Chapter 1

Functions

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1.1

Review of Functions

Relationships among quantities, or variables.

DEFINITION Function

A **function** f is a rule that assigns to each value x in a set D a *unique* value denoted $f(x)$. The set D is the **domain** of the function. The **range** is the set of all values of $f(x)$ produced as x varies over the entire domain (Figure 1.1).

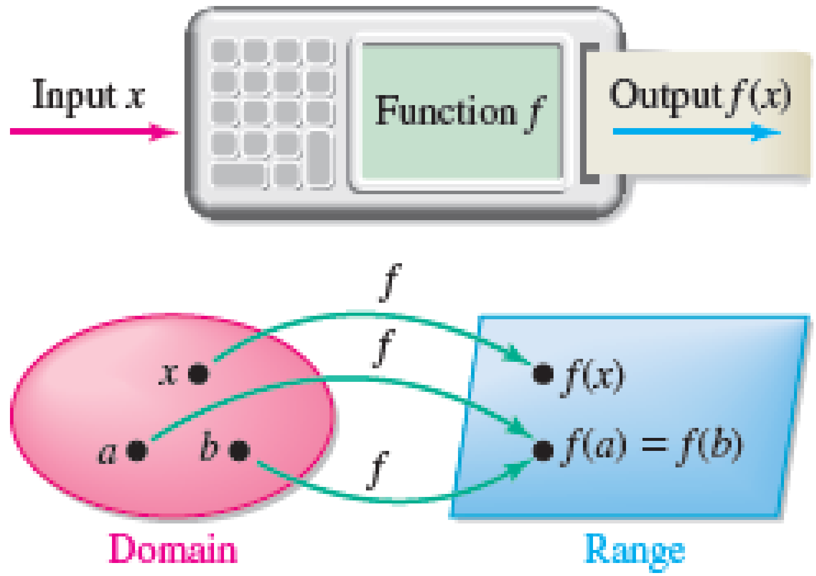


Figure 1.1

Independent variable is the variable associated with the domain.

Dependent variable belongs to the range.

The **graph** of a function f is the set of all points (x, y) in the xy-pane that satisfy the equation $y = f(x)$.

The **argument** of a function is the expression on which the function works.

Vertical Line Test

A graph represents a function if and only if it passes the **vertical line test**: Every vertical line intersects the graph at most once. A graph that fails this test does not represent a function.

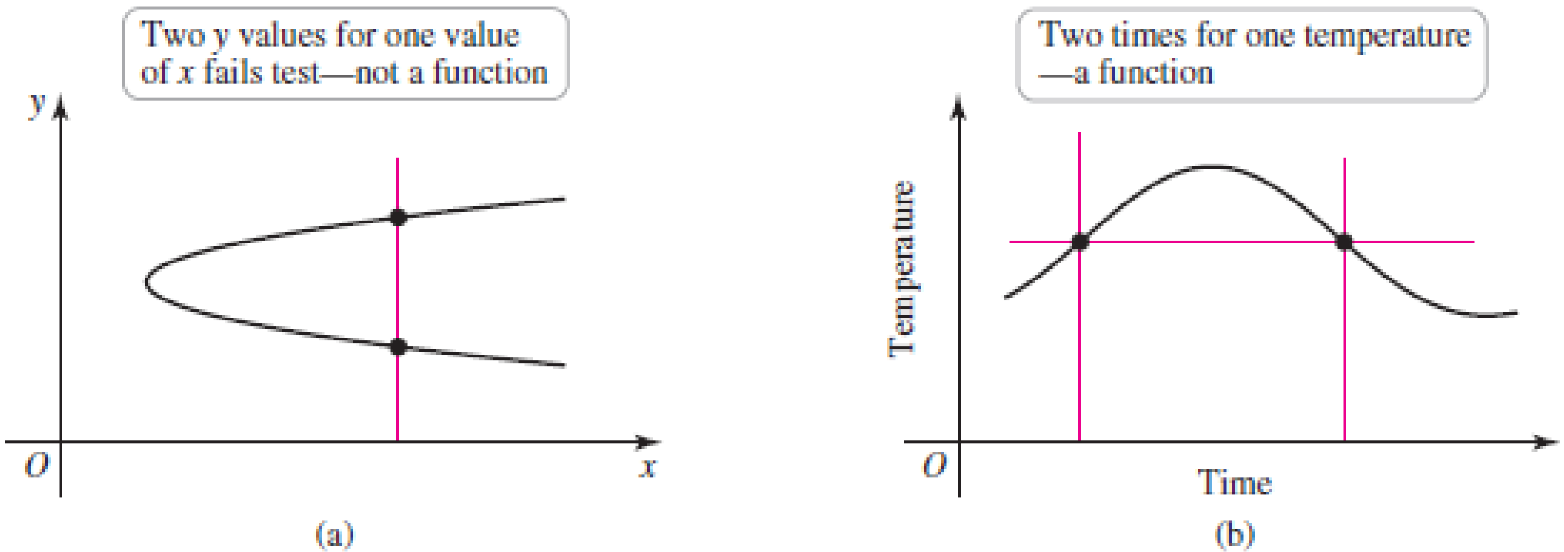


Figure 1.2

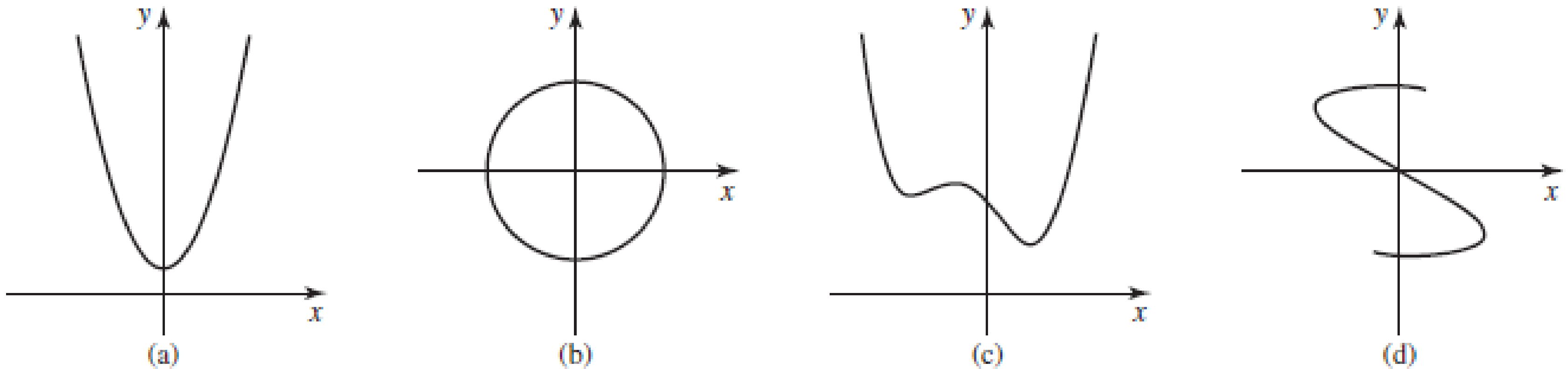


Figure 1.3

EXAMPLE 2 **Domain and range** Graph each function with a graphing utility using the given window. Then state the domain and range of the function.

a. $y = f(x) = x^2 + 1$; $[-3, 3] \times [-1, 5]$

b. $z = g(t) = \sqrt{4 - t^2}$; $[-3, 3] \times [-1, 3]$

c. $w = h(u) = \dfrac{1}{u - 1}$; $[-3, 5] \times [-4, 4]$

EXAMPLE 3 **Domain and range in context** At time $t = 0$, a stone is thrown vertically upward from the ground at a speed of 30 m/s. Its height above the ground in meters (neglecting air resistance) is approximated by the function $h = f(t) = 30t - 5t^2$, where t is measured in seconds. Find the domain and range of f in the context of this particular problem.

DEFINITION Composite Functions

Given two functions f and g, the composite function $f \circ g$ is defined by $(f \circ g)(x) = f(g(x))$. It is evaluated in two steps: $y = f(u)$, where $u = g(x)$. The domain of $f \circ g$ consists of all x in the domain of g such that $u = g(x)$ is in the domain of f (Figure 1.8).

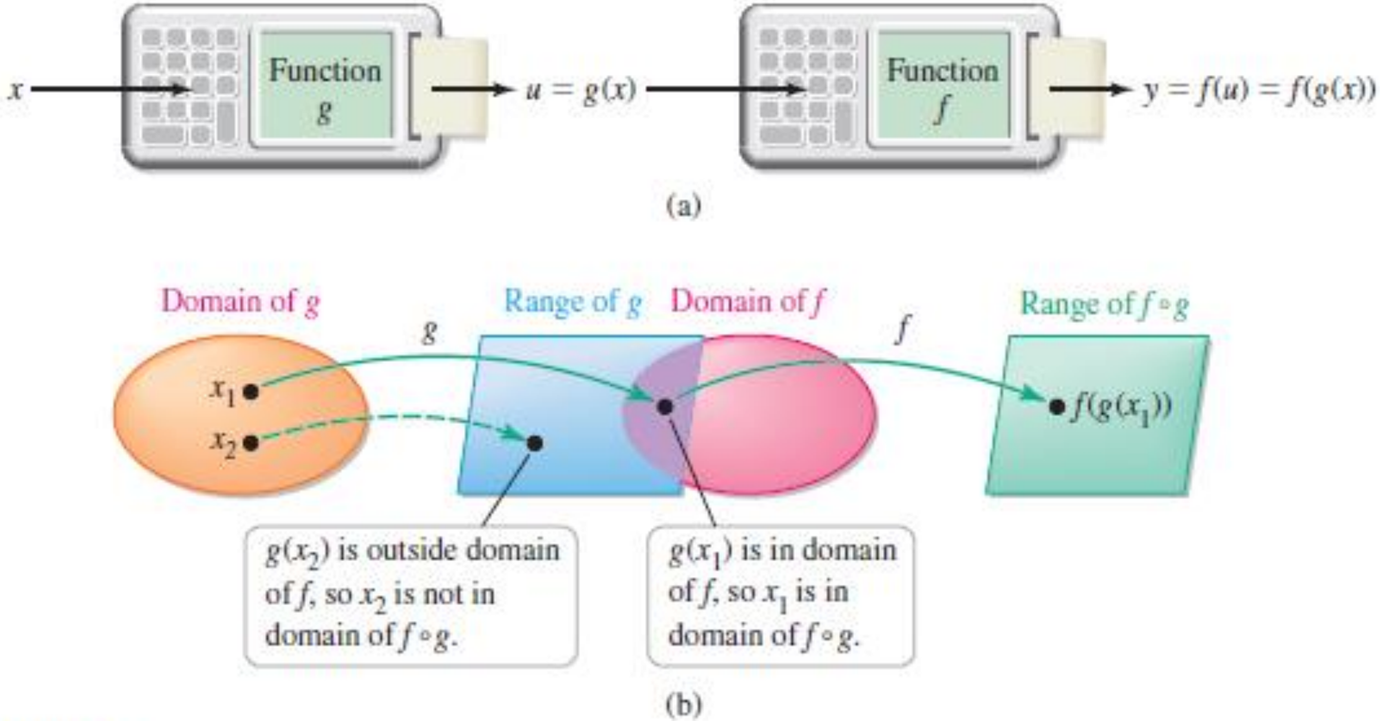


Figure 1.8

Example 4 **Using graphs to evaluate composite functions**.

Use the graphs of f and g in Fig. 1.9 to find the following values.

a. $f(g(3))$ **b.** $g(f(3))$ **c.** $f(f(4))$ **d.** $f(g(f(8)))$

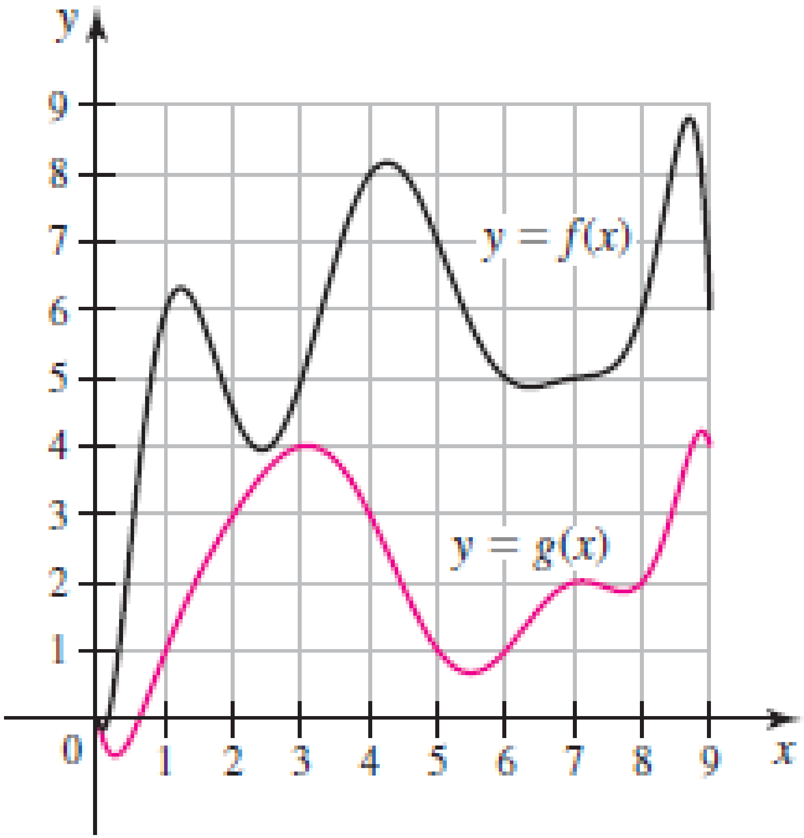


Figure 1.9

Example 5 Using a table to evaluate composite functions.

Use the function values in the table to evaluate the following values.

a. $(f \circ g)(0)$ **b.** $g(f(-1))$ **c.** $f(g(g(-1)))$

x	-2	-1	0	1	2
***f* (*x*)**	0	1	3	4	2
***g* (*x*)**	-1	0	-2	-3	-4

EXAMPLE 6 **Composite functions and notation** Let $f(x) = 3x^2 - x$ and $g(x) = 1/x$. Simplify the following expressions.

a. $f(5p + 1)$ **b.** $g(1/x)$ **c.** $f(g(x))$ **d.** $g(f(x))$

EXAMPLE 7 **Working with composite functions** Identify possible choices for the inner and outer functions in the following composite functions. Give the domain of the composite function.

a. $h(x) = \sqrt{9x - x^2}$ **b.** $h(x) = \dfrac{2}{(x^2 - 1)^3}$

EXAMPLE 8 **More composite functions** Given $f(x) = \sqrt[3]{x}$ and $g(x) = x^2 - x - 6$, find (a) $g \circ f$ and (b) $g \circ g$, and their domains.

Secant Lines and the Difference Quotient

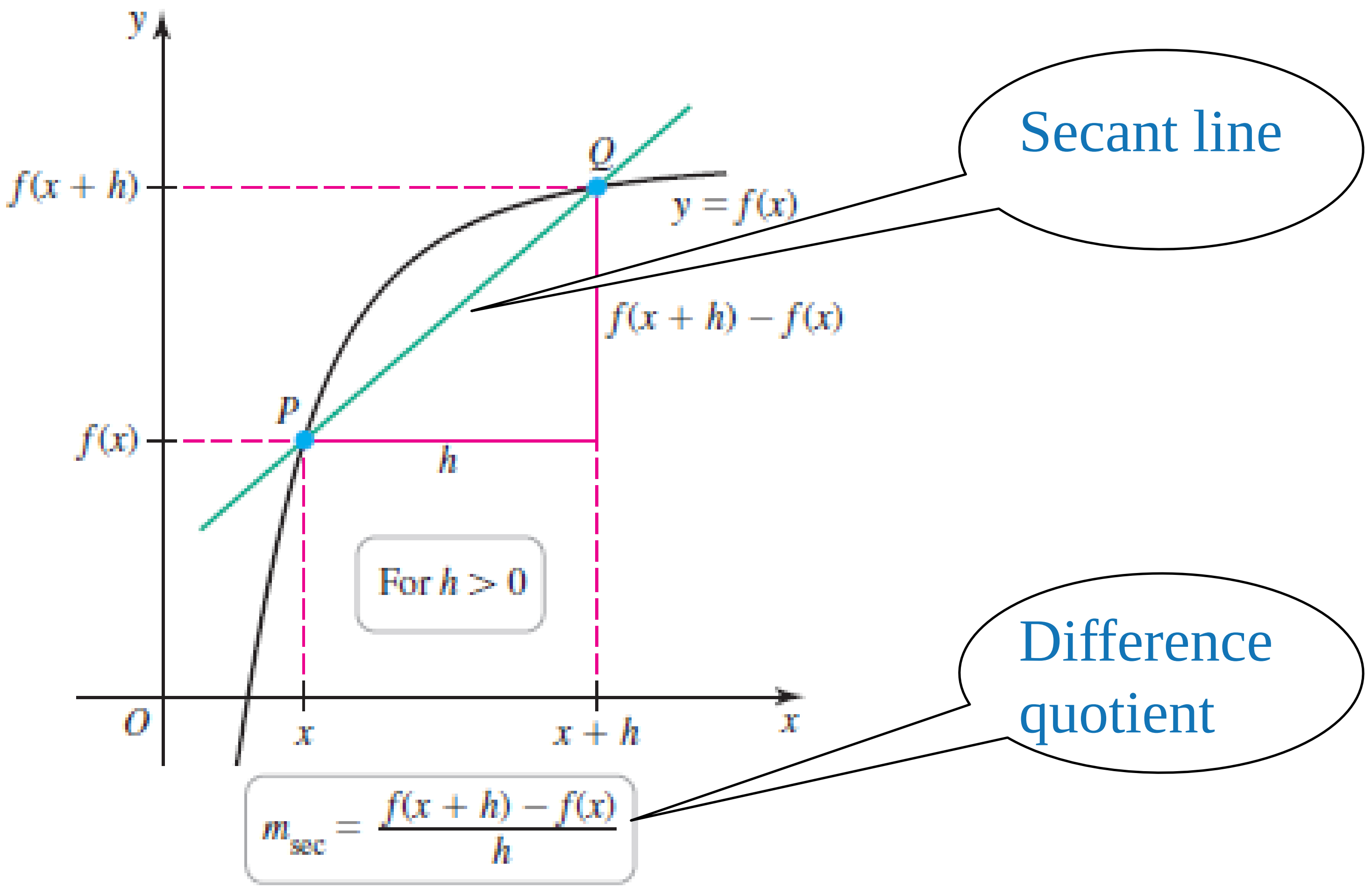


Figure 1.10

Difference quotient: the slope of the secant line through P and Q, is given by

$$m_{\text{sec}} = \frac{\text{change in } y}{\text{change in } x} = \frac{f(x+h) - f(x)}{(x+h) - x} = \frac{f(x+h) - f(x)}{h}.$$

Can be expressed in several different ways depending on the coordinates of P and Q. For example,

$$m_{\text{sec}} = \frac{f(x) - f(a)}{x - a}.$$

Can be interpreted as the ***average rate of change*** of f over the interval $[a, x]$.

EXAMPLE 9 **Working with difference quotients**

a. Simplify the difference quotient $\dfrac{f(x+h)-f(x)}{h}$, for $f(x) = 3x^2 - x$.

b. Simplify the difference quotient $\dfrac{f(x)-f(a)}{x-a}$, for $f(x) = x^3$.

EXAMPLE 10 **Interpreting the slope of the secant line** Sound intensity I, measured in watts per square meter $(\mathrm{W/m^2})$, at a point r meters from a sound source with acoustic power P is given by $I(r) = \dfrac{P}{4\pi r^2}$.

a. Find the sound intensity at two points $r_1 = 10$ m and $r_2 = 15$ m from a sound source with power $P = 100$ W. Then find the slope of the secant line through the points $(10, I(10))$ and $(15, I(15))$ on the graph of the intensity function and interpret the result.

b. Find the slope of the secant line through any two points $(r_1, I(r_1))$ and $(r_2, I(r_2))$ on the graph of the intensity function with acoustic power P.

Symmetry

DEFINITION Symmetry in Graphs

A graph is **symmetric with respect to the *y*-axis** if whenever the point (x, y) is on the graph, the point $(-x, y)$ is also on the graph. This property means that the graph is unchanged when reflected across the y-axis (Figure 1.13a).

A graph is **symmetric with respect to the *x*-axis** if whenever the point (x, y) is on the graph, the point $(x, -y)$ is also on the graph. This property means that the graph is unchanged when reflected across the x-axis (Figure 1.13b).

A graph is **symmetric with respect to the origin** if whenever the point (x, y) is on the graph, the point $(-x, -y)$ is also on the graph (Figure 1.13c). Symmetry about both the x- and y-axes implies symmetry about the origin, but not vice versa.

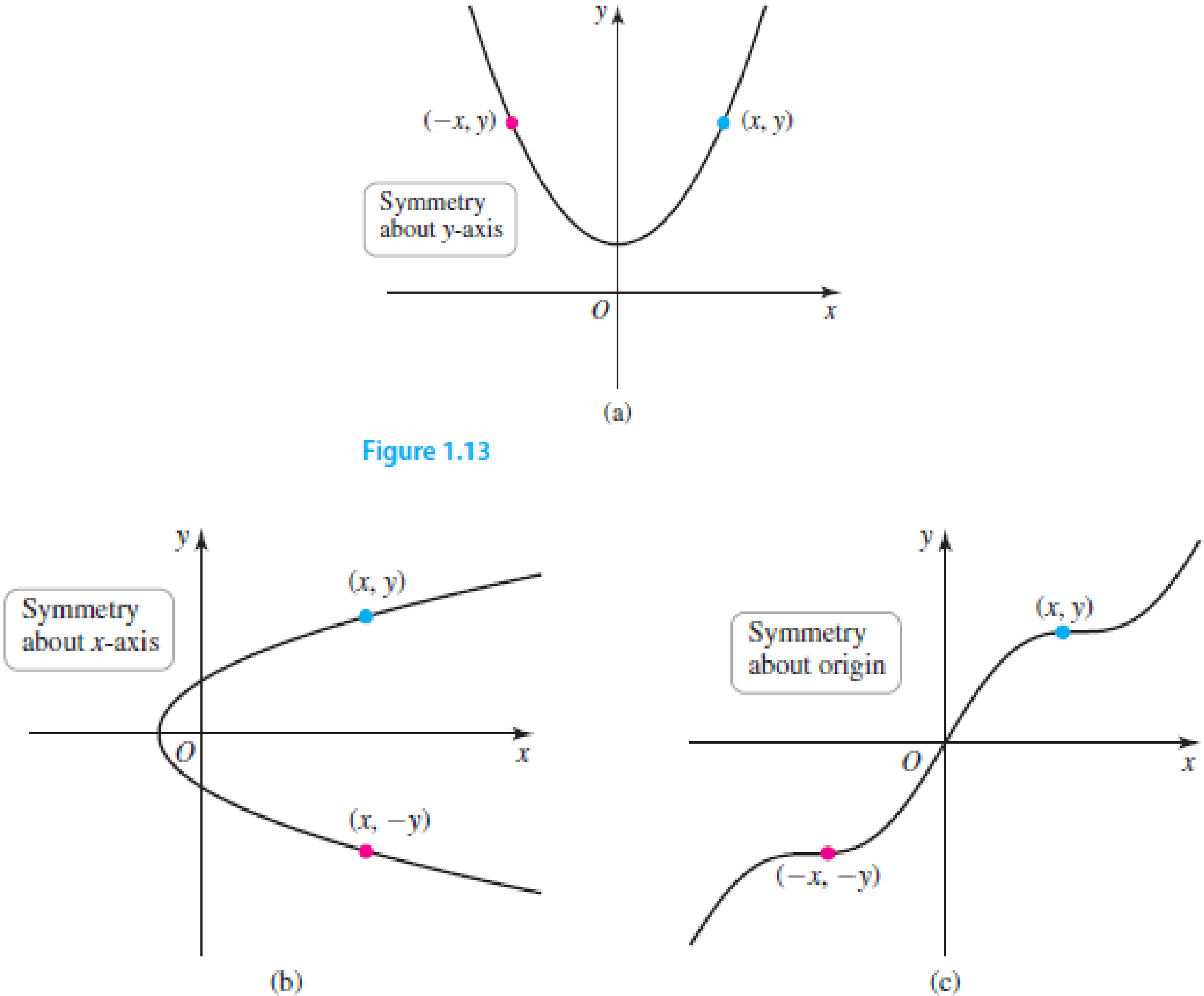

y
(−x, y)
(x, y)
Symmetry about y-axis
O
x
(a)
y
(x, y)
Symmetry about x-axis
O
x
(x, −y)
(b)
y
(x, y)
Symmetry about origin
O
x
(−x, −y)
(c)

Figure 1.13

DEFINITION Symmetry in Functions

An even function f has the property that $f(-x) = f(x)$, for all x in the domain. The graph of an even function is symmetric about the y-axis.

An odd function f has the property that $f(-x) = -f(x)$, for all x in the domain. The graph of an odd function is symmetric about the origin.

Polynomials consisting of only even (odd) powers of the variable are even (odd) functions.

EXAMPLE 11 Identifying symmetry in functions Identify the symmetry, if any, in the following functions.

a. $f(x) = x^4 - 2x^2 - 20$ **b.** $g(x) = x^3 - 3x + 1$ **c.** $h(x) = \dfrac{1}{x^3 - x}$

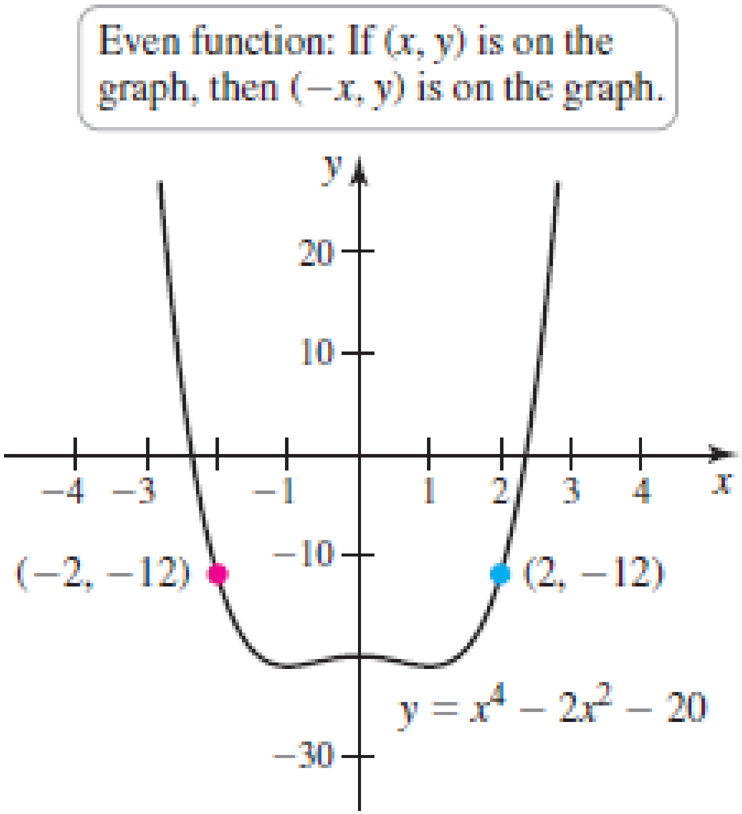


Figure 1.14

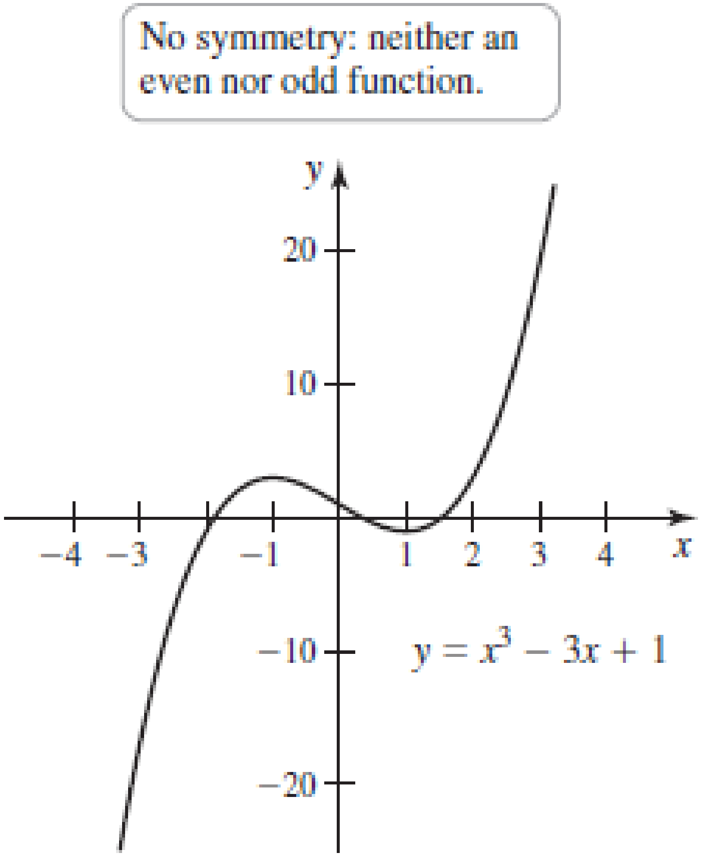
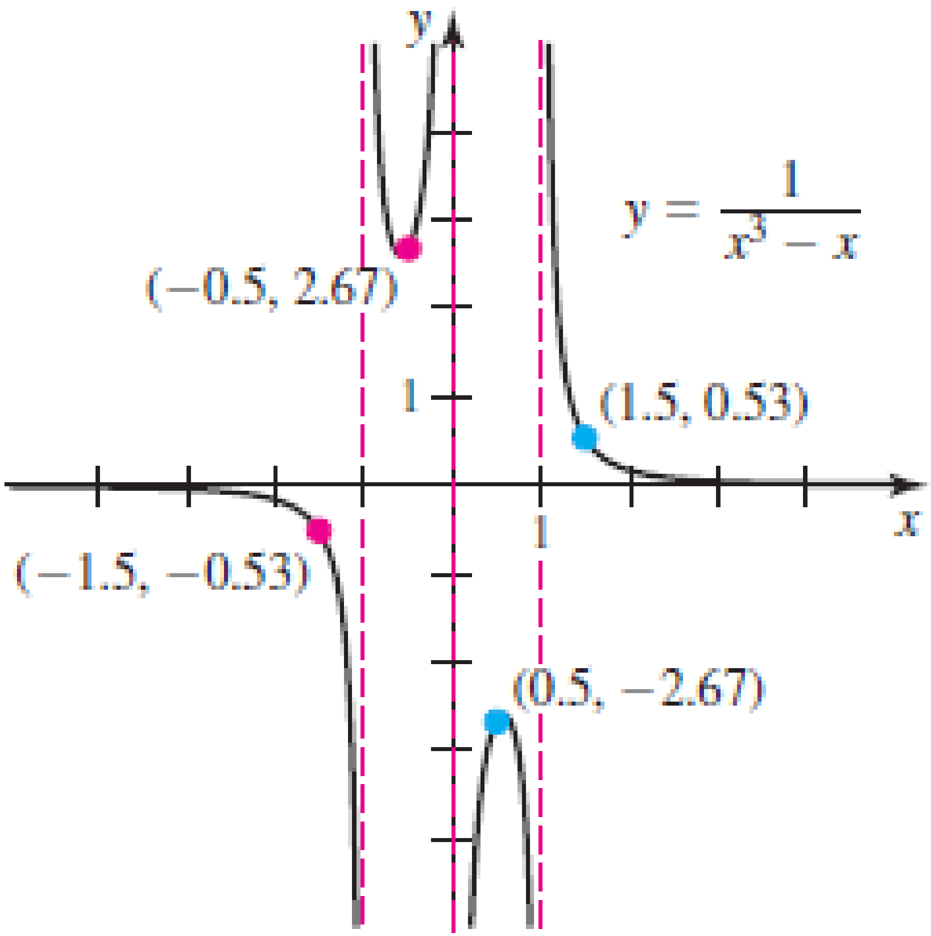


Figure 1.15

Odd function: If (x, y) is on the graph, then $(-x, -y)$ is on the graph.



1.2

Representing Functions

Representations of Functions

There are four possible ways to represent a function:

- verbally (by a description in words)
- numerically (by a table of values)
- visually (by a graph)
- algebraically (by an explicit formula)

Using Formulas

1. **Polynomials**

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

Coefficients $a_0, a_1, \ldots, a_n$ are real numbers with $a_n \neq 0$

Nonnegative integer n is the ***degree*** of the polynomial.

An n-th polynomial can have as many as n real ***zeros*** or ***roots*** (values of x at which $p(x)=0$).

Fundamental Theorem of Algebra

2. **Rational functions** are ratios of the form

$$f(x) = p(x)/q(x)$$

where p and q are polynomials, and $q \neq 0$.

3. **Algebraic functions** are constructed using the operations of algebra: addition, subtraction, multiplication, division, and roots.

4. **Exponential functions** have the form

$$f(x) = b^x$$

where $b \neq 1$ is a positive real number.

Logarithmic functions are closed associated with exponential functions of the form

$$f(x) = \log_b x$$

where $b > 0$ and $b \neq 1$.

The base of **Natural exponential function** is $b = e$

$$f(x) = e^x$$

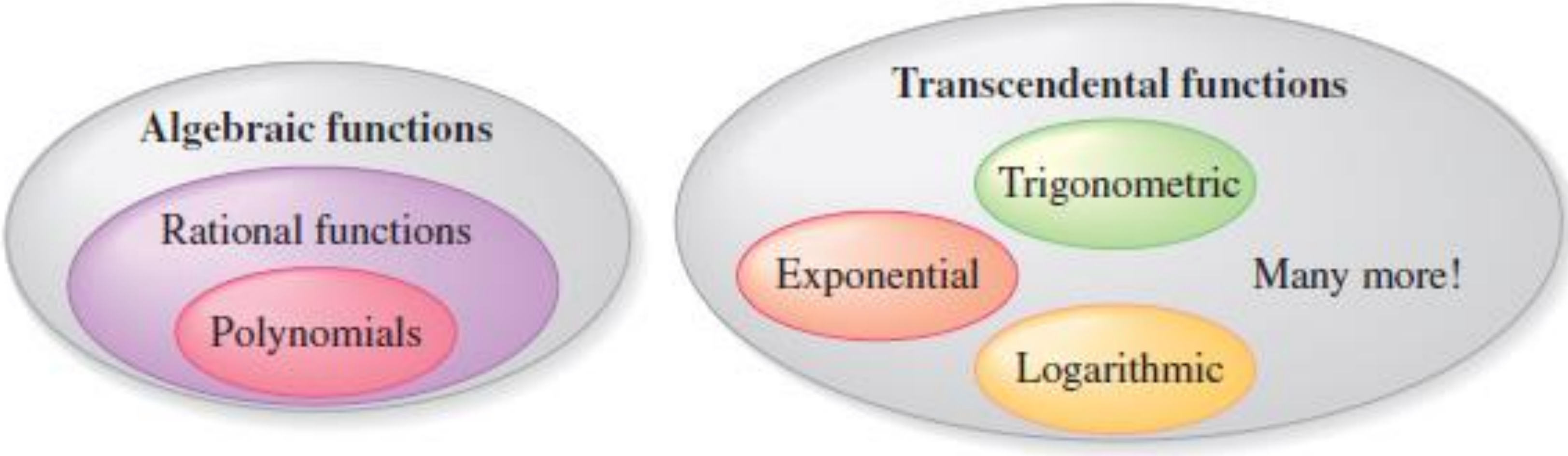
The associated **Natural logarithm function** is

$$f(x) = \ln x$$

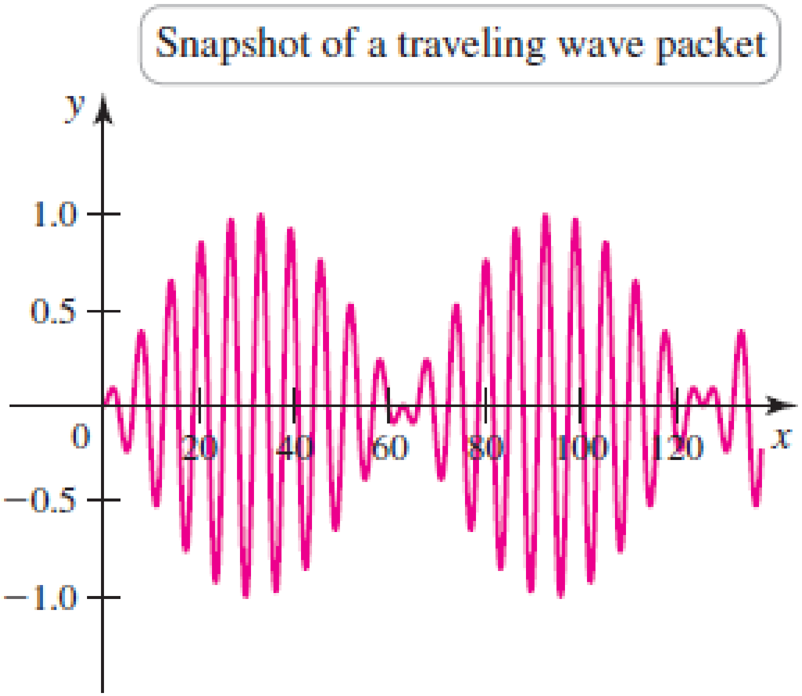
5. **Trigonometric functions** are $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$, and $\csc x$.

Inverse trigonometric functions

6. **Transcendental functions** including trigonometric, exponential, and logarithmic functions, and many more.



Using Graphs, the most illuminating representations



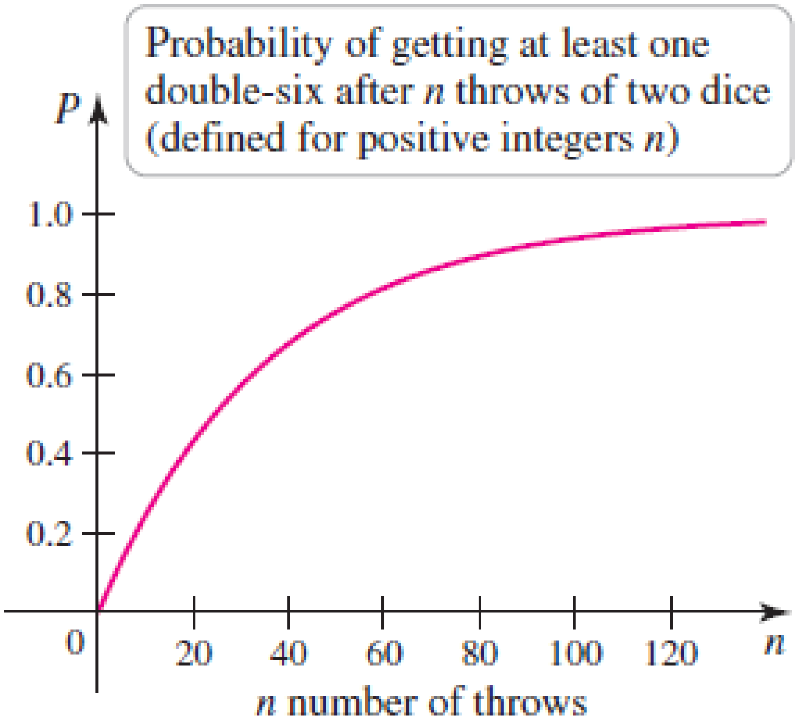


Figure 1.18

Two approaches to **graphing functions**

- **Technologies** including graphing calculators, tablets, and software
- **Analytical methods** (pencil-and-paper methods) to analyze functions and make accurate graphs by hand
- Analytical methods rely heavily on calculus
- Both technology and analytical methods are essential and must be used together in an integrated way

Linear functions

The function $f(x) = mx + b$ has a straight-line graph and is called a **linear function**.

EXAMPLE 1 **Linear functions and their graphs** Determine the function represented by the line in Figure 1.19.

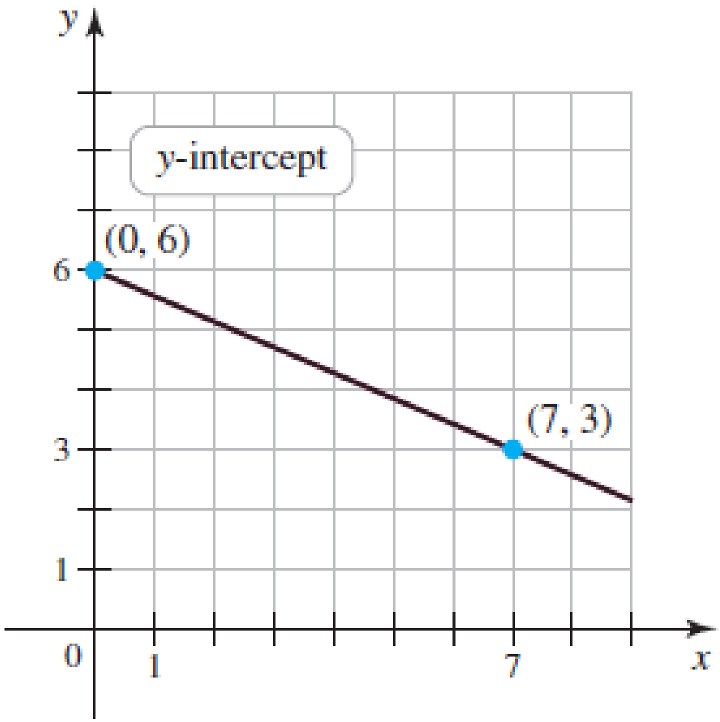


Figure 1.19

EXAMPLE 2 **Demand function for pizzas** After studying sales for several months, the owner of a pizza chain knows that the number of two-topping pizzas sold in a week (called the *demand*) decreases as the price increases. Specifically, her data indicate that at a price of \$14 per pizza, an average of 400 pizzas are sold per week, while at a price of \$17 per pizza, an average of 250 pizzas are sold per week. Assume that the demand d is a *linear* function of the price p.

a. Find the constants m and b in the demand function $d = f(p) = mp + b$. Then graph f.

b. According to this model, how many pizzas (on average) are sold per week at a price of \$20?

Piecewise Functions

Functions that have different definitions on different parts of their domain are called **piecewise functions**.

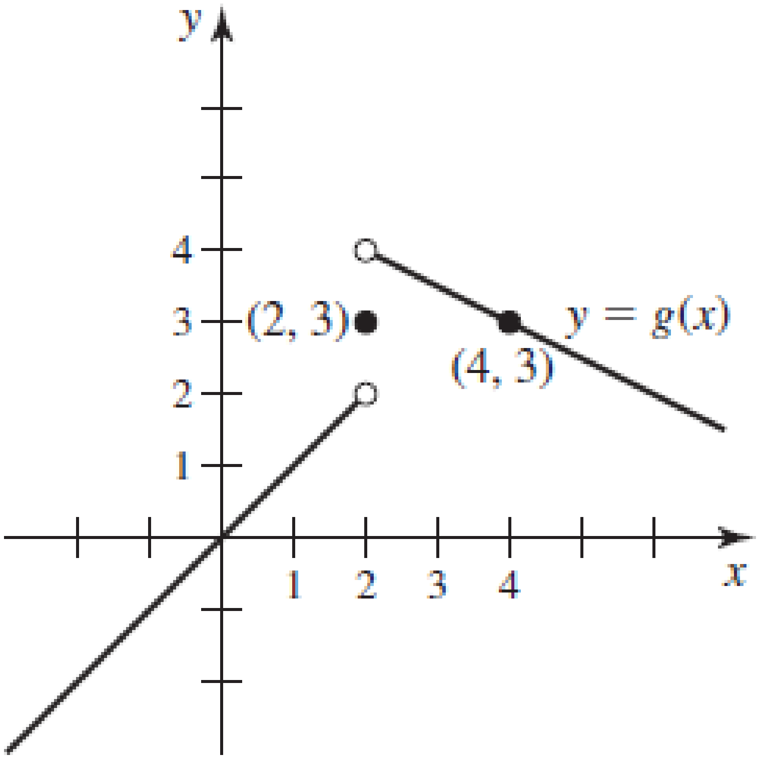
If all the pieces are linear, the function is **piecewise linear**.

EXAMPLE 4 **Graphing piecewise functions** Graph the following functions.

a. $f(x) = \begin{cases} \dfrac{x^2 - 5x + 6}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$

b. $f(x) = |x|$, the absolute value function

EXAMPLE 3 **Defining a piecewise function** The graph of a piecewise linear function g is shown in Figure 1.21. Find a formula for the function.



Power Functions

Special case of polynomials. They have the form $f(x) = x^n$

When n is an even integer, the function values are nonnegative and the graph passes through the origin, opening upward

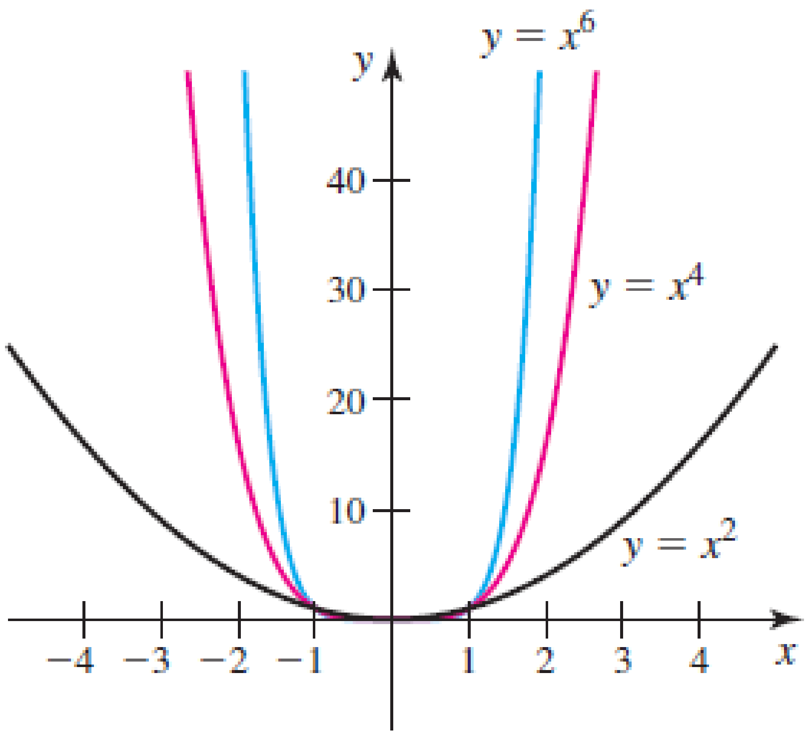


Figure 1.24

When n is an odd integer, the power function has values that are positive when x is positive and negative when x is negative

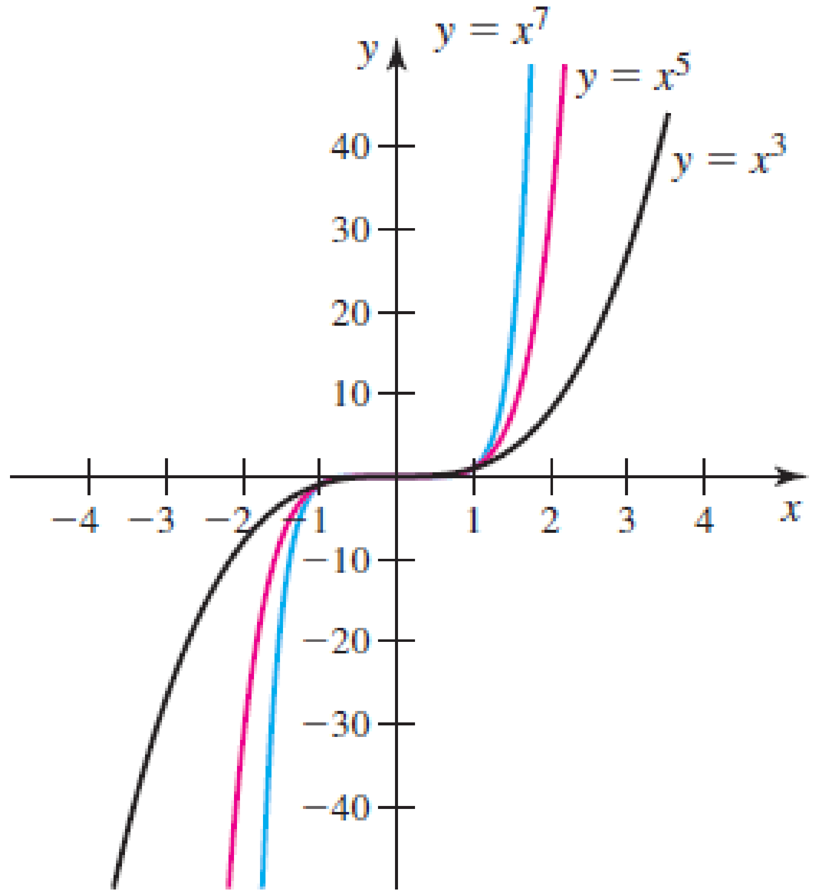


Figure 1.25

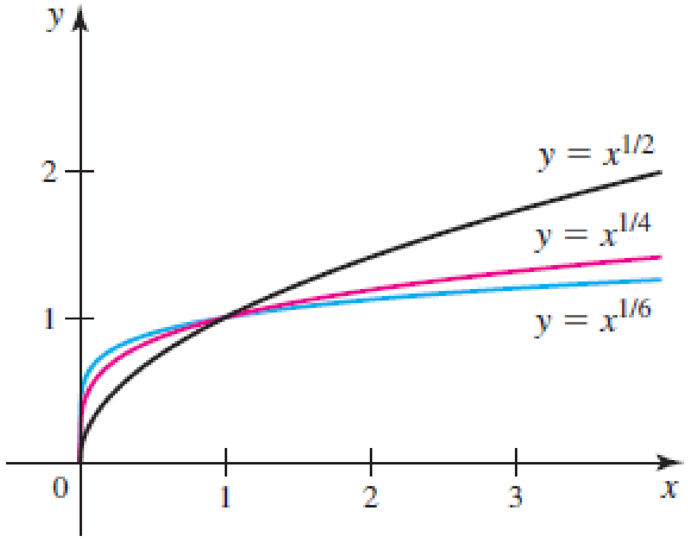
Root Functions

Special case of algebraic functions. They have the form

$$f(x) = x^{1/n},$$

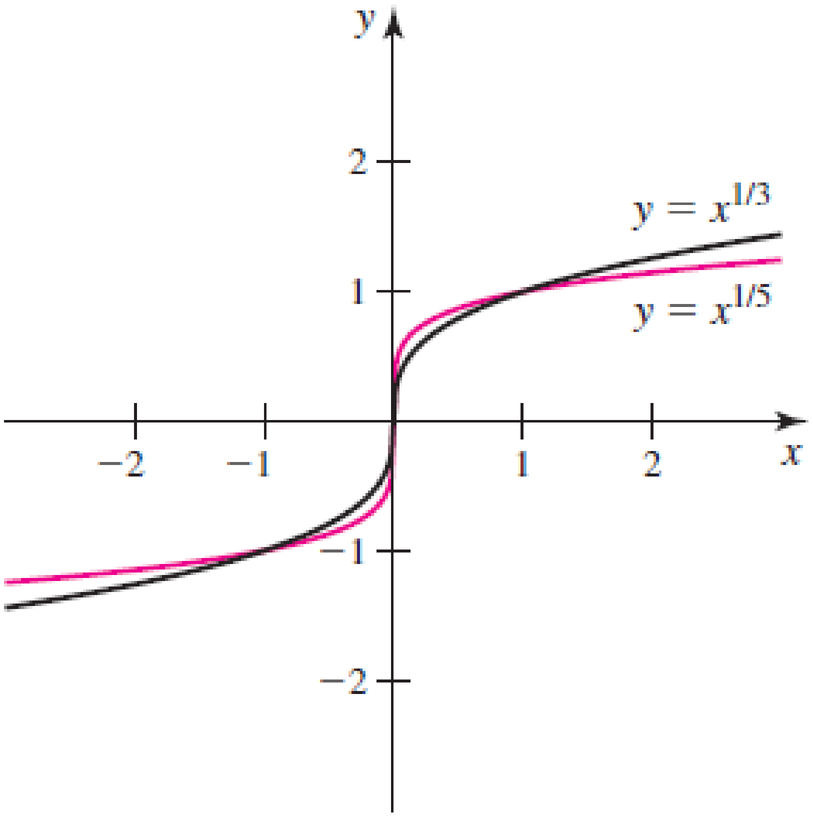
where $n > 1$ is a positive integer.

When n is even (square roots, fourth roots, and so forth), the domain and range consist of nonnegative numbers



The odd root functions (cube roots, fifth roots, and so forth) have domain and range consisting of all real numbers

Their graphs pass through the origin, open upward for $x<0$ and downward for $x>0$, and flatten out as x increases in magnitude.



Rational Functions

Analysis and technology work together

EXAMPLE 5 **Technology and analysis** Consider the rational function

$$f(x) = \frac{3x^3 - x - 1}{x^3 + 2x^2 - 6}.$$

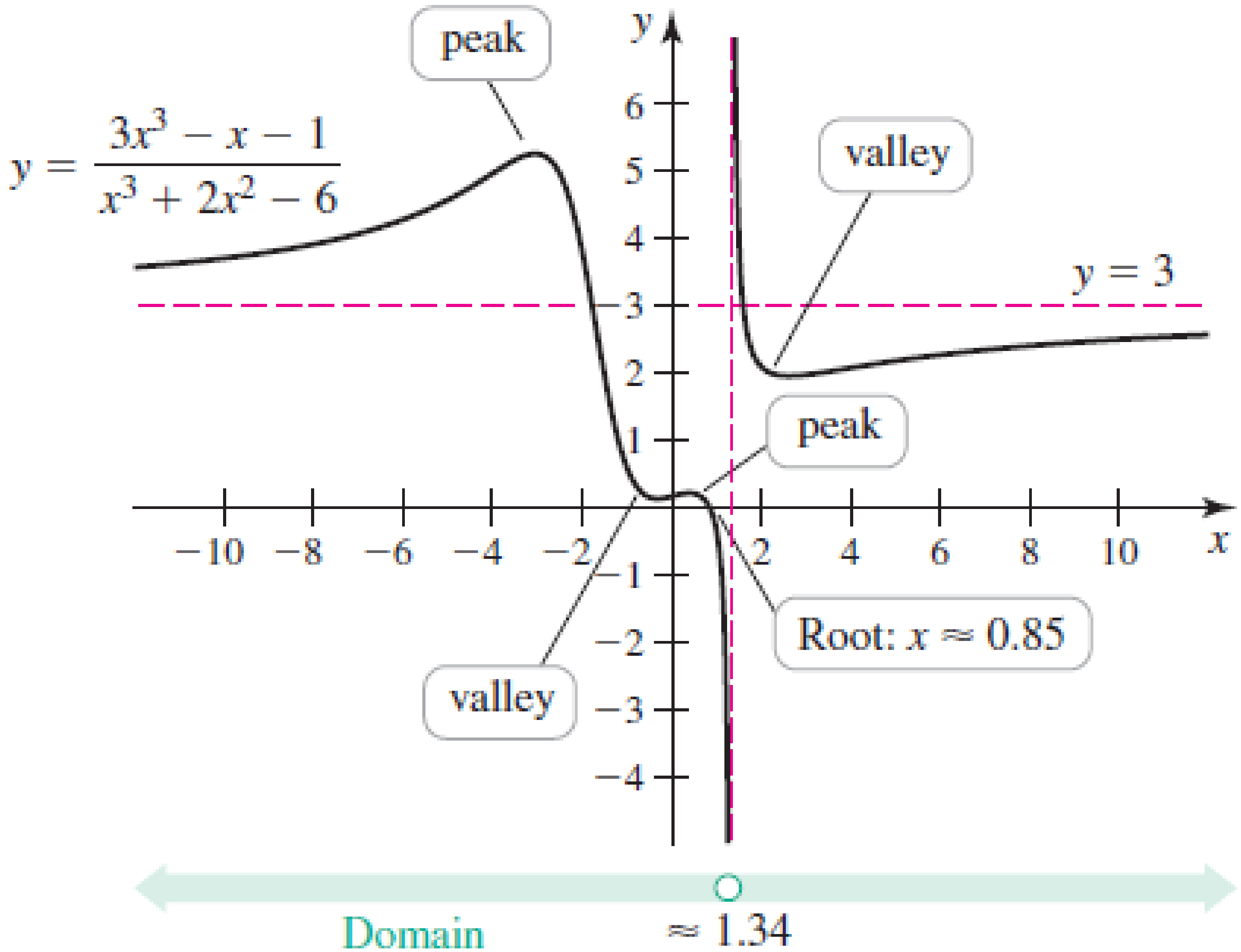
a. What is the domain of f?

b. Find the roots (zeros) of f.

c. Graph the function using a graphing utility.

d. At what points does the function have peaks and valleys?

e. How does f behave as x grows large in magnitude?



$y = \dfrac{3x^3 - x - 1}{x^3 + 2x^2 - 6}$
peak
valley
$y = 3$
peak
valley
Root: $x \approx 0.85$
y
x
6
5
4
3
2
1
−1
−2
−3
−4
−10
−8
−6
−4
−2
2
4
6
8
10
Domain
≈ 1.34

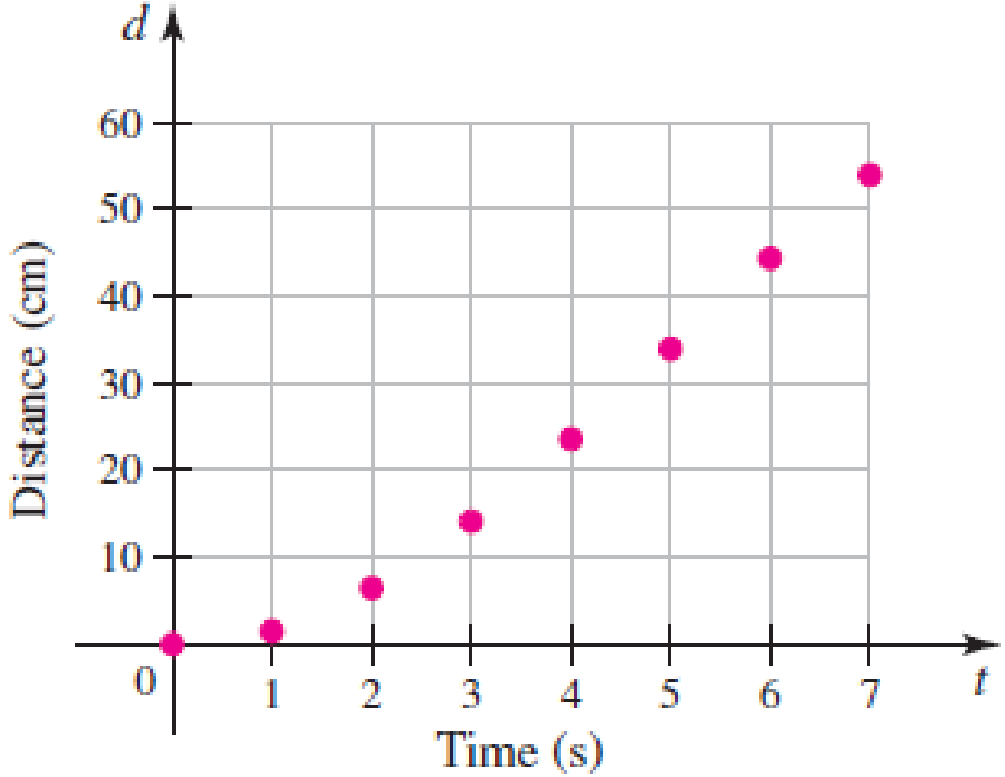
Figure 1.28

Using Tables

Experimental data; discrete

Table 1.1

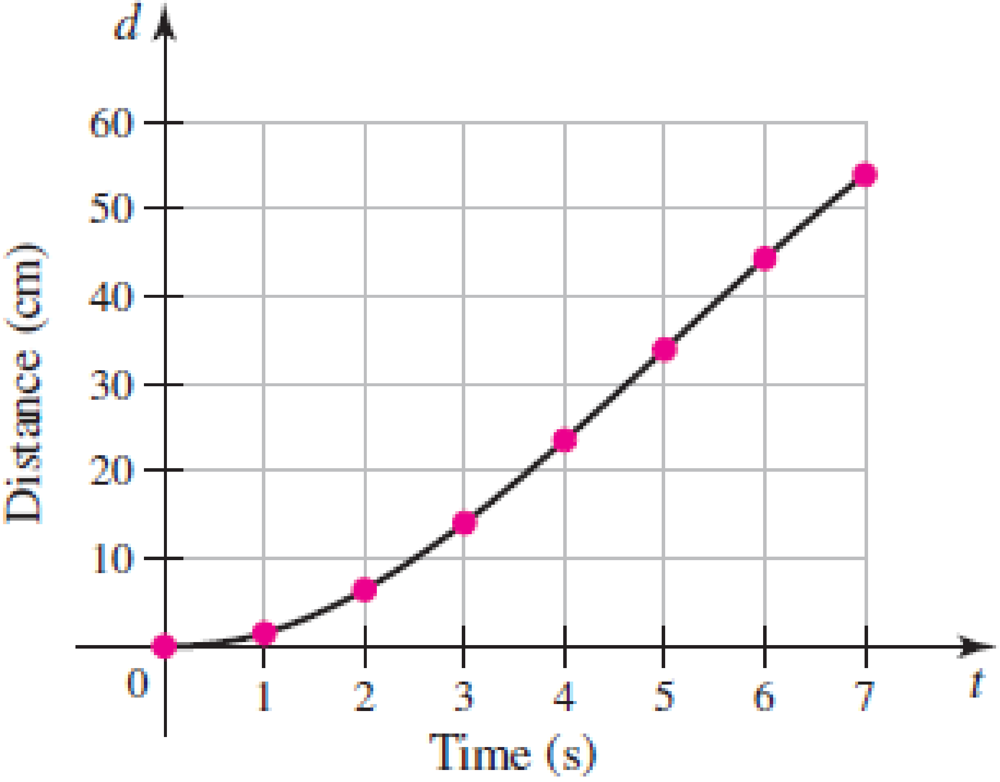
t (s)	d (cm)
0	0
1	2
2	6
3	14
4	24
5	34
6	44
7	54



Data points plot

The data points suggest that there is a function

$$d = f(t)$$



Using Words

Often the way in which functions originate

EXAMPLE 6 **A slope function** Let g be the **slope function** for a given function f. In words, this means that $g(x)$ is the slope of the curve $y = f(x)$ at the point $(x, f(x))$. Find and graph the slope function for the function f in Figure 1.31.

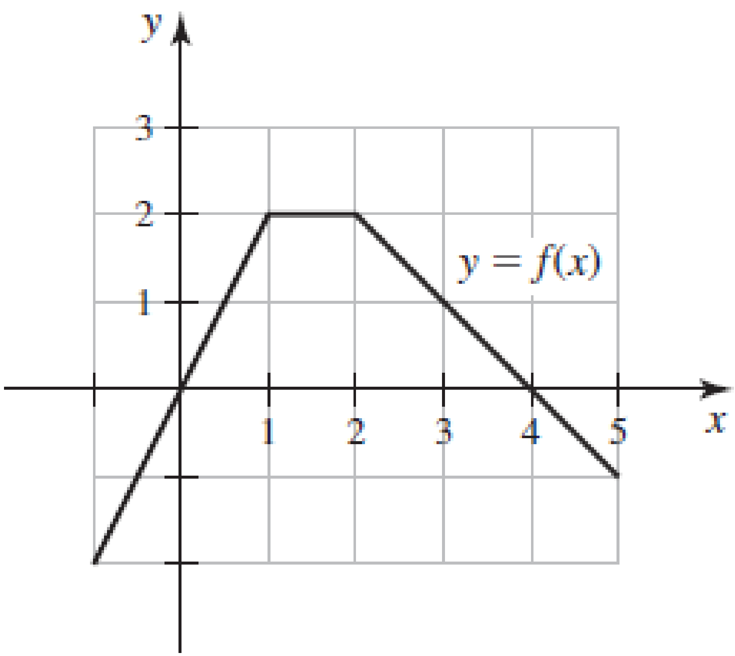


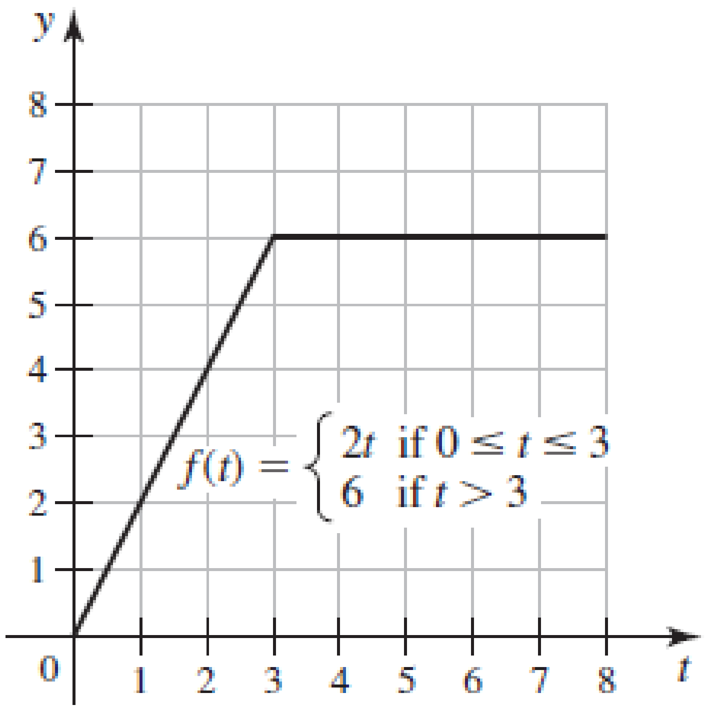
Figure 1.31

EXAMPLE 7 **An area function** Let A be an **area function** for a positive function f. In words, this means that $A(x)$ is the area of the region bounded by the graph of f and the t-axis from $t = 0$ to $t = x$. Consider the function (Figure 1.33)

$$f(t) = \begin{cases} 2t & \text{if } 0 \le t \le 3 \\ 6 & \text{if } t > 3. \end{cases}$$

a. Find $A(2)$ and $A(5)$.

b. Find a piecewise formula for the area function for f.



Transformation of Functions and Graphs

- Transform the graph of a function to produce graphs of new functions.
- Four common transformations:
  - ***shifts*** of x- and y-directions
  - ***scalings*** in the x- and y-directions

The graph of $y = f(x) + d$ is the graph of $y = f(x)$ shifted vertically by d units (up if $d > 0$ and down if $d < 0$).

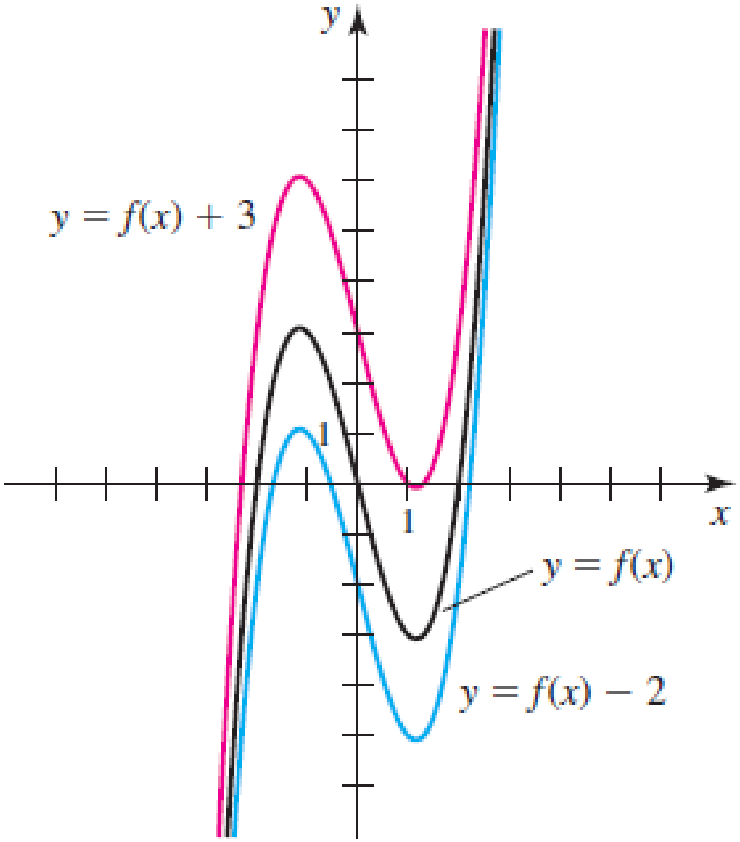


Figure 1.37

The graph of $y = f(x - b)$ is the graph of $y = f(x)$ shifted horizontally by b units (right if $b > 0$ and left if $b < 0$).

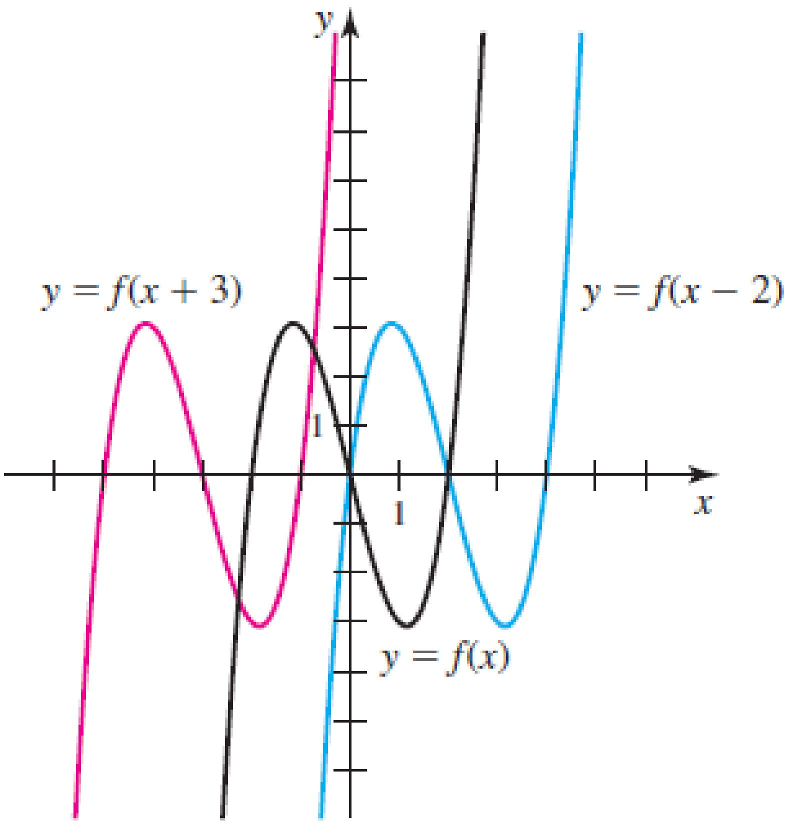


Figure 1.38

For $c > 0$, the graph of $y = cf(x)$ is the graph of $y = f(x)$ scaled vertically by a factor of c (wider if $0 < c < 1$ and narrower if $c > 1$).

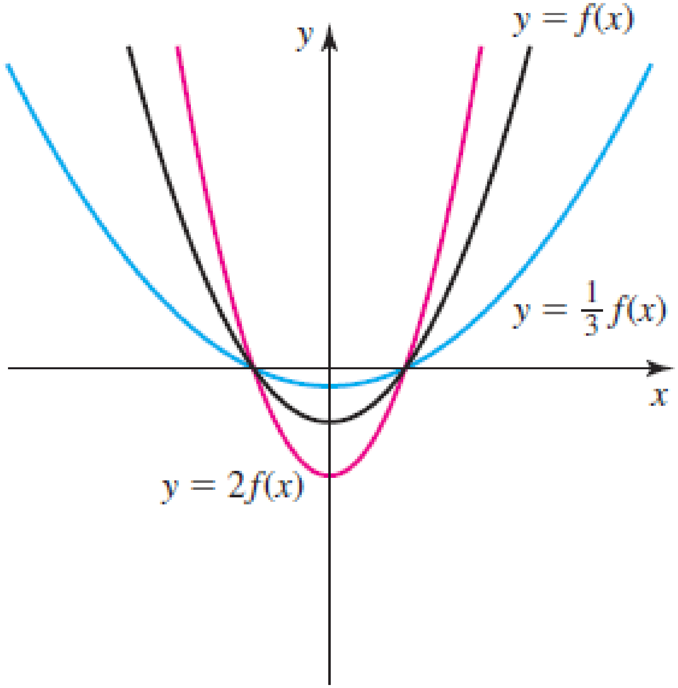


Figure 1.39

For $c < 0$, the graph of $y = cf(x)$ is the graph of $y = f(x)$ scaled vertically by a factor of $|c|$ and reflected across the x-axis (wider if $-1 < c < 0$ and narrower if $c < -1$).

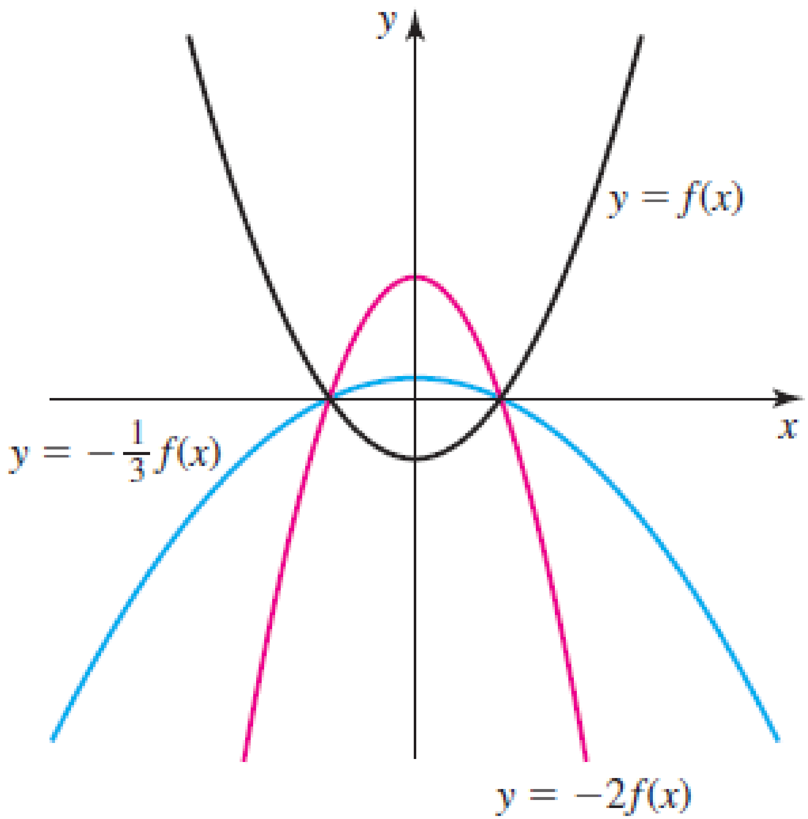


Figure 1.40

For $a > 0$, the graph of $y = f(ax)$ is the graph of $y = f(x)$ scaled horizontally by a factor of a (wider if $0 < a < 1$ and narrower if $a > 1$).

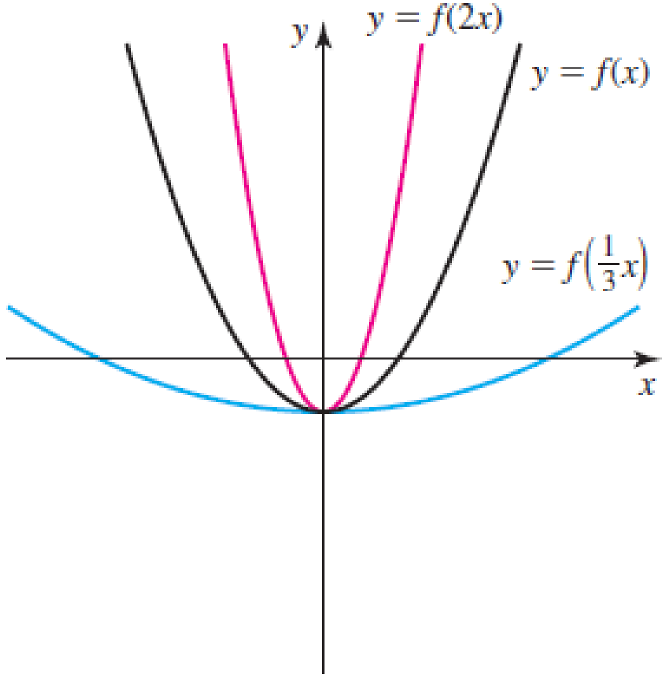


Figure 1.41

For $a < 0$, the graph of $y = f(ax)$ is the graph of $y = f(x)$ scaled horizontally by a factor of $|a|$ and reflected across the y-axis (wider if $-1 < a < 0$ and narrower if $a < -1$).

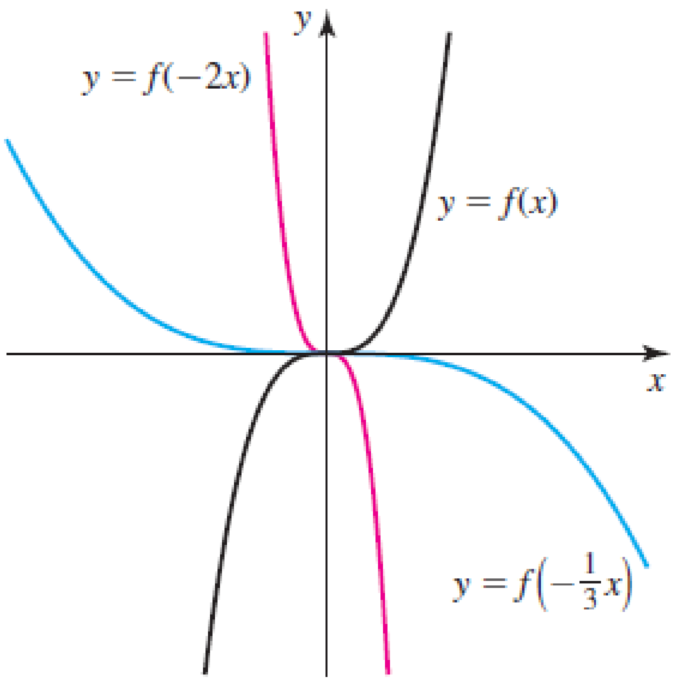


Figure 1.42

EXAMPLE 8 **Shifting parabolas** The graphs A, B, and C in Figure 1.43 are obtained from the graph of $f(x) = x^2$ using shifts and scalings. Find the function that describes each graph.

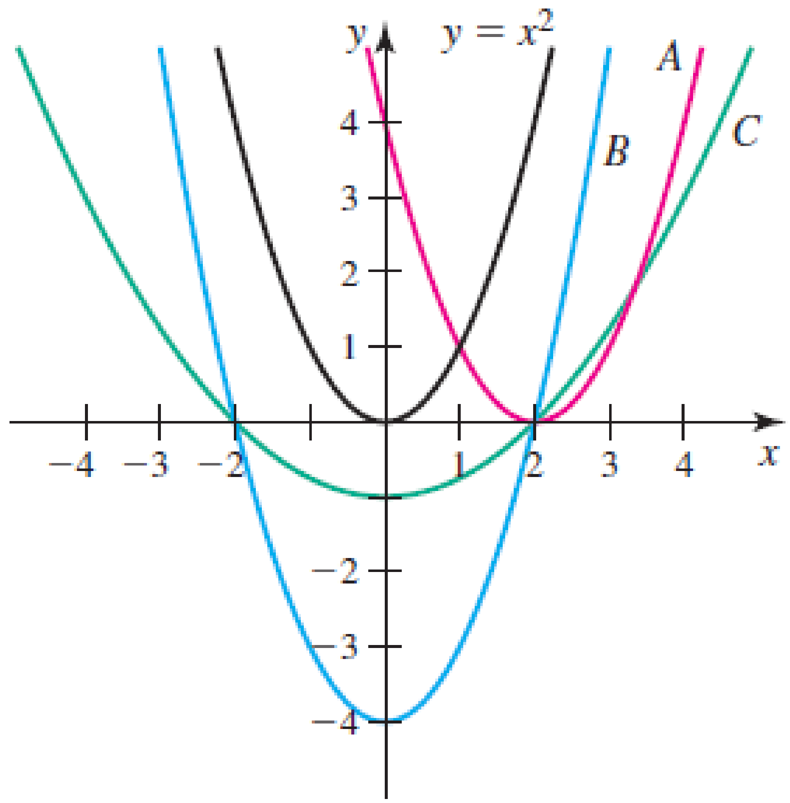
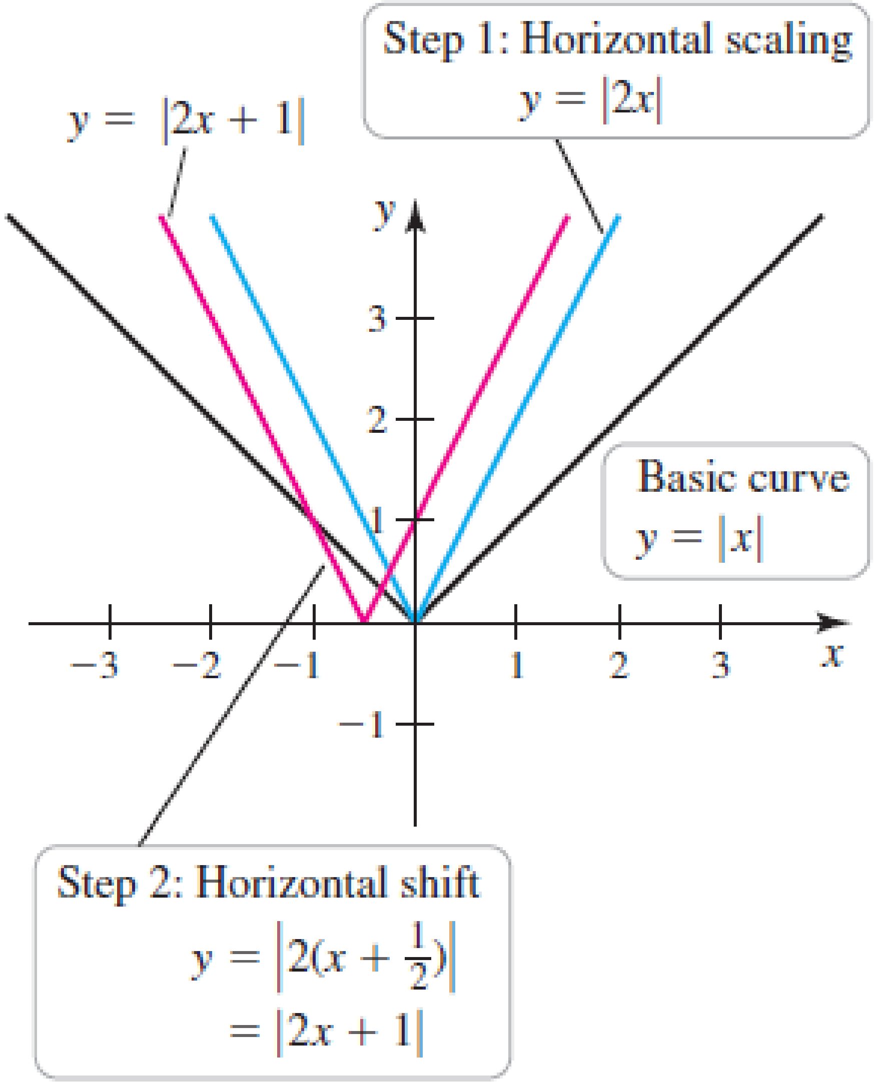


Figure 1.43

EXAMPLE 9 **Scaling and shifting** Graph $g(x) = |2x + 1|$.



SUMMARY **Transformations**

Given the real numbers a, b, c, and d and the function f, the graph of $y = cf(a(x - b)) + d$ can be obtained from the graph of $y = f(x)$ in the following steps.

$$y = f(x) \xrightarrow[\text{by a factor of } |a|]{\text{horizontal scaling}} y = f(ax)$$

$$\xrightarrow[\text{by } b \text{ units}]{\text{horizontal shift}} y = f(a(x - b))$$

$$\xrightarrow[\text{by a factor of } |c|]{\text{vertical scaling}} y = cf(a(x - b))$$

$$\xrightarrow[\text{by } d \text{ units}]{\text{vertical shift}} y = cf(a(x - b)) + d$$

1.3

Inverse, Exponential, and Logarithmic Functions

Exponential Functions

Exponential functions have the form $f(x) = b^x$,

where $b \neq 1$ is a positive real number.

When x is rational, the numerator and denominator are interpreted as power and root respectively

$$16^{3/4} = 16^{\overset{\text{power}}{3}/\underset{\text{root}}{4}} = \Big(\underbrace{\sqrt[4]{16}}_{2}\Big)^3 = 8.$$

Question: How to compute b^x when x is irrational?

Conclusion: The domain of an exponential function is all real numbers, which will be discussed in Section 7.1.

Properties of Exponential Functions $f(x) = b^x$

1. Because b^x is defined for all real numbers, the domain of f is $\{x: -\infty < x < \infty\}$. Because $b^x > 0$ for all values of x, the range of f is $\{y: 0 < y < \infty\}$.
2. For all $b > 0$, $b^0 = 1$, and therefore $f(0) = 1$.
3. If $b > 1$, then f is an increasing function of x (Figure 1.45). For example, if $b = 2$, then $2^x > 2^y$ whenever $x > y$.
4. If $0 < b < 1$, then f is a decreasing function of x. For example, if $b = \frac{1}{2}$,

$$f(x) = \left(\frac{1}{2}\right)^x = \frac{1}{2^x} = 2^{-x},$$

and because 2^x increases with x, 2^{-x} decreases with x (Figure 1.46).

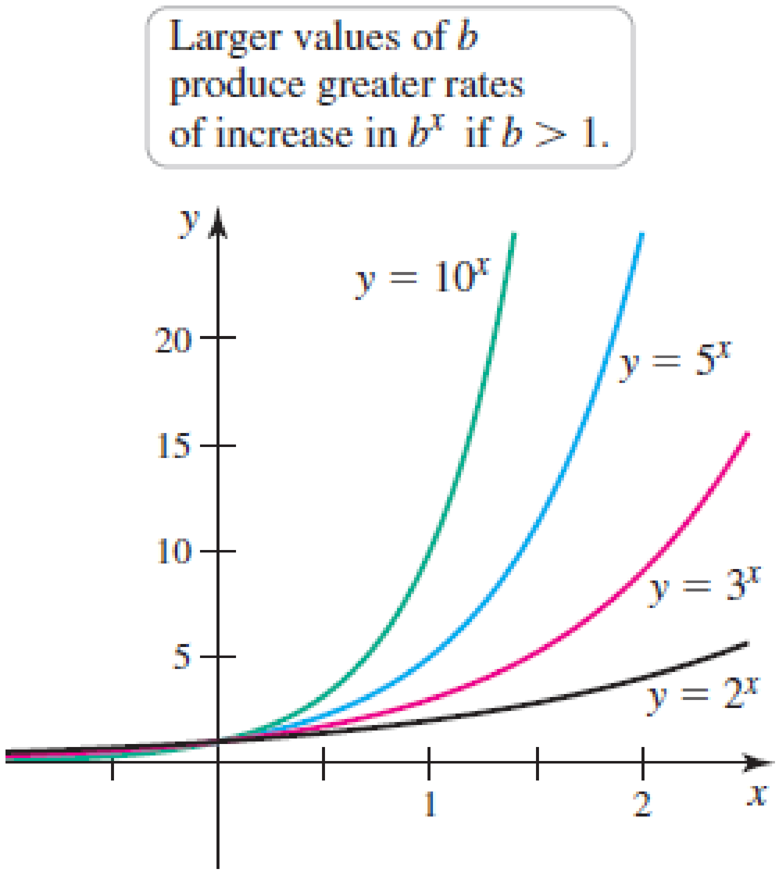


Figure 1.45

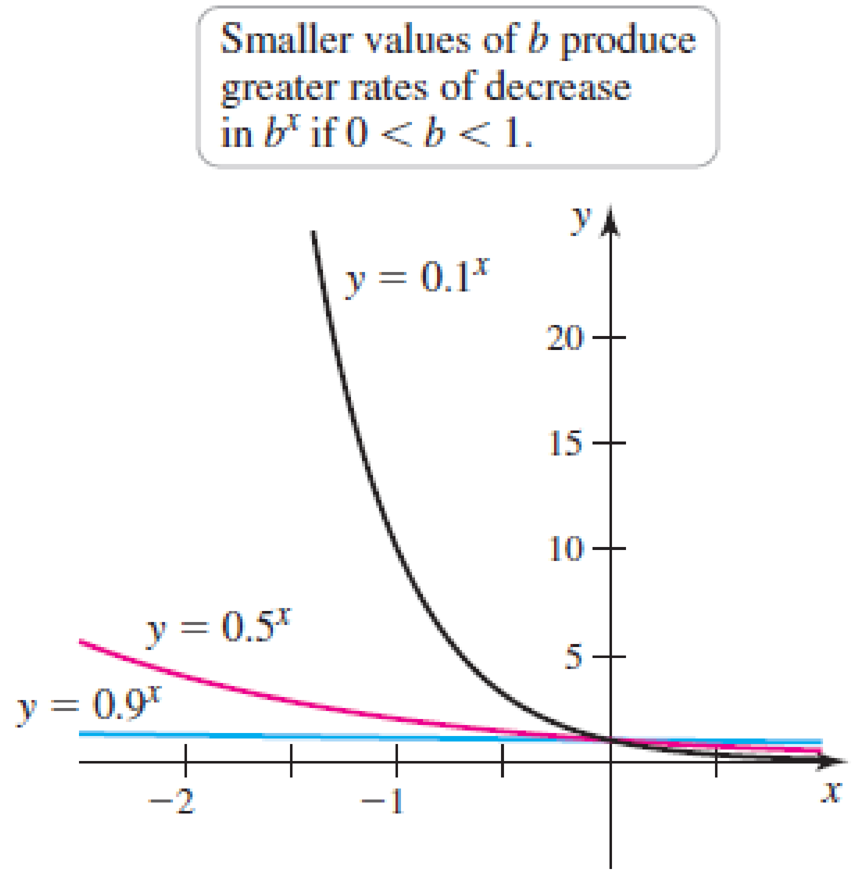


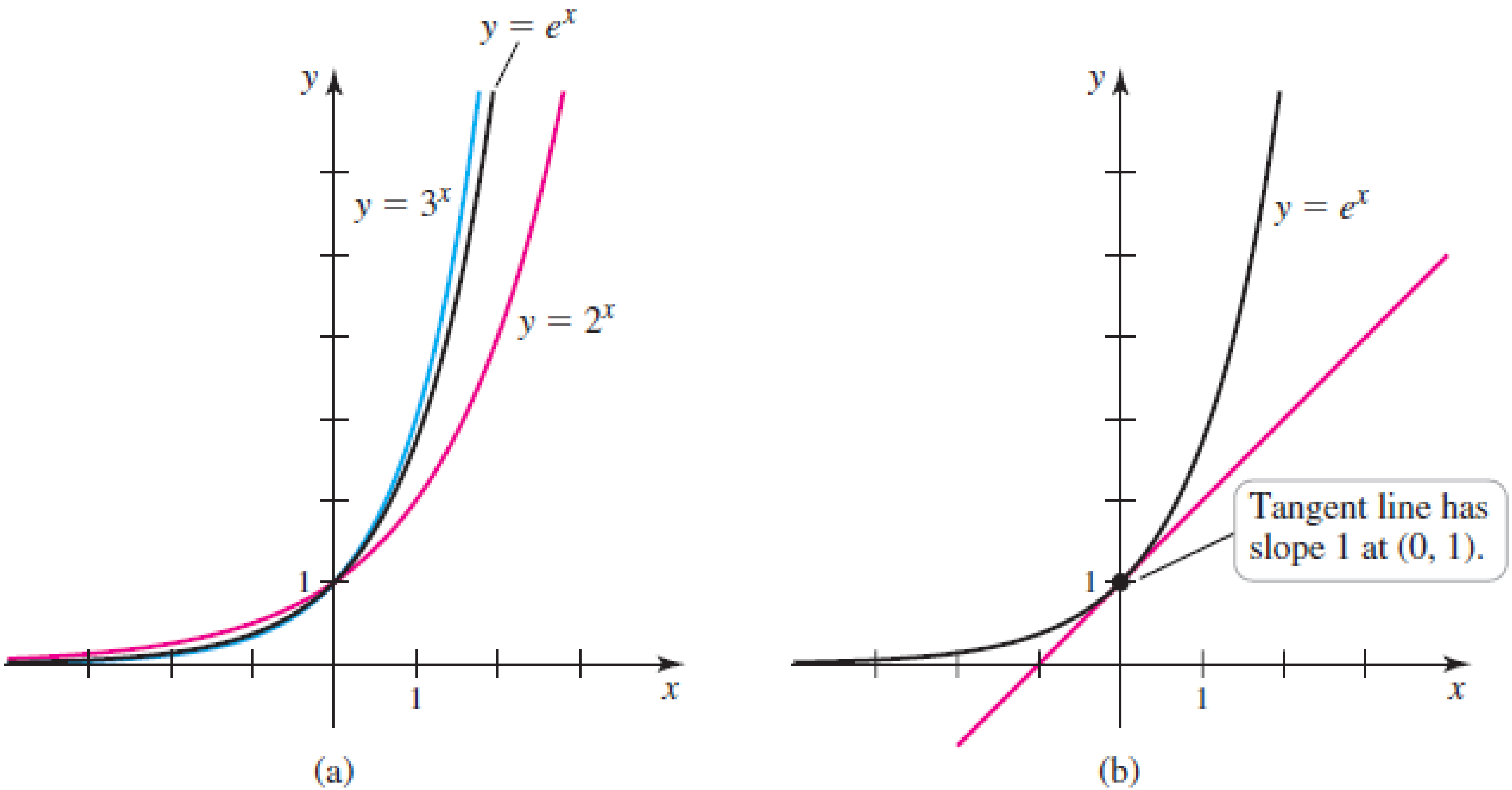
Figure 1.46

Natural Exponential Function

A special case with a special base *e*.

> **DEFINITION The Natural Exponential Function**
>
> The natural exponential function is $f(x) = e^x$, which has the base $e = 2.718281828459\ldots$.

- At every point on the graph of $y = e^x$, it is possible to draw a *tangent line* that touches the graph only at that point.
- It is the only exponential function with the property that the slope of the tangent line at x=0 is 1; therefore, e^x has *both value and slope equal to 1 at x=0*.


y = e^x
y
y = 3^x
y = 2^x
1
1
x
(a)
y
y = e^x
Tangent line has
slope 1 at (0, 1).
1
1
x
(b)

Inverse Functions

DEFINITION Inverse Function

Given a function f, its inverse (if it exists) is a function f^{-1} such that whenever $y = f(x)$, then $f^{-1}(y) = x$ (Figure 1.48).

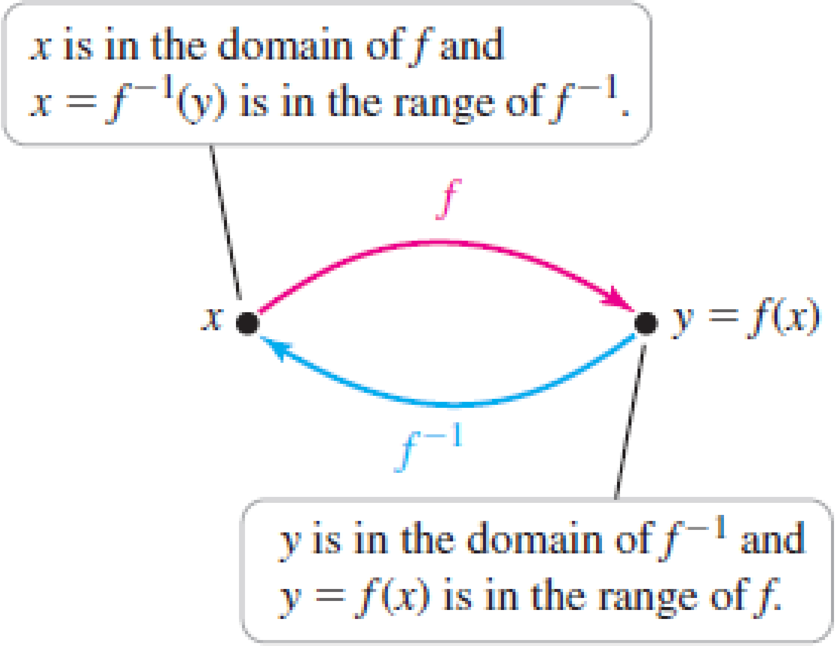
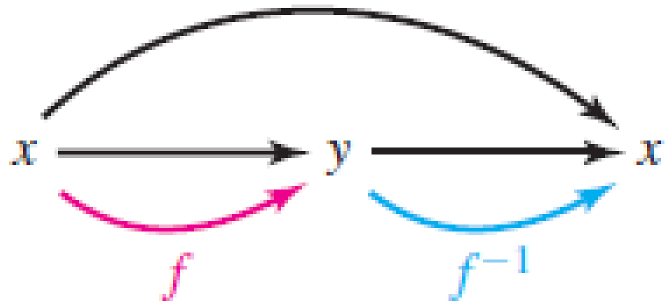


Figure 1.48

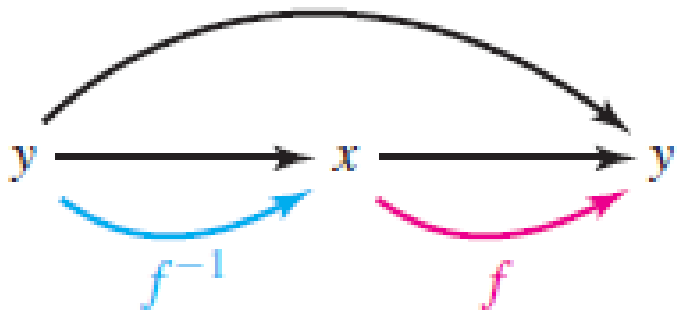
Note that $f^{-1}(x)$ is not the reciprocal, $1/f(x) = (f(x))^{-1}$.

The inverse “undoes” the original function.

$$f^{-1}(f(x)) = x.$$



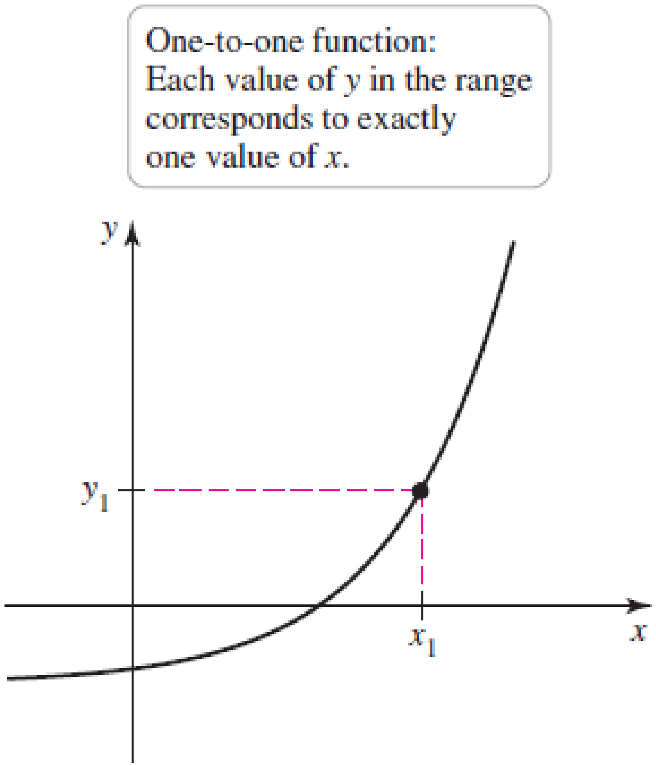
$$f(f^{-1}(y)) = y.$$

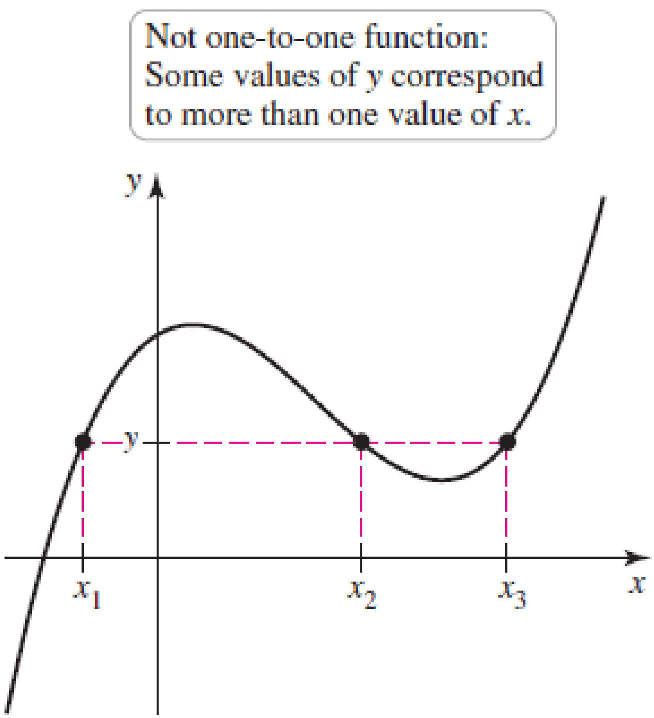


One-to-One Functions

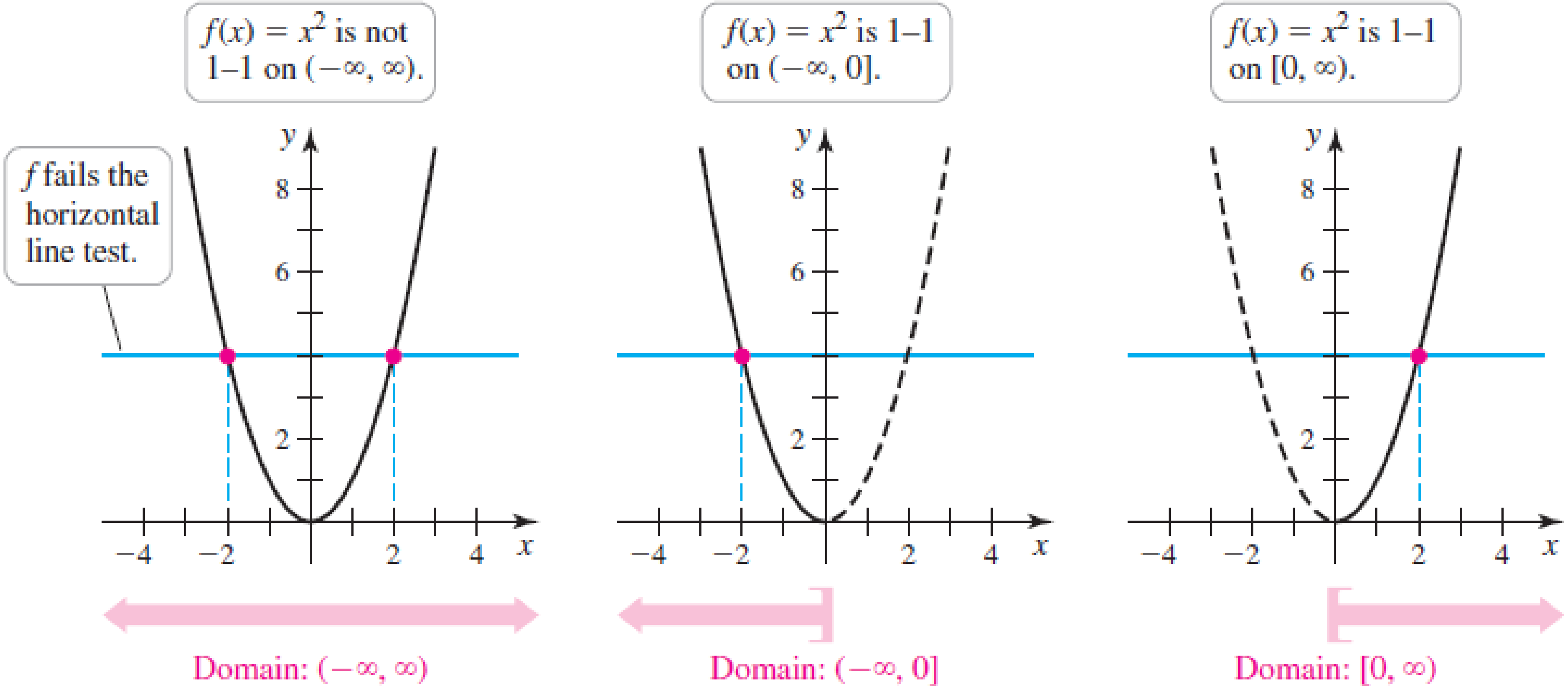
DEFINITION One-to-One Functions and the Horizontal Line Test

A function f is **one-to-one** on a domain D if each value of $f(x)$ corresponds to exactly one value of x in D. More precisely, f is one-to-one on D if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$, for x_1 and x_2 in D. The **horizontal line test** says that every horizontal line intersects the graph of a one-to-one function at most once (Figure 1.49).





For some function f that does not have an inverse function on the interval $(-\infty, \infty)$, we can restrict the domain of f to one of the intervals $(-\infty, 0]$ or $[0, \infty)$.



EXAMPLE 1 **One-to-one functions** Determine the (largest possible) intervals on which the function $f(x) = 2x^2 - x^4$ (Figure 1.51) is one-to-one.

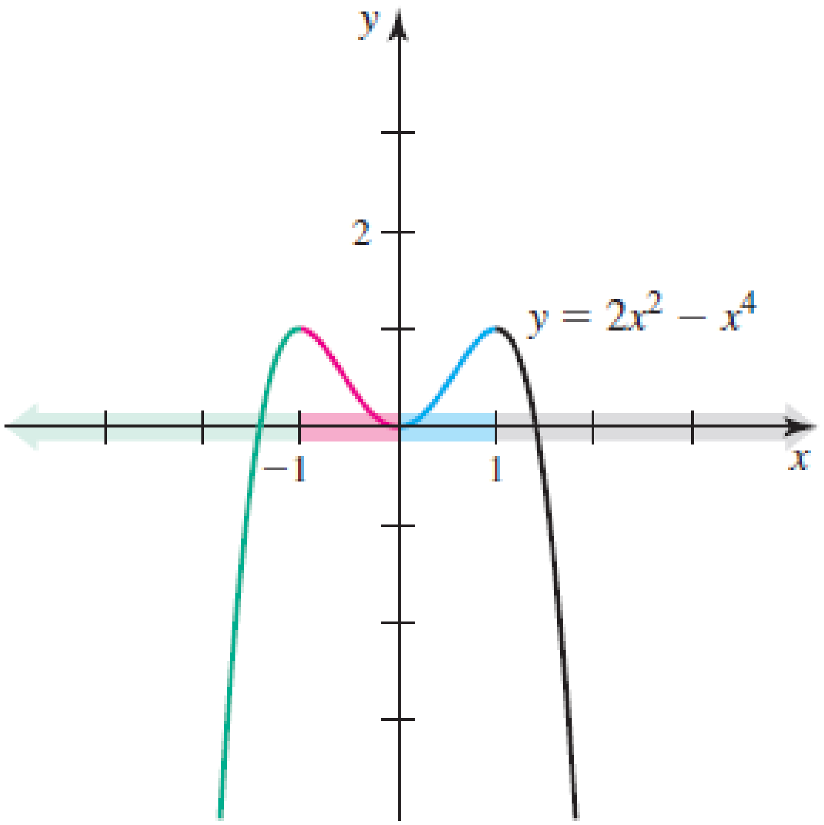
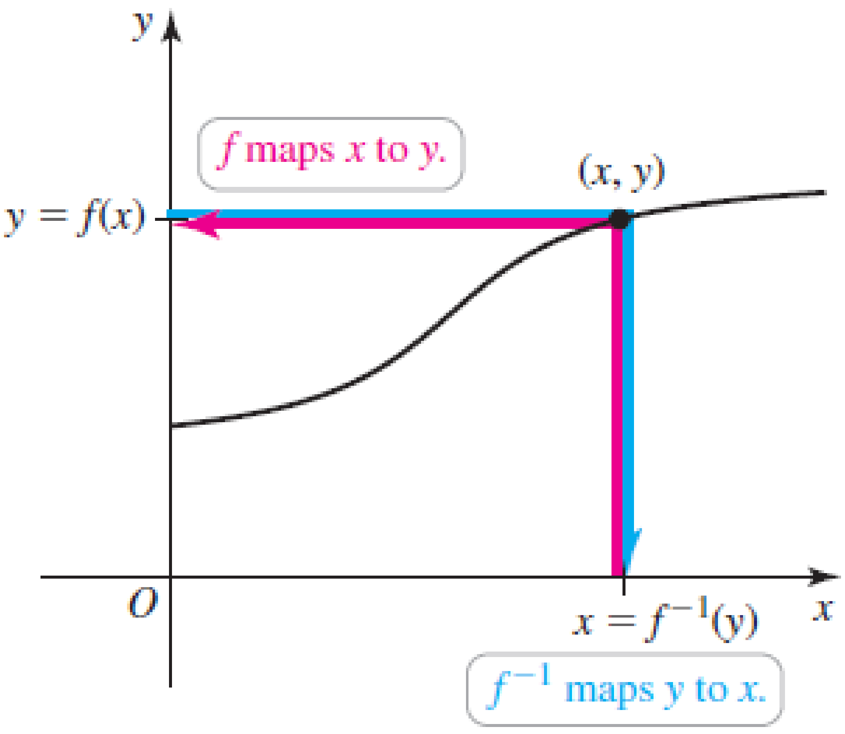


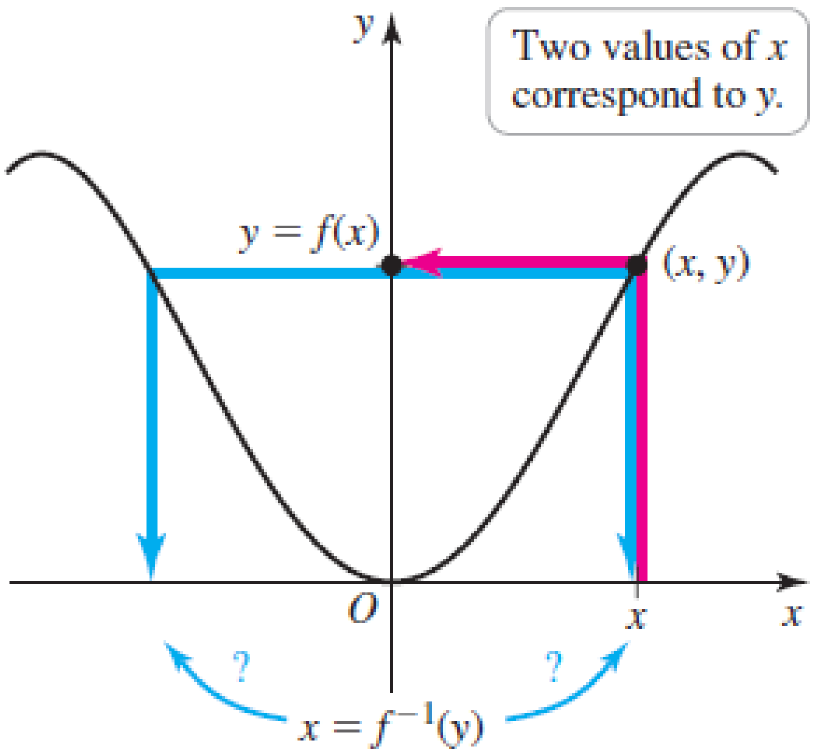
Figure 1.51

Existence of Inverse Functions

Existence of Inverse Functions Figure 1.52a illustrates the actions of a one-to-one function f and its inverse f^{-1}. We see that f maps a value of x to a unique value of y. In turn, f^{-1} maps that value of y back to the original value of x. This procedure cannot be carried out if f is *not* one-to-one (Figure 1.52b).



(a)



(b)

THEOREM 1.1 Existence of Inverse Functions

Let f be a one-to-one function on a domain D with a range R. Then f has a unique inverse f^{-1} with domain R and range D such that

$$f^{-1}(f(x)) = x \quad \text{and} \quad f(f^{-1}(y)) = y,$$

where x is in D and y is in R.

EXAMPLE 2 Does an inverse exist? Determine the largest intervals on which $f(x) = x^2 - 1$ has an inverse function.

Finding Inverse Functions

PROCEDURE **Finding an Inverse Function**

Suppose f is one-to-one on an interval I. To find f^{-1}:

1. Solve $y = f(x)$ for x. If necessary, choose the function that corresponds to I.
2. Interchange x and y and write $y = f^{-1}(x)$.

EXAMPLE 3 **Finding inverse functions** Find the inverse(s) of the following functions. Restrict the domain of f if necessary.

a. $f(x) = 2x + 6$

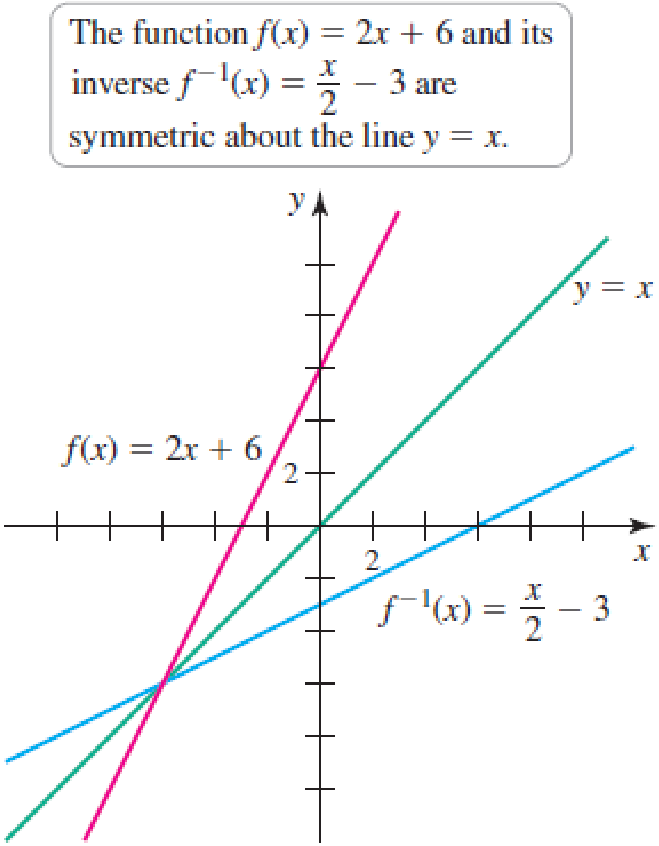
b. $f(x) = x^2 - 1$

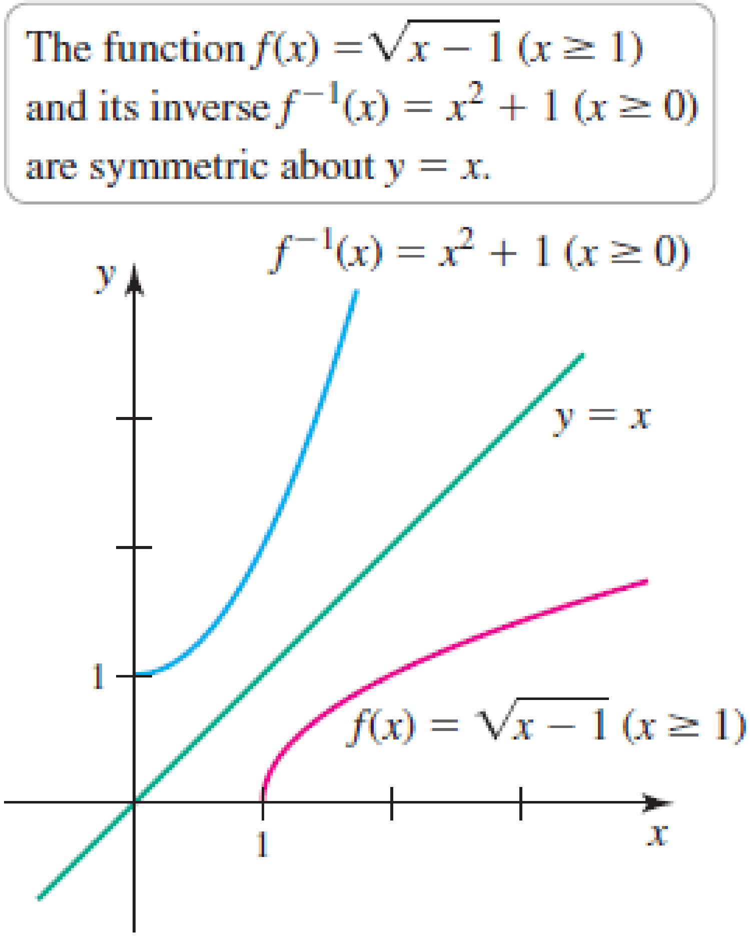
Graphing Inverse Functions

EXAMPLE 4 **Graphing inverse functions** Plot f and f^{-1} on the same coordinate axes.

a. $f(x) = 2x + 6$

b. $f(x) = \sqrt{x - 1}$

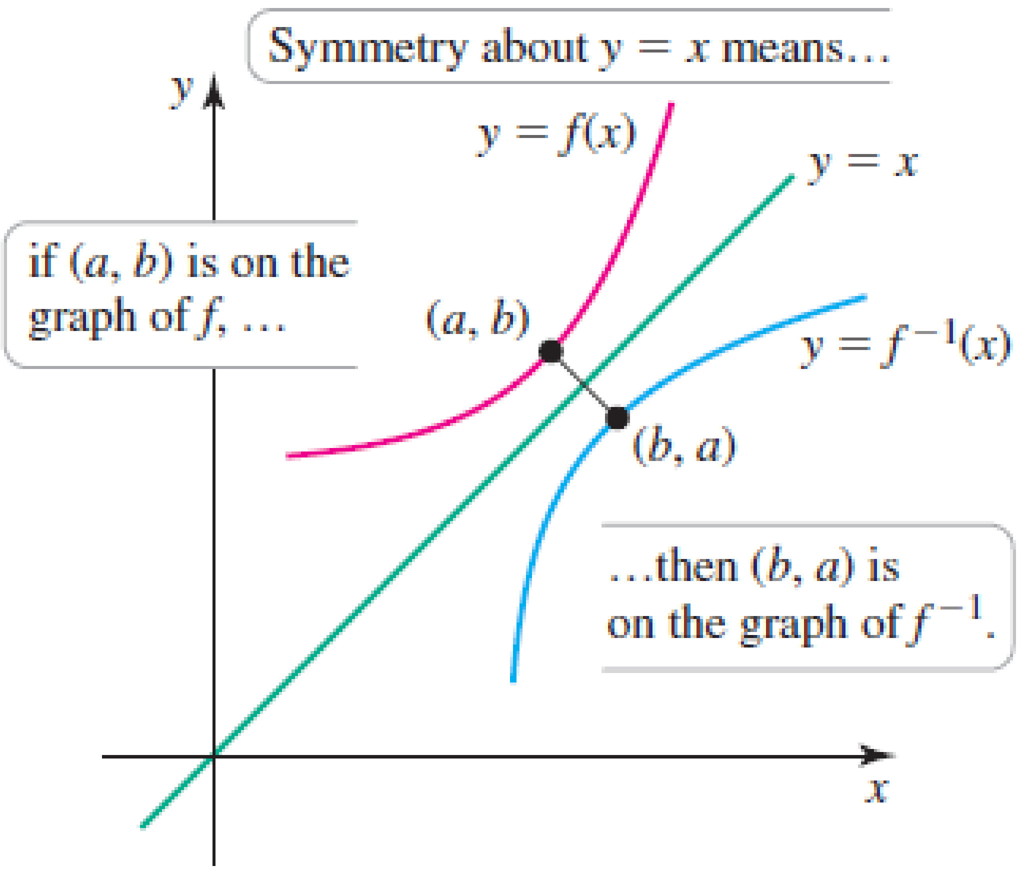




A symmetry always occurs when a function and its inverse are plotted on the same set of axes.

One curve is the reflection of the other curve across $y = x$.

The curves have ***symmetry about the line $y = x$***.



Logarithmic Functions

DEFINITION Logarithmic Function Base b

For any base $b > 0$, with $b \neq 1$, the **logarithmic function base b**, denoted $y = \log_b x$, is the inverse of the exponential function $y = b^x$. The inverse of the natural exponential function with base $b = e$ is the **natural logarithm function**, denoted $y = \ln x$.

$$y = \log_b x \quad \text{if and only if} \quad b^y = x.$$

Inverse Relations for Exponential and Logarithmic Functions

For any base $b > 0$, with $b \neq 1$, the following inverse relations hold.

I1. $b^{\log_b x} = x$, for $x > 0$

I2. $\log_b b^x = x$, for real values of x

Properties of Logarithmic Functions

Logarithm Rules

For any base $b > 0$ $(b \neq 1)$, positive real numbers x and y, and real numbers z, the following relations hold:

L1. $\log_b xy = \log_b x + \log_b y$

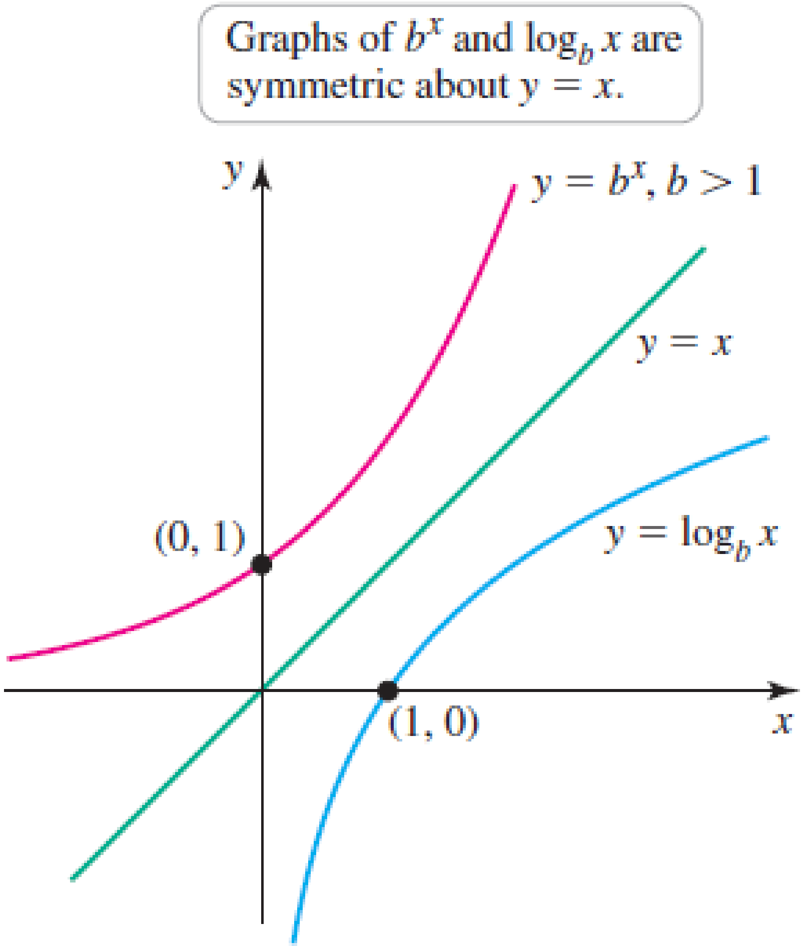
L2. $\log_b \dfrac{x}{y} = \log_b x - \log_b y$

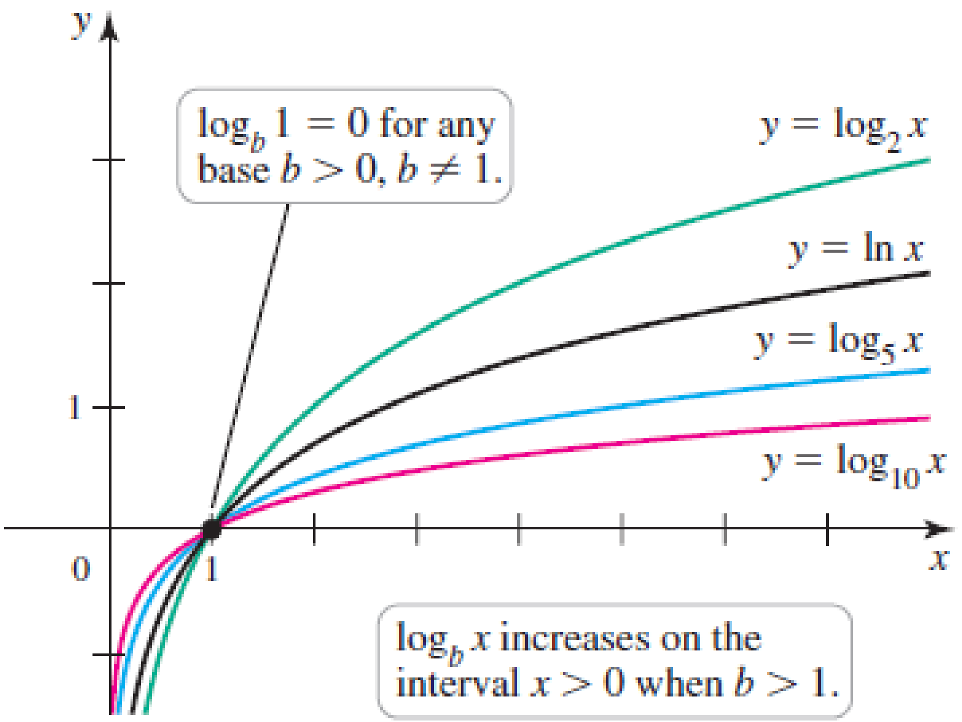
$$\left(\text{includes } \log_b \frac{1}{y} = -\log_b y\right)$$

L3. $\log_b x^z = z \log_b x$

L4. $\log_b b = 1$

Graph of logarithmic function using the symmetry of the graphs of a function and its inverse





Logarithmic functions with base $b>0$ satisfy properties that parallel the properties of the exponential functions

1. Because the range of b^x is $(0, \infty)$ the domain of $\log_b x$ is $(0, \infty)$.
2. The domain of b^x is $(-\infty, \infty)$, which implies that the range of $\log_b x$ is $(-\infty, \infty)$.
3. Because $b^0 = 1$, it follows that $\log_b 1 = 0$.
4. If $b > 1$, then $\log_b x$ is an increasing function of x. For example, if $b = e$, then $\ln x > \ln y$ whenever $x > y$ (Figure 1.59).

EXAMPLE 5 **Using inverse relations** One thousand grams of a particular radioactive substance decays according to the function $m(t) = 1000e^{-t/850}$, where $t \geq 0$ measures time in years. When does the mass of the substance reach the safe level deemed to be 1 g?

Change of Base

It doesn't matter in principle, just for practical reasons.

Change-of-Base Rules

Let b be a positive real number with $b \neq 1$. Then

$$b^x = e^{x \ln b}, \text{ for all } x \quad \text{and} \quad \log_b x = \frac{\ln x}{\ln b}, \text{ for } x > 0.$$

More generally, if c is a positive real number with $c \neq 1$, then

$$b^x = c^{x \log_c b}, \text{ for all } x \quad \text{and} \quad \log_b x = \frac{\log_c x}{\log_c b}, \text{ for } x > 0.$$

EXAMPLE 6 **Changing bases**

a. Express 2^{x+4} as an exponential function with base e.

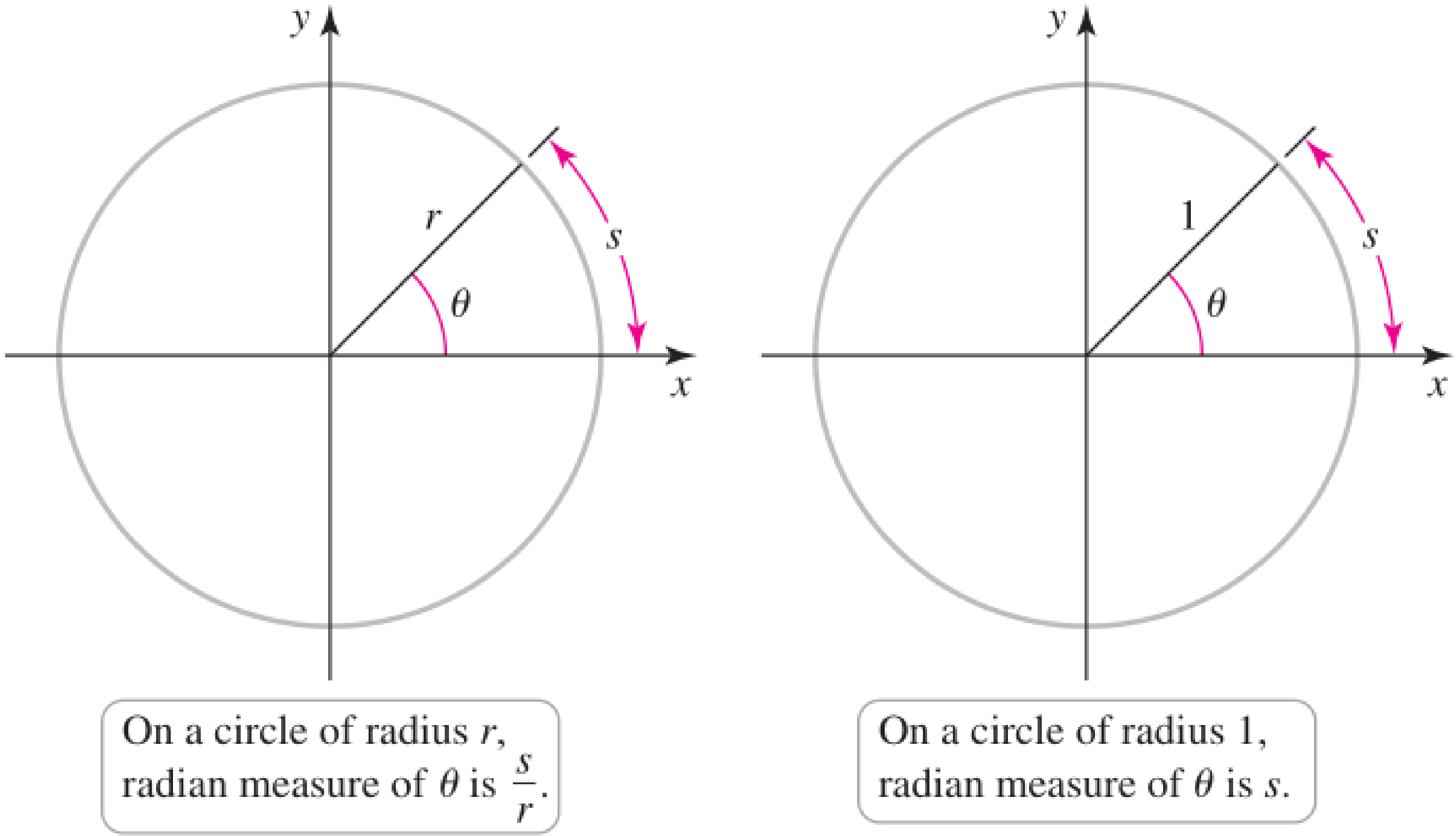
b. Express $\log_2 x$ using base e and base 32.

1.4

Trigonometric Functions and Their Inverses

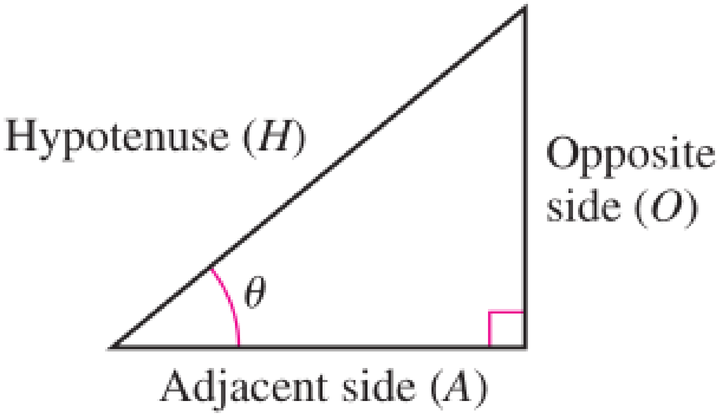
Radian Measure

It doesn’t matter in principle, but useful in practice.



(a) (b)

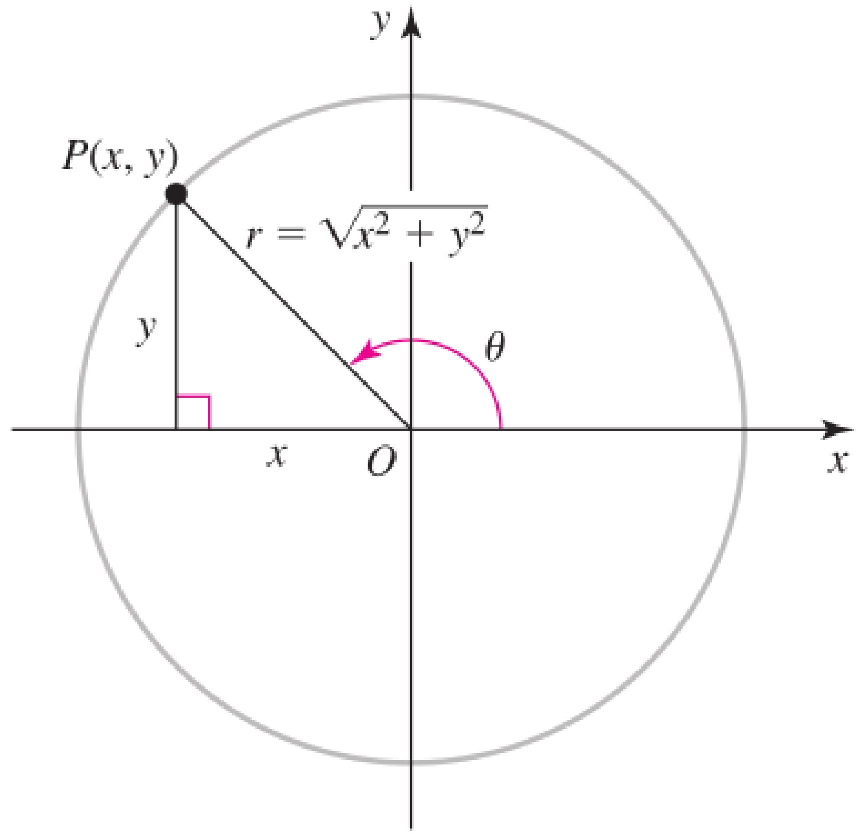
Trigonometric Functions



$$\sin\theta = \frac{O}{H} \qquad \cos\theta = \frac{A}{H}$$

$$\tan\theta = \frac{O}{A} \qquad \cot\theta = \frac{A}{O}$$

$$\sec\theta = \frac{H}{A} \qquad \csc\theta = \frac{H}{O}$$



A positive angle θ results from a counterclockwise rotation.

DEFINITION Trigonometric Functions

Let $P(x, y)$ be a point on a circle of radius r associated with the angle θ. Then

$$\sin\theta = \frac{y}{r}, \qquad \cos\theta = \frac{x}{r}, \qquad \tan\theta = \frac{y}{x},$$

$$\cot\theta = \frac{x}{y}, \qquad \sec\theta = \frac{r}{x}, \qquad \csc\theta = \frac{r}{y}.$$

Radian measure of standard angles, $\pi = 180^{\circ}$

EXAMPLE 1 **Evaluating trigonometric functions** Evaluate the following expressions.

a. $\sin(8\pi/3)$ **b.** $\csc(-11\pi/3)$

Trigonometric Identities (Properties)

Trigonometric Identities

Reciprocal Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$$

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta}$$

Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1 \qquad 1 + \cot^2\theta = \csc^2\theta \qquad \tan^2\theta + 1 = \sec^2\theta$$

Double- and Half-Angle Formulas

$$\sin 2\theta = 2\sin\theta\cos\theta \qquad \cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2} \qquad \sin^2\theta = \frac{1 - \cos 2\theta}{2}$$

EXAMPLE 2 **Solving trigonometric equations** Solve the following equations.

a. $\sqrt{2}\sin x + 1 = 0$ **b.** $\cos 2x = \sin 2x$, where $0 \le x < 2\pi$

Graphs of Trigonometric Functions

Periodic functions: Their values repeat over every interval of some fixed length.

Period of Trigonometric Functions

The functions $\sin\theta$, $\cos\theta$, $\sec\theta$, and $\csc\theta$ have a period of 2π:

$$\sin(\theta + 2\pi) = \sin\theta \qquad \cos(\theta + 2\pi) = \cos\theta$$
$$\sec(\theta + 2\pi) = \sec\theta \qquad \csc(\theta + 2\pi) = \csc\theta,$$

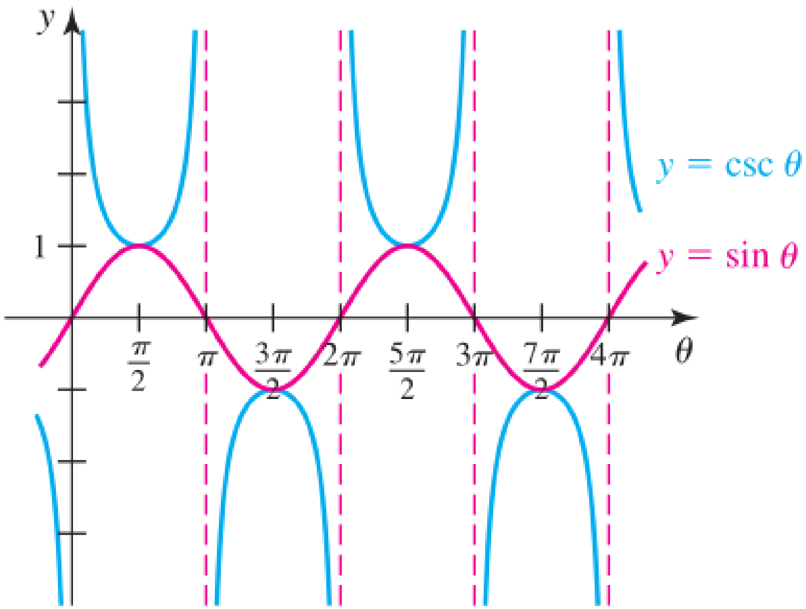
for all θ in the domain.

The functions $\tan\theta$ and $\cot\theta$ have a period of π:

$$\tan(\theta + \pi) = \tan\theta \qquad \cot(\theta + \pi) = \cot\theta,$$

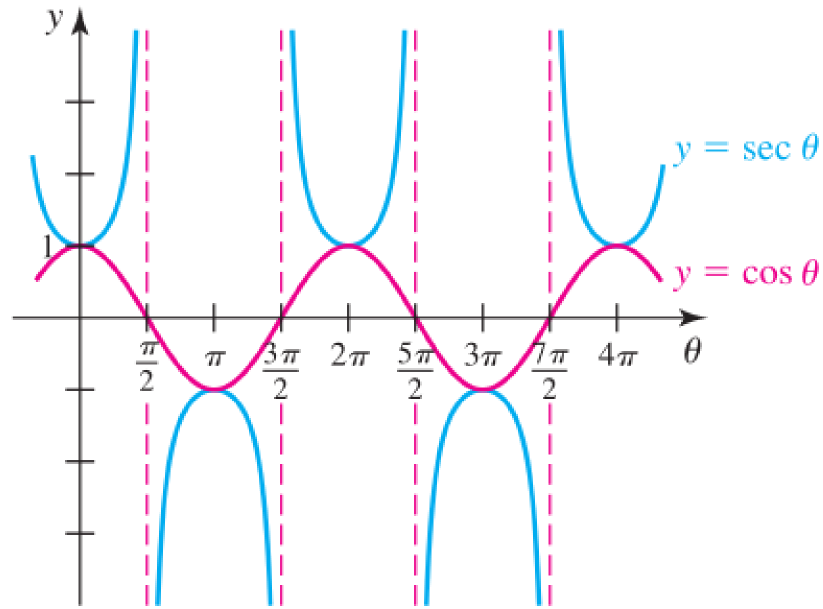
for all θ in the domain.

The graphs of $y = \sin\theta$ and its reciprocal, $y = \csc\theta$



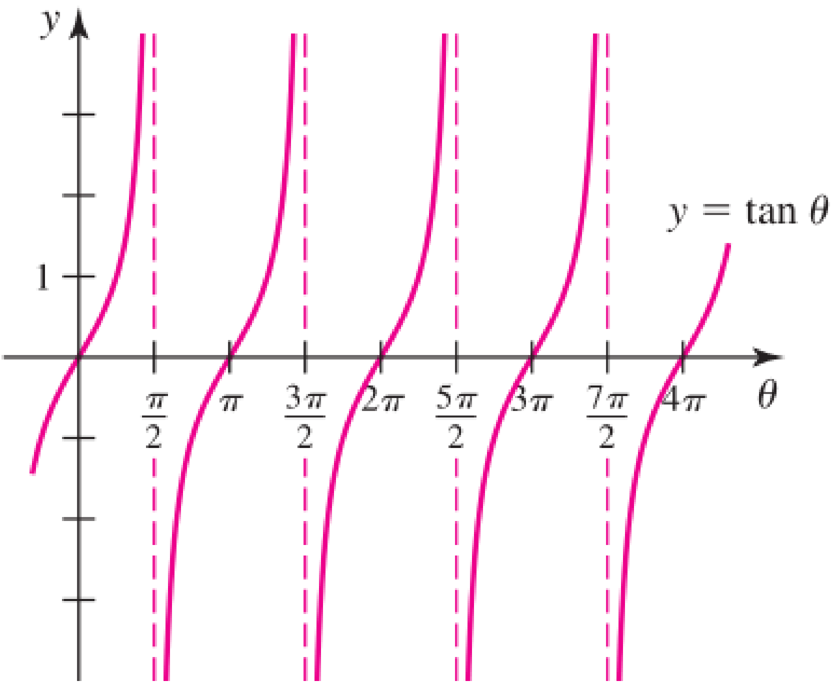
(a)

The graphs of $y = \cos\theta$ and its reciprocal, $y = \sec\theta$



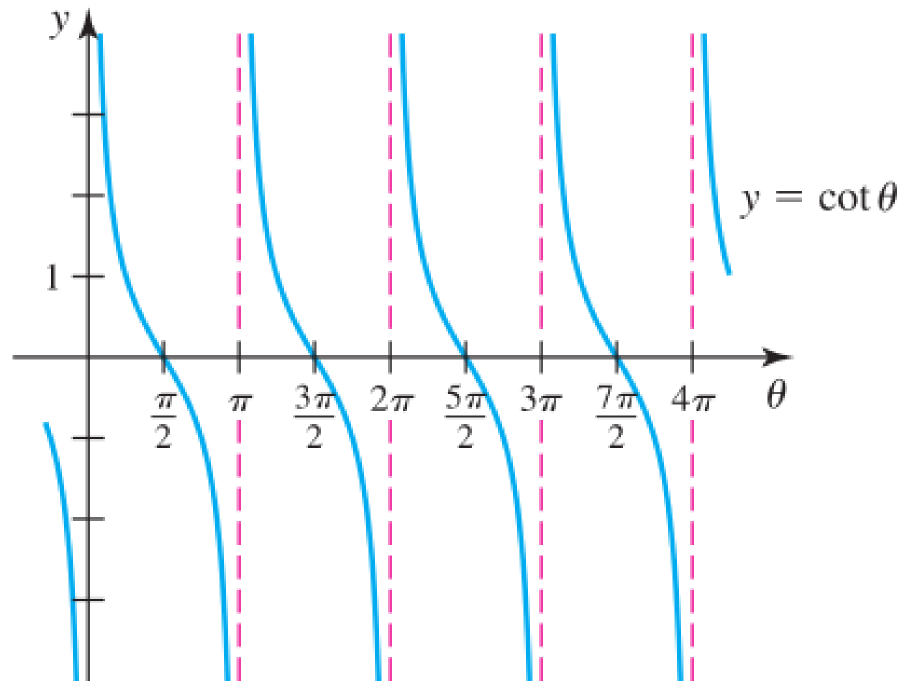
(b)

The graph of $y = \tan\theta$ has period π.



(a)

The graph of $y = \cot\theta$ has period π.



(b)

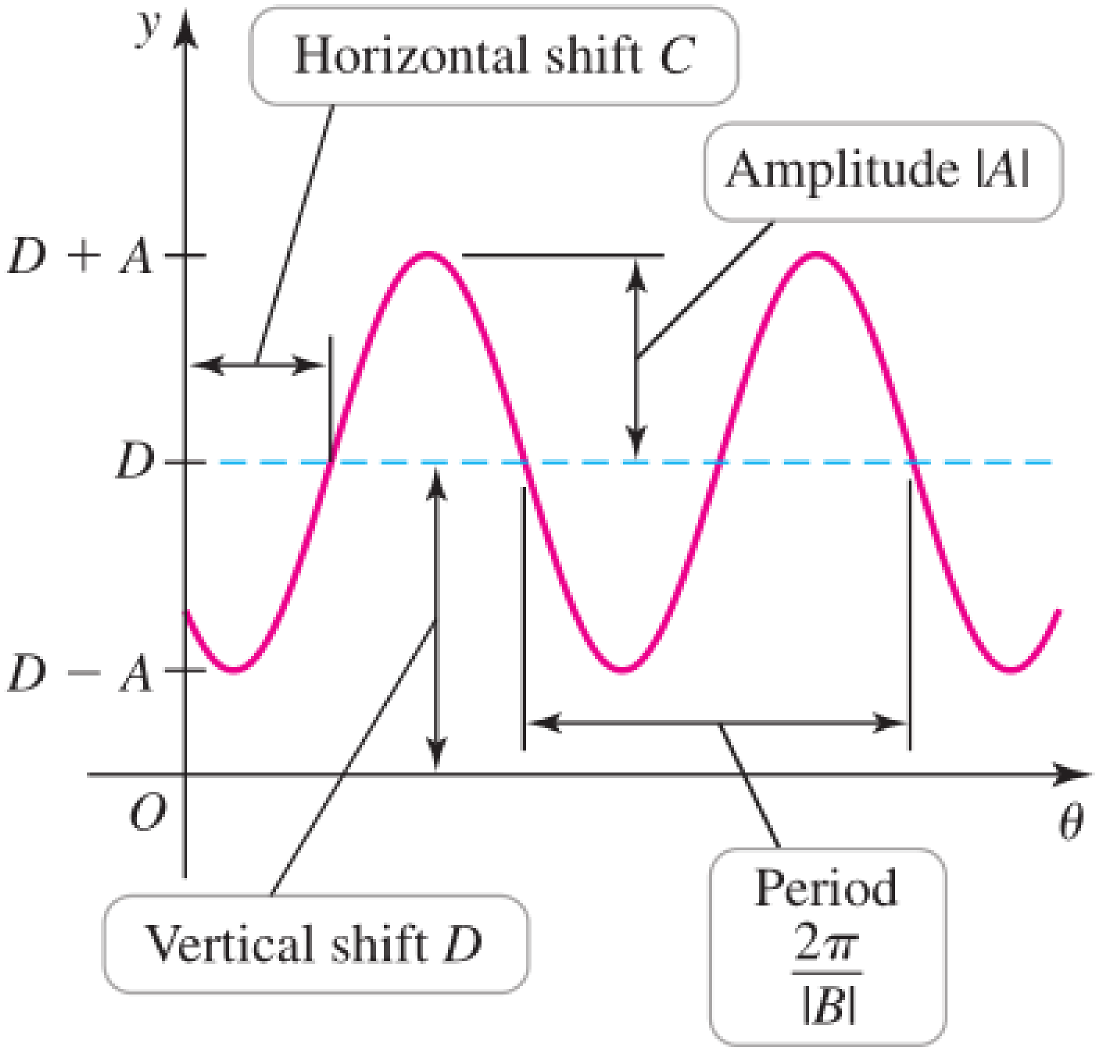
Transforming Graphs

Many physical phenomena can be modeled using trigonometric functions

$$y = A\sin(B(\theta - C)) + D \quad \text{and} \quad y = A\cos(B(\theta - C)) + D,$$

when compared to the graphs of $y = \sin\theta$ and $y = \cos\theta$, have a vertical stretch (or **amplitude**) of $|A|$, a period of $2\pi/|B|$, a horizontal shift (or **phase shift**) of C, and a **vertical shift** of D (Figure 1.68).

$$y = A\sin(B(\theta - C)) + D$$



Inverse Trigonometric Functions

Inverse questions: Given a number y, what is the angle x such that $\sin x = y$ or $\cos x = y$?

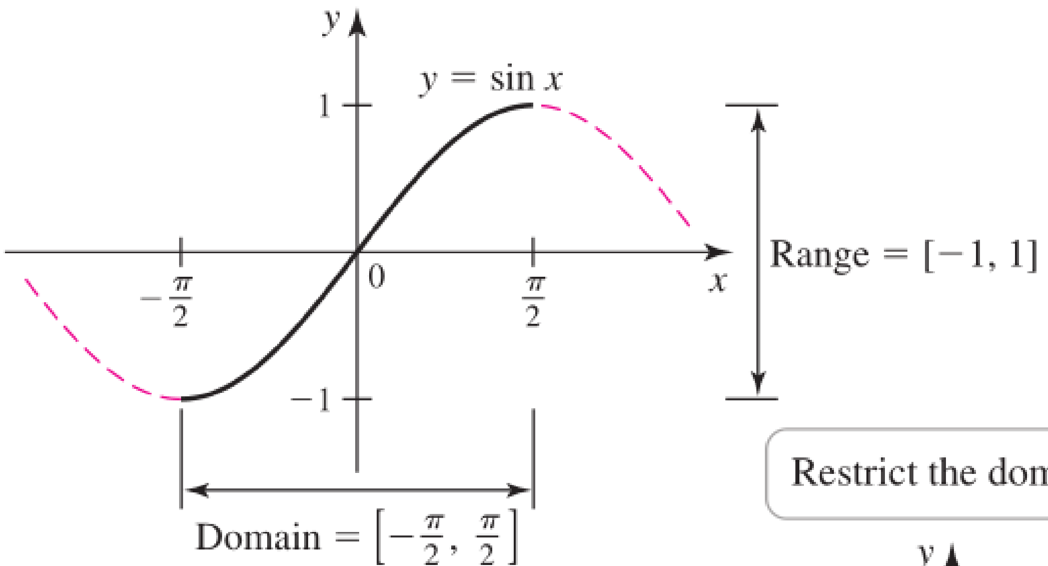
These inverse questions do not have unique answers because $\sin x$ and $\cos x$ are not one-to-one on their domains.

To define their inverses, these functions are restricted to the intervals on which they are one-to-one.

For sine function, the standard choice is $[-\pi/2, \pi/2]$; for cosine, it is $[0, \pi]$.

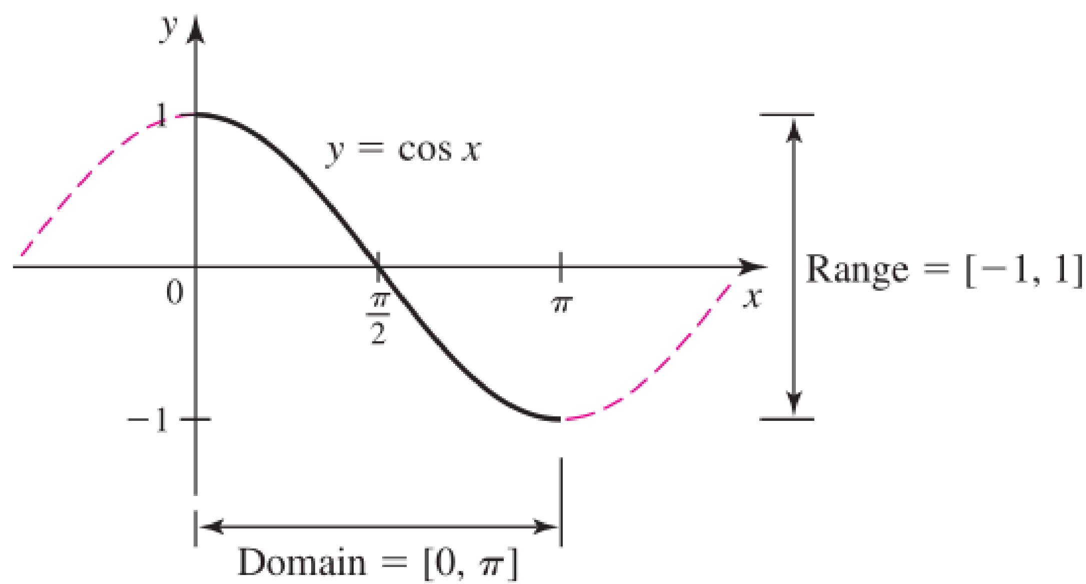
Inverse sine, $y = \arcsin x$; **inverse cosine**, $y = \arccos x$.

Restrict the domain of $y = \sin x$ to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.



(a)

Restrict the domain of $y = \cos x$ to $[0, \pi]$.



(b)

DEFINITION Inverse Sine and Cosine

$y = \sin^{-1} x$ is the value of y such that $x = \sin y$, where $-\pi/2 \le y \le \pi/2$.
$y = \cos^{-1} x$ is the value of y such that $x = \cos y$, where $0 \le y \le \pi$.

The domain of both $\sin^{-1} x$ and $\cos^{-1} x$ is $\{x: -1 \le x \le 1\}$.

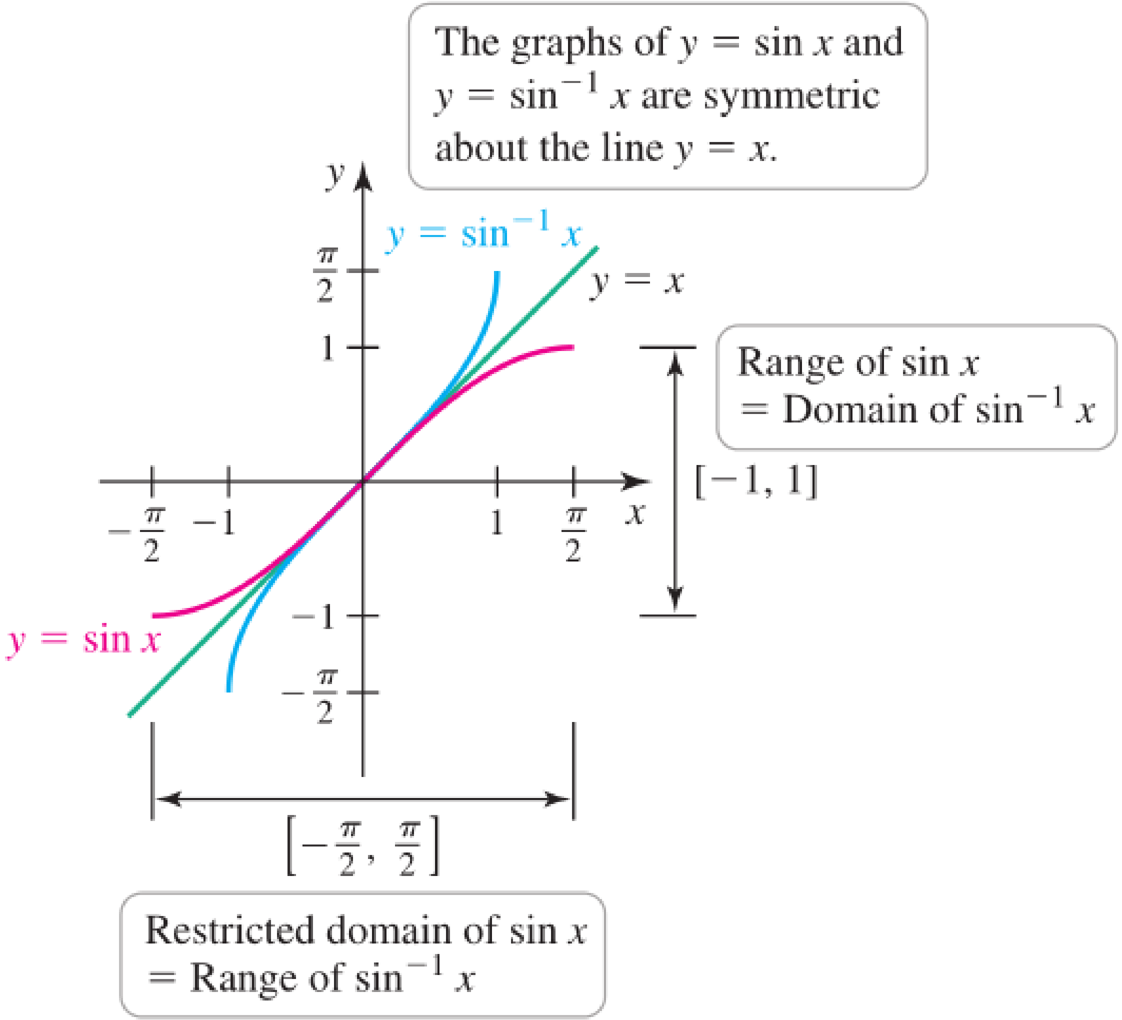
- $\sin(\sin^{-1} x) = x$ and $\cos(\cos^{-1} x) = x$, for $-1 \le x \le 1$.
- $\sin^{-1}(\sin y) = y$, for $-\pi/2 \le y \le \pi/2$.
- $\cos^{-1}(\cos y) = y$, for $0 \le y \le \pi$.

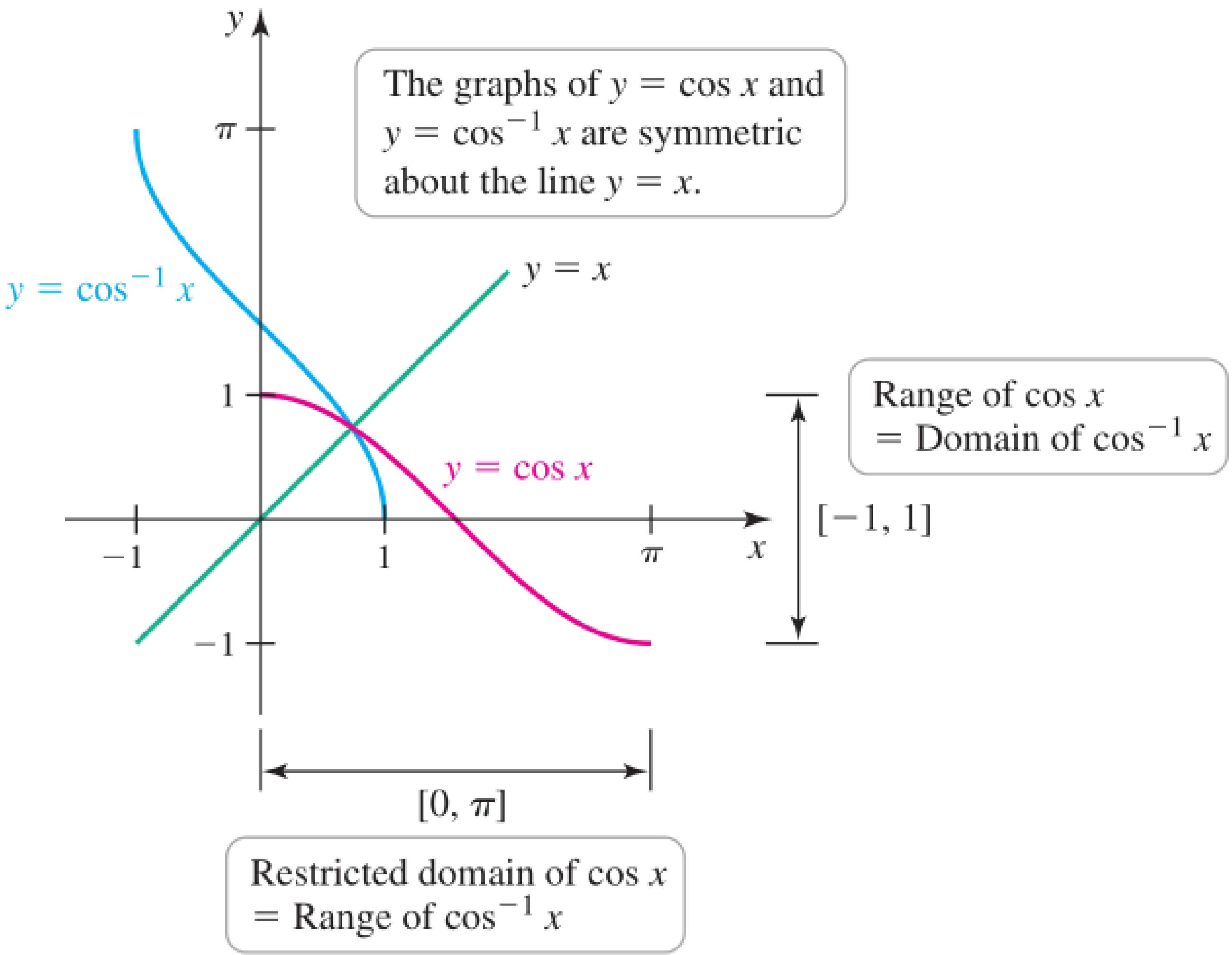
EXAMPLE 3 **Working with inverse sine and cosine** Evaluate the following expressions.

a. $\sin^{-1}(\sqrt{3}/2)$ **b.** $\cos^{-1}(-\sqrt{3}/2)$ **c.** $\cos^{-1}(\cos 3\pi)$ **d.** $\sin^{-1}\left(\sin \frac{3\pi}{4}\right)$

Graphs and Properties

Reflecting about the line $y = x$.



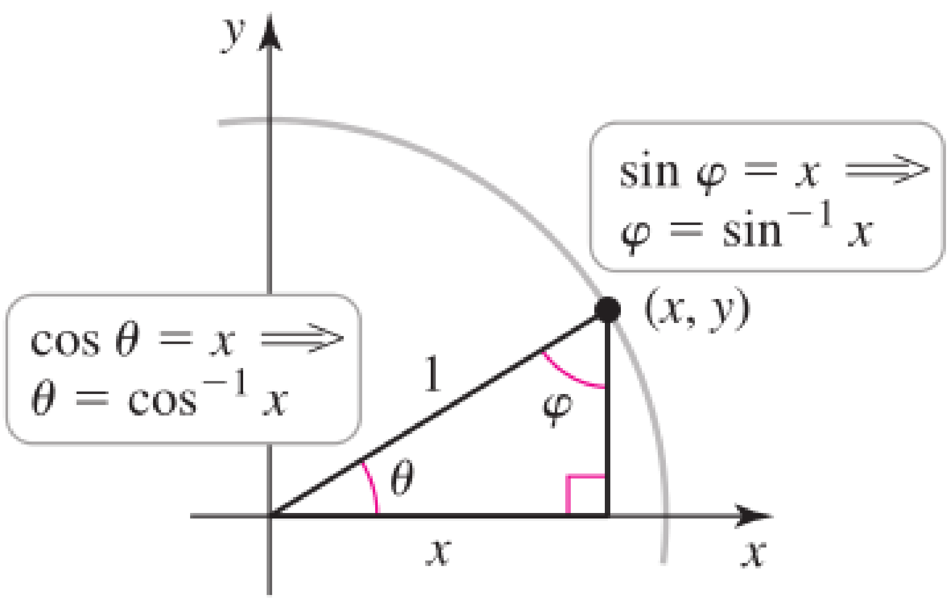

y
The graphs of y = cos x and y = cos⁻¹ x are symmetric about the line y = x.
π
y = cos⁻¹ x
y = x
1
Range of cos x = Domain of cos⁻¹ x
y = cos x
−1
1
π
x
[−1, 1]
−1
[0, π]
Restricted domain of cos x = Range of cos⁻¹ x

EXAMPLE 4 **Right-triangle relationships**

a. Suppose $\theta = \sin^{-1}(2/5)$. Find $\cos\theta$ and $\tan\theta$.

b. Find an alternative form for $\cot(\cos^{-1}(x/4))$ in terms of x.

EXAMPLE 5 **A useful identity** Use right triangles to explain why $\cos^{-1}x + \sin^{-1}x = \pi/2$.



Other Inverse Trigonometric Functions

Restrictions be imposed to ensure that an inverse exists.

DEFINITION Other Inverse Trigonometric Functions

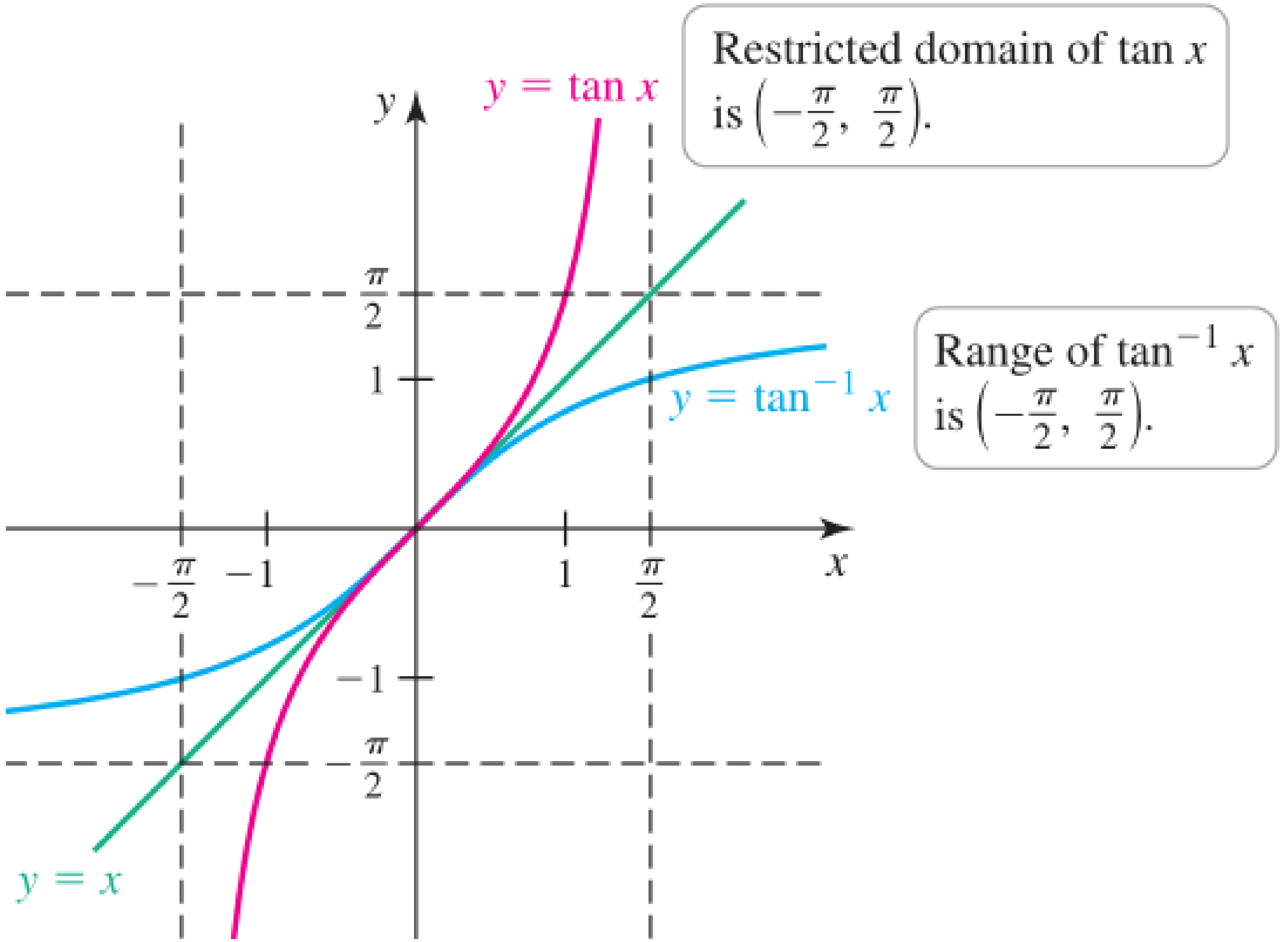
$y = \tan^{-1} x$ is the value of y such that $x = \tan y$, where $-\pi/2 < y < \pi/2$.
$y = \cot^{-1} x$ is the value of y such that $x = \cot y$, where $0 < y < \pi$.

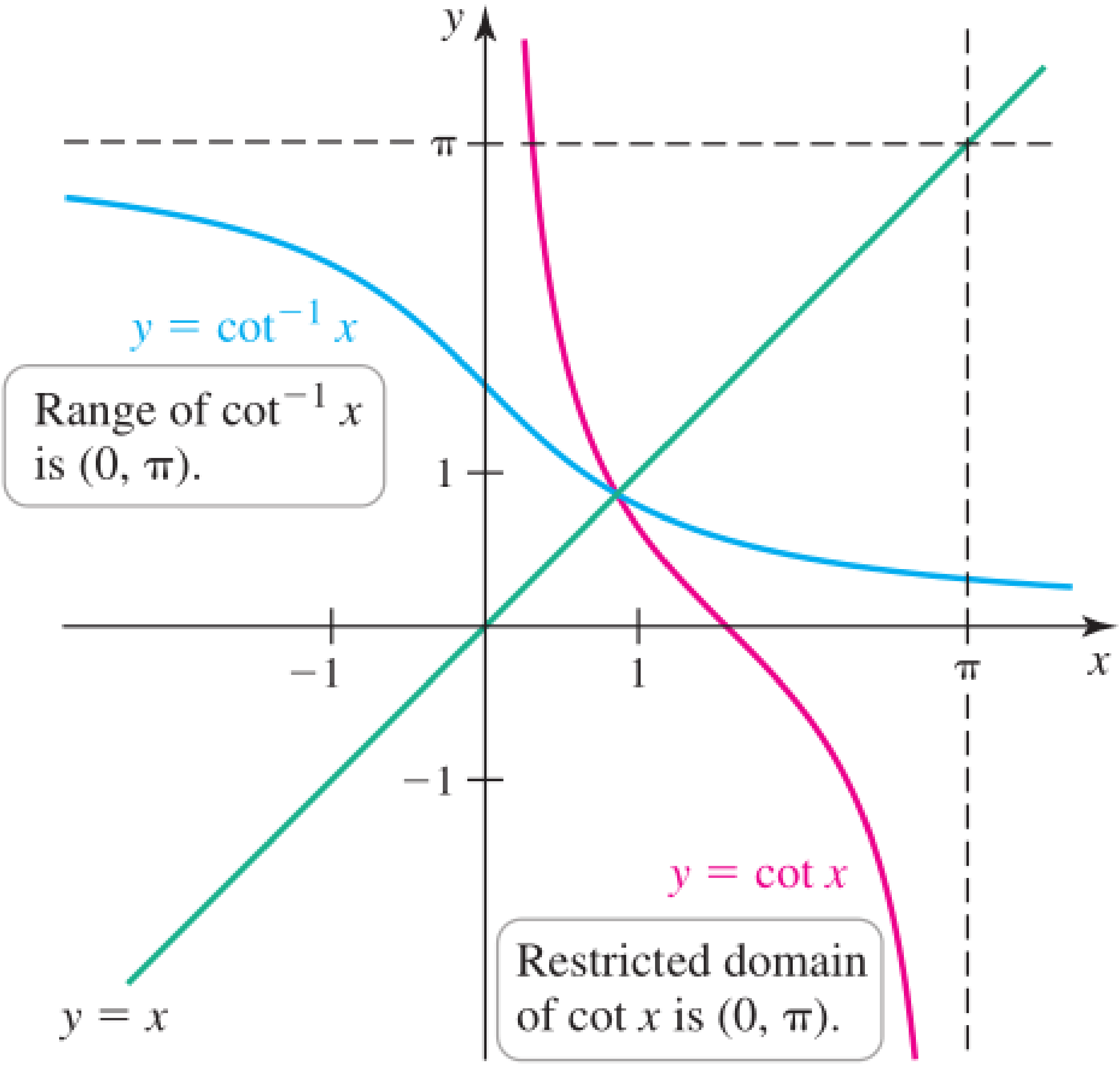
The domain of both $\tan^{-1} x$ and $\cot^{-1} x$ is $\{x: -\infty < x < \infty\}$.

$y = \sec^{-1} x$ is the value of y such that $x = \sec y$, where $0 \le y \le \pi$, with $y \ne \pi/2$.
$y = \csc^{-1} x$ is the value of y such that $x = \csc y$, where $-\pi/2 \le y \le \pi/2$, with $y \ne 0$.

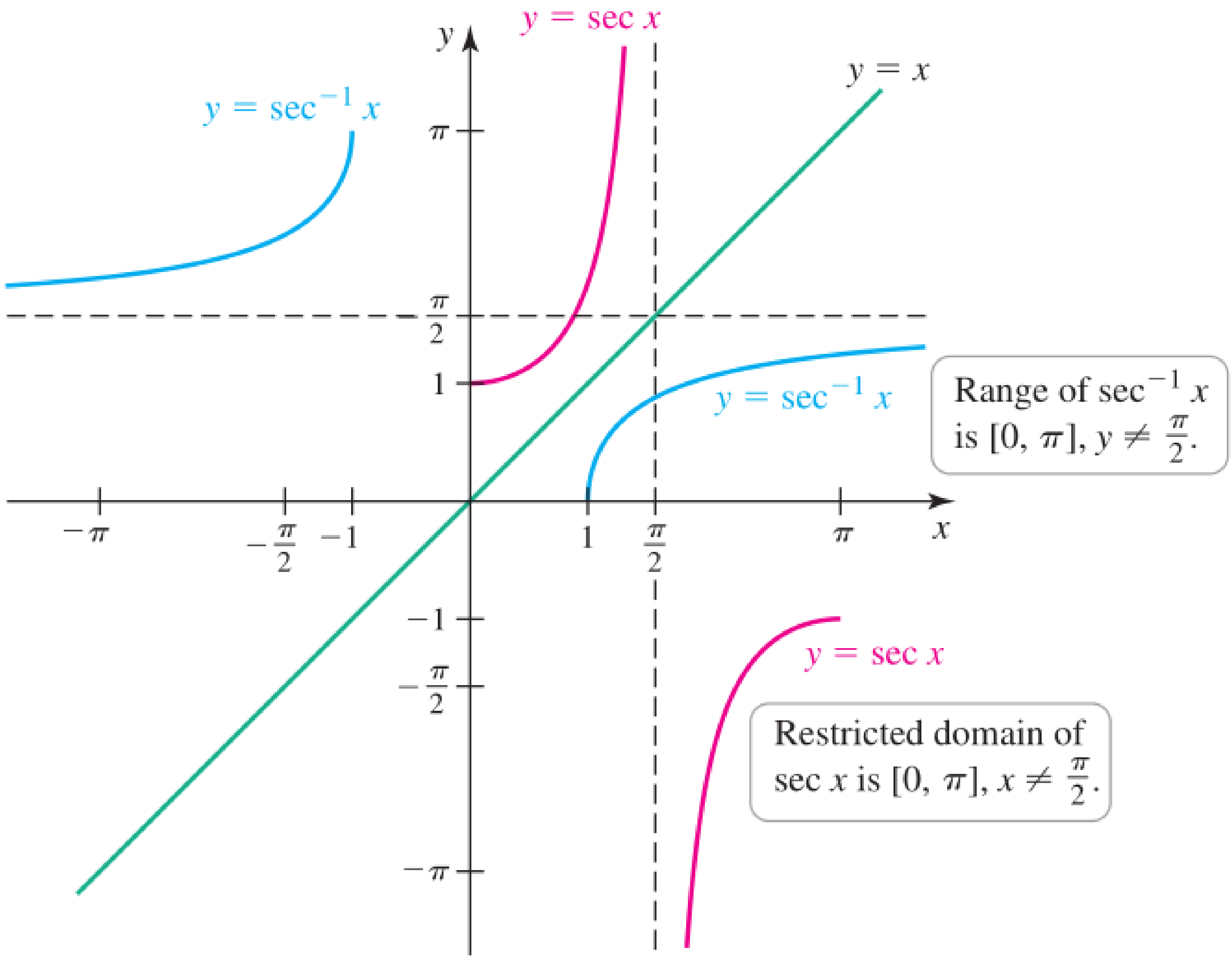
The domain of both $\sec^{-1} x$ and $\csc^{-1} x$ is $\{x: |x| \ge 1\}$.

Graphs are obtained by reflecting about the line $y = x$.

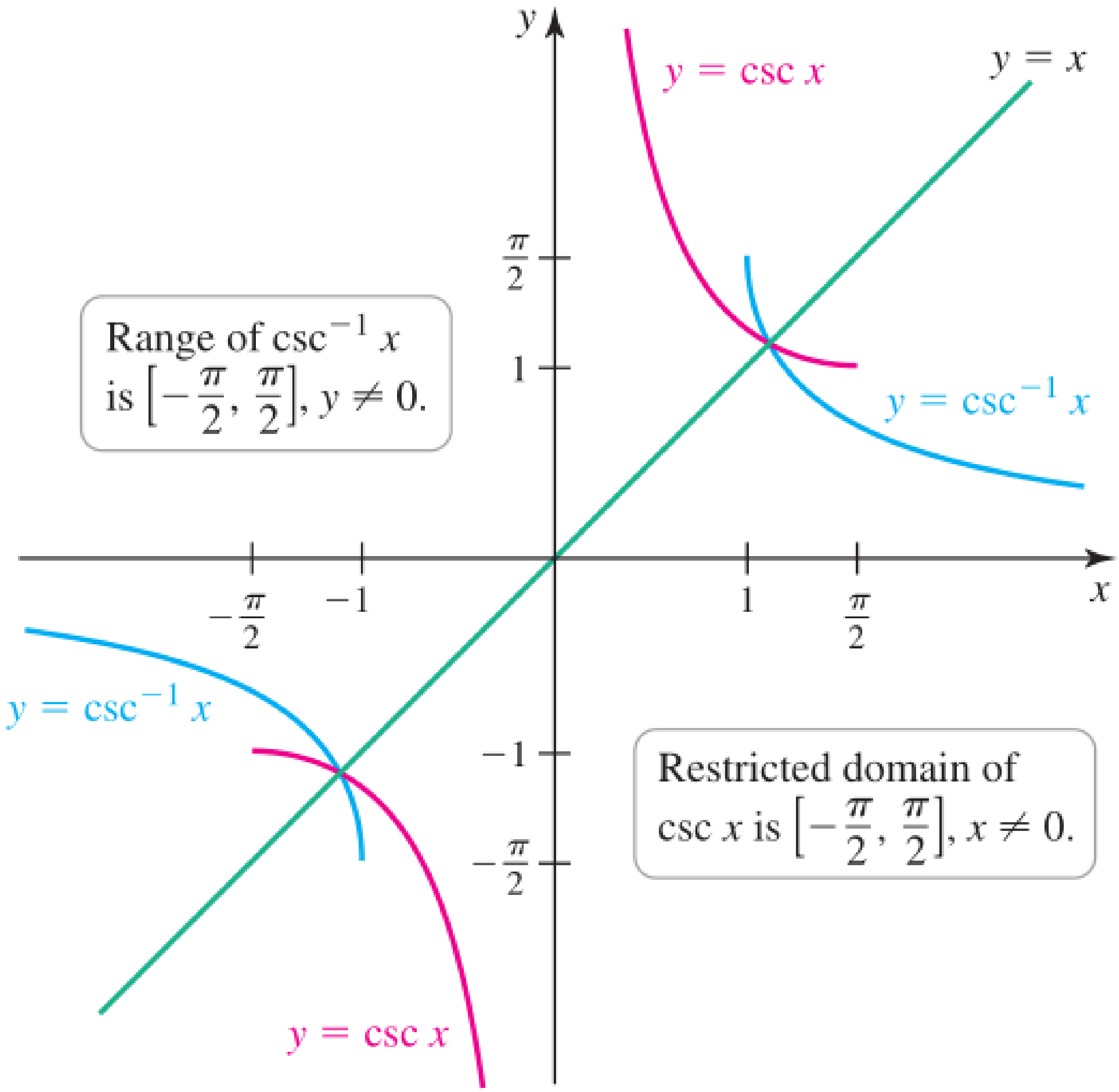




y
π
$y = \cot^{-1} x$
Range of $\cot^{-1} x$ is $(0, \pi)$.
1
−1
1
π
x
−1
$y = \cot x$
$y = x$
Restricted domain of $\cot x$ is $(0, \pi)$.



y = sec x
y = x
y = sec⁻¹ x
π
π/2
1
y = sec⁻¹ x
Range of sec⁻¹ x is [0, π], y ≠ π/2.
−π
−π/2
−1
1
π/2
π
x
y
−1
−π/2
y = sec x
Restricted domain of sec x is [0, π], x ≠ π/2.
−π



y
x
y = csc x
y = x
y = csc⁻¹ x
y = csc⁻¹ x
y = csc x
π/2
1
−1
−π/2
−π/2
−1
1
π/2
Range of csc⁻¹ x is [−π/2, π/2], y ≠ 0.
Restricted domain of csc x is [−π/2, π/2], x ≠ 0.

EXAMPLE 6 **Working with inverse trigonometric functions** Evaluate or simplify the following expressions.

a. $\tan^{-1}(-1/\sqrt{3})$ **b.** $\sec^{-1}(-2)$ **c.** $\sin(\tan^{-1} x)$

Chapter 1

Functions

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