MATH 2233 Differential Equations

Chapter 2 First-Order Differential Equations

Section 2.2 Separable Equations

Goal of this section

1. understand what is a separable equation.
2. learn how to solve separable equations.

Example 1. *Solve the differential equation*

$$\frac{dy}{dx} = e^{3x+2y}.$$

Definition of Separable Equations If the right-hand side of the equation $\frac{dy}{dx} = f(x, y)$ can be expressed as

Example 2. *Are the following equations separable?*

$$\frac{dy}{dx} = \frac{2x + xy}{y^2 + 1}, \qquad \frac{dy}{dx} = 1 + xy$$

Method for Solving Separable Equations

To solve

$$\frac{dy}{dx} = g(x)p(y)$$

1. Multiply by dx and by $h(y) = 1/p(y)$ to obtain

2. Integrate both side

3. If possible, solve the functional equation to obtain the explicit solution

Example 3. *Is the differential equation separable? Find its solution.*

$$y' = \frac{x - 5}{y^2}$$

Example 4. *Solve the initial value problem*

$$\frac{dy}{dx} = \frac{y-1}{x+3}, \quad y(-1) = 0.$$

Example 5. ***(partial fraction)*** *Solve the following differential equation*

$$\frac{dy}{dx} = y^2 - 4.$$

Example 6. ***(implicit solution)*** *Solve the initial value problem*

$$\frac{dy}{dx} = \frac{6x^5 - 2x + 1}{\cos(y) + e^y}, \quad y(0) = 0.$$

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Section 2.3   Linear Equations

Goal of this section

1. understand how to solve first-order linear equations.

Review. The **general form** of a first-order linear differential equation is

The **standard form** of the first-order equation is

Example 1. *Find the solution of the differential equation*

$$(4 + x^2)\frac{dy}{dx} + 2xy = 4x.$$

Question: How to solve general first-order linear equations?

Method of Integrating Factors

Example 2. *Solve the differential equation*

$$\frac{dy}{dx} + \frac{1}{2}y = \frac{1}{2}e^{\frac{x}{3}}.$$

Method for Solving Linear Equations

Step 1. We write the equation in standard form

$$\frac{dy}{dx} + P(x)y = Q(x).$$

Step 2. Calculate the integrating factor

Step 3. Multiply

Step 4. Integrate

Example 3. *Solve the differential equation*

$$\frac{1}{x}\frac{dy}{dx} - \frac{2y}{x^2} = x\cos(x), \quad x > 0.$$

Example 4. ***(An equation is both linear and separable)***

Solve the initial value problem

$$\frac{dy}{dx} + 2xy = 0, \quad y(0) = 2.$$

Method #1: Solve as a separable equation

Method #2: Solve as a linear equation

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Section 2.4   Exact Equations

Goal of this section

1. understand what is an exact equation, and how to test for exactness.
2. understand how to solve exact equations.

Example 1. *Solve the differential equation*

$$2x + y^2 + 2xyy' = 0.$$

Definition of Exact Equations

Consider the differential equation

$$M(x, y) + N(x, y)y' = 0.$$

If we can identify a function $F(x, y)$ such that

Definition. A differential equation $M(x, y)+N(x, y)y' = 0$ is called an ________________, if

Test for Exactness

The differential equation

$$M(x, y) + N(x, y)y' = 0.$$

is **exact** if and only if

Method for Solving Exact Equations

Step 1. If $M(x, y)dx + N(x, y)dy = 0$ is exact, then

Step 2. To determine $g(y)$, we take

Step 3. Integrate

Step 4. The solution is implicitly given by

Example 2. *Solve the differential equation*

$$(2xy - \sec^2(x))dx + (x^2 + 2y)dy = 0.$$

Remark

- As a check on our work, we observe that when we solve for $g'(y)$, we must
- When constructing $F(x, y)$, we can also

Example 3. *Solve the initial value problem*

$$\frac{dy}{dx} = \frac{xy^2 - \cos(x)\sin(x)}{y(1-x^2)}, \quad y(0) = 2.$$

Example 4. *Show that*

$$(x + 3x^3 \sin(y))dx + (x^4 \cos(y))dy = 0$$

is not exact but multiplying this equations by the factor x^{-1} *yields an exact equation. Use this fact to solve the equation.*

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Section 1.4 Numerical Approximation: Euler's Method

Goal of this section

1. understand how to use Euler's method to approximate solution of an IVP.
2. get familiar to computer program of Euler's Method.

Although the analytical techniques in Chapter 2 were useful for solving several 1st-order differential equations, the majority of the differential equations encountered in applications cannot be solved analytically. In this section, we introduce a numerical method for approximating the solution to an initial value problem for a first-order equation:

$$\frac{dy}{dx} = f(x, y), \qquad y(x_0) = y_0.$$

The method is called ________________________.

How to use tangent lines to approximate the solution $y = \phi(x)$**?**

Starting with the initial point (x_0, y_0),

The **Euler's Method** can be summarized by the recursive formulas:

$$x_{n+1} =$$
$$y_{n+1} =$$

Example 1. *Use Euler's method with* $h = 0.1$ *to approximate the solution to the initial value problem*

$$y' = x\sqrt{y}, \quad y(1) = 4$$

at the points $x = 1.1$*,* 1.2*, and* 1.3*. Compare them with the corresponding values of the true solution.*

Example 2. *Use Euler's method to find approximations to the solution of the initial value problem*

$$y' = y, \quad y(0) = 1.$$

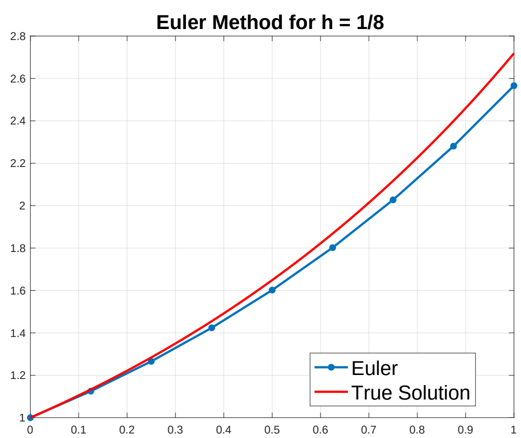
at the point $x = 1$*, taking* 1*,* 2*,* 4*,* 8*, and* 16 *steps.*

MATLAB Implementation of Euler's Method

- To test Example 2 with MATLAB, we write a "driver file"

```
%Example 1.4.2 in the Textbook
clc
%% Inputs
fun = @(x,y) y; % define a two-variable function.
x0 = 0; y0 = 1;  % initial condition

xE = 1; % final
N = 8; % number of steps.
h = (xE-x0)/N; % step size
X = zeros(N+1,1); Y = zeros(N+1,1);
X(1) = x0; Y(1) = y0;

for i = 1:N
    X(i+1) = X(i) + h;
    Y(i+1) = Y(i) + h*fun(X(i),Y(i));
end

disp('     x         Euler     True')
disp('------------------------------')
trueY = exp(X);
disp([X,Y,trueY]);

%% Plot Solution
plot(X,Y,'*-','linewidth',2)
hold on
fplot(@(x) exp(x), [x0,xE],'r-','linewidth',2)
hold off
legend('Euler','True Solution','fontSize',18,'location','best')
title(['Euler Method for h = 1/',int2str(N)],'fontSize',18)
grid on
```

- Numerical Results



x	Euler	True
0	1.0000	1.0000
0.1250	1.1250	1.1331
0.2500	1.2656	1.2840
0.3750	1.4238	1.4550
0.5000	1.6018	1.6487
0.6250	1.8020	1.8682
0.7500	2.0273	2.1170
0.8750	2.2807	2.3989
1.0000	2.5658	2.7183