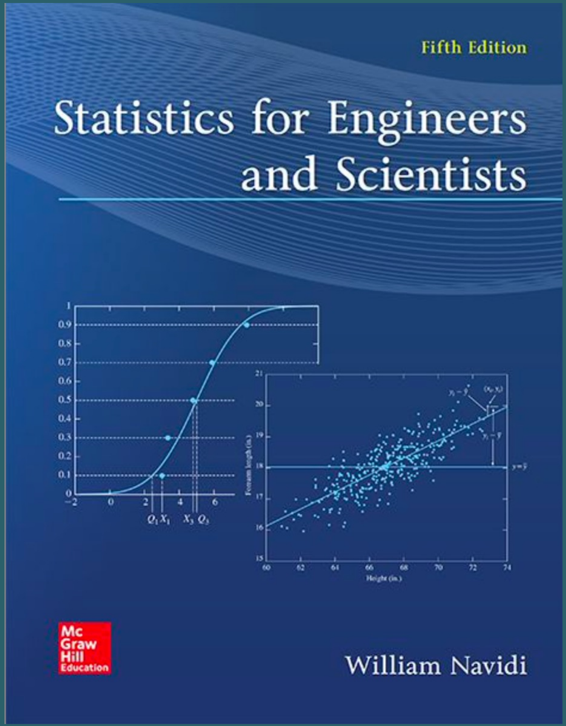

Ch 3: Propagation of Error…

(not all sections are required)

Chapter 3 Overview

3-1 Measurement Error

3-2 Linear Combinations of Measurements

3-3 Uncertainties for Functions of One Measurement

~~*3-4 Uncertainties for Several Measurements*~~

Introduction

- ✓ In many cases we wish to estimate the uncertainty in a nonlinear function of a measurement.
- ✓ For example, if the radius R of a circle is measured to be 5.00 ± 0.01 cm, what is the uncertainty in the area A?
- ✓ In statistical terms, we know that the standard deviation σ_R is 0.01 cm, and we must calculate the standard deviation of A, where A is the function of R given by $\mathbf{A = \pi R^2}$.

Problem

- Given a random variable X, with known standard deviation σ_X, and given a function $U = U(X)$, how do we compute the standard deviation σ_U?

Solution

- If U is a linear function, the methods of Section 3.2 apply.
- If U is not linear, we can still approximate σ_U, by multiplying σ_X by the absolute value of the derivative dU/dX.
- The approximation will be good so long as σ_X is small.

If X is a measurement whose uncertainty σ_X is small, and if U is a function of X, then

$$\sigma_U \approx \left|\frac{dU}{dX}\right| \sigma_X \tag{3.10}$$

In practice, we evaluate the derivative dU/dX at the observed measurement X.

- This is **the propagation of error** formula

Rounding!!!

- **Propagation of Error Uncertainties Are Only Approximate**

- The uncertainties computed by using Equation (3.10) are often only rough approximations.

- For this reason, these uncertainties should be expressed with **no more than two significant digits**. Indeed, some authors suggest using only one significant digit.

Nonlinear Functions Are Bias!!!

- If X is an unbiased measurement of a true value μ_X, and if the function U = U(X) is a nonlinear function of X, then in most cases **U will be biased for the true value $U(\mu_X)$**.

- In practice this bias is usually ignored.

- As long as the uncertainty **σ_X is small**, the bias in U will in general be small as well, except for some fairly unusual circumstances when the second derivative is quite large.

- If X is a measurement with non-negligible bias, then the bias in U may be large.

Example 3.14 (p.181)

The radius R of a circle is measured to be 5.00 ± 0.01 cm.

Estimate the area of the circle and find the uncertainty in this estimate.

Solution

The area A is given by $A = \pi R^2$. The estimate of A is $\pi(5.00\text{ cm})^2 = 78.5\text{ cm}^2$. Now $\sigma_R = 0.01$ cm, and $dA/dR = 2\pi R = 10\pi$ cm. We can now find the uncertainty in A:

$$\begin{aligned}\sigma_A &= \left|\frac{dA}{dR}\right|\sigma_R \\ &= (10\pi\text{ cm})(0.01\text{ cm}) \\ &= 0.31\text{ cm}^2\end{aligned}$$

We estimate the area of the circle to be $78.5 \pm 0.3\text{ cm}^2$.

Example 3.15 (p.181)

- A rock identified as cobble-sized quartzite has a mass m of 674.0 g.
- Assume this measurement has negligible uncertainty.
- The volume V of the rock will be measured by placing it in a graduated cylinder partially filled with water and measuring the volume of water displaced.
- The density D of the rock will be computed as D = m/V. Assume the volume of displaced water is 261.0 ± 0.1 mL.
- Estimate the density of the rock and find the uncertainty in this estimate.

Example 3.15 (p.181) - SOLUTION

- Substituting V = 261.0 mL, the estimate of the density D is 674.0/261.0 = 2.582 g/mL
- Treating m = 674.0 as a known constant,

$$dD/dV = -674.0/V^2 = -674.0/(261.0)^2 = -0.010 \text{ g/mL}^2$$

- We know that $\sigma_V = 0.1$ mL. The uncertainty in D is therefore

$$\begin{aligned}\sigma_D &= \left|\frac{dD}{dV}\right|\sigma_V \\ &= |-0.010|(0.1 \text{ g/mL}) \\ &= 0.001 \text{ g/mL}\end{aligned}$$

- We estimate the density to be 2.582 ± 0.001 g/mL

Relative Uncertainties for Functions of One Measurement

- The standard deviation σ_U of a measurement U is referred to as the uncertainty in U.
- A more complete name for σ_U is **the absolute uncertainty**, because it is expressed in the same units as the measurement U.
- Sometimes we wish to express **the uncertainty as a fraction of the true value**, which (assuming no bias) is the mean measurement μ_U. This is called the **relative uncertainty in U**.
- The relative uncertainty can also be called the coefficient of variation.
- In practice, since μ_U is unknown, the measured value U is used in its place when computing the relative uncertainty.

Relative Uncertainties for Functions of One Measurement…

Summary

If U is a measurement whose true value is μ_U, and whose uncertainty is σ_U, the relative uncertainty in U is the quantity σ_U/μ_U.

The relative uncertainty is a unitless quantity. It is frequently expressed as a percent. In practice μ_U is unknown, so if the bias is negligible, we estimate the relative uncertainty with σ_U/U.

Relative Uncertainties for Functions of One Measurement…

There are two methods for approximating the relative uncertainty σ_U/U in a function $U = U(X)$:

1. Compute $\boldsymbol{\sigma_U}$ using Equation (3.10), and then **divide by U**
2. Compute ***ln*** **U** and use Equation (3.10) to find $\boldsymbol{\sigma_{\ln U}}$, which is equal to $\boldsymbol{\sigma_U/U}$

Both methods work in every instance.

This choice is usually dictated by whether it is **easier** to compute the derivative of U or of ln U.

Example 3.16 (p.183)

- The radius of a circle is measured to be 5.00 ± 0.01 cm.
- Estimate the area, and find the relative uncertainty in the estimate.

Example 3.16 (p.183) - SOLUTION

- In Example 3.14 the area $\mathbf{A = \pi R^2}$ was computed to be $\mathbf{78.5 \pm 0.3\ cm^2}$.
- The absolute uncertainty is $\boldsymbol{\sigma_A = 0.3\ cm^2}$
- The relative uncertainty is $\boldsymbol{\sigma_A/A = 0.3/78.5 = 0.004}$
- We can therefore express the area as

A = 78.5 cm2 ± 0.4%

Example 3.16 (p.183) - SOLUTION

- If we had not already computed σ_A, it would be easier to compute the relative uncertainty by computing the absolute uncertainty in ***ln* A**.
- Since *ln* A = *ln* π + 2*ln* R

  d*ln* A/dR = 2/R = 0.4

- The relative uncertainty in A is therefore

$$\frac{\sigma_A}{A} = \sigma_{\ln A}$$
$$= \left|\frac{d\ln A}{dR}\right|\sigma_R$$
$$= 0.4\sigma_R$$
$$= (0.4)(0.01)$$
$$= 0.4\%$$

Example 3.17 (p.183)

- The acceleration of a mass down a frictionless inclined plane is given by $a = g \sin \theta$, where g is the acceleration due to gravity and θ is the angle of inclination of the plane.
- Assume the uncertainty in g is negligible.
- If $\theta = 0.60 \pm 0.01$ rad, find the relative uncertainty in a.

Example 3.17 (p.183) - SOLUTION

- The relative uncertainty in a is the absolute uncertainty in ln a.
- Now ln a =ln g + ln(sin θ), where ln g is constant.
- Therefore,

d ln a/dθ = d ln(sin θ)/dθ = cos θ/sin θ = cot θ = cot(0.60) = 1.46

- The uncertainty in θ is $\sigma_\theta = 0.01$.
- The relative uncertainty in a is therefore

$$\begin{aligned}\frac{\sigma_a}{a} &= \sigma_{\ln a} \\ &= \left|\frac{d \ln a}{d\theta}\right| \sigma_\theta \\ &= (1.46)(0.01) \\ &= 1.5\%\end{aligned}$$

End of Section 3-3