Chapter 7

Logarithmic, Exponential and Hyperbolic Functions

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7.1

Logarithmic and Exponential Functions Revisited

Objective

Earlier in the text, we made several claims about exponential and logarithmic functions (e.g., they are continuous and differentiable on their domains), but did not prove them.

The objective in this section is to place these important functions on a solid foundation by presenting a more rigorous development of their properties.

Roadmap for This Section

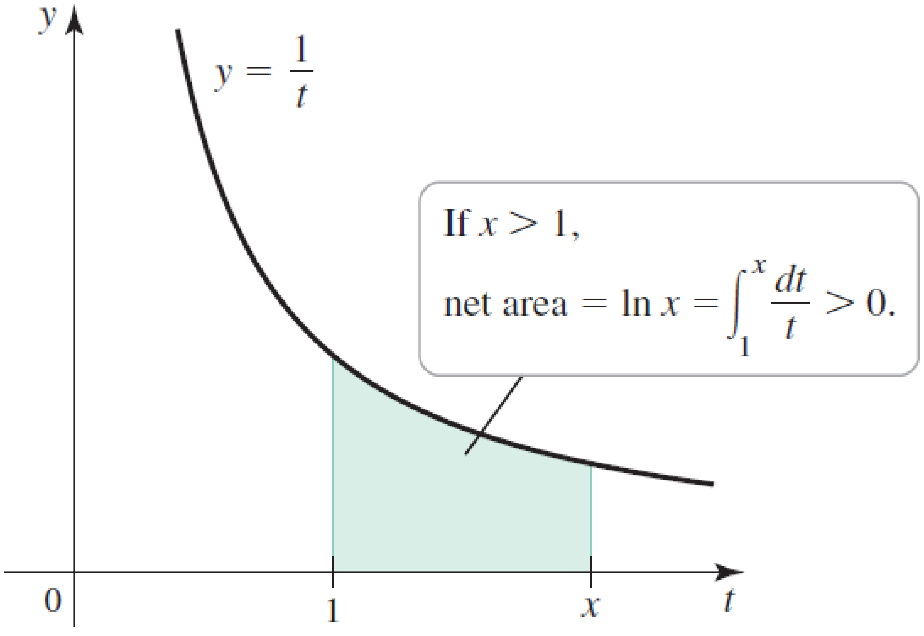
- First define the natural logarithm function in terms of an integral, and then derive the properties of $\ln x$.
- The natural exponential function e^x is introduced as the inverse of $\ln x$, and the properties of e^x are developed.
- Next, define the general exponential function b^x in terms of e^x, and the general logarithmic function $\log_b x$ in terms of $\ln x$.
- Finally, revisit the General Power Rule (Section 3.9) and derive a limit that can be used to approximate e.

$$
\begin{tikzcd}
\ln x \arrow[r] \arrow[d] & e^x \arrow[d] \\
\log_b x & b^x
\end{tikzcd}
$$

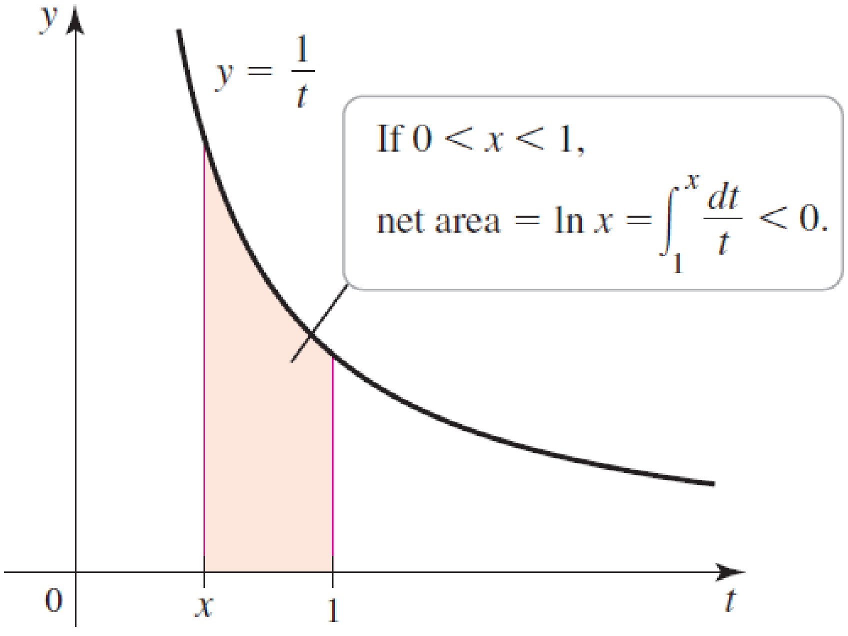
Step 1: The Natural Logarithm

DEFINITION The Natural Logarithm

The **natural logarithm** of a number $x > 0$, is $\ln x = \displaystyle\int_1^x \frac{1}{t}\,dt.$



(a)



(b)

Properties of the Natural Logarithm

Domain, range, and sign

The integrand is undefined at $t = 0$, so the domain of $\ln x$ is $(0, \infty)$.

Its value is the net area under the curve $y = 1/t$ between $t = 1$ and $t = x$.

On the interval $(1, \infty)$, $\ln x$ is positive.

On $(0,1)$, we have $\int_1^x \frac{1}{t}\,dt = -\int_x^1 \frac{1}{t}\,dt$, so $\ln x$ is negative.

when $x = 1$, we have $\ln 1 = \int_1^1 \frac{1}{t}\,dt = 0$

The net area interpretation of $\ln x$ implies that the range of $\ln x$ is $(-\infty, \infty)$.

Derivative

According to the Fundamental Theorem of Calculus

$$\frac{d}{dx}(\ln x) = \frac{d}{dx}\int_1^x \frac{1}{t}\,dt = \frac{1}{x}$$

for $x > 0$.

Two important consequences:

- Because its derivative is defined for $x > 0$, $\ln x$ is a differentiable function for $x > 0$, which means it is continuous on its domain (Theorem 3.1).
- Because $1/x > 0$ for $x > 0$, $\ln x$ is strictly increasing and one-to-one on its domain; therefore, it has a well-defined inverse.

According to the Chain Rule, when $x < 0$,

$$\frac{d}{dx}(\ln(-x)) = \frac{-1}{-x} = \frac{1}{x}$$

So,

$$\frac{d}{dx}(\ln|x|) = \frac{1}{x}$$

More generally,

$$\frac{d}{dx}(\ln|u(x)|) = \frac{1}{u(x)}u'(x) = \frac{u'(x)}{u(x)}$$

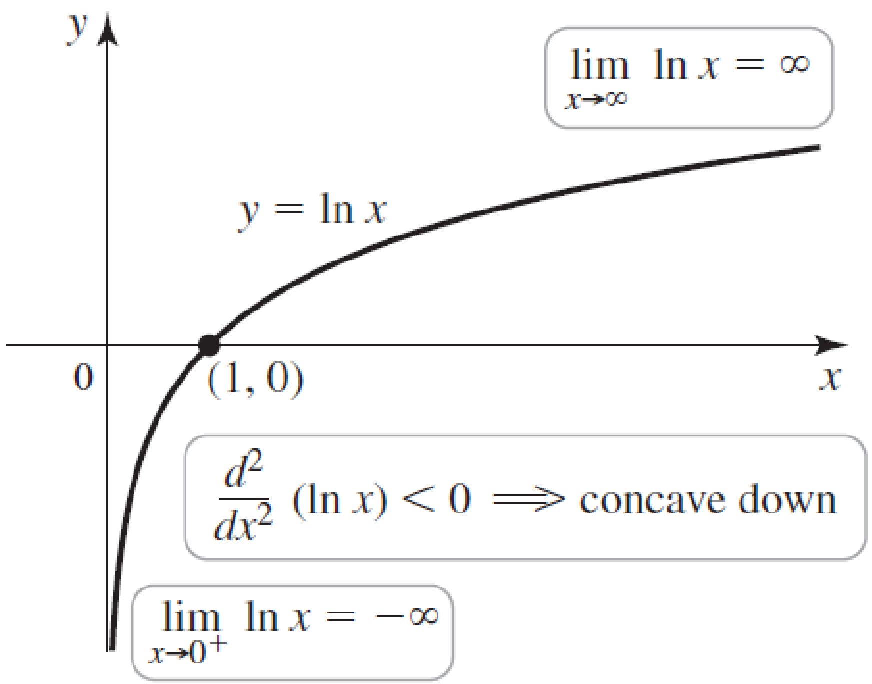
Graph of $\ln x$

$\ln x$ is continuous and strictly increasing for $x > 0$.

The second derivative $\frac{d^2}{dx^2}(\ln x) = -\frac{1}{x^2} < 0$ for $x > 0$, so the graph of $\ln x$ is concave down for $x > 0$.

Furthermore, $\lim_{x\to\infty} \ln x = \infty$, and $\lim_{x\to 0^+} \ln x = -\infty$.

Coupling with $\ln 1 = 0$, gives the graph of $y = \ln x$.



Logarithm of a product

$$\ln xy = \ln x + \ln y$$

may be proved using the integral definition.

$$\ln xy = \int_1^{xy} \frac{dt}{t} = \int_1^{x} \frac{dt}{t} + \int_x^{xy} \frac{dt}{t} = \int_1^{x} \frac{dt}{t} + \int_1^{y} \frac{du}{u}$$

Logarithm of a quotient

By assuming $x > 0$, $y > 0$, the product property and a bit of algebra give

$$\ln x = \ln\left(y \cdot \frac{x}{y}\right) = \ln y + \ln \frac{x}{y}$$

Solving for $\ln \frac{x}{y}$,

$$\ln \frac{x}{y} = \ln x - \ln y$$

Logarithm of a power

Assuming $x > 0$, and p is rational

$$\ln x^p = \int_1^{x^p} \frac{dt}{t} = p \int_1^{x} \frac{du}{u} = p \ln x$$

By letting $t = u^p$, $dt = pu^{p-1} du$.

We prove later that $\ln x^p = p \ln x$, for all real values of p.

Integrals

By derivative $\frac{d}{dx}(\ln |x|) = \frac{1}{x}$, we have

$$\int \frac{1}{x} dx = \ln |x| + C$$

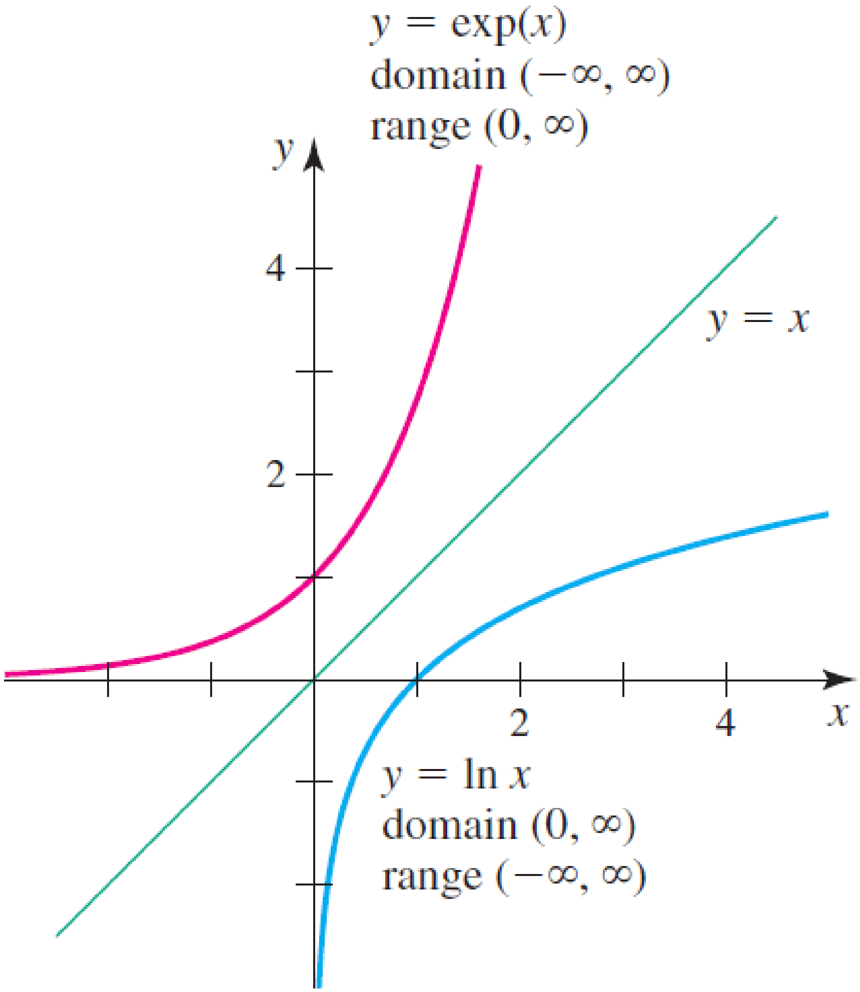
THEOREM 7.1 Properties of the Natural Logarithm

1. The domain and range of $\ln x$ are $(0, \infty)$ and $(-\infty, \infty)$, respectively.
2. $\ln xy = \ln x + \ln y$, for $x > 0$ and $y > 0$
3. $\ln (x/y) = \ln x - \ln y$, for $x > 0$ and $y > 0$
4. $\ln x^p = p \ln x$, for $x > 0$ and p a rational number
5. $\dfrac{d}{dx}(\ln |x|) = \dfrac{1}{x}$, for $x \neq 0$
6. $\dfrac{d}{dx}(\ln |u(x)|) = \dfrac{u'(x)}{u(x)}$, for $u(x) \neq 0$
7. $\displaystyle\int \frac{1}{x}\, dx = \ln |x| + C$

EXAMPLE 1 **Integrals with ln x** Evaluate $\int_0^4 \frac{x}{x^2 + 9}\,dx$.

Step 2: The Exponential Function

$f(x) = \ln x$ is continuous, increasing on the interval $(0, \infty)$, and therefore one-to-one and its inverse function exists on $(0, \infty)$, denoted as $f^{-1}(x) = \exp(x)$.
Its graph is obtained by reflecting the graph of $f(x) = \ln x$ about the line $y = x$. Domain and range are obtained accordingly.



Properties of the Exponential Function

The usual relationships between a function and its inverse also hold

- $y = \exp(x)$, if and only if $x = \ln y$.
- $\exp(\ln x) = x$, for $x > 0$, and $\ln(\exp(x)) = x$ for all x.

$$\begin{aligned}
\exp(x_1 + x_2) &= \exp(\underbrace{\ln y_1 + \ln y_2}_{\ln y_1 y_2}) && \text{Substitute } x_1 = \ln y_1, x_2 = \ln y_2. \\
&= \exp(\ln y_1 y_2) && \text{Properties of logarithms} \\
&= y_1 y_2 && \text{Inverse property of } \exp(x) \text{ and } \ln x \\
&= \exp(x_1)\exp(x_2). && y_1 = \exp(x_1), y_2 = \exp(x_2)
\end{aligned}$$

Therefore, $\exp(x)$ satisfies the property of exponential functions $b^{x_1+x_2} = b^{x_1} b^{x_2}$.

Similar arguments show that $\exp(x)$ satisfies other characteristic properties of exponential functions.

- $\exp(0) = 1$,
- $\exp(x_1 - x_2) = \dfrac{\exp(x_1)}{\exp(x_2)}$, and
- $\left(\exp(x)\right)^p = \exp(px)$, for rational numbers p.

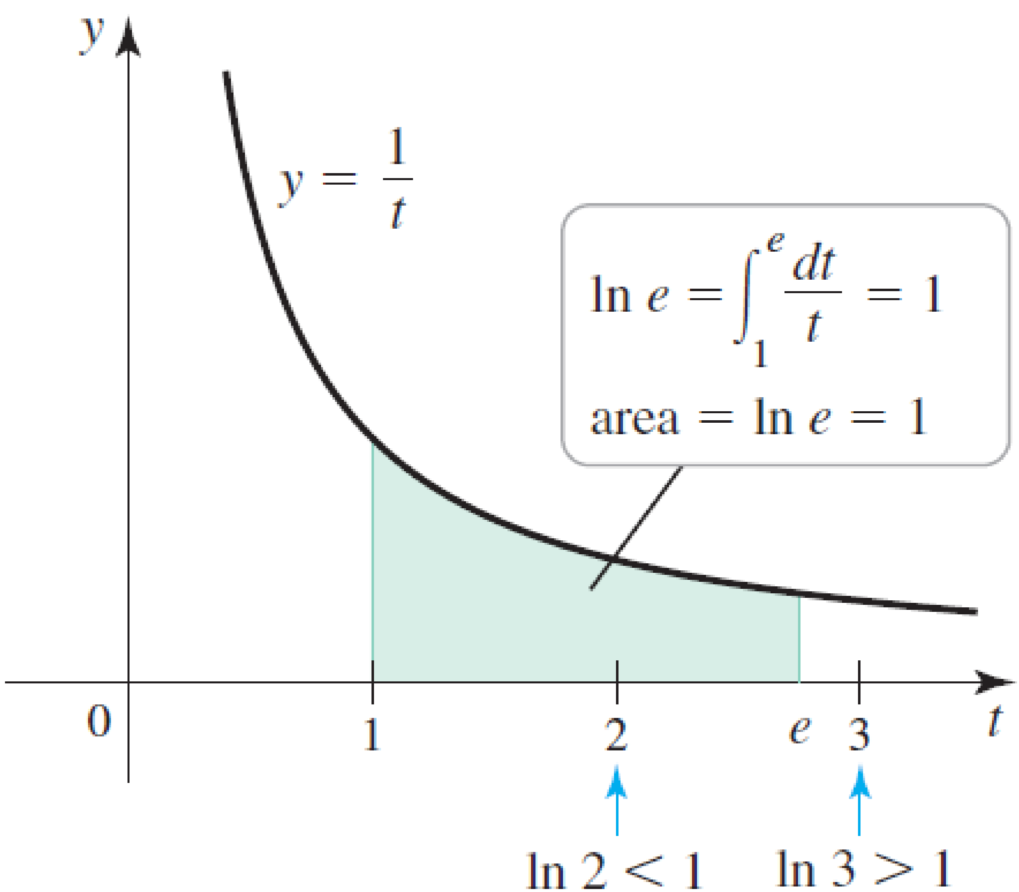
Suspecting that $\exp(x)$ is an exponential function, then identify its base, called forethought as e.

The inverse relationship between $\ln x$ and $\exp(x)$ implies that

$$\text{if } e = \exp(1), \text{ then } \ln e = \ln(\exp(1)) = 1.$$

DEFINITION **The Number *e***

The number e is the real number that satisfies $\ln e = \int_1^e \frac{dt}{t} = 1.$



Area of the region bounded by the graph of $y = 1/t$ and the t-axis on interval $[1, e]$ is 1.

$\ln 2 < 1$ and $\ln 3 > 1$.

By Intermediate Value Theorem, $2 < e < 3$ such that $\ln e = 1$.

Show $\exp(x)$ is indeed the exponential function e^x.

Assume that p is a rational number, and $e^p > 0$

$$\ln e^p = p \ln e = p$$

Using the inverse relationship between $\ln x$ and $\exp(x)$

$$\ln(\exp(p)) = p$$

Equating these two expressions, we conclude that

$$\ln e^p = \ln(\exp(p))$$

Because $\ln x$ is a one-to-one function,

$$e^p = \exp(p), \text{ for all rational numbers } p.$$

We conclude that $\exp(x)$ is the exponential function with base e.

The range of $x = \ln y$ is all real numbers, and so the domain of its inverse $y = \exp(x)$ is all real numbers.

DEFINITION The Exponential Function

For any real number x, $y = e^x = \exp(x)$, where $x = \ln y$.

THEOREM 7.2 Properties of e^x

The exponential function e^x satisfies the following properties, all of which result from the integral definition of $\ln x$. Let x and y be real numbers.

1. $e^{x+y} = e^x e^y$
2. $e^{x-y} = e^x / e^y$
3. $(e^x)^p = e^{xp}$, where p is a rational number
4. $\ln(e^x) = x$
5. $e^{\ln x} = x$, for $x > 0$

Derivatives and Integrals of $\exp(x)$

$\ln(e^x) = x$, differentiate both sides with respect to x:

$$\frac{d}{dx}(\ln e^x) = \underbrace{\frac{d}{dx}(x)}_{1}$$

$$\frac{1}{e^x}\frac{d}{dx}(e^x) = 1 \qquad \frac{d}{dx}(\ln u(x)) = \frac{u'(x)}{u(x)} \text{ (Chain Rule)}$$

$$\frac{d}{dx}(e^x) = e^x. \qquad \text{Solve for } \frac{d}{dx}(e^x).$$

THEOREM 7.3 Derivative and Integral of the Exponential Function

For real numbers x,

$$\frac{d}{dx}\left(e^{u(x)}\right) = e^{u(x)}u'(x) \quad \text{and} \quad \int e^x\,dx = e^x + C.$$

EXAMPLE 2 **Integrals with e^x** Evaluate $\displaystyle\int \frac{e^x}{1 + e^x}\,dx.$

Step 3: General Logarithmic and Exponential Functions

Turn to exponential and logarithmic functions with a general positive base b.

From Th. 7.2, if x is a rational number, then

$$b^x = \left(e^{\ln b}\right)^x = e^{x \ln b}$$

Because e^x is defined for all real x, we have

DEFINITION Exponential Functions with General Bases

Let b be a positive real number with $b \neq 1$. Then for all real x,

$$b^x = e^{x \ln b}.$$

Property 4 of Theorem 7.1 ($\ln x^p = p \ln x$) for real powers.

Use the definition of b^x to write

$$x^p = e^{p \ln x}, \text{ for } x > 0 \text{ and } p \text{ real.}$$

Taking the natural logarithm of both sides and using the inverse relationship between e^x and $\ln x$,

$$\ln x^p = \ln e^{p \ln x} = p \ln x, \text{ for } x > 0 \text{ and } p \text{ real}$$

Express logarithms with base $b > 1$ and $b \neq 1$ in terms of $\ln x$.

All that is needed is the change of base formula

$$\log_b x = \frac{\ln x}{\ln b}$$

SUMMARY Derivatives and Integrals with Other Bases

Let $b > 0$ and $b \neq 1$. Then

$$\frac{d}{dx}(\log_b |u(x)|) = \frac{u'(x)}{u(x)\ln b}, \text{ for } u(x) \neq 0 \text{ and } \frac{d}{dx}(b^{u(x)}) = (\ln b)b^{u(x)}u'(x).$$

For $b > 0$ and $b \neq 1$, $\displaystyle\int b^x\,dx = \frac{1}{\ln b}b^x + C.$

EXAMPLE 3 Integrals involving exponentials with other bases Evaluate the following integrals.

a. $\displaystyle\int x\,3^{x^2}\,dx$ **b.** $\displaystyle\int_1^4 \frac{6^{-\sqrt{x}}}{\sqrt{x}}\,dx$

Step 4: General Power Rule

With $x^p = e^{p \ln x}$, extend the Power Rule to all real numbers

Differentiating $x^p = e^{p \ln x}$:

$$\frac{d}{dx}(x^p) = \frac{d}{dx}(e^{p \ln x}) \qquad x^p = e^{p \ln x}$$

$$= \underbrace{e^{p \ln x}}_{x^p} \frac{p}{x} \qquad \text{Chain Rule}$$

$$= x^p \frac{p}{x} \qquad e^{p \ln x} = x^p$$

$$= p x^{p-1}. \qquad \text{Simplify.}$$

THEOREM 7.4 General Power Rule

For any real number p,

$$\frac{d}{dx}(x^p) = px^{p-1} \quad \text{and} \quad \frac{d}{dx}(u(x)^p) = pu(x)^{p-1}u'(x).$$

EXAMPLE 4 **Derivative of a tower function** Evaluate the derivative of $f(x) = x^{2x}$.

Computing *e*

Approximate the value of *e*

Recall that the derivative of $\ln x$ at $x = 1$ is 1. By the definition of derivative,

$$1 = \frac{d}{dx}(\ln x)\bigg|_{x=1} = \lim_{h \to 0} \frac{\ln(1+h) - \ln 1}{h} \qquad \text{Derivative of } \ln x \text{ at } x = 1$$

$$= \lim_{h \to 0} \frac{\ln(1+h)}{h} \qquad \ln 1 = 0$$

$$= \lim_{h \to 0} \ln(1+h)^{1/h}. \qquad p \ln x = \ln x^p$$

The natural logarithm is continuous for $x > 0$, so

$$\ln \underbrace{\left(\lim_{h \to 0} (1+h)^{1/h} \right)}_{e} = 1.$$

Table 6.2

h	$(1+h)^{1/h}$	h	$(1+h)^{1/h}$
10^{-1}	2.593742	-10^{-1}	2.867972
10^{-2}	2.704814	-10^{-2}	2.731999
10^{-3}	2.716924	-10^{-3}	2.719642
10^{-4}	2.718146	-10^{-4}	2.718418
10^{-5}	2.718268	-10^{-5}	2.718295
10^{-6}	2.718280	-10^{-6}	2.718283
10^{-7}	2.718282	-10^{-7}	2.718282

Note that $\ln e = 1$, and only one number satisfies this equation. Therefore,

$$e = \lim_{h \to 0} (1+h)^{1/h}$$

7.2

Exponential Models

Exponential Growth

Exponential growth functions have the form $y(t) = Ce^{kt}$

C is a constant, and the ***rate constant*** k is positive.

Take derivative, we have $y'(t) = ky$

$$y'(t) = \frac{d}{dt}(Ce^{kt}) = C \cdot ke^{kt} = k(\underbrace{Ce^{kt}}_{y});$$

First insight: *The rate of change is proportional to their value*

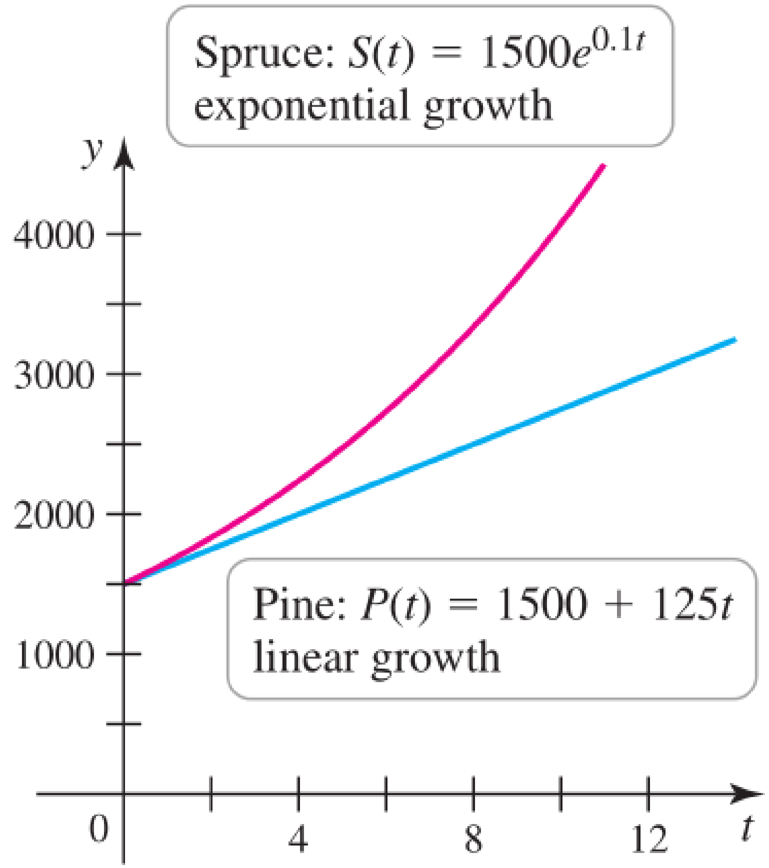
If y represents a population, then $y'(t)$ is the **growth rate**.
The larger the population, the faster its growth

Relative growth rate: the growth rate divided by the current value of that quantity, or $y'(t)/y(t)$

Write the equation $y'(t) = ky$ in the form $\frac{y'(t)}{y} = k$

Another interpretation: *a quantity that grows exponentially has a constant relative growth rate.*
Constant relative or percentage change is the hallmark of exponential growth.

EXAMPLE 1 **Linear versus exponential growth** Suppose the population of the town of Pine is given by $P(t) = 1500 + 125t$, while the population of the town of Spruce is given by $S(t) = 1500e^{0.1t}$, where $t \geq 0$ is measured in years. Find the growth rate and the relative growth rate of each town.



Let $t = 0$, we have $y(0) = C$, i.e., C is the initial value y_0.

The initial value and the rate constant determine an exponential growth function completely.

Exponential Growth Functions

Exponential growth is described by functions of the form $y(t) = y_0 e^{kt}$. The initial value of y at $t = 0$ is $y(0) = y_0$, and the **rate constant** $k > 0$ determines the rate of growth. Exponential growth is characterized by a constant relative growth rate.

DEFINITION Doubling Time

The quantity described by the function $y(t) = y_0 e^{kt}$, for $k > 0$, has a constant **doubling time** of $T_2 = \dfrac{\ln 2}{k}$, with the same units as t.

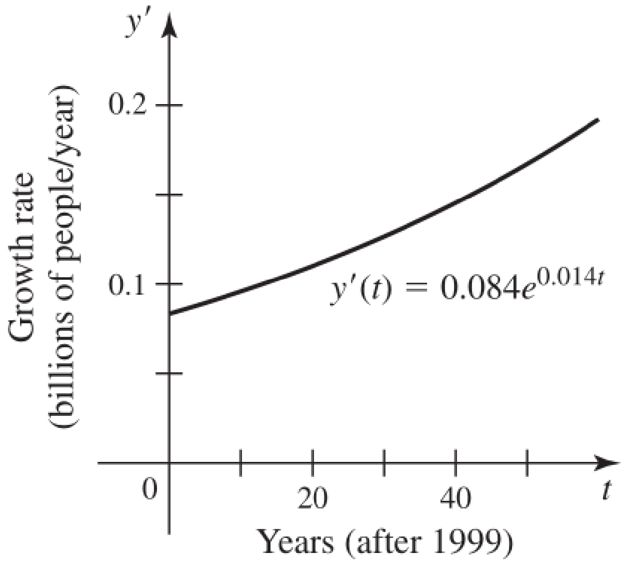
EXAMPLE 2 **World population** Human population growth rates vary geographically and fluctuate over time. The overall growth rate for world population peaked at an annual rate of 2.1% per year in the 1960s. Assume a world population of 6.0 billion in 1999 ($t = 0$) and 6.9 billion in 2009 ($t = 10$).

a. Find an exponential growth function for the world population that fits the two data points.

b. Find the doubling time for the world population using the model in part (a).

c. Find the (absolute) growth rate $y'(t)$ and graph it, for $0 \le t \le 50$.

d. How fast was the population growing in 2014 ($t = 15$)?



the rate constant $k = \dfrac{\ln(6.9/6)}{10} \approx 0.013976 \approx 0.014 \text{ year}^{-1}$.

$$y(t) = 6e^{0.014t}.$$

$$T_2 = \frac{\ln 2}{k} \approx \frac{\ln 2}{0.014} \approx 50 \text{ years.}$$

A Financial Model

Considering simple saving account in which an initial deposit earns interest that is reinvested in the account.

The balance in the account increases exponentially at a rate that can be determined from the advertised *annual percentage yield* (or **APY**) of the account.

$y(t) = y_0 e^{kt}$, where y_0 is the initial deposit, t is measured in years, and k is determined by the APY.

EXAMPLE 3 **Compounding** The APY of a savings account is the percentage increase in the balance over the course of a year. Suppose you deposit \$500 in a savings account that has an APY of 6.18% per year. Assume that the interest rate remains constant and that no additional deposits or withdrawals are made. How long will it take the balance to reach \$2500?

Solving the rate constant k

$$y(1) = 1.0618\, y_0 = y_0 e^k.$$

$$k = \ln 1.0618 \approx 0.060\ \mathrm{yr}^{-1}.$$

Therefore, the balance is $y(t) = 500e^{0.060t}$

Solve the equation $y(t) = 500e^{0.060t} = 2500.$

The balance reaches \$2500 in $t = (\ln 5)/0.060 \approx 26.8$ yr.

Resource Consumption

Total energy: $E(t)$.
Power $P(t)$ is the rate at which energy is used, $P(t) = E'(t)$.

The basic unit of energy is the **joule** (J), and the basic unit of power is the **watt** (W), where $1W = 1\,J/s$.
A more useful measure of energy for large quantities is the **kilowatt-hour** (kWh)

The total amount of energy used between the times $t = a$ and $t = b$ is

$$\text{total energy used} = \int_a^b E'(t)\,dt = \int_a^b P(t)\,dt.$$

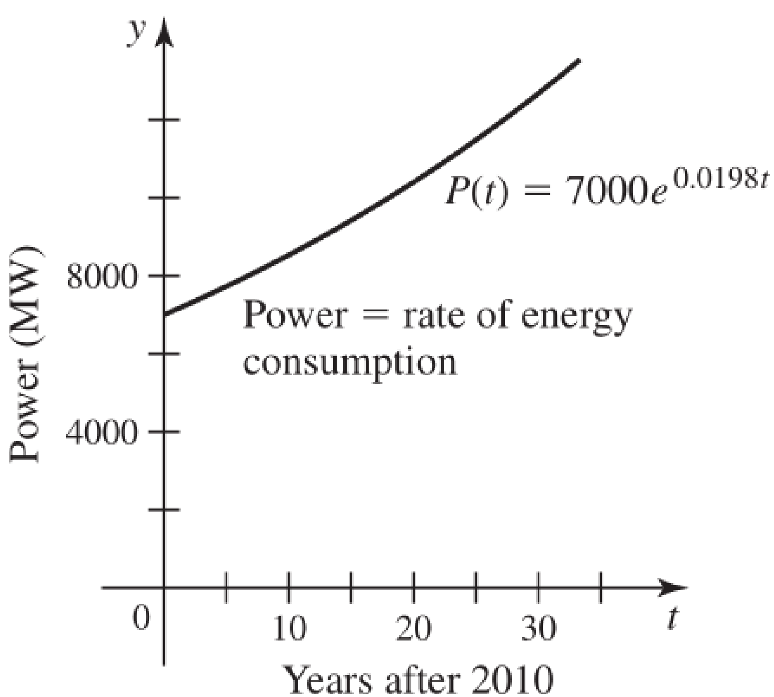
So, energy is the area under the power curve.

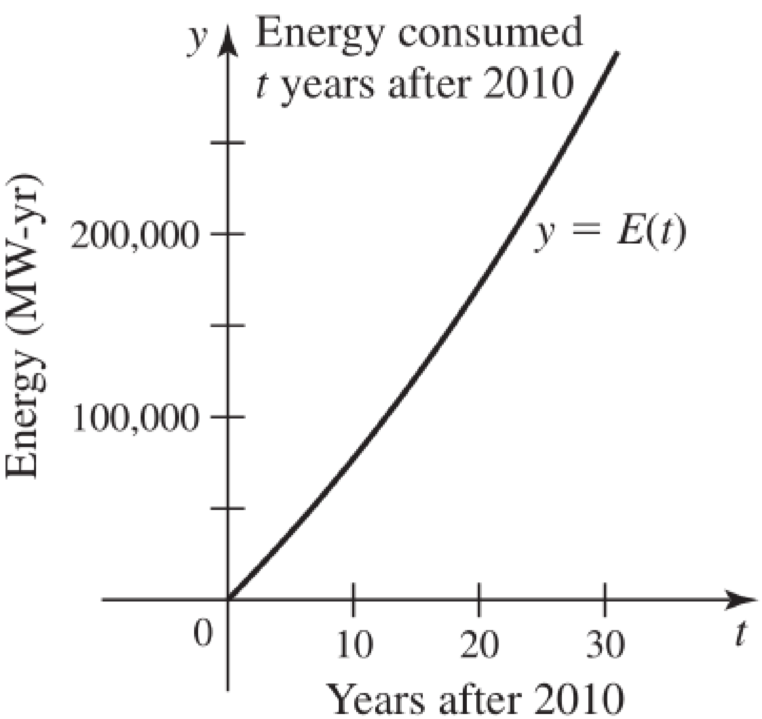
EXAMPLE 4 **Energy consumption** At the beginning of 2010, the rate of energy consumption for the city of Denver was 7000 megawatts (MW), where 1 MW $= 10^6$ W. That rate is expected to increase at an annual growth rate of 2% per year.

a. Find the function that gives the power or rate of energy consumption for all times after the beginning of 2010.

b. Find the total amount of energy used during 2014.

c. Find the function that gives the total (cumulative) amount of energy used by the city between 2010 and any time $t \geq 0$.





Exponential Decay

A function that decreases exponentially: $y(t) = y_0 e^{-kt}$, where $y_0 = y(0)$ is the initial value and $k > 0$ is the rate constant.

Exponential decay is characterized by a constant relative decay rate and by a constant *half-life*.

To compute the half-life, we solve $y_0 e^{-kt} = y_0/2$ for t.

$$e^{-kt} = \frac{1}{2} \quad \Rightarrow \quad -kt = \ln\frac{1}{2} = -\ln 2 \quad \Rightarrow \quad t = \frac{\ln 2}{k}.$$

Exponential Decay Functions

Exponential decay is described by functions of the form $y(t) = y_0 e^{-kt}$. The initial value of y is $y(0) = y_0$, and the rate constant $k > 0$ determines the rate of decay. Exponential decay is characterized by a constant relative decay rate. The constant **half-life** is $T_{1/2} = \frac{\ln 2}{k}$, with the same units as t.

Radiometric Dating

EXAMPLE 5 **Radiometric dating** Researchers determine that a fossilized bone has 30% of the C-14 of a live bone. Estimate the age of the bone. Assume a half-life for C-14 of 5730 years.

The exponential decay function, $y(t) = y_0\, e^{-kt}$ represents the amount of C-14 in the bone t years after its owner died.

By the half-life formula, $T_{1/2} = (\ln 2)/k$ for t, substituting $T_{1/2} = 5730$, the rate constant is

$$k = \frac{\ln 2}{T_{1/2}} = \frac{\ln 2}{5730\text{ yr}} \approx 0.000121\text{ yr}^{-1}.$$

Using the decay function, $0.3y_0 = y_0 e^{-0.000121t}$.
Solving for t,

the age of the bone (in years) is $t = \dfrac{\ln 0.3}{-0.000121} \approx 9950.$

Pharmacokinetics

Pharmacokinetics describes the processes by which drugs are assimilated by the body.

EXAMPLE 6 **Pharmacokinetics** An exponential decay function $y(t) = y_0e^{-kt}$ models the amount of drug in the blood t hr after an initial dose of $y_0 = 100$ mg is administered. Assume the half-life of the drug is 16 hours.

a. Find the exponential decay function that governs the amount of drug in the blood.

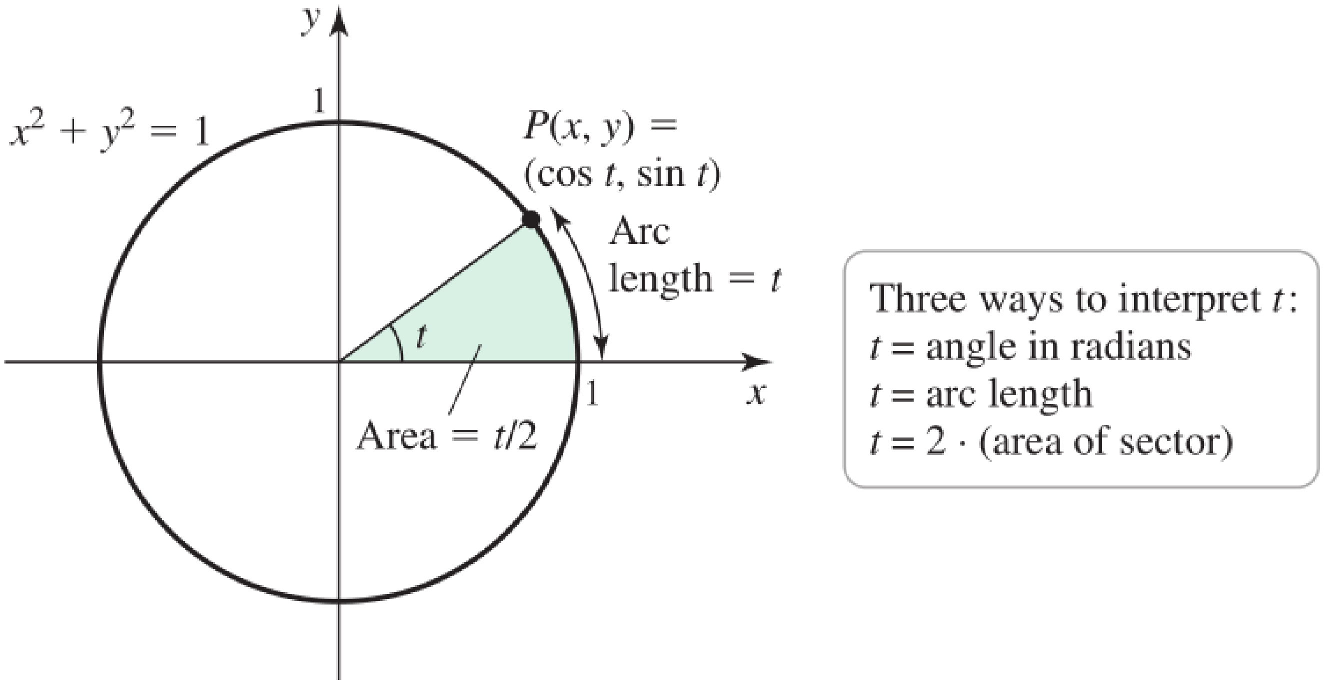
b. How much time is required for the drug to reach 1% of the initial dose (1 mg)?

c. If a second 100-mg dose is given 12 hr after the first dose, how much time is required for the drug level to reach 1 mg?

7.3

Hyperbolic Functions

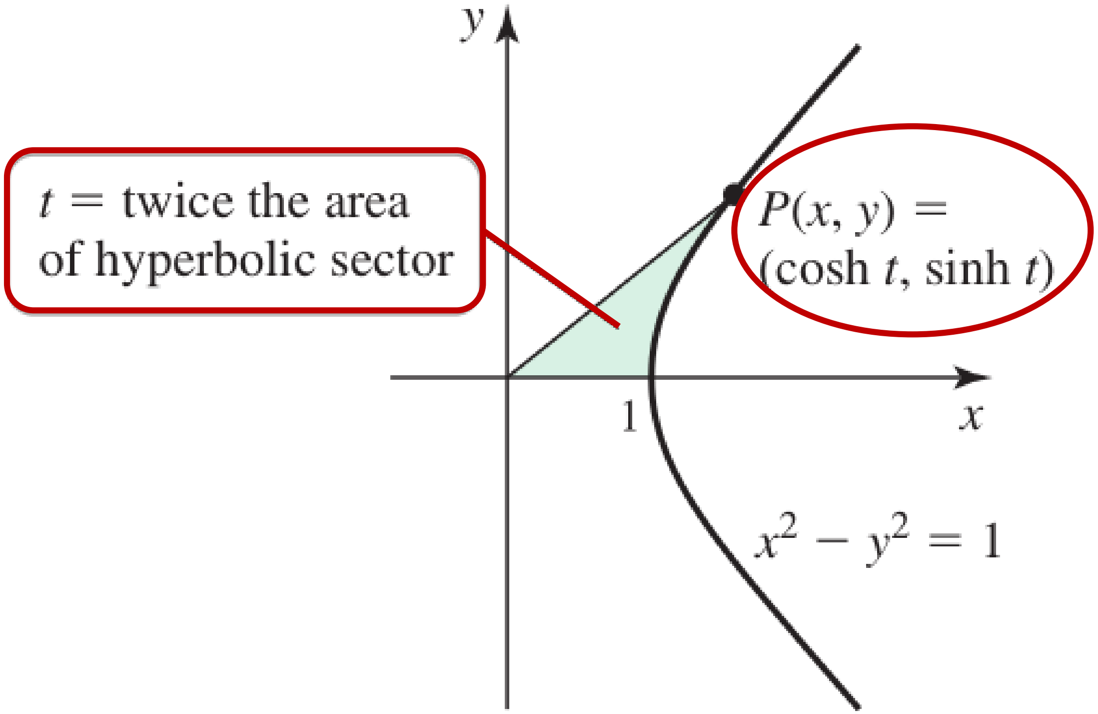
Relationship Between Trigonometric and Hyperbolic Functions



The third interpretation that links the trigonometric and hyperbolic functions.

That associates P with a sector whose area is one-half of t.

The *hyperbolic cosine* and *hyperbolic sine* are defined in an analogous fashion using the hyperbola $x^2 - y^2 = 1$ instead of the circle $x^2 + y^2 = 1$.



$$x = \cosh t = \frac{e^t + e^{-t}}{2} \text{ and } y = \sinh t = \frac{e^t - e^{-t}}{2}.$$

Definitions, Identities, and Graphs of the Hyperbolic Functions

DEFINITION **Hyperbolic Functions**

Hyperbolic cosine

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

Hyperbolic sine

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

Hyperbolic tangent

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Hyperbolic cotangent

$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

Hyperbolic secant

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$

Hyperbolic cosecant

$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

Fundamental identity: $\cosh^2 x - \sinh^2 x = 1.$

Hyperbolic Identities

$$\cosh^2 x - \sinh^2 x = 1 \qquad \cosh(-x) = \cosh x$$

$$1 - \tanh^2 x = \operatorname{sech}^2 x \qquad \sinh(-x) = -\sinh x$$

$$\coth^2 x - 1 = \operatorname{csch}^2 x \qquad \tanh(-x) = -\tanh x$$

$$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$$

$$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x \qquad \sinh 2x = 2 \sinh x \cosh x$$

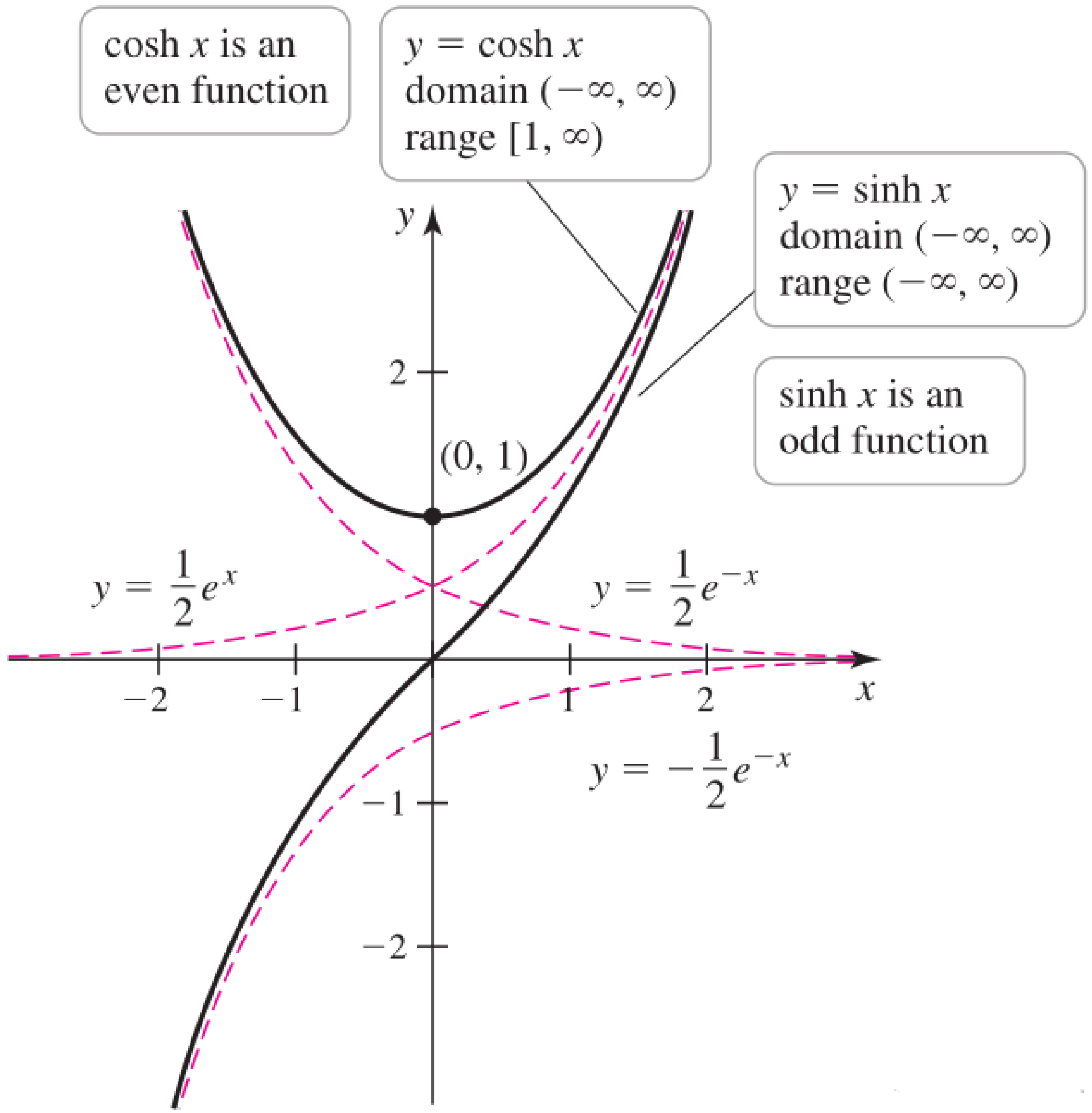
$$\cosh^2 x = \frac{\cosh 2x + 1}{2} \qquad \sinh^2 x = \frac{\cosh 2x - 1}{2}$$

EXAMPLE 1 **Deriving hyperbolic identities**

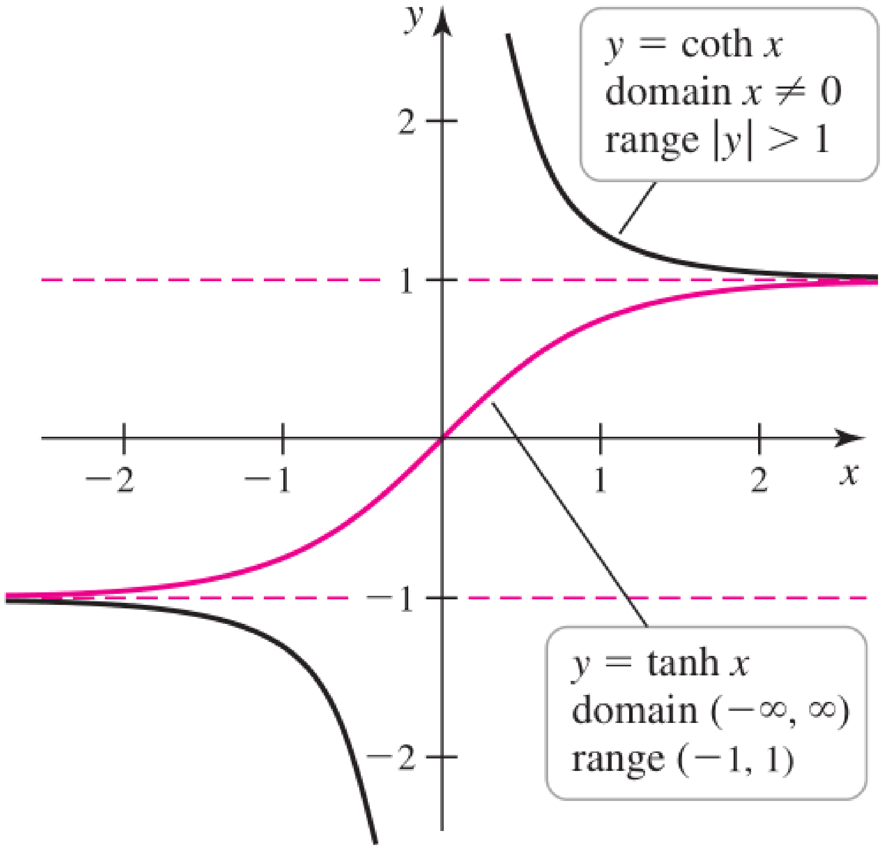
a. Use the fundamental identity $\cosh^2 x - \sinh^2 x = 1$ to prove that $1 - \tanh^2 x = \operatorname{sech}^2 x$.

b. Derive the identity $\sinh 2x = 2 \sinh x \cosh x$.

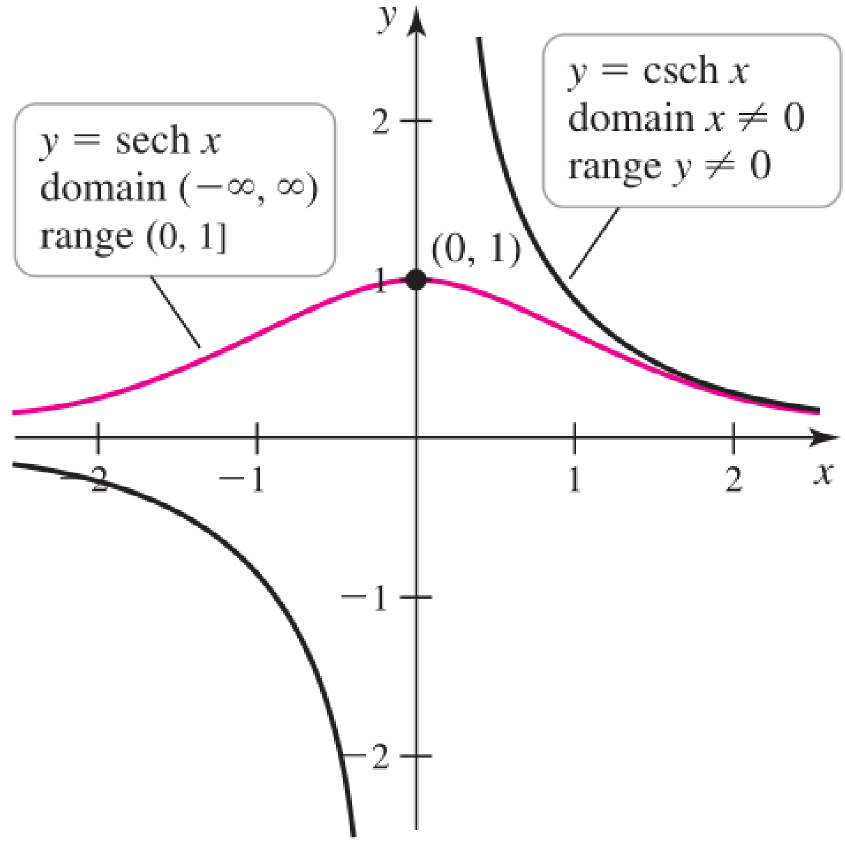
Graphs



cosh x is an even function
$y = \cosh x$
domain $(-\infty, \infty)$
range $[1, \infty)$
$y = \sinh x$
domain $(-\infty, \infty)$
range $(-\infty, \infty)$
sinh x is an odd function
y
2
(0, 1)
$y = \frac{1}{2}e^{x}$
$y = \frac{1}{2}e^{-x}$
−2
−1
1
2
x
$y = -\frac{1}{2}e^{-x}$
−1
−2



y
$y = \coth x$
domain $x \neq 0$
range $|y| > 1$
2
1
−1
−2
−2
−1
1
2
x
$y = \tanh x$
domain $(-\infty, \infty)$
range $(-1, 1)$



y
$y = \text{csch}\, x$
domain $x \neq 0$
range $y \neq 0$
$y = \text{sech}\, x$
domain $(-\infty, \infty)$
range $(0, 1]$
(0, 1)
2
1
−1
−2
−2
−1
1
2
x

Derivatives and Integrals of Hyperbolic Functions

THEOREM **Derivative and Integral Formulas**

1. $\frac{d}{dx}(\cosh x) = \sinh x \quad \Rightarrow \quad \int \sinh x \, dx = \cosh x + C$

2. $\frac{d}{dx}(\sinh x) = \cosh x \quad \Rightarrow \quad \int \cosh x \, dx = \sinh x + C$

3. $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x \quad \Rightarrow \quad \int \operatorname{sech}^2 x \, dx = \tanh x + C$

4. $\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x \quad \Rightarrow \quad \int \operatorname{csch}^2 x \, dx = -\coth x + C$

5. $\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x \quad \Rightarrow \quad \int \operatorname{sech} x \tanh x \, dx = -\operatorname{sech} x + C$

6. $\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x \quad \Rightarrow \quad \int \operatorname{csch} x \coth x \, dx = -\operatorname{csch} x + C$

EXAMPLE 2 **Derivatives and integrals of hyperbolic functions** Evaluate the following derivatives and integrals.

a. $\dfrac{d}{dx}(\operatorname{sech} 3x)$

b. $\dfrac{d^2}{dx^2}(\operatorname{sech} 3x)$

c. $\displaystyle\int \frac{\operatorname{csch}^2 \sqrt{x}}{\sqrt{x}}\, dx$

d. $\displaystyle\int_0^{\ln 3} \sinh^3 x \cosh x \, dx$

THEOREM 7.6 Integrals of Hyperbolic Functions

1. $\displaystyle\int \tanh x\, dx = \ln \cosh x + C$

2. $\displaystyle\int \coth x\, dx = \ln|\sinh x| + C$

3. $\displaystyle\int \operatorname{sech} x\, dx = \tan^{-1}(\sinh x) + C$

4. $\displaystyle\int \operatorname{csch} x\, dx = \ln|\tanh(x/2)| + C$

EXAMPLE 3 **Integrals involving hyperbolic functions** Determine the indefinite integral $\int x \coth(x^2)\, dx$.

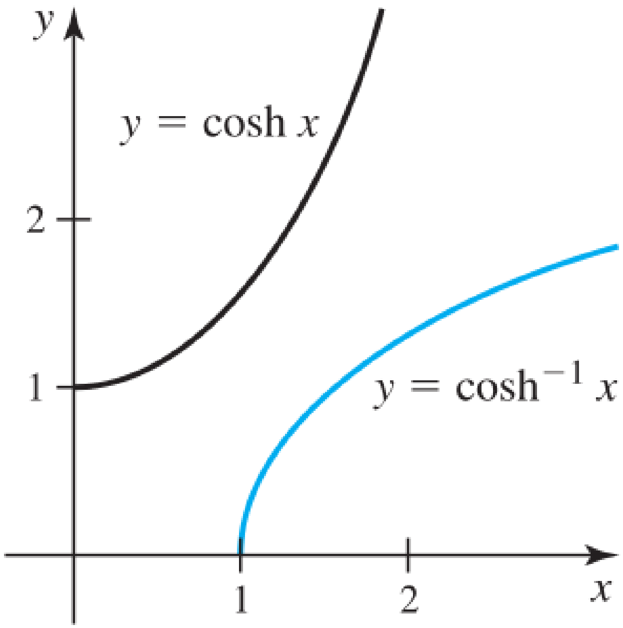
Inverse Hyperbolic Functions

Helpful to discover new integration formulas, e.g., $\int \frac{dx}{\sqrt{x^2+4}}$.

Useful for solving equations involving hyperbolic functions.

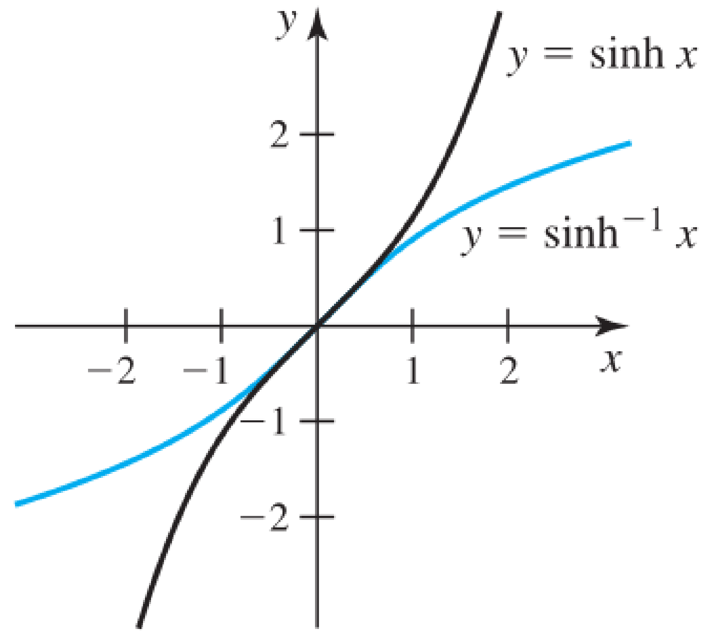
$\sinh x$, $\tanh x$, $\coth x$, and $\operatorname{csch} x$ are all one-to-one.

When $y = \cosh x$ is restricted to $[0, \infty]$, it is one-to-one.



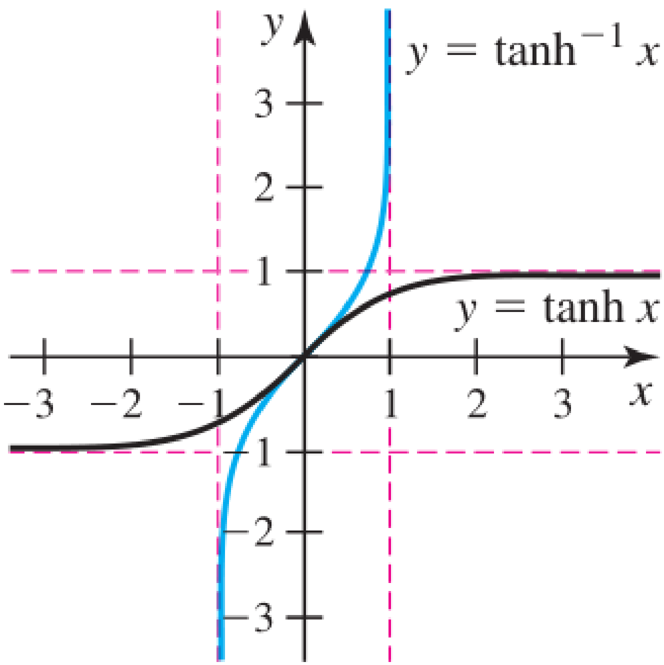
$$y = \cosh^{-1} x \Leftrightarrow x = \cosh y$$
for $x \geq 1$ and $0 \leq y < \infty$

(a)



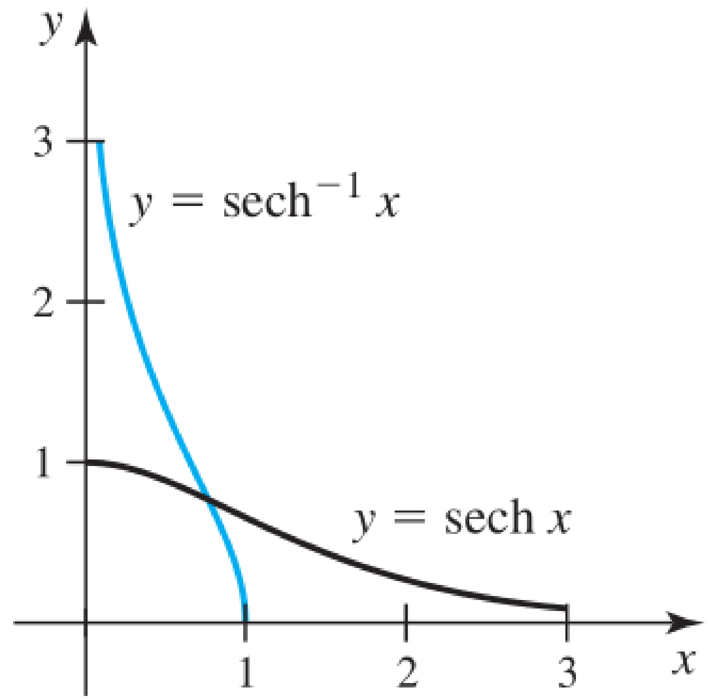
$$y = \sinh^{-1} x \Leftrightarrow x = \sinh y$$
for $-\infty < x < \infty$ and $-\infty < y < \infty$

(b)



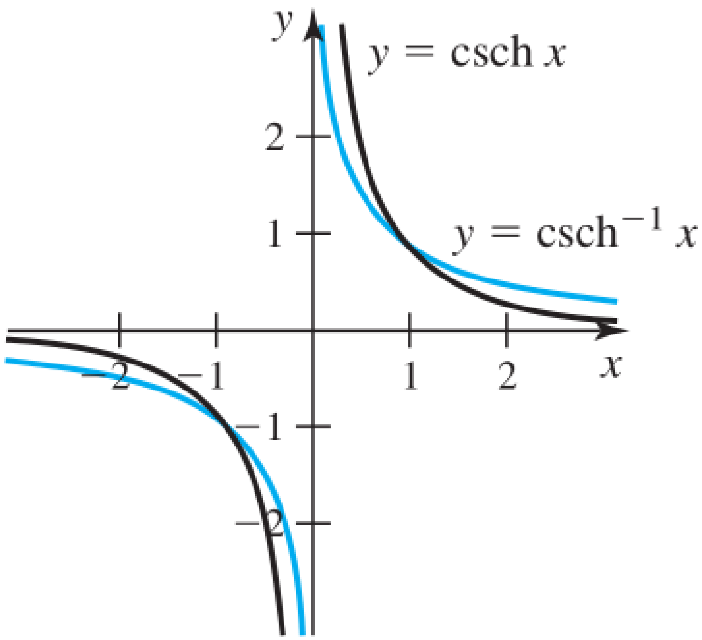
$y = \tanh^{-1} x \Leftrightarrow x = \tanh y$
for $-1 < x < 1$ and $-\infty < y < \infty$

(c)



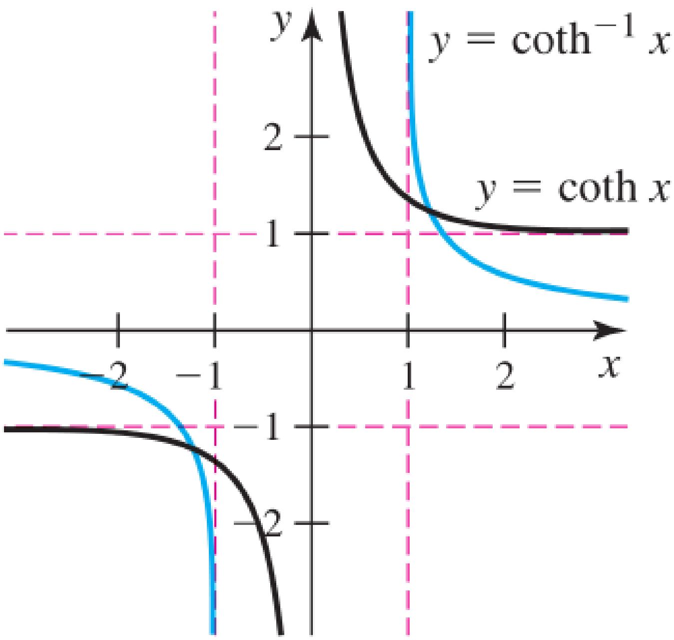
$y = \operatorname{sech}^{-1} x \Leftrightarrow x = \operatorname{sech} y$
for $0 < x \leq 1$ and $0 \leq y < \infty$

(d)



$y = \operatorname{csch}^{-1} x \Leftrightarrow x = \operatorname{csch} y$
for $x \neq 0$ and $y \neq 0$

(e)



$y = \coth^{-1} x \Leftrightarrow x = \coth y$
for $|x| > 1$ and $y \neq 0$

(f)

Solve for inverse hyperbolic functions

$$y = \sinh^{-1}x \Leftrightarrow x = \sinh y = \frac{e^y - e^{-y}}{2}$$

$$x = \frac{e^y - e^{-y}}{2} \quad \Rightarrow \quad e^y - 2x - e^{-y} = 0 \qquad \text{Rearrange equation.}$$

$$\Rightarrow \quad (e^y)^2 - 2xe^y - 1 = 0. \qquad \text{Multiply by } e^y.$$

$$e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2} = x \pm \sqrt{x^2 + 1} = \underbrace{x + \sqrt{x^2 + 1}}_{\text{choose positive root}}.$$

$$e^y = x + \sqrt{x^2 + 1} \quad \Rightarrow \quad y = \ln(x + \sqrt{x^2 + 1}).$$

That is, $y = \sinh^{-1}x = \ln\left(x + \sqrt{x^2 + 1}\right)$

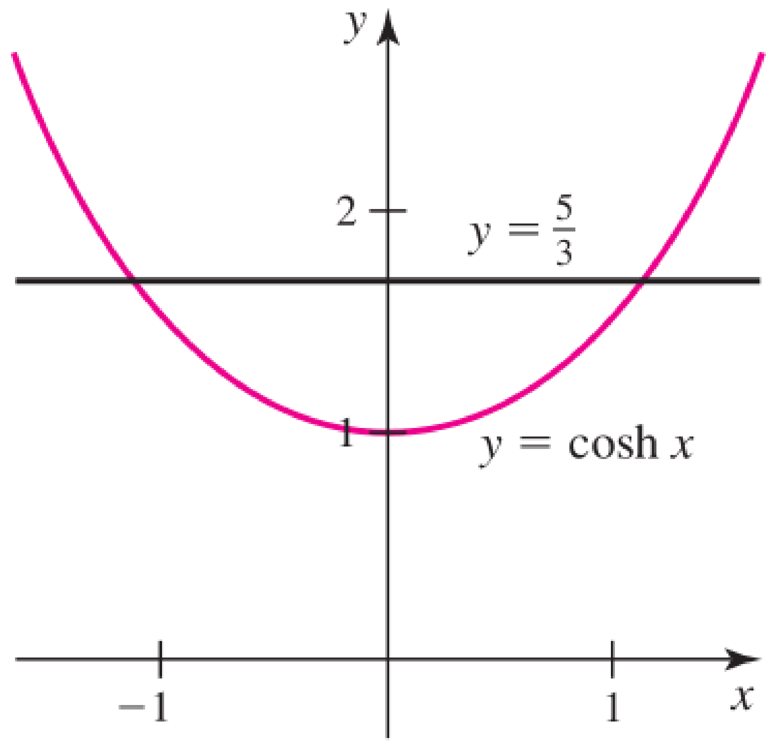
THEOREM 7.7 Inverses of the Hyperbolic Functions Expressed as Logarithms

$$\cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right) \quad (x \geq 1) \qquad \operatorname{sech}^{-1} x = \cosh^{-1} \frac{1}{x} \quad (0 < x \leq 1)$$

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right) \qquad \operatorname{csch}^{-1} x = \sinh^{-1} \frac{1}{x} \quad (x \neq 0)$$

$$\tanh^{-1} x = \frac{1}{2} \ln\left(\frac{1 + x}{1 - x}\right) \quad (|x| < 1) \qquad \coth^{-1} x = \tanh^{-1} \frac{1}{x} \quad (|x| > 1)$$

EXAMPLE 4 **Points of intersection** Find the points at which the curves $y = \cosh x$ and $y = \frac{5}{3}$ intersect (Figure).



Derivatives of Inverse Hyperbolic Functions and Related Integral Formulas

Can be computed directly from the logarithmic formulas

But more efficient to use the definitions

$$x = \sinh y \qquad y = \sinh^{-1} x \iff x = \sinh y$$

$$1 = (\cosh y)\frac{dy}{dx} \qquad \text{Use implicit differentiation.}$$

$$\frac{dy}{dx} = \frac{1}{\cosh y} \qquad \text{Solve for } dy/dx.$$

$$\frac{dy}{dx} = \frac{1}{\pm\sqrt{\sinh^2 y + 1}} \qquad \cosh^2 y - \sinh^2 y = 1$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2 + 1}}. \qquad x = \sinh y \qquad \cosh y > 0$$

THEOREM 7.8 Derivatives of the Inverse Hyperbolic Functions

$$\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2 - 1}} \quad (x > 1) \qquad \frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{x^2 + 1}}$$

$$\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1 - x^2} \quad (|x| < 1) \qquad \frac{d}{dx}(\coth^{-1} x) = \frac{1}{1 - x^2} \quad (|x| > 1)$$

$$\frac{d}{dx}(\operatorname{sech}^{-1} x) = -\frac{1}{x\sqrt{1 - x^2}} \quad (0 < x < 1) \qquad \frac{d}{dx}(\operatorname{csch}^{-1} x) = -\frac{1}{|x|\sqrt{1 + x^2}} \quad (x \neq 0)$$

The restrictions associated with the formulas are a direct consequence of the domains of the invers functions

Reversal of the derivative formulas

THEOREM 7.9 Integral Formulas

1. $$\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\frac{x}{a} + C, \text{ for } x > a$$

2. $$\int \frac{dx}{\sqrt{x^2 + a^2}} = \sinh^{-1}\frac{x}{a} + C, \text{ for all } x$$

3. $$\int \frac{dx}{a^2 - x^2} = \begin{cases} \dfrac{1}{a}\tanh^{-1}\dfrac{x}{a} + C, \text{ for } |x| < a \\[2ex] \dfrac{1}{a}\coth^{-1}\dfrac{x}{a} + C, \text{ for } |x| > a \end{cases}$$

4. $$\int \frac{dx}{x\sqrt{a^2 - x^2}} = -\frac{1}{a}\operatorname{sech}^{-1}\frac{x}{a} + C, \text{ for } 0 < x < a$$

5. $$\int \frac{dx}{x\sqrt{a^2 + x^2}} = -\frac{1}{a}\operatorname{csch}^{-1}\frac{|x|}{a} + C, \text{ for } x \neq 0$$

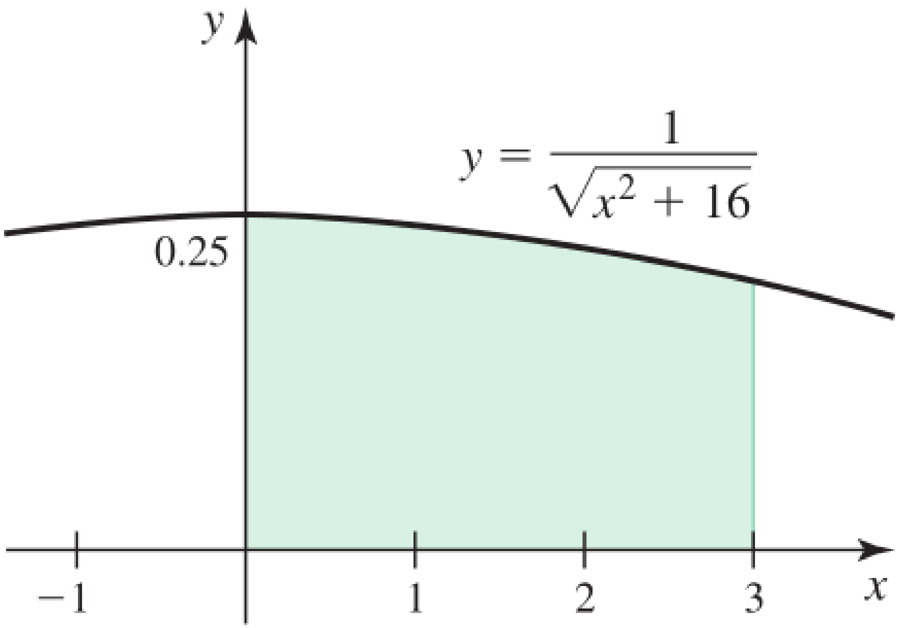
EXAMPLE 5 **Derivatives of inverse hyperbolic functions** Compute dy/dx for each function.

a. $y = \tanh^{-1} 3x$ **b.** $y = x^2 \sinh^{-1} x$

EXAMPLE 6 Integral computations

a. Compute the area of the region bounded by $y = 1/\sqrt{x^2 + 16}$ over the interval $[0, 3]$.

b. Evaluate $\displaystyle\int_9^{25} \frac{dx}{\sqrt{x}(4 - x)}$.

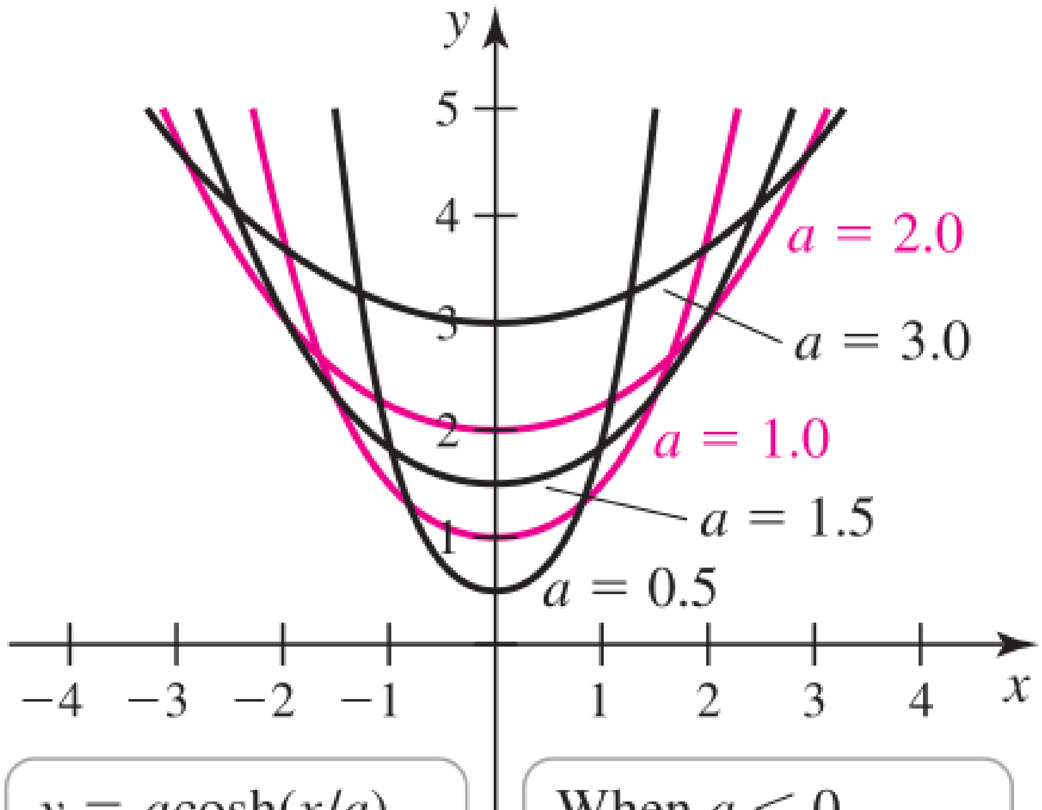


Applications of Hyperbolic Functions

The Catenary

When a free-hanging rope or flexible cable supporting only its own weight is attached to two points of equal height, it takes the shape of a curve known as a *catenary*.

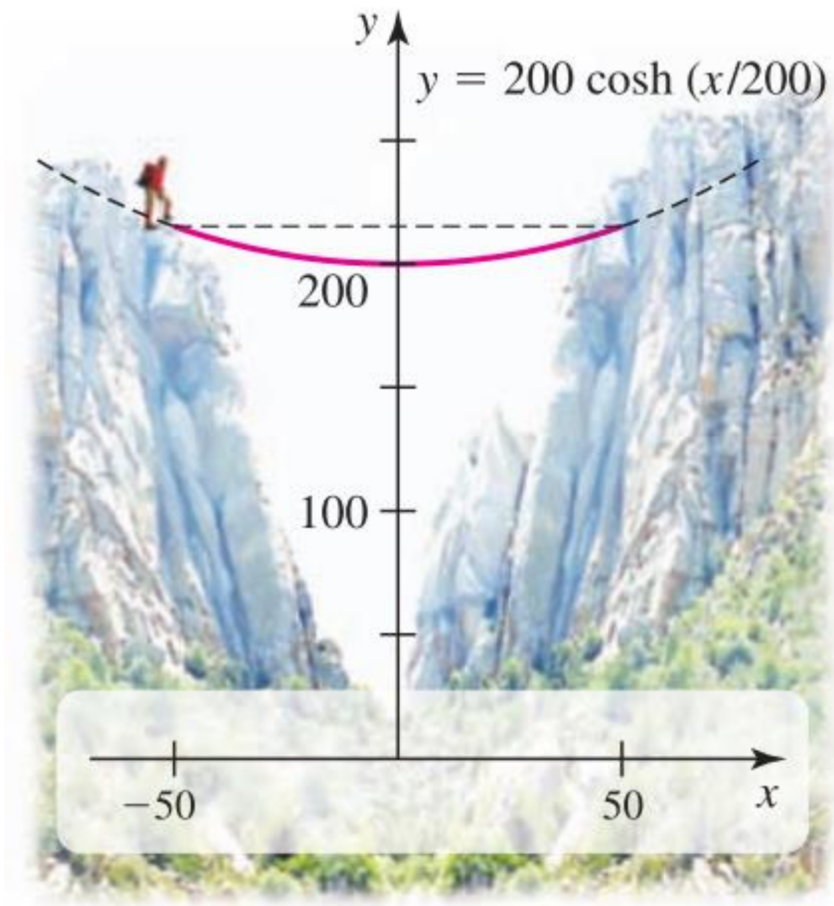
The equation for a general catenary is $y = a\cosh(x/a)$. When $a < 0$, the curve is called an *inverted catenary*, sometimes used in the design of arches.



$y = a\cosh(x/a)$ for several values of $a > 0$

When $a < 0$, inverted catenaries result (reflections across the x-axis)

EXAMPLE 7 **Length of a catenary** A climber anchors a rope at two points of equal height, separated by a distance of 100 ft, in order to perform a *Tyrolean traverse*. The rope follows the catenary $f(x) = 200\cosh(x/200)$ over the interval $[-50, 50]$ (Figure 6.98). Find the length of the rope between the two anchor points.



Velocity of a Wave

To describe the characteristics of a traveling wave, researchers formulate a wave equation that reflects the known (or hypothesized) properties of the wave and that often takes the form of a differential equation.

Solving a wave equation produces additional information about the wave, and it turns out that hyperbolic functions may arise naturally in this context.

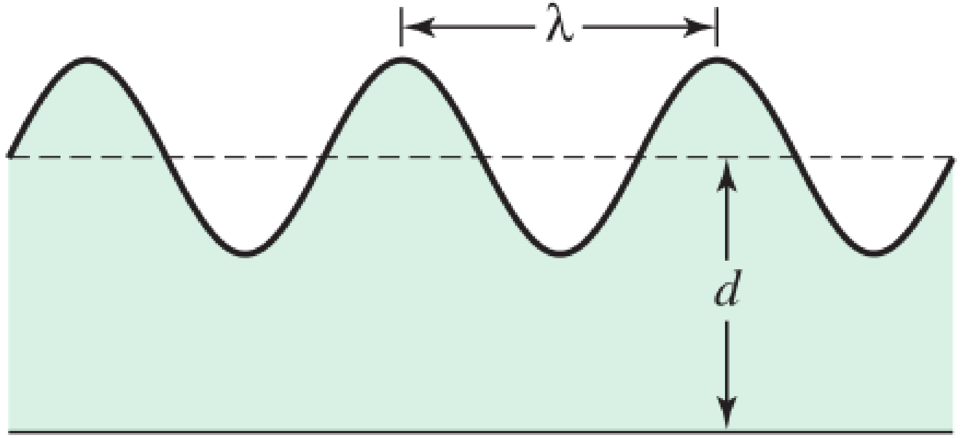
EXAMPLE 8 **Velocity of an ocean wave** The velocity v (in meters/second) of an idealized surface wave traveling on the ocean is modeled by the equation

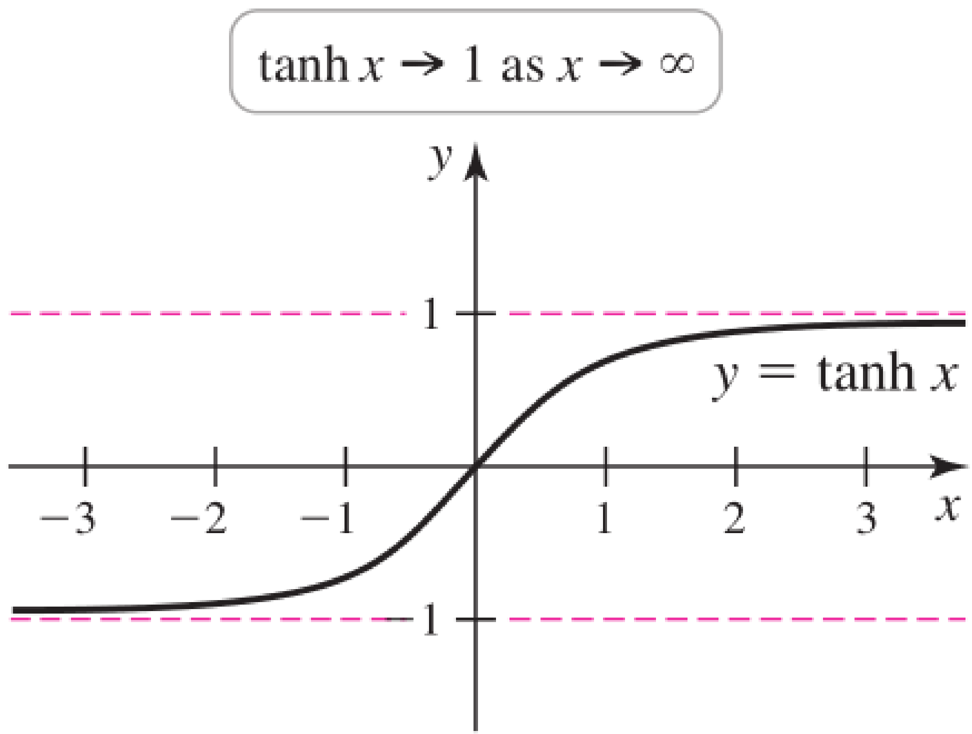
$$v = \sqrt{\frac{g\lambda}{2\pi}\tanh\left(\frac{2\pi d}{\lambda}\right)},$$

where $g = 9.8\ \mathrm{m/s^2}$ is the acceleration due to gravity, λ is the wavelength measured in meters from crest to crest, and d is the depth of the undisturbed water, also measured in meters (Figure 6.99).

a. A sea kayaker observes several waves that pass beneath her kayak, and she estimates that $\lambda = 12$ m and $v = 4$ m/s. How deep is the water in which she is kayaking?

b. The *deep-water* equation for wave velocity is $v = \sqrt{\frac{g\lambda}{2\pi}}$, which is an approximation to the velocity formula given above. Waves are said to be in deep water if the depth-to-wavelength ratio d/λ is greater than $\frac{1}{2}$. Explain why $v = \sqrt{\frac{g\lambda}{2\pi}}$ is a good approximation when $d/\lambda > \frac{1}{2}$.


λ
d


tanh x → 1 as x → ∞
y
1
y = tanh x
−3
−2
−1
1
2
3
x
−1

Chapter 7

Hyperbolic Functions

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