Chapter 3

Derivatives (II)

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3.6

Derivatives as Rates of Change

One-Dimensional Motion

Everything is in a state of change

Start from the motion in one dimension, i.e., move along a line

Position and Velocity Suppose an object moves along a straight line and its location at time t is given by the **position function** $s = f(t)$. All positions are measured relative to a reference point $s = 0$. The **displacement** of the object between $t = a$ and $t = a + \Delta t$ is $\Delta s = f(a + \Delta t) - f(a)$, where the elapsed time is Δt units (Figure 3.33).

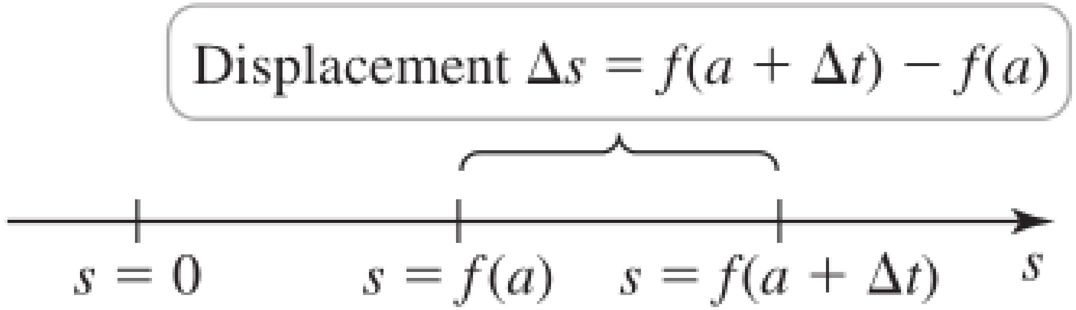
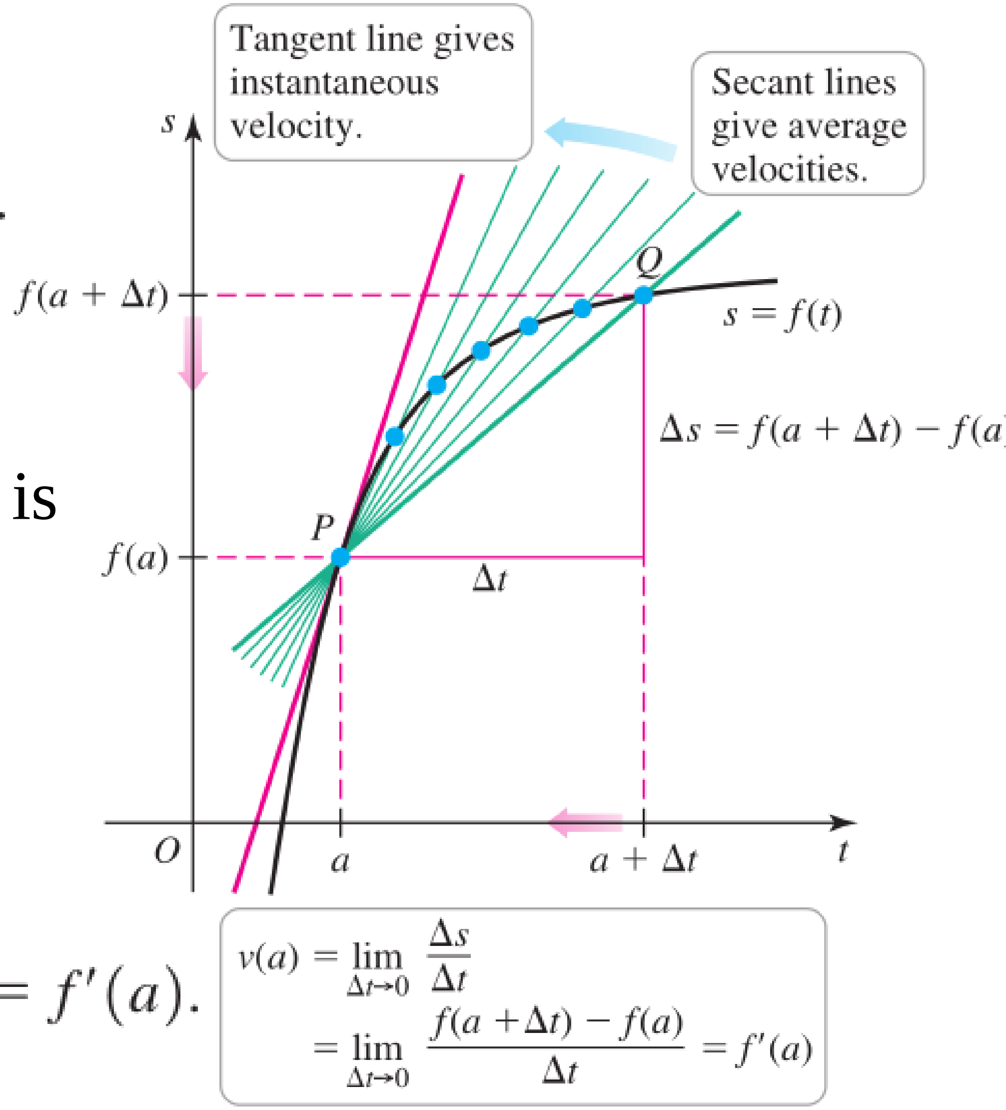


Figure 3.33

Average velocity over the interval $[a, a + \Delta t]$

$$v_{av} = \frac{\Delta s}{\Delta t} = \frac{f(a + \Delta t) - f(a)}{\Delta t}.$$



Let $\Delta t \to 0$, the limiting value is

Instantanous velocity at a

$$v(a) = \lim_{\Delta t \to 0} \frac{f(a + \Delta t) - f(a)}{\Delta t} = f'(a).$$

DEFINITION Average and Instantaneous Velocity

Let $s = f(t)$ be the position function of an object moving along a line. The **average velocity** of the object over the time interval $[a, a + \Delta t]$ is the slope of the secant line between $(a, f(a))$ and $(a + \Delta t, f(a + \Delta t))$:

$$v_{\text{av}} = \frac{f(a + \Delta t) - f(a)}{\Delta t}.$$

The **instantaneous velocity** at a is the slope of the line tangent to the position curve, which is the derivative of the position function:

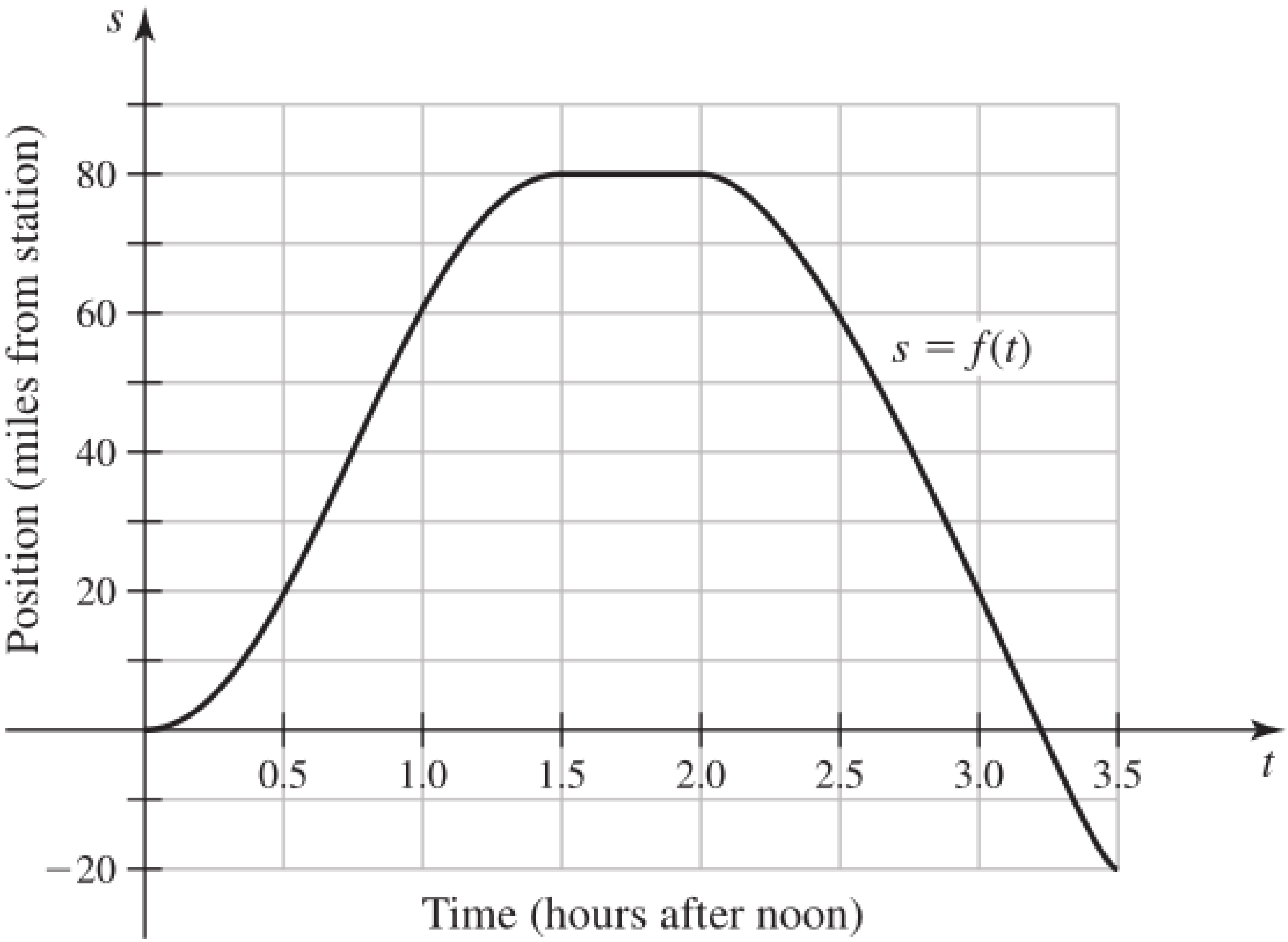
$$v(a) = \lim_{\Delta t \to 0} \frac{f(a + \Delta t) - f(a)}{\Delta t} = f'(a).$$

EXAMPLE 1 **Position and velocity of a patrol car** Assume a police station is located along a straight east-west freeway. At noon ($t = 0$), a patrol car leaves the station heading east. The position function of the car $s = f(t)$ gives the location of the car in miles east ($s > 0$) or west ($s < 0$) of the station t hours after noon (Figure 3.35).

a. Describe the location of the patrol car during the first 3.5 hr of the trip.

b. Calculate the displacement and average velocity of the car between 2:00 P.M. and 3:30 P.M. ($2 \le t \le 3.5$).

c. At what time(s) is the instantaneous velocity greatest *as the car travels east*?



s
80
60
40
20
−20
Position (miles from station)
0.5
1.0
1.5
2.0
2.5
3.0
3.5
t
s = f(t)
Time (hours after noon)

Speed and Acceleration

Only the magnitude of the velocity, we use ***speed***, $speed = |v|$

Acceleration: the rate of change of the velocity

DEFINITION Velocity, Speed, and Acceleration

Suppose an object moves along a line with position $s = f(t)$. Then

the **velocity** at time t is $\quad v = \dfrac{ds}{dt} = f'(t)$,

the **speed** at time t is $\quad |v| = |f'(t)|$, and

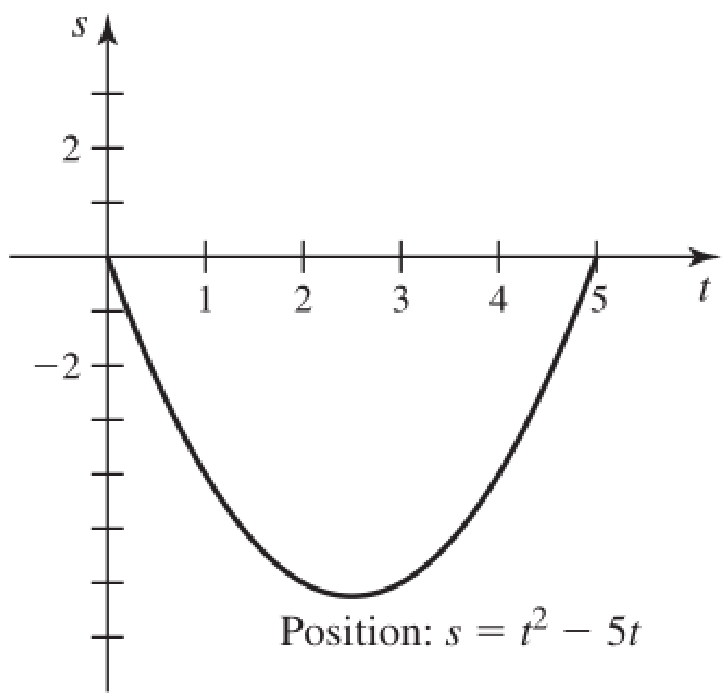
the **acceleration** at time t is $\quad a = \dfrac{dv}{dt} = \dfrac{d^2s}{dt^2} = f''(t)$.

EXAMPLE 2 **Velocity and acceleration** Suppose the position (in feet) of an object moving horizontally at time t (in seconds) is $s = t^2 - 5t$, for $0 \leq t \leq 5$ (Figure 3.37). Assume that positive values of s correspond to positions to the right of $s = 0$.

a. Graph the velocity function on the interval $0 \leq t \leq 5$ and determine when the object is stationary, moving to the left, and moving to the right.

b. Graph the acceleration function on the interval $0 \leq t \leq 5$.

c. Describe the motion of the object.



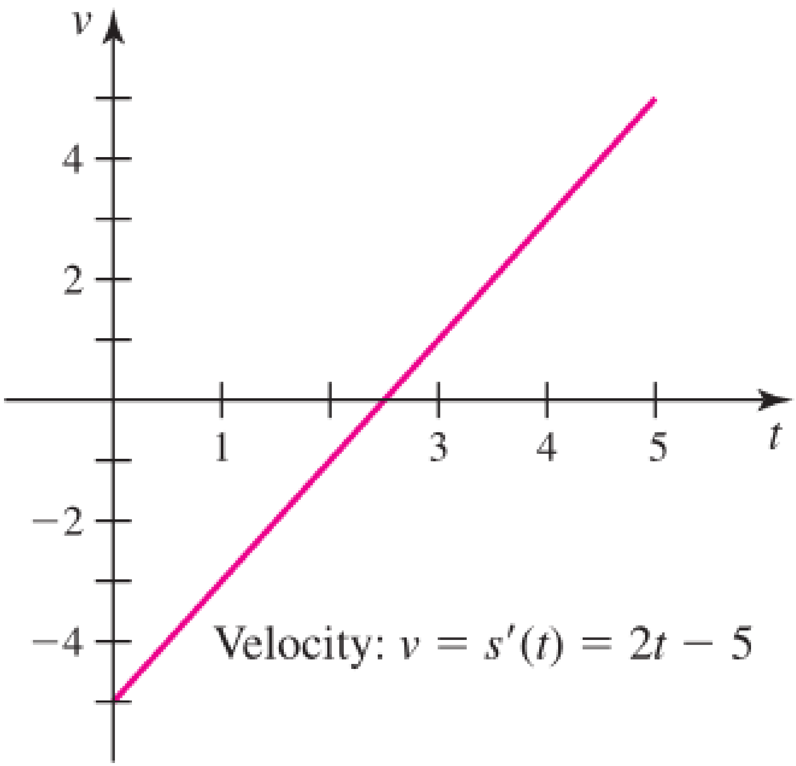


Figure 3.38

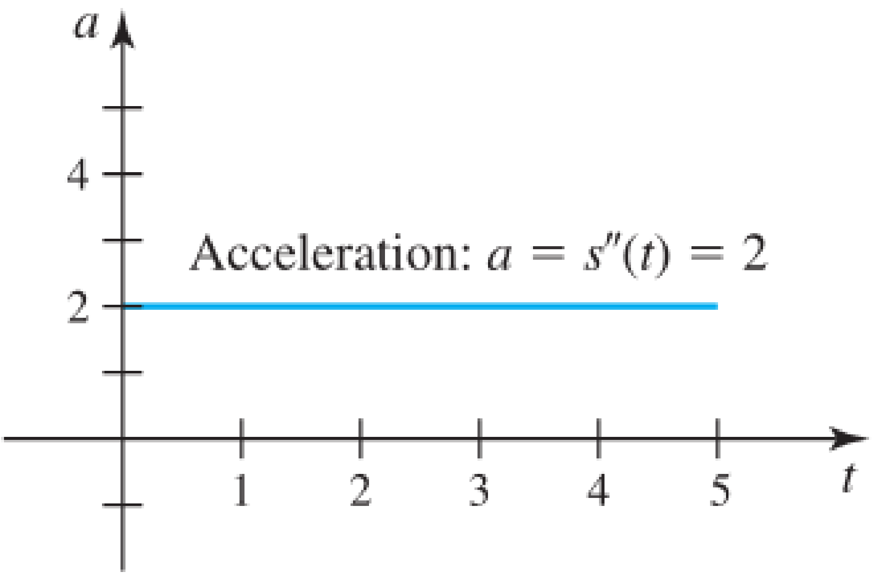


Figure 3.39

Free Fall

An object moves vertically in Earth's gravitational field, assuming that no other forces (e.g., air resistance)

EXAMPLE 3 **Motion in a gravitational field** Suppose a stone is thrown vertically upward with an initial velocity of 64 ft/s from a bridge 96 ft above a river. By Newton's laws of motion, the position of the stone (measured as the height above the river) after t seconds is

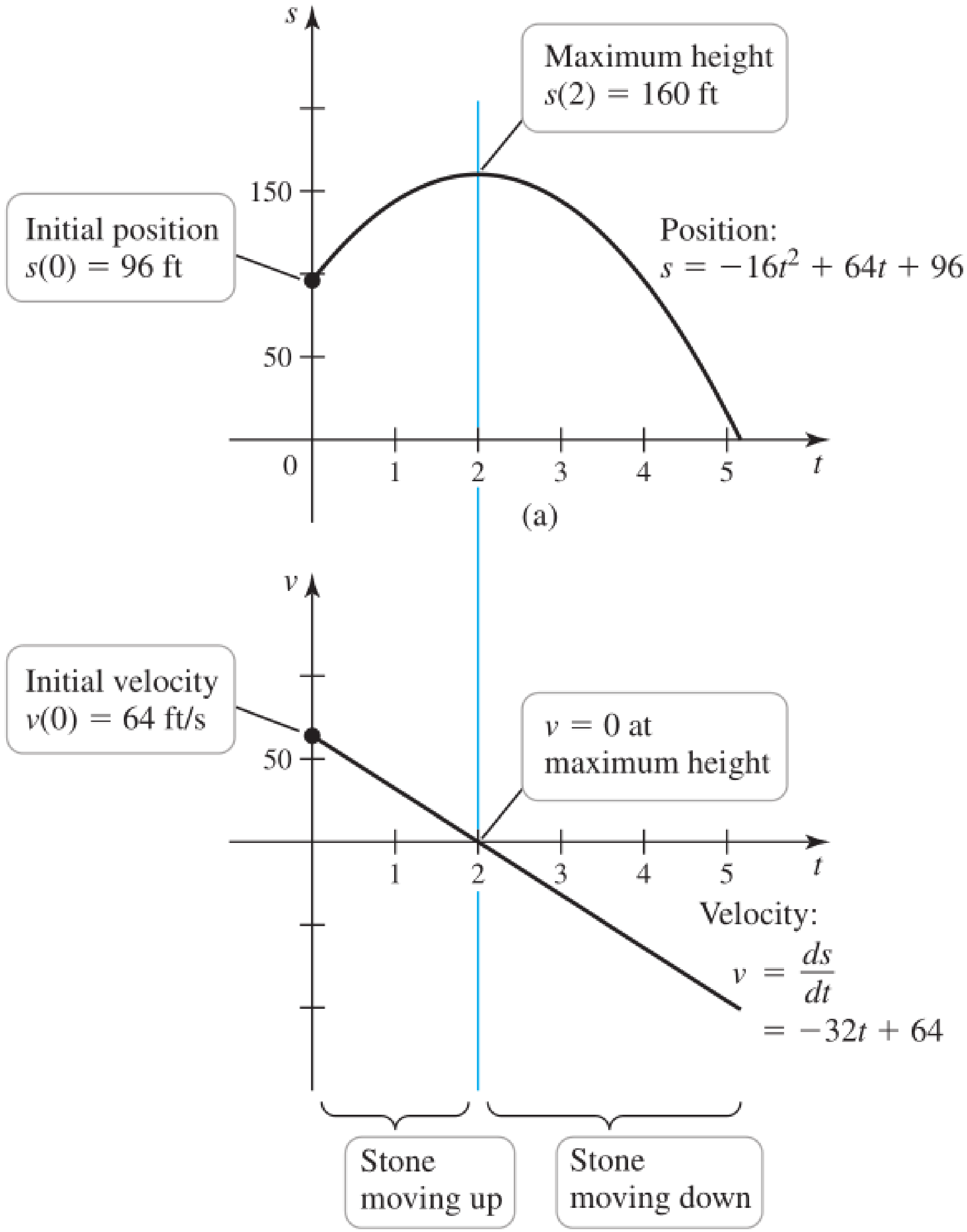
$$s(t) = -16t^2 + 64t + 96,$$

where $s = 0$ is the level of the river (Figure 3.40a).

a. Find the velocity and acceleration functions.

b. What is the highest point above the river reached by the stone?

c. With what velocity will the stone strike the river?


Maximum height
$s(2) = 160$ ft
Initial position
$s(0) = 96$ ft
150
50
0
1
2
3
4
5
s
t
Position:
$s = -16t^2 + 64t + 96$
(a)
Initial velocity
$v(0) = 64$ ft/s
$v = 0$ at
maximum height
50
v
t
Velocity:
$v = \dfrac{ds}{dt}$
$= -32t + 64$
Stone
moving up
Stone
moving down

Growth Models

- Populations, prices, and computer networks tend to increase
- Often leads to an understanding of underlying processes and allows for predictions
- Let $p = f(t)$ be the measure of a quantity of interest
- **Average growth rate** of p between $t = a$ and $t = a + \Delta t$

$$\frac{\Delta p}{\Delta t} = \frac{f(a + \Delta t) - f(a)}{\Delta t}$$

- **Instantaneous growth rate** of p
- Let $\Delta t \to 0$, then $\frac{\Delta p}{\Delta t} \to \frac{dp}{dt}$, the derivative is the growth rate

$$\frac{dp}{dt} = \lim_{\Delta t \to 0} \frac{\Delta p}{\Delta t}$$

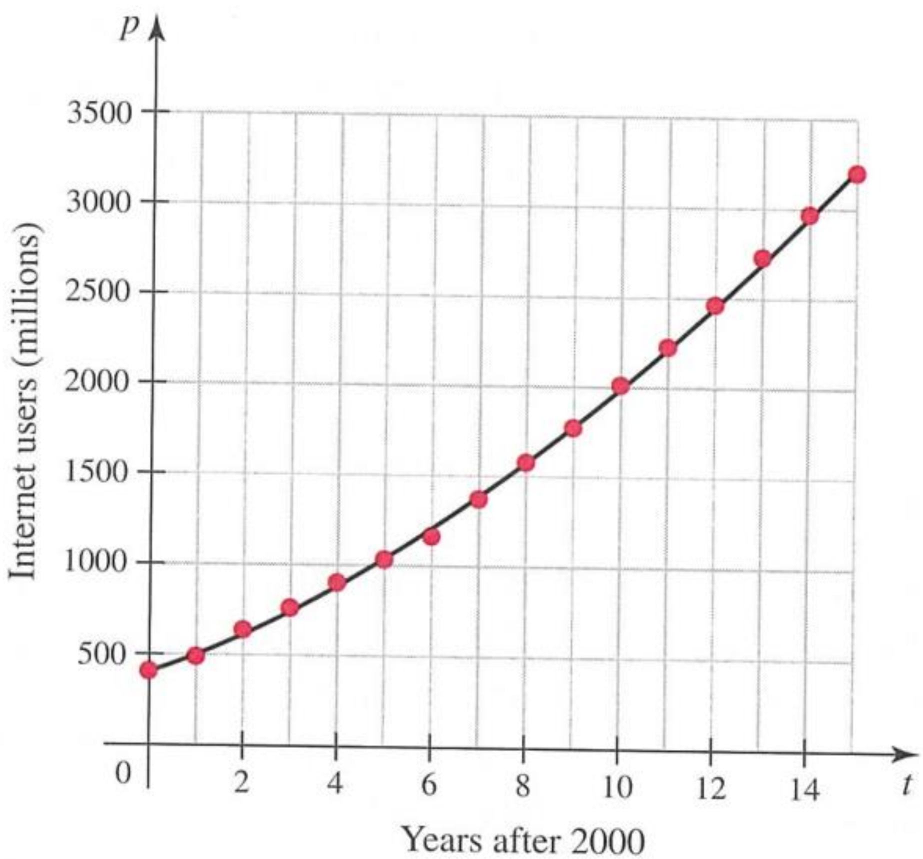
EXAMPLE 4 **Internet growth** The number of worldwide Internet users between 2000 and 2015 is shown in Figure 3.46. A reasonable fit to the data is given by the function $p(t) = 6t^2 + 98t + 431.2$, where t is measured in years after 2000.

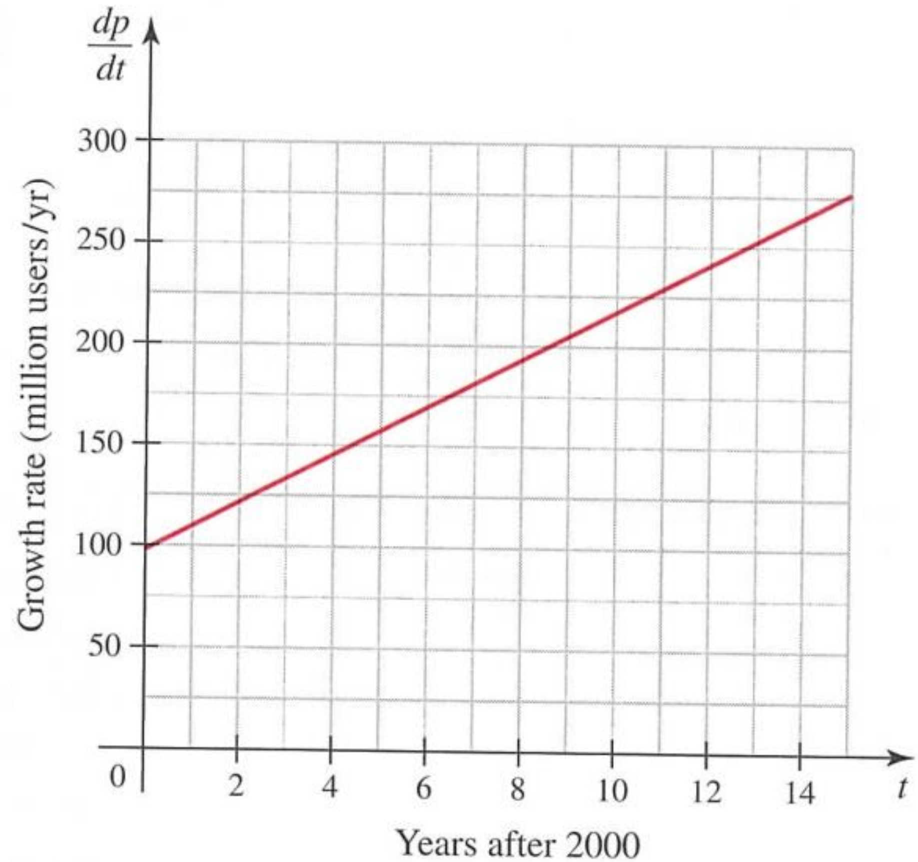
a. Use the function p to approximate the average growth rate of Internet users from 2005 $(t = 5)$ to 2010 $(t = 10)$.

b. What was the instantaneous growth rate of Internet users in 2011?

c. Use a graphing utility to plot the growth rate dp/dt. What does the graph tell you about the growth rate between 2000 and 2015?

d. Assuming the growth function can be extended beyond 2015, what is the predicted number of Internet users in 2020 $(t = 20)$?





The growth rate at time t is $p'(t) = 12t + 98$, whose graph is shown above. In 2011, the growth rate was $p'(11) = 230$ million

Economics and Business

The mathematics of derivatives is the same, while the vocabulary and interpretation are quite different.

Average and marginal cost

A company manufactures large quantities of a product

Cost function $C(x)$ gives the cost of manufacturing the first x items of the product.

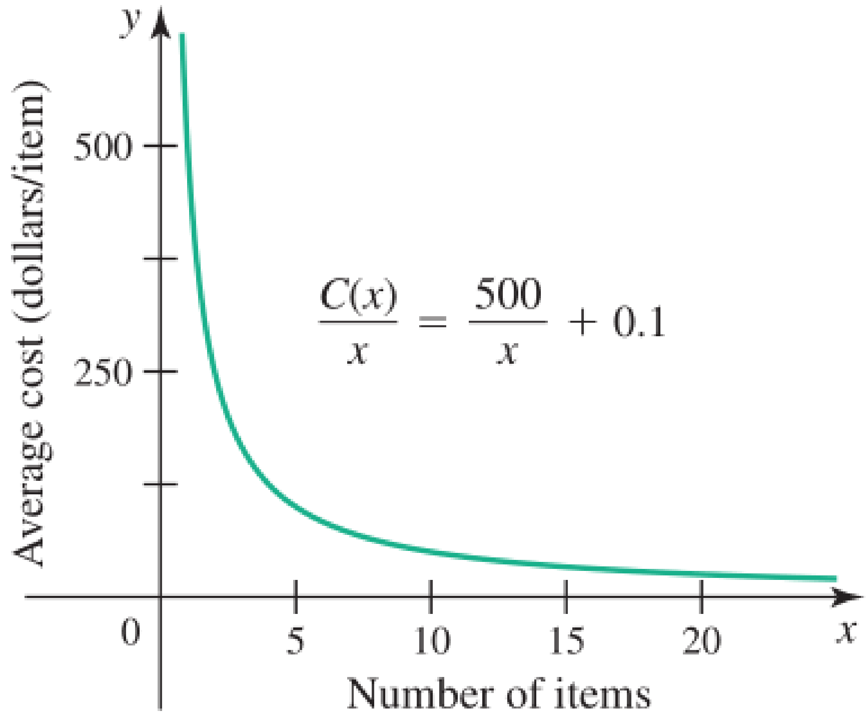
A simple cost function might be $y = C(x) = 500 + 0.1x$.

It includes a *fixed cost* of \$500 that is independent of the number of items produced, and a *unit cost* or *variable cost*, of \$0.10 per item produced.

The *average cost* is $\frac{C(x)}{x}$ per item.

For the cost function $C(x) = 500 + 0.1x$, the average cost is

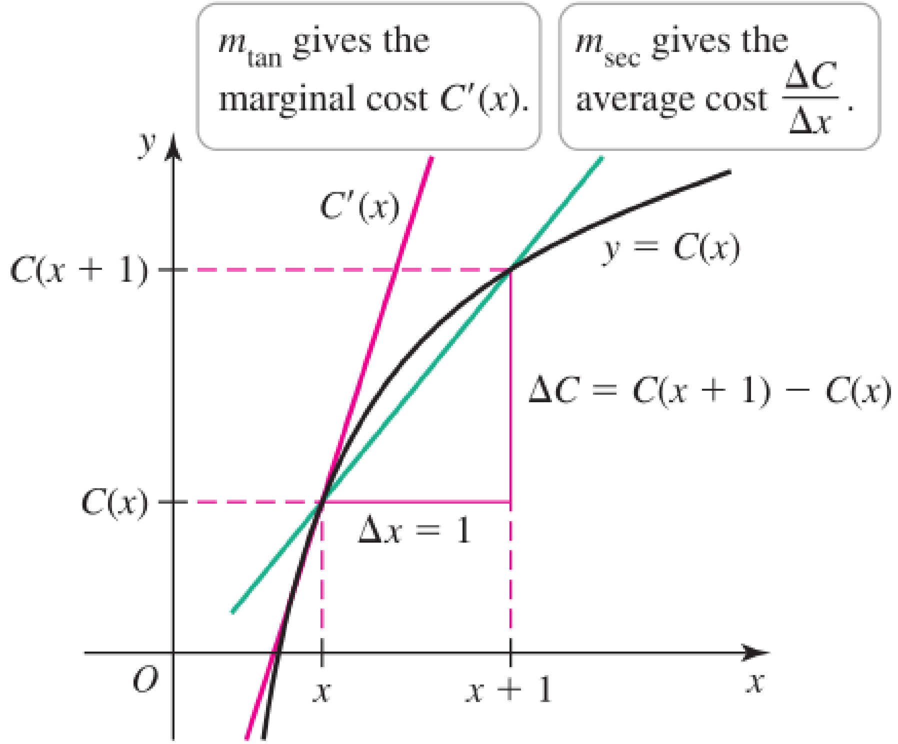
$$\frac{C(x)}{x} = \frac{500 + 0.1x}{x} = \frac{500}{x} + 0.1$$



How about the cost of producing additional (Δx) items?

$$\frac{C(x+\Delta x)-C(x)}{\Delta x}=\frac{\Delta C}{\Delta x}$$

Marginal cost Let $\Delta x \to 0$, then $\lim_{\Delta x \to 0} \frac{\Delta C}{\Delta x} = C'(x)$



In the preceding example, we have $C'(x) = 0.1$

DEFINITION Average and Marginal Cost

The **cost function** $C(x)$ gives the cost to produce the first x items in a manufacturing process. The **average cost** to produce x items is $\overline{C}(x) = C(x)/x$. The **marginal cost** $C'(x)$ is the approximate cost to produce one additional item after producing x items.

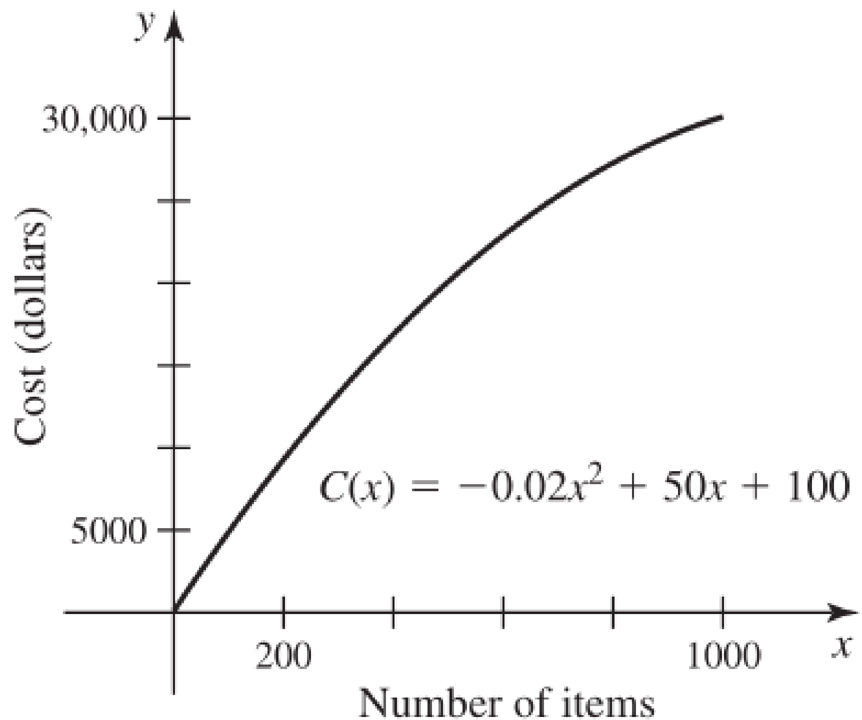
EXAMPLE 5 **Average and marginal costs** Suppose the cost of producing x items is given by the function (Figure 3.46)

$$C(x) = -0.02x^2 + 50x + 100, \quad \text{for} \quad 0 \leq x \leq 1000.$$

a. Determine the average and marginal cost functions.

b. Determine the average and marginal cost for the first 100 tems and interpret these values.

c. Determine the average and marginal cost for the first 900 items and interpret these values.



Elasticity in Economics

Elasticity is applied to prices, income, capital, labor and other variables in system with input and output.
Elasticity describes how changes in the input to a system is related to changes in the output.

For example, sales at a gas station has a linear demand function $D(p) = 1200 - 200p$, where $D(p)$ is the number of gallons sold per day at a price p.
If the price is increased, sales decrease.

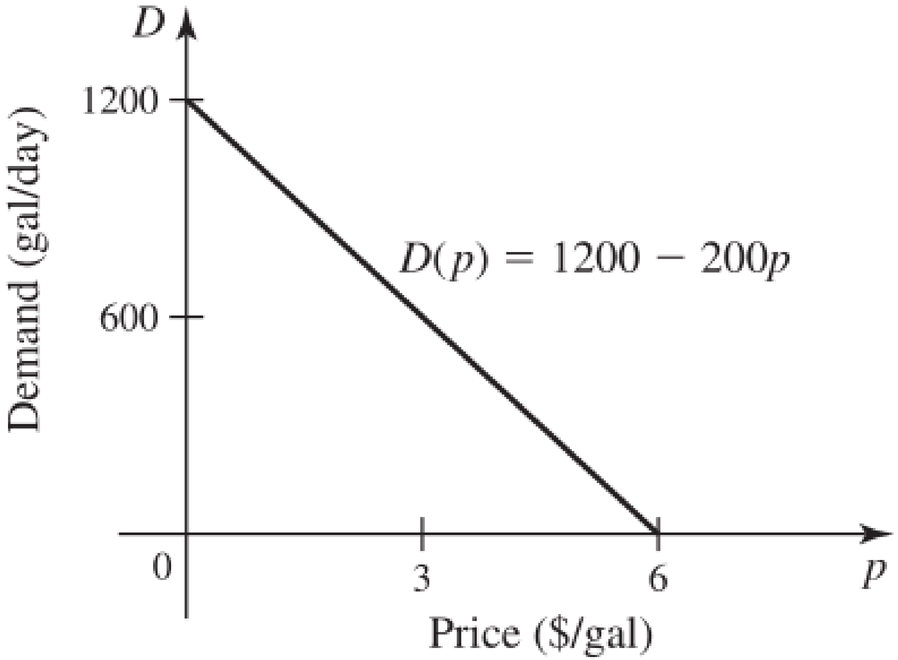
Suppose that price increases from \$3.60 to \$3.96 per gallon, call this change $\Delta p = \$0.36$.
The resulting change in number of gallons sold is $\Delta D = -72$.

Percentage change is of interest, for example,

$$\frac{\Delta p}{p} = \frac{\$0.36}{\$3.6} = 10\%$$

The corresponding percentage change in the gallons sold is

$$\frac{\Delta D}{D} = \frac{-72}{480} = -15\%$$



The *price elasticity of the demand* (or simply, *elasticity*) is the ratio of the percentage change in demand to the percentage change in price; i.e., $E = \frac{\Delta D/D}{\Delta p/p}$.

For the above example, the elasticity is $\frac{-15\%}{10\%} = -1.5$.

More useful when expressed as the function of the price.

$$E(p) = \lim_{\Delta p \to 0} \frac{\Delta D/D}{\Delta p/p} = \lim_{\Delta p \to 0} \frac{\Delta D}{\Delta p}\left(\frac{p}{D}\right) = \frac{dD}{dp}\frac{p}{D}$$

For the gas demand example,

$$E(p) = \frac{dD}{dp}\frac{p}{D} = \frac{d}{dp}(1200 - 1200p)\frac{p}{1200 - 1200p} = \frac{p}{p-6}$$

Given a particular price, the elasticity is interpreted as the percentage change in the demand that results for every 1% change in the price.

DEFINITION Elasticity

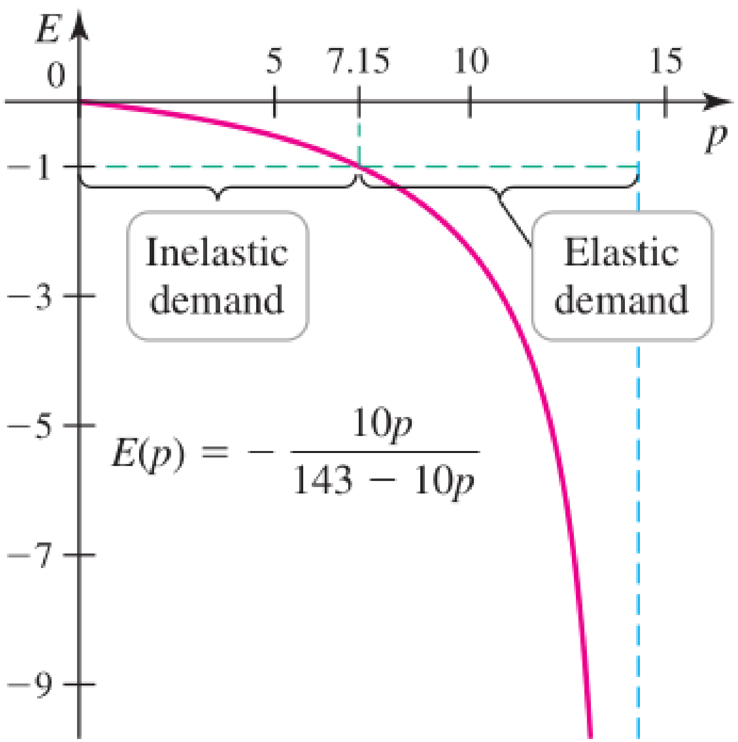
If the demand for a product varies with the price according the function $D = f(p)$, then the **price elasticity of the demand** is $E(p) = \frac{dD}{dp}\frac{p}{D}$.

EXAMPLE 6 **Elasticity in pork prices** The demand for processed pork in Canada is described by the function $D(p) = 286 - 20p$. (*Source: Microeconomics*, J. Perloff, Addison Wesley, 2012)

a. Compute and graph the price elasticity of the demand.

b. When $-\infty < E < -1$, the demand is said to be **elastic**. When $-1 < E < 0$, the demand is said to be **inelastic**. Interpret these terms.

c. For what prices is the demand for pork elastic? Inelastic?



3.7

The Chain Rule

Chain Rule Formulas

Finding the derivative of *composite functions*, e.g., $(5x+4)^{100}$.

Example, Yancey picks (y) apples three times faster than Uri (u) , which means the rate at which Yancey picks apples with respect to Uri is $\frac{dy}{du} = 3$.

Uri picks apples twice as fast as Xan, so $\frac{du}{dx} = 2$. Therefore, Yancey picks apples at a rate that is $3 \cdot 2 \ = 6$ times greater than Xan's rate, i.e., $\frac{dy}{dx} = 6$.

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 3 \cdot 2 = 6$$

Version 1 of the Chain Rule the composite function $f = g(u)$, and $u = g(x)$.

THEOREM 3.14 **The Chain Rule**

Suppose $y = f(u)$ is differentiable at $u = g(x)$ and $u = g(x)$ is differentiable at x. The composite function $y = f(g(x))$ is differentiable at x, and its derivative can be expressed in two equivalent ways.

Version 1 $$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Version 2 $$\frac{d}{dx}\big(f(g(x))\big) = f'(g(x)) \cdot g'(x)$$

Proof is given at the end of the section.

The key to using the Chain Rule is identifying the *inner* and *outer functions*.

PROCEDURE Using the Chain Rule

Assume the differentiable function $y = f(g(x))$ is given.

1. Identify an outer function f and an inner function g, and let $u = g(x)$.
2. Replace $g(x)$ with u to express y in terms of u:

$$y = f(\underbrace{g(x)}_{u}) \Rightarrow y = f(u).$$

3. Calculate the product $\dfrac{dy}{du} \cdot \dfrac{du}{dx}$.
4. Replace u with $g(x)$ in $\dfrac{dy}{du}$ to obtain $\dfrac{dy}{dx}$.

EXAMPLE 1 **Version 1 of the Chain Rule** For each of the following composite functions, find the inner function $u = g(x)$ and the outer function $y = f(u)$. Use Version 1 of the Chain Rule to find $\frac{dy}{dx}$.

a. $y = (5x + 4)^3$ **b.** $y = \sin^3 x$ **c.** $y = \sin x^3$

EXAMPLE 2 **Version 2 of the Chain Rule** Use Version 2 of the Chain Rule to calculate the derivatives of the following functions.

a. $(6x^3 + 3x + 1)^{10}$ **b.** $\sqrt{5x^2 + 1}$ **c.** $\left(\frac{5t^2}{3t^2 + 2}\right)^3$ **d.** e^{-3x}

The Chain Rule is also used to calculate the derivative of a composite function for a specific value of the variable.

If $h(x) = f(g(x))$ and a is a real number, then $h'(a) = f'(g(a))g'(a)$, provided the necessary derivatives exist.

Therefore, $h'(a)$ is the derivative of f evaluated at $g(a)$ multiplied by the derivative of g evaluated at a.

EXAMPLE 3 **Calculating derivatives at a point** Let $h(x) = f(g(x))$. Use the values in Table 3.3 to calculate $h'(1)$ and $h'(2)$.

Table 3.3

x	$f'(x)$	$g(x)$	$g'(x)$
1	5	2	3
2	7	1	4

EXAMPLE 4 **Applying the Chain Rule** A trail runner programs her GPS unit to record her altitude a (in feet) every 10 minutes during a training run in the mountains; the resulting data are shown in Table 3.4. Meanwhile, at a nearby weather station, a weather probe records the atmospheric pressure p (in hectopascals, or hPa) at various altitudes, shown in Table 3.5.

Table 3.4

t (minutes)	0	10	20	30	40	50	60	70	80
$a(t)$ (altitude)	10,000	10,220	10,510	10,980	11,660	12,330	12,710	13,330	13,440

Table 3.5

a (altitude)	5485	7795	10,260	11,330	12,330	13,330	14,330	15,830	16,230
$p(a)$ (pressure)	1000	925	840	821	793	765	738	700	690

Use the Chain Rule to estimate the rate of change in pressure per unit time experienced by the trail runner when she is 50 minutes into her run.

Chain Rule for Powers

THEOREM 3.15 Chain Rule for Powers
If g is differentiable for all x in its domain and n is an integer, then

$$\frac{d}{dx}\left((g(x))^n\right) = n(g(x))^{n-1}g'(x).$$

EXAMPLE 5 Chain Rule for powers Find $\frac{d}{dx}(\tan x + 10)^{21}$.

The Composition of Three or More Functions

Applying the Chain Rule repeatedly

EXAMPLE 6 **Composition of three functions** Calculate the derivative of $\sin\left(e^{\cos x}\right)$.

EXAMPLE 7 **Combining rules** Find $\dfrac{d}{dx}\left(x^2\sqrt{x^2+1}\right)$.

Proof of the Chain Rule

Suppose f is differentiable at $u = g(a)$, g is differentiable at a, and $h(x) = f(g(x))$. By definition,

$$h'(a) = \lim_{x \to a} \frac{h(x) - h(a)}{x - a} = \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{x - a}$$

Assume that $g(x) \neq g(a)$ for x near a but not equal to a.

Multiply the right side by $\frac{g(x)-g(a)}{g(x)-g(a)}$, and let $v = g(x)$ and $u = g(a)$, we have

$$h'(a) = \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \cdot \frac{g(x) - g(a)}{x - a}$$

$$= \lim_{x \to a} \frac{f(v) - f(u)}{v - u} \cdot \frac{g(x) - g(a)}{x - a}$$

g is continuous at a, i.e., $\lim_{x \to a} g(x) = g(a)$, so $v \to u$ as $x \to a$.

$$h'(a) = \underbrace{\lim_{v \to u} \frac{f(v) - f(u)}{v - u}}_{f'(u)} \cdot \underbrace{\lim_{x \to a} \frac{g(x) - g(a)}{x - a}}_{g'(a)} = f'(u)g'(a).$$

Because f and g are differentiable at u and a, the two limits in this expression exist; therefore, $h'(a)$ exists.

Noting that $u = g(a)$, we have $h'(a) = f'(g(a))g'(a)$.

Replacing a with x gives the *Chain Rule* $h'(x) = f'(g(x))g'(x)$.

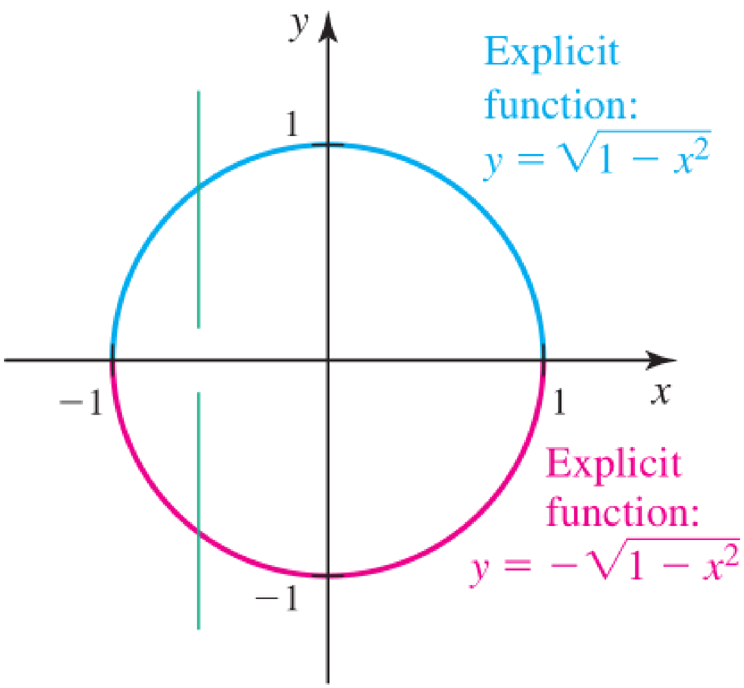
3.8

Implicit Differentiation

Implicit Function

Functions whose relations between variables are expressed implicitly, e.g., equation of the unit circle $x^2 + y^2 = 1$

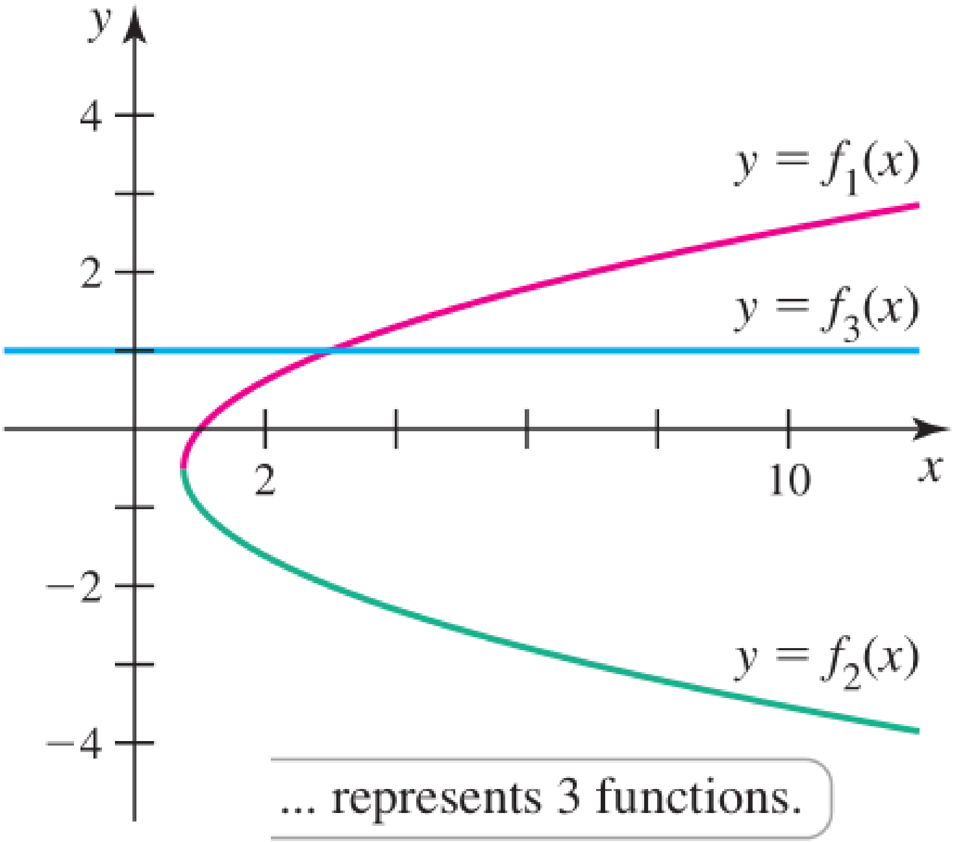
Each semicircle passes the vertical line test.



If $y = \sqrt{1 - x^2}$,
then $\frac{dy}{dx} = -\frac{x}{\sqrt{1-x^2}}$.

If $y = -\sqrt{1 - x^2}$,
then $\frac{dy}{dx} = \frac{x}{\sqrt{1-x^2}}$.

It is sometimes *difficult* or *impossible* to solve equations for y. For example, $x + y^3 - xy = 1$ represents three functions, while solving to obtain these three functions is challenging. Derivative for each function must be calculated *separately*.



To find a *single* expression for the derivative *directly* from the equation without solving for y, called **implicit differentiation**.

EXAMPLE 1 Implicit differentiation

a. Calculate $\frac{dy}{dx}$ directly from the equation for the unit circle $x^2 + y^2 = 1$.

b. Find the slope of the unit circle at $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$.

SOLUTION

Replace the variable y with $y(x)$: $x^2 + (y(x))^2 = 1$

Take derivative *with respect to x*:

$$\underbrace{\frac{d}{dx}(x^2)}_{2x} + \underbrace{\frac{d}{dx}(y(x))^2}_{\text{Use the Chain Rule}} = \underbrace{\frac{d}{dx}(1)}_{0}.$$

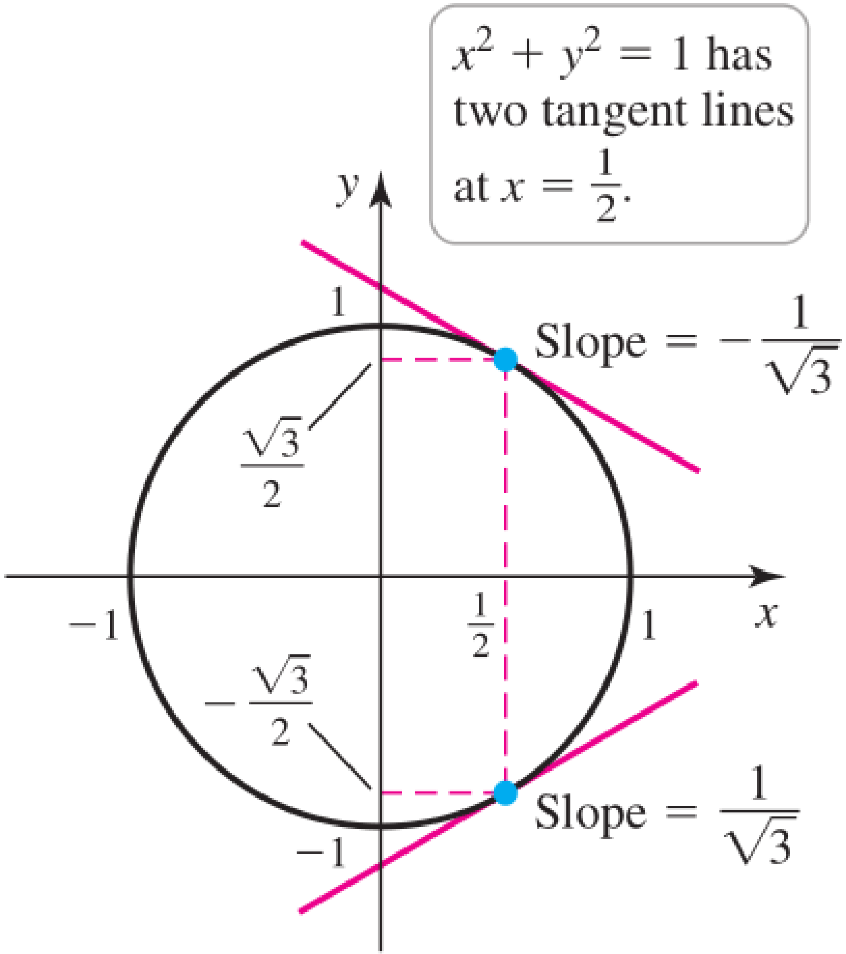
By the Chain Rule, $\dfrac{d}{dx}(y(x))^2 = 2y(x)y'(x)$

Substituting this result, $2x + 2y\dfrac{dy}{dx} = 0.$

Solve for $\dfrac{dy}{dx}$: $\dfrac{dy}{dx} = -\dfrac{x}{y}$ $(y \neq 0).$

At the points (1,0) and (-1,0), the circle has vertical tangent lines.

The derivative $\frac{dy}{dx} = -\frac{x}{y}$ depends on *both x and y*.



EXAMPLE 2 **Implicit differentiation** Find $y'(x)$ when $\sin xy = x^2 + y$.

Slopes of Tangent Lines (Implicit Function)

Derivatives obtained by implicit differentiation typically depend on x and y.
Therefore, the slope of a curve at a particular point (a, b) requires both the $x-$ and $y-$coordinates of the point.

EXAMPLE 3 **Finding tangent lines with implicit functions** Find an equation of the line tangent to the curve $x^2 + xy - y^3 = 7$ at $(3, 2)$.

Higher-Order Derivatives of Implicit Functions

The same approach

EXAMPLE 4 **A second derivative** Find $\dfrac{d^2y}{dx^2}$ if $x^2 + y^2 = 1$.

SOLUTION The first derivative $\dfrac{dy}{dx} = -\dfrac{x}{y}$ was computed in Example 1.

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(-\frac{x}{y}\right)$$ Take derivatives with respect to x.

$$\frac{d^2y}{dx^2} = -\frac{y \cdot 1 - x\dfrac{dy}{dx}}{y^2}$$ Quotient Rule

$$= -\frac{y - x\left(-\dfrac{x}{y}\right)}{y^2}$$ Substitute for $\dfrac{dy}{dx}$.

$$= -\frac{x^2 + y^2}{y^3}$$ Simplify.

$$= -\frac{1}{y^3}.$$ $x^2 + y^2 = 1$

3.9

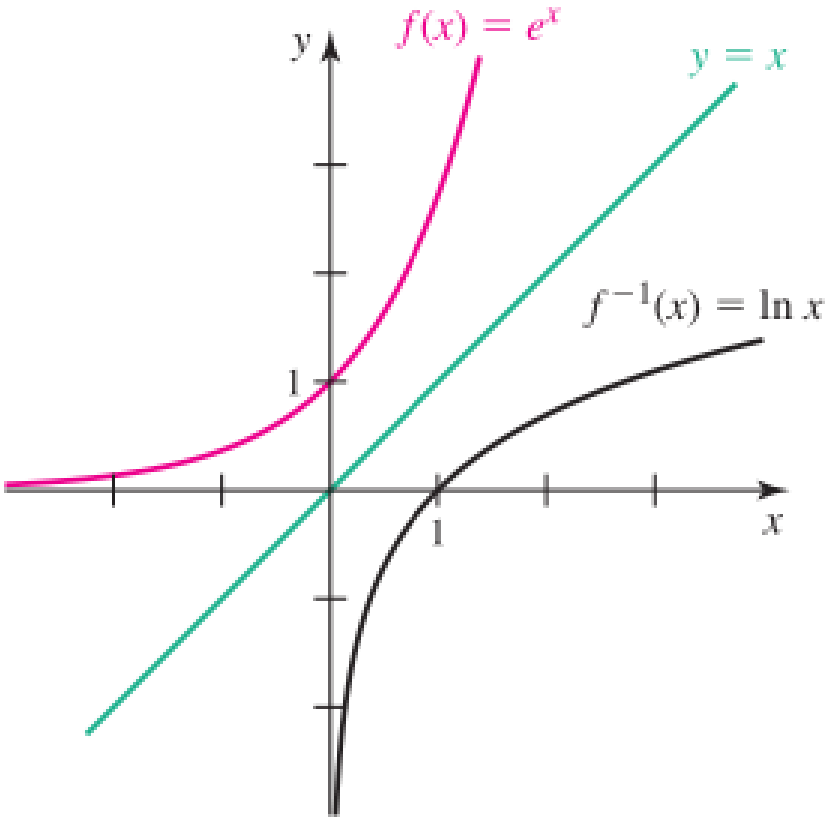
Derivatives of Logarithmic and Exponential Functions

The Derivative of $y = \ln x$

The inverse function of natural exponential function $f(x) = e^x$ is natural logarithmic function $f^{-1}(x) = \ln x$, whose domain is $(0, \infty)$.

Inverse Properties for e^x and $\ln x$

1. $e^{\ln x} = x$, for $x > 0$, and $\ln(e^x) = x$, for al
2. $y = \ln x$ if and only if $x = e^y$.
3. For real numbers x and $b > 0$, $b^x = e^{\ln b^x}$



To find the derivative of $y = \ln x$, begin with the inverse property, and write $x = e^y \;\; (x > 0)$.

Compute $\frac{dy}{dx}$ with implicit differentiation and Chain Rule

$$x = e^y \qquad y = \ln x \text{ if and only if } x = e^y$$

$$1 = e^y \cdot \frac{dy}{dx} \qquad \text{Differentiate both sides with respect to } x.$$

$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}. \qquad \text{Solve for } dy/dx \text{ and use } x = e^y.$$

Therefore,

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

An important extension, consider $\ln|x|$ $(x \neq 0)$

$$\ln|x| = \begin{cases} \ln x & x > 0 \\ \ln(-x) & x < 0 \end{cases}$$

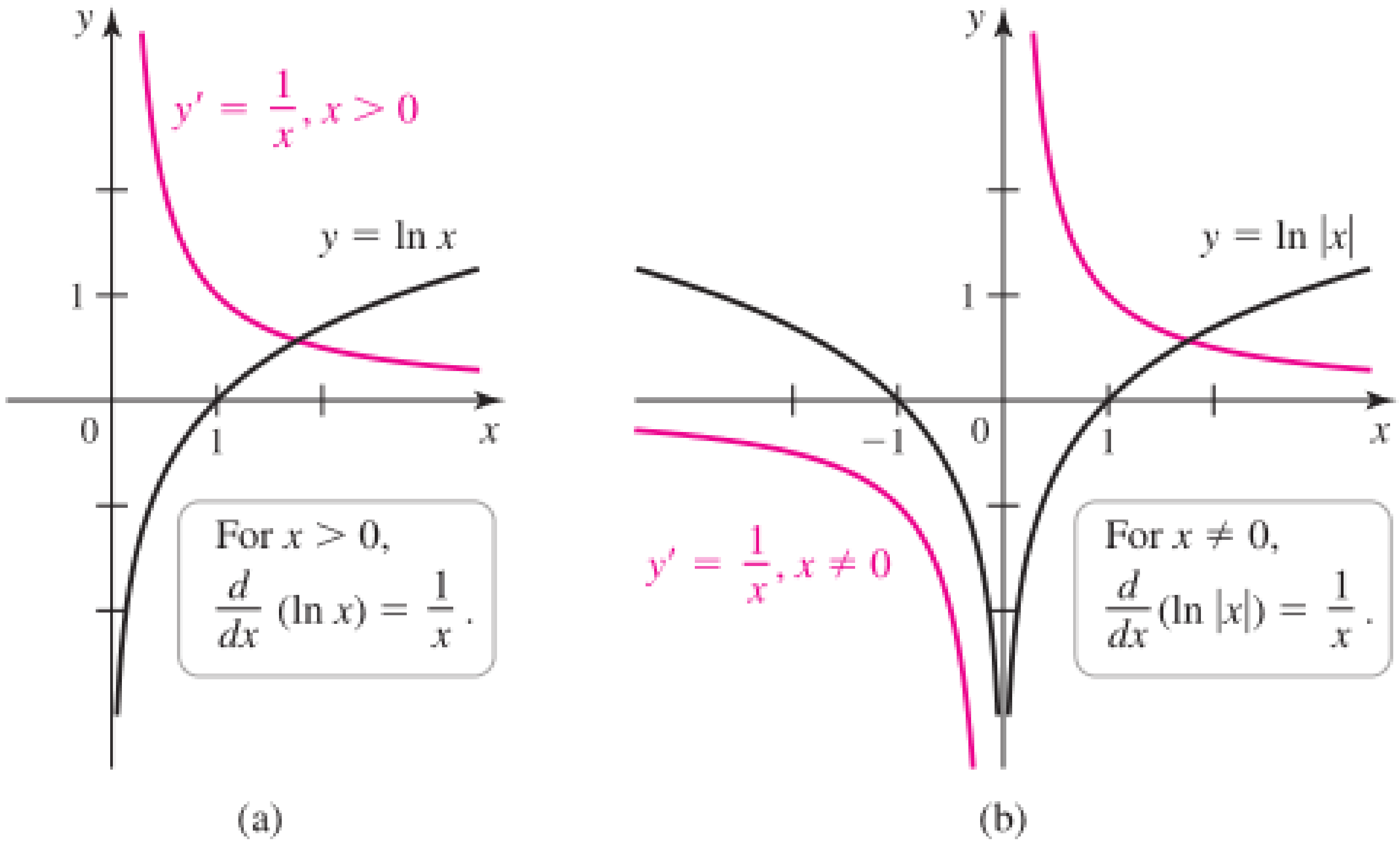
For $x > 0$, $\frac{d}{dx}(\ln|x|) = \frac{d}{dx}(\ln x) = \frac{1}{x}$

THEOREM 3.17 **Derivative of $\ln x$**

$$\frac{d}{dx}(\ln x) = \frac{1}{x}, \text{ for } x > 0 \qquad \frac{d}{dx}(\ln|x|) = \frac{1}{x}, \text{ for } x \neq 0$$

If u is differentiable at x and $u(x) \neq 0$, then

$$\frac{d}{dx}(\ln|u(x)|) = \frac{u'(x)}{u(x)}.$$



y
$y' = \frac{1}{x}, x > 0$
$y = \ln x$
1
0
1
x
For $x > 0$,
$\frac{d}{dx}(\ln x) = \frac{1}{x}$.
(a)
y
$y = \ln |x|$
1
$y' = \frac{1}{x}, x \neq 0$
-1
0
1
x
For $x \neq 0$,
$\frac{d}{dx}(\ln |x|) = \frac{1}{x}$.
(b)

EXAMPLE 1 **Derivatives involving ln x** Find $\frac{dy}{dx}$ for the following functions.

a. $y = \ln 4x$ **b.** $y = x \ln x$ **c.** $y = \ln |\sec x|$ **d.** $y = \frac{\ln x^2}{x^2}$

The Derivative of b^x

Because $b^x = e^{x \ln b}$,

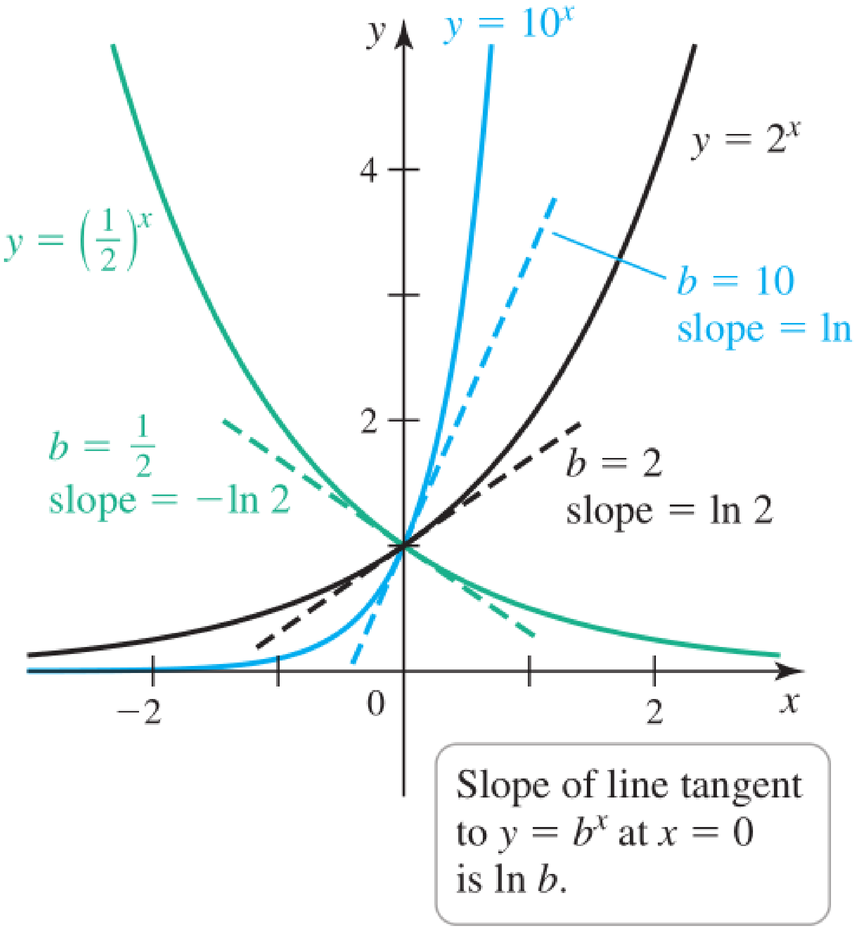
$$\frac{d}{dx}(b^x) = \frac{d}{dx}(e^{x \ln b}) = \underbrace{e^{x \ln b}}_{b^x} \cdot \ln b. \quad \text{Chain Rule with } \frac{d}{dx}(x \ln b) = \ln b$$

THEOREM 3.18 Derivative of b^x

If $b > 0$ and $b \neq 1$, then for all x,

$$\frac{d}{dx}(b^x) = b^x \ln b.$$

When $b > 1$, $\ln b > 0$ and the graph of $y = b^x$ has tangent lines with positive slopes for all x.
When $0 < b < 1$, $\ln b < 0$ and the graph of $y = b^x$ has tangent lines with negative slopes for all x.
In either case, the tangent line at $(0, 1)$ has slope $\ln b$.



EXAMPLE 2 **Derivatives with b^x** Find the derivative of the following functions.

a. $f(x) = 3^x$ **b.** $g(t) = 108 \cdot 2^{t/12}$

EXAMPLE 3 **An exponential model** Table 3.6 and Figure 3.60 show how the incidence of Down syndrome in newborn infants increases with the age of the mother. The data can be modeled with the exponential function $P(a) = \frac{1}{1{,}613{,}000} 1.2733^a$, where a is the age of the mother (in years) and $P(a)$ is the incidence (number of Down syndrome children per total births).

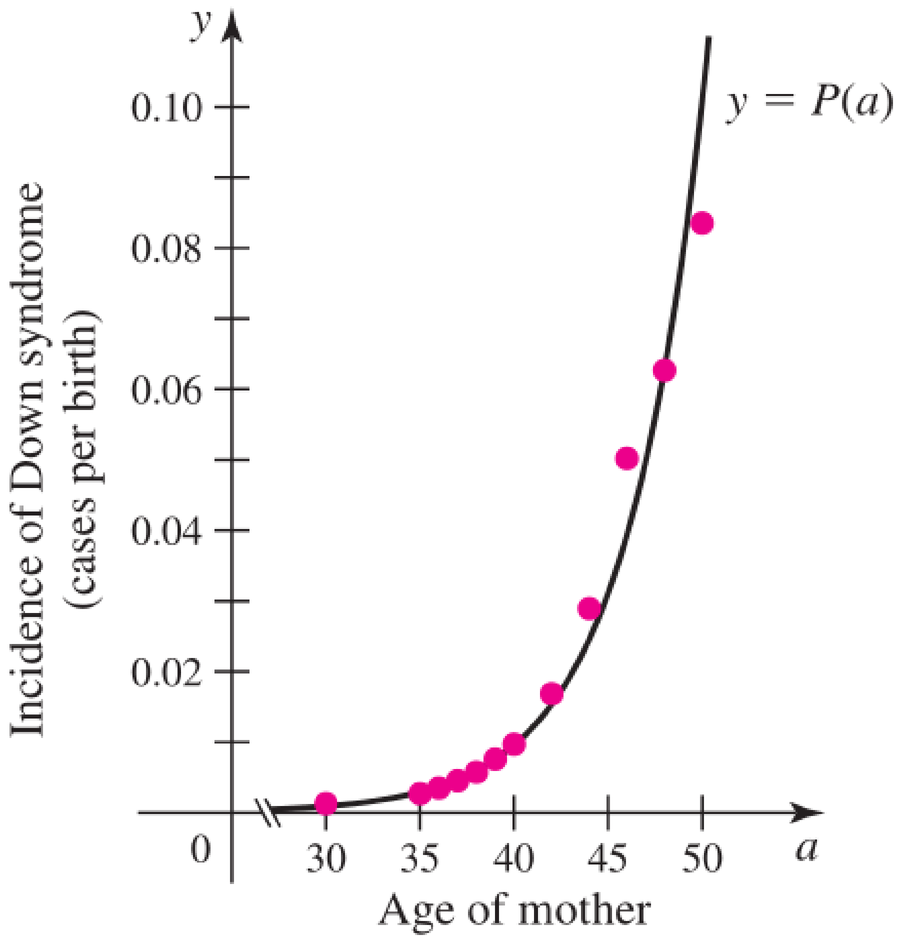
a. According to the model, at what age is the incidence of Down syndrome equal to 0.01 (that is, 1 in 100)?

b. Compute $P'(a)$.

c. Find $P'(35)$ and $P'(46)$, and interpret each.

Table 3.6

Mother's Age	Incidence of Down Syndrome	Decimal Equivalent
30	1 in 900	0.00111
35	1 in 400	0.00250
36	1 in 300	0.00333
37	1 in 230	0.00435
38	1 in 180	0.00556
39	1 in 135	0.00741
40	1 in 105	0.00952
42	1 in 60	0.01667
44	1 in 35	0.02875
46	1 in 20	0.05000
48	1 in 16	0.06250
49	1 in 12	0.08333



The General Power Rule

As it stands now, $\frac{d}{dx}(x^p) = px^{p-1}$ for integers p.
Actually, it can be extended to all real powers.

THEOREM 3.19 General Power Rule
For real numbers p and for $x > 0$,

$$\frac{d}{dx}(x^p) = px^{p-1}.$$

Furthermore, if u is a positive differentiable function on its domain, then

$$\frac{d}{dx}(u(x)^p) = p(u(x))^{p-1} \cdot u'(x).$$

Proof: For $x > 0$ and real numbers p, $x^p = e^{p \ln x}$. Therefore,

$$\begin{aligned}
\frac{d}{dx}(x^p) &= \frac{d}{dx}(e^{p \ln x}) && \text{Inverse property (3)} \\
&= e^{p \ln x} \cdot \frac{p}{x} && \text{Chain Rule, } \frac{d}{dx}(p \ln x) = \frac{p}{x} \\
&= x^p \cdot \frac{p}{x} && e^{p \ln x} = x^p \\
&= p x^{p-1}. && \text{Simplify.}
\end{aligned}$$

EXAMPLE 4 **Computing derivatives** Find the derivative of the following functions.

a. $y = x^{\pi}$ **b.** $y = \pi^{x}$ **c.** $y = (x^2 + 4)^e$

Functions of the form, $f(x) = (g(x))^{h(x)}$, where both g and h are nonconstant functions, are neither exponential functions nor power functions (sometimes called *tower functions*) .

To compute their derivatives, use the identity $b^x = e^{x \ln b}$ to rewrite it as

$$f(x) = (g(x))^{h(x)} = e^{h(x) \ln g(x)}$$

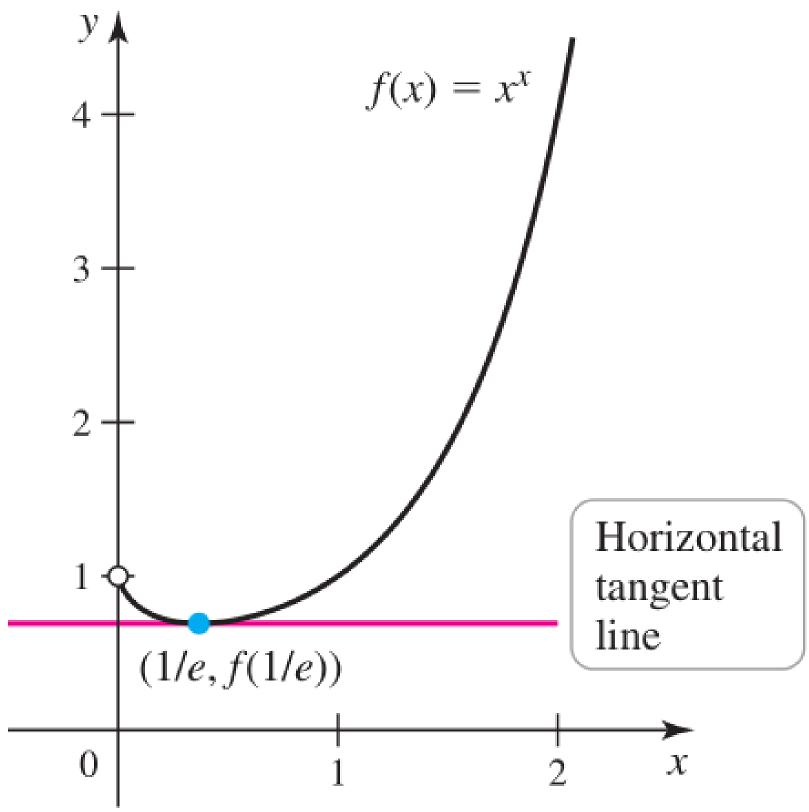
Then, compute the derivatives using the above methods.

EXAMPLE 5 **General exponential functions** Let $f(x) = x^{\sin x}$, for $x \geq 0$.

a. Find $f'(x)$. **b.** Evaluate $f'\left(\frac{\pi}{2}\right)$.

EXAMPLE 6 **Finding a horizontal tangent line** Determine whether the graph of $f(x) = x^x$, for $x > 0$, has any horizontal tangent lines.

$$f(x) = x^x = e^{x \ln x}$$



Derivatives of General Logarithmic Functions

The technique used to differentiate the natural logarithm applies to the general logarithm function, $f(x) = \log_b x$.

$$y = \log_b x \Leftrightarrow x = b^y, \text{ where } x > 0.$$

$$1 = b^y \ln b \cdot \frac{dy}{dx} \quad \text{Implicit differentiation}$$

$$\frac{dy}{dx} = \frac{1}{b^y \ln b} \quad \text{Solve for } \frac{dy}{dx}.$$

$$\frac{dy}{dx} = \frac{1}{x \ln b}. \quad b^y = x.$$

THEOREM 3.20 Derivative of $\log_b x$

If $b > 0$ and $b \neq 1$, then

$$\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}, \text{ for } x > 0 \quad \text{and} \quad \frac{d}{dx}(\log_b |x|) = \frac{1}{x \ln b}, \text{ for } x \neq 0.$$

EXAMPLE 7 Derivatives with general logarithms Compute the derivative of the following functions.

a. $f(x) = \log_5 (2x + 1)$ **b.** $T(n) = n \log_2 n$

Logarithmic Differentiation

Direct computation of a derivative might be very tedious, e.g.,

$$f(x) = \frac{(x^3 - 1)^4 \sqrt{3x - 1}}{x^2 + 4}$$

We would need the Quotient, Product, and Chain Rule to compute $f'(x)$, and simplifying the result

- The properties of logarithms needed for logarithmic differentiation (where $x > 0$ and $y > 0$):

1. $\ln xy = \ln x + \ln y$
2. $\ln (x/y) = \ln x - \ln y$
3. $\ln x^y = y \ln x$

EXAMPLE 8 **Logarithmic differentiation** Let $f(x) = \dfrac{(x^3 - 1)^4\sqrt{3x - 1}}{x^2 + 4}$ and compute $f'(x)$.

$$\begin{aligned}
\ln(f(x)) &= \ln\left(\frac{(x^3 - 1)^4\sqrt{3x - 1}}{x^2 + 4}\right) \\
&= \ln(x^3 - 1)^4 + \ln\sqrt{3x - 1} - \ln(x^2 + 4) && \log xy = \log x + \log y \\
&= 4\ln(x^3 - 1) + \tfrac{1}{2}\ln(3x - 1) - \ln(x^2 + 4). && \log x^y = y\log x
\end{aligned}$$

$$\frac{f'(x)}{f(x)} = 4 \cdot \frac{1}{x^3 - 1} \cdot 3x^2 + \frac{1}{2} \cdot \frac{1}{3x - 1} \cdot 3 - \frac{1}{x^2 + 4} \cdot 2x.$$

$$f'(x) = f(x)\left(\frac{12x^2}{x^3 - 1} + \frac{3}{2(3x - 1)} - \frac{2x}{x^2 + 4}\right).$$

Logarithmic Differentiation also provides an alternative method for finding the derivative of tower function $g(x)^{h(x)}$.

$$\begin{aligned}
f(x) &= x^x \\
\ln(f(x)) &= \ln(x^x) = x\ln x && \text{Take logarithms of both sides; use properties.} \\
\frac{1}{f(x)} f'(x) &= 1 \cdot \ln x + x \cdot \frac{1}{x} && \text{Differentiate both sides.} \\
f'(x) &= f(x)(\ln x + 1) && \text{Solve for } f'(x) \text{ and simplify.} \\
f'(x) &= x^x(\ln x + 1). && \text{Replace } f(x) \text{ with } x^x.
\end{aligned}$$

3.10

Derivatives of Inverse Trigonometric Functions

Inverse Sine and Its Derivative

Recall: $y = \sin^{-1} x$ is the value of y such that $x = \sin y$, where $-\pi/2 \le y \le \pi/2$, $-1 \le x \le 1$ (domain).
The derivative of $y = \sin^{-1} x$ is obtained as follows.

$$x = \sin y \qquad y = \sin^{-1} x \iff x = \sin y$$

$$\frac{d}{dx}(x) = \frac{d}{dx}(\sin y) \qquad \text{Differentiate with respect to } x.$$

$$1 = (\cos y)\frac{dy}{dx} \qquad \text{Chain Rule on the right side}$$

$$\frac{dy}{dx} = \frac{1}{\cos y}. \qquad \text{Solve for } \frac{dy}{dx}.$$

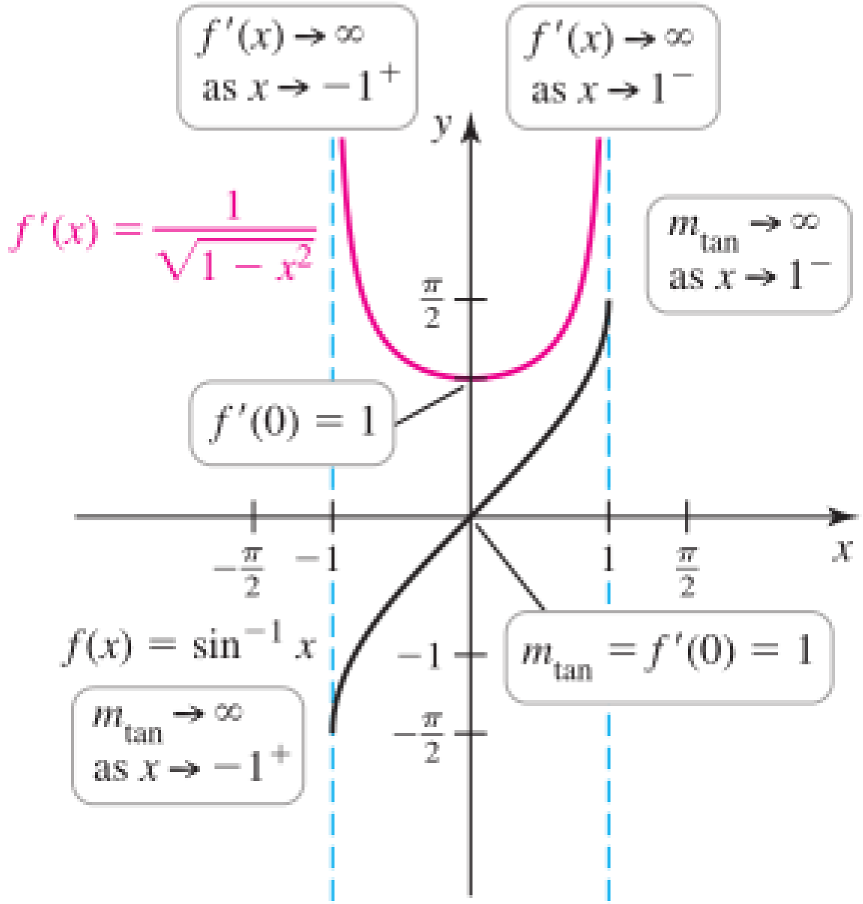
$$\cos y = \pm\sqrt{1 - \underbrace{\sin^2 y}_{x^2}} \qquad x = \sin y \Rightarrow x^2 = \sin^2 y$$

$$= \pm\sqrt{1 - x^2}.$$

As y is restricted to the interval $-\pi/2 \leq y \leq \pi/2$, we have $\cos y \geq 0$. Therefore, we have

THEOREM 3.21 **Derivative of Inverse Sine**

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}, \text{ for } -1 < x < 1$$



EXAMPLE 1 **Derivatives involving the inverse sine** Compute the following derivatives.

a. $\frac{d}{dx}(\sin^{-1}(x^2 - 1))$ **b.** $\frac{d}{dx}(\cos(\sin^{-1} x))$

Derivatives of Inverse Tangent and Secant

Inverse Tangent: $y = \tan^{-1} x$ is the value of y such that $x = \tan y$, where $-\frac{\pi}{2} < y < \pi/2$, $-\infty \leq x \leq \infty$ (domain).

Similarly, we have

$$x = \tan y \qquad y = \tan^{-1} x \Leftrightarrow x = \tan y$$

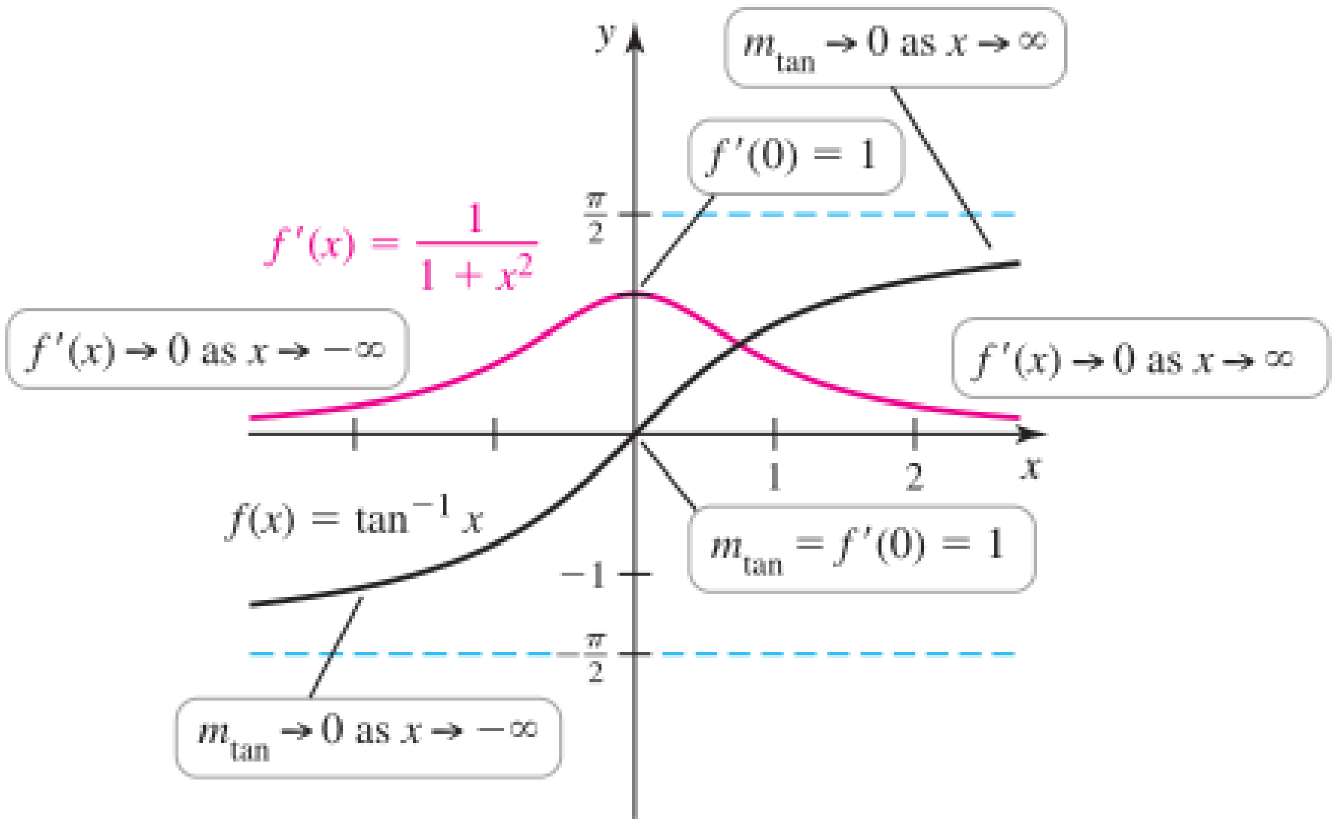
$$\frac{d}{dx}(x) = \frac{d}{dx}(\tan y) \qquad \text{Differentiate with respect to } x.$$

$$1 = \sec^2 y \cdot \frac{dy}{dx} \qquad \text{Chain Rule}$$

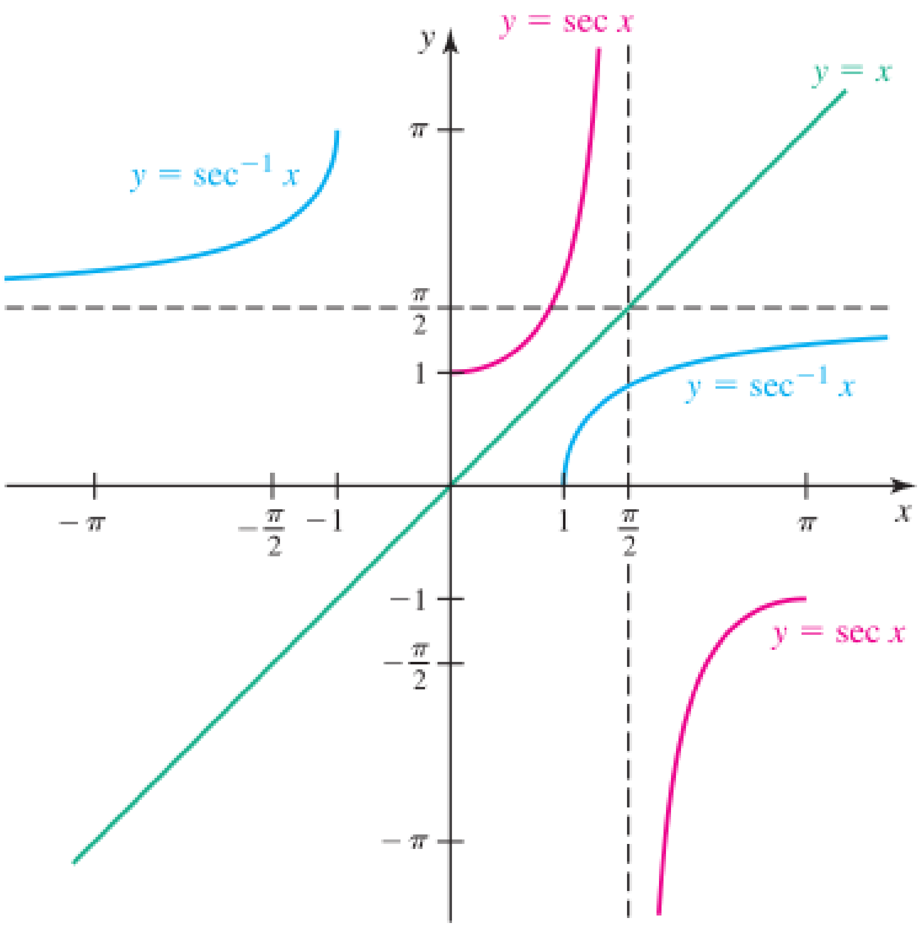
$$\frac{dy}{dx} = \frac{1}{\sec^2 y}. \qquad \text{Solve for } \frac{dy}{dx}.$$

Combining the trigonometric identity $\sec^2 y = 1 + \tan^2 y$ and $x = \tan y$ to obtain $\sec^2 y = 1 + x^2$, which follows that

$$\frac{dy}{dx} = \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1 + x^2}.$$



Inverse Secant: $y = \sec^{-1} x$ is the value of y such that $x = \sec y$, where $0 \le y \le \pi$, with $y \le \pi/2$. $|x| \ge 1$ (domain).



Domain of $\sec^{-1} x$: $|x| \ge 1$
Range of $\sec^{-1} x$: $0 \le y \le \pi$, $y \ne \frac{\pi}{2}$

Similarly, we have $$1 = \sec y \tan y \frac{dy}{dx}$$

Solving for $\frac{dy}{dx}$ $$\frac{dy}{dx} = \frac{d}{dx}(\sec^{-1} x) = \frac{1}{\sec y \tan y}$$

Using the trigonometric identity $\sec^2 y = 1 + \tan^2 y$, we have

$$\tan y = \pm\sqrt{\sec^2 y - 1} = \pm\sqrt{x^2 - 1}$$

- By the definition of $y = \sec^{-1} x$, if $x \geq 1$, then $0 \leq y < \pi/2$ and $\tan y > 0$. In this case, we choose the positive branch and take $\tan y = \sqrt{x^2 - 1}$.
- However, if $x \leq -1$, then $\pi/2 < y \leq \pi$ and $\tan y < 0$. Now we choose the negative branch.

$$\frac{d}{dx}(\sec^{-1} x) = \begin{cases} \dfrac{1}{x\sqrt{x^2 - 1}} & \text{if } x > 1 \\ -\dfrac{1}{x\sqrt{x^2 - 1}} & \text{if } x < -1, \end{cases}$$

Or

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2 - 1}}, \text{ for } |x| > 1.$$

Derivatives of Other Inverse Trigonometric Functions

$y = \cos^{-1} x$, its derivative results from the identity

$$\cos^{-1} x + \sin^{-1} x = \frac{\pi}{2}$$

$$\frac{d}{dx}(\cos^{-1} x) + \underbrace{\frac{d}{dx}(\sin^{-1} x)}_{1/\sqrt{1-x^2}} = \underbrace{\frac{d}{dx}\left(\frac{\pi}{2}\right)}_{0}.$$

$$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}.$$

Similarly, based on the analogous identities

$$\cot^{-1} x + \tan^{-1} x = \frac{\pi}{2} \text{ and } \csc^{-1} x + \sec^{-1} x = \frac{\pi}{2}$$

Derivatives of $\cot^{-1} x$ and $\csc^{-1} x$ are the negative of the derivatives of $\tan^{-1} x$ and $\sec^{-1} x$, respectively.

THEOREM 3.22 Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}, \text{ for } -1 < x < 1$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}, \text{ for } -\infty < x < \infty$$

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}} \qquad \frac{d}{dx}(\csc^{-1} x) = -\frac{1}{|x|\sqrt{x^2-1}}, \text{ for } |x| > 1$$

EXAMPLE 2 **Derivatives of inverse trigonometric functions**

a. Evaluate $f'(2\sqrt{3})$, where $f(x) = x \tan^{-1}(x/2)$.

b. Find an equation of the line tangent to the graph of $g(x) = \sec^{-1} 2x$ at the point $(1, \pi/3)$.

EXAMPLE 3 **Shadows in a ballpark** As the sun descends behind the 150-ft grandstand of a baseball stadium, the shadow of the stadium moves across the field (Figure 3.67). Let ℓ be the line segment between the edge of the shadow and the sun, and let θ be the angle of elevation of the sun—the angle between ℓ and the horizontal. The length of the shadow s is the distance between the edge of the shadow and the base of the grandstand.

a. Express θ as a function of the shadow length s.

b. Compute $d\theta/ds$ when $s = 200$ ft and explain what this rate of change measures.

Derivatives of Inverse Functions in General

Using implicit differentiation does not always work.

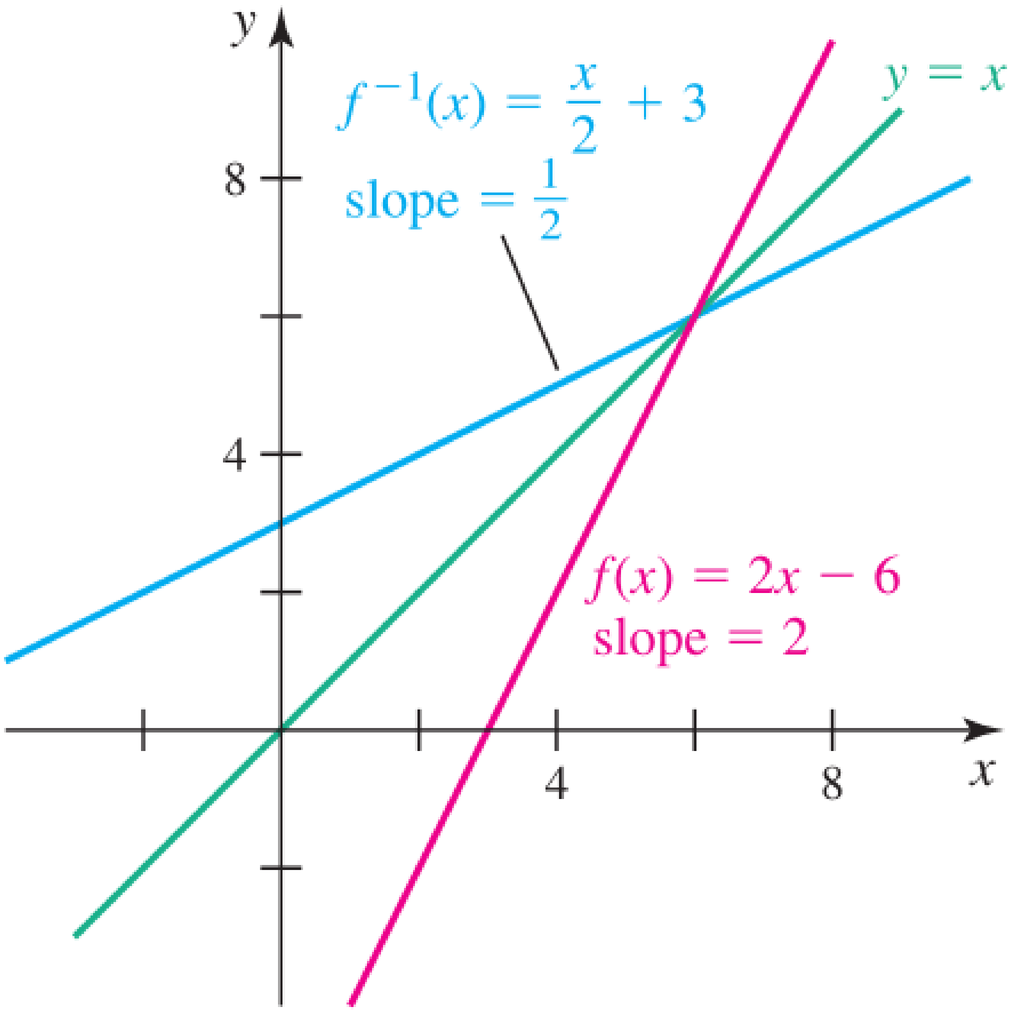
The key to finding the derivative of inverse function lies in the symmetry of the graph of f and f^{-1}.

EXAMPLE 4 **Linear functions, inverses, and derivatives** Consider the general linear function $y = f(x) = mx + b$, where $m \neq 0$ and b are constants.

a. Write the inverse of f in the form $y = f^{-1}(x)$.

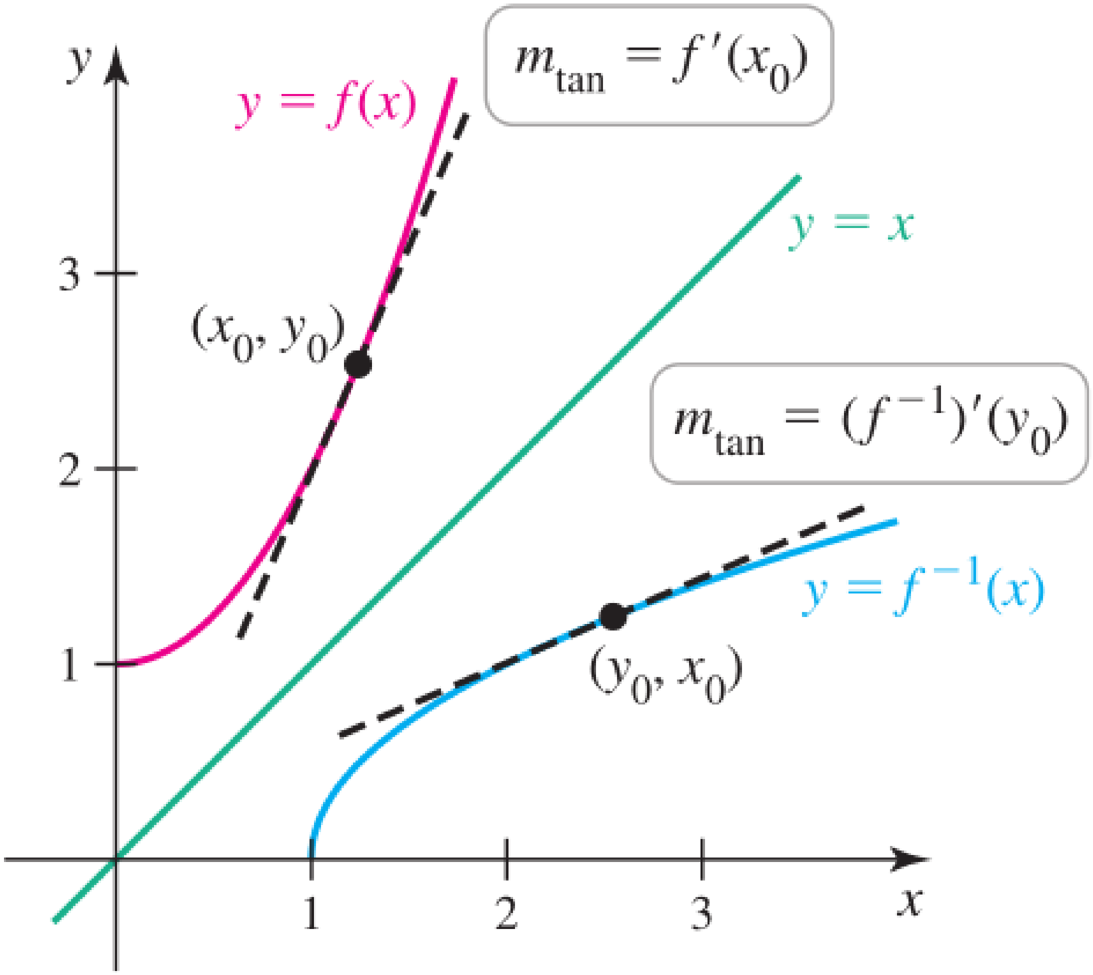
b. Find the derivative of the inverse $\frac{d}{dx}(f^{-1}(x))$.

c. Consider the specific case $f(x) = 2x - 6$. Graph f and f^{-1}, and find the slope of each line.



y
x
$f^{-1}(x) = \frac{x}{2} + 3$
slope $= \frac{1}{2}$
$y = x$
$f(x) = 2x - 6$
slope $= 2$
8
4
4
8

The *reciprocal property* obeyed by f' and $(f^{-1})'$ in Example 4 holds for all functions with inverses as shown in the Fig.



THEOREM 3.23 Derivative of the Inverse Function

Let f be differentiable and have an inverse on an interval I. If x_0 is a point of I at which $f'(x_0) \neq 0$, then f^{-1} is differentiable at $y_0 = f(x_0)$ and

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}, \text{ where } \quad y_0 = f(x_0).$$

Or, $\dfrac{dy}{dx} = \dfrac{1}{\frac{dx}{dy}}$

The slope of the line tangent to the graph of f^{-1} at the point (y_0, x_0) is the reciprocal of the slope of the line tangent to the graph of f at the point (x_0, y_0).

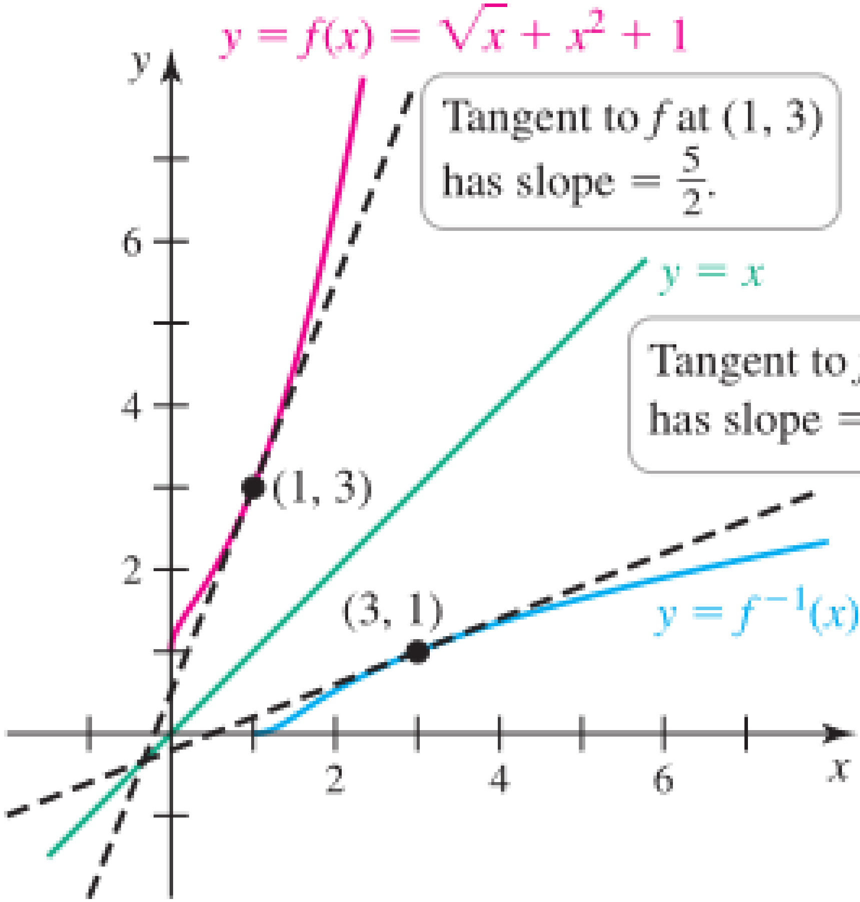
We can evaluate the derivative of the inverse function without finding the inverse function itself.

Proof.

- At a point x_0 where f is differentiable, $y_0 = f(x_0)$ and $x_0 = f^{-1}(y_0)$.
- f is continuous at x_0, f^{-1} is continuous at y_0. Therefore, as $y \to y_0$, $x \to x_0$.

$$
\begin{aligned}
(f^{-1})'(y_0) &= \lim_{y \to y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} && \text{Definition of derivative of } f^{-1} \\
&= \lim_{x \to x_0} \frac{x - x_0}{f(x) - f(x_0)} && y = f(x) \text{ and } x = f^{-1}(y);\ x \to x_0 \text{ as } y \to y_0 \\
&= \lim_{x \to x_0} \frac{1}{\dfrac{f(x) - f(x_0)}{x - x_0}} && \frac{a}{b} = \frac{1}{b/a} \\
&= \frac{1}{f'(x_0)}. && \text{Definition of derivative of } f
\end{aligned}
$$

EXAMPLE 5 **Derivative of an inverse function** The function $f(x) = \sqrt{x} + x^2 + 1$ is one-to-one, for $x \geq 0$, and has an inverse on that interval. Find the slope of the curve $y = f^{-1}(x)$ at the point $(3, 1)$.



EXAMPLE 6 **Derivatives of an inverse function** Use the values of a one-to-one differentiable function f in Table 3.7 to compute the indicated derivatives or state that the derivative cannot be determined.

Table 3.7

x	-1	0	1	2	3
$f(x)$	2	3	5	6	7
$f'(x)$	$1/2$	2	$3/2$	1	$2/3$

a. $(f^{-1})'(5)$ **b.** $(f^{-1})'(2)$ **c.** $(f^{-1})'(1)$

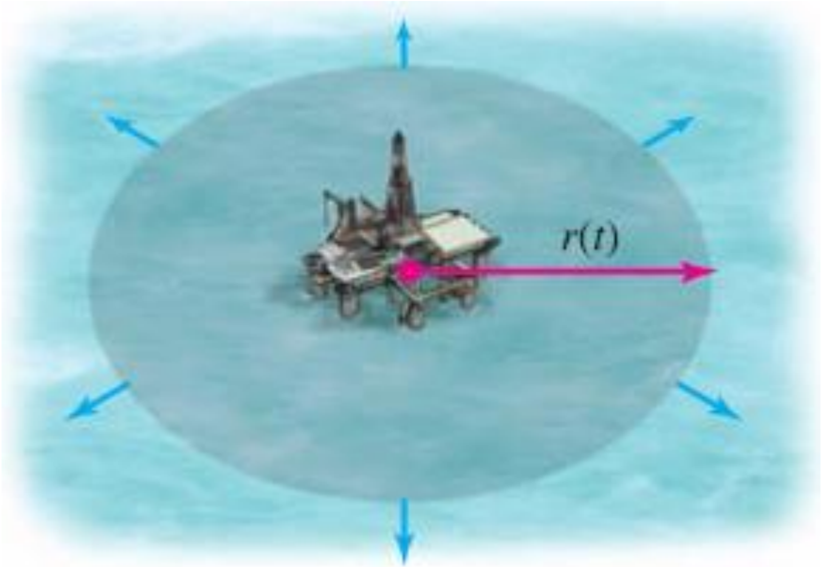
3.11

Related Rates

The essential feature of the derivative problems is that two or more variables, which are related in a known way, are themselves changing in time.

Related rates

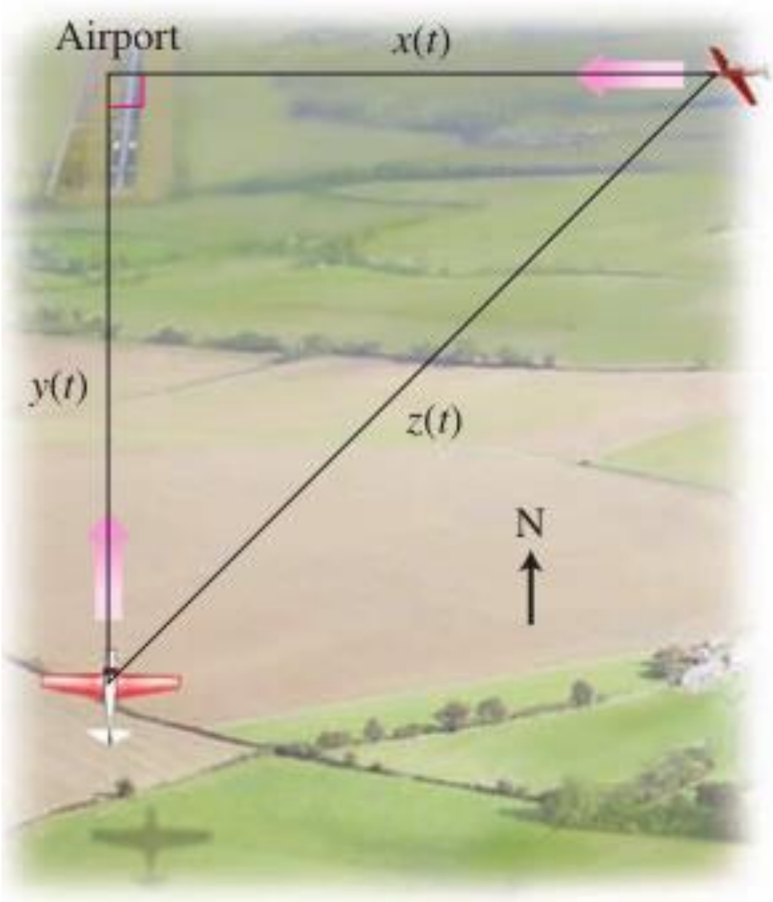
EXAMPLE 1 **Spreading oil** An oil rig springs a leak in calm seas, and the oil spreads in a circular patch around the rig. If the radius of the oil patch increases at a rate of 30 m/hr, how fast is the area of the patch increasing when the patch has a radius of 100 meters (Figure 3.72)?



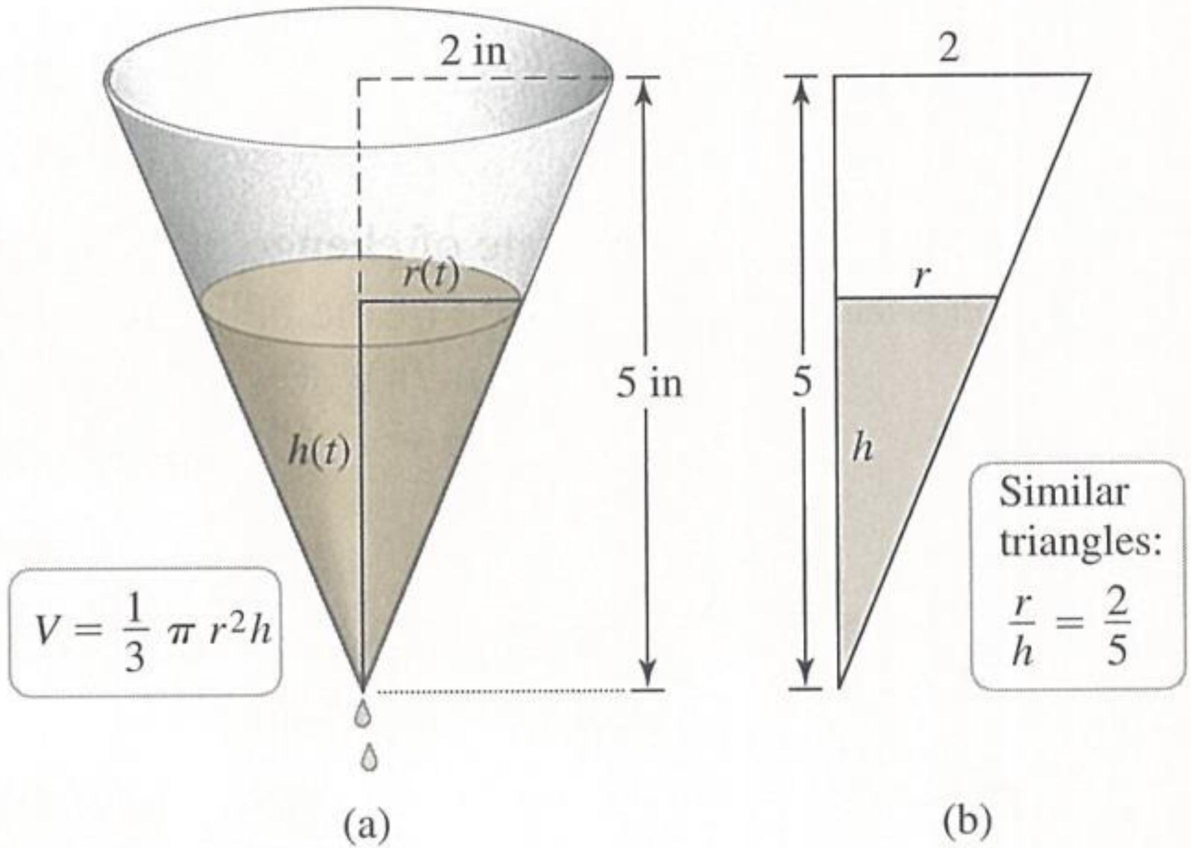
PROCEDURE **Steps for Related-Rate Problems**

1. Read the problem carefully, making a sketch to organize the given information. Identify the rates that are given and the rate that is to be determined.
2. Write one or more equations that express the basic relationships among the variables.
3. Introduce rates of change by differentiating the appropriate equation(s) with respect to time t.
4. Substitute known values and solve for the desired quantity.
5. Check that units are consistent and the answer is reasonable. (For example, does it have the correct sign?)

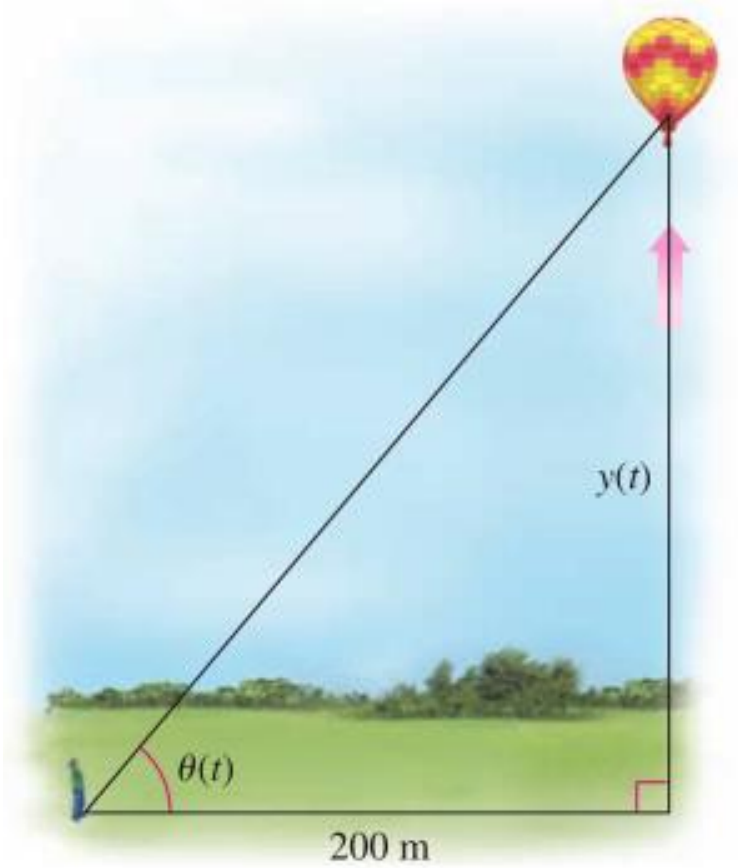
EXAMPLE 2 **Converging airplanes** Two small planes approach an airport, one flying due west at 120 mi/hr and the other flying due north at 150 mi/hr. Assuming they fly at the same constant elevation, how fast is the distance between the planes changing when the westbound plane is 180 miles from the airport and the northbound plane is 225 miles from the airport?

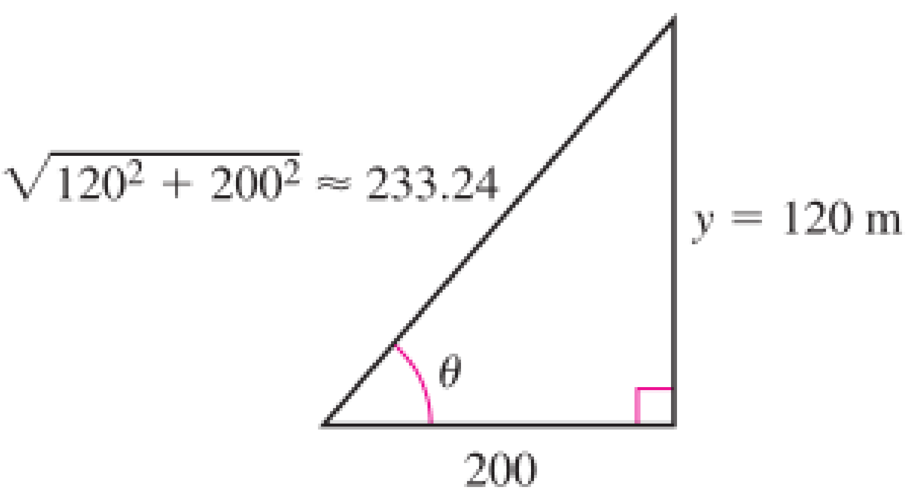


EXAMPLE 3 **Morning coffee** Coffee is draining out of a conical filter at a rate of $2.25 \text{ in}^3/\text{min}$. If the cone is 5 in tall and has a radius of 2 in, how fast is the coffee level dropping when the coffee is 3 in deep?



EXAMPLE 4 **Observing a launch** An observer stands 200 meters from the launch site of a hot-air balloon. The balloon rises vertically at a constant rate of 4 m/s. How fast is the angle of elevation of the balloon increasing 30 seconds after the launch? (The angle of elevation is the angle between the ground and the observer's line of sight to the balloon.)





$$\cos \theta \approx \frac{200}{233.24} \approx 0.86$$

Chapter 3

Derivatives (II)

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