MATH 2233 Differential Equations

Chapter 4 Linear Second-Order Equations

Section 4.2 Homogeneous Linear Equations

Goal of this section

- determine auxiliary equation of constant-coefficient homogeneous linear equation.
- understand linear dependence of two functions and the Wronskian.
- solve the equation whose auxiliary equation has two distinct real roots or repeated real roots.

Recall The general form of the second-order linear equations is

If $f(t) \equiv 0$ (i.e., f is identically equal to zero), then the equation is called

We begin our study of linear, second-order, constant-coefficient, homogeneous equation

$$ay'' + by' + cy = 0$$

where a, b, c are real constants.

Example 1. *Verify that $y_1 = e^{2t}$ and $y_2 = e^{7t}$ are both solutions of homogeneous equation*

$$y'' - 9y' + 14y = 0.$$

Then, verify that $y = c_1e^{2t} + c_2e^{7t}$ is also the solution for any constants c_1 and c_2.

From the previous example, we can see that

- Exponential functions of the form ____________ are possible solutions.
- After obtaining two solutions $y_1(t)$ and $y_2(t)$, we can create infinitely many solutions by

1. Auxiliary Equation

Assume that $y = e^{rt}$ is a solution of the equation

$$ay'' + by' + cy = 0$$

The **auxiliary equation** (or **characteristic equation**) of the differential equation is

There are two roots of the auxiliary equation:

$$r_1 = \qquad\qquad\qquad\qquad r_2 =$$

- If $b^2 - 4ac > 0$,
- If $b^2 - 4ac = 0$,
- If $b^2 - 4ac < 0$,

2. Case I: Distinct Real Roots

Example 2. *Find a pair of solutions of*

$$y'' - 3y' + 2y = 0.$$

Remark. Consider the homogeneous linear equation $ay'' + by' + cy = 0$.

- If its auxiliary equation has two real distinct roots r_1 and r_2, then the **general solution** of the differential equation is

- The two constants C_1 and C_2 can be determined by initial conditions

Example 3. *Find the solution of the initial value problem*

$$y'' + y' - 2y = 0, \qquad y(0) = 0, \qquad y'(0) = -1.$$

3. Linear Dependence of Two Functions

Definition.

- A pair of functions $y_1(t)$ and $y_2(t)$ are **linearly independent** on the interval I if and only if

- A pair of functions $y_1(t)$ and $y_2(t)$ are **linearly dependent** on I if

A Condition for Linear Dependence of Solutions.
A pair of functions $y_1(t)$ and $y_2(t)$ are linearly dependent on I if and only if

$$W(y_1, y_2) =$$

Here, $W(y_1, y_2)$ is called

Example 4. *Determine if the following functions are linearly independent*

- $y_1(t) = \cos(t)\sin(t)$, $y_2(t) = \sin(2t)$
- $y_1(t) = e^{2t}$, $y_2(t) = te^{2t}$
- $y_1(t) = 0$, $y_2(t) = te^t$

Remark. Consider the homogeneous linear equation $ay'' + by' + cy = 0$.

- If $y_1(t)$ and $y_2(t)$ are solutions, and they are linearly independent, then

- In this case, the functions $y_1(t)$ and $y_2(t)$ are said to form a

4. Case II: Repeated Real Roots

Example 5. (Reduction of Orders) *Given that* $y_1(t) = e^{2t}$ *is a solution to* $y'' - 4y' + 4y = 0$. *(Verify). Find a second solution* $y_2(t)$ *that is linearly independent with* $y_1(t)$.

Repeated Real Roots

If the auxiliary equation has two repeated real roots $r_1 = r_2$, then the general solution of the differential equation is

Example 6. *Solve the initial value problem*

$$y'' + 2y' + y = 0, \qquad y(0) = 1, \qquad y'(0) = 3.$$

Example 7 (Higher Order Equation). *Solve the differential equation*

$$y''' + 3y'' - y' - 3y = 0.$$

MATH 2233 Differential Equations

Section 4.3 Auxiliary Equations with Complex Roots

Goal of this section

- solve the linear differential equation whose auxiliary equation has complex-value roots

Recall: The auxiliary equation of the linear homogeneous equation $ay'' + by' + cy = 0$ is

We have studied two possible cases:

- **Case I: if $b^2 - 4ac > 0$, we have distinct real roots**

- **Case II: if $b^2 - 4ac = 0$, we have repeated real roots**

In this section, we study the third case when $b^2 - 4ac < 0$.

1. Case III: Complex Conjugated Roots

The auxiliary equation has two complex roots

$r_1 =$ $\qquad\qquad\qquad$ $r_2 =$

To clarify the the meaning of $e^{i\beta t}$, we introduce **Euler's formula:** $e^{i\theta} = \cos(\theta) + i\sin(\theta)$

Using Euler's formula, we can express the general solution as

$$
\begin{aligned}
y(t) &= c_1 y_1(t) + c_2 y_2(t) \\
\\
&=
\end{aligned}
$$

However, solutions in this form are expressed in complex-value functions. It is desirable to seek an alternative pair of solutions which have simpler form and don't require complex values.

Complex Conjugated Roots

Consider the linear homogeneous differential equation $ay'' + by' + cy = 0$

- If the auxiliary equation has two complex roots $r_{1,2} = \alpha \pm i\beta$, then two linearly independent solutions are

- In this case, the general solution of the differential equation is

Example 1. *Find the general solution of*

$$y'' + 2y' + 4y = 0.$$

Example 2. *Solve the initial value problem*

$$y'' + 9y = 0, \quad y(0) = 1, \quad y'(0) = 1.$$

Example 3. *Solve the differential equation*

$$y^{(4)} + 2y'' + y = 0.$$

MATH 2233 Differential Equations

Section 4.4 Method of Undetermined Coefficients

Goal of this section

- construct a particular solution using method of undetermined coefficients.

In this section, we use "judicious guessing" to derive a simple procedure for finding a solution to a **nonhomogeneous** differential equation

$$ay'' + by' + cy = f(t).$$

The solution is often called **particular solution**, denoted by $y_p(t)$.

Example 1. *Guess a solution $y_p(t)$ that satisfies the equation*

$$y'' + 3y' + 2y = 10e^{3t}.$$

Example 2. *Find a particular solution of*

$$y'' + 3y' + 2y = 3t.$$

Example 3. *Find a particular solution of*

$$y'' + 3y' + 2y = \sin(t).$$

Example 4. *Find a particular solution of*

$$y'' + 4y = 5t^2 e^t.$$

Let us summarize what we have learned so far in the next example.

Example 5. *Fill in the table for the appropriate form of the particular solution*

No.	$f(t)$	*A particular solution* y_p
1.	1	
2.	$5t+7$	
3.	$3t^2-2$	
4.	t^3-t+1	
5.	$\sin(4t)$	
6.	$\cos(2t)$	
7.	e^{5t}	
8.	$(9t-2)e^{5t}$	
9.	t^2e^{5t}	
10.	$e^{3t}\sin(4t)$	
11.	$5t^2\sin(4t)$	
12.	$te^{3t}\cos(4t)$	

Example 6. ***(A glitch in the method)*** *Find a particular solution of*

$$y'' - 5y' + 4y = 8e^t.$$

Remark

- The particular solution that we guess must not overlap with
- If there is an overlap, we need to
- When we guess the form of the particular solution, we need to take into account of the solution of the corresponding homogeneous equation.

Example 7. *Find the proper form of a particular solution to*

$$y'' + 2y' - 3y = 2te^t.$$

Example 8. *Find the proper form of a particular solution to*

$$y'' - 2y' + y = 2te^t.$$

Example 9. *Find the proper form of a particular solution to*

$$y^{(4)} + 2y'' + y = \sin(t).$$

MATH 2233 Differential Equations

Section 4.5 Superposition Principle

Goal of this section

- understant the superposition principle, which extends the applicability of the method of undetermined coefficients.

Recall: The method of undetermined coefficient can be applied to find a particular solution of a nonhomogeneous equation, if the right hand side is

1. exponential functions, 2. polynomials, 3. sine or cosine functions, 4. their products

Superposition Principle

If $y_1(t)$ is a solution to the differential equation

$$ay'' + by' + cy = f_1(t),$$

and $y_2(t)$ is a solution to the differential equation

$$ay'' + by' + cy = f_2(t),$$

then,

Example 1. *Find a particular solution of the following differential equation*

$$y'' + 3y' + 2y = 6t + 2e^{3t}.$$

Example 2. *Determine the proper form of a particular solution of*

$$y'' - y' = t + \sin(t).$$

Example 3. ***(Higher Order DE)*** *Determine the form of a particular solution of*

$$y^{(4)} + y'' = 2t - 1 + \sin(t)$$

General Solution to Nonhomogeneous Equations

The general solution $y(t)$ to the nonhomogeneous equation

$$ay'' + by' + cy = f(t)$$

is $y(t) =$ ________________, where

- $y_h(t) =$
- $y_p(t)$ is

To see this, by superposition principle, we have

$$ay'' + by' + cy =$$

Example 4. *Find the general solution of the differential equation*

$$y'' - 2y' - 3y = 3t^2 - 5$$

Solution Procedure for Initial Value Problems of Nonhomogeneous Equations

$$ay'' + by' + cy = f(t), \quad y(t_0) = y_0, \quad y'(t_0) = y_0'$$

Step 1 Find the homogeneous solution $y_h(t) = c_1 y_1(t) + c_2 y_2(t)$.

Step 2 Find the particular solution $y_p(t)$. Then the general solution is $y(t) = y_h(t) + y_p(t)$.

Step 3 Use initial condition to determine constants c_1 and c_2 in $y(t)$.

Example 5. ***(Nonhomogeneous IVP)*** *Solve the initial value problem*

$$y'' + y = 4t + 10\sin(t), \quad y(\pi) = 0, \ \ y'(\pi) = 2.$$

MATH 2233 Differential Equations

Section 4.6 Variation of Parameters

Goal of this section

- introduce a more general method for finding particular solutions.

Recall: There are two limitations of the method of undetermined coefficients

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Example 1. *Find a particular solution of*

$$y'' + y = \tan(t).$$

General Framework

Consider the nonhomogeneous linear second-order equation

$$ay'' + by' + cy = f(t).$$

Assume the homogenous (complementary) solution is

$$y_h(t) =$$

We assume a particular solution has the form

$$y_p(t) =$$

Taking the derivative of $y_p(t)$,

$$y_p'(t) =$$

As the example above, we impose the condition

Then y_p' becomes

$$y_p' =$$

and

$$y_p'' =$$

Plug in to the equation:

We can solve for u_1' and u_2' as

$$u_1'(t) = \qquad\qquad\qquad u_2'(t) =$$

In summary, we have

$$u_1(t) = \qquad\qquad\qquad u_2(t) =$$

Example 2. *Find the general solution of*

$$y'' - 4y' + 4y = (t+1)e^{2t}.$$

Non-constant coefficient linear equations

The method of variation of parameter can be used for solving linear equations whose coefficient a, b, and c are not constants, but are functions of t. However, the method of undetermined coefficients cannot be used in general.

Example 3. *Find the general solution of*

$$t^2y'' - 4ty' + 6y = 4t^3, \quad t > 0$$

given that $y_1(t) = t^2$ *and* $y_2(t) = t^3$ *are solutions to the corresponding homogeneous equation.*

MATH 2233 Differential Equations

Section 4.7 Cauchy-Euler Equation

Goal of this section

- understand the definition of Cauchy-Euler Equations.
- solve homogeneous and nonhomogeneous Cauchy-Euler Equations

The linear 2nd-order differential equations with variable coefficients have the form

$$a(t)y'' + b(t)y' + c(t)y = f(t),$$

where $a(t)$, $b(t)$, and $c(t)$ are functions of t.
In this section we consider the **Cauchy-Euler equations**. They have the following form

Assume ________________________ is a solution to the homogeneous C-E equation

$$at^2y'' + bty' + cy = 0,$$

then

The **auxiliary equation** of the Cauchy-Euler equation is

Example 1. ***(Distinct roots)*** *Find the general solution of*

$$t^2y'' - 2ty' - 4y = 0.$$

Repeated Roots: Method of Reduction of Orders

Example 2. ***(Repeated Roots)*** *Find the general solution of*

$$4t^2y'' + 8ty' + y = 0.$$

General Case: Repeated Roots

If the auxiliary equation of a Cauchy-Euler equation has repeated roots $m_1 = m_2$, then

Example 3. ***(Reduction of Order)*** *A differential equation and a nontrivial solution $y_1(t) = e^t$ are given below. Find a second linearly independent solution using reduction of order.*

$$ty'' - (t+1)y' + y = 0, \quad t > 0.$$

Complex Roots

If the auxiliary equation of a Cauchy-Euler equation has a pair of conjugated complex roots

$$m_1 = \alpha + i\beta, \quad m_2 = \alpha - i\beta.$$

Comparison of constant coefficient equations and Cauchy-Euler equations

Diff Equation	$ay'' + by' + cy = 0$	$at^2y'' + bty' + cy = 0$
Aux. Equ		
$r_1 \neq r_2$		
$r_1 = r_2$		
$r_{1,2} = \alpha \pm \beta i$		

Example 4. *Find the general solution of*

$$4t^2y'' + 17y = 0.$$

Nonhomogeneous Cauchy-Euler Equation

Since C-E equations are not constant-coefficient, we should use ______________________________ to seek particular solutions of nonhomogeneous Cauthy-Euler equations.

Example 5. ***(Nonhomogeneous Equation)*** *Find the general solution of*

$$t^2y'' - 3ty' + 3y = 2t^4e^t.$$

MATH 2233 Differential Equations

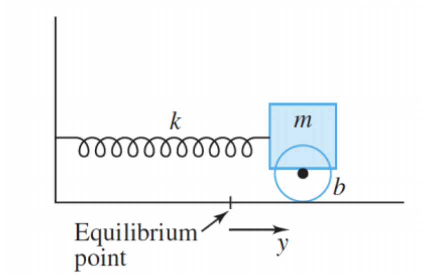
Section 4.9&4.10   Modeling with Second-Order Equations

Goal of this section

- Study the mass-spring model using second-order linear differential equations.

Mass-Spring System

Consider a mass m attached to a horizontal spring fixed at one end.



When the spring is unstretched and the mass m is still, the system is at ________________

We are interested in the motion of the mass when an external force is acted on, or it is initially displaced from the equilibrium position.
In this dynamical problem, there are several forces we should consider.

- **spring force (due to Hooke's Law)**

- **damping force (due to friction, air resistence)**

- **external force**

The Newton's second law of motion yields

1. Free Undamped Motion

Example 1. *A $1/8$-kg mass is attached to a spring with stiffness constant $k = 8$ N/m. The mass is displaced $1/2$ m to the right of the equilibrium point and given an outward velocity (to the right) of 2m/sec. Neglect the damping and external forces. Determine the motion of the mass.*

In Example 1, there is no damping ($b = 0$), and free of external forces ($F(t) = 0$), the equation of the Mass-Spring system can be written as

$$my'' + ky = 0.$$

For this simple motion, we introduce the following physical quantities:

- **Amplitude**

- **Period**

- **Frequency**

Example 2. *In the setting of Example 1, find out the amplitude, period, and the natural frequency of the motion. How long after release does the mass pass through the equilibrium point.*

2. Free Damped Motion

In most applications, there is some type of frictional or damping forces affecting the vibration. The free damped motion is

$$my'' + by' + ky = 0.$$

Example 3. *A* $1/4$*-kg mass is attached to a spring with a stiffness* 4 *N/m. The damping constant* $b = 1$ *(N-sec/m). If the mass is displaced* $1/2$*m to the left and given an initial velocity of* 1 *m/sec to the left, find the equation of motion. What is the maximum displacement that the mass will attain?*

3. Forced Mechanical Motion (Section 4.10)

We now consider the vibrations of a mass–spring system when an external force $F(t)$ is applied.

Example 4. *An* 8 *kg mass is attached to a spring hanging from the ceiling, thereby causing the spring to stretch* 1.96 *m upon coming to rest at equilibrium. At time* $t = 0$*, an external force of* $F(t) = \cos(t)$ *N is applied to the system. The damping constant for the system is* 16 *N-sec/m. Determine the motion of the system.*