MATH 2233 Differential Equations
Chapter 7 Laplace Transform

Section 7.2 Definition of Laplace Transform

Goal of this section

- Caculate the Laplace transform of a function using the definition.

Motivation: Many practical engineering problems involve mechanical or electrical systems acted on by *discontinuous or impulsive forcing terms*. Methods introduced in previous chapters are often rather awkward to use. In this chapter, we consider a new approach based on Laplace transform.

Definition: Laplace transform

Let $f(t)$ be a function on $(0, \infty)$. The **Laplace transform** of a function $f(t)$ is a function $F(s)$ defined by the integral

Remark The integral above is an **improper** integral defined by

Example 1. *Determine the Laplace transform of the constant function* $f(t) = 1$.

Example 2. *Find the Laplace transform of* $f(t) = e^{at}$*, where* a *is a constant.*

Example 3. *Find* $\mathcal{L}\{\sin(bt)\}$ *where* b *is a constant.*

Example 4. *Find the Laplace transform of*

$$f(t) = \begin{cases} 2, & 0 \le t < 5, \\ 0, & 5 < t < 10, \\ e^{4t}, & t > 10, \end{cases}$$

A Brief Table of Laplace Transforms.

$f(t)$	$F(s) = \mathcal{L}\{f\}(s)$
1	
t^n	
e^{at}	
$e^{at}t^n$	
$\sin(bt)$	
$\cos(bt)$	
$e^{at}\sin(bt)$	
$e^{at}\cos(bt)$	

Linearity of Laplace Transforms.

Let f, f_1, f_2 be functions whose Laplace transform exist. Let c be a constant. Then

- $\mathcal{L}\{f_1 + f_2\} =$

- $\mathcal{L}\{cf\} =$

Example 5. *Determine* $\mathcal{L}\{11 + 5e^{4t} - 6\sin(2t)\}$ *and* $\mathcal{L}\{5t^2e^{-3t} - e^{12t}\cos(8t)\}$.

Existence of the Transform

There are functions for which the improper integral fails to converge for any value of s. For example, this is the case for the functions $f(t) = 1/t$ which grows too fast near zero. Similarly, function like $f(t) = e^{t^2}$ grows increase very rapidly as $t \to \infty$.

Type of discontinuity: Removable, Jump, Infinity Discontinuity

Piecewise Continuity

A function $f(t)$ is called piecewise continuous on a finite interval $[a, b]$ if

A function $f(t)$ is called piecewise continuous on $[0, \infty)$ if

Example 6. *Show that*

$$f(t) = \begin{cases} t, & 0 < t < 1, \\ 2, & 1 < t < 2, \\ (t-2)^2, & 2 \le t < 3, \end{cases}$$

is piecewise continuous on $[0, 3]$.

Exponential Order α

A function $f(t)$ is said to be **exponential order** α if there exist positive constants

Example 7. *Determine if the following functions are of exponential order.*

$$(a)\ e^{4t}, \quad (b)\ e^{t^2}, \quad (c)\ \sin(t^2) + t^3e^{3t}.$$

Conditions for Existence of the Laplace Transform

If $f(t)$ is piecewise continuous on $[0, \infty)$ and of exponential order α, then