Chapter 5

Integration

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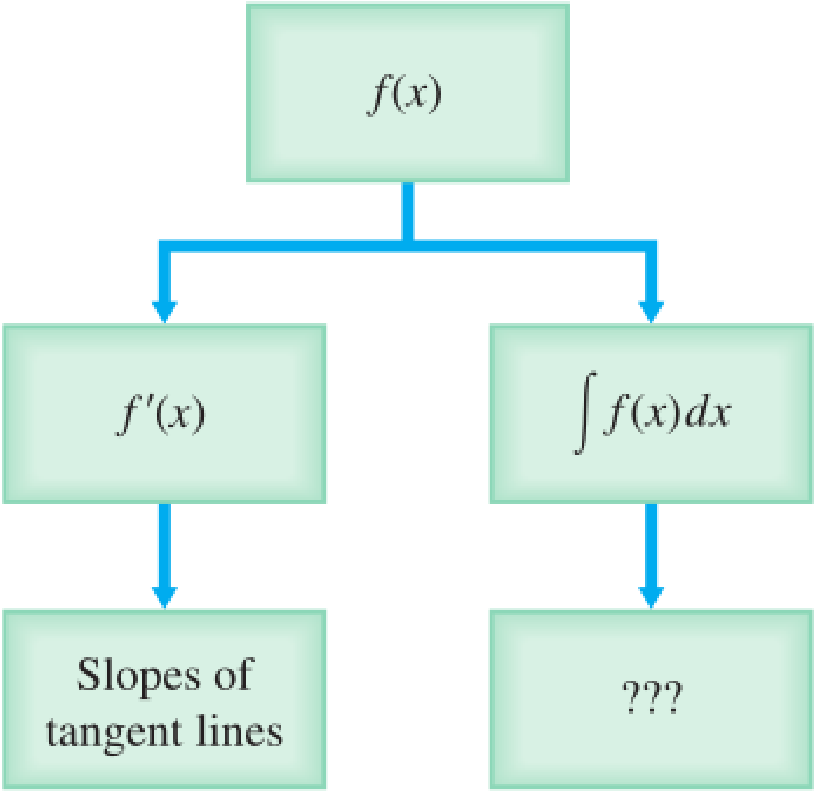
5.1

Approximating Areas under Curves

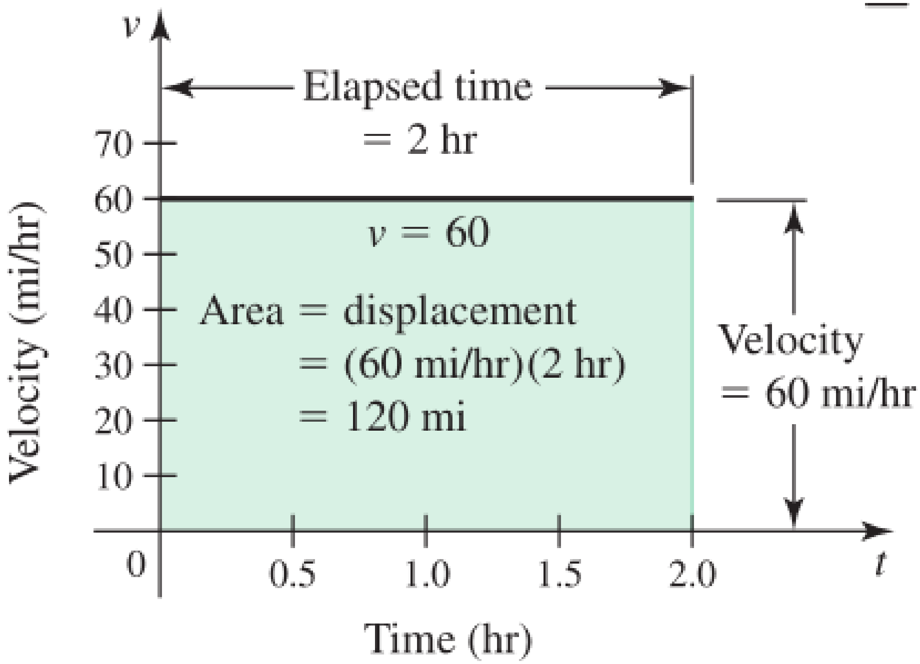
Area under a Velocity Curve

What is the geometric meaning of the integral?



Imagine a car traveling at a *constant velocity* of 60 mi/hr along a straight highway over a two-hour period. The graph of the velocity function $v = 60$ on the interval $0 \leq t \leq 2$ is a horizontal line. The displacement of the car between $t = 0$ and $t = 2$ is found by a familiar formula:

$$\begin{aligned} \text{displacement} &= \text{rate} \cdot \text{time} \\ &= 60 \text{ mi/hr} \cdot 2 \text{ hr} = 120 \text{ mi}. \end{aligned}$$



Extend to positive velocities that *change over an interval of time*. One strategy is to *divide* the time interval into many subintervals and *approximate* the velocity on each subinterval with a constant velocity. Then the displacements on each subinterval are calculated and *summed*.

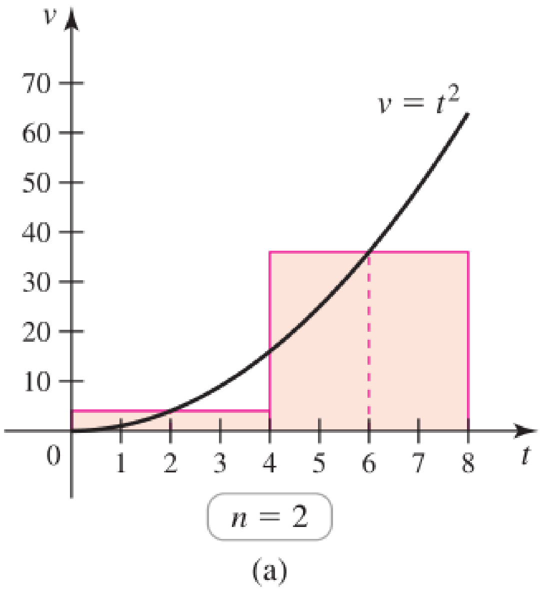
EXAMPLE 1 **Approximating the displacement** Suppose the velocity in m/s of an object moving along a line is given by the function $v = t^2$, where $0 \le t \le 8$. Approximate the displacement of the object by dividing the time interval $[0, 8]$ into n subintervals of equal length. On each subinterval, approximate the velocity with a constant equal to the value of v evaluated at the midpoint of the subinterval.

a. Begin by dividing $[0, 8]$ into $n = 2$ subintervals: $[0, 4]$ and $[4, 8]$.

b. Divide $[0, 8]$ into $n = 4$ subintervals: $[0, 2]$, $[2, 4]$, $[4, 6]$, and $[6, 8]$.

c. Divide $[0, 8]$ into $n = 8$ subintervals of equal length.

The midpoint of each subinterval is used to approximate the velocity over that subinterval.



The progression may be continued.
Larger values of n mean more rectangles.
A better fit to the region under the curve.

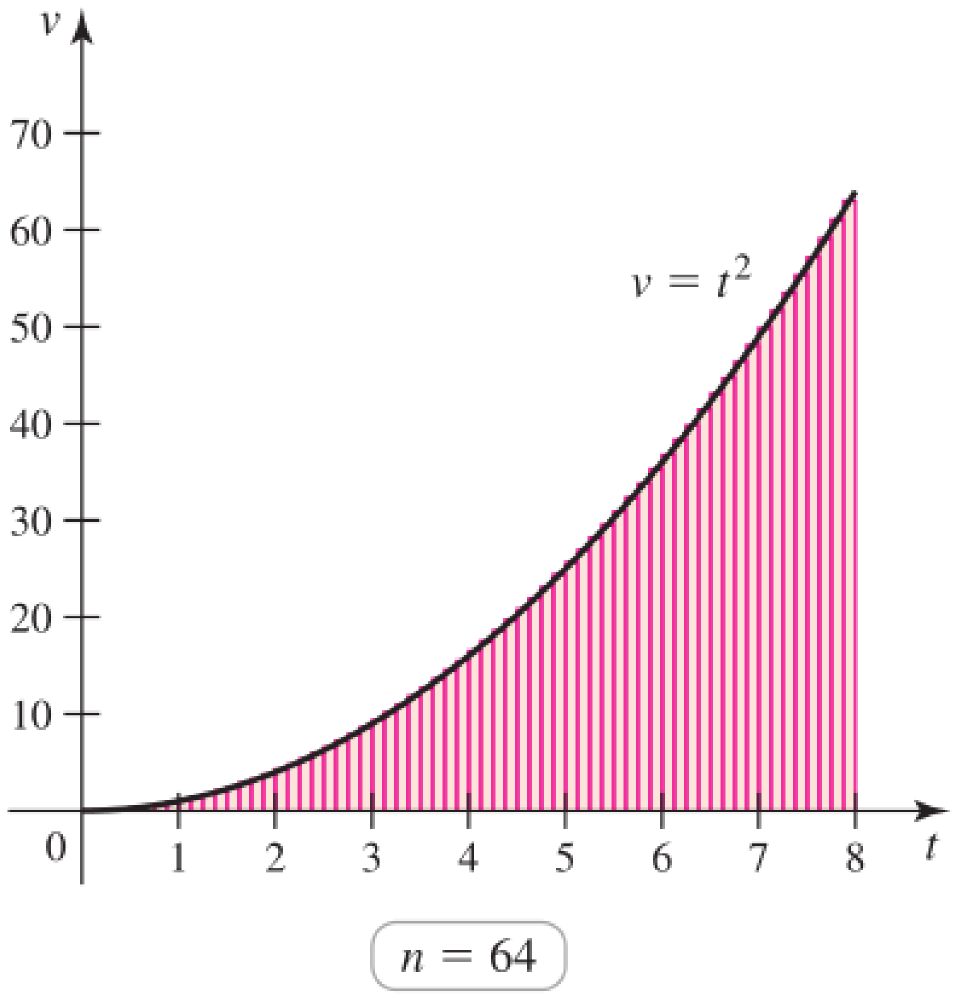
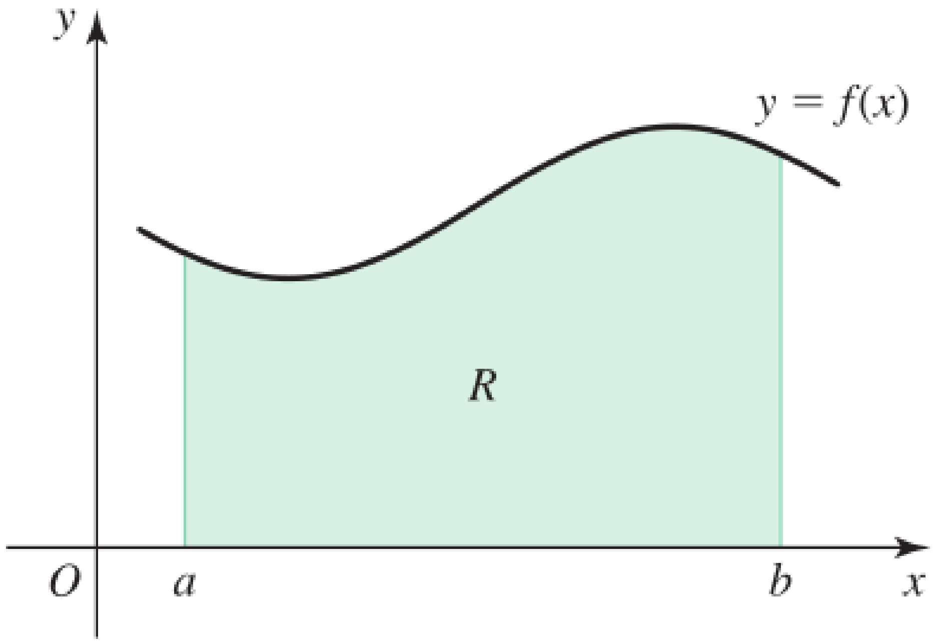


Table 5.1 Approximations to the area under the velocity curve $v = t^2$ on $[0, 8]$

Number of subintervals	Length of each subinterval	Approximate displacement (area under curve)
1	8 s	128.0 m
2	4 s	160.0 m
4	2 s	168.0 m
8	1 s	170.0 m
16	0.5 s	170.5 m
32	0.25 s	170.625 m
64	0.125 s	170.65625 m

Approximating Areas by Riemann Sums

Consider a function f that is continuous and nonnegative on an interval $[a, b]$. The goal is to approximate the area of the region R bounded by the graph of f and the x-axis from $x = a$ to $x = b$.



Divide the interval $[a, b]$ into n subintervals of equal length.

$$[a = x_0, x_1], [x_1, x_2], \dots, [x_{n-1}, x_n = b]$$

The length of each subinterval, denoted Δx, is found by dividing the length of the entire interval by n: $\Delta x = \frac{b-a}{n}$

DEFINITION Regular Partition

Suppose $[a, b]$ is a closed interval containing n subintervals

$$[x_0, x_1], [x_1, x_2], \dots, [x_{n-1}, x_n]$$

of equal length $\Delta x = \dfrac{b - a}{n}$ with $a = x_0$ and $b = x_n$. The endpoints $x_0, x_1, x_2, \dots,$ x_{n-1}, x_n of the subintervals are called **grid points**, and they create a **regular partition** of the interval $[a, b]$. In general, the kth grid point is

$$x_k = a + k\Delta x, \text{ for } k = 0, 1, 2, \dots, n.$$

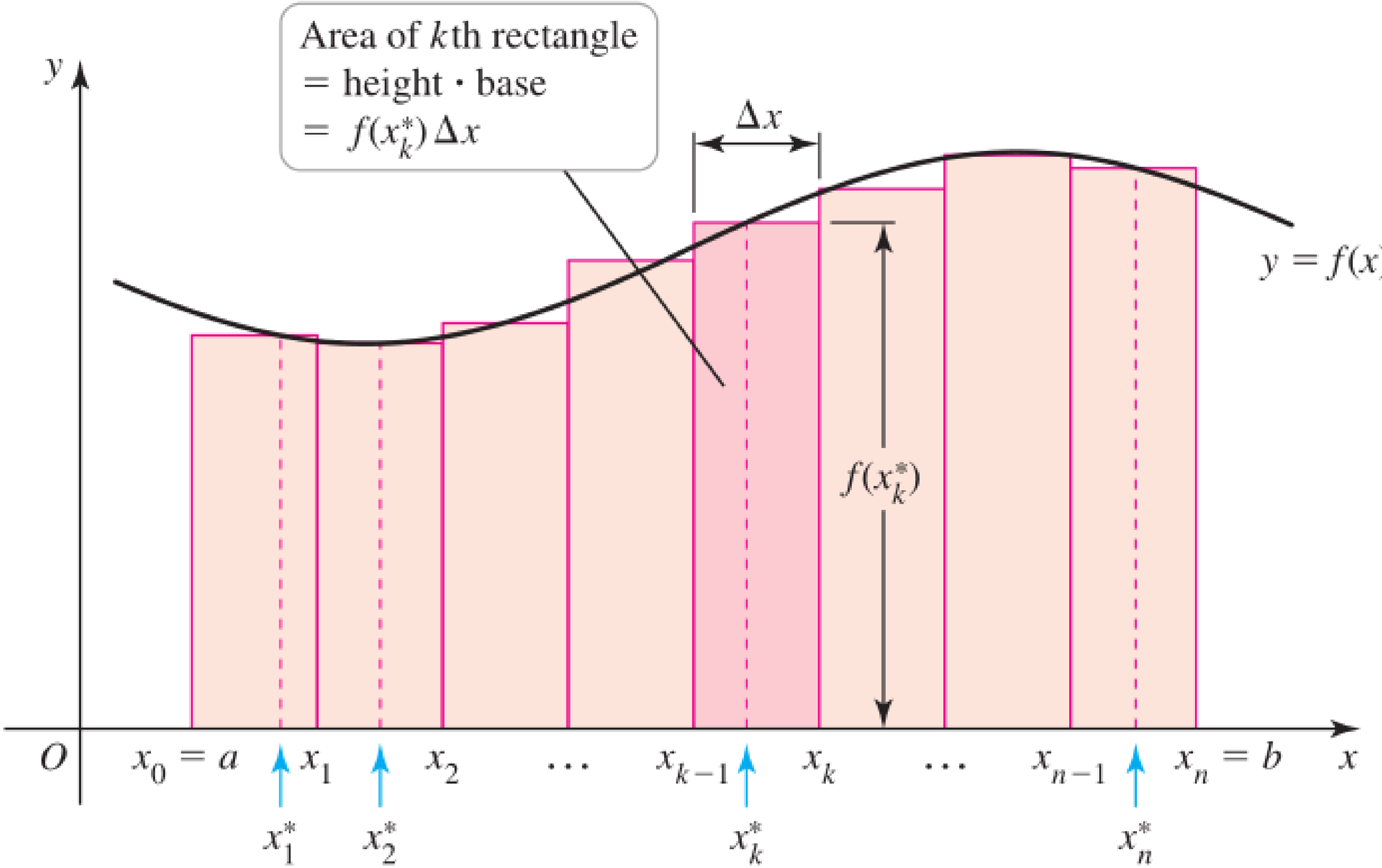
Choose any point $x_k^* \in [x_{k-1}, x_k]$ and **build a rectangle**.
The area of the rectangle on the kth subinterval is

$$\text{height} \cdot \text{base} = f(x_k^*)\Delta x, \qquad \text{where } k = 1, 2, \ldots, n.$$

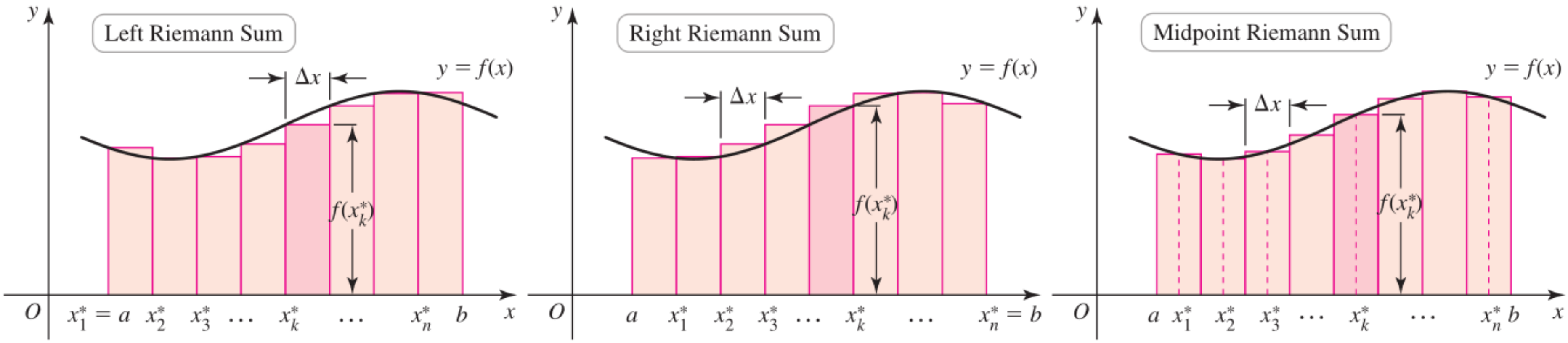
Summing the areas of the rectangles to obtain an approximation to the area of R, which is called a **Riemann sum**:

$$f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x.$$

Three notable Riemann sums: *left, right, and midpoint*



Area of kth rectangle
$= \text{height} \cdot \text{base}$
$= f(x_k^*)\,\Delta x$
y
Δx
$y = f(x)$
$f(x_k^*)$
O
$x_0 = a$
x_1
x_2
$\dots$
x_{k-1}
x_k
$\dots$
x_{n-1}
$x_n = b$
x
x_1^*
x_2^*
x_k^*
x_n^*


Left Riemann Sum
Right Riemann Sum
Midpoint Riemann Sum
$y = f(x)$
Δx
$f(x_k^*)$
O
$x_1^* = a$ x_2^* x_3^* $\cdots$ x_k^* $\cdots$ x_n^* b x
a x_1^* x_2^* x_3^* $\cdots$ x_k^* $\cdots$ $x_n^* = b$
a x_1^* x_2^* x_3^* $\cdots$ x_k^* $\cdots$ x_n^* b x

DEFINITION Riemann Sum

Suppose f is defined on a closed interval $[a, b]$, which is divided into n subintervals of equal length Δx. If x_k^* is any point in the kth subinterval $[x_{k-1}, x_k]$, for $k = 1, 2, \ldots, n$, then

$$f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x$$

is called a **Riemann sum** for f on $[a, b]$. This sum is called

- a **left Riemann sum** if x_k^* is the left endpoint of $[x_{k-1}, x_k]$ (Figure 5.9);
- a **right Riemann sum** if x_k^* is the right endpoint of $[x_{k-1}, x_k]$ (Figure 5.10); and
- a **midpoint Riemann sum** if x_k^* is the midpoint of $[x_{k-1}, x_k]$ (Figure 5.11), for $k = 1, 2, \ldots, n$.

EXAMPLE 2 **Left and right Riemann sums** Let R be the region bounded by the graph of $f(x) = \sin x$ and the x-axis between $x = 0$ and $x = \pi/2$.

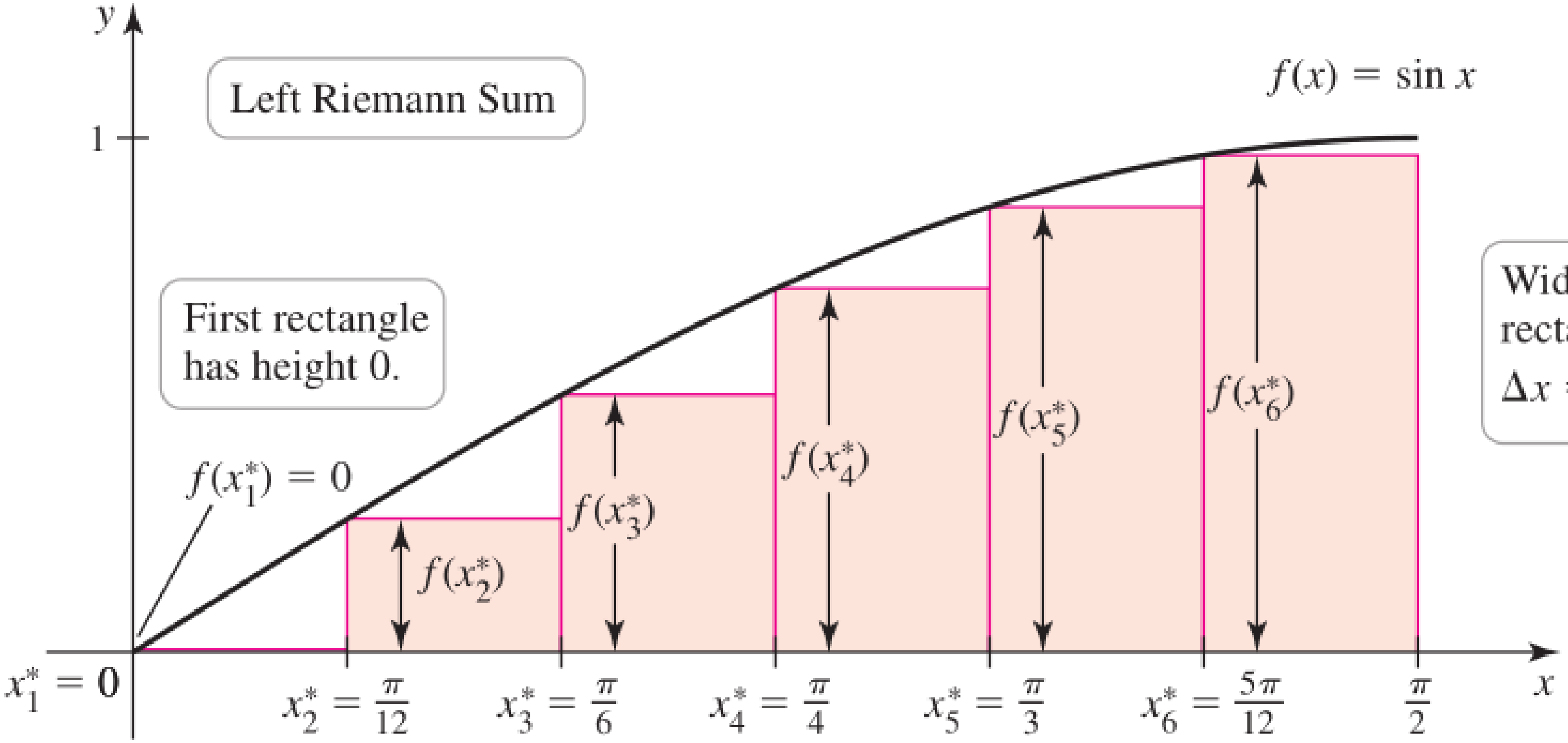
a. Approximate the area of R using a left Riemann sum with $n = 6$ subintervals. Illustrate the sum with the appropriate rectangles.

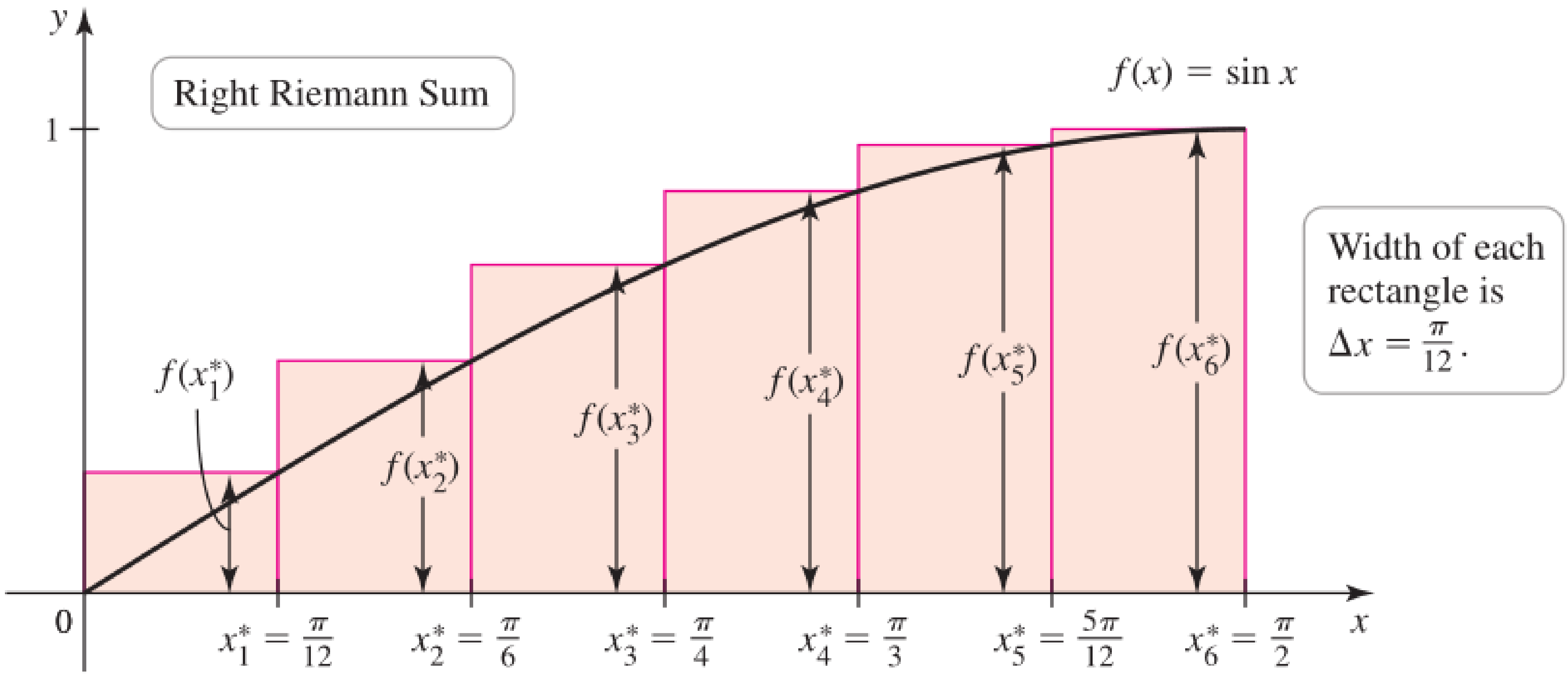
b. Approximate the area of R using a right Riemann sum with $n = 6$ subintervals. Illustrate the sum with the appropriate rectangles.

c. Do the area approximations in parts (a) and (b) underestimate or overestimate the actual area under the curve?

Solution

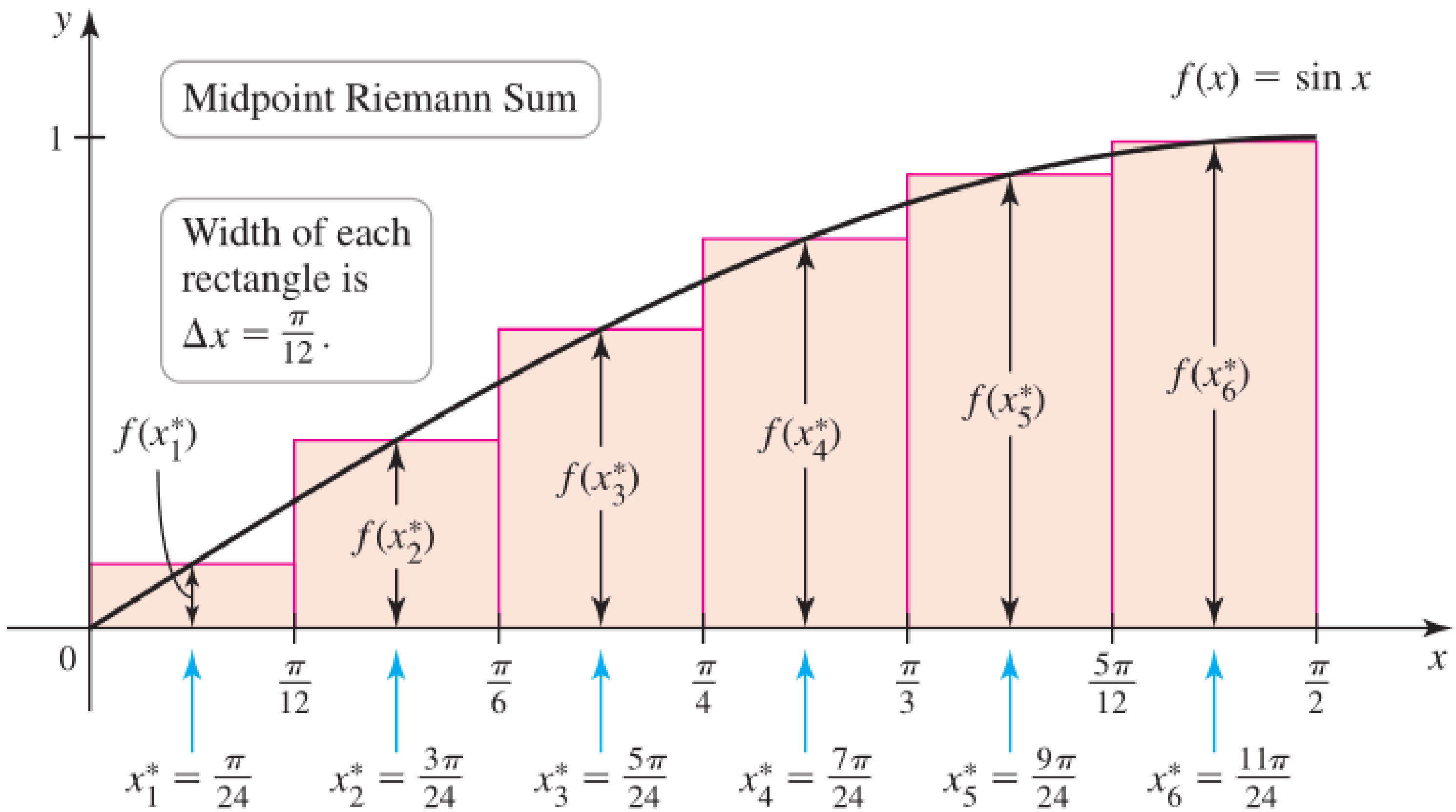
$$\Delta x = \frac{b-a}{n} = \frac{\pi/2 - 0}{6} = \frac{\pi}{12}.$$





y
Right Riemann Sum
$f(x) = \sin x$
1
Width of each rectangle is $\Delta x = \frac{\pi}{12}$.
$f(x_1^*)$
$f(x_2^*)$
$f(x_3^*)$
$f(x_4^*)$
$f(x_5^*)$
$f(x_6^*)$
0
$x_1^* = \frac{\pi}{12}$
$x_2^* = \frac{\pi}{6}$
$x_3^* = \frac{\pi}{4}$
$x_4^* = \frac{\pi}{3}$
$x_5^* = \frac{5\pi}{12}$
$x_6^* = \frac{\pi}{2}$
x

EXAMPLE 3 **A midpoint Riemann sum** Let R be the region bounded by the graph of $f(x) = \sin x$ and the x-axis between $x = 0$ and $x = \pi/2$. Approximate the area of R using a midpoint Riemann sum with $n = 6$ subintervals. Illustrate the sum with the appropriate rectangles.



EXAMPLE 4 **Riemann sums from tables** Estimate the area A under the graph of f on the interval $[0, 2]$ using left and right Riemann sums with $n = 4$, where f is continuous but known only at the points in Table 5.2.

Table 5.2

x	$f(x)$
0	1
0.5	3
1.0	4.5
1.5	5.5
2.0	6.0

Sigma (Summation) Notation

Sigma (or **summation**) **notation** is used to express sums in a compact way, e.g., $\sum_{k=1}^{10} k$, with *index* k.

$$\sum_{k=0}^{3} k^2 = 0^2 + 1^2 + 2^2 + 3^2 = 14 \qquad \sum_{k=1}^{4} (2k+1) = 3 + 5 + 7 + 9 = 24$$

$$\sum_{k=-1}^{2} (k^2 + k) = ((-1)^2 + (-1)) + (0^2 + 0) + (1^2 + 1) + (2^2 + 2) = 8.$$

The index in a sum is a *dummy variable*. $$\sum_{k=1}^{99} k = \sum_{n=1}^{99} n = \sum_{p=1}^{99} p.$$

Constant Multiple Rule $$\sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k.$$

Addition Rule $$\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k.$$

THEOREM 5.1 Sums of Powers of Integers

Let n be a positive integer and c a real number.

$$\sum_{k=1}^{n} c = cn \qquad \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

Riemann Sums Using Sigma Notation

$$f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x = \sum_{k=1}^{n} f(x_k^*)\Delta x.$$

DEFINITION Left, Right, and Midpoint Riemann Sums in Sigma Notation

Suppose f is defined on a closed interval $[a, b]$, which is divided into n subintervals of equal length Δx. If x_k^* is a point in the kth subinterval $[x_{k-1}, x_k]$, for $k = 1, 2, \ldots, n$, then the **Riemann sum** for f on $[a, b]$ is $\sum_{k=1}^{n} f(x_k^*)\Delta x$. Three cases arise in practice.

- **Left Riemann sum** if $x_k^* = a + (k - 1)\Delta x$
- **Right Riemann sum** if $x_k^* = a + k\Delta x$
- **Midpoint Riemann sum** if $x_k^* = a + (k - \frac{1}{2})\Delta x$

EXAMPLE 5 **Calculating Riemann sums** Evaluate the left, right, and midpoint Riemann sums for $f(x) = x^3 + 1$ between $a = 0$ and $b = 2$ using $n = 50$ subintervals. Make a conjecture about the exact area of the region under the curve (Figure 5.15).

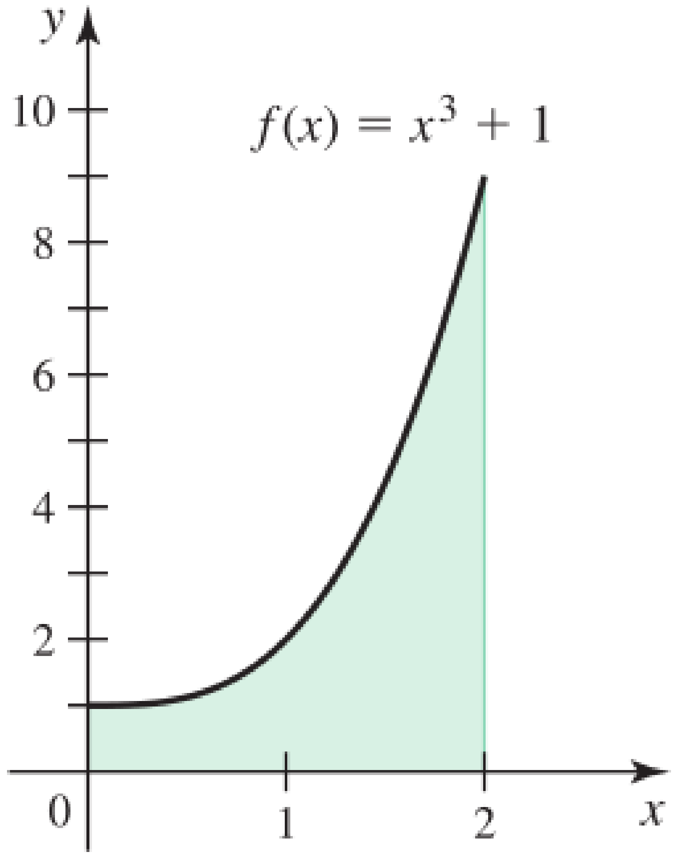


Table 5.3 Left, right, and midpoint Riemann sum approximations

n	L_n	R_n	M_n
20	5.61	6.41	5.995
40	5.8025	6.2025	5.99875
60	5.86778	6.13444	5.99944
80	5.90063	6.10063	5.99969
100	5.9204	6.0804	5.9998
120	5.93361	6.06694	5.99986
140	5.94306	6.05735	5.9999
160	5.95016	6.05016	5.99992
180	5.95568	6.04457	5.99994
200	5.9601	6.0401	5.99995

5.2

Definite Integrals

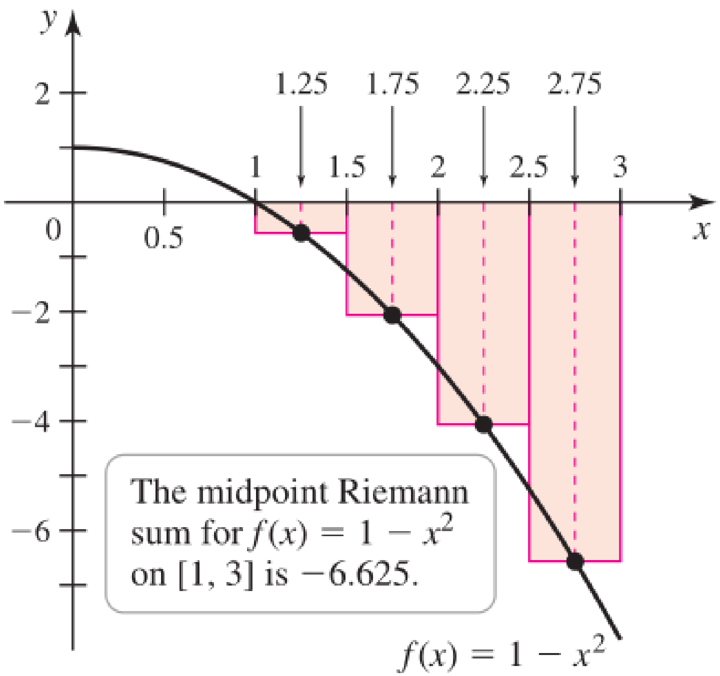
Net Area

The geometric meaning of Riemann sums when f is negative on some or all of $[a, b]$.

EXAMPLE 1 **Interpreting Riemann sums** Evaluate and interpret the following Riemann sums for $f(x) = 1 - x^2$ on the interval $[a, b]$ with n equally spaced subintervals.

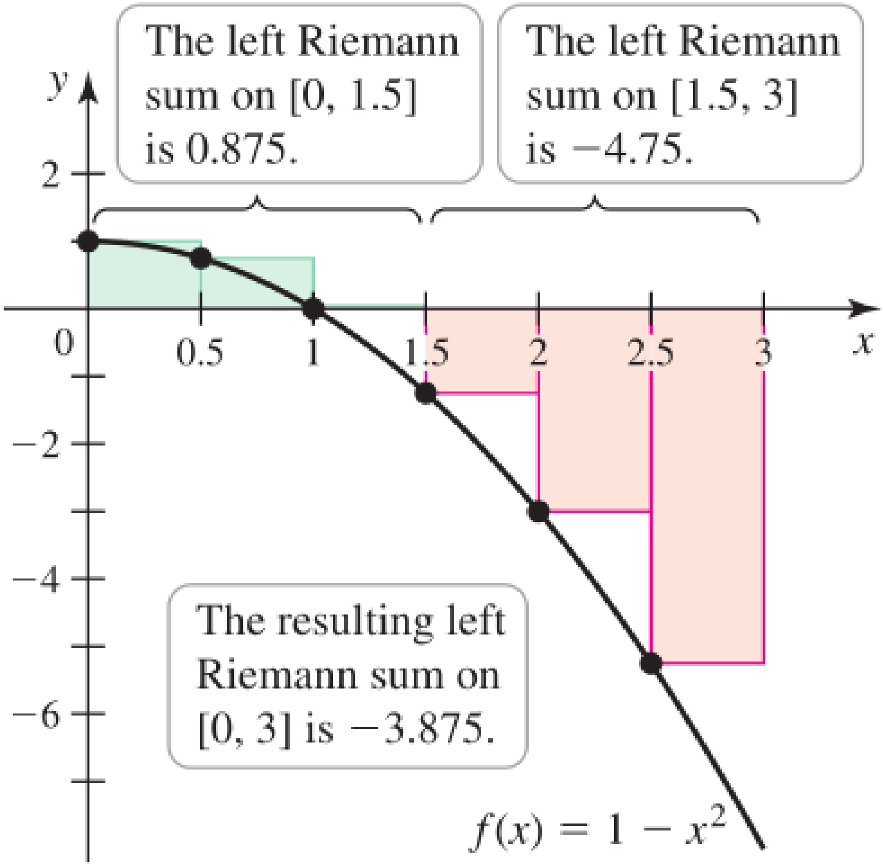
a. A midpoint Riemann sum with $[a, b] = [1, 3]$ and $n = 4$

b. A left Riemann sum with $[a, b] = [0, 3]$ and $n = 6$



Midpoint Riemann sum

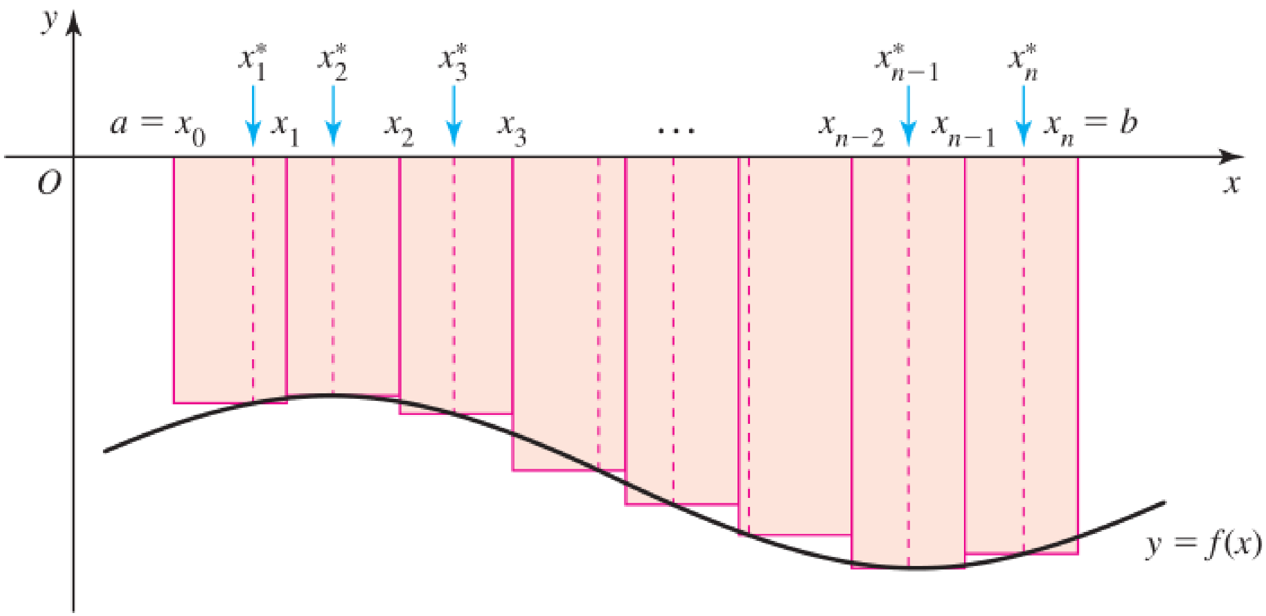
$$\sum_{k=1}^{n} f(x_k^*)\,\Delta x = -6.625$$



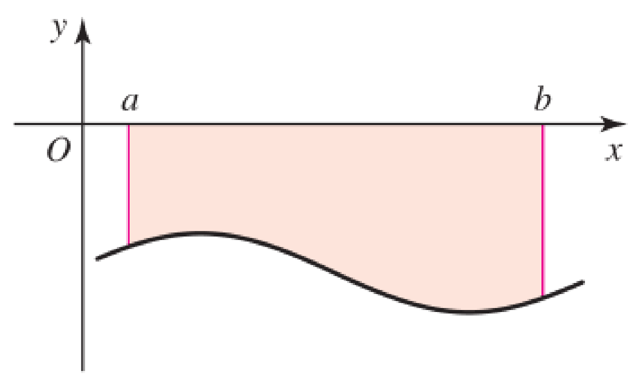
Left Riemann sum

$$\sum_{k=1}^{n} f(x_k^*)\,\Delta x = -3.875$$

On intervals where $f(x) < 0$, Riemann sums approximate the negative of the area of the region bounded by the curve.



The Riemann sum $\sum_{k=1}^{n} f(x_k^*)\Delta x$ approximates the negative of the area of the region bounded by the x-axis and the curve.



In the more general case

DEFINITION **Net Area**

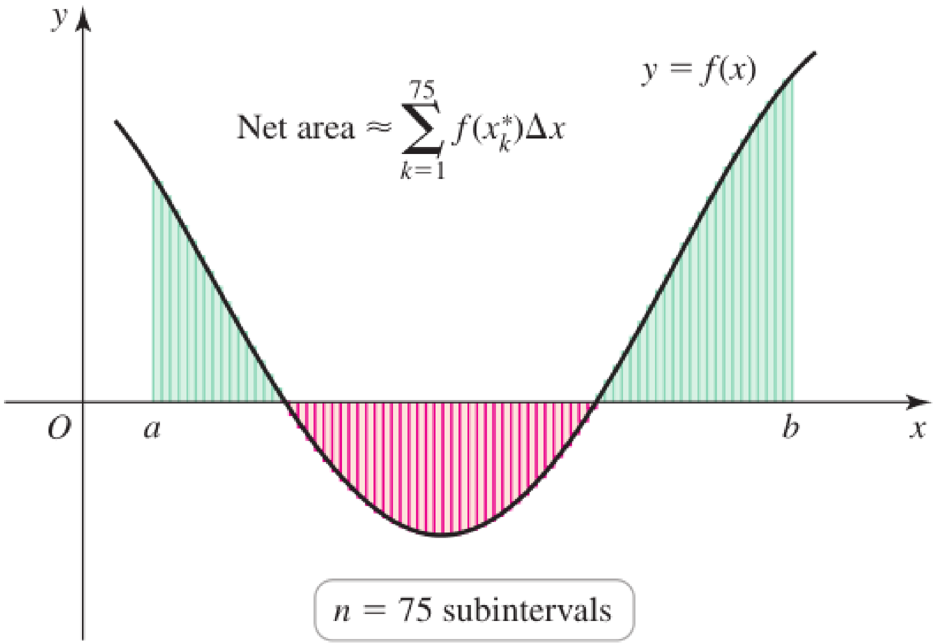
Consider the region R bounded by the graph of a continuous function f and the x-axis between $x = a$ and $x = b$. The **net area** of R is the sum of the areas of the parts of R that lie above the x-axis *minus* the sum of the areas of the parts of R that lie below the x-axis on $[a, b]$.

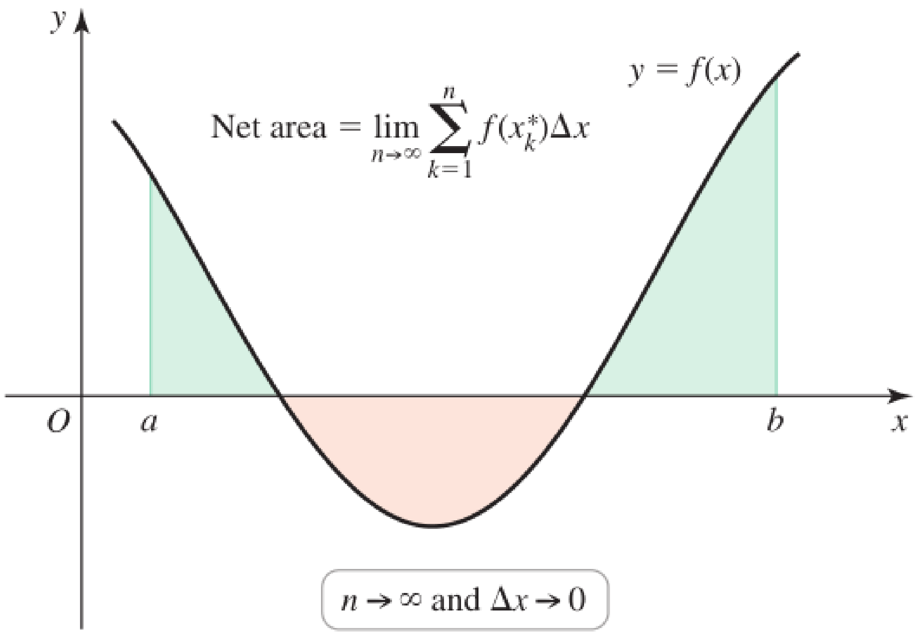
The Definite Integral

Riemann sums for f on $[a, b]$ give approximations to the net area.

How can we *make these approximations exact*?

$$\text{Net area} = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\,\Delta x$$





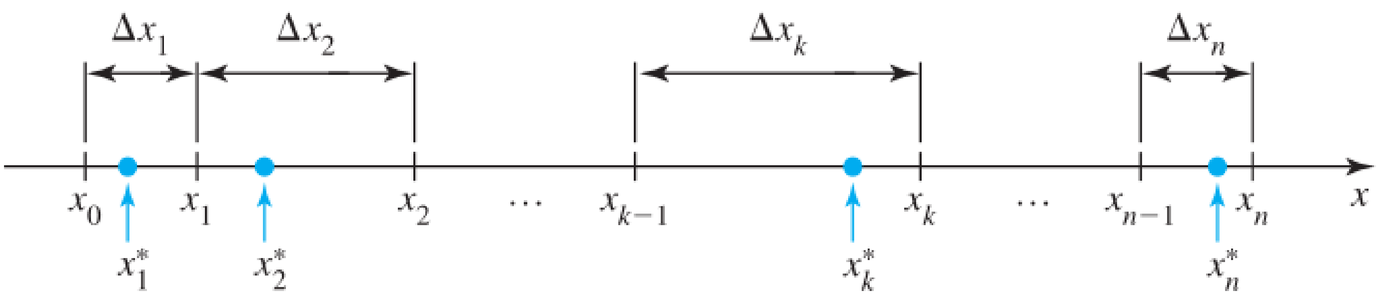
A **general partition** of $[a, b]$, not equal partition

DEFINITION General Riemann Sum

Suppose $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$ are subintervals of $[a, b]$ with

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

Let Δx_k be the length of the subinterval $[x_{k-1}, x_k]$ and let x_k^* be any point in $[x_{k-1}, x_k]$, for $k = 1, 2, \ldots, n$.



If f is defined on $[a, b]$, the sum

$$\sum_{k=1}^{n} f(x_k^*)\,\Delta x_k = f(x_1^*)\,\Delta x_1 + f(x_2^*)\,\Delta x_2 + \cdots + f(x_n^*)\,\Delta x_n$$

is called a **general Riemann sum for f on $[a, b]$**.

DEFINITION **Definite Integral**

A function f defined on $[a, b]$ is **integrable** on $[a, b]$ if $\lim\limits_{\Delta \to 0} \sum\limits_{k=1}^{n} f(x_k^*)\Delta x_k$ exists and is unique over all partitions of $[a, b]$ and all choices of x_k^* on a partition. This limit is the **definite integral of f from a to b**, which we write

$$\int_a^b f(x)\,dx = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k.$$

Indefinite integral $\int f(x)dx$ is a family of functions of x (the antiderivatives of f).

Definite integral $\int_a^b f(x)dx$ is a real number (the net area of a regionbounded by the graph of f and the x-axis on $[a, b]$).

Notation explanation

Direct match between the notation on either side of the equation in the definition. In the limit as $\Delta \to 0$, the finite sum, denoted Σ, becomes a sum with an infinite number of terms, denoted $\int$.

Upper limit of integration

Upper limit of summation

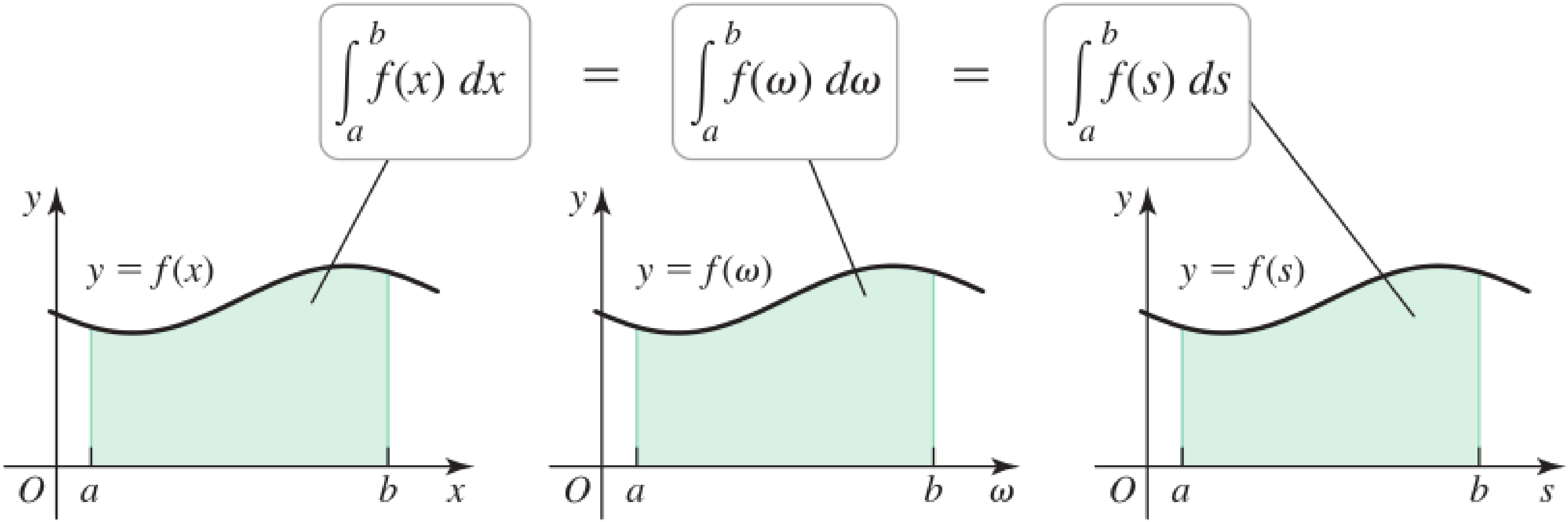
$$\int_a^b \underbrace{f(x)}_{\text{Integrand}}\, dx = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

Lower limit of integration

Lower limit of summation

x is the variable of integration.

The variable of integration is a *dummy variable* that is completely internal to the integral.



Slice-and-sum method

The strategy of *slicing* a region into smaller parts, *summing* the results from the parts, and *taking a limit* is used repeatedly in calculus and its applications. It often results in a Riemann sum whose limit is a definite integral.

Evaluating Definite Integrals

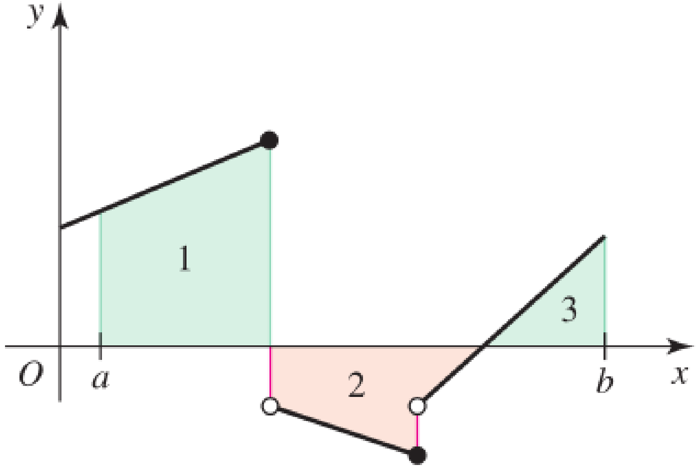
THEOREM 5.2 Integrable Functions

If f is continuous on $[a, b]$ or bounded on $[a, b]$ with a finite number of discontinuities, then f is integrable on $[a, b]$.

Net area $= \int_a^b f(x)\,dx$

$=$ area above x-axis (Regions 1 and 3)

$-$ area below x-axis (Region 2)

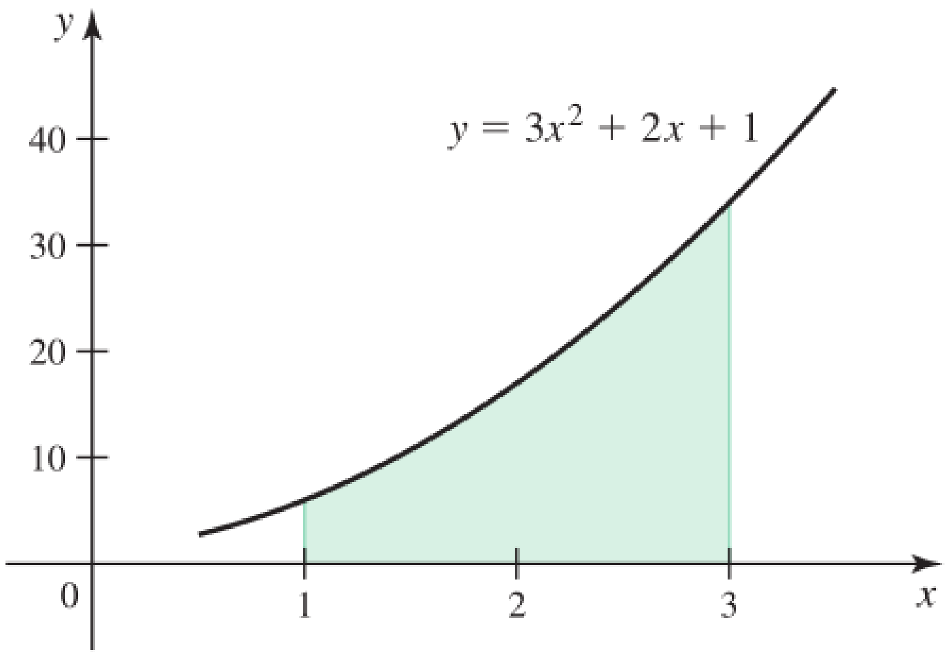


A bounded piecewise continuous function is integrable.

EXAMPLE 2 **Identifying the limit of a sum** Assume that

$$\lim_{\Delta \to 0} \sum_{k=1}^{n} (3x_k^{*2} + 2x_k^* + 1)\Delta x_k$$

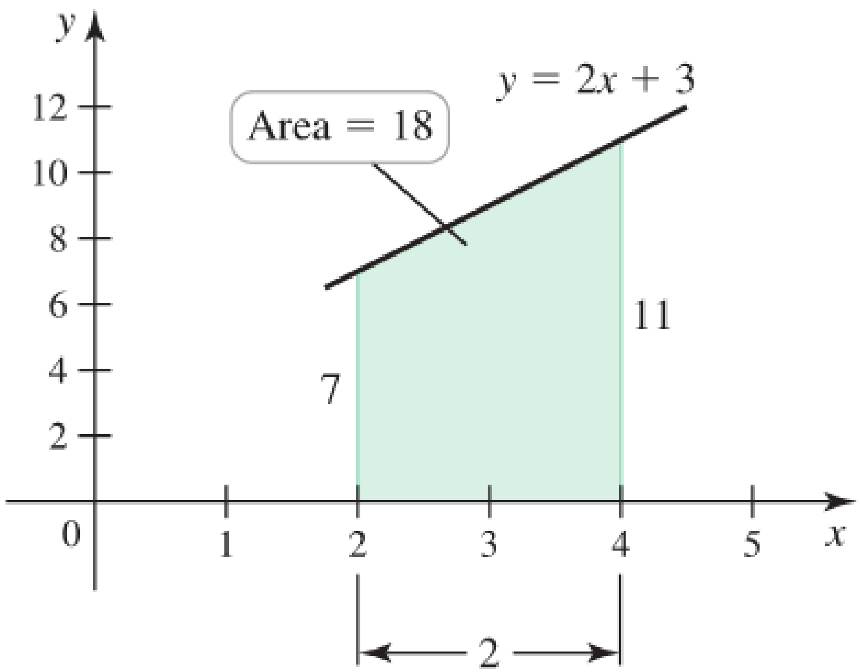
is the limit of a Riemann sum for a function f on $[1, 3]$. Identify the function f and express the limit as a definite integral. What does the definite integral represent geometrically?

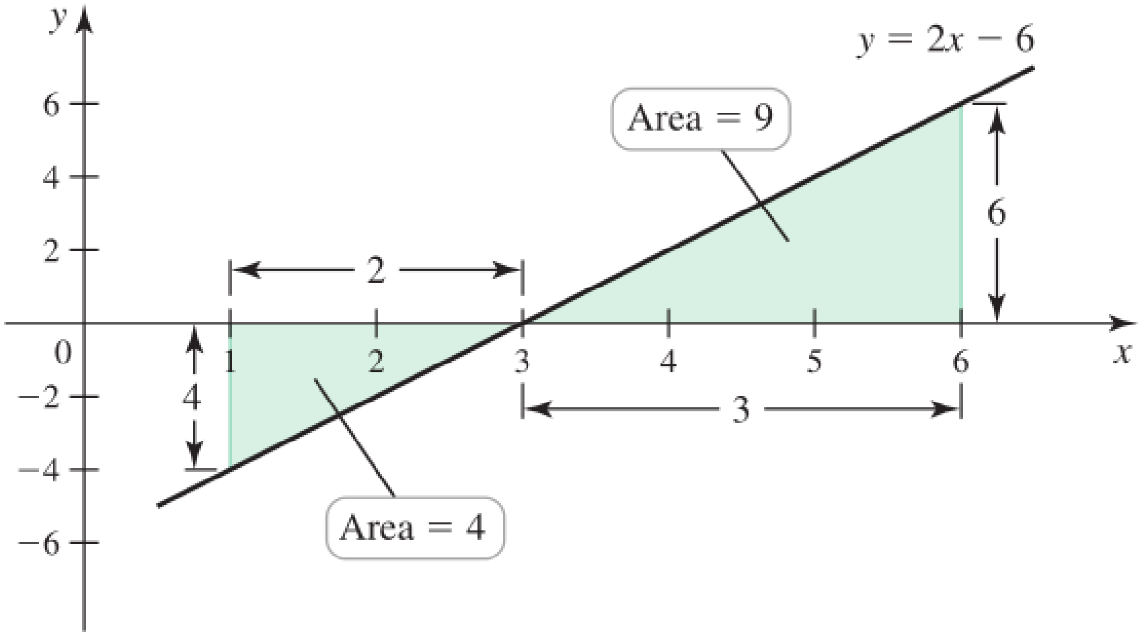


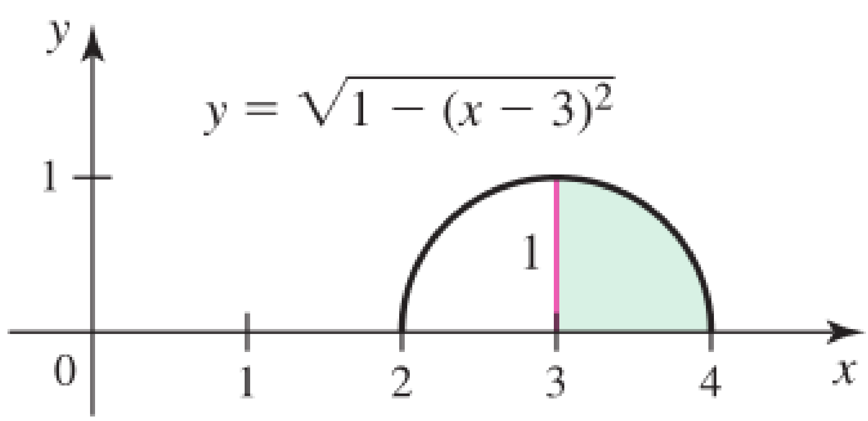
$$\lim_{\Delta \to 0} \sum_{k=1}^{n} (3x_k^{*2} + 2x_k^* + 1)\Delta x_k = \int_1^3 (3x^2 + 2x + 1)\,dx$$

EXAMPLE 3 **Evaluating definite integrals using geometry** Use familiar area formulas to evaluate the following definite integrals.

a. $\displaystyle\int_2^4 (2x + 3)\, dx$ **b.** $\displaystyle\int_1^6 (2x - 6)\, dx$ **c.** $\displaystyle\int_3^4 \sqrt{1 - (x - 3)^2}\, dx$



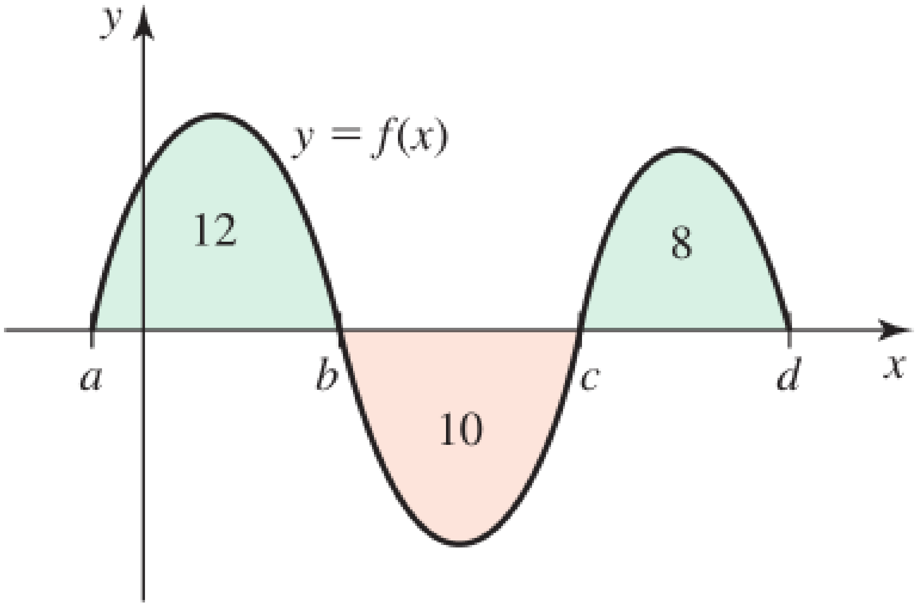




Area of shaded region $= \frac{1}{4}\pi(1)^2 = \frac{1}{4}\pi$

EXAMPLE 4 **Definite integrals from graphs** Figure 5.28 shows the graph of a function f with the areas of the regions bounded by its graph and the x-axis given. Find the values of the following definite integrals.

a. $\int_a^b f(x)\,dx$ **b.** $\int_b^c f(x)\,dx$ **c.** $\int_a^c f(x)\,dx$ **d.** $\int_b^d f(x)\,dx$



Properties of Definite Integrals

DEFINITION Reversing Limits and Identical Limits of Integration

Suppose f is integrable on $[a, b]$.

1. $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ **2.** $\int_a^a f(x)\,dx = 0$

Quick Check Evaluate $\int_a^b f(x)dx + \int_b^a f(x)dx$ assuming f is integrable on $[a, b]$).

Integral of a Sum

Assume f and g are integrable on $[a, b]$), then their sum f + g is integrable on $[a, b]$ and the integral of their sum is the sum of their integrals:

$$\int_a^b (f(x) + g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

Constants in Integrals

If f is integrable on $[a, b]$ and c is a constant, then cf is integrable on $[a, b]$ and

$$\int_a^b cf(x)dx = c\int_a^b f(x)dx$$

Integral over Subintervals

If the point p lies between a and b, then the integral on $[a, b]$ may be split into two integrals.

$$\int_a^b f(x)dx = \int_a^p f(x)dx + \int_p^b g(x)dx$$

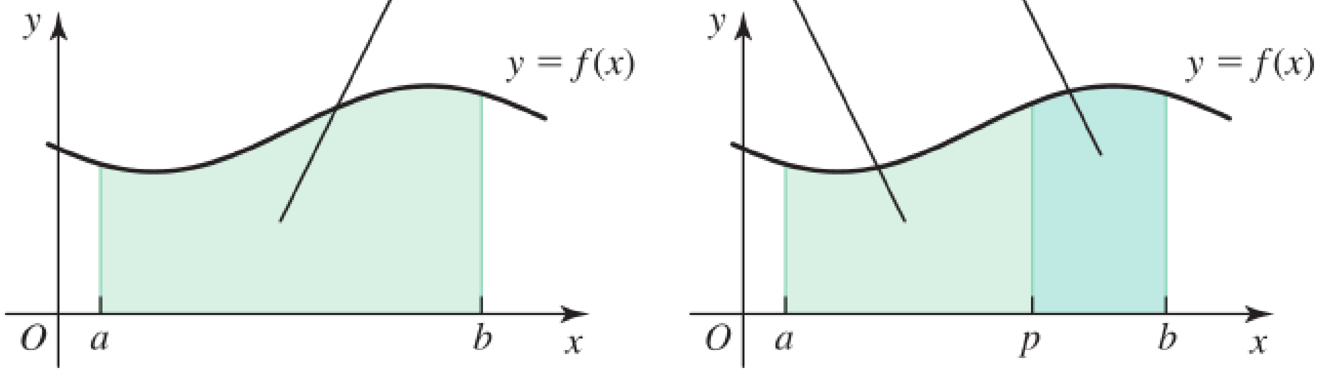
It is surprising that this property also holds when p lies outside the interval $[a, b]$, $a < b < p$.

$$\int_a^b f(x)dx = \int_a^p f(x)dx - \int_b^p g(x)dx$$

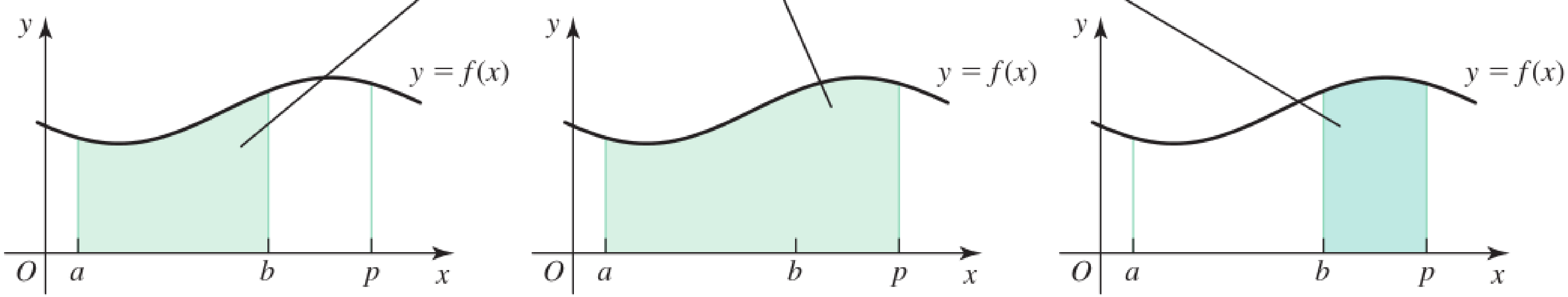
As $\int_p^b g(x)dx = -\int_b^p g(x)dx$, we have the original property

$$\int_a^b f(x)dx = \int_a^p f(x)dx + \int_p^b g(x)dx$$

$$\int_a^b f(x)\,dx = \int_a^p f(x)\,dx + \int_p^b f(x)\,dx.$$

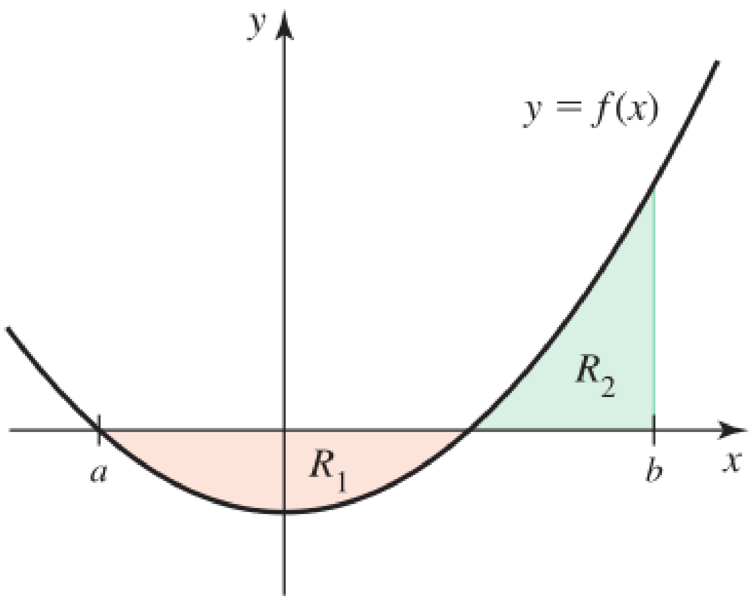


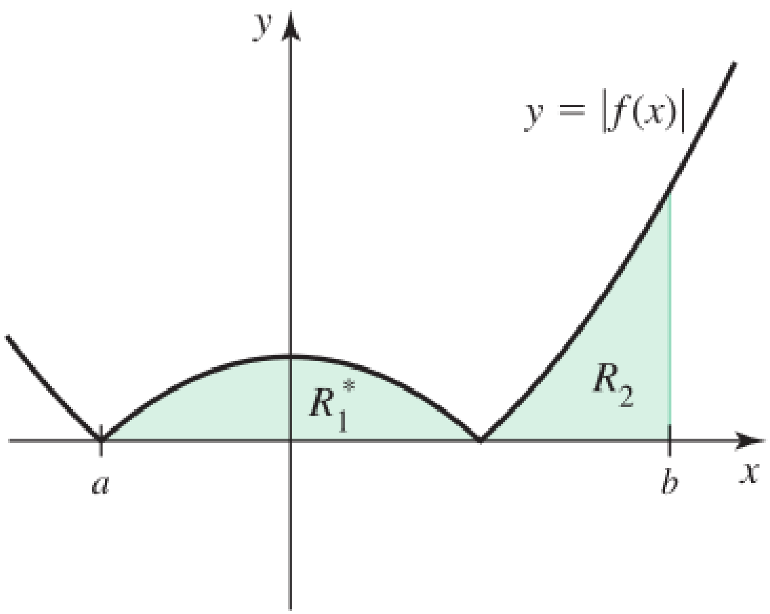
$$\int_a^b f(x)\,dx = \int_a^p f(x)\,dx - \int_b^p f(x)\,dx.$$



Integral of Absolute Values

$\int_a^b |f(x)|dx$ is the area of the entire region (above and below the x-axis) that lies between the graph of f and the x -axis on $[a, b]$.





$$\begin{aligned}\int_a^b |f(x)|\,dx &= \text{area of } R_1^* + \text{area of } R_2 \\ &= \text{area of } R_1 + \text{area of } R_2\end{aligned}$$

Table 5.4 Properties of definite integrals

Let f and g be integrable functions on an interval that contains a, b, and p.

1. $\int_a^a f(x)\,dx = 0$ Definition
2. $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ Definition
3. $\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
4. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$ For any constant c
5. $\int_a^b f(x)\,dx = \int_a^p f(x)\,dx + \int_p^b f(x)\,dx$
6. The function $|f|$ is integrable on $[a, b]$, and $\int_a^b |f(x)|\,dx$ is the sum of the areas of the regions bounded by the graph of f and the x-axis on $[a, b]$.

EXAMPLE 5 **Properties of integrals** Assume that $\int_0^5 f(x)\,dx = 3$ and $\int_0^7 f(x)\,dx = -10$. Evaluate the following integrals, if possible.

a. $\int_0^7 2f(x)\,dx$ **b.** $\int_5^7 f(x)\,dx$ **c.** $\int_5^0 f(x)\,dx$ **d.** $\int_7^0 6f(x)\,dx$ **e.** $\int_0^7 |f(x)|\,dx$

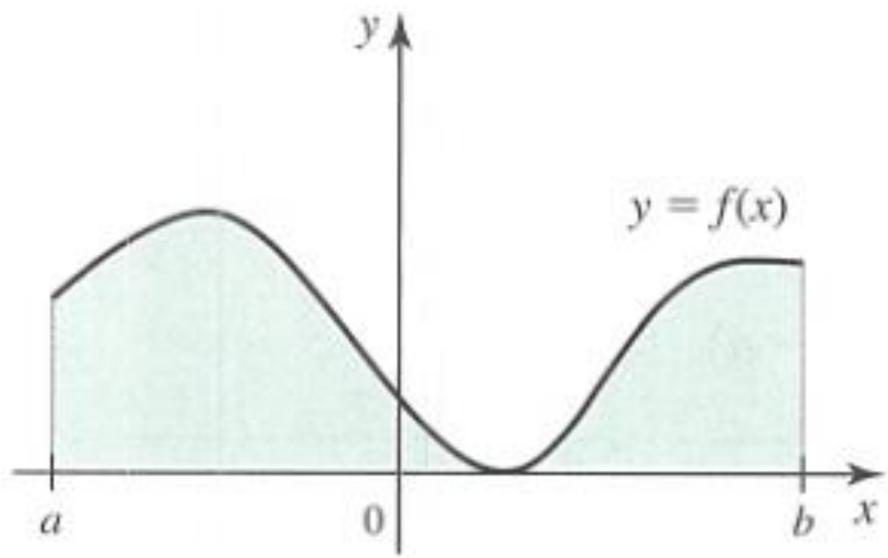
Bounds on Definite Integrals

Assume f and g are continuous on $[a, b]$.

Nonnegative Integrand

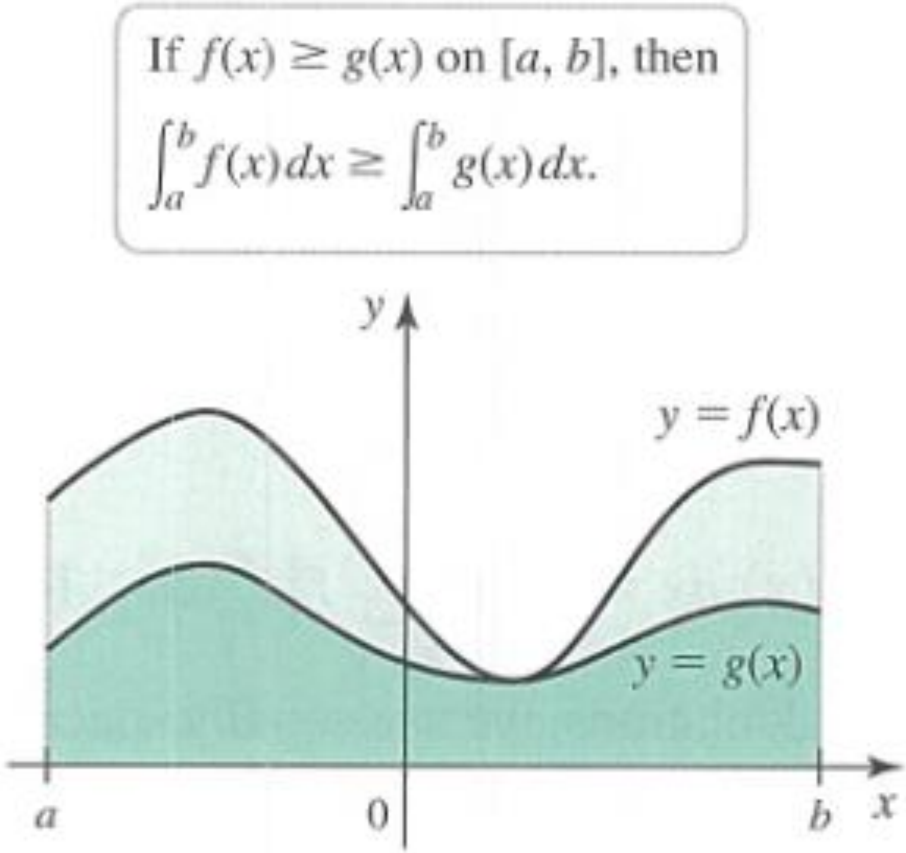
If $f(x) \geq 0$ on $[a, b]$, then $\int_a^b f(x)dx \geq 0$.

$\int_a^b f(x)\,dx =$ net area under the curve $y = f(x)$;
when $f(x) \geq 0$, net area ≥ 0.



Comparing Definite Integrals

If $f(x) \geq g(x)$ on $[a, b]$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$.



Lower and Upper Bounds

As f is continuous on $[a, b]$, it attains an absolute minimum value m and an absolute minimum value M on $[a, b]$.

Then $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$.

$$\underbrace{m(b-a)}_{\text{area of small rectangle}} \leq \underbrace{\int_a^b f(x)\,dx}_{\text{area under curve}} \leq \underbrace{M(b-a)}_{\text{area of large rectangle}}$$

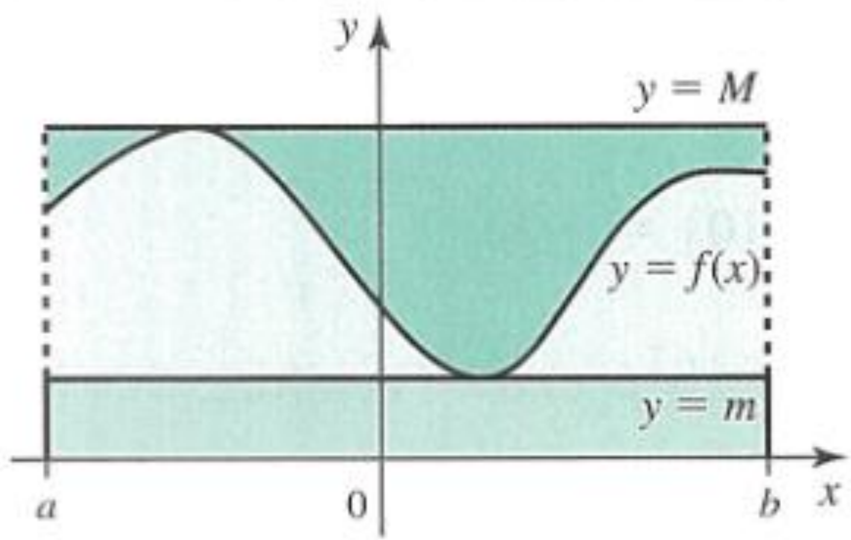


Table 5.5 Additional properties of definite integrals

Let f and g be integrable functions on $[a, b]$, where $b > a$.

7. If $f(x) \geq 0$ on $[a, b]$, then $\int_a^b f(x)\,dx \geq 0$.

8. If $f(x) \geq g(x)$ on $[a, b]$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.

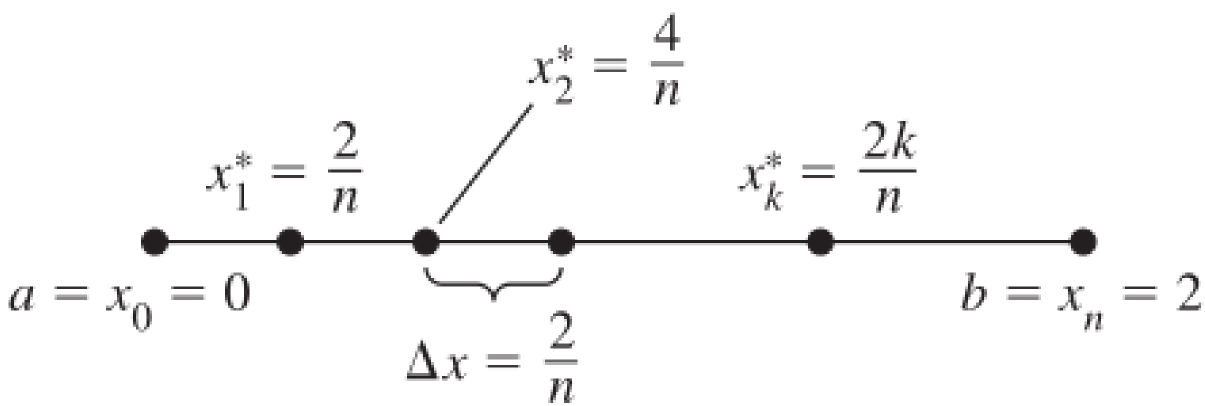
9. If $m \leq f(x) \leq M$, then $m(b - a) \leq \int_a^b f(x)\,dx \leq M(b - a)$.

Evaluating Definite Integrals Using Limits

Regions bounded by more general functions have curved boundaries for which conventional geometrical methods do not work. At the moment, the only way to handle such integrals is to appeal to the definition of the definite integral。

$$\int_a^b f(x)\,dx = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k = \lim_{n \to \infty} \sum_{k=1}^{n} f(a + k\Delta x)\Delta x.$$

EXAMPLE 6 **Evaluating definite integrals** Find the value of $\int_0^2 (x^3 + 1)\, dx$ by evaluating a right Riemann sum and letting $n \to \infty$.



$$\boxed{\begin{array}{l} x_k^* = a + k\Delta x = \dfrac{2k}{n} \\ k = 1, \ldots, n \end{array}}$$

$$\sum_{k=1}^{n} f(x_k^*)\Delta x = \sum_{k=1}^{n}\left(\left(\frac{2k}{n}\right)^3 + 1\right)\frac{2}{n}$$

$$= \frac{2}{n}\sum_{k=1}^{n}\left(\frac{8k^3}{n^3} + 1\right) \qquad \sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k$$

$$= \frac{2}{n}\left(\frac{8}{n^3}\sum_{k=1}^{n} k^3 + \sum_{k=1}^{n} 1\right) \qquad \sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$$

$$= \frac{2}{n}\left(\frac{8}{n^3}\left(\frac{n^2(n+1)^2}{4}\right) + n\right) \qquad \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4} \text{ and } \sum_{k=1}^{n} 1 = n;$$

Theorem 5.1

$$= \frac{4(n^2 + 2n + 1)}{n^2} + 2.$$

Simplify.

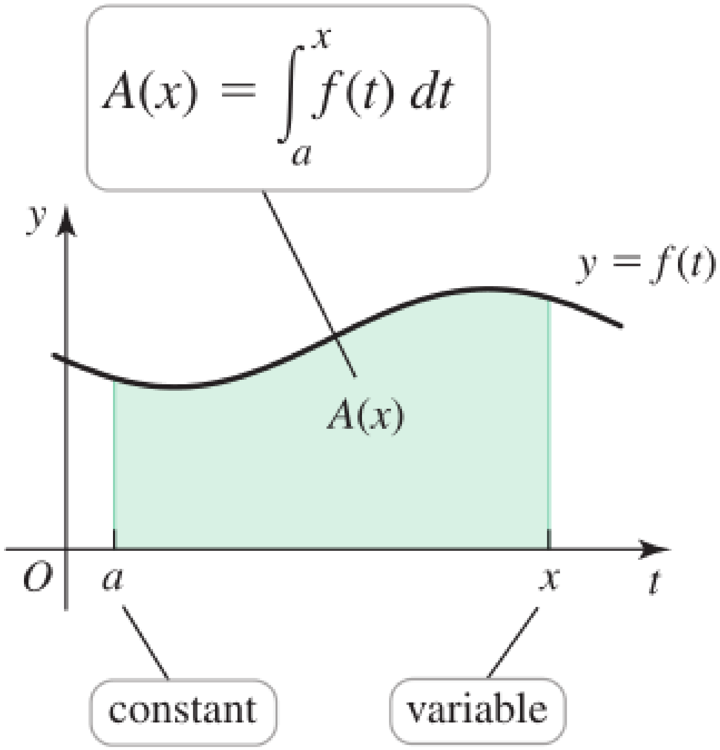
$$\begin{aligned}
\int_0^2 (x^3 + 1)\,dx &= \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x \\
&= \lim_{n\to\infty} \left(\frac{4(n^2 + 2n + 1)}{n^2} + 2 \right) \\
&= \underbrace{4 \lim_{n\to\infty} \left(\frac{n^2 + 2n + 1}{n^2} \right)}_{1} + \lim_{n\to\infty} 2 \\
&= 4(1) + 2 = 6.
\end{aligned}$$

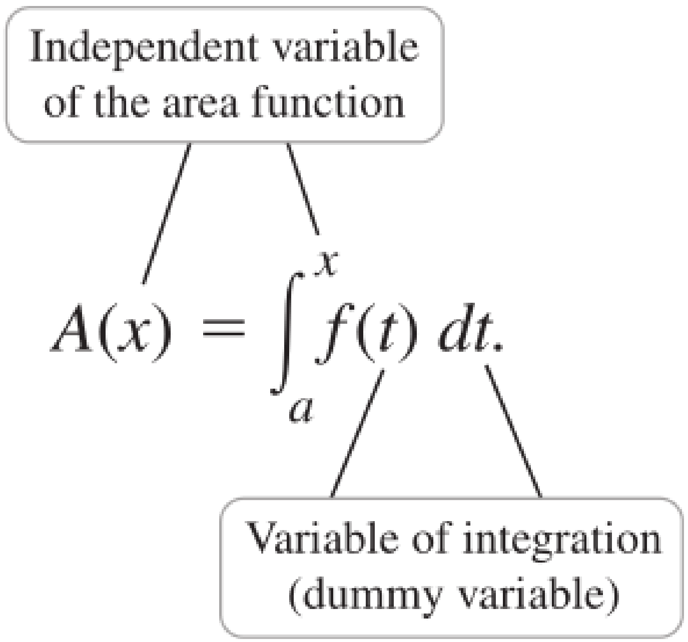
5.3

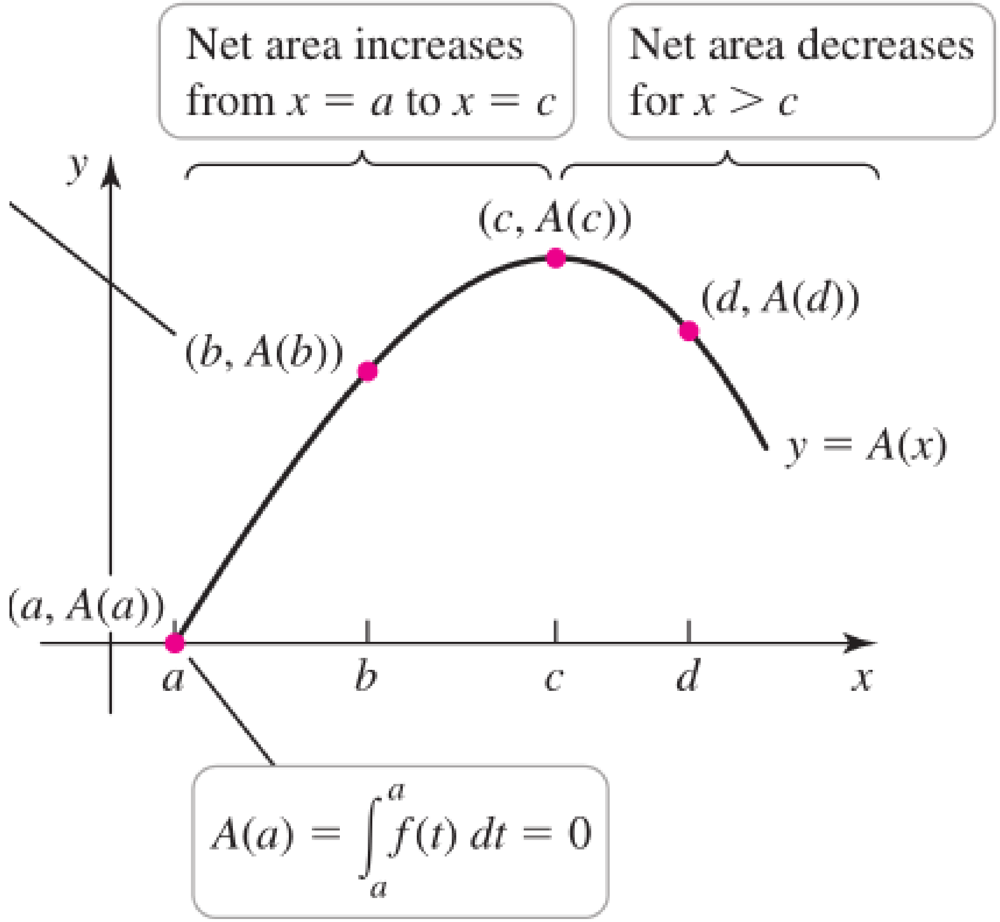
Fundamental Theorem of Calculus

Area Functions

The area function for f with left endpoint a is denoted $A(x)$







Net area increases from $x = a$ to $x = c$
Net area decreases for $x > c$
y
$(c, A(c))$
$(d, A(d))$
$(b, A(b))$
$y = A(x)$
$(a, A(a))$
a
b
c
d
x
$A(a) = \int_a^a f(t)\,dt = 0$

DEFINITION **Area Function**

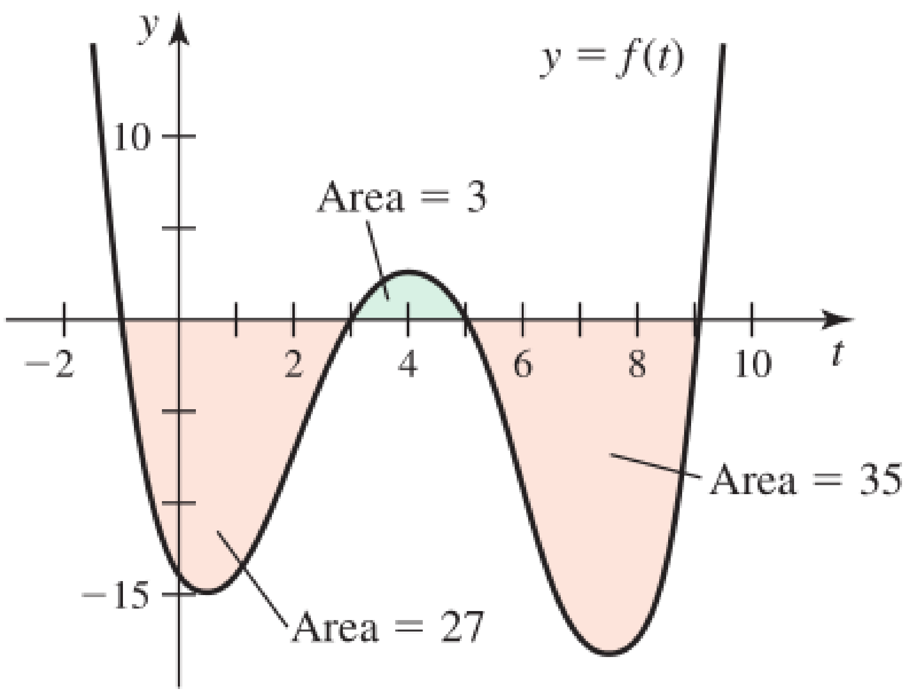
Let f be a continuous function, for $t \geq a$. The **area function for f with left endpoint a** is

$$A(x) = \int_a^x f(t)\, dt,$$

where $x \geq a$. The area function gives the net area of the region bounded by the graph of f and the t-axis on the interval $[a, x]$.

EXAMPLE 1 **Comparing area functions** The graph of f is shown in Figure 5.34 with areas of various regions marked. Let $A(x) = \int_{-1}^{x} f(t)\,dt$ and $F(x) = \int_{3}^{x} f(t)\,dt$ be two area functions for f (note the different left endpoints). Evaluate the following area functions.

a. $A(3)$ and $F(3)$ **b.** $A(5)$ and $F(5)$ **c.** $A(9)$ and $F(9)$



It illustrates the important fact that
two area functions of the same function differ by a constant.

EXAMPLE 2 **Area of a trapezoid** Consider the trapezoid bounded by the line $f(t) = 2t + 3$ and the t-axis from $t = 2$ to $t = x$ (Figure 5.35). The area function $A(x) = \int_2^x f(t)\,dt$ gives the area of the trapezoid, for $x \geq 2$.

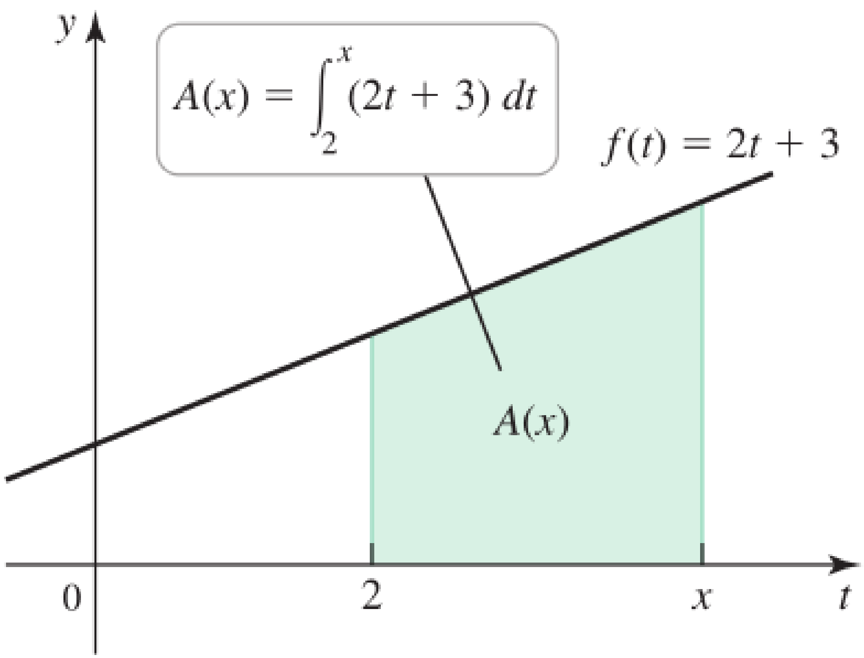
a. Evaluate $A(2)$.

b. Evaluate $A(5)$.

c. Find and graph the area function $y = A(x)$, for $x \geq 2$.

d. Compare the derivative of A to f.

SOLUTION



$$A(x) = \frac{1}{2}\underbrace{(x-2)}_{\substack{\text{distance between}\\ \text{parallel sides}}} \cdot \underbrace{(f(2) + f(x))}_{\substack{\text{sum of parallel}\\ \text{side lengths}}}$$

$$\begin{aligned} &= \frac{1}{2}(x-2)(7 + 2x + 3) \\ &= (x-2)(x+5) \\ &= x^2 + 3x - 10. \end{aligned}$$

$$A(x) = \int_2^x (2t + 3)\, dt = x^2 + 3x - 10.$$

$$A'(x) = \frac{d}{dx}(x^2 + 3x - 10) = 2x + 3 = f(x).$$

If $h > 0$ is small, the region is nearly rectangular with a base of length h and a height $f(x)$: $A(x+h) - A(x) \approx hf(x)$

Dividing by h, we have

$$\frac{A(x+h) - A(x)}{h} \approx f(x)$$

An analogous argument can be made with $h < 0$.
In the limit as $h \to 0$, we have

$$\underbrace{\lim_{h \to 0} \frac{A(x+h) - A(x)}{h}}_{A'(x)} = \underbrace{\lim_{h \to 0} f(x)}_{f(x)}.$$

Rewritten as

$$A'(x) = \frac{d}{dx} \underbrace{\int_a^x f(t)\,dt}_{A(x)} = f(x),$$

THEOREM 5.3 (PART 1) Fundamental Theorem of Calculus

If f is continuous on $[a, b]$, then the area function

$$A(x) = \int_a^x f(t)\, dt, \quad \text{for} \quad a \leq x \leq b,$$

is continuous on $[a, b]$ and differentiable on (a, b). The area function satisfies $A'(x) = f(x)$. Equivalently,

$$A'(x) = \frac{d}{dx}\int_a^x f(t)\, dt = f(x),$$

which means that the area function of f is an antiderivative of f on $[a, b]$.

It is one short step to a powerful method for evaluating definite integrals.

Assuming that F is any other antiderivative of f on $[a, b]$, then

$$F(x) = A(x) + C$$

Noting that $A(a) = 0$, it follows that

$$F(b) - F(a) = A(b) + C - (A(a) + C) = A(b)$$

Writing $A(b)$ in terms of a definite integral

$$A(b) = \int_a^b f(x)dx = F(b) - F(a)$$

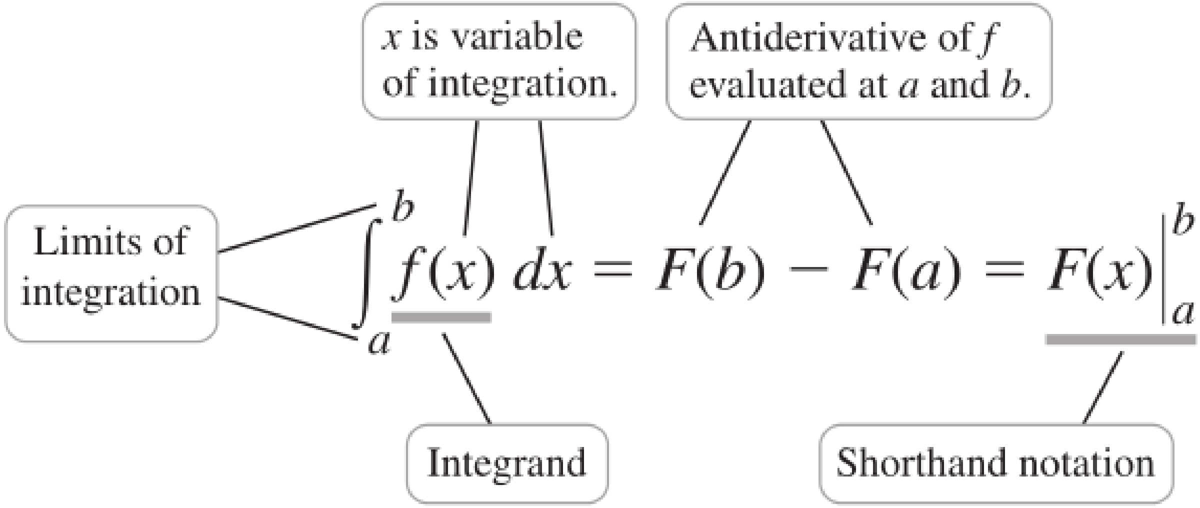
THEOREM 5.3 (PART 2) Fundamental Theorem of Calculus
If f is continuous on $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

To evaluate a definite integral of f, we

- find any antiderivative of f, which we call F; and
- compute $F(b) - F(a)$, the difference in the values of F between the upper and lower limits of integration.

It is customary and convenient to denote the difference $F(b) - F(a)$ by $F(x)|_a^b$



Inverse Relationship between Differentiation and Integration

Fundamental Theorem of Calculus says that

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

$$\int_a^b f'(x)dx = f(b) - f(a)$$

In other words, the integral "undoes" the derivative.

This last relationship is important because it expresses the *integral as an accumulation operation*.

The Fundamental Theorem says that we can integrate (that is, sum or accumulate) the rate of change over that interval and the result is simply the difference in f evaluated at the endpoints.

EXAMPLE 3 **Evaluating definite integrals** Evaluate the following definite integrals using the Fundamental Theorem of Calculus, Part 2. Interpret each result geometrically.

a. $\displaystyle\int_0^{10} (60x - 6x^2)\, dx$ **b.** $\displaystyle\int_0^{2\pi} 3 \sin x\, dx$ **c.** $\displaystyle\int_{1/16}^{1/4} \frac{\sqrt{t} - 1}{t}\, dt$

SOLUTION

$$\int_0^{10} (60x - 6x^2)\, dx = \left(30x^2 - 2x^3\right)\Big|_0^{10} \quad \text{Fundamental Theorem}$$

$$= (30 \cdot 10^2 - 2 \cdot 10^3) - (30 \cdot 0^2 - 2 \cdot 0^3) \quad \text{Evaluate at } x = 10 \text{ and } x = 0.$$

$$= (3000 - 2000) - 0$$

$$= 1000. \quad \text{Simplify.}$$

$$\int_0^{2\pi} 3 \sin x\, dx = -3 \cos x\Big|_0^{2\pi} \quad \text{Fundamental Theorem}$$

$$= (-3 \cos(2\pi)) - (-3 \cos(0)) \quad \text{Substitute.}$$

$$= -3 - (-3) = 0. \quad \text{Simplify.}$$

$$\int_{1/16}^{1/4} \frac{\sqrt{t}-1}{t}\,dt = \int_{1/16}^{1/4}\left(t^{-1/2}-\frac{1}{t}\right)dt$$

Simplify the integrand.

$$= \left(2t^{1/2} - \ln|t|\right)\Big|_{1/16}^{1/4}$$

Fundamental Theorem

$$= \left(2\left(\frac{1}{4}\right)^{1/2} - \ln\frac{1}{4}\right) - \left(2\left(\frac{1}{16}\right)^{1/2} - \ln\frac{1}{16}\right)$$

Evaluate.

$$= 1 - \ln\frac{1}{4} - \frac{1}{2} + \ln\frac{1}{16}$$

Simplify.

$$= \frac{1}{2} - \ln 4 \approx -0.8863.$$

EXAMPLE 4 **Net areas and definite integrals** The graph of $f(x) = 6x(x + 1)(x - 2)$ is shown in Figure 5.44. The region R_1 is bounded by the curve and the x-axis on the interval $[-1, 0]$, and R_2 is bounded by the curve and the x-axis on the interval $[0, 2]$.

a. Find the *net area* of the region between the curve and the x-axis on $[-1, 2]$.

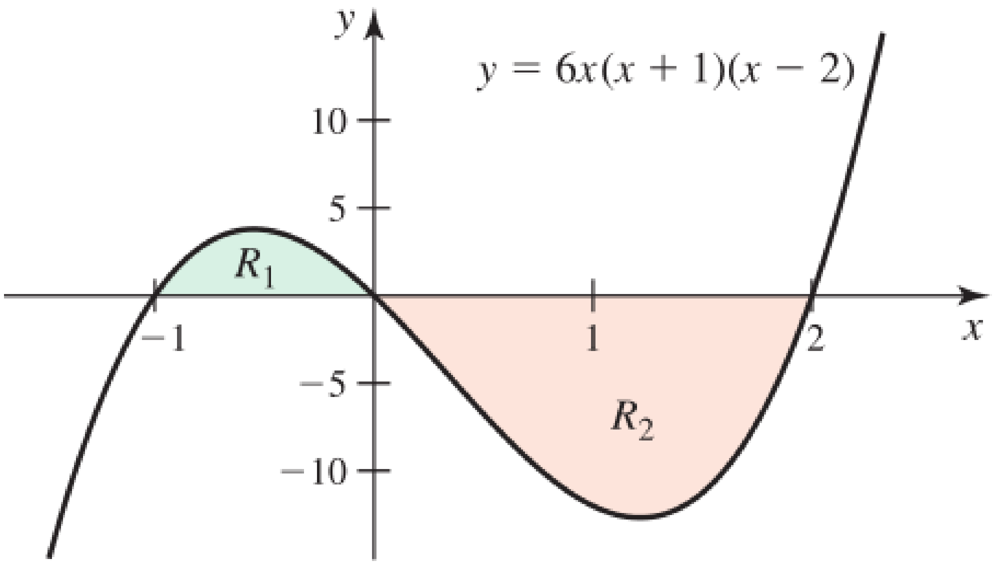
b. Find the *area* of the region between the curve and the x-axis on $[-1, 2]$.

SOLUTION

$$\int_{-1}^{2} f(x)\,dx = \int_{-1}^{2} (6x^3 - 6x^2 - 12x)\,dx. \qquad \text{Expand } f.$$

$$= \left(\frac{3}{2}x^4 - 2x^3 - 6x^2\right)\Bigg|_{-1}^{2} \qquad \text{Fundamental Theorem}$$

$$= -\frac{27}{2}. \qquad \text{Simplify.}$$



The region R_1 lies above the x-axis, so its area is

$$\int_{-1}^{0} (6x^3 - 6x^2 - 12x)\, dx = \left(\frac{3}{2}x^4 - 2x^3 - 6x^2\right)\Big|_{-1}^{0} = \frac{5}{2}.$$

The region R_2 lies below the x-axis, so its net area is

$$\int_{0}^{2} (6x^3 - 6x^2 - 12x)\, dx = \left(\frac{3}{2}x^4 - 2x^3 - 6x^2\right)\Big|_{0}^{2} = -16.$$

EXAMPLE 5 **Derivatives of integrals** Use Part 1 of the Fundamental Theorem to simplify the following expressions.

a. $\dfrac{d}{dx}\displaystyle\int_1^x \sin^2 t\, dt$ **b.** $\dfrac{d}{dx}\displaystyle\int_x^5 \sqrt{t^2+1}\, dt$ **c.** $\dfrac{d}{dx}\displaystyle\int_0^{x^2} \cos t^2\, dt$

SOLUTION

$$\frac{d}{dx}\int_1^x \sin^2 t\, dt = \sin^2 x.$$

$$\frac{d}{dx}\int_x^5 \sqrt{t^2+1}\, dt = -\frac{d}{dx}\int_5^x \sqrt{t^2+1}\, dt = -\sqrt{x^2+1}.$$

Treat as a composite function

$$y = g(u) = \int_0^u \cos t^2 \, dt.$$

$$\frac{d}{dx}\int_0^{x^2} \cos t^2 \, dt = \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$ Chain Rule

$$= \left(\frac{d}{du}\int_0^u \cos t^2 \, dt\right)(2x)$$ Substitute for g; note that $u'(x) = 2x$.

$$= (\cos u^2)(2x)$$ Fundamental Theorem

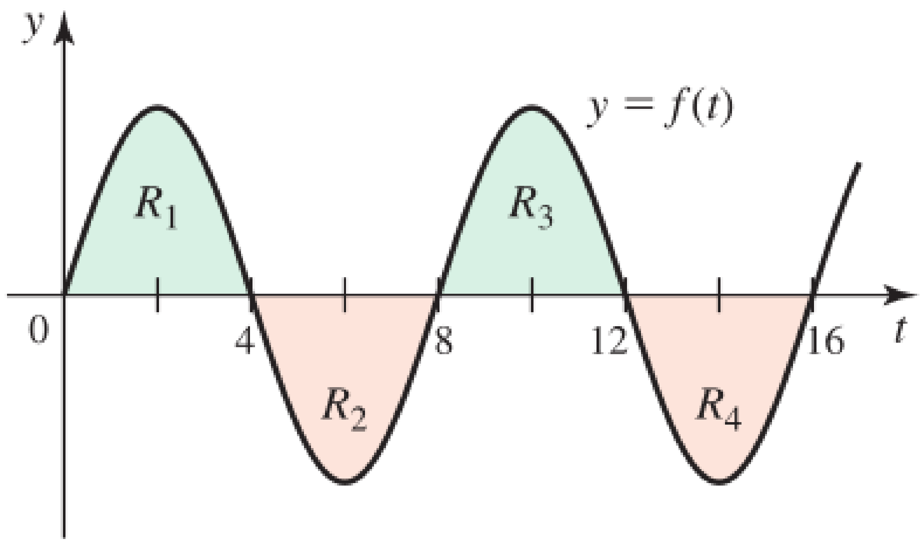
$$= 2x \cos x^4.$$ Substitute $u = x^2$.

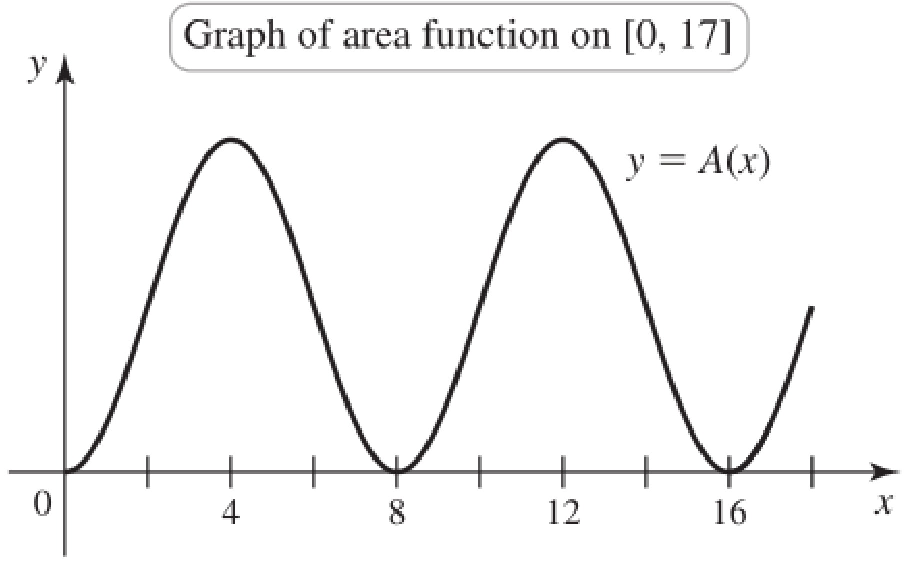
EXAMPLE 6 **Working with area functions** Consider the function f shown in Figure 5.45 and its area function $A(x) = \int_0^x f(t)\,dt$, for $0 \le x \le 17$. Assume that the four regions R_1, R_2, R_3, and R_4 have the same area. Based on the graph of f, do the following.

a. Find the zeros of A on $[0, 17]$.

b. Find the points on $[0, 17]$ at which A has local maxima or local minima.

c. Sketch a graph of A, for $0 \le x \le 17$.

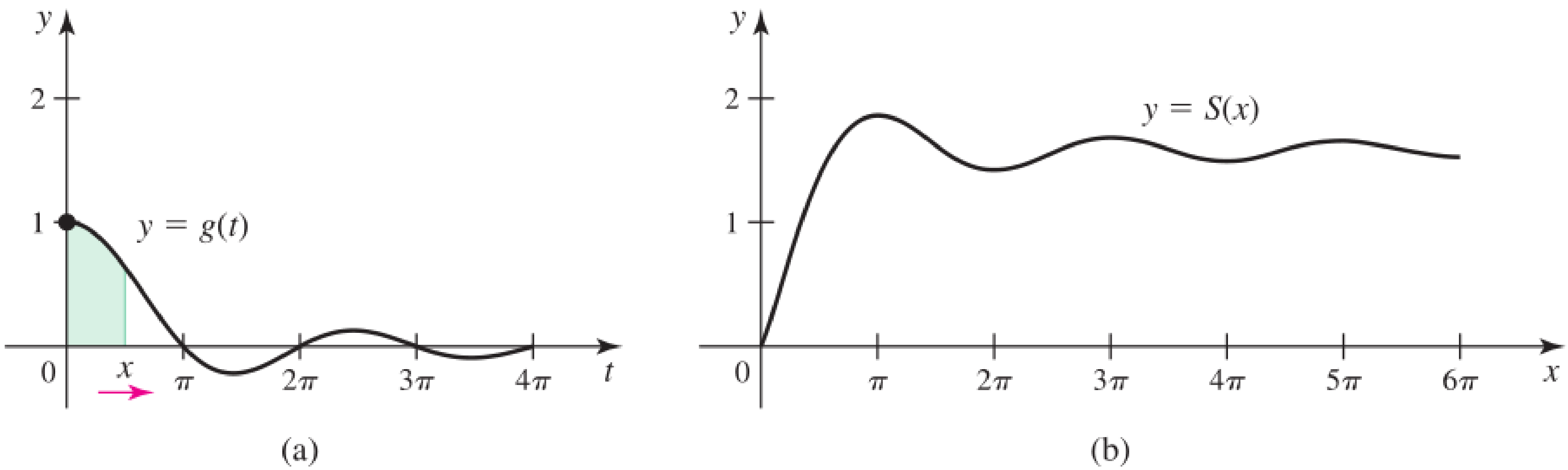




EXAMPLE 7 **The sine integral function** Let

$$g(t) = \begin{cases} \dfrac{\sin t}{t} & \text{if } t > 0 \\ 1 & \text{if } t = 0. \end{cases}$$

Graph the *sine integral function* $S(x) = \int_0^x g(t)\,dt$, for $x \geq 0$.



$$S'(x) = \frac{d}{dx}\int_0^x g(t)\,dt = \frac{\sin x}{x}, \text{ for } x > 0.$$

As anticipated, the derivative of S changes sign at integer multiples of π. Specifically, S' is positive and S increases on the intervals $(0, \pi)$, $(2\pi, 3\pi)$, . . . , $(2n\pi, (2n + 1)\pi)$, . . . , while S' is negative and S decreases on the remaining intervals. Clearly, S has local maxima at $x = \pi, 3\pi, 5\pi, \ldots$, and it has local minima at $x = 2\pi, 4\pi, 6\pi, \ldots$.

One more observation, although S oscillates for increasing x, its graph gradually flattens out and approaches a horizontal asymptote.

Proof of the Fundamental Theorem:

Let f be continuous on $[a, b]$ and let A be the area function for f with left endpoint a.

Step 1. We assume that $a < x < b$ and use the definition of the derivative,

$$A'(x) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h}.$$

First assume that $h > 0$. Using Figure 5.48 and Property 5 of Table 5.4, we have

$$A(x+h) - A(x) = \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt = \int_x^{x+h} f(t)\,dt.$$

That is, $A(x+h) - A(x)$ is the net area of the region bounded by the curve on the interval $[x, x+h]$.

Let m and M be the minimum and maximum values of f on $[x, x+h]$, respectively, which exist by the continuity of f. Suppose $x = t_0 < t_1 < t_2 < \ldots < t_n = x + h$ is a general partition of $[x, x+h]$ and let $\sum_{k=1}^{n} f(t_k^*)\Delta t_k$ be a corresponding general Riemann sum, where $\Delta t_k = t_k - t_{k-1}$. Because $m \leq f(t) \leq M$ on $[x, x+h]$, it follows that

$$\underbrace{\sum_{k=1}^{n} m\Delta t_k}_{mh} \leq \sum_{k=1}^{n} f(t_k^*)\Delta t_k \leq \underbrace{\sum_{k=1}^{n} M\Delta t_k}_{Mh}$$

or

$$mh \leq \sum_{k=1}^{n} f(t_k^*)\Delta t_k \leq Mh.$$

We have used the facts that $\sum_{k=1}^{n} m\Delta t_k = m\sum_{k=1}^{n} \Delta t_k = mh$ and similarly, $\sum_{k=1}^{n} M\Delta t_k = Mh$.

Notice that these inequalities hold for every Riemann sum for f on $[x, x + h]$; that is, for all partitions and for all n. Therefore, we are justified in taking the limit as $n \to \infty$ across these inequalities to obtain

$$\lim_{n\to\infty} mh \leq \underbrace{\lim_{n\to\infty} \sum_{k=1}^{n} f(t_k^*)\Delta t_k}_{\int_x^{x+h} f(t)\,dt} \leq \lim_{n\to\infty} Mh.$$

Evaluating each of these three limits results in

$$mh \leq \underbrace{\int_x^{x+h} f(t)\,dt}_{A(x+h)-A(x)} \leq Mh.$$

Substituting for the integral, we find that

$$mh \leq A(x + h) - A(x) \leq Mh.$$

Dividing these inequalities by $h > 0$, we have

$$m \leq \frac{A(x + h) - A(x)}{h} \leq M.$$

The case $h < 0$ is handled similarly and leads to the same conclusion

We now take the limit as $h \to 0$ across these inequalities. As $h \to 0$, m and M approach $f(x)$, because f is continuous at x. At the same time, as $h \to 0$, the quotient that is sandwiched between m and M approaches $A'(x)$:

$$\underbrace{\lim_{h\to 0} m}_{f(x)} = \underbrace{\lim_{h\to 0} \frac{A(x+h) - A(x)}{h}}_{A'(x)} = \underbrace{\lim_{h\to 0} M}_{f(x)}.$$

By the Squeeze Theorem (Theorem 2.5), we conclude that $A'(x)$ exists and A is differentiable for $a < x < b$. Furthermore, $A'(x) = f(x)$. Finally, because A is differentiable on (a, b), A is continuous on (a, b) by Theorem 3.1. Exercise 116 shows that A is also right- and left-continuous at the endpoints a and b, respectively.

Step 2. Having established that the area function A is an antiderivative of f, we know that $F(x) = A(x) + C$, where F is any antiderivative of f and C is a constant. Noting that $A(a) = 0$, it follows that

$$F(b) - F(a) = (A(b) + C) - (A(a) + C) = A(b).$$

Writing $A(b)$ in terms of a definite integral, we have

$$A(b) = \int_a^b f(x)\,dx = F(b) - F(a),$$

which is Part 2 of the Fundamental Theorem.

5.4

Working with Integrals

Integrating Even and Odd Functions

The role of *symmetry* in integrals

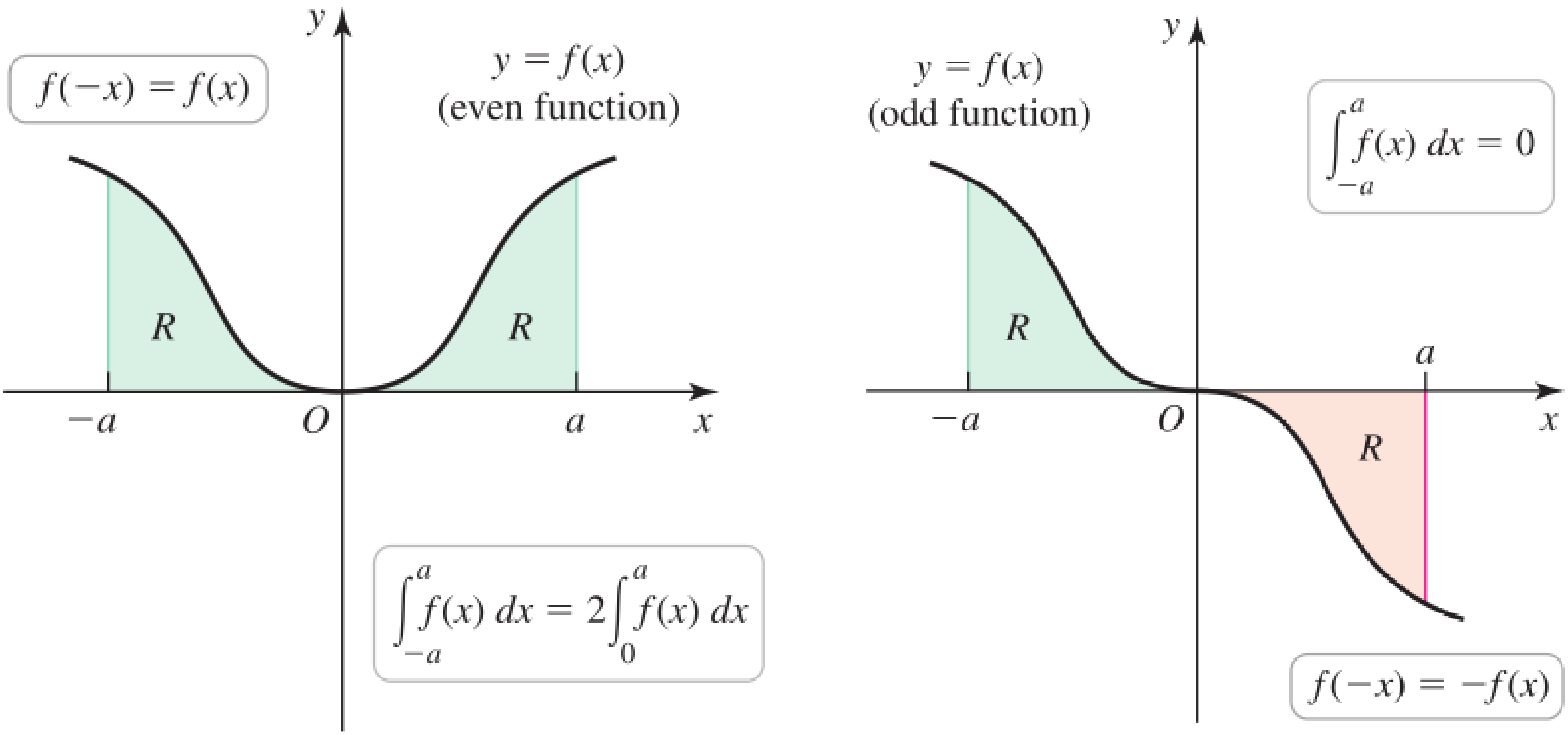
Applying the slice-and-sum strategy to define the average value of a function, leading to the *Mean Value Theorem for Integrals*.

Special things happen when we integrate even and odd functions on intervals centered at the origin.

THEOREM 5.4 Integrals of Even and Odd Functions

Let a be a positive real number and let f be an integrable function on the interval $[-a, a]$.

- If f is even, $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.
- If f is odd, $\int_{-a}^{a} f(x)\,dx = 0$.



$f(-x) = f(x)$
$y = f(x)$
(even function)
R
R
$-a$
O
a
x
y
$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$
$y = f(x)$
(odd function)
$\int_{-a}^{a} f(x)\,dx = 0$
R
R
$-a$
O
a
x
y
$f(-x) = -f(x)$

EXAMPLE 1 **Integrating symmetric functions** Evaluate the following integrals using symmetry arguments.

a. $\displaystyle\int_{-2}^{2} (x^4 - 3x^3)\, dx$

b. $\displaystyle\int_{-\pi/2}^{\pi/2} (\cos x - 4 \sin^3 x)\, dx$

Average Value of a Function

Consider a function f that is continuous on $[a, b]$.

A regular partition $x_0 = a, x_1, x_2, \cdots, x_n = b$, with $\Delta x = \frac{b-a}{n}$
Select a point x_k^* in each subinterval and compute $f(x_k^*)$
The average of these function values is

$$\frac{f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)}{n}$$

Write the average of the n sample values as the Riemann sum

$$\frac{f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)}{(b-a)/\Delta x} = \frac{1}{b-a}\sum_{k=1}^{n} f(x_k^*)\Delta x$$

The limit of this sum is a definite integral that gives the average value $\bar{f}$ on $[a, b]$:

$$\bar{f} = \frac{1}{b-a} \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x = \frac{1}{b-a} \int_a^b f(t)dt$$

DEFINITION Average Value of a Function

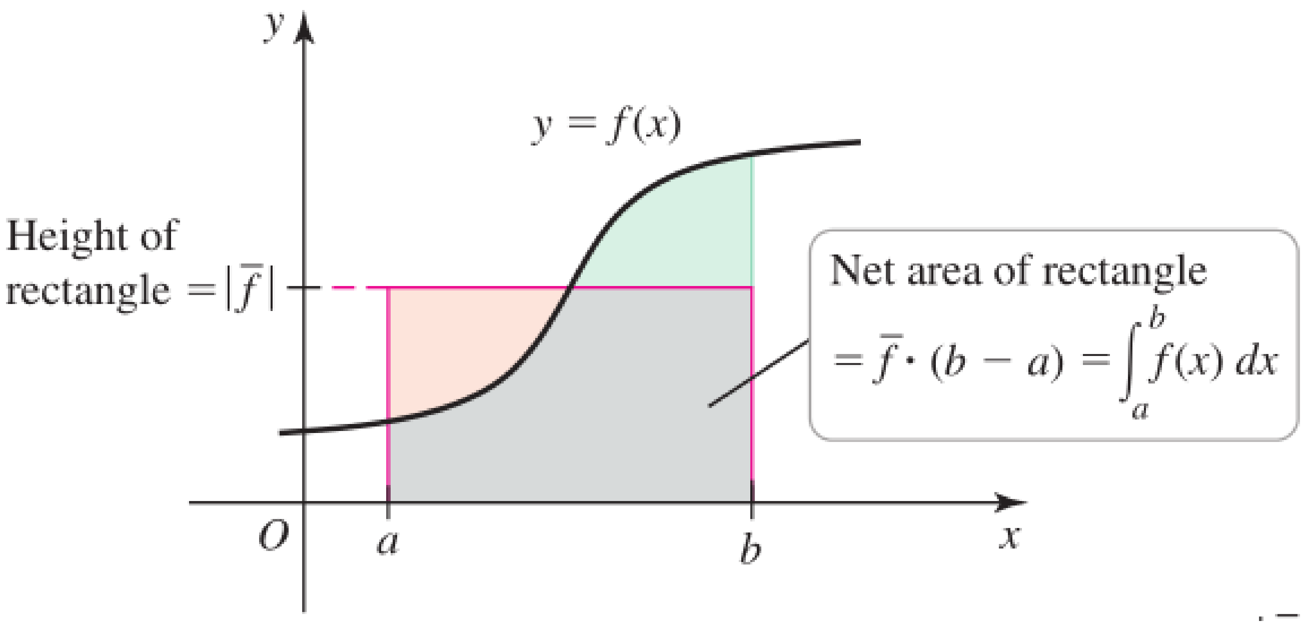
The average value of an integrable function f on the interval $[a, b]$ is

$$\bar{f} = \frac{1}{b-a} \int_a^b f(x)\, dx.$$

Geometrical interpretation of the average value of a function f on an interval $[a, b]$.

Multiplying both sides of the average value by $(b - a)$,

$$\underbrace{(b-a)\bar{f}}_{\text{net area of rectangle}} = \underbrace{\int_a^b f(x)\,dx.}_{\text{net area of region bounded by curve}}$$

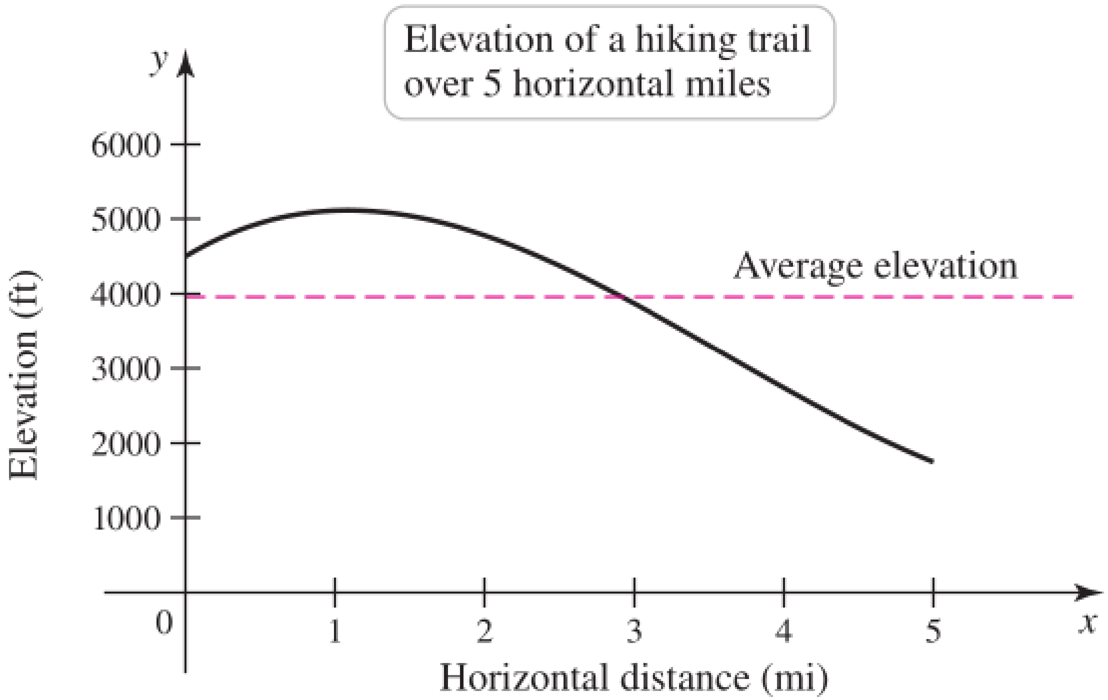


Noting that $\bar{f}$ may be zero or negative

EXAMPLE 2 **Average elevation** A hiking trail has an elevation given by

$$f(x) = 60x^3 - 650x^2 + 1200x + 4500,$$

where f is measured in feet above sea level and x represents horizontal distance along the trail in miles, with $0 \leq x \leq 5$. What is the average elevation of the trail?



Mean Value Theorem for Integrals

THEOREM 5.5 Mean Value Theorem for Integrals
Let f be continuous on the interval $[a, b]$. There exists a point c in (a, b) such that

$$f(c) = \bar{f} = \frac{1}{b-a}\int_a^b f(t)\,dt.$$

Proof:

Letting $F(x) = \int_a^x f(t)dt$,

F is continuous on $[a, b]$ and differentiable on (a, b) (Fundamental Theorem, Part 1).

Apply the Mean Value Theorem for derivatives to F and conclude that there exists at least one point c in (a, b) such that

$$F'(c) = \frac{F(b) - F(a)}{b - a}$$

We know that

$$F'(c) = f(c)$$

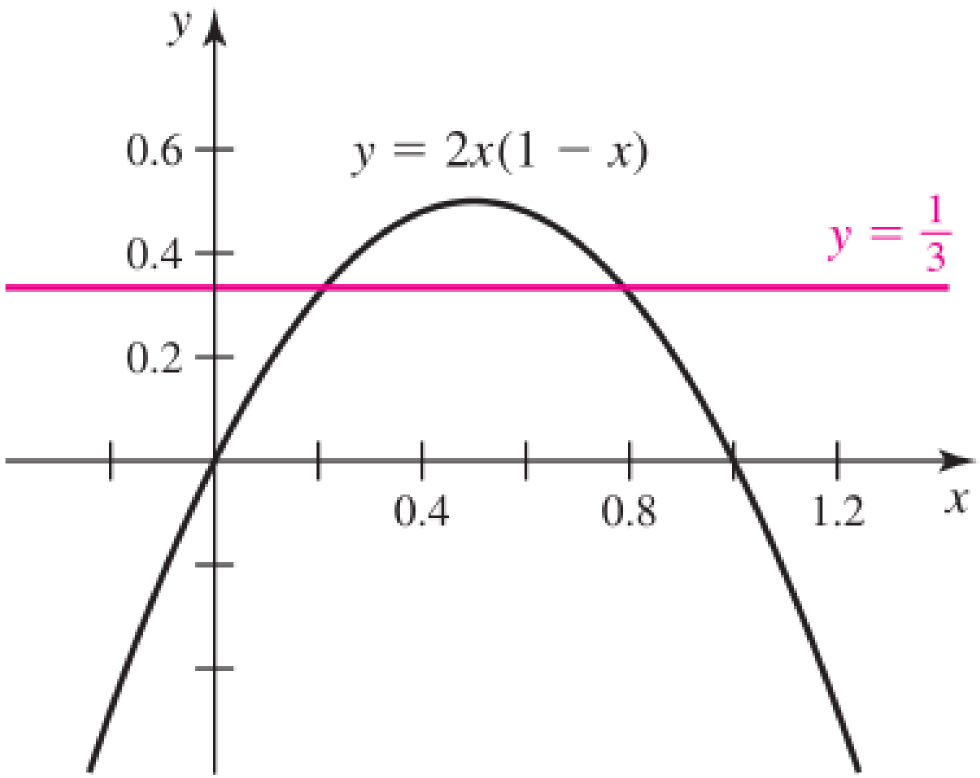
By Fundamental Theorem, Part 2

$$F(b) - F(a) = \int_a^b f(x)dx$$

Combining these observations, we have

$$f(c) = \frac{1}{b - a}\int_a^b f(x)dx$$

EXAMPLE 3 **Average value equals function value** Find the point(s) on the interval $(0, 1)$ at which $f(x) = 2x(1 - x)$ equals its average value on $[0, 1]$.



5.5

Substitution Rule

Indefinite Integrals

One way to find new antiderivative rules is to start with familiar derivative rules and work backward.
When applied to the Chain Rule, leads to the Substitution Rule.

EXAMPLE 1 **Antiderivatives by trial and error** Find $\int \cos 2x \, dx$.

$$\frac{d}{dx}\left(\frac{1}{2}\sin 2x\right) = \frac{1}{2} \cdot 2\cos 2x = \cos 2x.$$

$$\int \cos 2x \, dx = \frac{1}{2}\sin 2x + C.$$

Using the Chain Rule to differentiate the composite function $F(g(x))$

$$\frac{d}{dx}\big(F(g(x))\big) = \underbrace{F'(g(x))}_{f(g(x))}g'(x) = f(g(x))g'(x).$$

This equation says that $F(g(x))$ is an antiderivative of $f(g(x))g'(x)$

$$\int f(g(x))g'(x)\,dx = F(g(x)) + C,$$

Identify the "inner function" as $u = g(x)$, which implies that $du = g'(x)dx$.

$$\int \underbrace{f(g(x))}_{f(u)}\underbrace{g'(x)dx}_{du} = \int f(u)\,du = F(u) + C.$$

THEOREM 5.6 Substitution Rule for Indefinite Integrals

Let $u = g(x)$, where g' is continuous on an interval, and let f be continuous on the corresponding range of g. On that interval,

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

PROCEDURE **Substitution Rule (Change of Variables)**

1. Given an indefinite integral involving a composite function $f(g(x))$, identify an inner function $u = g(x)$ such that a constant multiple of $g'(x)$ appears in the integrand.
2. Substitute $u = g(x)$ and $du = g'(x)\,dx$ in the integral.
3. Evaluate the new indefinite integral with respect to u.
4. Write the result in terms of x using $u = g(x)$.

Disclaimer: Not all integrals yield to the Substitution Rule.

EXAMPLE 2 **Perfect substitutions** Use the Substitution Rule to find the following indefinite integrals. Check your work by differentiating.

a. $\int 2(2x + 1)^3\,dx$ **b.** $\int 10e^{10x}\,dx$

EXAMPLE 3 **Introducing a constant** Find the following indefinite integrals.

a. $\int x^4(x^5 + 6)^9\,dx$ **b.** $\int \cos^3 x \sin x\,dx$

EXAMPLE 4 **Variations on the substitution method** Find $\displaystyle\int \frac{x}{\sqrt{x+1}}\,dx$.

General Formulas for Indefinite Integrals

Assume that $a \neq 0$ is a real number and that C is an arbitrary constant.

Table 4.9 **Indefinite Integrals of Trigonometric Functions**

1. $\dfrac{d}{dx}(\sin ax) = a \cos ax \quad \Rightarrow \quad \displaystyle\int \cos ax\, dx = \frac{1}{a}\sin ax + C$
2. $\dfrac{d}{dx}(\cos ax) = -a \sin ax \quad \Rightarrow \quad \displaystyle\int \sin ax\, dx = -\frac{1}{a}\cos ax + C$
3. $\dfrac{d}{dx}(\tan ax) = a \sec^2 ax \quad \Rightarrow \quad \displaystyle\int \sec^2 ax\, dx = \frac{1}{a}\tan ax + C$
4. $\dfrac{d}{dx}(\cot ax) = -a \csc^2 ax \quad \Rightarrow \quad \displaystyle\int \csc^2 ax\, dx = -\frac{1}{a}\cot ax + C$
5. $\dfrac{d}{dx}(\sec ax) = a \sec ax \tan ax \quad \Rightarrow \quad \displaystyle\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax + C$
6. $\dfrac{d}{dx}(\csc ax) = -a \csc ax \cot ax \quad \Rightarrow \quad \displaystyle\int \csc ax \cot ax\, dx = -\frac{1}{a}\csc ax + C$

Other Indefinite Integrals

Table 4.10 Other Indefinite Integrals

7. $\dfrac{d}{dx}(e^{ax}) = ae^{ax} \Rightarrow \displaystyle\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$

8. $\dfrac{d}{dx}(b^{x}) = b^{x}\ln b \Rightarrow \displaystyle\int b^{x}\,dx = \frac{1}{\ln b}b^{x} + C,\ b > 0,\ b \neq 1$

9. $\dfrac{d}{dx}(\ln|x|) = \dfrac{1}{x} \Rightarrow \displaystyle\int \frac{dx}{x} = \ln|x| + C$

10. $\dfrac{d}{dx}\left(\sin^{-1}\dfrac{x}{a}\right) = \dfrac{1}{\sqrt{a^2 - x^2}} \Rightarrow \displaystyle\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C$

11. $\dfrac{d}{dx}\left(\tan^{-1}\dfrac{x}{a}\right) = \dfrac{a}{a^2 + x^2} \Rightarrow \displaystyle\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

12. $\dfrac{d}{dx}\left(\sec^{-1}\left|\dfrac{x}{a}\right|\right) = \dfrac{a}{x\sqrt{x^2 - a^2}} \Rightarrow \displaystyle\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C,\ a > 0$

EXAMPLE 5 **Additional indefinite integrals** Evaluate the following indefinite integrals. Assume a is a nonzero real number.

a. $\displaystyle\int \frac{dx}{x}$ **b.** $\displaystyle\int e^{ax}\,dx$ **c.** $\displaystyle\int \frac{dx}{a^2 + x^2}$

EXAMPLE 6 **Indefinite integrals** Determine the following indefinite integrals.

a. $\displaystyle\int e^{-10t}\,dt$ **b.** $\displaystyle\int \frac{4}{\sqrt{9 - x^2}}\,dx$ **c.** $\displaystyle\int \frac{dx}{16x^2 + 1}$

Definite Integrals

Two ways to use Substitution Rule for definite integrals

- Use the Substitution Rule to find an antiderivative F and then use the Fundamental Theorem to evaluate $F(b) - F(a)$.
- Alternatively, once you have changed variables from x to u, you also may change the limits of integration and complete the integration with respect to u.
  - ✓ If $u = g(x)$, the lower limit $x = a$ is replaced with $u = g(a)$ and the upper limit $x = b$ is replaced with $u = g(b)$.

The second option tends to be more efficient, and we use it whenever possible.

THEOREM 5.7 Substitution Rule for Definite Integrals
Let $u = g(x)$, where g' is continuous on $[a, b]$, and let f be continuous on the range of g. Then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

EXAMPLE 5 Definite integrals Evaluate the following integrals.

a. $\displaystyle\int_0^2 \frac{dx}{(x+3)^3}$ **b.** $\displaystyle\int_0^4 \frac{x}{x^2+1}\,dx$ **c.** $\displaystyle\int_0^{\pi/2} \sin^4 x \cos x\,dx$

- When the integrand has the form $f(ax + b)$, the substitution $u = ax + b$ is often effective.

EXAMPLE 6 **Integral of $\cos^2\theta$** Evaluate $\displaystyle\int_0^{\pi/2} \cos^2\theta\, d\theta$.

$$\int \cos^2\theta\, d\theta = \int \frac{1+\cos 2\theta}{2}\, d\theta = \frac{1}{2}\int d\theta + \frac{1}{2}\int \cos 2\theta\, d\theta.$$

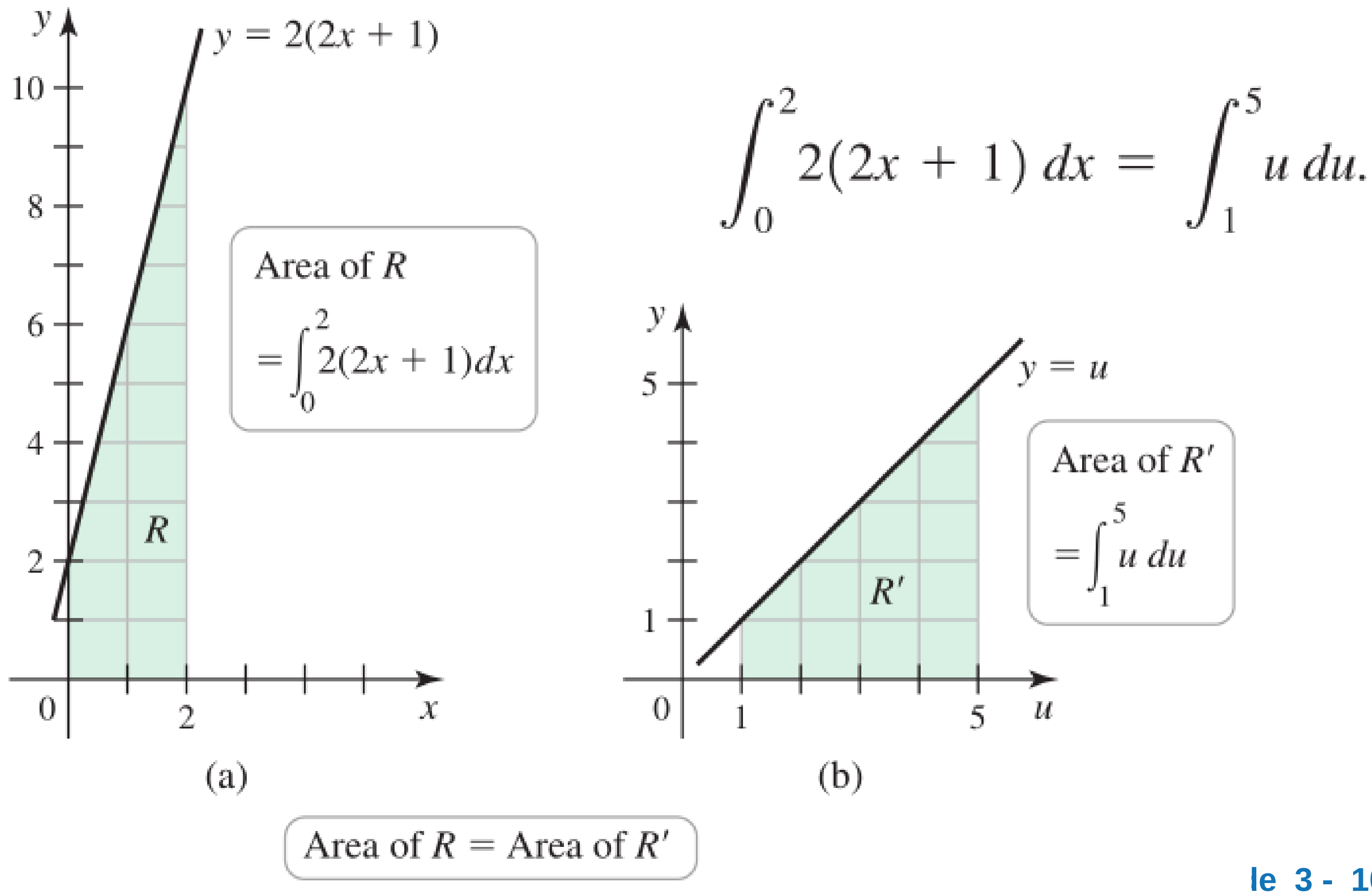
$$= \frac{\theta}{2} + \frac{1}{4}\sin 2\theta + C.$$

$$\int \sin^2 ax\, dx = \frac{x}{2} - \frac{\sin(2ax)}{4a} + C \quad \text{and}$$

$$\int \cos^2 ax\, dx = \frac{x}{2} + \frac{\sin(2ax)}{4a} + C.$$

Geometry of Substitution

Consider the integral $\int_0^2 2(2x+1)dx$



Chapter 5

Integration

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