第三节 矩阵的分块

1. 设 $A = \begin{pmatrix} -1 & 2 & 0 & 0 & 0 \\ 4 & 1 & 0 & 1 & 0 \\ 0 & 5 & 0 & 0 & 1 \\ 3 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \end{pmatrix}$,

$B = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 3 \\ 2 & 1 & -3 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & 4 & 0 \end{pmatrix}$，利用分块矩阵计算 AB.

【解题过程】

令 $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$,

其中 $A_{11} = \begin{pmatrix} -1 & 2 \\ 4 & 1 \\ 0 & 5 \end{pmatrix}$, $A_{12} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$,

$A_{21} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$, $A_{22} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ ；

$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$,

其中 $B_{11} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $B_{12} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$,

$B_{21} = \begin{pmatrix} 2 & 1 & -3 \\ 1 & -2 & 1 \\ 0 & 1 & 4 \end{pmatrix}$, $B_{22} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$；则

$$AB=\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

$$=\begin{pmatrix} A_{11}B_{11}+A_{12}B_{21} & A_{11}B_{12}+A_{12}B_{22} \\ A_{21}B_{11}+A_{22}B_{21} & A_{21}B_{12}+A_{22}B_{22} \end{pmatrix}.$$

$\because\ A_{11}B_{11}+A_{12}B_{21}$

$$=\begin{pmatrix} -1 & 2 \\ 4 & 1 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}+\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 & -3 \\ 1 & -2 & 1 \\ 0 & 1 & 4 \end{pmatrix}$$

$$=\begin{pmatrix} 0 & 0 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & 4 \end{pmatrix}$$

$A_{11}B_{12}+A_{12}B_{22}$

$$=\begin{pmatrix} -1 & 2 \\ 4 & 1 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix}+\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$=\begin{pmatrix} 4 \\ 11 \\ 15 \end{pmatrix}$$

$A_{21}B_{11}+A_{22}B_{21}$

$$=\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}+\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 & -3 \\ 1 & -2 & 1 \\ 0 & 1 & 4 \end{pmatrix}$$

$$=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$A_{21}B_{12}+A_{22}B_{22}$

$$=\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix}+\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 6 \\ 9 \end{pmatrix}$$

$$\therefore\ AB = \begin{pmatrix} 0 & 0 & 0 & 4 \\ 1 & -2 & 1 & 11 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 9 \end{pmatrix}.$$

2.设 $\boldsymbol{A} = \begin{pmatrix} 3 & 4 & 1 & 0 \\ 4 & -3 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 2 \end{pmatrix}$，计算 $\boldsymbol{A}^4$.

【解题过程】 令 $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$，其中

$A_{11} = \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}, A_{12} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, A_{21} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$,

$A_{22} = \begin{pmatrix} 2 & 0 \\ 2 & 2 \end{pmatrix}$,

$$A^4 = A^2A^2 = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}^2 \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}^2$$

$$= \begin{pmatrix} A_{11}^2 + A_{12}A_{21} & A_{11}A_{12} + A_{12}A_{22} \\ A_{21}A_{11} + A_{22}A_{21} & A_{21}A_{12} + A_{22}^2 \end{pmatrix} \begin{pmatrix} A_{11}^2 + A_{12}A_{21} & A_{11}A_{12} + A_{12}A_{22} \\ A_{21}A_{11} + A_{22}A_{21} & A_{21}A_{12} + A_{22}^2 \end{pmatrix}$$

$$= \begin{pmatrix} 25 & 0 & 5 & 4 \\ 0 & 25 & 6 & -1 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 8 & 4 \end{pmatrix} \begin{pmatrix} 25 & 0 & 5 & 4 \\ 0 & 25 & 6 & -1 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 8 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 625 & 0 & 177 & 116 \\ 0 & 625 & 166 & -29 \\ 0 & 0 & 16 & 0 \\ 0 & 0 & 64 & 16 \end{pmatrix}.$$

3.设 A 为 $\boldsymbol{n}$ 阶方阵，若对任意的 n 维列向量 α 均有 $A\alpha = O$,证明 $A = O$.

【解题过程】

设 $A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$,若对任意的

n 维列向量 α 均有 $A\alpha=O$,则取

$\alpha=\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}$，得 $a_{11}=a_{21}=\cdots=a_{n1}=0$；

取 $\alpha=\begin{pmatrix}0\\1\\0\\\vdots\\0\end{pmatrix}$，得 $a_{12}=a_{22}=\cdots=a_{n2}=0$;

$\cdots$;取 $\alpha=\begin{pmatrix}0\\0\\\vdots\\0\\1\end{pmatrix}$,

得 $a_{1n}=a_{2n}=\cdots=a_{nn}=0$.

由此可知，$A=O$.