# Chapter 14

## Vector-Valued Functions

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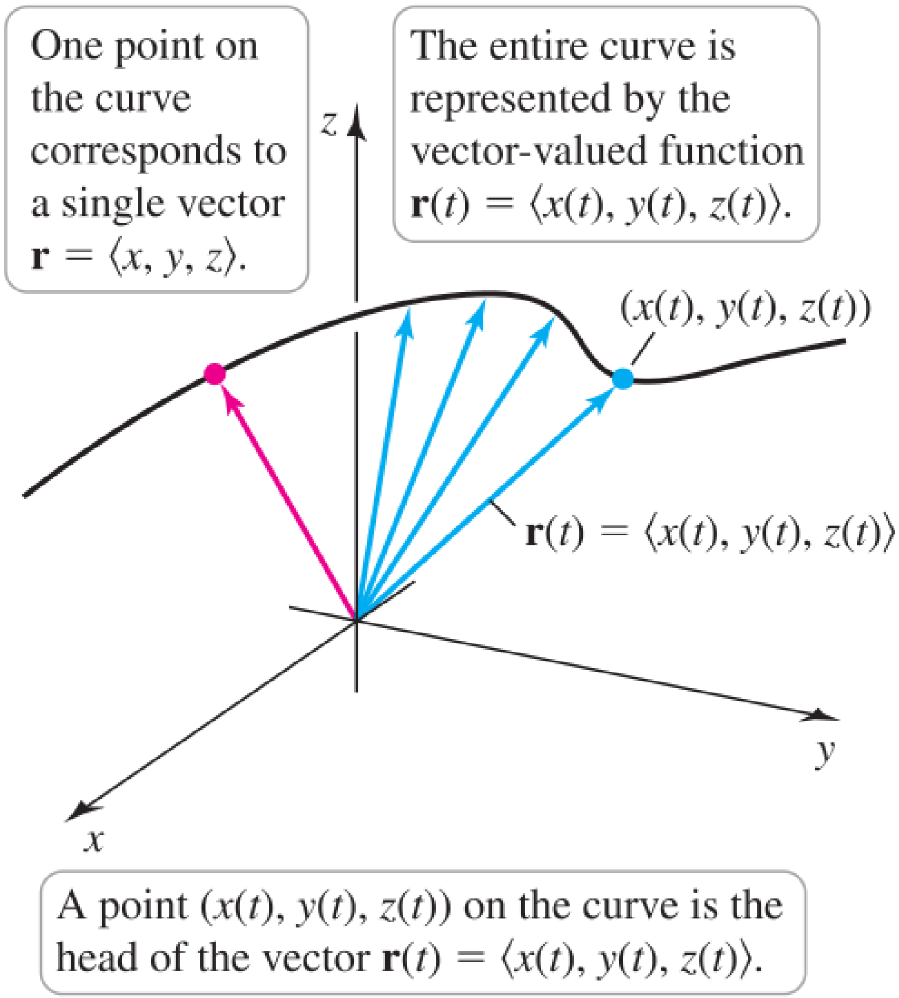
# 14.1

## Vector-Valued Functions

# Vector-Valued Functions

A function of the form $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ may be viewed in two ways.

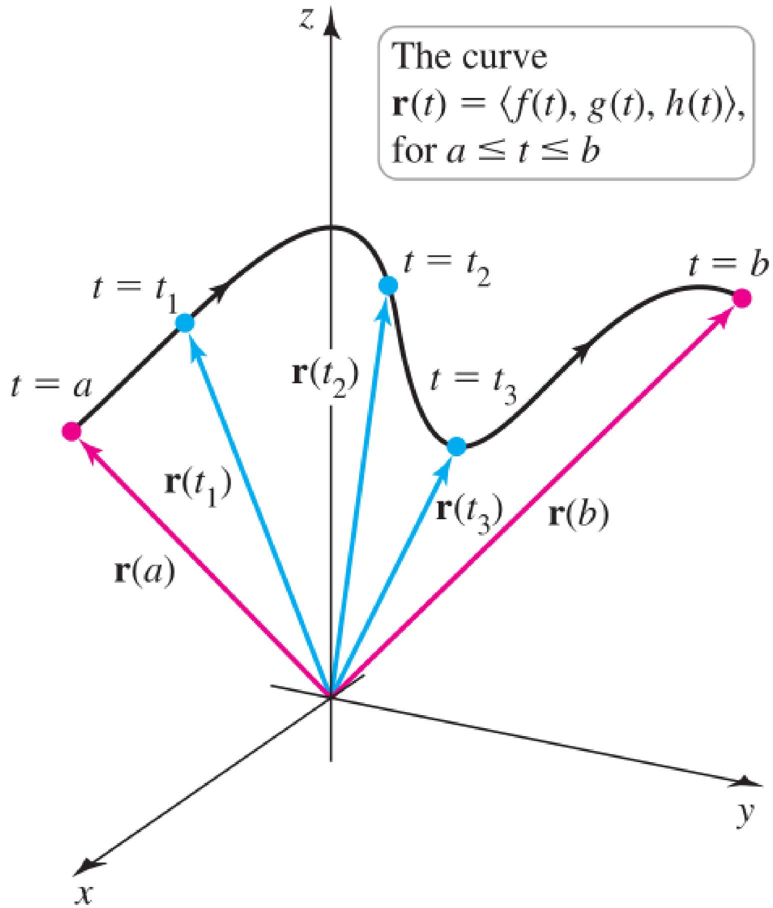
- A set of three parametric equations that describe a curve in space.
- A *vector-valued function*, meaning the three dependent variables $x, y, z$ are the components of $\mathbf{r}$, and each component varies with respect to a single independent variable $t$ (time).



## Curves in Space

$$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

The domain of **r** is the largest set of values of $t$ on which all of $f$, $g$, and $h$ are defined.

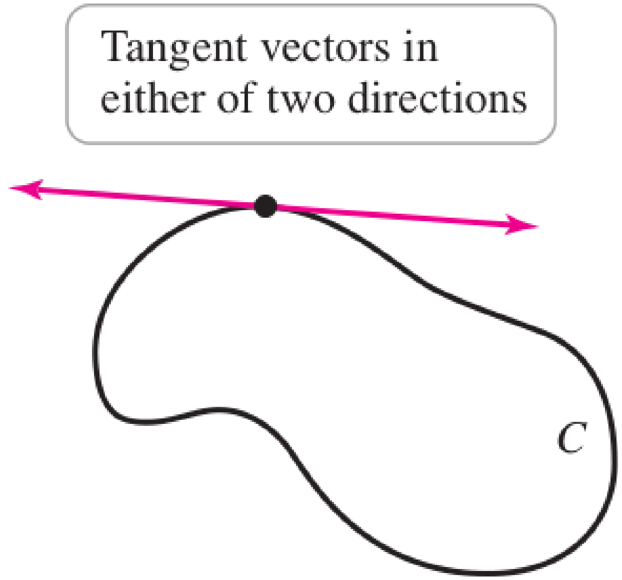



As parameter $t$ varies over the interval $a \le t \le b$, each value of $t$ produces a position vector that corresponds to a point on the curve,
Starting at the initial vector $\mathbf{r}(a)$
Ending at the terminal vector $\mathbf{r}(b)$

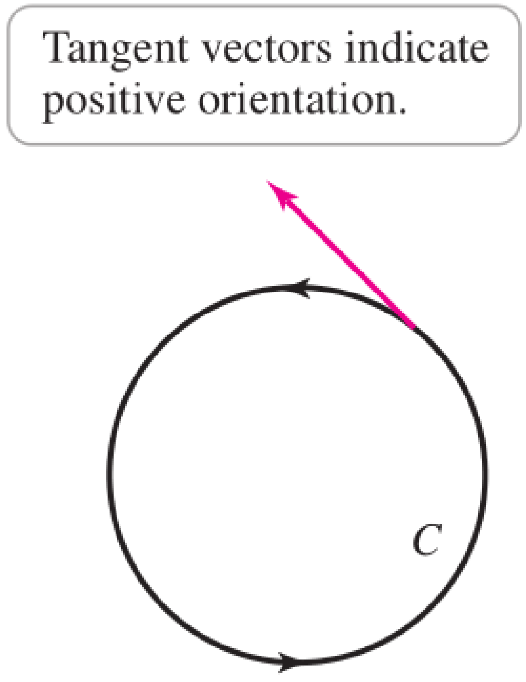
# Orientation of Curves

The *positive orientation* is the direction in which the curve is generated as the parameter increases from $a$ to $b$

The orientation of a parameterized curve and its tangent vectors are consistent




Unparameterized curve

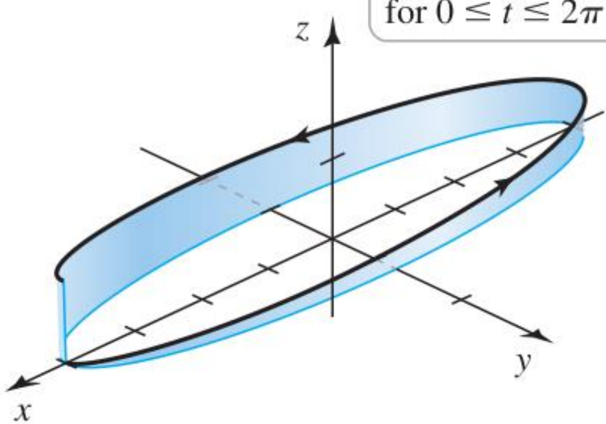



Parameterized curve

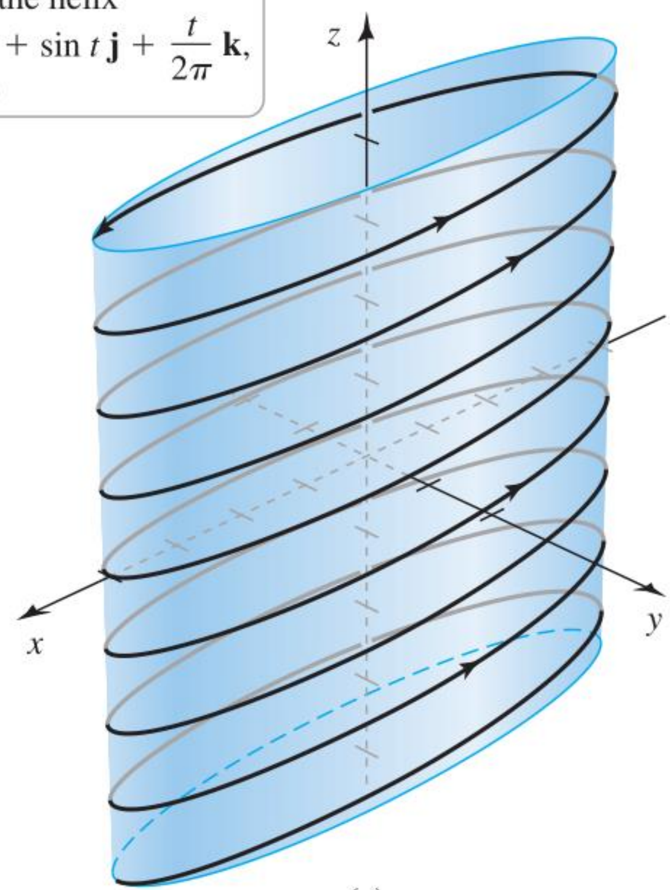
**EXAMPLE 4** **A helix** Graph the curve described by the equation

$$\mathbf{r}(t) = 4\cos t\,\mathbf{i} + \sin t\,\mathbf{j} + \frac{t}{2\pi}\,\mathbf{k},$$

where (a) $0 \le t \le 2\pi$ and (b) $-\infty < t < \infty$.



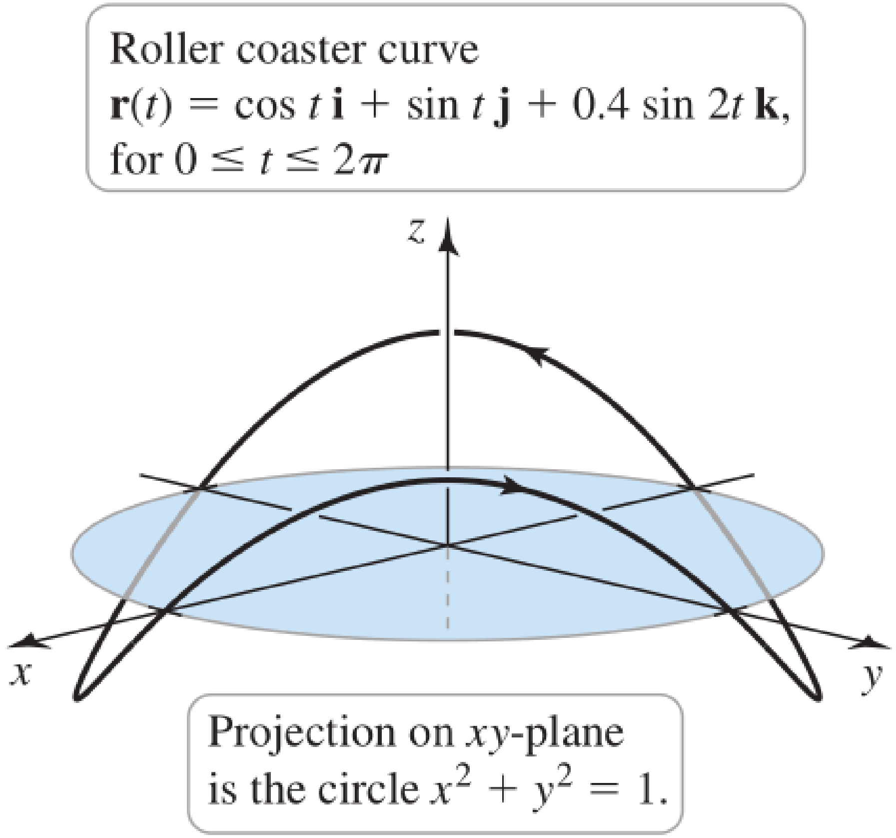

One loop of the helix
$\mathbf{r}(t) = 4\cos t\,\mathbf{i} + \sin t\,\mathbf{j} + \frac{t}{2\pi}\,\mathbf{k}$,
for $0 \le t \le 2\pi$




Eight loops of the helix
$\mathbf{r}(t) = 4\cos t\,\mathbf{i} + \sin t\,\mathbf{j} + \frac{t}{2\pi}\,\mathbf{k}$,
for $-\infty < t < \infty$

**EXAMPLE 5** **Roller coaster curve** Graph the curve

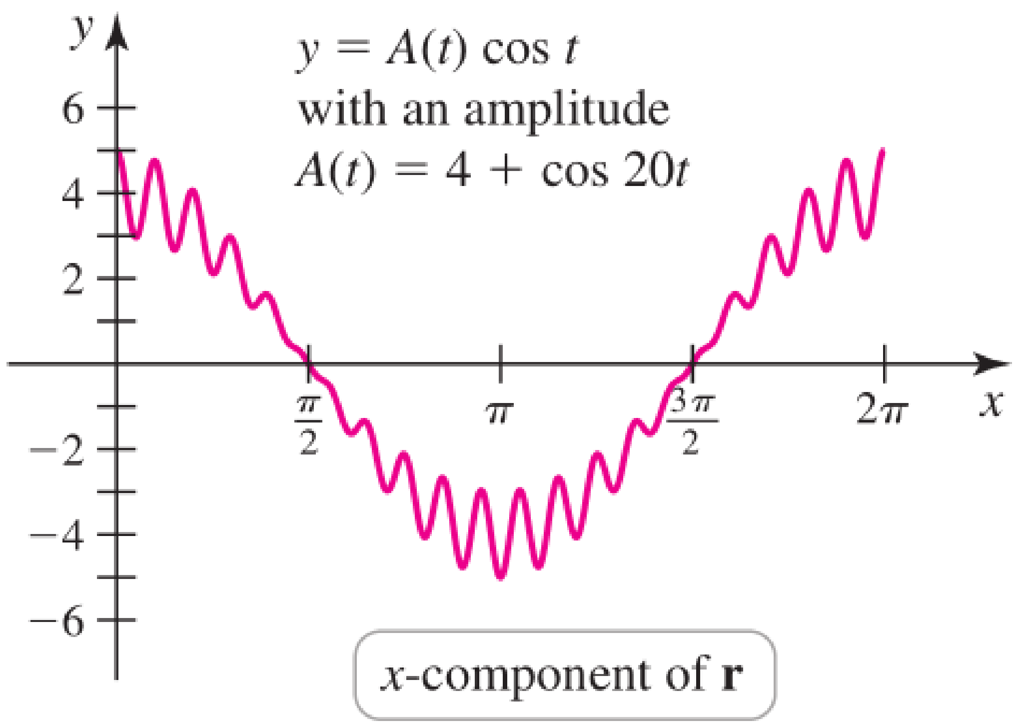
$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + 0.4\sin 2t\,\mathbf{k}, \quad \text{for } 0 \le t \le 2\pi.$$

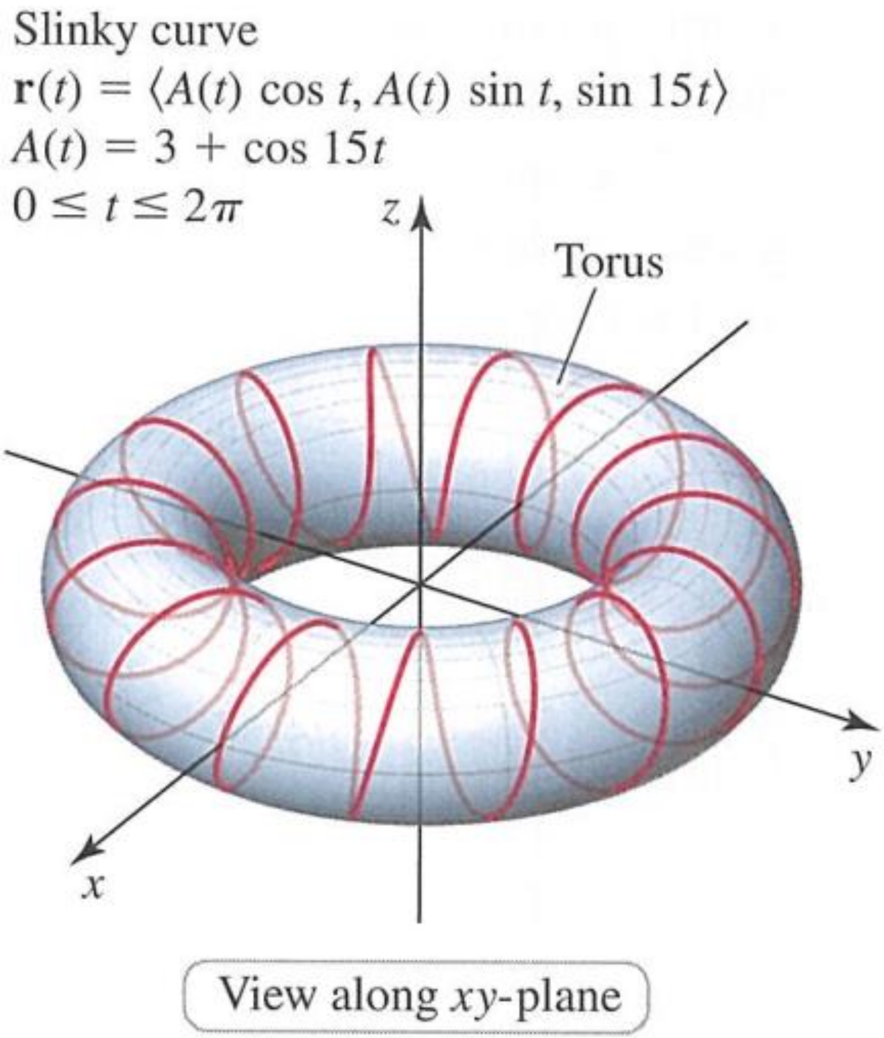


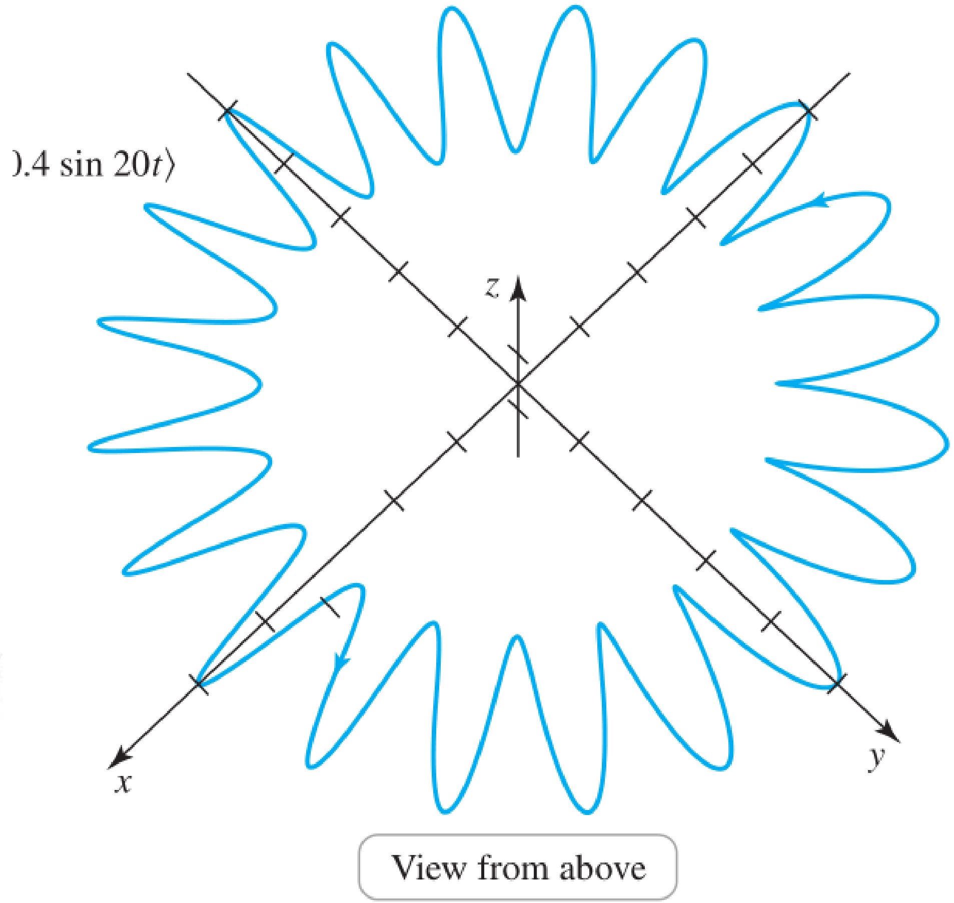
**EXAMPLE 6** **Slinky curve** Graph the curve

$$\mathbf{r}(t) = (4 + \cos 20t)\cos t\,\mathbf{i} + (4 + \cos 20t)\sin t\,\mathbf{j} + 0.4\sin 20t\,\mathbf{k},$$

for $0 \le t \le 2\pi$.




Slinky curve
$\mathbf{r}(t) = \langle A(t)\cos t, A(t)\sin t, \sin 15t\rangle$
$A(t) = 3 + \cos 15t$
$0 \le t \le 2\pi$
z
Torus
x
y
View along xy-plane


).4 sin 20t⟩
z
x
y
View from above

# Limits and Continuity for Vector-Valued Functions

The concepts of limits, derivatives, integrals of vector-valued functions are direct extensions of what you have already learned.

**DEFINITION Limit of a Vector-Valued Function**

A vector-valued function $\mathbf{r}$ approaches the limit $\mathbf{L}$ as $t$ approaches $a$, written $\lim_{t \to a} \mathbf{r}(t) = \mathbf{L}$, provided $\lim_{t \to a} |\mathbf{r}(t) - \mathbf{L}| = 0$.

Given $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, suppose that

$$\lim_{t\to a} f(t) = L_1, \lim_{t\to a} g(t) = L_2, \lim_{t\to a} h(t) = L_3$$

Then,

$$\lim_{t\to a} \mathbf{r}(t) = \left\langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t) \right\rangle = \langle L_1, L_2, L_3 \rangle$$

The limit of $\mathbf{r}(t)$ is determined by computing the limits of its components.

The limits laws in Chapter 2 have analogs for vector-valued functions. For example,

$$\lim_{t\to a}(\mathbf{r}(t) + \mathbf{s}(t)) = \lim_{t\to a} \mathbf{r}(t) + \lim_{t\to a} \mathbf{s}(t)$$

$$\lim_{t\to a} c\mathbf{r}(t) = c \lim_{t\to a} \mathbf{r}(t)$$

## Continuity

A function $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ is continuous at $a$ provided

$$\lim_{t\to a} \mathbf{r}(t) = \mathbf{r}(a)$$

If the component functions $f$, $g$, and $h$ of $\mathbf{r}(t)$ are continuous at $a$, then $\mathbf{r}$ is also continuous at $a$ and vice versa.

The function $\mathbf{r}$ is continuous on an interval $I$ if it is continuous for all $t$ in $I$.

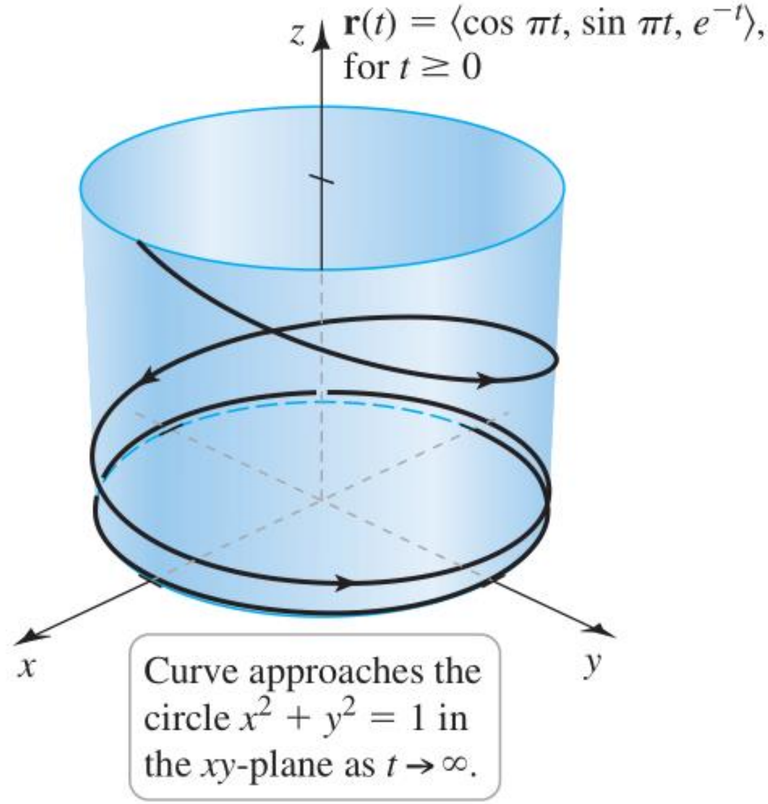
**EXAMPLE 7** **Limits and continuity** Consider the function

$$\mathbf{r}(t) = \cos \pi t\,\mathbf{i} + \sin \pi t\,\mathbf{j} + e^{-t}\,\mathbf{k}, \quad \text{for } t \geq 0.$$

**a.** Evaluate $\lim_{t \to 2} \mathbf{r}(t)$.

**b.** Evaluate $\lim_{t \to \infty} \mathbf{r}(t)$.

**c.** At what points is $\mathbf{r}$ continuous?



# 14.2

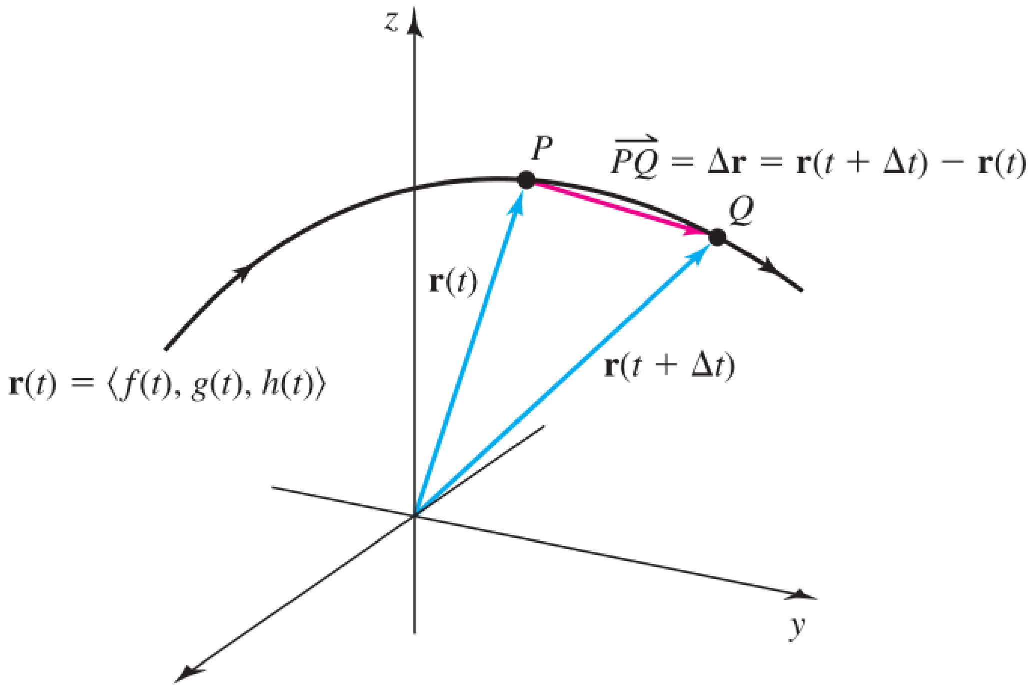
## Calculus of Vector-Valued Functions

# The Derivative and Tangent Vector

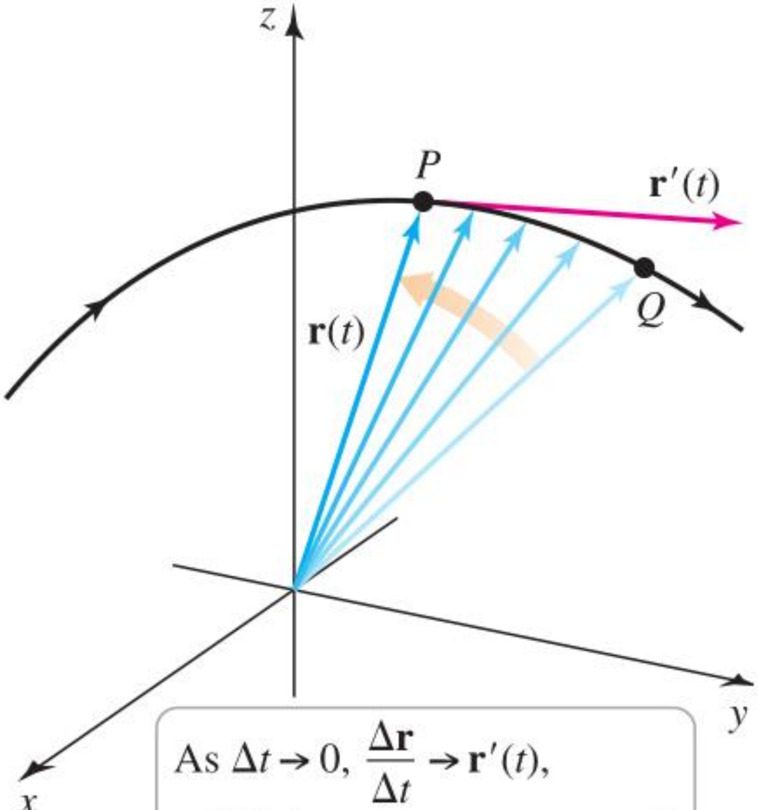
A function $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, where $f$, $g$, and $h$ are differentiable functions on an interval $a < t < b$

The derivative of the vector-valued function $\boldsymbol{r}(t)$

$$\mathbf{r}'(t) = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}$$




(a)




As $\Delta t \to 0$, $\dfrac{\Delta \mathbf{r}}{\Delta t} \to \mathbf{r}'(t)$,
which is a tangent vector at $P$.

This new function $\mathbf{r}'(t)$ has two important interpretations.

- The vector $\mathbf{r}'(t)$ points in the direction of the curve at $P$. For this reason, $\mathbf{r}'(t)$ is a *tangent vector* at $P$.
- The vector $\mathbf{r}'(t)$ is the *derivative* of $\mathbf{r}$ with respect to $t$; it gives the rate of change of the function $\mathbf{r}(t)$.
- $\mathbf{r}'(t)$ is the velocity vector of a moving object with position $\mathbf{r}(t)$, pointing in the direction of motion, and $|\mathbf{r}'(t)|$ is the speed of the object.

Evaluate the limit that defines $\mathbf{r}'(t)$ in terms of its components

$$\mathbf{r}'(t) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{(f(t + \Delta t)\,\mathbf{i} + g(t + \Delta t)\,\mathbf{j} + h(t + \Delta t)\,\mathbf{k}) - (f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k})}{\Delta t}$$

Substitute components of $\mathbf{r}$.

$$= \lim_{\Delta t \to 0} \left( \frac{f(t + \Delta t) - f(t)}{\Delta t}\mathbf{i} + \frac{g(t + \Delta t) - g(t)}{\Delta t}\mathbf{j} + \frac{h(t + \Delta t) - h(t)}{\Delta t}\mathbf{k} \right)$$

Rearrange terms inside of limit.

$$= \underbrace{\lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}}_{f'(t)}\mathbf{i} + \underbrace{\lim_{\Delta t \to 0} \frac{g(t + \Delta t) - g(t)}{\Delta t}}_{g'(t)}\mathbf{j} + \underbrace{\lim_{\Delta t \to 0} \frac{h(t + \Delta t) - h(t)}{\Delta t}}_{h'(t)}\mathbf{k}$$

Limit of sum equals sum of limits functions

**DEFINITION Derivative and Tangent Vector**

Let $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$, where $f$, $g$, and $h$ are differentiable functions on $(a, b)$. Then $\mathbf{r}$ has a **derivative** (or is **differentiable**) on $(a, b)$ and

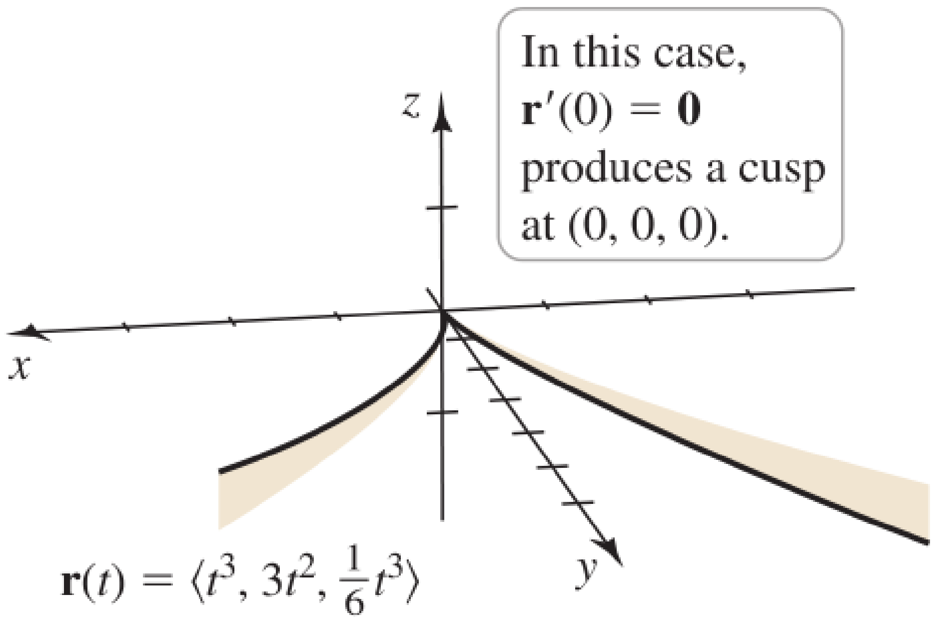
$$\mathbf{r}'(t) = f'(t)\,\mathbf{i} + g'(t)\,\mathbf{j} + h'(t)\,\mathbf{k}.$$

Provided $\mathbf{r}'(t) \neq \mathbf{0}$, $\mathbf{r}'(t)$ is a **tangent vector** at the point corresponding to $\mathbf{r}(t)$.

**EXAMPLE 1** **Derivative of vector functions** Compute the derivative of the following functions.

**a.** $\mathbf{r}(t) = \langle t^3, 3t^2, t^3/6 \rangle$

**b.** $\mathbf{r}(t) = e^{-t}\,\mathbf{i} + 10\sqrt{t}\,\mathbf{j} + 2\cos 3t\,\mathbf{k}$



# Unit Tangent Vector

*Unit tangent vector* applies when only the direction (but not the length) of the tangent vector is of interest.

**DEFINITION Unit Tangent Vector**

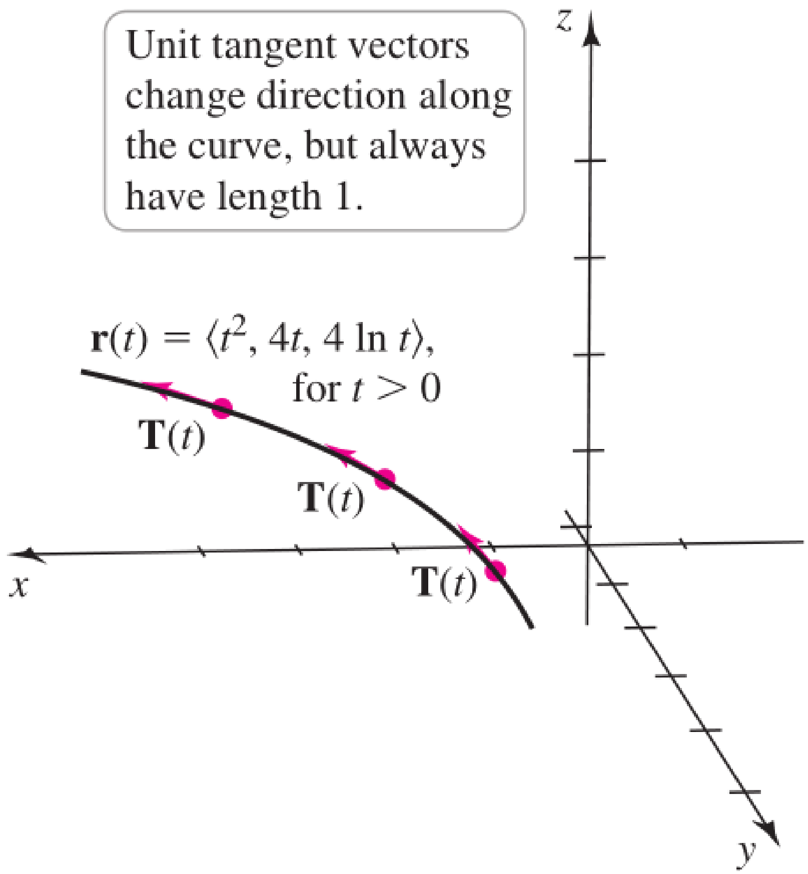
Let $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$ be a smooth parameterized curve, for $a \leq t \leq b$. The **unit tangent vector** for a particular value of $t$ is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}.$$

**EXAMPLE 2** **Unit tangent vectors** Find the unit tangent vectors for the following parameterized curves.

**a.** $\mathbf{r}(t) = \langle t^2, 4t, 4 \ln t \rangle, \quad \text{for } t > 0$

**b.** $\mathbf{r}(t) = \langle 10, 3 \cos t, 3 \sin t \rangle, \quad \text{for } 0 \leq t \leq 2\pi$



**THEOREM 7 Derivative Rules**

Let $\mathbf{u}$ and $\mathbf{v}$ be differentiable vector-valued functions and let $f$ be a differentiable scalar-valued function, all at a point $t$. Let $\mathbf{c}$ be a constant vector. The following rules apply.

1. $\dfrac{d}{dt}(\mathbf{c}) = \mathbf{0}$ Constant Rule

2. $\dfrac{d}{dt}(\mathbf{u}(t) + \mathbf{v}(t)) = \mathbf{u}'(t) + \mathbf{v}'(t)$ Sum Rule

3. $\dfrac{d}{dt}(f(t)\mathbf{u}(t)) = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$ Product Rule

4. $\dfrac{d}{dt}\big(\mathbf{u}(f(t))\big) = \mathbf{u}'(f(t))f'(t)$ Chain Rule

5. $\dfrac{d}{dt}(\mathbf{u}(t) \cdot \mathbf{v}(t)) = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$ Dot Product Rule

6. $\dfrac{d}{dt}(\mathbf{u}(t) \times \mathbf{v}(t)) = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$ Cross Product Rule

**EXAMPLE 3** **Derivative rules** Compute the following derivatives, where

$$\mathbf{u}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} - t^3\,\mathbf{k} \quad \text{and} \quad \mathbf{v}(t) = \sin t\,\mathbf{i} + 2\cos t\,\mathbf{j} + \cos t\,\mathbf{k}.$$

**a.** $\dfrac{d}{dt}(\mathbf{v}(t^2))$ **b.** $\dfrac{d}{dt}(t^2\,\mathbf{v}(t))$ **c.** $\dfrac{d}{dt}(\mathbf{u}(t)\cdot\mathbf{v}(t))$

## Higher-Order Derivatives

Simply differentiate each component multiple times.

**EXAMPLE 4** **Higher-order derivatives** Compute the first, second, and third derivative of $\mathbf{r}(t) = \langle t^2, 8 \ln t, 3e^{-2t} \rangle$.

# Integrals of Vector-Valued Functions

An **antiderivative** of the vector function $\mathbf{r}$ is a function $\mathbf{R}$ such that $\mathbf{R}' = \mathbf{r}$.

**DEFINITION Indefinite Integral of a Vector-Valued Function**

Let $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$ be a vector function and let $\mathbf{R}(t) = F(t)\,\mathbf{i} + G(t)\,\mathbf{j} + H(t)\,\mathbf{k}$, where $F$, $G$, and $H$ are antiderivatives of $f$, $g$, and $h$, respectively. The **indefinite integral** of $\mathbf{r}$ is

$$\int \mathbf{r}(t)\,dt = \mathbf{R}(t) + \mathbf{C},$$

where $\mathbf{C}$ is an arbitrary constant vector. Alternatively, in component form,

$$\int \langle f(t), g(t), h(t) \rangle\,dt = \langle F(t), G(t), H(t) \rangle + \langle C_1, C_2, C_3 \rangle.$$

**EXAMPLE 5** **Indefinite integrals** Compute

$$\int \left( \frac{t}{\sqrt{t^2 + 2}} \mathbf{i} + e^{-3t} \mathbf{j} + (\sin 4t + 1) \mathbf{k} \right) dt.$$

**EXAMPLE 6** **Finding one antiderivative** Find $\mathbf{r}(t)$ such that $\mathbf{r}'(t) = \langle 10, \sin t, t \rangle$ and $\mathbf{r}(0) = \mathbf{j}$.

**DEFINITION Definite Integral of a Vector-Valued Function**

Let $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$, where $f$, $g$, and $h$ are integrable on the interval $[a, b]$. The **definite integral** of $\mathbf{r}$ on $[a, b]$ is

$$\int_a^b \mathbf{r}(t)\,dt = \left(\int_a^b f(t)\,dt\right)\mathbf{i} + \left(\int_a^b g(t)\,dt\right)\mathbf{j} + \left(\int_a^b h(t)\,dt\right)\mathbf{k}$$

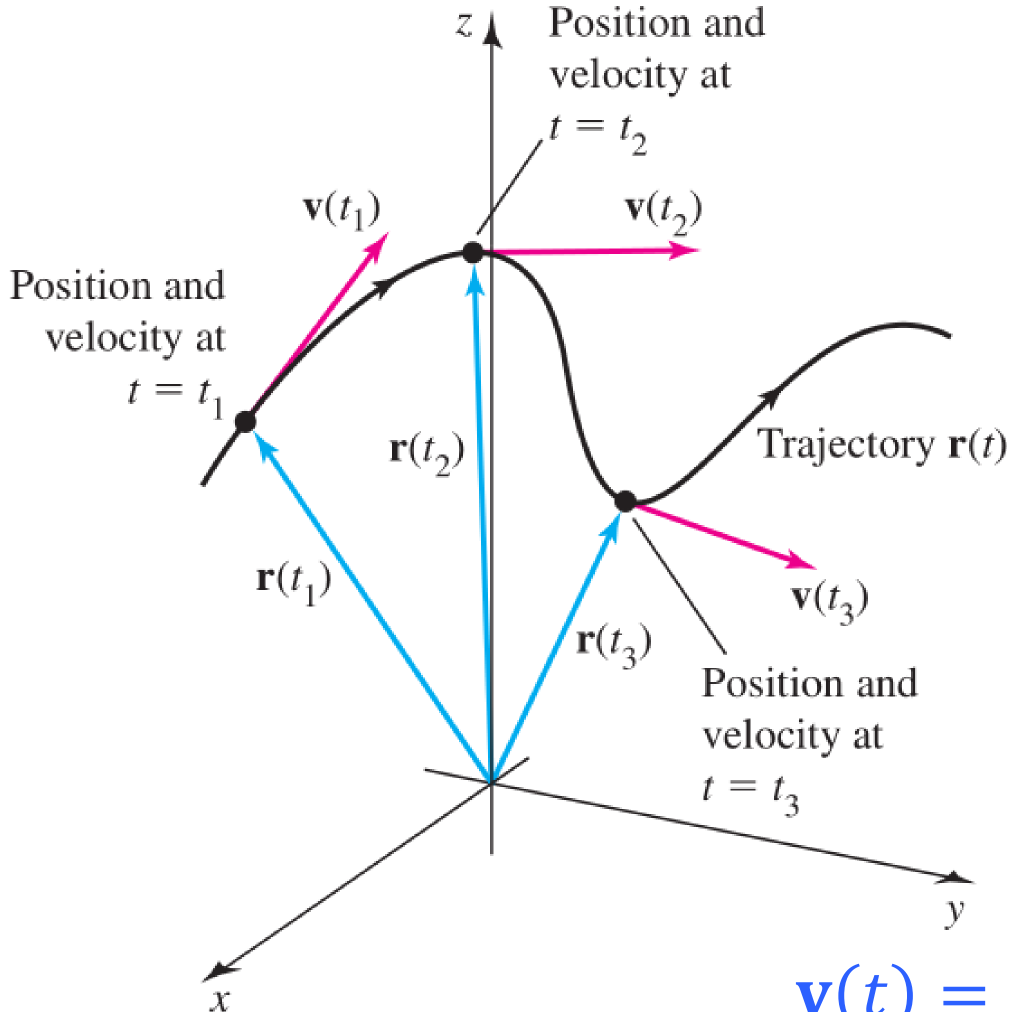
**EXAMPLE 7 Definite integrals** Evaluate

$$\int_0^\pi \left(\mathbf{i} + 3\cos\frac{t}{2}\,\mathbf{j} - 4t\,\mathbf{k}\right) dt.$$

# 14.3

## Motion in Space

# Position, Velocity, Speed, Acceleration




$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$, describe the position of a moving object

The curve described by $\mathbf{r}$ is the *path or trajectory* of the object

$\mathbf{r}'(t)$ is the *instantaneous velocity* of the object,

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$$

**DEFINITION Position, Velocity, Speed, Acceleration**

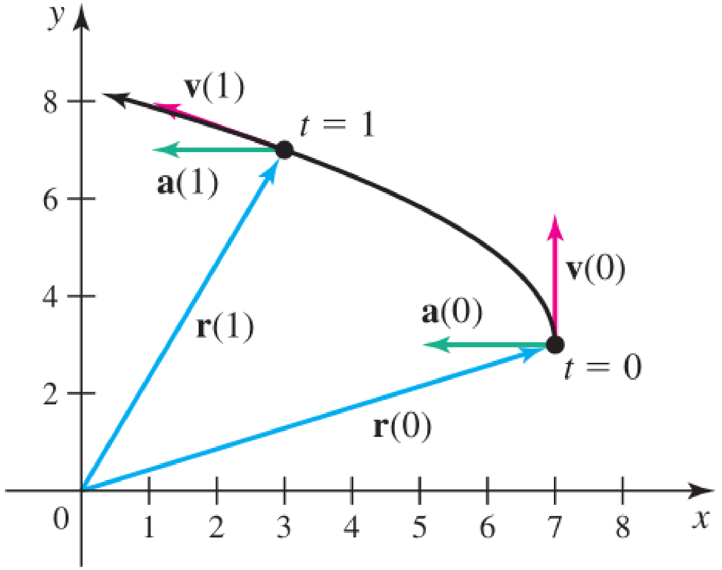
Let the **position** of an object moving in three-dimensional space be given by $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$, for $t \geq 0$. The **velocity** of the object is

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle.$$

The **speed** of the object is the scalar function

$$|\mathbf{v}(t)| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}.$$

The **acceleration** of the object is $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$.



**EXAMPLE 1** **Velocity and acceleration for circular motion** Consider the two-dimensional motion given by the position vector

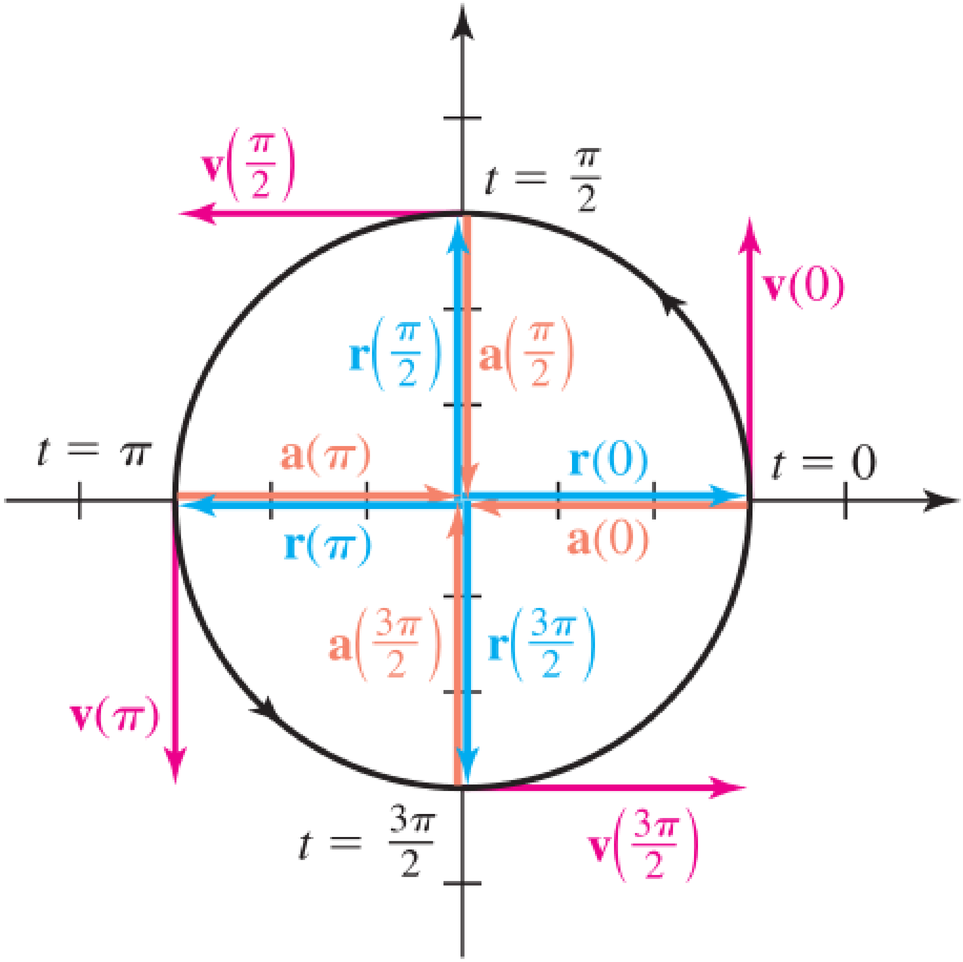
$$\mathbf{r}(t) = \langle x(t), y(t) \rangle = \langle 3 \cos t, 3 \sin t \rangle, \quad \text{for } 0 \leq t \leq 2\pi.$$

**a.** Sketch the trajectory of the object.

**b.** Find the velocity and speed of the object.

**c.** Find the acceleration of the object.

**d.** Sketch the position, velocity, and acceleration vectors, for $t = 0, \pi/2, \pi$, and $3\pi/2$.




Circular motion: At all times $\mathbf{a}(t) = -\mathbf{r}(t)$ and $\mathbf{v}(t)$ is orthogonal to $\mathbf{r}(t)$ and $\mathbf{a}(t)$.

**EXAMPLE 2** **Comparing trajectories** Consider the trajectories described by the position functions

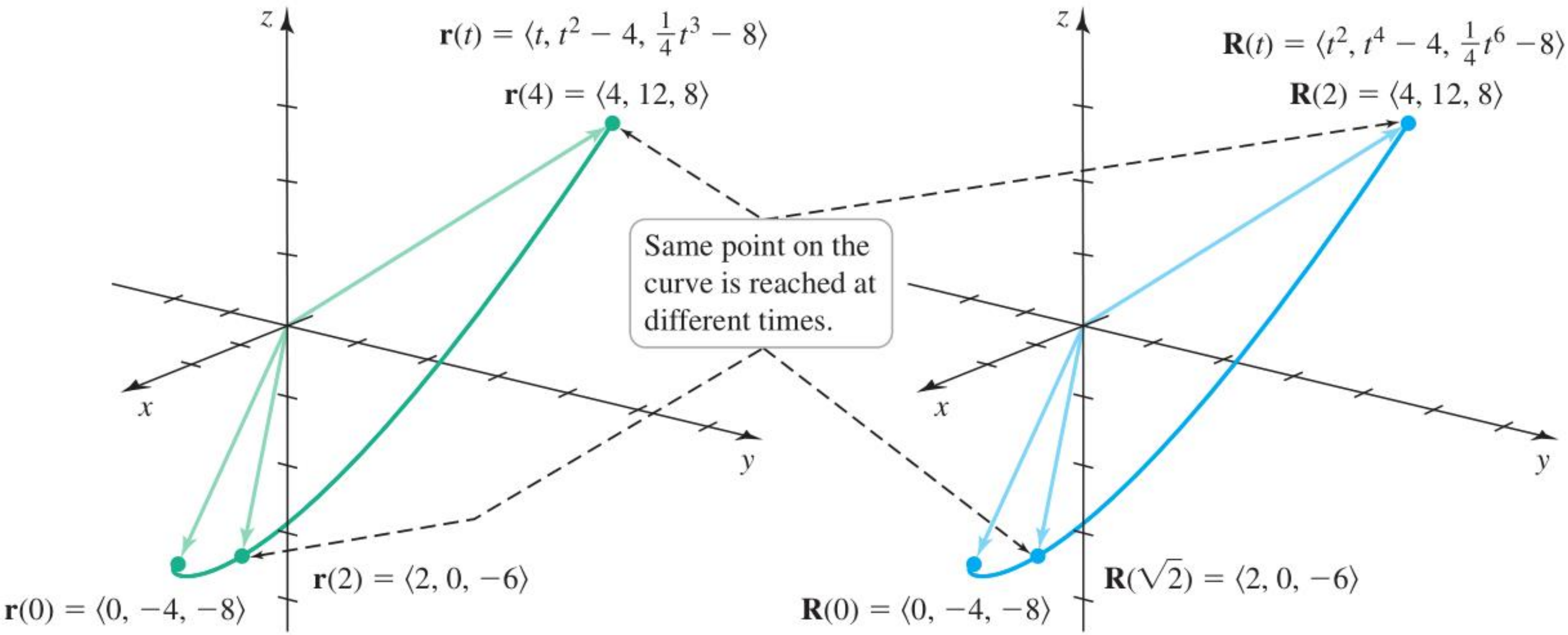
$$\mathbf{r}(t) = \left\langle t, t^2 - 4, \frac{t^3}{4} - 8 \right\rangle, \quad \text{for } t \geq 0, \text{ and}$$

$$\mathbf{R}(t) = \left\langle t^2, t^4 - 4, \frac{t^6}{4} - 8 \right\rangle, \quad \text{for } t \geq 0,$$

where $t$ is measured in the same time units for both functions.

**a.** Graph and compare the trajectories using a graphing utility.

**b.** Find the velocity vectors associated with the position functions.


$\mathbf{r}(t) = \langle t, t^2 - 4, \frac{1}{4}t^3 - 8 \rangle$
$\mathbf{r}(4) = \langle 4, 12, 8 \rangle$
$\mathbf{R}(t) = \langle t^2, t^4 - 4, \frac{1}{4}t^6 - 8 \rangle$
$\mathbf{R}(2) = \langle 4, 12, 8 \rangle$
Same point on the curve is reached at different times.
$\mathbf{r}(2) = \langle 2, 0, -6 \rangle$
$\mathbf{r}(0) = \langle 0, -4, -8 \rangle$
$\mathbf{R}(\sqrt{2}) = \langle 2, 0, -6 \rangle$
$\mathbf{R}(0) = \langle 0, -4, -8 \rangle$
x
y
z
x
y
z

# Straight-Line and Circular Motion

Two types of motion in space

- **Uniform** (constant velocity) **straight-line motion**

$$\mathbf{r}(t) = \langle x_0 + at, y_0 + bt, z_0 + ct \rangle, \qquad t \geq 0$$

A straight-line trajectory with an initial point $\langle x_0, y_0, z_0 \rangle$ and a direction given by the vector $\langle a, b, c \rangle$

- **Circular motion**

$$\mathbf{r}(t) = \langle A \cos t, A \sin t \rangle, \qquad 0 \leq t \leq 2\pi$$

The velocity and acceleration vectors are

$$\boldsymbol{v}(t) = \langle -A \sin t, A \cos t \rangle$$

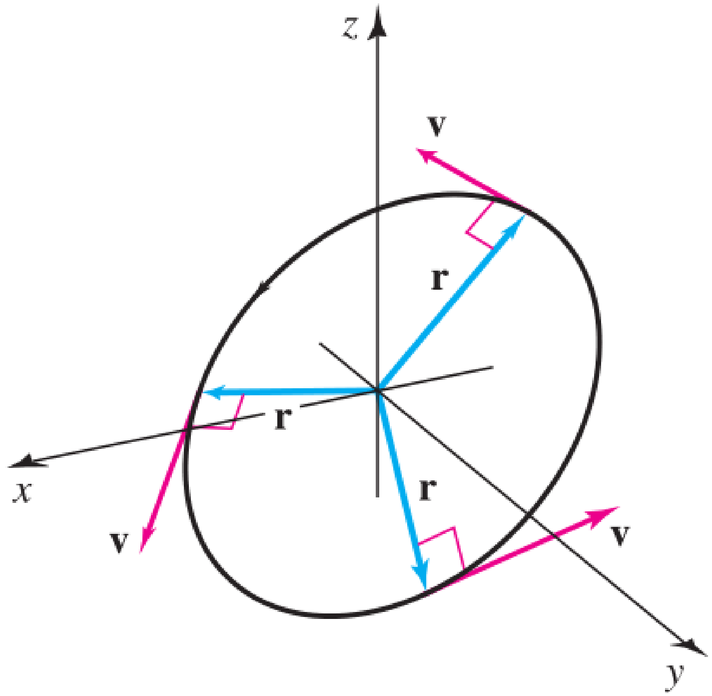
$$\mathbf{a}(t) = \langle -A \cos t, -A \sin t \rangle = -\mathbf{r}(t)$$

**r** and **a** are parallel, but point in opposite directions.

$\mathbf{r} \cdot \mathbf{v} = \mathbf{a} \cdot \mathbf{v} = 0$; the position and acceleration vectors are both orthogonal to the velocity vectors at any given point

**THEOREM 8 Motion with Constant $|\mathbf{r}|$**

Let $\mathbf{r}$ describe a path on which $|\mathbf{r}|$ is constant (motion on a circle or sphere centered at the origin). Then $\mathbf{r} \cdot \mathbf{v} = 0$, which means the position vector and the velocity vector are orthogonal at all times for which the functions are defined.



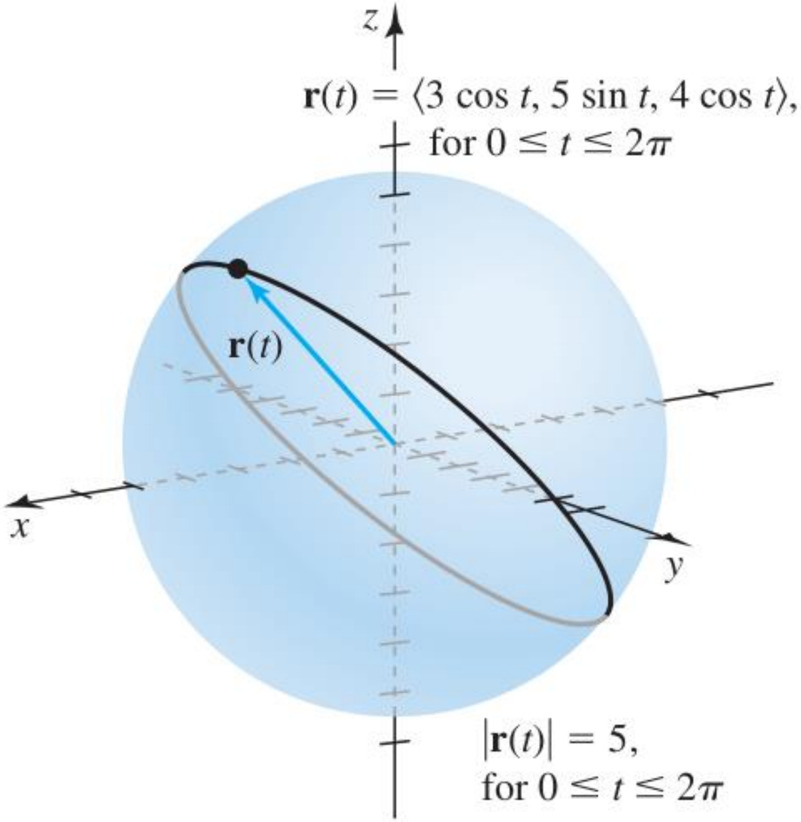

On a trajectory on which $|\mathbf{r}|$ is constant, $\mathbf{v}$ is orthogonal to $\mathbf{r}$ at all points.

**EXAMPLE 3** **Path on a sphere** An object moves on a trajectory described by

$$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle = \langle 3\cos t, 5\sin t, 4\cos t \rangle, \quad \text{for } 0 \le t \le 2\pi.$$

**a.** Show that the object moves on a sphere and find the radius of the sphere.

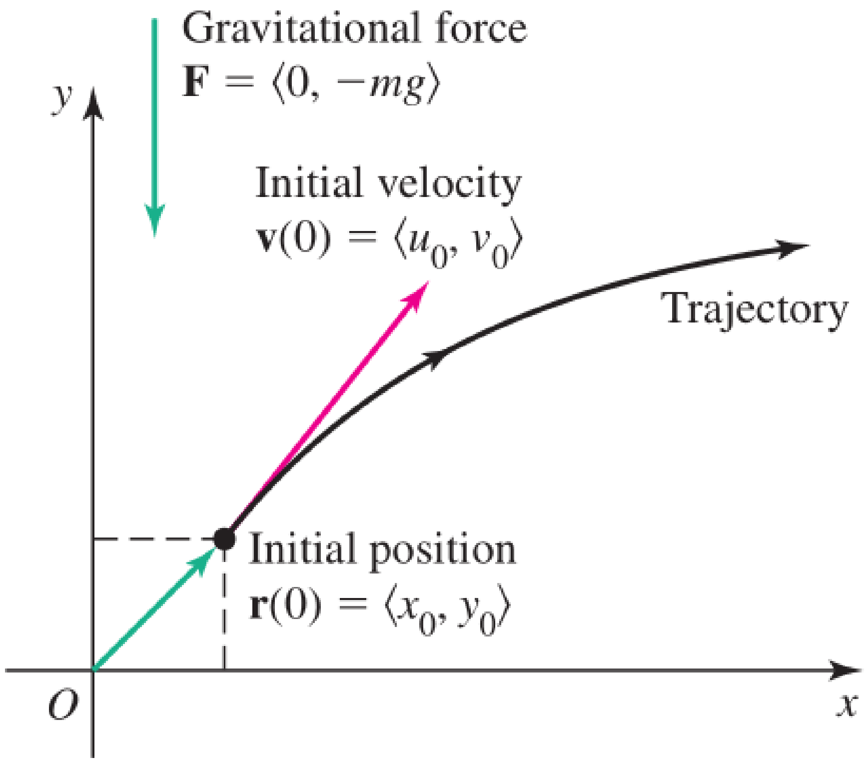
**b.** Find the velocity and speed of the object.



# Two-Dimensional Motion in a Gravitational Field

$$\underbrace{\text{mass}}_{m} \cdot \underbrace{\text{acceleration}}_{\mathbf{a}(t) = \mathbf{r}''(t)} = \underbrace{\text{sum of all forces.}}_{\sum \mathbf{F}_k}$$

## Finding Velocity and Position from Acceleration



From Newton's Second Law,

$$m\boldsymbol{a}(t) = \boldsymbol{F} = \langle 0, -mg\rangle$$

$$\boldsymbol{a}(t) = \langle 0, -g\rangle$$

Initial conditions: $\boldsymbol{v}(0) = \langle u_0, v_0\rangle$ and $\boldsymbol{r}(0) = \langle x_0, y_0\rangle$

1. Solve for the velocity

$$\boldsymbol{v}(t) = \int \boldsymbol{a}(t)dt = \int \langle 0, -g\rangle dt = \langle 0, -gt\rangle + \boldsymbol{C}$$

Then $v(0) = \langle 0, 0\rangle + \boldsymbol{C} = \langle u_0, v_0\rangle$, or $\boldsymbol{C} = \langle u_0, v_0\rangle$

2. Solve for the position

$$\boldsymbol{r}(t) = \int \boldsymbol{v}(t)dt = \int \langle u_0, -gt + v_0\rangle dt$$

$$= \left\langle u_0 t, -\frac{1}{2}gt^2 + v_0 t\right\rangle + \boldsymbol{C}$$

**SUMMARY Two-Dimensional Motion in a Gravitational Field**

Consider an object moving in a plane with a horizontal $x$-axis and a vertical $y$-axis, subject only to the force of gravity. Given the initial velocity $\mathbf{v}(0) = \langle u_0, v_0 \rangle$ and the initial position $\mathbf{r}(0) = \langle x_0, y_0 \rangle$, the velocity of the object, for $t \geq 0$, is

$$\mathbf{v}(t) = \langle x'(t), y'(t) \rangle = \langle u_0, -gt + v_0 \rangle$$

and the position is

$$\mathbf{r}(t) = \langle x(t), y(t) \rangle = \left\langle u_0 t + x_0, -\frac{1}{2} g t^2 + v_0 t + y_0 \right\rangle.$$

**EXAMPLE 4** **Flight of a baseball** A baseball is hit from 3 ft above home plate with an initial velocity in ft/s of $\mathbf{v}(0) = \langle u_0, v_0 \rangle = \langle 80, 80 \rangle$. Neglect all forces other than gravity.

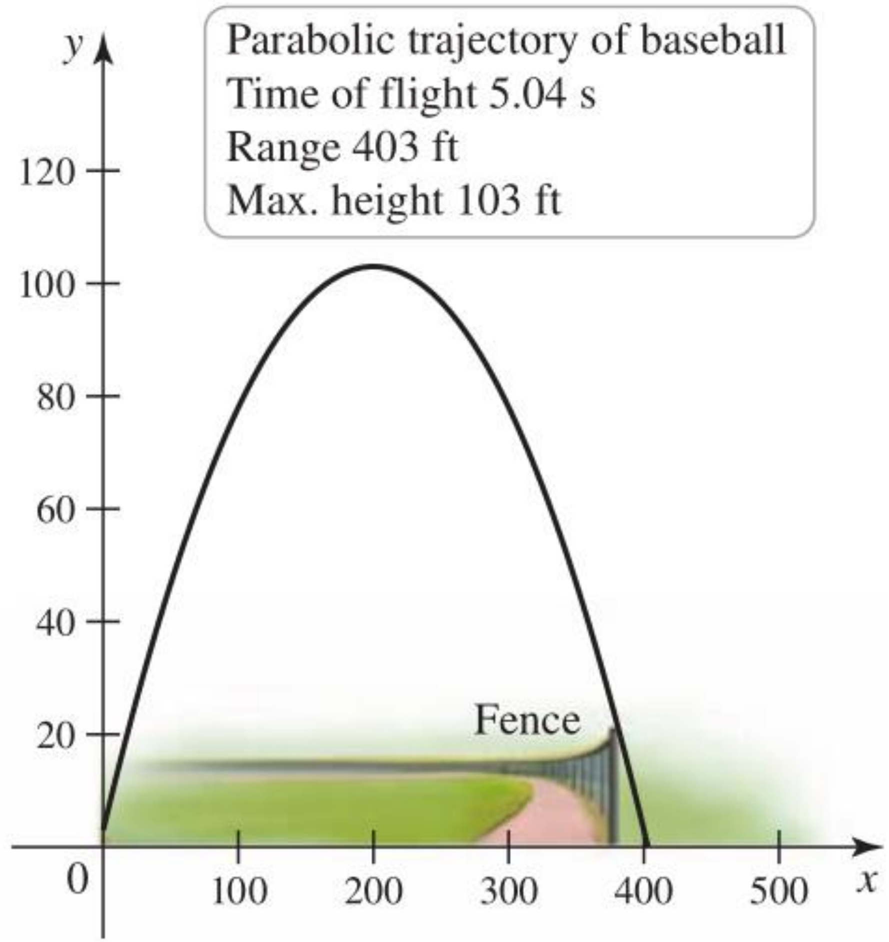
**a.** Find the position and velocity of the ball between the time it is hit and the time it first hits the ground.

**b.** Show that the trajectory of the ball is a segment of a parabola.

**c.** Assuming a flat playing field, how far does the ball travel horizontally? Plot the trajectory of the ball.

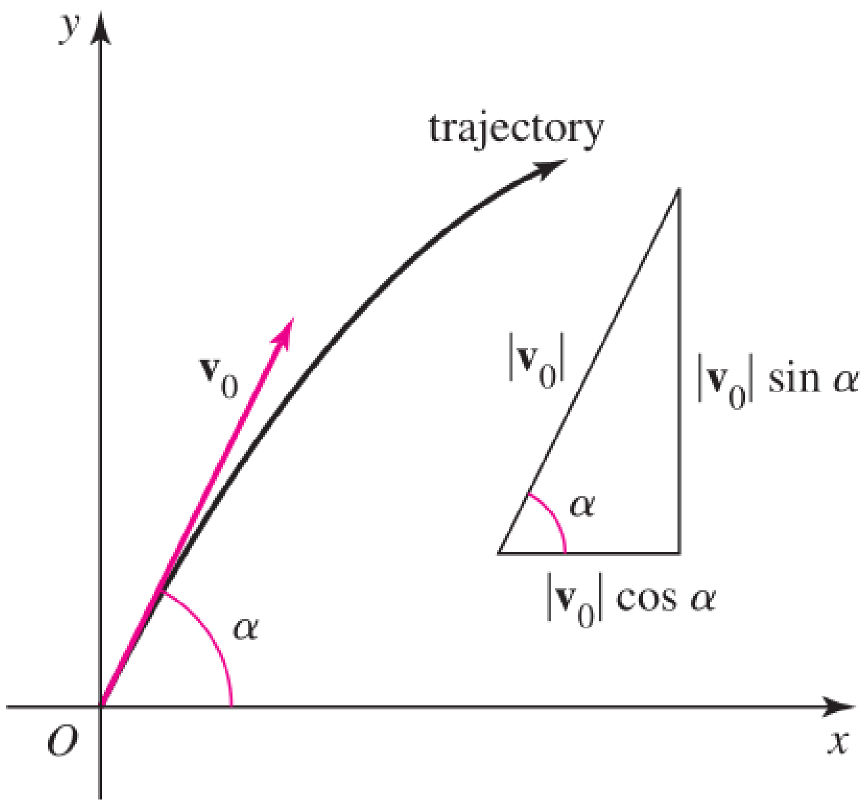
**d.** What is the maximum height of the ball?

**e.** Does the ball clear a 20-ft fence that is 380 ft from home plate (directly under the path of the ball)?


Parabolic trajectory of baseball
Time of flight 5.04 s
Range 403 ft
Max. height 103 ft
y
120
100
80
60
40
20
0
100
200
300
400
500
x
Fence

# Range, Time of Flight, Maximum Height

Object launched at the origin ($x_0 = y_0 = 0$) with an angle of $\alpha$ ($0 \leq \alpha \leq \pi/2$) above the horizontal and an initial speed $|\mathbf{v}_0|$. The initial velocity $\langle u_0, v_0 \rangle = \langle |\mathbf{v}_0| \cos\alpha\,, |\mathbf{v}_0| \sin\alpha \rangle$



Then, we have

The velocity

$$\mathbf{v}(t) = \langle u_0, -gt + v_0 \rangle = \langle |\mathbf{v}_0| \cos\alpha\,, -gt + |\mathbf{v}_0| \sin\alpha \rangle$$

The position

$$\mathbf{r}(t) = \left\langle u_0 t + x_0, -\frac{1}{2} g t^2 + v_0 t + y_0 \right\rangle$$

$$= \left\langle (|\mathbf{v}_0| \cos\alpha) t, -\frac{1}{2} g t^2 + (|\mathbf{v}_0| \sin\alpha) t \right\rangle$$

The motion is determined entirely by the parameters $|\mathbf{v}_0|$ and $\alpha$

1. Time of flight $T$, i.e., the time elapsed returning to the ground

Solving $y(t) = -\frac{1}{2}gt^2 + (|\mathbf{v}_0| \sin\alpha)t = 0$, we have

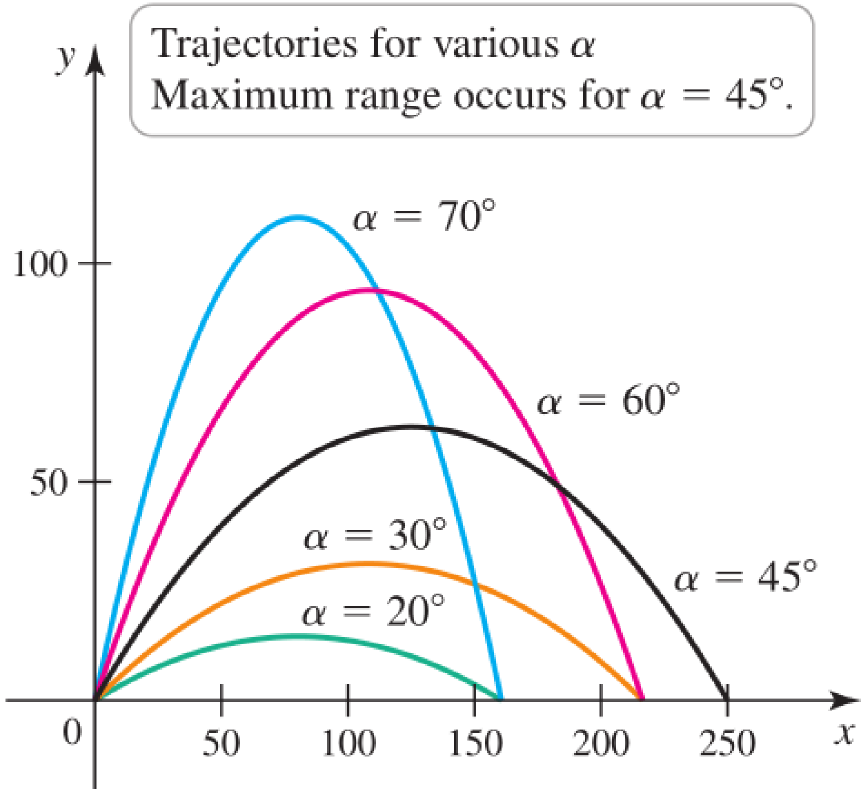
$$T = 2|\mathbf{v}_0| \sin\alpha / g$$

2. **Range** of the object, i.e., the horizontal distance it travels, when $t = T$, that is

$$x(T) = (|\mathbf{v}_0| \cos\alpha)T = \frac{|\mathbf{v}_0|^2 \sin 2\alpha}{g}$$

The maximum range is $\frac{|\mathbf{v}_0|^2}{g}$ when $\alpha = \pi/4$.

The ranges obtained with the angles $\alpha$ and $\frac{\pi}{2} - \alpha$ are equal



1. The maximum height of the object is reached when the vertical velocity is zero, $y'(t) = -gt + |\mathbf{v}_0| \sin\alpha = 0$. Solving for it, we have

$$t = \frac{|\mathbf{v}_0| \sin\alpha}{g} = T/2$$

The maximum height is

$$y(T/2) = \frac{(|\mathbf{v}_0| \sin\alpha)^2}{2g}$$

4. The trajectory of the object is a segment of a parabola.

## SUMMARY Two-Dimensional Motion

Assume an object traveling over horizontal ground, acted on only by the gravitational force, has an initial position $\langle x_0, y_0 \rangle = \langle 0, 0 \rangle$ and initial velocity $\langle u_0, v_0 \rangle = \langle |\mathbf{v}_0| \cos \alpha, |\mathbf{v}_0| \sin \alpha \rangle$. The trajectory, which is a segment of a parabola, has the following properties.

$$\text{time of flight} = T = \frac{2|\mathbf{v}_0| \sin \alpha}{g}$$

$$\text{range} = \frac{|\mathbf{v}_0|^2 \sin 2\alpha}{g}$$

$$\text{maximum height} = y\left(\frac{T}{2}\right) = \frac{(|\mathbf{v}_0| \sin \alpha)^2}{2g}$$

**EXAMPLE 5** **Flight of a golf ball** A golf ball is driven down a horizontal fairway with an initial speed of 55 m/s at an initial angle of 25° (from a tee with negligible height). Neglect all forces except gravity and assume that the ball's trajectory lies in a plane.

**a.** How far does the ball travel horizontally and when does it land?

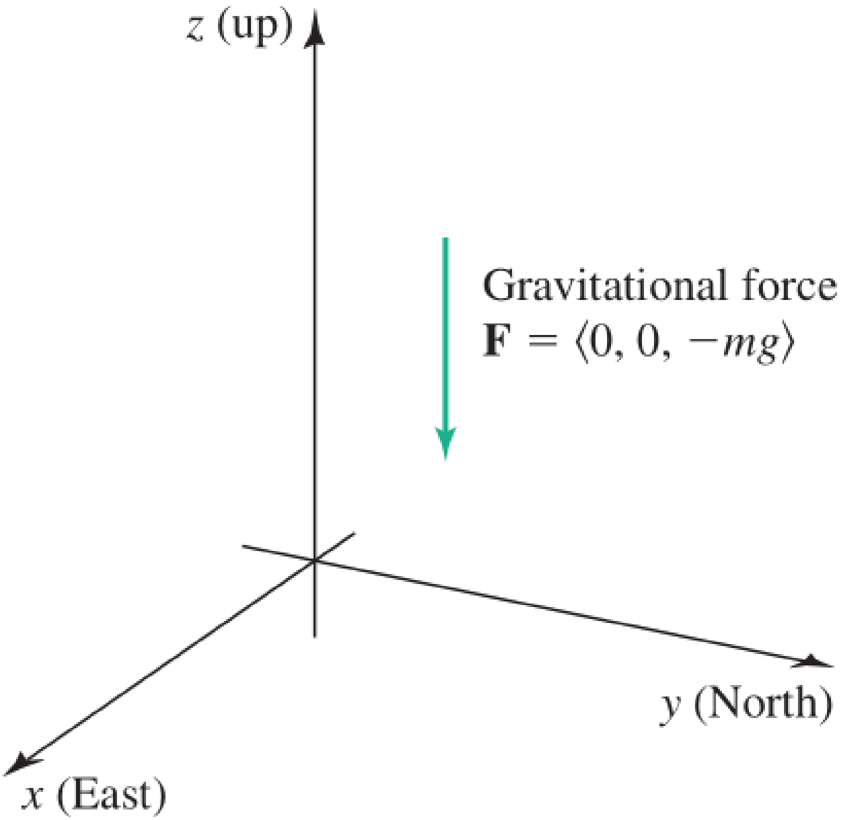
**b.** What is the maximum height of the ball?

**c.** At what angles should the ball be hit to reach a green that is 300 m from the tee?

# Three-Dimensional Motion

From Newton's Second Law,

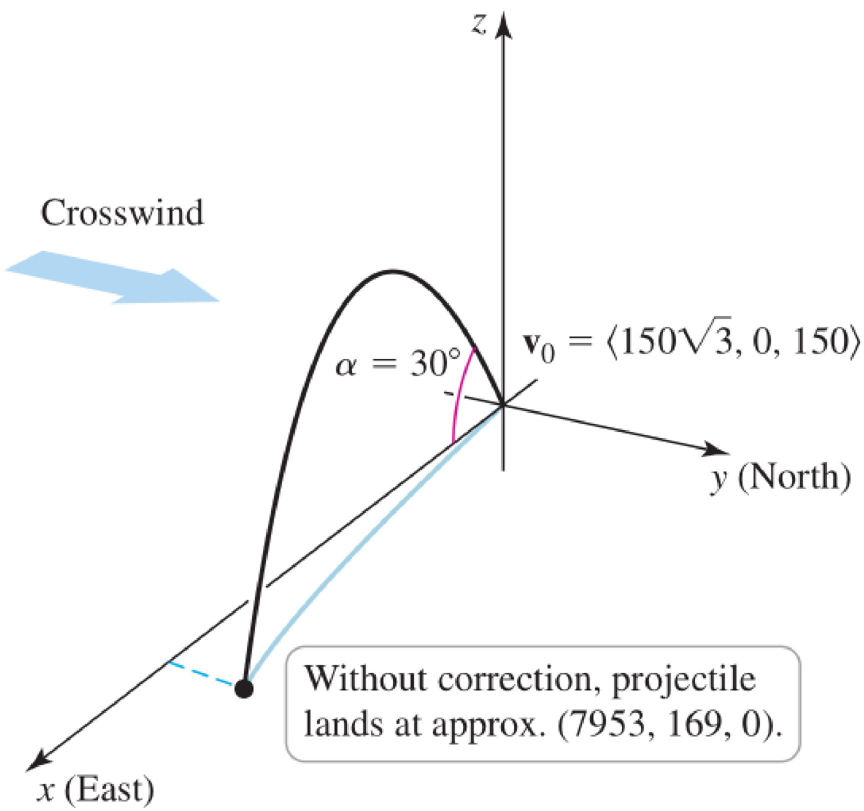
$$m\mathbf{a}(t) = \langle mx''(t), my''(t), mz''(t) \rangle = \mathbf{F}$$



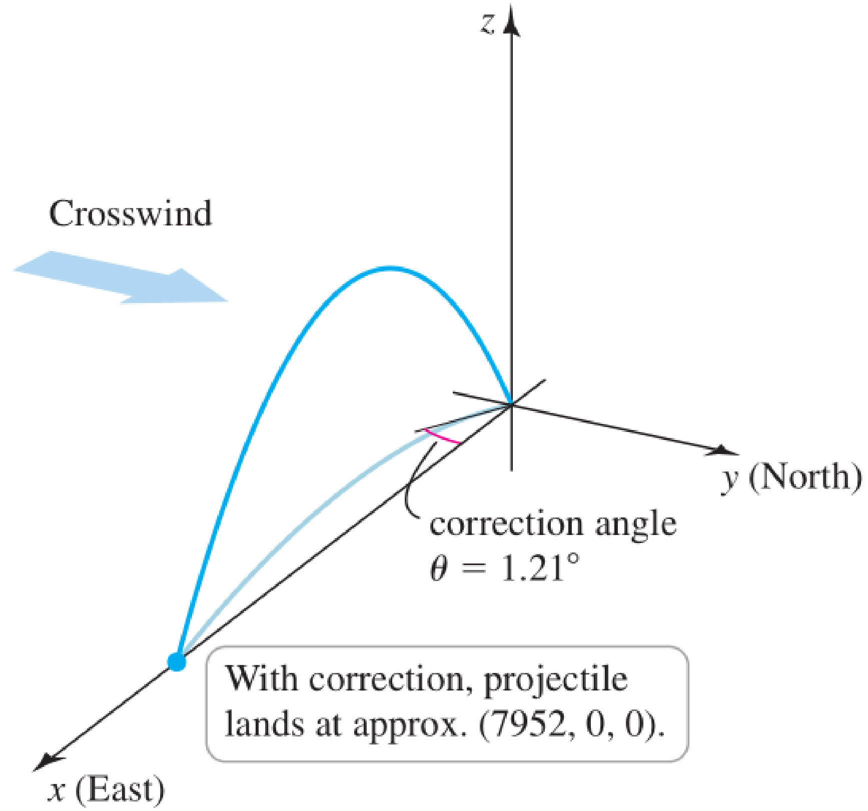
**EXAMPLE 6** **Projectile motion** A small projectile is fired over horizontal ground in an easterly direction with an initial speed of $|\mathbf{v}_0| = 300$ m/s at an angle of $\alpha = 30°$ above the horizontal. A crosswind blows from south to north, producing an acceleration of the projectile of $0.36$ m/s$^2$ to the north.

**a.** Where does the projectile land? How far does it land from its launch site?

**b.** In order to correct for the crosswind and make the projectile land due east of the launch site, at what angle from due east must the projectile be fired? Assume the initial speed $|\mathbf{v}_0| = 300$ m/s and the angle of elevation $\alpha = 30°$ are the same as in part (a).



z
Crosswind
$\alpha = 30°$
$\mathbf{v}_0 = \langle 150\sqrt{3}, 0, 150 \rangle$
y (North)
x (East)
Without correction, projectile lands at approx. (7953, 169, 0).

z
Crosswind
y (North)
correction angle
$\theta = 1.21°$
x (East)
With correction, projectile lands at approx. (7952, 0, 0).

# 14.4

## Length of Curves

## Arc Length

Recall: the length of the two-dimensional curve $\boldsymbol{r}(t) = \langle f(t), g(t)\rangle$, for $a < t < b$

$$L = \int_a^b \sqrt{f'(t)^2 + g'(t)^2}\,dt$$

**DEFINITION Arc Length for Vector Functions**

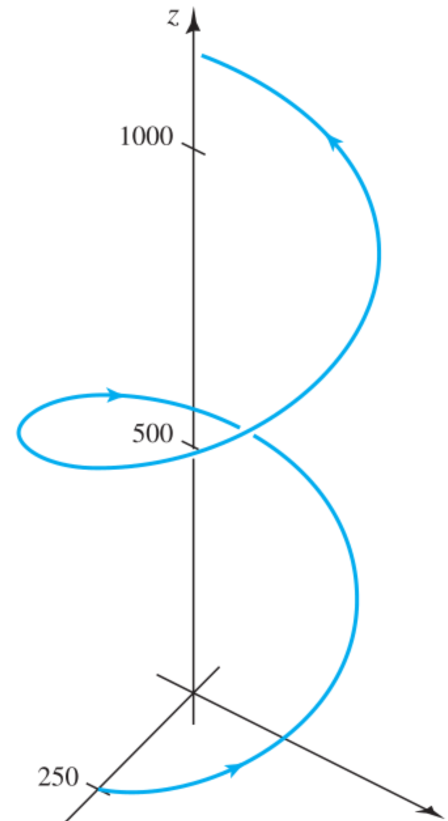
Consider the parameterized curve $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, where $f'$, $g'$, and $h'$ are continuous, and the curve is traversed once for $a \leq t \leq b$. The **arc length** of the curve between $(f(a), g(a), h(a))$ and $(f(b), g(b), h(b))$ is

$$L = \int_a^b \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2}\, dt = \int_a^b |\mathbf{r}'(t)|\, dt.$$

**EXAMPLE 3** **Flight of an eagle** An eagle rises at a rate of 100 vertical ft/min on a helical path given by

$$\mathbf{r}(t) = \langle 250 \cos t, 250 \sin t, 100t \rangle$$

(Figure 97), where $\mathbf{r}$ is measured in feet and $t$ is measured in minutes. How far does it travel in 10 min?




Helix
$\mathbf{r}(t) = \langle 250 \cos t, 250 \sin t, 100t \rangle$

**EXAMPLE 4** **Lengths of planetary orbits** According to Kepler's first law, the planets revolve about the sun in elliptical orbits. A vector function that describes an ellipse in the $xy$-plane is

$$\mathbf{r}(t) = \langle a\cos t, b\sin t\rangle, \qquad \text{where } 0 \le t \le 2\pi.$$

If $a > b > 0$, then $2a$ is the length of the major axis and $2b$ is the length of the minor axis (Figure 98). Verify the lengths of the planetary orbits given in Table 1. Distances are given in terms of the astronomical unit (AU), which is the length of the semimajor axis of Earth's orbit, or about 93 million miles.

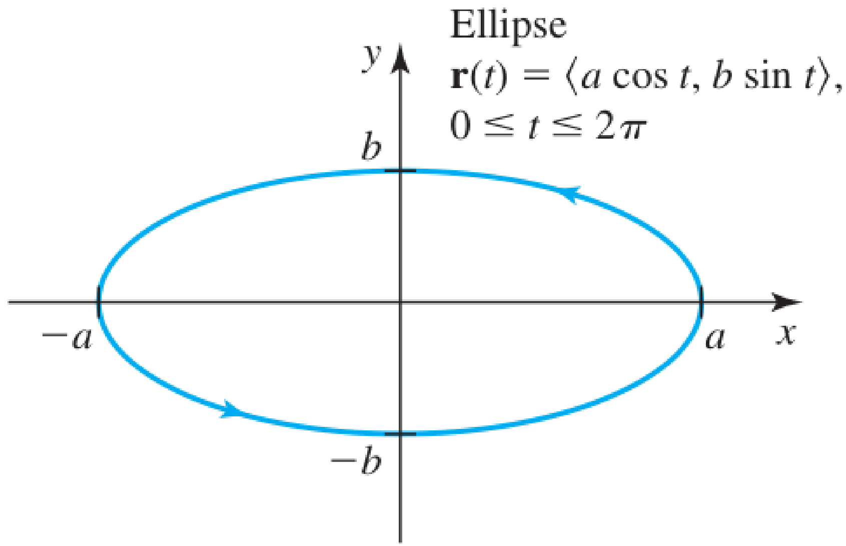


**Table 1**

| Planet | Semimajor axis, $a$ (AU) | Semiminor axis, $b$ (AU) | $\alpha = b/a$ | Orbit length (AU) |
|---|---|---|---|---|
| Mercury | 0.387 | 0.379 | 0.979 | 2.407 |
| Venus | 0.723 | 0.723 | 1.000 | 4.543 |
| Earth | 1.000 | 0.999 | 0.999 | 6.280 |
| Mars | 1.524 | 1.517 | 0.995 | 9.554 |
| Jupiter | 5.203 | 5.179 | 0.995 | 32.616 |
| Saturn | 9.539 | 9.524 | 0.998 | 59.888 |
| Uranus | 19.182 | 19.161 | 0.999 | 120.458 |
| Neptune | 30.058 | 30.057 | 1.000 | 188.857 |

# Arc Length of a Polar Curve

**Arc Length of a Polar Curve**

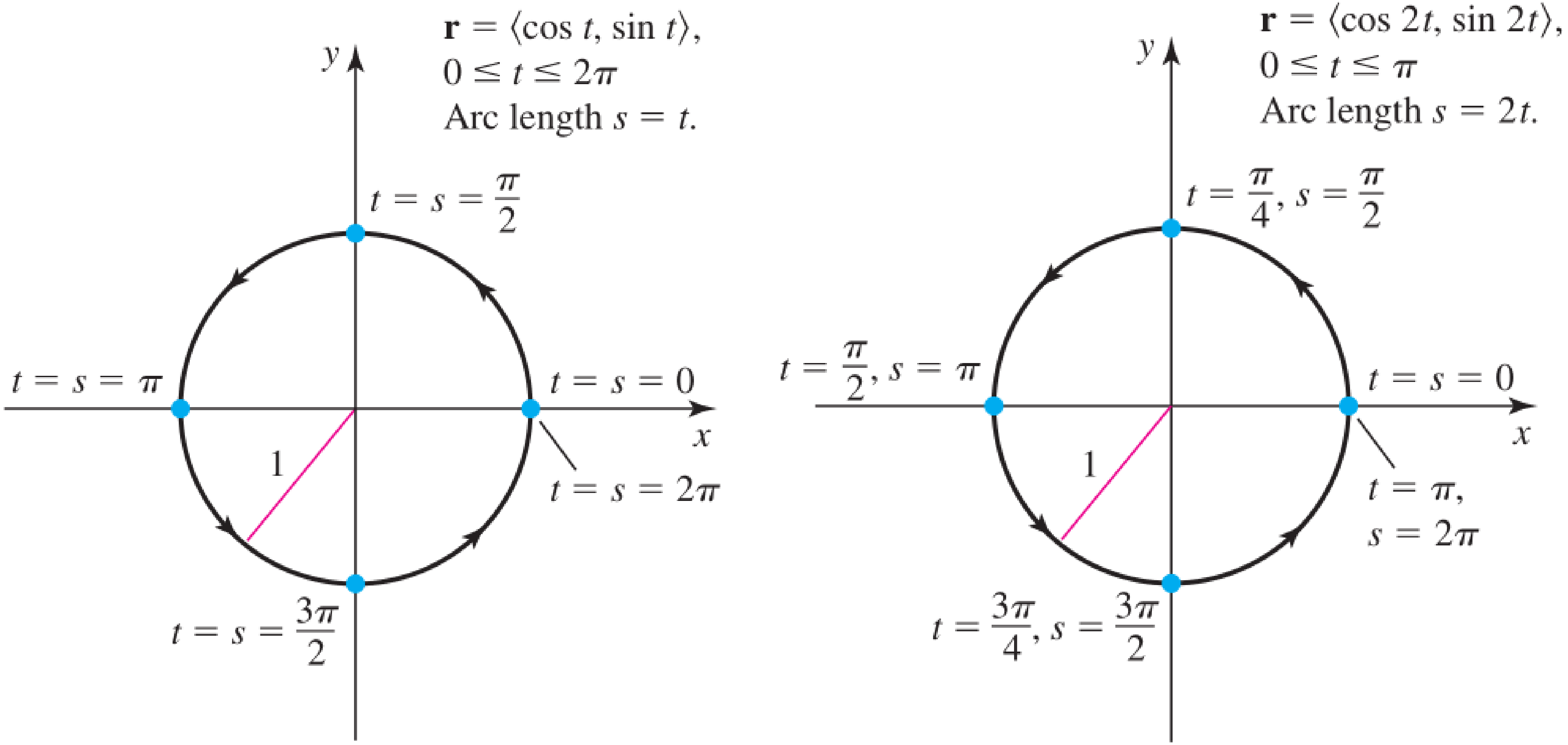
Let $f$ have a continuous derivative on the interval $[\alpha, \beta]$. The **arc length** of the polar curve $r = f(\theta)$ on $[\alpha, \beta]$ is

$$L = \int_{\alpha}^{\beta} \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta.$$

# Arc Length as a Parameter

Consider the following two characterizations of the unit circle

- $\langle \cos t, \sin t \rangle$, for $0 \le t \le 2\pi$
- $\langle \cos 2t, \sin 2t \rangle$, for $0 \le t \le \pi$



## The Arc Length Function

A smooth curve represented by $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$

The length of the curve from $\mathbf{r}(a)$ to $\mathbf{r}(t)$ is

$$s(t) = \int_a^t \sqrt{f'(u)^2 + g'(u)^2 + h'(u)^2}\,du = \int_a^t |\mathbf{v}(u)|\,du$$

An important consequence

$$\frac{ds}{dt} = \frac{d}{dt}\left(\int_a^t |\mathbf{v}(u)|\,du\right) = |\mathbf{v}(t)| > 0$$

If $\mathbf{r}(t)$ is a curve on which $|\mathbf{v}(t)| = 1$, then

$$s(t) = \int_a^t |\mathbf{v}(u)|\,du = \int_a^t 1\,du = t - a$$

**THEOREM 9 Arc Length as a Function of a Parameter**
Let $\mathbf{r}(t)$ describe a smooth curve, for $t \geq a$. The arc length is given by

$$s(t) = \int_a^t |\mathbf{v}(u)|\, du,$$

where $|\mathbf{v}| = |\mathbf{r}'|$. Equivalently, $\dfrac{ds}{dt} = |\mathbf{v}(t)|$. If $|\mathbf{v}(t)| = 1$, for all $t \geq a$, then the parameter $t$ corresponds to arc length.

**EXAMPLE 6** **Arc length parameterization** Consider the helix $\mathbf{r}(t) = \langle 2\cos t, 2\sin t, 4t \rangle$, for $t \geq 0$.

**a.** Find the arc length function $s(t)$.

**b.** Find another description of the helix that uses arc length as the parameter.

# Chapter 14

## Vector-Valued Functions

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