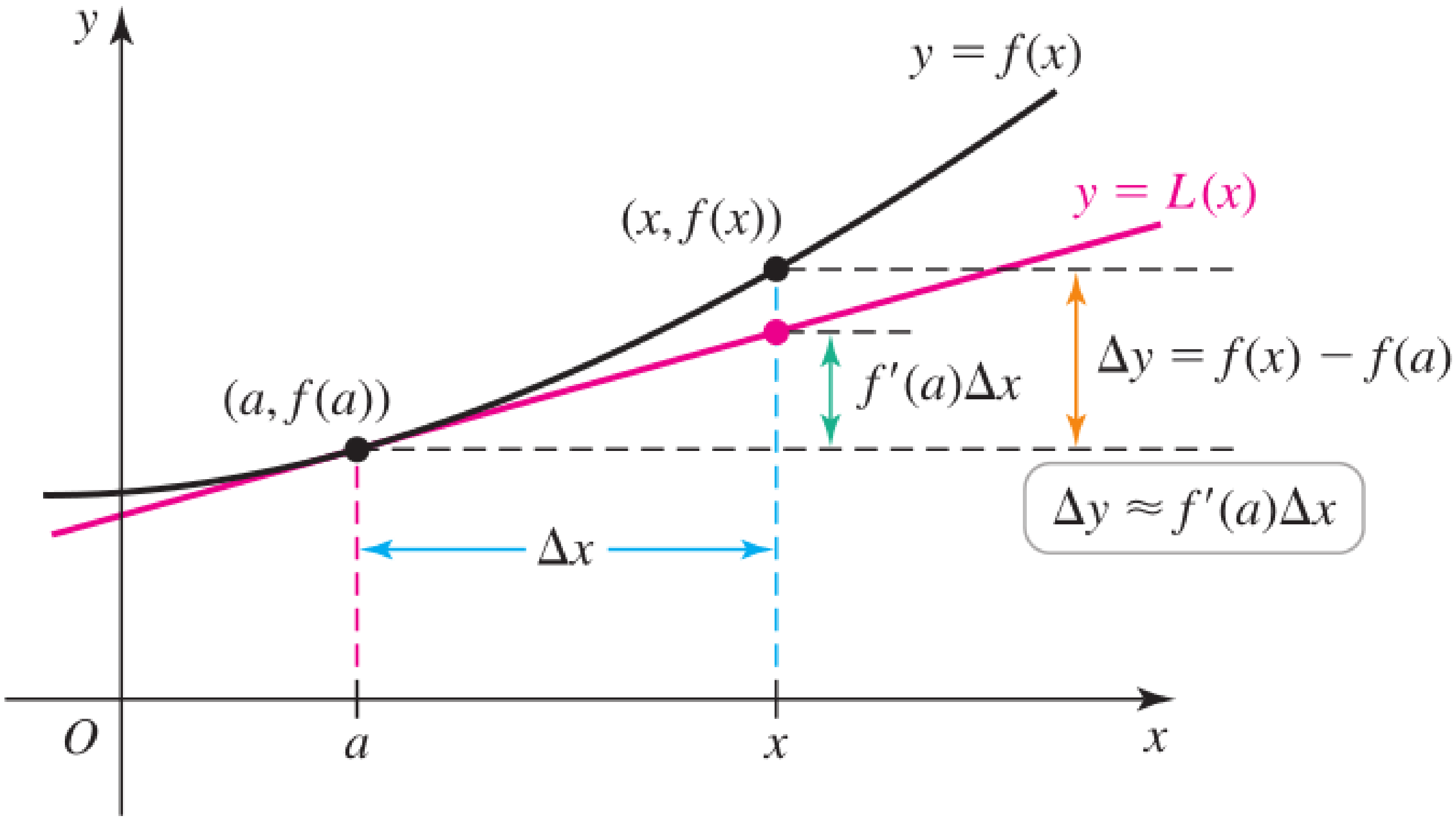
Rewrite this expression as

$$\underbrace{f(x) - f(a)}_{\Delta y} \approx f'(a)\underbrace{(x - a)}_{\Delta x}.$$

Use the notation Δ to denote a change

$$\Delta y \approx f'(a)\Delta x$$

In other words, a change in y (the function value) can be approximated by the corresponding change in x magnified or diminished by a factor of $f'(a)$.


y
$y = f(x)$
$y = L(x)$
$(x, f(x))$
$(a, f(a))$
$f'(a)\Delta x$
$\Delta y = f(x) - f(a)$
$\Delta y \approx f'(a)\Delta x$
Δx
O
a
x
x

Relationship Between Δx and Δy

Suppose f is differentiable on an interval I containing the point a. The change in the value of f between two points a and $a + \Delta x$ is approximately

$$\Delta y \approx f'(a)\,\Delta x,$$

where $a + \Delta x$ is in I.

EXAMPLE 5 **Estimating changes with linear approximations**

a. Approximate the change in $y = f(x) = x^9 - 2x + 1$ when x changes from 1.00 to 1.05.

b. Approximate the change in the surface area of a spherical hot-air balloon when the radius decreases from 4 m to 3.9 m.

SUMMARY Uses of Linear Approximation

- To approximate f near $x = a$, use

$$f(x) \approx L(x) = f(a) + f'(a)(x - a).$$

- To approximate the change Δy in the dependent variable when x changes from a to $a + \Delta x$, use

$$\Delta y \approx f'(a)\,\Delta x.$$

Differentials

An important concept to distinguish two related quantities:

- the change in the function $y = f(x)$ as x changes from a to $a + \Delta x$ (which we call Δy, as before), and
- the change in the linear approximation $y = L(x)$ as x changes from a to $a + \Delta x$ (which we call the differential dy)

$$\begin{aligned}\Delta L &= L(a + \Delta x) - L(a)\\ &= \underbrace{(f(a) + f'(a)(a + \Delta x - a))}_{L(a + \Delta x)} - \underbrace{(f(a) + f'(a)(a - a))}_{L(a)}\\ &= f'(a)\,\Delta x.\end{aligned}$$

$$\Delta L = \underbrace{dy}_{\substack{\text{same}\\ \text{as } \Delta L}} = f'(a)\,\Delta x = f'(a)\,\underbrace{dx}_{\substack{\text{same}\\ \text{as } \Delta x}}.$$

DEFINITION Differentials

Let f be differentiable on an interval containing x. A small change in x is denoted by the **differential** dx. The corresponding change in f is approximated by the **differential** $dy = f'(x)\,dx$; that is,

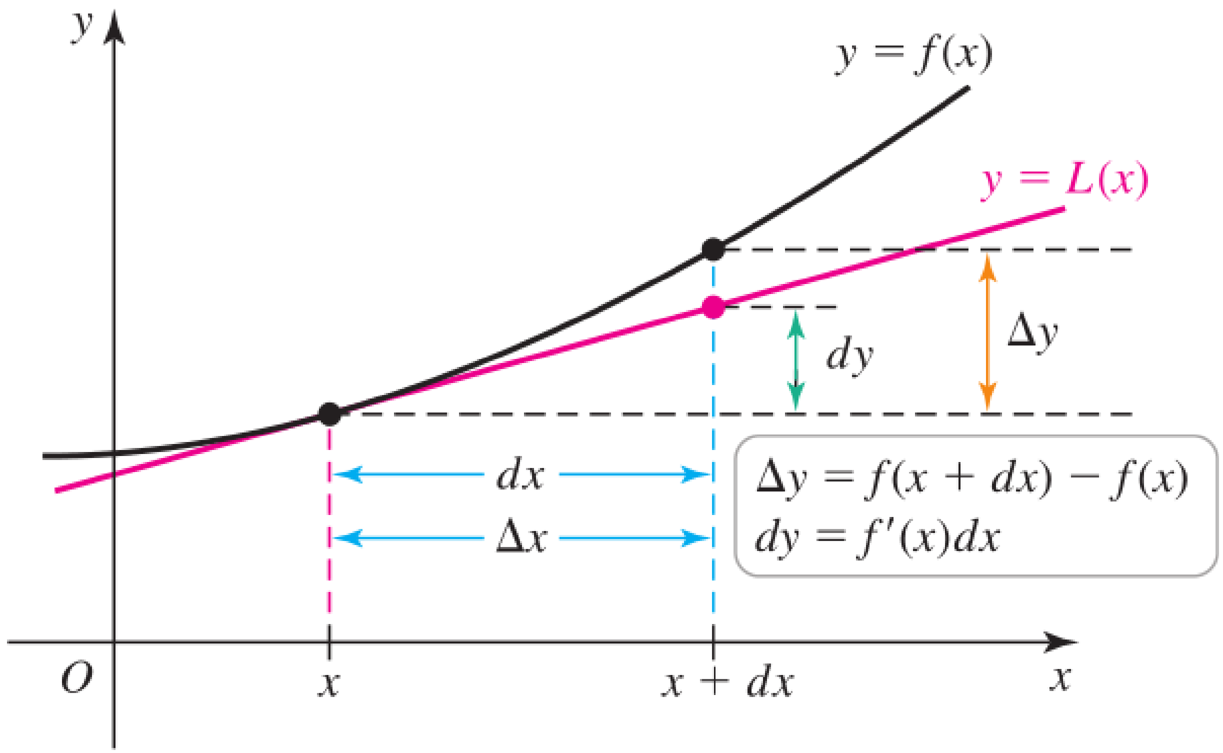
$$\Delta y = f(x + dx) - f(x) \approx dy = f'(x)\,dx.$$

The notation for differentials is consistent with the notation for the derivative:

If we divide both sides of $dy = f'(x)dx$ by dx, we have

$$\frac{dy}{dx} = \frac{f'(x)dx}{dx} = f'(x)$$

If $\Delta x = dx$ is small, then the change in f, which is Δy, is well approximated by the change in the linear approximation, which is dy. Furthermore, the approximation $\Delta y \approx dy$ improves as dx approaches 0.



EXAMPLE 6 **Differentials as change** Use the notation of differentials to write the approximate change in $f(x) = 3\cos^2 x$ given a small change dx.

4.7

L'Hôpital's Rule

Indeterminate Forms

Some limits, called *indeterminate forms*, cannot generally be evaluated using the techniques presented in Chapter 2.

Such as the limit $\lim_{x \to 0} \frac{\sin x}{x} = 1$.
If we attempt to substitute $x = 0$ into ${}^{\sin x}/_{x}$, we get $0/0$, which has no meaning. Yet we proved that the limit is 1.

Another example is $\lim_{x \to \infty} \frac{ax}{x+1}$, where $a \neq 0$
This limit has the indeterminate form ∞/∞.
The actual value of the limit is a.

In general, a limit with the form $0/0$ or ∞/∞ can have any value.

L'Hôpital's Rule for the Form 0/0

THEOREM 4.13 **L'Hôpital's Rule**

Suppose f and g are differentiable on an open interval I containing a with $g'(x) \neq 0$ on I when $x \neq a$. If $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

provided the limit on the right exists (or is $\pm\infty$). The rule also applies if $x \to a$ is replaced with $x \to \pm\infty$, $x \to a^+$, or $x \to a^-$.

Proof (special case):

The proof of this theorem relies on the Generalized Mean Value Theorem.

We prove a special case of the theorem in which we assume that f' and g' are continuous at a, $f(a) = g(a) = 0$, and $g'(a) \neq 0$.

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)} \qquad \text{Continuity of } f' \text{ and } g'$$

$$= \frac{\displaystyle\lim_{x \to a} \frac{f(x) - f(a)}{x - a}}{\displaystyle\lim_{x \to a} \frac{g(x) - g(a)}{x - a}} \qquad \text{Definition of } f'(a) \text{ and } g'(a)$$

$$= \lim_{x \to a} \frac{\dfrac{f(x) - f(a)}{x - a}}{\dfrac{g(x) - g(a)}{x - a}} \qquad \text{Limit of a quotient, } g'(a) \neq 0$$

$$= \lim_{x \to a} \frac{f(x) - f(a)}{g(x) - g(a)} \qquad \text{Cancel } x - a.$$

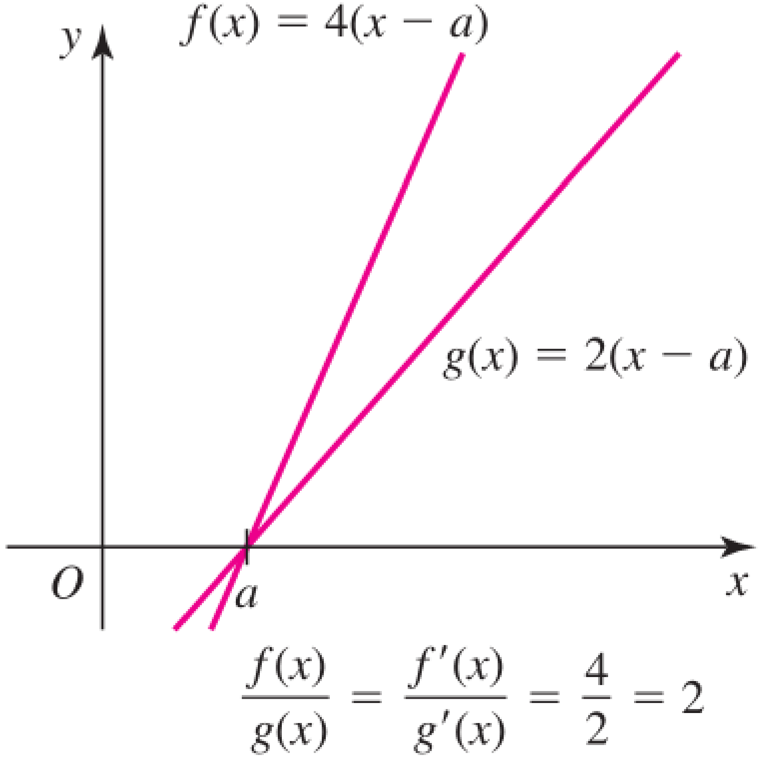
$$= \lim_{x \to a} \frac{f(x)}{g(x)}. \qquad f(a) = g(a) = 0$$

- The definition of the derivative provides an example of an indeterminate form. Assuming f is differentiable at x,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

has the form $0/0$.

The geometry of l’Hôpital's Rule offers some insight.



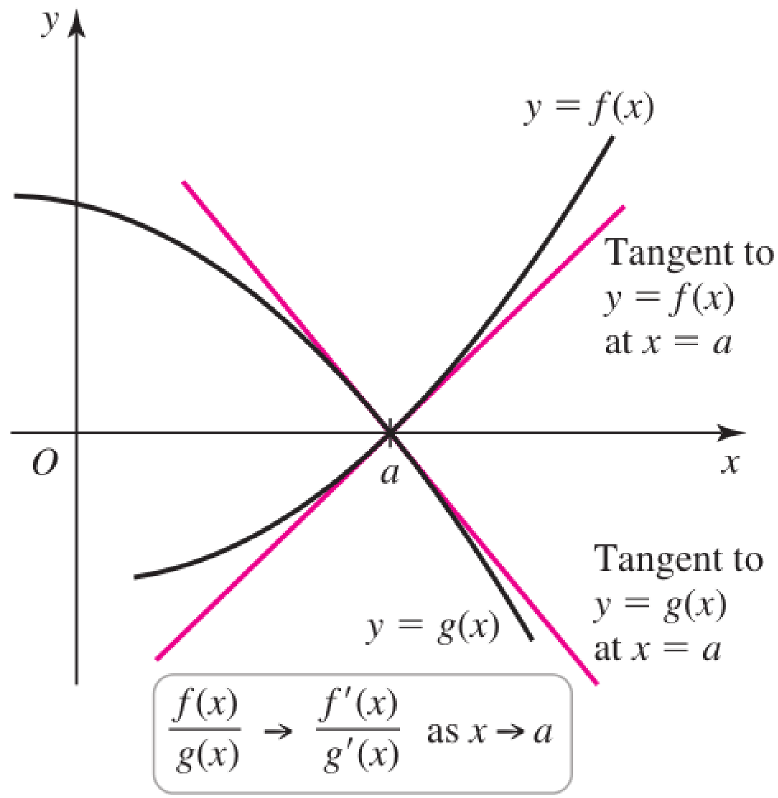
For any linear functions f and g with $f(a) = g(a) = 0$,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

For nonlinear functions f and g, replace them with their linear approximations at a. Near $x = a$, $\frac{f(x)}{g(x)} \approx \frac{f'(a)(x-a)}{g'(a)(x-a)} = \frac{f'(a)}{g'(a)}$.

Therefore,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$



EXAMPLE 1 **Using l'Hôpital's Rule** Evaluate the following limits.

a. $\displaystyle\lim_{x\to 1} \frac{x^3 + x^2 - 2x}{x - 1}$ **b.** $\displaystyle\lim_{x\to 0} \frac{\sqrt{9 + 3x} - 3}{x}$

L'Hôpital's Rule requires evaluating $\displaystyle\lim_{x\to a} \frac{f'(x)}{g'(x)}$. It may happen that this second limit is another indeterminate form to which L'Hôpital's Rule may again be applied.

EXAMPLE 2 **L'Hôpital's Rule repeated** Evaluate the following limits.

a. $\displaystyle\lim_{x\to 0} \frac{e^x - x - 1}{x^2}$ **b.** $\displaystyle\lim_{x\to 2} \frac{x^3 - 3x^2 + 4}{x^4 - 4x^3 + 7x^2 - 12x + 12}$

Indeterminate Form ∞/∞

THEOREM 4.14 **L'Hôpital's Rule (∞/∞)**
Suppose that f and g are differentiable on an open interval I containing a, with $g'(x) \neq 0$ on I when $x \neq a$. If $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

provided the limit on the right exists (or is $\pm\infty$). The rule also applies for $x \to \pm\infty$, $x \to a^+$, or $x \to a^-$.

EXAMPLE 3 **L'Hôpital's Rule for ∞/∞** Evaluate the following limits.

a. $\displaystyle\lim_{x\to\infty} \frac{4x^3 - 6x^2 + 1}{2x^3 - 10x + 3}$ **b.** $\displaystyle\lim_{x\to\pi/2^-} \frac{1 + \tan x}{\sec x}$

Related Indeterminate Forms: $0 \cdot \infty$ and $\infty - \infty$

L'Hôpital's Rule *cannot be directly applied* to a $0 \cdot \infty$ form, $\lim_{x \to a} f(x)g(x)$, where $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = \pm\infty$

A limit of the form $0 \cdot \infty$, in which the two functions compete with each other, may have any value or may not exist.

E.g., $f(x) = x$ and $g(x) = \frac{1}{x^2}$; or $f(x) = x$ and $g(x) = \frac{1}{\sqrt{x}}$

This indeterminate form can be recast in the form $0/0$ or ∞/∞, and then apply l'Hôpital's Rule.

EXAMPLE 4 **L'Hôpital's Rule for $0 \cdot \infty$** Evaluate $\lim_{x \to \infty} x^2 \sin\left(\frac{1}{4x^2}\right)$.

Limits of the form $\lim_{x \to a} f(x) - g(x)$ where $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = \infty$.

L'Hôpital's Rule *cannot be directly applied* to an $\infty - \infty$ form. For example, $\lim_{x \to \infty}((3x + 5) - (3x)) = 5$, while $\lim_{x \to \infty}(3x - 2x) = \lim_{x \to \infty} x = \infty$

Change of variables, e.g., to evaluate $\lim_{x \to \infty} f(x)$, define $t = {}^{1}\!/_{x}$ and note that as $x \to \infty$, $t \to 0^+$. Then

$$\lim_{x \to \infty} f(x) = \lim_{t \to 0^+} f\left(\frac{1}{t}\right)$$

EXAMPLE 5 **L'Hôpital's Rule for $\infty - \infty$** Evaluate $\lim_{x\to\infty}\left(x - \sqrt{x^2 - 3x}\right)$.

Solution

$$\lim_{x\to\infty}\left(x - \sqrt{x^2 - 3x}\right) = \lim_{x\to\infty}\left(x - \sqrt{x^2(1 - 3/x)}\right) \quad \text{Factor } x^2 \text{ under square root.}$$

$$= \lim_{x\to\infty} x\left(1 - \sqrt{1 - 3/x}\right) \quad x > 0, \text{ so } \sqrt{x^2} = x$$

$$= \lim_{x\to\infty} \frac{1 - \sqrt{1 - 3/x}}{1/x}. \quad \text{Write } 0 \cdot \infty \text{ form as } 0/0 \text{ form; } x = \frac{1}{1/x}.$$

This new limit has the form $0/0$, and l'Hôpital's Rule *may be applied*. One way is to use the change of variables $t = {}^{1}/_{x}$

$$\lim_{x\to\infty} \frac{1 - \sqrt{1 - 3/x}}{1/x} = \lim_{t\to 0^+} \frac{1 - \sqrt{1 - 3t}}{t} \quad \text{Let } t = 1/x; \text{ replace } \lim_{x\to\infty} \text{ with } \lim_{t\to 0^+}.$$

$$= \lim_{t\to 0^+} \frac{\dfrac{3}{2\sqrt{1 - 3t}}}{1} \quad \text{L'Hôpital's Rule}$$

$$= \frac{3}{2}. \quad \text{Evaluate limit.}$$

Indeterminate Forms: 1^∞, 0^0 and ∞^0

All arise in limits of the form $\lim_{x\to a} f(x)^{g(x)}$, where $x \to a$ could be replaced with $x \to a^\pm$ or $x \to \pm\infty$.
Must first be expressed in the form $0/0$ or ∞/∞.

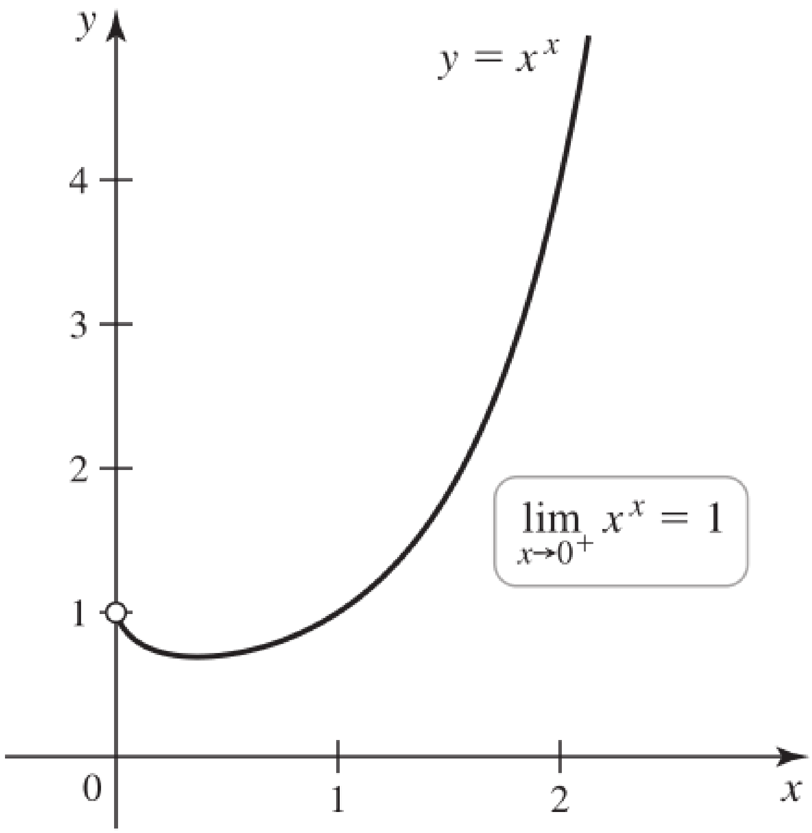
PROCEDURE **Indeterminate forms 1^∞, 0^0, and ∞^0**

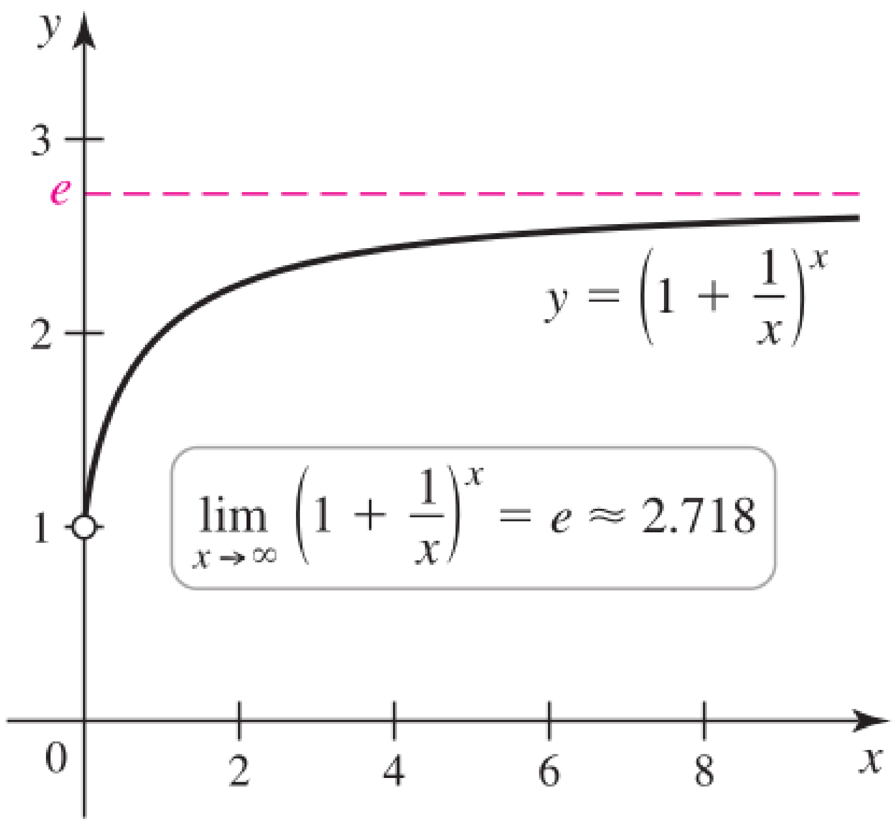
Assume $\lim_{x\to a} f(x)^{g(x)}$ has the indeterminate form 1^∞, 0^0, or ∞^0.

1. Analyze $L = \lim_{x\to a} g(x) \ln f(x)$. This limit can be put in the form $0/0$ or ∞/∞, both of which are handled by l'Hôpital's Rule.
2. When L is finite, $\lim_{x\to a} f(x)^{g(x)} = e^L$. If $L = \infty$ or $-\infty$, then $\lim_{x\to a} f(x)^{g(x)} = \infty$ or $\lim_{x\to a} f(x)^{g(x)} = 0$, respectively.

EXAMPLE 6 **Indeterminate forms 0^0 and 1^∞** Evaluate the following limits.

a. $\lim_{x \to 0^+} x^x$ **b.** $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$





Growth Rates of Functions

- A particular theory for modeling the spread of an epidemic predicts that the number of infected people t days after the start of the epidemic is given by the function.

$$N(t) = 2.5t^2e^{-0.01t} = 2.5\,\frac{t^2}{e^{0.01t}}$$

Question: In the long run (as $t \to \infty$), does the epidemic spread or does it die out?

- A prime number is an integer $p \geq 2$ that has only two divisors, 1 and itself. The first few prime numbers are 2, 3, 5, 7, and 11. The number of prime numbers less than x is approximately $P(x) = \dfrac{x}{\ln x}$, for large values of x.

Question: According to this theorem, is the number of prime numbers infinite?

The above two questions involve a comparison of two functions.
Goal: to obtain a ranking of the following families of functions based on their growth rates:

- mx, where $m > 0$ (represents *linear functions*)
- x^p, where $p > 0$ (represents *polynomials* and *algebraic functions*)
- x^x (sometimes called a *superexponential* or *tower function*)
- $\ln x$ (represents *logarithmic* functions)
- $\ln^q x$, where $q > 0$ (represents *powers* of logarithmic functions)
- $x^p \ln x$, where $p > 0$ (a *combination* of powers and logarithms)
- e^x (represents *exponential* functions)

Growth rates and what it means for f to grow faster than g as $x \to \infty$.

DEFINITION Growth Rates of Functions (as $x \to \infty$)

Suppose f and g are functions with $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} g(x) = \infty$. Then f **grows faster than g** as $x \to \infty$ if

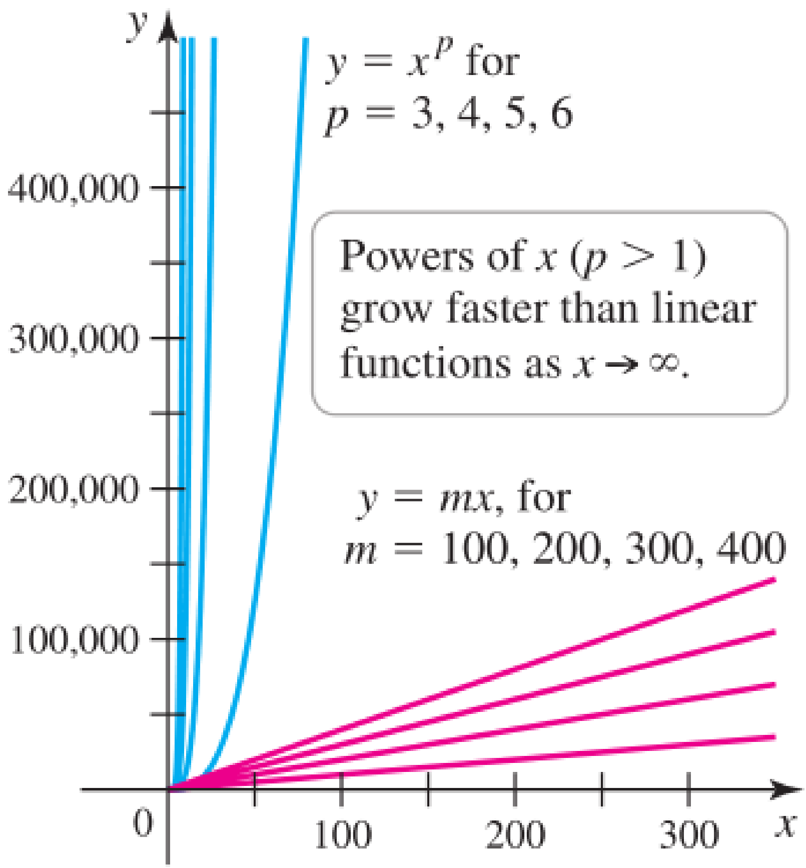
$$\lim_{x \to \infty} \frac{g(x)}{f(x)} = 0 \quad \text{or, equivalently, if} \quad \lim_{x \to \infty} \frac{f(x)}{g(x)} = \infty.$$

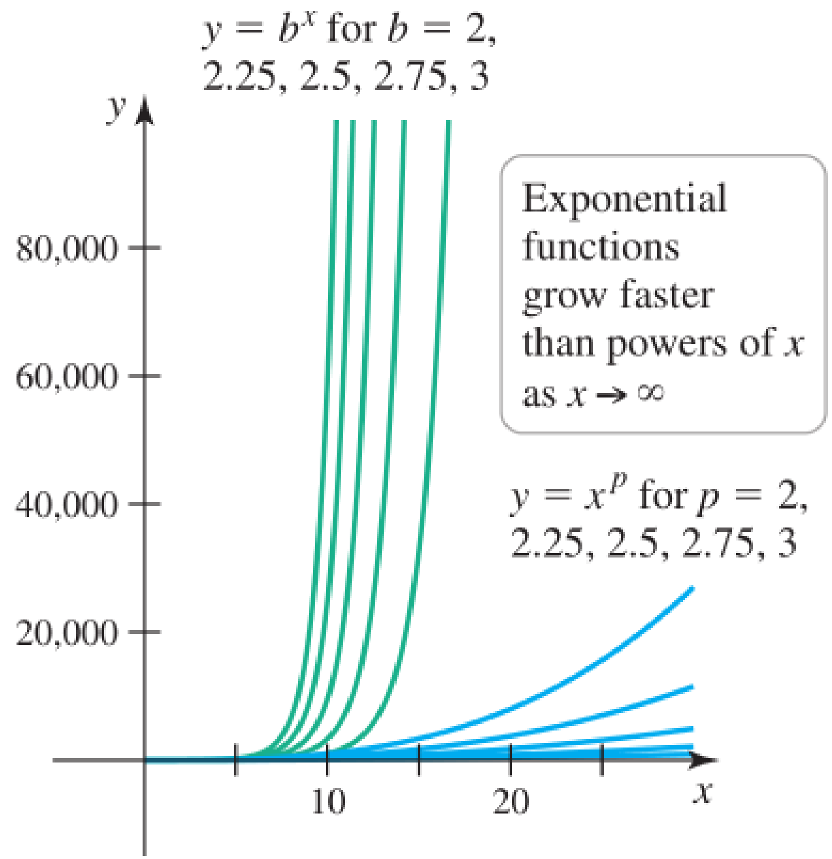
The functions f and g have **comparable growth rates** if

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = M,$$

where $0 < M < \infty$ (M is nonzero and finite).

Powers of x grow faster (their curves rise at a greater rate) than the linear functions as $x \to \infty$.





Exponential functions $y = b^x$, where $b > 1$, grow faster than powers of x of the form $y = x^p$, where $p > 0$, as $x \to \infty$.

EXAMPLE 7 **Powers of x vs. powers of $\ln x$** Compare the growth rates as $x \to \infty$ of the following pairs of functions.

a. $f(x) = \ln x$ and $g(x) = x^p$, where $p > 0$

b. $f(x) = \ln^q x$ and $g(x) = x^p$, where $p > 0$ and $q > 0$

Solution

a. Evaluate the limit of the ratio of the two functions

$$\lim_{x \to \infty} \frac{\ln x}{x^p} = \lim_{x \to \infty} \frac{1/x}{px^{p-1}} \quad \text{L'Hôpital's Rule}$$

$$= \lim_{x \to \infty} \frac{1}{px^p} \quad \text{Simplify.}$$

$$= 0. \quad \text{Evaluate the limit.}$$

We see that any positive power of x grows faster than $\ln x$.

b. Compare $\ln^q x$ and x^p by observing that

$$\lim_{x\to\infty} \frac{\ln^q x}{x^p} = \lim_{x\to\infty} \left(\frac{\ln x}{x^{p/q}}\right)^q = \underbrace{\left(\lim_{x\to\infty} \frac{\ln x}{x^{p/q}}\right)^q}_{0}.$$

Compare $\ln^q x$ and x^p by observing that

By part (**a**), $\lim\limits_{x\to\infty} \dfrac{\ln x}{x^{p/q}}$ (because $\dfrac{p}{q} > 0$). Therefore, $\lim\limits_{x\to\infty} \dfrac{\ln^q x}{x^p}$ (because $q > 0$).

We conclude that any positive power of x grows faster than any positive power of $\ln x$.

EXAMPLE 8 **Powers of x vs. exponentials** Compare the rates of growth of $f(x) = x^p$ and $g(x) = e^x$ as $x \to \infty$, where p is a positive real number.

Solution

let $x = \ln t$ and note that as $x \to \infty$, we also have $t \to \infty$.

Substitution, then $x^p = \ln^p t$ and $e^x = e^{\ln t} = t$. Therefore,

$$\lim_{x\to\infty} \frac{x^p}{e^x} = \lim_{t\to\infty} \frac{\ln^p t}{t} = 0. \quad \text{Example 7}$$

THEOREM 4.15 **Ranking Growth Rates as $x \to \infty$**

Let $f \ll g$ mean that g grows faster than f as $x \to \infty$. With positive real numbers p, q, r, and s and $b > 1$,

$$\ln^q x \ll x^p \ll x^p \ln^r x \ll x^{p+s} \ll b^x \ll x^x.$$

Pitfalls in Using l'Hôpital's Rule

- L'Hôpital's Rule says $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$, not
$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \left(\frac{f(x)}{g(x)}\right)' \text{ or } \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \left(\frac{1}{g(x)}\right)' f'(x)$$

- Be sure that the given limit involves the indeterminate form $0/0$ or ∞/∞ before applying l'Hôpital's Rule. For example,
$$\lim_{x \to 0} \frac{1 - \sin x}{\cos x} = \lim_{x \to 0} \frac{-\cos x}{\sin x} \qquad (\times)$$

- When using l'Hôpital's Rule repeatedly, be sure to simplify expressions as much as possible at each step and evaluate the limit as soon as the limit is no longer an indeterminate form.

- Repeated use of l'Hôpital's Rule occasionally leads to unending cycles, in which case other methods must be used. For example, limits of the form $\lim_{x \to \infty} \frac{\sqrt{ax+1}}{\sqrt{bx+1}}$

- Be sure that the limit produced by l'Hôpital's Rule exists. Consider $\lim_{x \to \infty} \frac{3x+\cos x}{x}$, which has the form ∞/∞. Applying l'Hôpital's Rule, we have

$$\lim_{x \to \infty} \frac{3x+\cos x}{x} = \lim_{x \to \infty} \frac{3-\sin x}{1} \quad \text{(does not exist)}$$

In fact, the original limit has a value of 3 (divide numerator and denominator by x). To reach a conclusion from l'Hôpital's Rule, the limit produced by l'Hôpital's Rule must exist (or be $\pm\infty$).

4.8

Newton's Method

Why Approximate?

Common problem: finding the *roots* or *zeros* of a function. *Newton's method*, is one of the most effective methods for approximating the roots of a function.

Fact: Analytical methods do not give the roots of most functions. For example, the root of $f(x) = e^{-x} - x$ cannot be found exactly using analytical method, which is the majority of cases.

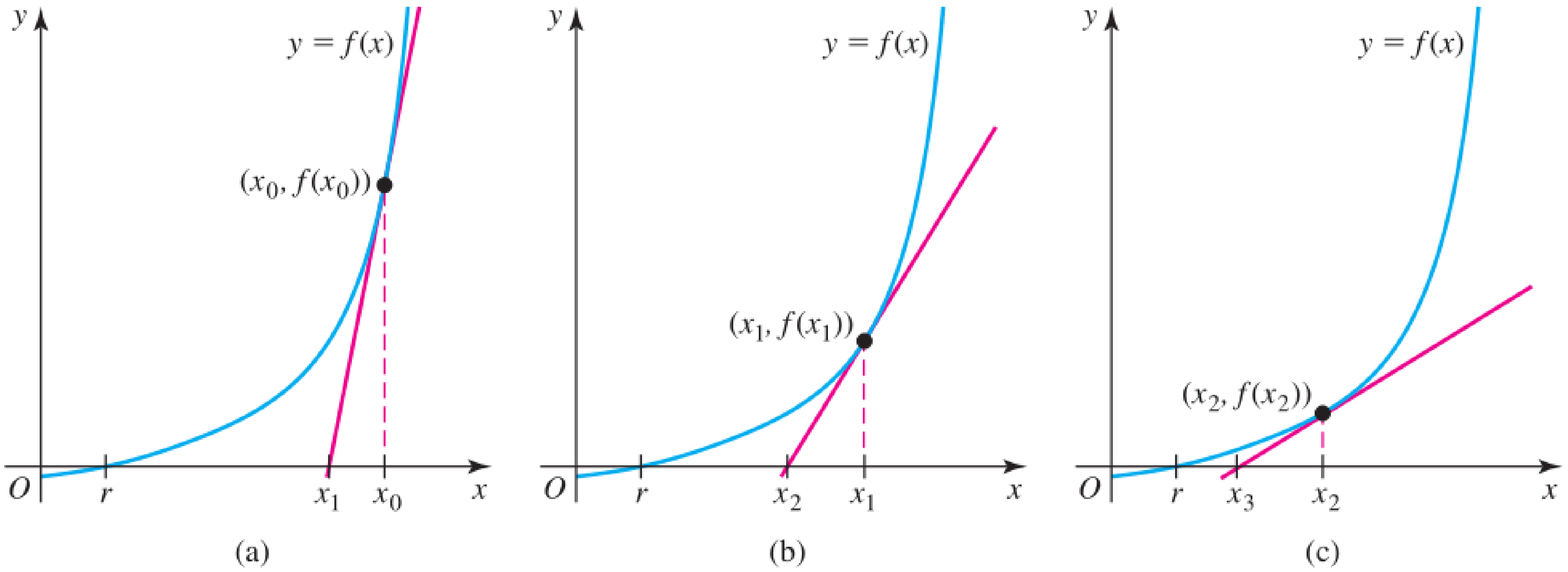
Another approach is to approximate roots using *numerical methods*, such as Newton's method.

Deriving Newton's Method

Assume that r is a root of f that we wish to approximate; this means that $f(r) = 0$. Assume also f is differentiable on some interval containing r.

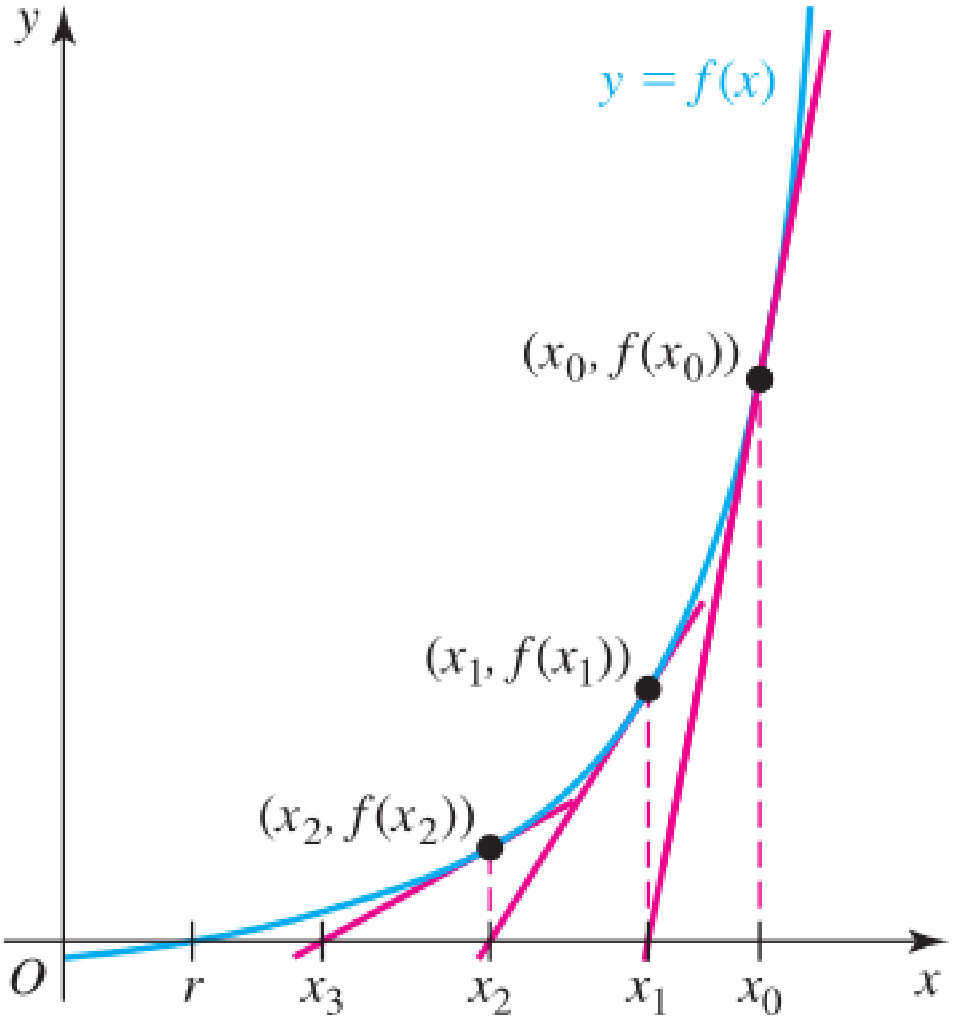
Suppose x_0 is an initial approximation to r that is generally obtained by some preliminary analysis. A better approximation to r is often obtained by carrying out the following two step:

- A line tangent to the curve $y = f(x)$ at the point $(x_0, f(x_0))$ is drawn.
- The point $(x_1, 0)$ at which the tangent line intersects the x-axis is found and x_1 becomes the new approximation to r.



It can be seen that x_1 is a better approximation to r than x_0.

To improve the approximation x_1, repeat the two-step process, using x_1 to determine the next estimate x_2, and then $x_3 \ldots$ Continuing in this fashion, we obtain a *sequence* of approximations $\{x_1, x_2, x_3, \cdots\}$, that ideally get closer and closer, or *converge*, to the root r.



y
$y = f(x)$
$(x_0, f(x_0))$
$(x_1, f(x_1))$
$(x_2, f(x_2))$
O
r
x_3
x_2
x_1
x_0
x

A formula that captures the process.
The tangent line at $(x_n, f(x_n))$

$$y - f(x_n) = \underbrace{f'(x_n)}_{m}(x - x_n).$$

Find the point at which this line intersects the x-axis by setting $y = 0$ in the equation of the line and solving for x.

This value of x becomes the new approximation x_{n+1}.

$$\underbrace{x_{n+1}}_{\substack{\text{new} \\ \text{approximation}}} = \underbrace{x_n}_{\substack{\text{current} \\ \text{approximation}}} - \frac{f(x_n)}{f'(x_n)}, \quad \text{provided } f'(x_n) \neq 0.$$

PROCEDURE Newton's Method for Approximating Roots of $f(x) = 0$

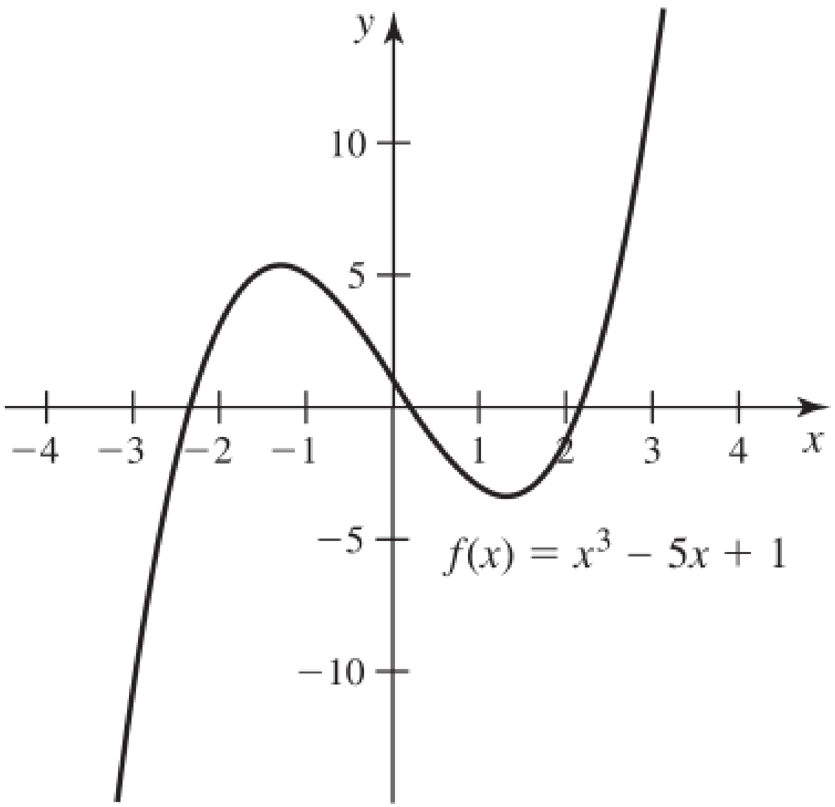
1. Choose an initial approximation x_0 as close to a root as possible.
2. For $n = 0, 1, 2, \ldots$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

provided $f'(x_n) \neq 0$.

3. End the calculations when a termination condition is met.

EXAMPLE 1 **Applying Newton's method** Approximate the roots of $f(x) = x^3 - 5x + 1$ (Figure 4.82) using seven steps of Newton's method. Use $x_0 = -3$, $x_0 = 1$, and $x_0 = 4$ as initial approximations.

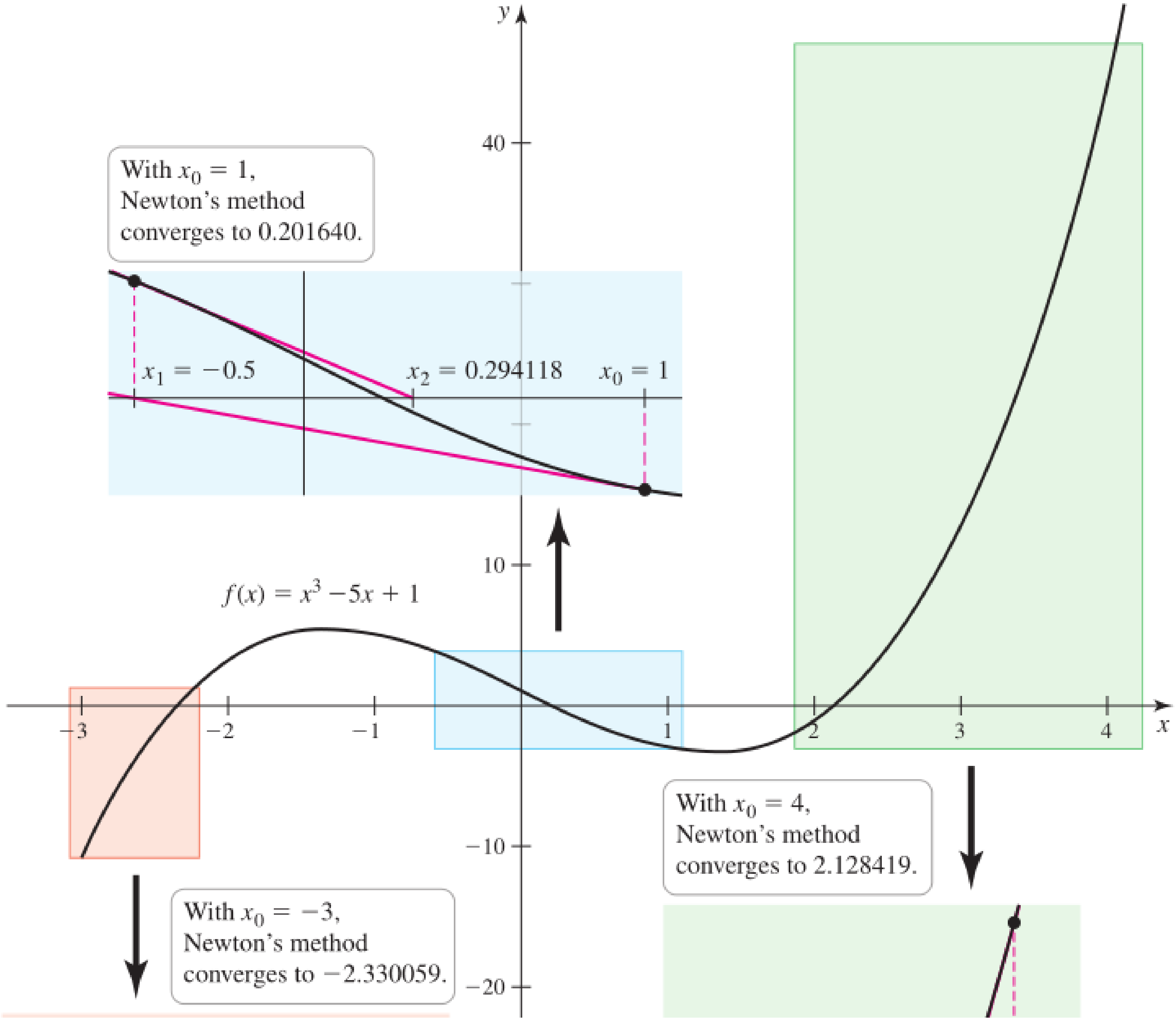


$$x_{n+1} = x_n - \frac{\overbrace{x_n^3 - 5x_n + 1}^{f(x_n)}}{\underbrace{3x_n^2 - 5}_{f'(x_n)}} = \frac{2x_n^3 - 1}{3x_n^2 - 5},$$

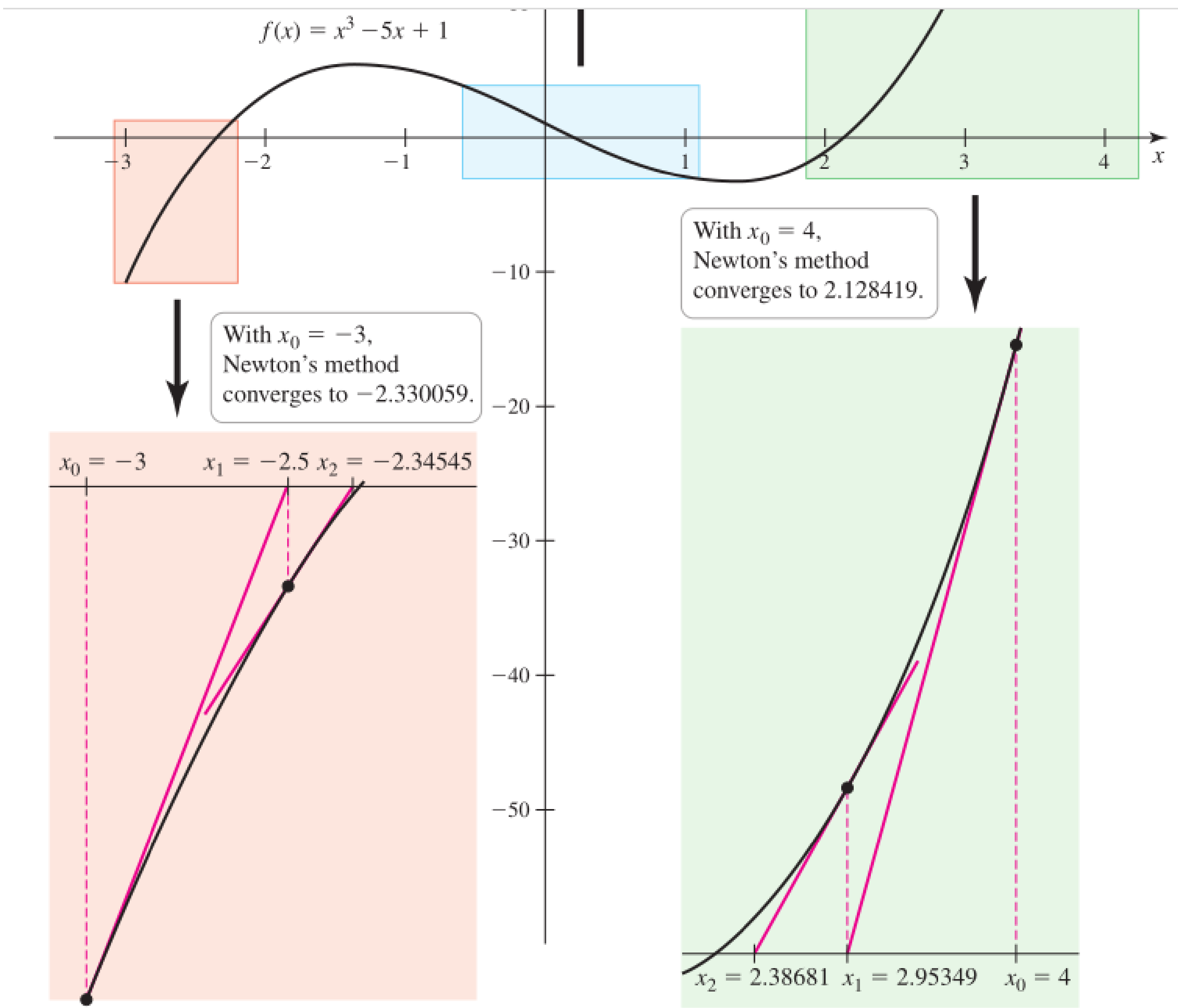
Table 4.5

n	x_n	x_n	x_n
0	−3	1	4
1	−2.500000	−0.500000	2.953488
2	−2.345455	0.294118	2.386813
3	−2.330203	0.200215	2.166534
4	−2.330059	0.201639	2.129453
5	−2.330059	0.201640	2.128420
6	−2.330059	0.201640	2.128419
7	−2.330059	0.201640	2.128419

It can be concluded that -2.330059, 0.201640, and 2.128419 are approximations to the roots of f with at least six digits of accuracy.



y
40
10
−10
−20
x
−3
−2
−1
1
2
3
4
With $x_0 = 1$,
Newton's method
converges to 0.201640.
$x_1 = -0.5$
$x_2 = 0.294118$
$x_0 = 1$
$f(x) = x^3 - 5x + 1$
With $x_0 = 4$,
Newton's method
converges to 2.128419.
With $x_0 = -3$,
Newton's method
converges to −2.330059.



$f(x) = x^3 - 5x + 1$
−3
−2
−1
1
2
3
4
x
−10
−20
−30
−40
−50
With $x_0 = 4$,
Newton's method
converges to 2.128419.
With $x_0 = -3$,
Newton's method
converges to −2.330059.
$x_0 = -3$
$x_1 = -2.5$
$x_2 = -2.34545$
$x_2 = 2.38681$
$x_1 = 2.95349$
$x_0 = 4$

When Do You Stop?

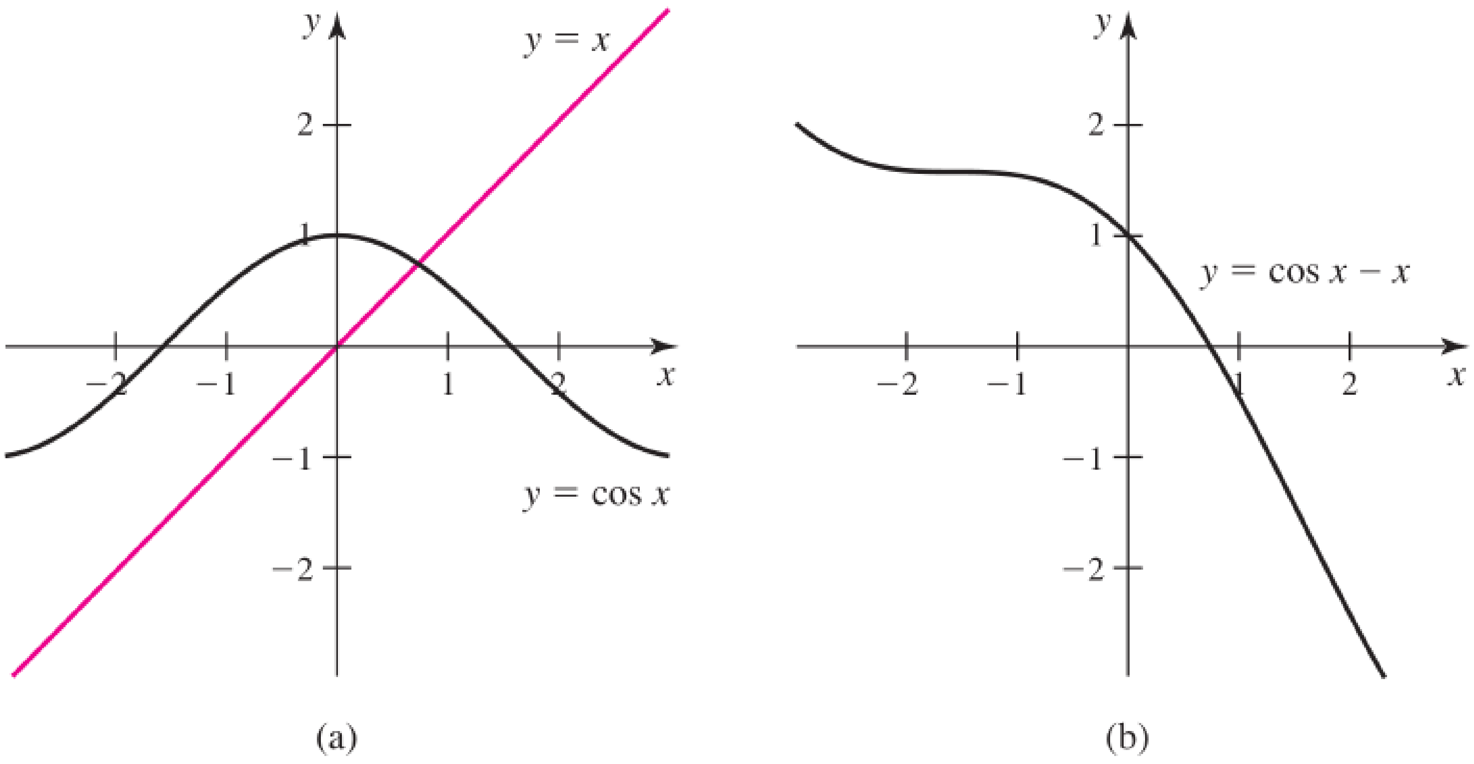
How many Newton approximations should you compute? Ideally, we would like to compute the ***error*** in x_n as an approximation to the root r, which is the quantity $|x_n - r|$. Unfortunately, we don't know r in practice.

- A general rule of thumb is that if two successive approximations agree to, say, seven digits, then those common digits are accurate (as an approximation to the root).
- As the approximations x_n approach the root r, $f(x_n)$ should approach zero. The quantity $f(x_n)$ is called a ***residual***, and small residuals usually (but not always) suggest that the approximations have small errors.

EXAMPLE 2 **Finding intersection points** Find the points at which the curves $y = \cos x$ and $y = x$ intersect.

Solution

Let $f(x) = g(x) - h(x) = \cos x - x$



(a) (b)

A reasonable initial approximation is $x_0 = 0.5$.

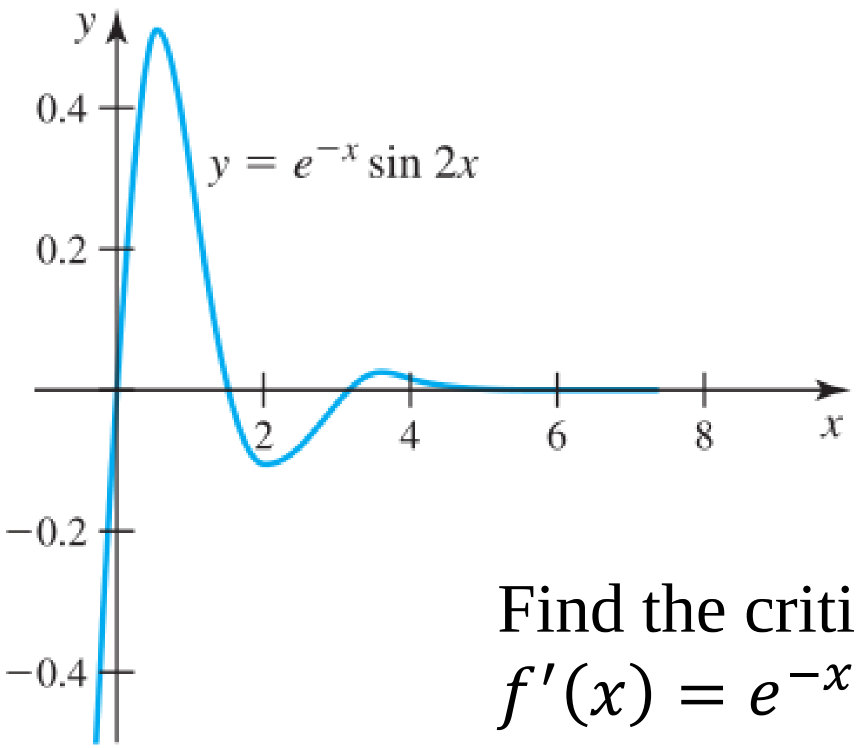
Newton's method takes the form

$$x_{n+1} = x_n - \frac{\overbrace{\cos x_n - x_n}^{f(x_n)}}{\underbrace{-\sin x_n - 1}_{f'(x_n)}} = \frac{x_n \sin x_n + \cos x_n}{\sin x_n + 1}.$$

Table 4.6

n	x_n	Residual
0	0.5	0.377583
1	0.755222	−0.0271033
2	0.739142	−0.0000946154
3	0.739085	-1.18098×10^{-9}
4	0.739085	0
5	0.739085	0

EXAMPLE 3 **Finding local extrema** Find the x-coordinate of the first local maximum and the first local minimum of the function $f(x) = e^{-x} \sin 2x$ on the interval $(0, \infty)$.



From the figure, the first local maximum occurs on the interval [0,1], and the first local minimum occurs on the interval [2,3].

Find the critical point by solving $f'(x) = e^{-x}(2\cos 2x - \sin 2x) = 0$.

We apply Newton's method to this equation.

Table 4.7

n	x_n	x_n
0	0.200000	2.500000
1	0.499372	1.623915
2	0.550979	2.062202
3	0.553568	2.121018
4	0.553574	2.124360
5	0.553574	2.124371
6	0.553574	2.124371

It can be concluded that the first local maximum occurs at $x \approx 0.553574$ and the first local minimum occurs at $x \approx 2.124371$.

Pitfalls of Newton's Method

The term $f'(x_n)$ appears in a denominator, so if at any step $f'(x_n) = 0$, then the method breaks down.
Furthermore, if $f'(x_n)$ is close to zero at any step, then the method may converge slowly or may fail to converge.

EXAMPLE 4 **Difficulties with Newton's method** Find the root of $f(x) = \dfrac{8x^2}{3x^2 + 1}$ using Newton's method with initial approximations of $x_0 = 1$, $x_0 = 0.15$, and $x_0 = 1.1$.

The formula for Newton's method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = \frac{x_n}{2}(1 - 3x_n^2)$$

Table 4.8

n	x_n	x_n	x_n
0	1	0.15	1.1
1	−1	0.0699375	−1.4465
2	1	0.0344556	3.81665
3	−1	0.0171665	−81.4865
4	1	0.00857564	8.11572×10^5
5	−1	0.00428687	-8.01692×10^{17}

The three cases in this example illustrate several ways that Newton's method may fail to converge at its usual rate: The approximations may *cycle* or *wander*, they may *converge slowly*, or they may *diverge* (often at a rapid rate).

4.9

Antiderivatives

Thinking Backward

The reverse process of differentiation, called *antidifferentiation*, is equally important: Given a function f, we look for an *antiderivative* function F whose derivative is f; that is, a function F such that $F' = f$.

DEFINITION Antiderivative

A function F is an **antiderivative** of f on an interval I provided $F'(x) = f(x)$, for all x in I.

Examples: an antiderivative of $f(x) = 1$ is $F(x) = x$
An antiderivative of $f(x) = 2x$ is $F(x) = x^2$ and
an antiderivative of $f(x) = \cos x$ is $F(x) = \sin x$.

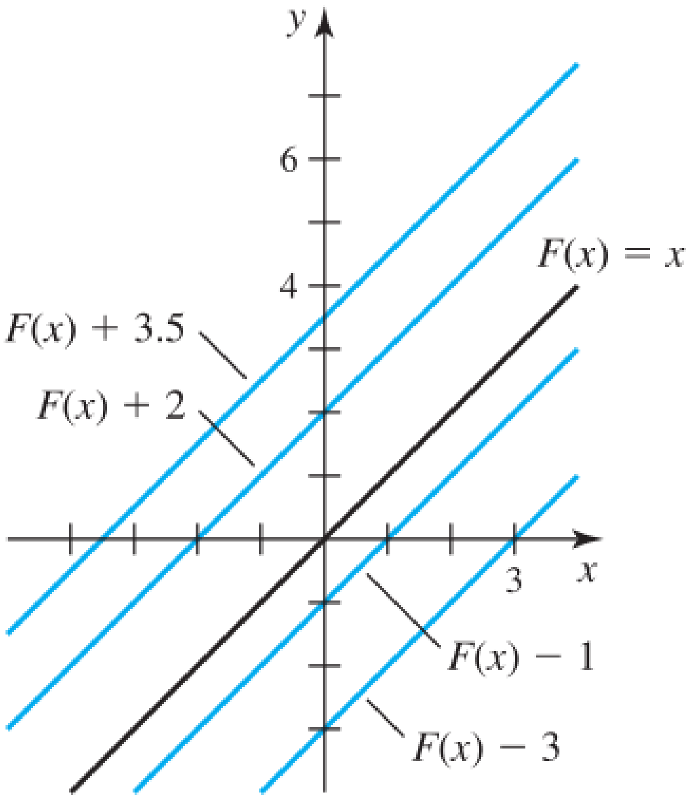
Question: Does a function have more than one antiderivative?

Example: for $f(x) = 1$, $F(x) = x + C$ is also its antiderivative
Therefore, $f(x) = 1$ actually has an infinite number of antiderivatives.

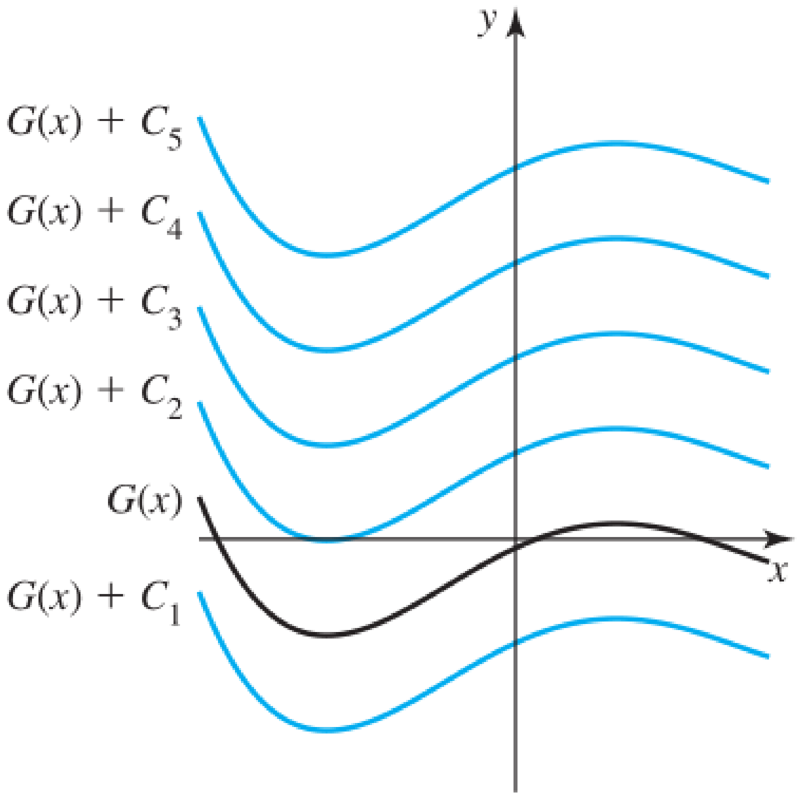
THEOREM 4.16 The Family of Antiderivatives
Let F be any antiderivative of f on an interval I. Then *all* the antiderivatives of f on I have the form $F + C$, where C is an arbitrary constant.

Proof: Suppose that F and G are antiderivatives of f on an interval I. Then $F' = f$ and $G' = f$, which implies that $F' = G'$ on I. Theorem 4.11 states that functions with equal derivatives differ by a constant. Therefore, $G = F + C$, and all antiderivatives of f have the form $F + C$, where C is an arbitrary constant. ◄

There are infinitely many antiderivatives of a function, they are all of one family, namely, those functions of the form $F(x) + C$



Several antiderivatives of $f(x) = 1$ from the family $F(x) + C = x + C$



If G is any antiderivative of g, the graphs of the antiderivatives $G + C$ are vertical translations of one another.

EXAMPLE 1 **Finding antiderivatives** Use what you know about derivatives to find all antiderivatives of the following functions.

a. $f(x) = 3x^2$ **b.** $f(x) = \dfrac{1}{1 + x^2}$ **c.** $f(t) = \sin t$

Indefinite Integrals

Analogous notation for antiderivatives: $\boldsymbol{\int f(x)dx}$, called **indefinite integral**, the following function $\boldsymbol{f}$ is called the **integrand**, followed by the differential dx.
For now, dx simply means that x is the independent variable, or the **variable of integration**.
The notation $\int f(x)dx$ represents all the antiderivatives of f.

For example,
$\int 3x^2dx = x^3 + C$, $\int \frac{1}{1+x^2}dx = \tan^{-1}x + C$, and $\int \sin t dt = -\cos t + C$
where C is an arbitrary constant called a **constant of integration**.

THEOREM 4.17 **Power Rule for Indefinite Integrals**

$$\int x^p \, dx = \frac{x^{p+1}}{p+1} + C,$$

where $p \neq -1$ is a real number and C is an arbitrary constant.

THEOREM 4.18 **Constant Multiple and Sum Rules**

Constant Multiple Rule: $\int cf(x)\, dx = c \int f(x)\, dx$, for real numbers c

Sum Rule: $\int (f(x) + g(x))\, dx = \int f(x)\, dx + \int g(x)\, dx$

EXAMPLE 2 **Indefinite integrals** Determine the following indefinite integrals.

a. $\int (3x^5 + 2 - 5\sqrt{x})\, dx$ **b.** $\int \left(\frac{4x^{19} - 5x^{-8}}{x^2} \right) dx$ **c.** $\int (z^2 + 1)(2z - 5)\, dz$

Indefinite Integrals of Trigonometric Functions

Two goals:

- Write the familiar derivative results for trigonometric functions as indefinite integrals.
- Show how these results are generalized using the Chain Rule.

EXAMPLE 3 **Indefinite integrals of trigonometric functions** Evaluate the following indefinite integrals.

a. $\int \sec^2 x \, dx$ **b.** $\int \sin 3x \, dx$

c. $\int \sec ax \tan ax \, dx$, where $a \neq 0$ is a real number

Assume that $a \neq 0$ is a real number and that C is an arbitrary constant.

Table 4.9 **Indefinite Integrals of Trigonometric Functions**

1. $\frac{d}{dx}(\sin ax) = a \cos ax \quad \Rightarrow \quad \int \cos ax\, dx = \frac{1}{a} \sin ax + C$

2. $\frac{d}{dx}(\cos ax) = -a \sin ax \quad \Rightarrow \quad \int \sin ax\, dx = -\frac{1}{a} \cos ax + C$

3. $\frac{d}{dx}(\tan ax) = a \sec^2 ax \quad \Rightarrow \quad \int \sec^2 ax\, dx = \frac{1}{a} \tan ax + C$

4. $\frac{d}{dx}(\cot ax) = -a \csc^2 ax \quad \Rightarrow \quad \int \csc^2 ax\, dx = -\frac{1}{a} \cot ax + C$

5. $\frac{d}{dx}(\sec ax) = a \sec ax \tan ax \quad \Rightarrow \quad \int \sec ax \tan ax\, dx = \frac{1}{a} \sec ax + C$

6. $\frac{d}{dx}(\csc ax) = -a \csc ax \cot ax \quad \Rightarrow \quad \int \csc ax \cot ax\, dx = -\frac{1}{a} \csc ax + C$

EXAMPLE 4 **Indefinite integrals of trigonometric functions** Determine the following indefinite integrals.

a. $\int \sec^2 3x\, dx$ **b.** $\int \cos \frac{x}{2}\, dx$

Other Indefinite Integrals

Table 4.10 Other Indefinite Integrals

7. $\dfrac{d}{dx}(e^{ax}) = ae^{ax} \quad \Rightarrow \quad \displaystyle\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$

8. $\dfrac{d}{dx}(b^{x}) = b^{x}\ln b \Rightarrow \displaystyle\int b^{x}\,dx = \frac{1}{\ln b}b^{x} + C, b > 0, b \neq 1$

9. $\dfrac{d}{dx}(\ln|x|) = \dfrac{1}{x} \quad \Rightarrow \quad \displaystyle\int \frac{dx}{x} = \ln|x| + C$

10. $\dfrac{d}{dx}\left(\sin^{-1}\dfrac{x}{a}\right) = \dfrac{1}{\sqrt{a^2 - x^2}} \quad \Rightarrow \quad \displaystyle\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C$

11. $\dfrac{d}{dx}\left(\tan^{-1}\dfrac{x}{a}\right) = \dfrac{a}{a^2 + x^2} \quad \Rightarrow \quad \displaystyle\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

12. $\dfrac{d}{dx}\left(\sec^{-1}\left|\dfrac{x}{a}\right|\right) = \dfrac{a}{x\sqrt{x^2 - a^2}} \quad \Rightarrow \quad \displaystyle\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C, a > 0$

EXAMPLE 5 **Additional indefinite integrals** Evaluate the following indefinite integrals. Assume a is a nonzero real number.

a. $\displaystyle\int \frac{dx}{x}$ **b.** $\displaystyle\int e^{ax}\,dx$ **c.** $\displaystyle\int \frac{dx}{a^2 + x^2}$

EXAMPLE 6 **Indefinite integrals** Determine the following indefinite integrals.

a. $\displaystyle\int e^{-10t}\,dt$ **b.** $\displaystyle\int \frac{4}{\sqrt{9 - x^2}}\,dx$ **c.** $\displaystyle\int \frac{dx}{16x^2 + 1}$

Introduction to Differential Equations

An equation involving an unknown function and its derivatives is called a **differential equation**. For example,

$$f'(x) = 2x + 10$$

Its solution is $f(x) = x^2 + 10x + C$.

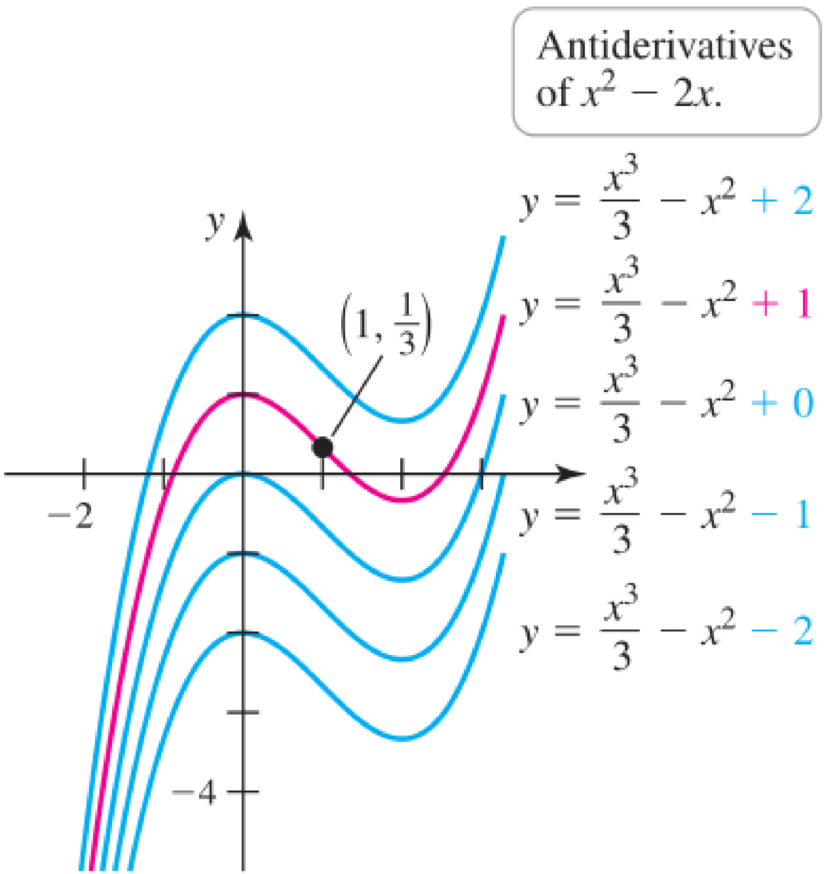
For a more general form $f'(x) = g(x)$
The solution consists of the antiderivative of g, involving an arbitrary constant.

In most practical cases, the differential equation is accompanied by an **initial condition** that allows us to determine the arbitrary constant.

A differential equation coupled with an initial condition is called an **initial value problem**.

$$f'(x) = g(x), \quad \text{where } g \text{ is given, and} \qquad \text{Differential equation}$$
$$f(a) = b, \quad \text{where } a \text{ and } b \text{ are given.} \qquad \text{Initial condition}$$

EXAMPLE 7 **An initial value problem** Solve the initial value problem $f'(x) = x^2 - 2x$ with $f(1) = \frac{1}{3}$.



$$f(x) = \frac{x^3}{3} - x^2 + C$$

This family of functions, called the **general solution**.

The solution to the initial value problem is

$$f(x) = \frac{x^3}{3} - x^2 + 1$$

Motion Problems Revisited

Suppose position function of an object is $s(t)$, then the velocity of the object is $v(t) = s'(t)$.
The position function is an antiderivative of the velocity.
Given the velocity function of an object and its position at a particular time, we can determine its position at all future times by solving an initial value problem.

Similarly, acceleration $a(t)$ of an object is the rate of change of the velocity, i.e., $a(t) = v'(t)$.
The velocity is an antiderivative of the acceleration.
Given the acceleration of an object and its velocity at a particular time, we can determine its velocity at all future times.

Initial Value Problems for Velocity and Position

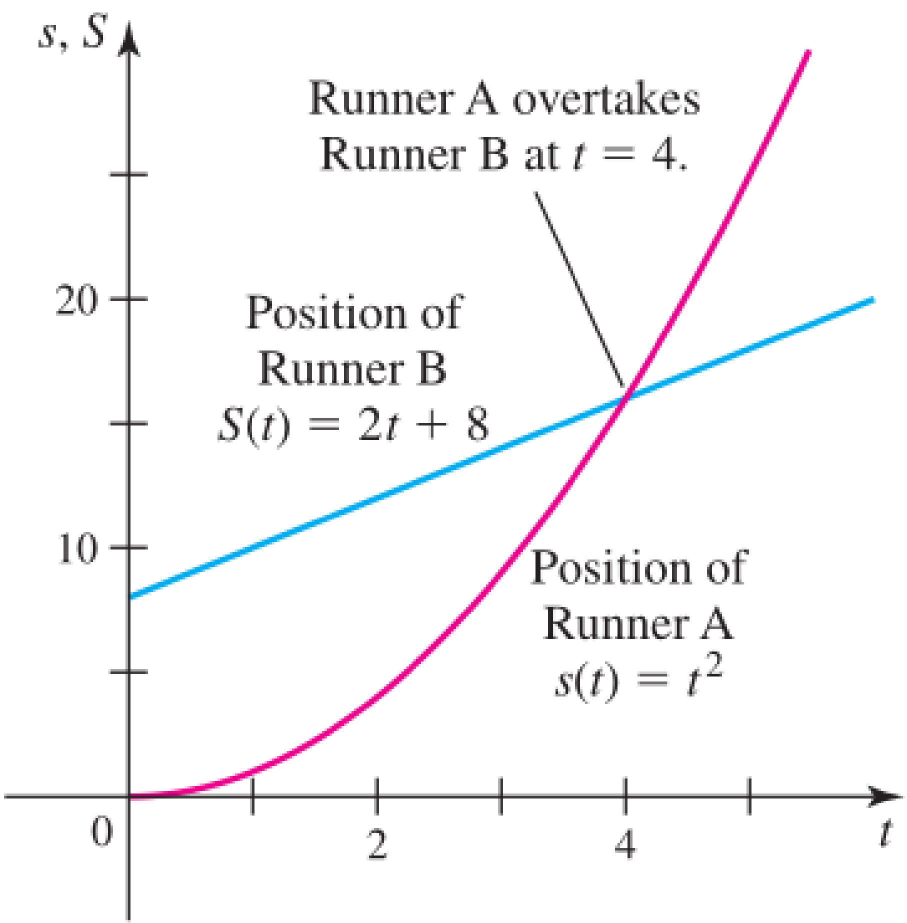
Suppose an object moves along a line with a (known) velocity $v(t)$, for $t \geq 0$. Then its position is found by solving the initial value problem

$$s'(t) = v(t), \; s(0) = s_0, \text{ where } s_0 \text{ is the (known) initial position.}$$

If the (known) acceleration of the object $a(t)$ is given, then its velocity is found by solving the initial value problem

$$v'(t) = a(t), \; v(0) = v_0, \text{ where } v_0 \text{ is the (known) initial velocity.}$$

EXAMPLE 8 **A race** Runner A begins at the point $s(0) = 0$ and runs with velocity $v(t) = 2t$. Runner B begins with a head start at the point $S(0) = 8$ and runs with velocity $V(t) = 2$. Find the positions of the runners for $t \geq 0$ and determine who is ahead at $t = 6$ time units.



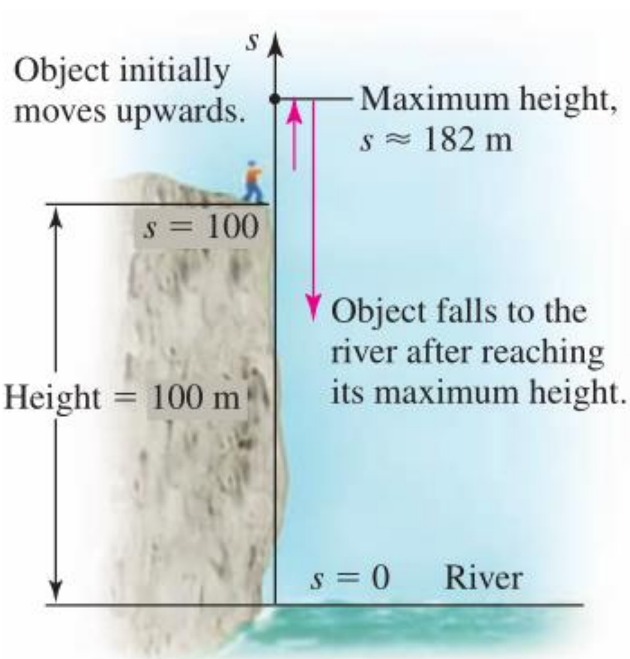
EXAMPLE 9 **Motion with gravity** Neglecting air resistance, the motion of an object moving vertically near Earth's surface is determined by the acceleration due to gravity, which is approximately $9.8\ \mathrm{m/s^2}$. Suppose a stone is thrown vertically upward at $t = 0$ with a velocity of $40\ \mathrm{m/s}$ from the edge of a cliff that is 100 m above a river.

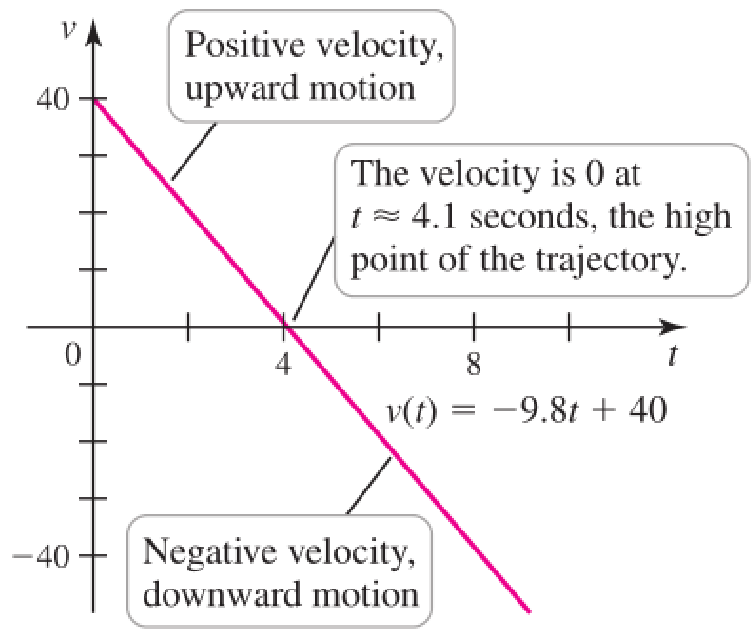
a. Find the velocity $v(t)$ of the object, for $t \geq 0$.

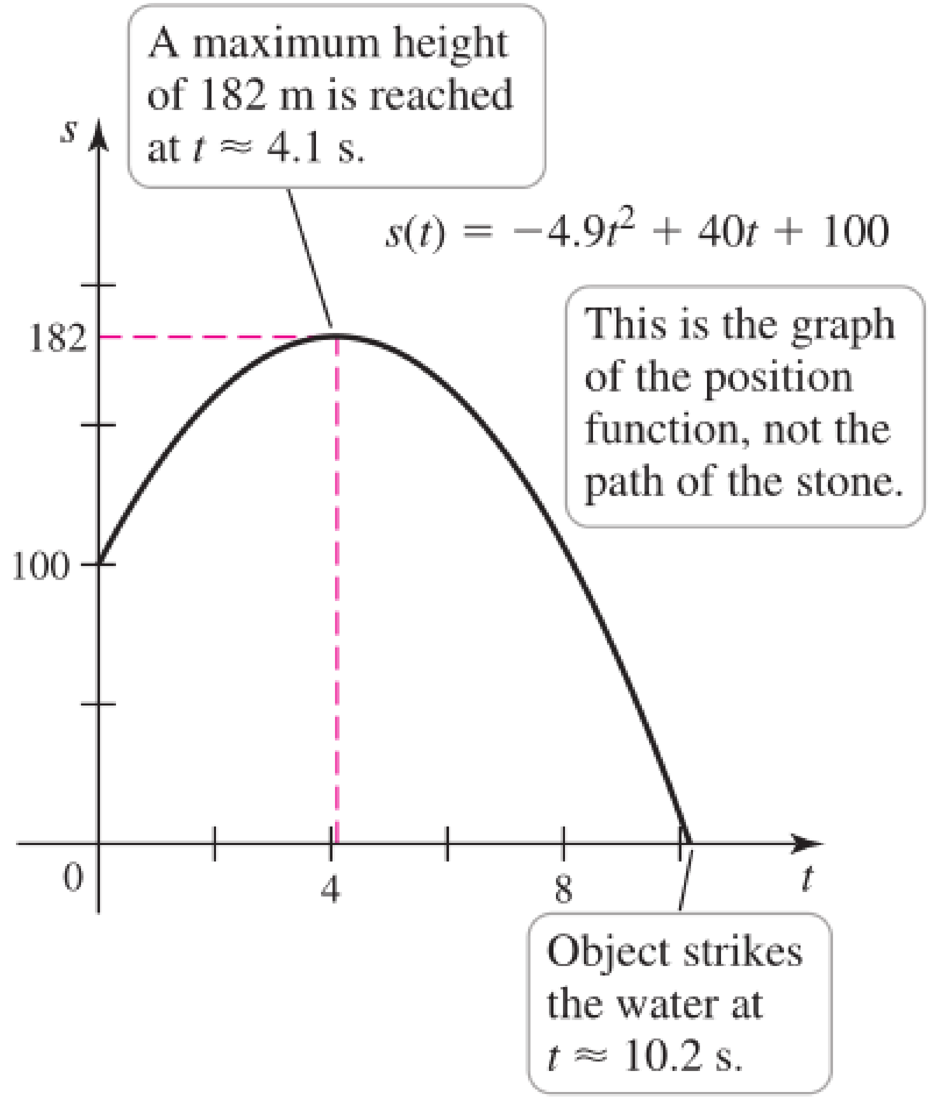
b. Find the position $s(t)$ of the object, for $t \geq 0$.

c. Find the maximum height of the object above the river.

d. With what speed does the object strike the river?






A maximum height of 182 m is reached at $t \approx 4.1$ s.
$s(t) = -4.9t^2 + 40t + 100$
This is the graph of the position function, not the path of the stone.
Object strikes the water at $t \approx 10.2$ s.
s
182
100
0
4
8
t

Chapter 4

Applications of the Derivative (II)

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