27. $y''(x) - y'(x) - 2y(x) = \cos x - \sin 2x,$

$y(0) = -\frac{7}{20}, \quad y'(0) = \frac{1}{5}$

Solution. A.E. $r^2 - r - 2 = 0 \Rightarrow r_1 = -1, r_2 = 2$

$$y_h(x) = C_1 e^{-x} + C_2 e^{2x}$$

Assume $y_p(x) = A\sin x + B\cos x + C\sin 2x + D\cos 2x$

$$y_p'(x) = A\cos x - B\sin x + 2C\cos 2x - 2D\sin 2x$$

$$y_p''(x) = -A\sin x - B\cos x - 4C\sin 2x - 4D\cos 2x$$

Plug in,

$$\begin{cases} -3A + B = 0 \\ -A - 3B = 1 \\ -6C + 2D = -1 \\ -2C - 6D = 0 \end{cases} \Rightarrow \begin{cases} A = -\frac{1}{10} \\ B = -\frac{3}{10} \\ C = \frac{3}{20} \\ D = -\frac{1}{20} \end{cases}$$

$\therefore y(x) = C_1 e^{-x} + C_2 e^{2x} - \frac{1}{10}\sin x - \frac{3}{10}\cos x + \frac{3}{20}\sin 2x - \frac{1}{20}\cos 2x$

$y'(x) = -C_1 e^{-x} + 2C_2 e^{2x} - \frac{1}{10}\cos x + \frac{3}{10}\sin x + \frac{3}{10}\cos 2x + \frac{1}{10}\sin 2x$

I.C.

$$\begin{cases} y(0) = C_1 + C_2 - \frac{3}{10} - \frac{1}{20} = -\frac{7}{20} \\ y'(0) = -C_1 + 2C_2 - \frac{1}{10} + \frac{3}{10} = \frac{1}{5} \end{cases} \Rightarrow \begin{cases} C_1 = 0 \\ C_2 = 0 \end{cases}$$

$\therefore y(x) = -\frac{1}{10}\sin x - \frac{3}{10}\cos x + \frac{3}{20}\sin 2x - \frac{1}{20}\cos 2x.$

29. $y''(\theta) - y(\theta) = \sin\theta - e^{2\theta}, \quad y(0) = 1, \quad y'(0) = -1.$

Solution. A.E. $r^2 - 1 = 0 \Rightarrow r_1 = -1, r_2 = 1$

$$y_h(\theta) = C_1 e^{-\theta} + C_2 e^{\theta}$$

Assume $y_p(\theta) = A\sin\theta + B\cos\theta + Ce^{2\theta}$

$$y_p'(\theta) = A\cos\theta - B\sin\theta + 2Ce^{2\theta}$$

$$y_p''(\theta) = -A\sin\theta - B\cos\theta + 4Ce^{2\theta}$$

Plug in, $-2A\sin\theta - 2B\cos\theta + 3Ce^{2\theta} = \sin\theta - e^{2\theta}$

$\therefore A = -\frac{1}{2}, \quad B = 0, \quad C = \frac{-1}{3}$

$\therefore y(\theta) = C_1 e^{-\theta} + C_2 e^{\theta} - \frac{1}{2}\sin\theta - \frac{1}{3}e^{2\theta}$

$y'(\theta) = -C_1 e^{-\theta} + C_2 e^{\theta} - \frac{1}{2}\cos\theta - \frac{2}{3}e^{2\theta}$

I.C.

$$\left.\begin{array}{l} y(0) = C_1 + C_2 - \frac{1}{3} = 1 \\ y'(0) = -C_1 + C_2 - \frac{1}{2} - \frac{2}{3} = -1 \end{array}\right\} \Rightarrow \begin{cases} C_1 = \frac{7}{12} \\ C_2 = \frac{3}{4} \end{cases}$$

$\therefore y(\theta) = \frac{7}{12}e^{-\theta} + \frac{3}{4}e^{\theta} - \frac{1}{2}\sin\theta - \frac{1}{3}e^{2\theta}$

38. $y^{(4)} - 5y'' + 4y = 10\cos t - 20\sin t$

Solution. A.E. $r^4 - 5r^2 + 4 = 0 \Rightarrow r_{1,2} = \pm 1, r_{3,4} = \pm 2$

$$y_h(t) = C_1 e^{-t} + C_2 e^{t} + C_3 e^{-2t} + C_4 e^{2t}$$

Assume $y_p(t) = A\sin(t) + B\cos(t)$

$$y_p'(t) = A\cos(t) - B\sin(t)$$

$$y_p''(t) = -A\sin t - B\cos t$$

$$y_p^{(4)}(t) = A\sin t + B\cos t$$

Plug in, $\begin{cases} 10A = -20 \\ 10B = 10 \end{cases} \Rightarrow \begin{cases} A = -2 \\ B = 1 \end{cases}$

$\therefore y_p(t) = -2\sin t + \cos t.$

39. $y''' + y'' - 2y = te^t + 1$

Solution. A.E. $r^3 + r^2 - 2 = 0$

$$(r-1)(r^2 + 2r + 2) = 0$$

$$r_1 = 1, \quad r_{2,3} = -1 \pm i$$

$$y_h(t) = C_1 e^{t} + C_2 e^{-t}\cos t + C_3 e^{-t}\sin t$$

Assume $y_p(t) = (At^2 + Bt)e^t + C$

$$y_p'(t) = e^t(At^2 + Bt + 2At + B)$$

$$y_p''(t) = e^t(At^2 + Bt + 4At + 2B + 2A)$$

$$y_p'''(t) = e^t(At^2 + Bt + 6At + 3B + 6A)$$

Plug in, $\begin{cases} 10At + 5B + 8A = t \\ -2C = 1 \end{cases}$

$\therefore A = \frac{1}{10}, \quad B = -\frac{4}{25}, \quad C = -\frac{1}{2}$

$\therefore y_p(t) = (\frac{1}{10}t^2 - \frac{4}{25}t)e^t - \frac{1}{2}$

41. (a) A.E. $r^2+2r+5=0$

$r_{1,2}=-1\pm 2i$

$y_h(t)=C_1e^{-t}\cos(2t)+C_2e^{-t}\sin(2t)$

Assume $y_p(t)=A$

Plug in $5A=10 \Rightarrow A=2$

$\therefore y(t)=C_1e^{-t}\cos(2t)+C_2e^{-t}\sin(2t)+2$

$y'(t)=(-C_1+2C_2)e^{-t}\cos(2t)+(-2C_1-C_2)e^{-t}\sin(2t)$

I.C. $\begin{cases} y(0)=C_1+2=0 \\ y'(0)=-C_1+2C_2=0 \end{cases} \Rightarrow \begin{cases} C_1=-2 \\ C_2=-1 \end{cases}$

$\therefore y(t)=-2e^{-t}\cos(2t)-e^{-t}\sin(2t)+2, \quad 0\le t\le \frac{3}{2}\pi$

(b) $y(t)=C_1e^{-t}\cos(2t)+C_2e^{-t}\sin(2t), \quad t>\frac{3}{2}\pi$

(c) $y(t)=\begin{cases} -2e^{-t}\cos(2t)-e^{-t}\sin(2t)+2, & 0\le t\le \frac{3}{2}\pi \\ C_1e^{-t}\cos(2t)+C_2e^{-t}\sin(2t), & t>\frac{3}{2}\pi \end{cases}$

$y(t)$ is continuous at $t=\frac{3}{2}\pi$

$y(\frac{3}{2}\pi) = -C_1e^{-\frac{3\pi}{2}} = 2e^{-\frac{3\pi}{2}}+2 \quad (*)$

$y'(t)$ is continuous at $t=\frac{3}{2}\pi$

$y'(\frac{3}{2}\pi)=C_1e^{-\frac{3\pi}{2}}-2C_2e^{-\frac{3}{2}\pi}=0 \quad (**)$

A combination of (*) (**) implies

$\begin{cases} C_1=-2-2e^{\frac{3}{2}\pi} \\ C_2=-1-e^{\frac{3}{2}\pi} \end{cases}$

42. (a) A.E. $mr^2+br+k=0$

$r_{1,2}=-\dfrac{b\pm\sqrt{4mk-b^2}\,i}{2m}$

$y_h(t)=C_1e^{-\frac{b}{2m}t}\cos\left(\frac{\sqrt{4mk-b^2}}{2m}t\right)+C_2e^{-\frac{b}{2m}t}\sin\left(\frac{\sqrt{4mk-b^2}}{2m}t\right)$

Assume $y_p(t)=A\sin\beta t+B\cos\beta t$

$y_p'(t)=A\beta\cos\beta t-B\beta\sin\beta t$

$y_p''(t)=-A\beta^2\sin\beta t-B\beta^2\cos\beta t$

Plug in,

$\begin{cases} -m\beta^2A-b\beta B+kA=1 \\ -m\beta^2B+b\beta A+kB=0 \end{cases}$

$\therefore \begin{cases} A=\dfrac{-m\beta^2+k}{(-m\beta^2+k)^2+b^2\beta^2} \\ B=\dfrac{-b\beta}{(-m\beta^2+k)^2+b^2\beta^2} \end{cases}$

$\therefore y(t)=C_1e^{-\frac{b}{2m}t}\cos\left(\frac{\sqrt{4mk-b^2}}{2m}t\right)+C_2e^{-\frac{b}{2m}t}\sin\left(\frac{\sqrt{4mk-b^2}}{2m}t\right)+A\sin\beta t+B\cos\beta t$

(b) When $t\to+\infty$, $e^{-\frac{b}{2m}}\to 0$

$y(t)\longrightarrow A\sin\beta t+B\cos\beta t$

4.6 Exercises 3. 7. 9. 11. 23

3. $y''-2y'+y=t^{-1}e^t$

Solution. A.E. $r^2-2r+1=0$

$r_1=r_2=1$

$y_h(t)=C_1e^t+C_2te^t$

Assume $y_p(t)=u_1(t)e^t+u_2(t)te^t$

$$W(y_1,y_2)=\begin{vmatrix} e^t & te^t \\ e^t & e^t+te^t \end{vmatrix}=e^{2t}$$

$$u_1(t)=\int -\frac{t^{-1}e^t\,te^t}{e^{2t}}dt=\int -1\,dt=-t$$

$$u_2(t)=\int \frac{t^{-1}e^te^t}{e^{2t}}dt=\int t^{-1}dt=\ln|t|$$

$\therefore y_p(t)=-te^t+te^t\ln|t|$

$\therefore y(t)=y_h(t)+y_p(t)$

$=C_1e^t+C_2te^t-te^t+te^t\ln|t|$

$=C_1e^t+C_2te^t+te^t\ln|t|$

7. $y''+4y'+4y=e^{-2t}\ln t$

Solution. A.E. $r^2+4r+4=0$

$r_1=r_2=-2$

$y_h(t)=C_1e^{-2t}+C_2te^{-2t}$

Assume $y_p(t)=u_1(t)e^{-2t}+u_2(t)te^{-2t}$

$$W(y_1,y_2)=\begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & e^{-2t}-2te^{-2t} \end{vmatrix}=e^{-4t}$$

$$u_1(t)=\int \frac{-e^{-2t}\ln t\cdot te^{-2t}}{e^{-4t}}dt=-\int t\ln t\,dt$$

$$=\frac{1}{4}t^2-\frac{1}{2}t^2\ln t$$

$$u_2(t)=\int \frac{e^{-2t}\ln t\cdot e^{-2t}}{e^{-4t}}dt=\int \ln t\,dt=t\ln t-t$$

$\therefore y_p(t)=(\frac{1}{4}-\frac{1}{2}\ln t)t^2e^{-2t}+(\ln t-1)t^2e^{-2t}$

$=(\frac{1}{2}\ln t-\frac{3}{4})t^2e^{-2t}$

$\therefore y(t)=y_h(t)+y_p(t)$

$=C_1e^{-2t}+C_2te^{-2t}+(\frac{1}{2}\ln t-\frac{3}{4})t^2e^{-2t}$

9. $y''-y=2t+4$

Method 1. A.E. $r^2-1=0$

$r_1=-1,\ r_2=1$

$y_h(t)=C_1e^{-t}+C_2e^t$

Assume $y_p(t)=At+B$

$y_p''(t)=0$

Plug in, $-At-B=2t+4$

$\therefore A=-2,\ B=-4$

$\therefore y_p(t)=-2t-4$

Method 2. $y_h(t)=C_1e^{-t}+C_2e^t$

Assume. $y_p(t)=u_1(t)e^{-t}+u_2(t)e^t$

$$W(y_1,y_2)=\begin{vmatrix} e^{-t} & e^t \\ -e^{-t} & e^t \end{vmatrix}=2$$

$$u_1(t)=\int -\frac{(2t+4)e^t}{2}dt=-\int(t+2)e^tdt$$

$=-(t+1)e^t$

$$u_2(t)=\int \frac{(2t+4)e^{-t}}{2}dt=\int(t+2)e^{-t}dt$$

$=-(t+3)e^{-t}$

$\therefore y_p(t)=-(t+1)e^t\cdot e^{-t}-(t+3)e^{-t}\cdot e^t$

$=-(t+1)-(t+3)$

$=-(2t+4)$

Undetermined coefficients method was quicker.

11. $y'' + y = \tan t + e^{3t} - 1$

Solution. A. E. $r^2 + 1 = 0$

$r_1 = -i,\ r_2 = i$

$y_h(t) = C_1 \cos t + C_2 \sin t$

for $f_1(t) = \tan t$,

Assume $y_{p_1}(t) = u_1(t)\cos t + u_2(t)\sin t$

$$W(y_1, y_2) = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = 1$$

$$u_1(t) = \int \frac{-\tan t \cdot \sin t}{1} dt = -\int \frac{\sin^2 t}{\cos t} dt$$

$$= \int (\cos t - \sec t) dt = \sin t - \ln|\sec t + \tan t|$$

$$u_2(t) = \int \frac{\tan t \cdot \cos t}{1} dt = \int \sin t\, dt$$

$$= -\cos t\ d$$

$\therefore y_{p_1}(t) = -\cos t \ln|\sec t + \tan t|$

for $f_2(t) = e^{3t} - 1$

assume $y_{p_2}(t) = Ae^{3t} + B$

$y_{p_2}''(t) = 9Ae^{3t}$

Plug in, $10Ae^{3t} + B = e^{3t} - 1$

$\therefore A = \frac{1}{10},\ B = -1$

$\therefore y_{p_2}(t) = \frac{1}{10} e^{3t} - 1$

$\therefore y(t) = C_1 \cos t + C_2 \sin t - \cos t \ln|\sec t + \tan t| + \frac{1}{10} e^{3t} - 1.$

23. $ty'' - (t+1)y' + y = t^2$

$y_1 = e^t,\quad y_2 = t+1$

Solution. Assume

$y_p(t) = u_1(t)e^t + u_2(t)(t+1)$

$$W(y_1, y_2) = \begin{vmatrix} e^t & t+1 \\ e^t & 1 \end{vmatrix} = -te^t$$

$$u_1(t) = \int -\frac{t^2(t+1)}{-te^t} dt$$

$$= \int (t+1)e^{-t} dt$$

$$= -(t+2)e^{-t}$$

$$u_2(t) = \int \frac{t^2 e^t}{-t \cdot te^t} dt$$

$$= \int -1\, dt = -t$$

$\therefore y_p(t) = -(t+2)e^{-t} \cdot e^t - t(t+1)$

$= -(t^2 + 2t + 2)$

$\therefore y(t) = C_1 e^t + C_2(t+1) - (t^2 + 2t + 2)$

$= C_1 e^t + C_2(t+1) - t^2.$

4.7 Exercises 9, 11, 12, 14, 20, 30, 35, 38, 39, 43.

9. $t^2y''+7ty'-7y=0$, $t>0$

Solution. A.E. $m(m-1)+7m-7=0$

$$m^2+6m-7=0$$

$$(m+7)(m-1)=0$$

$$m_1=-7,\ m_2=1$$

$$\therefore y(t)=C_1t^{-7}+C_2t.$$

11. $t^2\dfrac{d^2z}{dt^2}+5t\dfrac{dz}{dt}+4z=0$, $t>0$

Solution. A.E. $m(m-1)+5m+4=0$

$$m^2+4m+4=0$$

$$(m+2)^2=0$$

$$m_1=m_2=-2$$

$$\therefore y(t)=C_1t^{-2}+C_2t^{-2}\ln t$$

12. $\dfrac{d^2w}{dt^2}+\dfrac{6}{t}\dfrac{dw}{dt}+\dfrac{4}{t^2}w=0$

Solution.

$t^2w''+6tw'+4w=0$ Cauchy Euler

A.E. $m(m-1)+6m+4=0$

$$m^2+5m+4=0$$

$$(m+4)(m+1)=0$$

$$m_1=-4,\ m_2=-1$$

$$\therefore y(t)=C_1t^{-4}+C_2t^{-1}$$

14. $t^2y''-3ty'+4y=0$

Solution. A.E. $m(m-1)-3m+4=0$

$$m^2-4m+4=0$$

$$(m-2)^2=0$$

$$m_1=m_2=2$$

$$\therefore y(t)=C_1t^2+C_2t^2\ln t.$$

20. $t^2y''(t)+7ty'(t)+5y(t)=0$

$y(1)=-1,\ y'(1)=13$

Solution. A.E. $m(m-1)+7m+5=0$

$$m^2+6m+5=0$$

$$(m+5)(m+1)=0$$

$$m_1=-5,\ m_2=-1$$

$$\therefore y(t)=C_1t^{-5}+C_2t^{-1}$$

$$y'(t)=-5C_1t^{-6}-C_2t^{-2}$$

I.C. $\begin{cases} y(1)=C_1+C_2=-1 \\ y'(1)=-5C_1-C_2=13 \end{cases}$

$$\Rightarrow C_1=-3,\ C_2=2$$

$$\therefore y(t)=-3t^{-5}+2t^{-1}.$$

30. Proof.

$$\begin{cases} y_1''(t)+p(t)y_1'(t)+q(t)y_1(t)=g_1(t) \\ y_2''(t)+p(t)y_2'(t)+q(t)y_2(t)=g_2(t) \end{cases}$$

Let $y(t)=k_1y_1(t)+k_2y_2(t)$,

then $y''(t)+p(t)y'(t)+q(t)y(t)$

$$\begin{aligned}
&=k_1y_1''(t)+k_2y_2''(t) \\
&\quad+k_1p(t)y_1'(t)+k_2p(t)y_2'(t) \\
&\quad+k_1q(t)y_1(t)+k_2q(t)y_2(t) \\
&=k_1\left(y_1''(t)+p(t)y_1'(t)+q(t)y_1(t)\right) \\
&\quad+k_2\left(y_2''(t)+p(t)y_2'(t)+q(t)y_2(t)\right) \\
&=k_1g_1(t)+k_2g_2(t).
\end{aligned}$$

So $y(t)=k_1y_1(t)+k_2y_2(t)$ is a solution to

$y''(t)+p(t)y'(t)+q(t)y(t)=k_1g_1(t)+k_2g_2(t)$.

35. Solution.

$\because$ $1+t$, $1+2t$, $1+3t^2$ are solutions to $y''(t)+p(t)y'(t)+q(t)y(t)=g(t)$

$\therefore$ $(1+2t)-(1+t)=t$

$(1+3t^2)-(1+t)=3t^2-t$

$(1+3t^2)-(1+2t)=3t^2-2t$

are solutions to $y''(t)+p(t)y'(t)+q(t)y(t)=0$.

Since t, $3t^2-t$ are linearly independent,

$y(t)=C_1t+C_2(3t^2-t)+(1+t)$ is a general solution to $y''(t)+p(t)y'(t)+q(t)y(t)=g(t)$

$y'(t)=C_1+6C_2t-C_2+1$

I.C. $\begin{cases} y(1)=C_1+2C_2+2=2 \\ y'(1)=C_1+5C_2+1=0 \end{cases}$

$\Rightarrow C_1=\frac{2}{3},\quad C_2=-\frac{1}{3}$

$\therefore$ $y(t)=\frac{2}{3}t-\frac{1}{3}(3t^2-t)+(1+t)$

$=-t^2+2t+1$

38. $t^2y''+3ty'+y=t^{-1}$

Solution. A.E. $m(m-1)+3m+1=0$

$m^2+2m+1=0 \Rightarrow m_1=m_2=-1$

$\therefore$ $y_h(t)=C_1t^{-1}+C_2t^{-1}\ln t$

Assume $y_p(t)=u_1(t)\,t^{-1}+u_2(t)\,t^{-1}\ln t$

$W(y_1,y_2)=\begin{vmatrix} t^{-1} & t^{-1}\ln t \\ -t^{-2} & t^{-2}-t^{-2}\ln t \end{vmatrix}=t^{-3}$

$u_1(t)=\int \frac{-t^{-1}\,t^{-1}\ln t}{t^2\,t^{-3}}dt=-\int t^{-1}\ln t\,dt=-\frac{\ln^2 t}{2}$

$u_2(t)=\int \frac{t^{-1}\,t^{-1}}{t^2\,t^{-3}}dt=\int t^{-1}dt=\ln t$

$\therefore$ $y_p(t)=-\frac{1}{2}t^{-1}\ln^2 t+t^{-1}\ln^2 t=\frac{1}{2}t^{-1}\ln^2 t$

$\therefore$ $y(t)=C_1t^{-1}+C_2t^{-1}\ln t+\frac{1}{2}t^{-1}\ln^2 t$

39. $t^2z''-tz'+z=t\left(1+\frac{3}{\ln t}\right)$

Solution. A.E. $m(m-1)-m+1=0$

$m^2-2m+1=0 \Rightarrow m_1=m_2=1$

$\therefore$ $y_h(t)=C_1t+C_2t\ln t$

Assume $y_p(t)=u_1(t)\,t+u_2(t)\,t\ln t$

$W(y_1,y_2)=\begin{vmatrix} t & t\ln t \\ 1 & \ln t+1 \end{vmatrix}=t$

$u_1(t)=\int -\frac{\ln t\cdot t^2\left(1+\frac{3}{\ln t}\right)}{t^2\cdot t}dt$

$=-\left(\int \frac{\ln t}{t}dt+\int \frac{3}{t}dt\right)=-\left(\frac{\ln^2 t}{2}+3\ln t\right)$

$u_2(t)=\int \frac{t^2\left(1+\frac{3}{\ln t}\right)}{t^2\cdot t}dt$

$=\int \frac{1}{t}dt+\int \frac{3}{t\ln t}dt=\ln t+3\ln|\ln t|$

$\therefore$ $y_p(t)=-\left(\frac{\ln^2 t}{2}+3\ln t\right)t+(\ln t+3\ln|\ln t|)\,t\ln t$

$=\frac{1}{2}t\ln^2 t-3t\ln t+3t\ln t\cdot\ln|\ln t|$

$\therefore$ $y(t)=C_1t+C_2t\ln t+\frac{1}{2}t\ln^2 t$

$+3t\ln t\cdot\ln|\ln t|$.

4.9 Exercises 1, 10, 16

1. $m=2,\ k=50,\ b=0,\ f(t)=0$

$y(0)=-\frac{1}{4},\ y'(0)=-1$

$$\therefore \begin{cases} 2y''(t)+50y(t)=0 \\ y(0)=-\frac{1}{4},\ y'(0)=-1 \end{cases}$$

A.E. $r^2+25r=0 \Rightarrow r_{1,2}=\pm 5i$

$y(t)=C_1\cos 5t+C_2\sin 5t$

I.C. $\Rightarrow C_1=-\frac{1}{4},\ C_2=-\frac{1}{5}$

$$\therefore y(t)=-\frac{1}{4}\cos 5t-\frac{1}{5}\sin 5t$$

$$=\frac{\sqrt{41}}{20}\sin(5t+\varphi)$$

$\varphi=\arctan\frac{5}{4}+\pi=4.0376$

$$\therefore y(t)=\frac{\sqrt{41}}{20}\sin(5t+4.0376)$$

with

amplitude $A=\frac{\sqrt{41}}{20}$

period $T=\frac{2\pi}{5}=\frac{2}{5}\pi$

frequency $\frac{1}{T}=\frac{5}{2\pi}$

Let $y(t)=\frac{\sqrt{41}}{20}\sin(5t+4.0376)=0$

$5t+4.0376=2\pi,\ 3\pi,\ \cdots$

$$\therefore t=\frac{2\pi-4.0376}{5}=0.4491 \text{ (sec)}.$$

10. $m=\frac{1}{4},\ k=8,\ b=\frac{1}{4},\ f(t)=0$

$y(0)=-1,\ y'(0)=0.$

$$\therefore \begin{cases} \frac{1}{4}y''(t)+\frac{1}{4}y'(t)+8y(t)=0 \\ y(0)=-1,\ y'(0)=0. \end{cases}$$

A.E. $\frac{1}{4}r^2+\frac{1}{4}r+8=0$

$(r+\frac{1}{2})^2=-\frac{127}{4}$

$r_{1,2}=-\frac{1}{2}\pm\frac{\sqrt{127}}{2}i$

$$\therefore y(t)=C_1e^{-\frac{1}{2}t}\cos\frac{\sqrt{127}}{2}t+C_2e^{-\frac{1}{2}t}\sin\frac{\sqrt{127}}{2}t$$

I.C. $\Rightarrow C_1=-1,\ C_2=-\frac{1}{\sqrt{127}}$

$$\therefore y(t)=-e^{-\frac{1}{2}t}\left(\cos\frac{\sqrt{127}}{2}t+\frac{1}{\sqrt{127}}\sin\frac{\sqrt{127}}{2}t\right)$$

$$=\sqrt{\frac{128}{127}}e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{127}}{2}t+\varphi\right)$$

$\varphi=\pi+\arctan\sqrt{127}=4.6239$

$$\therefore y(t)=\sqrt{\frac{128}{127}}e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{127}}{2}t+4.6239\right)$$

$$y'(t)=\sqrt{\frac{128}{127}}e^{-\frac{1}{2}t}\left[-\frac{1}{2}\sin\left(\frac{\sqrt{127}}{2}t+4.6239\right)+\frac{\sqrt{127}}{2}\cos\left(\frac{\sqrt{127}}{2}t+4.6239\right)\right]$$

$\Rightarrow \tan\left(\frac{\sqrt{127}}{2}t+4.6239\right)=\sqrt{127}$

$\Rightarrow \frac{\sqrt{127}}{2}t+4.6239=\arctan\sqrt{127}+k\pi$

$(k=2,4,6,\cdots)$

when $k=2$,

$\frac{\sqrt{127}}{2}t=\pi \Rightarrow t=0.5575$ (sec)

$$\boxed{y_{max}=\cancel{\sqrt{\frac{128}{127}}}\,\cancel{e^{-\frac{1}{2}t}}=0.\cancel{7597}\,(m)}$$

$$y_{max}=\sqrt{\frac{128}{127}}e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{127}}{2}t+\varphi\right)$$

$$=e^{-\frac{1}{2}t}=0.7567\,(m)$$

4.9 Exercises

16. $b=0,\ f(t)=0$

$$my''+ky=0$$

A.E. $mr^2+k=0 \Rightarrow r_{1,2} = \pm\sqrt{\frac{k}{m}}\,i$

$$y(t) = C_1\cos\left(\sqrt{\tfrac{k}{m}}t\right) + C_2\sin\left(\sqrt{\tfrac{k}{m}}t\right)$$

$$= \sqrt{C_1^2+C_2^2}\,\sin\left(\sqrt{\tfrac{k}{m}}t+\varphi\right)$$

$$\begin{cases} T_1 = \dfrac{2\pi}{\sqrt{\frac{k}{m}}} = 2\pi\sqrt{\dfrac{m}{k}} = 3 \\ T_2 = \dfrac{2\pi}{\sqrt{\frac{k}{m+2}}} = 2\pi\sqrt{\dfrac{m+2}{k}} = 4 \end{cases}$$

$$\Rightarrow 4\sqrt{m} = 3\sqrt{m+2}$$

$$\Rightarrow m = \frac{18}{7}\ (kg).$$

4.10 Exercises

14. $m=8,\ k=40,\ b=3$

$f(t) = 2\sin\left(2t+\frac{\pi}{4}\right)$

$$8y''+3y'+40y = 2\sin\left(2t+\frac{\pi}{4}\right)$$

Assume $y_p(t) = A\cos(2t)+B\sin(2t)$

$y_p'(t) = -2A\sin(2t)+2B\cos(2t)$

$y_p''(t) = -4A\cos(2t)-4B\sin(2t)$

Plug in. $A=\frac{\sqrt{2}}{50},\ B=\frac{7\sqrt{2}}{50}$

$\therefore y_p(t) = \frac{\sqrt{2}}{50}\cos(2t) + \frac{7\sqrt{2}}{50}\sin(2t)$

$= \frac{1}{5}\sin(2t+\varphi)$

$\therefore$ amplitude $A=\frac{1}{5}$

frequency $f=\frac{2}{2\pi}=\frac{1}{\pi}$

5.2 Exercises 1, 2, 9, 11, 19, 21, 31

1. (a) $A[y] = (D-1)[y] = y' - y$

$= -t^3 + 3t^2 + 8$

(b) $B[A[y]] = (D+2)[-t^3+3t^2+8]$

$= -2t^3 + 3t^2 + 6t + 16$

(c) $B[y] = (D+2)[y] = y' + 2y$

$= 2t^3 + 3t^2 - 16$

(d) $A[B[y]] = (D-1)[2t^3 + 3t^2 - 16]$

$= -2t^3 + 3t^2 + 6t + 16$

(e) $C[y] = (D^2 + D - 2)[y] = y'' + y' - 2y$

$= -2t^3 + 3t^2 + 6t + 16$

2. Solution.

$(D-1)(D+2)[y] = (D-1)[y' + 2y]$

$= y'' + 2y' - y' - 2y$

$= y'' + y' - 2y$

$= (D^2 + D - 2)[y]$

So $(D-1)(D+2) = D^2 + D - 2$

9. $x' + y' + 2x = 0$

$x' + y' - x - y = \sin t$

Solution.

$$\begin{cases} (D+2)[x] + Dy = 0 & ① \\ (D-1)[x] + (D-1)[y] = \sin t & ② \end{cases}$$

$D[②] - (D-1)[①]$

$\Rightarrow (D(D-1) - (D-1)(D+2))[x] = \cos t$

$\Rightarrow (2 - 2D)[x] = \cos t$

$\Rightarrow x' - x = -\frac{1}{2}\cos t$

$\mu(t) = e^{\int -1 dt} = e^{-t}$

$(\mu(t)x)' = -\frac{1}{2}e^{-t}\cos t$

$e^{-t}x = -\frac{1}{2}\int e^{-t}\cos t \, dt$

$= -\frac{1}{2}\cdot\frac{1}{2}e^{-t}(\sin t - \cos t) + C$

$\therefore x(t) = Ce^t + \frac{1}{4}\cos t - \frac{1}{4}\sin t$

$① - ② \Rightarrow 3x + y = -\sin t$

$\Rightarrow y(t) = -3x(t) - \sin t$

$= -3Ce^t - \frac{3}{4}\cos t - \frac{1}{4}\sin t$

11. $(D^2-1)[u]+5v=e^t$ ①

$2u+(D^2+2)[v]=0$ ②

Solution. $(D^2-1)[②]-2①$

$\Rightarrow ((D^2-1)(D^2+2)-10)[v]=-2e^t$

$\Rightarrow (D^4+D^2-12)[v]=-2e^t$

i.e. $v^{(4)}+v''-12v=-2e^t$

A.E. $r^4+r^2-12=0$

$(r^2+4)(r^2-3)=0$

$r_{1,2}=\pm 2i,\ r_{3,4}=\pm\sqrt{3}$

$v_h(t)=C_1\cos 2t+C_2\sin 2t+C_3e^{\sqrt{3}t}+C_4e^{-\sqrt{3}t}$

Assume $v_p(t)=Ae^t$

$v_p^{(2)}(t)=v_p^{(4)}(t)=Ae^t$

Plug in, $(A+A-12A)e^t=-2e^t$

$\therefore A=\frac{1}{5}$

$\therefore v_p(t)=\frac{1}{5}e^t$

$\therefore v(t)=C_1\cos 2t+C_2\sin 2t+C_3e^{\sqrt{3}t}+C_4e^{-\sqrt{3}t}+\frac{1}{5}e^t$

$u(t)=-\frac{1}{2}(D^2+2)[v]$

$=C_1\cos 2t+C_2\sin 2t-\frac{5}{2}C_3e^{\sqrt{3}t}-\frac{5}{2}C_4e^{-\sqrt{3}t}-\frac{3}{10}e^t.$

19. $\frac{dx}{dt}=4x+y,\quad x(0)=1$

$\frac{dy}{dt}=-2x+y,\quad y(0)=0$

Solution.

$$\begin{cases}(D-4)[x]-y=0 & ①\\ 2x+(D-1)[y]=0 & ②\end{cases}$$

$(D-1)[①]+②$

$\Rightarrow ((D-1)(D-4)+2)[x]=0$

$\Rightarrow (D^2-5D+6)[x]=0$

$x''-5x'+6x=0$

A.E. $r^2-5r+6=0$

$(r-2)(r-3)=0$

$r_1=2,\ r_2=3$

$\therefore x(t)=C_1e^{2t}+C_2e^{3t}$

$y(t)=(D-4)[x]$

$=-2C_1e^{2t}-C_2e^{3t}$

I.C. $\left.\begin{array}{l}x(0)=C_1+C_2=1\\ y(0)=-2C_1-C_2=0\end{array}\right\}\Rightarrow\begin{cases}C_1=-1\\ C_2=2\end{cases}$

$$\therefore\begin{cases}x(t)=-e^{2t}+2e^{3t}\\ y(t)=2e^{2t}-2e^{3t}\end{cases}$$

21. $\frac{d^2x}{dt^2}=y$, $x(0)=3$, $x'(0)=1$

$\frac{d^2y}{dt^2}=x$, $y(0)=1$, $y'(0)=-1$

Solution. $\frac{d^4x}{dt^4}=x$

A.E. $r^4-1=0$

$r_{1,2}=\pm i$, $r_{3,4}=\pm 1$

$\therefore x(t)=C_1\cos t+C_2\sin t+C_3e^{t}+C_4e^{-t}$

$\therefore y(t)=\frac{d^2x}{dt^2}$

$=-C_1\cos t-C_2\sin t+C_3e^{t}+C_4e^{-t}$

I.C.
$$\begin{cases} x(0)=C_1\cancel{+C_2}+C_3+C_4=3 \\ x'(0)=C_2+C_3-C_4=1 \\ y(0)=-C_1+C_3+C_4=1 \\ y'(0)=-C_2+C_3-C_4=-1 \end{cases}$$

$\Rightarrow C_1=1,\ C_2=1,\ C_3=1,\ C_4=1$

$\therefore x(t)=\cos t+\sin t+e^{t}+e^{-t}$

$y(t)=-\cos t-\sin t+e^{t}+e^{-t}$

31.
$$\begin{cases} \frac{dx}{dt}=1.2+\frac{y}{100}-\frac{x}{100}\cdot 7, & x(0)=0 \\ \frac{dy}{dt}=\frac{3x}{100}-\frac{3y}{100}, & y(0)=20 \end{cases}$$

$$\begin{cases} (D+\frac{7}{100})[x]-\frac{1}{100}y=1.2 & ① \\ -\frac{3}{100}x+(D+\frac{3}{100})[y]=0 & ② \end{cases}$$

$(D+\frac{3}{100})[①]+\frac{1}{100}\cdot ②$

$\Rightarrow \left((D+\frac{3}{100})(D+\frac{7}{100})-\frac{3}{100}\cdot\frac{1}{100}\right)[x]=\frac{3.6}{100}$

$\Rightarrow \left(D^2+\frac{1}{10}D+\frac{18}{10000}\right)[x]=\frac{3.6}{100}$

i.e. $x''+\frac{1}{10}x'+\frac{18}{10000}x=\frac{3.6}{100}$

A.E. $r^2+\frac{1}{10}r+\frac{18}{10000}=0$

$r_{1,2}=-\frac{1}{20}\pm\frac{\sqrt{7}}{100}$

$(r_1=-\frac{1}{20}+\frac{\sqrt{7}}{100},\ r_2=-\frac{1}{20}-\frac{\sqrt{7}}{100})$

~~Assum~~ $\therefore x_h(t)=C_1e^{r_1t}+C_2e^{r_2t}$

Assume $x_p(t)=A$

Plug in, $A=20$

$\therefore x(t)=C_1e^{r_1t}+C_2e^{r_2t}+20$

$\therefore y(t)=100(D+\frac{7}{100})[x]-120$

$=(100r_1+7)C_1e^{r_1t}+(100r_2+7)C_2e^{r_2t}+20$

I.C.
$$\begin{cases} x(0)=C_1+C_2+20=0 \\ y(0)=(100r_1+7)C_1+(100r_2+7)C_2+20=20 \end{cases}$$

$\Rightarrow C_1=-10+\frac{20}{\sqrt{7}},\quad C_2=-10-\frac{20}{\sqrt{7}}$

$r_1=-\frac{1}{20}+\frac{\sqrt{7}}{100},\quad r_2=-\frac{1}{20}-\frac{\sqrt{7}}{100}$

$$\begin{cases} x(t)=C_1e^{r_1t}+C_2e^{r_2t}+20 \\ y(t)=(100r_1+7)C_1e^{r_1t}+(100r_2+7)C_2e^{r_2t}+20 \end{cases}$$

$y(t)=-\frac{30}{\sqrt{7}}e^{r_1t}+\frac{30}{\sqrt{7}}e^{r_2t}+20$