Chapter 16

Multiple Integration

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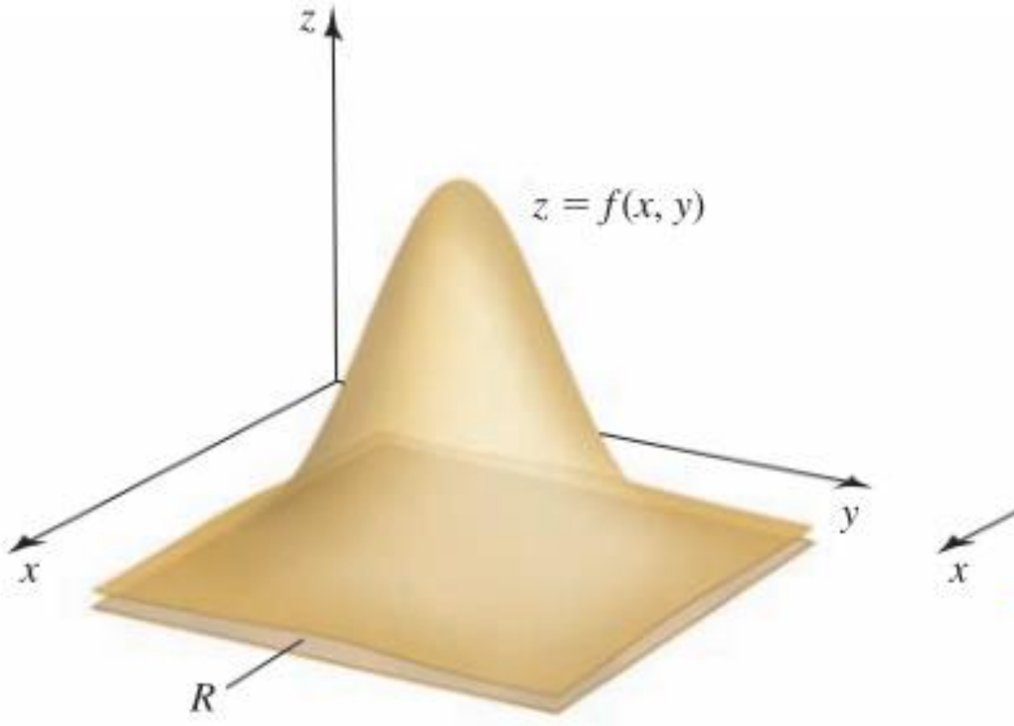
16.1

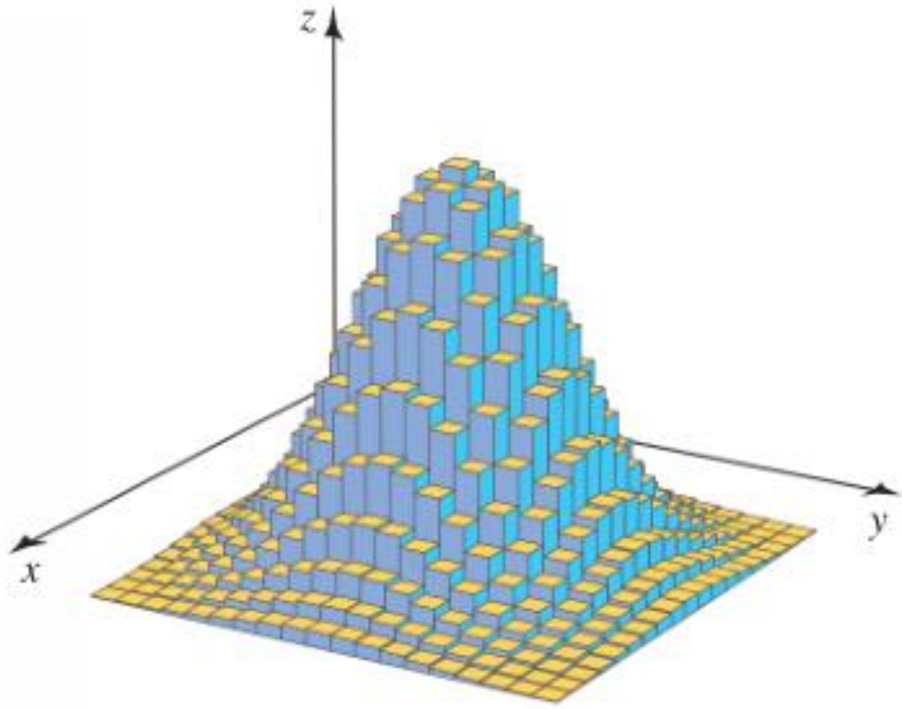
Table 1

	Derivatives	Integrals
Single variable: $f(x)$	$f'(x)$	$\int_a^b f(x)\,dx$
Several variables: $f(x, y)$ and $f(x, y, z)$	$\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, \dfrac{\partial f}{\partial z}$	$\iint_R f(x, y)\,dA, \iiint_D f(x, y, z)\,dV$

Volumes of Solids

A three-dimensional solid bounded by $z = f(x, y)$ and a region R in the xy-plane is approximated by a collection of boxes.

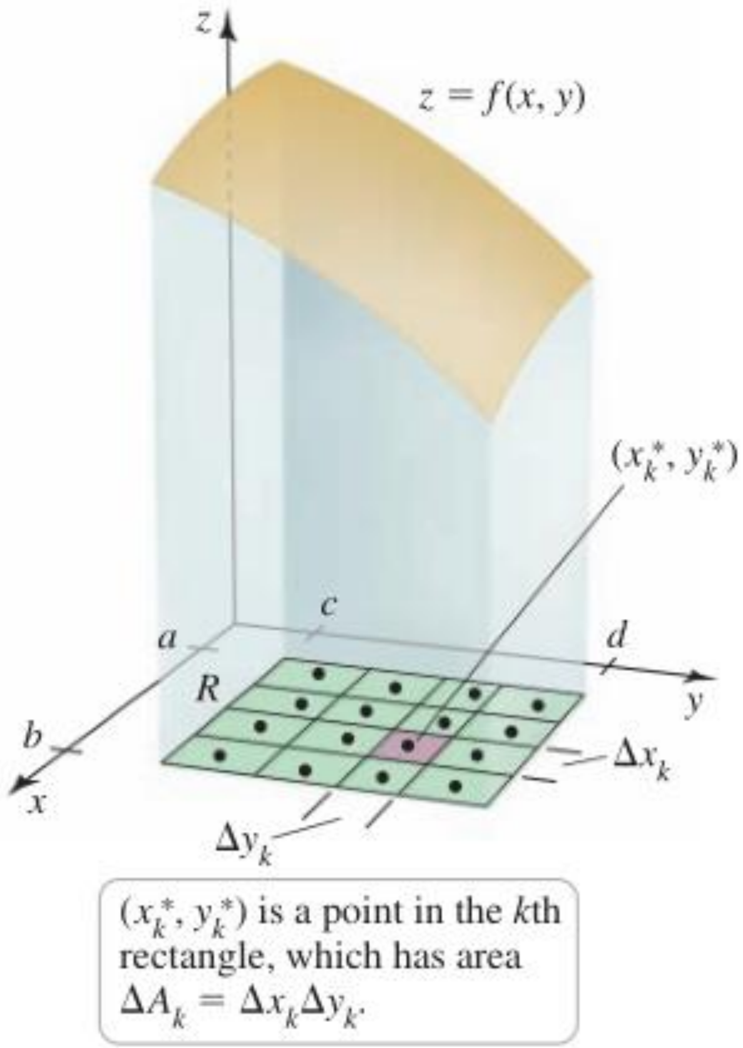


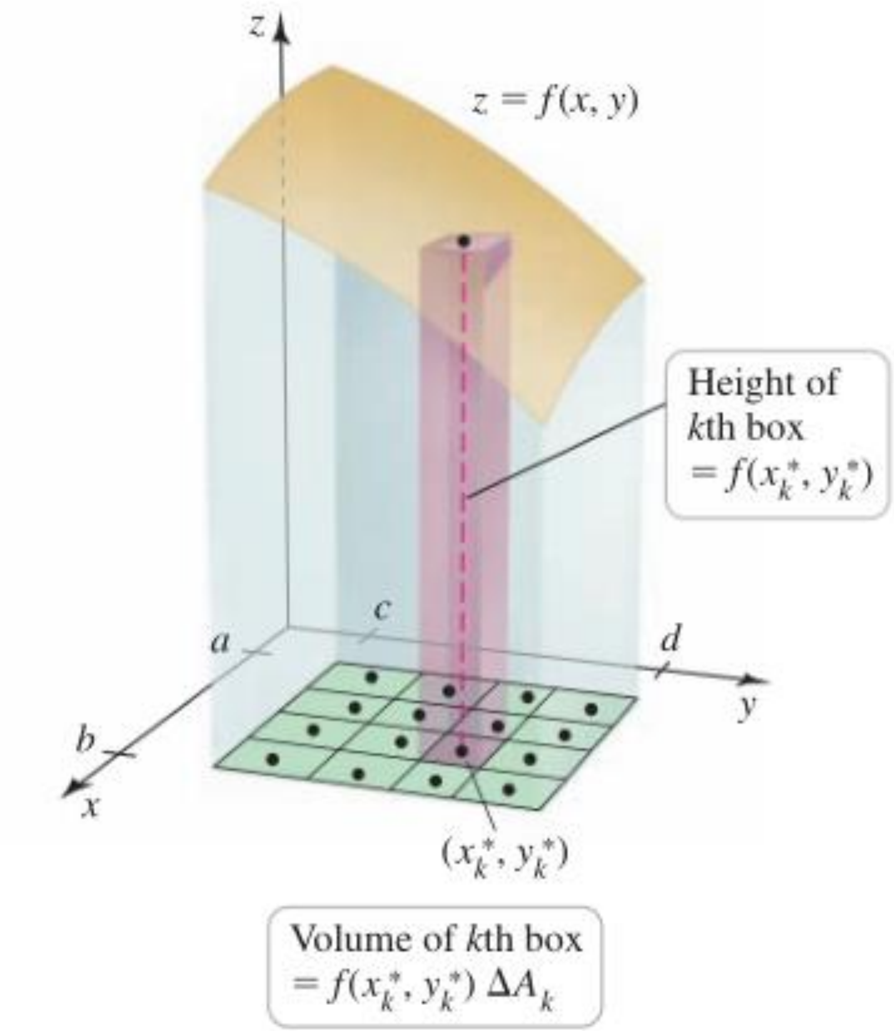


As the number of boxes increases, the approximations converge to the value of a *double integral*, which is the volume of the solid

Assume a *nonnegative* function $z = f(x, y)$ defined on a *rectangular* region $R = \{(x, y): a \leq x \leq b, c \leq y \leq d\}$

A **partition** of R



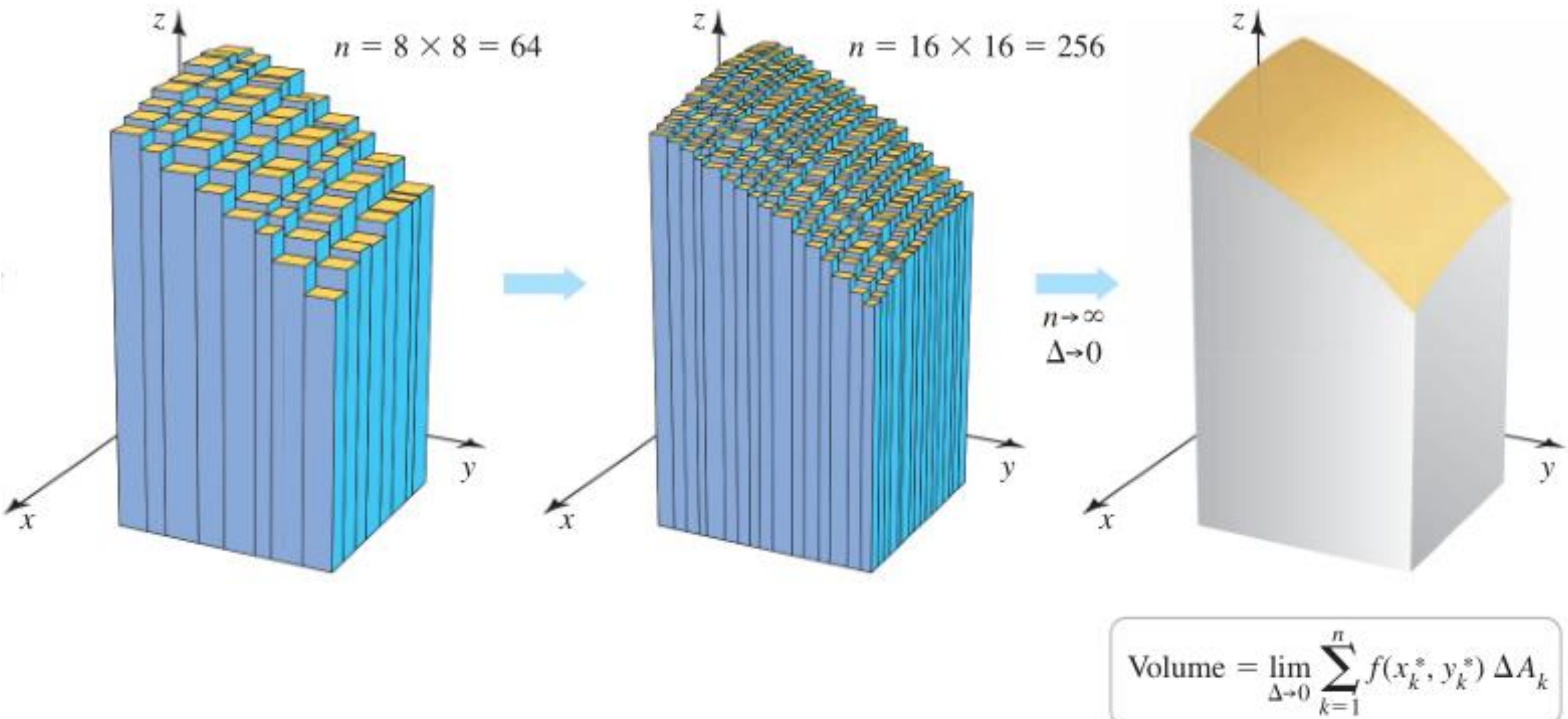


Approximation: the volume of the kth box

$$f(x_k^*, y_k^*)\Delta A_k = f(x_k^*, y_k^*)\Delta x_k \Delta y_k$$

Summing of the volumes of the n boxes

$$V \approx \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k$$



DEFINITION Double Integrals

A function f defined on a rectangular region R in the xy-plane is **integrable** on R if

$\lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*) \Delta A_k$ exists for all partitions of R and for all choices of (x_k^*, y_k^*)

within those partitions. The limit is the **double integral of f over R**, which we write

$$\iint_R f(x, y)\, dA = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*) \Delta A_k.$$

Iterated Integrals

To compute the volume of the solid region bounded by the plane $z = f(x,y) = 6 - 2x - y$ over the rectangular region $R = \{(x,y): 0 \le x \le 1, 0 \le y \le 2\}$

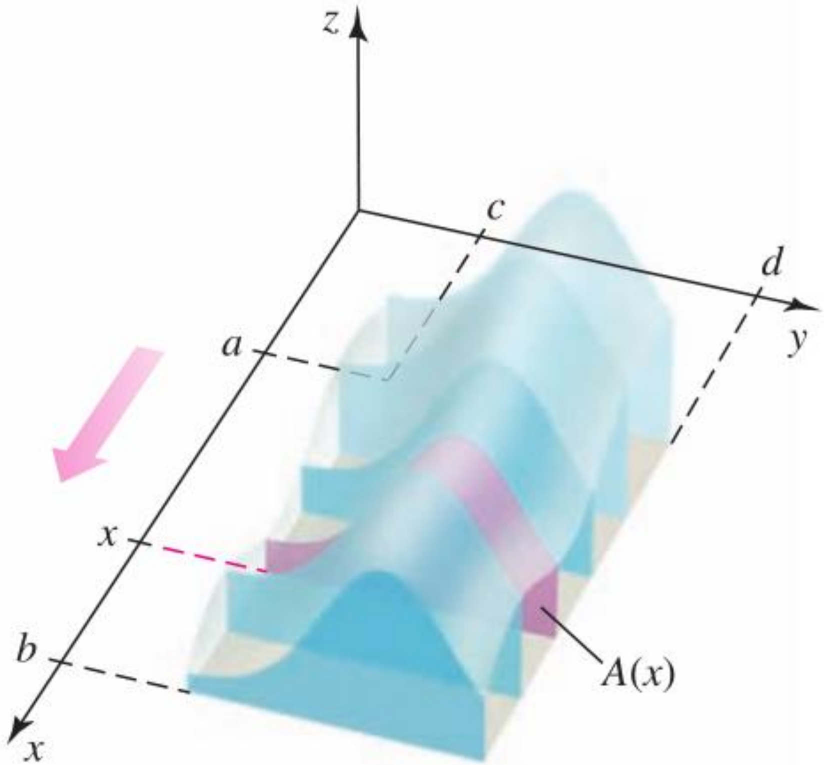
By definition, $V = \iint_R f(x,y)dA = \iint_R (6 - 2x - y)dA$

By slicing method $V = \int_0^1 A(x)dx$, while

$$A(x) = \int_0^2 (6 - 2x - y)dy$$

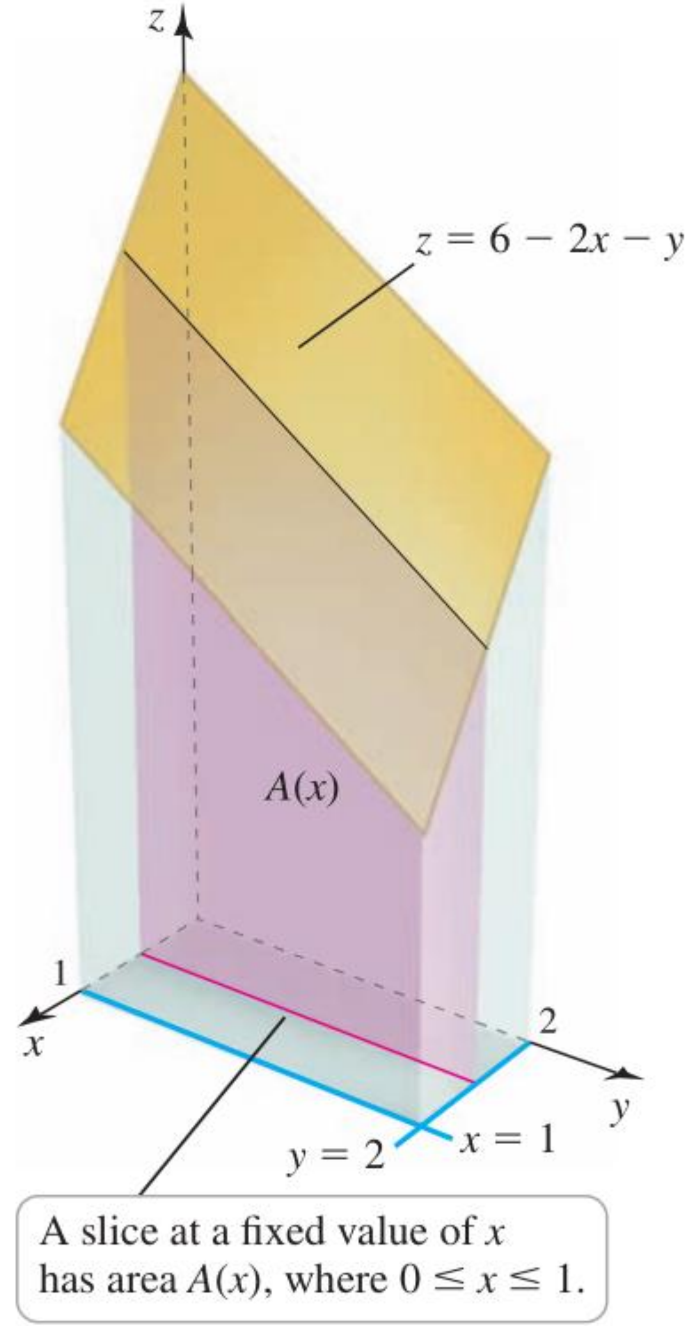
Therefore,

$$V = \int_0^1 A(x)dx = \int_0^1 \left(\int_0^2 (6 - 2x - y)dy \right) dx$$



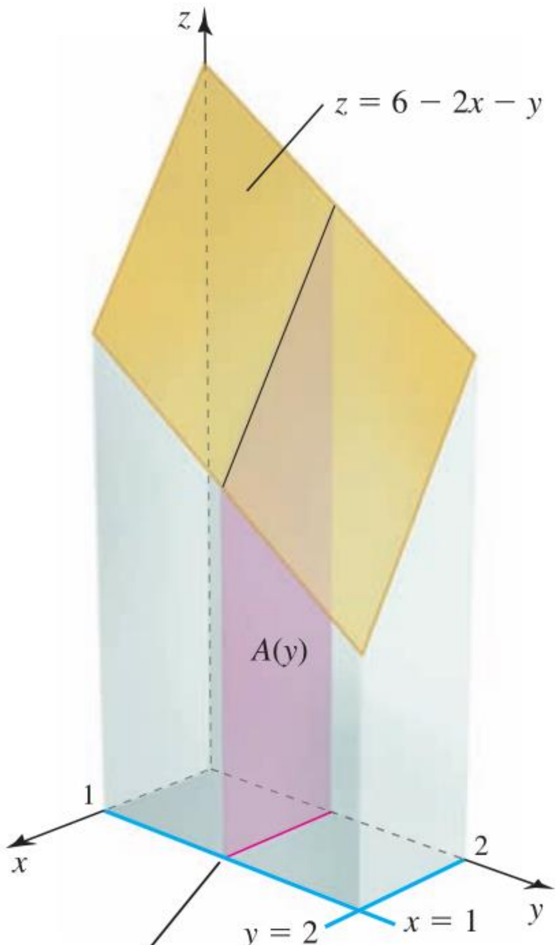
If a solid is sliced parallel to the y-axis and perpendicular to the xy-plane, and the cross-sectional area of the slice at the point x is $A(x)$, then the volume of the solid region is

$$V = \int_a^b A(x)\,dx.$$



EXAMPLE 1 **Evaluating an iterated integral** Evaluate $V = \int_0^1 A(x)\,dx$, where $A(x) = \int_0^2 (6 - 2x - y)\,dy$.

EXAMPLE 2 **Same double integral, different order** Example 1 used slices through the solid parallel to the yz-plane. Compute the volume of the same solid using vertical slices through the solid parallel to the xz-plane, for $0 \le y \le 2$



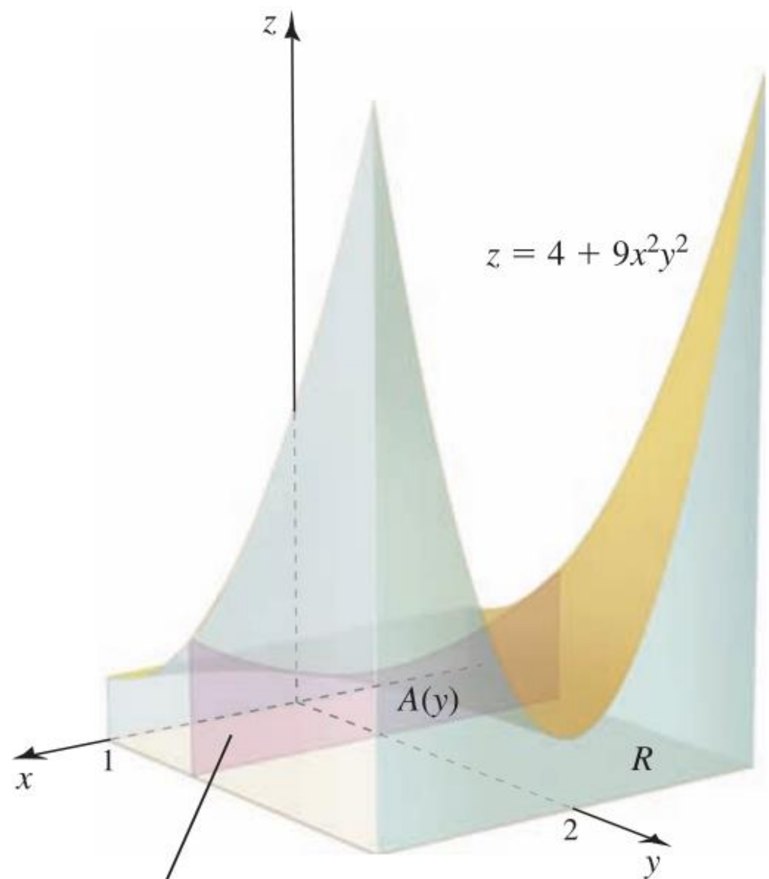
A slice at a fixed value of y has area $A(y)$, where $0 \le y \le 2$.

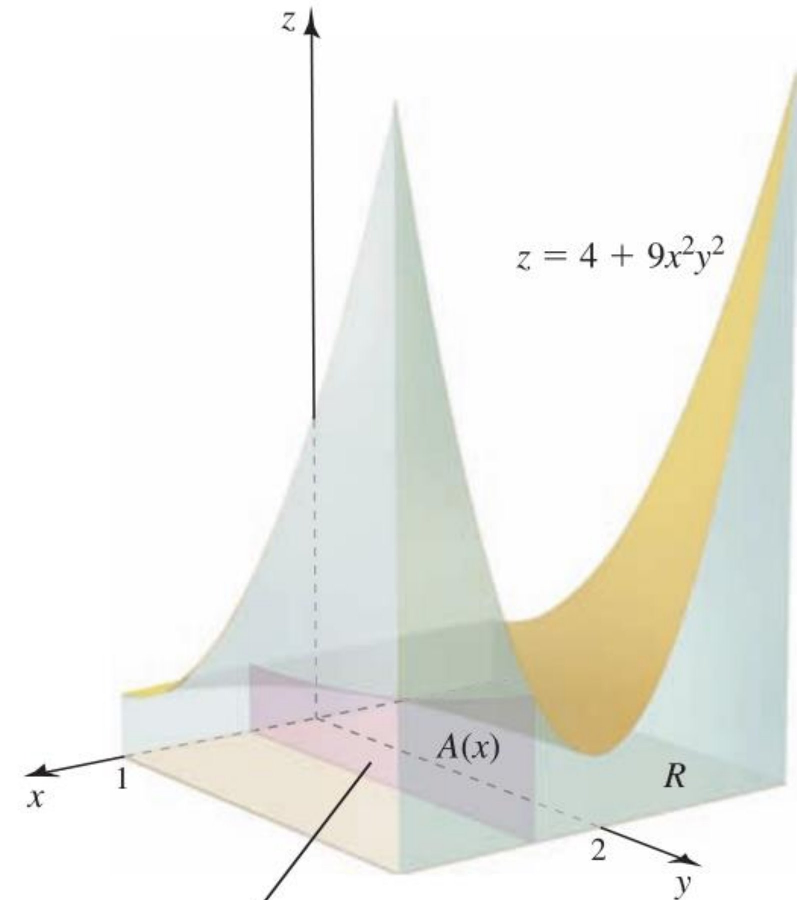
THEOREM 1 (Fubini) Double Integrals on Rectangular Regions

Let f be continuous on the rectangular region $R = \{(x, y): a \leq x \leq b, c \leq y \leq d\}$. The double integral of f over R may be evaluated by either of two iterated integrals:

$$\iint_R f(x, y)\, dA = \int_c^d \int_a^b f(x, y)\, dx\, dy = \int_a^b \int_c^d f(x, y)\, dy\, dx.$$

EXAMPLE 3 **A double integral** Find the volume of the solid bounded by the surface $f(x, y) = 4 + 9x^2y^2$ over the region $R = \{(x, y): -1 \le x \le 1, 0 \le y \le 2\}$. Use both possible orders of integration.





$$A(y) = \int_{-1}^{1} (4 + 9x^2y^2)dx$$

$$V = \int_0^2 \int_{-1}^{1} (4 + 9x^2y^2)\, dx\, dy$$

$$A(x) = \int_0^2 (4 + 9x^2y^2)dy$$

$$V = \int_{-1}^{1} \int_0^2 (4 + 9x^2y^2)\, dy\, dx$$

EXAMPLE 4 **Choosing a convenient order of integration** Evaluate $\iint_R ye^{xy}\,dA$, where $R = \{(x, y)\colon 0 \le x \le 1, 0 \le y \le \ln 2\}$.

Average Value

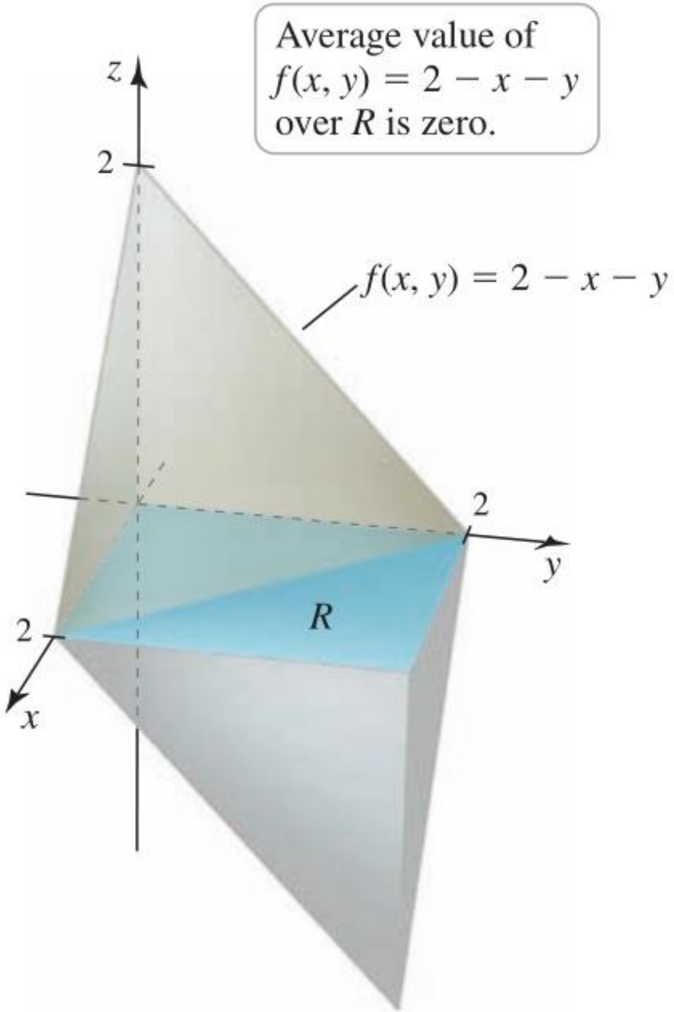
Recall the average value of the integrable function f over the interval $[a, b]$

DEFINITION Average Value of a Function over a Plane Region

The **average value** of an integrable function f over a region R is

$$\bar{f} = \frac{1}{\text{area of } R} \iint_R f(x, y)\, dA.$$

EXAMPLE 5 **Average value** Find the average value of the quantity $2 - x - y$ over the square $R = \{(x, y): 0 \le x \le 2, 0 \le y \le 2\}$



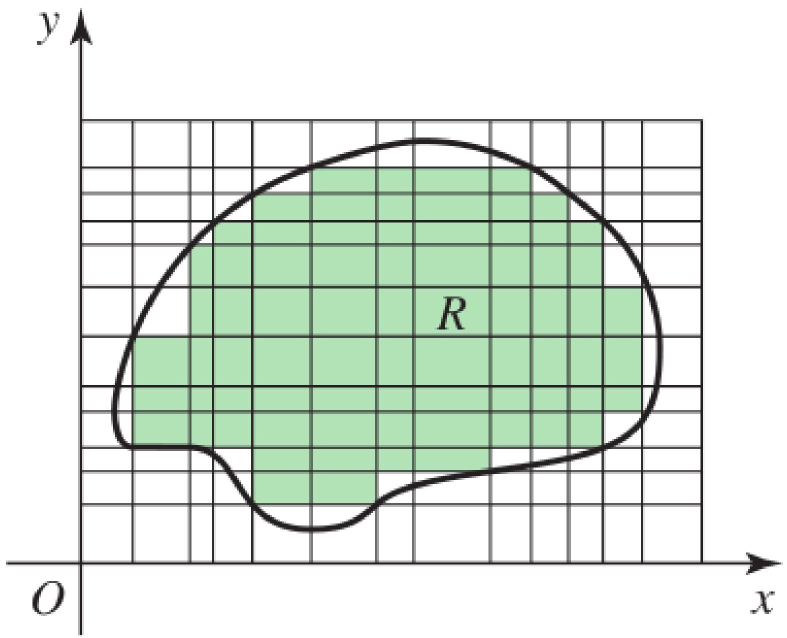
16.2

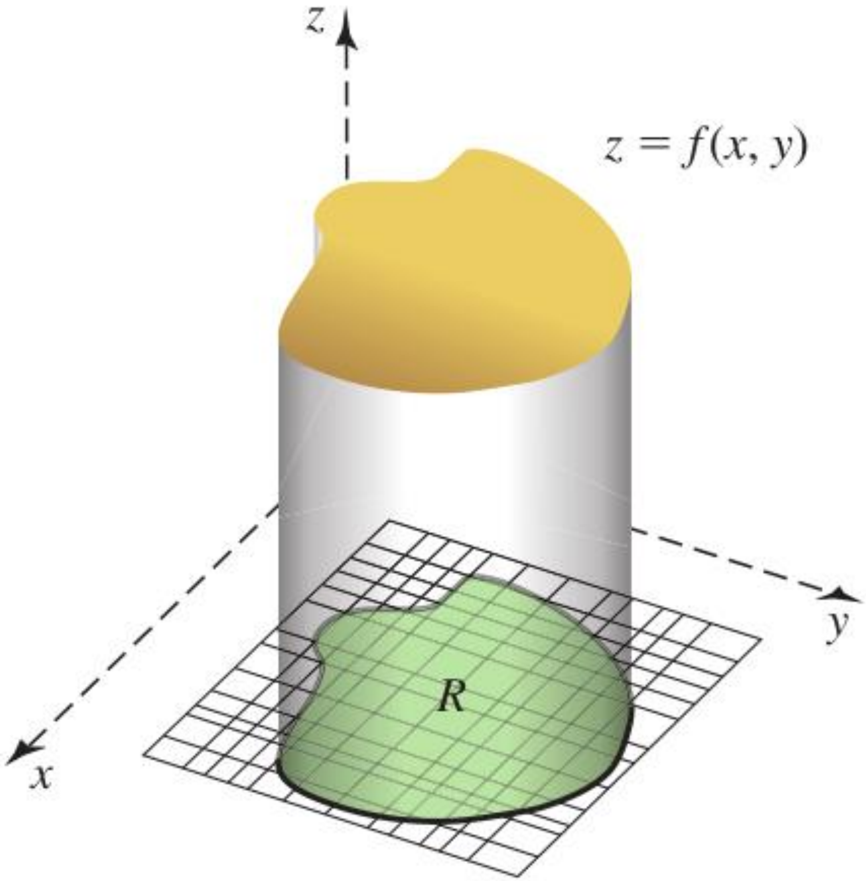
Double Integrals over General Regions

General Regions of Integration

Only the n rectangles that lie entirely within R are considered to be in the partition.
Assume f is *nonnegative* on R.
The volume of the solid bounded by the surface $z = f(x, y)$ and the *xy*-plane over R is approximated by the Riemann Sum

$$V \approx \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k, \text{ where } \Delta A_k = \Delta x_k \Delta y_k$$





$$\text{Volume of solid} = \iint_R f(x, y)\, dA = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*)\, \Delta A_k$$

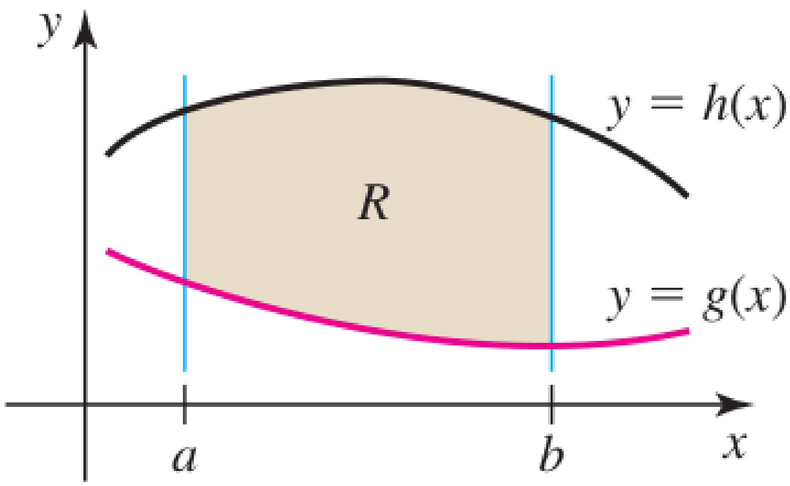
The limit approached by the Riemann sums is the double integral of f over R

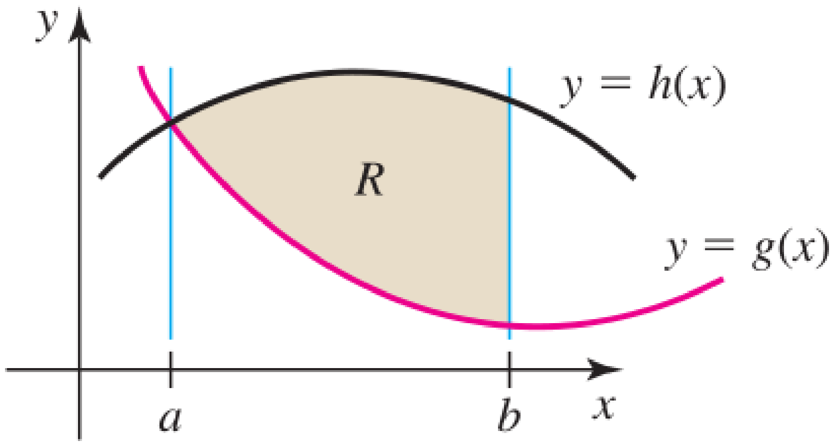
When this limit exists, f is integrable over R

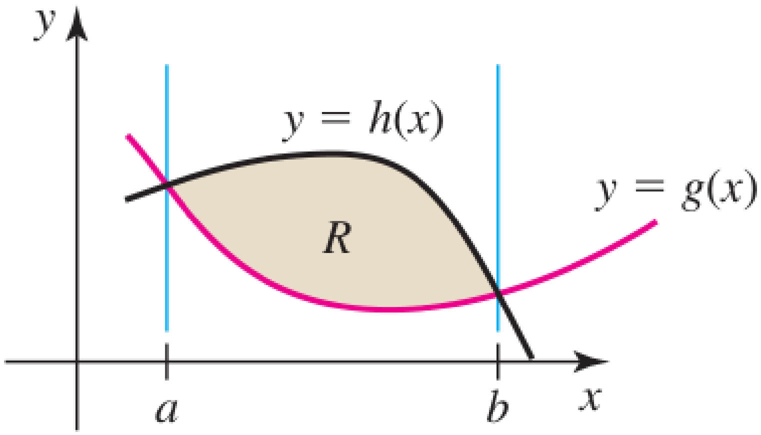
Iterated Integrals

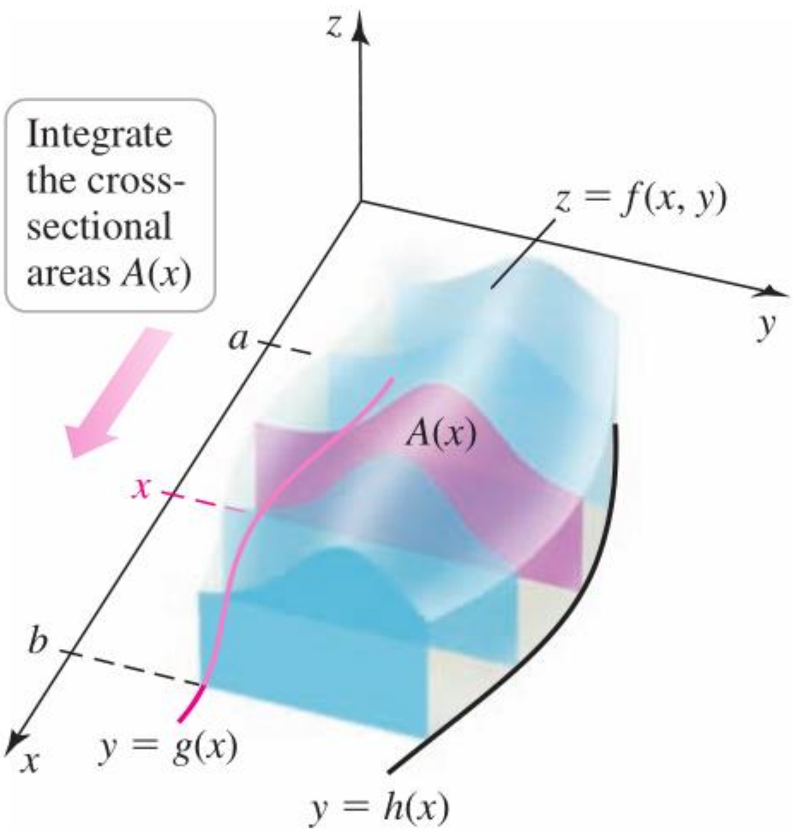
In this more general setting, the order of integration is critical.

The first type of region has the property that its lower and upper boundaries are the graphs of continuous functions $y = g(x)$ and $y = h(x)$, respectively, for $a \leq x \leq b$.









The general slicing method

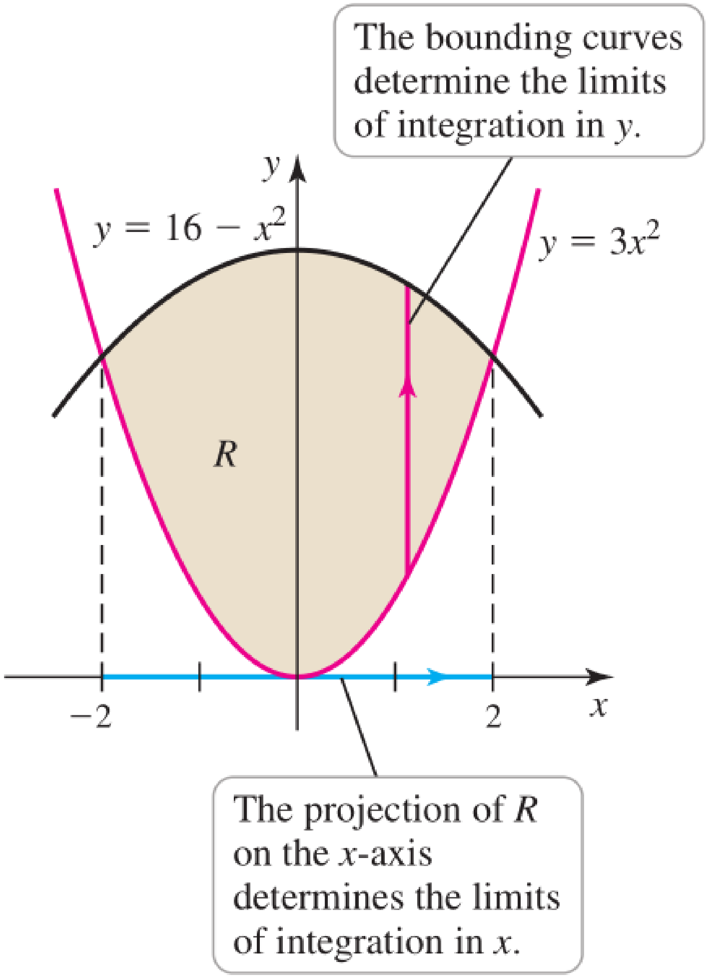
The area of that cross section is

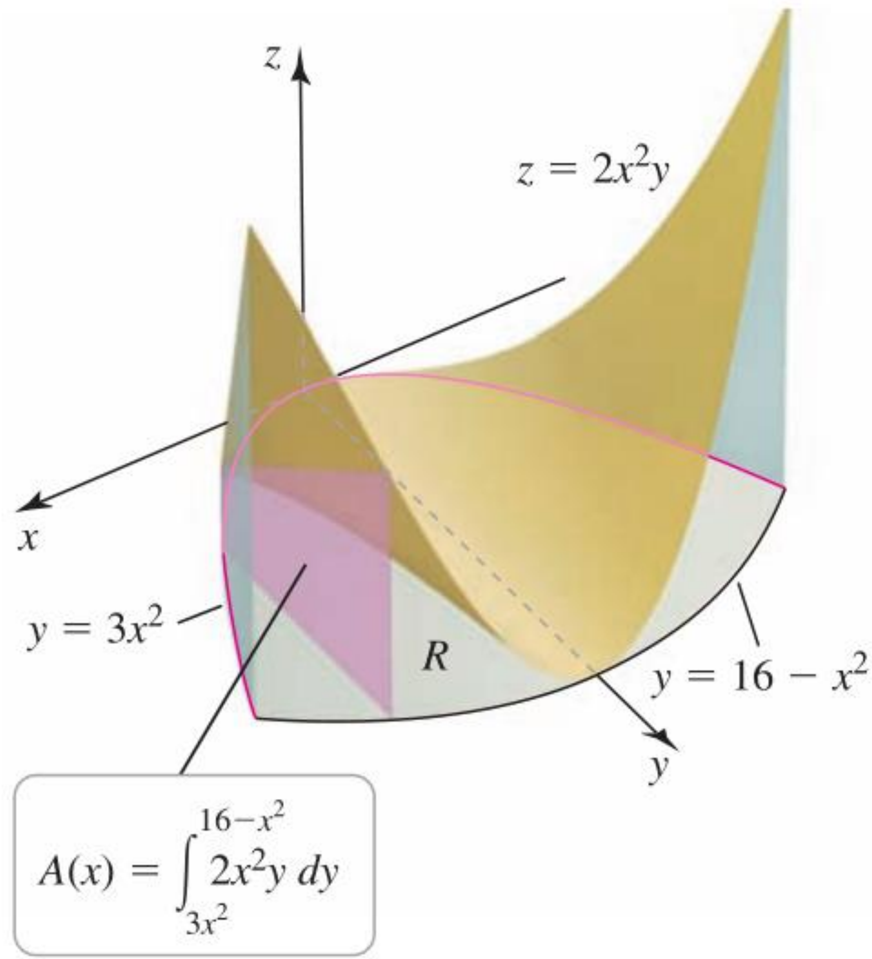
$$A(x) = \int_{g(x)}^{h(x)} f(x,y)dy$$

The volume is given by a double integral, which is evaluated by

$$V = \iint_R f(x,y)dA$$
$$= \int_a^b \left(\int_{g(x)}^{h(x)} f(x,y)dy \right) dx$$

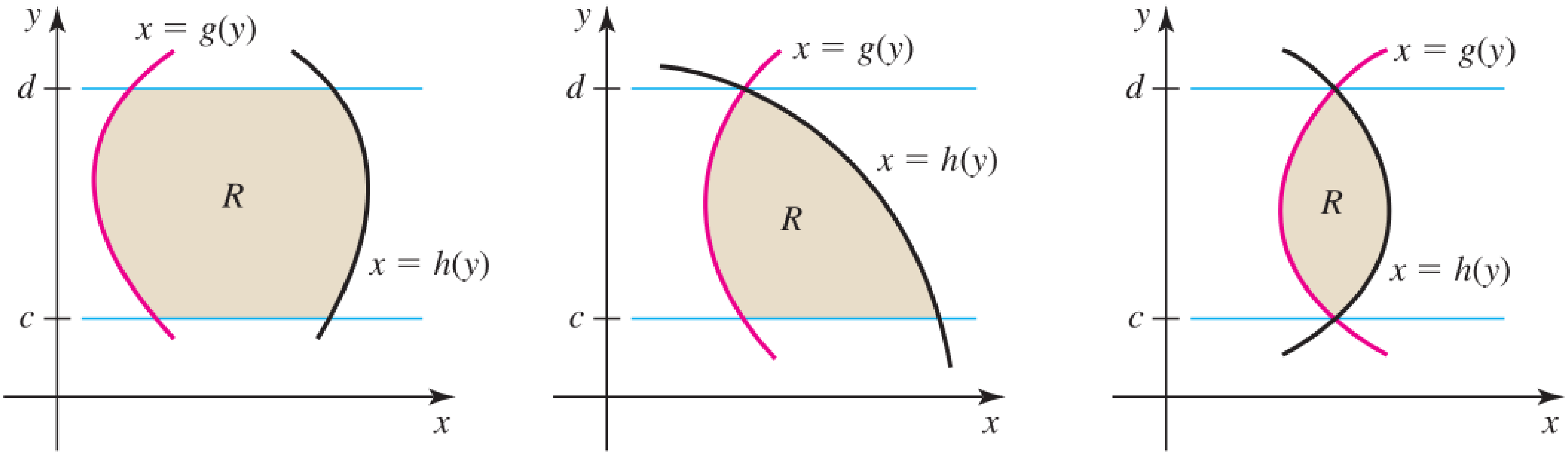
EXAMPLE 1 **Evaluating a double integral** Express the integral $\iint_R 2x^2y\,dA$ as an iterated integral, where R is the region bounded by the parabolas $y = 3x^2$ and $y = 16 - x^2$. Then evaluate the integral.





Change of Perspective

The second type of region R is bounded on the left and right by the graphs of continuous functions $x = g(y)$ and $x = h(y)$, respectively, for $c \leq y \leq d$.



THEOREM 2 Double Integrals over Nonrectangular Regions

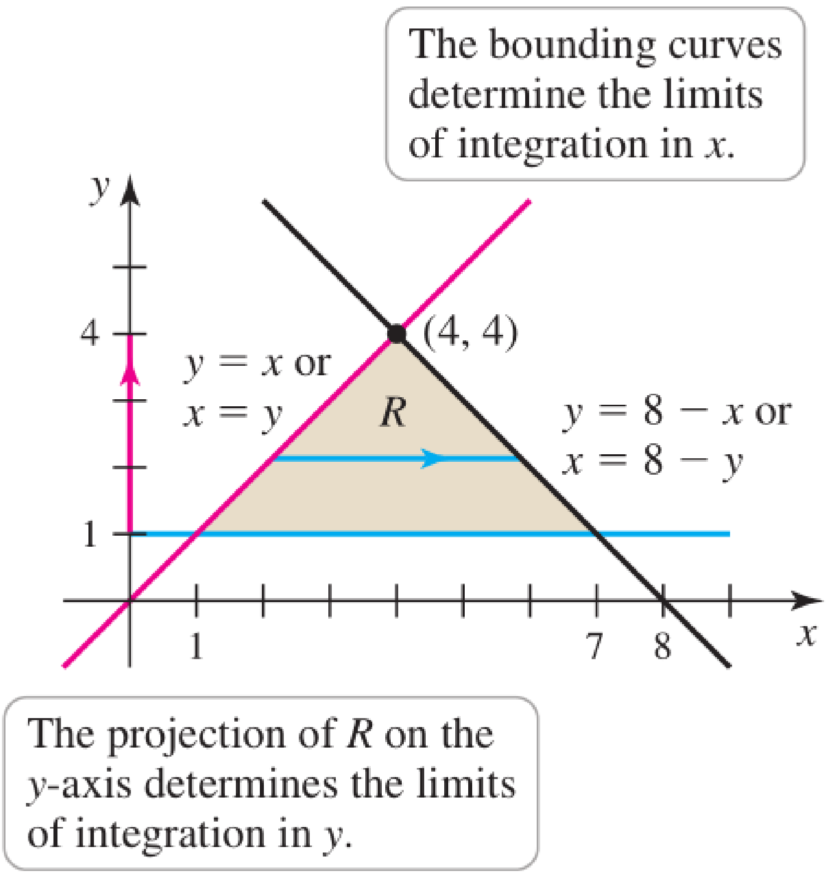
Let R be a region bounded below and above by the graphs of the continuous functions $y = g(x)$ and $y = h(x)$, respectively, and by the lines $x = a$ and $x = b$ (Figure 11). If f is continuous on R, then

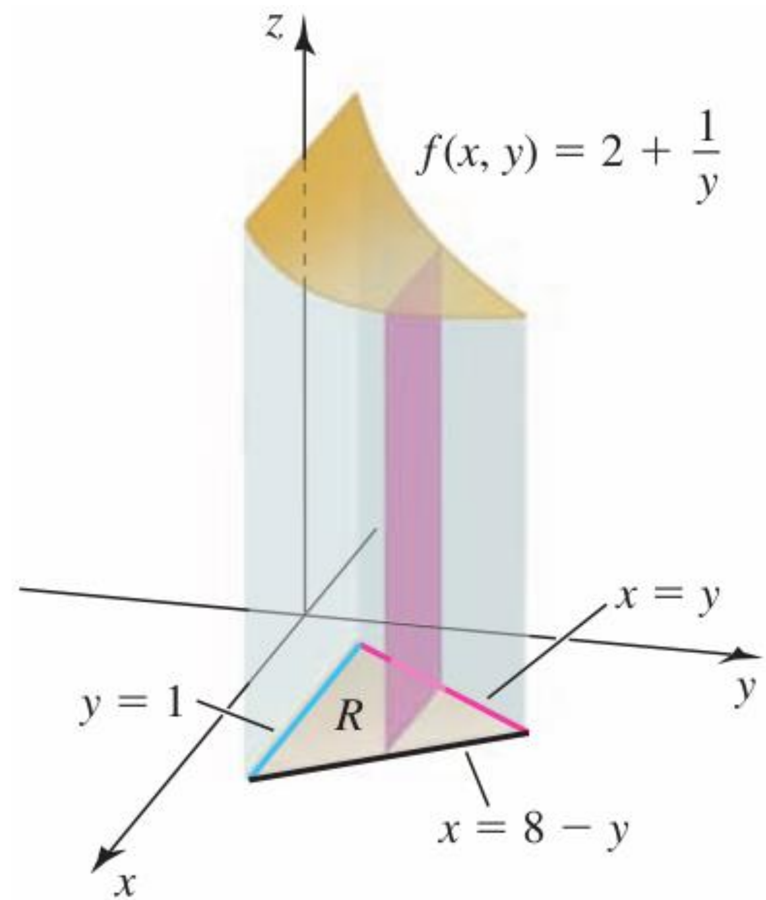
$$\iint_R f(x, y)\, dA = \int_a^b \int_{g(x)}^{h(x)} f(x, y)\, dy\, dx.$$

Let R be a region bounded on the left and right by the graphs of the continuous functions $x = g(y)$ and $x = h(y)$, respectively, and the lines $y = c$ and $y = d$ (Figure 13.15). If f is continuous on R, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_{g(y)}^{h(y)} f(x, y)\, dx\, dy.$$

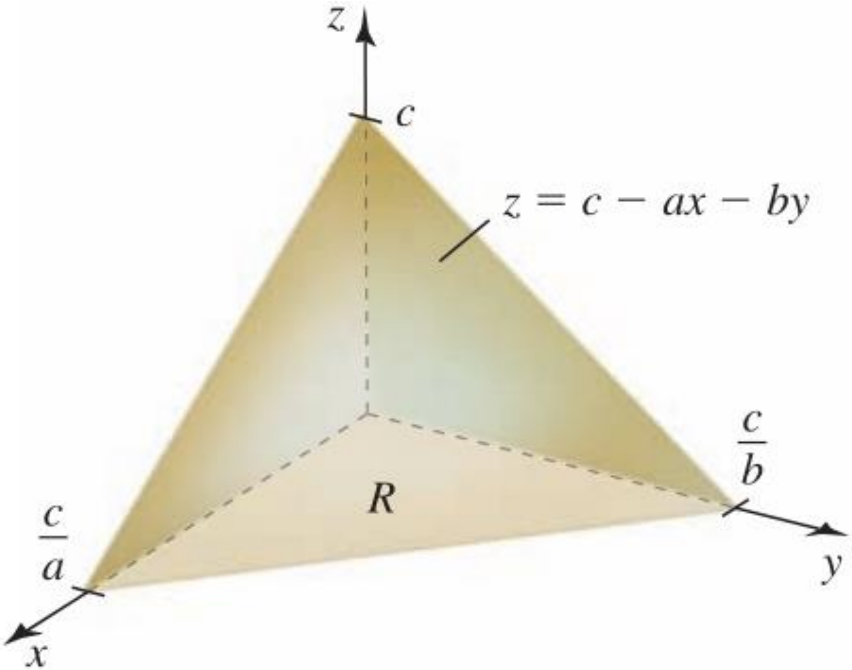
EXAMPLE 2 **Computing a volume** Find the volume of the solid below the surface $f(x, y) = 2 + \frac{1}{y}$ and above the region R in the xy-plane bounded by the lines $y = x$, $y = 8 - x$, and $y = 1$. Notice that $f(x, y) > 0$ on R.

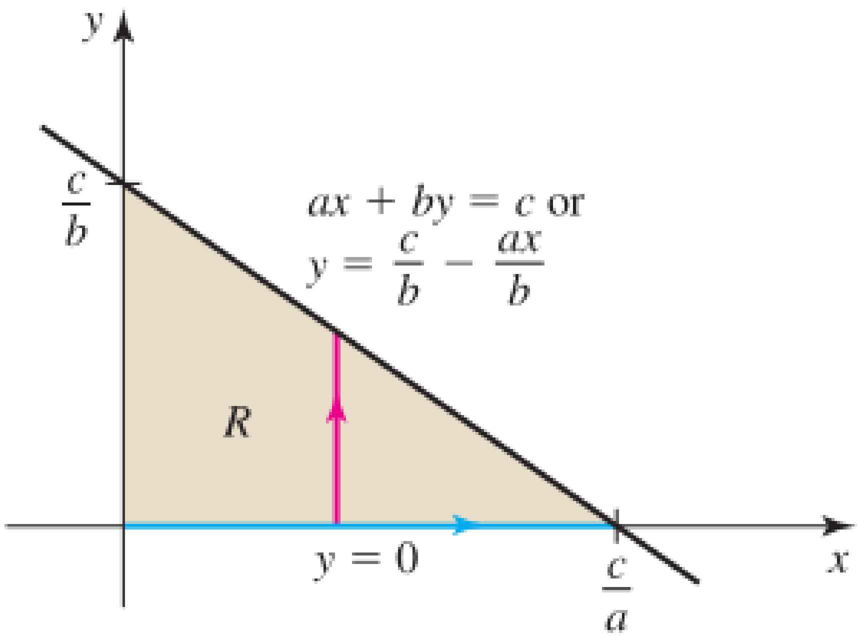




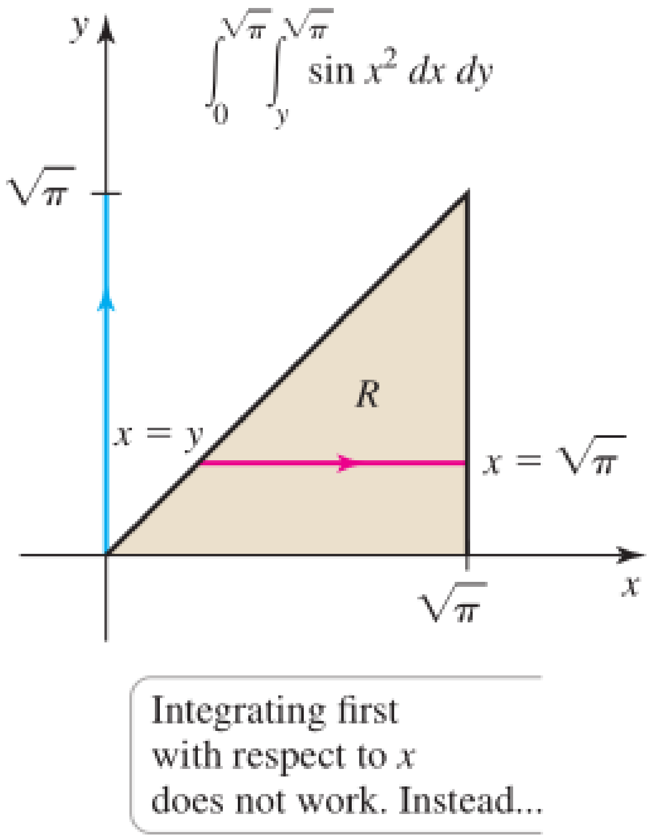
Choosing and Changing the Order of Integration

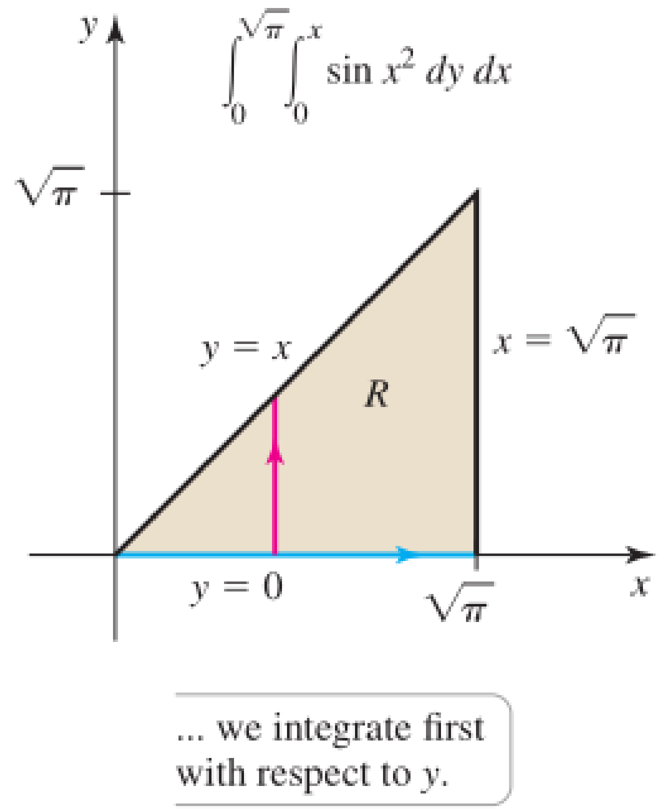
EXAMPLE 3 **Volume of a tetrahedron** Find the volume of the tetrahedron (pyramid with four triangular faces) in the first octant bounded by the plane $z = c - ax - by$ and the coordinate planes ($x = 0$, $y = 0$, and $z = 0$). Assume a, b, and c are positive real numbers (Figure 19).



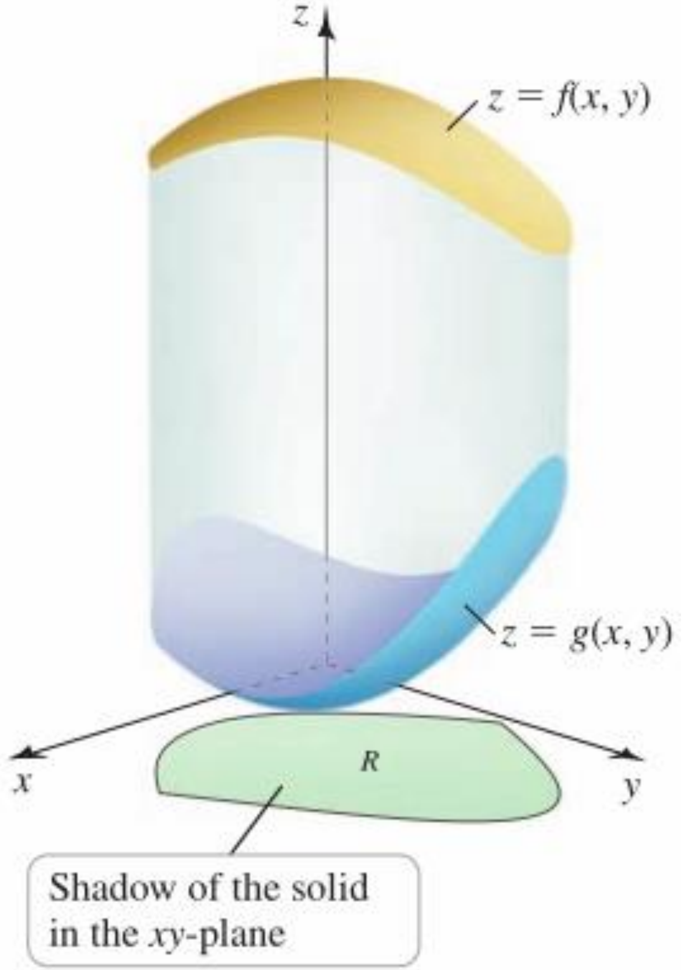


EXAMPLE 4 **Changing the order of integration** Consider the iterated integral $\int_0^{\sqrt{\pi}} \int_y^{\sqrt{\pi}} \sin x^2 \, dx \, dy$. Sketch the region of integration determined by the limits of integration and then evaluate the iterated integral.



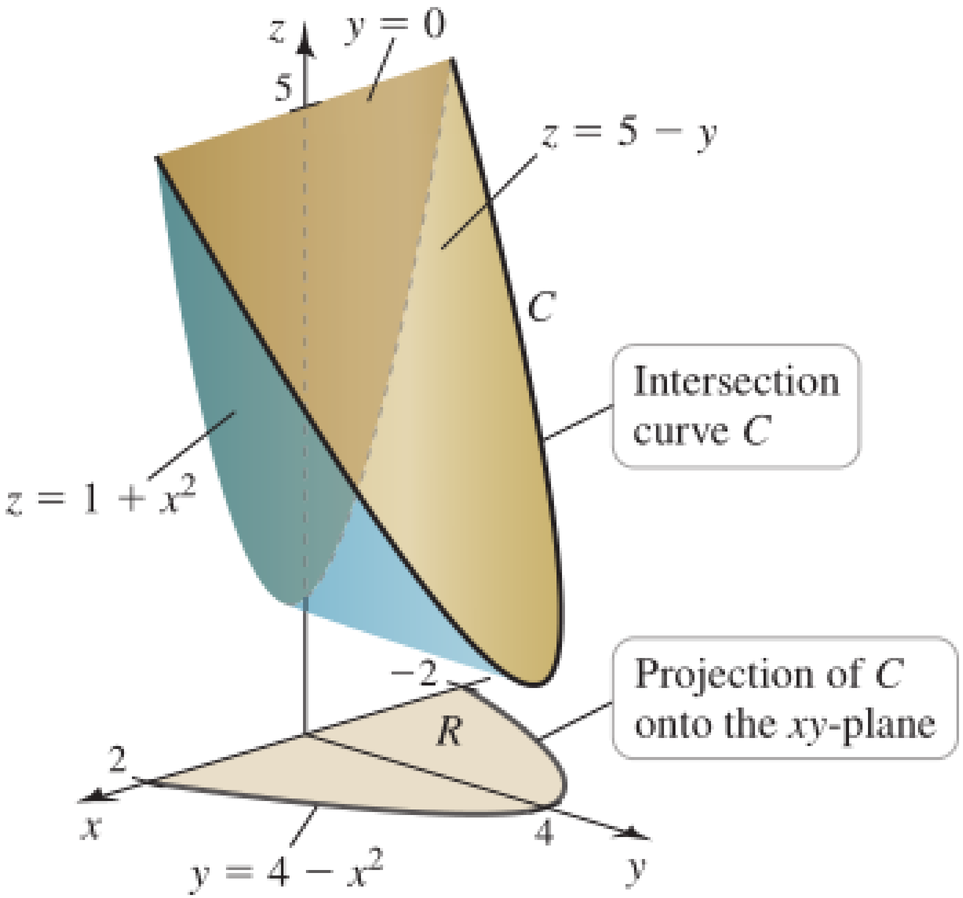


Regions Between Two Surfaces



$$\text{Volume} = \iint_R (f(x, y) - g(x, y))\, dA$$

EXAMPLE 5 **Region bounded by two surfaces** Find the volume of the solid bounded by the parabolic cylinder $z = 1 + x^2$ and the planes $z = 5 - y$ and $y = 0$

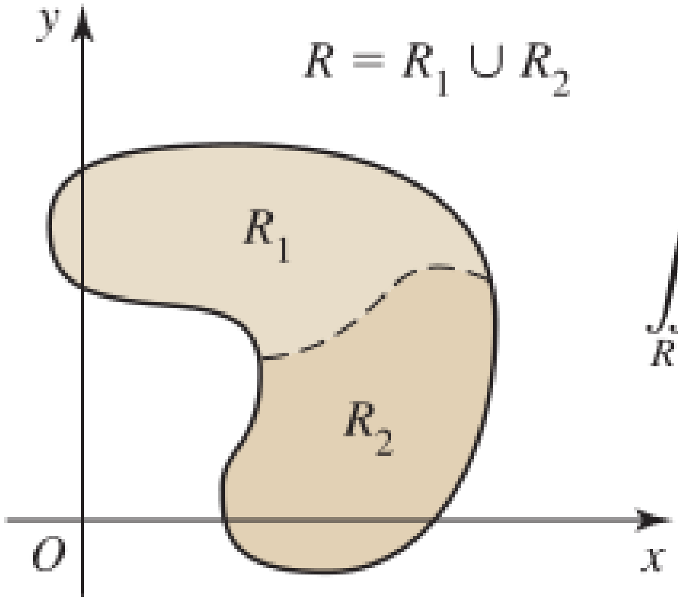


Decomposition of Regions

More complicated regions

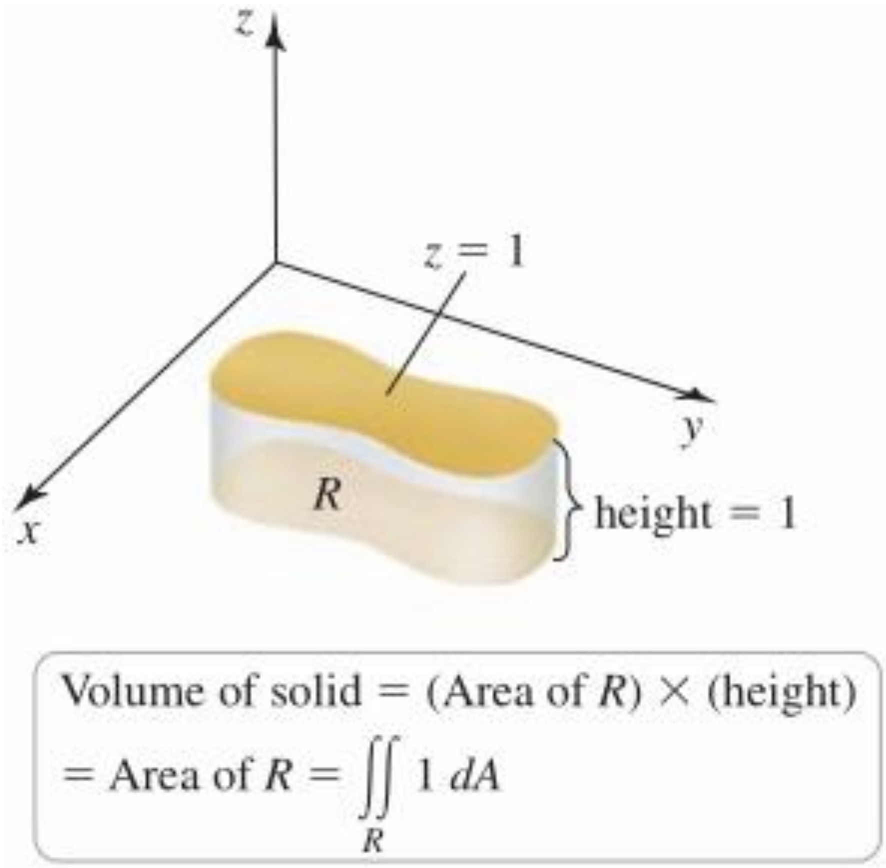
Technique called *decomposition*: subdivide a region of integration into two (or more) subregions.

E.g., the region R in Figure is divided into two nonoverlapping subregions R_1 and R_2



$$\iint_R f(x, y)\, dA = \iint_{R_1} f(x, y)\, dA + \iint_{R_2} f(x, y)\, dA.$$

Finding Area by Double Integrals



The integral $\iint 1\, dA$ gives the volume of the solid between the horizontal plane $z = 1$ and the region R.

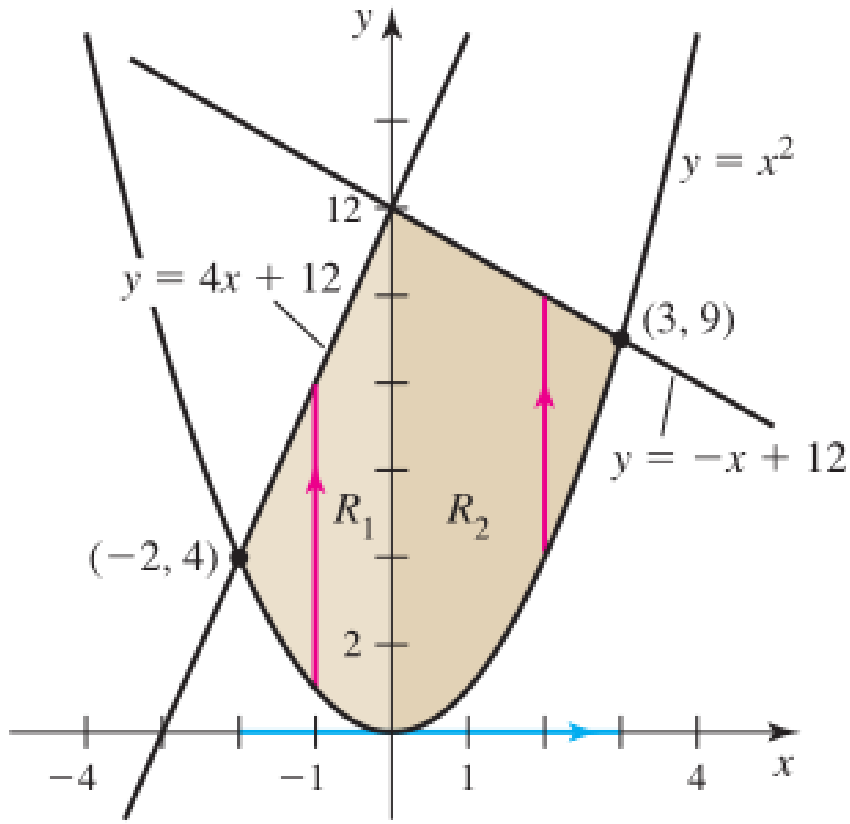
Because the height of this solid is 1, its volume equals (numerically) the area of R

Areas of Regions by Double Integrals

Let R be a region in the xy-plane. Then

$$\text{area of } R = \iint\limits_R dA.$$

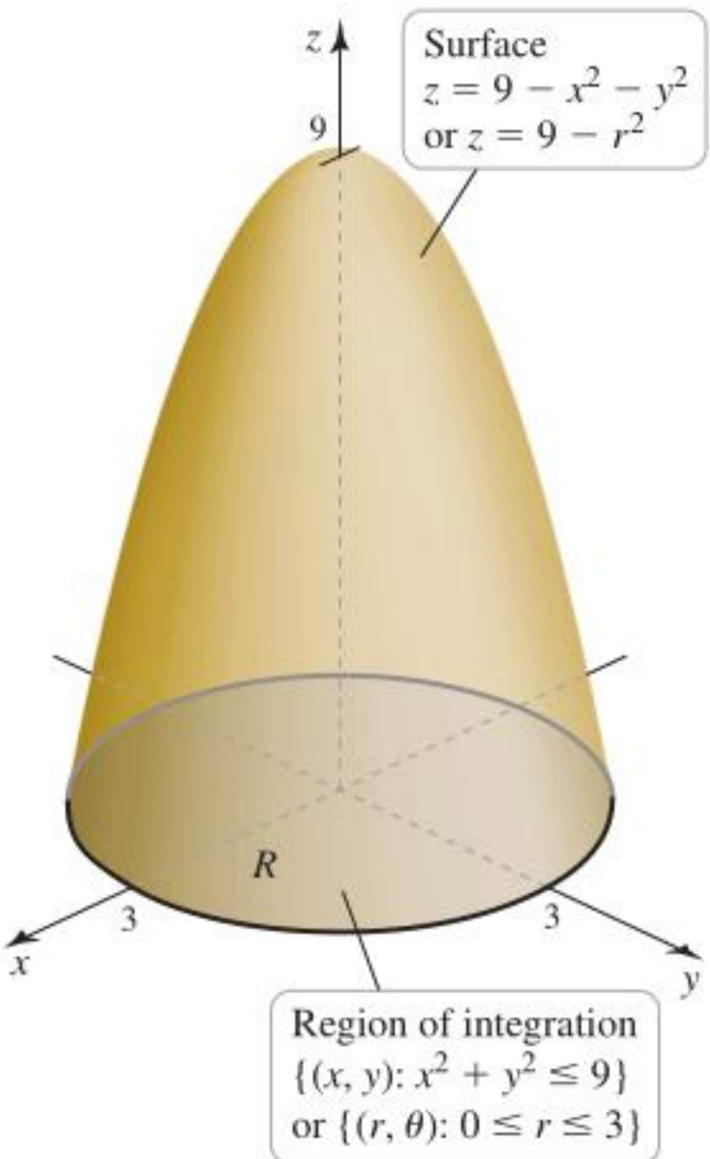
EXAMPLE 6 **Area of a plane region** Find the area of the region R bounded by $y = x^2$, $y = -x + 12$, and $y = 4x + 12$



16.3

Double Integrals in Polar Coordinates

Polar Rectangular Regions



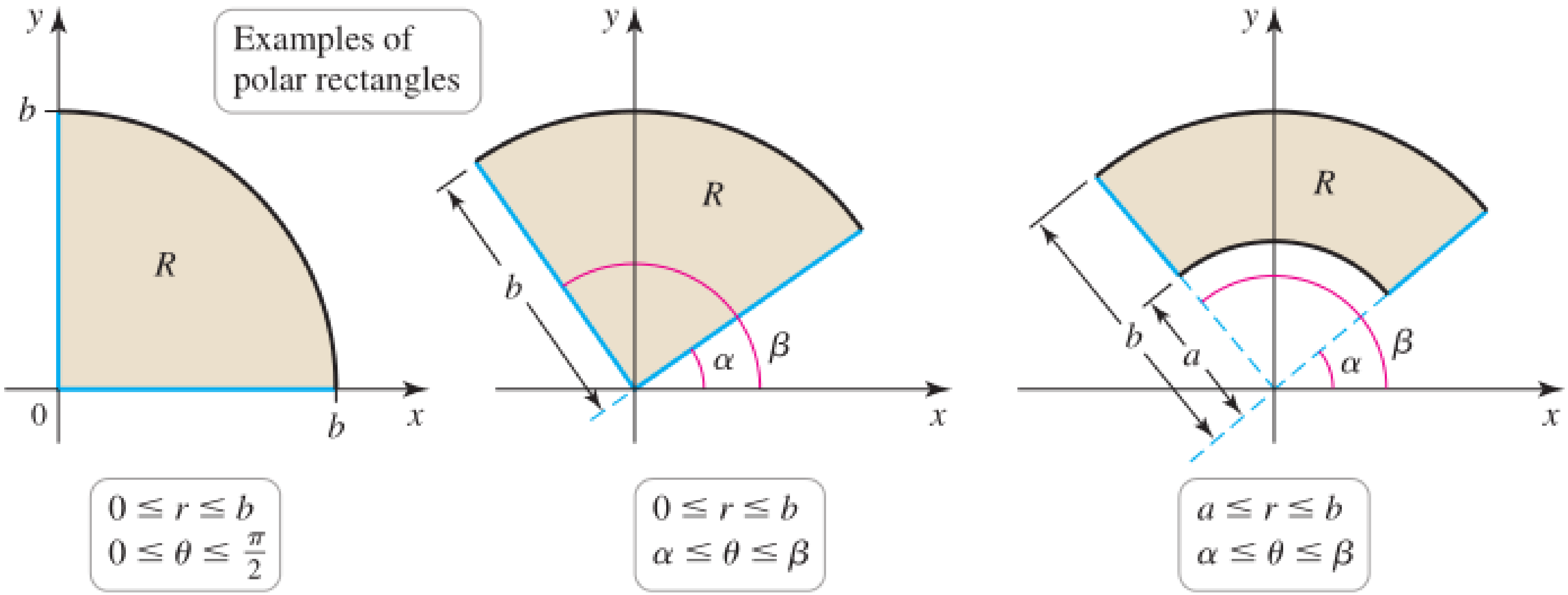
Recall the conversions between Cartesian and polar coordinates

$x = r\cos\theta$, $y = r\sin\theta$, or

$r^2 = x^2 + y^2$, $\tan\theta = y/x$

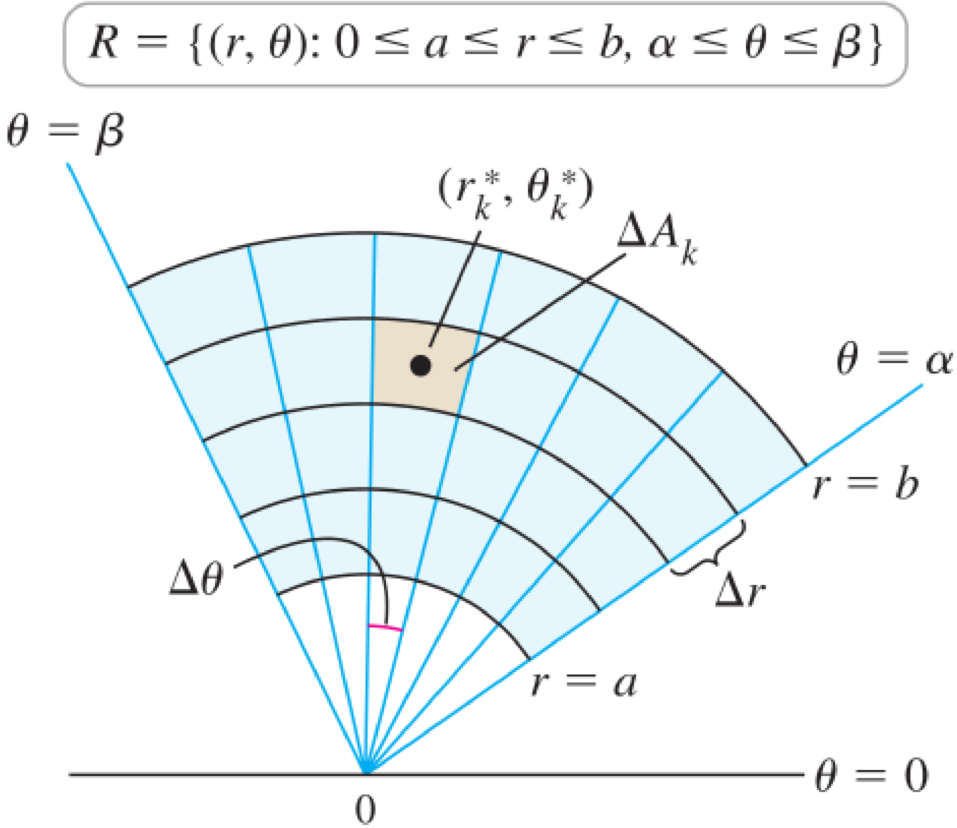
The region of integration is an example of a **polar rectangle**

$$R = \{(r,\theta): 0 \le a \le r \le b, \alpha \le \theta \le \beta\}, \beta - \alpha \le 2\pi$$



Approach: divide $[a, b]$ into M subintervals of equal length $\Delta r = (b - a)/M$

Similarly, divide $[\alpha, \beta]$ into m subintervals of equal length $\Delta\theta = (\beta - \alpha)/m$.



The region R is divided into $n = Mm$ polar rectangles.
ΔA_k: the area of the kth rectangle,
(r_k^*, θ_k^*): an arbitrary point in that rectangle
The volume of that rectangle is approximated as $f(r_k^*, \theta_k^*)\Delta A_k$

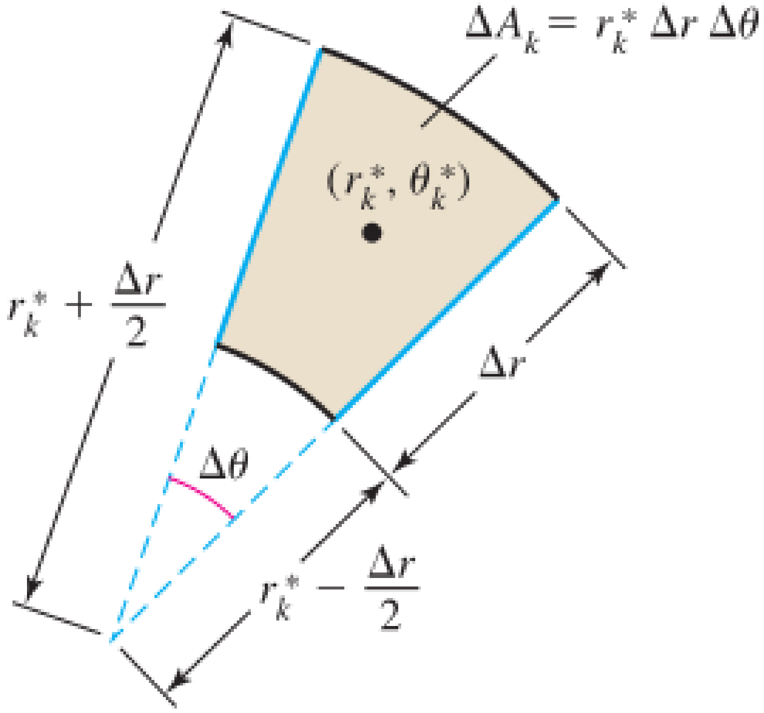
Summing of the volumes of the n “boxes”

$$V \approx \sum_{k=1}^{n} f(r_k^*, \theta_k^*)\Delta A_k$$

Taking limit

$$\iint_R f(r, \theta)dA = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(r_k^*, \theta_k^*)\Delta A_k$$

Iterated integral: Write ΔA_k in terms of Δr and $\Delta\theta$



Choose the point (r_k^*, θ_k^*) so that

$$\Delta A_k = \frac{1}{2}\left(r_k^* + \frac{\Delta r}{2}\right)^2 \Delta\theta - \frac{1}{2}\left(r_k^* - \frac{\Delta r}{2}\right)^2 \Delta\theta = r_k^* \Delta r \Delta\theta$$

So,

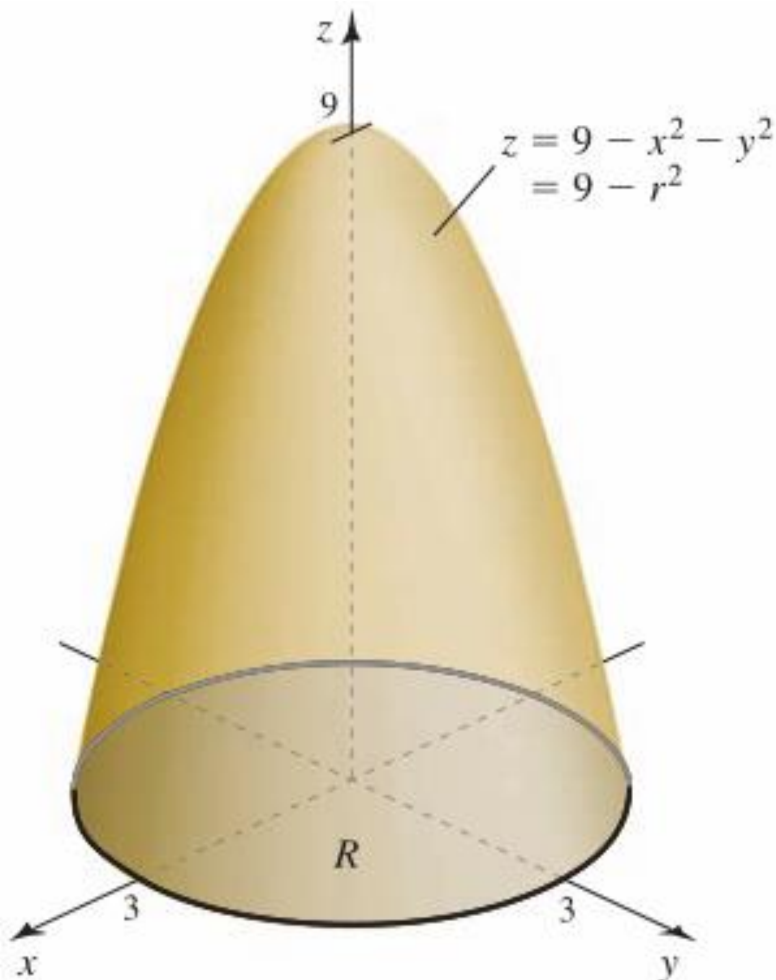
$$\iint_R f(r,\theta)\,dA = \lim_{\Delta\to 0} \sum_{k=1}^{n} f(r_k^*, \theta_k^*)\, r_k^* \Delta r \Delta\theta$$

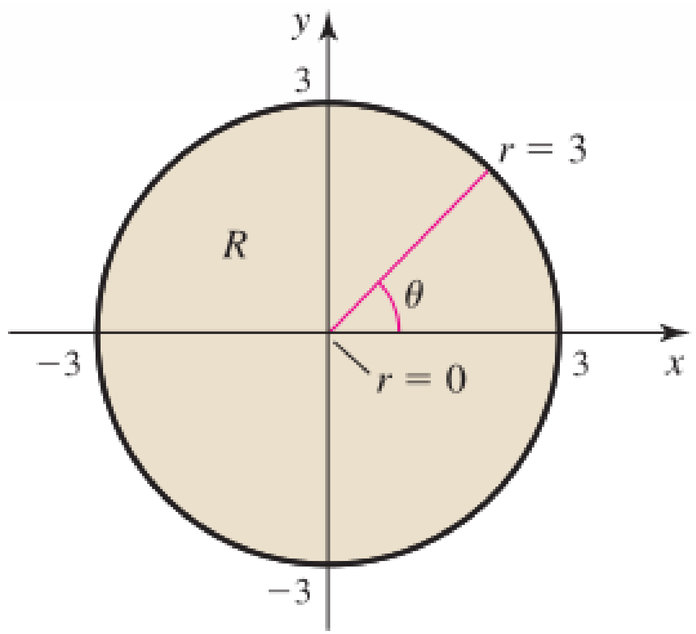
THEOREM 3 Double Integrals over Polar Rectangular Regions

Let f be continuous on the region in the xy-plane $R = \{(r, \theta): 0 \leq a \leq r \leq b, \alpha \leq \theta \leq \beta\}$, where $\beta - \alpha \leq 2\pi$. Then

$$\iint_R f(r, \theta)\, dA = \int_\alpha^\beta \int_a^b f(r, \theta)\, r\, dr\, d\theta.$$

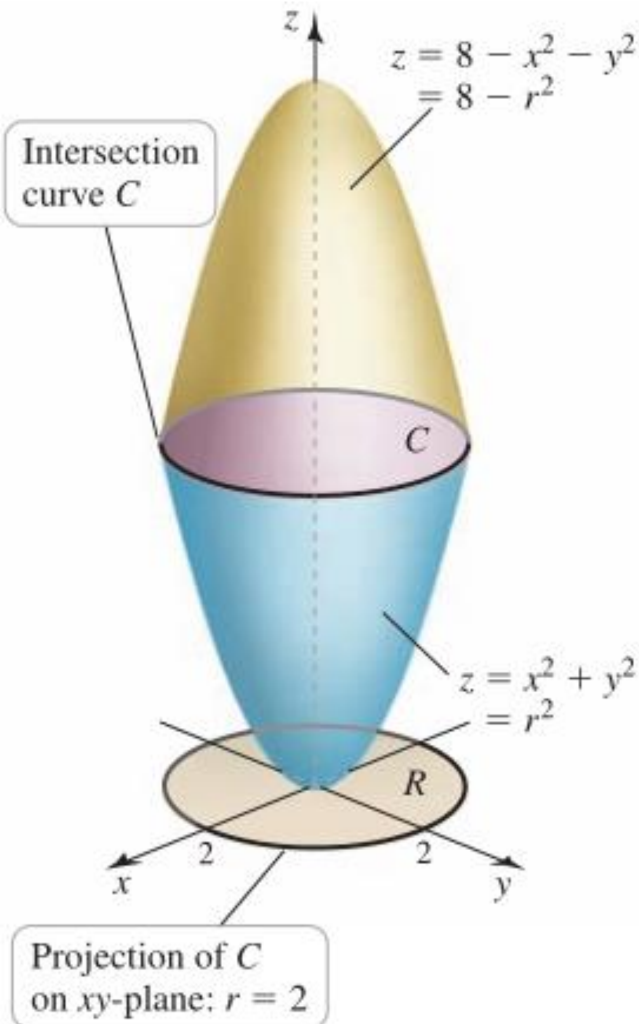
EXAMPLE 1 **Volume of a paraboloid cap** Find the volume of the solid bounded by the paraboloid $z = 9 - x^2 - y^2$ and the xy-plane.





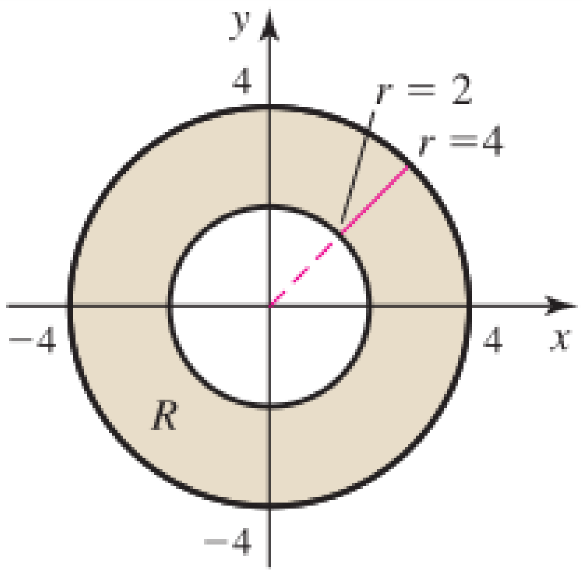
$R = \{(r, \theta): 0 \le r \le 3, 0 \le \theta \le 2\pi\}$

EXAMPLE 2 **Region bounded by two surfaces** Find the volume of the region bounded by the paraboloids $z = x^2 + y^2$ and $z = 8 - x^2 - y^2$.



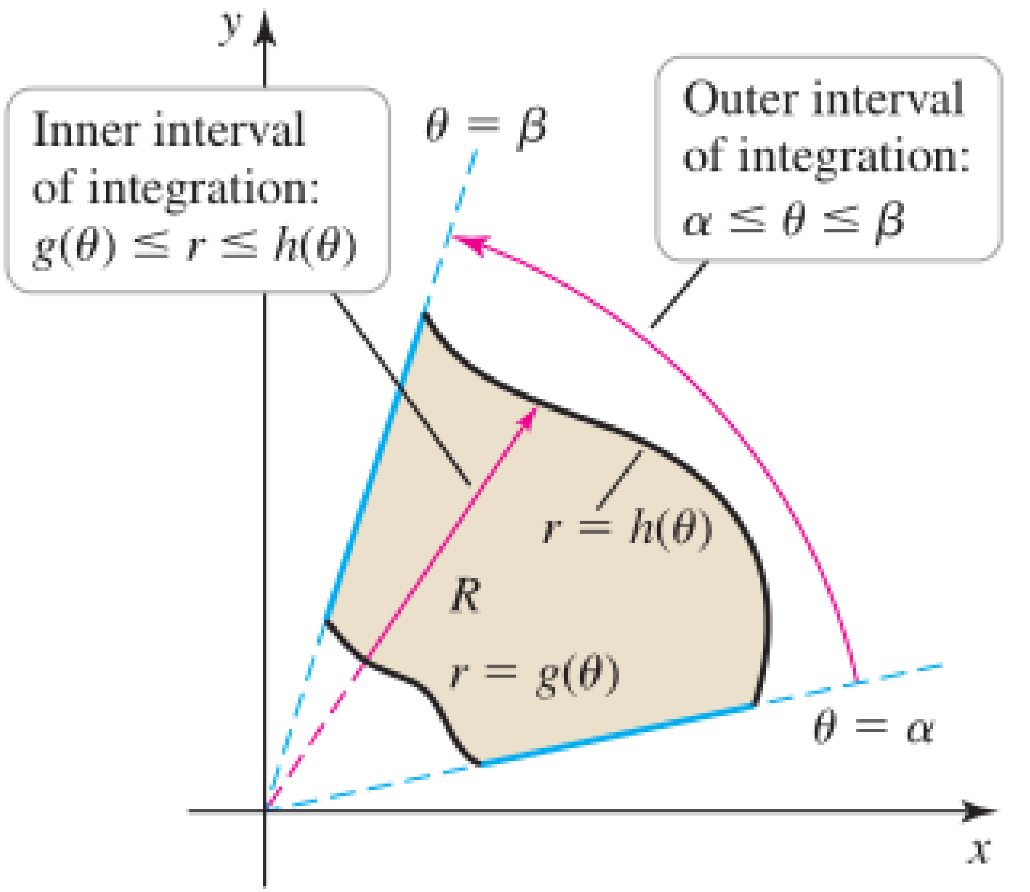
Compare the difference of the integrals in polar coordinates and in rectangular coordinates.

EXAMPLE 3 **Annular region** Find the volume of the region beneath the surface $z = xy + 10$ and above the annular region $R = \{(r, \theta): 2 \le r \le 4, 0 \le \theta \le 2\pi\}$. (An *annulus* is the region between two concentric circles.)



$R = \{(r, \theta): 2 \le r \le 4, 0 \le \theta \le 2\pi\}$

More General Polar Regions



$$R = \{(r, \theta): 0 \leq g(\theta) \leq r \leq h(\theta),\ \alpha \leq \theta \leq \beta\}$$

THEOREM 4 Double Integrals over More General Polar Regions

Let f be continuous on the region in the xy-plane

$$R = \{(r, \theta): 0 \leq g(\theta) \leq r \leq h(\theta), \alpha \leq \theta \leq \beta\},$$

where $0 < \beta - \alpha \leq 2\pi$. Then

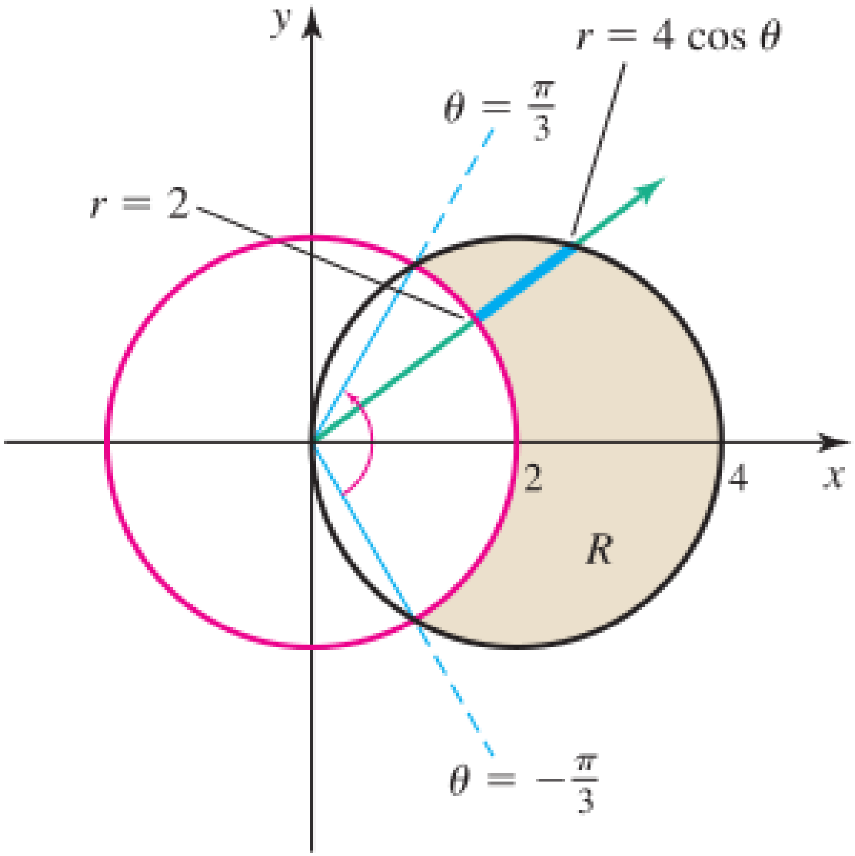
$$\iint_R f(r, \theta)\, dA = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} f(r, \theta)\, r\, dr\, d\theta.$$

EXAMPLE 4 **Specifying regions** Write an iterated integral for $\iint_R f(r, \theta)\, dA$ for the following regions R in the xy-plane.

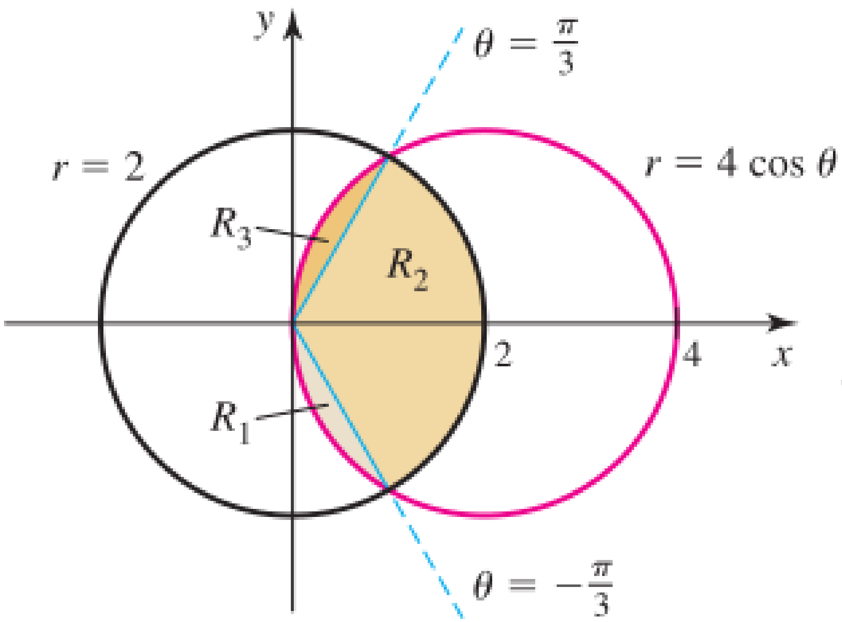
a. The region outside the circle $r = 2$ (with radius 2 centered at $(0, 0)$) and inside the circle $r = 4\cos\theta$ (with radius 2 centered at $(2, 0)$)

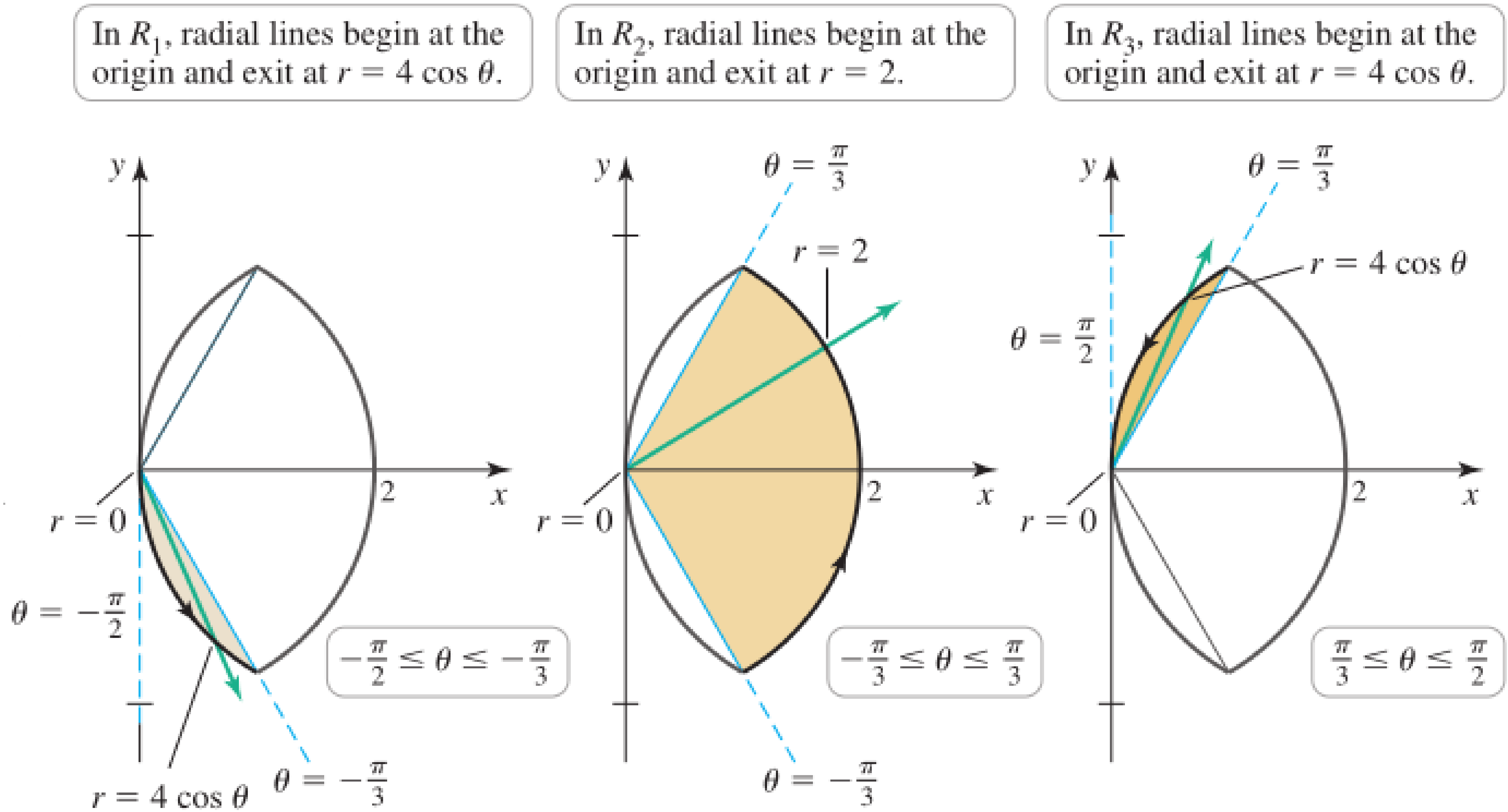
b. The region inside both circles of part (a)

Radial lines enter the region R at $r = 2$ and exit the region at $r = 4\cos\theta$.



The inner and outer boundaries of R are traversed as θ varies from $-\frac{\pi}{3}$ to $\frac{\pi}{3}$.




In R_1, radial lines begin at the origin and exit at $r = 4\cos\theta$.
In R_2, radial lines begin at the origin and exit at $r = 2$.
In R_3, radial lines begin at the origin and exit at $r = 4\cos\theta$.
y
x
2
$r = 0$
$\theta = -\frac{\pi}{2}$
$r = 4\cos\theta$
$\theta = -\frac{\pi}{3}$
$-\frac{\pi}{2} \le \theta \le -\frac{\pi}{3}$
y
$\theta = \frac{\pi}{3}$
$r = 2$
x
2
$r = 0$
$\theta = -\frac{\pi}{3}$
$-\frac{\pi}{3} \le \theta \le \frac{\pi}{3}$
y
$\theta = \frac{\pi}{3}$
$r = 4\cos\theta$
$\theta = \frac{\pi}{2}$
x
2
$r = 0$
$\frac{\pi}{3} \le \theta \le \frac{\pi}{2}$

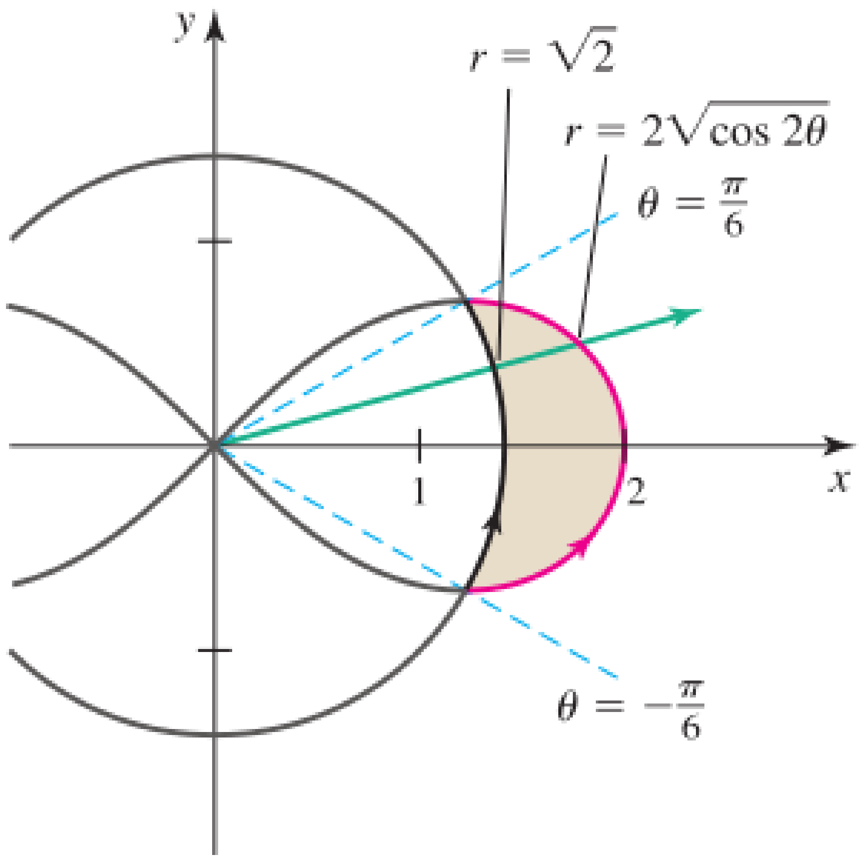
Areas of Regions

Area of Polar Regions

The area of the region $R = \{(r, \theta): 0 \leq g(\theta) \leq r \leq h(\theta), \alpha \leq \theta \leq \beta\}$, where $0 < \beta - \alpha \leq 2\pi$, is

$$A = \iint_R dA = \int_{\alpha}^{\beta} \int_{g(\theta)}^{h(\theta)} r\, dr\, d\theta.$$

EXAMPLE 5 **Area within a lemniscate** Compute the area of the region in the first and fourth quadrants outside the circle $r = \sqrt{2}$ and inside the lemniscate $r^2 = 4\cos 2\theta$



Average Value over a Planar Polar Region

EXAMPLE 6 **Average y-coordinate** Find the average value of the y-coordinates of the points in the semicircular disk of radius a given by $R = \{(r, \theta): 0 \leq r \leq a, 0 \leq \theta \leq \pi\}$.

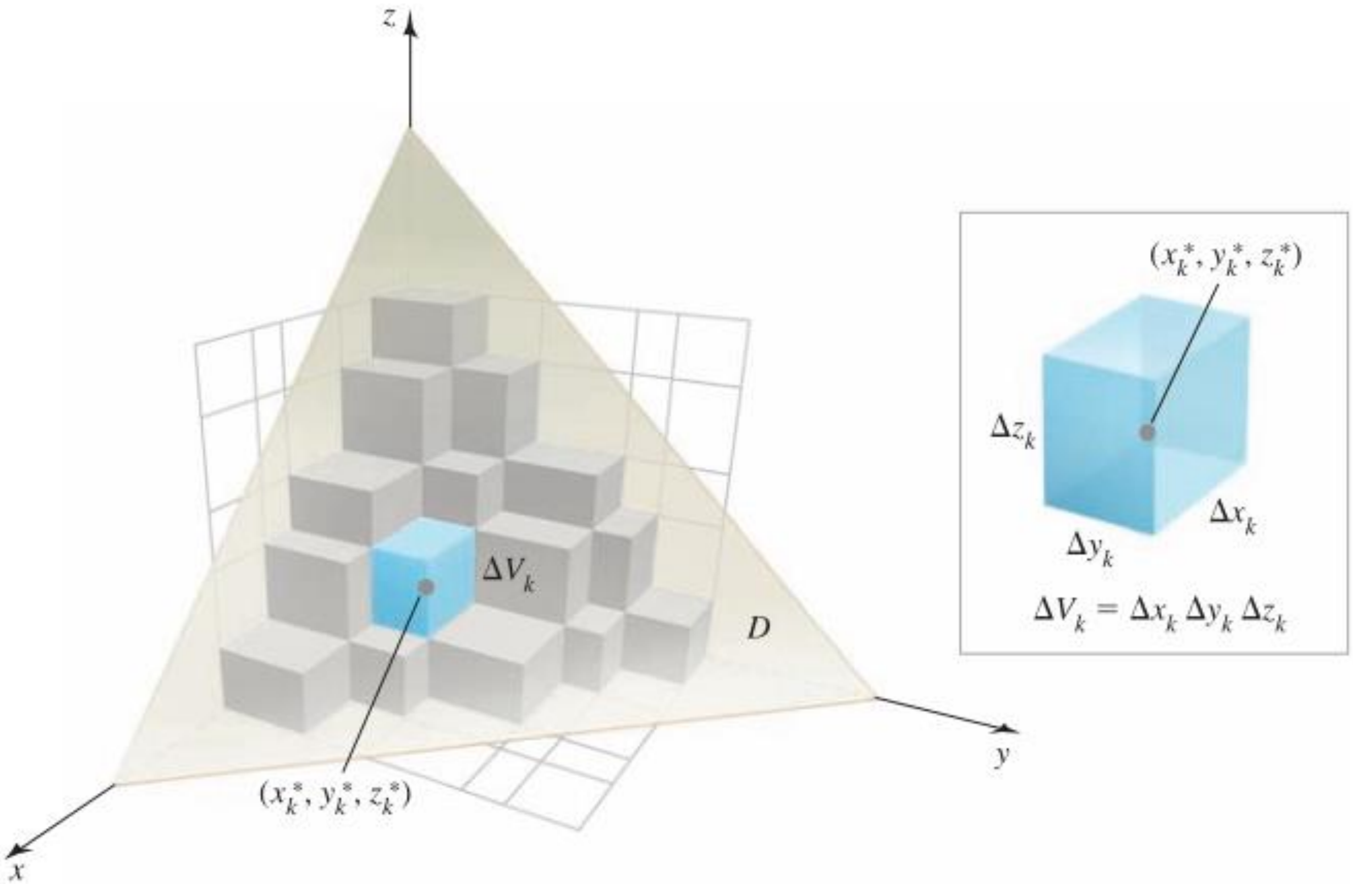
$$\bar{y} = \frac{1}{\pi a^2/2} \int_0^{\pi} \int_0^{a} r \sin\theta \, r \, dr \, d\theta$$

16.4

Triple Integrals

Triple Integrals in Rectangular Coordinates

A function $w = f(x, y, z)$, defined on a closed and bounded region D of $\mathbb{R}^3$, whose graph lies in four-dimensional space. The integral of f over D.



A Riemann sum is formed, in which the kth term is the function value $f(x_k^*, y_k^*, z_k^*)$ multiplied by the volume of the kth box:

$$\sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\Delta V_k$$

where $\Delta V_k = \Delta x_k \Delta y_k \Delta z_k$

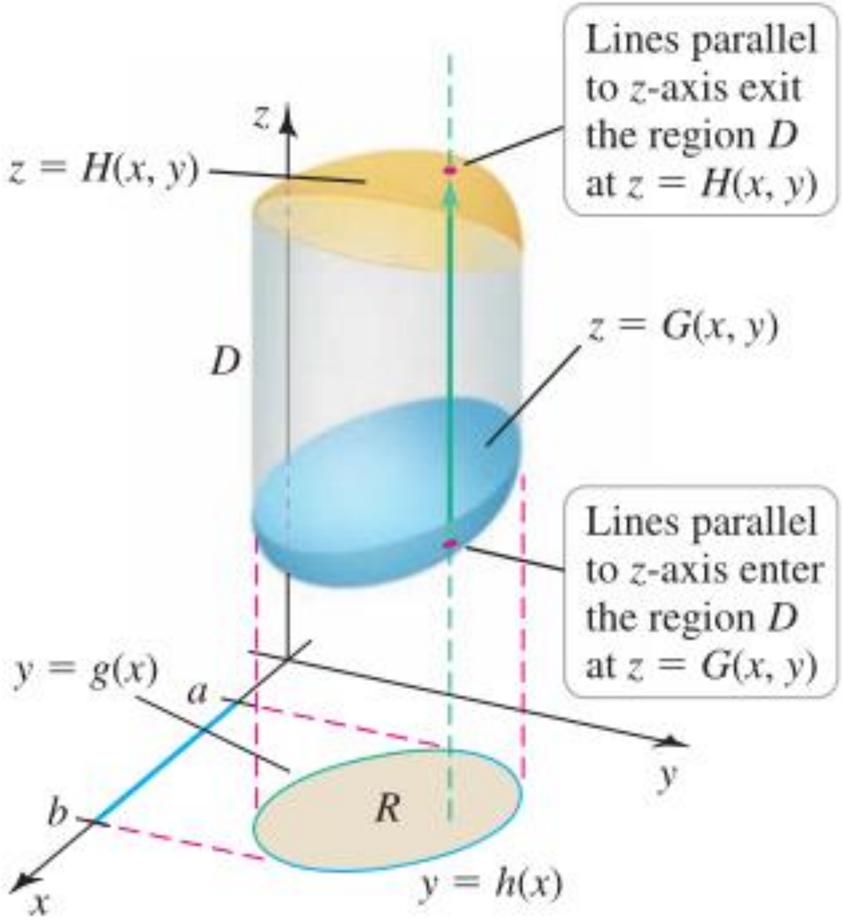
Taking limit to get the **triple integral** of f over D

$$\iiint_D f(x,y,z)\,dV = \lim_{\Delta\to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\Delta V_k$$

If $f(x,y,z) = 1$, the triple integral gives the volume of region D

If $f(x,y,z)$ is the density of a solid object D, the triple integral gives the mass of the object.

Finding Limits of integration



$$\iiint_D f(x, y, z)\, dV = \iint_R \left(\int_{G(x,y)}^{H(x,y)} f(x, y, z)\, dz \right) dA$$

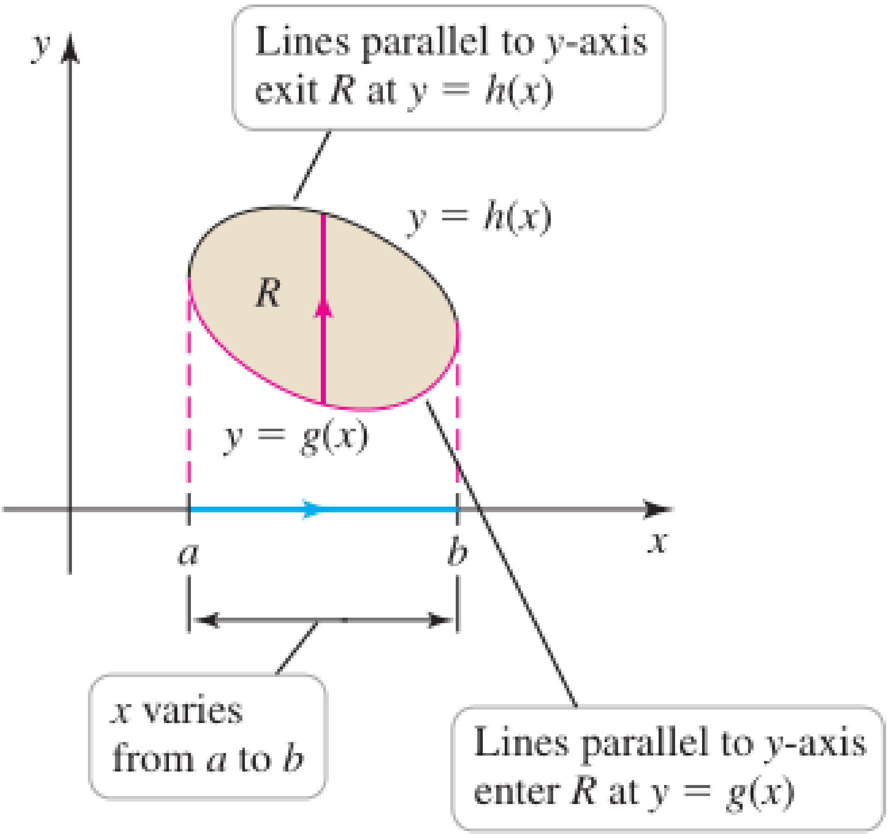
Suppose the region D in $\mathbb{R}^3$ is bounded above by $z = H(x, y)$ and below by $z = G(x, y)$.

Project the region D onto the xy-plane to form a region R.

$$\iiint_D f(x, y, z) dV$$

$$= \iint_R \left(\int_{G(x,y)}^{H(x,y)} f(x, y, z) dz \right) dA$$

Next step



Suppose R is bounded above and below by the curves $y = h(x)$ and $y = g(x)$, and bounded on the right and left by the lines $x = a$ and $x = b$.

The remaining integration over R is carried out as a double integral.

THEOREM 5 Triple Integrals

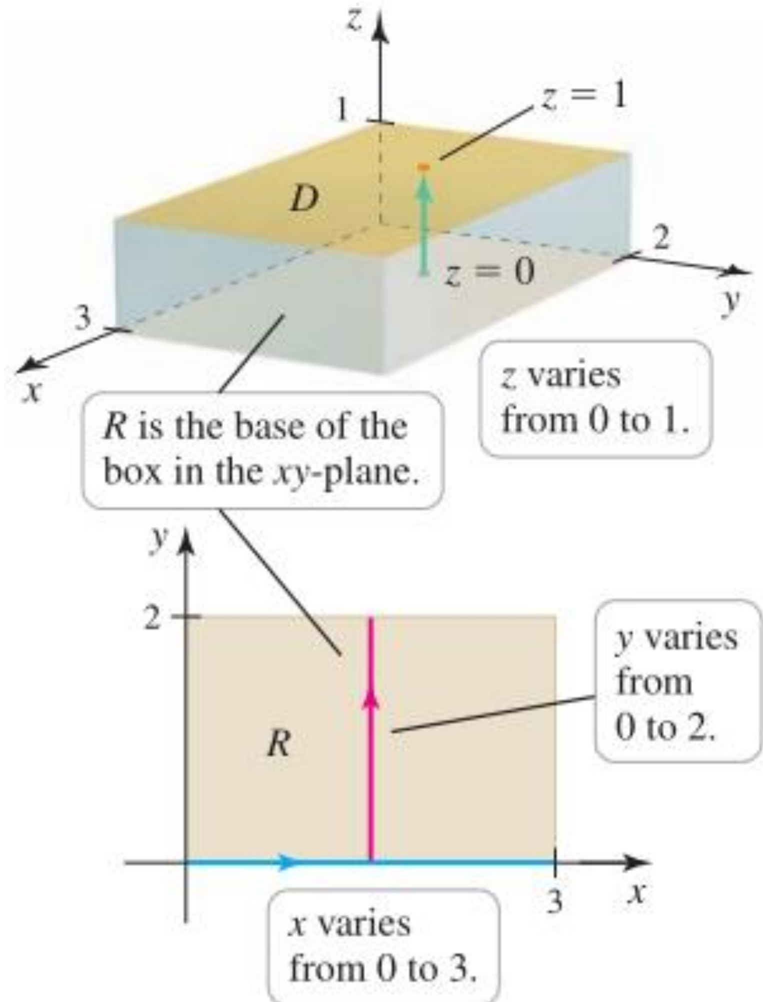
Let f be continuous over the region

$$D = \{(x, y, z): a \le x \le b, g(x) \le y \le h(x), G(x, y) \le z \le H(x, y)\},$$

where g, h, G, and H are continuous functions. Then f is integrable over D and the triple integral is evaluated as the iterated integral

$$\iiint_D f(x, y, z)\, dV = \int_a^b \int_{g(x)}^{h(x)} \int_{G(x, y)}^{H(x, y)} f(x, y, z)\, dz\, dy\, dx.$$

EXAMPLE 1 **Mass of a box** A solid box D is bounded by the planes $x = 0$, $x = 3$, $y = 0$, $y = 2$, $z = 0$, and $z = 1$. The density of the box decreases linearly in the positive z-direction and is given by $f(x, y, z) = 2 - z$. Find the mass of the box.

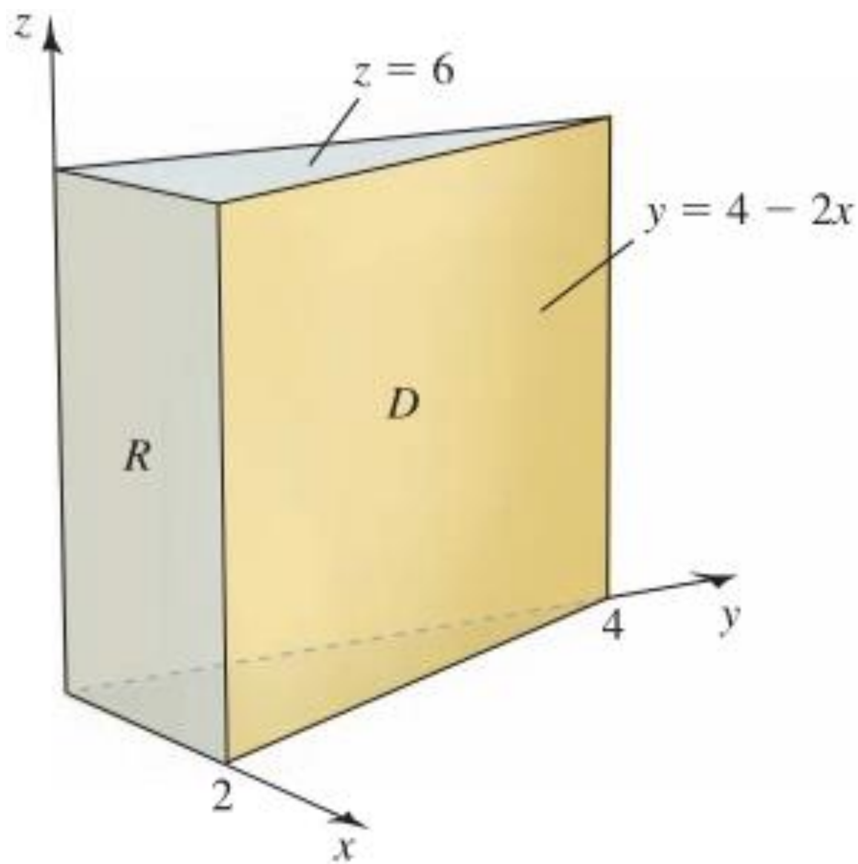


$$M = \iiint_D (2 - z)\,dV$$

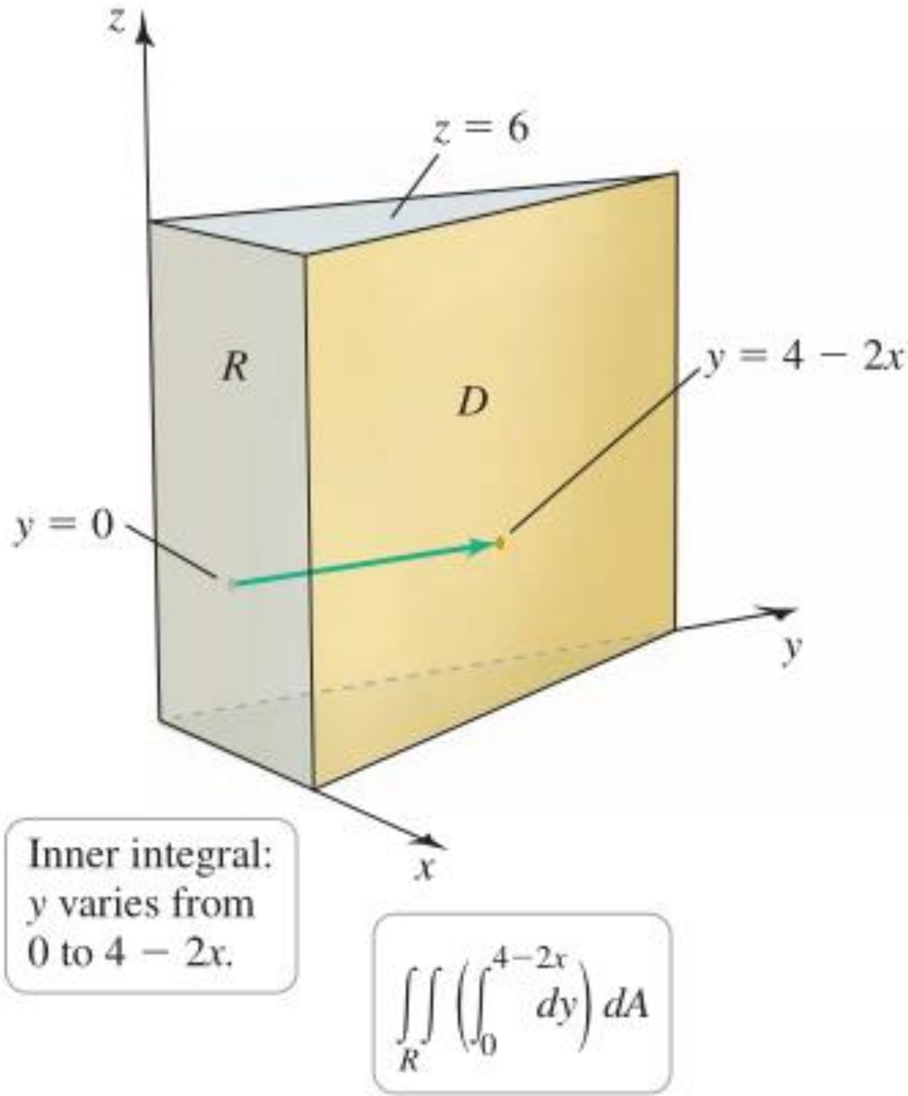
$$= \int_0^3 \int_0^2 \int_0^1 (2 - z)\,dz\,dy\,dx$$

Any other order of integration produces the same result.

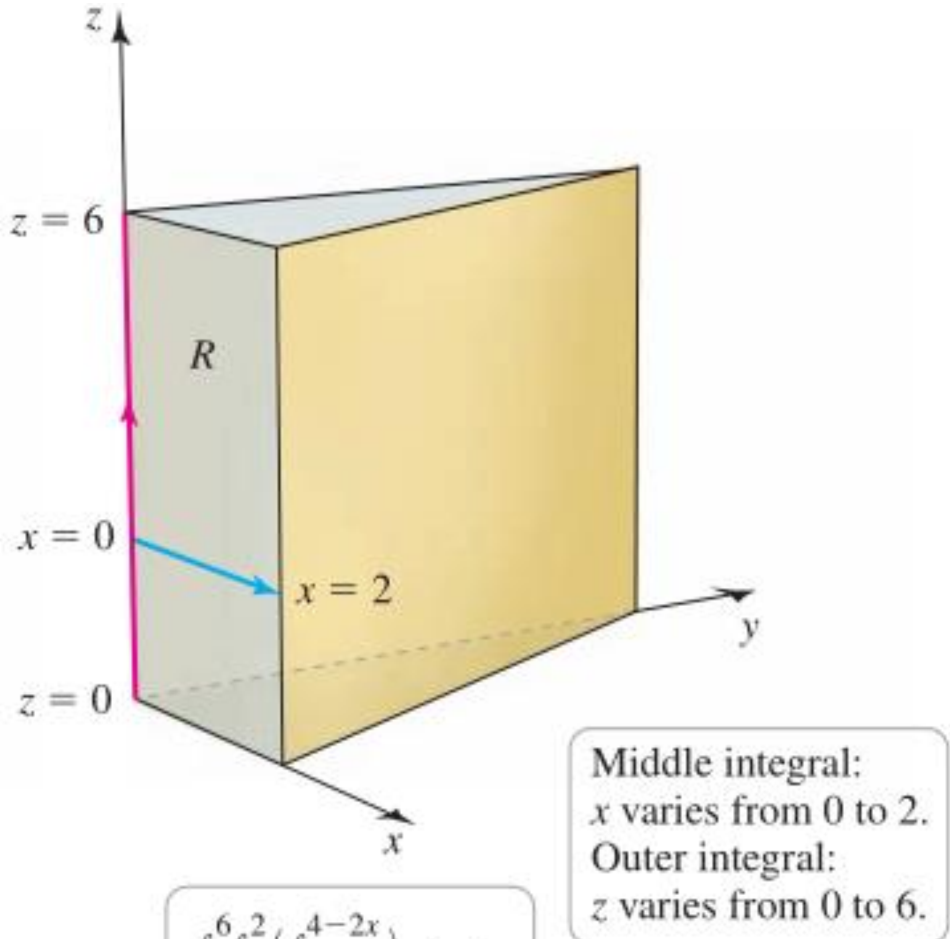
EXAMPLE 2 **Volume of a prism** Find the volume of the prism D in the first octant bounded by the planes $y = 4 - 2x$ and $z = 6$



Inner integral with respect to y

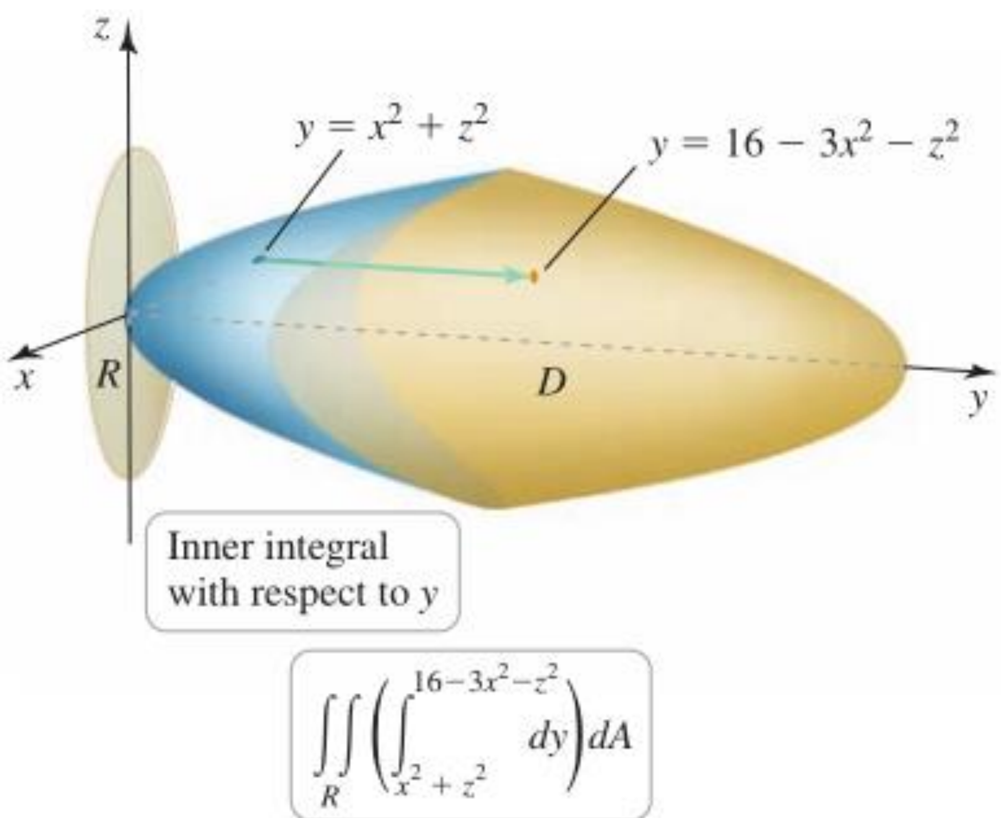


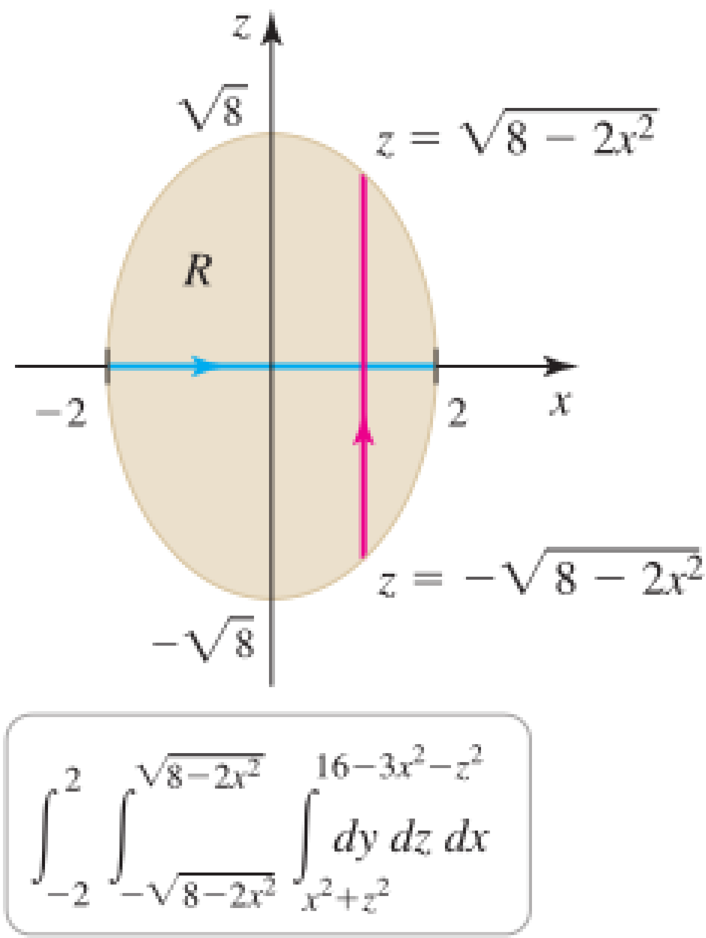
Middle integral with respect to x
Outer integral with respect to z



$$\int_0^6 \int_0^2 \left(\int_0^{4-2x} dy \right) dx\, dz$$

EXAMPLE 3 **A volume integral** Find the volume of the region D bounded by the paraboloids $y = x^2 + z^2$ and $y = 16 - 3x^2 - z^2$





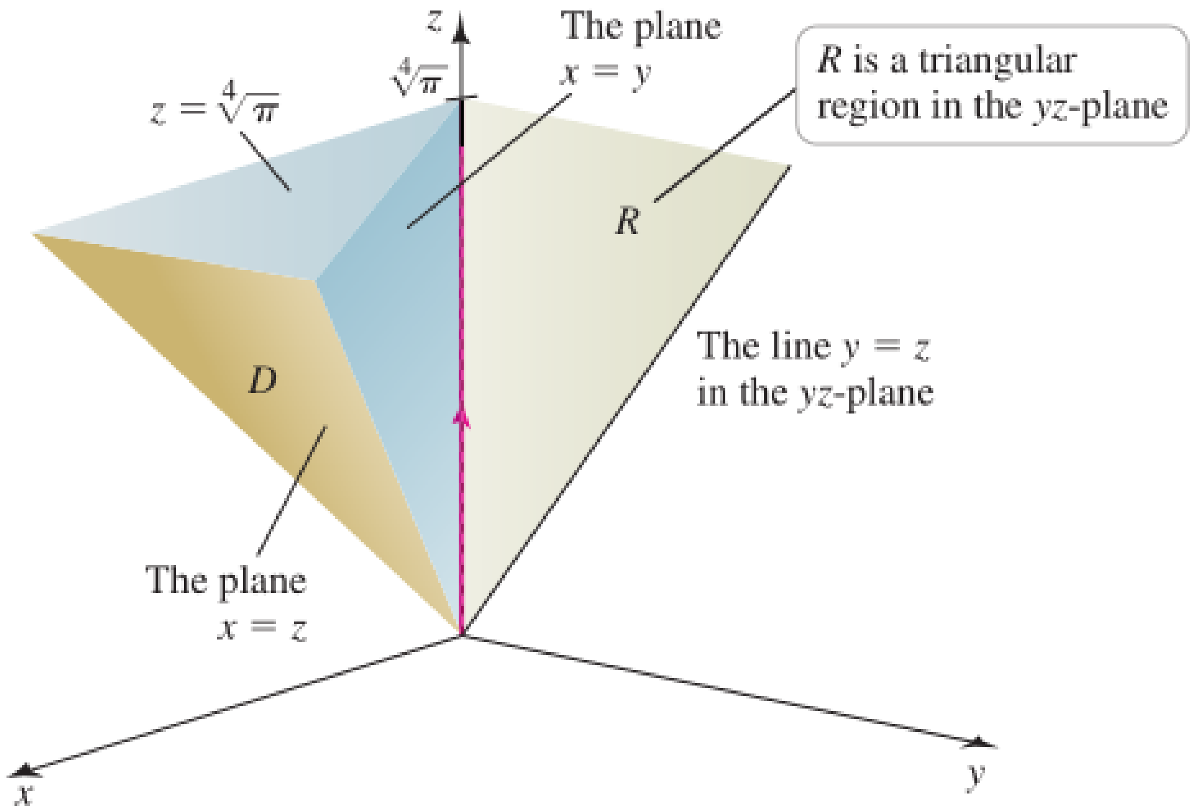
Changing the Order of Integration

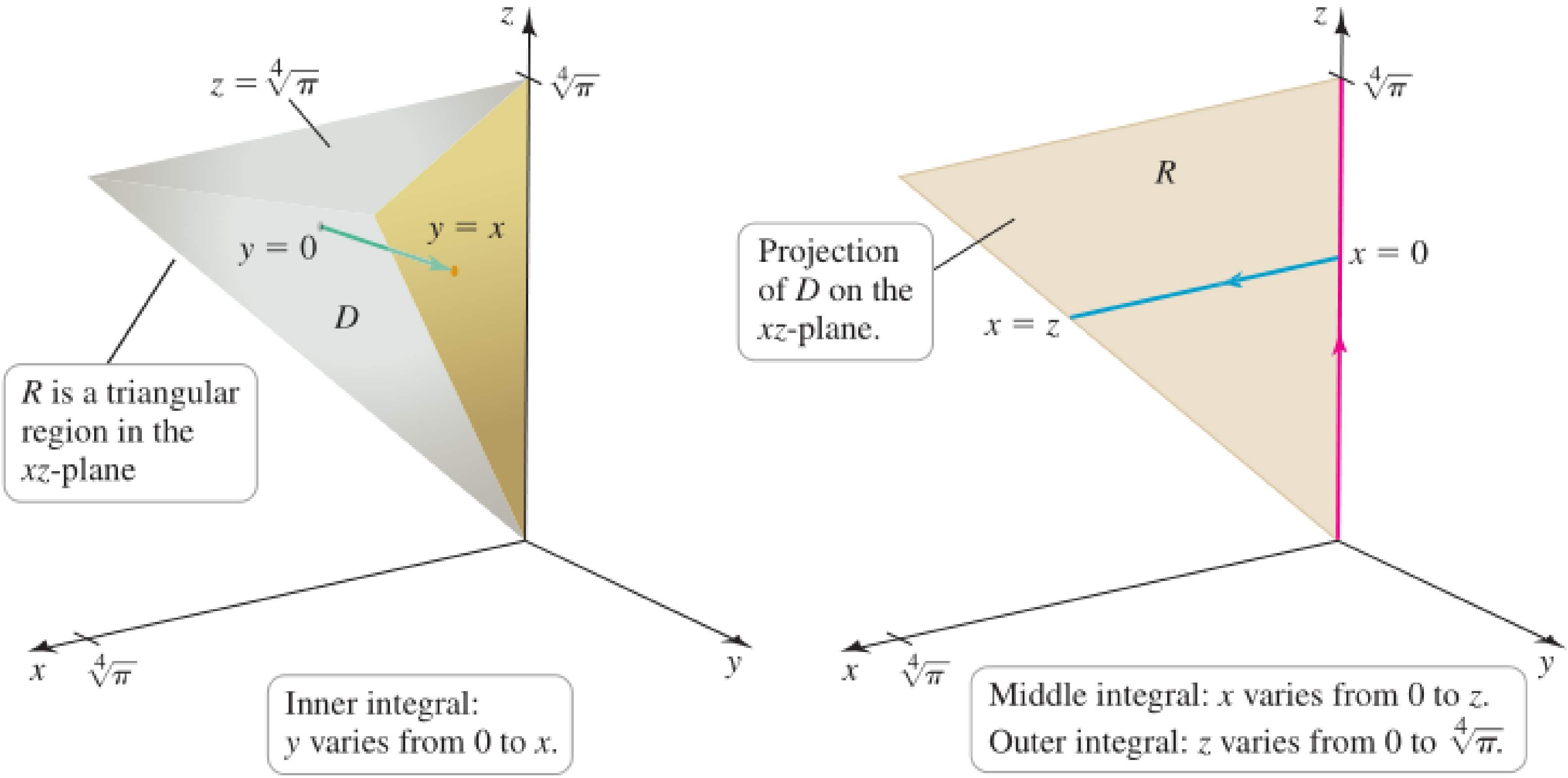
EXAMPLE 4 **Changing the order of integration** Consider the integral

$$\int_0^{\sqrt[4]{\pi}} \int_0^z \int_y^z 12y^2 z^3 \sin x^4 \, dx\, dy\, dz.$$

a. Sketch the region of integration D.

b. Evaluate the integral by changing the order of integration.


z
$\sqrt[4]{\pi}$
The plane
$x = y$
R is a triangular
region in the yz-plane
$z = \sqrt[4]{\pi}$
R
D
The line $y = z$
in the yz-plane
The plane
$x = z$
x
y



z
$z = \sqrt[4]{\pi}$
$\sqrt[4]{\pi}$
$y = 0$
$y = x$
D
R is a triangular region in the xz-plane
x
$\sqrt[4]{\pi}$
y
Inner integral: y varies from 0 to x.
z
$\sqrt[4]{\pi}$
R
Projection of D on the xz-plane.
$x = 0$
$x = z$
x
$\sqrt[4]{\pi}$
y
Middle integral: x varies from 0 to z.
Outer integral: z varies from 0 to $\sqrt[4]{\pi}$.

Average Value of a Function of Three Variables

DEFINITION **Average Value of a Function of Three Variables**

If f is continuous on a region D of $\mathbb{R}^3$, then the average value of f over D is

$$\bar{f} = \frac{1}{\text{volume }(D)} \iiint_D f(x, y, z)\, dV.$$

EXAMPLE 5 **Average temperature** Consider a block of a conducting material occupying the region

$$D = \{(x, y, z): 0 \le x \le 2, 0 \le y \le 2, 0 \le z \le 1\}.$$

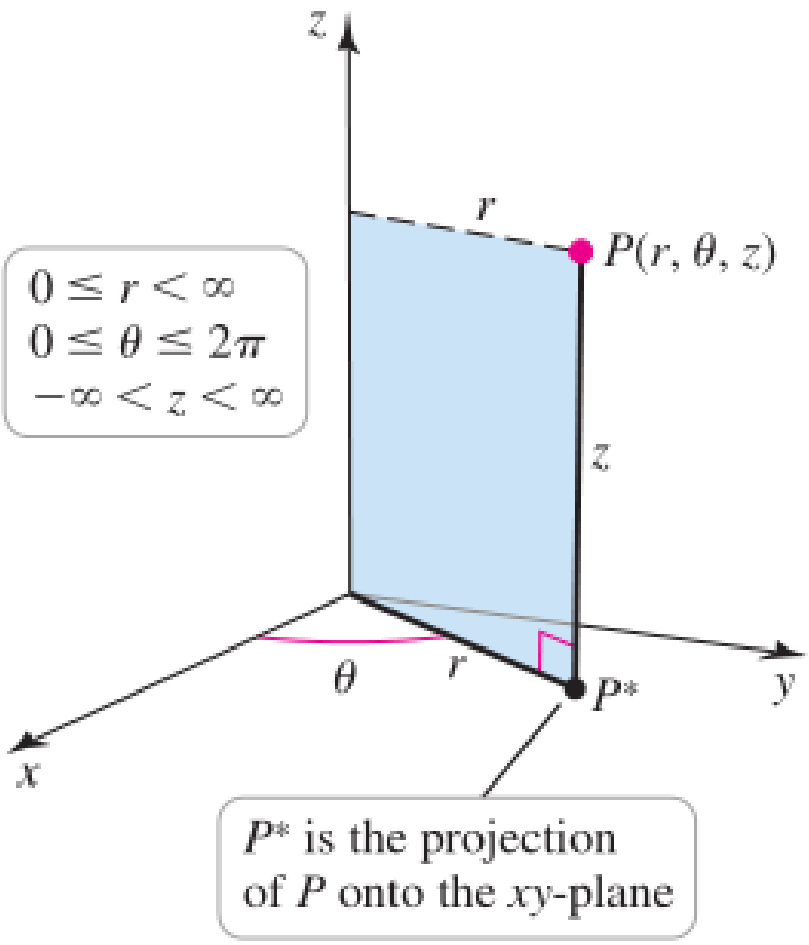
Due to heat sources on its boundaries, the temperature in the block is given by $T(x, y, z) = 250xy \sin \pi z$. Find the average temperature of the block.

16.5

Triple Integrals in Cylindrical and Spherical Coordinates

Cylindrical Coordinates

Extend polar coordinates from $\mathbb{R}^2$ to $\mathbb{R}^3$



In cylindrical coordinate system, a point P has coordinates (r, θ, z), where r and θ are polar coordinates for the point P^*.

$0 \le r \le \infty$, $0 \le \theta \le 2\pi$ and $-\infty \le z \le \infty$.

Table 4

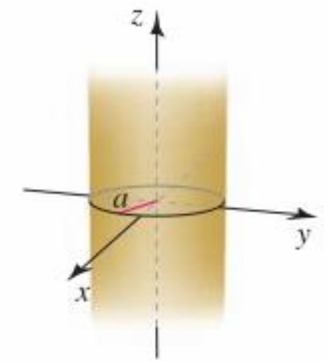
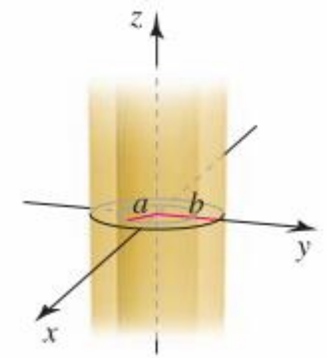
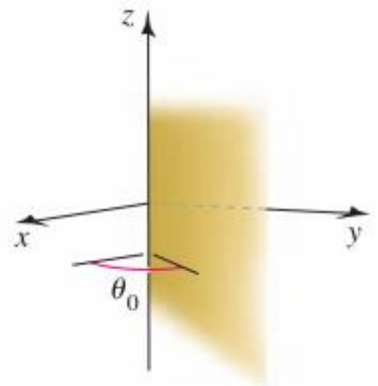
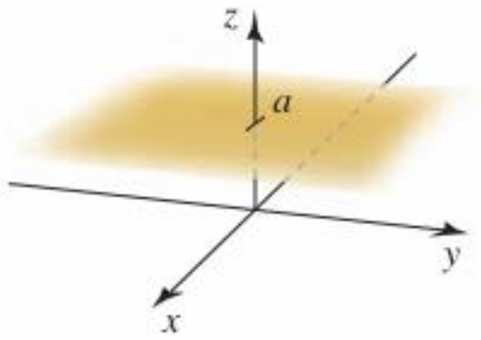
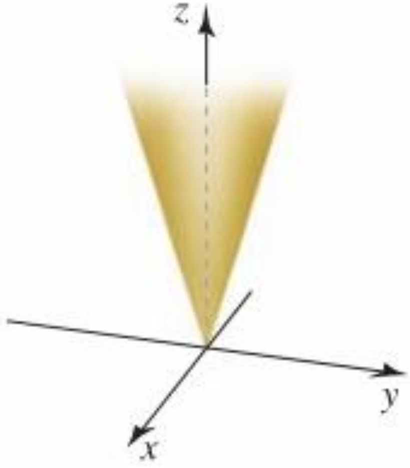
Name	Description	Example
Cylinder	$\{(r, \theta, z): r = a\}, a > 0$	
Cylindrical shell	$\{(r, \theta, z): 0 < a \le r \le b\}$	
Vertical half plane	$\{(r, \theta, z): \theta = \theta_0\}$	

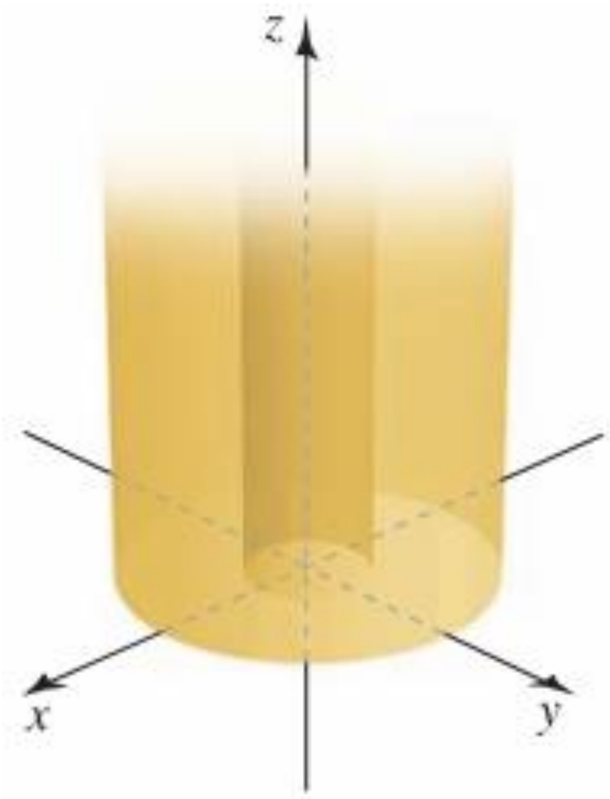
Table 4 (Continued)

Name	Description	Example
Horizontal plane	$\{(r, \theta, z) : z = a\}$	
Cone	$\{(r, \theta, z) : z = ar\}, a \neq 0$	

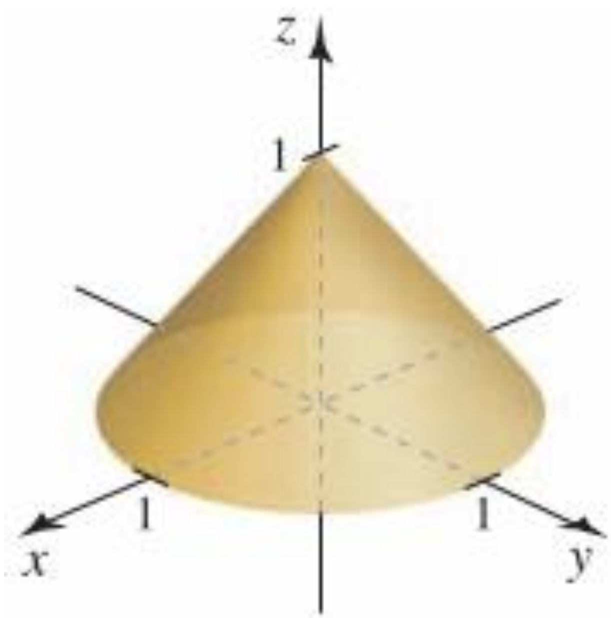
EXAMPLE 1 **Sets in cylindrical coordinates** Identify and sketch the following sets in cylindrical coordinates.

a. $Q = \{(r, \theta, z): 1 \leq r \leq 3, z \geq 0\}$

b. $S = \{(r, \theta, z): z = 1 - r, 0 \leq r \leq 1\}$



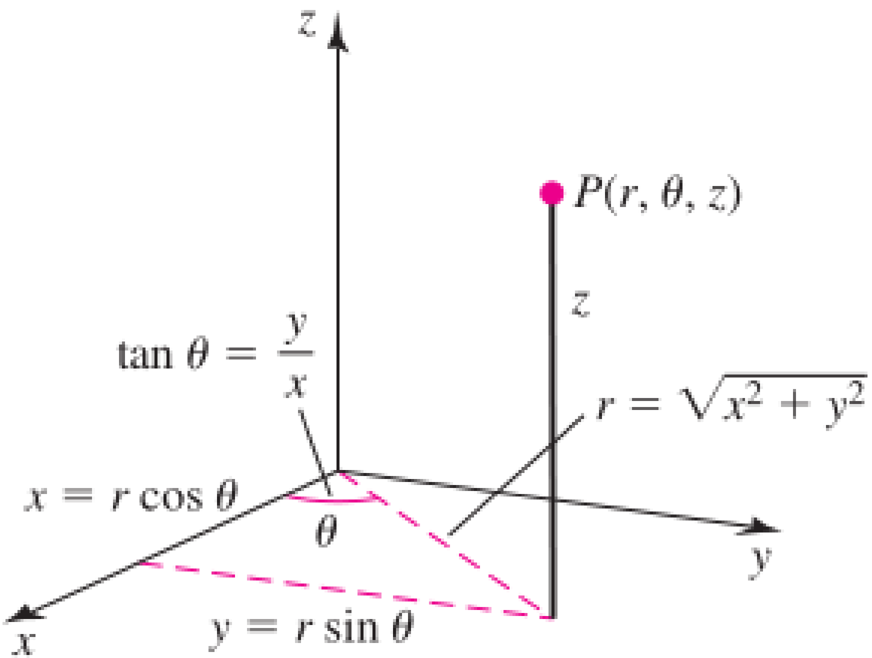
(a)



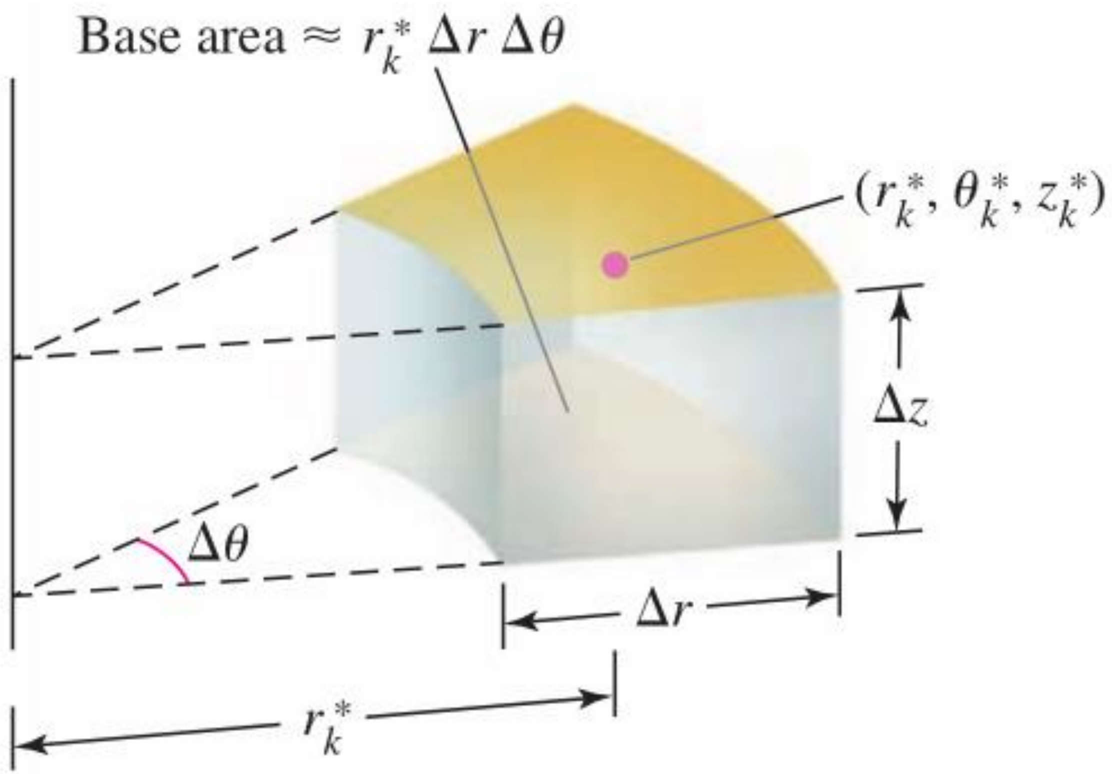
(b)

Transformations Between Cylindrical and Rectangular Coordinates

Rectangular → Cylindrical	Cylindrical → Rectangular
$r^2 = x^2 + y^2$	$x = r\cos\theta$
$\tan\theta = y/x$	$y = r\sin\theta$
$z = z$	$z = z$



Integration in Cylindrical Coordinates



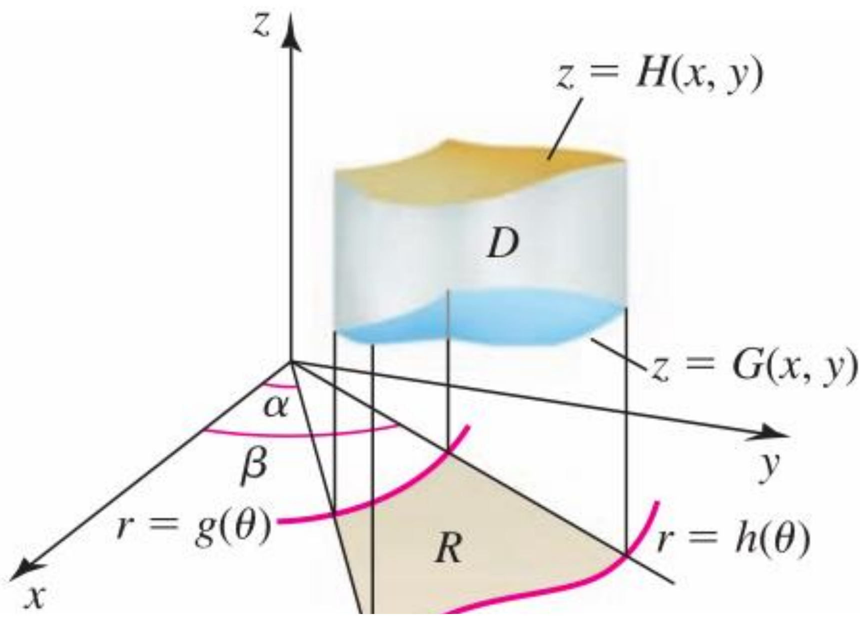
Approximate volume $\Delta V_k \approx r_k^* \, \Delta r \, \Delta\theta \, \Delta z$

Assume that f is continuous on D and form a Riemann sum over the region by adding function values multiplied by the corresponding approximate volumes

$$\sum_{k=1}^{n} f(r_k^*, \theta_k^*, z_k^*)\Delta V_k = \sum_{k=1}^{n} f(r_k^*, \theta_k^*, z_k^*) r_k^* \Delta r \Delta\theta \Delta z$$

Taking limit to get the **triple integral in cylindrical coordinates**

$$\iiint_D f(r, \theta, z) dV = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(r_k^*, \theta_k^*, z_k^*) r_k^* \Delta r \Delta\theta \Delta z$$



THEOREM 6 Triple Integrals in Cylindrical Coordinates
Let f be continuous over the region

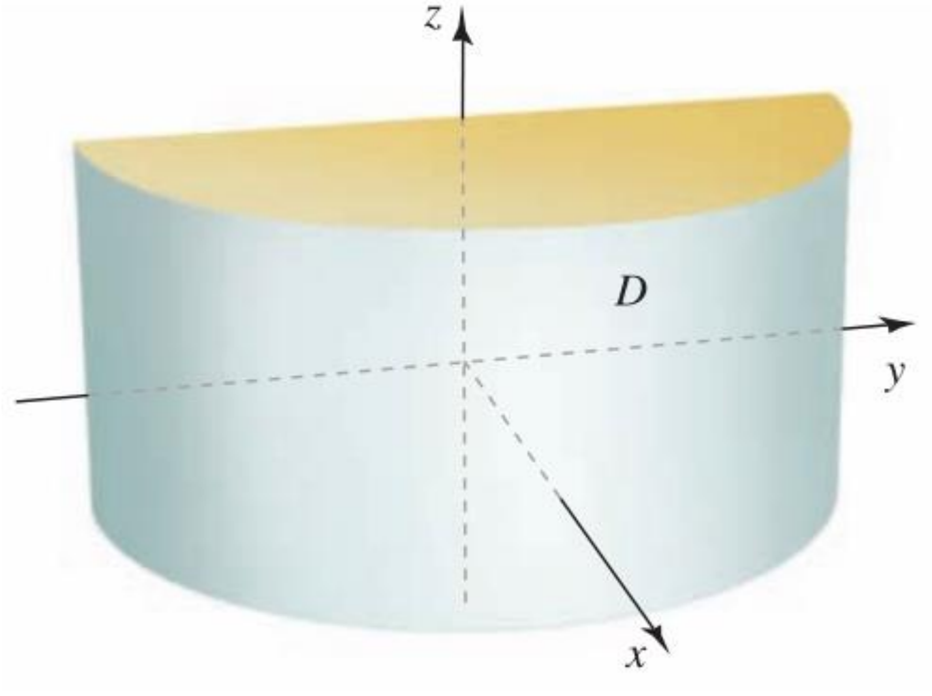
$$D = \{(r, \theta, z) \colon 0 \leq g(\theta) \leq r \leq h(\theta), \alpha \leq \theta \leq \beta, G(x, y) \leq z \leq H(x, y)\}.$$

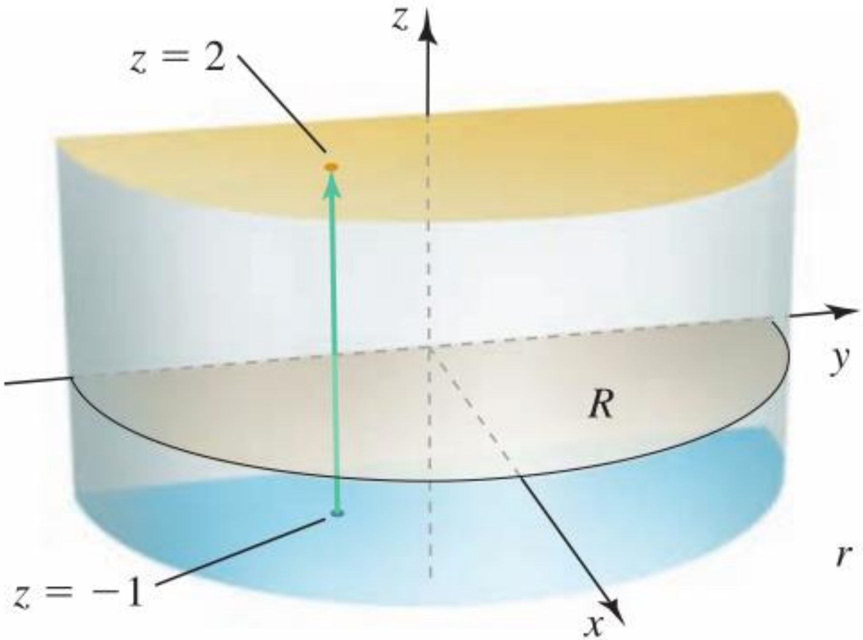
Then f is integrable over D, and the triple integral of f over D in cylindrical coordinates is

$$\iiint_D f(x, y, z)\, dV = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} \int_{G(r\cos\theta,\, r\sin\theta)}^{H(r\cos\theta,\, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, dz\, r\, dr\, d\theta.$$

EXAMPLE 2 **Switching coordinate systems** Evaluate the integral

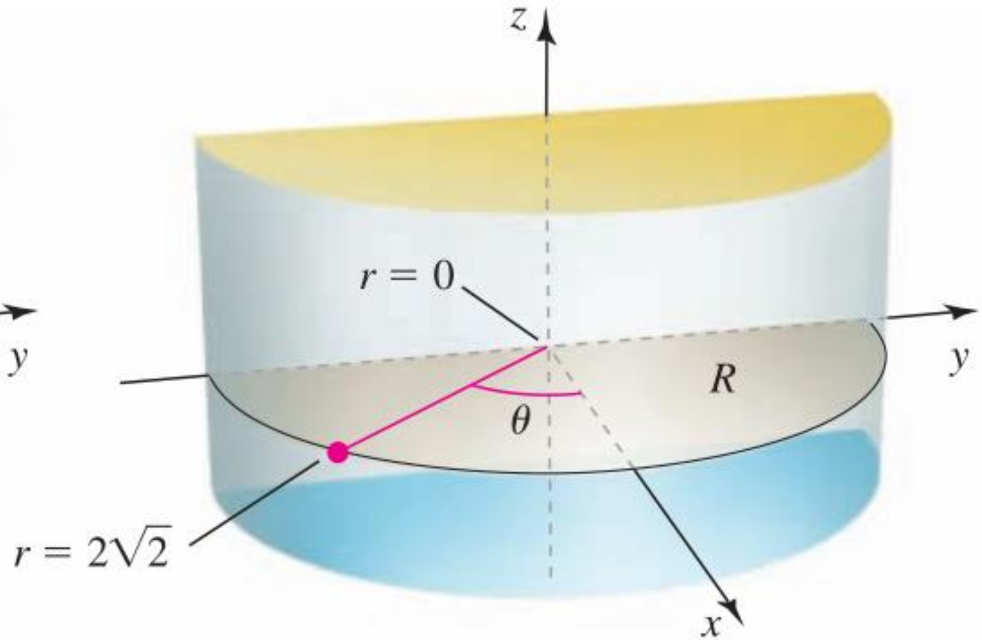
$$I = \int_{0}^{2\sqrt{2}} \int_{-\sqrt{8-x^2}}^{\sqrt{8-x^2}} \int_{-1}^{2} \sqrt{1 + x^2 + y^2}\, dz\, dy\, dx.$$





$$\iint_R \left(\int_{-1}^{2} \sqrt{1 + r^2}\, dz \right) dA$$

In cylindrical coordinates, integrate in z with $-1 \le z \le 2$;...



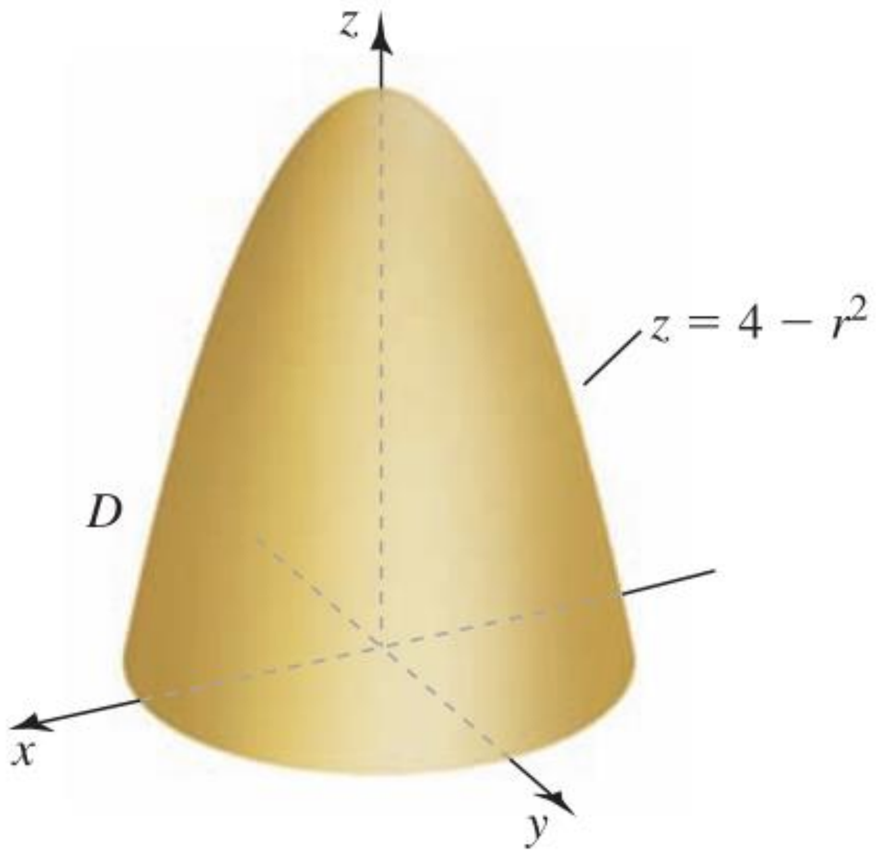
$$\int_{-\pi/2}^{\pi/2} \int_{0}^{2\sqrt{2}} \int_{-1}^{2} \sqrt{1 + r^2}\, dz\, r\, dr\, d\theta$$

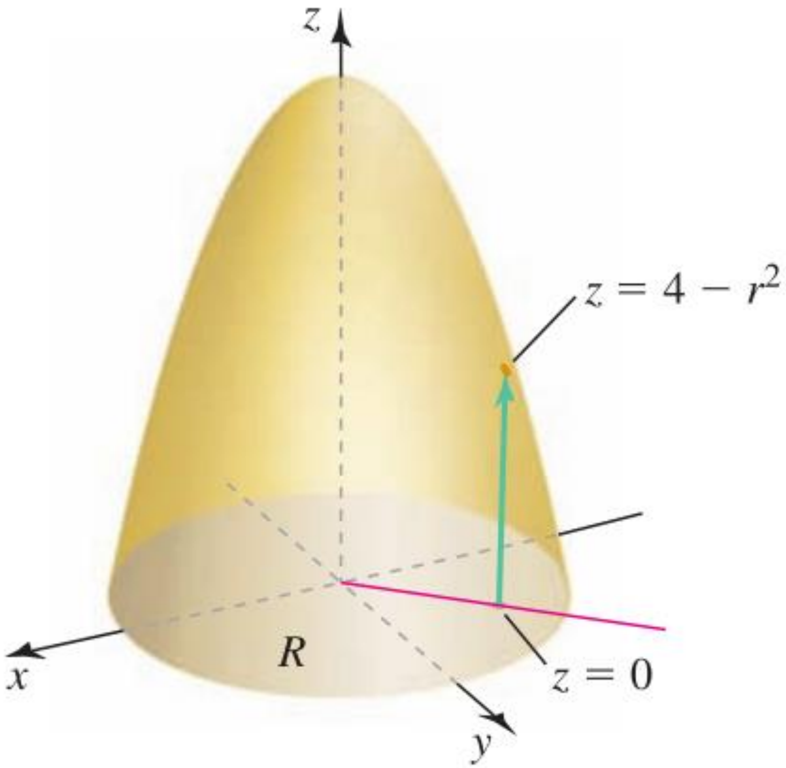
... then integrate over R with $0 \le r \le 2\sqrt{2}$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

Tips to choose the best coordinate system for a particular integral

- In which coordinate system is the region of integration most easily described?
- In which coordinate system is the integrand most easily expressed?
- In which coordinate system is the triple integral most easily evaluated?

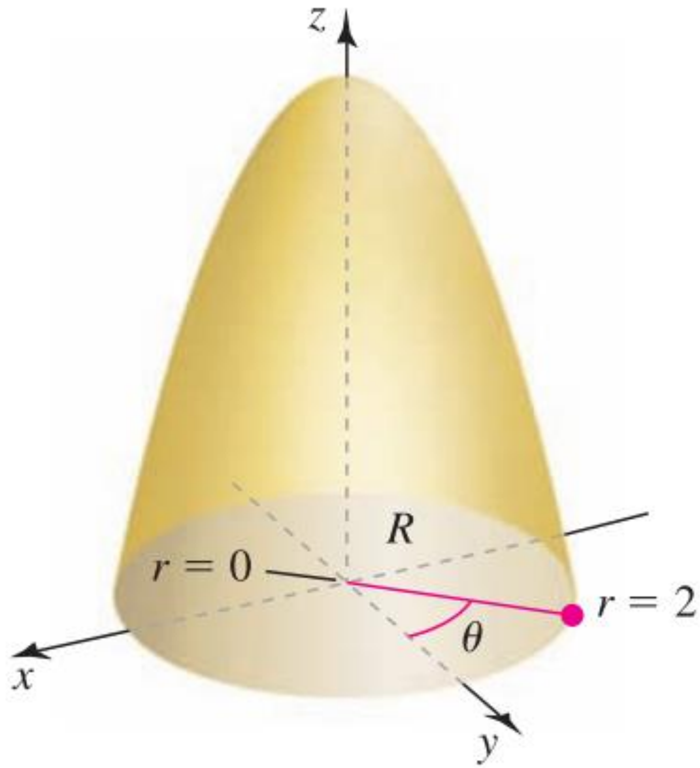
EXAMPLE 3 **Mass of a solid paraboloid** Find the mass of the solid D bounded by the paraboloid $z = 4 - r^2$ and the plane $z = 0$ (Figure 53a), where the density of the solid is $f(r, \theta, z) = 5 - z$ (heavy near the base and light near the vertex).





$$\iint_R \left(\int_0^{4-r^2} (5-z)\,dz \right) dA$$

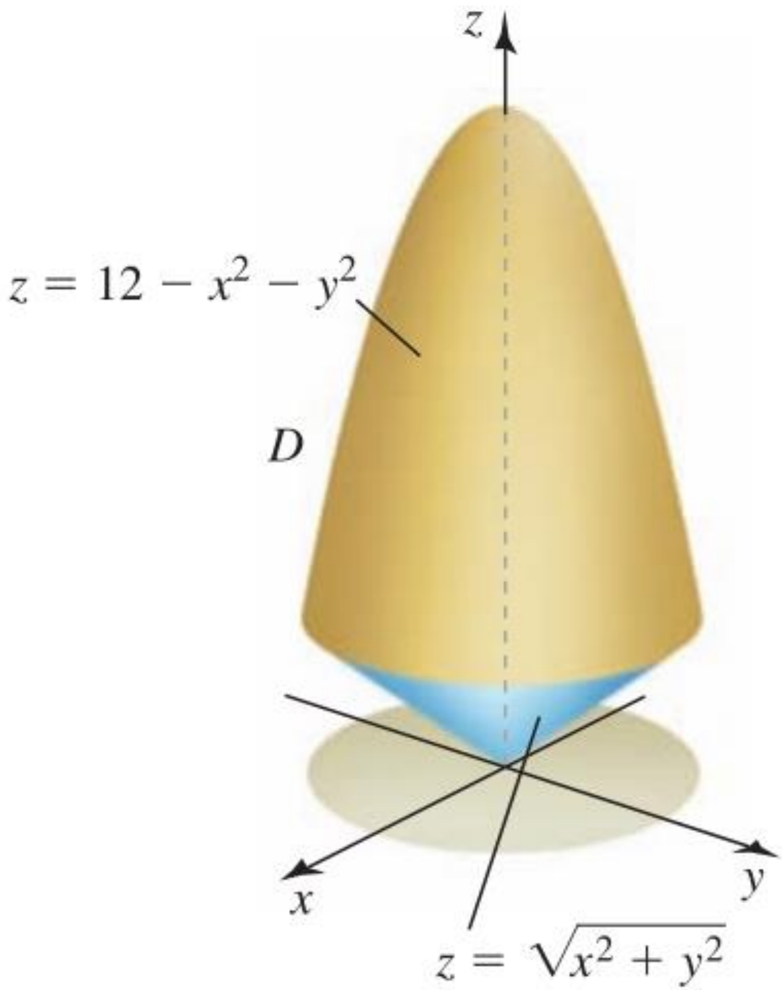
Integrate first in z
with $0 \le z \le 4 - r^2$;...

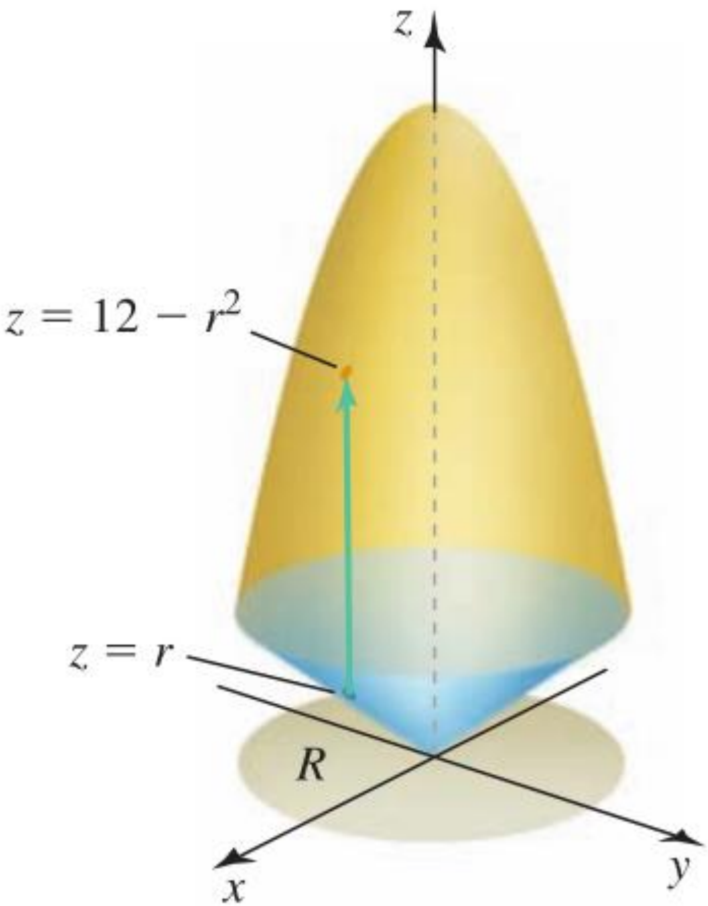


$$\int_0^{2\pi}\int_0^{2}\int_0^{4-r^2} (5-z)\,dz\,r\,dr\,d\theta$$

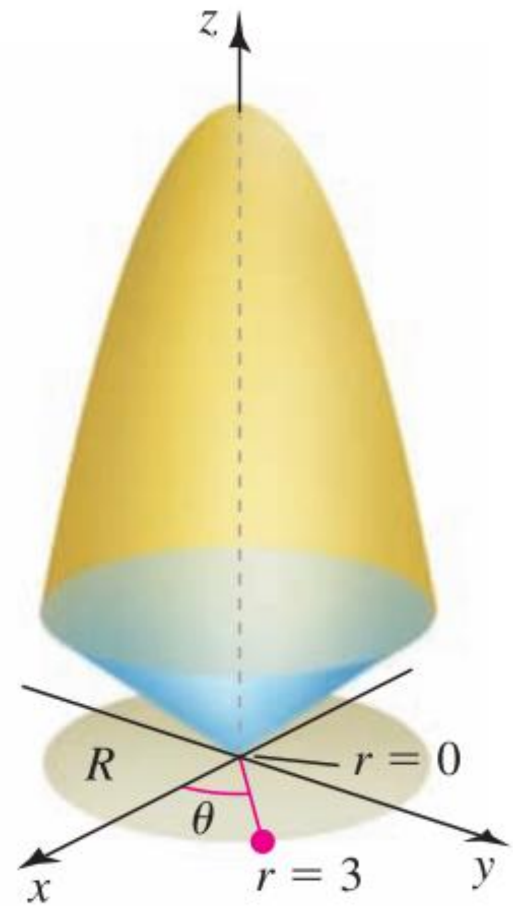
... then integrate over R
with $0 \le r \le 2$, $0 \le \theta \le 2\pi$.

EXAMPLE 4 **Volume between two surfaces** Find the volume of the solid D between the cone $z = \sqrt{x^2 + y^2}$ and the inverted paraboloid $z = 12 - x^2 - y^2$ (Figure 54a).





Integrate first in z
with $r \le z \le 12 - r^2$;...

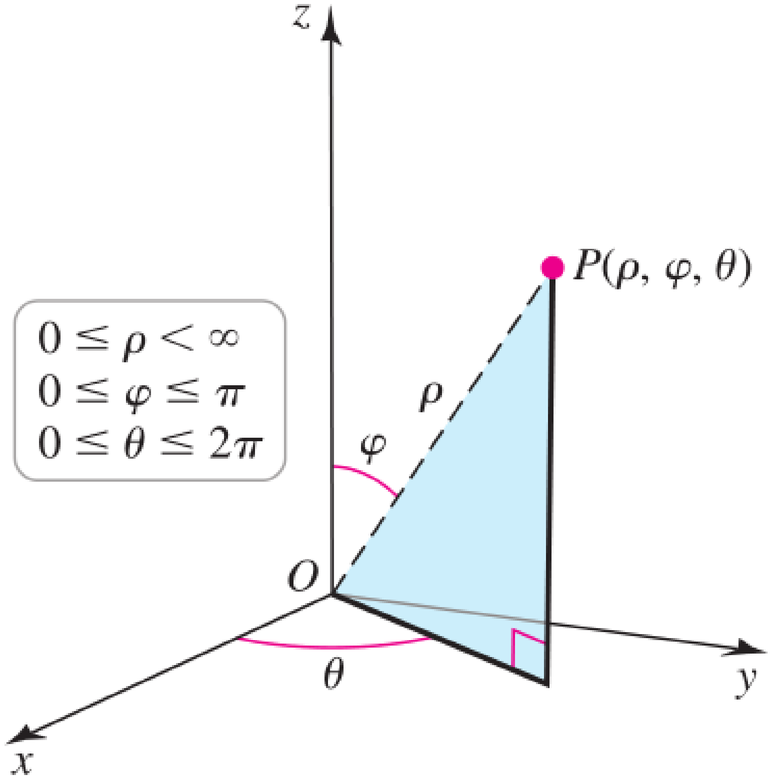


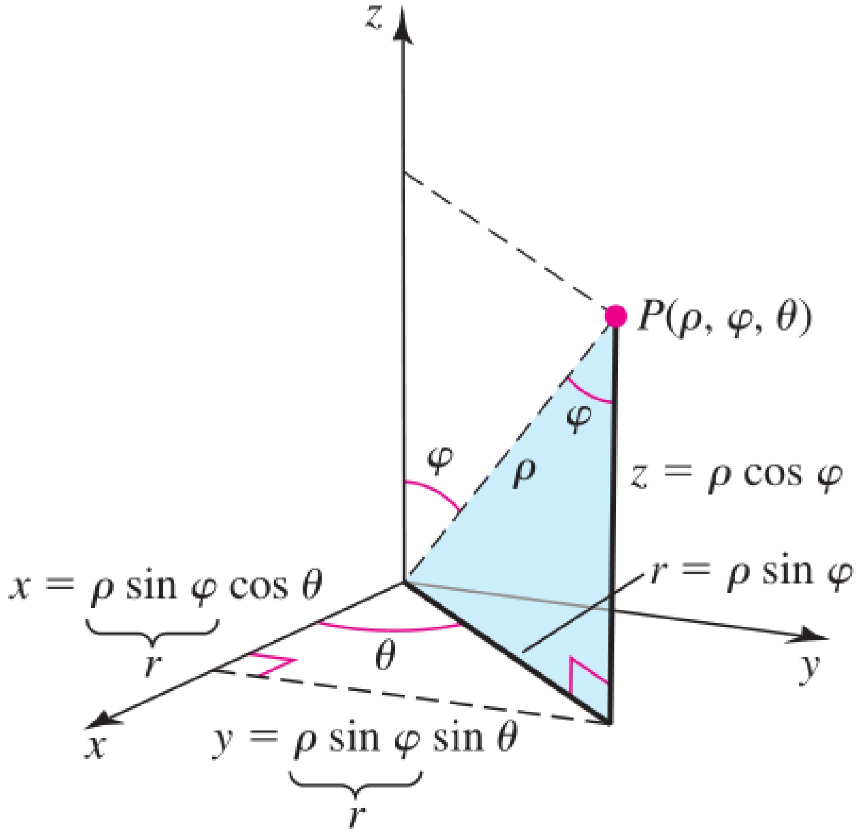
... then integrate over R
with $0 \le r \le 3$, $0 \le \theta \le 2\pi$.

Spherical Coordinates

A point P in $\mathbb{R}^3$ is represented by three coordinates (ρ, φ, θ)

- ρ is the distance from the origin to P.
- φ is the angle between the positive z-axis and the line OP.
- θ is the same angle as in cylindrical coordinates.





Transformations Between Spherical and Rectangular Coordinates

Rectangular → Spherical

$$\rho^2 = x^2 + y^2 + z^2$$

Use trigonometry to find

φ and θ

Spherical → Rectangular

$$x = \rho \sin \varphi \cos \theta$$

$$y = \rho \sin \varphi \sin \theta$$

$$z = \rho \cos \varphi$$

EXAMPLE 5 **Sets in spherical coordinates** Express the following sets in rectangular coordinates and identify the set. Assume that a is a positive real number.

a. $\{(\rho, \varphi, \theta): \rho = 2a \cos \varphi, 0 \le \varphi \le \pi/2, 0 \le \theta \le 2\pi\}$

b. $\{(\rho, \varphi, \theta): \rho = 4 \sec \varphi, 0 \le \varphi < \pi/2, 0 \le \theta \le 2\pi\}$

Table 5

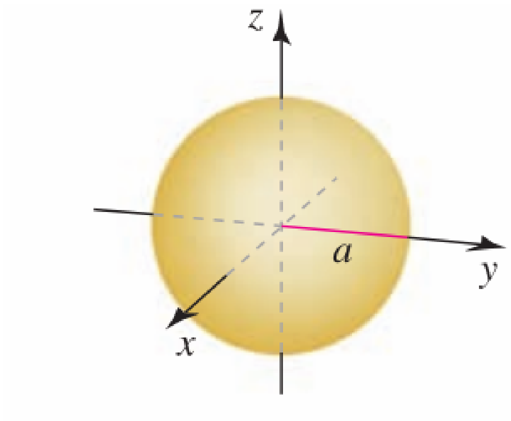
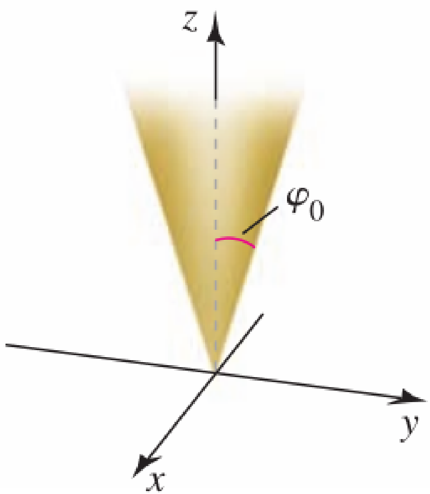
Name	Description	Example
Sphere, radius a, center $(0, 0, 0)$	$\{(\rho, \varphi, \theta): \rho = a\}, a > 0$	
Cone	$\{(\rho, \varphi, \theta): \varphi = \varphi_0\}, \varphi_0 \neq 0, \pi/2, \pi$	

Table 5 (Continued)

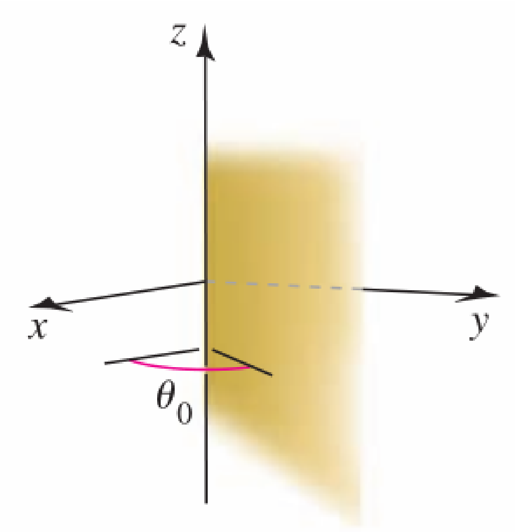
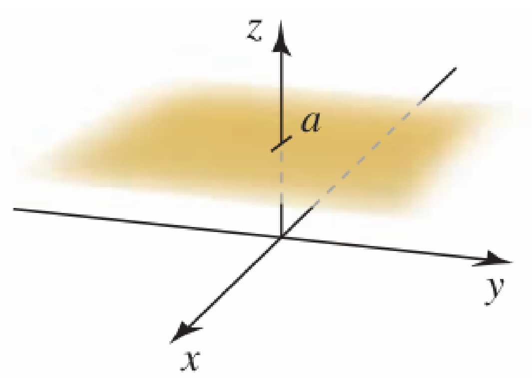
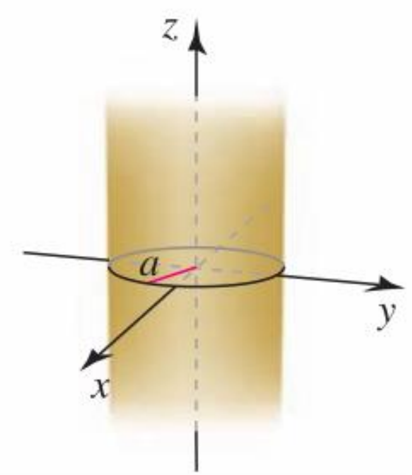
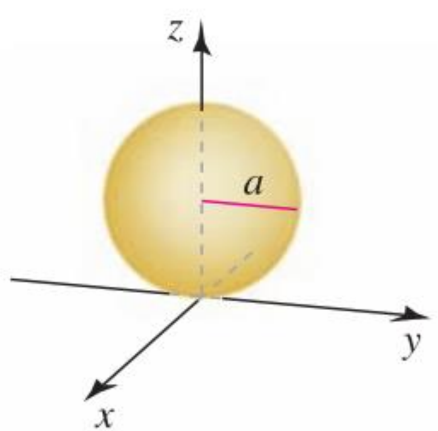
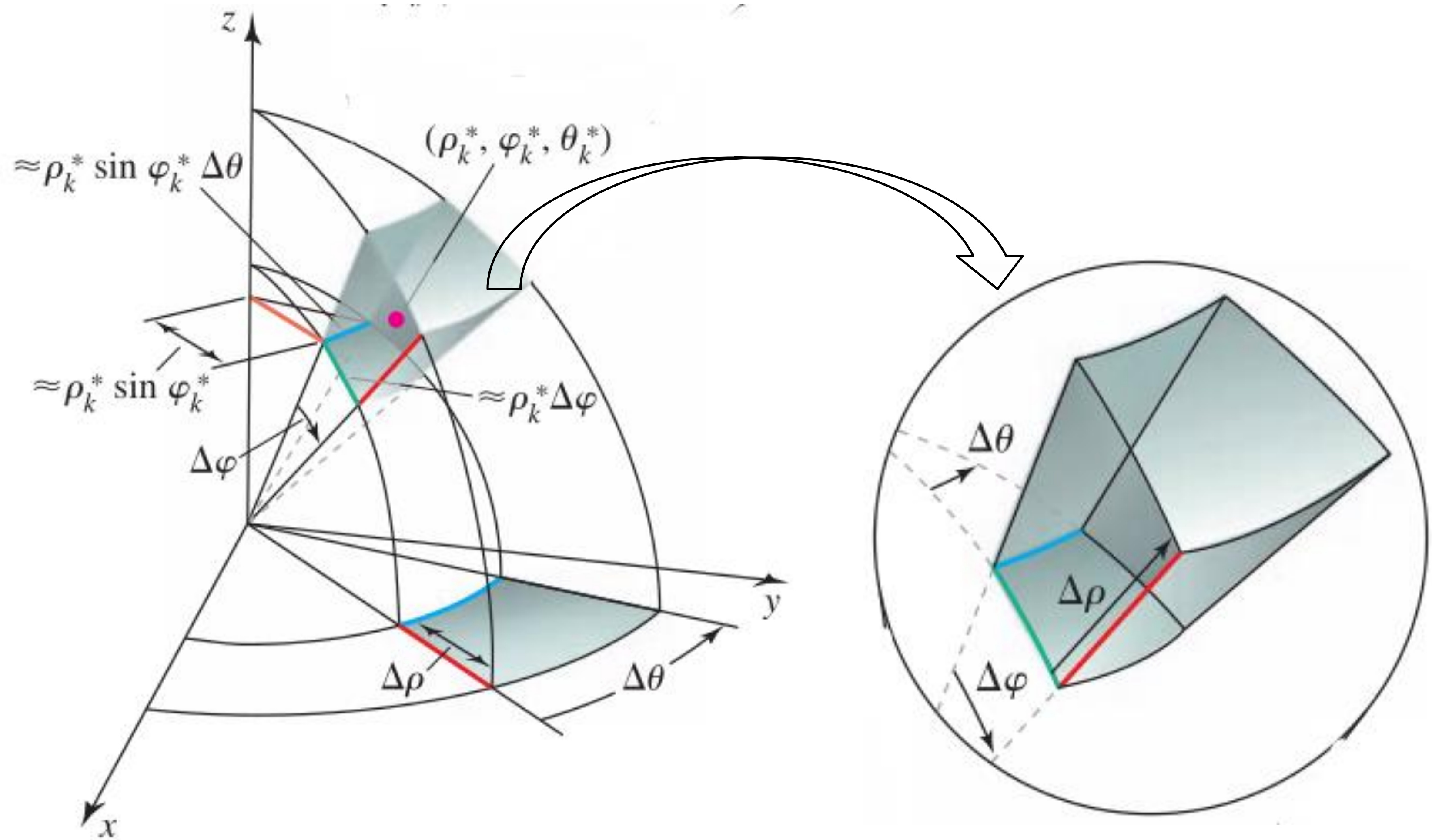
Name	Description	Example
Vertical half plane	$\{(\rho, \varphi, \theta): \theta = \theta_0\}$	
Horizontal plane, $z = a$	$a > 0$: $\{(\rho, \varphi, \theta): \rho = a \sec \varphi, 0 \leq \varphi < \pi/2\}$ $a < 0$: $\{(\rho, \varphi, \theta): \rho = a \sec \varphi, \pi/2 < \varphi \leq \pi\}$	

Table 5 (Continued)

Name	Description	Example
Cylinder, radius $a > 0$	$\{(\rho, \varphi, \theta): \rho = a \csc \varphi, 0 < \varphi < \pi\}$	
Sphere, radius $a > 0$, center $(0, 0, a)$	$\{(\rho, \varphi, \theta): \rho = 2a \cos \varphi, 0 \leq \varphi \leq \pi/2\}$	

Integration in Spherical Coordinates



Approximate volume =
$\Delta V_k \approx \rho_k^{*2} \sin \varphi_k^* \, \Delta\rho \, \Delta\varphi \, \Delta\theta$

To approximate the volume of this typical box

- The length of the box in the ρ-direction is $\Delta\rho$.
- The approximate length in the θ-direction is $\rho_k^* \sin\varphi_k^* \Delta\theta$.
- The approximate length in the φ-direction is $\rho_k^* \Delta\varphi$.

So, the approximate volume of the kth spherical box is

$$\Delta V_k = \rho_k^{*2} \sin\varphi_k^* \Delta\rho\Delta\varphi\Delta\theta$$

The corresponding approximate volumes

$$\sum_{k=1}^{n} f(\rho_k^*, \varphi_k^*, \theta_k^*)\Delta V_k = \sum_{k=1}^{n} f(\rho_k^*, \varphi_k^*, \theta_k^*)\rho_k^{*2} \sin\varphi_k^* \Delta\rho\Delta\varphi\Delta\theta$$

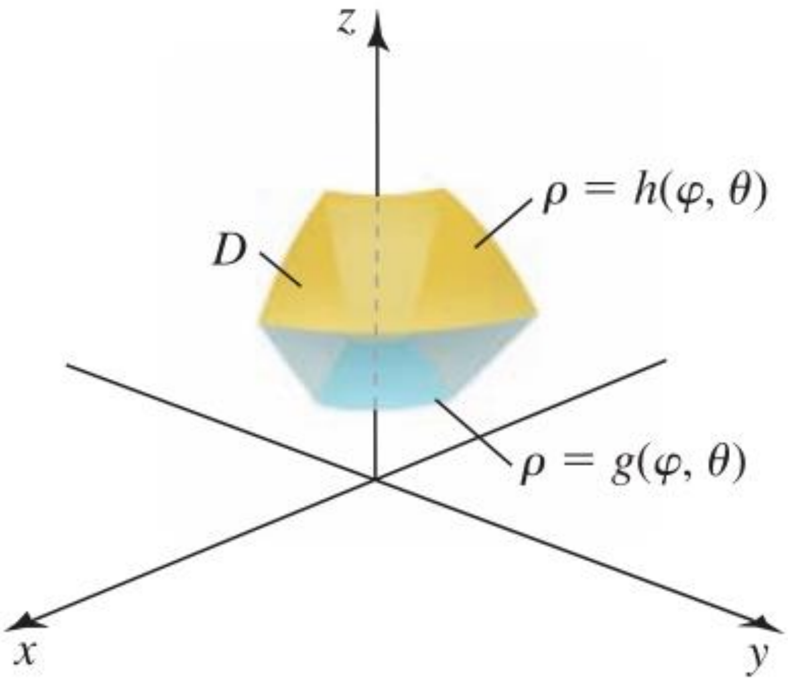
Taking limit to get the **triple integral in spherical coordinates**

$$\iiint_D f(\rho, \varphi, \theta)\,dV = \lim_{\Delta\to 0} \sum_{k=1}^{n} f(\rho_k^*, \varphi_k^*, \theta_k^*)\rho_k^{*2} \sin\varphi_k^* \Delta\rho\Delta\varphi\Delta\theta$$

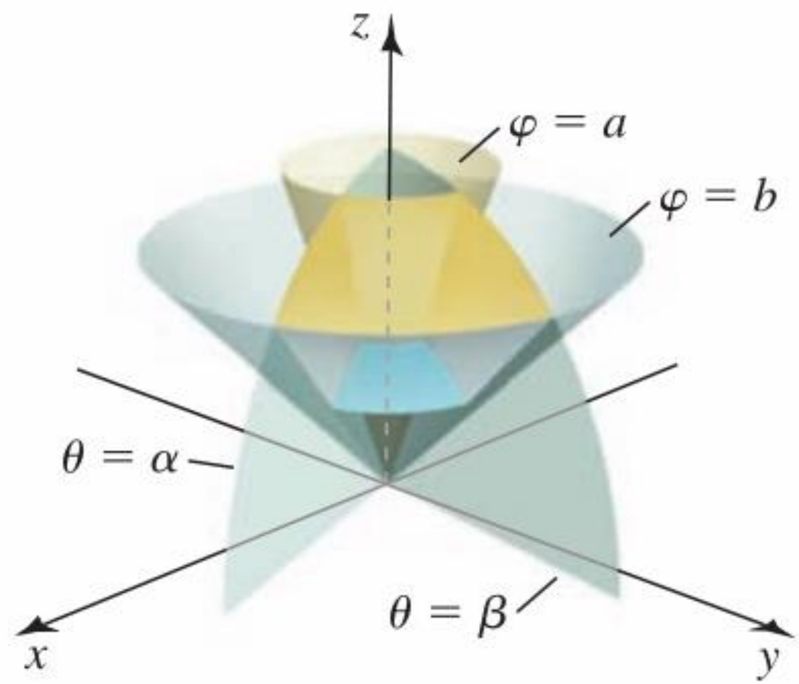
Finding Limits of integration

Consider a common situation

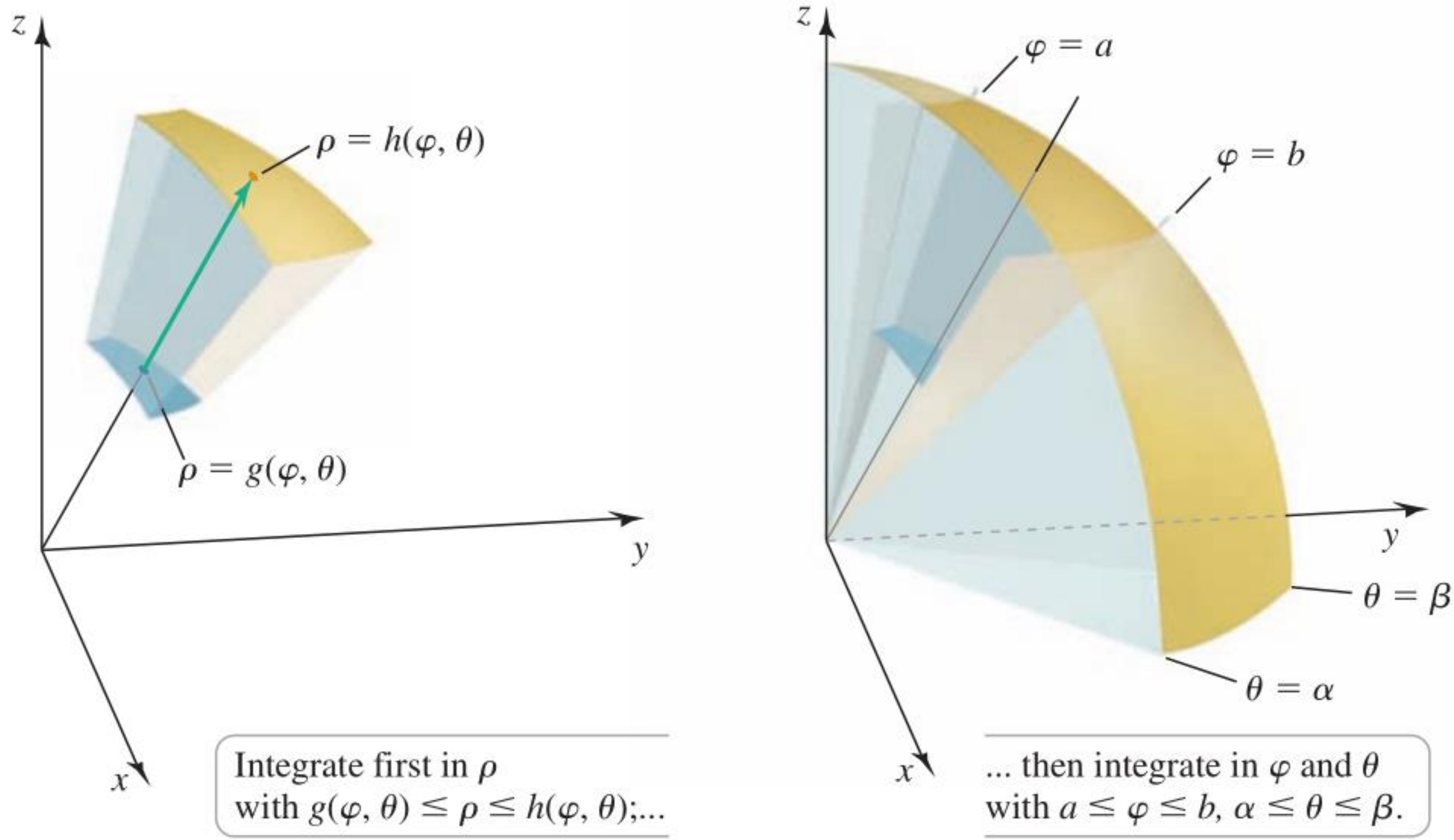
$D = \{(\rho, \varphi, \theta): 0 \le g(\varphi, \theta) \le v \le h(\varphi, \theta), a \le \varphi \le b, \alpha \le \theta \le \beta\}$



$g(\varphi, \theta) \le \rho \le h(\varphi, \theta)$



$\alpha \le \theta \le \beta, a \le \varphi \le b$


z
$\rho = h(\varphi, \theta)$
$\rho = g(\varphi, \theta)$
y
x
Integrate first in ρ
with $g(\varphi, \theta) \le \rho \le h(\varphi, \theta)$;...
z
$\varphi = a$
$\varphi = b$
y
$\theta = \beta$
$\theta = \alpha$
x
... then integrate in φ and θ
with $a \le \varphi \le b$, $\alpha \le \theta \le \beta$.

THEOREM 7 Triple Integrals in Spherical Coordinates

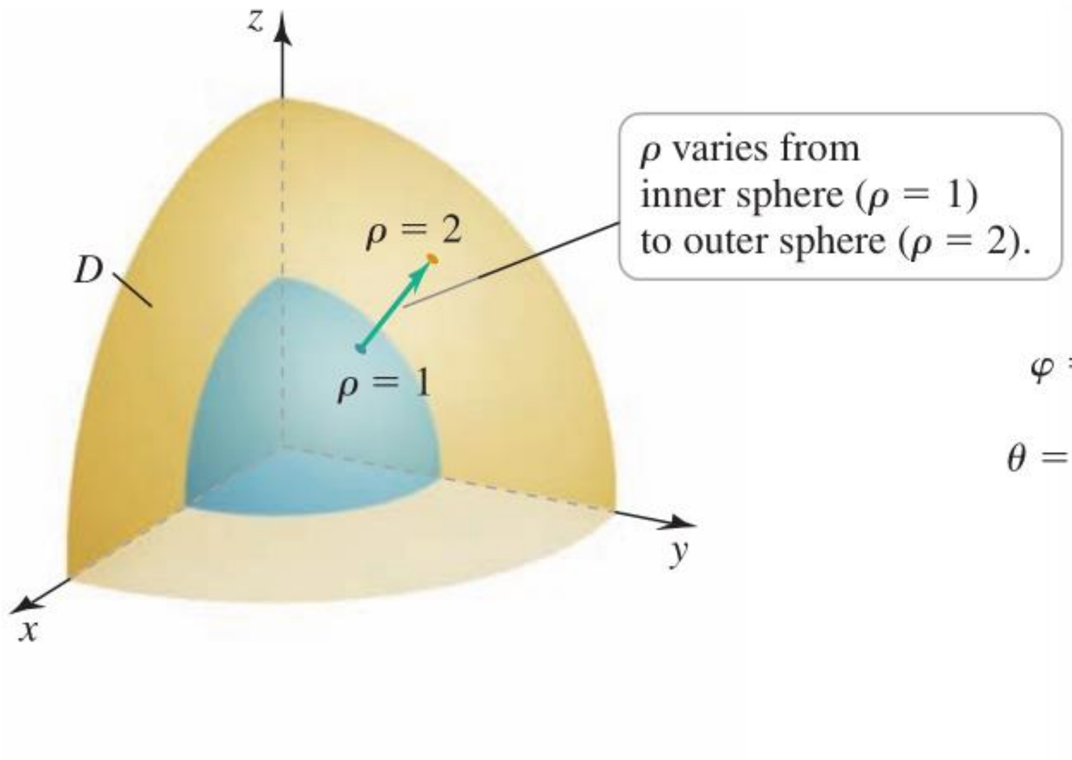
Let f be continuous over the region

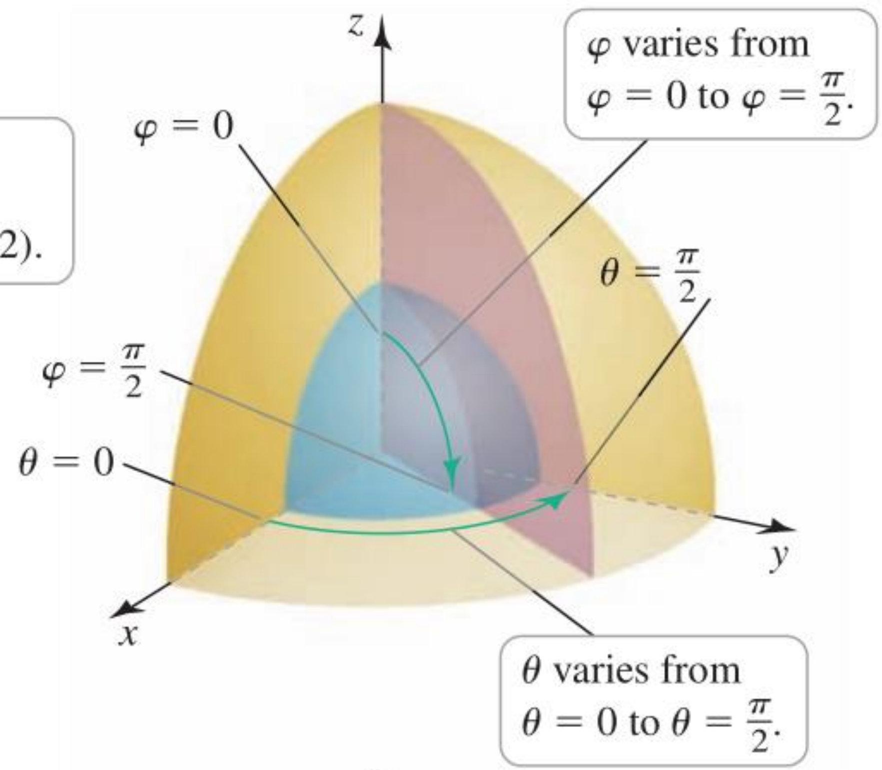
$$D = \{(\rho, \varphi, \theta): 0 \le g(\varphi, \theta) \le \rho \le h(\varphi, \theta), a \le \varphi \le b, \alpha \le \theta \le \beta\}.$$

Then f is integrable over D, and the triple integral of f over D in spherical coordinates is

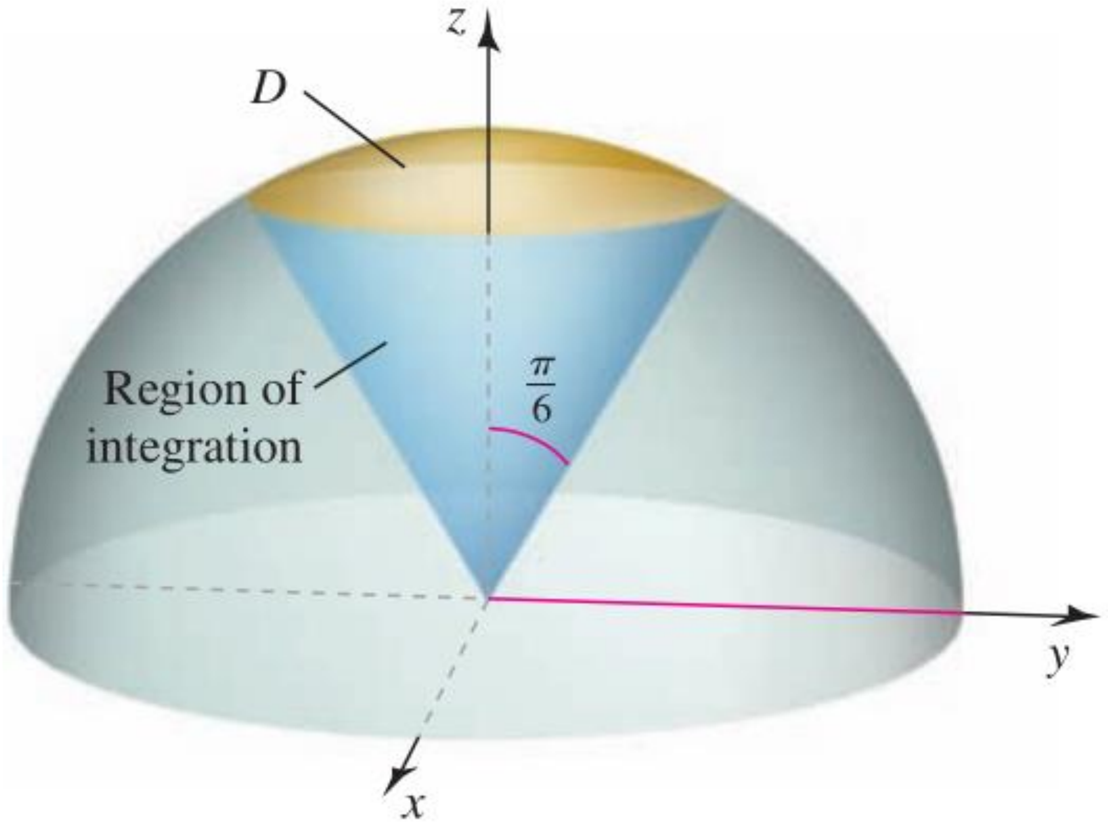
$$\iiint_D f(\rho, \varphi, \theta)\, dV = \int_\alpha^\beta \int_a^b \int_{g(\varphi, \theta)}^{h(\varphi, \theta)} f(\rho, \varphi, \theta)\, \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta.$$

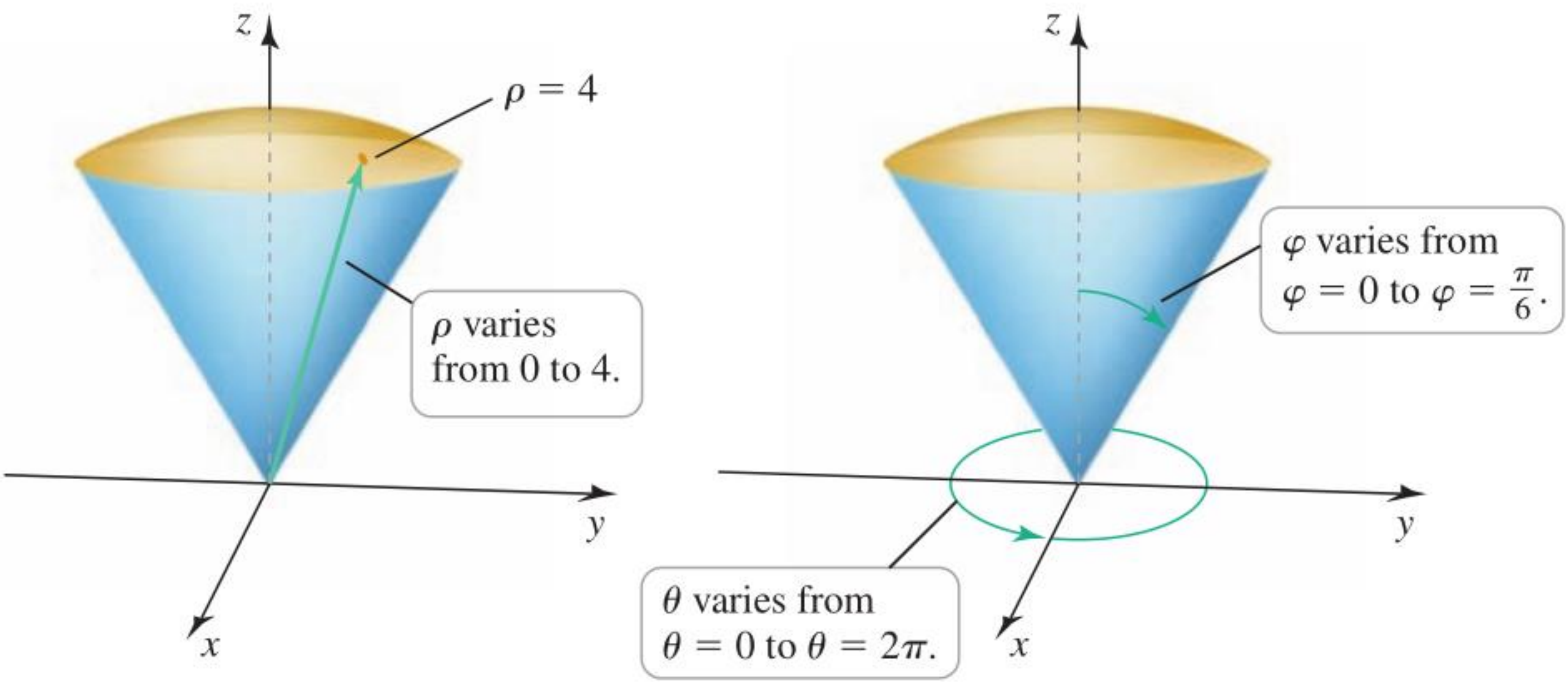
EXAMPLE 6 **A triple integral** Evaluate $\iiint_D (x^2 + y^2 + z^2)^{-3/2}\, dV$, where D is the region in the first octant between two spheres of radius 1 and 2 centered at the origin.





EXAMPLE 7 **Ice cream cone** Find the volume of the solid region D that lies inside the cone $\varphi = \pi/6$ and inside the sphere $\rho = 4$ (Figure 62a).





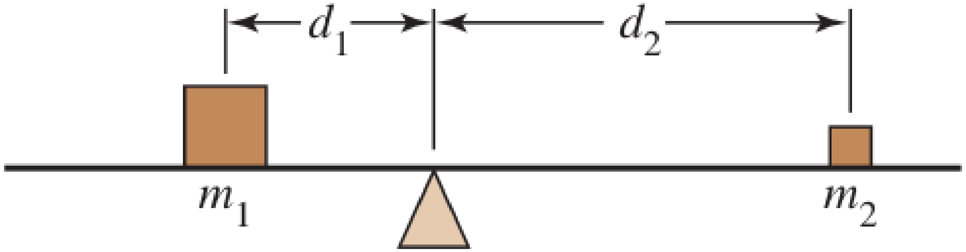
z
$\rho = 4$
ρ varies from 0 to 4.
y
x
z
φ varies from $\varphi = 0$ to $\varphi = \frac{\pi}{6}$.
y
θ varies from $\theta = 0$ to $\theta = 2\pi$.
x

16.6

Integrals for Mass Calculations

Find the *center of mass* of an object

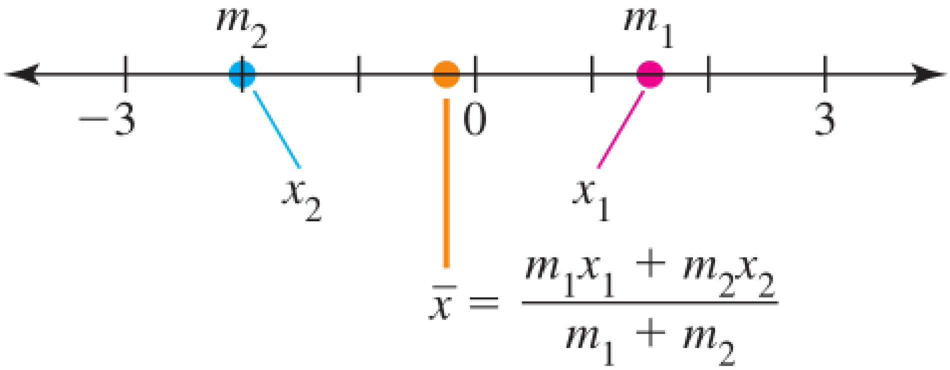
Sets of Individual Objects



$$m_1 d_1 = m_2 d_2$$

In order to find the balance point $\bar{x}$, introduce a coordinate system with the origin at $x = 0$,.

The balance equation is $m_1(x_1 - \bar{x}) + m_2(x_2 - \bar{x}) = 0$



Several Objects on a Line

Generalize it to n objects having masses $m_1, m_2, \cdots$, and m_n with coordinates $x_1, x_2, \cdots$, and x_n.

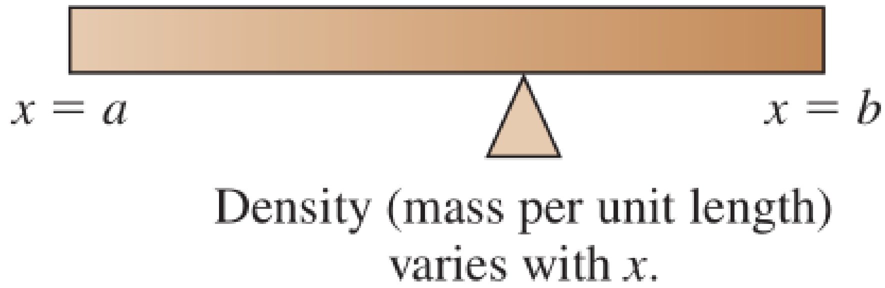
The balance equation becomes

$$m_1(x_1 - \bar{x}) + \cdots + m_n(x_n - \bar{x}) = \sum_{k=1}^{n} m_k(x_k - \bar{x}) = 0$$

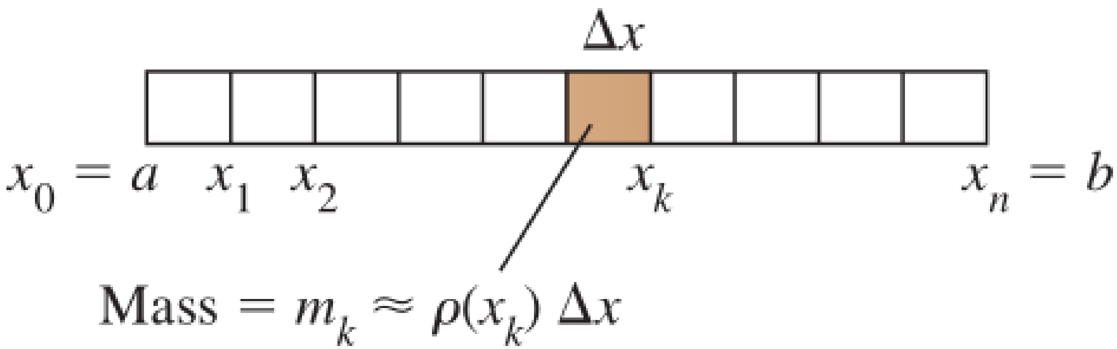
Solving it for the location of the center of mass

$$\bar{x} = \frac{\sum_{k=1}^{n} m_k x_k}{\sum_{k=1}^{n} m_k}$$

Continuous Objects in One Dimension



Using the slice-and-sum strategy



$$\bar{x} = \lim_{\Delta x \to 0} \frac{\sum_{k=1}^{n} (\rho(x_k)\Delta x)\, x_k}{\sum_{k=1}^{n} \rho(x_k)\Delta x} = \frac{\lim_{\Delta x \to 0} \sum_{k=1}^{n} x_k\, \rho(x_k)\Delta x}{\lim_{\Delta x \to 0} \sum_{k=1}^{n} \rho(x_k)\Delta x} = \frac{\int_a^b x\rho(x)\, dx}{\int_a^b \rho(x)\, dx}.$$

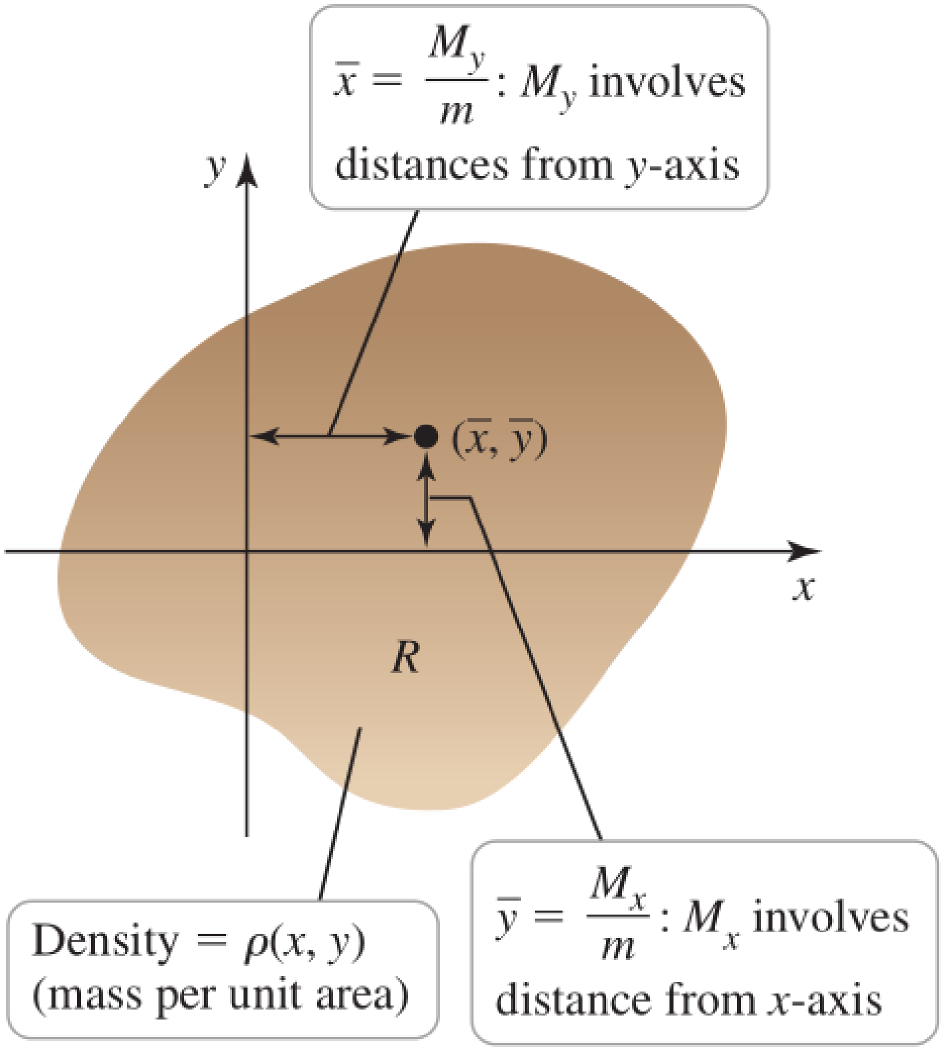
DEFINITION Center of Mass in One Dimension

Let ρ be an integrable density function on the interval $[a, b]$ (which represents a thin rod or wire). The **center of mass** is located at the point $\bar{x} = \dfrac{M}{m}$, where the **total moment** M and mass m are

$$M = \int_a^b x\rho(x)\, dx \quad \text{and} \quad m = \int_a^b \rho(x)\, dx.$$

EXAMPLE 2 **Center of mass of a one-dimensional object** Suppose a thin 2-m bar is made of an alloy whose density in kg/m is $\rho(x) = 1 + x^2$, where $0 \le x \le 2$. Find the center of mass of the bar.

Two-Dimensional Objects



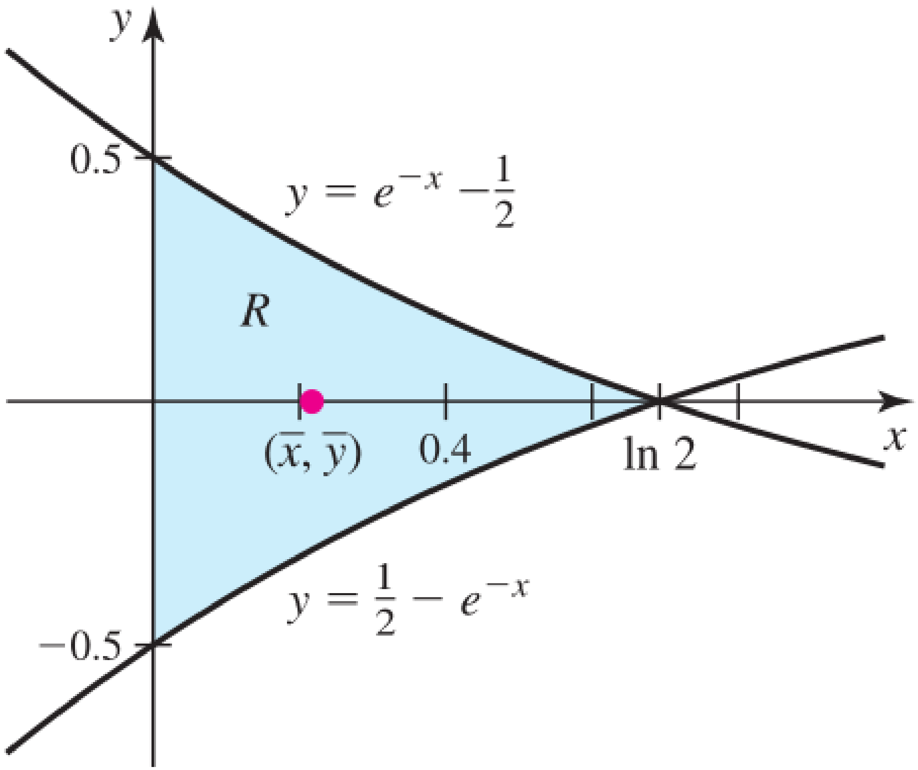
DEFINITION **Center of Mass in Two Dimensions**

Let ρ be an integrable area density function defined over a closed bounded region R in $\mathbb{R}^2$. The coordinates of the center of mass of the object represented by R are

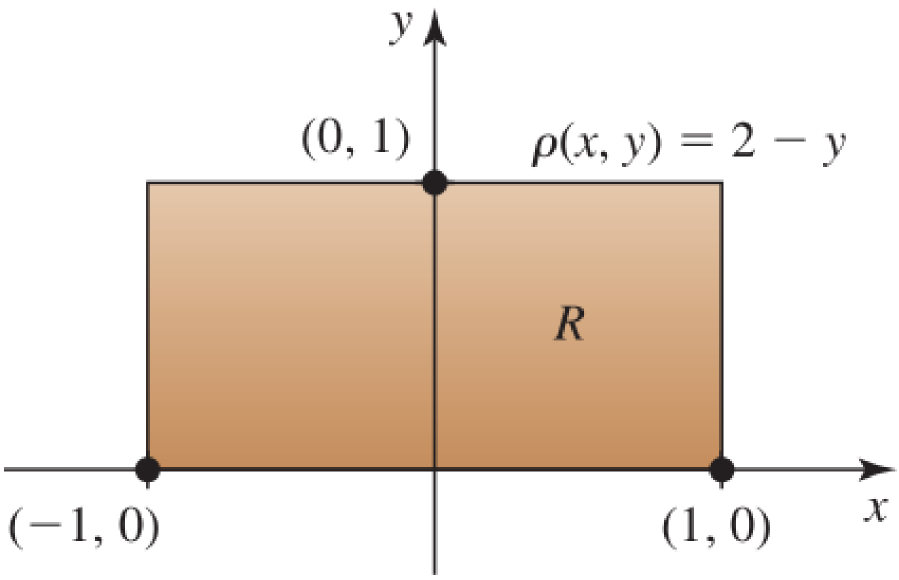
$$\bar{x} = \frac{M_y}{m} = \frac{1}{m}\iint_R x\rho(x, y)\, dA \quad \text{and} \quad \bar{y} = \frac{M_x}{m} = \frac{1}{m}\iint_R y\rho(x, y)\, dA,$$

where $m = \iint_R \rho(x, y)\, dA$ is the mass, and M_y and M_x are the moments with respect to the y-axis and x-axis, respectively. If ρ is constant, the center of mass is called the **centroid** and is independent of the density.

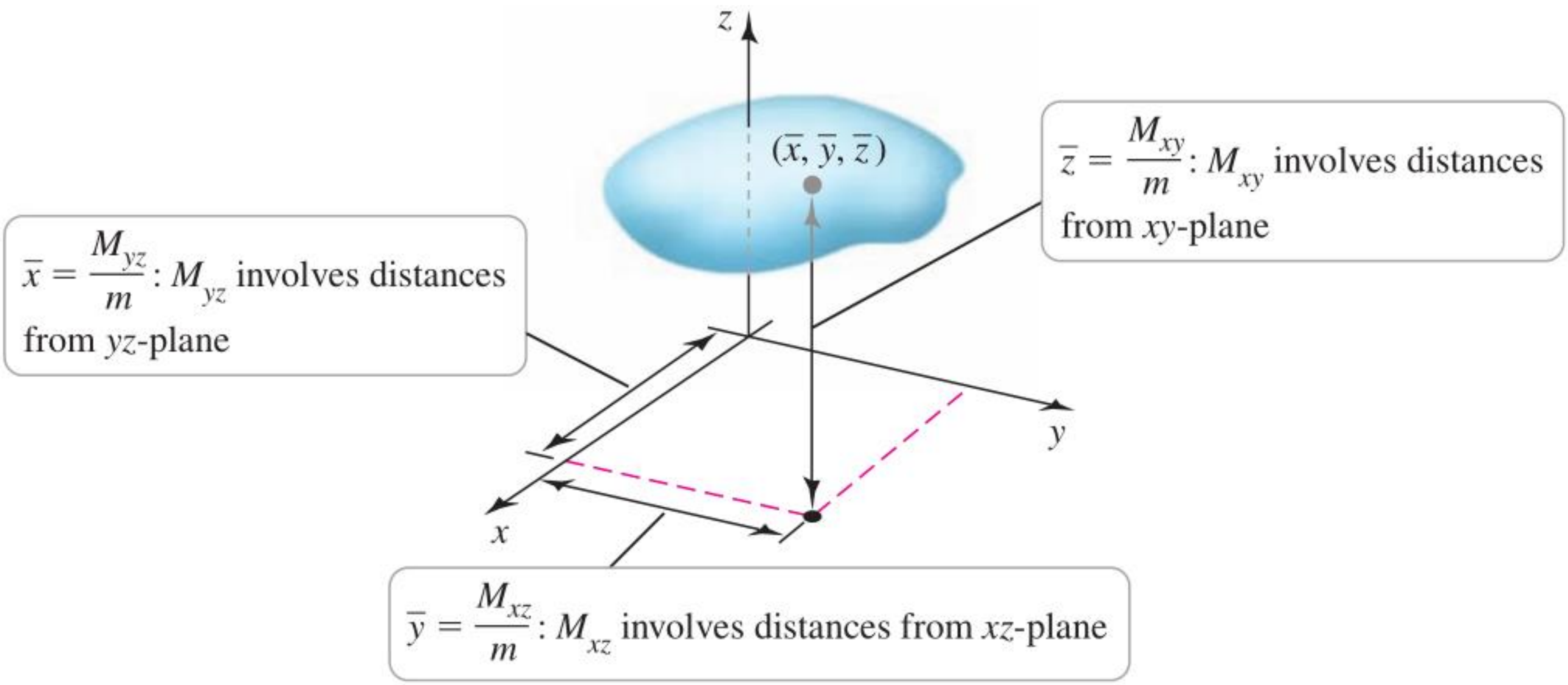
EXAMPLE 3 **Centroid calculation** Find the centroid (center of mass) of the unit density, dart-shaped region bounded by the y-axis and the curves $y = e^{-x} - \frac{1}{2}$ and $y = \frac{1}{2} - e^{-x}$ (Figure 71).



EXAMPLE 4 **Variable-density plate** Find the center of mass of the rectangular plate $R = \{(x, y): -1 \le x \le 1, 0 \le y \le 1\}$ with a density of $\rho(x, y) = 2 - y$ (heavy at the lower edge and light at the top edge; Figure 72).



Three-Dimensional Objects



DEFINITION **Center of Mass in Three Dimensions**

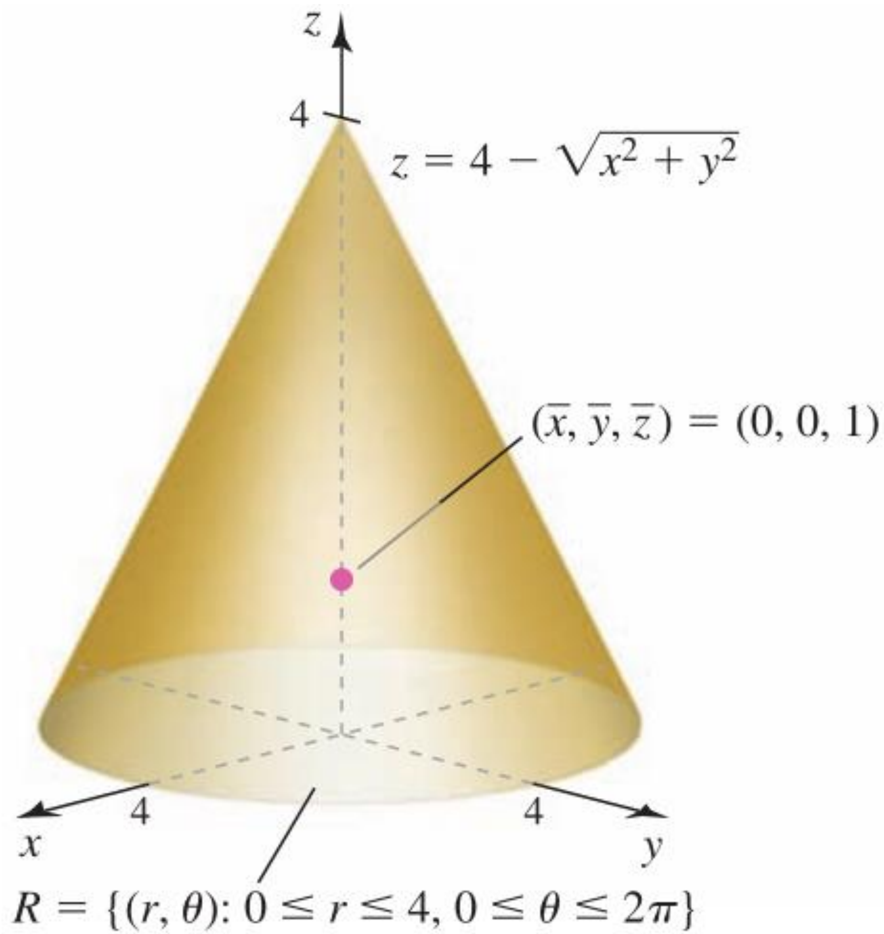
Let ρ be an integrable density function on a closed bounded region D in $\mathbb{R}^3$. The coordinates of the center of mass of the region are

$$\bar{x} = \frac{M_{yz}}{m} = \frac{1}{m} \iiint_D x\rho(x, y, z)\, dV, \quad \bar{y} = \frac{M_{xz}}{m} = \frac{1}{m} \iiint_D y\rho(x, y, z)\, dV, \text{ and}$$

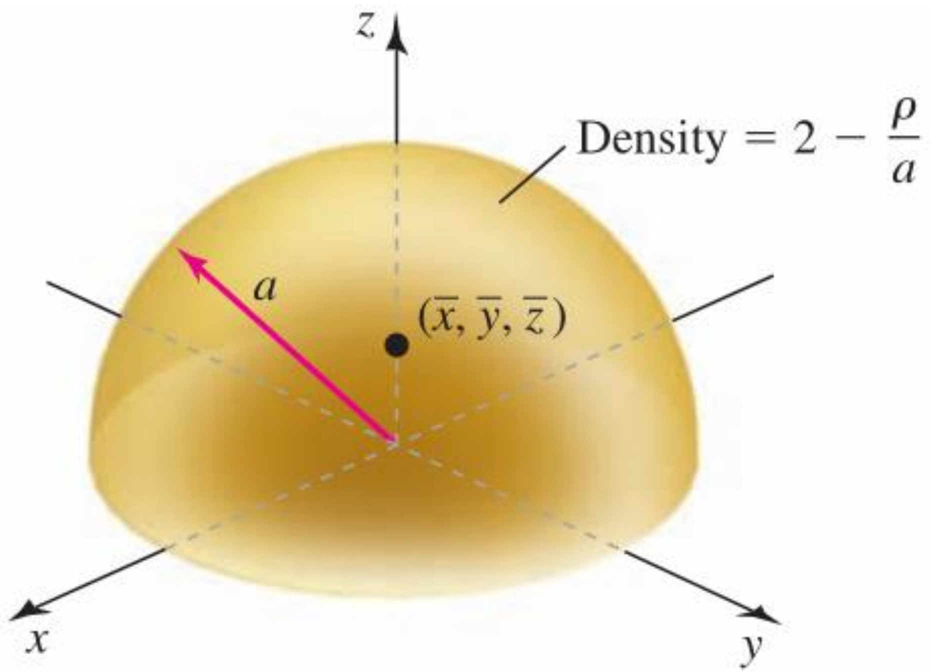
$$\bar{z} = \frac{M_{xy}}{m} = \frac{1}{m} \iiint_D z\rho(x, y, z)\, dV,$$

where $m = \iiint_D \rho(x, y, z)\, dV$ is the mass, and M_{yz}, M_{xz}, and M_{xy} are the moments with respect to the coordinate planes.

EXAMPLE 5 **Center of mass with constant density** Find the center of mass of the constant-density solid cone D bounded by the surface $z = 4 - \sqrt{x^2 + y^2}$ and $z = 0$



EXAMPLE 6 **Center of mass with variable density** Find the center of mass of the interior of the hemisphere D of radius a with its base on the xy-plane. The density of the object is $f(\rho, \varphi, \theta) = 2 - \rho/a$ (heavy near the center and light near the outer surface;



Chapter 16

Multiple Integration

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