Differential Equations

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Outline

1. Definition of Differential Equations

Example 1 (Types of equations)

1. Find x in $x^2 + 2x + 1 = 0$.
   Algebraic Equation, the solution is a number $x = -1$
2. Find $f(t)$ in $f(t)e^t + \sin(t) = \cos(t)$.
   Functional Equation,
   the solution is a function $f(t) = e^{-t}(cos(t) - sin(t))$
3. Find $y(t)$ in $y''(t) + 3y'(t) = e^t$.
   Differential Equation, the solution is a function

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Definition

A **differential equation** is

- an equation, and
- involves the derivatives of unknown functions

Remark

Remark. If a differential equation involves the derivative of one variable with respect to another, then the former is called a dependent variable, and the latter an independent variable.
For example, in the equation

$$\frac{d^2x}{dt^2} + a\frac{dx}{dt} + kx = 0,$$

- x: dependent variable
- t: independent variable
- a and k: coefficient

2. Classification of Differential Equations

2.1. By type

- If a differential equation involves only *ordinary derivatives* of the unknown function, it is called an ordinary differential equation (ODE)

  e.g. $\frac{dx}{dt} = e^x$

- If a differential equation involves *partial derivatives*, it is called a partial differential equation (PDE)

  e.g. Solve $u(t, x)$ in $\frac{\partial u}{\partial t} - 2\frac{\partial^2 u}{\partial x^2} = 3tx$

2.2. By order

- The order of a differential equation is the order of the highest-order derivatives of the unknown in the equation.

Example 2. Determine the type and the order of the following differential equations.

- $y'' + 4y' = e^x$
- $y'' + 4(y')^3 + 5y = e^x$
- $u_t - 2u_{xx} = 0$
- $t^2y''' - t^3y'' + ty^4 = sin(t)$

2.3. By Linearity

- An n-th order ordinary differential equation $F(x, y, y', \cdots, y^{(n)}) = 0$ is called linear, if F is linear with respect to $y, y', y'', \cdots, y^{(n)}$. More explicitly, an ODE is **linear**, if it can be written as

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x).$$

- If a differential equation is not linear, it is called nonlinear.

Remark

There are two special cases that we are going to discuss throughout this semester

- linear first-order equations: $a_1(x)y' + a_0(x)y = g(x)$.
- linear second-order equations: $a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$.

Example 3. Determine the type, order, and linearity of the following differential equations.

- $x^3y''' + xy' - 5y = e^x$
- $\dfrac{dy}{dx} = \dfrac{x}{y+2}$
- $\dfrac{d^2y}{dx^2} = \dfrac{y+2}{\sqrt{x^2+1}}$
- $(1-y)y' + 2y = e^x$
- $y'' + \sin(x)y = 0$
- $y'' + x\sin(y) = 0$

1. Explicit Solution

The general form of n-th order ODEs with x independent, y dependent, can be expressed as

$$F(x, y, y', \cdots, y^{(n)}) = 0.$$

In many cases, we can isolate the highest-order term and write the equation as

$$y^{(n)} = f(x, y, y', \cdots, y^{(n-1)}).$$

Definition.

A function $\phi(x)$ is called an <u>explicit solution</u> of an ODE if the equation becomes an identity when substituting y by $\phi(x)$.

Example 1. Verify that $\phi(x) = x^2 - x^{-1}$ is an explicit solution to the differential equation

$$\frac{d^2y}{dx^2} - \frac{2}{x^2}y = 0$$

Example 2. Show that for any choice of the constant c_1 and c_2, the function $\phi(x) = c_1 e^{-x} + c_2 e^{2x}$ is an explicit solution to the linear equation

$$y'' - y' - 2y = 0.$$

Example 3. Determine for which values of m the function $\phi(x) = x^m$ is a solution to the differential equation

$$5x^2y'' - 11xy' + 3y = 0.$$

2. Implicit Solution

As we will see in following chapters, the methods for solving differential equations do not always yield an explicit solution. We may have to settle for a solution that is defined implicitly.

Example 4. Show that the relation $y^2 - x^3 + 8 = 0$ implicitly defines a solution to the nonlinear equation

$$\frac{dy}{dx} = \frac{3x^2}{2y}$$

on the interval $(2, \infty)$.

Definition.

A relation $G(x, y)$ is said to be an implicit solution of an ODE on the interval I if it defines one or more explicit solutions on I.

Example 5. Show that the relation $x + y + e^{xy} = 0$ is an implicit solution to the nonlinear equation

$$(1 + xe^{xy})\frac{dy}{dx} + 1 + ye^{xy} = 0.$$

Remark

For brevity, from now on we use the term **solution** to mean either explicit or implicit solution.

3. Initial Value Problem

As indicated in Example 2, a differential equation usually has infinitely many solutions. To uniquely determine a solution, we often impose additional conditions.

Example 6.

- Find all solutions of the differential equation $\frac{dy}{dt} = y$.
- In addition to the differential equation, we also require $y(0) = 3$. What can we say about the solution?

Remark

- The additional condition $y(0) = 3$ is often called initial condition (IC), since the independent variable t often represents time in many physical applications.
- A differential equation together with an initial condition is called an Initial Value Problem (IVP).

Remark.

- The IVP for a first-order differential equation is

$$F(t, y, y') = 0, y(t_0) = y_0, \text{ where } t_0, y_0 \text{ are given.}$$

- The IVP for a second-order differential equation is

$$F(t, y, y', y'') = 0, y(t_0) = y_0, y'(t_0) = y_1, \text{ where } t_0, y_0, y_1 \text{ are given.}$$

Example 7. As shown in Example 2, the function $\phi(x) = c_1 e^{-x} + c_2 e^{2x}$ is a solution to

$$y'' - y' - 2y = 0$$

for any choice of constants c_1 and c_2. Determine c_1 and c_2 so that the initial conditions

$$y(0) = 2 \quad \text{and} \quad y'(0) = -3$$

are satisfied.

1. Direction Fields

Definition

The **direction field** of the first order differential equation

$$\frac{dy}{dx} = f(x, y)$$

is a plot of short line segments drawn at various points in the xy-plane showing the slope of the solution curve.

Example 1. Sketch the direction field of the differential equation

$$\frac{dy}{dx} = 1 - \frac{y}{5}$$

2. Use Direction Field to Analyze the Solution

Example 2. The following differential equation

$$\frac{dp}{dt} = p(2 - p)$$

is a logistic equation for modeling the growth of population. Here, p (in thousand) is the population at time t. Sketch the direction field of this equation. Then answer the following questions.

1. If the initial population is 3000, what can you say about the population in a long time?
2. Can a population of 1000 ever decline to 500?
3. Can a population of 1000 ever increase to 3000?

3. Online Direction Field Softwares

Computer softwares can be used to sketch direction fields of more complicated differential equations accurately. For example,

Geogebra: https://www.geogebra.org/m/W7dAdgqc

Example 3. Use a computer software to sketch the direction field of

$$\frac{dy}{dx} = x^2 - y.$$

From the direction field, sketch the solutions with initial conditions $y(0) = 0, 1$, and -1.