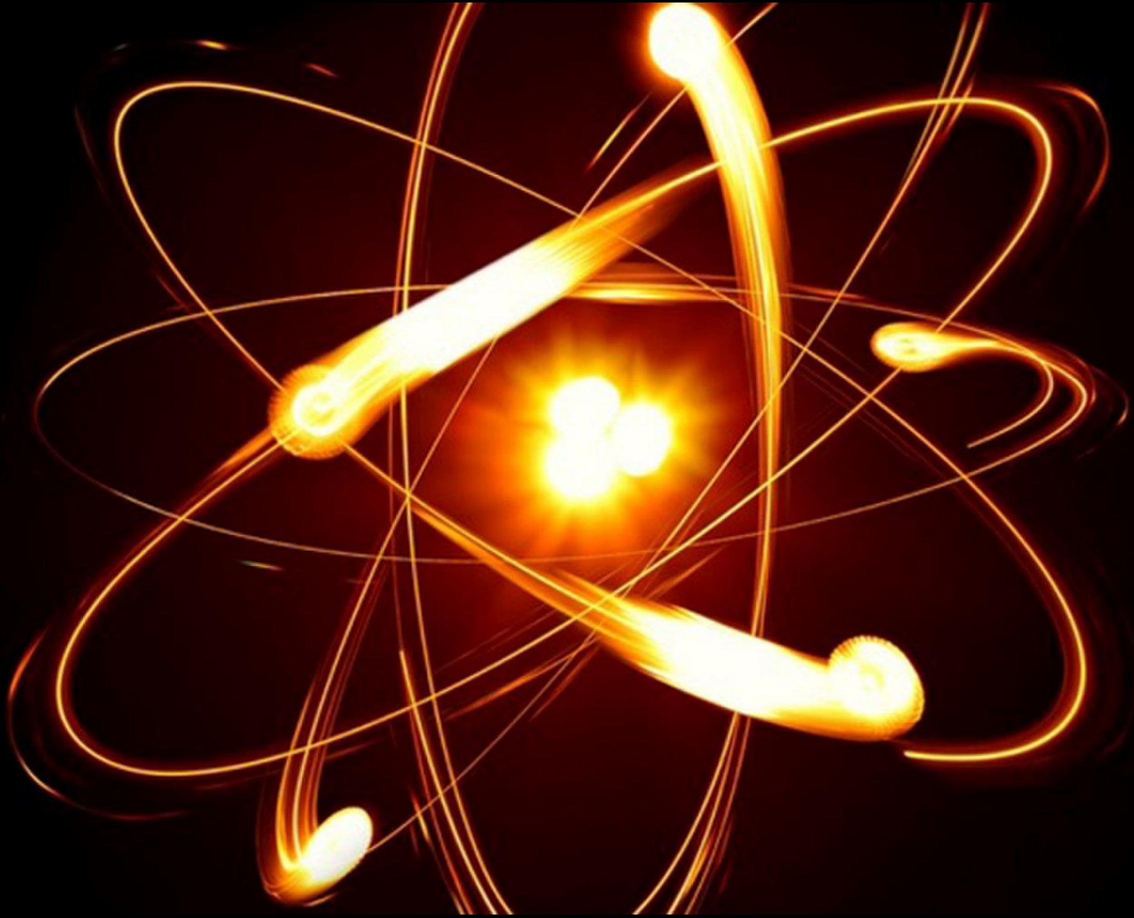


Physics 1: **Mechanics and Waves**

Week 7 – Circular motion

2023.4

Review

1. Acceleration in curvilinear Motion

$$\vec{a} = \frac{\mathrm{d}v}{\mathrm{d}t}\hat{t} + \frac{v^2}{\rho}\hat{n} = \vec{a}_t + \vec{a}_n$$

2. Nonuniform circular motion

$$\vec{a} = \frac{\mathrm{d}v}{\mathrm{d}t}\hat{t} + \frac{v^2}{r}\hat{n} = \vec{a}_t + \vec{a}_c$$

3. Acceleration in uniform circular motion

$$\vec{a} = \frac{\mathrm{d}v}{\mathrm{d}t}\hat{t} + \frac{v^2}{r}\hat{n} = \frac{v^2}{r}\hat{n}$$

Review

4. Description of the circular motion with angular coordinates

① Angular position (coordinate) $\theta(t)$

② Angular displacement $\Delta\theta(t)$

③ angular speed and angular velocity vector

$$\vec{\omega}(t)=\omega_z(t)\hat{k}=\frac{\mathrm{d}\theta(t)}{\mathrm{d}t}\hat{k}$$

SI： rad/s

④ angular acceleration

$$\vec{\alpha}(t)=\frac{\mathrm{d}\vec{\omega}(t)}{\mathrm{d}t}=\alpha_z(t)\hat{k}$$

SI： rad/s^2

Review

5. The relationship between angular and linear quantities in circular motion

$$\vec{r}(t) = r\cos\theta\hat{i} + r\sin\theta\hat{j}$$

$$\Delta s = r\Delta\theta$$

$$v = \frac{\mathrm{d}s(t)}{\mathrm{d}t} = r\frac{\mathrm{d}\theta(t)}{\mathrm{d}t} = r\omega$$

$$\vec{v}(t) = \vec{\omega}(t)\times\vec{r}(t)$$

$$\begin{aligned}\vec{a}(t) &= \vec{\alpha}(t)\times\vec{r}(t) + \vec{\omega}(t)\times\vec{v}(t)\\ &= r\alpha_z\hat{t} + r\omega^2\hat{n}\\ &= \vec{a}_t + \vec{a}_c\end{aligned}$$

$$a_t = r\alpha_z$$

$$a_n = r\omega^2$$

Nonuniform circular motion - with a constant angular acceleration

Initial time: $\alpha_z = \text{constant}, \quad \omega(t_i) = \omega_{z0}, \quad \theta(t_i) = \theta_0$

From $\alpha_z = \dfrac{d\omega_z}{dt} = \text{costant}$

Integrate in both sides $\displaystyle\int_{\omega_{z0}}^{\omega_z} d\omega_z = \int_0^t \alpha_z dt, \quad \omega_z(t) = \omega_{z0} + \alpha_z t$

From $\omega_z(t) = \dfrac{d\theta(t)}{dt}$

Integrate in both sides $\displaystyle\int_{\theta_0}^{\theta} d\theta = \int_0^t \omega_z dt = \int_0^t (\omega_{z0} + \alpha_z t) dt$

$$\theta(t) = \theta_0 + \omega_{z0} t + \frac{1}{2}\alpha_z t^2$$

Nonuniform circular motion - with a constant angular acceleration

Comparison : nonuniform circular motion with a constant angular acceleration and the rectilinear motion with a constant acceleration

$$\begin{cases} \alpha_z = \dfrac{\mathrm{d}\omega_z}{\mathrm{d}t} = \mathbf{costant} \\ \omega_z(t) = \omega_{z0} + \alpha_z t \\ \theta(t) = \theta_0 + \omega_{z0} t + \dfrac{1}{2}\alpha_z t^2 \\ \omega_z{}^2 - \omega_{z0}^2 = 2\alpha_z \Delta\theta \end{cases}$$

$$\begin{cases} a_x = \dfrac{\mathrm{d}v_x}{\mathrm{d}t} = \mathbf{costant} \\ v_x(t) = v_{x0} + a_x t \\ x(t) = x_0 + v_{x0} t + \dfrac{1}{2} a_x t^2 \\ v^2 - v_0^2 = 2a\Delta x \end{cases}$$

Conceptual example

(1) **If the motion function of a particle is $x = x(t)$, $y = y(t)$, then which one is correct in the following expressions, why?**

$$(a)\ r = \sqrt{x^2 + y^2} \qquad (b)\ v = \frac{dr}{dt} \qquad (c)\ a = \frac{d^2 r}{dt^2}$$

(2) **Judge what is correct answer for a particle moves in a curve path in the following expressions:**

$$(a)\ \frac{dv}{dt} = a \qquad (b)\ \frac{dr}{dt} = v \qquad (c)\ \frac{ds}{dt} = v \qquad (d)\ \left|\frac{d\vec{v}}{dt}\right| = a_t$$

The circular motion - Exercise 1

Your antique stereo turntable of radius 13.7 cm, initially spinning at 33.0 revolutions per minute, is shut off. The turntable coasts to a stop after 120 s. Assume a constant angular acceleration. Calculate the angular acceleration of the turntable and the number of revolutions through which it spins as it stops.

The circular motion - Exercise 1

Your antique stereo turntable of radius 13.7 cm, initially spinning at 33.0 revolutions per minute, is shut off. The turntable coasts to a stop after 120 s. Assume a constant angular acceleration. Calculate the angular acceleration of the turntable and the number of revolutions through which it spins as it stops.

Solution:

$$\omega_z(t) = \omega_{z0} + \alpha_z t \Rightarrow \alpha_z = \frac{-\omega_{z0}}{t}$$

$$\theta(t) = \theta_0 + \omega_{z0} t + \frac{1}{2}\alpha_z t^2$$

$$\Rightarrow \theta = \omega_{z0} t - \frac{1}{2}\left(\frac{\omega_{z0}}{t}\right)t^2 = \frac{1}{2}\omega_{z0} t$$

The circular motion - Exercise 2

A particle begins at rest on a circular track and is subjected to a constant angular acceleration of magnitude α beginning when t = 0 s.

(a) Show that the magnitudes of the tangential and centripetal accelerations of the particle are equal when $t = (1/\alpha)^{1/2}$ independent of the radius of the circular track.

(b) What is the angle that the total acceleration vector makes with the radial direction at this time?

The circular motion

Solution:

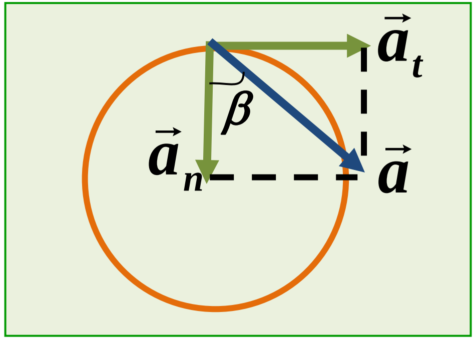
$$\omega_0 = 0$$

(a) $$a_t = r\alpha, \quad a_n = r\omega^2, \quad \omega = \omega_0 + \alpha t$$

$$a_t = a_n, \quad r\alpha = r(\alpha t)^2 \Rightarrow t = \sqrt{\frac{1}{\alpha}}$$

(b)

$$\beta = \tan^{-1}\frac{a_t}{a_n} = \tan^{-1} 1 = 45^\circ$$



The circular motion - Exercise 3

A point P on the edge of a rotational disk with radius *R*, the distance traveling by the point P is

$$\boldsymbol{s = v_0 t + \frac{1}{2} b t^2}$$

where v_0 and *b* are constants, what are the speed and the magnitude of the acceleration of the point P at any instant *t*?

The circular motion - Exercise 3

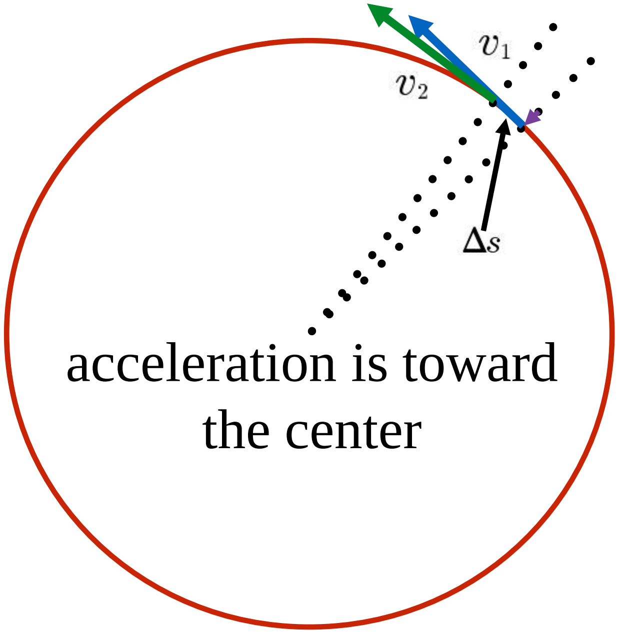
Solution:

$$v = \frac{\mathrm{d}s}{\mathrm{d}t} = v_0 + bt$$

$$a_t = \frac{\mathrm{d}v}{\mathrm{d}t} = b, \quad a_c = \frac{v^2}{R} = \frac{(v_0 + bt)^2}{R}$$

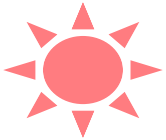
$$a = \sqrt{a_t^2 + a_c^2} = \sqrt{b^2 + \frac{(v_0 + bt)^4}{R^2}}$$

Uniform Circular motion - Centripetal acceleration



acceleration is toward the center

Direction and magnitude



$$a_c = \frac{v^2}{r}$$

$$v = \frac{2\pi r}{T}$$

Number of revolutions per second $f = 1/T$ *Time for one complete revolution*

Correspondence

	Linear	Circular	Relationship
Displacement	x	θ	$\theta = \frac{x}{r}$
Velocity	v	ω	$\omega = \frac{v}{r}$
Acceleration	a_τ	α	$\alpha = \frac{a}{r}$

Why

How to make sure the speed limit when a car rounds a curve?

Why a people in a satellite orbit close to the Earth will experience "weightlessness"
There is no gravity in space?
What is the net force on an astronaut at rest inside the space station?

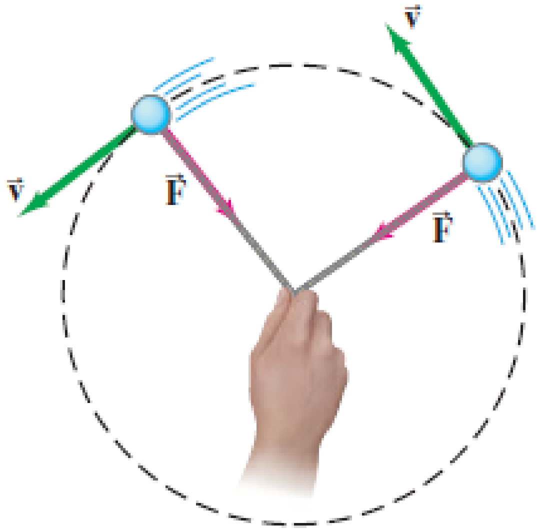
Dynamics of Uniform Circular Motion

Centripetal force :

If the particle has a mass *m* and is being accelerated, it must be subject to a force.

$$F = ma$$

This force is acting towards the center of the circle – it is centripetal (meaning = center seeking)



$$F = ma_{cent} = \frac{mv^2}{r}$$

Dynamics of Uniform Circular Motion

What is the normal force at the top and at the bottom ?

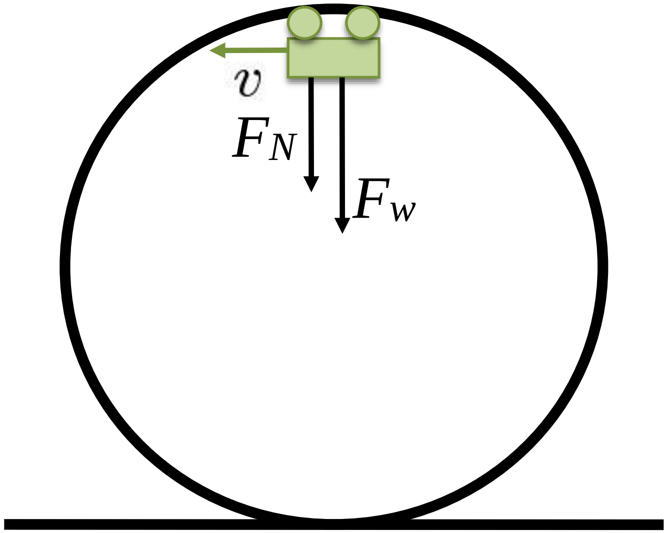
Roller Coaster

speed = 15 m/s

diameter = 40 m

total mass = 1200 kg

Example



On top of the loop

$$F_N + F_w = \frac{mv^2}{r}$$

$$F_N = \frac{mv^2}{r} - mg$$

$$F_N = 1500\text{N}$$

At the bottom

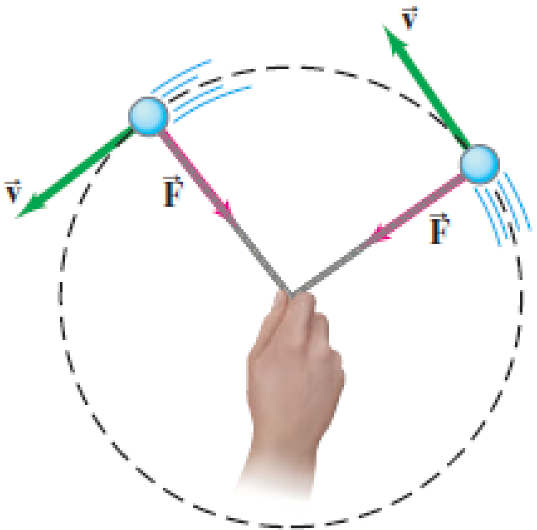
$$F_N - F_w = \frac{mv^2}{r}$$

$$F_N = \frac{mv^2}{r} + mg$$

$$F_N = 25{,}500\ \text{N}$$

Centripetal Force

What will happen without this force?



Centripetal Force

EXERCISE D To negotiate a flat (unbanked) curve at a *faster* speed, a driver puts a couple of sand bags in his van aiming to increase the force of friction between the tires and the road. Will the sand bags help?

Increasing the friction force to increase a and v

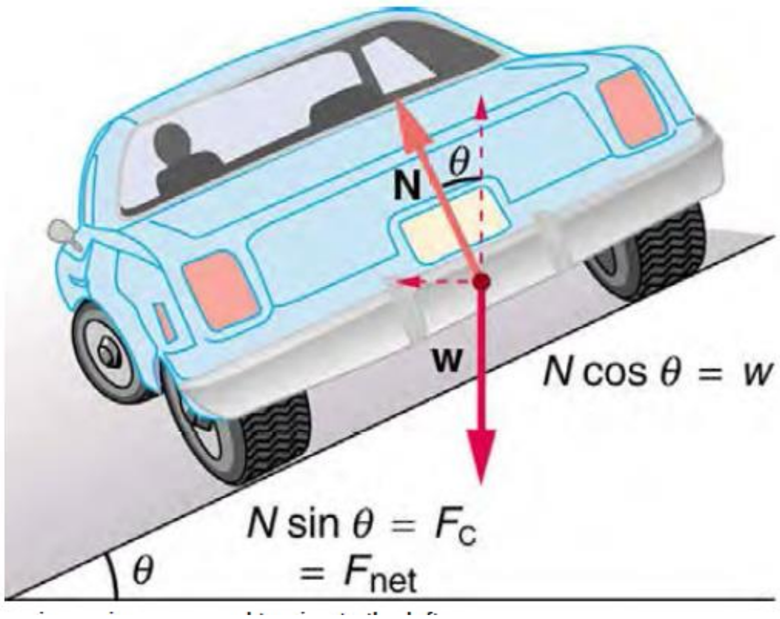
For a faster speed, will the sand bags help?

What will help?

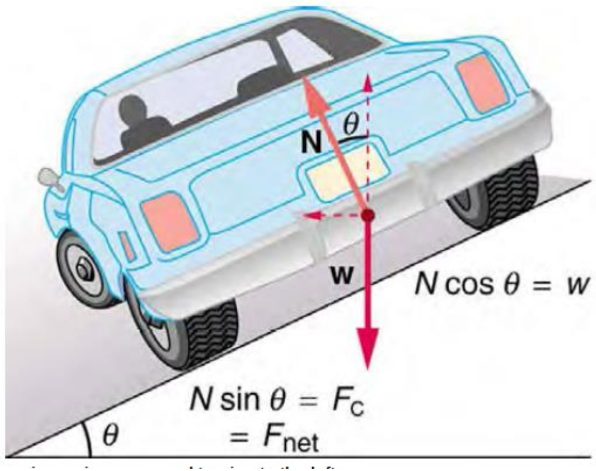
Banked curves

To reduce the chance of skidding, banking road

What’s the ideal angle for *v*



Banked curves



We want: $$N \sin\theta = \frac{mv^2}{r}.$$

But also: $$N \cos\theta = mg.$$

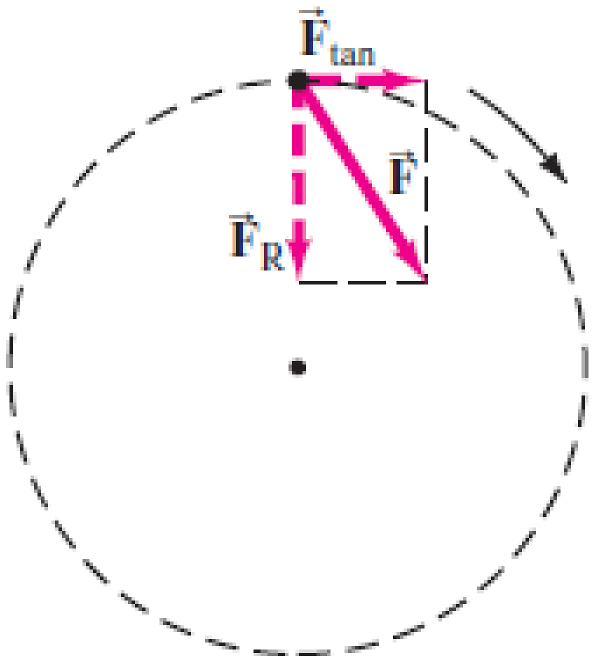
$$mg\frac{\sin\theta}{\cos\theta} = \frac{mv^2}{r}$$

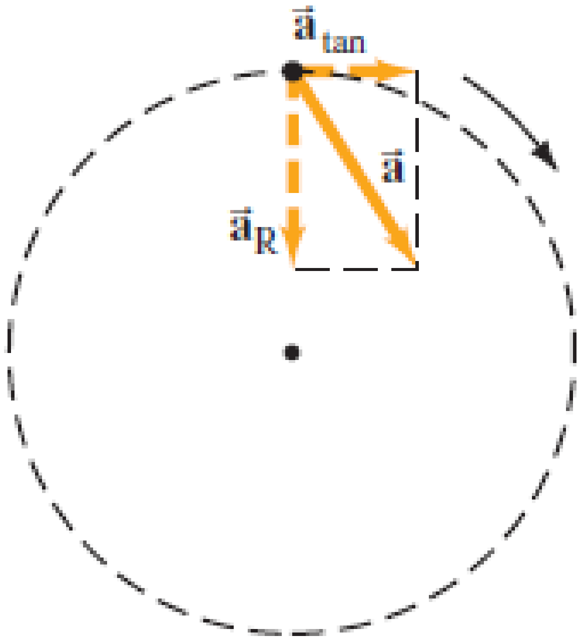
$$mg\tan(\theta) = \frac{mv^2}{r}$$

Does not depend on the mass

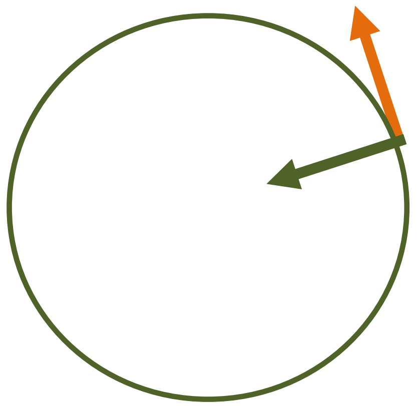
'Ideal' angle: $$\tan\theta = \frac{v^2}{rg}.$$

Nonuniform Circular Motion





Nonuniform Circular Motion - Angular acceleration



$$\vec{a} = \vec{a}_{\tan} + \vec{a}_R$$

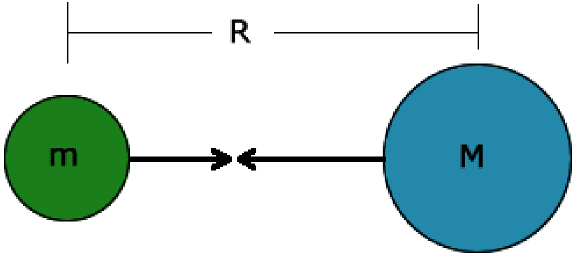
Angular acceleration

Tangential and centripetal (radial) acceleration

Newton's law of Universal Gravitation

Group work : thinking map

$$F_G = \frac{Gm_1 m_2}{r^2}$$



READING P 119 - 5-5

Process

Apple fall

Attract at a distance

a_r for moon

$F \propto \frac{1}{r^2}$

Comparison

1/3600

a_r on earth

60times further

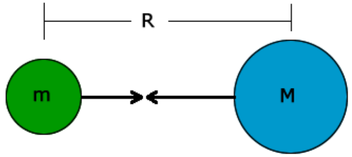
Depend on mass

Third law

Both mass

Measure G

Newton's law of Universal Gravitation



Every particle in the universe ***attracts*** every other particle with a force that is

proportional to the product of their masses and

inversely proportional to the square of the distance between them.

This force acts ***along the line*** joining the two particles.

$$F_G = \frac{G m_1 m_2}{r^2}$$

$$G = 6.67 \times 10^{-11}\,\mathrm{N\,m^2/kg^2}$$

Universal Gravitational Constant ($m^3/kg \cdot s^2$)

Example

Estimate the gravitational force of attraction between two students sitting next to each other.

Gravitational constant: $G = 6.67\times10^{-11}\ \mathrm{N\cdot m^2\cdot kg^{-2}}$)

Assume the mass of each student to be $m = 60$ kg ($m_1 = m_2$)

Assume two students are sitting 0.5 m apart in classroom

Example

Estimate the gravitational force of attraction between two students sitting next to each other.

Gravitational constant: $G = 6.67\times10^{-11}\ \mathrm{N{\cdot}m^2{\cdot}kg^{-2}}$)

Assume the mass of each student to be m = 60 kg ($m_1 = m_2$)

Assume two students are sitting 0.5 m apart in classroom

$$F = 9.6\times10^{-7}N \approx 1\mu N$$

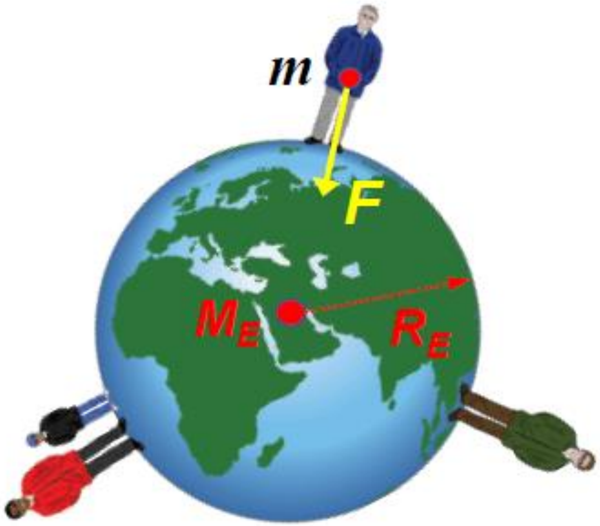
Small but measurable

The gravity of the earth

For a body with ***spherical symmetry***, such as the Earth, all the mass can be regarded to be at the center of mass, when calculating the gravitational force.

For a body with ***spherical symmetry***, such as the Earth, all the mass can be regarded to be at the ***center of mass***, when calculating the gravitational force.

$M_E = 5.97 \times 10^{24}$ kg
$m = 60$ kg
$R_E = 6.37 \times 10^{6}$ m
$G = 6.67 \times 10^{-11} m^3 \cdot kg^{-1} \cdot s^{-2}$



$$F_G = \frac{G m_1 m_2}{r^2}$$

$$F = 588.8N$$

Gravitational force of a uniform sphere on a particle

1. Shell theorem #1

A uniformly dense spherical shell attracts an external particle as if all the mass of the shell were concentrated at its center.

2. Shell theorem #2

A uniformly dense spherical shell exerts no gravitational force on a particle located anywhere inside it.

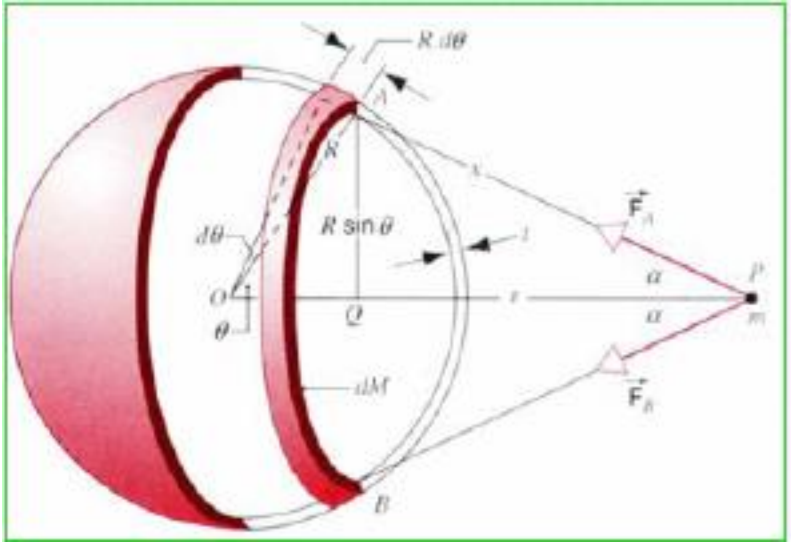
Proof of the shell theorems

$$dM = 2\pi R\sin\theta \cdot Rd\theta \cdot t \cdot \rho = 2\pi\rho t R^2 \sin\theta d\theta$$

$$\cos\alpha = \frac{r - R\cos\theta}{x}$$

$$R\cos\theta = \frac{r^2 + R^2 - x^2}{2r}$$

$$\sin\theta d\theta = \frac{x}{rR}dx$$



The force exerted by the circular ring d*M* on *m*: $dF = \frac{\pi G t\rho m R}{r^2}(\frac{r^2 - R^2}{x^2} + 1)dx$

Proof of the shell theorems

The total force on *m* due to the entire shell:

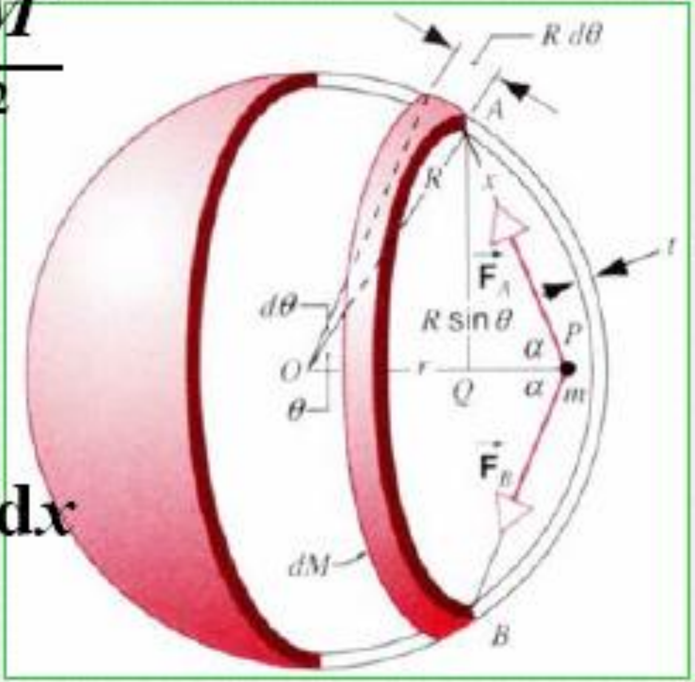
$$F = \int dF = \frac{\pi G t \rho m R}{r^2} \int_{r-R}^{r+R} \left(\frac{r^2 - R^2}{x^2} + 1\right) dx$$

Proof of the shell theorems

$$F = \frac{\pi G t \rho m R}{r^2}(4R) = G\frac{mM}{r^2}$$

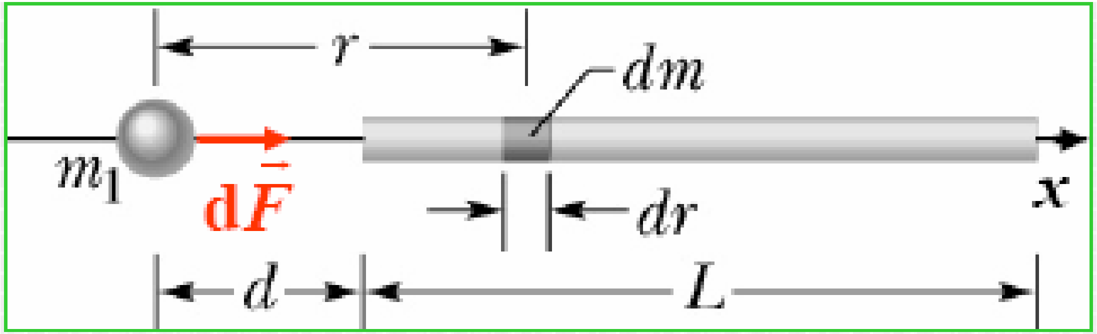
Inside the shell:

$$F = \int \mathrm{d}F = \frac{\pi G t \rho m R}{r^2}\int_{R-r}^{r+R}\left(\frac{r^2 - R^2}{x^2} + 1\right)\mathrm{d}x = 0$$



Example

Find the gravitational force between The small ball of mass *m*1 and the thin staff of mass *m* and length *L*.



Solution:

$$d\vec{F} = G\frac{m_1 dm}{r^2}\hat{i}$$

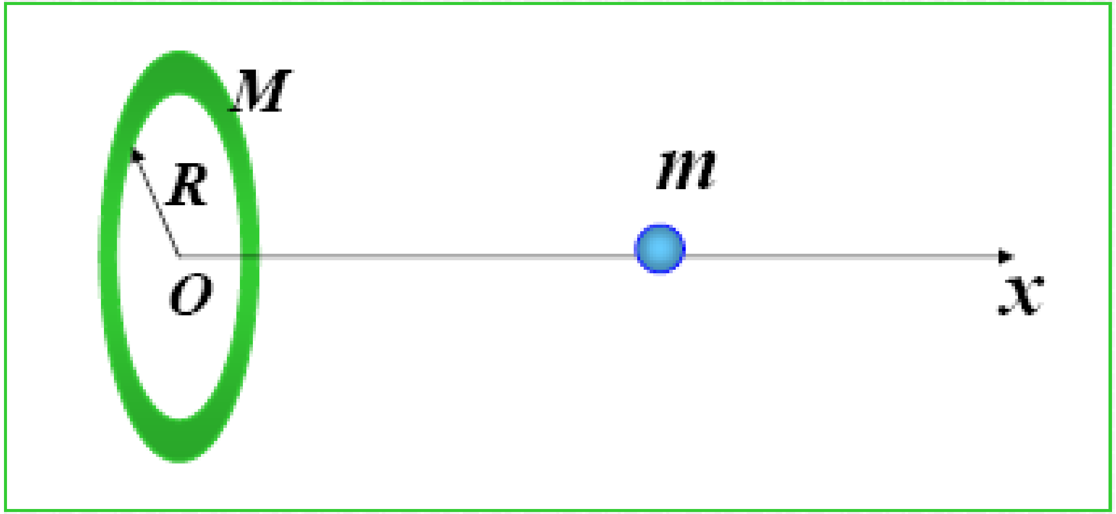
$$dm = \lambda dr = \frac{m}{L}dr$$

$$\vec{F} = \int d\vec{F} = \int_d^{d+L} G\frac{m_1 m}{L}\frac{dr}{r^2}\hat{i} = G\frac{m_1 m}{L}\left(\frac{1}{d} - \frac{1}{d+L}\right)\hat{i}$$

$$\vec{F} = G\frac{m_1 m}{L}\left(\frac{1}{d} - \frac{1}{d+L}\right)\hat{i}$$

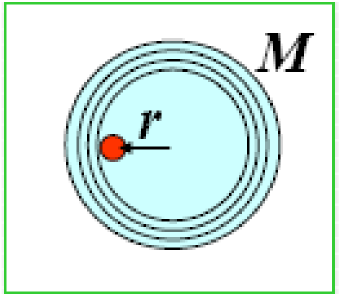
Exercise 1

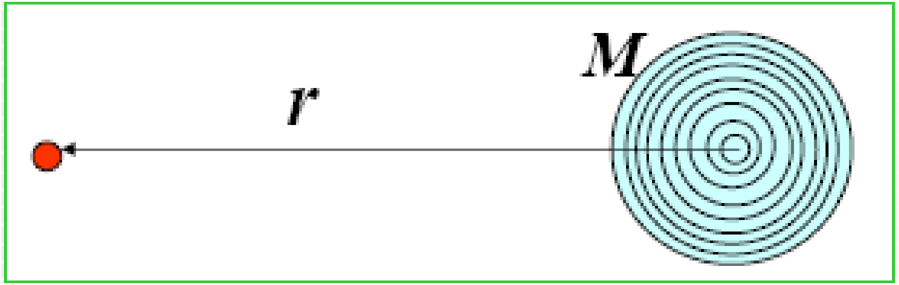
a particle of mass *m* is placed on the axis of a circular ring of mass *M* and radius *R*. Find the gravitational force exerted by the ring on the particle located a distance from the center of the ring.

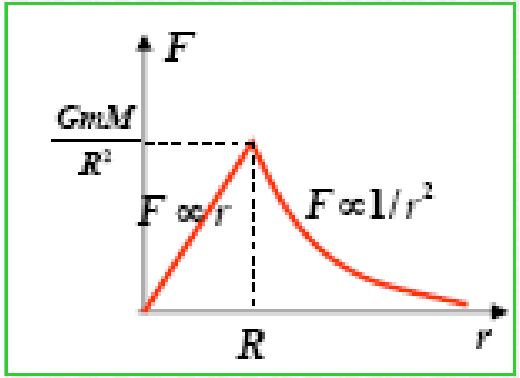


Exercise 2

(a) A point mass *m* located outside the sphere;
(b) A point mass *m* located within the volume enclosed by the sphere itself.

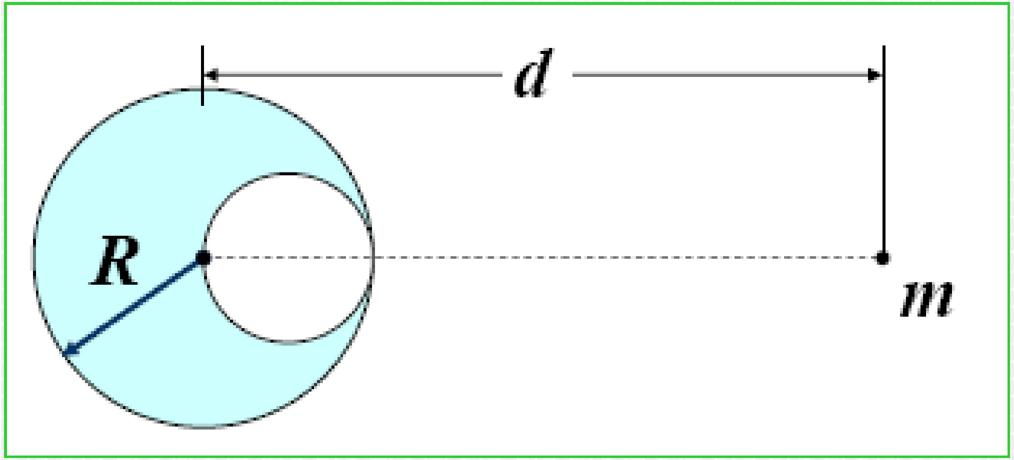






Exercise 3

A spherical hollow is made in a lead sphere of radius *R*, such that its surface touches the outside surface of the lead sphere and passes through its center. The mass of the sphere before hollowing was *M*. What is the attracting force between a small sphere of mass *m* and the hollowing sphere.



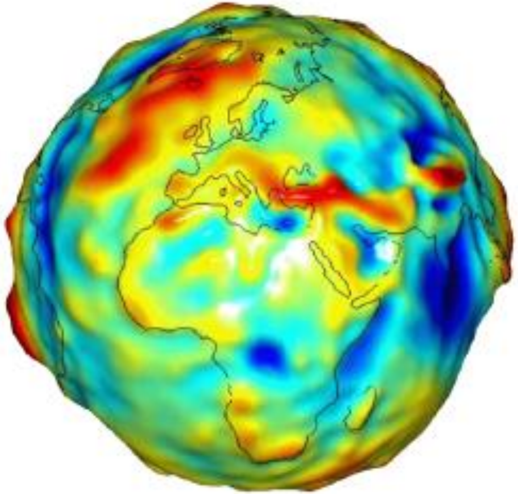
Gravity Near the Earth's Surface

$$F_G = \frac{Gm_1 m_2}{r^2}$$

$$F_G = mg$$

$$g = G\frac{m_E}{{r_E}^2}$$

Map of Map of g



Gravity not exactly the same on the surface of Earth

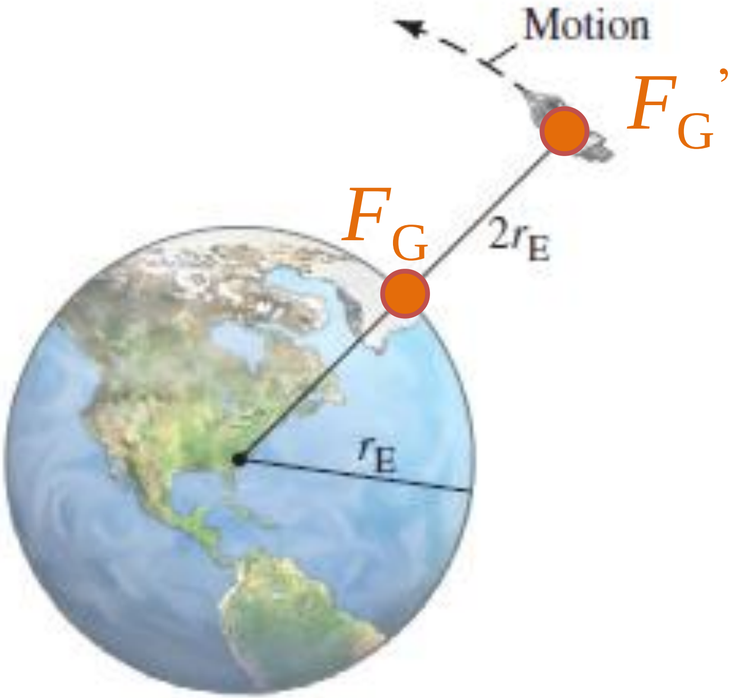
The gravity of the earth

$$F_G = \frac{G m_1 m_2}{r^2}$$

For a body with ***spherical symmetry***, such as the Earth, all the mass can be regarded to be at the ***center of mass***, when calculating the gravitational force.

What is F_G for an object at the surface of the earth?

How about F_G' for two Earth radii?



Example

The mass of the Moon is 7.35×10^{22} kg and the radius of the Moon is 1740 km. What is ***g*** on the surface of the Moon, $\boldsymbol{g}_{\text{Moon}}$?

$$g_{Moon} = \frac{GM_{Moon}}{r_{Moon}^2} = \frac{6.67\times10^{-11}\times7.35\times10^{22}}{\left(1740\times10^3\right)^2} = 1.62\,ms^{-2}$$

$$\frac{g_{Earth}}{g_{Moon}} = \frac{9.80}{1.62} = 6.0$$

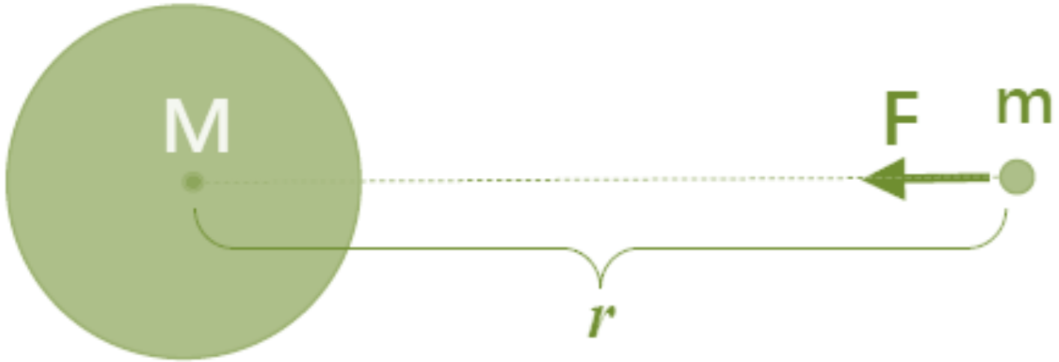
•How does your mass change on the moon?

•How does your weight change on the moon?

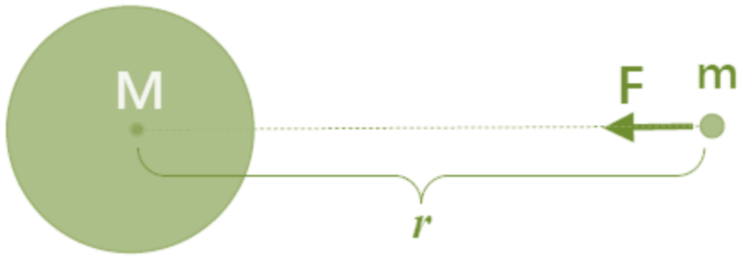
Gravitational potential energy

Suppose the Earth is stationary.

What is the potential energy of an object a distance *r* away from the center of Earth?



Gravitational potential energy



Let’s move the object to

infinitely far away .

The work done by the gravitational force is :

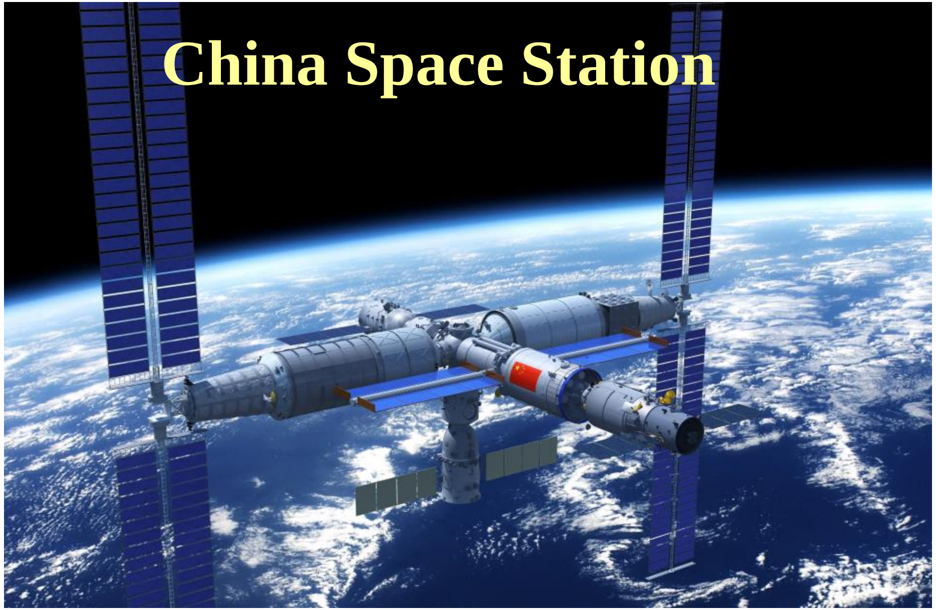
$$W = \int_r^{\infty} \vec{F} \cdot d\vec{r} = -\int_r^{\infty} F\,dr = -\int_r^{\infty} G\frac{Mm}{r^2}dr$$

$$\mathrm{W} = \frac{-GMm}{r}$$

Negative work = increase of potential energy

Gravity in orbit

380 km above the ground



r = 6751 km

g = 8.75 ms^{-2}

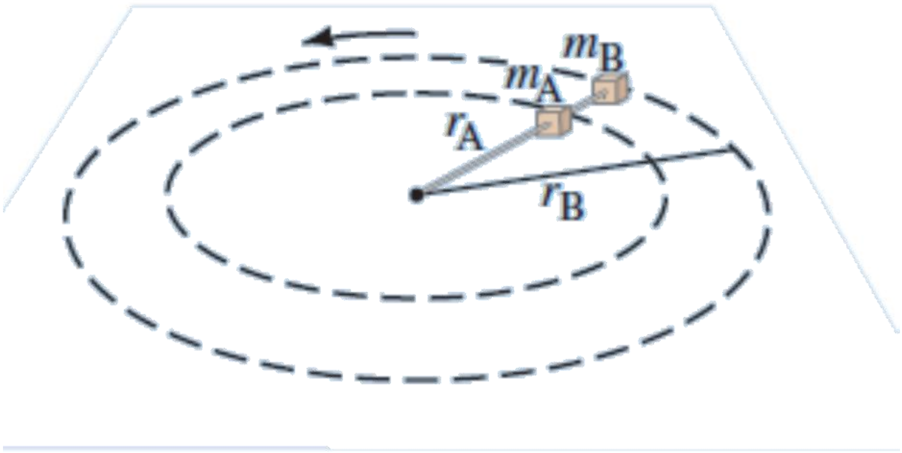
Satellites

Which needs a faster speed?

Close to the earth ?

Far away from the earth?

Try to find speed and period.



Satellites

$$F_G = F_{cent}$$

$$\frac{GM_{earth}m_{moon}}{r^2} = \frac{m_{moon}v^2}{r}$$

$$\Rightarrow v^2 = G(M_{earth}/r)$$

Orbital speed: $v = \sqrt{\frac{GM}{r}} = \sqrt{gr}$

Does not depend on its own mass!

All orbiting objects at the same distance must have the same *Speed and period*

Period of an orbit

$$T = \frac{4\pi^2 r^3}{GM}$$

Kepler's Third Law of planetary motion


near-Earth orbits
geostationary orbit

Near-Earth orbits

Low near-Earth orbits are usually from **160 to 2000 km** above earth.

With a very rough approximation, we assume **the orbit radius is the radius of the Earth** (which could be true for planets without atmosphere)

Orbital speed: $v = \sqrt{\boldsymbol{gr}} = \sqrt{\mathbf{9.8} \times \mathbf{6.37} \times \mathbf{106}} = \mathbf{7910}\ \boldsymbol{m/s}$

The first cosmic speed

The orbital period is about 84 mins

Example

If a satellite is travelling at 7 km/s, what is the radius of its orbit?

Orbital Speed: $v = \sqrt{\frac{GM}{r}}$ $\qquad r = \frac{GM}{v^2}$

$$r = \frac{6.67\times10^{-11}\times5.98\times10^{24}}{(7000)^2} = 8.14\times10^6 m$$

(1760 km above the Earth's surface)

What is the total energy of the satellite?

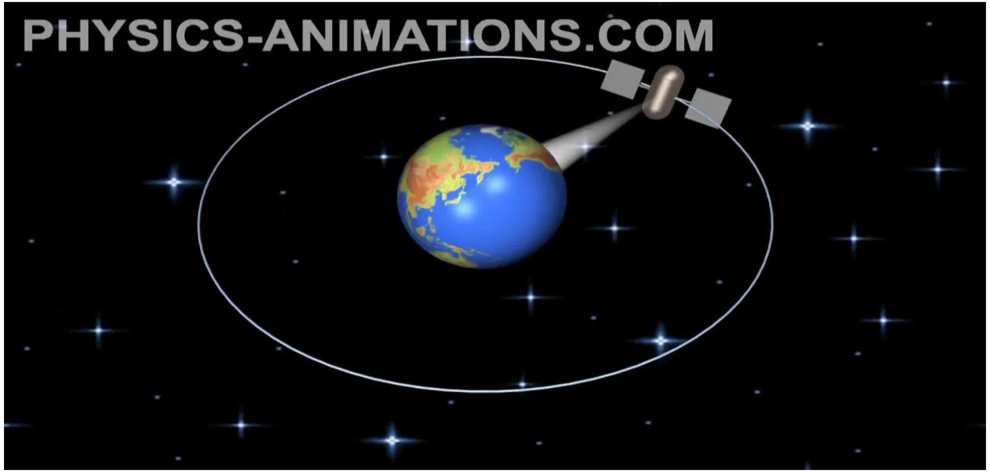
$$E_k = \frac{1}{2}mv^2 \qquad E_p = -G\frac{Mm}{r} = -m\frac{GM}{r} = -mv^2$$

$$E_k + E_p = -\frac{1}{2}mv^2 = -\frac{GMm}{2r}$$

Negative and inversely proportional to the radius

Geostationary satellite (Geosynchronous)

The orbit's period is equal to the Earth's rotation period



What is the height of this kind of satellite?

Geostationary satellite

The period of the satellite equals the spinning period of the Earth (24 h)

What is the height of this kind of satellite?

$$\frac{GM_{earth}m_{sat}}{r^2} = \frac{m_{sat}v^2}{r}$$

$$\frac{GM_{earth}}{r} = v^2$$

$$v = \frac{2\pi r}{T}$$

$$\frac{GM_{earth}}{r} = \frac{4\pi^2 r^2}{T^2}$$

$$4\pi^2 r^3 = GM_{earth}T^2$$

$$r = \sqrt[3]{\frac{GM_{earth}T^2}{4\pi^2}}$$

$$h \cong 6r_E = 36{,}000 \text{ km}$$

How about the gravity at the satellite?

Orbit of the satellite

Weightlessness

Why a people in a satellite orbit close to the Earth will experience “weightlessness”?

There is no gravity in space?

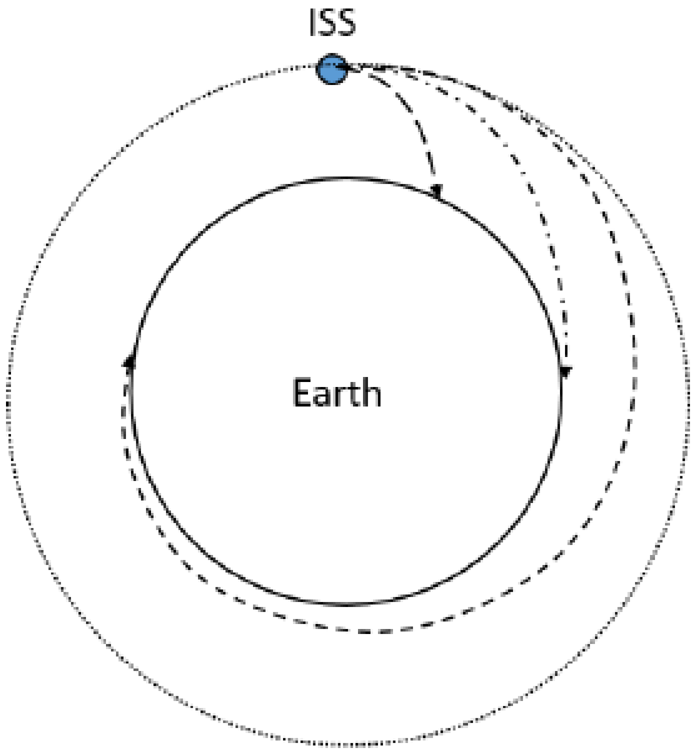
What is the net force on an astronaut at rest inside the space station?

Weightlessness

The station and astronauts are both in ***free -fall*** . But due to the horizontal speed, the projectile motion might be ***a circular motion***.

Gravity provides the ***centripetal force*** to keep the space station orbiting around the Earth.

For a uniform circular motion, the ***centripetal*** acceleration equals the ***gravitational*** acceleration.



Apparent weightlessness

A person on a scale in an elevator.

If the elevator fall freely, the ***reading of the scale would be 0***

***Apparent* (not real)**

1. Don’t seem to have weight
2. Gravity doesn’t disappear

Weightlessness

Apparent weightlessness?