MATH 2233 Differential Equations Chapter 7 Laplace Transform

Section 7.2 Definition of Laplace Transform

Goal of this section

• Caculate the Laplace transform of a function using the definition.

Motiviation: Many practical engineering problems involve mechanical or electrical systems acted on by *discontinuous or impulsive forcing terms*. Methods introduced in previous chapters are often rather awkward to use. In this chapter, we consider a new approach based on Laplace transform.

Definition: Laplace transform

Let f(t) be a function on $(0, \infty)$. The **Laplace transform** of a function f(t) is a function F(s) defined by the integral

Remark The integral above is an **improper** integral defined by

Example 1. Determine the Laplace transform of the constant function f(t) = 1.

Example 2. Find the Laplace transform of $f(t) = e^{at}$, where a is a constant.	
Example 3. Find $\mathcal{L}\{\sin(bt)\}$ where b is a constant.	

Example 4. Find the Laplace transform of

$$f(t) = \begin{cases} 2, & 0 \le t < 5, \\ 0, & 5 < t < 10, \\ e^{4t}, & t > 10, \end{cases}$$

A Brief Table of Laplace Transforms.

f(t)	$F(s) = \mathcal{L}\{f\}(s)$
1	
t^n	
e^{at}	
$e^{at}t^n$	
$\sin(bt)$	
$\cos(bt)$	
$e^{at}\sin(bt)$	
$e^{at}\cos(bt)$	

Linearity of Laplace Transforms.

Let f, f_1 , f_2 be functions whose Laplace transform exist. Let c be a constant. Then

•
$$\mathcal{L}\{f_1+f_2\}=$$

•
$$\mathcal{L}\{cf\} =$$

Example 5. Determine $\mathcal{L}\{11 + 5e^{4t} - 6\sin(2t)\}$ and $\mathcal{L}\{5t^2e^{-3t} - e^{12t}\cos(8t)\}$.

Section 7.3 Properties of Laplace Transform

Goal of this section

• study three basic properties of Laplace Transform Recall The Laplace Transform of a function f(t) is

Property (I): Translation in s

If
$$\mathcal{L}{f(t)} = F(s)$$
, then $\mathcal{L}{e^{at}f(t)} =$

A one-line proof

Example 1. Determine the Laplace transform of $e^{at} \sin(bt)$.

Property (II): Laplace Transform of Derivatives

Let f(t) be continuous on $[0, \infty)$. If $\mathcal{L}\{f(t)\} = F(s)$, then

$$\mathcal{L}\{f'(t)\} =$$

$$\mathcal{L}\{f''(t)\} =$$

In general, we have for higher-order derivatives

$$\mathcal{L}\{f^{(n)}(t)\} =$$

Example 2.	Using the	Property (II)	and the	fact that
				<i>J</i>

$$\mathcal{L}\{\sin(bt)\}(s) = \frac{b}{s^2 + b^2}$$

determine $\mathcal{L}\{\cos(bt)\}$

Property (III): Derivatives of Laplace Transform

Let f(t) be piecewise continuous on $[0,\infty)$. If $\mathcal{L}\{f(t)\}=F(s)$, then

$$\mathcal{L}\{t^n f(t)\} =$$

Example 3. Determine $\mathcal{L}\{t\sin(bt)\}$

A Summary of Properties of Laplace Transform

•
$$\mathcal{L}{f+g} =$$

•
$$\mathcal{L}\{cf\} =$$

•
$$\mathcal{L}\{e^{at}f(t)\}=$$

•
$$\mathcal{L}\{f'\}(s) =$$

•
$$\mathcal{L}{f''}(s) =$$

•
$$\mathcal{L}\{f^{(n)}\}(s) =$$

•
$$\mathcal{L}\{t^n f(t)\}(s) =$$

Section 7.4 Inverse Laplace Transform

Goal of this section

• Given a function F(s), compute its inverse Laplace Transform f(t).

Inverse Laplace Transform

If $F(s) = \mathcal{L}\{f(t)\}$, then we say f(t) is

Example 1. Compute $\mathcal{L}^{-1}{F}$ where

(a).
$$F(s) = \frac{2}{s^3}$$
,

(b).
$$F(s) = \frac{3}{s^2 + 9}$$

(a).
$$F(s) = \frac{2}{s^3}$$
, (b). $F(s) = \frac{3}{s^2 + 9}$, (c). $F(s) = \frac{s - 1}{s^2 - 2s + 5}$.

Linearity of Inverse Transform

•
$$\mathcal{L}^{-1}\{F_1 + F_2\} =$$

•
$$\mathcal{L}^{-1}\{cF\} =$$

Example 2. Compute $\mathcal{L}^{-1} \left\{ \frac{5}{s-6} - \frac{6s}{s^2+9} + \frac{3}{2s^2+8s+10} \right\}$

Example 3. Compute $\mathcal{L}^{-1}\left\{\frac{5}{(s+2)^4}\right\}$

Example 4. Compute $\mathcal{L}^{-1}\left\{ \frac{3s+2}{s^2+2s+10} \right\}$

Method of Partial Fraction

In general, how to compute the inverse transform of $\mathcal{L}^{-1}\left\{\frac{P(s)}{Q(s)}\right\}$? Here, P(s) and Q(s) are polynomials with the degree of P less than the degree of Q.

1. Non-repeated Linear Factors

If $Q(s) = (s - r_1)(s - r_2) \cdots (s - r_n)$, the partial fraction expansion has the form

$$\frac{P(s)}{Q(s)} = \frac{A_1}{s - r_1} + \frac{A_2}{s - r_2} + \dots + \frac{A_n}{s - r_n}.$$

Example 5. Determine $\mathcal{L}^{-1}\left\{\frac{7s-1}{(s+1)(s+2)(s-3)}\right\}$

2. Repeated Linear Factors

Let (s-r) be a factor of Q(s) and suppose $(s-r)^m$ is the highest power of s-r that divides Q(s). Then the portion corresponding to the term $(s-r)^m$ is

$$\frac{A_1}{s-r} + \frac{A_2}{(s-r)^2} + \dots + \frac{A_m}{(s-r)^m}.$$

Example 6. Determine
$$\mathcal{L}^{-1}\left\{\frac{s^2+9s+2}{(s-1)^2(s+3)}\right\}$$

3. Quadratic Factors

Let $(s-\alpha)^2+\beta^2$ be a quadratic factor of Q(s) that cannot be reduced to linear factors with real coefficients. Suppose m is the is the highest power of $(s-\alpha)^2+\beta^2$ that divides Q(s). Then the portion corresponding to the term $[(s-\alpha)^2+\beta^2]^m$ is

$$\frac{C_1s + D_1}{(s - \alpha)^2 + \beta^2} + \frac{C_2s + D_2}{[(s - \alpha)^2 + \beta^2]^2} + \dots + \frac{C_ms + D_m}{[(s - \alpha)^2 + \beta^2]^m}.$$

Example 7. Determine
$$\mathcal{L}^{-1}\left\{\frac{2s^2+10s}{(s^2-2s+5)(s+1)}\right\}$$

Section 7.5 Solve Initial Value Problems

Goal of this section

• use Laplace transform to solve initial value problems of linear differential equations.

Method of Laplace transform

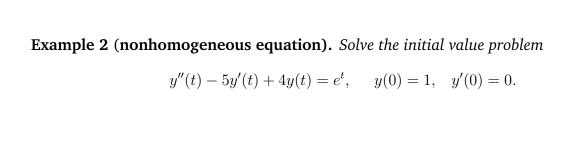
Step 1 Take Laplace transforms on both sides of the DE.

Step 2 Solve the algebraic equation of Y(s).

Step 3 Take the Inverse transform to resolve y(t).

Example 1. Use Laplace Transform to solve the initial value problem

$$y'' - 2y' + 5y = 0$$
, $y(0) = 2$, $y'(0) = 4$.



The method of Laplace transform can solve IVPs in which the initial condition is not at t=0.

Example 3 (Initial condition not at zero). Use the Laplace transform to solve the initial value problem

$$w''(t) - 2w'(t) + 5w(t) = -8e^{\pi - t}, \quad w(\pi) = 2, \quad w'(\pi) = 12.$$

Example 4 (high order equation). <i>problem</i>		n). Use the Laplace transform to solve the initial value			
prootent	y''' - y'' + y' - y = 0,	y(0) = 1,	y'(0) = 1,	y''(0) = 3.	

Section 7.6 Transforms of Discontinuous Functions

Goal of this section

• Use Laplace transform to solve IVPs with discontinuous nonhomogeneous function.

Unit Step Function

The **unit step function** u(t) is defined by

By shifting u(t), the jump can be moved to a different location.

$$u(t-a) =$$

Example 1. Sketch the graph of y = h(t), where

$$h(t) = u(t - \pi) - u(t - 2\pi)$$

Example 2. Write the piecewisely continuous function

$$f(t) = \begin{cases} 3, & t < 2, \\ 1, & 2 < t < 5 \\ t, & 5 < t < 8 \\ t^2/10, & t > 8 \end{cases}$$

in terms of the unit step functions.

Laplace Transform of Step functions

$$\mathcal{L}(u(t-a)) =$$

Assume y = f(t) is defined for $t \ge 0$. Then the function

$$g(t) = f(t - a)u(t - a)$$

Translation in t

Let $\mathcal{L}\{f(t)\} = F(s)$. Then

$$\mathcal{L}\{f(t-a)u(t-a)\} =$$

The inverse form: If $f(t) = \mathcal{L}^{-1}\{F(s)\}$, then

$$\mathcal{L}^{-1}\left\{e^{-as}F(s)\right\} =$$

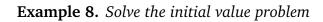
Example 3. Determine $\mathcal{L}\left\{t\,u(t-1)\right\}$

Example 4. Determine $\mathcal{L}\{t^2u(t-1)\}$

Example 5. Determine $\mathcal{L}\{\cos(t)u(t-\pi)\}$

Example 6. Determine $\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2}\right\}$

Example 7. Determine $\mathcal{L}^{-1}\left\{\frac{se^{-3s}}{s^2+4s+5}\right\}$



$$y' + y = f(t), \quad y(0) = 5, \text{ where } f(t) = \begin{cases} 0 & 0 \le t < \pi, \\ 3\cos(t) & t \ge \pi. \end{cases}$$

Section 7.10 Solve Linear Systems with Laplace Transforms

Goal: use Laplace transform to solve a system of differential equations with initial conditions.

Example 1. Solve the initial value problem

$$x'(t) - 2y(t) = 4t,$$
 $x(0) = 4,$
 $y'(t) + 2y(t) - 4x(t) = -4t - 2,$ $y(0) = -5.$