# **MATH 2233 Differential Equations**

# **Chapter 1 Introduction**

# Section 1.1 Background

### Goal of this section

- 1. understand what is a differential equation.
- 2. understand the classification of differential equations.

## 1. Definition of Differential Equations

## Example 1. (Types of equations)

- 1. Find x in  $x^2 + 2x + 1 = 0$ .
- 2. Find f(t) in  $f(t)e^t + \sin(t) = \cos(t)$ .
- 3. Find y(t) in  $y''(t) + 3y'(t) = e^t$ .

# **Definition.** A differential equation is

•

•

**Remark.** If a differential equation involves the derivative of one variable with respect to another, then the former is called a \_\_\_\_\_\_, and the latter an \_\_\_\_\_.

For example, in the equation

$$\frac{d^2x}{dt^2} + a\frac{dx}{dt} + kx = 0,$$

- *x*:
- *t*:
- *a* and *k*:

# 2. Classification of Differential Equations

### 2.1. By Type

• If a differential equation involves only *ordinary derivatives* of the unknown function, it is called an

e.g.

• If a differential equation involves *partial derivatives*, it is called a \_\_\_\_\_\_

e.g.

### 2.2 By Order

The \_\_\_\_\_\_ of a differential equation is the order of the highest-order derivatives of the unknown in the equation.

**Example 2.** Determine the type and the order of the following differential equations.

$$y'' + 4y' = e^x$$

• 
$$y'' + 4(y')^3 + 5y = e^x$$

$$\bullet \ u_t - 2u_{xx} = 0$$

• 
$$t^2y''' - t^3y'' + ty^4 = \sin(t)$$

## 2.3. By Linearity

An n-th order ordinary differential equation  $F(x,y,y',\cdots,y^{(n)})=0$  is called \_\_\_\_\_\_, if

More explicitly, an ODE is linear, if it can be written as

If a differential equation is not linear, it is called \_\_\_\_\_.

**Remark:** There are two special cases that we are going to discuss throughout this semester

- linear first-order equations:
- linear second-order equations:

**Example 3.** Determine the type, order, and linearity of the following differential equations.

$$\bullet \ x^3y''' + xy' - 5y = e^x$$

$$\bullet \ \frac{dy}{dx} = \frac{x}{y+2}$$

$$\bullet \ \frac{d^2y}{dx^2} = \frac{y+2}{\sqrt{x^2+1}}$$

$$\bullet \ (1-y)y' + 2y = e^x$$

$$\bullet \ y'' + \sin(x)y = 0$$

$$y'' + x\sin(y) = 0$$

# **MATH 2233 Differential Equations**

## **Section 1.2 Solutions and Initial Value Problems**

#### Goal of this section

- 1. understand the explicit and implicit solution of a differential equation.
- 2. understand the initial value problem for a differential equation.
- 3. understand the existence and uniqueness of solution.

## 1. Explicit Solution

The general form of n-th order ODEs with x independent, y dependent, can be expressed as

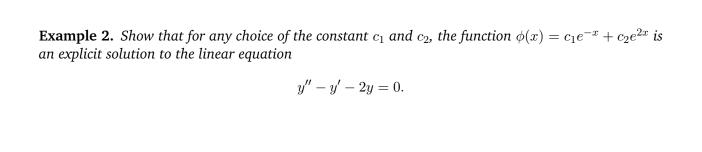
In many cases, we can isolate the highest-order term and write the equation as

<u>**Definition.**</u> A function  $\phi(x)$  is called an \_\_\_\_\_ of an ODE if the equation becomes an identity when substituting y by  $\phi(x)$ .

**Example 1.** Verify that  $\phi(x) = x^2 - x^{-1}$  is an explicit solution to the differential equation

$$\frac{d^2y}{dx^2} - \frac{2}{x^2}y = 0,$$

but  $\psi(x) = x^3$  is not.



**Example 3.** Determine for which values of m the function  $\phi(x) = x^m$  is a solution to the differential equation

$$5x^2y'' - 11xy' + 3y = 0.$$

# 2. Implicit Solution

As we will see in following chapters, the methods for solving differential equations do not always yield an explicit solution. We may have to settle for a solution that is defined implicitly.

**Example 4.** Show that the relation  $y^2 - x^3 + 8 = 0$  implicitly defines a solution to the nonlinear equation

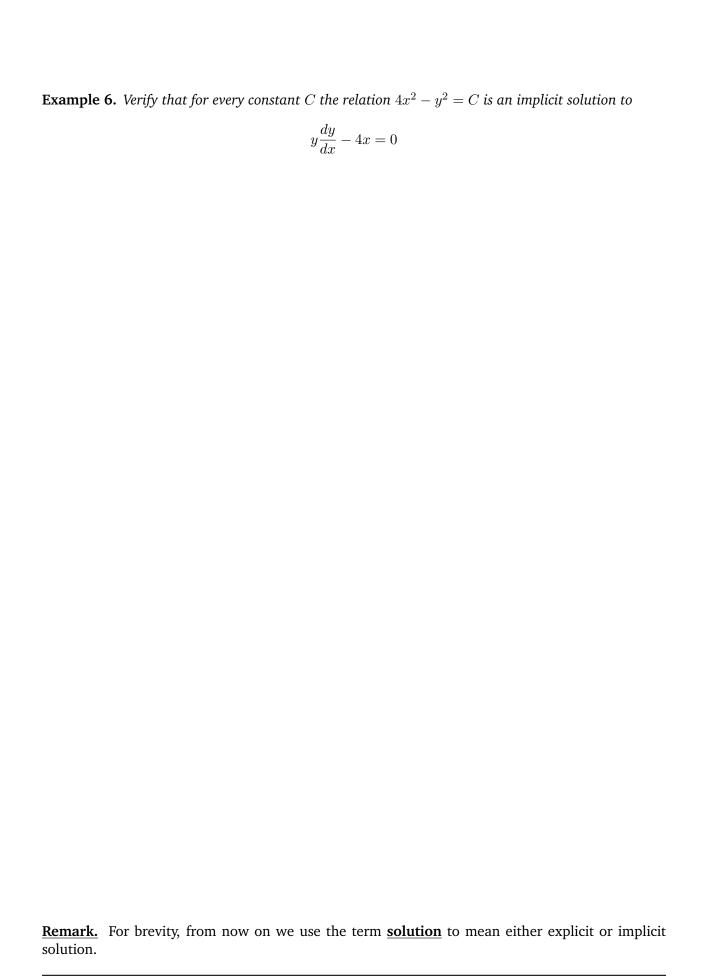
$$\frac{dy}{dx} = \frac{3x^2}{2y}$$

on the interval  $(2, \infty)$ .

**<u>Definition.</u>** A relation G(x,y) is said to be an \_\_\_\_\_\_ of an ODE on the interval I if it defines one or more explicit solutions on I.

**Example 5.** Show that the relation  $x + y + e^{xy} = 0$  is an implicit solution to the nonlinear equation

$$(1 + xe^{xy})\frac{dy}{dx} + 1 + ye^{xy} = 0.$$



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As indicated in Example 2, a differential equation usually has infinitely many solutions. To uniquely determine a solution, we often impose additional conditions.

### Example 7.

- ullet Find  $\underline{all}$  solutions of the differential equation  $\frac{dy}{dt}=y.$
- In addition to the differential equation, we also require y(0) = 3. What can we say about the solution?

#### Remark.

- ullet The additional condition y(0)=3 is often called \_\_\_\_\_\_\_, since the independent variable t often represents time in many physical applications.

#### Remark.

- The IVP for a first-order differential equation is
- The IVP for a second-order differential equation is

**Example 8.** As shown in Example 2, the function  $\phi(x) = c_1 e^{-x} + c_2 e^{2x}$  is a solution to

$$y'' - y' - 2y = 0$$

for any choice of constants  $c_1$  and  $c_2$ . Determine  $c_1$  and  $c_2$  so that the initial conditions

$$y(0) = 2$$
 and  $y'(0) = -3$ 

are satisfied.

# 4. Existence and Uniqueness of IVP

### Theorem (Existence and Uniqueness of Solution)

Consider the initial value problem

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0.$$

If f and  $\frac{\partial f}{\partial y}$  are continuous functions in some rectangle

$$R = \{(x, y) : a < x < b, c < y < d\}$$

that contains the point  $(x_0, y_0)$ , then the initial value problem has a unique solution  $\phi(x)$  in some interval  $x_0 - \delta < x < x_0 + \delta$ , where  $\delta$  is a positive number.

**Example 9.** Show that there exists a unique solution to the initial value problem

$$3\frac{dy}{dx} = x^2 - xy^3, \quad y(1) = 6.$$

# **MATH 2233 Differential Equations**

## **Section 1.3 Direction Fields**

### Goal of this section

- 1. draw a direction field to see the solution of 1st-order equations without solving them
- 2. use the direction field to analyze properties of the solution
- 3. introduce an online software to sketch more accurate direction fields

### 1. Direction Field

**<u>Definition.</u>** The **direction field** of the first order differential equation

$$\frac{dy}{dx} = f(x, y)$$

is a plot of short line segments drawn at various points in the xy-plane showing the slope of the solution curve.

**Example 1.** Sketch the direction field of the differential equation

$$\frac{dy}{dx} = 1 - \frac{y}{5}$$

# 2. Use Direction Field to Analyze the Solution

**Example 2.** The following differential equation

$$\frac{dp}{dt} = p(2-p)$$

is a <u>logistic equation</u> for modeling the growth of population. Here, p (in thousand) is the population at time t. Sketch the direction field of this equation. Then answer the following questions.

- 1. If the initial population is 3000, what can you say about the the population in a long time?
- 2. Can a population of 1000 ever decline to 500?
- 3. Can a population of 1000 ever increase to 3000?

### 3. Online Direction Field Softwares

Computer softwares can be used to sketch direction fields of more complicated differential equations accurately. For example,

Geogebra: https://www.geogebra.org/m/W7dAdgqc

**Example 3.** Use a computer software to sketch the direction field of

$$\frac{dy}{dx} = x^2 - y.$$

From the direction field, sketch the solutions with initial conditions y(0) = 0, 1, and -1.