# Calculus Class Test I (共 70 分)

1. Evaluate the indefinite integral **[32 分]**

(1) $\int \frac{\sin^4 x}{\cos^6 x} dx$ **[6 分]** (2) $\int \frac{x^2+1}{(x+1)^2(x-1)} dx$ **[10 分]**

(3) $\int \sin x \ln(\tan x) dx$ **[8 分]** (4) $\int \frac{(1+e^x)^2}{1+e^{2x}} dx$ **[8 分]**

(1) $\int \frac{\sin^4 x}{\cos^6 x} dx = \int \frac{\sin^4 x}{\cos^4 x} \cdot \sec^2 x dx = \int \tan^4 x d(\tan x) = \frac{1}{5}\tan^5 x + C$

(2) Let $\frac{x^2+1}{(x+1)^2(x-1)} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{(x-1)}$,

**then** $x^2 + 1 = A(x+1)(x-1) + B(x-1) + C(x+1)^2$.

**Let** $x = -1$, **we obtain** $B = -1$; **Let** $x = 1$, **we obtain** $C = \frac{1}{2}$.

**Compare the coefficient of** $x^2$, **we have** $A + C = 1$, **and so** $A = \frac{1}{2}$.

**Or, the resulted linear equation is**

$$\begin{cases} A + C = 1 \\ B + 2C = 0 \\ C - A - B = 1 \end{cases}$$

Solve it to obtain $A = \frac{1}{2}, B = -1, C = \frac{1}{2}$.

So, $\int \frac{x^2+1}{(x+1)^2(x-1)} dx = \int \left(\frac{1}{2(x+1)} - \frac{1}{(x+1)^2} + \frac{1}{2(x-1)}\right) dx = \frac{1}{2}\ln|x^2 - 1| + \frac{1}{(x+1)} + C.$

(3) 
$$\begin{aligned} \int \sin x \ln(\tan x)\, dx &= -\int \ln(\tan x)\, d(\cos x) \\ &= -\cos x \ln(\tan x) + \int \cos x \cot x \sec^2 x\, dx \\ &= -\cos x \ln(\tan x) + \int \frac{1}{\sin x} dx \\ &= -\cos x \ln(\tan x) - \ln|\csc x + \cot x| + C \end{aligned}$$

(4) 
$$\begin{aligned} \int \frac{(1+e^x)^2}{1+e^{2x}} dx &= \int \frac{1+e^{2x}+2e^x}{1+e^{2x}} dx = \int 1 + \frac{2e^x}{1+e^{2x}} dx \\ &= x + 2\int \frac{1}{1+e^{2x}} de^x = x + 2\tan^{-1} e^x + C \end{aligned}$$

**2.** Evaluate the definite integral **[38 分]**

(1) $\int_{-1}^{1} \frac{x}{x^2+x+1} dx$ **[8 分]** (2) $\int_0^{\pi} (x \sin x)^2 dx$ **[10 分]**

(3) $\int_1^9 x\sqrt[3]{1-x}dx$ [8 分] (4) $\int_0^{3\sqrt{3}/2}\frac{x^3}{(4x^2+9)^{3/2}}dx$ [12 分]

(1) $\int_{-1}^{1}\frac{x}{x^2+x+1}dx=\frac{1}{2}\int_{-1}^{1}\frac{2x+1}{x^2+x+1}dx-\frac{1}{2}\int_{-1}^{1}\frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}dx$

$$=\frac{1}{2}\int_{-1}^{1}\frac{1}{x^2+x+1}d(x^2+x+1)-\frac{1}{2}\int_{-1}^{1}\frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}d\left(x+\frac{1}{2}\right)$$

$$=\frac{1}{2}\ln(x^2+x+1)\Big|_{-1}^{1}-\frac{1}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}\Big|_{-1}^{1}=\frac{1}{2}\ln 3-\frac{\pi}{2\sqrt{3}}$$

(2) $\int_0^{\pi}(x\sin x)^2dx=\frac{1}{2}\int_0^{\pi}x^2(1-\cos 2x)dx=\frac{1}{6}x^3\Big|_0^{\pi}-\frac{1}{2}\int_0^{\pi}x^2\cos 2x\,dx$

$$=\frac{\pi^3}{6}-\frac{x^2}{4}\sin 2x\Big|_0^{\pi}+\frac{1}{2}\int_0^{\pi}x\sin 2x\,dx$$

$$=\frac{\pi^3}{6}-\left(\frac{x}{4}\cos 2x\right)\Big|_0^{\pi}+\frac{1}{4}\int_0^{\pi}\cos 2x\,dx$$

$$=\frac{\pi^3}{6}-\frac{\pi}{4}$$

(3) $\int_1^9 x\sqrt[3]{1-x}dx$

Let $\sqrt[3]{1-x}=t$, then $x=1-t^3$, $dx=-3t^2dt$; $t=0$, as $x=1$, and $t=-2$, as $x=9$.

So, $\int_1^9 x\sqrt[3]{1-x}dx=-3\int_0^{-2}(t^3-t^6)dt$

$$=\left(\frac{3}{4}t^4-\frac{3}{7}t^7\right)\Big|_{-2}^{0}=-\frac{468}{7}=-66\frac{6}{7}.$$

(4) $\int_0^{3\sqrt{3}/2}\frac{x^3}{(4x^2+9)^{3/2}}dx$

Let $x=\frac{3}{2}\tan\theta$, then $dx=\frac{3}{2}\sec^2\theta\,d\theta$; $\theta=0$, as $x=0$, and $\theta=\pi/3$, as $x=3\sqrt{3}/2$.

So, $\int_0^{3\sqrt{3}/2}\frac{x^3}{(4x^2+9)^{3/2}}dx=\int_0^{\pi/3}\frac{\frac{27}{8}\tan^3\theta}{27\sec^3\theta}\frac{3}{2}\sec^2\theta\,d\theta$

$$=\frac{3}{16}\int_0^{\pi/3}\frac{\tan^3\theta}{\sec\theta}d\theta=\frac{3}{16}\int_0^{\pi/3}\frac{\sin^3\theta}{\cos^2\theta}d\theta$$

$$=-\frac{3}{16}\int_0^{\pi/3}\frac{1-\cos^2\theta}{\cos^2\theta}d(\cos\theta)=-\frac{3}{16}\int_0^{\pi/3}\left(\frac{1}{\cos^2\theta}-1\right)d(\cos\theta)$$

$$=\frac{3}{16}\left(\cos\theta+\frac{1}{\cos\theta}\right)\Big|_0^{\pi/3}=\frac{3}{32}$$