Chapter 13 Exam

Calculus III

Fall 2022

Name: KEY

1. Find $f_y(x, y)$ for $f(x, y) = e^{2xy}(-\cos x \sin y)$.

$$f_y(x,y) = \frac{\partial}{\partial y}\left[e^{2xy}\right](-\cos x \sin y) + e^{2xy}\frac{\partial}{\partial y}\left[-\cos x \sin y\right]$$

$$= 2xe^{2xy}(-\cos x \sin y) + e^{2xy}(-\cos x \cos y)$$

$$= (-\cos x)e^{2xy}(2x \sin y + \cos y)$$

2. Find $\frac{\partial z}{\partial x}$ if $3x^2 - (3y^2 + z) = 0$

$$f(x,y,z) = 3x^2 - (3y^2 + z)$$

$$\frac{\partial z}{\partial x} = \frac{-f_x(x,y,z)}{f_z(x,y,z)}$$

$f_x = 6x$ $\qquad$ $f_z = -1$

$$\frac{\partial z}{\partial x} = 6x$$

3. Find $f_{xx}(x, y)$ for $f(x, y) = \frac{4x^2}{y} + \frac{y^2}{2x}$.

$$f_x(x,y) = \frac{8x}{y} - \frac{y^2}{2x^2} = \frac{8x}{y} - \frac{1}{2}y^2x^{-2}$$

$$f_{xx}(x,y) = \frac{8}{y} + \frac{y^2}{x^3}$$

4. Let $f(x, y) = -2x^3e^{-y}$. Find ∇f at the point $(-2, 0)$.

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$$

$$f_x = -6x^2e^{-y}$$

$$f_y = 2x^3e^{-y}$$

$$\nabla f(x,y) = \langle -6x^2e^{-y}, 2x^3e^{-y} \rangle$$

$$\nabla f(-2,0) = \langle -6(-2)^2e^0, 2(-2)^3e^0 \rangle$$

$$= \langle -24, -16 \rangle$$

5. Calculate the directional derivative of $f(x,y) = -2x^3e^{-y}$. at the point $(-2,0)$ in the direction of $\vec{v} = 3\mathbf{i} - \mathbf{j}$.

Need unit vector for

$\vec{v} = \langle 3,-1 \rangle$

$\|\vec{v}\| = \sqrt{10}$

$\hat{u} = \langle \frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}} \rangle$

$D_{\hat{u}} f(x,y) = \nabla f \cdot \hat{u}$

$f(x,y) = -2x^3e^{-y}$

$f_x = -6x^2e^{-y}$

$f_y = 2x^3e^{-y}$

$\nabla f(x,y) = \langle -6x^2e^{-y}, 2x^3e^{-y} \rangle$

$\nabla f(-2,0) = \langle -24, -16 \rangle$

$$D_{\hat{u}} f(-2,0) = \langle -24,-16 \rangle \cdot \langle \frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}} \rangle$$

$$= -\frac{72}{\sqrt{10}} + \frac{16}{\sqrt{10}}$$

$$= \frac{-56}{\sqrt{10}}$$

$$= \frac{-28\sqrt{10}}{5}$$

6. For the function $f(x, y, z) = -z^2 e^{3xy}$, find the maximum value of the directional derivative at the point $(-2, 1, 3)$.

$$\text{Max}\left(D_{\hat{u}} f(x,y,z)\right) = \|\nabla f(x,y,z)\|$$

$$f_x = -3yz^2 e^{3xy} \to f_x(-2,1,3) = -3(1)(3)^2 e^{-6} = -27e^{-6}$$

$$f_y = -3xz^2 e^{3xy} \to f_y(-2,1,3) = 54e^{-6}$$

$$f_z = -2ze^{3xy} \to f_z(-2,1,3) = -6e^{-6}$$

$$\nabla f(-2,1,3) = \langle -27e^{-6}, 54e^{-6}, -6e^{-6} \rangle$$

$$\|\nabla f(-2,1,3)\| = e^{-6}\sqrt{\cancel{3528}}$$

$$3681$$

$$\approx \cancel{0.1422}$$

$$\approx 0.1504$$

7. Find an equation for the tangent plane to the surface given by $f(x, y) = x^2y^3$ at the point $(6, 7, 10)$.

Need normal and point.

Normal to tangent plane is ∇f.

$f(x,y,z) = x^2y^3 - z$

$f_x = 2xy^3 \quad f_y = 3x^2y^2 \quad f_z = -1$

$\nabla f(x,y,z) = \langle 2xy^3, 3x^2y^2, -1 \rangle$

$\nabla f(6,7,10) = \langle 2(6)(7)^3, 3(6)^2(7)^2, -1 \rangle$

$= \langle 4116, 5292, -1 \rangle$

Eq. of tangent plane:

$4116(x-6) + 5292(y-7) - (z-10) = 0$

8. Find the saddle point for $f(x, y) = x^2 - y^2 - 2x - 6y - 3$.

$f_x(x,y) = 2x - 2$ $\quad$ $f_{xx}(x,y) = 2$

$f_y(x,y) = -2y - 6$ $\quad$ $f_{yy}(x,y) = -2$

$f_{xy}(x,y) = 0$

$$\begin{cases} 2x - 2 = 0 \\ -2y - 6 = 0 \end{cases} \rightarrow \begin{matrix} x = 1 \\ y = -3 \end{matrix} \quad \boxed{CP: (1, -3)}$$

$D(x,y) = f_{xx} \cdot f_{yy} - [f_{xy}]^2$

$= -4$

$\therefore D(1,-3) = -4 \rightarrow$ Saddle

9. Use Lagrange Multiplies to maximize $f(x, y, z) = 4x^2 + y^2 + z^2$ with the constraint that $2x - y + z = 4$.

$f(x,y,z) = 4x^2 + y^2 + z^2$

$g(x,y,z) = 2x - y + z$

$f_x = 8x \qquad g_x = 2$

$f_y = 2y \qquad g_y = -1$

$f_z = 2z \qquad g_z = 1$

$\nabla f(x,y,z) = \lambda \cdot g(x,y,z)$

$\langle 8x, 2y, 2z \rangle = \lambda \langle 2, -1, 1 \rangle$

$8x = 2\lambda \qquad 2y = -\lambda \qquad 2z = 1\lambda$

$x = \frac{\lambda}{4} \qquad y = -\frac{\lambda}{2} \qquad z = \frac{\lambda}{2}$

$2x - y + z = 4 \implies 2\left(\frac{\lambda}{4}\right) + \frac{\lambda}{2} + \frac{\lambda}{2} = 4$

$\lambda + \lambda + \lambda = 8$

$3\lambda = 8$

$\lambda = \frac{8}{3}$

$\boxed{CP: \left(\frac{2}{3}, \frac{-4}{3}, \frac{4}{3}\right)}$ $\boxed{f\left(\frac{2}{3}, -\frac{4}{3}, \frac{4}{3}\right) = \frac{16}{3}}$

10. A function f has continuous second partial derivatives on an open region containing the critical point (a, b). If $f_{xx}(a, b)$ and $f_{yy}(a, b)$ have opposites signs, what is implied? Explain. If nothing can be concluded, state why.

Assume $f_{xx} > 0$ $f_{yy} < 0$.

Then $f_{xx} \cdot f_{yy} < 0$

Also, $-[f_{xy}]^2 \leq 0$.

Thus, $f_{xx} \cdot f_{yy} - [f_{xy}]^2 < 0$

Making (a,b) a saddle point.

Chapter 13 Exam Grade

Question	Score
1	/10
2	/10
3	/10
4	/10
5	/10
6	/10
7	/10
8	/10
9	/10
10	/10
TOTAL	**/100**