Finite Control Volume Analysis

Jamie F. Townsend

1st April 2024

Homework No. 1

- It must be submitted **individually on paper** before the lecture on 8 April.
- **ALL** necessary **calculations** must be shown.
- 40 points in total = **10%** of the final grade.

Fluid Mechanics – Homework No. 1

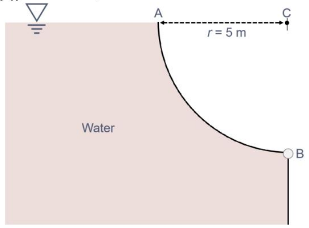
Issued: **Monday, 25 March 2024** **Submission deadline: Monday, 8 April 2024**

For ALL the problems show ALL your calculations and state clearly the assumptions, if necessary.

Question 1 (4 points)
The gage pressure at the bottom of a gasoline storage tank is 94 kPa. If it is filled and vented, how tall is the tank? Assume the specific weight of gasoline is γ_{gas} = 6.67 kN/m³.

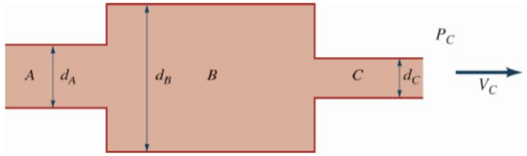
Question 2 (8 points)
A scale indicates that an object weighs 206 N when suspended in air, but only 57 N when it is submerged in water. What is the object's specific weight (γ)?

Question 3 (12 points)
Calculate the total (resultant) hydrostatic pressure force on the gate AB per meter of unit depth (into the page). Find the angle α of the resultant force with respect to the free surface. Assume the specific weight of water is γ_W = 9.8 kN/m³.



Question 4 (10 points)
A velocity field is described by $V = [(A + By^2)\hat{\imath} + C\hat{\jmath}]$ m/s, where A = 1 m/s, B = 1 m/s, C = 2 m/s, and *x* and *y* are in meters. Determine an expression for the streamlines and plot the streamline that passes through the point with the coordinates (*x*, *y*) = (2,1). Determine the *y*-coordinate and the speed along the streamline at *x* = -2m, 2m, and 4m.

Question 5 (6 points)
Oil (ρ = 790 kg/m³) flows through the horizontal expansion and contraction sections as sketched in the figure below, in which d_A = 2cm, d_B = 5cm, and d_C =1cm. In section C, the velocity is 10 m/s and the gage pressure is 2 kPa. Calculate the static pressures (kPa, gage) in sections A and B. Assume steady, incompressible, and inviscid flow.

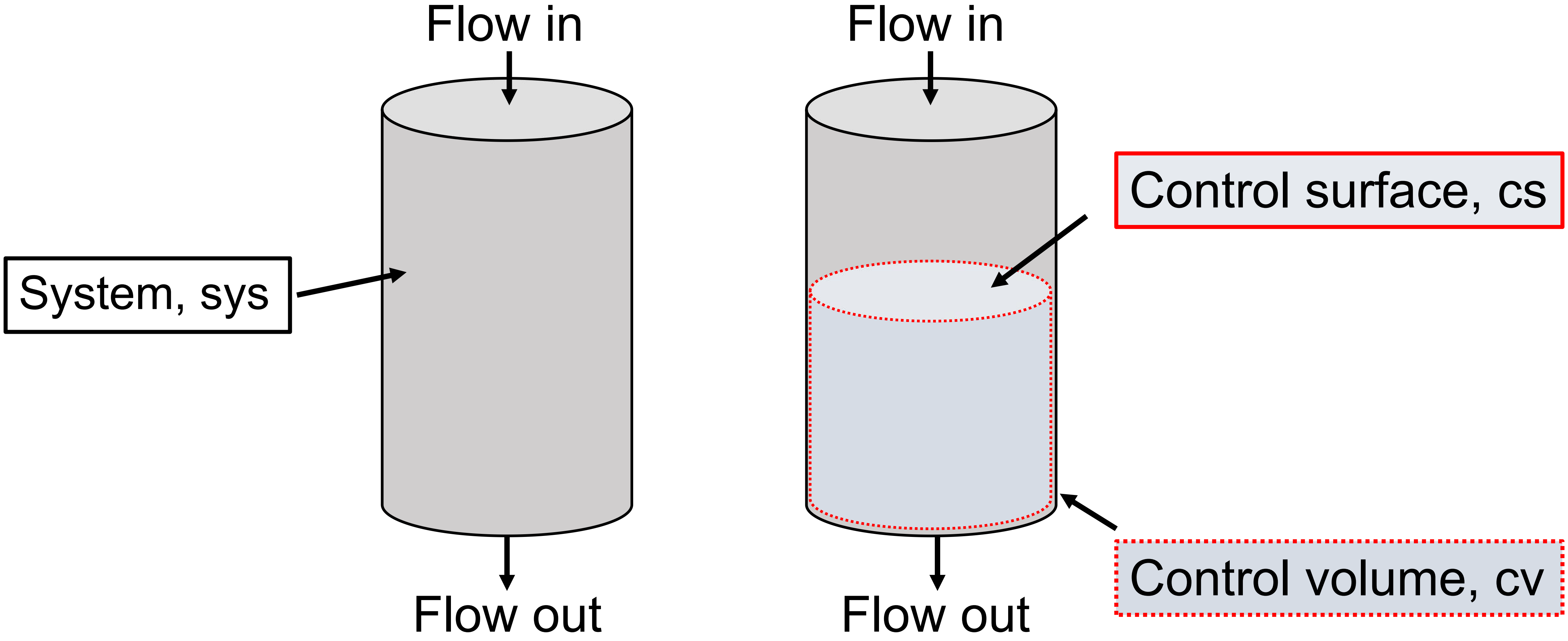


[Total points = 40]

Learning objectives

- Understand the concept of a control volume.
- Select an appropriate control volume to solve fluid mechanics problems.
- Understand principles of conservation of mass and momentum for a fluid.
- Apply the principles of mass and momentum conservation to study different fluid systems.
- Understand the concept of energy conservation within a fluid system.

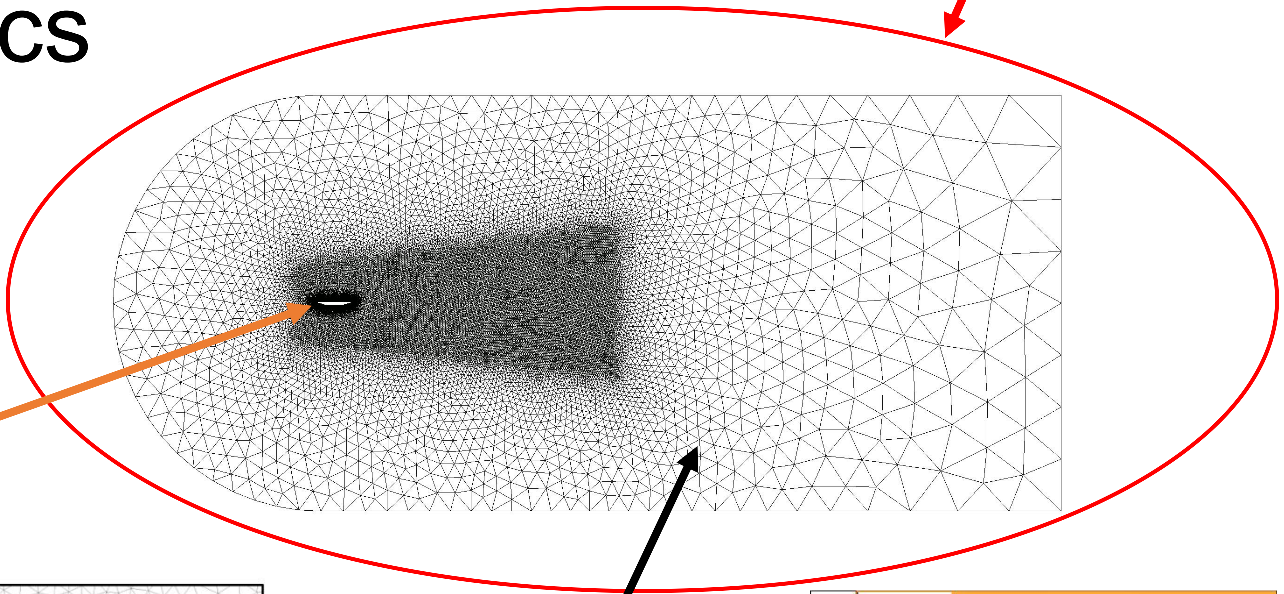
Control volume



The amount of mass in a system does not change with time. → Time rate of change of the system mass = 0

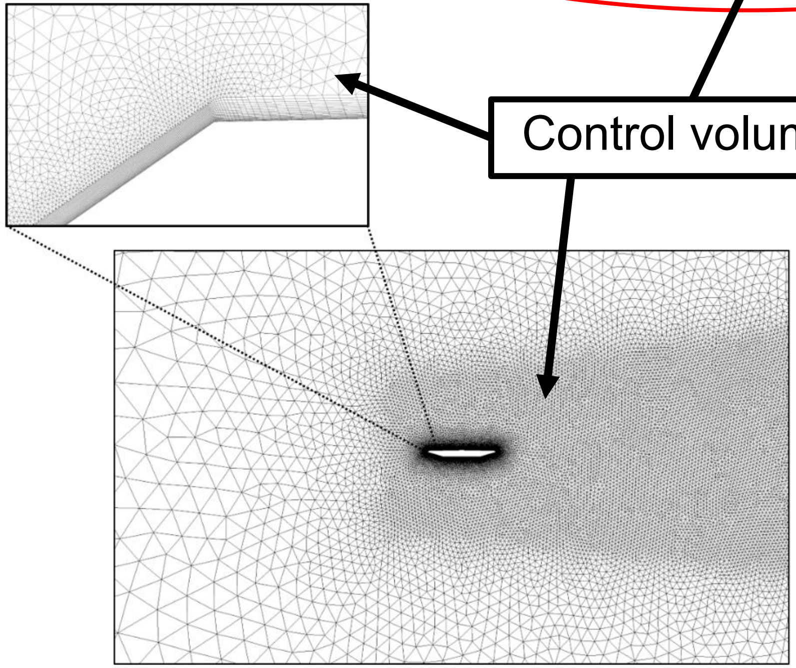
Control volume example: Computational Fluid Dynamics

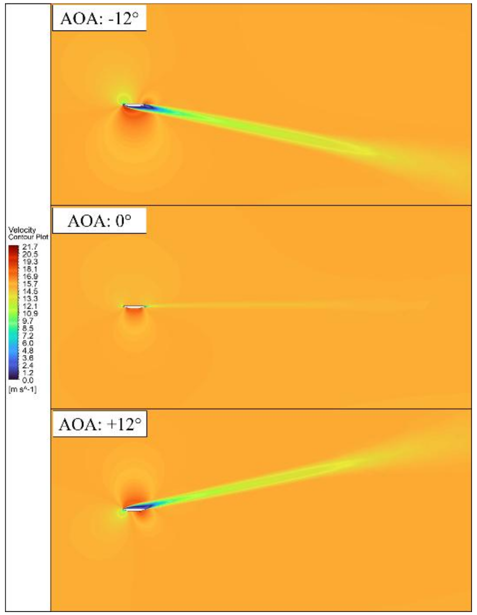
System



Bridge deck

Control volumes





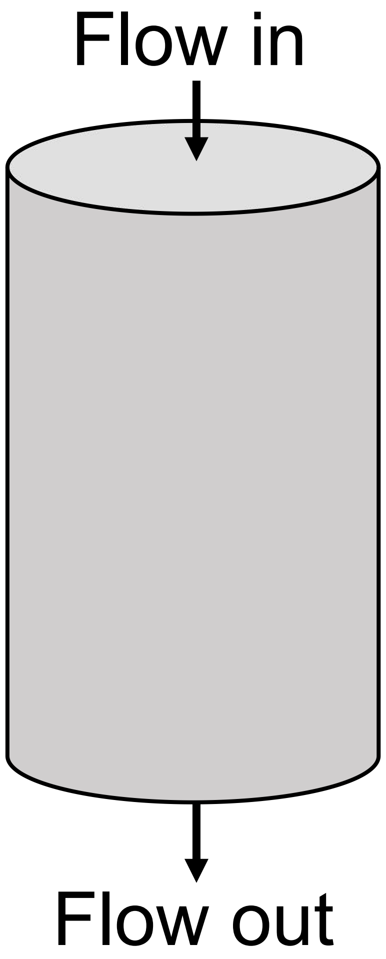
Conservation of mass: continuity equation

The amount of mass in a system does not change with time. → Time rate of change of the system mass = 0

Total mass in the system:

$$M_{\text{sys}} = \int_{\text{sys}} \rho \, dV$$

System, sys



$$\frac{DM_{\text{sys}}}{Dt} = 0$$

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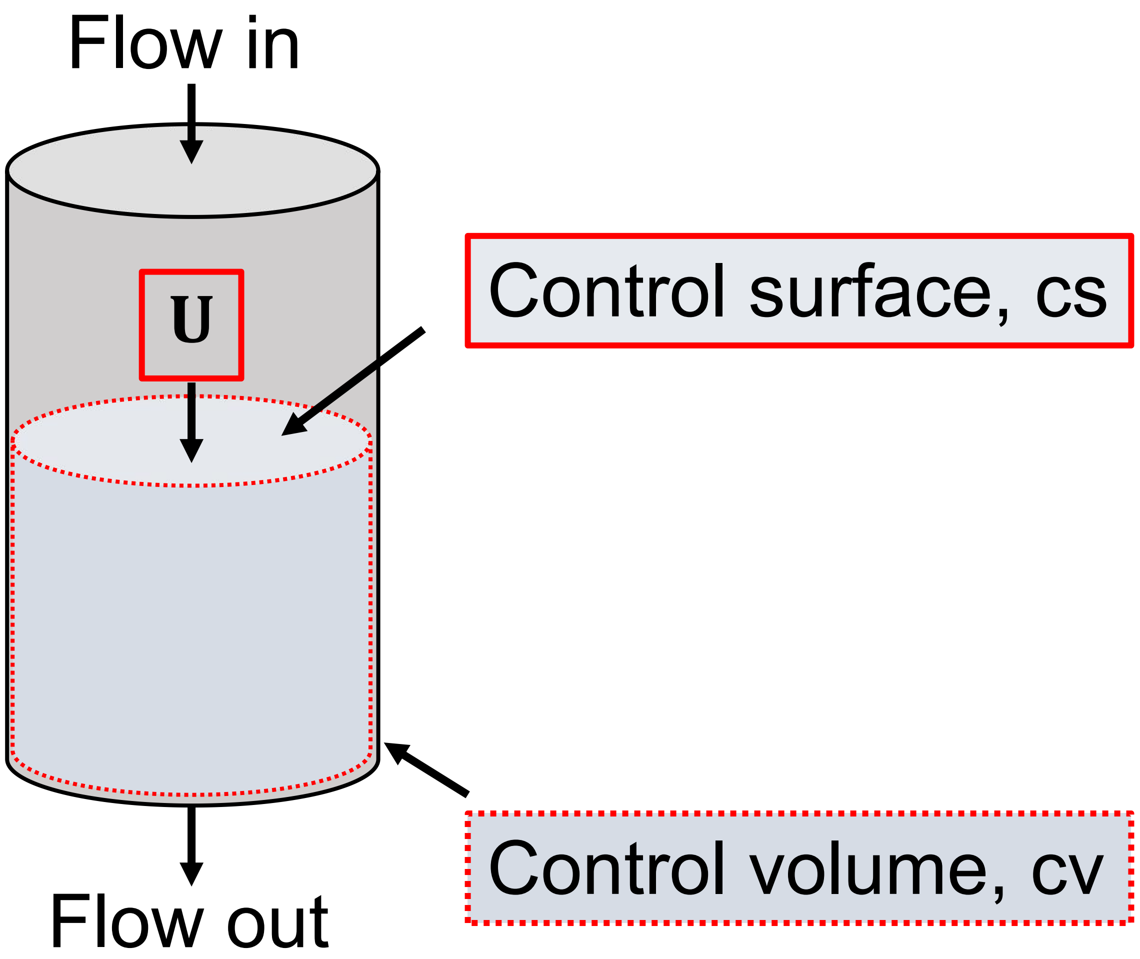
Conservation of mass: continuity equation

The amount of mass in a system does not change with time.

$$\underbrace{\frac{D}{Dt}\int_{\mathrm{sys}} \rho\, dV}_{\substack{\text{Time rate of}\\ \text{change of the}\\ \text{mass in the system}}} = \underbrace{\frac{\partial}{\partial t}\int_{\mathrm{cv}} \rho\, dV}_{\substack{\text{Time rate of}\\ \text{change of the}\\ \text{mass of the}\\ \text{contents in the}\\ \text{control volume}}} + \underbrace{\overbrace{\int_{\mathrm{cs}} \rho\, \mathbf{U}\cdot\hat{\mathbf{n}}\, dA}}_{\substack{\text{Net rate of flow of}\\ \text{mass through the}\\ \text{control surface}}}$$



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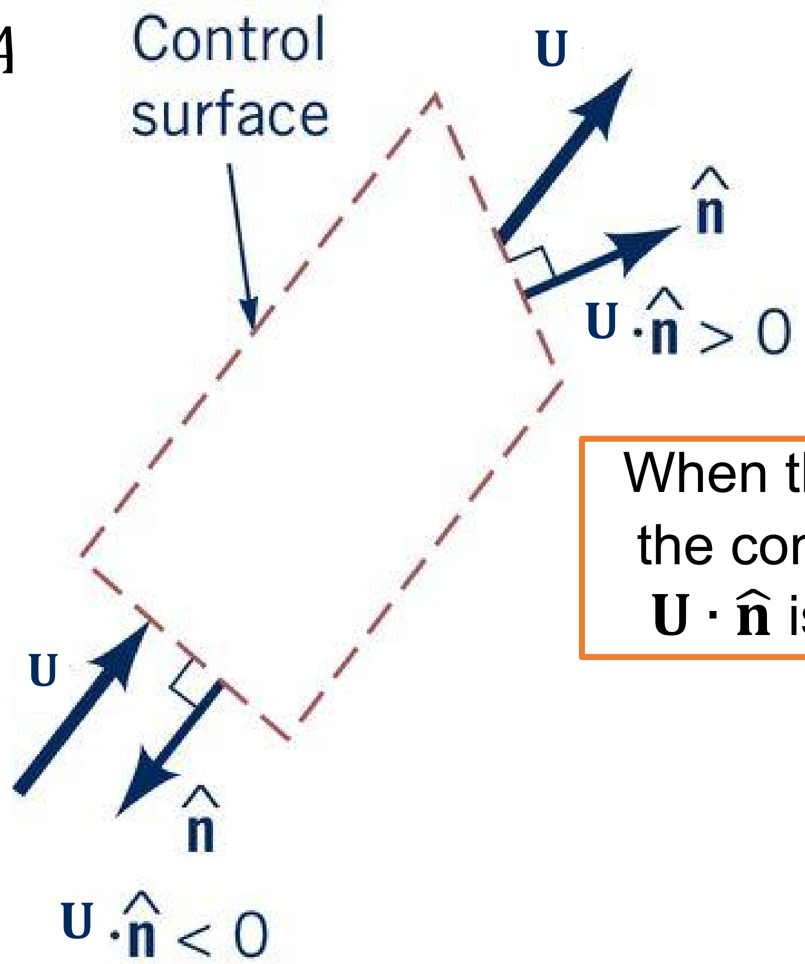


Time rate of change of the mass in the control volume

$$\frac{\partial}{\partial t}\int_{\text{cv}} \rho\, dV$$

The net rate of mass flow through the control surface

$$\int_{\text{cs}} \rho\, \mathbf{U}\cdot\hat{\mathbf{n}}\, dA$$

When the flow exits the control volume $\mathbf{U}\cdot\hat{\mathbf{n}}$ is positive +

When the flow enters the control volume $\mathbf{U}\cdot\hat{\mathbf{n}}$ is negative -

Conservation of mass: continuity equation

An expression for the conservation of mass within a control volume is found by combing the aforementioned expressions:

$$\frac{DM_{\text{sys}}}{Dt} = 0$$

$$M_{\text{sys}} = \int_{\text{sys}} \rho \, dV$$

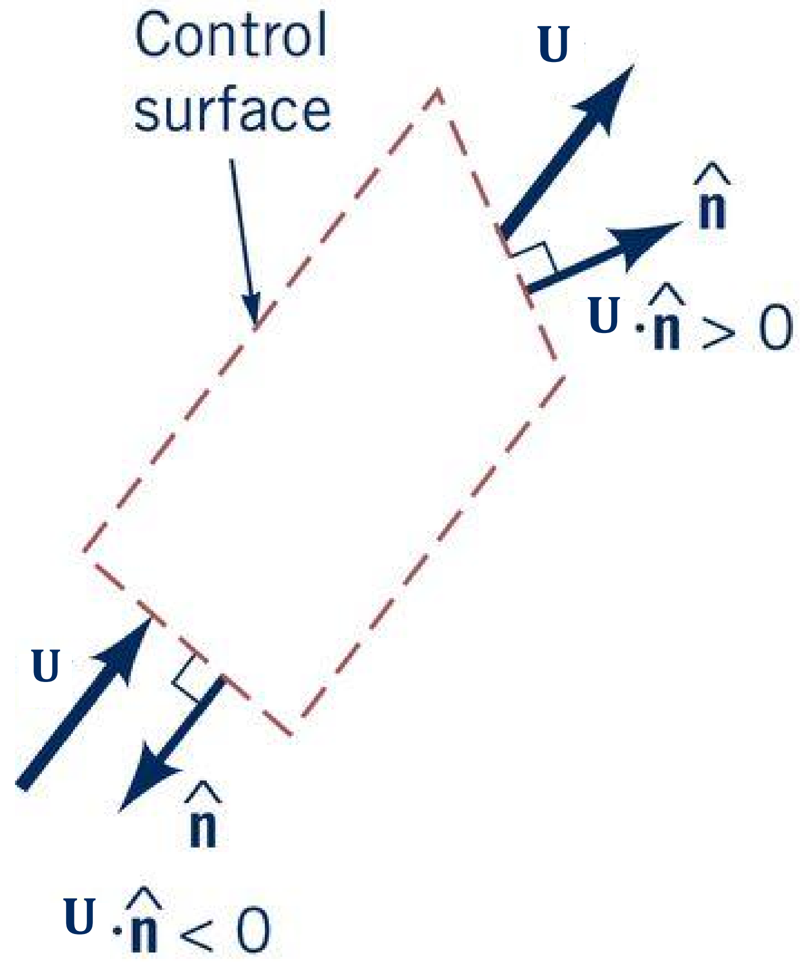
$$\frac{D}{Dt}\int_{\text{sys}} \rho \, dV = \frac{\partial}{\partial t}\int_{\text{cv}} \rho \, dV + \int_{\text{cs}} \rho \, \mathbf{U} \cdot \hat{\mathbf{n}} \, dA$$

The continuity equation for a fixed control volume:

$$\frac{\partial}{\partial t}\int_{\text{cv}} \rho \, dV + \int_{\text{cs}} \rho \, \mathbf{U} \cdot \hat{\mathbf{n}} \, dA = 0$$

The continuity equation is a statement that mass is conserved.

Conservation of mass: continuity equation



When all of the differential quantities, $\rho\, \mathbf{U} \cdot \hat{\mathbf{n}}$, are summed over the entire control surface:

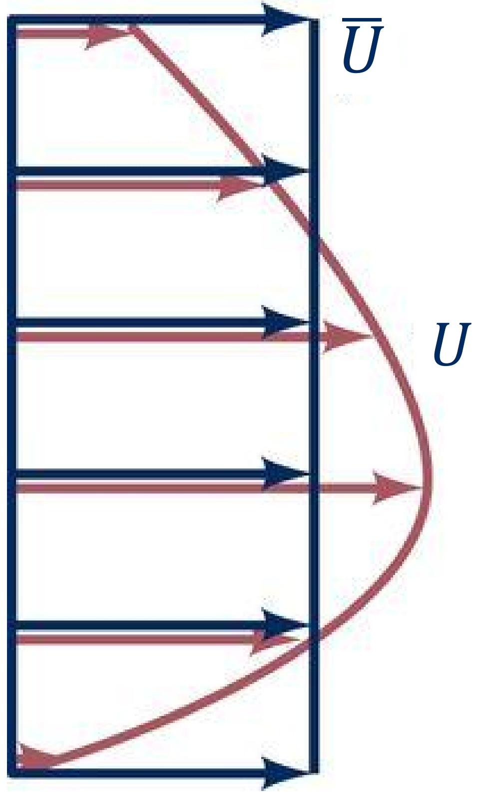
$$\int_{\text{cs}} \rho\, \mathbf{U} \cdot \hat{\mathbf{n}}\, dA$$

the result is the net mass flow rate through the control surface:

$$\int_{\text{cs}} \rho\, \mathbf{U} \cdot \hat{\mathbf{n}}\, dA = \sum \dot{m}_{\text{out}} - \sum \dot{m}_{\text{in}}$$

where $\dot{m}$ is the mass flow rate (kg/s).

Conservation of mass: mass flowrate



Mass flowrate is a useful quanitity following from the conservation of mass principles discussed:

$$\int_A \rho \, \mathbf{U} \cdot \hat{\mathbf{n}} \, dA = \dot{m}$$

We can using an average, or representative, velocity when appropriate, such that:

$$\dot{m} = \rho U A = \rho Q$$

Volume flowrate = Q

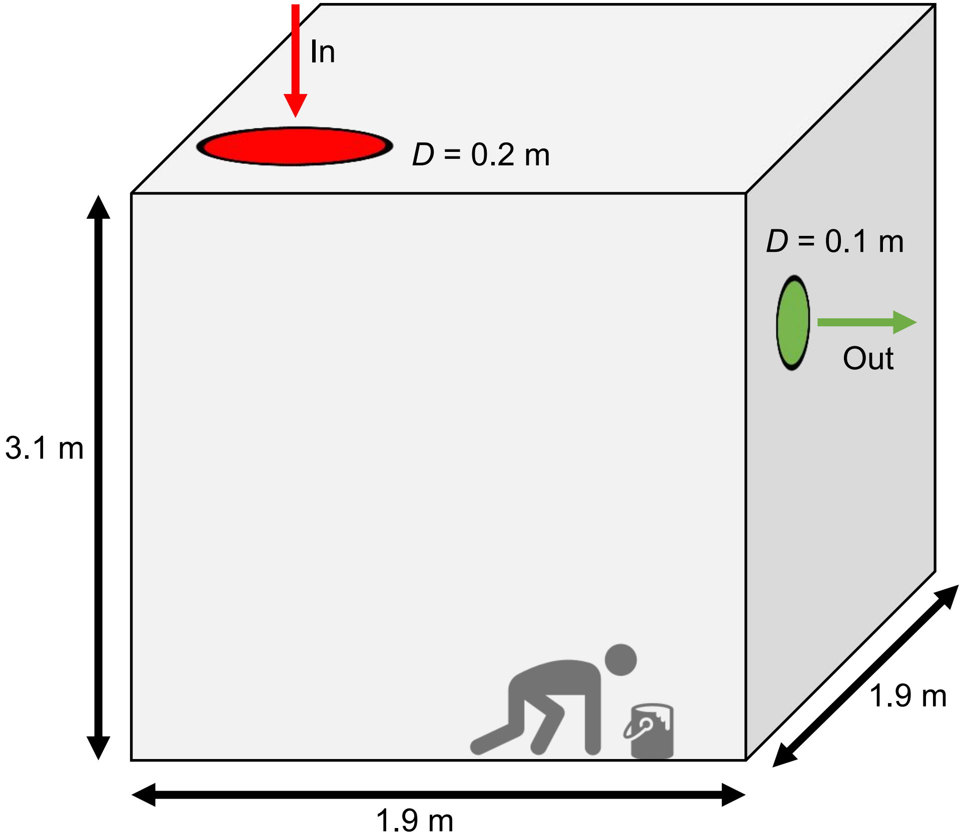
The average velocity, $\overline{U}$, is found through:

$$\overline{U} = \frac{\int_A \rho \, \mathbf{U} \cdot \hat{\mathbf{n}} \, dA}{\rho A} = U$$

If the velocity is uniformly distributed, then it is often just written as U.

Example 1 (steady flow, fixed control volume): A worker is performing maintenance in a small rectangular tank.
Fresh air enters (red hole), and exits (green hole). The flow is steady and incompressible. Find:

(a) the exchange rate needed (m³/kg) for the tank such that the air is replaced every 3 minutes.
(b) the velocity of air entering and exiting the tank at this exchange rate.



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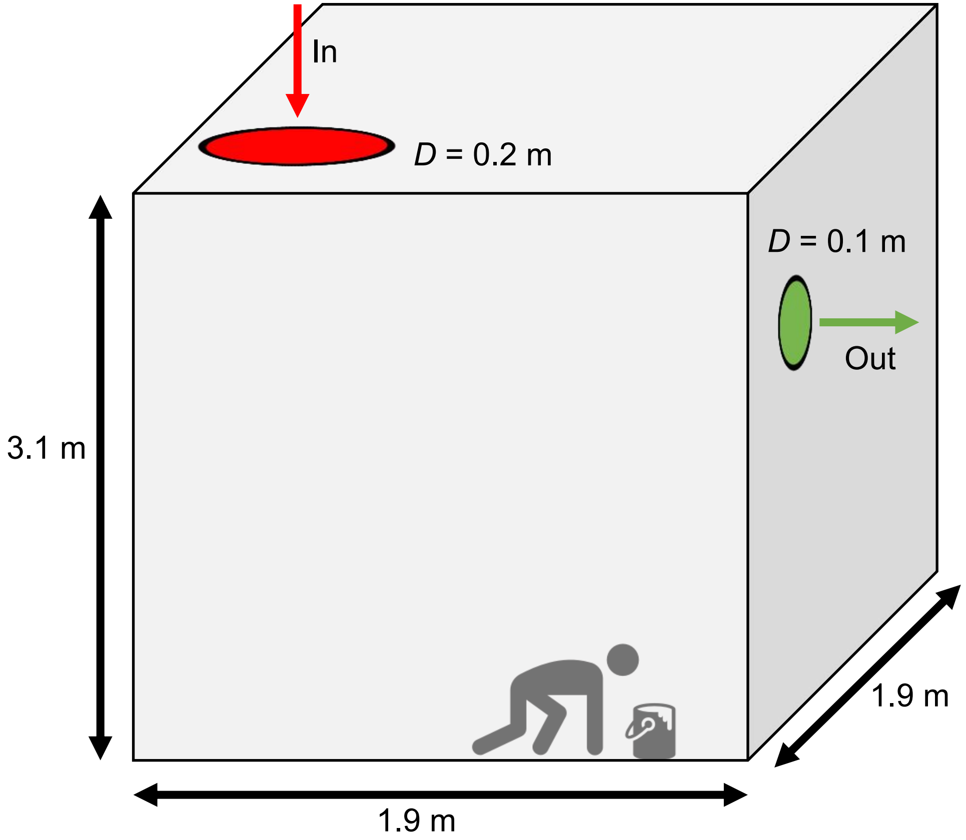
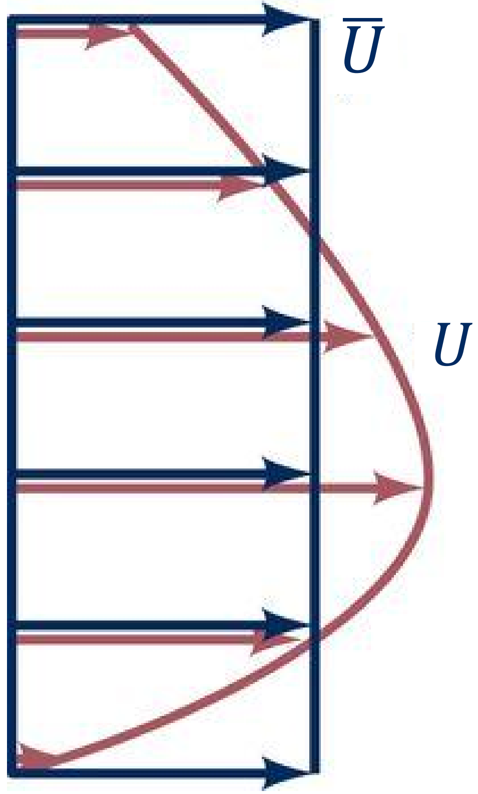


Conservation of mass: mass flowrate



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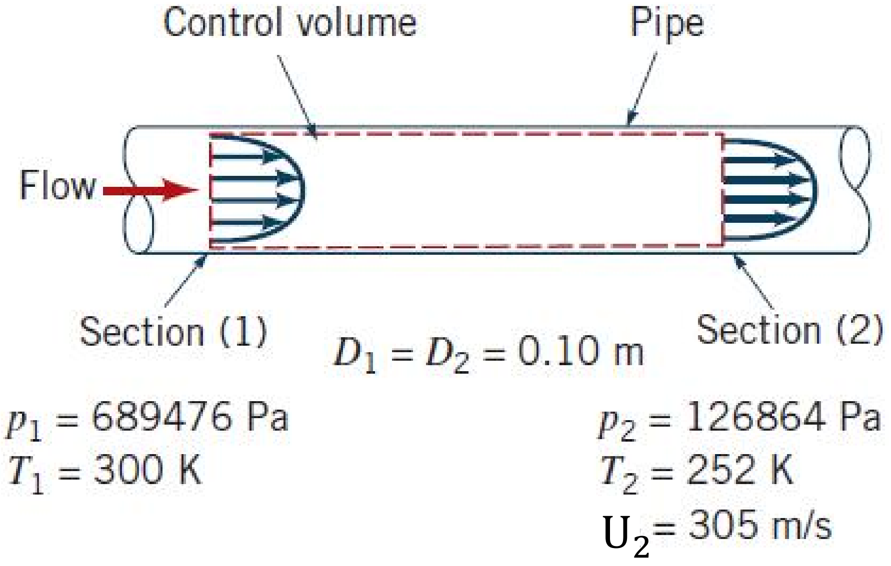
If the velocity is uniformly distributed, then it is often just written as U.

Example 2 (compressible flow, non-uniform velocity): Air flows steadily between two sections in a long straight pipe of diameter $D = 0.1$ m. The fluid pressure and temperature are uniformly distributed. The average air velocity (non-uniform distribution) at section (2) is 305 m/s.

Calculate the average air velocity at section (1).



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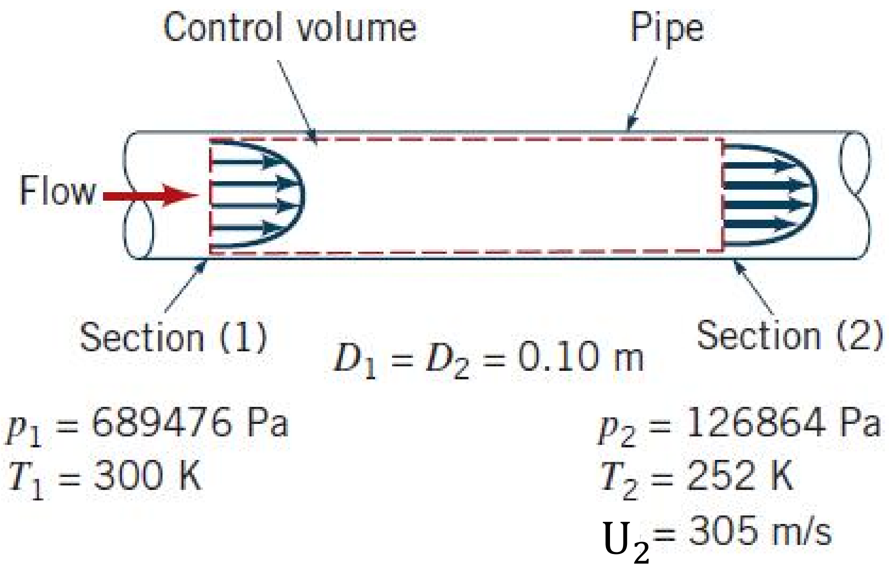
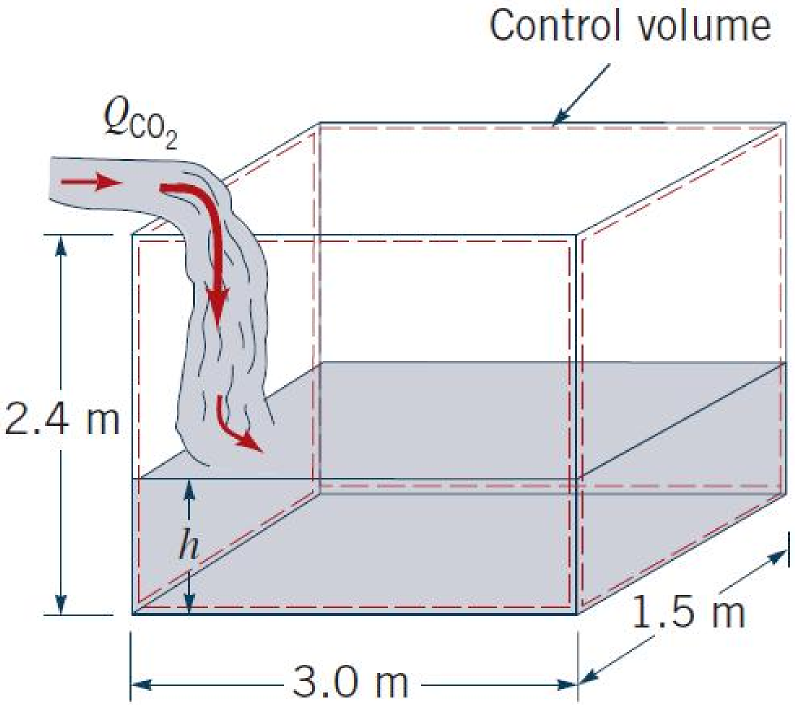
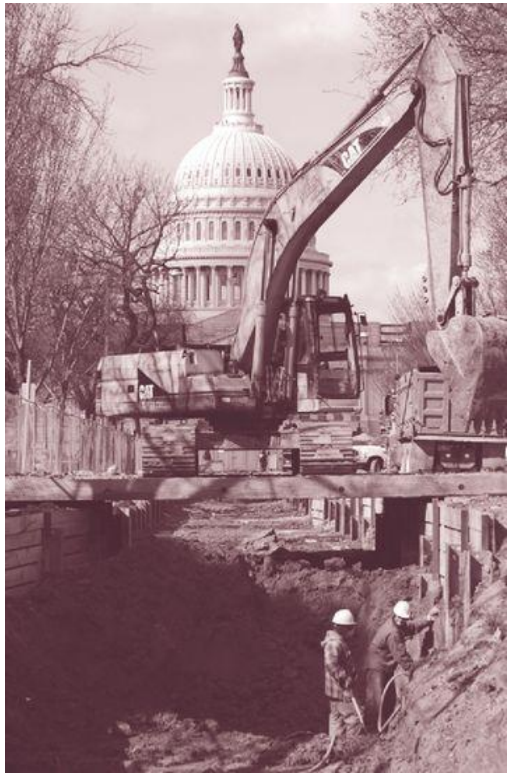


Example 3 (unsteady flow, fixed control volume): Construction workers in a trench are installing a new waterline. The trench is 3 m long, 1.5 m wide, and 2.4 m deep. The trench is near a road, so carbon dioxide from nearby vehicle exhausts enters the trench at a rate of 2.8 m³/min. Carbon dioxide has a greater density than oxygen, so it will settle at the bottom of the trench where the workers are. Assume that the oxygen and carbon dioxide do not mix.

(a) estimate the time rate of change of the depth of carbon dioxide in the trench, $\partial h/\partial t$, in meter per minute at any instant.
(b) Calculate the time, $t_{h=1.8}$, it would take for the level of carbon dioxide to reach a height of 1.8 m (the approximate height of the workers).





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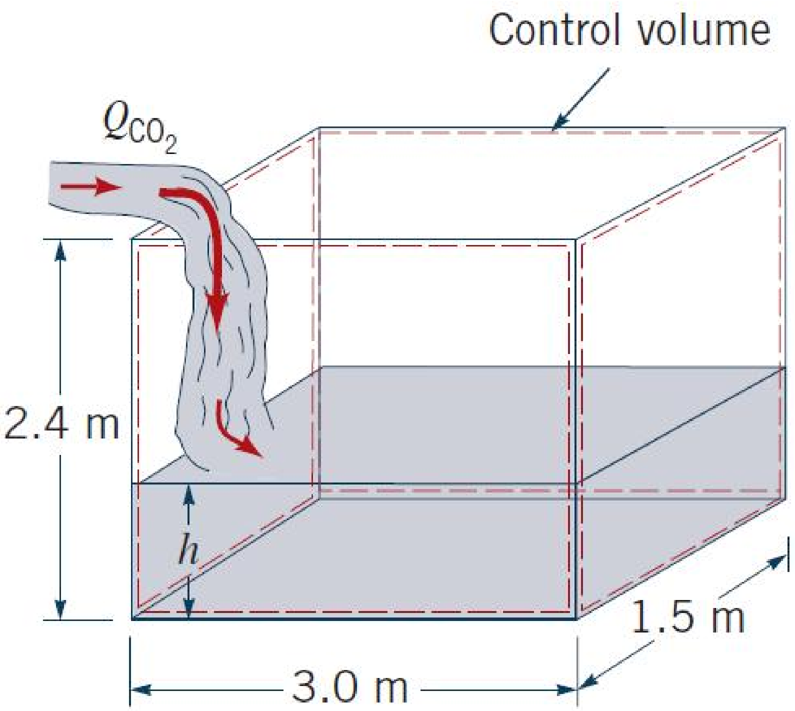

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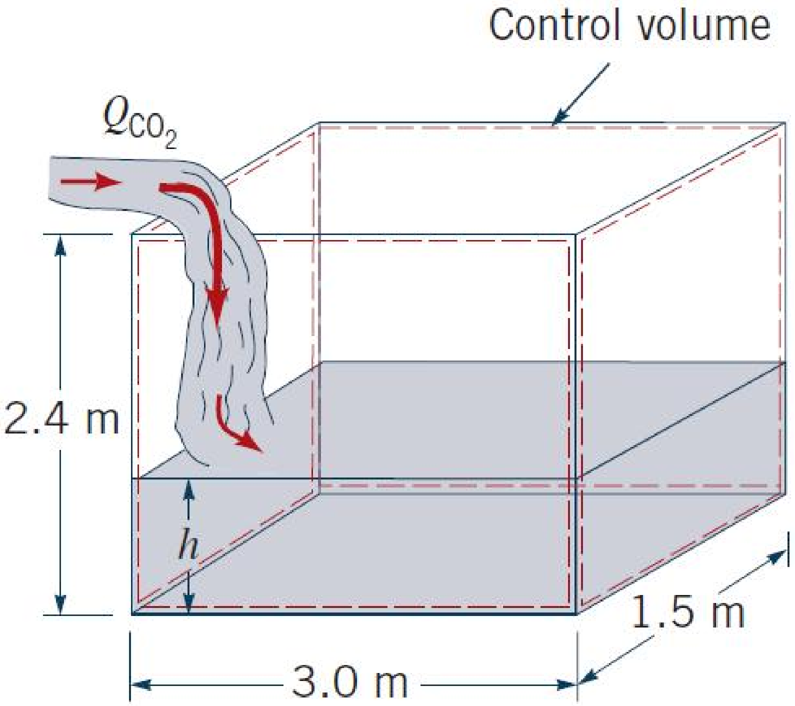


The UK has a fluid mechanics problem…

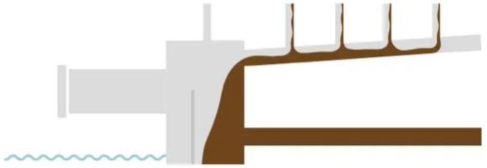
Record sewage spills into England rivers and seas in 2023

9 minutes ago BBC, 27 March 2024

By Esme Stallard and Jonah Fisher,
BBC News Climate and Science

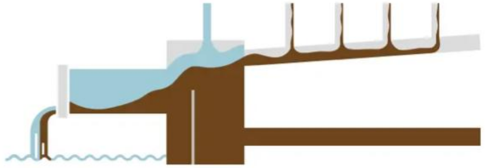
Normal operation

When the system is functioning normally, wastewater flows smoothly through the system to the sewage treatment plant.



Legal spill

Water companies are allowed to spill untreated wastewater under certain conditions, such as heavy or prolonged rainfall.



Source: Getty

BBC

Outrage over record discharges of sewage into rivers and seas

Demand for urgent action over water firms' failure to tackle problem

Sandra Laville
Helena Horton
Alex Clark

Water companies in England have faced a barrage of criticism after data revealed raw sewage was discharged for more than 3.6m hours into rivers and seas last year, more than double the hours in the previous 12 months.

The data made 2023 the worst year for storm water pollution. Early figures seen by the Guardian put the total at more than 4m hours, but officials said this was an early estimate.

The Liberal Democrat leader, Ed Davey, said the scandal of raw sewage pouring into waterways should be declared a national environmental emergency. He called on the government to convene an urgent meeting of the Scientific Advisory Group for Emergencies (Sage) to look at the impact on people's health.

Total discharges from the 14,000 storm overflows owned by English water companies that release untreated sewage into rivers and coastal waters were up 54% to 464,056, according to industry data submitted to the Environment Agency. Senior industry figures highlighted the heavy 6

▲ *Cambridge University rowers train on the Thames. Rowers have been warned not to enter the water after Saturday's Boat Races over E coli fears*

"…national environmental emergency."

Deforming control volume

Before: fixed → $\frac{D}{Dt}\int_{\mathrm{sys}} \rho \, dV = \frac{\partial}{\partial t}\int_{\mathrm{cv}} \rho \, dV + \int_{\mathrm{cs}} \rho \, \mathbf{U} \cdot \hat{\mathbf{n}} \, dA$

After: deforming → $\frac{D}{Dt}\int_{\mathrm{sys}} \rho \, dV = \frac{\partial}{\partial t}\int_{\mathrm{cv}} \rho \, dV + \int_{\mathrm{cs}} \rho \, \mathbf{W} \cdot \hat{\mathbf{n}} \, dA$

$\frac{\partial}{\partial t}\int_{\mathrm{cv}} \rho \, dV$ is usually non-zero.

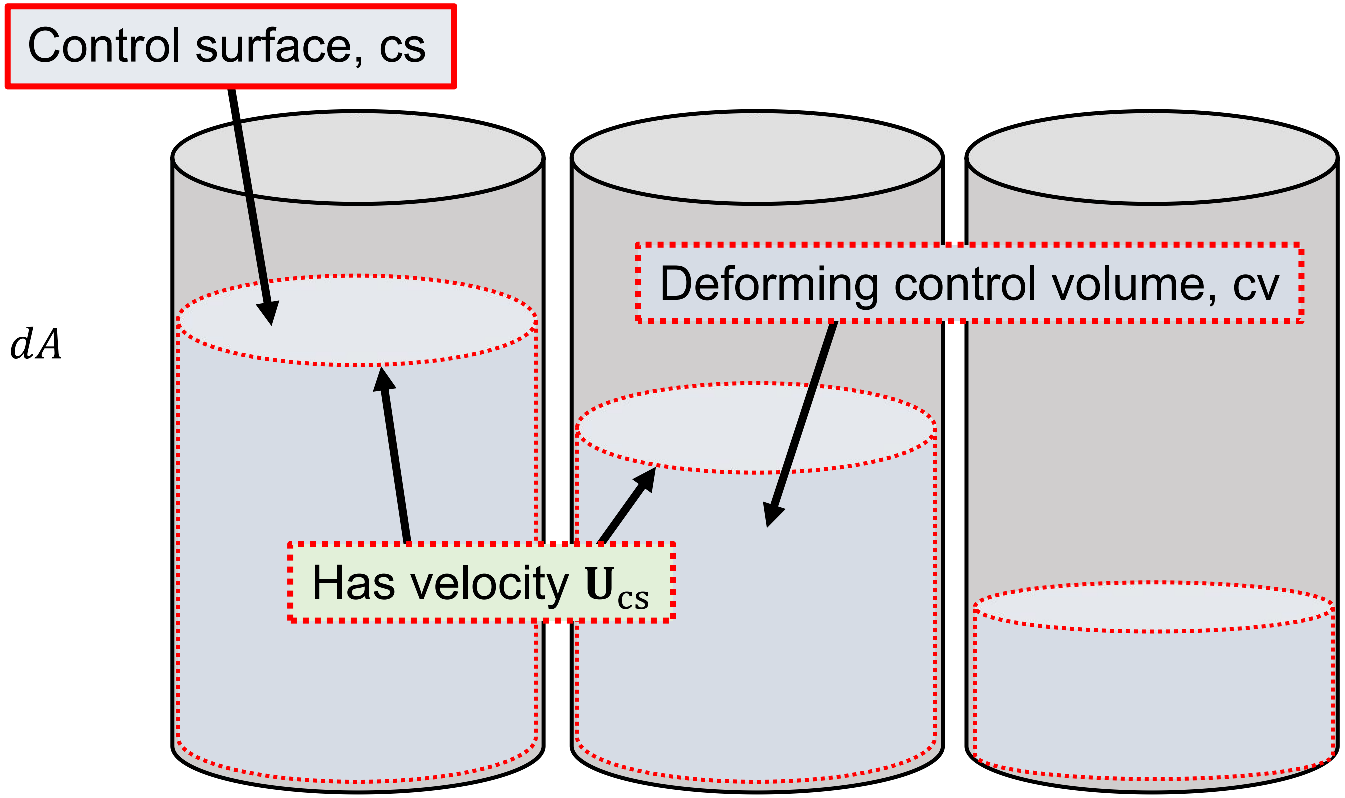
The mass flowrate term requires a relative velocity, $\mathbf{W}$:

$$\int_{\mathrm{cs}} \rho \, \mathbf{W} \cdot \hat{\mathbf{n}} \, dA$$

This is defined relative to the velocity of the control surface.

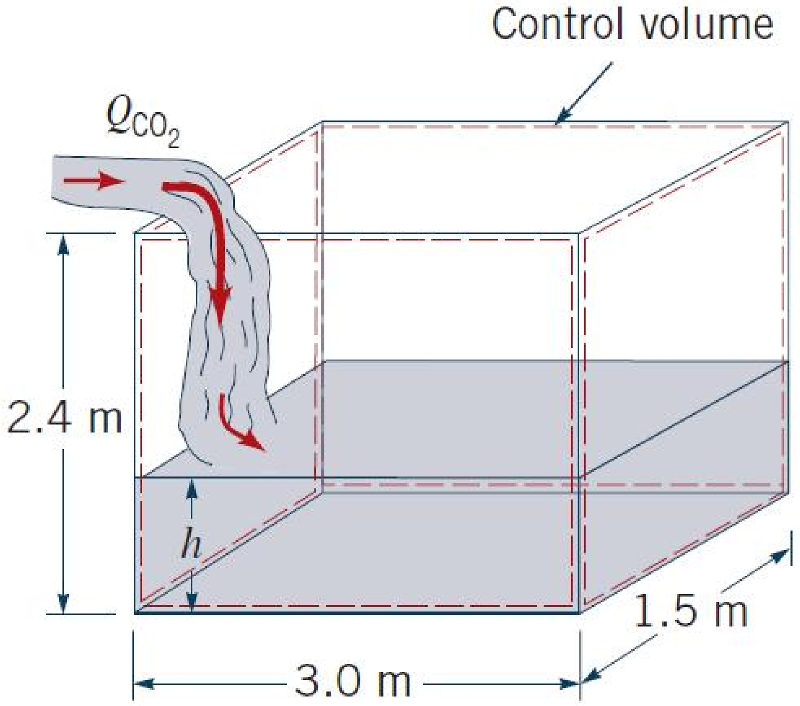
$$\mathbf{U} = \mathbf{W} + \mathbf{U}_{\mathrm{cs}}$$

The velocity of the surface of a deforming control volume is not the same at all points on the surface.



For a deforming control volume, the flowrate is calculated using the velocity relative to the control volume.

Example 4 (same as Example 3, but using unsteady flow, deforming control volume): Estimate the time rate of change of the depth of carbon dioxide in the trench, $\partial h/\partial t$, in meter per minute at any instant.



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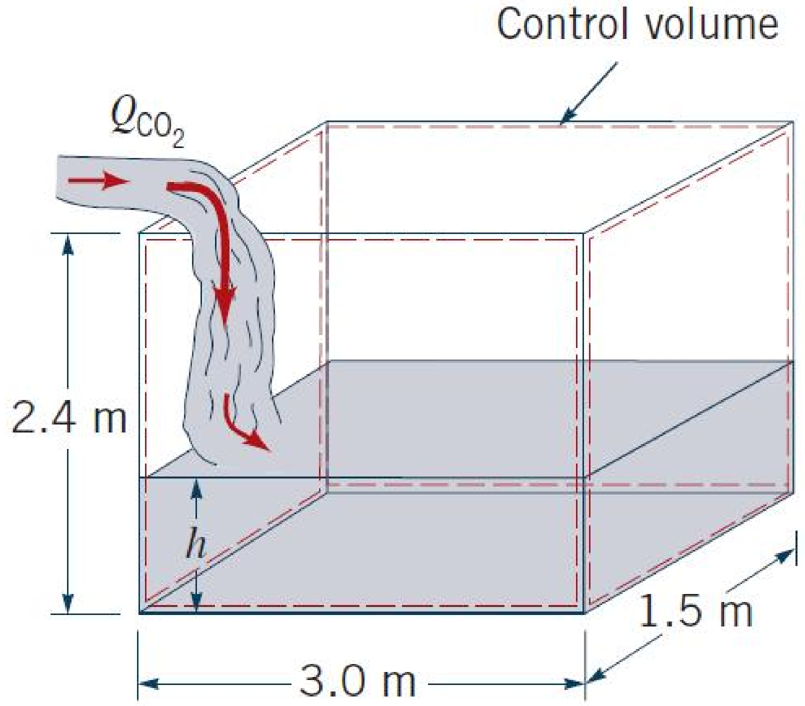
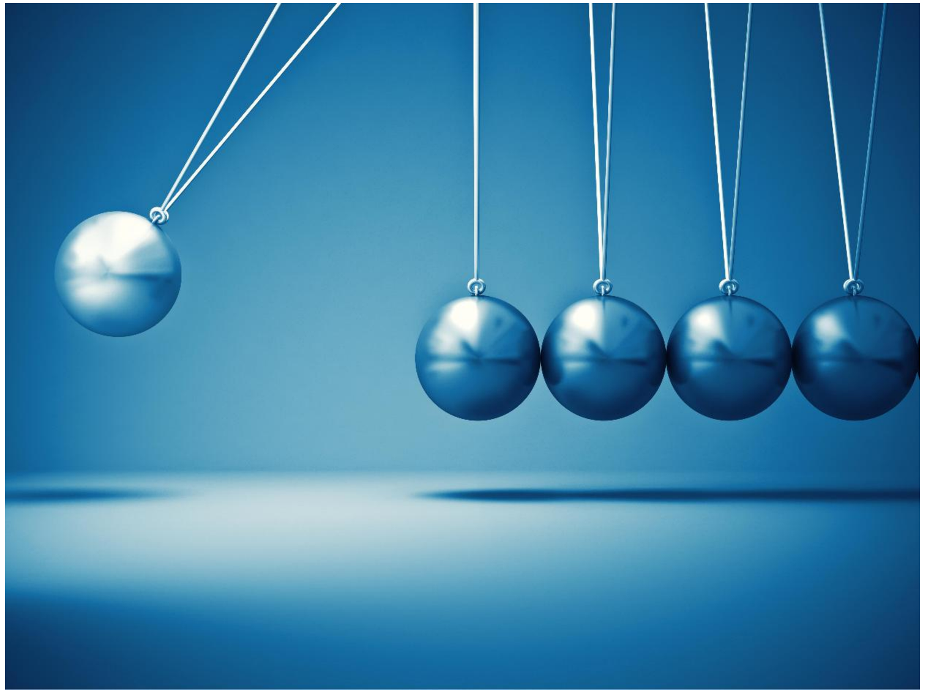


Newton’s second law: linear momentum equation

What is linear momentum?

Linear momentum is the product of a systems’ mass and its velocity.



Mathematically, this is written as:

$$\textit{Linear momentum} = m\,\mathbf{U} = \mathbf{U}\rho\, dV$$

Newton’s second law states that:

Time rate of change of the linear momentum in the system	=	Sum of the external forces acting on the system

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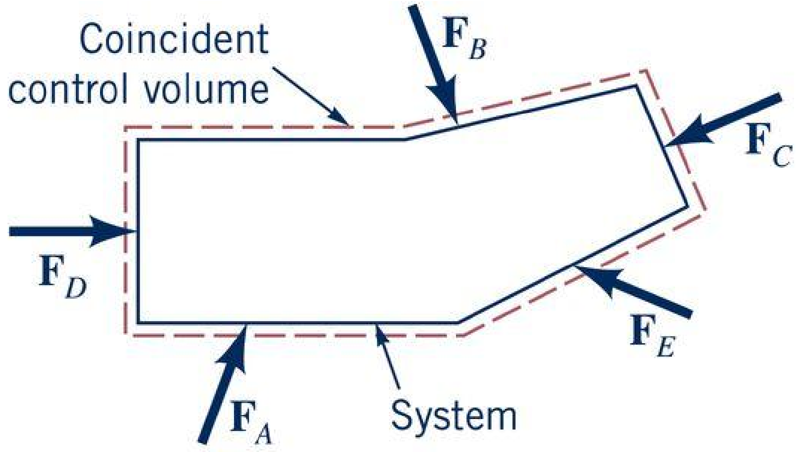
Newton's second law: linear momentum equation

Time rate of change of the linear momentum in the system = Sum of the external forces acting on the system

$$\frac{D}{Dt}\int_{\text{sys}} \mathbf{U}\,\rho\, dV = \sum \mathbf{F}_{\text{sys}}$$

Assume the control volume is coincident with the system, i.e., they are the same:

$$\sum \mathbf{F}_{\text{sys}} = \sum \mathbf{F}_{\substack{\text{contents of the}\\ \text{control volume}}}$$



Newton's second law: linear momentum equation

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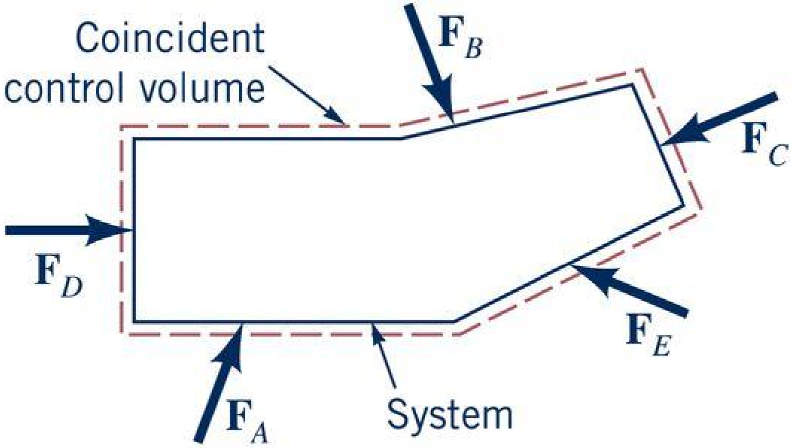

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$$\sum \mathbf{F}_{\text{sys}} = \sum \mathbf{F}_{\substack{\text{contents of the}\\ \text{control volume}}}$$

Recall the Reynolds Transport Theorem (from Dariusz's lecture, Chapter 3):

$$\frac{D\mathrm{B}_{\text{sys}}}{Dt} = \frac{\partial}{\partial t}\int_{\text{cv}} \rho\, b\, dV + \int_{\text{cs}} \rho b\, \mathbf{U}\cdot\hat{\mathbf{n}}\, dA$$

Substituting B_{sys} as the momentum of the system, and b as the velocity:

$$\frac{D}{Dt}\int_{\text{sys}} \mathbf{U}\,\rho\,dV = \frac{\partial}{\partial t}\int_{\text{cv}} \mathbf{U}\,\rho\,dV + \int_{\text{cs}} \mathbf{U}\,\rho\,\mathbf{U}\cdot\hat{\mathbf{n}}\,dA$$

Forces acting on a flowing fluid can change its velocity magnitude and/or direction.

Newton's second law: linear momentum equation

$$\underbrace{\frac{D}{Dt}\int_{\text{sys}} \mathbf{U}\,\rho\, dV}_{\text{Time rate of change of the linear momentum in the system}} = \underbrace{\frac{\partial}{\partial t}\int_{\text{cv}} \mathbf{U}\,\rho\, dV}_{\text{Time rate of change of the linear momentum of the contents in the control volume}} + \underbrace{\int_{\text{cs}} \mathbf{U}\,\rho\, \mathbf{U}\cdot\hat{\mathbf{n}}\, dA}_{\text{Net rate of flow of linear momentum through the control surface}}$$

$$\overbrace{\frac{D}{Dt}\int_{\text{sys}} \mathbf{U}\,\rho\, dV} = \underbrace{\sum \mathbf{F}_{\text{sys}}}$$

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Newton’s second law: linear momentum equation

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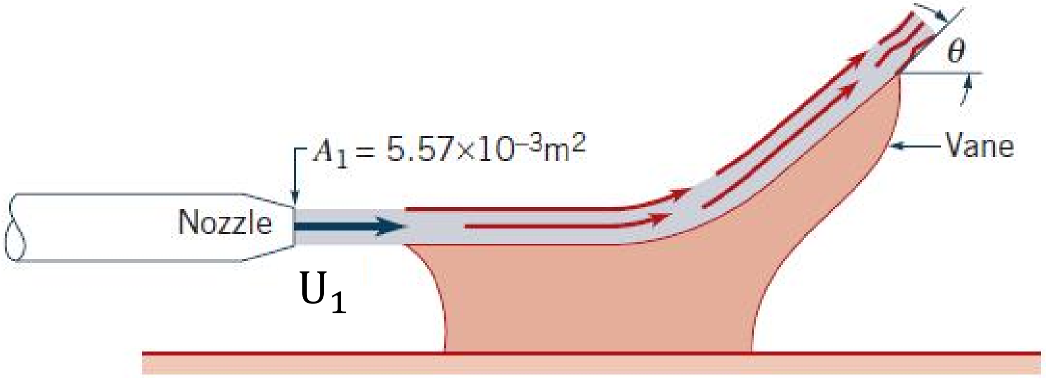
$$\sum \mathbf{F}_{\text{sys}} = \sum \mathbf{F}_{\substack{\text{contents of the}\\ \text{control volume}}}$$

The linear momentum equation for a fixed control volume is then given by:

$$\frac{\partial}{\partial t}\int_{\text{cv}} \mathbf{U}\,\rho\,dV + \int_{\text{cs}} \mathbf{U}\,\rho\,\mathbf{U}\cdot\hat{\mathbf{n}}\,dA = \sum \mathbf{F}_{\substack{\text{contents of the}\\ \text{control volume}}}$$

Example 5 (change in flow direction): A horizontal jet of water exits a nozzle with a uniform speed of $V_1 = 3.1\ \mathrm{m}/s$, strikes a vane, and is turned through an angle, θ.

Determine the anchoring force needed to hold the vane stationary if gravity and viscous effects are negligible.



(a)

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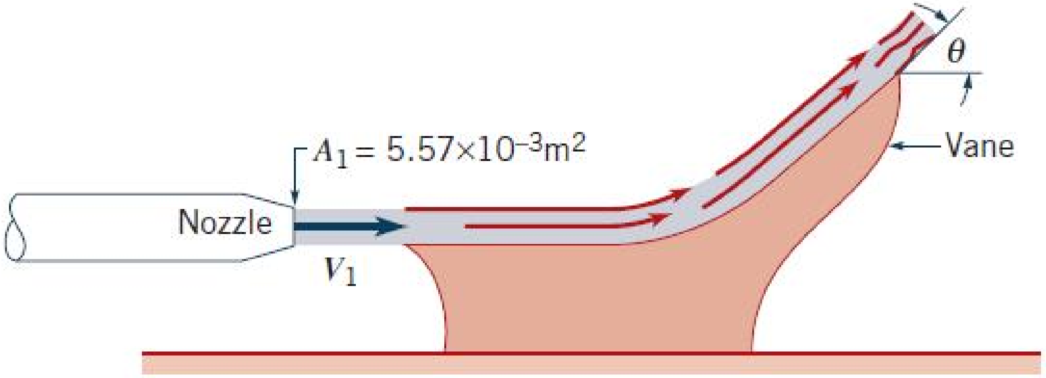

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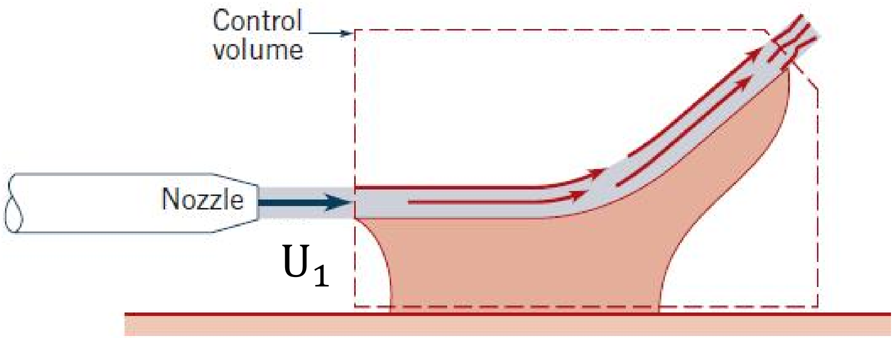

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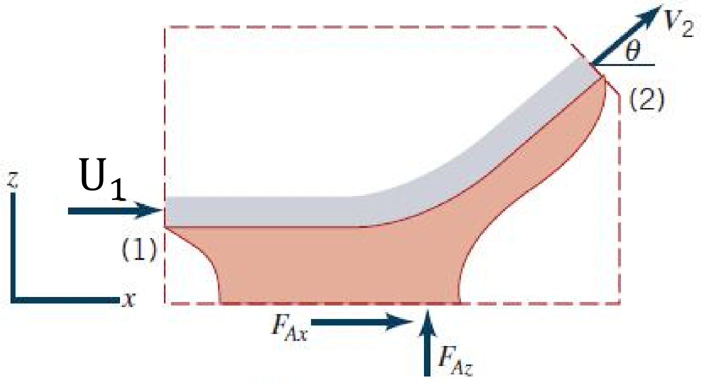
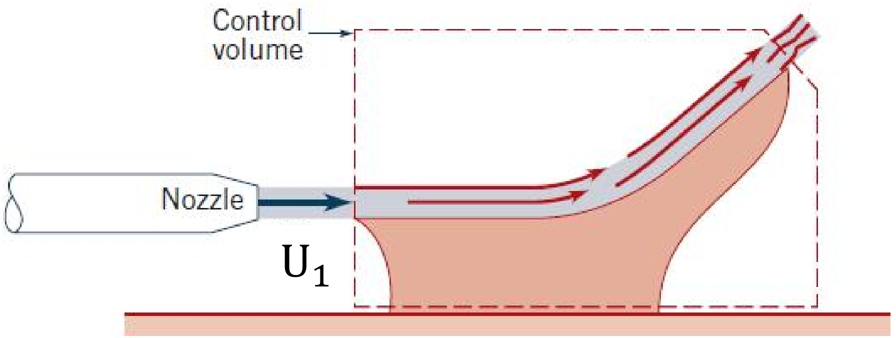


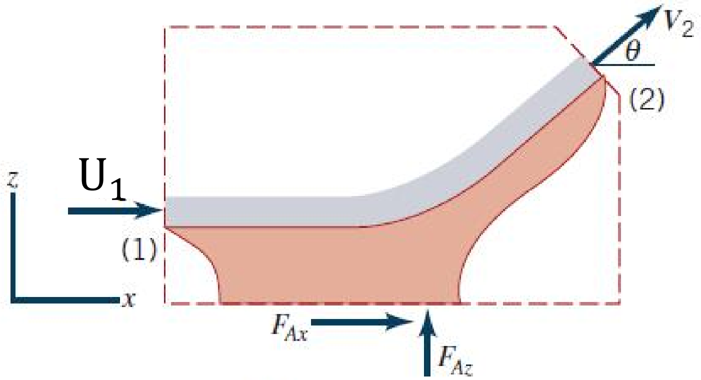
(c)

Example 5 (change in flow direction): A horizontal jet of water exits a nozzle with a uniform speed of $V_1 = 3.1\ \mathrm{m/s}$, strikes a vane, and is turned through an angle, θ.

Determine the anchoring force needed to hold the vane stationary if gravity and viscous effects are negligible.

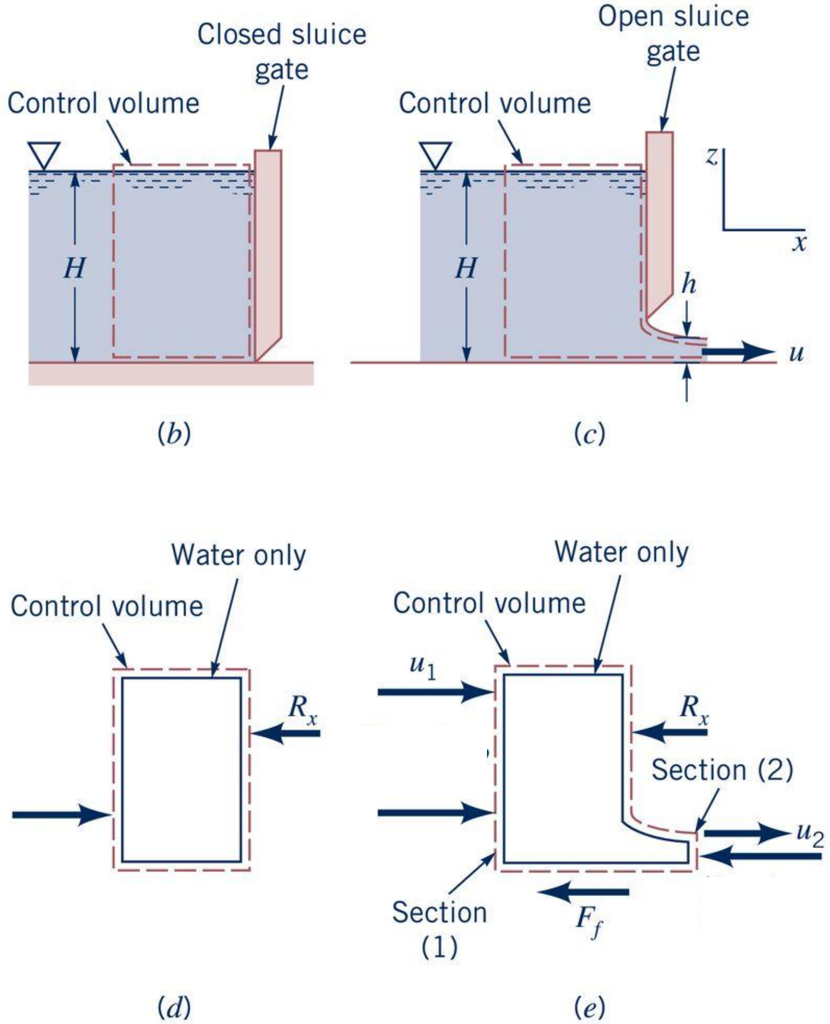


(b)

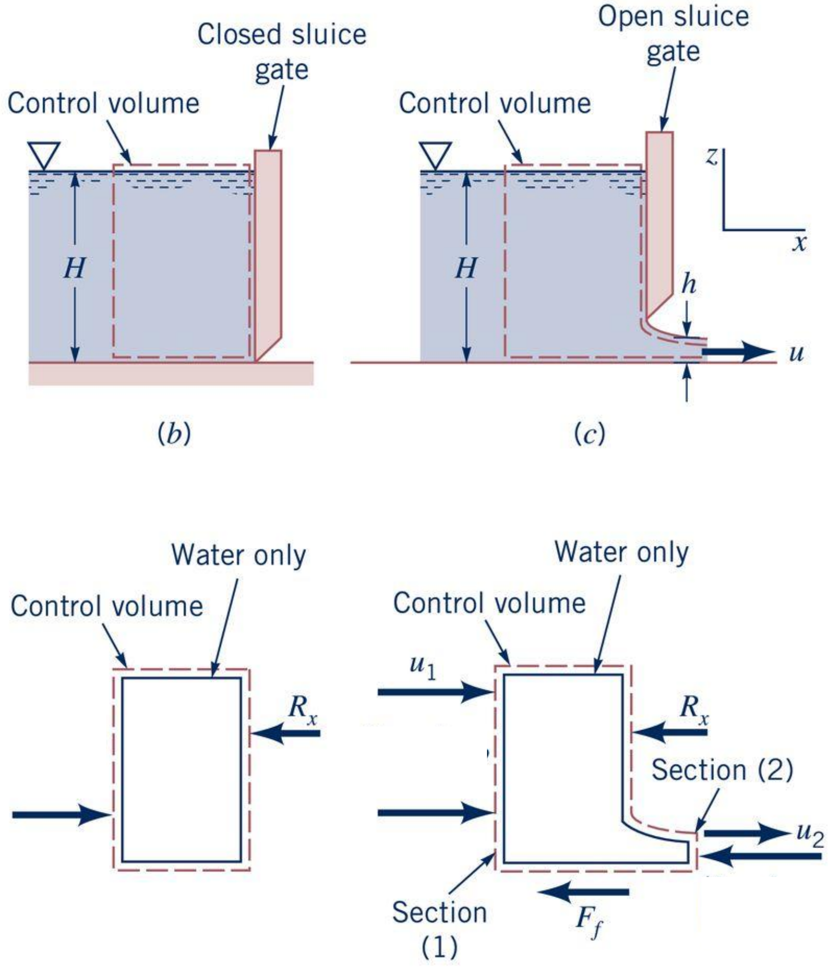


(c)

Example 6 (non-uniform pressure): A sluice gate across a channel of width b is shown in open and closed positions, Figs. (b) and (c), respectively. Determine the horizontal reaction forces R_x when the gate is open and closed.



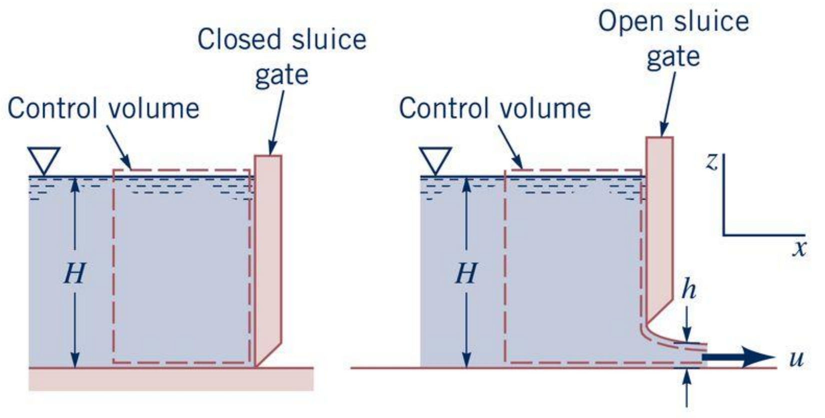
Example 6 (non-uniform pressure): A sluice gate across a channel of width b is shown in open and closed positions, Figs. (b) and (c), respectively. Determine the horizontal reaction forces R_x when the gate is open and closed.



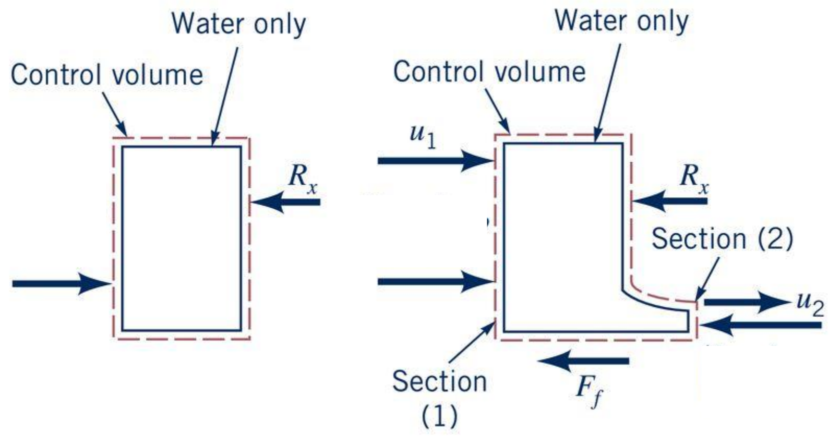
(b) (c)

(d) (e)

Example 6 (non-uniform pressure): A sluice gate across a channel of width b is shown in open and closed positions, Figs. (b) and (c), respectively. Determine the horizontal reaction forces R_x when the gate is open and closed.



(b) (c)

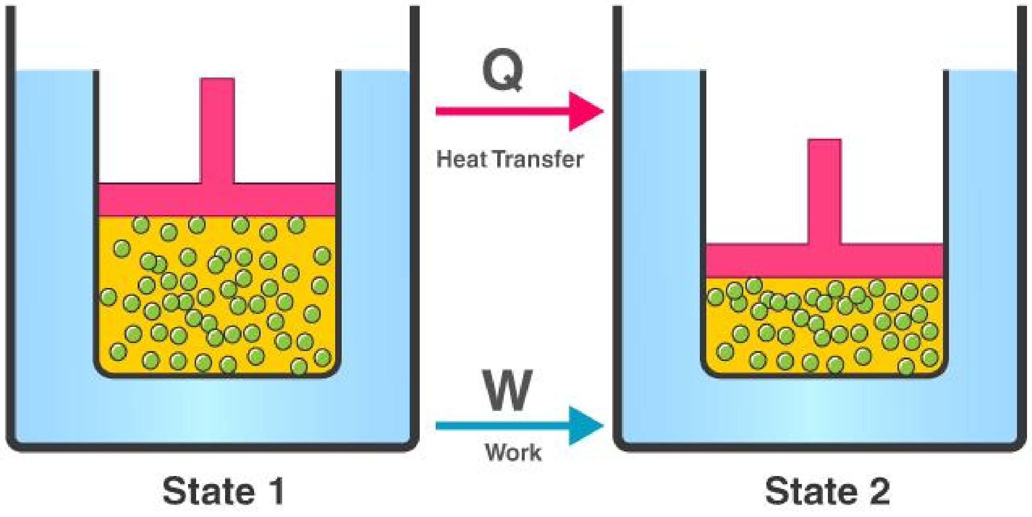


(d) (e)

Fluids and energy

There is a relationship between the energy stored in a fluid and the rate of heat transfer and work done.

First law of thermodynamics: the energy equation

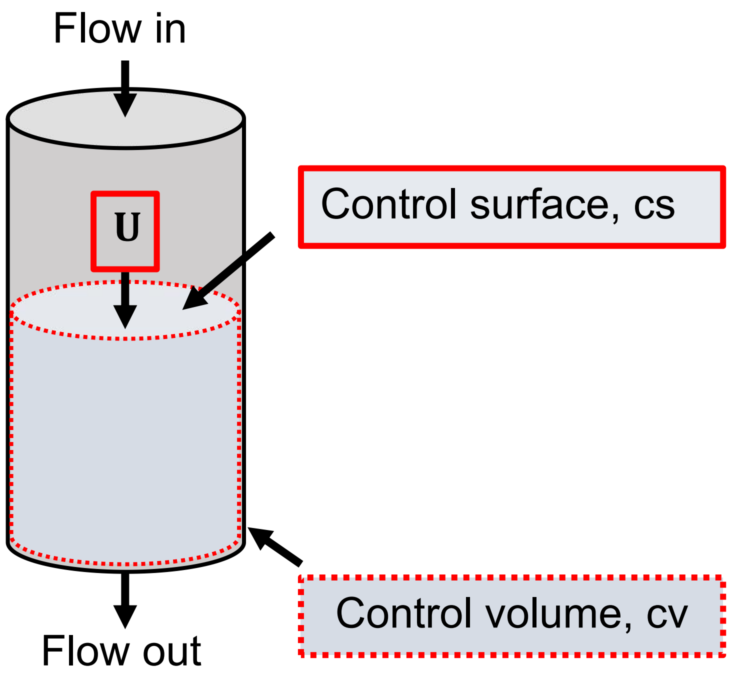


$$\underbrace{\frac{D}{Dt}\int_{\mathrm{sys}} e\,\rho\,dV}_{\text{Time rate of increase of the total energy stored in the system}} = \underbrace{\left(\sum \dot{Q}_{\mathrm{in}} - \sum \dot{Q}_{\mathrm{out}}\right)_{\mathrm{sys}}}_{\text{Net time rate of energy addition by heat transfer into the system}} + \underbrace{\left(\sum \dot{W}_{\mathrm{in}} - \sum \dot{W}_{\mathrm{out}}\right)_{\mathrm{sys}}}_{\text{Net time rate of energy addition by work transfer into the system}}$$

The first law of thermodynamics is a statement of conversation of energy.

Key concepts

Control volume: can be part of the system, or the same as the system (coincident).



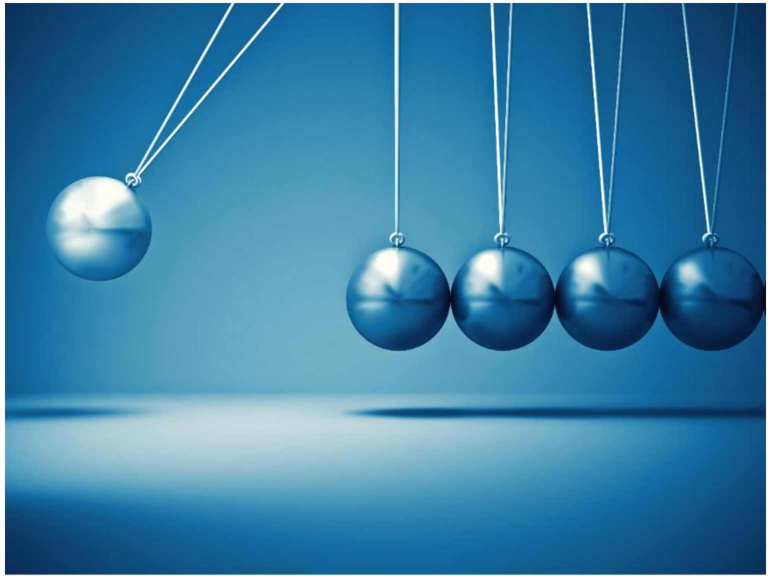
Conservation Laws

Conservation of mass: mass cannot be created or destroyed within a system.

Conservation of linear momentum

Conservation of energy

(no calculations required in this course)

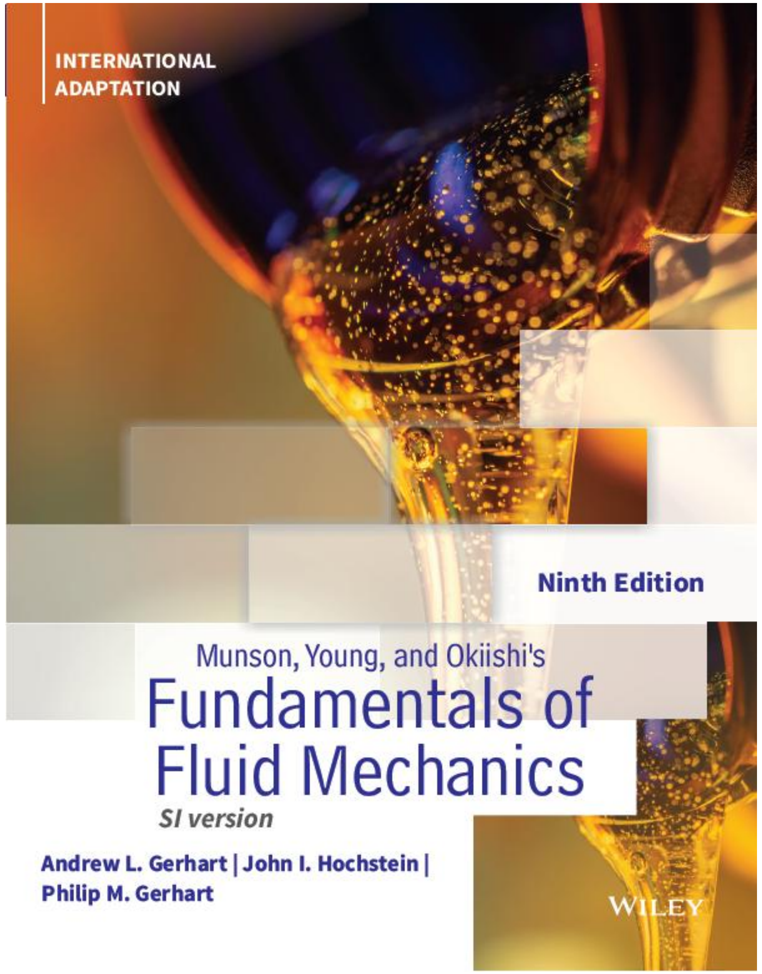


Deeper reading from textbook

5.1.1 Derivation of the Continuity Equation
5.1.2 Fixed, Nonderforming Control Volume
5.1.4 Deforming Control Volume

5.2.1 Derivtion of the Linear Momentum Equation
5.2.2 Application of the Linear Momentum Equation

5.3.1 Derivation of the Engery Equation



Key equations

Mass is conserved within a system: $\frac{DM_{\text{sys}}}{Dt} = 0$

Conservation of mass: $\frac{\partial}{\partial t}\int_{\text{cv}} \rho \, dV + \int_{\text{cs}} \rho \, \mathbf{U} \cdot \hat{\mathbf{n}} \, dA = 0$

Mass flowrate: $\dot{m} = \rho U A = \rho Q$

Average velocity: $\bar{U} = \frac{\int_A \rho \, \mathbf{U} \cdot \hat{\mathbf{n}} \, dA}{\rho A}$

Steady-flow mass conservation: $\sum \dot{m}_{\text{out}} - \sum \dot{m}_{\text{in}}$

Force related to change in linear momentum: $\frac{\partial}{\partial t}\int_{\text{cv}} \mathbf{U} \, \rho \, dV + \int_{\text{cs}} \mathbf{U} \, \rho \, \mathbf{U} \cdot \hat{\mathbf{n}} \, dA = \sum \mathbf{F}_{\substack{\text{contents of the} \\ \text{control volume}}}$

Conservation of energy (*learn concept*): $\frac{D}{Dt}\int_{\text{sys}} e \, \rho \, dV = \left(\sum \dot{Q}_{\text{in}} - \sum \dot{Q}_{\text{out}}\right)_{\text{sys}} + \left(\sum \dot{W}_{\text{in}} - \sum \dot{W}_{\text{out}}\right)_{\text{sys}}$