Calculus Class Test II (共 50 分)

1. Evaluate the limit of the sequences

(1) $a_n = \ln(n+1) - \ln n$ (2) $a_n = n\sin\left(\frac{1}{n}\right)$

(3) $a_n = \sqrt{\left(1+\frac{1}{2n}\right)^n}$ (4) $a_n = \sqrt[n]{2^{1+3n}}$

Solution

(1) $\lim\limits_{n\to\infty} \boldsymbol{a_n} = \lim\limits_{n\to\infty} \ln\left(\frac{\boldsymbol{n+1}}{\boldsymbol{n}}\right) = \lim\limits_{n\to\infty} \ln\left(\mathbf{1}+\frac{\mathbf{1}}{\boldsymbol{n}}\right) = \ln \mathbf{1} = \mathbf{0}$

(2) $\lim\limits_{n\to\infty} \boldsymbol{a_n} = \lim\limits_{n\to\infty} \boldsymbol{n}\sin\left(\frac{1}{n}\right) = \lim\limits_{n\to\infty} \sin\left(\frac{1}{n}\right)/\frac{1}{n} = 1$

(3) $\lim\limits_{n\to\infty} \boldsymbol{a_n} = \lim\limits_{n\to\infty} \sqrt{\left(\mathbf{1}+\frac{\mathbf{1}}{\mathbf{2}\boldsymbol{n}}\right)^{\boldsymbol{n}}} = \sqrt{\lim\limits_{n\to\infty}\left(\mathbf{1}+\frac{\mathbf{1}}{\mathbf{2}\boldsymbol{n}}\right)^{\boldsymbol{n}}} = \sqrt{\lim\limits_{n\to\infty}\left(\mathbf{1}+\frac{\mathbf{1}}{\mathbf{2}\boldsymbol{n}}\right)^{\mathbf{2}\boldsymbol{n}\cdot\mathbf{1}/\mathbf{2}}} = \boldsymbol{e}^{\mathbf{1}/\mathbf{4}}$

(4) $\lim\limits_{n\to\infty} \boldsymbol{a_n} = \lim\limits_{n\to\infty} \sqrt[\boldsymbol{n}]{\mathbf{2}^{\mathbf{1}+\mathbf{3}\boldsymbol{n}}} = \mathbf{2}^{\mathbf{3}} \lim\limits_{n\to\infty} \sqrt[\boldsymbol{n}]{\mathbf{2}} = \mathbf{2}^{\mathbf{3}} \lim\limits_{n\to\infty} e^{\frac{1}{n}\ln 2} = \mathbf{2}^{\mathbf{3}} e^{\lim_{n\to\infty}\frac{1}{n}\ln 2} = \mathbf{2}^{\mathbf{3}}$

2. Test the infinite series for convergence or divergence

(1) $\sum_{n=1}^{\infty}\frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$ (2) $\sum_{n=1}^{\infty} ne^{-n^2}$

(3) $\sum_{n=1}^{\infty}\frac{n-1}{2n+1}$ (4) $\sum_{n=1}^{\infty}(-1)^n\frac{n^3}{n^4+1}$

(5) $\sum_{k=1}^{\infty}\frac{2^k}{k!}$ (6) $\sum_{\boldsymbol{n}=\mathbf{1}}^{\infty}\frac{\mathbf{1}}{\mathbf{2}+\mathbf{3}^{\boldsymbol{n}}}$

Solution

(1) $\sum_{n=1}^{\infty}\frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$

Since a_n is an algebraic function of n, we use Limit Comparison Test to compare the given series with a p-series $\sum b_n$, where

$$b_n = \frac{\sqrt{n^3}}{3n^3} = \frac{1}{3n^{3/2}}$$

As $a_n \le b_n$, and the p-series $\sum b_n$ is convergent ($p = \frac{3}{2} > 1$), so $\sum_{n=1}^{\infty}\frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$ is convergent.

(2) $\sum_{n=1}^{\infty} ne^{-n^2}$

Since the integral $\int_1^{\infty} xe^{-x^2}dx$ is easily evaluated, we use the Integral Test.

$$\int_1^{\infty} xe^{-x^2}dx = -\frac{1}{2}\int_1^{\infty} e^{-x^2}d(-x^2) = -\frac{1}{2}e^{-x^2}\Big|_1^{\infty} = \frac{1}{2}(e^{-1}-1)$$

So $\sum_{n=1}^{\infty} ne^{-n^2}$ is convergent.

The Ratio Test also works.

(3) $\sum_{n=1}^{\infty} \frac{n-1}{2n+1}$

Since $a_n \to \frac{1}{2} \neq 0$ as $n \to \infty$, according to Divergence Test, $\sum_{n=1}^{\infty} \frac{n-1}{2n+1}$ is divergent.

(4) $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{n^4+1}$

Since the series is alternating, we use the Alternating Series Test.

First to show $a_n = \frac{n^3}{n^4+1}$ is decreasing.

Let $f(x) = \frac{x^3}{x^4+1}$, we have $f'(x) = \frac{x^2(3-x^4)}{(x^4+1)^2}$.

When $x > \sqrt[4]{3}$, $3 - x^4 < 0$, $f'(x) < 0$. So, $a_{n+1} < a_n$ when $n \geq 2$.

Second, $\lim_{n\to\infty} \frac{n^3}{n^4+1} = \lim_{n\to\infty} \frac{1/n}{1+1/n^4} = 0$

Therefore, $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{n^4+1}$ is convergent.

(5) $\sum_{k=1}^{\infty} \frac{2^k}{k!}$

Since the series involves $k!$, we use the Ratio Test.

$$a_{k+1}/a_k = \frac{2^{k+1}}{(k+1)!} \cdot \frac{k!}{2^k} = \frac{2}{k+1} \to 0$$

So $\sum_{k=1}^{\infty} \frac{2^k}{k!}$ is convergent.

(6) $\sum_{n=1}^{\infty} \frac{1}{2+3^n}$

Since the series is closely related to the geometric series $\sum_{n=1}^{\infty} \frac{1}{3^n}$, we use the Comparison Test.

$$\frac{1}{2+3^n} < \frac{1}{3^n},$$

and $\sum_{n=1}^{\infty} \frac{1}{3^n}$ is convergent ($r = \frac{1}{3} < 1$), so $\sum_{n=1}^{\infty} \frac{1}{2+3^n}$ is convergent.