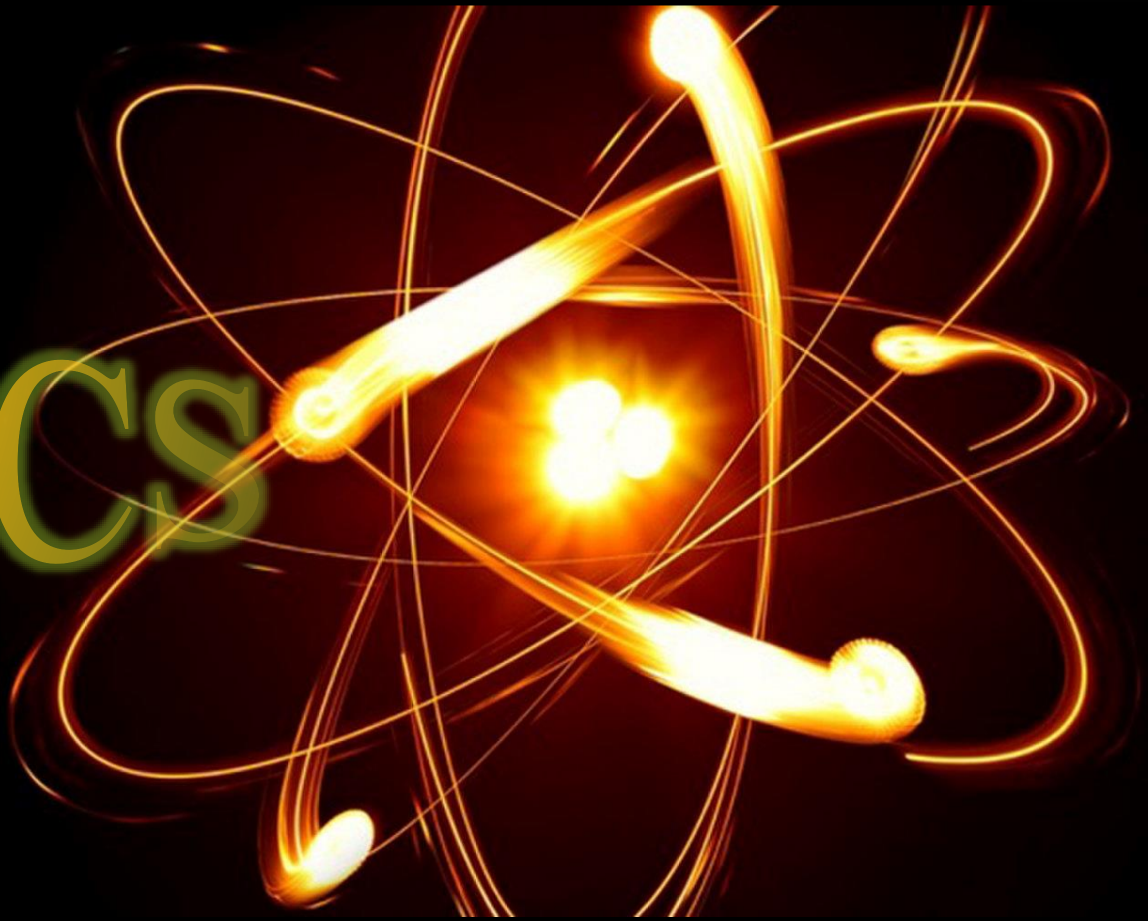
PHYSICS



Physics 1: **Mechanics and Waves**

Week 12 – Simple harmonic motion

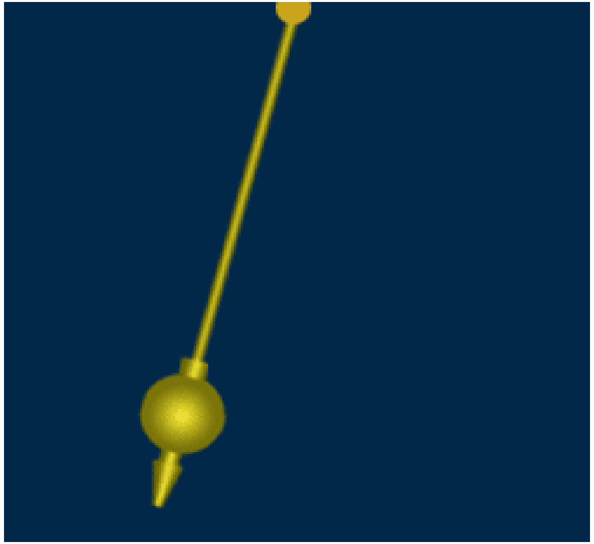
2023.5

QQ group: 776916994
cyjing@swjtu.edu.cn

Oscillation

- repeat itself
- back and forth
- over the same path
- the motion is periodic

Oscillation



Pendulum

Mass on a spring

Why learn?

Many objects vibrate or oscillate—

- an object on the end of a spring
- a tuning fork
- the balance wheel of an old watch
- a pendulum
- a plastic ruler held firmly over the edge of a table and gently struck
- the strings of a guitar or piano

Why learn?

- Spiders detect prey by the vibrations of their webs;
- cars oscillate up and down when they hit a bump;
- buildings and bridges vibrate when heavy trucks pass or the wind is fierce.
- Indeed, because most solids are elastic, they vibrate (at least briefly) when given an impulse.

Why learn?

- Electrical oscillations occur in radio and television sets.
- At the atomic level, atoms oscillate within a molecule
- the atoms of a solid oscillate about their relatively fixed positions.

Why learn?

Because it is so common in everyday life and occurs in so many areas of physics, oscillatory (or vibrational) motion is of great importance. Mechanical oscillations or vibrations are fully described on the basis of Newtonian mechanics.

Why learn?

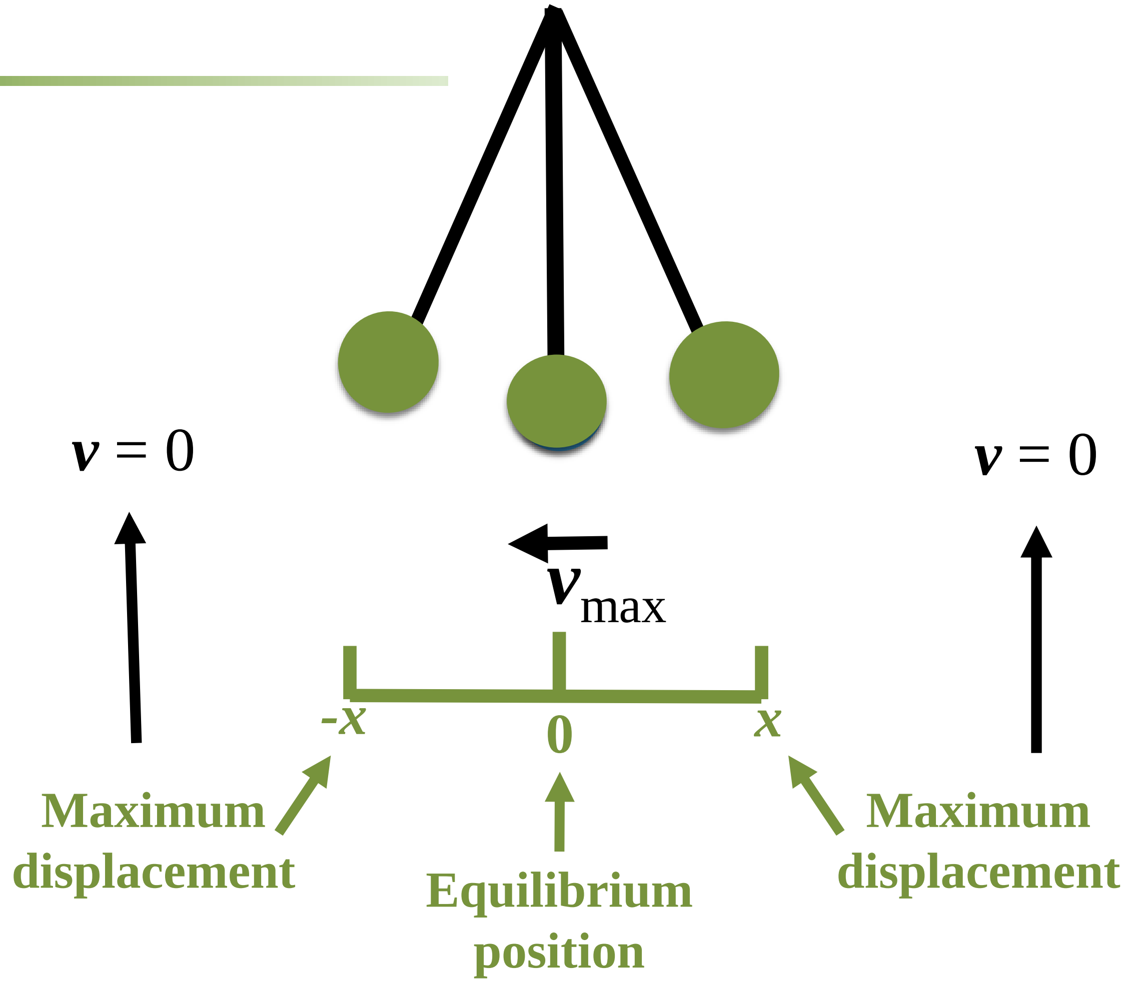
Vibrations and wave motion are intimately related.

- Waves—whether ocean waves, waves on a string, earthquake waves, or sound waves in air—have as their source a vibration.
- In the case of sound, not only is the source a vibrating object, but so is the detector—the eardrum or the membrane of a microphone.
- Indeed, when a wave travels through a medium, the medium oscillates (such as air for sound waves).

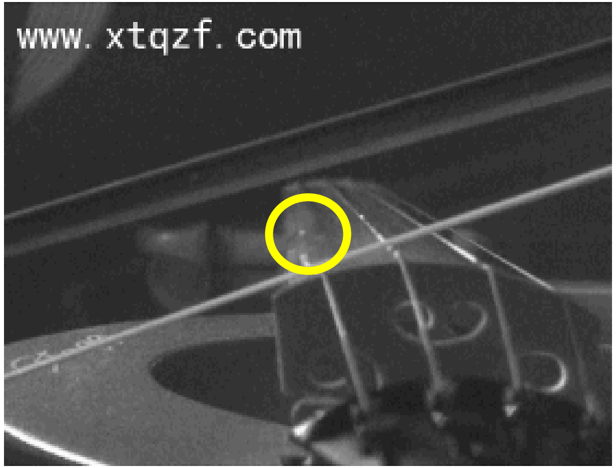
What after?

- After we discuss oscillations, we will discuss simple waves such as those on water or on a string.
- We will study sound waves, and in later Chapters we will encounter other forms of wave motion, including electromagnetic waves and light.

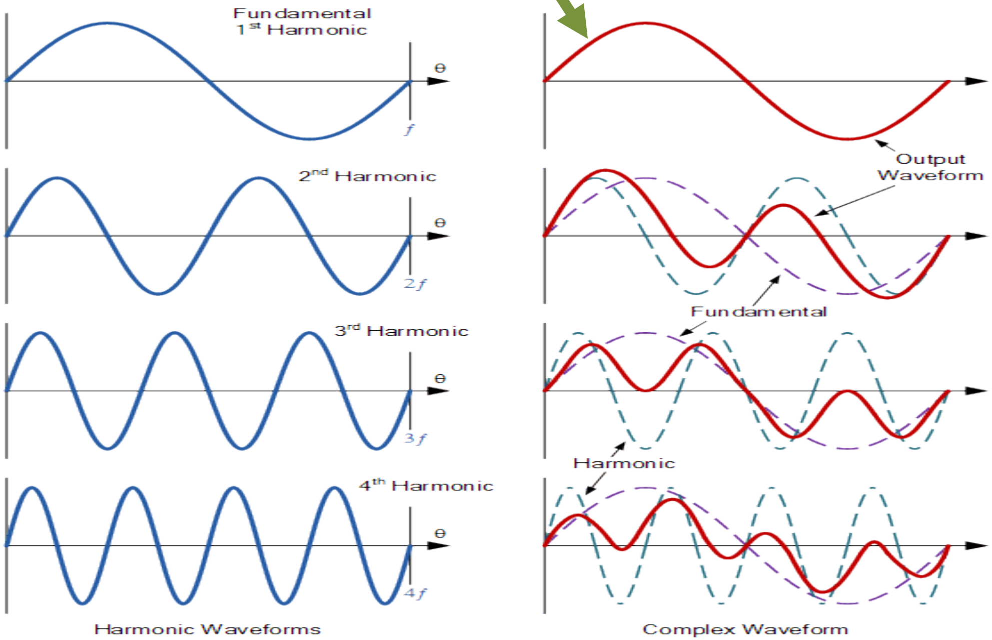
Oscillation



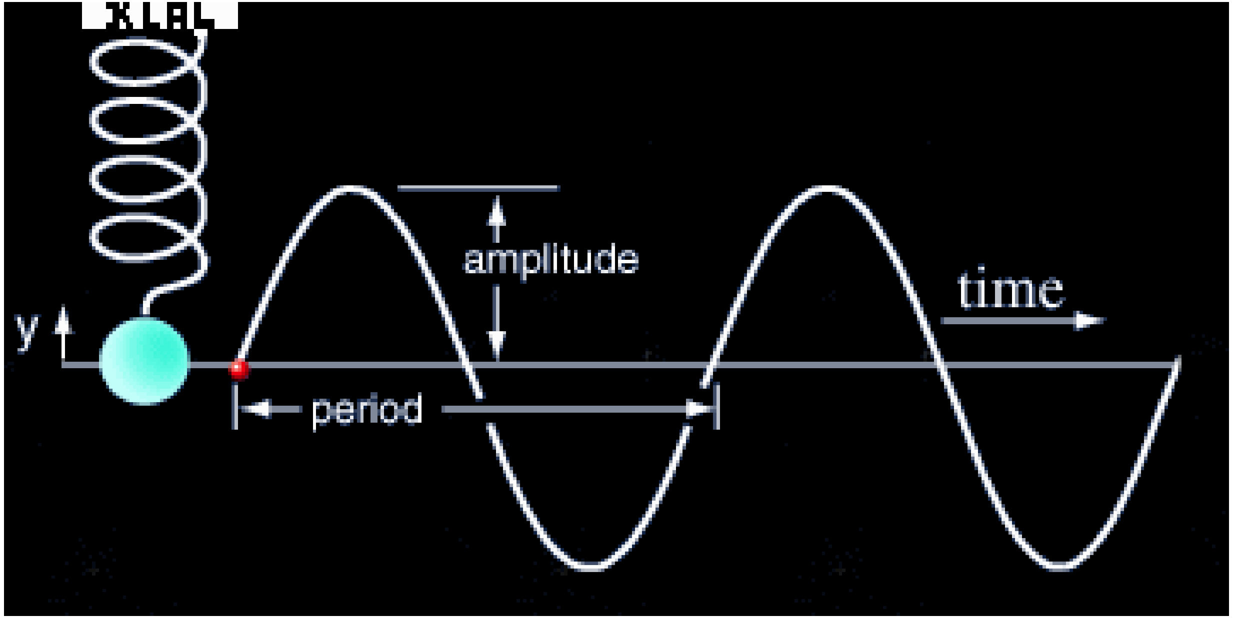
“Harmonic”



“Simple”



“Simple” Harmonic motion

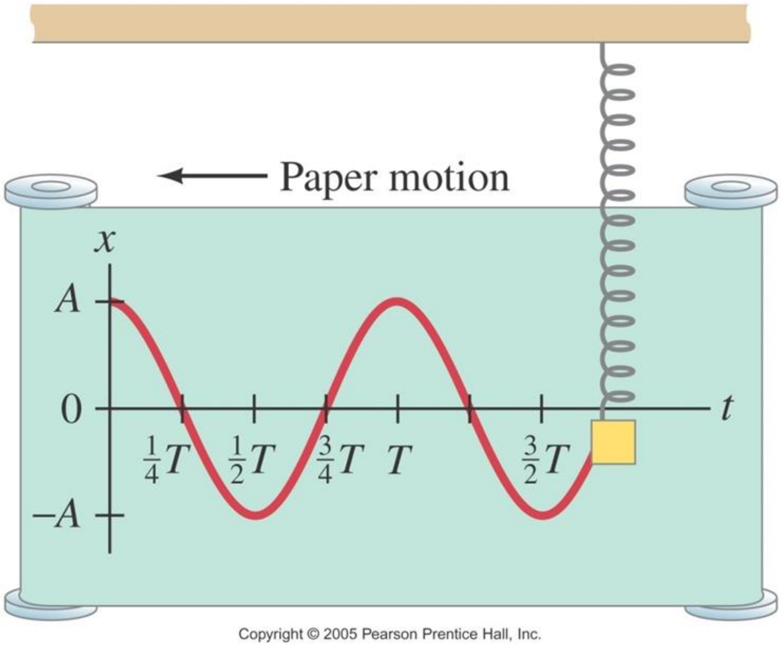


Position as a function of **time** is a sinusoidal function

“Simple” : only <u>**one**</u> sine/cosine function

"Simple" Harmonic motion

Vertical position *x* as a function of time *t*



$$x = A\cos\left(\frac{2\pi}{T}t + \varphi\right)$$

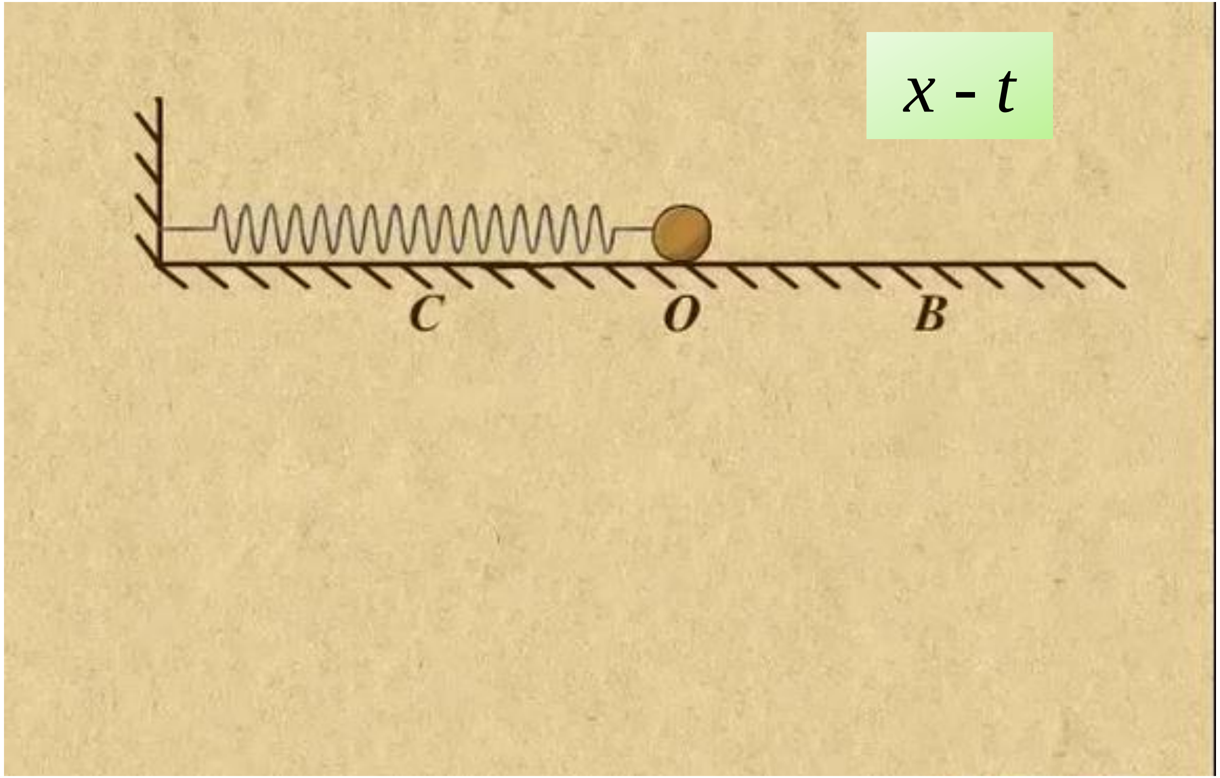
A is the amplitude

T is the period

φ is the initial phase angle

depends on the "initial conditions"

$x-t$ in SHM



Other ways of expressing *x(t)*

$$x = A\cos(2\pi f t + \varphi)$$

$$\boxed{x = A\cos(\omega t + \varphi)}$$

f is the frequency, cycles in a second $\qquad f = \frac{1}{T}$

ω is the **angular frequency** $\qquad \omega = 2\pi f$

A is the amplitude – maximum displacement $\qquad -A \le x \le A$

Simple Harmonic Motion

$$x = A\cos\left(\frac{2\pi}{T}t + \varphi\right)$$

$$x = A\cos(2\pi f t + \varphi)$$

$$x = A\cos(\omega t + \varphi)$$

T is the period

$$f = \frac{1}{T}$$

f is the frequency

ω is the **angular frequency**

$$\omega = 2\pi f$$

$\boldsymbol{A}$ is the amplitude – maximum displacement

$$-A \le x \le A$$

φ is the initial phase angle

$$\boldsymbol{A} = \sqrt{x_0^2 + \frac{v_0^2}{\omega^2}} = \sqrt{x^2 + \frac{v^2}{\omega^2}}$$

Initial phase angle

φ describe the *initial state* of the spring oscillator. It is called initial phase or phase constant.

A and φ is related to the initial states or conditions of the system.

$\omega t+\varphi$ is called the *phase of the motion*. It describes the states of the oscillation system.

Initial conditions

General solution: $x(t) = Acos(\omega t + \varphi)$

What is the position at time $t = 0$?

$$x(0) = Acos\varphi$$

If $x(0) = 0 \quad \Rightarrow \varphi = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \cdots \frac{(2n+1)\pi}{2}$

$x(0) = A \quad \Rightarrow \varphi = 0, 2\pi, 4\pi, \cdots 2n\pi$

$x(0) = -A \quad \Rightarrow \varphi = -\pi, \pi, 3\pi, 5\pi \cdots (2n-1)\pi$

What is the velocity?

$$v(t) = \frac{dx}{dt} = A\frac{d}{dt}\cos(\omega t + \varphi) \Rightarrow \boxed{v(t) = -A\omega\sin(\omega t + \varphi)}$$

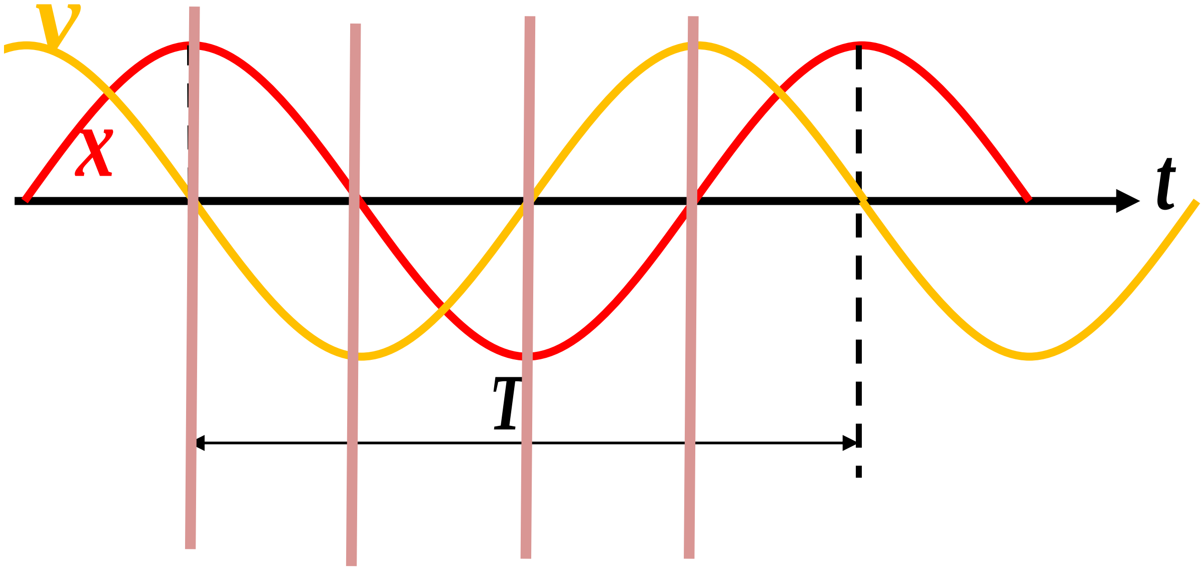
If $\boxed{v(0) = -A\omega\sin\varphi}$

$x(0) = A \quad \Rightarrow \varphi = 0, 2\pi \cdots 2n\pi \Rightarrow \boxed{v(0) = 0}$

$x(0) = -A \quad \Rightarrow \varphi = (2n-1)\pi \Rightarrow \boxed{v(0) = 0}$

$$x(0) = 0 \begin{cases} \varphi = \frac{\pi}{2}, \frac{5\pi}{2}, \cdots (2n+\frac{1}{2})\pi & \Rightarrow \boxed{v(0) = -A\omega} \\ \varphi = \frac{3\pi}{2}, \frac{7\pi}{2}, \cdots (2n+\frac{3}{2})\pi & \Rightarrow \boxed{v(0) = A\omega} \end{cases}$$

Velocity



$x = A, v = 0$

$x = 0, v = -A\omega$

$x = 0, v = A\omega$

$x = -A, v = 0$

Acceleration

$$\boxed{x(t) = Acos(\omega t + \varphi)}$$

$$\boxed{v(t) = -A\omega\sin(\omega t + \varphi)}$$

$$a(t) = \frac{dv}{dt} = -A\omega\frac{d}{dt}\sin(\omega t + \varphi)$$

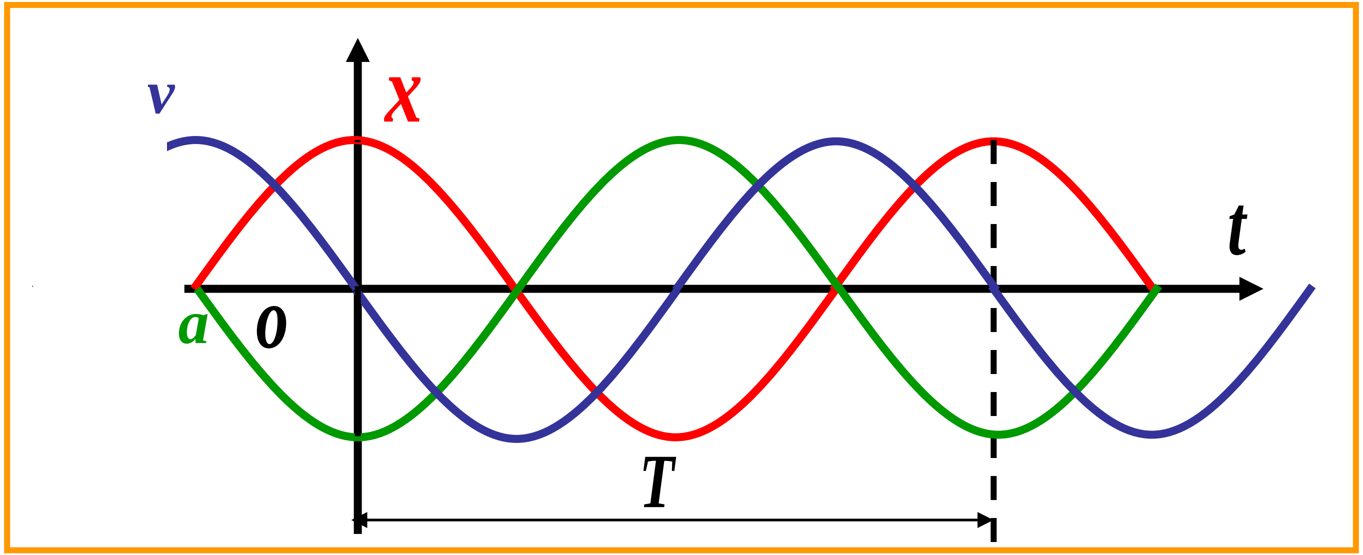
$$\boxed{a(t) = -A\omega^2 cos(\omega t + \varphi)}$$

Acceleration

$$x = A\cos(\omega t + \varphi_0)$$

$$v = -A\omega\sin(\omega t + \varphi_0)$$

$$a = -A\omega^2 \cos(\omega t + \varphi_0)$$



Why is a motion "*simple harmonic*" ?

$$x = A\cos(\omega t + \varphi_0)$$

$$a = -A\omega^2 cos(\omega t + \varphi_0)$$

$$a = -\omega^2 x$$

Newton's second law $F = ma$

$F = -(\omega^2 m)x$ or $F = -kx$

The net force on an object is **proportional** to its displacement, and oppose to the direction of the displacement

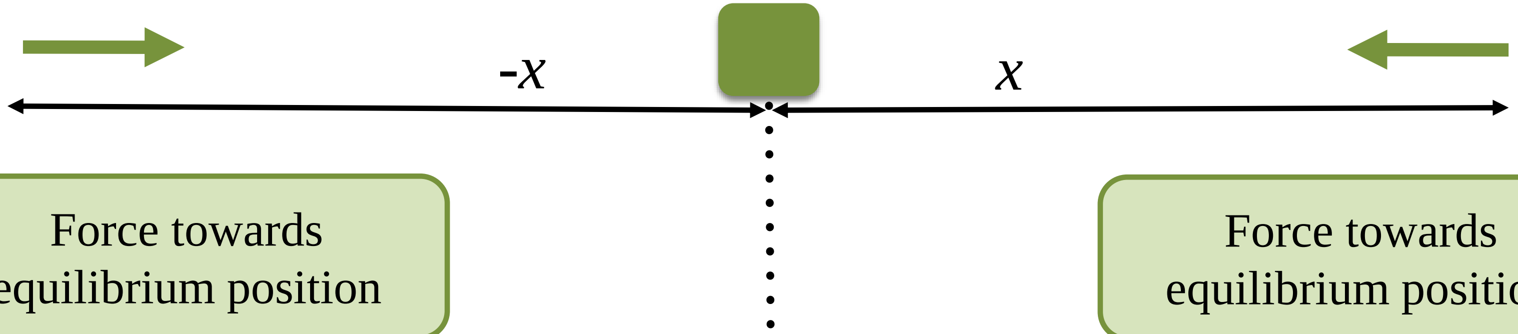
- Restoring Force

RESTORING FORCE

$$\boldsymbol{F = - k\,x}$$

$F = k\,x$

$F = - k\,x$



Force towards equilibrium position

Force towards equilibrium position

equilibrium position

No force!

Net restoring force is proportional to the negative displacement

Hooke's law

The extension of a spring is proportional to the force

$\boldsymbol{F = - k\, x}$ **Simple harmonic motion**

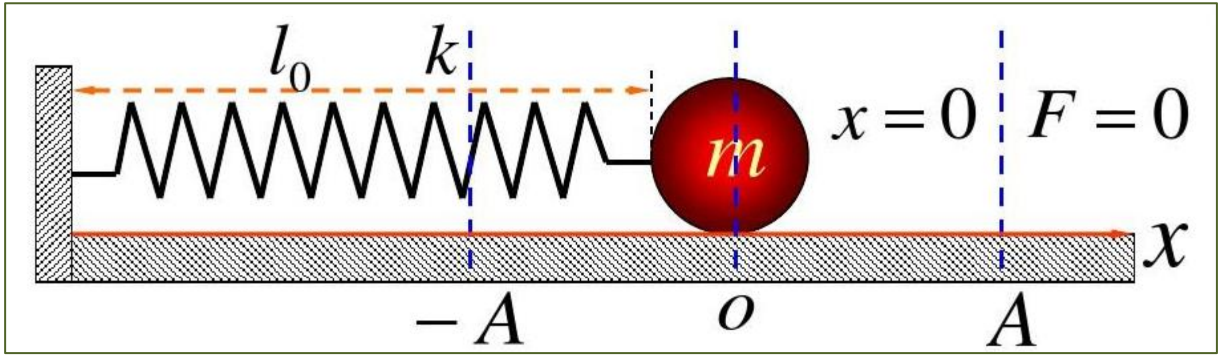
k is called the *spring stiffness constant*

$$\boldsymbol{F = -(\omega^2 m)x} \Rightarrow \boldsymbol{k = \omega^2 m} \Rightarrow \boldsymbol{\omega = ?}$$

The motion of the mass can be expressed as $\boldsymbol{x(t) = A\cos(\sqrt{\frac{k}{m}}t + \varphi)}$

Equilibrium position

Horizontal



Equilibrium position

Vertical

Period of a spring harmonic oscillator

Restoring force: $F = -kx$

Harmonic motion: $x = A\cos\omega t$ (suppose φ is 0)

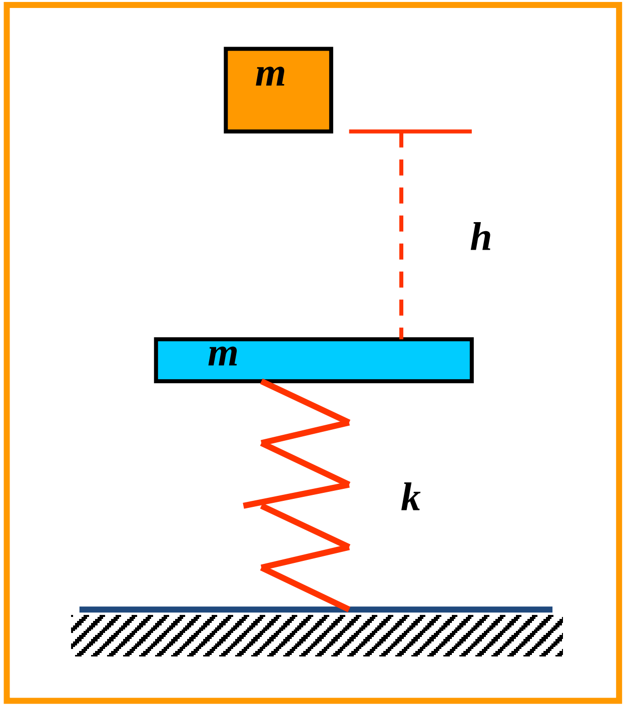
Newton's 2nd law: $M\frac{d}{dt}\left(\frac{d}{dt}x(t)\right) = -MA\omega^2\cos\omega t = -kA\cos\omega t$

$$\omega^2 = \frac{k}{M} \implies \omega = \sqrt{\frac{k}{M}}$$

$\omega = 2\pi f = \frac{2\pi}{T}$ $f = \frac{1}{2\pi}\sqrt{\frac{k}{M}}$ $T = 2\pi\sqrt{\frac{M}{k}}$

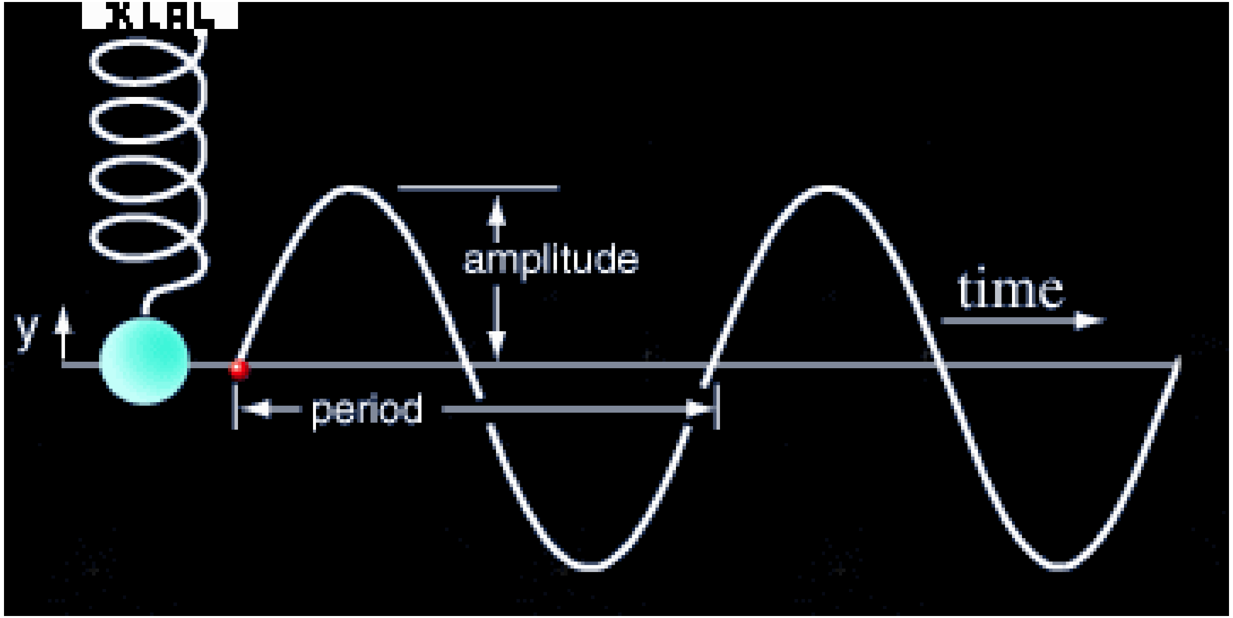
Example 1

As shown in Fig. 1, when the block of mass *m* falls freely and make a completely unelastic collision with the plate of mass *m*, the system will oscillate up and down. Find the T, A and φ of the motion.



Expression of SHM

$$x = A\cos(\omega t + \varphi)$$

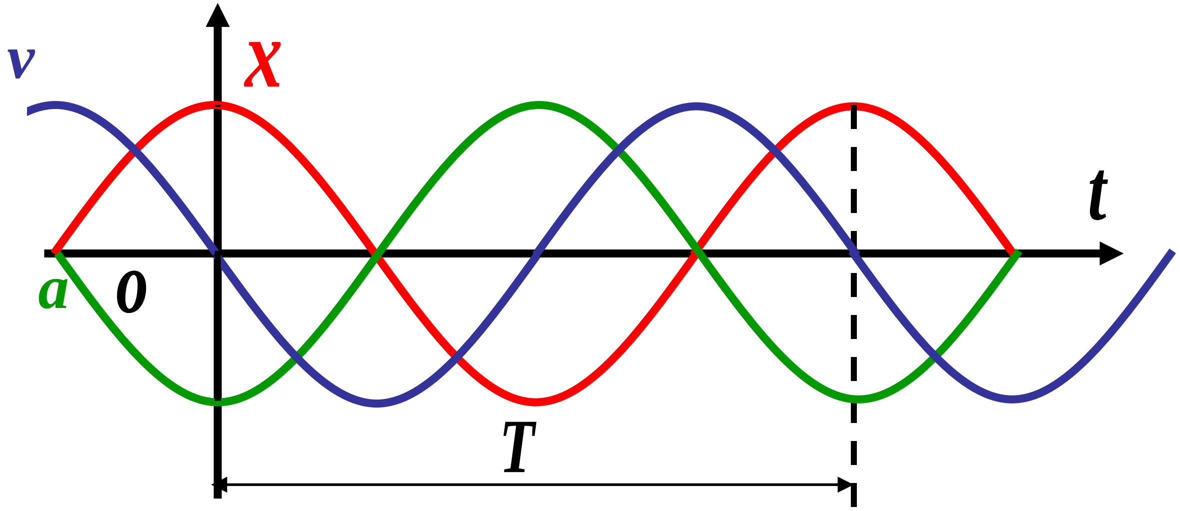


Expression of SHM

$$x = A\cos(\omega t + \varphi_0)$$

$$v = -A\omega\sin(\omega t + \varphi_0)$$

$$a = -A\omega^2 cos(\omega t + \varphi_0)$$



Expression of SHM

1. Analytical method

   Given expression $\Rightarrow$ A、T、φ

   Given A、T、φ $\Rightarrow$ expression

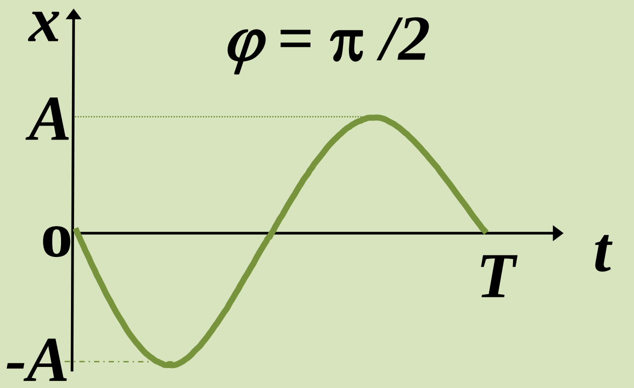
2. Curve method

   Given curve $\Rightarrow$ *A*、*T*、*φ*

   Given *A*、*T*、*φ* $\Rightarrow$ curve

3. Phasor – Rotating vector method

$$x = A\cos(\omega t + \varphi)$$



Expression of SHM

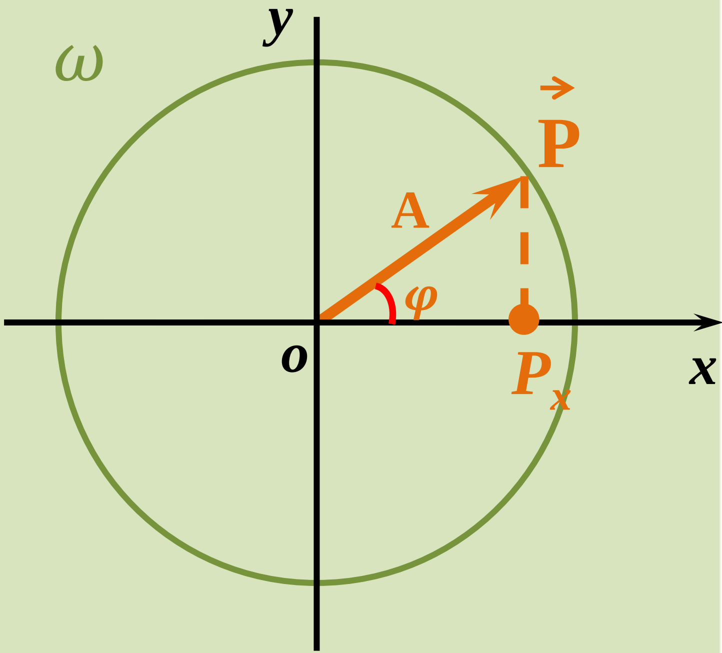
3. Phasor – Rotational vector method

$$P_x = A\cos\varphi$$

$$\overset{\omega}{\Longrightarrow}\begin{cases} P_x \rightarrow x(t) \\ \varphi \rightarrow \omega t + \varphi_0 \end{cases}$$

$$P_x = A\cos(\omega t + \varphi_0)$$

$$x = A\cos(\omega t + \varphi)$$



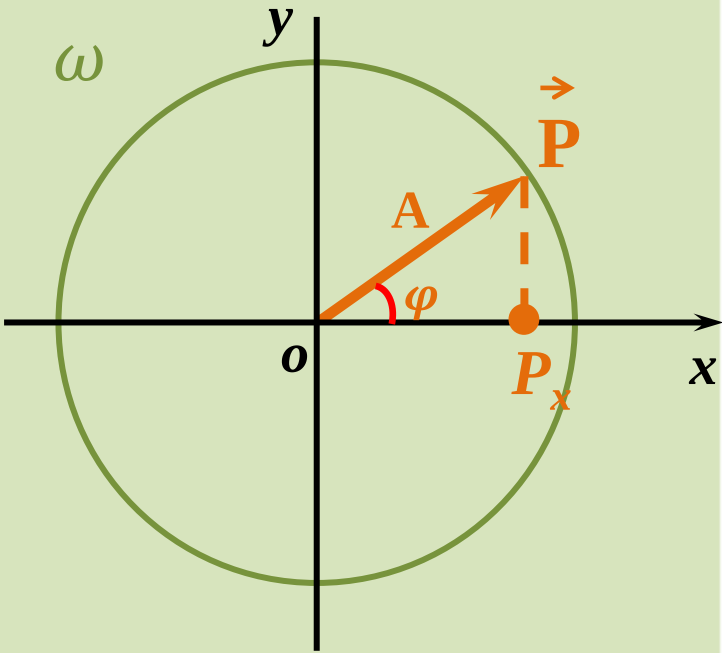
Expression of SHM

3. Phasor – Rotational vector method

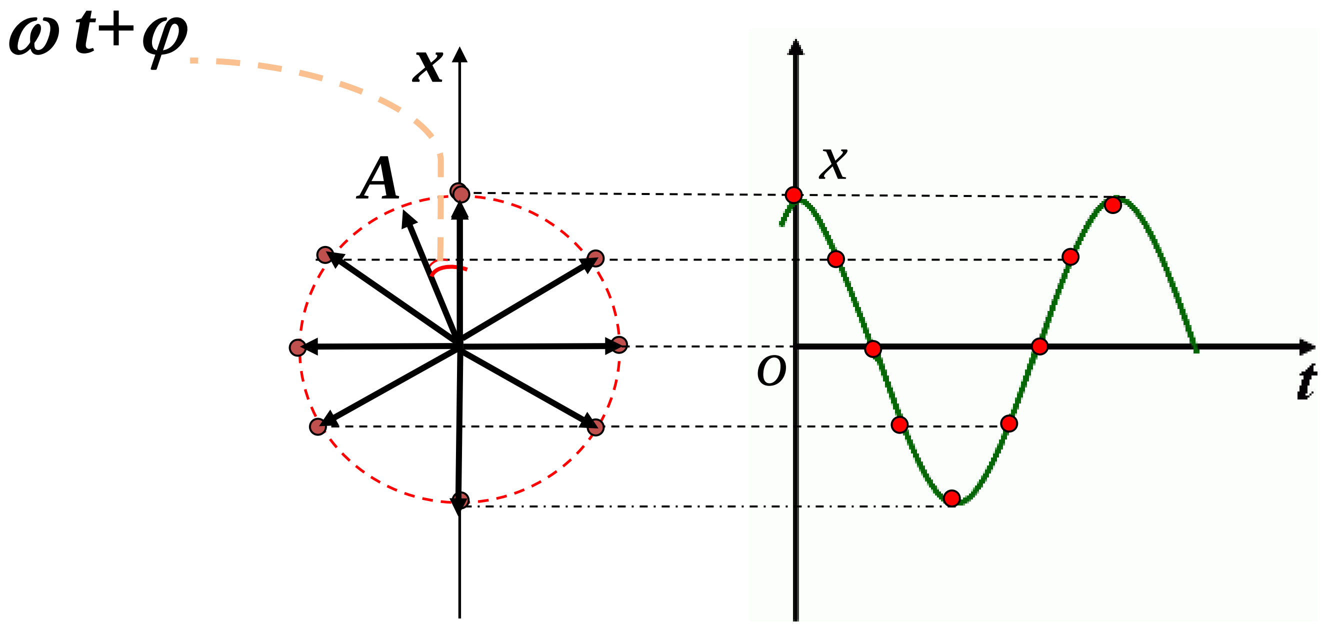
The projection in x axis of the position vector of point P_x moves in SHM along a diameter of circle, at t is :

$$P_x = A\cos(\omega t + \varphi_0)$$

$$x = A\cos(\omega t + \varphi)$$



Expression of SHM



Relation between the two ways of expressions

旋转矢量 A	简谐振动	符号或表达式
模	振幅	A
角速度	角频率	ω
$t=0$ 时，A 与 x 轴的夹角	初相	φ_0
旋转周期	振动周期	$T=2\pi/\omega$
t 时刻，A 与 x 轴的夹角	相位	$\omega t+\varphi_0$
A 在 x 轴上的投影	位移	$x=A\cos(\omega t+\varphi_0)$
A 端点的速度在 x 轴上的投影	速度	$v=-A\omega\sin(\omega t+\varphi_0)$
A 端点的加速度在 x 轴上的投影	加速度	$a=-A\omega^2\cos(\omega t+\varphi_0)$

Example

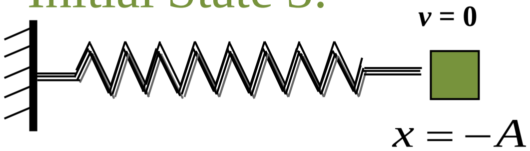
Initial State 1:

$$\Rightarrow \varphi_0 = 0$$

Initial State 2:

$$\Rightarrow \varphi_0 = \frac{\pi}{2}$$

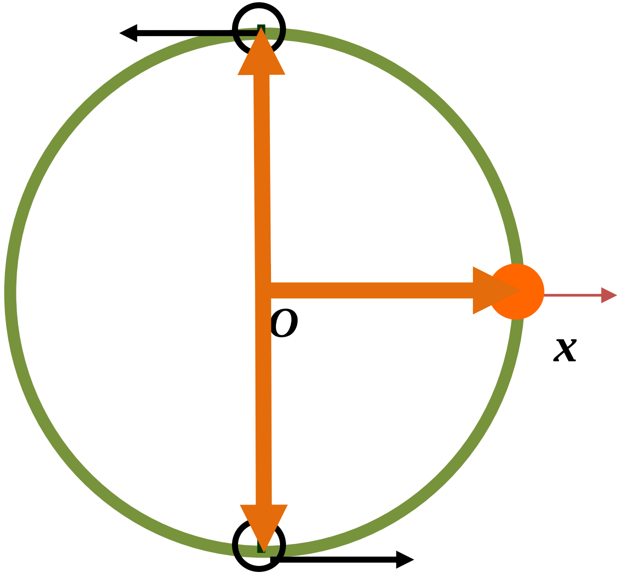
Initial State 3:



$$\Rightarrow \varphi_0 = \pi$$

Initial State 4:

$$\Rightarrow \varphi_0 = \frac{3\pi}{2}$$



Example 2

There is a simple harmonic oscillation of amplitude 0.24 m and period 3s. At initial time, t=0, x_0=0.12m, v_0<0. Find the initial phase and the shortest time interval in which the oscillator arrive at position x= −0.12m.

Example 3

A particle is in SHM along *x* axis, A = 0.12m, T = 2s. When t = 0, x_o= 0.06m, and v > 0 (moves along positive *x* direction). Try to find out:

(1) The expression of this SHM;

(2) $t = T/4$, x = ? v = ? and a = ?

(3) At what time will the particle pass the "*O*" first time（何时物体第一次通过平衡位置）?

Expression of SHM - phasor

The virtues of describing the simple harmonic motion by using the phasor

1. **Express the A, T and $\omega t+\varphi$ of simple harmonic motion;**

2. **Determine the initial phase of oscillation easily;**

3. **Make the superposition of several oscillations conveniently.**

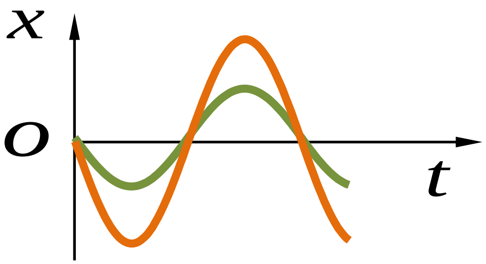
Superposition of oscillations

Two SHM in same direction with same frequencies :

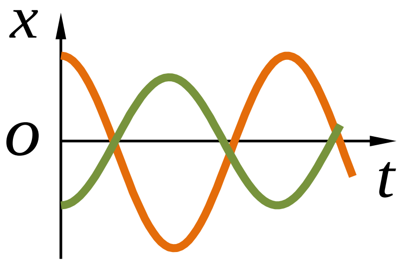
$$x_1 = A_1 \cos(\omega t + \varphi_1) \qquad x_2 = A_2 \cos(\omega t + \varphi_2)$$

$$\Delta\varphi = (\omega t + \varphi_2) - (\omega t + \varphi_1) \qquad \boxed{\Delta\varphi = \varphi_2 - \varphi_1}$$

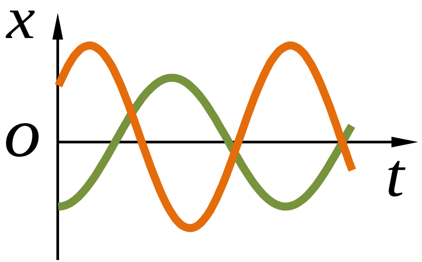
$\Delta\varphi = 0$



$\Delta\varphi = \pm\pi$



$\Delta\varphi \begin{cases} \text{lead} \\ \text{lag} \end{cases}$



Superposition of oscillations

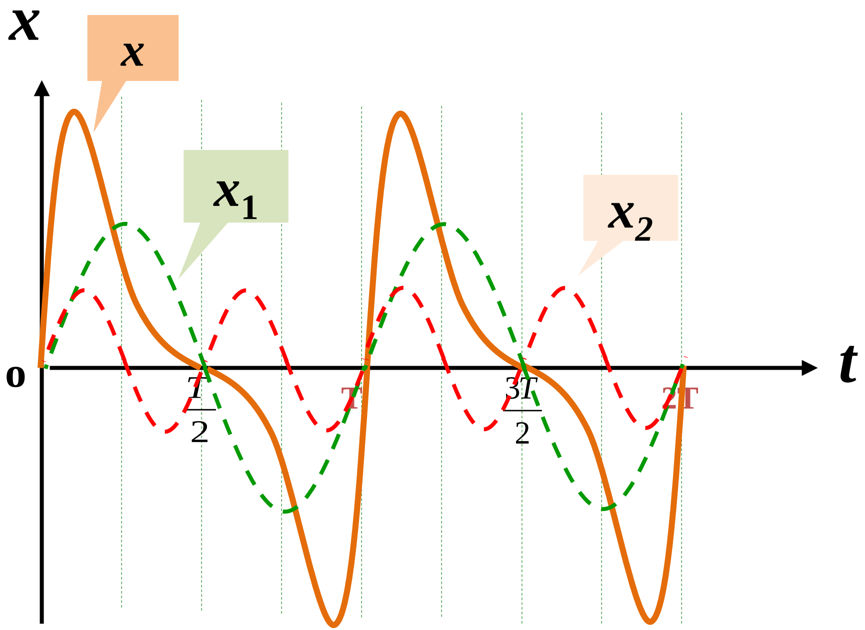
Two SHM in same direction with same frequencies :

$$x_1 = A_1 \cos(\omega t + \varphi_1)$$

$$x_2 = A_2 \cos(\omega t + \varphi_2)$$

$$x = x_1 + x_2$$

$$x = A \cos(\omega t + \varphi)$$



Superposition of oscillations

$$x_1 = A_1 \cos(\omega t + \varphi_1)$$

$$x_2 = A_2 \cos(\omega t + \varphi_2)$$

$$x = x_1 + x_2$$

$$x = A \cos(\omega t + \varphi)$$

$$\boldsymbol{A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos(\varphi_2 - \varphi_1)}}$$

$$\mathbf{tan}\boldsymbol{\varphi} = \frac{\boldsymbol{A_1 \sin\varphi_1 + A_2 \sin\varphi_2}}{\boldsymbol{A_1 \cos\varphi_1 + A_2 \cos\varphi_2}}$$

Superposition of oscillations

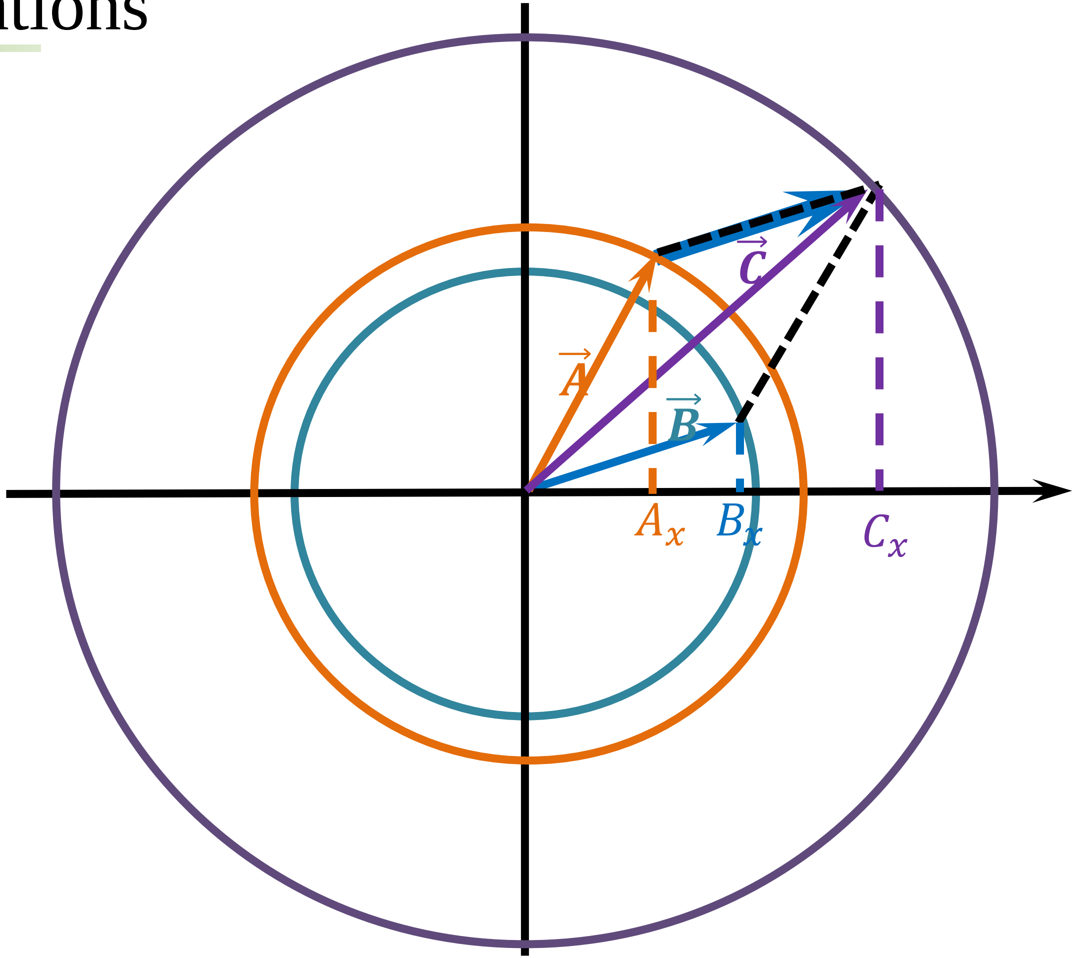
$$\vec{\boldsymbol{C}} = \vec{\boldsymbol{A}} + \vec{\boldsymbol{B}}$$

$$C_x = A_x + B_x$$

Phase difference?

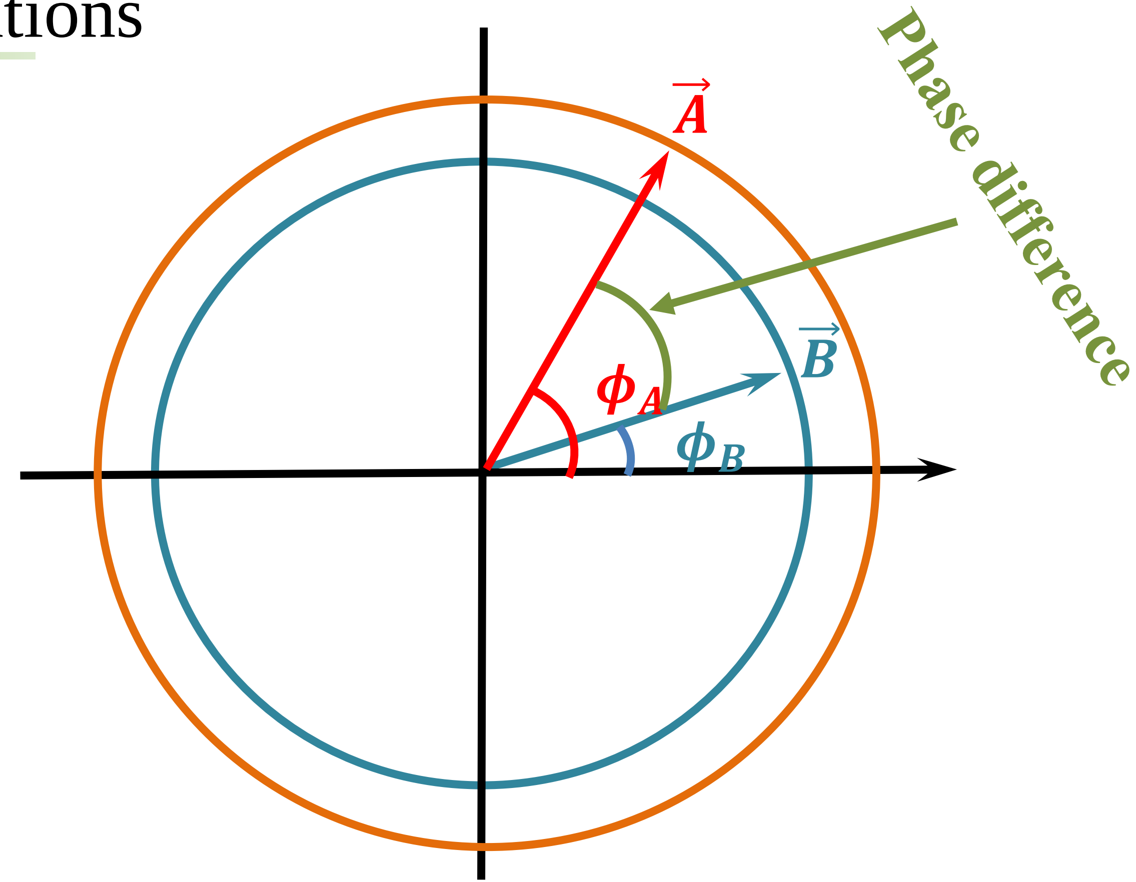
$$x_1 = A_1 \cos(\omega t + \varphi_1)$$

$$x_2 = A_2 \cos(\omega t + \varphi_2)$$



Superposition of oscillations

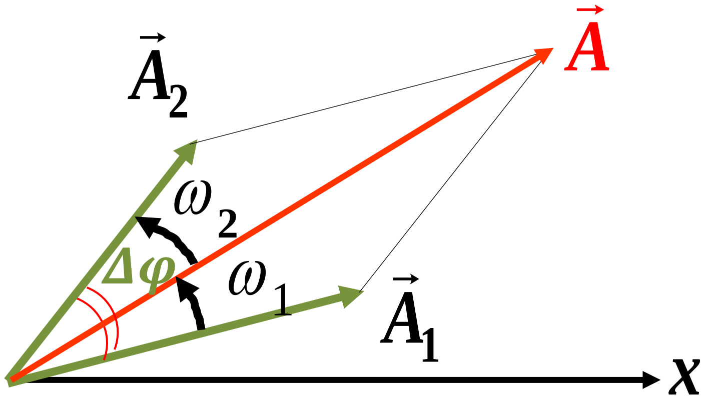
The **phase difference** between two phasors **won't change** during the rotation, if they have the **same** frequency.



Two SHM in same direction with same frequencies

Superposition of oscillations

Superposition of two SHM in same direction with different frequencies



If $\omega_1 = \omega_2$, $\Delta\phi =$ *constant* ;

if $\omega_1 \neq \omega_2$, $\Delta\phi$ **will vary,**

resultant oscillation isn't SHM

$$x_1 = A_1 \cos(\omega_1 t + \phi)$$

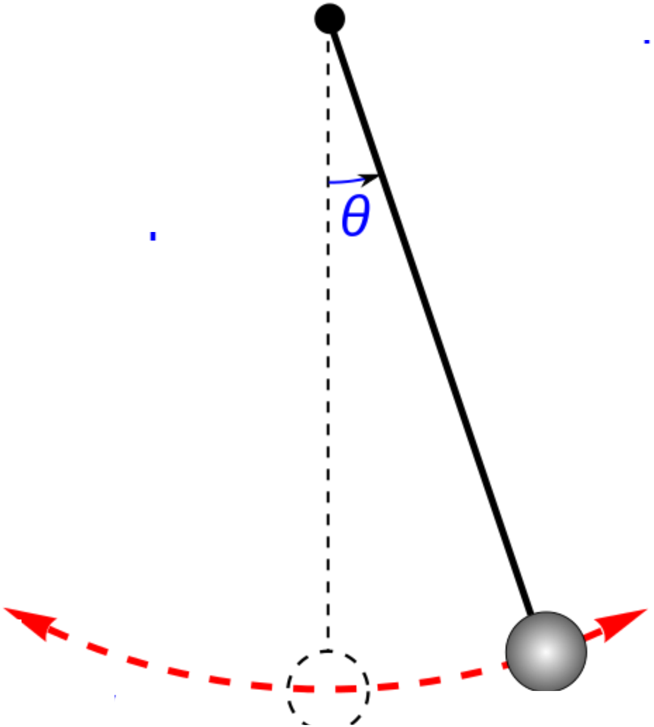
$$x_2 = A_2 \cos(\omega_2 t + \phi)$$

$$x = x_1 + x_2 = \boxed{2A\cos(\frac{\omega_2 - \omega_1}{2}t)}\boxed{\cos(\frac{\omega_2 + \omega_1}{2}t + \phi)}$$

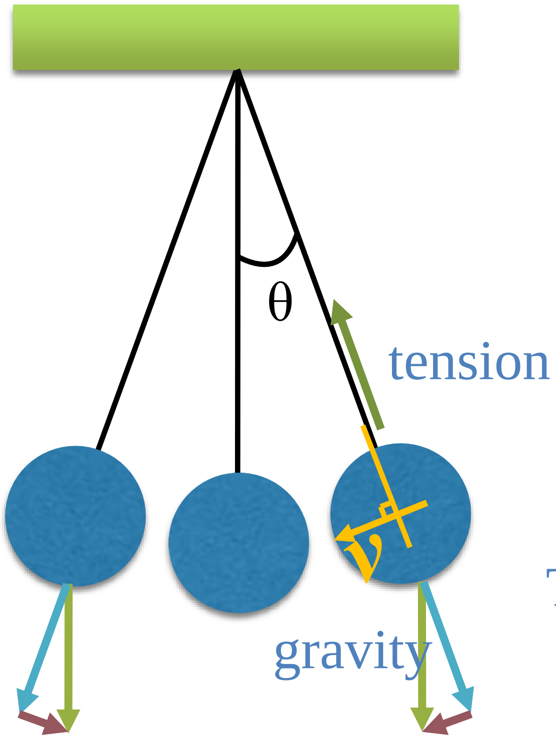
Assume $A_1 = A_2 = A$

The Pendulum

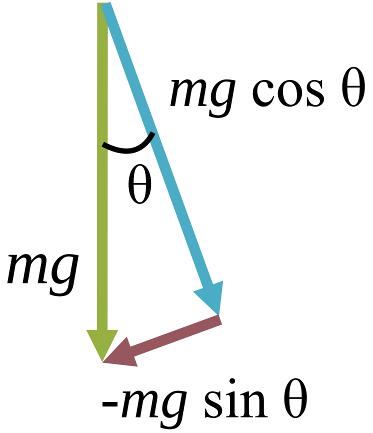
Radians



Forces on the pendulum



free body diagram



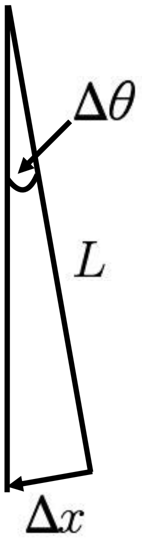
If θ is small $(\theta \ll 1) \Longrightarrow \sin\theta \approx \theta$

The *tangential* force on the mass is:

$$F \approx -mg\,\theta$$

The force is proportional to the distance from "equilibrium"

The displacement



$$\frac{\Delta x}{L} = \sin\Delta\theta \approx \Delta\theta$$

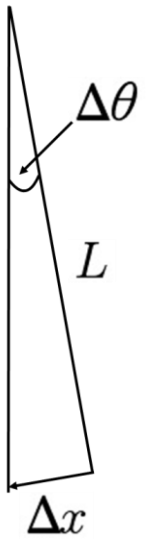
Speed: $\Delta x = L\,\Delta\theta \Rightarrow dx = L\,d\theta$

$$v = \frac{dx}{dt} = L\frac{d\theta}{dt}$$

$\frac{d\theta}{dt}$ is the "**angular velocity**"

Acceleration: $a = \frac{dv}{dt} = L\frac{d}{dt}\left(\frac{d\theta}{dt}\right) = L\frac{d^2\theta}{dt^2}$

tangential direction:



$$F = ma = -mg\,\theta = -\frac{mg}{L}\Delta x$$

$$mL\frac{d^2\theta(t)}{dt^2} = -mg\,\theta(t)$$

"Differential equation" $L\frac{d^2\theta(t)}{dt^2} = -g\,\theta(t)$

Try this solution: $\theta = \theta_m \cos\left(\sqrt{\frac{g}{L}}\,t + \varphi\right)$

$$\frac{d^2\theta}{dt^2} = -\theta_m \frac{g}{L}\cos\left(\sqrt{\frac{g}{L}}\,t + \varphi\right)$$

Simple harmonic motion!

Simple harmonic motion

$$x = A\cos(\omega t + \varphi_0)$$

$$a = -\omega^2 x$$

$$F = -(\omega^2 m)x$$

or $F = -kx$

Spring

$$\theta = \theta_m \cos\left(\sqrt{\frac{g}{L}}\, t + \varphi\right)$$

$$L\frac{d^2\theta(t)}{dt^2} = -g\,\theta(t)$$

$$F = -mg\theta = -\frac{mg}{L}\Delta x$$

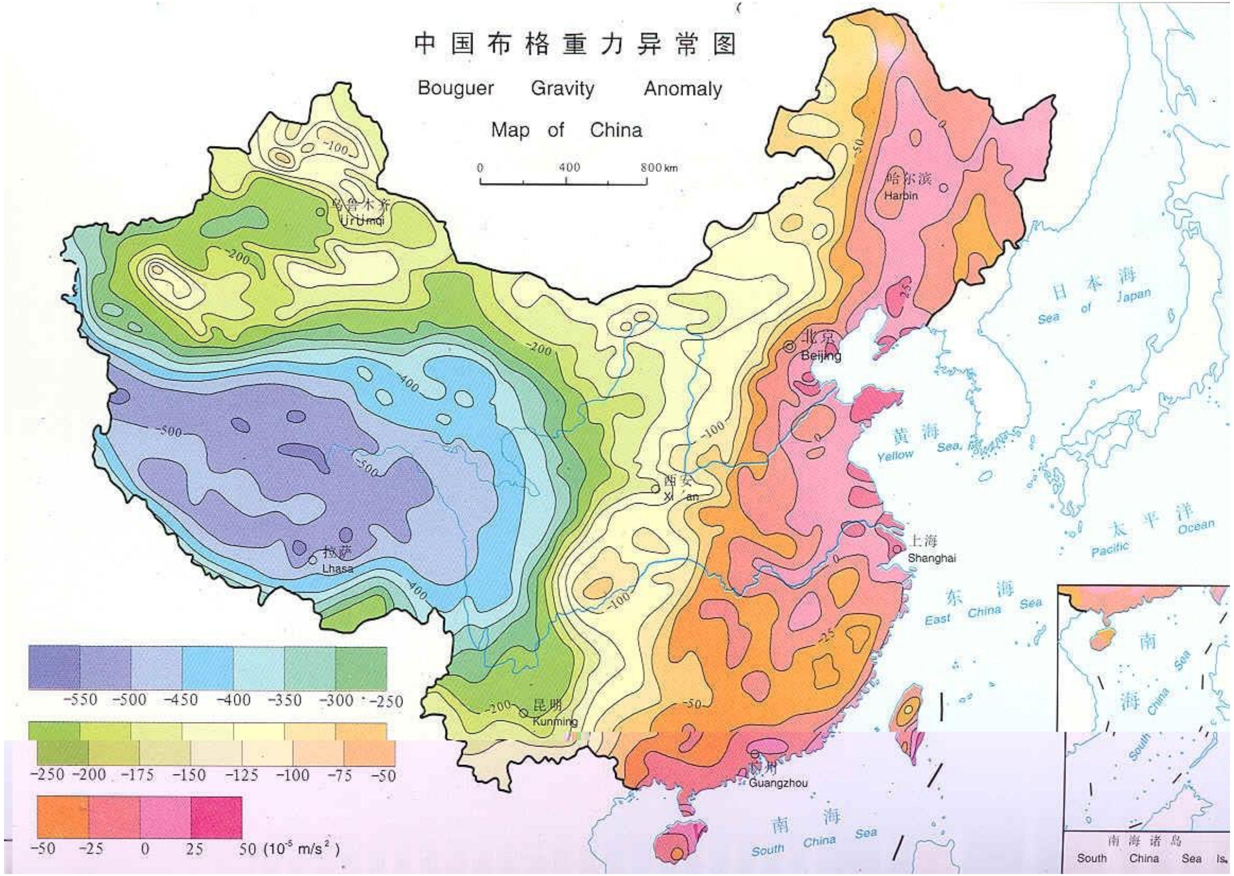
Pendulum

The period of the pendulum:

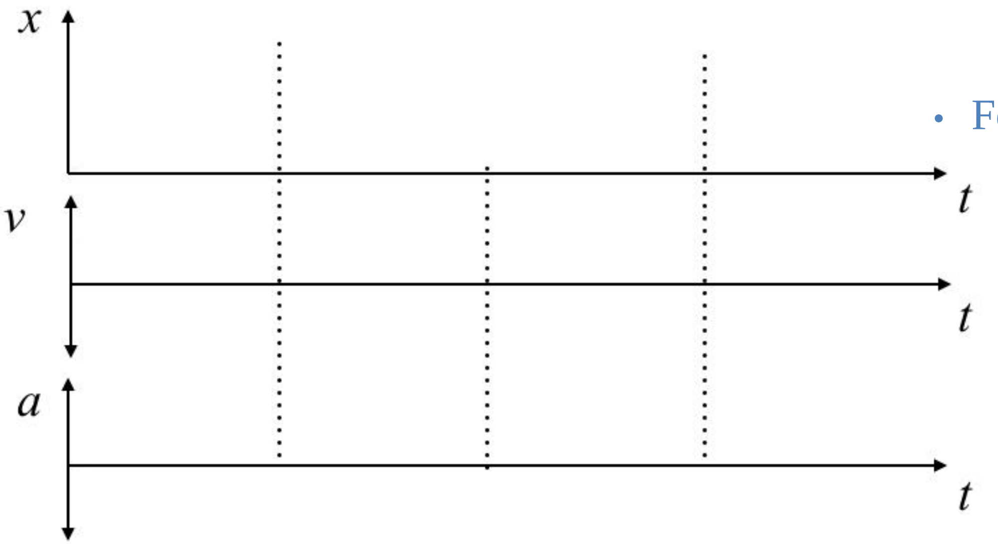
$$\boxed{T = 2\pi\sqrt{\frac{L}{g}}}$$

- ✓ The period is independent of the mass *m*
- ✓ The period is independent of the amplitude A
- ✓ The period depends on the length *L*
- ✓ The period also depends on gravity (*g*)
- ✓ The amplitude (angle) must be small

Using a pendulum to measure *g*



Ball bouncing back and forth



- Period just depends on the speed of the ball
- Force not prop. to *x*

Potential Energy in the harmonic oscillator

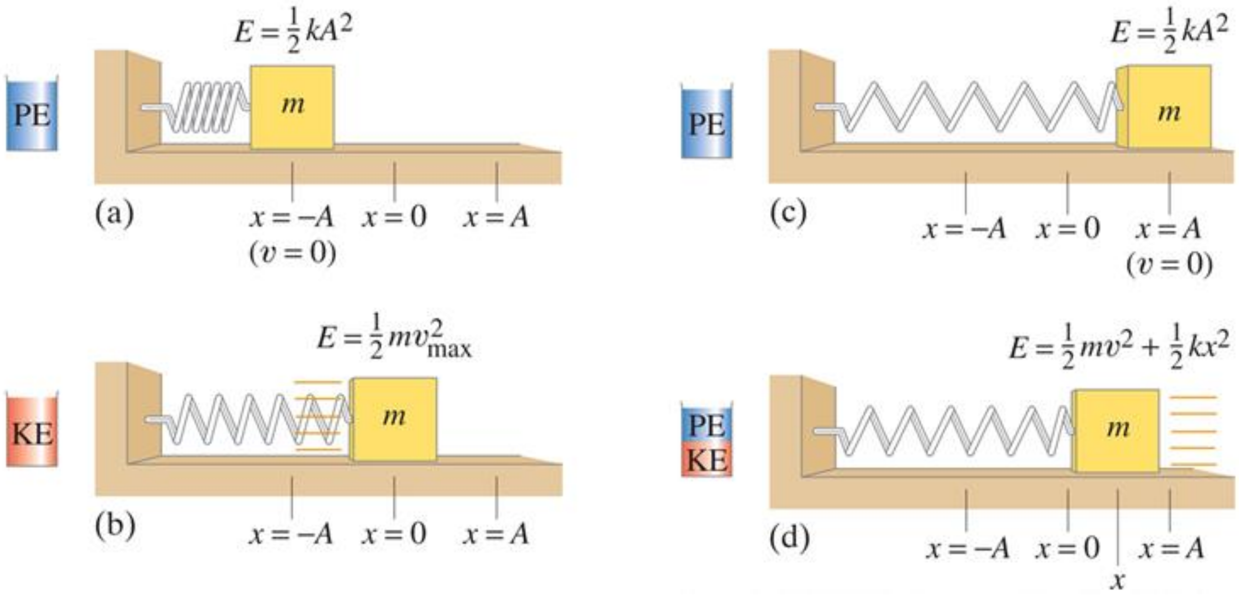
How to derive the potential energy in a derivative way?

Energy conversion in the harmonic oscillator

Ideally, total mechanical energy is constant

$$E_{tot} = U + K = \boxed{\frac{kx^2}{2}} + \boxed{\frac{mv^2}{2}}$$

potential energy　kinetic energy



Energy in SHM

Kinetic Energy of the mass

$$E_K = \frac{1}{2}mv^2 \qquad\qquad v(t) = -A\omega\sin(\omega t + \varphi)$$

Potential energy of the spring

$$E_s = \frac{1}{2}kx^2$$

$$k = m\omega^2 \qquad\qquad x(t) = Acos(\omega t + \varphi)$$

total energy $\quad E(t) = E_K(t) + E_s(t) = ?$

Energy in SHM

Kinetic Energy of the mass

$$E_K = \frac{1}{2}mv^2 \qquad v(t) = -A\omega\sin(\omega t + \varphi)$$

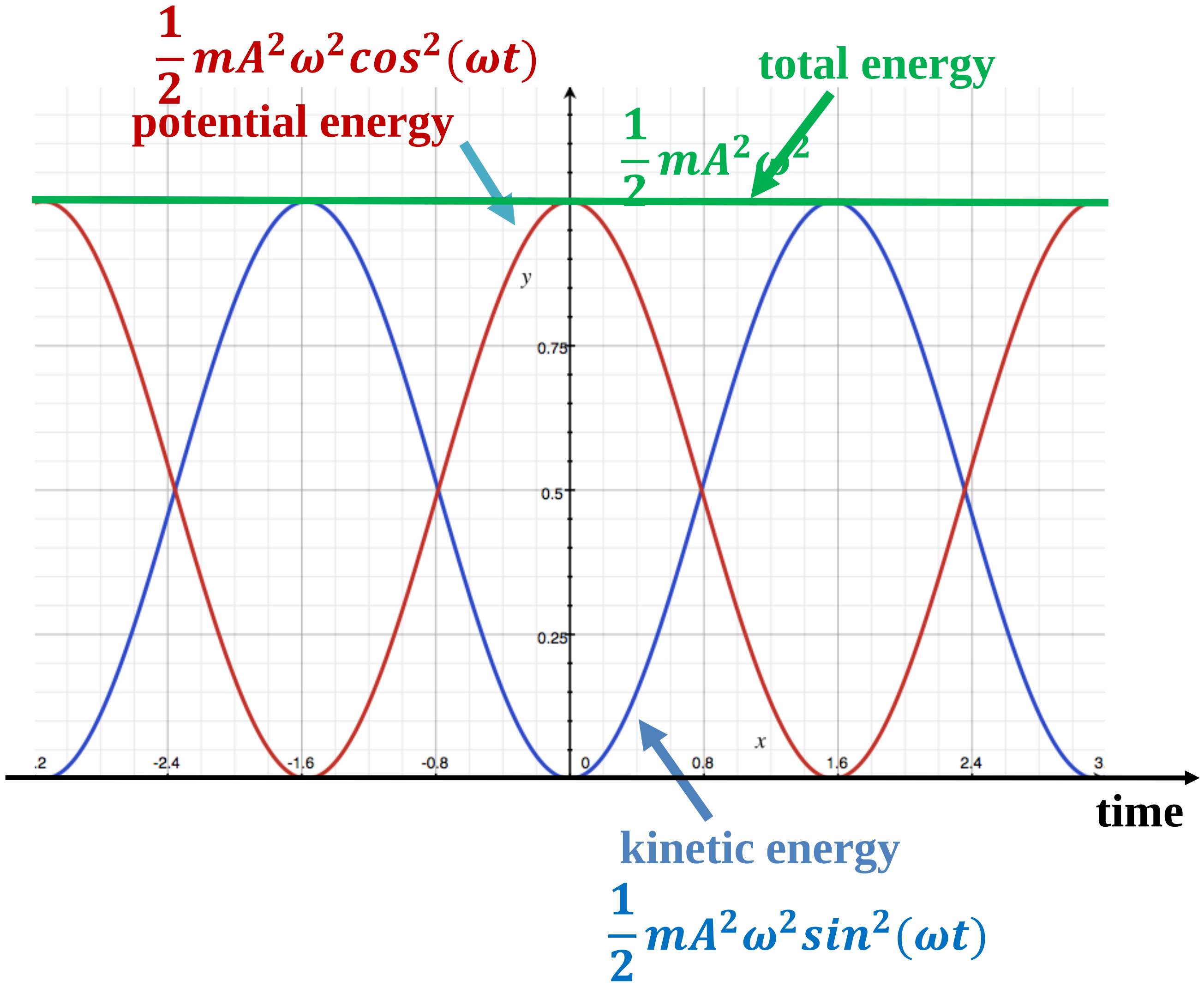
$$\boxed{E_K(t) = \frac{1}{2}mA^2\omega^2 sin^2(\omega t + \varphi)}$$

Potential energy of the spring

$$E_s = \frac{1}{2}kx^2 \qquad k = m\omega^2 \qquad x(t) = Acos(\omega t + \varphi)$$

$$\boxed{E_s(t) = \frac{1}{2}m\omega^2 A^2 cos^2(\omega t + \varphi)}$$

total energy $E(t) = E_K(t) + E_s(t) = \boxed{\frac{1}{2}mA^2\omega^2}$ ***Conserved!***



$\frac{1}{2}mA^2\omega^2cos^2(\omega t)$
potential energy
total energy
$\frac{1}{2}mA^2\omega^2$
y
0.75
0.5
0.25
x
.2
-2.4
-1.6
-0.8
0
0.8
1.6
2.4
3
time
kinetic energy
$\frac{1}{2}mA^2\omega^2sin^2(\omega t)$

Calculate the speed from the energy (finding *v*(*x*))

$$E_{tot} = U + K = \frac{kx^2}{2} + \frac{mv^2}{2} = \frac{kA^2}{2}$$?

$$\frac{mv^2}{2} = \frac{kA^2}{2} - \frac{kx^2}{2}$$

$$v^2 = \frac{k}{m}(A^2 - x^2)$$

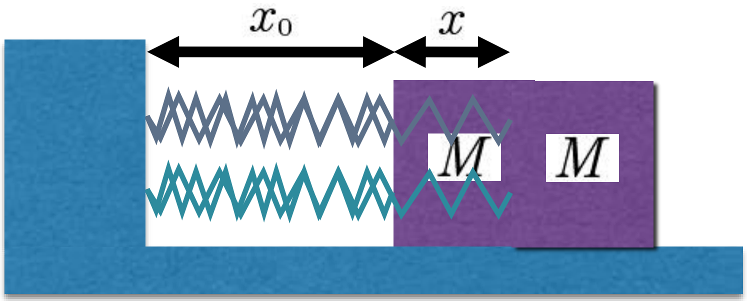
$$\omega = \sqrt{\frac{k}{M}}$$

$$\boxed{v(x) = \pm\sqrt{\frac{k}{m}(A^2 - x^2)}} \Rightarrow \boxed{v(x) = \pm\omega\sqrt{A^2 - x^2}}$$

$$\left.\begin{aligned} x &= A\cos(\omega t + \varphi) \\ v &= -A\omega\sin(\omega t + \varphi) \end{aligned}\right\}$$

Consistent!

What if you have two springs "in parallel"



$$F_1 = -k_1 x$$

$$F_2 = -k_2 x$$

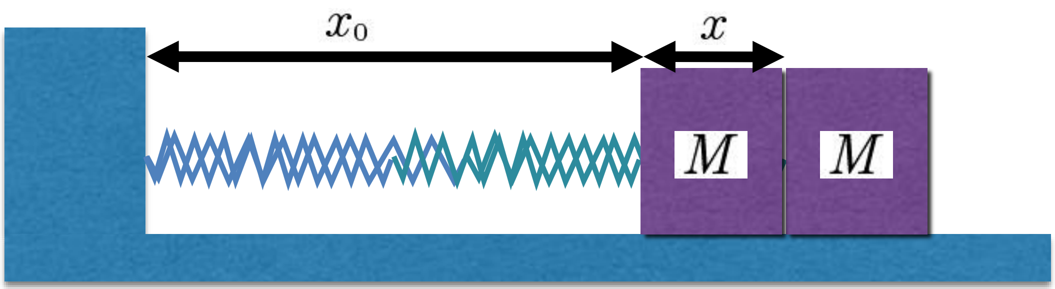
$$F_{tot} = -(k_1 + k_2)x$$

$$F_{tot} = -k_{eq} x$$

Find an "equivalent" spring constant

$$k_{eq} = (k_1 + k_2)$$

What if you have two springs “in series”



$$F_1 = -k_1 x_1 \qquad\qquad F_2 = -k_2 x_2$$

$$F_1 = F_2 \qquad \Longrightarrow \quad x_2 = \frac{k_1}{k_2} x_1$$

$$F_{tot} = -k_2 x_2 \quad = -k_{eq}(x_1 + x_2)$$

$$k_{eq} = k_2 \frac{x_2}{(x_1 + x_2)} = k_2 \frac{\frac{k_1}{k_2} x_1}{x_1\left(1 + \frac{k_1}{k_2}\right)} = \frac{k_1 k_2}{(k_1 + k_2)} \qquad \boxed{\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2}}$$