Chapter 13

Vectors and
Vector-Valued Functions (I)

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13.1

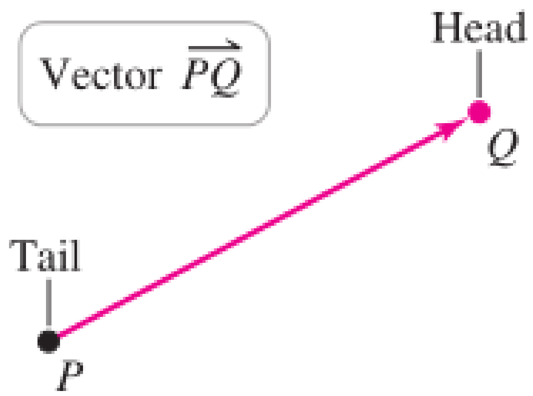
Vectors in the Plane

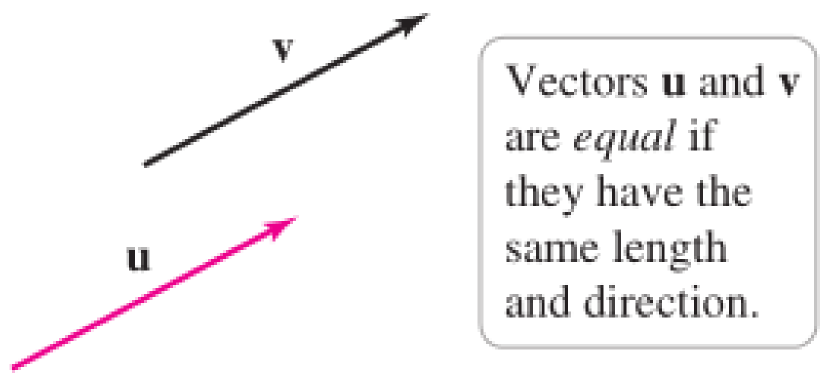
Basic Vector Operations

Vectors: quantities that have both *length* (or *magnitude*) and *direction*.

Scalars: quantities that have magnitude, but no direction

One exception is the zero vector, denoted **0**: It has length 0 and no direction.



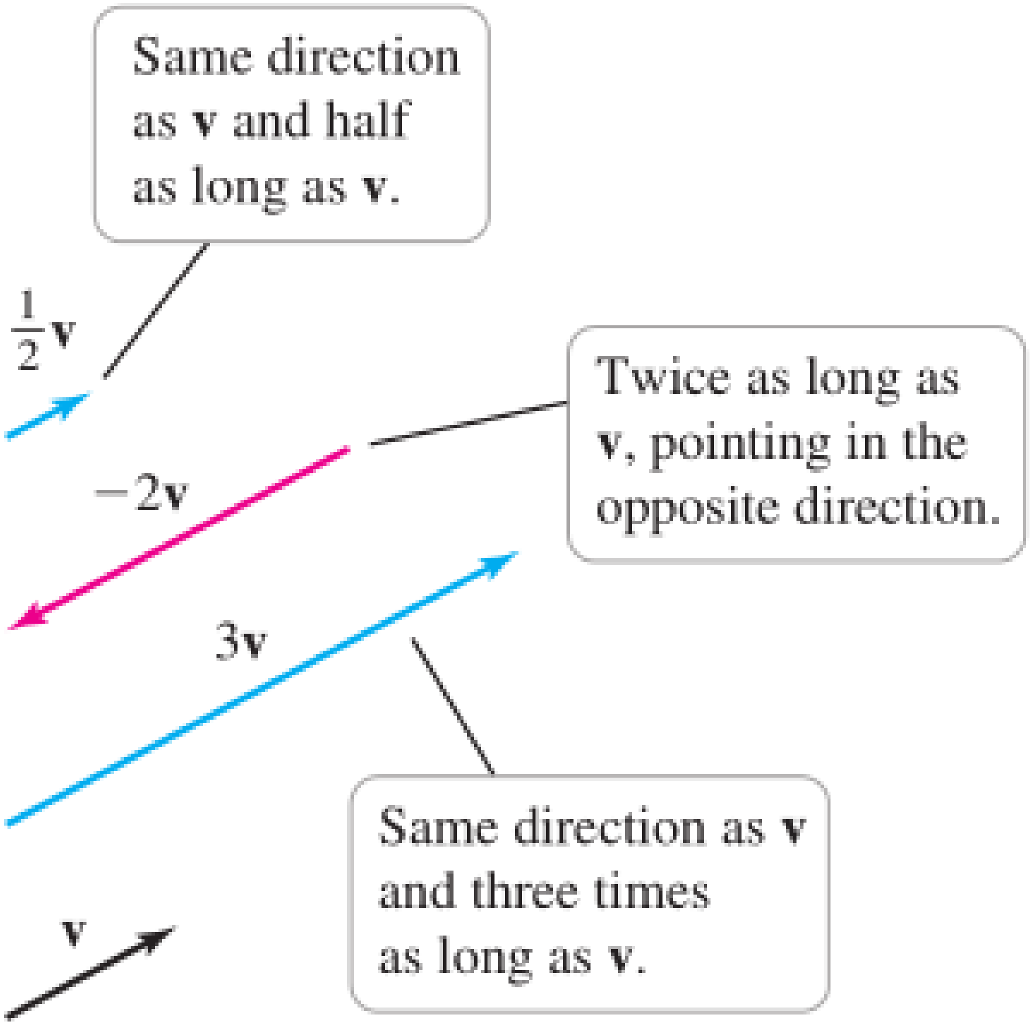


Scalar Multiplication

DEFINITION Scalar Multiples and Parallel Vectors

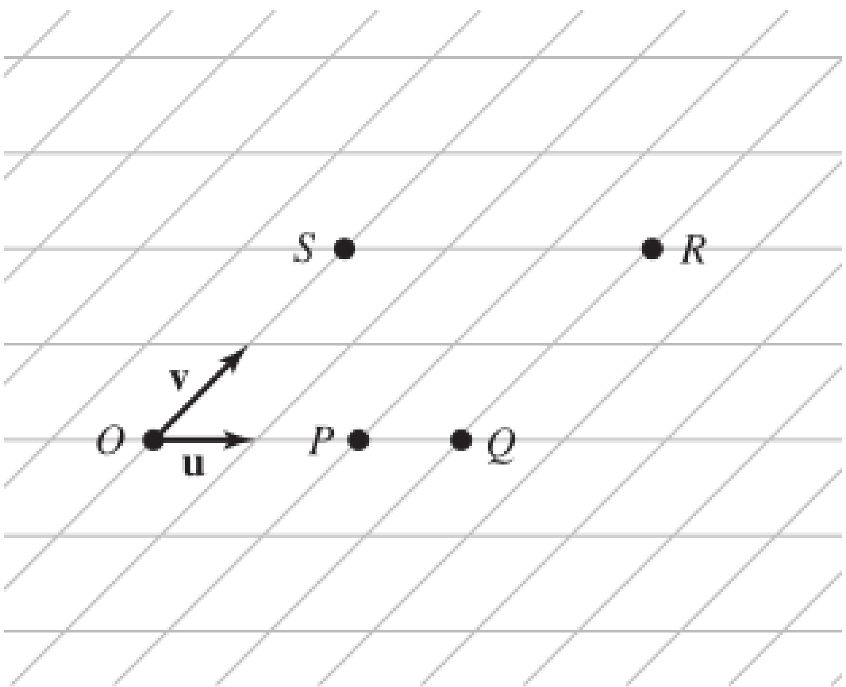
Given a scalar c and a vector $\mathbf{v}$, the **scalar multiple** $c\mathbf{v}$ is a vector whose magnitude is $|c|$ multiplied by the magnitude of $\mathbf{v}$. If $c > 0$, then $c\mathbf{v}$ has the same direction as $\mathbf{v}$. If $c < 0$, then $c\mathbf{v}$ and $\mathbf{v}$ point in opposite directions. Two vectors are **parallel** if they are scalar multiples of each other.

Zero vector is *parallel to all vectors*.

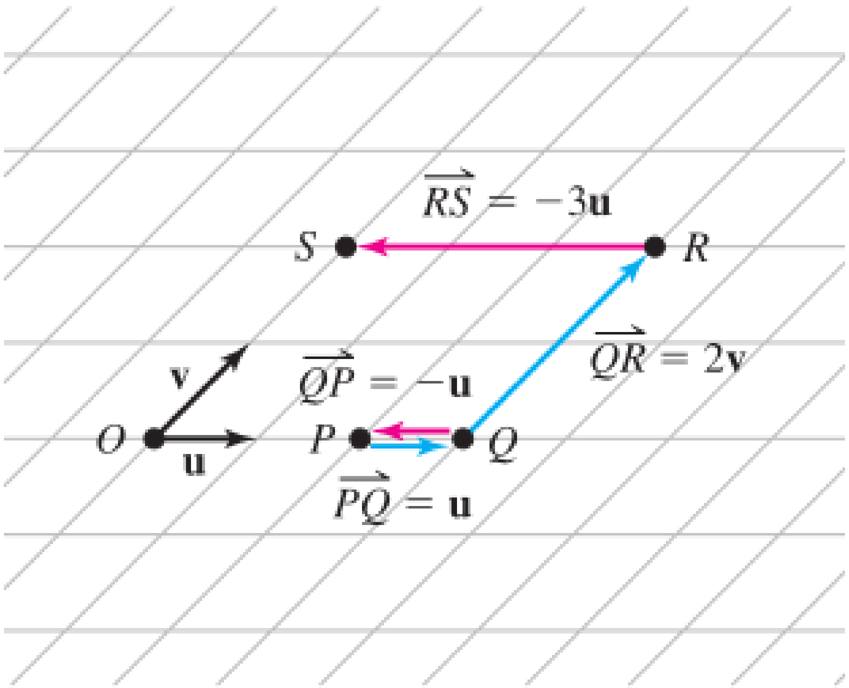

Same direction as v and half as long as v.
$\frac{1}{2}\mathbf{v}$
Twice as long as v, pointing in the opposite direction.
$-2\mathbf{v}$
$3\mathbf{v}$
Same direction as v and three times as long as v.
$\mathbf{v}$

EXAMPLE 1 **Parallel vectors** Using Figure 6a, write the following vectors in terms of **u** or **v**.

a. $\overrightarrow{PQ}$ **b.** $\overrightarrow{QP}$ **c.** $\overrightarrow{QR}$ **d.** $\overrightarrow{RS}$



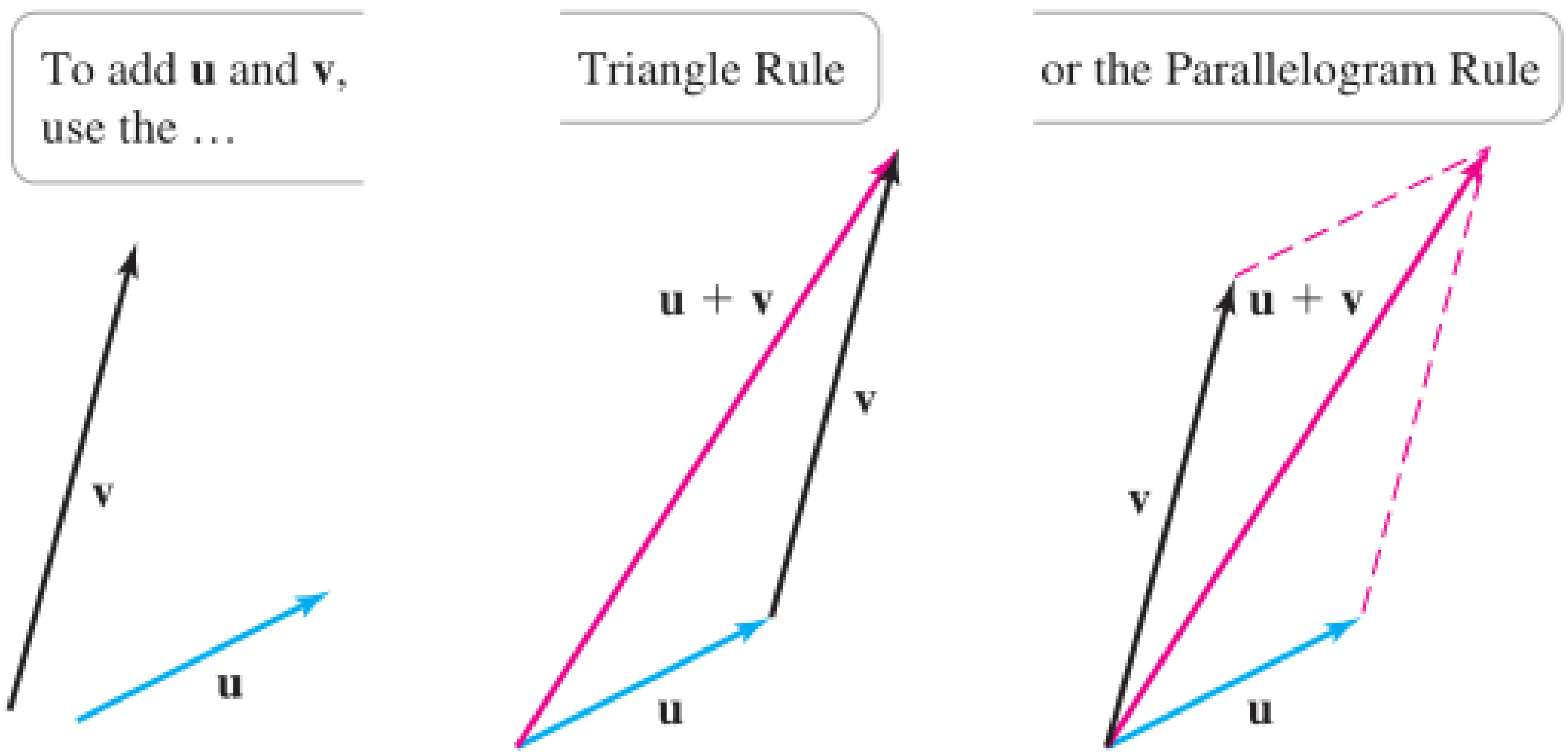
(a)



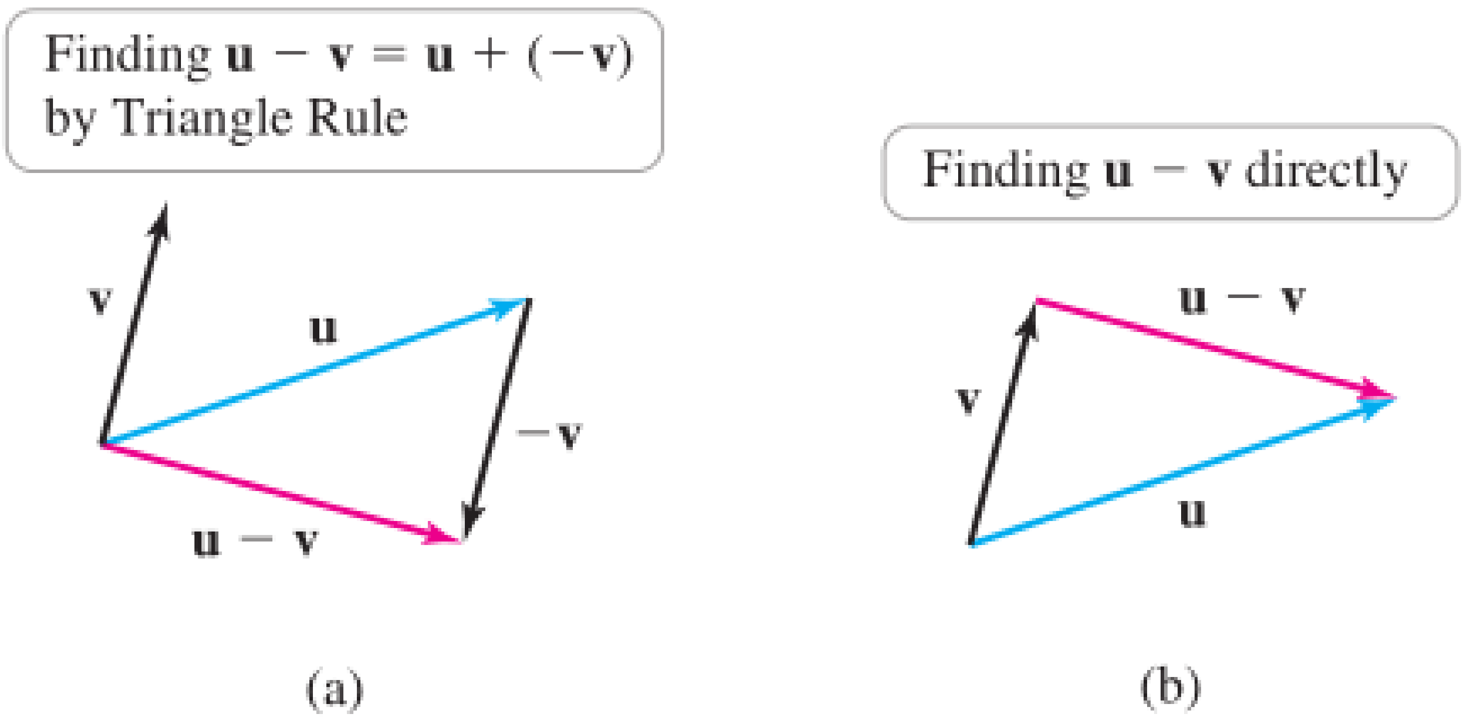
(b)

Vector Addition and Subtraction

Two ways to form the vector sum of two nonzero vectors u and v geometrically: **Triangle Rule** and **Parallelogram Rule**.

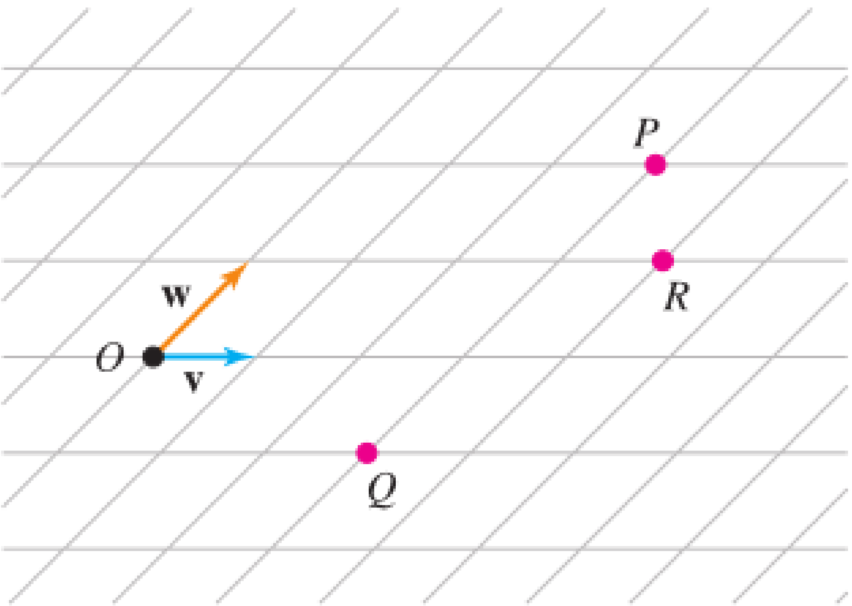


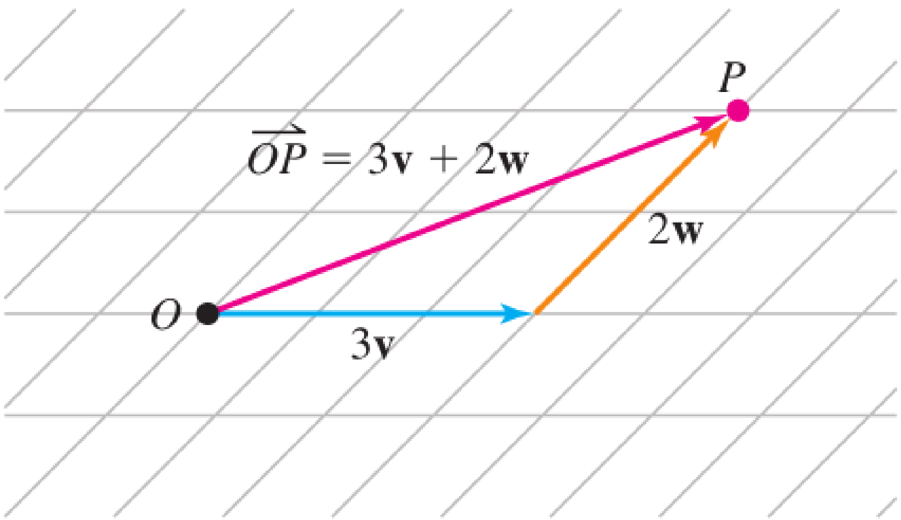
Difference $u - v$ is defined to be the sum $u + (-v)$.



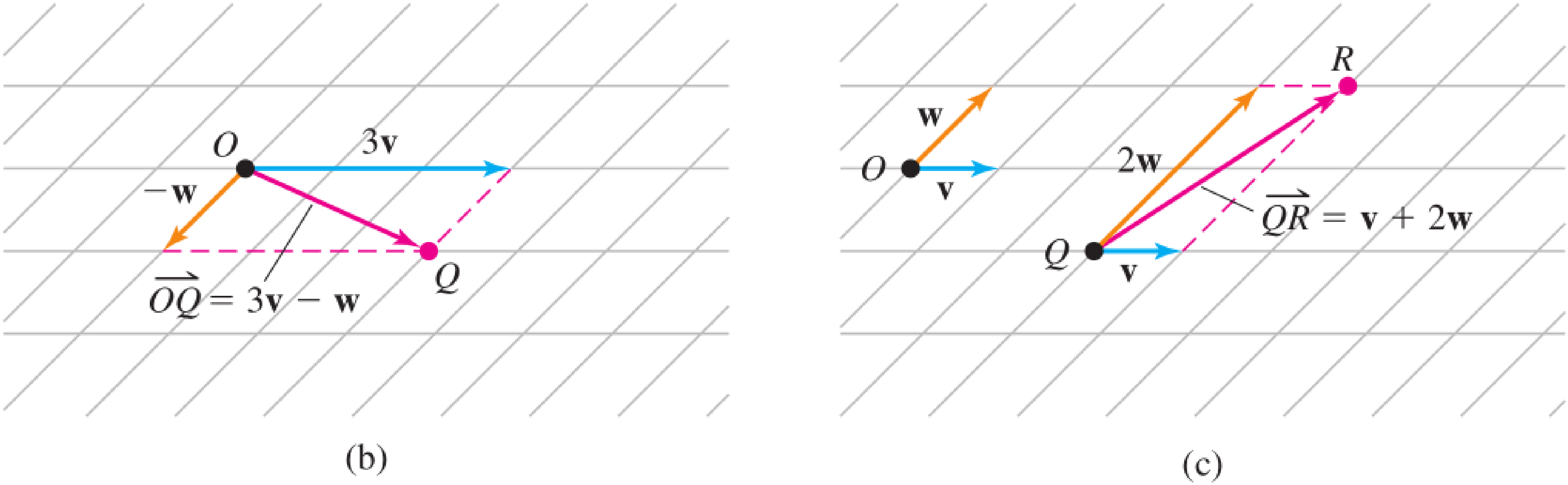
EXAMPLE 2 **Vector operations** Use Figure 10 to write the following vectors as sums of scalar multiples of **v** and **w**.

a. $\overrightarrow{OP}$ **b.** $\overrightarrow{OQ}$ **c.** $\overrightarrow{QR}$





(a)

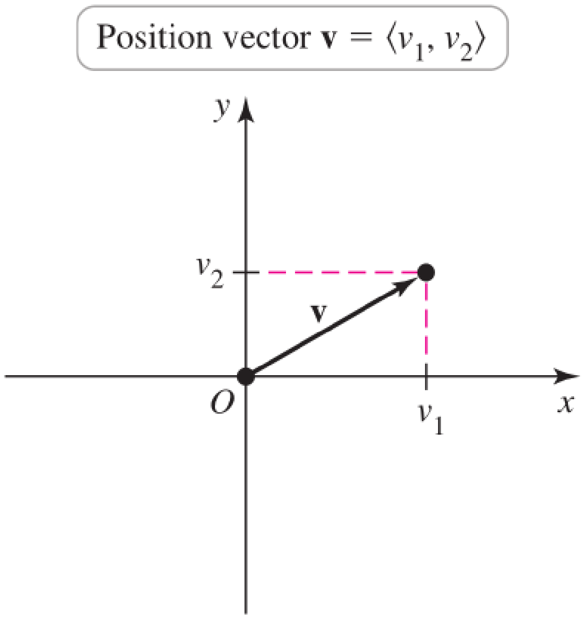


O
3v
−w
Q
$\overrightarrow{OQ} = 3\mathbf{v} - \mathbf{w}$
(b)
R
w
O
v
2w
$\overrightarrow{QR} = \mathbf{v} + 2\mathbf{w}$
Q
v
(c)

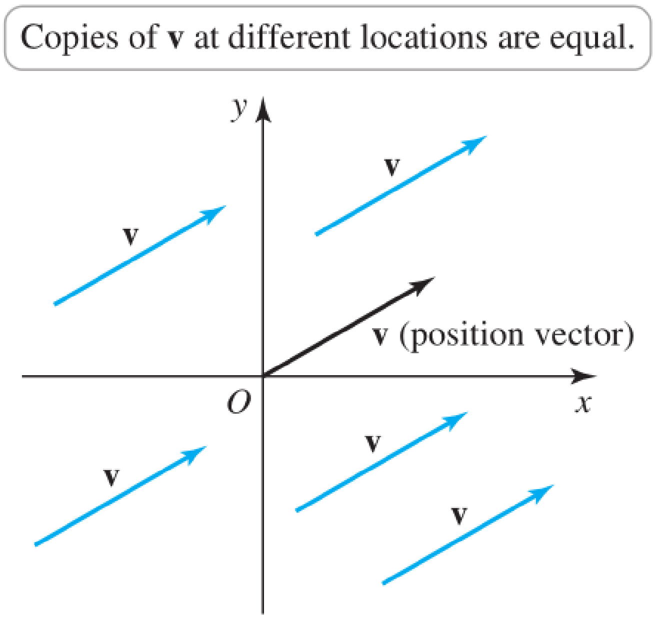
Vector Components

DEFINITION **Position Vectors and Vector Components**

A vector $\mathbf{v}$ with its tail at the origin and head at the point (v_1, v_2) is called a **position vector** (or is said to be in **standard position**) and is written $\langle v_1, v_2 \rangle$. The real numbers v_1 and v_2 are the x- and y-**components** of $\mathbf{v}$, respectively. The position vectors $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ are **equal** if and only if $u_1 = v_1$ and $u_2 = v_2$.

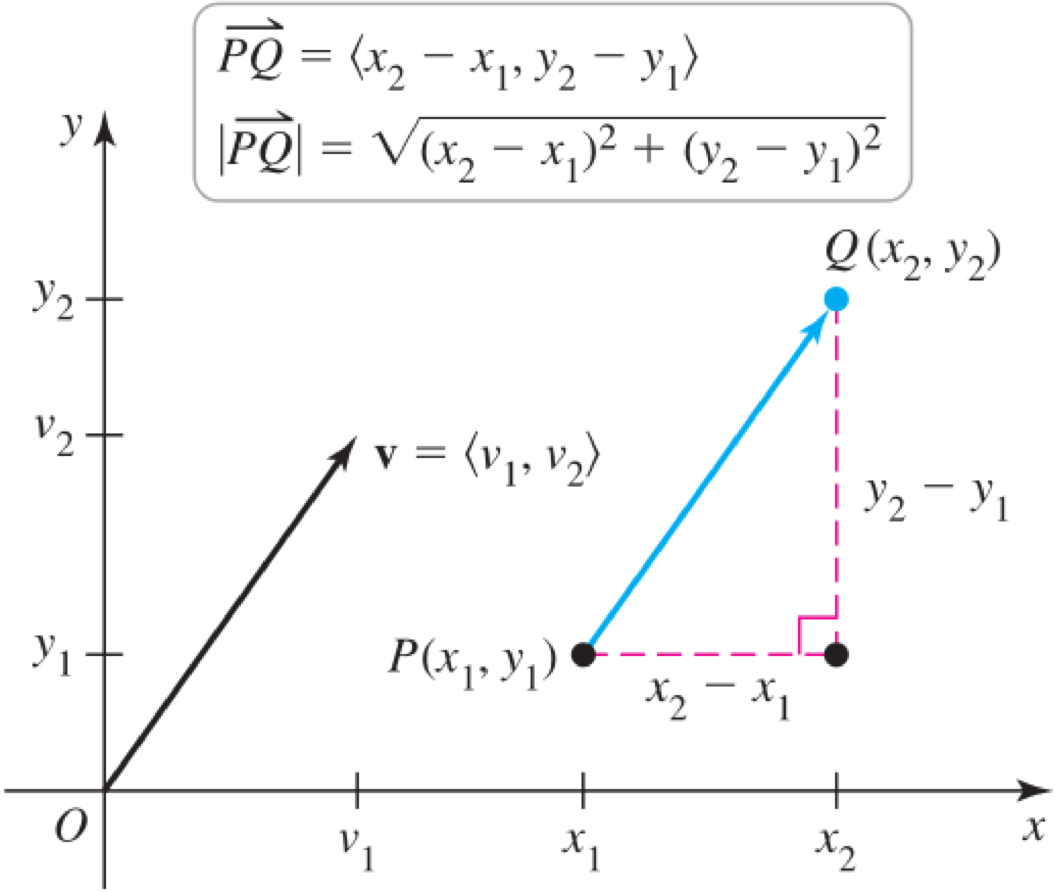


(a)



(b)

For vector $\overrightarrow{PQ}$ equal to v, but not in standard position



Magnitude

DEFINITION Magnitude of a Vector

Given the points $P(x_1, y_1)$ and $Q(x_2, y_2)$, the **magnitude**, or **length**, of $\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1 \rangle$, denoted $|\overrightarrow{PQ}|$, is the distance between P and Q:

$$|\overrightarrow{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

The magnitude of the position vector $\mathbf{v} = \langle v_1, v_2 \rangle$ is $|\mathbf{v}| = \sqrt{v_1^2 + v_2^2}$.

EXAMPLE 3 **Calculating components and magnitude** Given the points $O(0, 0)$, $P(-3, 4)$, and $Q(6, 5)$, find the components and magnitude of the following vectors.

a. $\overrightarrow{OP}$ **b.** $\overrightarrow{PQ}$

Vector Operations in Terms of Components

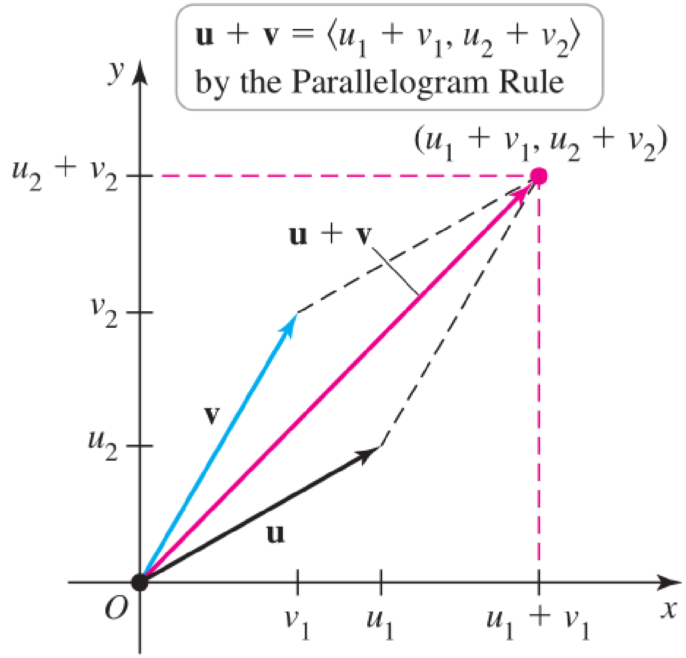
DEFINITION **Vector Operations in $\mathbb{R}^2$**

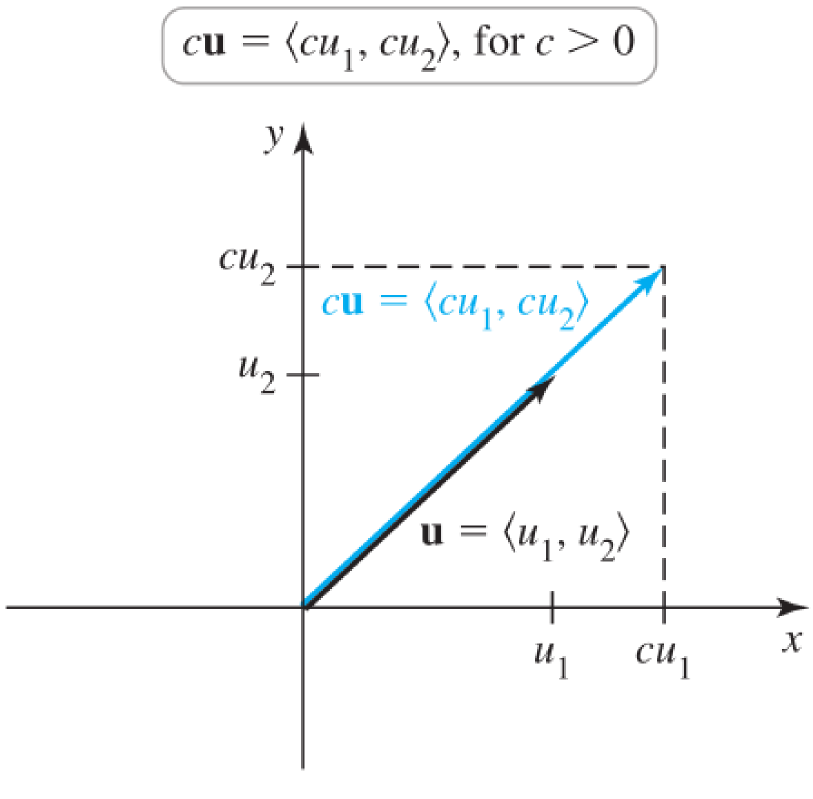
Suppose c is a scalar, $\mathbf{u} = \langle u_1, u_2 \rangle$, and $\mathbf{v} = \langle v_1, v_2 \rangle$.

$$\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2 \rangle \qquad \text{Vector addition}$$

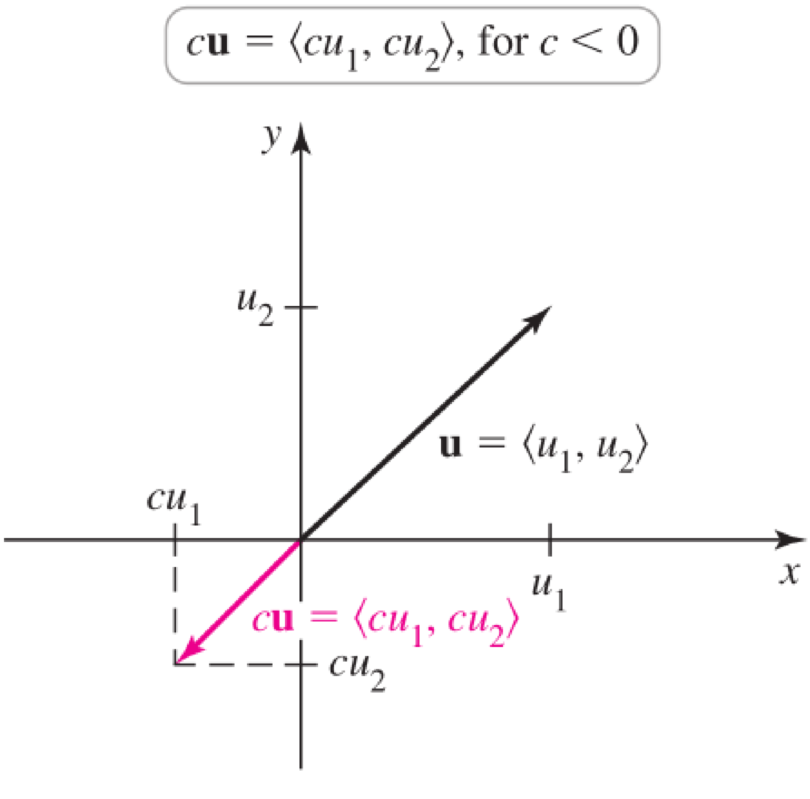
$$\mathbf{u} - \mathbf{v} = \langle u_1 - v_1, u_2 - v_2 \rangle \qquad \text{Vector subtraction}$$

$$c\mathbf{u} = \langle cu_1, cu_2 \rangle \qquad \text{Scalar multiplication}$$





(a)



(b)

EXAMPLE 4 **Vector operations** Let $\mathbf{u} = \langle -1, 2 \rangle$ and $\mathbf{v} = \langle 2, 3 \rangle$.

a. Evaluate $|\mathbf{u} + \mathbf{v}|$.

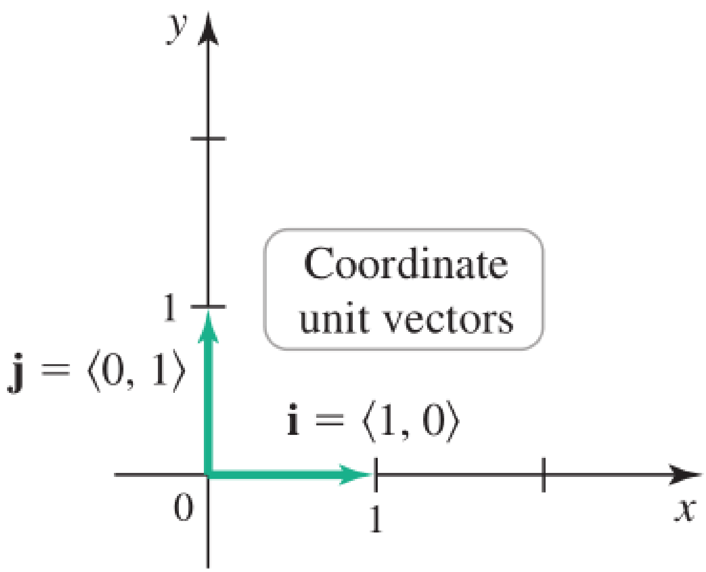
b. Simplify $2\mathbf{u} - 3\mathbf{v}$.

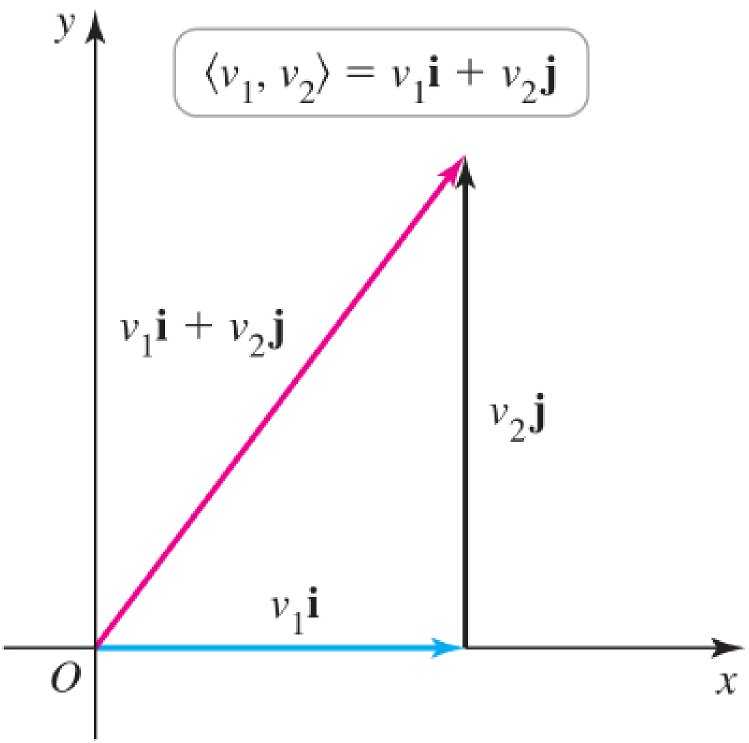
c. Find two vectors half as long as $\mathbf{u}$ and parallel to $\mathbf{u}$.

Unit Vectors

A unit vector is any vector with length 1

Coordinate unit vectors $\boldsymbol{i} = \langle 1,0 \rangle$ and $\boldsymbol{j} = \langle 0,1 \rangle$





In general, the vector $\boldsymbol{v} = \langle v_1, v_2 \rangle = v_1\boldsymbol{i} + v_2\boldsymbol{j}$

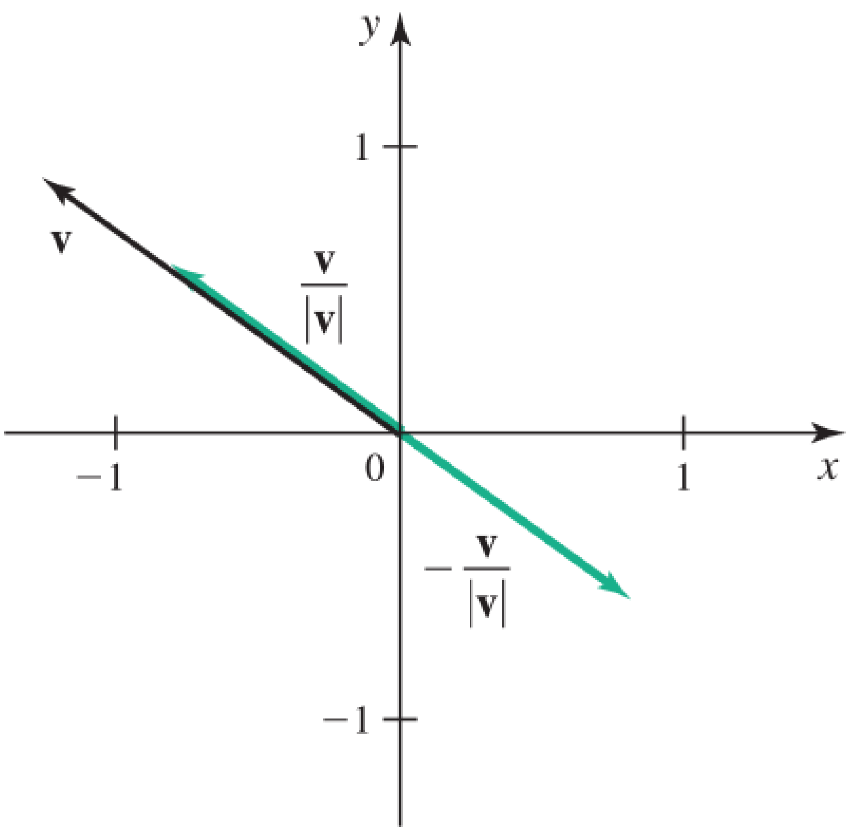
Construct a new vector parallel to $\boldsymbol{v}$ of a specified length, e.g., 1.

$$\boldsymbol{u} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|}$$

$\mathbf{u} = \dfrac{\mathbf{v}}{|\mathbf{v}|}$ and $-\mathbf{u} = -\dfrac{\mathbf{v}}{|\mathbf{v}|}$ have length 1.

$\boldsymbol{u}$ is a unit vector with the same direction as $\boldsymbol{v}$.

$-\boldsymbol{u} = -\dfrac{\boldsymbol{v}}{|\boldsymbol{v}|}$ is also a unit vector with the opposite direction



Similarly, form the vector $\dfrac{c\boldsymbol{v}}{|\boldsymbol{v}|}$

EXAMPLE 5 **Magnitude and unit vectors** Consider the points $P(1, -2)$ and $Q(6, 10)$.

a. Find $\overrightarrow{PQ}$ and two unit vectors parallel to $\overrightarrow{PQ}$.

b. Find two vectors of length 2 parallel to $\overrightarrow{PQ}$.

Properties of Vector Operations

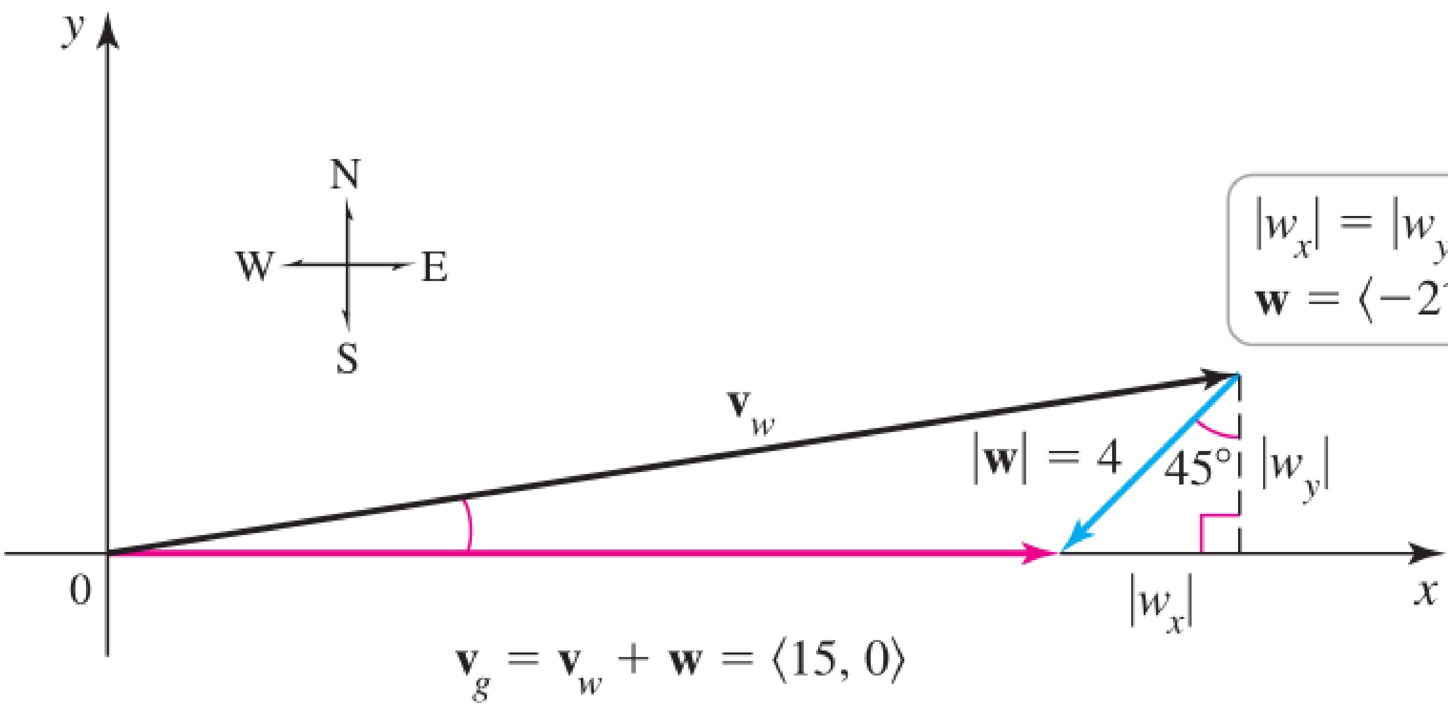
SUMMARY **Properties of Vector Operations**

Suppose $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors and a and c are scalars. Then the following properties hold (for vectors in any number of dimensions).

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$	Commutative property of addition
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$	Associative property of addition
3. $\mathbf{v} + \mathbf{0} = \mathbf{v}$	Additive identity
4. $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$	Additive inverse
5. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$	Distributive property 1
6. $(a + c)\mathbf{v} = a\mathbf{v} + c\mathbf{v}$	Distributive property 2
7. $0\mathbf{v} = \mathbf{0}$	Multiplication by zero scalar
8. $c\mathbf{0} = \mathbf{0}$	Multiplication by zero vector
9. $1\mathbf{v} = \mathbf{v}$	Multiplicative identity
10. $a(c\mathbf{v}) = (ac)\mathbf{v}$	Associative property of scalar multiplication

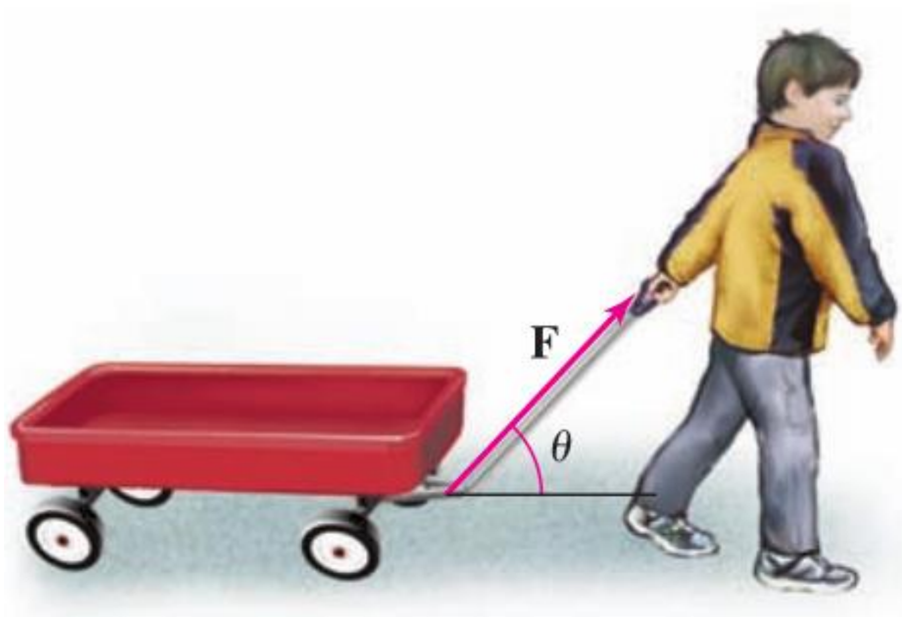
Velocity Vectors

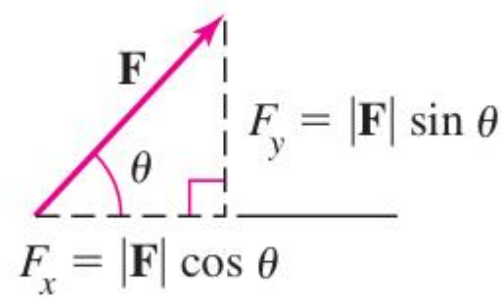
EXAMPLE 6 **Speed of a boat in a current** Suppose the water in a river moves southwest (45° west of south) at 4 mi/hr and a motorboat travels due east at 15 mi/hr relative to the shore. Determine the speed of the boat and its heading relative to the moving water



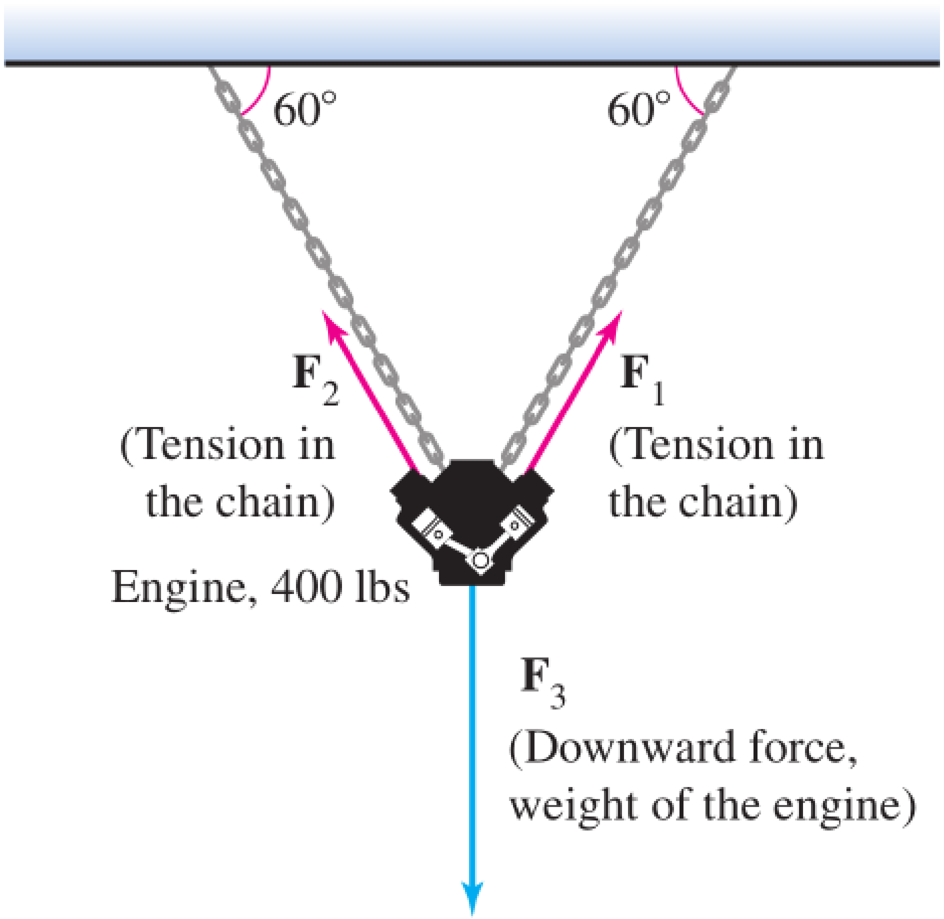
Force Vectors

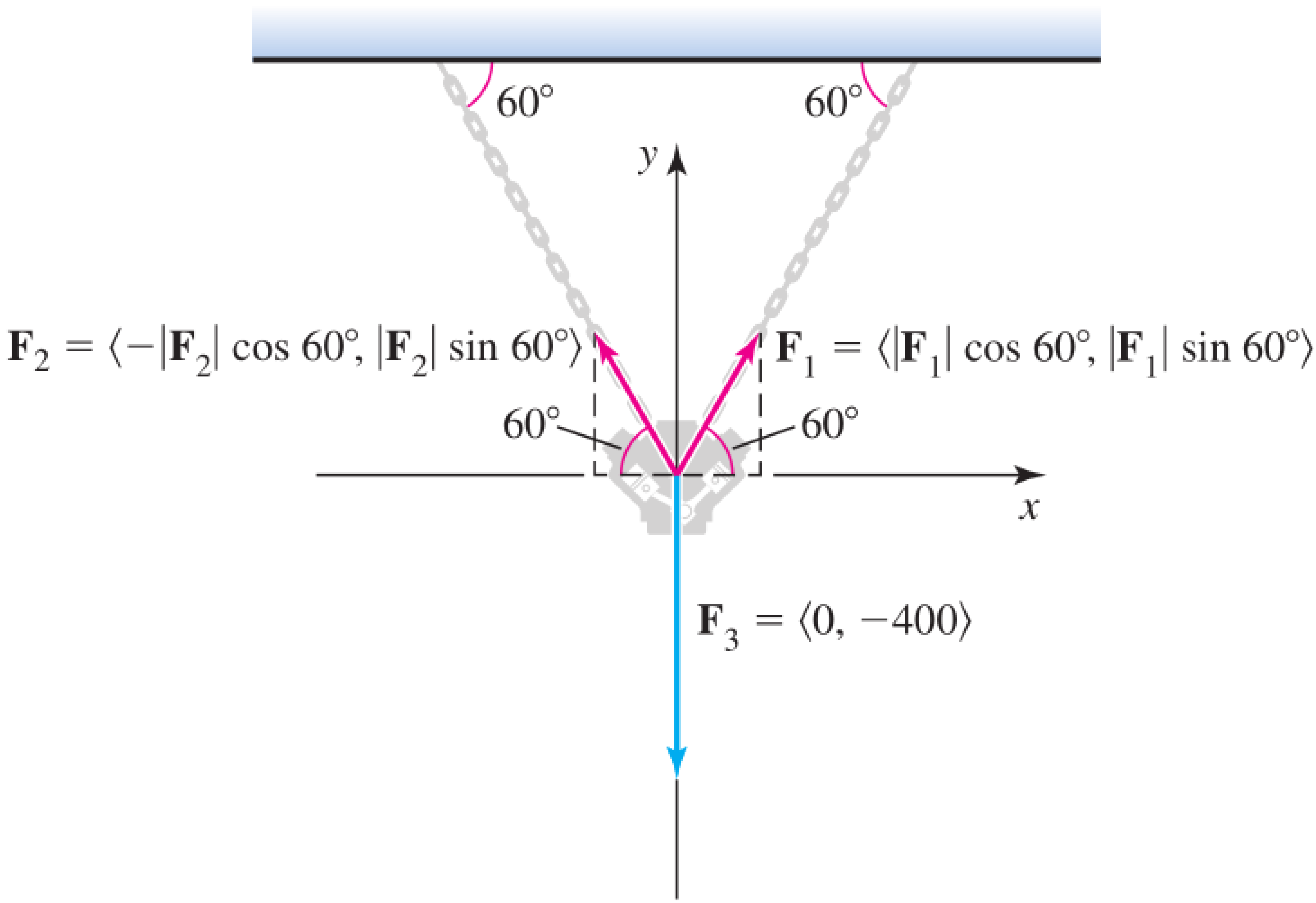
EXAMPLE 7 **Finding force vectors** A child pulls a wagon (Figure 21) with a force of $|\mathbf{F}| = 20$ lb at an angle of $\theta = 30°$ to the horizontal. Find the force vector $\mathbf{F}$.





EXAMPLE 8 **Balancing forces** A 400-lb engine is suspended from two chains that form 60° angles with a horizontal ceiling (Figure 23). How much weight does each chain support?




60°
60°
y
$\mathbf{F}_2 = \langle -|\mathbf{F}_2| \cos 60°, |\mathbf{F}_2| \sin 60° \rangle$
$\mathbf{F}_1 = \langle |\mathbf{F}_1| \cos 60°, |\mathbf{F}_1| \sin 60° \rangle$
60°
60°
x
$\mathbf{F}_3 = \langle 0, -400 \rangle$

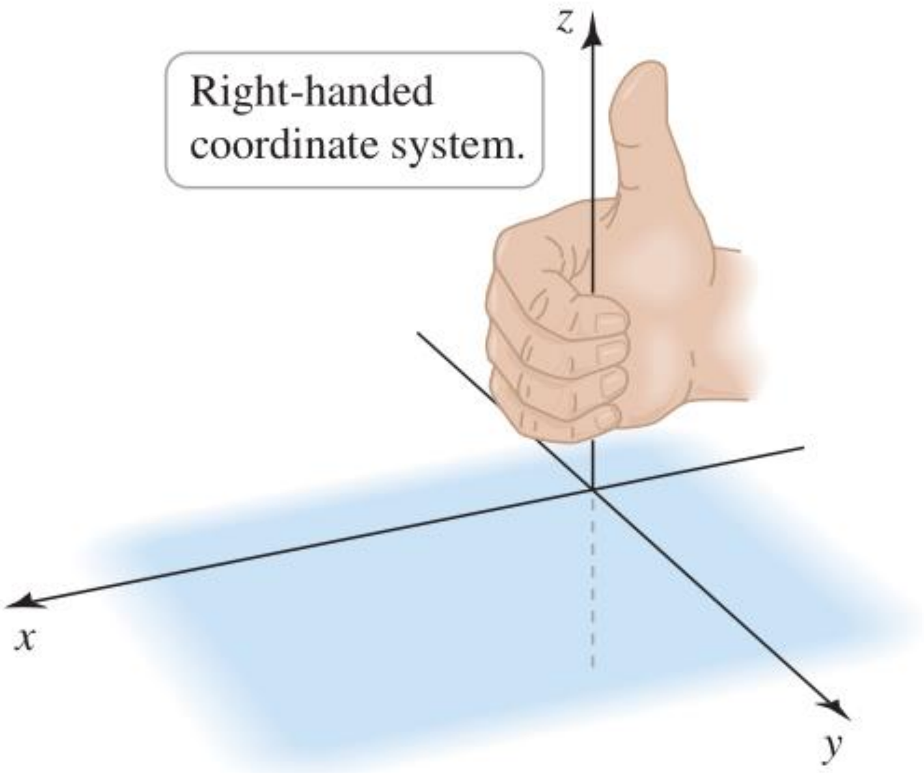
13.2

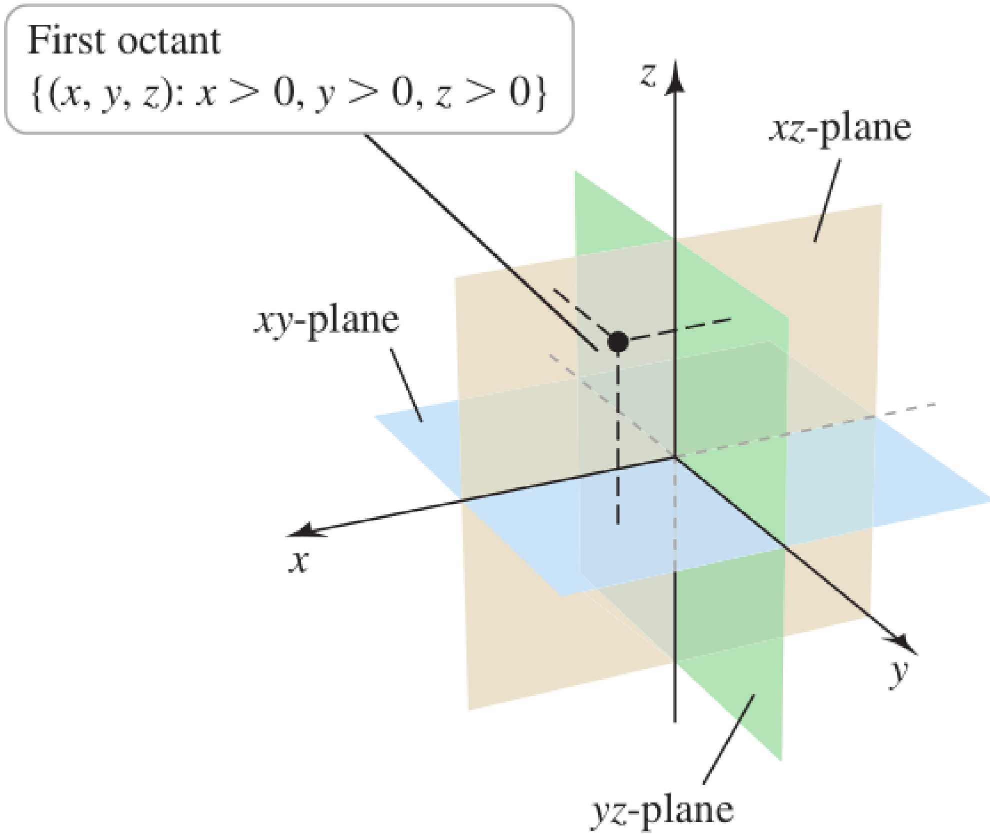
Vectors in Three Dimensions

The *xyz*-Coordinate System

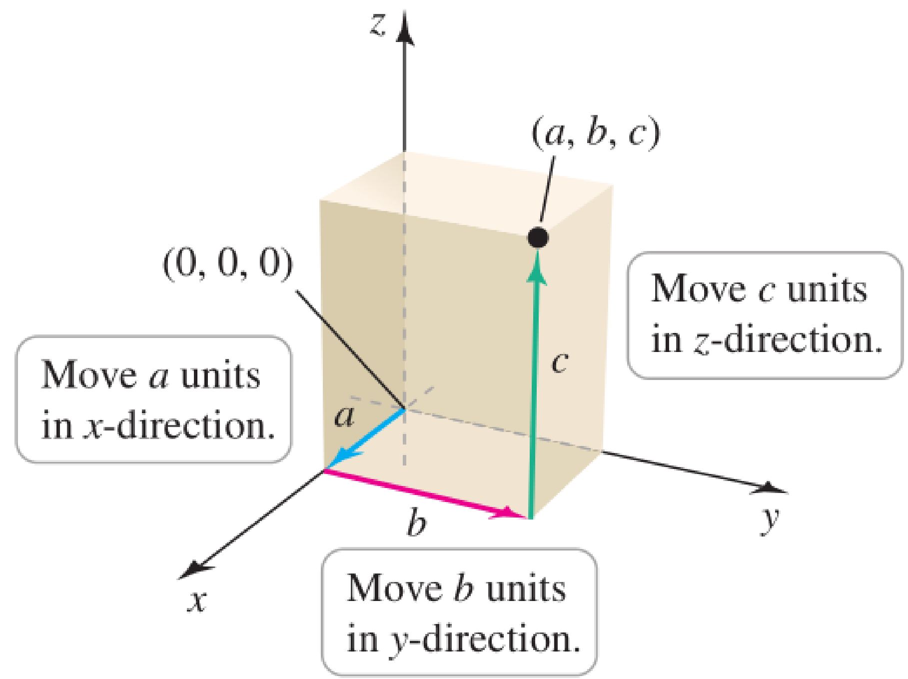
Three-dimensional rectangular coordinate system or the *xyz*-coordinate system.

Right-handed coordinate system:



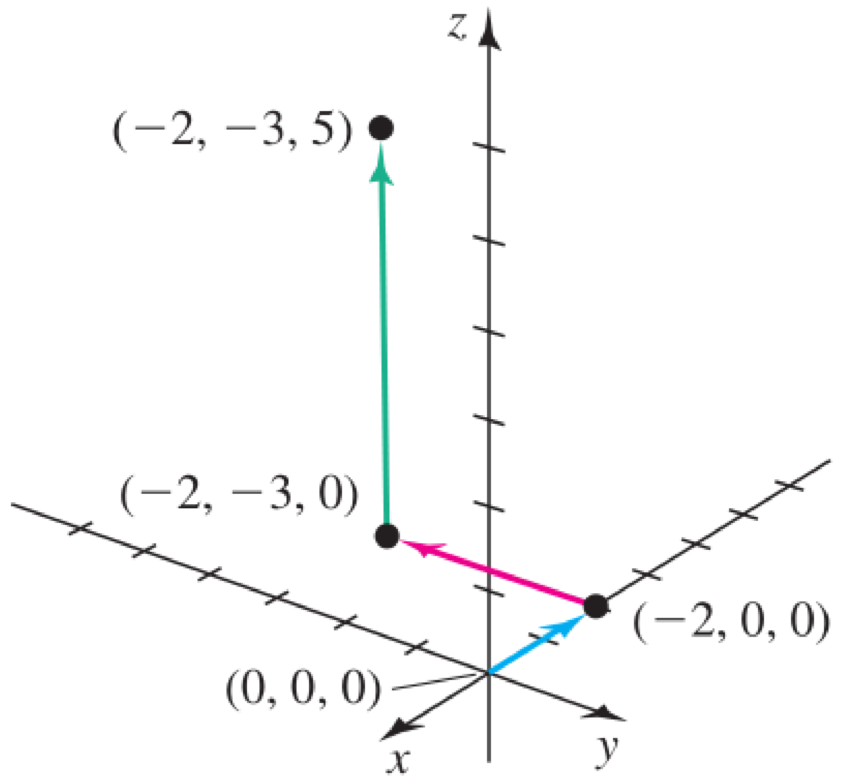


xyz-space is divided into octants.



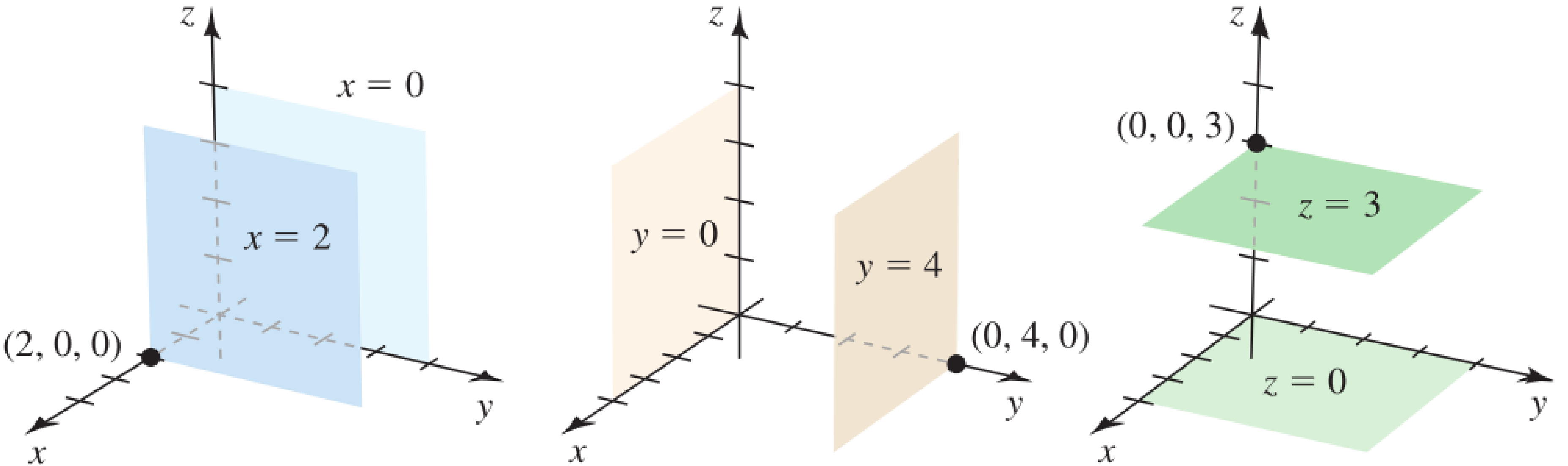
EXAMPLE 1 **Plotting points in *xyz*-space** Plot the following points.

a. $(3, 4, 5)$ **b.** $(-2, -3, 5)$

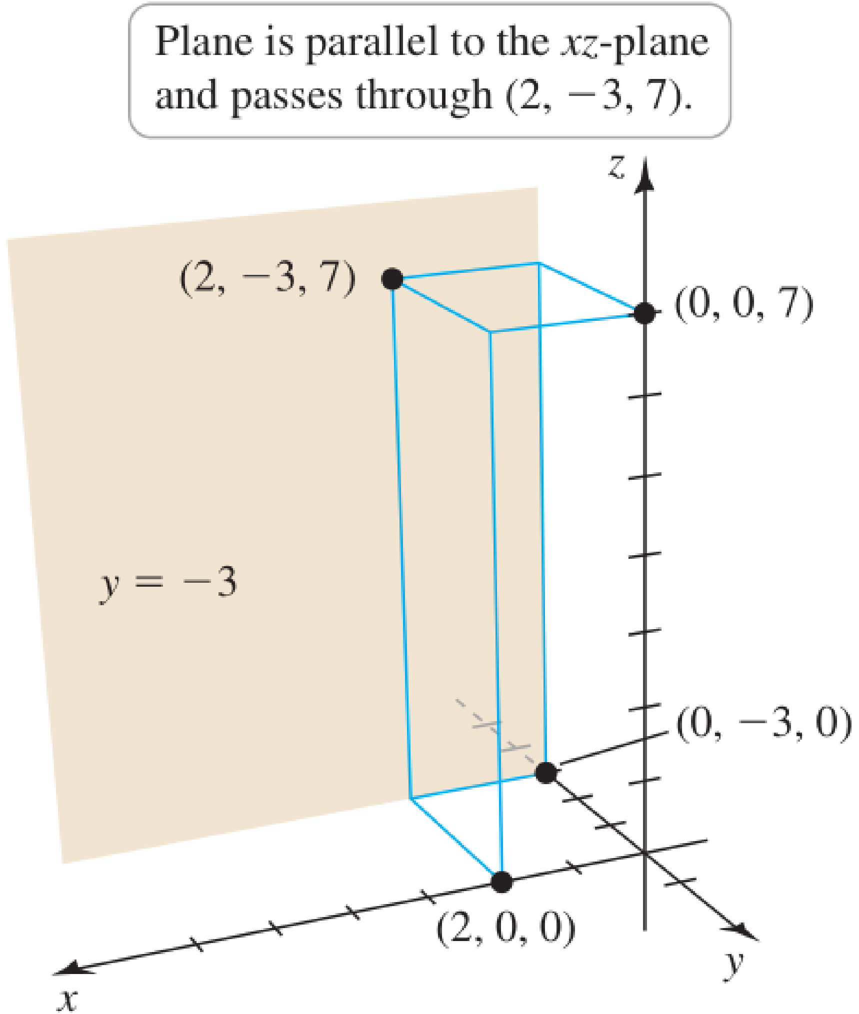


Plotting $(-2, -3, 5)$

Equations of Simple Planes



EXAMPLE 2 **Parallel planes** Determine the equation of the plane parallel to the xz-plane passing through the point $(2, -3, 7)$.

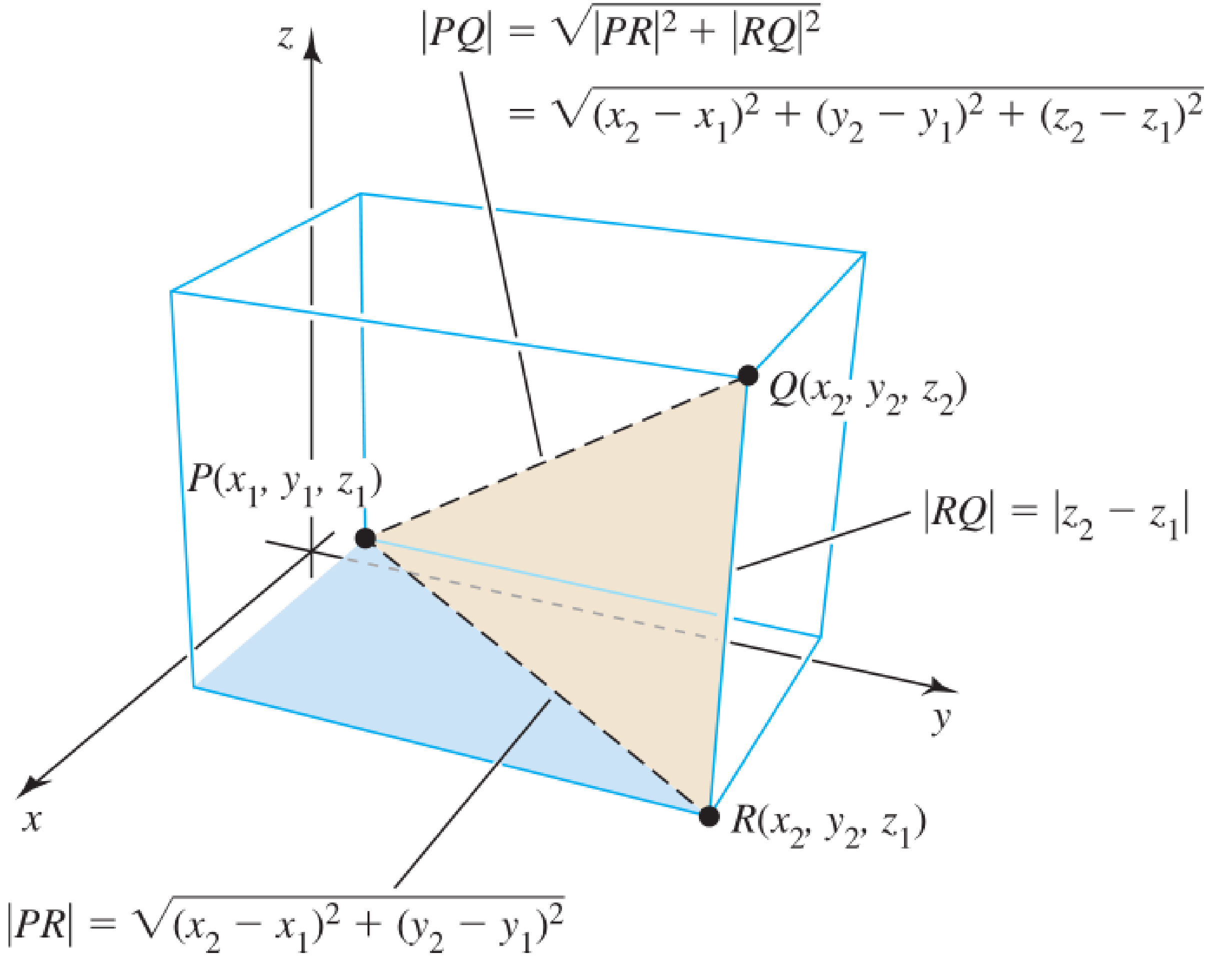


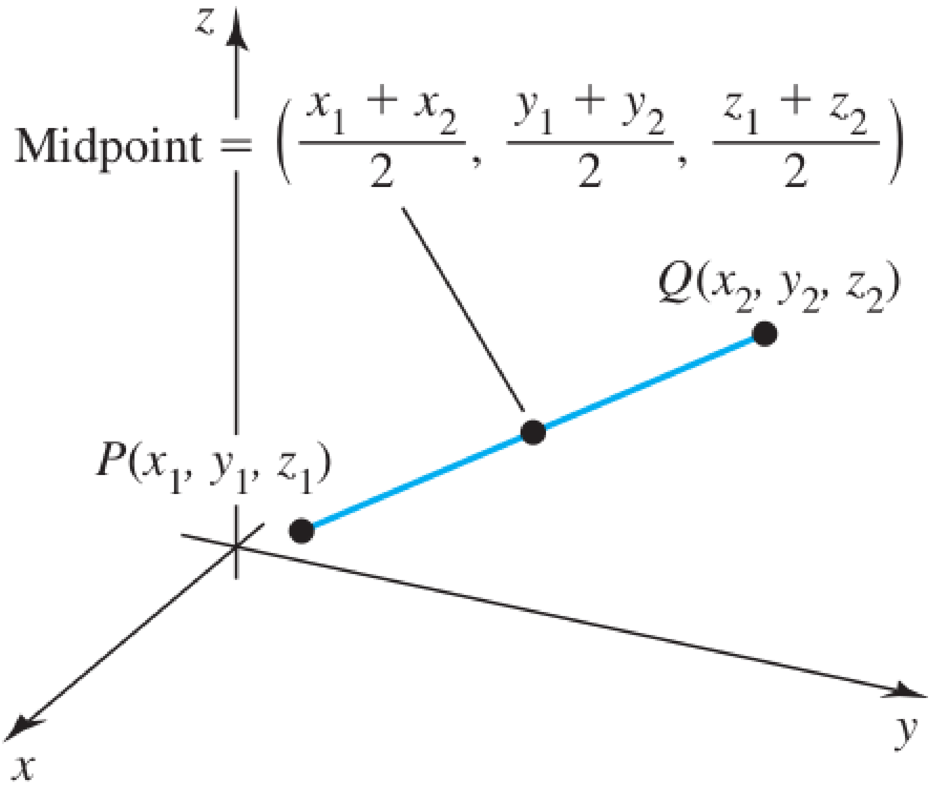
Distances in *xyz*-Space

Distance Formula in *xyz*-Space

The distance between the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$


$|PQ| = \sqrt{|PR|^2 + |RQ|^2}$
$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$
z
$Q(x_2, y_2, z_2)$
$P(x_1, y_1, z_1)$
$|RQ| = |z_2 - z_1|$
y
x
$R(x_2, y_2, z_1)$
$|PR| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$


z
Midpoint = $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$
$Q(x_2, y_2, z_2)$
$P(x_1, y_1, z_1)$
y
x

Equation of a Sphere

Spheres and Balls

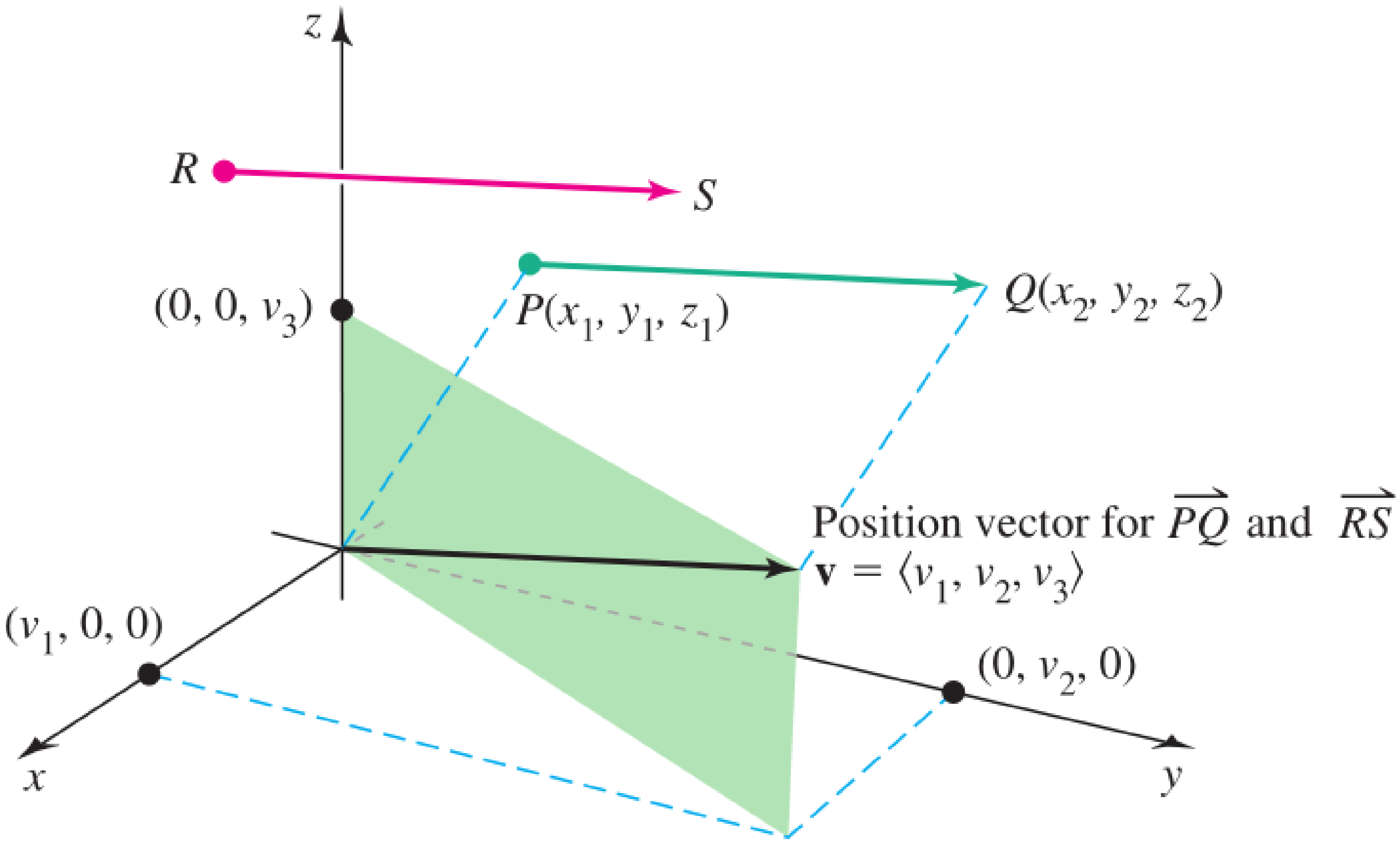
A **sphere** centered at (a, b, c) with radius r is the set of points satisfying the equation

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2.$$

A **ball** centered at (a, b, c) with radius r is the set of points satisfying the inequality

$$(x-a)^2+(y-b)^2+(z-c)^2\le r^2.$$

Vectors in $\mathbb{R}^3$



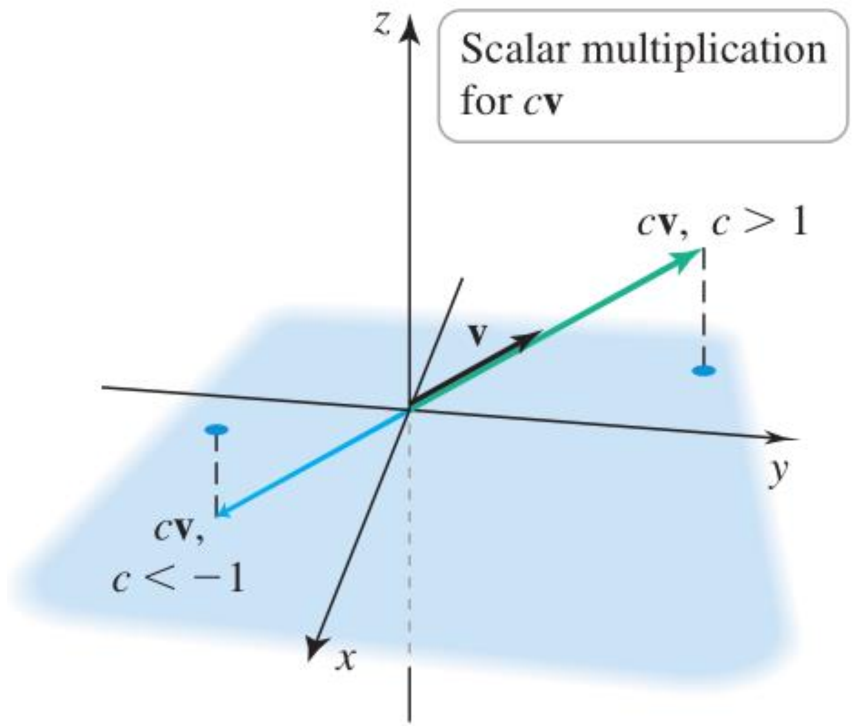
DEFINITION **Vector Operations in $\mathbb{R}^3$**

Let c be a scalar, $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$, and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$.

$$\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle \quad \text{Vector addition}$$

$$\mathbf{u} - \mathbf{v} = \langle u_1 - v_1, u_2 - v_2, u_3 - v_3 \rangle \quad \text{Vector subtraction}$$

$$c\mathbf{u} = \langle cu_1, cu_2, cu_3 \rangle \quad \text{Scalar multiplication}$$



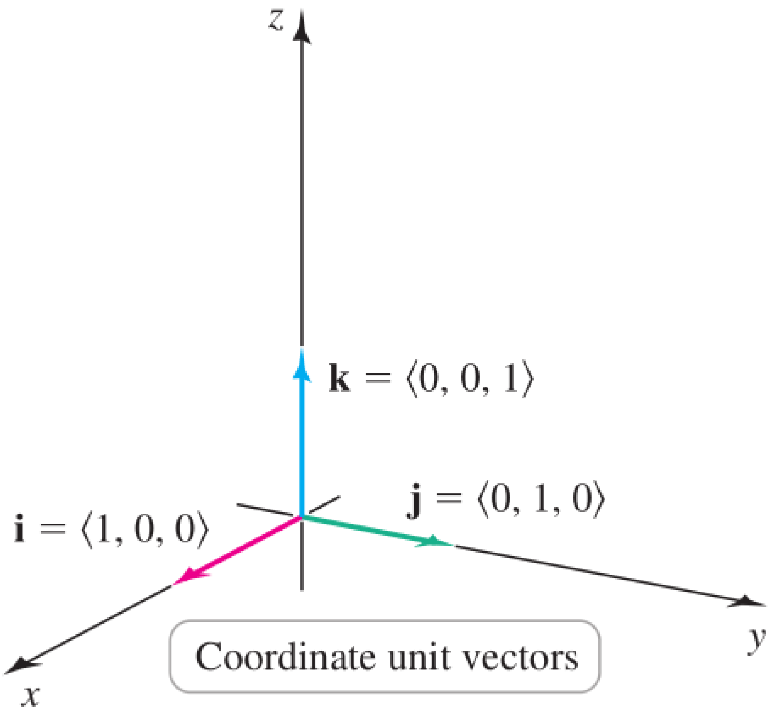
Magnitude and Unit Vectors

DEFINITION Magnitude of a Vector

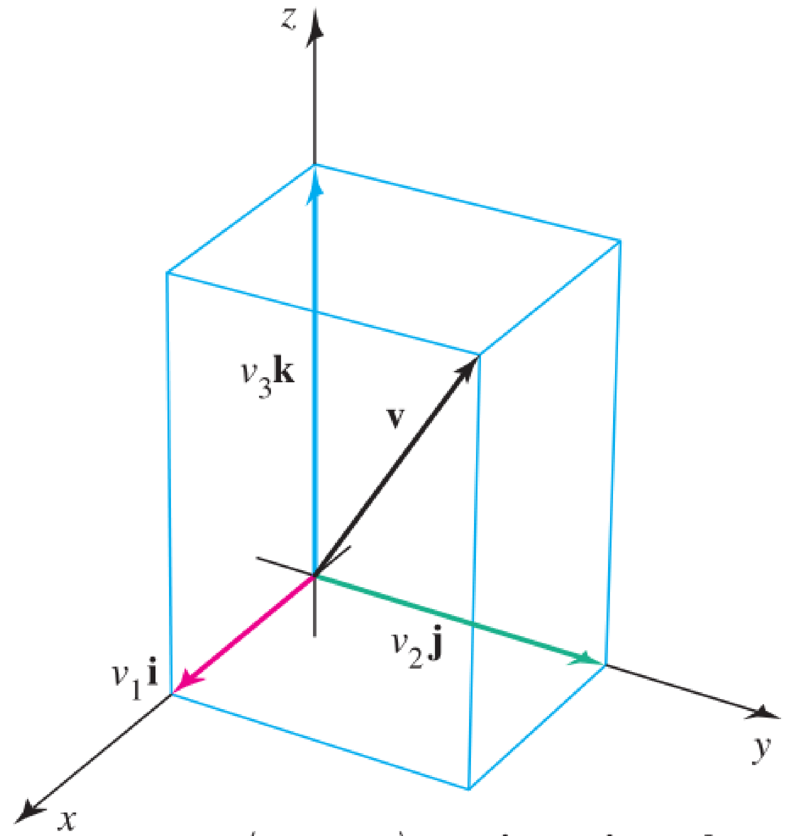
The **magnitude** (or **length**) of the vector $\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$ is the distance from $P(x_1, y_1, z_1)$ to $Q(x_2, y_2, z_2)$:

$$|\overrightarrow{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Coordinate unit vectors $\boldsymbol{i} = \langle 1,0,0 \rangle$, $\boldsymbol{j} = \langle 0,1,0 \rangle$ and $\boldsymbol{k} = \langle 0,0,1 \rangle$



Coordinate unit vectors



$\mathbf{v} = \langle v_1, v_2, v_3 \rangle = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$

In general, $\boldsymbol{v} = \langle v_1, v_2, v_3 \rangle = v_1\boldsymbol{i} + v_2\boldsymbol{j} + v_3\boldsymbol{k}$

EXAMPLE 6 **Magnitudes and unit vectors** Consider the points $P(5, 3, 1)$ and $Q(-7, 8, 1)$.

a. Express $\overrightarrow{PQ}$ in terms of the unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

b. Find the magnitude of $\overrightarrow{PQ}$.

c. Find the position vector of magnitude 10 in the direction of $\overrightarrow{PQ}$.

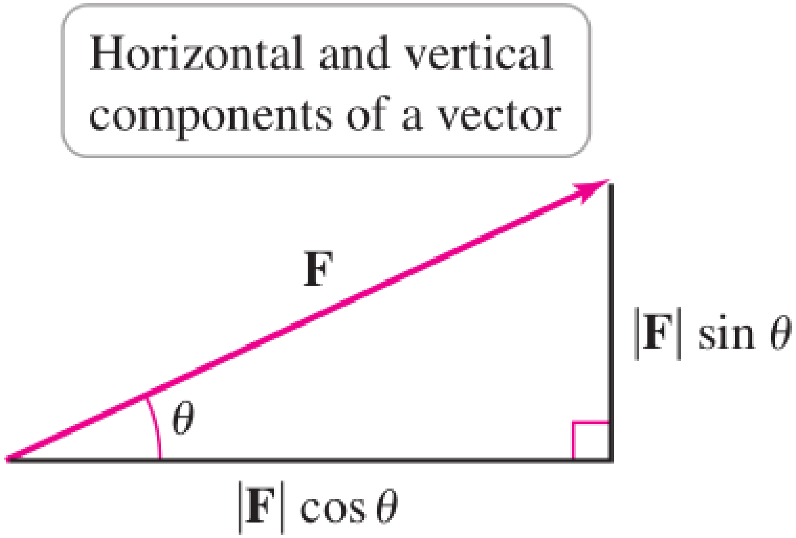
EXAMPLE 7 **Flight in crosswinds** A plane is flying horizontally due north in calm air at 300 mi/hr when it encounters a horizontal crosswind blowing southeast at 40 mi/hr and a downdraft blowing vertically downward at 30 mi/hr. What are the resulting speed and direction of the plane relative to the ground?

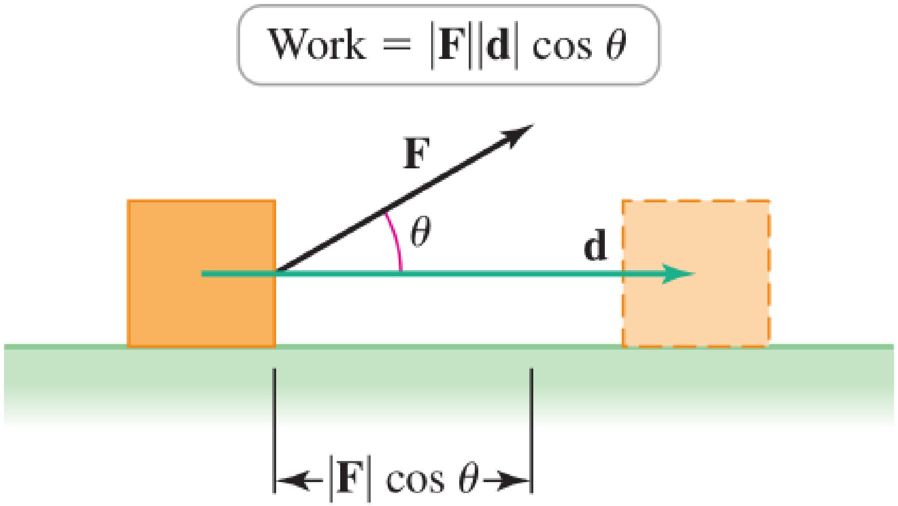
13.3

Dot Products

Two Forms of the Dot Product

Dot product, to determine the *angle* between two vectors, and calculate *projections*.





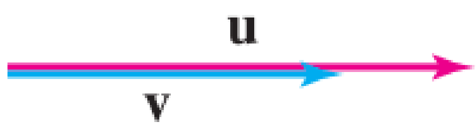
DEFINITION Dot Product

Given two nonzero vectors **u** and **v** in two or three dimensions, their **dot product** is

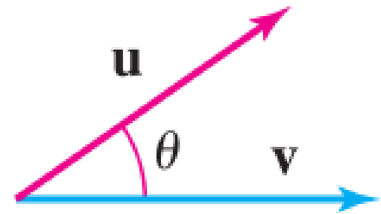
$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos \theta,$$

where θ is the angle between **u** and **v** with $0 \leq \theta \leq \pi$ (Figure 44). If $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$, then $\mathbf{u} \cdot \mathbf{v} = 0$, and θ is undefined.

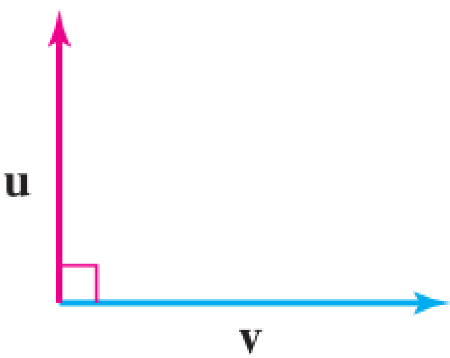
Cauchy–Schwarz Inequality *The definition* $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos \theta$ *implies that* $|\mathbf{u} \cdot \mathbf{v}| \leq |\mathbf{u}||\mathbf{v}|$ (*because* $|\cos \theta| \leq 1$). *This inequality, known as the Cauchy–Schwarz Inequality, holds in any number of dimensions and has many consequences.*



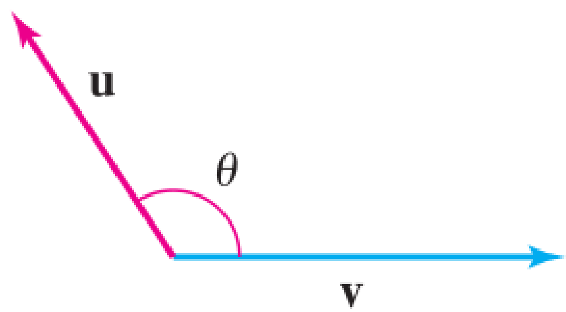
$$\theta = 0, \;\; \mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}|$$



$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos \theta > 0$$



$$\theta = \frac{\pi}{2}, \;\; \mathbf{u} \cdot \mathbf{v} = 0$$



$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos \theta < 0$$

$$\theta = \pi, \;\; \mathbf{u} \cdot \mathbf{v} = -|\mathbf{u}||\mathbf{v}|$$

Two special cases:

- $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel ($\theta = 0$ or $\theta = \pi$) if and only if $\boldsymbol{u} \cdot \boldsymbol{v} = \pm\, |\boldsymbol{u}||\,\boldsymbol{v}\,|$
- $\boldsymbol{u}$ and $\boldsymbol{v}$ are perpendicular ($\theta = \pi/2$) if and only if $\boldsymbol{u} \cdot \boldsymbol{v} = 0$

DEFINITION Orthogonal Vectors

Two vectors $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal** if and only if $\mathbf{u} \cdot \mathbf{v} = 0$. The zero vector is orthogonal to all vectors. In two or three dimensions, two nonzero orthogonal vectors are perpendicular to each other.

EXAMPLE 1 **Dot products** Compute the dot products of the following vectors.

a. $\mathbf{u} = 2\mathbf{i} - 6\mathbf{j}$ and $\mathbf{v} = 12\mathbf{k}$

b. $\mathbf{u} = \langle \sqrt{3}, 1 \rangle$ and $\mathbf{v} = \langle 0, 1 \rangle$

THEOREM 1 Dot Product

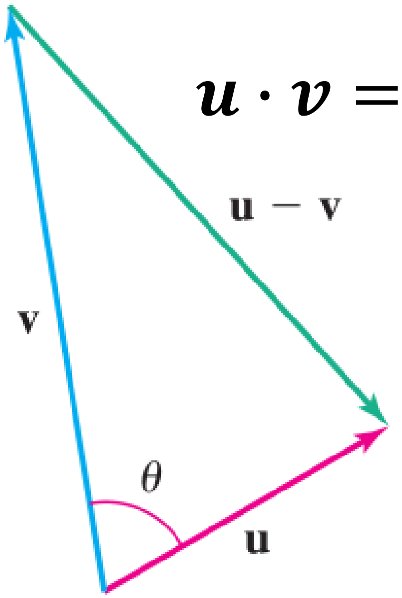
Given two vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$,

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

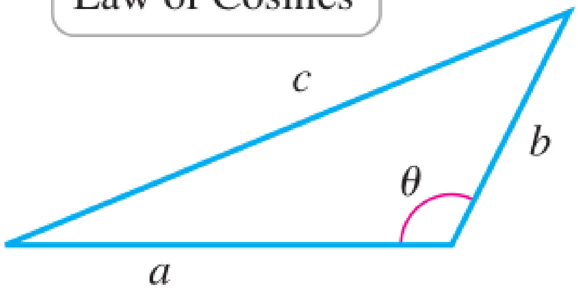
Proof: Apply the Law of Cosines to $\boldsymbol{u}$ and $\boldsymbol{v}$, where θ is the angle between them.

$$|\boldsymbol{u} - \boldsymbol{v}|^2 = |\boldsymbol{u}|^2 + |\boldsymbol{v}|^2 - 2|\boldsymbol{u}||\boldsymbol{v}| \cos\theta$$

$$\boldsymbol{u} \cdot \boldsymbol{v} = |\boldsymbol{u}||\boldsymbol{v}| \cos\theta = \frac{\mathbf{1}}{\mathbf{2}}\left(|\boldsymbol{u}|^2 + |\boldsymbol{v}|^2 - |\boldsymbol{u} - \boldsymbol{v}|^2\right)$$



Law of Cosines



$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

This new representation of $\boldsymbol{u} \cdot \boldsymbol{v}$ has two immediate consequences

1. Combining it with the definition of dot product gives

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 = |\boldsymbol{u}||\boldsymbol{v}| \cos\theta$$

A way to compute θ, given nonzero u and v:

$$\cos\theta = \frac{u_1 v_1 + u_2 v_2 + u_3 v_3}{|\boldsymbol{u}||\boldsymbol{v}|} = \frac{\boldsymbol{u} \cdot \boldsymbol{v}}{|\boldsymbol{u}||\boldsymbol{v}|}$$

2. A relationship between the dot product and the magnitude of a vector:

$$|\boldsymbol{u}| = \sqrt{\boldsymbol{u} \cdot \boldsymbol{u}} \text{ or } |\boldsymbol{u}|^2 = \boldsymbol{u} \cdot \boldsymbol{u} = u_1^2 + u_2^2 + u_3^2$$

EXAMPLE 2 **Dot products and angles** Let $\mathbf{u} = \langle \sqrt{3}, 1, 0 \rangle$, $\mathbf{v} = \langle 1, \sqrt{3}, 0 \rangle$, and $\mathbf{w} = \langle 1, \sqrt{3}, 2\sqrt{3} \rangle$.

a. Compute $\mathbf{u} \cdot \mathbf{v}$.

b. Find the angle between $\mathbf{u}$ and $\mathbf{v}$.

c. Find the angle between $\mathbf{u}$ and $\mathbf{w}$.

Based on Cauchy-Schwarz Inequality, we have

Triangle Inequality Given vectors $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{u}+\boldsymbol{v}$,

$$|\boldsymbol{u}+\boldsymbol{v}| \leq |\boldsymbol{u}| + |\boldsymbol{v}|$$

Algebra Inequality

$$(u_1+u_2+u_3)^2 \leq 3(u_1^2+u_2^2+u_3^2)$$

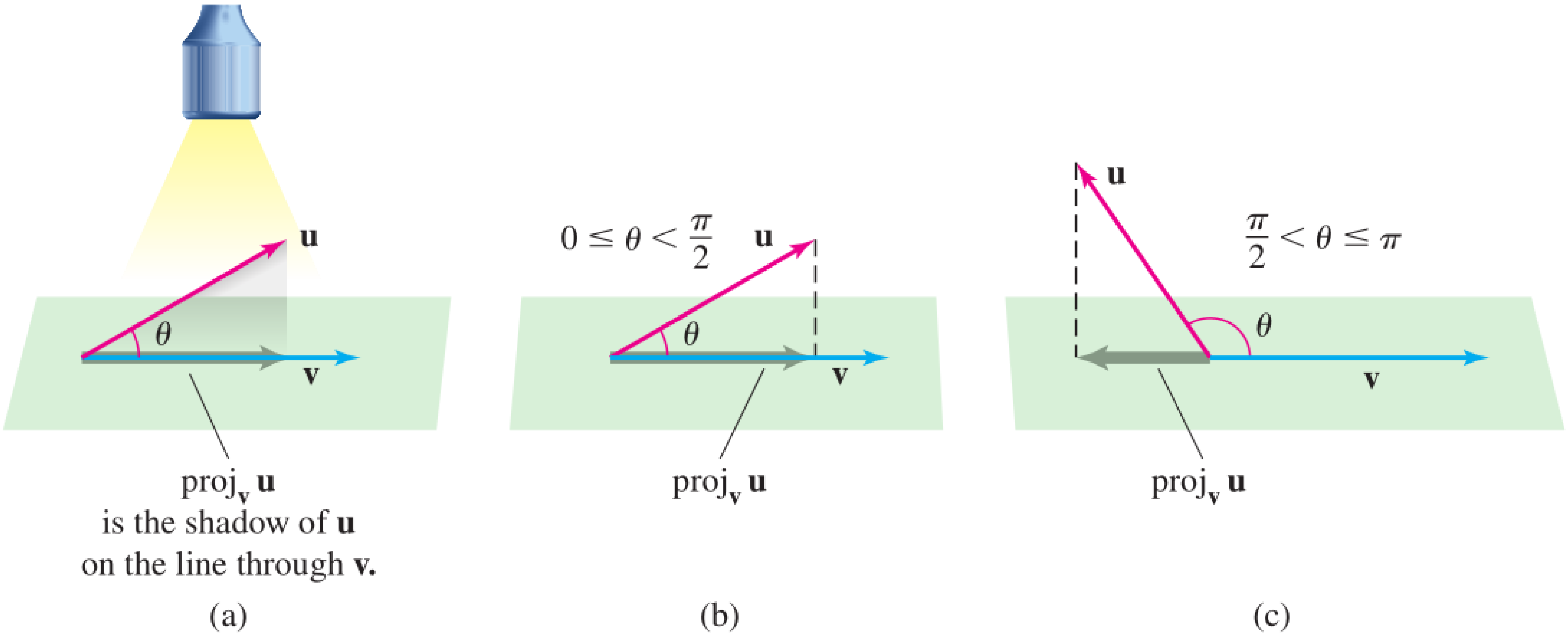
Properties of Dot Product

THEOREM 2 Properties of the Dot Product

Suppose $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors and let c be a scalar.

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ — Commutative property
2. $c(\mathbf{u} \cdot \mathbf{v}) = (c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v})$ — Associative property
3. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ — Distributive property

Orthogonal Projections



The vector $\text{proj}_{\boldsymbol{v}}\boldsymbol{u}$, is the *orthogonal projection* of $\boldsymbol{u}$ onto $\boldsymbol{v}$

1. If $0 \le \theta < \pi/2$, then $\text{proj}_{\boldsymbol{v}}\boldsymbol{u}$ has length $|\boldsymbol{u}|\cos\theta$ and points in the direction of the unit vector $\boldsymbol{v}/|\boldsymbol{v}|$. Therefore

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \underbrace{|\mathbf{u}|\cos\theta}_{\text{length}} \underbrace{\left(\frac{\mathbf{v}}{|\mathbf{v}|}\right)}_{\text{direction}}.$$

2. If $\pi/2 < \theta \le \pi$, then $\text{proj}_{\boldsymbol{v}}\boldsymbol{u}$ has length $-|\boldsymbol{u}|\cos\theta$ and points in the direction of $-\boldsymbol{v}/|\boldsymbol{v}|$. Therefore

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \underbrace{-|\mathbf{u}|\cos\theta}_{\text{length}} \underbrace{\left(-\frac{\mathbf{v}}{|\mathbf{v}|}\right)}_{\text{direction}} = |\mathbf{u}|\cos\theta\left(\frac{\mathbf{v}}{|\mathbf{v}|}\right)$$

DEFINITION **(Orthogonal) Projection of u onto v**

The **orthogonal projection of u onto v**, denoted $\text{proj}_{\mathbf{v}}\mathbf{u}$, where $\mathbf{v} \neq \mathbf{0}$, is

$$\text{proj}_{\mathbf{v}}\mathbf{u} = |\mathbf{u}| \cos\theta \left(\frac{\mathbf{v}}{|\mathbf{v}|}\right).$$

The orthogonal projection may also be computed with the formulas

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \text{scal}_{\mathbf{v}}\mathbf{u}\left(\frac{\mathbf{v}}{|\mathbf{v}|}\right) = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v},$$

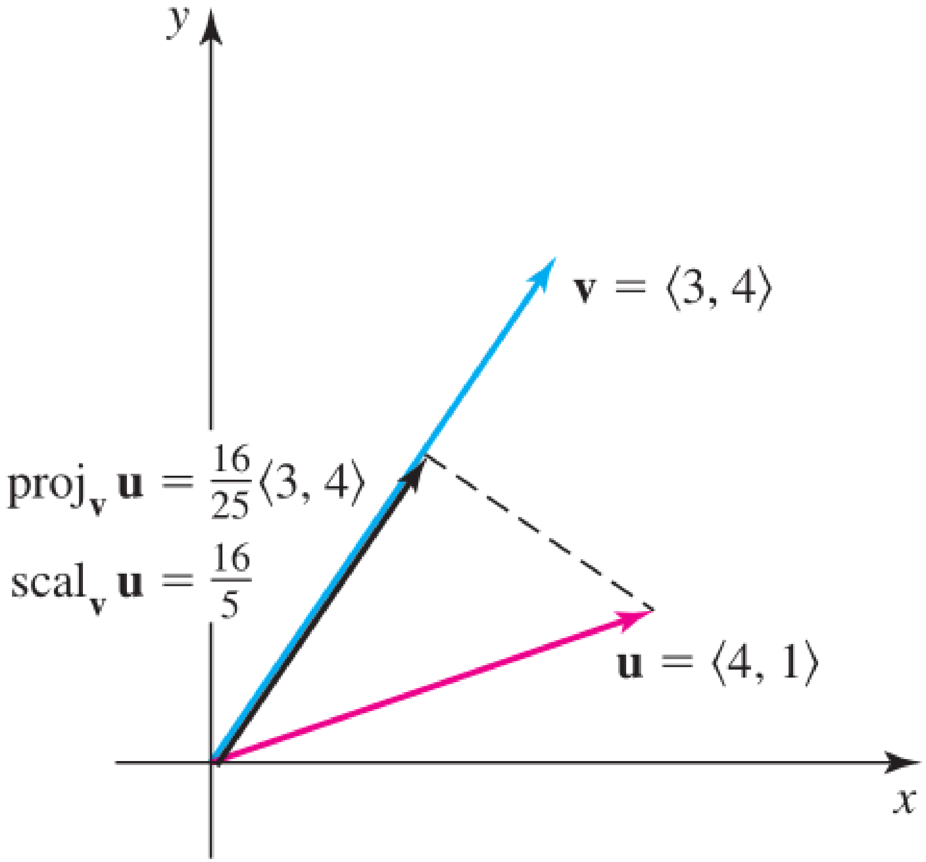
where the **scalar component of u in the direction of v** is

$$\text{scal}_{\mathbf{v}}\mathbf{u} = |\mathbf{u}| \cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|}.$$

EXAMPLE 3 **Orthogonal projections** Find $\text{proj}_{\mathbf{v}}\mathbf{u}$ and $\text{scal}_{\mathbf{v}}\mathbf{u}$ for the following vectors and illustrate each result.

a. $\mathbf{u} = \langle 4, 1 \rangle, \mathbf{v} = \langle 3, 4 \rangle$

b. $\mathbf{u} = \langle -4, -3 \rangle, \mathbf{v} = \langle 1, -1 \rangle$



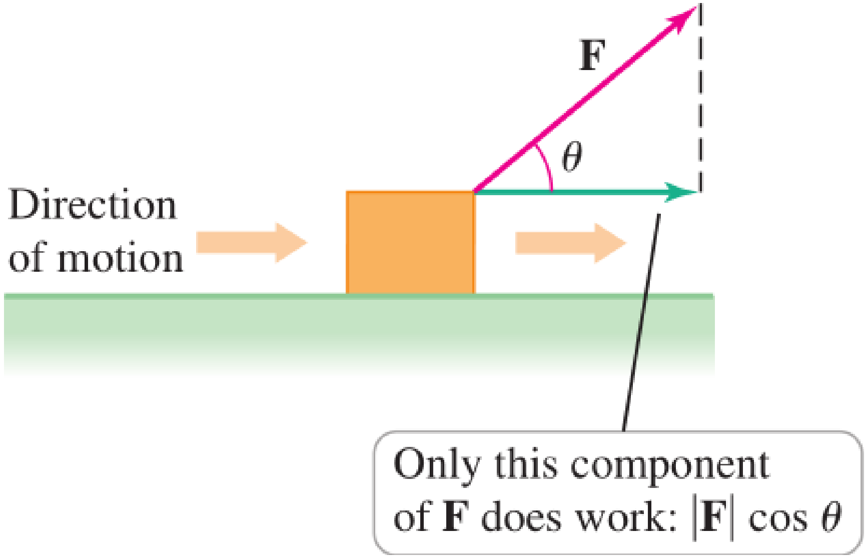
Applications of Dot Products

Work and force

DEFINITION Work

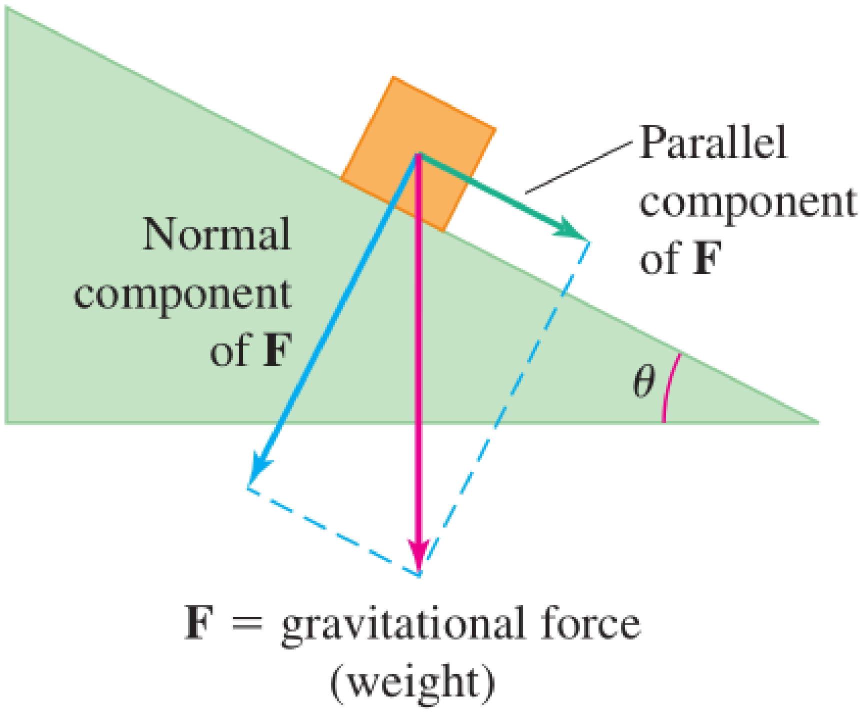
Let a constant force **F** be applied to an object, producing a displacement **d**. If the angle between **F** and **d** is θ, then the **work** done by the force is

$$W = |\mathbf{F}||\mathbf{d}|\cos\theta = \mathbf{F}\cdot\mathbf{d}.$$

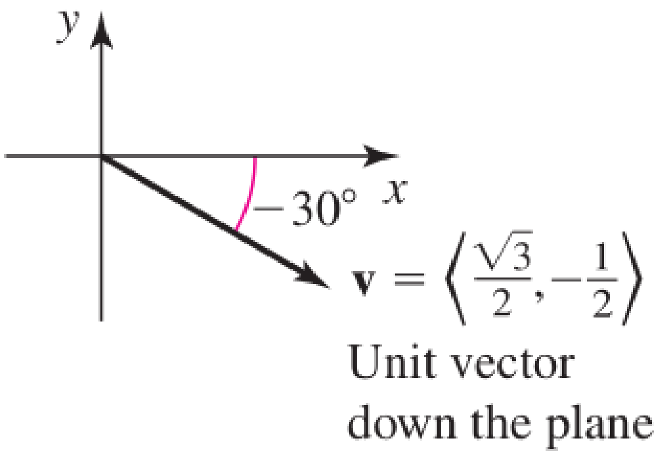


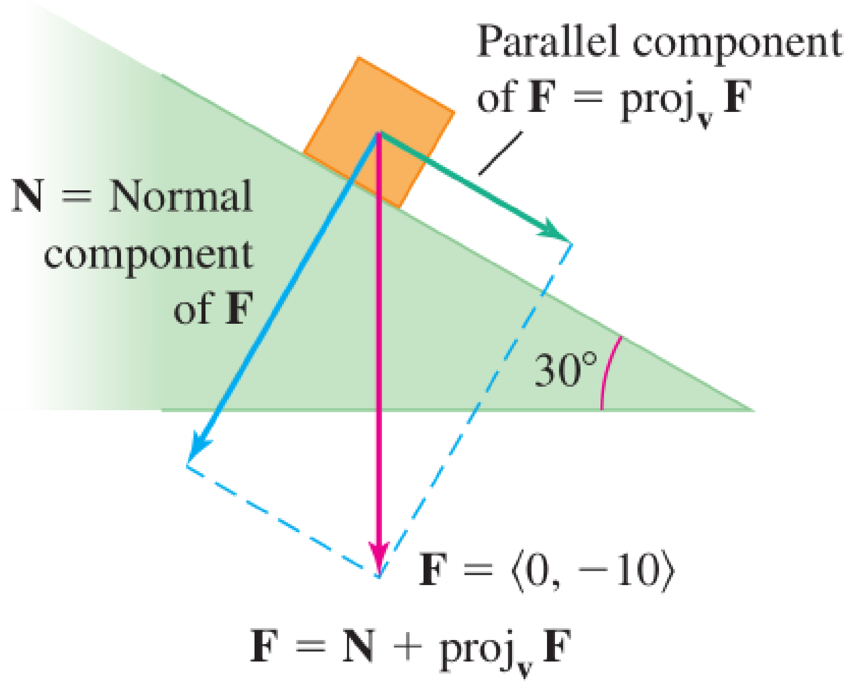
EXAMPLE 4 **Calculating work** A force $\mathbf{F} = \langle 3, 3, 2 \rangle$ (in newtons) moves an object along a line segment from $P(1, 1, 0)$ to $Q(6, 6, 0)$ (in meters). What is the work done by the force? Interpret the result.

Parallel and normal forces



EXAMPLE 5 **Components of a force** A 10-lb block rests on a plane that is inclined at 30° below the horizontal. Find the components of the gravitational force parallel and normal (perpendicular) to the plane.



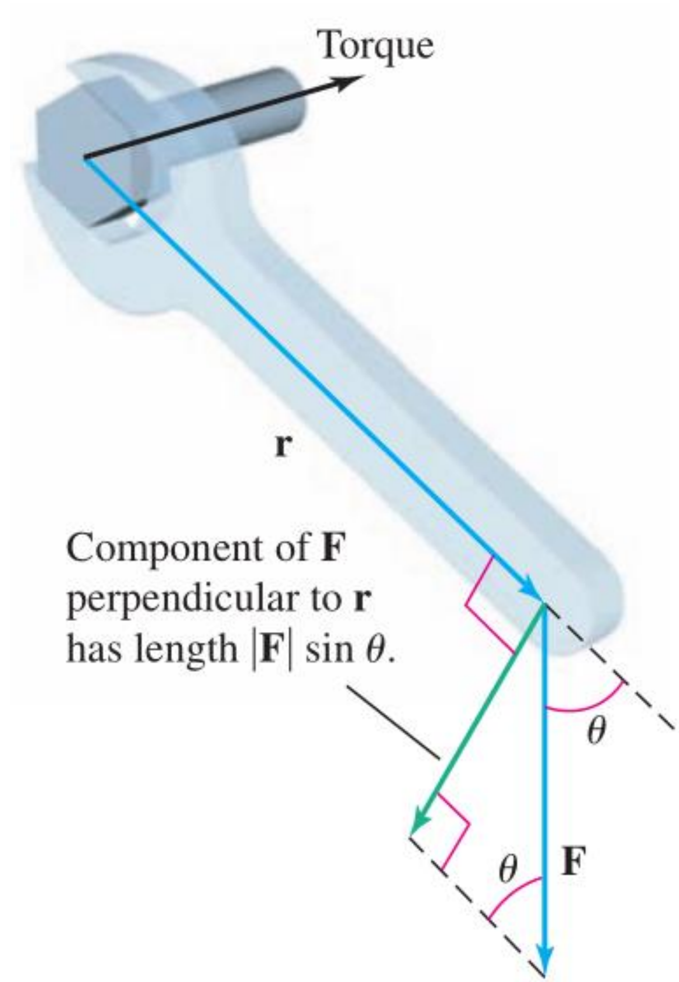


13.4

Cross Products

The Cross Product

Cross product (or **vector product**), combines two vectors in $\mathbb{R}^3$ to produce a vector result.

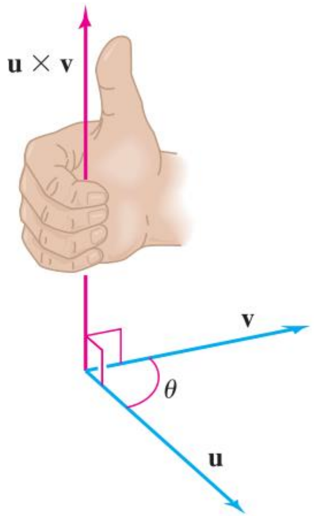


DEFINITION Cross Product

Given two nonzero vectors **u** and **v** in $\mathbb{R}^3$, the **cross product u $\times$ v** is a vector with magnitude

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin \theta,$$

where $0 \leq \theta \leq \pi$ is the angle between **u** and **v**. The direction of **u** $\times$ **v** is given by the **right-hand rule**: When you put the vectors tail to tail and let the fingers of your right hand curl from **u** to **v**, the direction of **u** $\times$ **v** is the direction of your thumb, orthogonal to both **u** and **v** (Figure 56). When $\mathbf{u} \times \mathbf{v} = \mathbf{0}$, the direction of **u** $\times$ **v** is undefined.



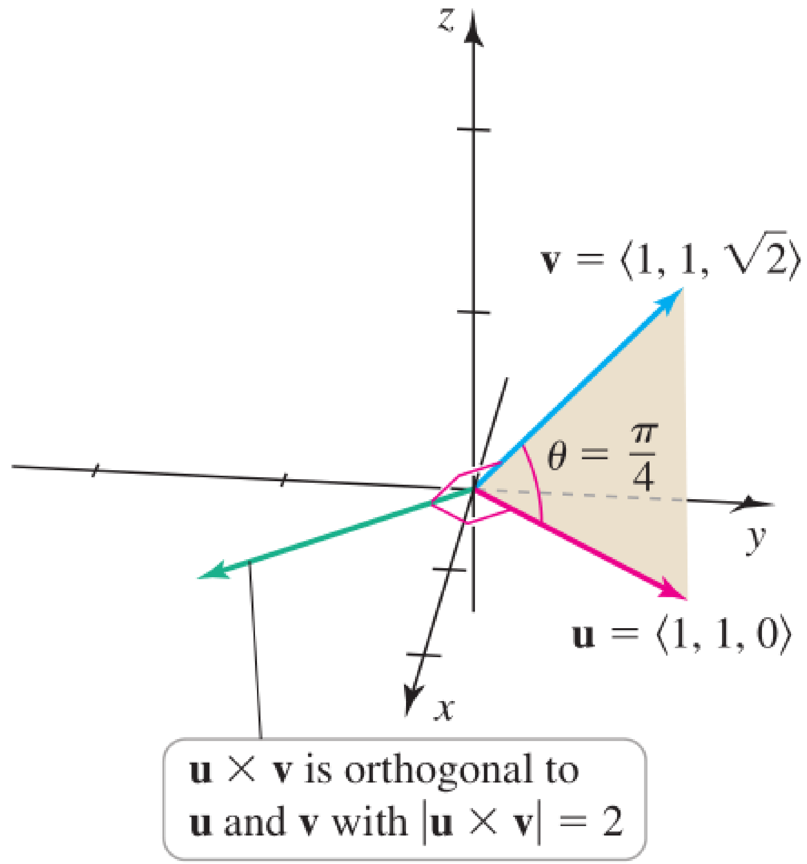
THEOREM 3 Geometry of the Cross Product

Let $\mathbf{u}$ and $\mathbf{v}$ be two nonzero vectors in $\mathbb{R}^3$.

1. The vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel ($\theta = 0$ or $\theta = \pi$) if and only if $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.
2. If $\mathbf{u}$ and $\mathbf{v}$ are two sides of a parallelogram (Figure 11.57), then the area of the parallelogram is

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin \theta.$$

EXAMPLE 1 **A cross product** Find the magnitude and direction of $\mathbf{u} \times \mathbf{v}$, where $\mathbf{u} = \langle 1, 1, 0 \rangle$ and $\mathbf{v} = \langle 1, 1, \sqrt{2} \rangle$.



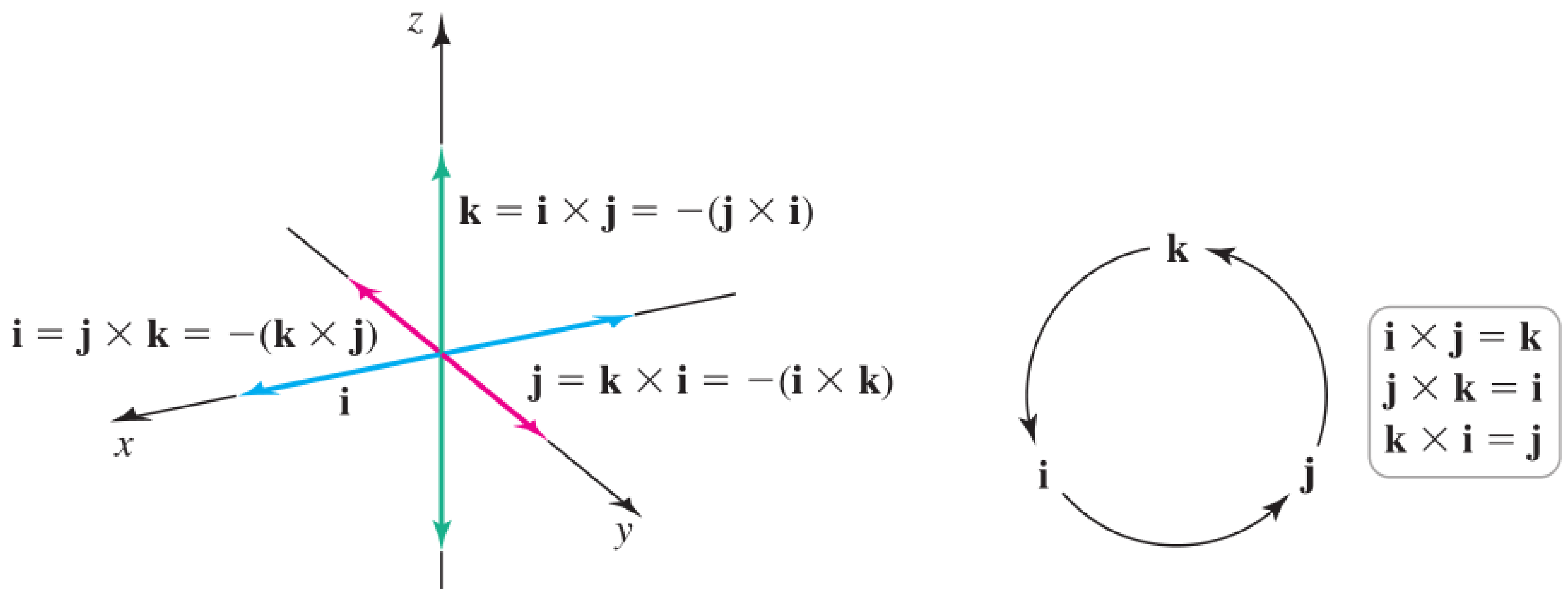
Properties of the Cross Product

THEOREM 4 Properties of the Cross Product

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be nonzero vectors in $\mathbb{R}^3$, and let a and b be scalars.

1. $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$ — Anticommutative property
2. $(a\mathbf{u}) \times (b\mathbf{v}) = ab(\mathbf{u} \times \mathbf{v})$ — Associative property
3. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$ — Distributive property
4. $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w})$ — Distributive property

EXAMPLE 2 **Cross products of unit vectors** Evaluate all the cross products among the coordinate unit vectors **i**, **j**, and **k**.



THEOREM 5 Cross Products of Coordinate Unit Vectors

$$\mathbf{i} \times \mathbf{j} = -(\mathbf{j} \times \mathbf{i}) = \mathbf{k} \qquad \mathbf{j} \times \mathbf{k} = -(\mathbf{k} \times \mathbf{j}) = \mathbf{i}$$

$$\mathbf{k} \times \mathbf{i} = -(\mathbf{i} \times \mathbf{k}) = \mathbf{j} \qquad \mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}$$

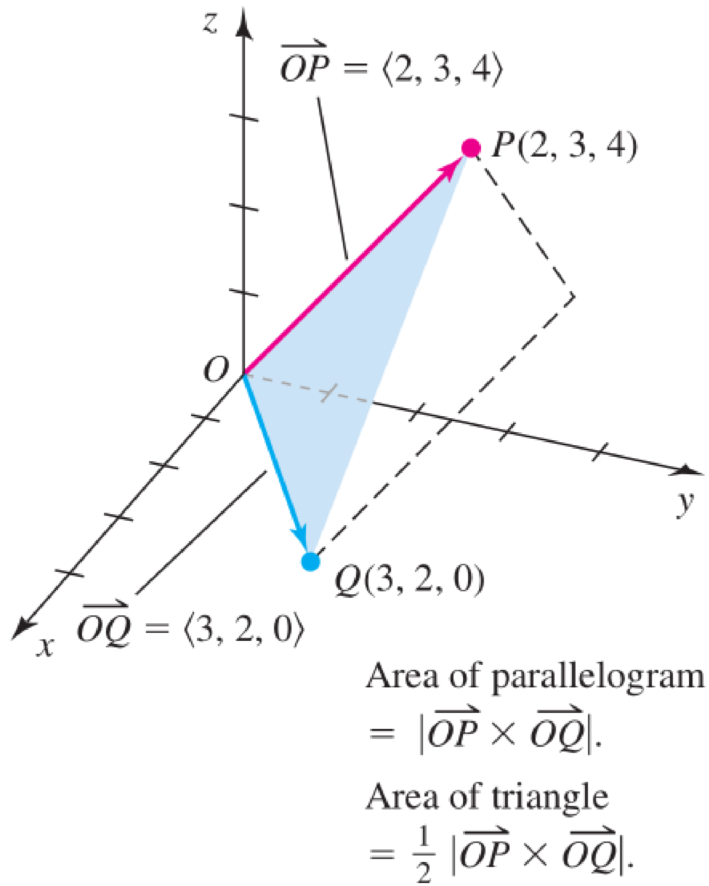
THEOREM 6 Evaluating the Cross Product

Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$. Then

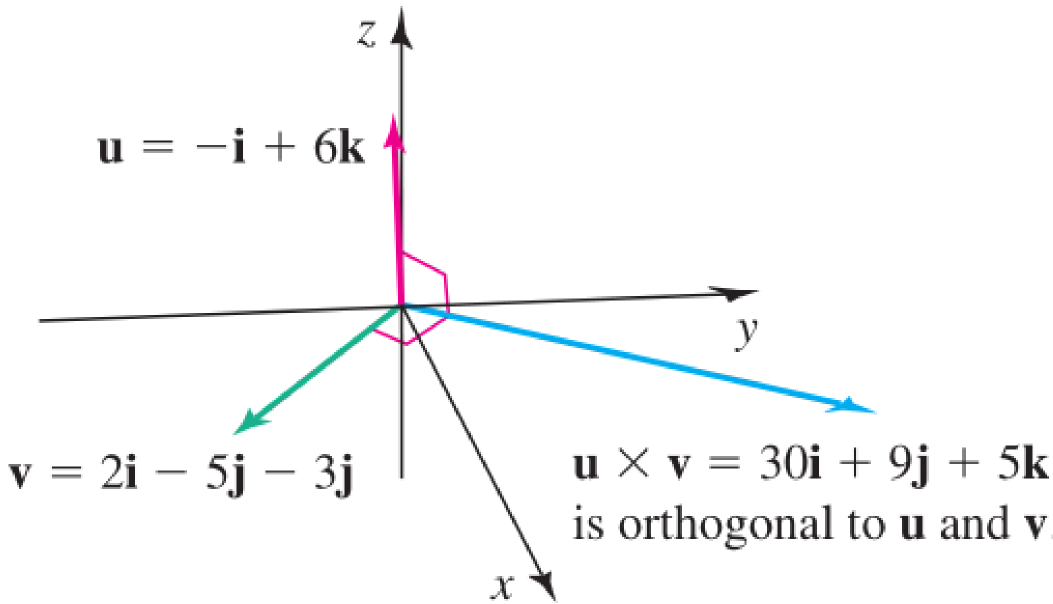
$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\mathbf{k}.$$

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) \\ &= u_1v_1\underbrace{(\mathbf{i} \times \mathbf{i})}_{\mathbf{0}} + u_1v_2\underbrace{(\mathbf{i} \times \mathbf{j})}_{\mathbf{k}} + u_1v_3\underbrace{(\mathbf{i} \times \mathbf{k})}_{-\mathbf{j}} \\ &\quad + u_2v_1\underbrace{(\mathbf{j} \times \mathbf{i})}_{-\mathbf{k}} + u_2v_2\underbrace{(\mathbf{j} \times \mathbf{j})}_{\mathbf{0}} + u_2v_3\underbrace{(\mathbf{j} \times \mathbf{k})}_{\mathbf{i}} \\ &\quad + u_3v_1\underbrace{(\mathbf{k} \times \mathbf{i})}_{\mathbf{j}} + u_3v_2\underbrace{(\mathbf{k} \times \mathbf{j})}_{-\mathbf{i}} + u_3v_3\underbrace{(\mathbf{k} \times \mathbf{k})}_{\mathbf{0}}. \end{aligned}$$

EXAMPLE 3 **Area of a triangle** Find the area of the triangle with vertices $O(0, 0, 0)$, $P(2, 3, 4)$, and $Q(3, 2, 0)$

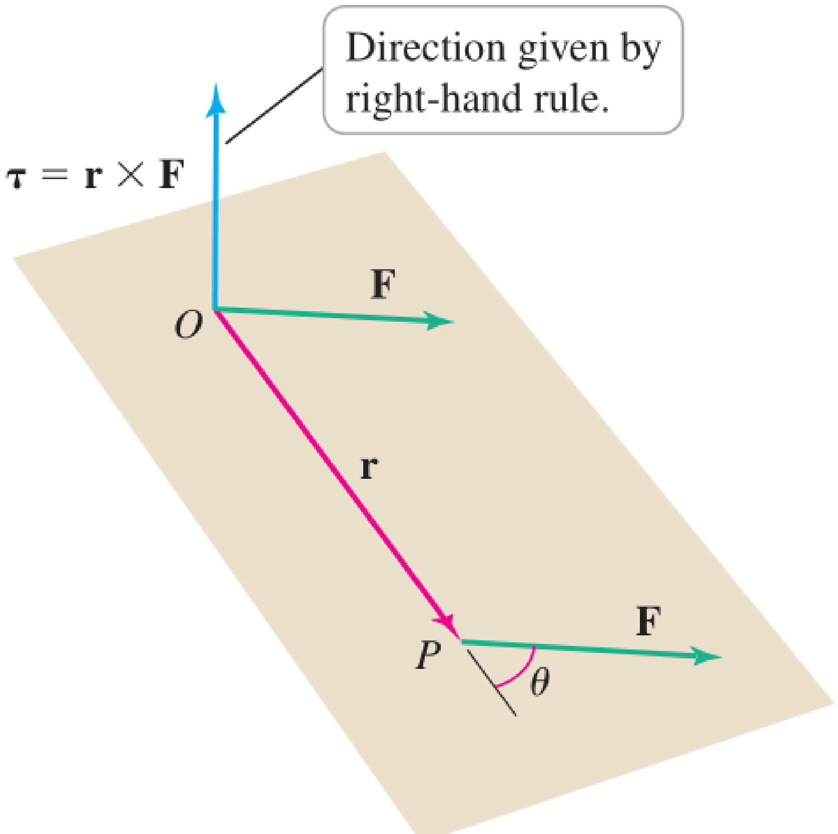


EXAMPLE 4 **Vector orthogonal to two vectors** Find a vector orthogonal to the two vectors $\mathbf{u} = -\mathbf{i} + 6\mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} - 5\mathbf{j} - 3\mathbf{k}$.



Applications of the Cross Product

Torque A force $\boldsymbol{F}$ is applied to the point P at the head of a vector $\boldsymbol{r} = \overrightarrow{\boldsymbol{OP}}$. The torque, or twisting effect, produced by the force about the point $\boldsymbol{O}$ is given by $\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F}$

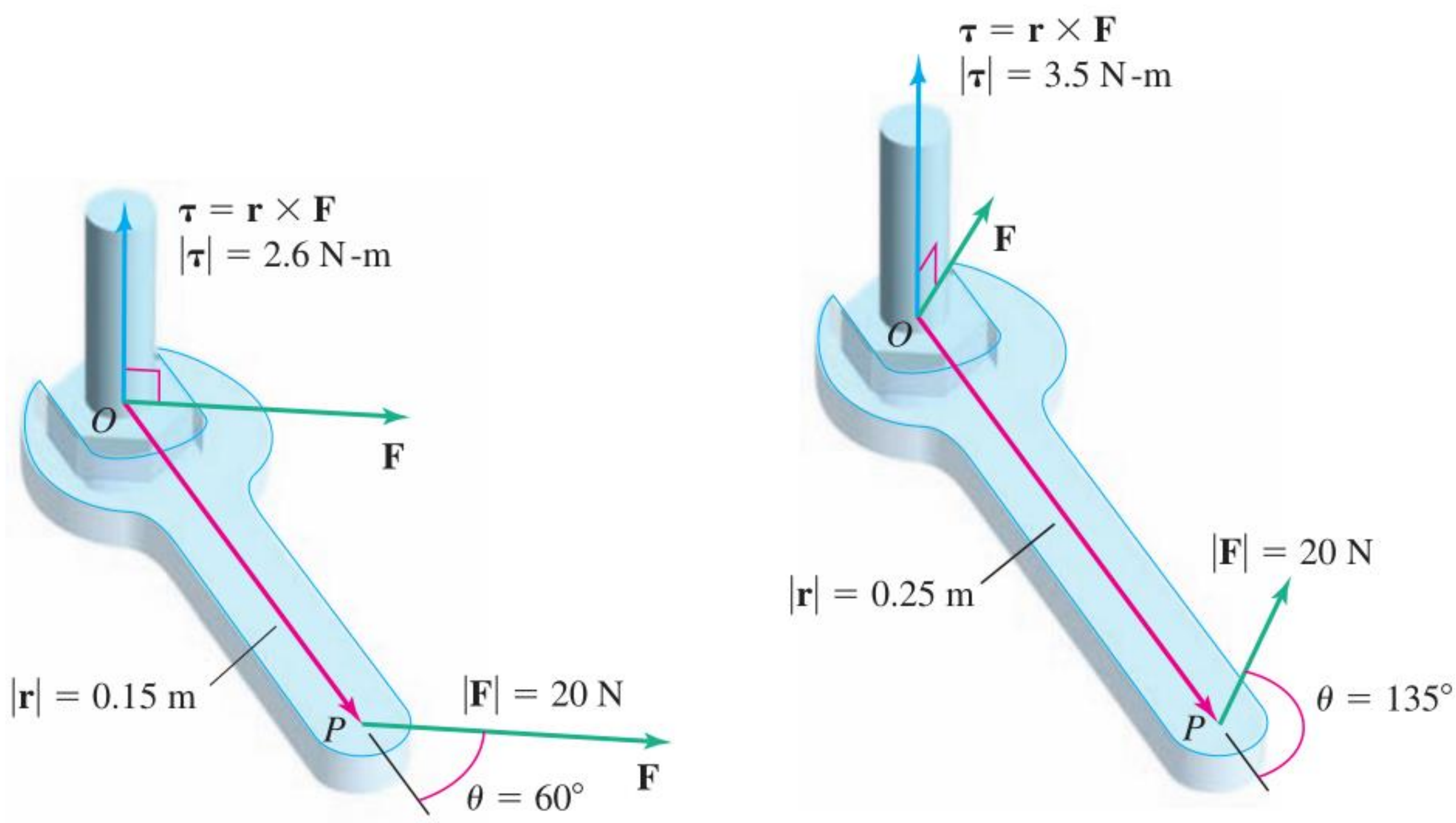


Magnitude of torque vector

$$|\boldsymbol{\tau}| = |\boldsymbol{r} \times \boldsymbol{F}| = |\boldsymbol{r}||\boldsymbol{F}| \sin\theta$$

Direction of the torque is given by the right-hand rule; Orthogonal to both $\boldsymbol{r}$ and $\boldsymbol{F}$

EXAMPLE 5 **Tightening a bolt** A force of 20 N is applied to a wrench attached to a bolt in a direction perpendicular to the bolt (Figure .63). Which produces more torque: applying the force at an angle of 60° on a wrench that is 0.15 m long or applying the force at an angle of 135° on a wrench that is 0.25 m long? In each case, what is the direction of the torque?



Magnetic force on a moving charge

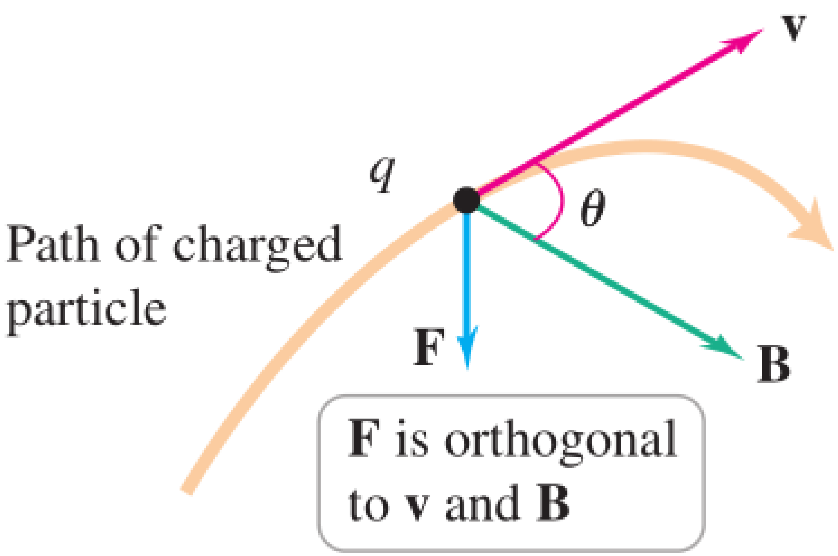
An isolated charge q, $\boldsymbol{v}$: its velocity, $\boldsymbol{B}$: the magnetic field

The force: $\boldsymbol{F} = q(\boldsymbol{v} \times \boldsymbol{B})$. The magnitude of the force is

$$|\boldsymbol{F}| = |q||\boldsymbol{v} \times \boldsymbol{B}| = |q||\boldsymbol{v}||\boldsymbol{B}| \sin\theta$$

θ: the angle between $\boldsymbol{v}$ and $\boldsymbol{B}$

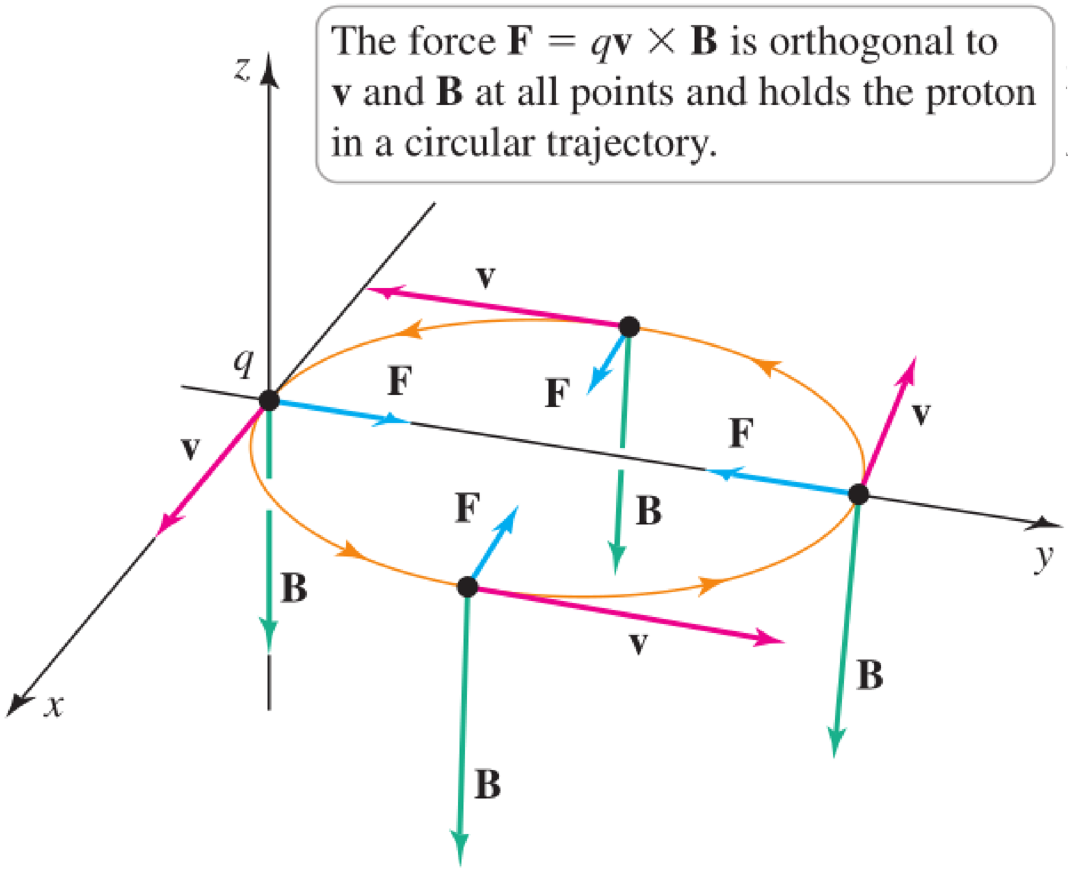
The sign of the charge determines the direction of the force.



EXAMPLE 6 **Force on a proton** A proton with a mass of 1.7×10^{-27} kg and a charge of $q = +1.6 \times 10^{-19}$ coulombs (C) moves along the x-axis with a speed of $|\mathbf{v}| = 9 \times 10^5$ m/s. When it reaches (0, 0, 0), a uniform magnetic field is turned on. The field has a constant strength of 1 tesla (1 T) and is directed along the negative z-axis

a. Find the magnitude and direction of the force on the proton at the instant it enters the magnetic field.

b. Assume that the … s as a *centripetal force* with magni… rcular orbit of radius R. Find th…



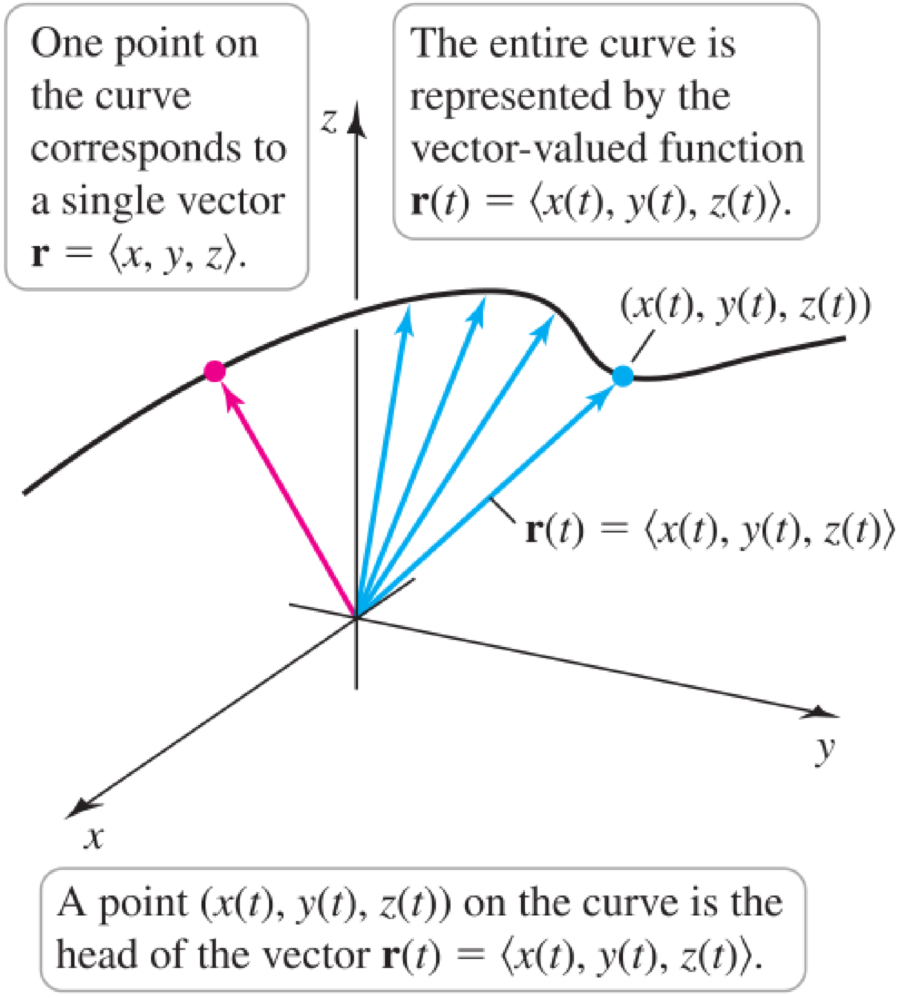
13.5

Lines and Curves in Space

Vector-Valued Functions

A function of the form $\boldsymbol{r}(t) = \langle x(t), y(t), z(t)\rangle$ may be viewed in two ways.

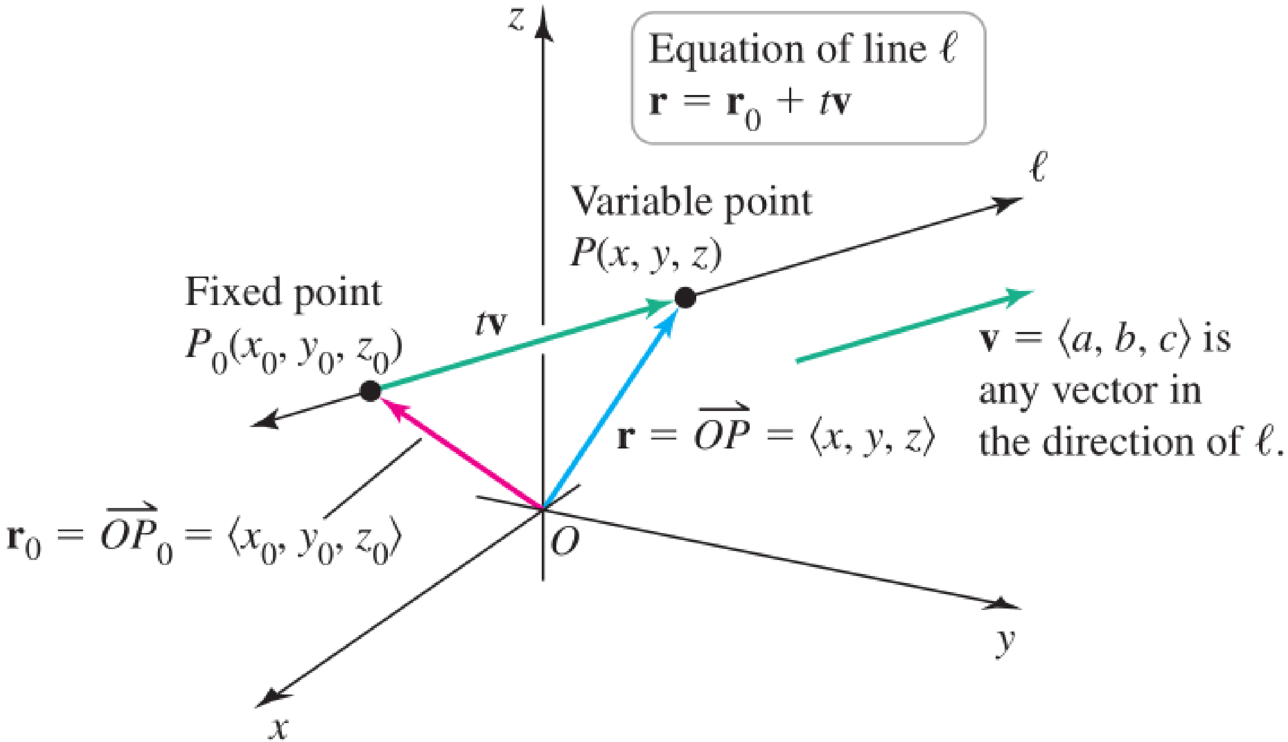
- A set of three parametric equations that describe a curve in space.
- A *vector-valued function*, meaning the three dependent variables x, y, z are the components of $\boldsymbol{r}$, and each component varies with respect to a single independent variable t (time).



Lines in Space

Two ways to determine a line in $\mathbb{R}^3$

- Two distinct points & One point and a direction



$$\underbrace{\langle x, y, z \rangle}_{\mathbf{r} = \overrightarrow{OP}} = \underbrace{\langle x_0, y_0, z_0 \rangle}_{\mathbf{r}_0 = \overrightarrow{OP}_0} + t\underbrace{\langle a, b, c \rangle}_{\mathbf{v}} \quad \text{or} \quad \mathbf{r} = \mathbf{r}_0 + t\mathbf{v}.$$

Equation of a Line

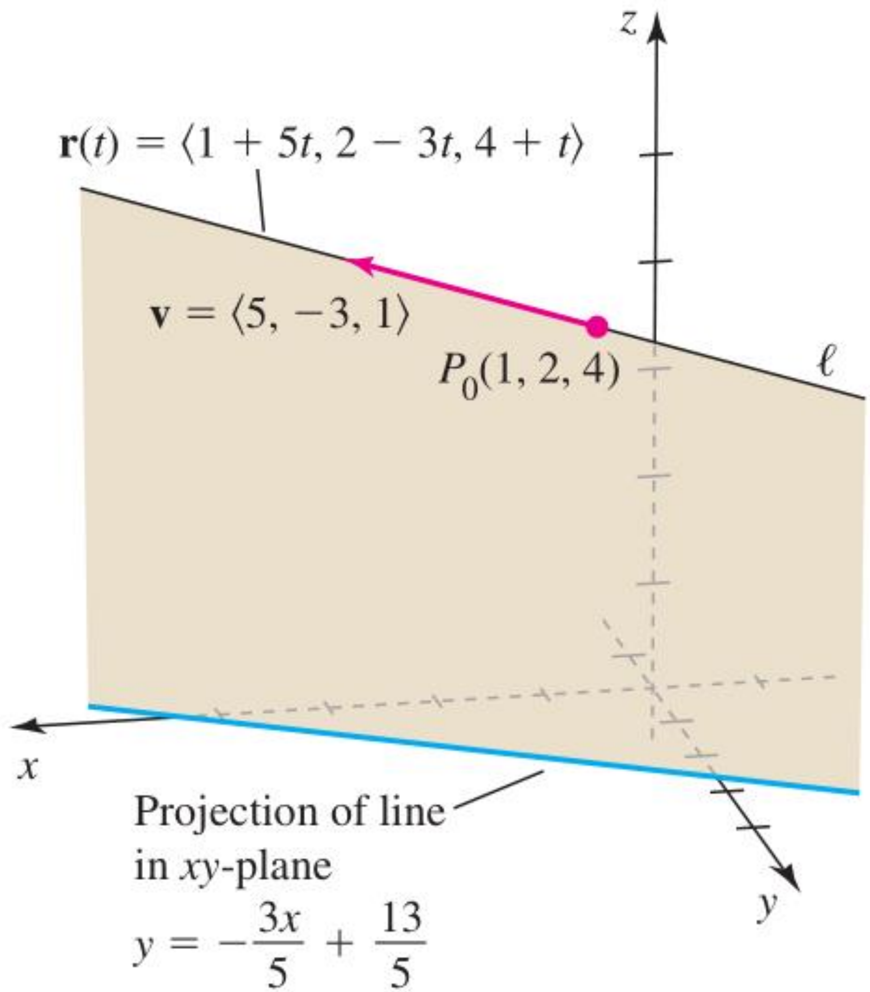
An **equation of the line** passing through the point $P_0(x_0, y_0, z_0)$ in the direction of the vector $\mathbf{v} = \langle a, b, c \rangle$ is $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$, or

$$\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle, \quad \text{for} \quad -\infty < t < \infty.$$

Equivalently, the corresponding parametric equations of the line are

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct, \quad \text{for} \quad -\infty < t < \infty.$$

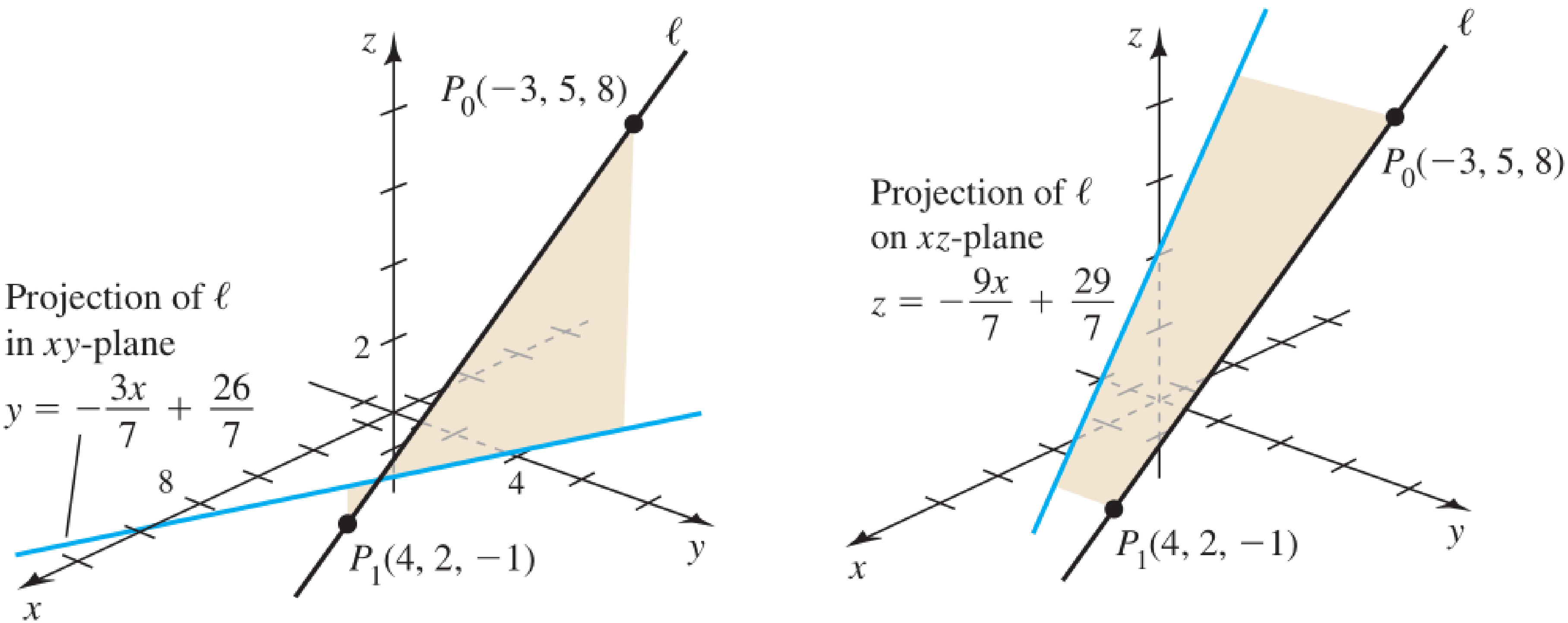
EXAMPLE 1 **Equations of lines** Find an equation of the line ℓ that passes through the point $P_0(1, 2, 4)$ in the direction of $\mathbf{v} = \langle 5, -3, 1 \rangle$.



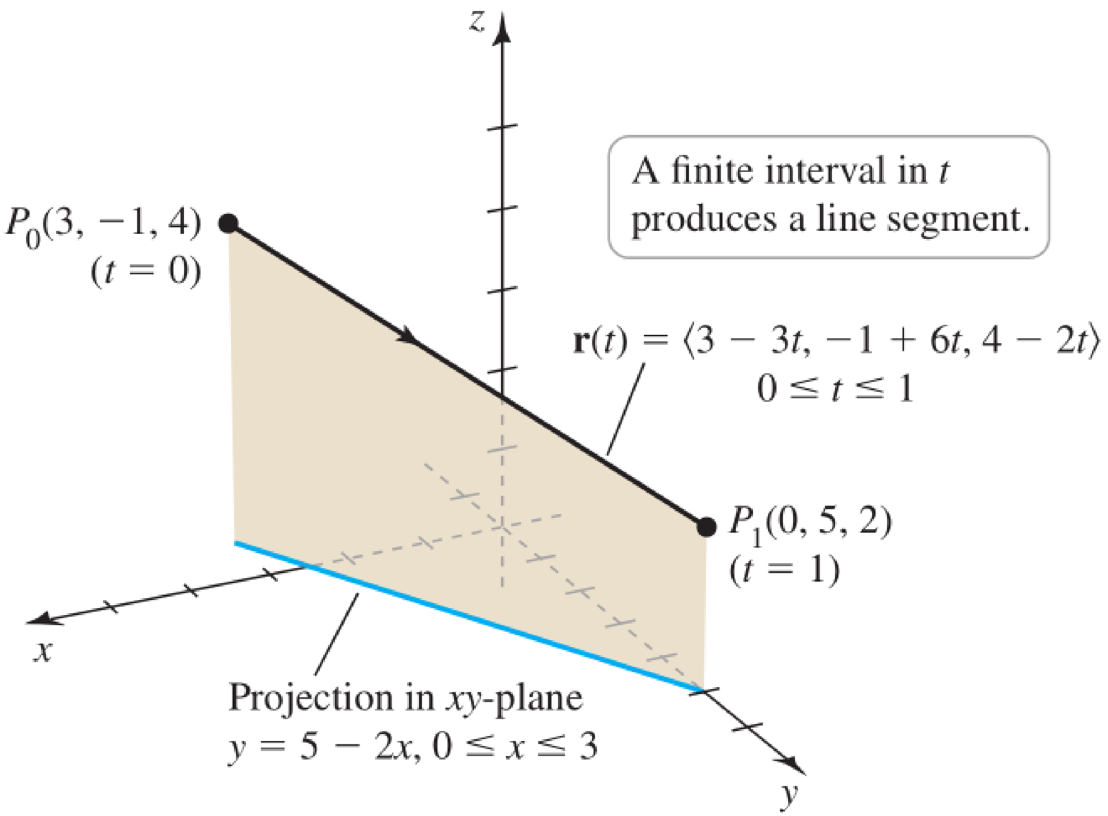
EXAMPLE 2 **Equations of lines** Let ℓ be the line that passes through the points $P_0(-3, 5, 8)$ and $P_1(4, 2, -1)$.

a. Find an equation of ℓ.

b. Find equations of the projections of ℓ on the xy- and xz-planes. Then graph those projection lines.



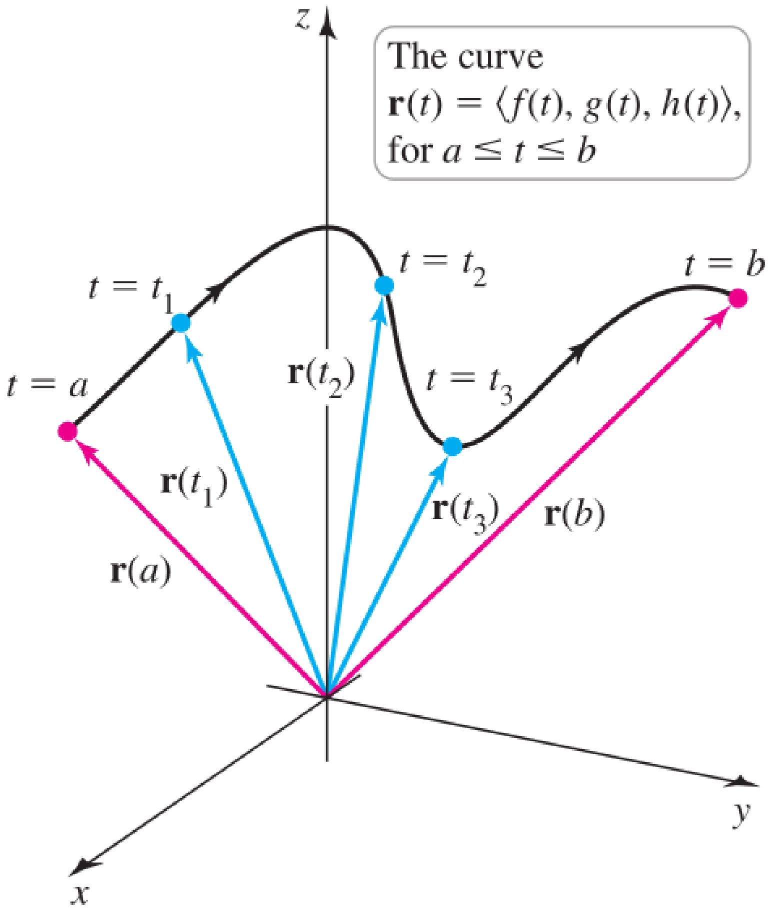
EXAMPLE 3 **Equation of a line segment** Find an equation of the line segment that extends from $P_0(3, -1, 4)$ to $P_1(0, 5, 2)$.



Curves in Space

$$\boldsymbol{r}(t) = \langle f(t), g(t), h(t)\rangle = f(t)\boldsymbol{i} + g(t)\boldsymbol{j} + h(t)\boldsymbol{k}$$

The domain of $\boldsymbol{r}$ is the largest set of values of t on which all of f, g, and h are defined.



As parameter t varies over the interval $a \le t \le b$, each value of t produces a position vector that corresponds to a point on the curve,

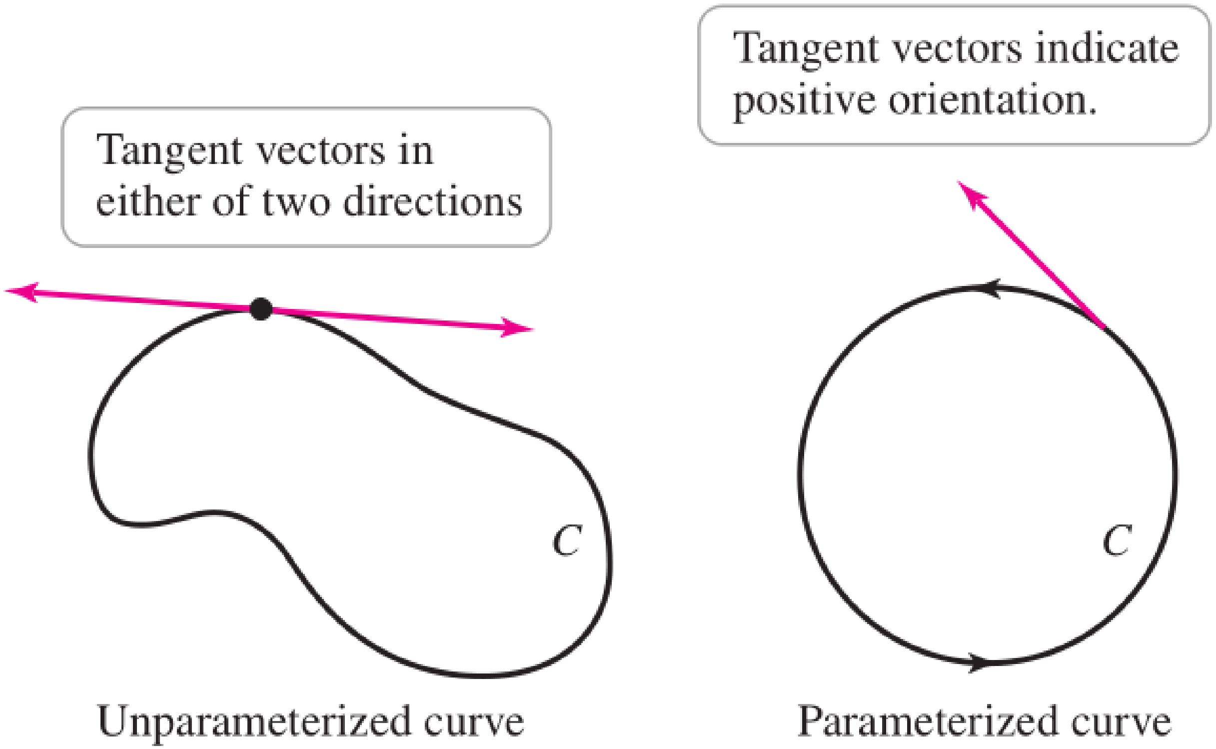
Starting at the initial vector $\boldsymbol{r}(a)$

Ending at the terminal vector $\boldsymbol{r}(b)$

Orientation of Curves

The *positive orientation* is the direction in which the curve is generated as the parameter increases from a to b

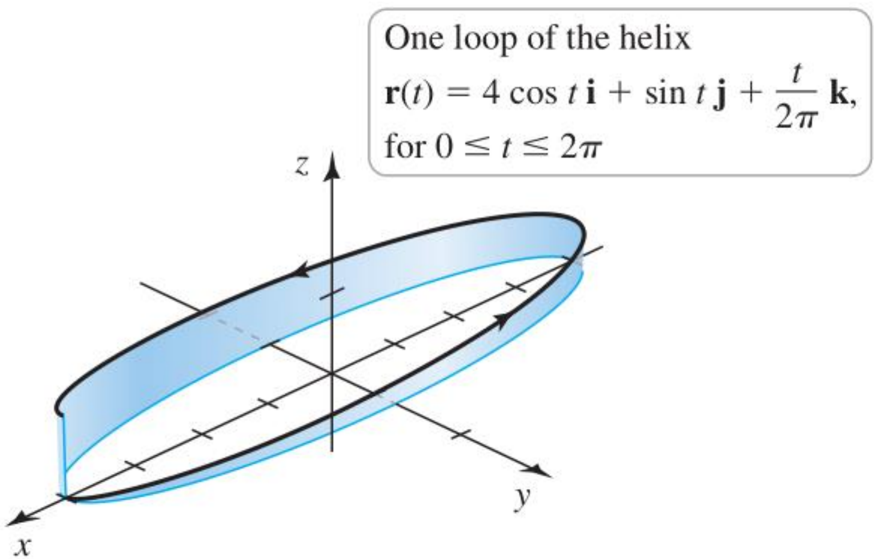
The orientation of a parameterized curve and its tangent vectors are consistent

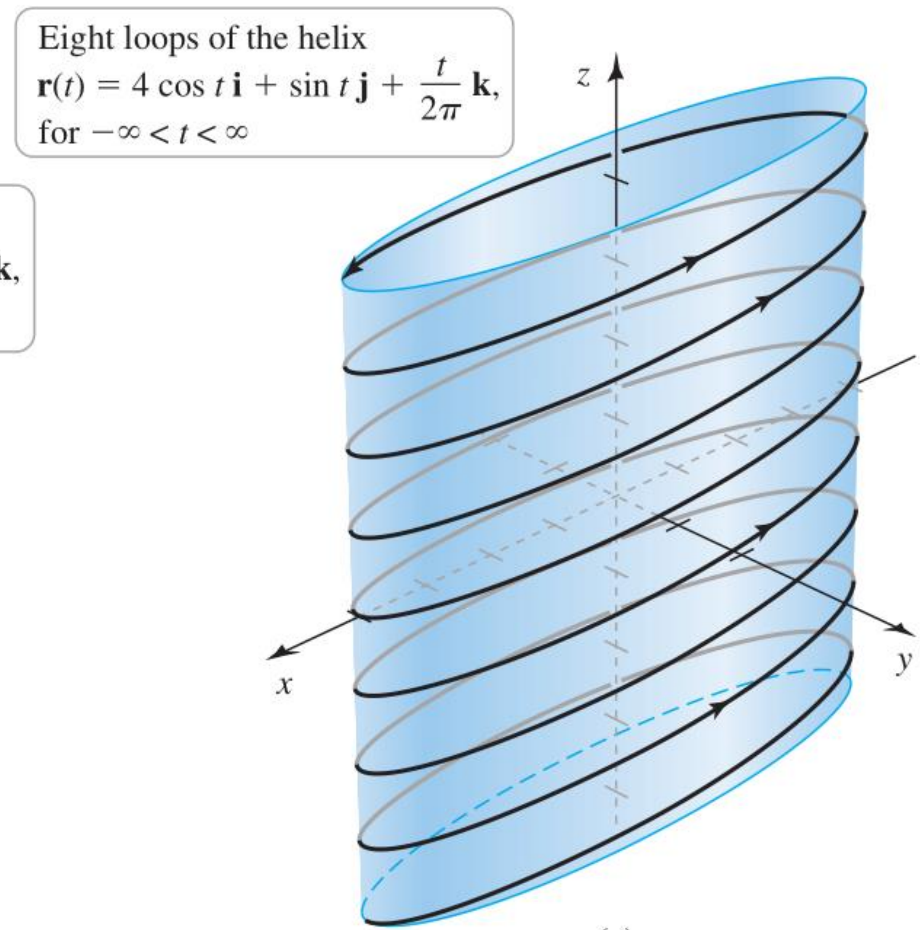


EXAMPLE 4 **A helix** Graph the curve described by the equation

$$\mathbf{r}(t) = 4\cos t\,\mathbf{i} + \sin t\,\mathbf{j} + \frac{t}{2\pi}\,\mathbf{k},$$

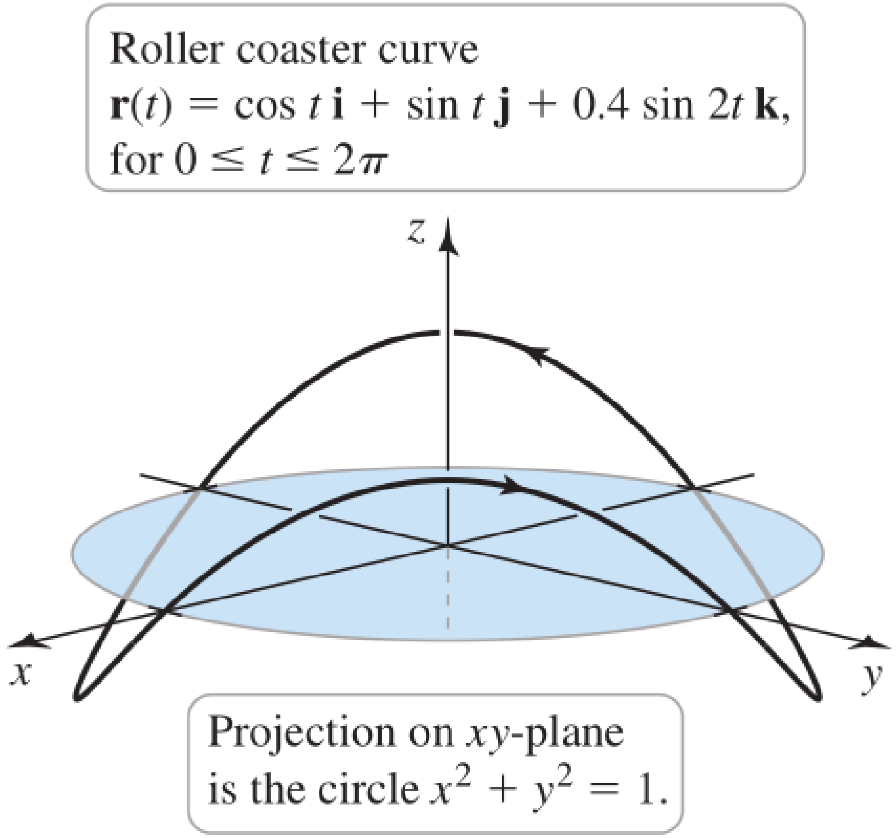
where (a) $0 \le t \le 2\pi$ and (b) $-\infty < t < \infty$.





EXAMPLE 5 **Roller coaster curve** Graph the curve

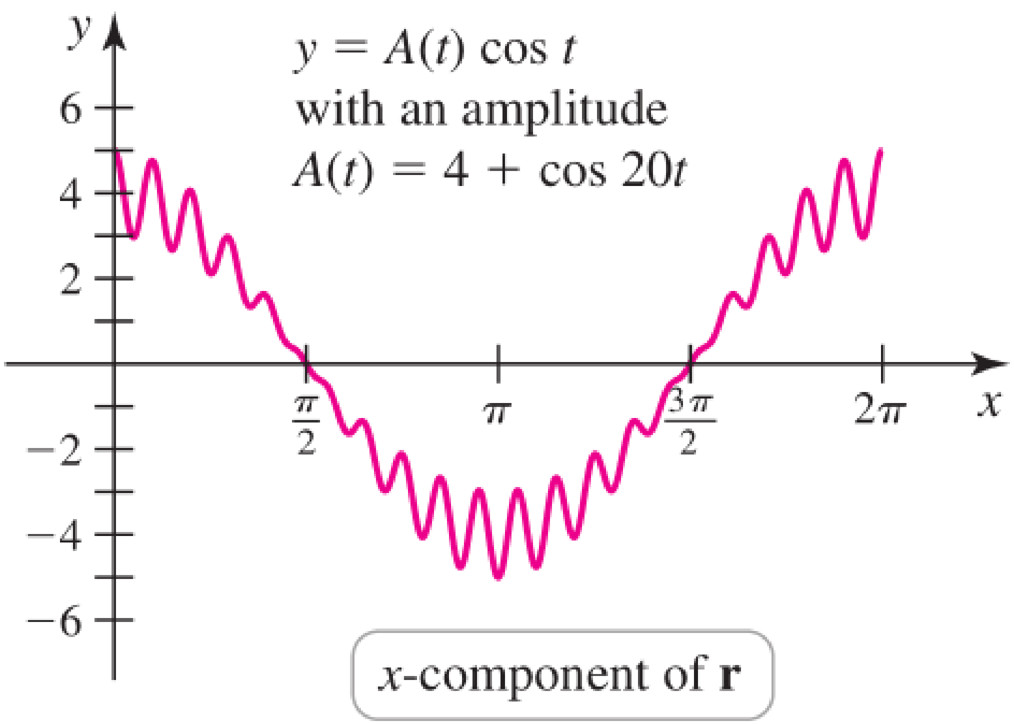
$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + 0.4\sin 2t\,\mathbf{k}, \quad \text{for } 0 \le t \le 2\pi.$$



EXAMPLE 6 **Slinky curve** Graph the curve

$$\mathbf{r}(t) = (4 + \cos 20t)\cos t\,\mathbf{i} + (4 + \cos 20t)\sin t\,\mathbf{j} + 0.4\sin 20t\,\mathbf{k},$$

for $0 \le t \le 2\pi$.

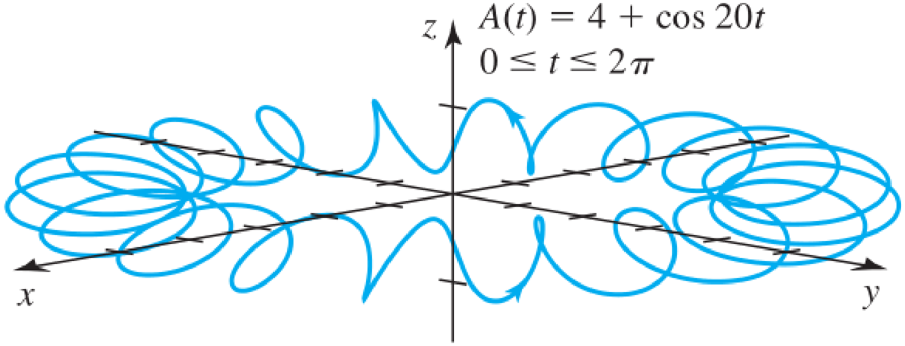


Slinky curve

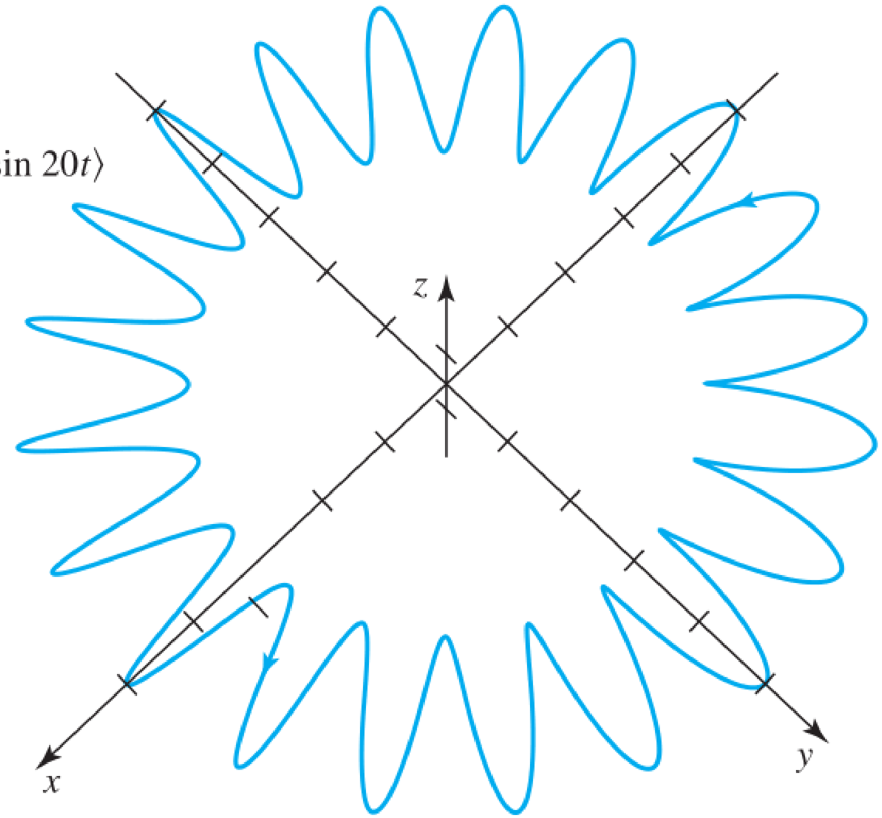
$\mathbf{r}(t) = \langle A(t) \cos t, A(t) \sin t, 0.4 \sin 20t \rangle$

$A(t) = 4 + \cos 20t$

$0 \leq t \leq 2\pi$



View along *xy*-plane



View from above

Limits and Continuity for Vector-Valued Functions

The concepts of limits, derivatives, integrals of vector-valued functions are direct extensions of what you have already learned.

DEFINITION Limit of a Vector-Valued Function

A vector-valued function $\mathbf{r}$ approaches the limit $\mathbf{L}$ as t approaches a, written $\lim_{t \to a} \mathbf{r}(t) = \mathbf{L}$, provided $\lim_{t \to a} |\mathbf{r}(t) - \mathbf{L}| = 0$.

Given $\boldsymbol{r}(t) = \langle f(t), g(t), h(t) \rangle$, suppose that

$$\lim_{t \to a} f(t) = L_1, \lim_{t \to a} g(t) = L_2, \lim_{t \to a} h(t) = L_3$$

Then,

$$\lim_{t \to a} \boldsymbol{r}(t) = \left\langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \right\rangle = \langle L_1, L_2, L_3 \rangle$$

The limit of $\boldsymbol{r}(t)$ is determined by computing the limits of its components.

The limits laws in Chapter 2 have analogs for vector-valued functions. For example,

$$\lim_{t \to a} (\boldsymbol{r}(t) + \boldsymbol{s}(t)) = \lim_{t \to a} \boldsymbol{r}(t) + \lim_{t \to a} \boldsymbol{s}(t)$$

$$\lim_{t \to a} c\boldsymbol{r}(t) = c \lim_{t \to a} \boldsymbol{r}(t)$$

Continuity

A function $\boldsymbol{r}(t) = f(t)\boldsymbol{i} + g(t)\boldsymbol{j} + h(t)\boldsymbol{k}$ is continuous at a provided

$$\lim_{t \to a} \boldsymbol{r}(t) = \boldsymbol{r}(a)$$

If the component functions f, g, and h of $\boldsymbol{r}(t)$ are continuous at a, then $\boldsymbol{r}$ is also continuous at a and vice versa.

The function $\boldsymbol{r}$ is continuous on an interval I if it is continuous for all t in I.

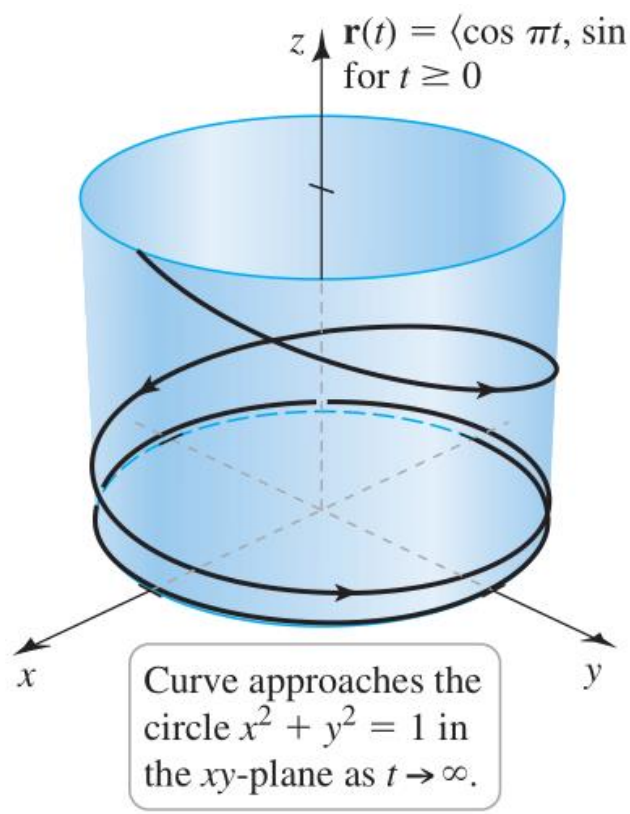
EXAMPLE 7 **Limits and continuity** Consider the function

$$\mathbf{r}(t) = \cos \pi t\, \mathbf{i} + \sin \pi t\, \mathbf{j} + e^{-t}\, \mathbf{k}, \quad \text{for } t \geq 0.$$

a. Evaluate $\lim_{t \to 2} \mathbf{r}(t)$.

b. Evaluate $\lim_{t \to \infty} \mathbf{r}(t)$.

c. At what points is $\mathbf{r}$ continuous?



Chapter 13

Vectors and Vector-Valued Functions (I)

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