Calculus Class Test III

1. Calculate the following limits

(1) $\lim\limits_{x\to 0}\frac{e^x-x-1}{3x^2+2x}$ (2) $\lim\limits_{\theta\to\pi/2^-}(\tan\theta-\sec\theta)$

(3) $\lim\limits_{x\to\infty}\frac{e^{3x}}{3e^{3x}+5}$ (4) $\lim\limits_{x\to\infty}\frac{\sin^4 x}{\sqrt{x}}$

(5) $\lim\limits_{x\to 0^+}\left(\frac{1}{x}\right)^{\tan x}$

Solution (注意解法不唯一)

(1) $\lim\limits_{x\to 0}\frac{e^x-x-1}{3x^2+2x}=\lim\limits_{x\to 0}\frac{e^x-1}{6x+2}=0$ (第二次不能再用咯必达法则)

(2) $\lim\limits_{\theta\to\pi/2^-}(\tan\theta-\sec\theta)=\lim\limits_{\theta\to\pi/2^-}(\frac{\sin\theta-1}{\cos\theta})=\lim\limits_{\theta\to\pi/2^-}(\frac{\cos\theta}{-\sin\theta})=0$

(3) $\lim\limits_{x\to\infty}\frac{e^{3x}}{3e^{3x}+5}=\lim\limits_{x\to\infty}\frac{3e^{3x}}{9e^{3x}}=\frac{1}{3}$ or $\lim\limits_{x\to\infty}\frac{e^{3x}}{3e^{3x}+5}=\lim\limits_{x\to\infty}\frac{1}{3+5/e^{3x}}=\frac{1}{3}$

(4) $\sin^4 x$ 有界，$\lim\limits_{x\to\infty}\frac{1}{\sqrt{x}}=0$，故 $\lim\limits_{x\to\infty}\frac{\sin^4 x}{\sqrt{x}}=0$

注：第(4)题不能用洛必达法则。

(5) $\lim\limits_{x\to 0^+}\left(\frac{1}{x}\right)^{\tan x}=e^{\lim\limits_{x\to 0^+}\tan x\ln\left(\frac{1}{x}\right)}$

$\lim\limits_{x\to 0^+}\tan x\ln\left(\frac{1}{x}\right)=\lim\limits_{x\to 0^+}\frac{-\ln x}{\cot x}=\lim\limits_{x\to 0^+}\frac{\sin^2 x}{x}=\lim\limits_{x\to 0^+}\sin x\ (\text{or } \lim\limits_{x\to 0^+}2\sin x\cos x)=0$

So, $\lim\limits_{x\to 0^+}\left(\frac{1}{x}\right)^{\tan x}=e^{\lim\limits_{x\to 0^+}\tan x\ln\left(\frac{1}{x}\right)}=e^0=1$

2. Suppose the derivative of a function f is $f'(x)=(x+1)^2(x-3)^5(x-6)^4$. On what interval f is increasing?

Solution

Let $f'(x)=0$, we can obtain $x=-1,3,6$ **[2 分]**

$(-\infty,-1)$	$(-1,3)$	$(3,6)$	$(6,\infty)$
$f'<0$	$f'<0$	$f'>0$	$f'>0$
f decreasing	f decreasing	f increasing	f increasing

So, f is increasing on the interval $(3,\infty)$.

3. Show that the curve $y=(1+x)/(1+x^2)$ has three points of inflection and they all lie on one straight line.

Solution

$y'=\frac{(1+x^2)-2x(1+x)}{(1+x^2)^2}=\frac{1-x^2-2x}{(1+x^2)^2}$

$y'' = \frac{-2(x+1)(1+x^2)^2 - 4x(1-x^2-2x)(1+x^2)}{(1+x^2)^4} = \frac{2(x^3+3x^2-3x-1)}{(1+x^2)^3}$

Let $y'' = 0$, i.e., $x^3 + 3x^2 - 3x - 1 = 0$, $(x-1)(x^2+4x+1) = 0$, we have $x = 1, -2 \pm \sqrt{3}$.

$(-\infty, -2-\sqrt{3})$	$(-2-\sqrt{3}, -2+\sqrt{3})$	$(-2+\sqrt{3}, 1)$	$(1, \infty)$
$y'' < 0$	$y'' > 0$	$y'' < 0$	$y'' > 0$

So, $(1,1), \left(-2+\sqrt{3}, \frac{1+\sqrt{3}}{4}\right), \left(-2-\sqrt{3}, \frac{1-\sqrt{3}}{4}\right)$ are all points of inflection of $y = (1+x)/(1+x^2)$.

The slope of the line joining $(1,1), \left(-2+\sqrt{3}, \frac{1+\sqrt{3}}{4}\right)$ is $\frac{-3+\sqrt{3}}{4}/(-3+\sqrt{3}) = \frac{1}{4}$.

The slope of the line joining $(1,1), \left(-2-\sqrt{3}, \frac{1-\sqrt{3}}{4}\right)$ is $\frac{-3-\sqrt{3}}{4}/(-3-\sqrt{3}) = \frac{1}{4}$.

Therefore, these three points lie on one straight line.

4. Find f

(1) $f''(x) = \frac{2}{3}x^{2/3}$ (2) $f'(x) = \sqrt{x}(6+5x)$, $f(1) = 10$

(3) $f''(t) = 3/\sqrt{t}$, $f(4) = 20$, $f'(4) = 7$

Solution

(1) $f'(x) = \int \frac{2}{3}x^{2/3}dx = \frac{2}{3} * \frac{3}{5}x^{5/3} + C_1 = \frac{2}{5}x^{5/3} + C_1$

$f(x) = \frac{2}{5} * \frac{3}{8}x^{8/3} + C_1x + C_2 = \frac{2}{20}x^{8/3} + C_1x + C_2$.

(2) $f(x) = f(1) + \int_1^x \sqrt{t}(6+5t)dt = 10 + (4t^{3/2} + 2t^{5/2})\big|_1^x = 4x^{3/2} + 2x^{5/2} + 4$

Method II: $f(x) = \int \sqrt{x}(6+5x)dx = 4x^{3/2} + 2x^{5/2} + C$,

From $10 = f(1) = 6 + C$, we have $C = 4$, and so $f(x) = 4x^{3/2} + 2x^{5/2} + 4$.

(3) $f'(t) = f'(4) + \int_4^t 3x^{-1/2}dx = 7 + 6x^{1/2}\big|_4^t = 6\sqrt{t} - 5$

$f(t) = f(4) + \int_4^t (6x^{1/2} - 5)dx = 20 + (4x^{3/2} - 5x)\big|_4^t = 4t^{3/2} - 5t + 8$.

5. Calculate the definite integrals

(1) $\int_0^\pi \sqrt{\sin^3 x - \sin^5 x}dx$ (2) $\int_0^4 \frac{x}{\sqrt{1+2x}}dx$

Solution (注意解法不唯一)

(1) $\int_0^\pi \sqrt{\sin^3 x - \sin^5 x}dx = \int_0^\pi \sqrt{\sin^3 x(1-\sin^2 x)}dx = \int_0^\pi \sqrt{\sin^3 x}|\cos x|dx$

$= \int_0^{\pi/2} \sqrt{\sin^3 x}\cos x\, dx - \int_{\pi/2}^{\pi} \sqrt{\sin^3 x}\cos x\, dx$

$= \int_0^{\pi/2} \sqrt{\sin^3 x} d \sin x - \int_{\pi/2}^{\pi} \sqrt{\sin^3 x} d \sin x = \frac{2}{5} (\sin x)^{5/2} \Big|_0^{\pi/2} - \frac{2}{5} (\sin x)^{5/2} \Big|_{\pi/2}^{\pi}$

$= \frac{4}{5}$

(2) Let $u = \sqrt{1+2x}$, then $x = \frac{u^2-1}{2}$, $dx = udu$, $u = 1$ when $x = 0$, $u = 3$ when $x = 4$,

We then have $\int_0^4 \frac{x}{\sqrt{1+2x}} dx = \int_1^3 \frac{u^2-1}{2u} udu = \frac{1}{2} (\frac{1}{3} u^3 - u) \Big|_1^3 = \frac{10}{3}$