Chapter 17

Vector Calculus (I)

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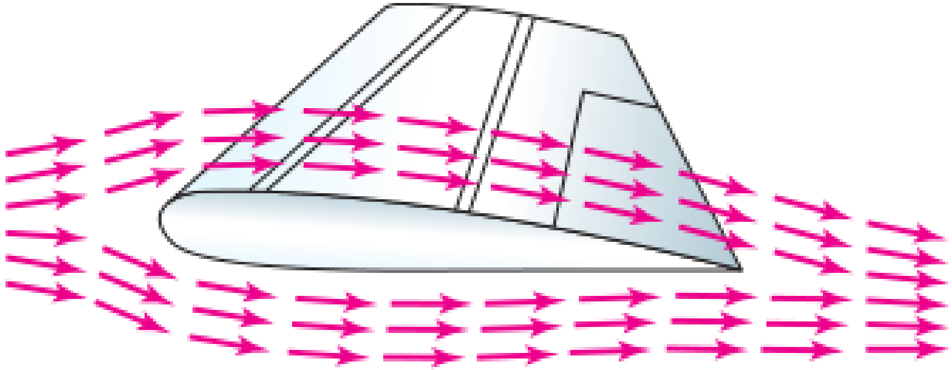
17.1

Vector Fields

Vector Fields in Two Dimensions



DEFINITION **Vector Fields in Two Dimensions**

Let f and g be defined on a region R of $\mathbb{R}^2$. A **vector field** in $\mathbb{R}^2$ is a function $\mathbf{F}$ that assigns to each point in R a vector $\langle f(x, y), g(x, y) \rangle$. The vector field is written as

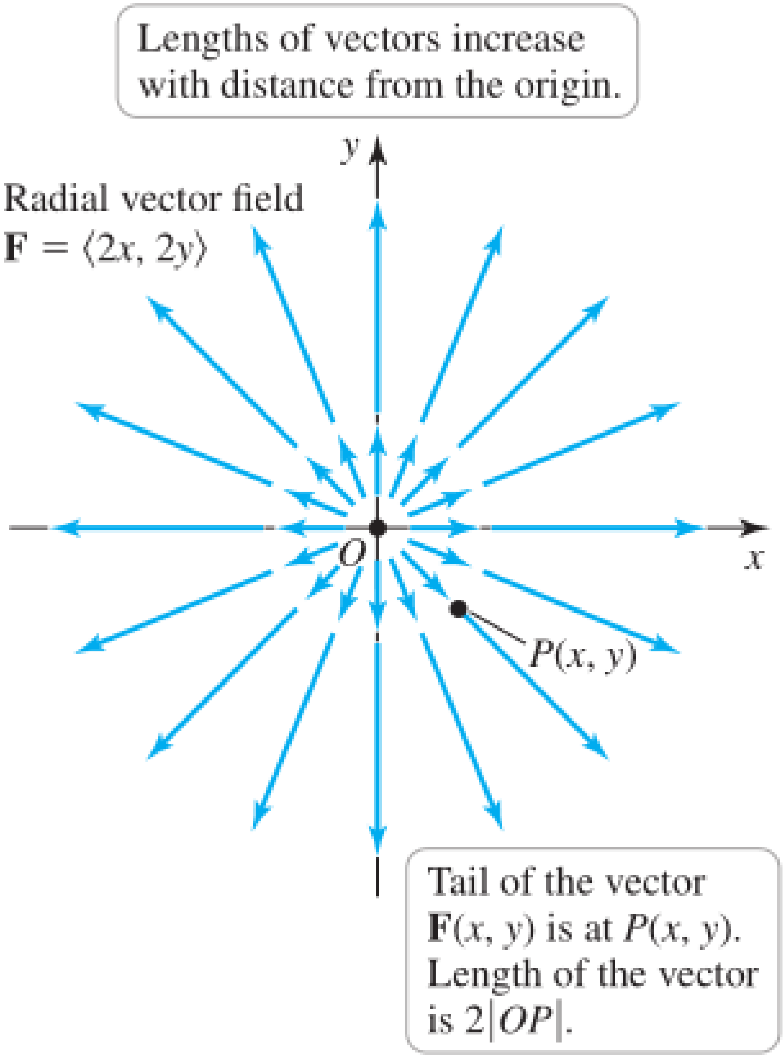
$$\mathbf{F}(x, y) = \langle f(x, y), g(x, y) \rangle \quad \text{or}$$
$$\mathbf{F}(x, y) = f(x, y)\,\mathbf{i} + g(x, y)\,\mathbf{j}.$$

A vector field $\mathbf{F} = \langle f, g \rangle$ is continuous or differentiable on a region R of $\mathbb{R}^2$ if f and g are continuous or differentiable on R, respectively.

A vector field cannot be represented graphically in its entirety

A representative sample $\mathbf{F}(x, y) = \langle 2x, 2y \rangle$

- For every (x, y) except $(0, 0)$, the vector $\mathbf{F}(x, y)$ points in the direction of $\langle 2x, 2y \rangle$, directly outward from the origin.
- The length of $\mathbf{F}(x, y)$ is $|\mathbf{F}| = |\langle 2x, 2y \rangle| = 2\sqrt{x^2 + y^2}$, which increases with distance from the origin.
- Vector fields are sometimes called *flows*

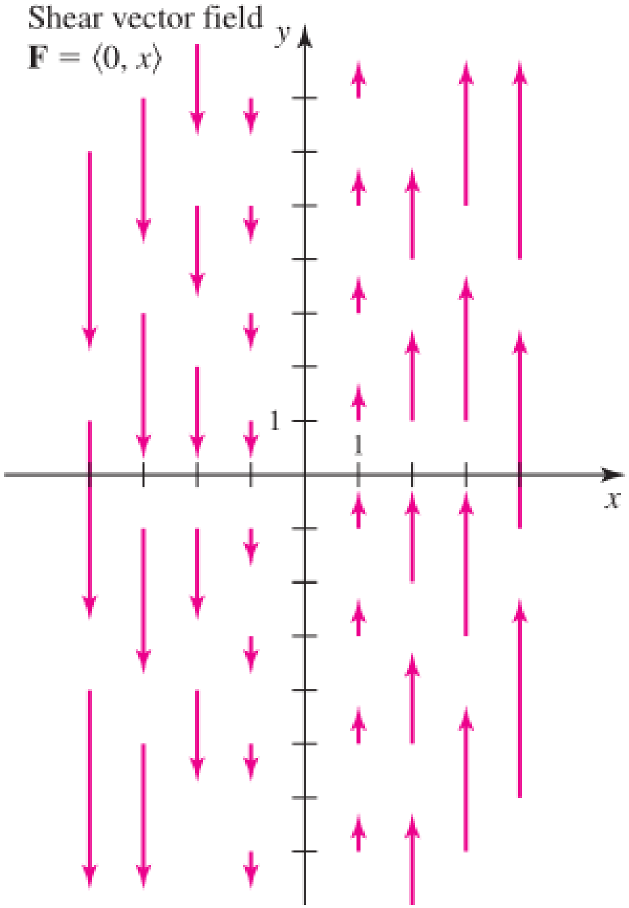


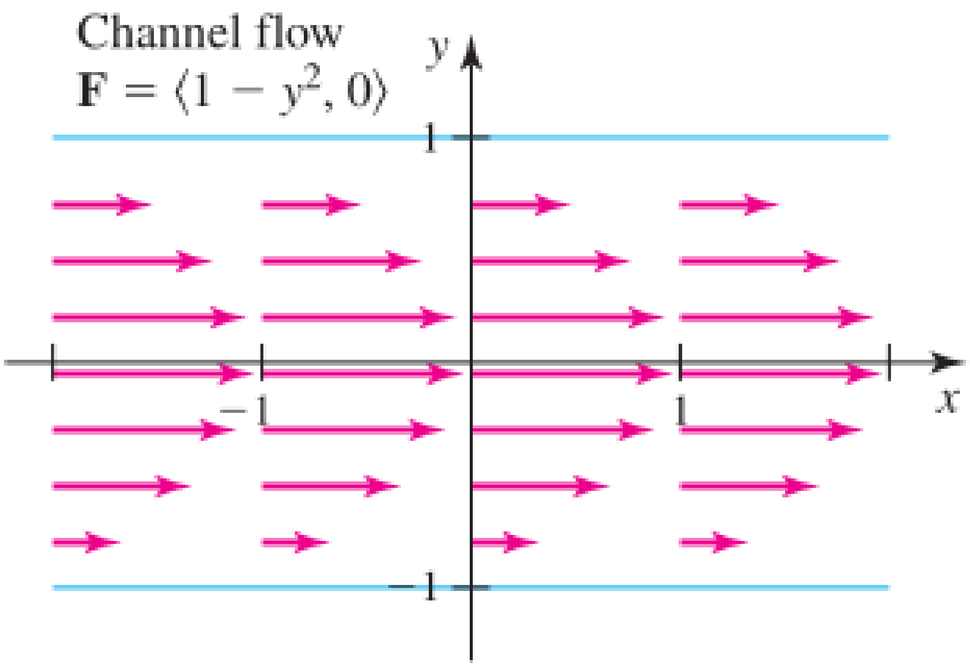
EXAMPLE 1 **Vector fields** Sketch representative vectors of the following vector fields.

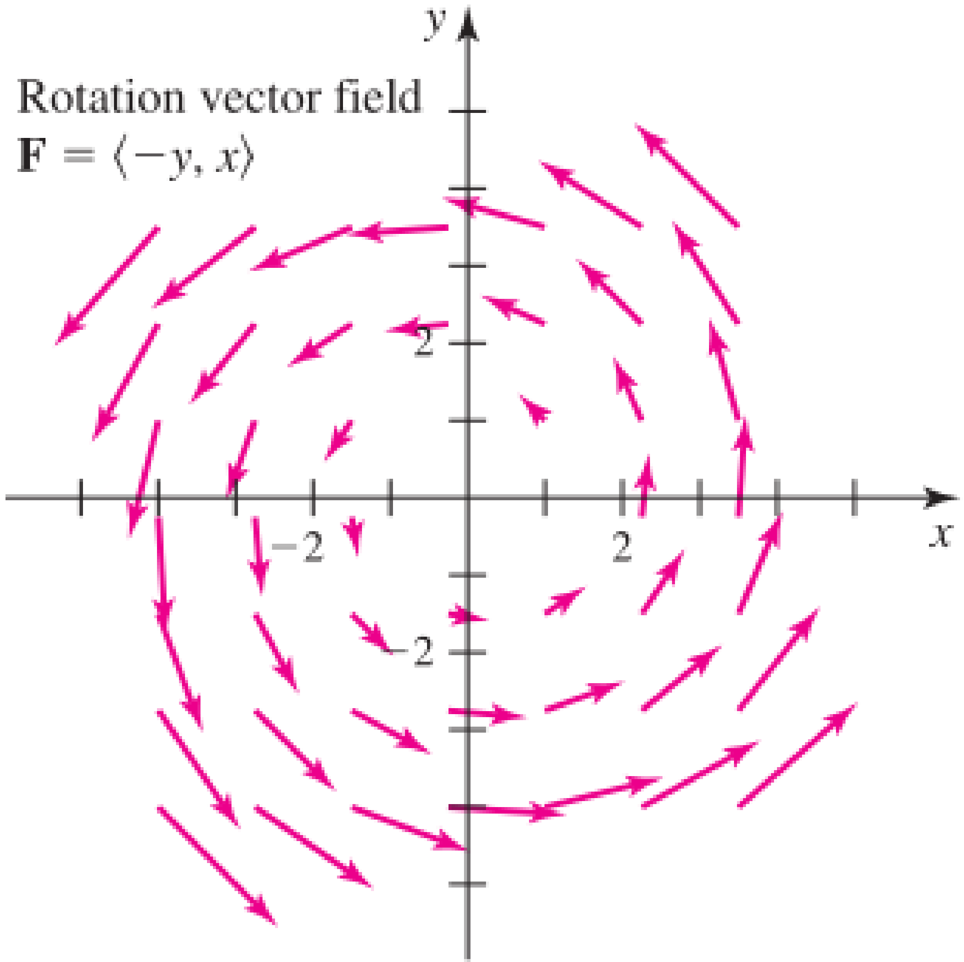
a. $\mathbf{F}(x, y) = \langle 0, x \rangle = x\,\mathbf{j}$ (a shear field)

b. $\mathbf{F}(x, y) = \langle 1 - y^2, 0 \rangle = (1 - y^2)\,\mathbf{i}$, for $|y| \leq 1$ (channel flow)

c. $\mathbf{F}(x, y) = \langle -y, x \rangle = -y\,\mathbf{i} + x\,\mathbf{j}$ (a rotation field)






y
Rotation vector field
$\mathbf{F} = \langle -y, x \rangle$
2
−2
2
−2
x

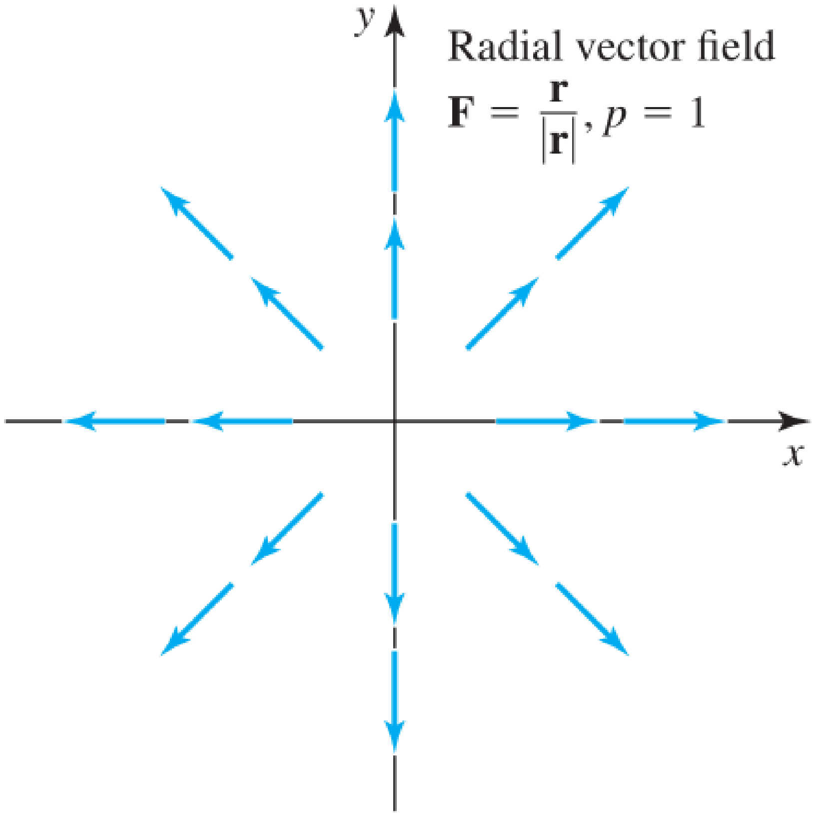
Radial Vector fields in $\mathbb{R}^2$

Its vectors point directly toward or away from the origin at all points, parallel to the position vectors $\mathbf{r} = \langle x, y \rangle$.

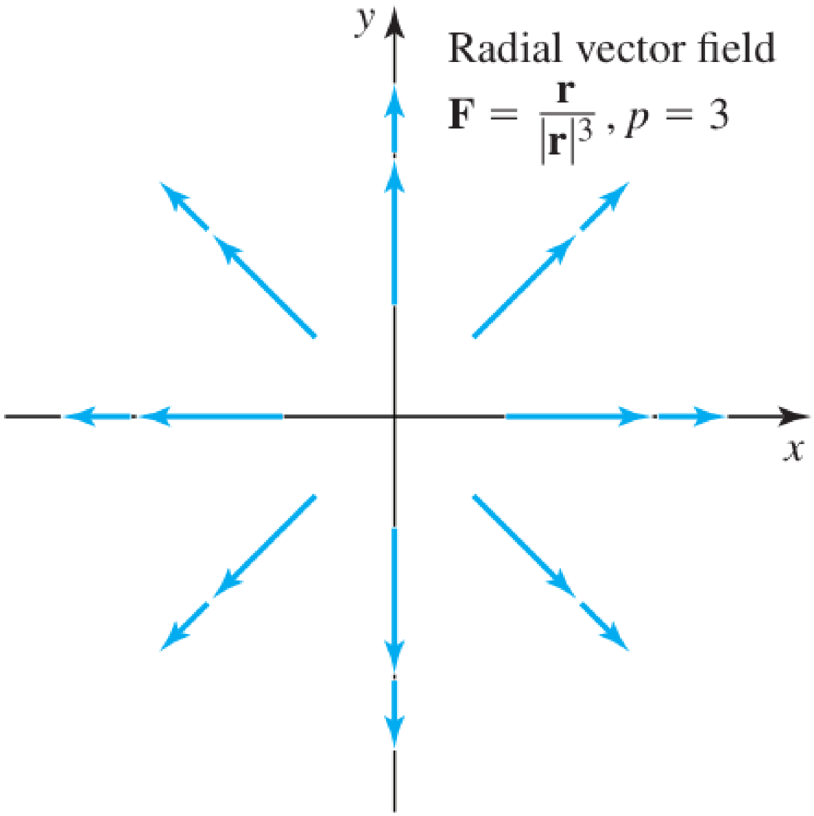
Mainly work with the form

$$\mathbf{F}(x, y) = \frac{\mathbf{r}}{|\mathbf{r}|^p} = \frac{\langle x, y \rangle}{|\mathbf{r}|^p} = \frac{\mathbf{r}}{|\mathbf{r}|} \frac{1}{|\mathbf{r}|^{p-1}}$$

Application example, central forces, such as gravitational or electrostatic forces between point masses or charges, are described by radial vector fields with $p = 3$.



Vectors have unit length.



Lengths of vectors decrease with distance from the origin.

DEFINITION **Radial Vector Fields in $\mathbb{R}^2$**

Let $\mathbf{r} = \langle x, y \rangle$. A vector field of the form $\mathbf{F} = f(x, y)\,\mathbf{r}$, where f is a scalar-valued function, is a **radial vector field**. Of specific interest are the radial vector fields

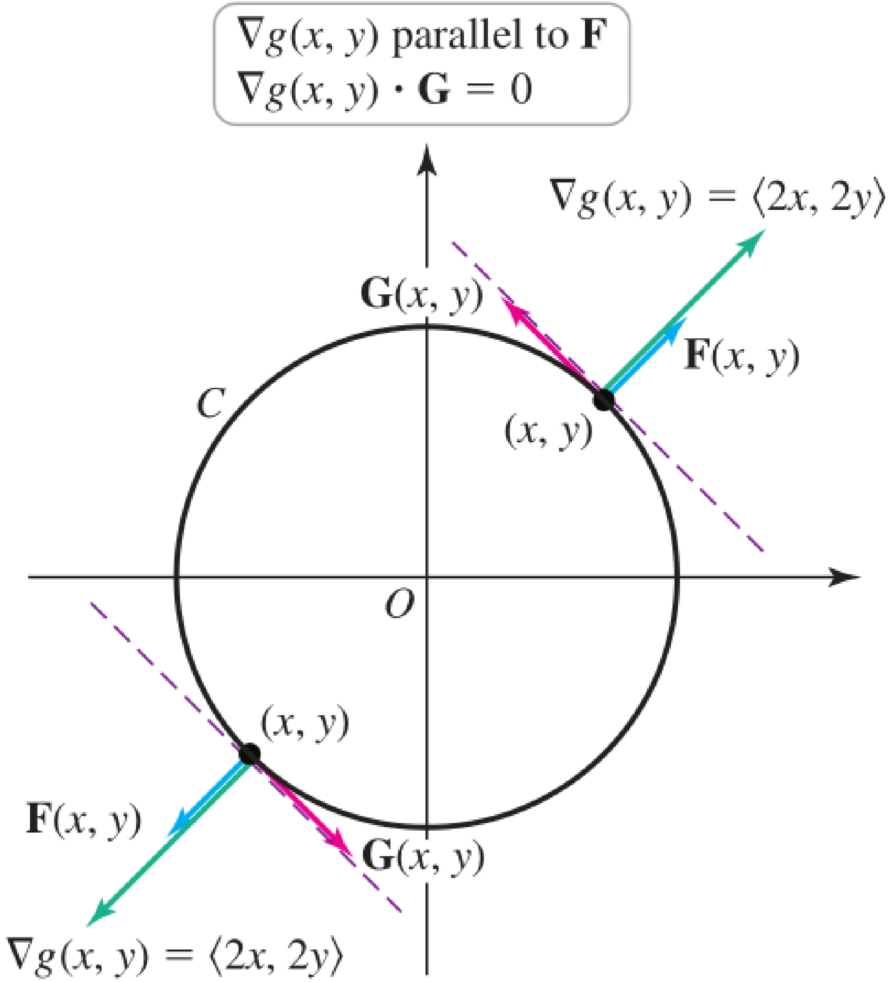
$$\mathbf{F}(x, y) = \frac{\mathbf{r}}{|\mathbf{r}|^p} = \frac{\langle x, y \rangle}{|\mathbf{r}|^p},$$

where p is a real number. At every point (except the origin), the vectors of this field are directed outward from the origin with a magnitude of $|\mathbf{F}| = \dfrac{1}{|\mathbf{r}|^{p-1}}$.

EXAMPLE 2 **Normal and tangent vectors** Let C be the circle $x^2 + y^2 = a^2$, where $a > 0$.

a. Show that at each point of C, the radial vector field $\mathbf{F}(x, y) = \frac{\mathbf{r}}{|\mathbf{r}|} = \frac{\langle x, y \rangle}{\sqrt{x^2 + y^2}}$ is orthogonal to the line tangent to C at that point.

b. Show that at each point of C, the rotation vector field $\mathbf{G}(x, y) = \frac{\langle -y, x \rangle}{\sqrt{x^2 + y^2}}$ is parallel to the line tangent to C at that point.



Radial field $\mathbf{F} = \dfrac{\langle x, y \rangle}{\sqrt{x^2 + y^2}}$

Rotation field $\mathbf{G} = \dfrac{\langle -y, x \rangle}{\sqrt{x^2 + y^2}}$

Vector Fields in Three Dimensions

DEFINITION Vector Fields and Radial Vector Fields in $\mathbb{R}^3$

Let f, g, and h be defined on a region D of $\mathbb{R}^3$. A **vector field** in $\mathbb{R}^3$ is a function $\mathbf{F}$ that assigns to each point in D a vector $\langle f(x, y, z), g(x, y, z), h(x, y, z) \rangle$. The vector field is written as

$$\mathbf{F}(x, y, z) = \langle f(x, y, z), g(x, y, z), h(x, y, z) \rangle \quad \text{or}$$
$$\mathbf{F}(x, y, z) = f(x, y, z)\,\mathbf{i} + g(x, y, z)\,\mathbf{j} + h(x, y, z)\,\mathbf{k}.$$

A vector field $\mathbf{F} = \langle f, g, h \rangle$ is continuous or differentiable on a region D of $\mathbb{R}^3$ if f, g, and h are continuous or differentiable on D, respectively. Of particular importance are the **radial vector fields**

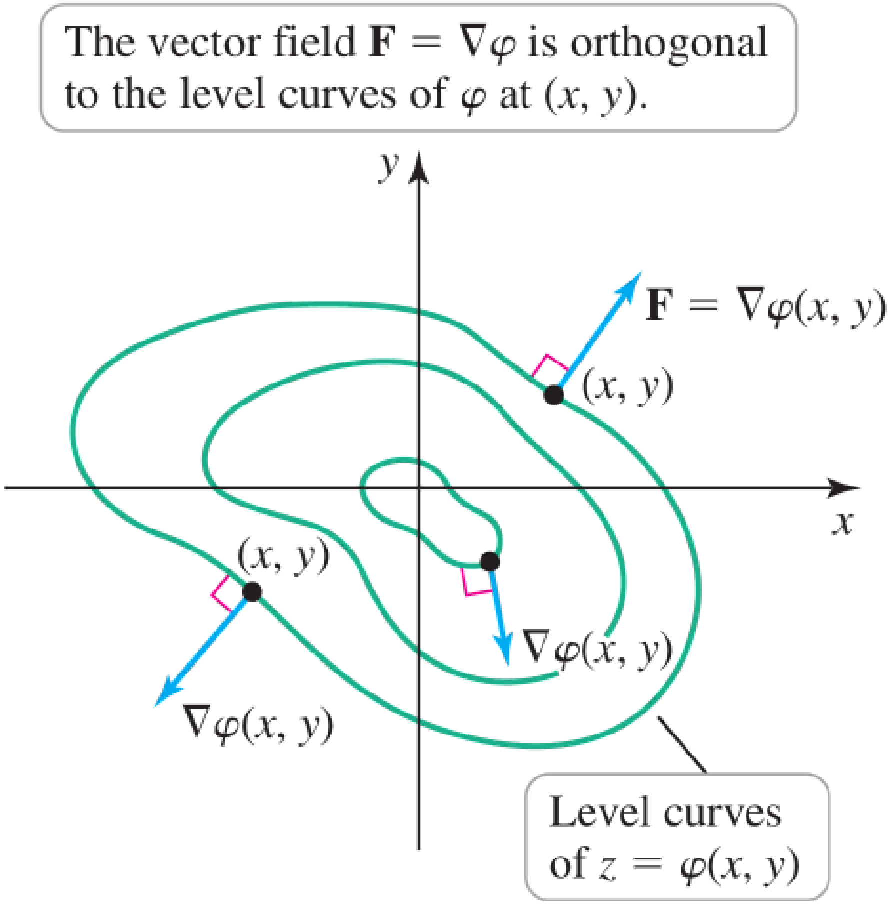
$$\mathbf{F}(x, y, z) = \frac{\mathbf{r}}{|\mathbf{r}|^p} = \frac{\langle x, y, z \rangle}{|\mathbf{r}|^p},$$

where p is a real number.

Gradient Fields and Potential Functions

DEFINITION Gradient Fields and Potential Functions

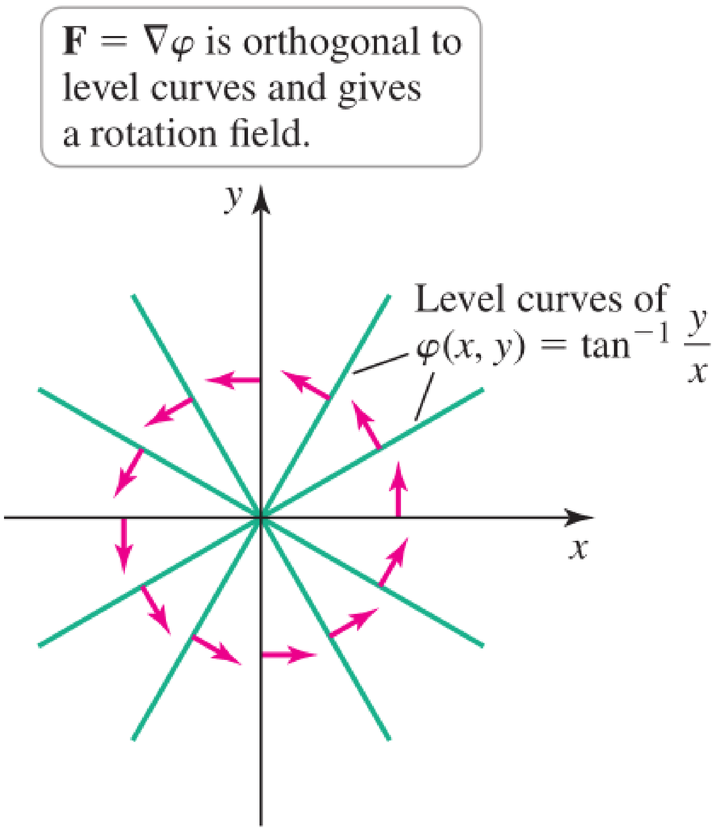
Let φ be differentiable on a region of $\mathbb{R}^2$ or $\mathbb{R}^3$. The vector field $\mathbf{F} = \nabla\varphi$ is a **gradient field** and the function φ is a **potential function** for $\mathbf{F}$.

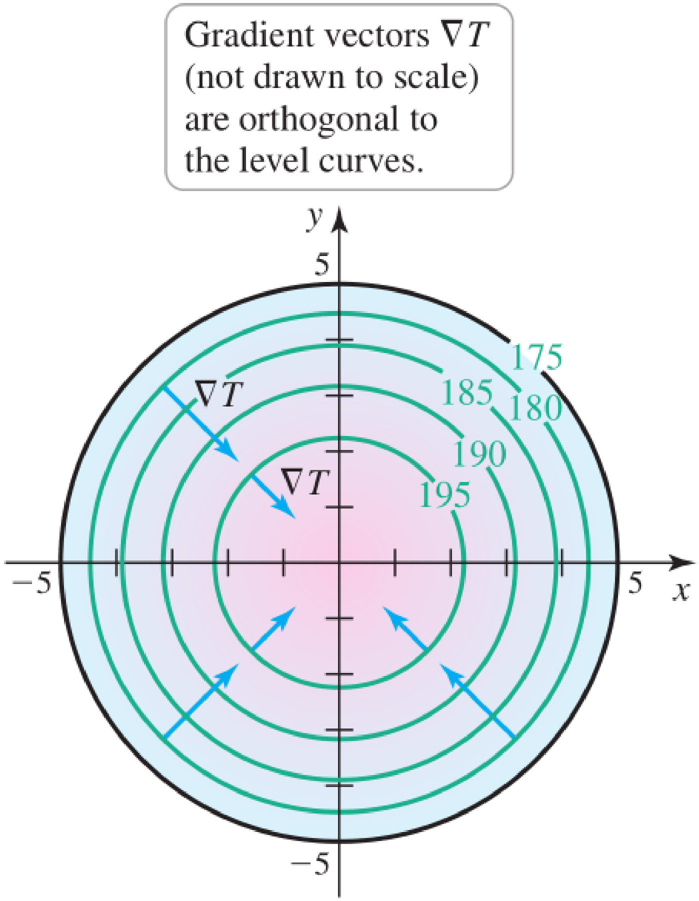


EXAMPLE 4 Gradient fields

a. Sketch and interpret the gradient field associated with the temperature function $T = 200 - x^2 - y^2$ on the circular plate $R = \{(x, y): x^2 + y^2 \leq 25\}$.

b. Sketch and interpret the gradient field associated with the velocity potential $\varphi = \tan^{-1}(y/x)$.

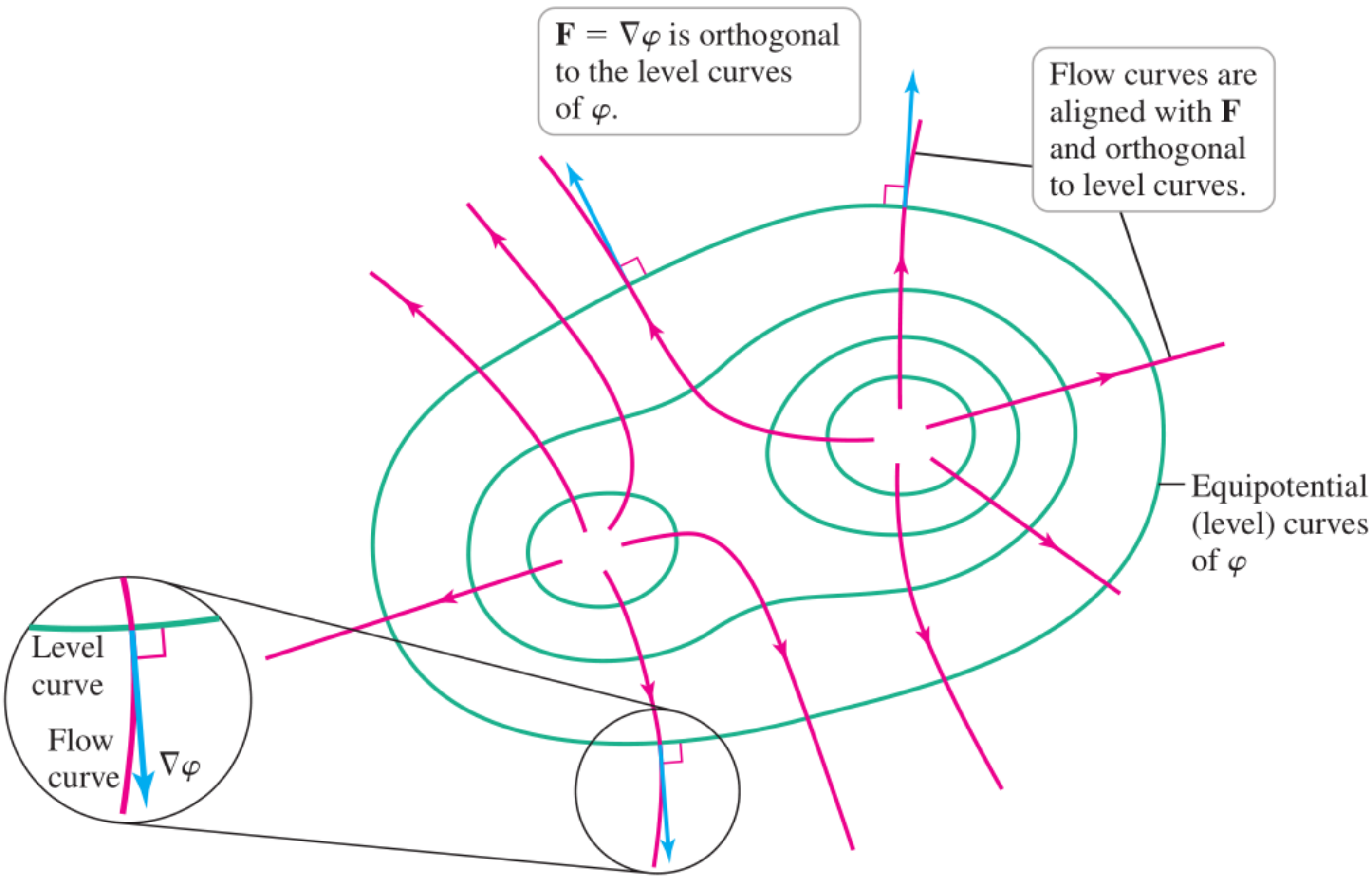




Level curves of $T(x, y) = 200 - x^2 - y^2$

Equipotential Curves and Surfaces

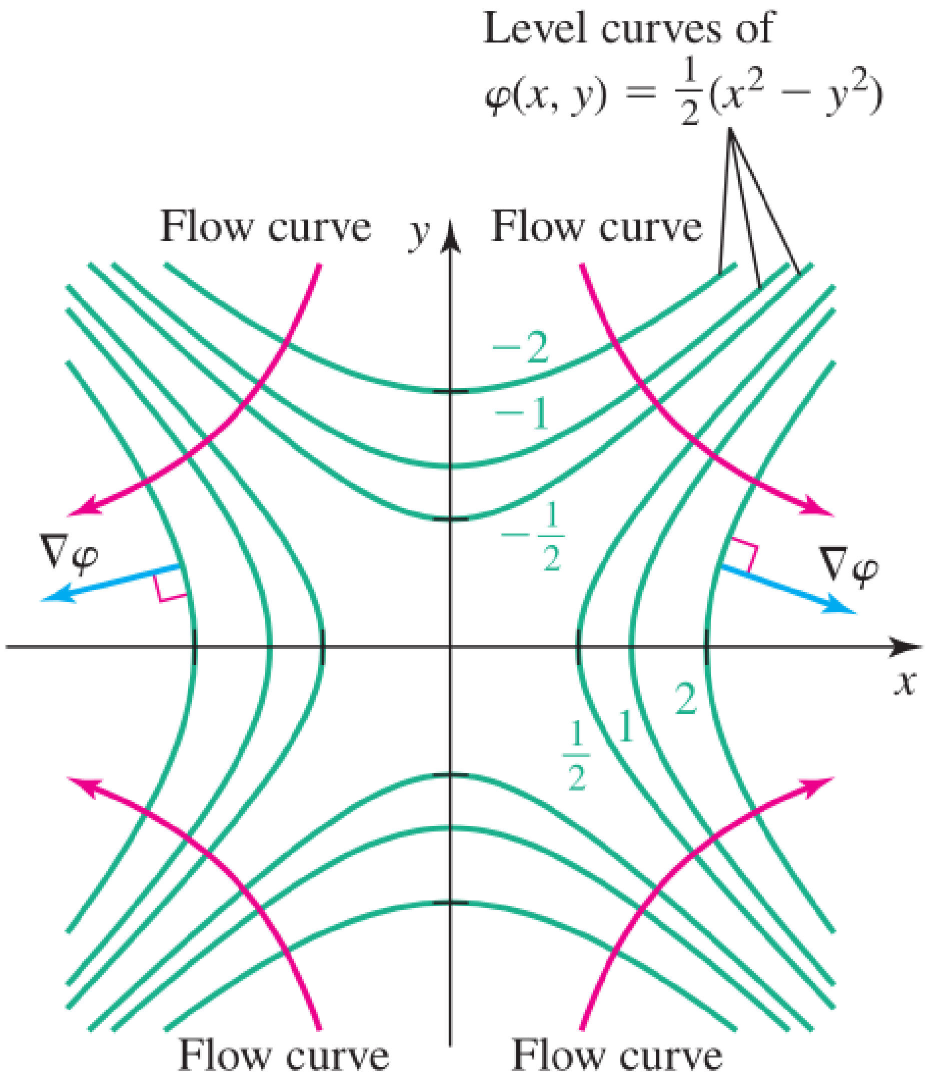
Gradient field $\mathbf{F} = \nabla\varphi$ of a potential function φ.
The level curves of a potential function are called **equipotential curves** (curves on which the potential function is constant).



EXAMPLE 5 **Equipotential curves** The equipotential curves for the potential function $\varphi(x, y) = (x^2 - y^2)/2$ are shown in green in Figure 14.15.

a. Find the gradient field associated with φ and verify that the gradient field is orthogonal to the equipotential curve at $(2, 1)$.

b. Verify that the vector field $\mathbf{F} = \nabla\varphi$ is orthogonal to the equipotential curves at all points (x, y).



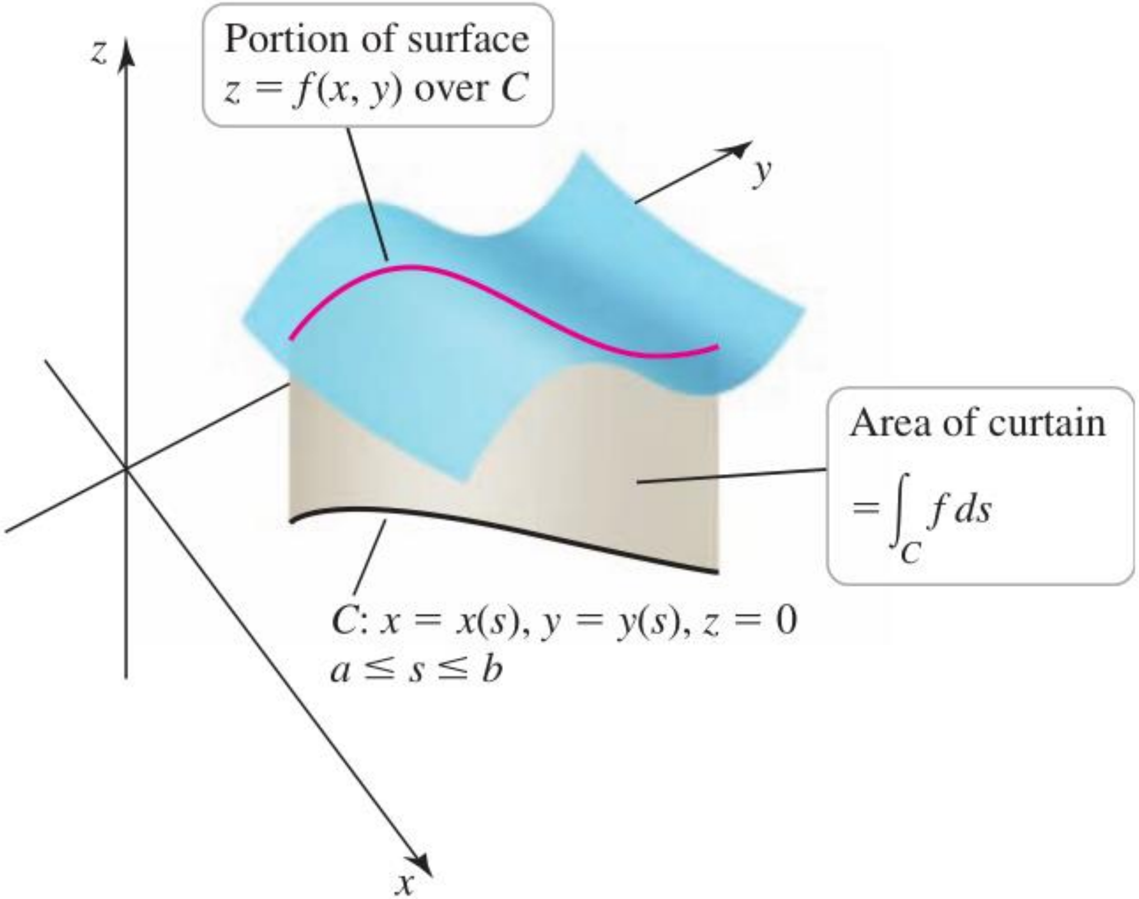
Flow curves of $\mathbf{F} = \nabla\varphi$ are orthogonal to level curves of φ everywhere.

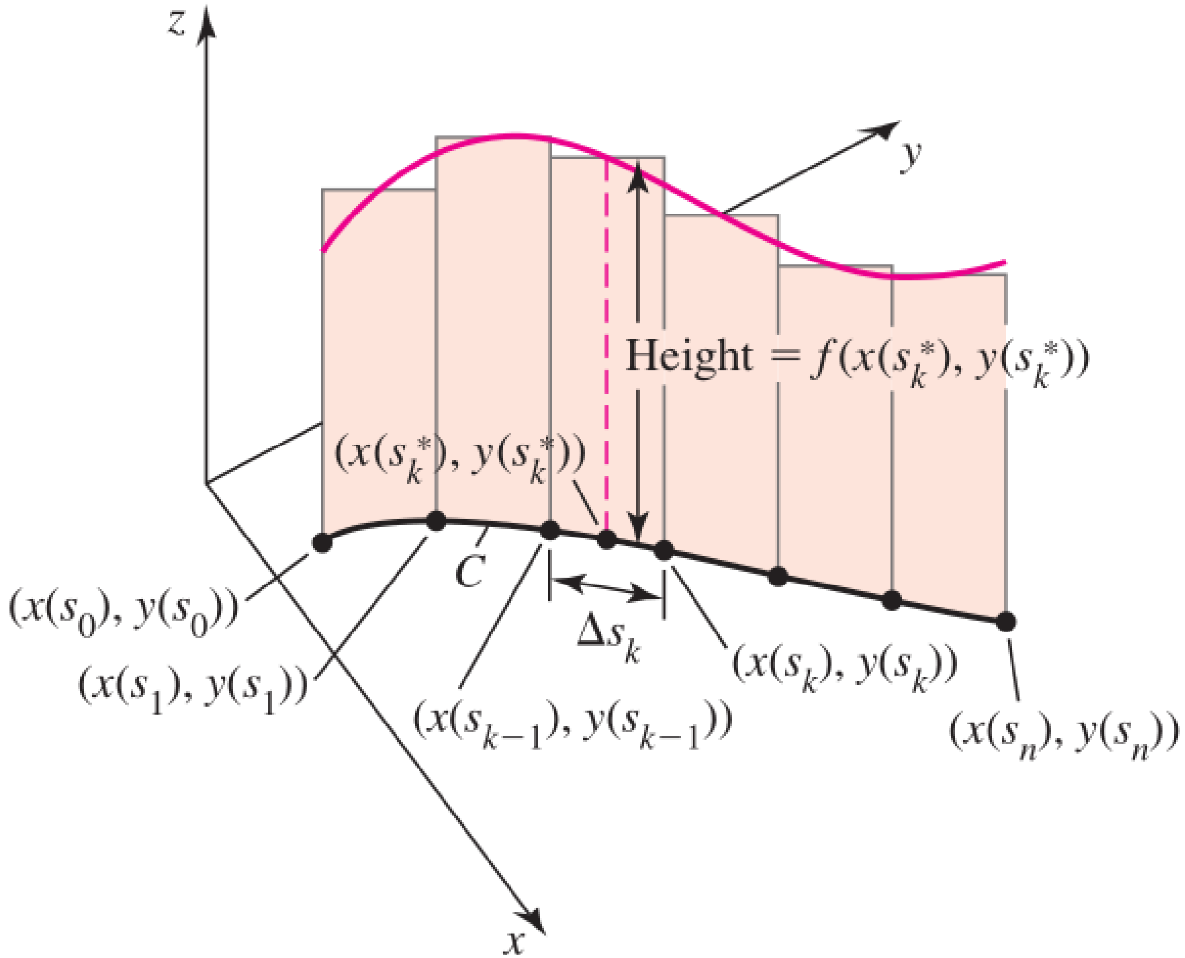
17.2

Line (Curve) Integrals

Scalar Line Integrals in the Plane

To integrate either scalar-valued functions or vector fields along curves.





$$\text{area} \approx \sum_{k=1}^{n} f(x(s_k^*), y(s_k^*))\,\Delta s_k.$$

DEFINITION Scalar Line Integral in the Plane, Arc Length Parameter

Suppose the scalar-valued function f is defined on the smooth curve $C: \mathbf{r}(s) = \langle x(s), y(s) \rangle$, parameterized by the arc length s. The **line integral of f over C** is

$$\int_C f(x(s), y(s))\, ds = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x(s_k^*), y(s_k^*))\Delta s_k,$$

provided this limit exists over all partitions of C. When the limit exists, f is said to be **integrable** on C.

More notations: $\int_C f(\mathbf{r}(s))ds$, $\int_C f(x,y)ds$ or $\int_C f ds$.

If f is continuous on a region containing C, then the line integral of f over C exists.

If $f(x,y) = 1$, $\int_C ds$ gives the length of the curve.

Parameters Other Than Arc Length

Assume C is described by $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, for $a \le t \le b$. The length of C over the interval $[a, t]$ is

$$s(t) = \int_a^t |\mathbf{r}'(u)| du$$

Differentiating both sides of this equation gives

$$ds = s'(t)dt = |\mathbf{r}'(t)|dt$$

Therefore,

$$\int_C f\, ds = \int_a^b f(x(t), y(t)) \underbrace{|\mathbf{r}'(t)|\, dt}_{ds}.$$

THEOREM 1 Evaluating Scalar Line Integrals in $\mathbb{R}^2$

Let f be continuous on a region containing a smooth curve C: $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, for $a \leq t \leq b$. Then

$$\int_C f\,ds = \int_a^b f(x(t), y(t))\,|\mathbf{r}'(t)|\,dt$$

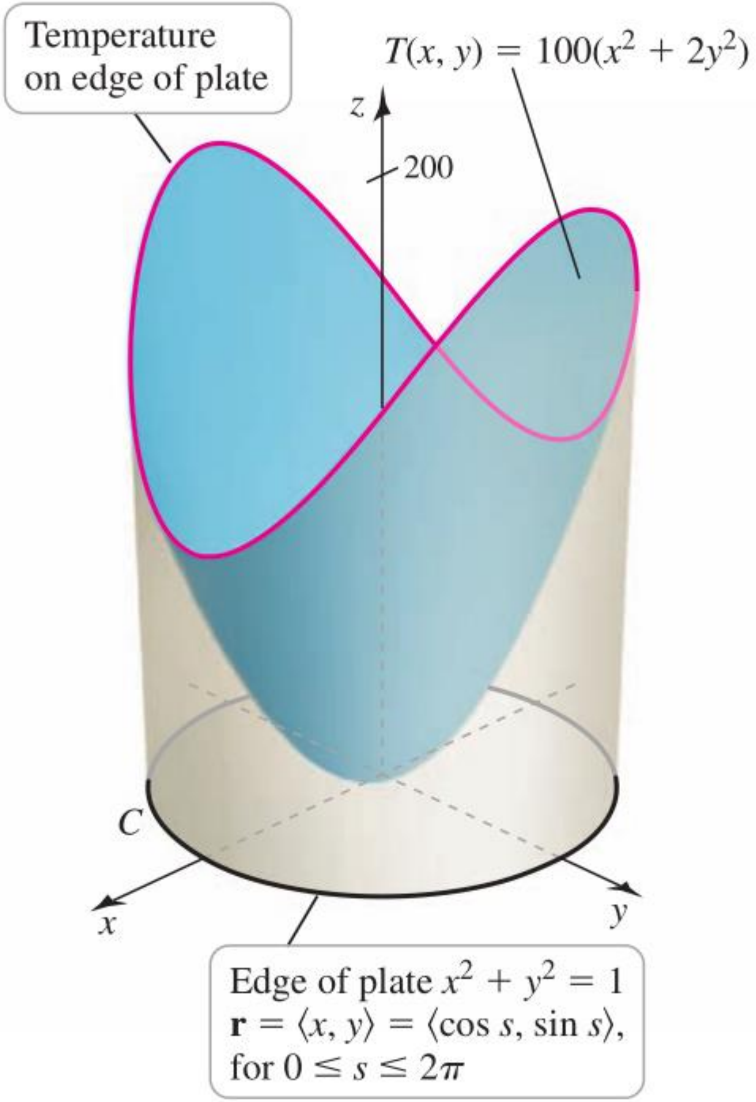
$$= \int_a^b f(x(t), y(t))\sqrt{x'(t)^2 + y'(t)^2}\,dt.$$

PROCEDURE **Evaluating the Line Integral $\int_C f\,ds$**

1. Find a parametric description of C in the form $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, for $a \le t \le b$.
2. Compute $|\mathbf{r}'(t)| = \sqrt{x'(t)^2 + y'(t)^2}$.
3. Make substitutions for x and y in the integrand and evaluate an ordinary integral:

$$\int_C f\,ds = \int_a^b f(x(t), y(t))\,|\mathbf{r}'(t)|\,dt.$$

EXAMPLE 1 **Average temperature on a circle** The temperature of the circular plate $R = \{(x, y): x^2 + y^2 \leq 1\}$ is $T(x, y) = 100(x^2 + 2y^2)$. Find the average temperature along the edge of the plate.



Line Integrals in $\mathbb{R}^3$

THEOREM 2 Evaluating Scalar Line Integrals in $\mathbb{R}^3$
Let f be continuous on a region containing a smooth curve $C: \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$, for $a \le t \le b$. Then

$$\int_C f\,ds = \int_a^b f(x(t), y(t), z(t))\,|\mathbf{r}'(t)|\,dt$$

$$= \int_a^b f(x(t), y(t), z(t))\sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\,dt.$$

EXAMPLE 3 **Line integrals in $\mathbb{R}^3$** Evaluate $\int_C (xy + 2z)\, ds$ on the following line segments.

a. The line segment from $P(1, 0, 0)$ to $Q(0, 1, 1)$

b. The line segment from $Q(0, 1, 1)$ to $P(1, 0, 0)$

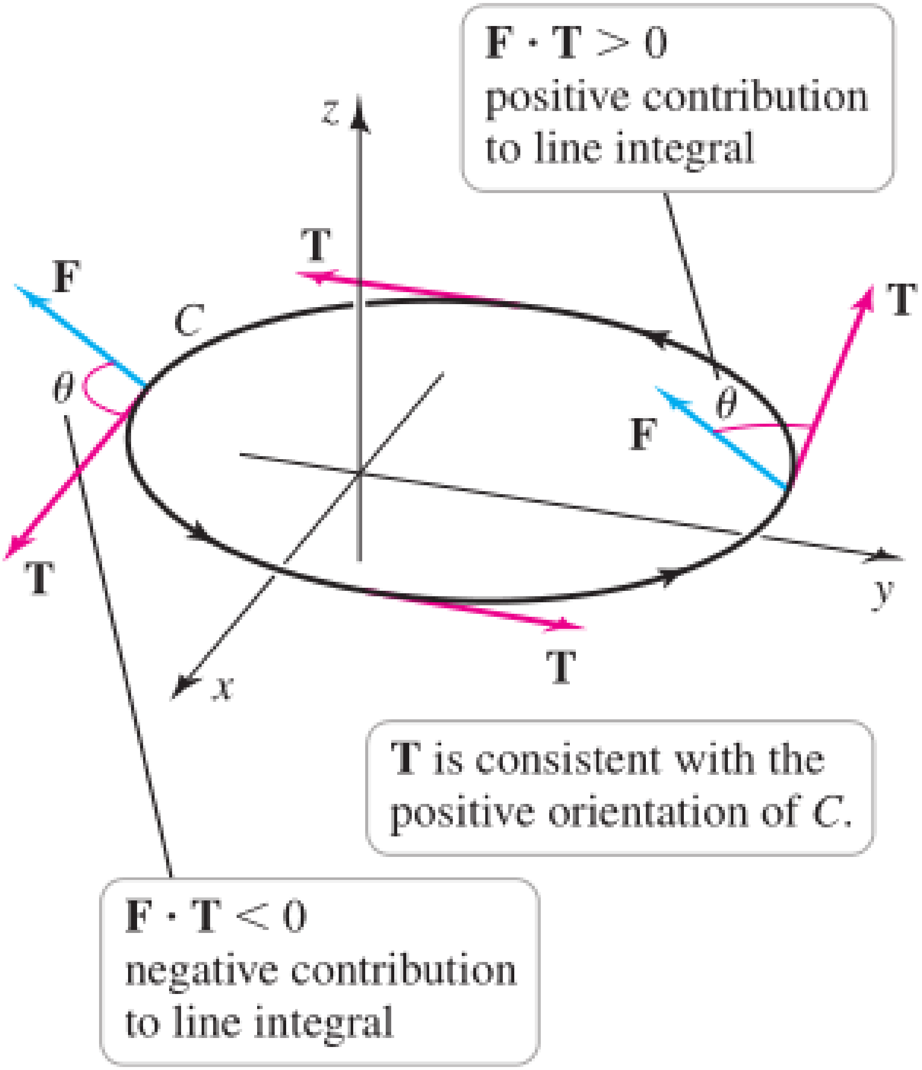
A scalar line integral is independent of the orientation and parameterization of the curve.

EXAMPLE 4 **Flight of an eagle** An eagle soars on the ascending spiral path

$$C: \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle = \left\langle 2400 \cos \frac{t}{2}, 2400 \sin \frac{t}{2}, 500t \right\rangle,$$

where x, y, and z are measured in feet and t is measured in minutes. How far does the eagle fly over the time interval $0 \leq t \leq 10$?

Line Integrals of Vector Fields



- *Oriented curve*: *positive orientation* is the direction where parameter increases, the unit tangent vector **T**
- The value of line integral depends on the orientation
- *Tangential component* of a vector field **F** in the direction of **T**, i.e., C
  $|\mathbf{F}| \cos \boldsymbol{\theta} = |\mathbf{F}||\mathbf{T}| \cos \boldsymbol{\theta}$
  $= \mathbf{F} \cdot \mathbf{T}$

Integrating $\mathbf{F} \cdot \mathbf{T}$ along C is to add up the components of **F** in the direction of C at each point of C

DEFINITION Line Integral of a Vector Field

Let $\mathbf{F}$ be a vector field that is continuous on a region containing a smooth oriented curve C parameterized by arc length. Let $\mathbf{T}$ be the unit tangent vector at each point of C consistent with the orientation. The line integral of $\mathbf{F}$ over C is $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$.

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_a^b \mathbf{F} \cdot \underbrace{\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}}_{\mathbf{T}} \underbrace{|\mathbf{r}'(t)|\, dt}_{ds} = \int_a^b \mathbf{F} \cdot \mathbf{r}'(t)\, dt.$$

Assume $\mathbf{F} = \langle f, g, h \rangle$

$$= \int_a^b \left(f(t)x'(t) + g(t)y'(t) + h(t)z'(t) \right) dt.$$

Let $d\mathbf{r} = \langle dx, dy, dz \rangle$

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_C \mathbf{F} \cdot d\mathbf{r}.$$

Different Forms of Line Integrals of Vector Fields

The line integral $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$ may be expressed in the following forms, where $\mathbf{F} = \langle f, g, h \rangle$ and C has a parameterization $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$, for $a \leq t \leq b$:

$$\begin{aligned}\int_a^b \mathbf{F} \cdot \mathbf{r}'(t)\, dt &= \int_a^b (f(t)x'(t) + g(t)y'(t) + h(t)z'(t))\, dt \\ &= \int_C f\, dx + g\, dy + h\, dz \\ &= \int_C \mathbf{F} \cdot d\mathbf{r}.\end{aligned}$$

For line integrals in the plane, we let $\mathbf{F} = \langle f, g \rangle$ and assume C is parameterized in the form $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, for $a \leq t \leq b$. Then

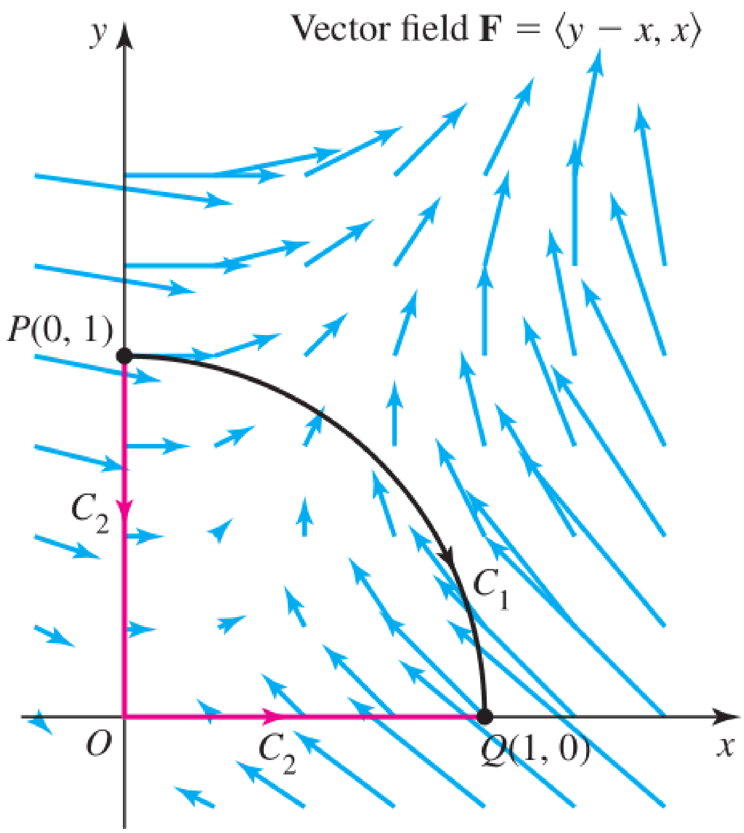
$$\int_a^b \mathbf{F} \cdot \mathbf{r}'(t)\, dt = \int_a^b (f(t)x'(t) + g(t)y'(t))\, dt = \int_C f\, dx + g\, dy = \int_C \mathbf{F} \cdot d\mathbf{r}.$$

EXAMPLE 5 **Different paths** Evaluate $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$ with $\mathbf{F} = \langle y - x, x \rangle$ on the following oriented paths in $\mathbb{R}^2$ (Figure 20).

a. The quarter circle C_1 from $P(0, 1)$ to $Q(1, 0)$

b. The quarter circle $-C_1$ from $Q(1, 0)$ to $P(0, 1)$

c. The path C_2 from $P(0, 1)$ to $Q(1, 0)$ via two line segments through $O(0, 0)$

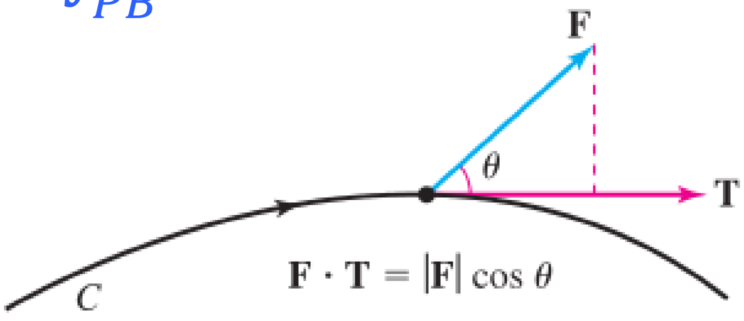


- The line integral of vector filed depends on the orientation of the curve.

- For what vector fields are the values of a line integral independent of path?

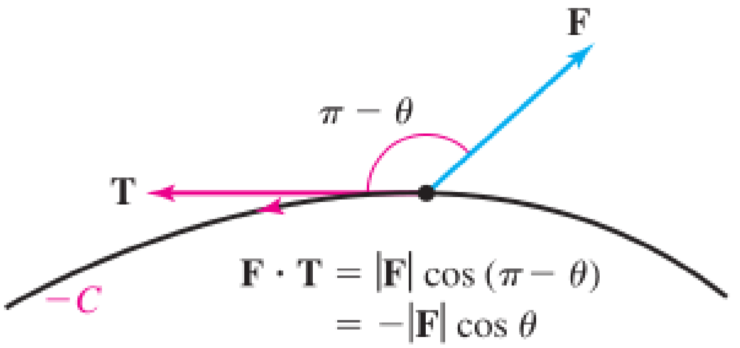
If C is a smooth curve from A to B and P is a point on C between A and B, then:

$$\int_{AB} \mathbf{F}\cdot d\mathbf{r} = \int_{AP} \mathbf{F}\cdot d\mathbf{r} + \int_{PB} \mathbf{F}\cdot d\mathbf{r}$$



The solutions to parts (a) and (b) of Example 5 illustrate a *general result* that applies to line integrals of vector fields:

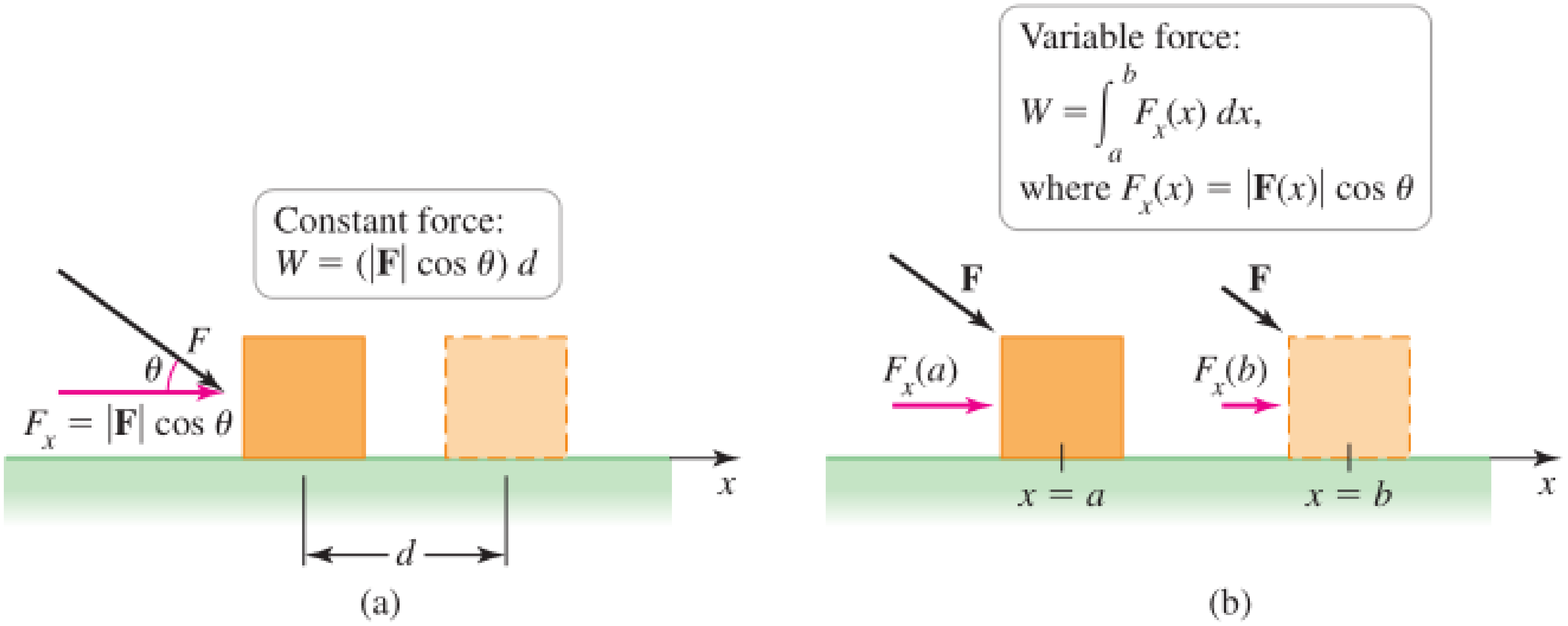
$$\int_{-C} \mathbf{F}\cdot\mathbf{T}ds = -\int_{C} \mathbf{F}\cdot\mathbf{T}ds$$



Reversing the orientation of C changes the sign of $\mathbf{F}\cdot\mathbf{T}$ at each point on C.

Work Integrals

If **F** is a *variable* force field, the work done in moving an object from $x = a$ to $x = b$ is $W = \int_a^b F_x(x)dx$ (in the direction of motion, x-axis here)



Take this progression *one step further*.

Let **F** is a *variable* force field defined in a region D of $\mathbb{R}^3$ and suppose C is a smooth, oriented curve in D, along which an object moves.

DEFINITION Work Done in a Force Field

Let $\mathbf{F}$ be a continuous force field in a region D of $\mathbb{R}^3$. Let C: $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$, for $a \leq t \leq b$, be a smooth curve in D with a unit tangent vector $\mathbf{T}$ consistent with the orientation. The work done in moving an object along C in the positive direction is

$$W = \int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_a^b \mathbf{F} \cdot \mathbf{r}'(t)\, dt.$$

EXAMPLE 6 **An inverse square force** Gravitational and electrical forces between point masses and point charges obey inverse square laws: They act along the line joining the centers and they vary as $1/r^2$, where r is the distance between the centers. The force of attraction (or repulsion) of an inverse square force field is given by the vector field $\mathbf{F} = \frac{k\langle x, y, z\rangle}{(x^2 + y^2 + z^2)^{3/2}}$, where k is a physical constant. Because $\mathbf{r} = \langle x, y, z\rangle$, this force may also be written $\mathbf{F} = \frac{k\mathbf{r}}{|\mathbf{r}|^3}$. Find the work done in moving an object along the following paths.

a. C_1 is the line segment from $(1, 1, 1)$ to (a, a, a), where $a > 1$.

b. C_2 is the extension of C_1 produced by letting $a \to \infty$.

A parametric description of C_1 consistent with the orientation $r(t) = \langle t, t, t\rangle$, with $r'(t) = \langle 1,1,1\rangle$

Circulation and Flux of a Vector Field

The **Circulation** of $\mathbf{F}$ along C is a measure of how much of the vector field points in the direction of C.

Imagine you are travelling along C in the positive direction, how much of the vector field is at your back and how much of it is in your face?

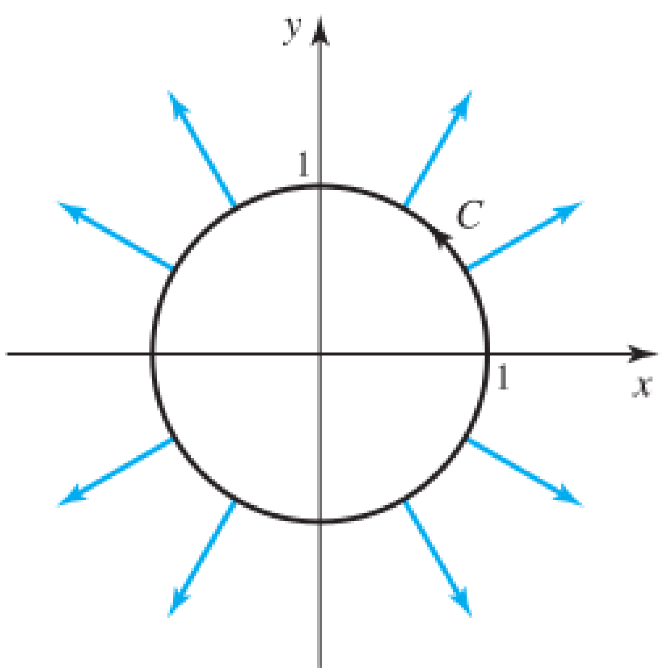
DEFINITION Circulation

Let $\mathbf{F}$ be a continuous vector field on a region D of $\mathbb{R}^3$ and let C be a closed smooth oriented curve in D. The **circulation** of $\mathbf{F}$ on C is $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$, where $\mathbf{T}$ is the unit vector tangent to C consistent with the orientation.

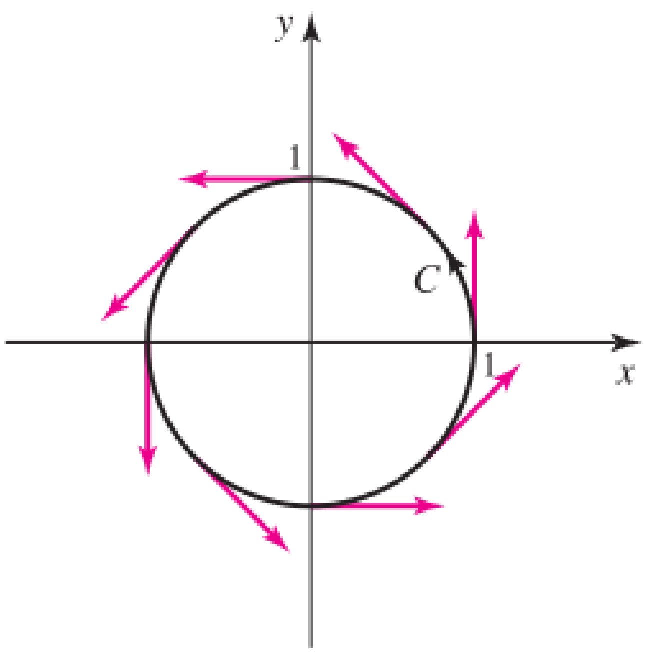
EXAMPLE 7 **Circulation of two-dimensional flows** Let C be the unit circle with counterclockwise orientation. Find the circulation on C of the following vector fields.

a. The radial vector field $\mathbf{F} = \langle x, y \rangle$

b. The rotation vector field $\mathbf{F} = \langle -y, x \rangle$

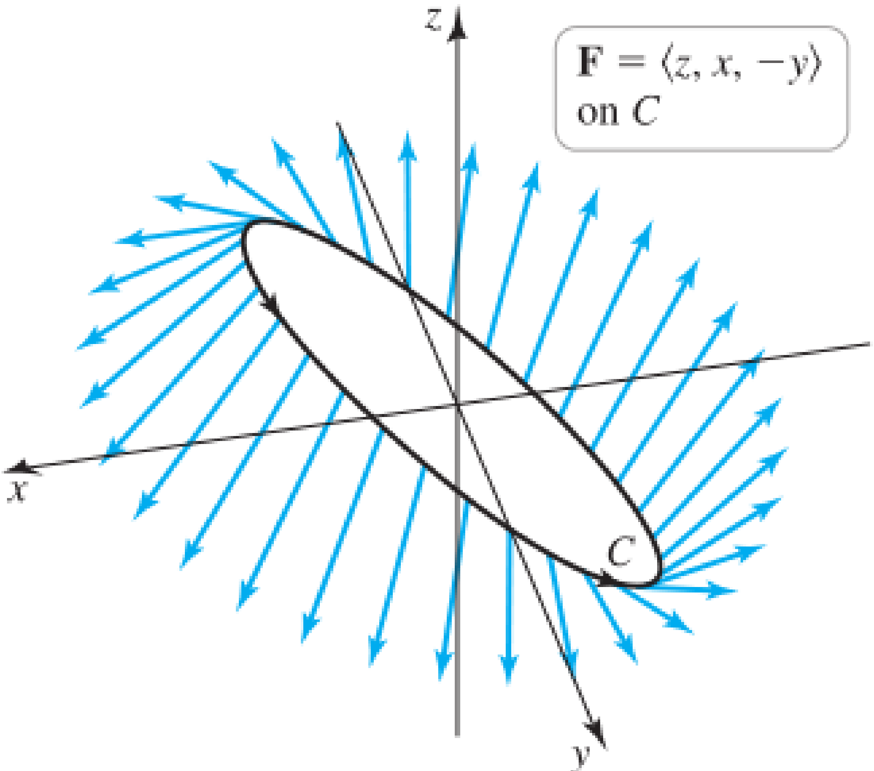


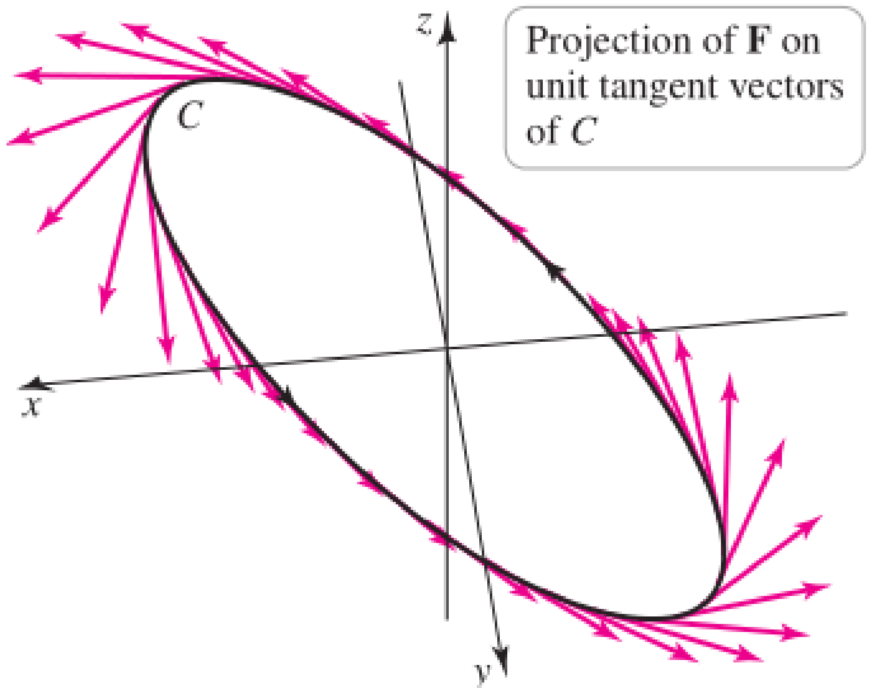
On the unit circle, $\mathbf{F} = \langle x, y \rangle$ is orthogonal to C and has zero circulation on C.



On the unit circle, $\mathbf{F} = \langle -y, x \rangle$ is tangent to C and has positive circulation on C.

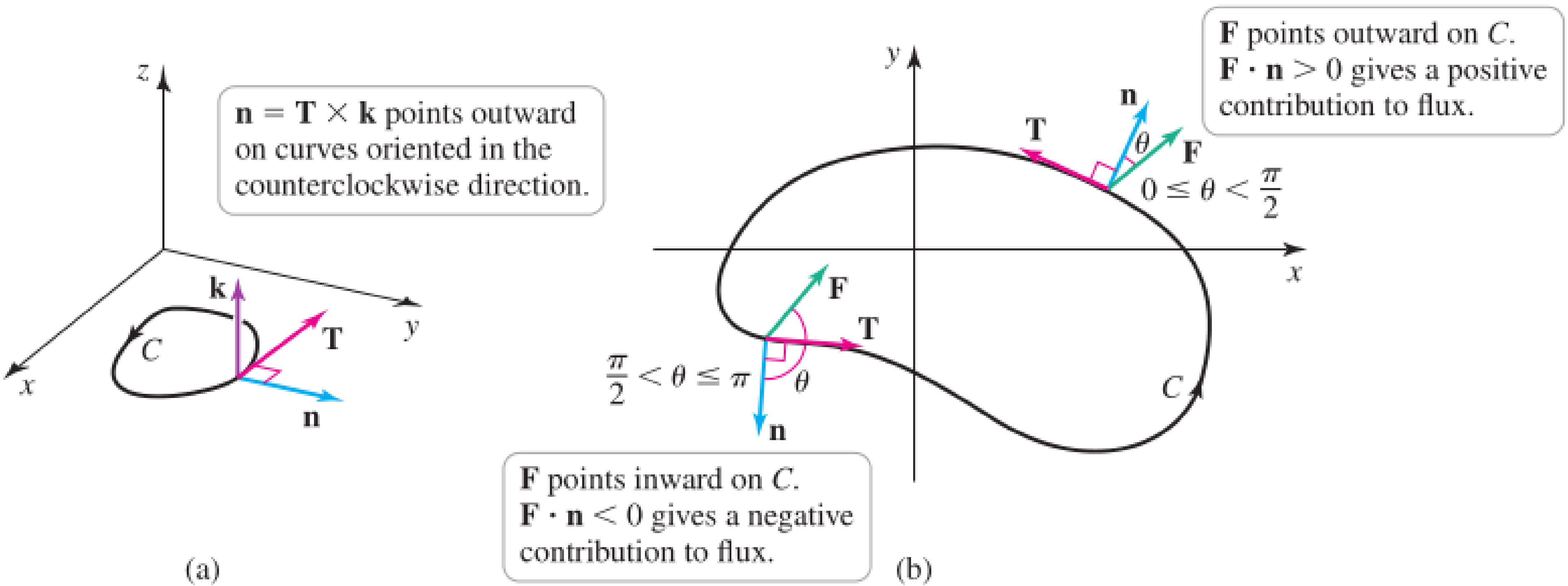
EXAMPLE 8 **Circulation of a three-dimensional flow** Find the circulation of the vector field $\mathbf{F} = \langle z, x, -y \rangle$ on the tilted ellipse C: $\mathbf{r}(t) = \langle \cos t, \sin t, \cos t \rangle$, for $0 \le t \le 2\pi$ (Figure 25a).





Flux of Two-Dimensional Vector Field

The **flux** of **F** across (outward) C is to "add up" the components of **F** *orthogonal or normal* to C at each point of C, $\int_C \mathbf{F}\cdot\mathbf{n}\,ds$



$$\mathbf{T} = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \langle T_x, T_y, 0 \rangle = \frac{\langle x'(t), y'(t), 0 \rangle}{|\mathbf{r}'(t)|}.$$

$$\mathbf{n} = \mathbf{T} \times \mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ T_x & T_y & 0 \\ 0 & 0 & 1 \end{vmatrix} = T_y\,\mathbf{i} - T_x\,\mathbf{j}.$$

$$\mathbf{n} = T_y\,\mathbf{i} - T_x\,\mathbf{j} = \frac{y'(t)}{|\mathbf{r}'(t)|}\mathbf{i} - \frac{x'(t)}{|\mathbf{r}'(t)|}\mathbf{j} = \frac{\langle y'(t), -x'(t) \rangle}{|\mathbf{r}'(t)|}.$$

$$\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_a^b \mathbf{F} \cdot \underbrace{\frac{\langle y'(t), -x'(t) \rangle}{|\mathbf{r}'(t)|}}_{\mathbf{n}} \underbrace{|\mathbf{r}'(t)|\, dt}_{ds} = \int_a^b (f(t)y'(t) - g(t)x'(t))\, dt.$$

$$= \int_C f\, dy - g\, dx.$$

DEFINITION Flux

Let $\mathbf{F} = \langle f, g \rangle$ be a continuous vector field on a region R of $\mathbb{R}^2$. Let C: $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, for $a \le t \le b$, be a smooth oriented curve in R that does not intersect itself. The **flux** of the vector field $\mathbf{F}$ across C is

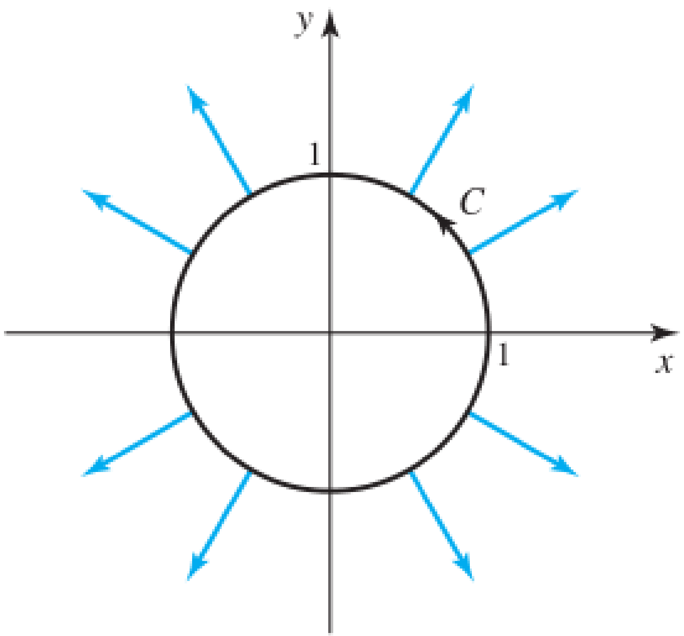
$$\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_a^b (f(t)y'(t) - g(t)x'(t))\, dt,$$

where $\mathbf{n} = \mathbf{T} \times \mathbf{k}$ is the unit normal vector and $\mathbf{T}$ is the unit tangent vector consistent with the orientation. If C is a closed curve with counterclockwise orientation, $\mathbf{n}$ is the outward normal vector and the flux integral gives the **outward flux** across C.

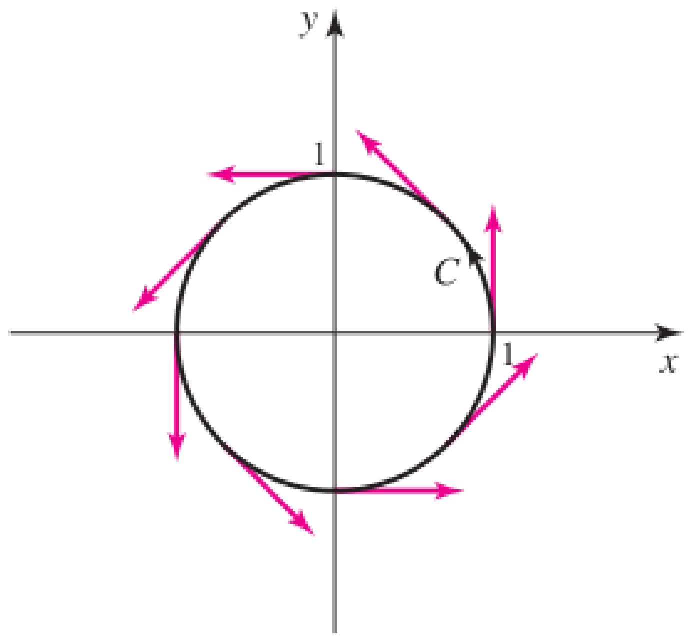
EXAMPLE 9 **Flux of two-dimensional flows** Find the outward flux across the unit circle with counterclockwise orientation for the following vector fields.

a. The radial vector field $\mathbf{F} = \langle x, y \rangle$

b. The rotation vector field $\mathbf{F} = \langle -y, x \rangle$



On the unit circle, $\mathbf{F} = \langle x, y \rangle$ is orthogonal to C and has positive outward flux on C.



On the unit circle, $\mathbf{F} = \langle -y, x \rangle$ is tangent to C and has zero outward flux on C.

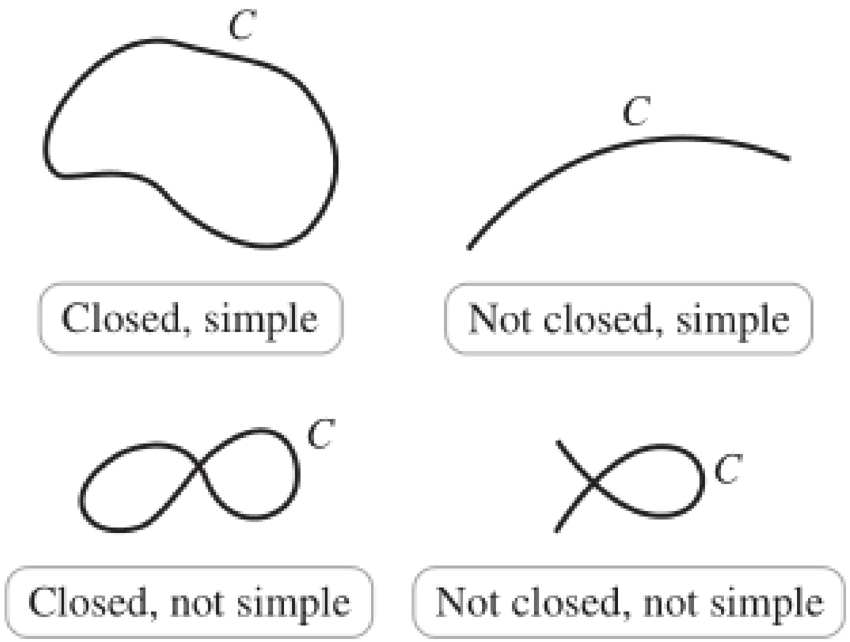
17.3

Conservative Vector Fields

Types of Curves and Regions

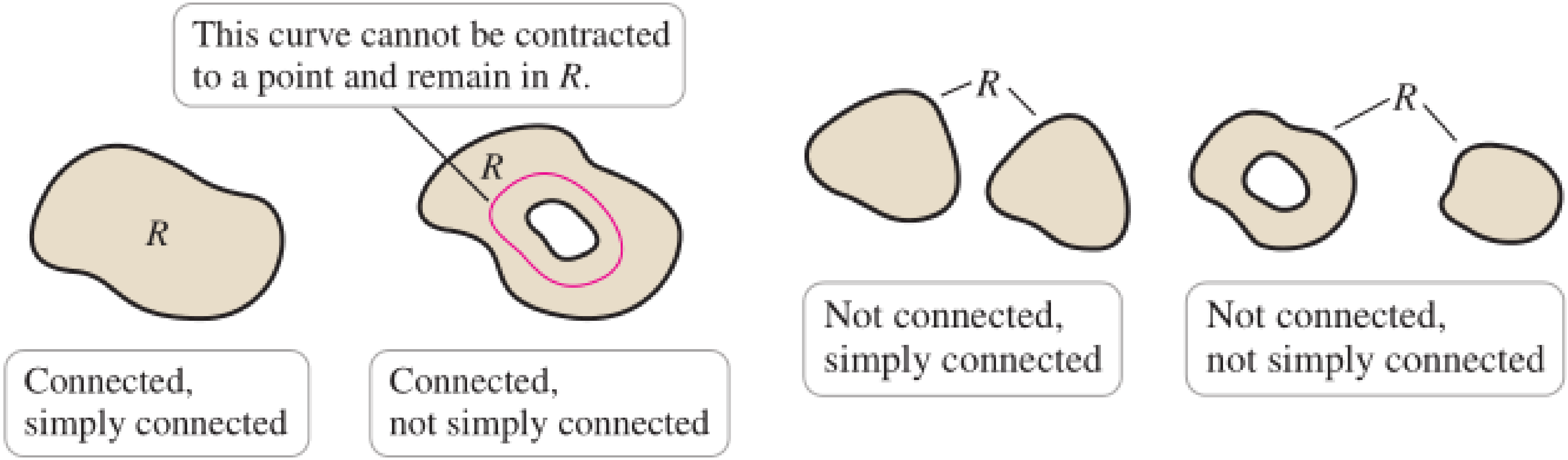
DEFINITION **Simple and Closed Curves**

Suppose a curve C (in $\mathbb{R}^2$ or $\mathbb{R}^3$) is described parametrically by $\mathbf{r}(t)$, where $a \leq t \leq b$. Then C is a **simple curve** if $\mathbf{r}(t_1) \neq \mathbf{r}(t_2)$ for all t_1 and t_2, with $a < t_1 < t_2 < b$; that is, C never intersects itself between its endpoints. The curve C is **closed** if $\mathbf{r}(a) = \mathbf{r}(b)$; that is, the initial and terminal points of C are the same (Figure 14.28).



DEFINITION Connected and Simply Connected Regions

An open region R in $\mathbb{R}^2$ (or D in $\mathbb{R}^3$) is **connected** if it is possible to connect any two points of R by a continuous curve lying in R. An open region R is **simply connected** if every closed simple curve in R can be deformed and contracted to a point in R (Figure 14.29).



Test for Conservative Vector Fields

DEFINITION Conservative Vector Field

A vector field $\mathbf{F}$ is said to be **conservative** on a region (in $\mathbb{R}^2$ or $\mathbb{R}^3$) if there exists a scalar function φ such that $\mathbf{F} = \nabla\varphi$ on that region.

THEOREM 14.3 Test for Conservative Vector Fields

Let $\mathbf{F} = \langle f, g, h \rangle$ be a vector field defined on a connected and simply connected region D of $\mathbb{R}^3$, where f, g, and h have continuous first partial derivatives on D. Then $\mathbf{F}$ is a conservative vector field on D (there is a potential function φ such that $\mathbf{F} = \nabla\varphi$) if and only if

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}, \qquad \frac{\partial f}{\partial z} = \frac{\partial h}{\partial x}, \quad \text{and} \quad \frac{\partial g}{\partial z} = \frac{\partial h}{\partial y}.$$

For vector fields in $\mathbb{R}^2$, we have the single condition $\dfrac{\partial f}{\partial y} = \dfrac{\partial g}{\partial x}$.

EXAMPLE 1 **Testing for conservative fields** Determine whether the following vector fields are conservative on $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively.

a. $\mathbf{F} = \langle e^x \cos y, -e^x \sin y \rangle$

b. $\mathbf{F} = \langle 2xy - z^2, x^2 + 2z, 2y - 2xz \rangle$

Finding Potential Functions

EXAMPLE 2 **Finding potential functions** Find a potential function for the conservative vector fields in Example 1.

a. $\mathbf{F} = \langle e^x \cos y, -e^x \sin y \rangle$

b. $\mathbf{F} = \langle 2xy - z^2, x^2 + 2z, 2y - 2xz \rangle$

PROCEDURE **Finding Potential Functions in $\mathbb{R}^3$**

Suppose $\mathbf{F} = \langle f, g, h \rangle$ is a conservative vector field. To find φ such that $\mathbf{F} = \nabla\varphi$, use the following steps:

1. Integrate $\varphi_x = f$ with respect to x to obtain φ, which includes an arbitrary function $c(y, z)$.
2. Compute φ_y and equate it to g to obtain an expression for $c_y(y, z)$.
3. Integrate $c_y(y, z)$ with respect to y to obtain $c(y, z)$, including an arbitrary function $d(z)$.
4. Compute φ_z and equate it to h to get $d(z)$.

A similar procedure beginning with $\varphi_y = g$ or $\varphi_z = h$ may be easier in some cases.

Fundamental Theorem for Line Integrals and Path Independence

THEOREM 4 Fundamental Theorem for Line Integrals

Let $\mathbf{F}$ be a continuous vector field on an open connected region R in $\mathbb{R}^2$ (or D in $\mathbb{R}^3$). There exists a potential function φ with $\mathbf{F} = \nabla\varphi$ (which means that $\mathbf{F}$ is conservative) if and only if

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_C \mathbf{F} \cdot d\mathbf{r} = \varphi(B) - \varphi(A),$$

for all points A and B in R (or D) and all piecewise-smooth oriented curves C in R (or D) from A to B.

Meaning of this theorem:

- If **F** is a conservative vector field, then the value of a line integral of **F** depends only on the endpoints of the path. More simply, *the line integral is independent of path*.

- If think of φ as an antiderivative of the vector field **F**, then the line integral of **F** is the difference of the values of φ evaluated at the endpoints, parallel to the Fundamental Theorem of Calculus.

EXAMPLE 3 **Verifying path independence** Consider the potential function $\varphi(x, y) = (x^2 - y^2)/2$ and its gradient field $\mathbf{F} = \langle x, -y \rangle$.

- Let C_1 be the quarter circle $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$, for $0 \leq t \leq \pi/2$, from $A(1, 0)$ to $B(0, 1)$.
- Let C_2 be the line $\mathbf{r}(t) = \langle 1 - t, t \rangle$, for $0 \leq t \leq 1$, also from A to B.

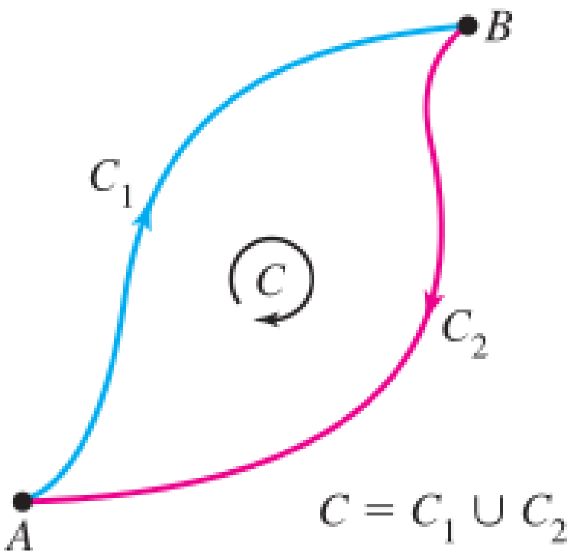
Evaluate the line integrals of $\mathbf{F}$ on C_1 and C_2, and show that both are equal to $\varphi(B) - \varphi(A)$.

EXAMPLE 4 **Line integral of a conservative vector field** Evaluate

$$\int_C ((2xy - z^2)\,\mathbf{i} + (x^2 + 2z)\,\mathbf{j} + (2y - 2xz)\,\mathbf{k}) \cdot d\mathbf{r},$$

where C is a simple curve from $A(-3, -2, -1)$ to $B(1, 2, 3)$.

Line Integrals on Closed Curves



$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

If **F** is a conservative vector field, then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \varphi(A) - \varphi(A) = 0$$

$$0 = \oint_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = -\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{-C_2} \mathbf{F} \cdot d\mathbf{r}$$

Where $-C_2$ is the curve C_2 traversed in the opposite direction.

THEOREM 5 Line Integrals on Closed Curves

Let R in $\mathbb{R}^2$ (or D in $\mathbb{R}^3$) be an open connected region. Then $\mathbf{F}$ is a conservative vector field on R if and only if $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ on all simple closed piecewise-smooth oriented curves C in R.

EXAMPLE 5 **A closed curve line integral in $\mathbb{R}^3$** Evaluate $\int_C \nabla(-xy + xz + yz) \cdot d\mathbf{r}$ on the curve C: $\mathbf{r}(t) = \langle \sin t, \cos t, \sin t \rangle$, for $0 \le t \le 2\pi$, without using Theorems 4 or 5.

Summary of the Properties of Conservative Vector Fields

Three equivalent properties of conservative vector fields F defined on an open connected region R in $\mathbb{R}^2$ (or D in $\mathbb{R}^3$)

- There exists a potential function φ such that $\mathbf{F} = \nabla\varphi$ (Definition)
- $\int_C \mathbf{F} \cdot d\mathbf{r} = \varphi(B) - \varphi(A)$ for all points A and B in R and all piecewise-smooth oriented curves C in R from A to B (Path Independence).
- $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ on all simple piecewise-smooth closed oriented curves C in R.

$$\text{Path independence} \overset{\text{Theorem 4}}{\Longleftrightarrow} \mathbf{F} \text{ is conservative } (\nabla\varphi = \mathbf{F}) \overset{\text{Theorem 5}}{\Longleftrightarrow} \oint_C \mathbf{F} \cdot d\mathbf{r} = 0.$$

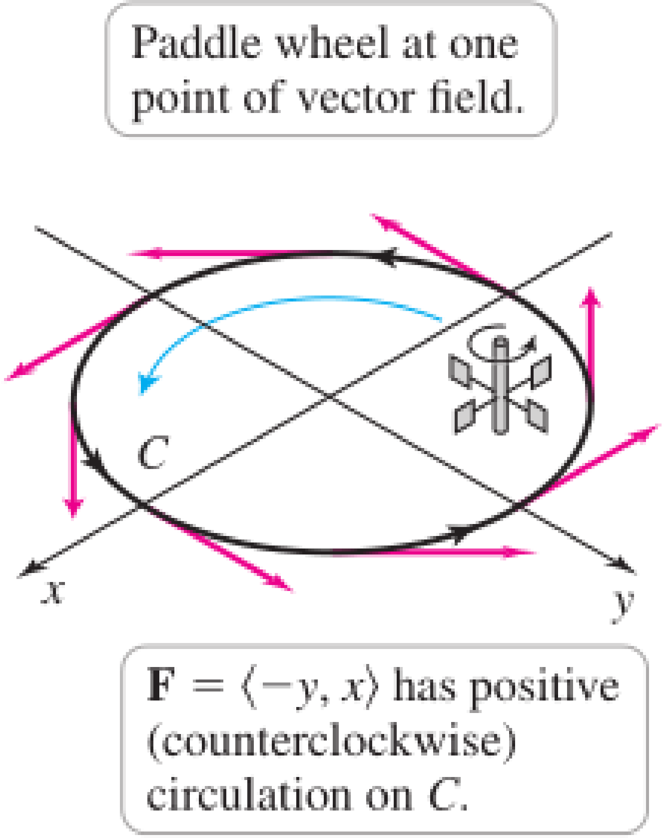
17.4

Green’s Theorem

Circulation Form of Green's Theorem

$$\int_a^b \frac{df}{dx}\,dx = f(b) - f(a)$$

$$\int_C \nabla\varphi \cdot d\mathbf{r} = \varphi(B) - \varphi(A)$$

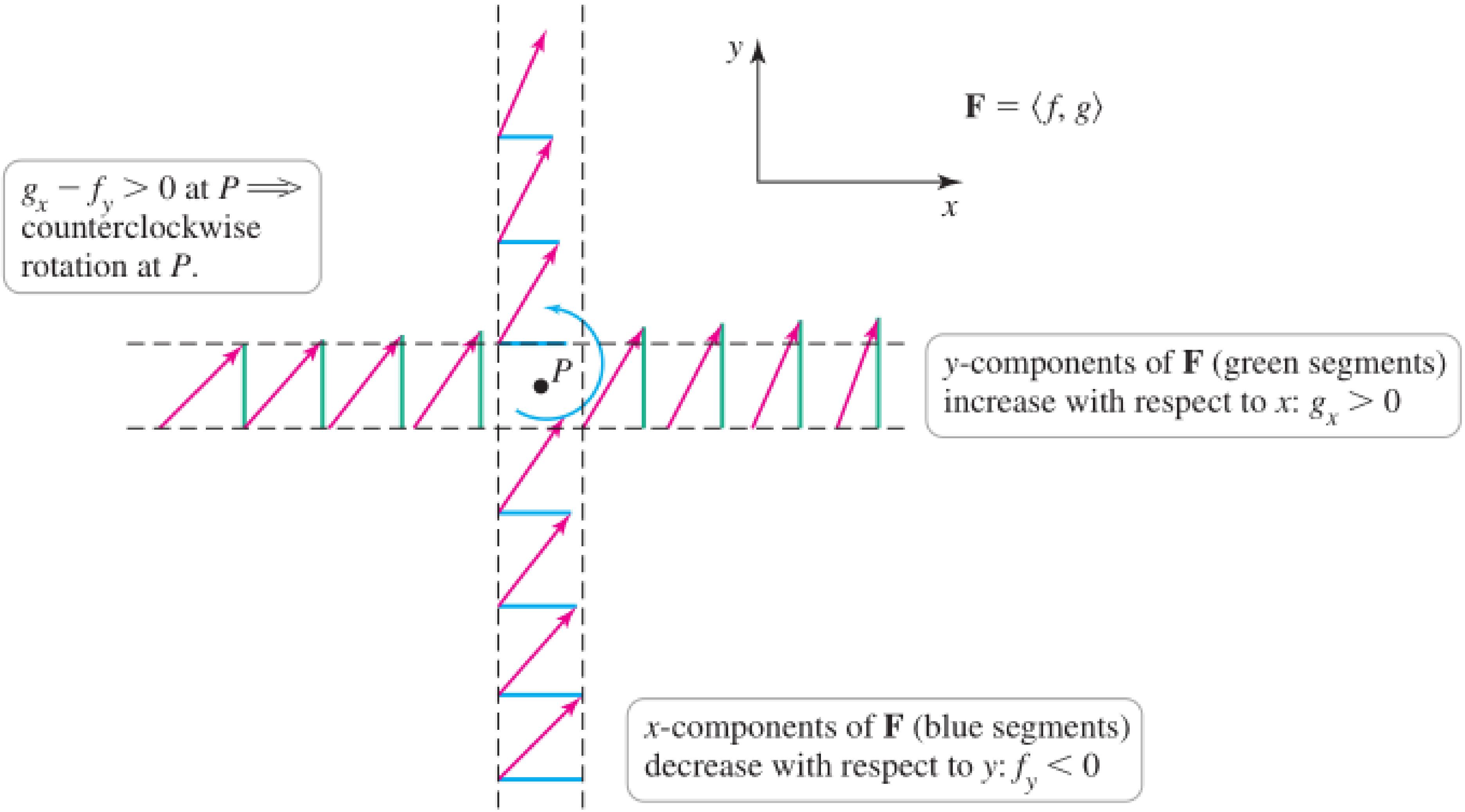


A nonzero circulation on a closed curve says that the vector field must have some property *inside* the curve that produces the circulation, can be thought of as a *net rotation*.

THEOREM 14.6 Green's Theorem—Circulation Form
Let C be a simple closed piecewise-smooth curve, oriented counterclockwise, that encloses a connected and simply connected region R in the plane. Assume $\mathbf{F} = \langle f, g \rangle$, where f and g have continuous first partial derivatives in R. Then

$$\underbrace{\oint_C \mathbf{F} \cdot d\mathbf{r}}_{\text{circulation}} = \underbrace{\oint_C f\,dx + g\,dy}_{\text{circulation}} = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA.$$

- The quantity $\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}$ is called the *two-dimensional curl* of the vector field, which describes the rotation of the vector field *within* C that produces the circulation *on* C.
- The theorem says that the net rotation throughout R equals the circulation on the boundary of R.

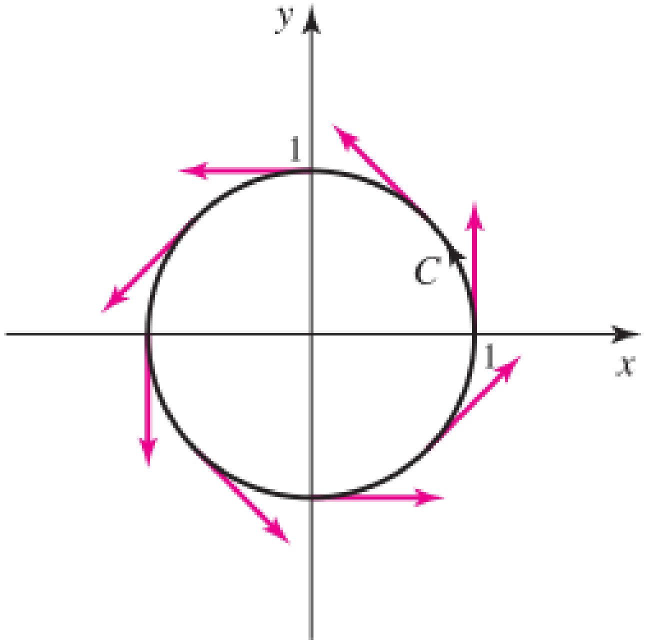

$g_x - f_y > 0$ at $P \Longrightarrow$ counterclockwise rotation at P.
$\mathbf{F} = \langle f, g \rangle$
y
x
P
y-components of $\mathbf{F}$ (green segments) increase with respect to x: $g_x > 0$
x-components of $\mathbf{F}$ (blue segments) decrease with respect to y: $f_y < 0$

DEFINITION **Two-Dimensional Curl**

The **two-dimensional curl** of the vector field $\mathbf{F} = \langle f, g \rangle$ is $\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}$. If the curl is zero throughout a region, the vector field is **irrotational** on that region.

- For a conservative vector field **F** in a region, the circulation $\oint_C \mathbf{F} \cdot d\mathbf{r}$ is zero on any simple closed curve in the region.
- A two-dimensional vector field $\mathbf{F} = \langle f, g \rangle$ for which $\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = 0$ at all points is said to be *irrotational*, because it produces zero circulation on closed curves in the region.
- Irrotational vector fields on simply connected regions in $\mathbb{R}^2$ are conservative.

EXAMPLE 1 **Circulation of a rotation field** Consider the rotation vector field $\mathbf{F} = \langle -y, x \rangle$ on the unit disk $R = \{(x, y): x^2 + y^2 \leq 1\}$ (Figure 31). In Example 7 of Section 2, we showed that $\oint_C \mathbf{F} \cdot d\mathbf{r} = 2\pi$, where C is the boundary of R oriented counterclockwise. Confirm this result using Green's Theorem.



On the unit circle, $\mathbf{F} = \langle -y, x \rangle$ is tangent to C and has positive circulation on C.

Calculating Area by Green's Theorem

Consider the vector fields $\mathbf{F} = \langle f, g\rangle = \langle 0, x\rangle$ and $\mathbf{F} = \langle y, 0\rangle$.

In the first case, $g_x = 1$ and $f_y = 0$; by Green's Theorem,

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C \underbrace{x\,dy}_{\mathbf{F}\cdot d\mathbf{r}} = \iint_R \underbrace{dA}_{\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = 1} = \text{area of } R.$$

In the second case, $g_x = 0$ and $f_y = 1$; by Green's Theorem,

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C y dx = -\iint_{\boldsymbol{R}} dA = -\text{area of } R$$

Area of a Plane Region by Line Integrals

Under the conditions of Green's Theorem, the area of a region R enclosed by a curve C is

$$\oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C (x\,dy - y\,dx).$$

EXAMPLE 2 **Area of an ellipse** Find the area of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

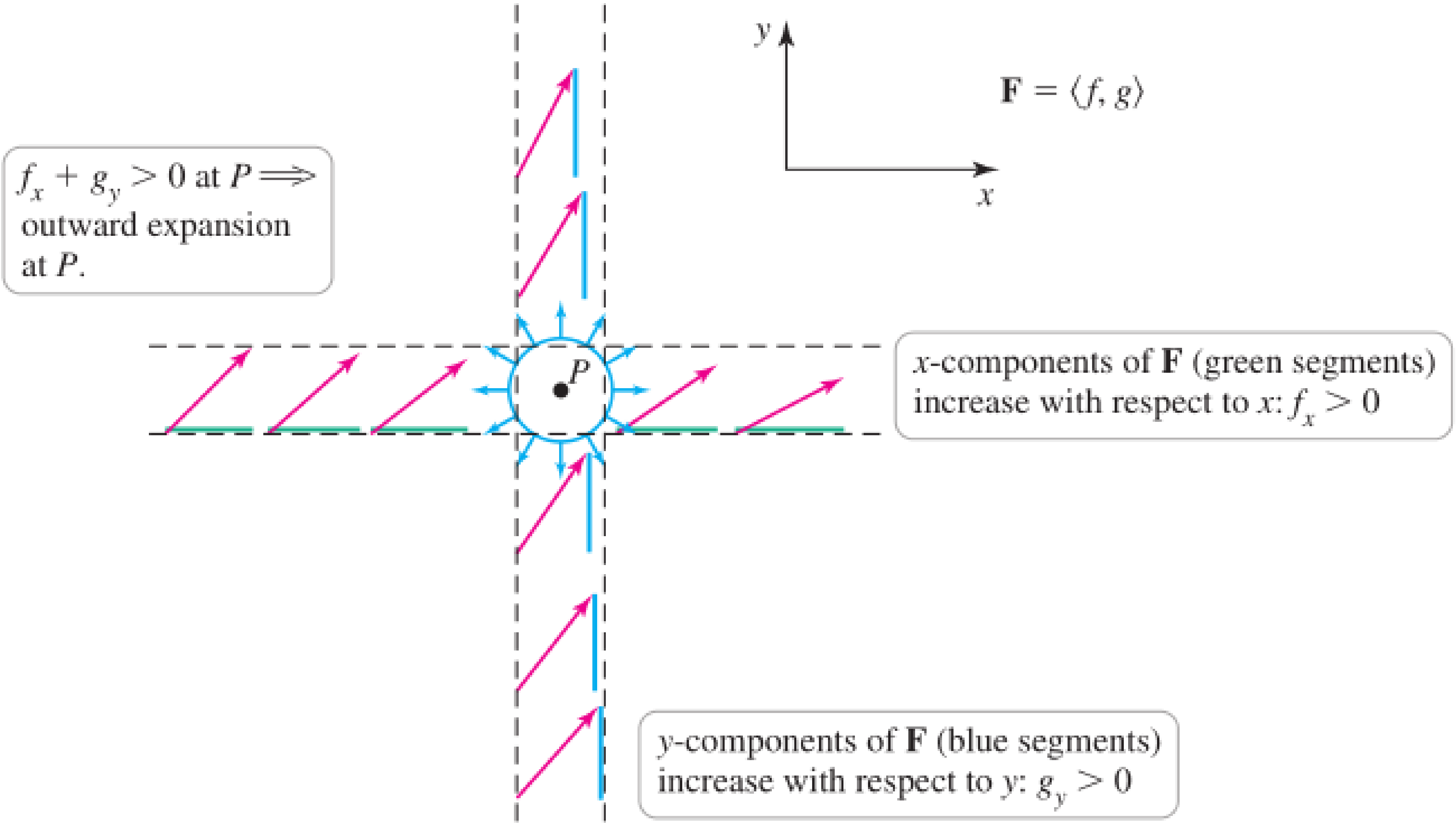
Flux Form of Green's Theorem

THEOREM 7 Green's Theorem, Flux Form
Let C be a simple closed piecewise-smooth curve, oriented counterclockwise, that encloses a connected and simply connected region R in the plane. Assume $\mathbf{F} = \langle f, g \rangle$, where f and g have continuous first partial derivatives in R. Then

$$\underbrace{\oint_C \mathbf{F} \cdot \mathbf{n}\, ds}_{\text{outward flux}} = \underbrace{\oint_C f\, dy - g\, dx}_{\text{outward flux}} = \iint_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \right) dA,$$

where $\mathbf{n}$ is the outward unit normal vector on the curve.

- The line integrals give the *outward flux* of the vector field across C.
- Quantity $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$, the *two-dimensional divergence* of the vector field, describes the property of the vector field that produces the flux *across* C.

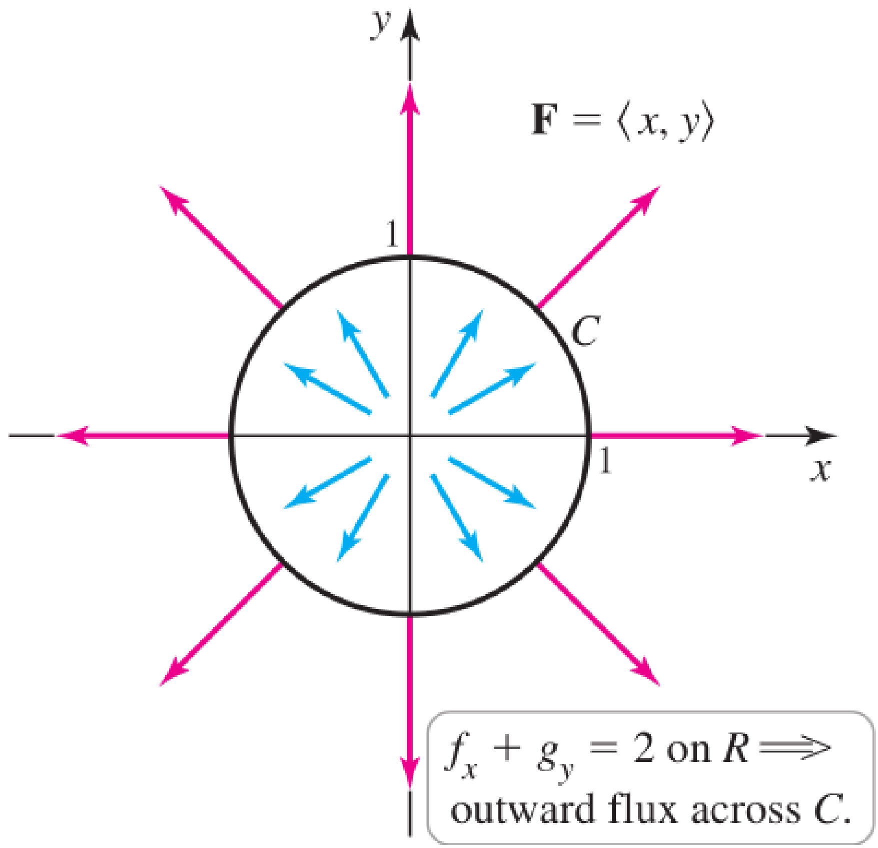


$f_x + g_y > 0$ at $P \Longrightarrow$ outward expansion at P.
y
x
$\mathbf{F} = \langle f, g \rangle$
P
x-components of $\mathbf{F}$ (green segments) increase with respect to x: $f_x > 0$
y-components of $\mathbf{F}$ (blue segments) increase with respect to y: $g_y > 0$

DEFINITION **Two-Dimensional Divergence**

The **two-dimensional divergence** of the vector field $\mathbf{F} = \langle f, g \rangle$ is $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$. If the divergence is zero throughout a region, the vector field is **source free** on that region.

EXAMPLE 3 **Outward flux of a radial field** Use Green's Theorem to compute the outward flux of the radial field $\mathbf{F} = \langle x, y \rangle$ across the unit circle $C = \{(x, y): x^2 + y^2 = 1\}$ (Figure 34). Interpret the result.



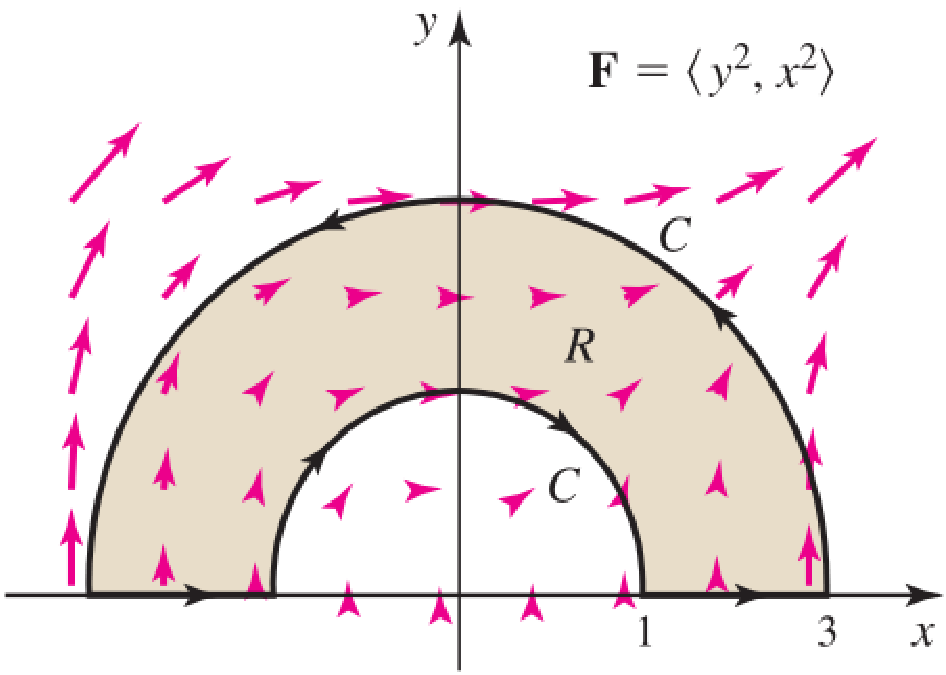
EXAMPLE 4 **Line integral as a double integral** Evaluate

$$\oint_C (4x^3 + \sin y^2)\, dy - (4y^3 + \cos x^2)\, dx,$$

where C is the boundary of the disk $R = \{(x, y): x^2 + y^2 \leq 4\}$ oriented counterclockwise.

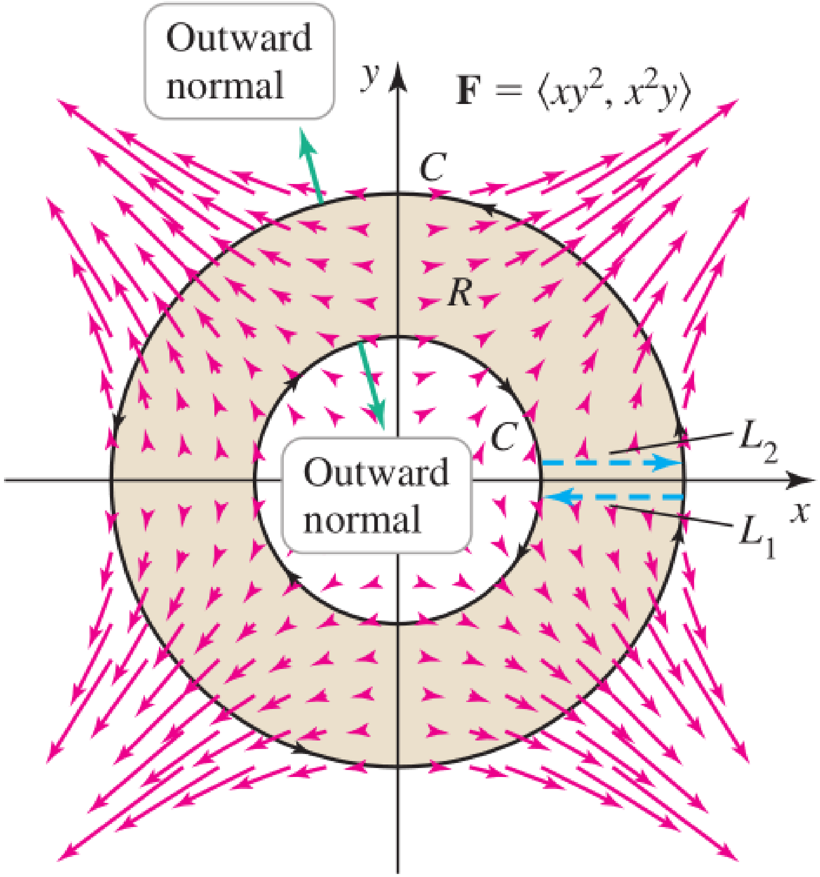
Circulation and Flux on More General Regions

EXAMPLE 5 **Circulation on a half annulus** Consider the vector field $\mathbf{F} = \langle y^2, x^2 \rangle$ on the half annulus $R = \{(x, y): 1 \leq x^2 + y^2 \leq 9, y \geq 0\}$, whose boundary is C. Find the circulation on C, assuming it has the orientation shown in Figure 35.



Circulation on boundary of R is negative.

EXAMPLE 6 **Flux across the boundary of an annulus** Find the outward flux of the vector field $\mathbf{F} = \langle xy^2, x^2y \rangle$ across the boundary of the annulus $R = \{(x, y): 1 \le x^2 + y^2 \le 4\} = \{(r, \theta): 1 \le r \le 2, 0 \le \theta \le 2\pi\}$ (Figure 36).



Net flux across boundary of R is positive.

Stream Functions

Vector field $\mathbf{F} = \langle f, g \rangle$, differentiable on a region R.

A **stream function** for the vector field is a function ψ that satisfies

$$\frac{\partial \psi}{\partial y} = f, \frac{\partial \psi}{\partial x} = -g$$

It plays the same role for source-free fields that the potential function plays for conservative fields.

Compute the divergence of a vector field $\mathbf{F} = \langle f, g \rangle$ that has stream function and use the fact that $\psi_{xy} = \psi_{yx}$, then

$$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = \underbrace{\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right)}_{\psi_{yx} = \psi_{xy}} = 0.$$

- The existence of a stream function guarantees that the vector field has zero divergence or, equivalently, is source free.
- Flux integrals of a source-free field are also independent of path.
- Vector fields that are both conservative (zero curl, $g_x - f_y = 0$) and source free (zero divergence, $f_x + g_y = 0$) are quite interesting.
- The potential function and the stream function both satisfy an important equation known as **Laplace's equation**:

$$\varphi_{xx} + \varphi_{yy} = 0 \text{ and } \psi_{xx} + \psi_{yy} = 0$$

- Any function satisfying Laplace's equation can be used as a potential function or stream function for a conservative, source-free vector field.

C is a simple piecewise-smooth oriented curve and is either closed or has endpoints A and B.

Table 1

Conservative Fields $\mathbf{F} = \langle f, g \rangle$	Source-Free Fields $\mathbf{F} = \langle f, g \rangle$
• curl $= \dfrac{\partial g}{\partial x} - \dfrac{\partial f}{\partial y} = 0$	• divergence $= \dfrac{\partial f}{\partial x} + \dfrac{\partial g}{\partial y} = 0$
• Potential function φ with $\mathbf{F} = \nabla\varphi$ or $f = \dfrac{\partial \varphi}{\partial x}, \quad g = \dfrac{\partial \varphi}{\partial y}$	• Stream function ψ with $f = \dfrac{\partial \psi}{\partial y}, \quad g = -\dfrac{\partial \psi}{\partial x}$
• Circulation $= \oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ on all closed curves C.	• Flux $= \oint_C \mathbf{F} \cdot \mathbf{n}\, ds = 0$ on all closed curves C.
• Path independence $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r} = \varphi(B) - \varphi(A)$	• Path independence $\displaystyle\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \psi(B) - \psi(A)$

Various cases that arise with line integrals of both the circulation and flux types.

Table 2

Circulation/work integrals: $\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C f\, dx + g\, dy$

	***C* closed**	***C* not closed**
F conservative $(\mathbf{F} = \nabla\varphi)$	$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$	$\int_C \mathbf{F} \cdot d\mathbf{r} = \varphi(B) - \varphi(A)$
F not conservative	Green's Theorem $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R (g_x - f_y)\, dA$	Direct evaluation $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b (fx' + gy')\, dt$

Flux integrals: $\int_C \mathbf{F} \cdot \mathbf{n} \, ds = \int_C f \, dy - g \, dx$

	***C* closed**	***C* not closed**
F source free $(f = \psi_y, g = -\psi_x)$	$\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = 0$	$\int_C \mathbf{F} \cdot \mathbf{n} \, ds = \psi(B) - \psi(A)$
F not source free	Green's Theorem $\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \iint_R (f_x + g_y) \, dA$	Direct evaluation $\int_C \mathbf{F} \cdot \mathbf{n} \, ds = \int_a^b (fy' - gx') \, dt$

Chapter 17

Vector Calculus (I)

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