MATH 2233 Differential Equations
Chapter 7 Laplace Transform

Section 7.2 Definition of Laplace Transform

Goal of this section

- Caculate the Laplace transform of a function using the definition.

Motivation: Many practical engineering problems involve mechanical or electrical systems acted on by *discontinuous or impulsive forcing terms*. Methods introduced in previous chapters are often rather awkward to use. In this chapter, we consider a new approach based on Laplace transform.

Definition: Laplace transform

Let $f(t)$ be a function on $(0, \infty)$. The **Laplace transform** of a function $f(t)$ is a function $F(s)$ defined by the integral

Remark The integral above is an **improper** integral defined by

Example 1. *Determine the Laplace transform of the constant function* $f(t) = 1$.

Example 2. *Find the Laplace transform of* $f(t) = e^{at}$*, where* a *is a constant.*

Example 3. *Find* $\mathcal{L}\{\sin(bt)\}$ *where* b *is a constant.*

Example 4. *Find the Laplace transform of*

$$f(t) = \begin{cases} 2, & 0 \le t < 5, \\ 0, & 5 < t < 10, \\ e^{4t}, & t > 10, \end{cases}$$

A Brief Table of Laplace Transforms.

$f(t)$	$F(s) = \mathcal{L}\{f\}(s)$
1	
t^n	
e^{at}	
$e^{at}t^n$	
$\sin(bt)$	
$\cos(bt)$	
$e^{at}\sin(bt)$	
$e^{at}\cos(bt)$	

Linearity of Laplace Transforms.

Let f, f_1, f_2 be functions whose Laplace transform exist. Let c be a constant. Then

- $\mathcal{L}\{f_1 + f_2\} =$
- $\mathcal{L}\{cf\} =$

Example 5. *Determine* $\mathcal{L}\{11 + 5e^{4t} - 6\sin(2t)\}$ *and* $\mathcal{L}\{5t^2e^{-3t} - e^{12t}\cos(8t)\}$.

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Section 7.3 Properties of Laplace Transform

Goal of this section

- study three basic properties of Laplace Transform

Recall The Laplace Transform of a function $f(t)$ is

Property (I): Translation in s

If $\mathcal{L}\{f(t)\} = F(s)$, then $\mathcal{L}\{e^{at}f(t)\} =$

A one-line proof

Example 1. *Determine the Laplace transform of* $e^{at}\sin(bt)$.

Property (II): Laplace Transform of Derivatives

Let $f(t)$ be continuous on $[0, \infty)$. If $\mathcal{L}\{f(t)\} = F(s)$, then

$$\mathcal{L}\{f'(t)\} =$$

$$\mathcal{L}\{f''(t)\} =$$

In general, we have for higher-order derivatives

$$\mathcal{L}\{f^{(n)}(t)\} =$$

Example 2. *Using the Property (II) and the fact that*

$$\mathcal{L}\{\sin(bt)\}(s) = \frac{b}{s^2 + b^2}$$

determine $\mathcal{L}\{\cos(bt)\}$

Property (III): Derivatives of Laplace Transform

Let $f(t)$ be piecewise continuous on $[0, \infty)$. If $\mathcal{L}\{f(t)\} = F(s)$, then

$$\mathcal{L}\{t^n f(t)\} =$$

Example 3. *Determine* $\mathcal{L}\{t\sin(bt)\}$

A Summary of Properties of Laplace Transform

- $\mathcal{L}\{f + g\} =$

- $\mathcal{L}\{cf\} =$

- $\mathcal{L}\{e^{at}f(t)\} =$

- $\mathcal{L}\{f'\}(s) =$

- $\mathcal{L}\{f''\}(s) =$

- $\mathcal{L}\{f^{(n)}\}(s) =$

- $\mathcal{L}\{t^n f(t)\}(s) =$

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Section 7.4 Inverse Laplace Transform

Goal of this section

- Given a function $F(s)$, compute its inverse Laplace Transform $f(t)$.

Inverse Laplace Transform

If $F(s) = \mathcal{L}\{f(t)\}$, then we say $f(t)$ is

Example 1. *Compute $\mathcal{L}^{-1}\{F\}$ where*

$$\textbf{(a).}\ F(s) = \frac{2}{s^3}, \qquad \textbf{(b).}\ F(s) = \frac{3}{s^2+9}, \qquad \textbf{(c).}\ F(s) = \frac{s-1}{s^2-2s+5}.$$

Linearity of Inverse Transform

- $\mathcal{L}^{-1}\{F_1 + F_2\} =$

- $\mathcal{L}^{-1}\{cF\} =$

Example 2. *Compute* $\mathcal{L}^{-1}\left\{\frac{5}{s-6}-\frac{6s}{s^2+9}+\frac{3}{2s^2+8s+10}\right\}$

Example 3. *Compute* $\mathcal{L}^{-1}\left\{\frac{5}{(s+2)^4}\right\}$

Example 4. *Compute* $\mathcal{L}^{-1}\left\{\dfrac{3s+2}{s^2+2s+10}\right\}$

Method of Partial Fraction

In general, how to compute the inverse transform of $\mathcal{L}^{-1}\left\{\dfrac{P(s)}{Q(s)}\right\}$? Here, $P(s)$ and $Q(s)$ are polynomials with the degree of P less than the degree of Q.

1. Non-repeated Linear Factors

If $Q(s) = (s - r_1)(s - r_2)\cdots(s - r_n)$, the partial fraction expansion has the form

$$\frac{P(s)}{Q(s)} = \frac{A_1}{s - r_1} + \frac{A_2}{s - r_2} + \cdots + \frac{A_n}{s - r_n}.$$

Example 5. *Determine* $\mathcal{L}^{-1}\left\{\dfrac{7s - 1}{(s + 1)(s + 2)(s - 3)}\right\}$

2. Repeated Linear Factors

Let $(s-r)$ be a factor of $Q(s)$ and suppose $(s-r)^m$ is the highest power of $s-r$ that divides $Q(s)$. Then the portion corresponding to the term $(s-r)^m$ is

$$\frac{A_1}{s-r} + \frac{A_2}{(s-r)^2} + \cdots + \frac{A_m}{(s-r)^m}.$$

Example 6. *Determine* $\mathcal{L}^{-1}\left\{\dfrac{s^2+9s+2}{(s-1)^2(s+3)}\right\}$

3. Quadratic Factors

Let $(s-\alpha)^2+\beta^2$ be a quadratic factor of $Q(s)$ that cannot be reduced to linear factors with real coefficients. Suppose m is the is the highest power of $(s-\alpha)^2+\beta^2$ that divides $Q(s)$. Then the portion corresponding to the term $[(s-\alpha)^2+\beta^2]^m$ is

$$\frac{C_1 s + D_1}{(s-\alpha)^2+\beta^2} + \frac{C_2 s + D_2}{[(s-\alpha)^2+\beta^2]^2} + \cdots + \frac{C_m s + D_m}{[(s-\alpha)^2+\beta^2]^m}.$$

Example 7. *Determine* $\mathcal{L}^{-1}\left\{\dfrac{2s^2+10s}{(s^2-2s+5)(s+1)}\right\}$

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Section 7.5 Solve Initial Value Problems

Goal of this section

- use Laplace transform to solve initial value problems of linear differential equations.

Method of Laplace transform

Step 1 Take Laplace transforms on both sides of the DE.

Step 2 Solve the algebraic equation of $Y(s)$.

Step 3 Take the Inverse transform to resolve $y(t)$.

Example 1. *Use Laplace Transform to solve the initial value problem*

$$y'' - 2y' + 5y = 0, \quad y(0) = 2, \quad y'(0) = 4.$$

Example 2 (nonhomogeneous equation). *Solve the initial value problem*

$$y''(t) - 5y'(t) + 4y(t) = e^t, \qquad y(0) = 1, \quad y'(0) = 0.$$

The method of Laplace transform can solve IVPs in which the initial condition is not at $t = 0$.

Example 3 (Initial condition not at zero). *Use the Laplace transform to solve the initial value problem*

$$w''(t) - 2w'(t) + 5w(t) = -8e^{\pi - t}, \quad w(\pi) = 2, \quad w'(\pi) = 12.$$

Example 4 (high order equation). *Use the Laplace transform to solve the initial value problem*

$$y''' - y'' + y' - y = 0, \quad y(0) = 1, \quad y'(0) = 1, \quad y''(0) = 3.$$

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Section 7.6 Transforms of Discontinuous Functions

Goal of this section

- Use Laplace transform to solve IVPs with discontinuous nonhomogeneous function.

Unit Step Function

The **unit step function** $u(t)$ is defined by

By shifting $u(t)$, the jump can be moved to a different location.

$$u(t-a) =$$

Example 1. *Sketch the graph of* $y = h(t)$*, where*

$$h(t) = u(t-\pi) - u(t-2\pi)$$

Example 2. *Write the piecewisely continuous function*

$$f(t) = \begin{cases} 3, & t < 2, \\ 1, & 2 < t < 5 \\ t, & 5 < t < 8 \\ t^2/10, & t > 8 \end{cases}$$

in terms of the unit step functions.

Laplace Transform of Step functions

$$\mathcal{L}(u(t-a)) =$$

Assume $y = f(t)$ is defined for $t \geq 0$. Then the function

$$g(t) = f(t-a)u(t-a)$$

Translation in t

Let $\mathcal{L}\{f(t)\} = F(s)$. Then

$$\mathcal{L}\{f(t-a)u(t-a)\} =$$

The inverse form: If $f(t) = \mathcal{L}^{-1}\{F(s)\}$, then

$$\mathcal{L}^{-1}\left\{e^{-as}F(s)\right\} =$$

Example 3. *Determine* $\mathcal{L}\{t\,u(t-1)\}$

Example 4. *Determine* $\mathcal{L}\{t^2u(t-1)\}$

Example 5. *Determine* $\mathcal{L}\{\cos(t)u(t-\pi)\}$

Example 6. *Determine* $\mathcal{L}^{-1}\left\{\dfrac{e^{-2s}}{s^2}\right\}$

Example 7. *Determine* $\mathcal{L}^{-1}\left\{\dfrac{se^{-3s}}{s^2+4s+5}\right\}$

Example 8. *Solve the initial value problem*

$$y' + y = f(t), \quad y(0) = 5, \ \textit{where } f(t) = \begin{cases} 0 & 0 \le t < \pi, \\ 3\cos(t) & t \ge \pi. \end{cases}$$

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Section 7.10 Solve Linear Systems with Laplace Transforms

Goal: use Laplace transform to solve a system of differential equations with initial conditions.

Example 1. *Solve the initial value problem*

$$\begin{aligned} x'(t) - 2y(t) &= 4t, & x(0) &= 4, \\ y'(t) + 2y(t) - 4x(t) &= -4t - 2, & y(0) &= -5. \end{aligned}$$