MATH 2233 Differential Equations Chapter 8 Series Solutions of Differential Equations

Section 8.1 Taylor Polynomial Approximation

Goal of this section

• Determine the Taylor polynomial approximation of the initial value problems of differential equations.

Taylor Polynomials

The n-th degree Taylor polynomial $p_n(x)$ of a given function f(x) near a particular point $x=x_0$ is

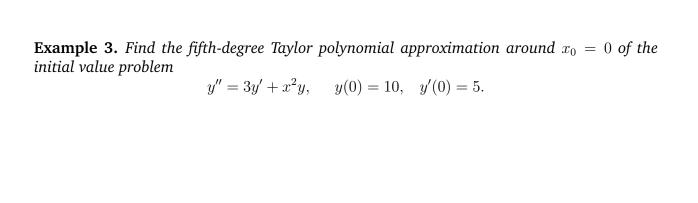
The Taylor polynomial is probably the best approximation of f(x) near $x=x_0$. In fact,

• $p_n(x)$ matches the value of f(x) and its derivatives:

• In particular, when n = 1,

Example 1. Find the first four Taylor polynomials for $f(x) = e^x$ around $x_0 = 0$.

Example 2. Determine the fourth-degree Taylor polynomials matching the functions e^x , $\cos(x)$, and $\sin(x)$ at $x_0 = 2$.



Example 4. Determine the Taylor polynomial of degree 3 for the solution of the initial value problem

$$y' = \frac{1}{x+y+1}, \quad y(0) = 0.$$

MATH 2233 Differential Equations

Section 8.2 Power Series and Analytic Functions

Goal of this section

- get to know the definition of power series
- find convergence region of power series
- get to know the definition of analytic functions.

Recall

Given a function y(x), the Taylor polynomial approximation is

When $n \to \infty$, it is called Taylor series

Question: Will the series always converge?

Power Series

A **power series** about the point x_0 is

We say that the series **converges** at the point x = c if the infinite series

If the limit does not exist, the power series is called to **diverge** at x = c.

Radius of Convergence

Clearly the power series converges at $x = x_0$,

In fact, the power series converges <i>absolutely</i> inside some interval $(x_0 - \rho, x_0 + \rho)$ centered at x_0 , but diverges outside this interval. The positive number ρ is called

Ratio Test

For the power series $\sum_{n=0}^{\infty} a_n (x-x_0)^n$, if for n large, the coefficient a_n are nonzero and satisfy

$$\lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = L \quad (0 \le L \le \infty),$$

then

Example 1. Determine the converge set of

$$\sum_{n=0}^{\infty} \frac{(-2)^n}{n+1} (x-3)^n$$

For each value of x for which the power series $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ converges, we can denote its sum by a function f(x). For example, we have the **geometric series**:

Remark

- If the power series $\sum_{n=0}^{\infty} a_n (x-x_0)^n = 0$ for all x in some open interval, then
- If the power series $f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$ has a positive convergence radius ρ , then

Example 2. Using the geometric series find a power series for each of the following functions:

(a)
$$\frac{1}{1+x^2}$$
 (b) $\frac{1}{(1-x)^2}$ (c) $\arctan(x)$.

Shifting the Summation Index

The index of summation in a power series is a dummy index just like the variable of integration. Thus,

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n =$$

Example 3. Express the series

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$$

as a series where the generic term is x^k instead of x^{n-2} .

Example 4. Find the power series expansion $\sum_{n=0}^{\infty} a_n x^n$ for f(x) + g(x) where

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n+5} x^n, \quad g(x) = \sum_{n=1}^{\infty} 3^{-n} x^{n-1}.$$

Example 5. Show that

$$x^{3} \sum_{n=0}^{\infty} n^{2}(n-2)a_{n}x^{n} = \sum_{n=3}^{\infty} (n-3)^{2}(n-5)a_{n-3}x^{n}.$$

Analytic Functions

Not all functions can be expressed as power series. Those distinguished functions that can be so represented are called **analytic**.

Definition (Analytic at a point)

A function f is said to be **analytic at** x_0 if

Example 6. The following functions are typical analytic functions:

- polynomial function $f(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_m x^m$
- rational function $f(x) = \frac{P(x)}{Q(x)}$, where P(x) and Q(x) are polynomials

- exponential function
- sine cosine functions

• logarithm function

Remark. If f(x) is analytic at x_0 , then

MATH 2233 Differential Equations

Section 8.3 Power Series Solutions to Linear Differential Equations

Goal of this section

• determine power series solution to a linear differential equation with polynomial coefficients.

Recall

A second-order linear differential equation has the general form

It can be written in the standard form

Definition: Ordinary and Singular Points

A point x_0 is called an **ordinary point** of the above differential equation if

If x_0 is not an ordinary point, it is called

Example 1. Determine all singular points of

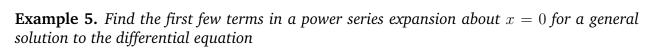
$$(x^2 - 7)y'' + 4y' + \sin(x)y = 0$$

Example 2. Determine all singular points of

$$xy'' + x(1-x)^{-1}y' + \sin(x)y = 0$$







$$(1+x^2)y'' - y' + y = 0.$$