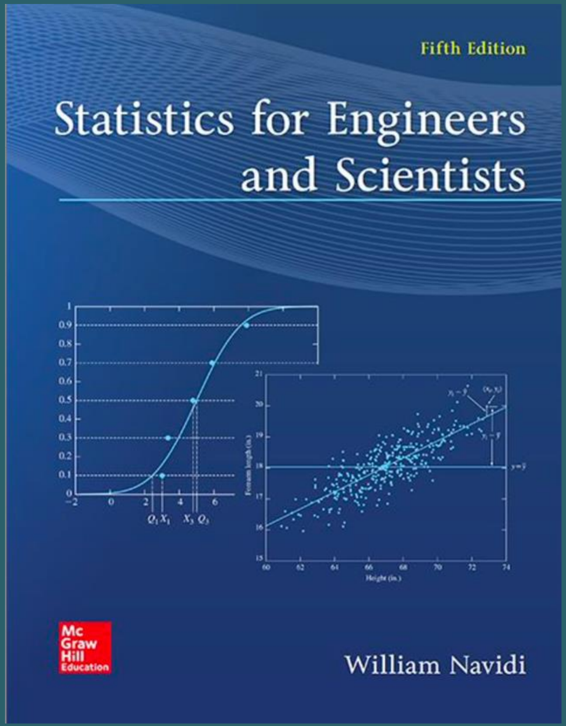

Ch 4: Commonly Used Distributions

(not all sections are required)

Ch 4: Overview (Required Sections)

4-1 The Bernoulli Distribution

4-2 The Binomial Distribution

4-3 The Poisson Distribution

4-5 The Normal Distribution

4-9 Some Principles of Point Estimation

4-10 Probability Plots

4-11 Central Limit Theorem

The Poisson Distribution (p.217)

- The Poisson distribution arises frequently in scientific work.
- It can be considered as an approximation to the binomial distribution when n is large and p is small.

Example of Poisson Distribution (p.217)

- A mass contains **10,000 atoms** of a radioactive substance.
- The probability that a given atom will decay in a one-minute time period is **0.0002**.
- Let X represent the number of atoms that decay in one minute.
- Now each atom can be thought of as a Bernoulli trial, where success occurs if the atom decays.
- Thus, X is the number of successes in 10,000 independent Bernoulli trials, each with success probability 0.0002, so the distribution of X is Bin(10,000, 0.0002).
- The mean of X is $\boldsymbol{\mu_X = n.p = (10{,}000)(0.0002) = 2}$

Example of Poisson Distribution (p.217)

- Another mass contains 5000 atoms, and each of these atoms has probability 0.0004 of decaying in a one-minute time interval.
- Let Y represent the number of atoms that decay in one minute from this mass.
- Thus,
- $Y \sim \text{Bin}(5000, 0.0004)$ and $\mu_Y = (5000)(0.0004) = 2$

Example of Poisson Distribution (p.217)

- In each of these cases, the number of trials n and the success probability p are different, but the mean number of successes, which is equal to the product np, is the same.
- Now assume that we wanted to compute the probability that exactly three atoms decay in one minute for each of these masses.
- Using the binomial probability mass function, we would compute as follows:

$$P(X = 3) = \frac{10{,}000!}{3!\,9997!}(0.0002)^3(0.9998)^{9997} = 0.180465091$$

$$P(Y = 3) = \frac{5000!}{3!\,4997!}(0.0004)^3(0.9996)^{4997} = 0.180483143$$

The Poisson Distribution (p.217)

- If n is large and p is small, and we let ***λ = np***
- The Poisson probability mass function can be defined by

$$p(x) = P(X = x) = \begin{cases} e^{-\lambda}\dfrac{\lambda^x}{x!} & \text{if } x \text{ is a non-negative integer} \\ 0 & \text{otherwise} \end{cases} \tag{4.9}$$

- If X is a random variable whose probability mass function is given by Equation (4.9), then X is said to have the Poisson distribution with parameter λ.
- The notation is **X ~ Poisson(λ)**

Example 4.15 (p.217)

If X ~ Poisson(3), compute

- P(X = 2)
- P(X = 10)
- P(X = 0)
- P(X = −1)
- P(X = 0.5)

$$P(X = 2) = e^{-3}\frac{3^2}{2!} = 0.2240$$

$$P(X = 10) = e^{-3}\frac{3^{10}}{10!} = 0.0008$$

$$P(X = 0) = e^{-3}\frac{3^0}{0!} = 0.0498$$

$P(X = -1) = 0$ because −1 is not a non-negative integer

$P(X = 0.5) = 0$ because 0.5 is not a non-negative integer

Example 4.16 (p.217)

If X ~ Poisson(4), compute P(X ≤ 2)

$$
\begin{aligned}
P(X \le 2) &= P(X = 0) + P(X = 1) + P(X = 2) \\
&= e^{-4}\frac{4^0}{0!} + e^{-4}\frac{4^1}{1!} + e^{-4}\frac{4^2}{2!} \\
&= 0.0183 + 0.0733 + 0.1465 \\
&= 0.2381
\end{aligned}
$$

Example 4.16 (p.217)

If X ~ Poisson(4), compute P(X > 1)

To find $P(X > 1)$, we might try to start by writing

$$P(X > 1) = P(X = 2) + P(X = 3) + \cdots$$

This leads to an infinite sum that is difficult to compute. Instead, we write

$$\begin{aligned} P(X > 1) &= 1 - P(X \leq 1) \\ &= 1 - [P(X = 0) + P(X = 1)] \\ &= 1 - \left(e^{-4}\frac{4^0}{0!} + e^{-4}\frac{4^1}{1!} \right) \\ &= 1 - (0.0183 + 0.0733) \\ &= 0.9084 \end{aligned}$$

The Poisson approximation to the binomial probability mass function

For the radioactive masses described at the beginning of this section,

$$P(X = 3) = \frac{10{,}000!}{3!\,9997!}(0.0002)^3(0.9998)^{9997} = 0.180465091$$

$$P(Y = 3) = \frac{5000!}{3!\,4997!}(0.0004)^3(0.9996)^{4997} = 0.180483143$$

we would use the Poisson mass function to approximate either P(X = x) or P(Y = x) by substituting λ = 2 into Equation (4.9)

$$p(x) = P(X = x) = \begin{cases} e^{-\lambda}\dfrac{\lambda^x}{x!} & \text{if } x \text{ is a non-negative integer} \\ 0 & \text{otherwise} \end{cases} \qquad (4.9)$$

Table 4.1 (page 218) shows that the approximation is excellent.

x	$P(X = x)$, $X \sim$ Bin(10,000, 0.0002)	$P(Y = x)$, $Y \sim$ Bin(5000, 0.0004)	Poisson Approximation, Poisson(2)
0	0.135308215	0.135281146	0.135335283
1	0.270670565	0.270670559	0.270670566
2	0.270697637	0.270724715	0.270670566
3	0.180465092	0.180483143	0.180447044
4	0.090223521	0.090223516	0.090223522
5	0.036082189	0.036074965	0.036089409
6	0.012023787	0.012017770	0.012029803
7	0.003433993	0.003430901	0.003437087
8	0.000858069	0.000856867	0.000859272
9	0.000190568	0.000190186	0.000190949

*When n is large and p is small, the Bin(n, p) probability mass function is well approximated by the Poisson (i) probability mass function (Equation 4.9), with $\lambda = np$. Here $X \sim$ Bin(10,000, 0.0002) and $Y \sim$ Bin(5000, 0.0004), so $\lambda = np = 2$, and the Poisson approximation is Poisson(2).

The Poisson Distribution (p.218)

Summary

If $X \sim \text{Poisson}(\lambda)$, then

- X is a discrete random variable whose possible values are the non-negative integers.
- The parameter λ is a positive constant.
- The probability mass function of X is

$$p(x) = P(X = x) = \begin{cases} e^{-\lambda}\dfrac{\lambda^x}{x!} & \text{if } x \text{ is a non-negative integer} \\ 0 & \text{otherwise} \end{cases}$$

- The Poisson probability mass function is very close to the binomial probability mass function when n is large, p is small, and $\lambda = np$.

The Mean and Variance of a Poisson Random Variable (p.219)

Summary

If $X \sim \text{Poisson}(\lambda)$, then the mean and variance of X are given by

$$\mu_X = \lambda \tag{4.10}$$

$$\sigma_X^2 = \lambda \tag{4.11}$$

The Mean and Variance of a Poisson Random Variable (p.219)…

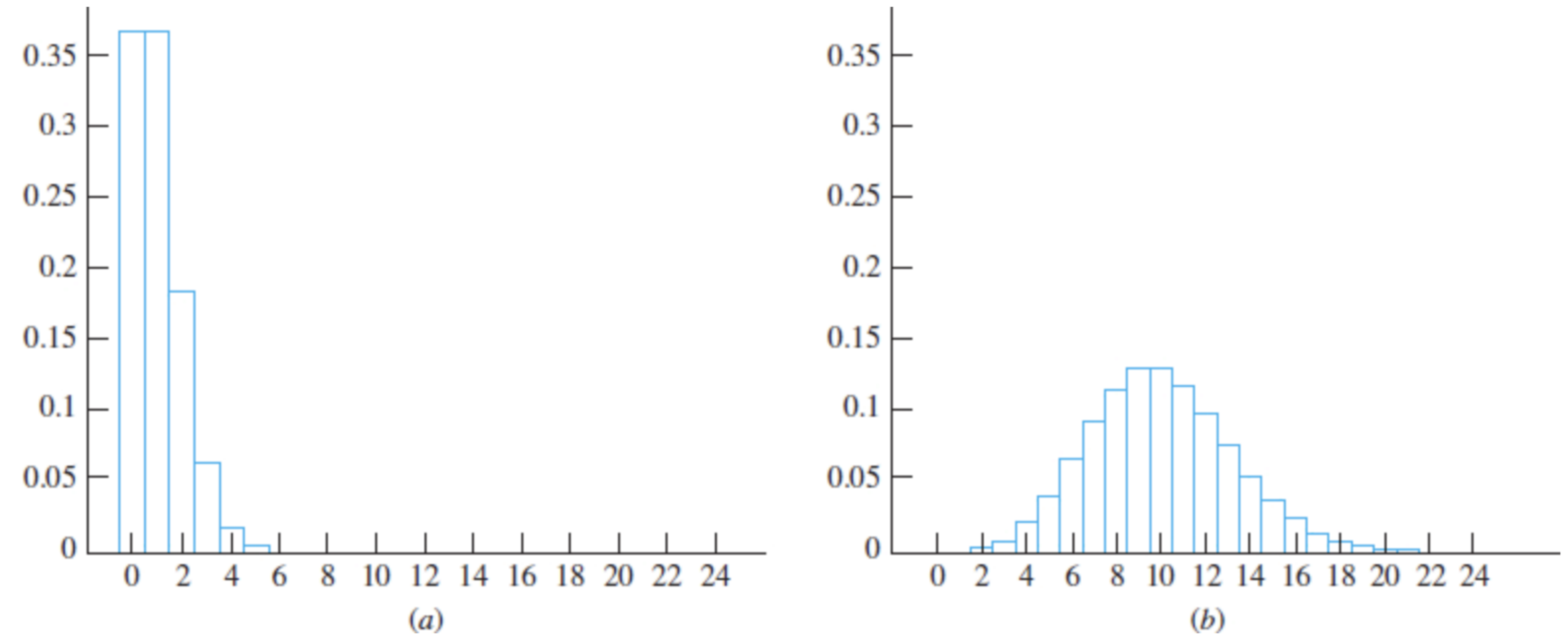

FIGURE 4.3 *(a)* The Poisson(1) probability histogram. *(b)* The Poisson(10) probability histogram.

Example 4.17 (p.220)

- Particles (e.g., yeast cells) are suspended in a liquid medium at a concentration of 10 particles per mL.
- A large volume of the suspension is thoroughly agitated, and then 1 mL is withdrawn.
- **What is the probability that exactly eight particles are withdrawn?**

Example 4.17 (p.220) - SOLUTION

- Let V be the total volume of the suspension, in mL.
- Then the total number of particles in the suspension is 10V.
- Think of each of the 10V particles as a Bernoulli trial. A particle "succeeds" if it is withdrawn.
- Now 1 mL out of the total of V mL is to be withdrawn.
- Therefore, the amount to be withdrawn is 1/V of the total, so it follows that each particle has probability 1/V of being withdrawn.

Example 4.17 (p.220) – SOLUTION…

- Let X denote the number of particles withdrawn.
- Then X represents the number of successes in 10V Bernoulli trials, each with probability 1/V of success.
- Therefore $X \sim \text{Bin}(10V, 1/V)$. Since V is large, 10V is large and 1/V is small.
- Thus, to a very close approximation, **$X \sim \text{Poisson}(10)$**.
- We compute $P(X = 8)$ with the Poisson probability mass function:

$$\mathbf{P(X = 8) = e^{-10}10^8/8! = 0.1126}$$

Example 4.18 (p.221)

- Particles are suspended in a liquid medium at a concentration of **6 particles per mL**.

- A large volume of the suspension is thoroughly agitated, and then **3 mL are withdrawn**.

- What is the probability that exactly **15 particles are withdrawn**?

Example 4.18 (p.221) - SOLUTION

- Let X represent the number of particles withdrawn.
- The mean number of particles in a 3 mL volume is 18.
- Therefore **X ~ Poisson(18)**.
- The probability that **exactly 15 particles** are withdrawn is

$$P(X = 15) = e^{-18}\frac{18^{15}}{15!} = 0.0786$$

Examples 4.17 and 4.18 - COMMENTS

- Note that for the solutions to Examples 4.17 and 4.18 to be correct, it is important that the amount of suspension withdrawn **not be too large** a fraction of the total.

- For example, if the total volume in Example 4.18 was 3 mL, so that the entire amount was withdrawn, it would be certain that all 18 particles would be withdrawn.
- So, the probability of withdrawing 15 particles would be **zero**.

Summary

If $X \sim \text{Poisson}(\lambda)$, then

- X is a discrete random variable whose possible values are the non-negative integers.
- The parameter λ is a positive constant.
- The probability mass function of X is

$$p(x) = P(X = x) = \begin{cases} e^{-\lambda}\dfrac{\lambda^x}{x!} & \text{if } x \text{ is a non-negative integer} \\ 0 & \text{otherwise} \end{cases}$$

- The Poisson probability mass function is very close to the binomial probability mass function when n is large, p is small, and $\lambda = np$.

Summary

If $X \sim \text{Poisson}(\lambda)$, then the mean and variance of X are given by

$$\mu_X = \lambda \tag{4.10}$$

$$\sigma_X^2 = \lambda \tag{4.11}$$

End of Section 4.3