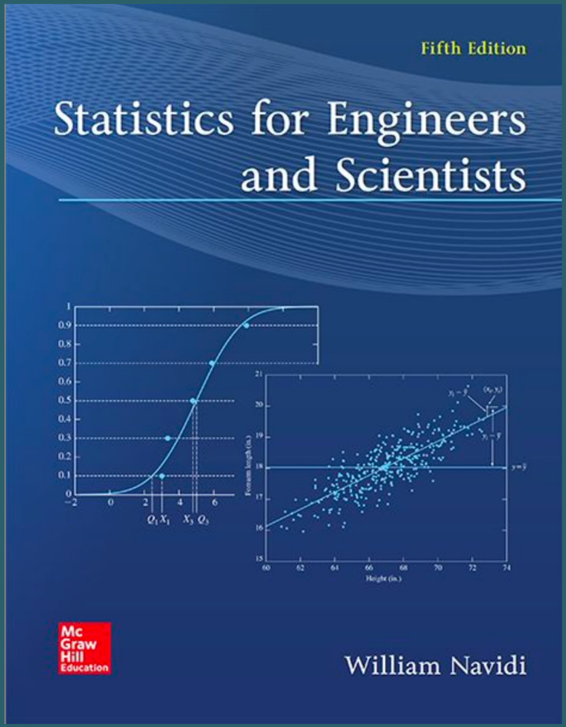
Mc Graw Hill Education



CHAPTER 6

Hypothesis Testing (part 4)

Ch. 6 - Hypothesis Testing

- 6-1 Large-Sample Tests for a Population Mean
- 6-2 Drawing Conclusions from the Results of Hypothesis Tests
- 6-3 Tests for a Population Proportion
- 6-4 Small-Sample Tests for a Population Mean
- 6-5 Large-Sample Tests for the Difference Between Two Means
- 6-7 Small-Sample Tests for the Difference Between Two Means
- 6-8 Tests with Paired Data
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Tests with Paired Data

- When the subjects are **paired** or matched in some way the amples are considered to be **dependent samples.**
- Dependent samples are sometimes called **paired samples**.
- When the samples are dependent, a special *t*-test for dependent means is used.
- This test employs **the difference in values of the matched pairs**.

Tests with Paired Data…

- The hypotheses are as follows:

Two-tailed	Left-tailed	Right-tailed
H_0: $\mu_D = 0$ H_1: $\mu_D \neq 0$	H_0: $\mu_D = 0$ H_1: $\mu_D < 0$	H_0: $\mu_D = 0$ H_1: $\mu_D > 0$

- Assumptions:

Assumptions for the *t* Test for Two Means When the Samples Are Dependent

1. The sample or samples are random.
2. The sample data are dependent.
3. When the sample size or sample sizes are less than 30, the population or populations must be normally or approximately normally distributed.

Tests with Paired Data (p.454)

Summary

Let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be a sample of ordered pairs whose differences $D_1, \ldots, D_n$ are a sample from a *normal* population with mean μ_D. Let s_D be the sample standard deviation of $D_1, \ldots, D_n$.

To test a null hypothesis of the form $H_0 : \mu_D \le \mu_0$, $H_0 : \mu_D \ge \mu_0$, or $H_0 : \mu_D = \mu_0$:

- Compute the test statistic $\boxed{t = \dfrac{\overline{D} - \mu_0}{s_D/\sqrt{n}}}$.
- Compute the *P*-value. The *P*-value is an area under the Student's *t* curve with $n - 1$ degrees of freedom, which depends on the alternate hypothesis as follows:

Alternate Hypothesis	***P*-value**
$H_1 : \mu_D > \mu_0$	Area to the right of t
$H_1 : \mu_D < \mu_0$	Area to the left of t
$H_1 : \mu_D \ne \mu_0$	Sum of the areas in the tails cut off by t and $-t$

- If the sample is large, the D_i need not be normally distributed, the test statistic is $z = \dfrac{\overline{D} - \mu_0}{s_D/\sqrt{n}}$, and a z test should be performed.

Tests with Paired Data…

- The test value is calculated in a different way.
- All other steps described earlier are the same.

a. Make a table, as shown.

X_1	X_2	A $D = X_1 - X_2$	B $D^2 = (X_1 - X_2)^2$
⋮	⋮	$\Sigma D =$ ______	$\Sigma D^2 =$ ______

b. Find the differences and place the results in column A.

$$D = X_1 - X_2$$

c. Find the mean of the differences.

$$\overline{D} = \frac{\Sigma D}{n}$$

d. Square the differences and place the results in column B. Complete the table.

$$D^2 = (X_1 - X_2)^2$$

e. Find the standard deviation of the differences.

$$s_D = \sqrt{\frac{n\Sigma D^2 - (\Sigma D)^2}{n(n-1)}}$$

f. Find the test value.

$$t = \frac{\overline{D} - \mu_D}{s_D/\sqrt{n}} \qquad \text{with d.f.} = n - 1$$

Example: Cholesterol Levels

A dietitian wishes to see if a person's cholesterol level will change if the diet is supplemented by a certain mineral. Six randomly selected subjects were pretested, and then they took the mineral supplement for a 6-week period. The results are shown in the table. (Cholesterol level is measured in milligrams per deciliter.) Can it be concluded that the cholesterol level has been changed at $\alpha = 0.10$? Assume the variable is approximately normally distributed.

Subject	1	2	3	4	5	6
Before (X_1)	210	235	208	190	172	244
After (X_2)	190	170	210	188	173	228

STEP 1: State the hypotheses and identify the claim.

If the diet is effective, the before cholesterol levels should be different from the after levels.

$$H_0\colon \mu_D = 0 \qquad \text{and} \qquad H_1\colon \mu_D \neq 0 \text{ (claim)}$$

Example: Cholesterol Levels – SOLUTION…

STEP 2: Compute the test value

- Make a table (*this is a suggestion not a requirement*)

Before (X_1)	After (X_2)	A $D = X_1 - X_2$	B $D^2 = (X_1 - X_2)^2$
210	190		
235	170		
208	210		
190	188		
172	173		
244	228		

- Then complete the table (*this is a suggestion not a requirement*)

Before (X_1)	After (X_2)	A $D = X_1 - X_2$	B $D^2 = (X_1 - X_2)^2$
210	190	20	400
235	170	65	4225
208	210	–2	4
190	188	2	4
172	173	–1	1
244	228	16	256
		$\Sigma D = 100$	$\Sigma D^2 = 4890$

Example: Cholesterol Levels – SOLUTION…

Before (X_1)	After (X_2)	A $D = X_1 - X_2$	B $D^2 = (X_1 - X_2)^2$
210	190	20	400
235	170	65	4225
208	210	–2	4
190	188	2	4
172	173	–1	1
244	228	16	256
		$\Sigma D = 100$	$\Sigma D^2 = 4890$

- Find the mean and the standard deviation of the differences

$$\overline{D} = \frac{\Sigma D}{n} = \frac{100}{6} = 16.7$$

$$s_D = \sqrt{\frac{n\Sigma D^2 - (\Sigma D)^2}{n(n-1)}}$$

$$= \sqrt{\frac{6 \cdot 4890 - 100^2}{6(6-1)}}$$

$$= \sqrt{\frac{29{,}340 - 10{,}000}{30}}$$

$$= 25.4$$

Example: Cholesterol Levels – SOLUTION…

$$\overline{D} = \frac{\Sigma D}{n} = \frac{100}{6} = 16.7$$

$$s_D = \sqrt{\frac{n\Sigma D^2 - (\Sigma D)^2}{n(n-1)}} = \sqrt{\frac{6 \cdot 4890 - 100^2}{6(6-1)}} = \sqrt{\frac{29,340 - 10,000}{30}} = 25.4$$

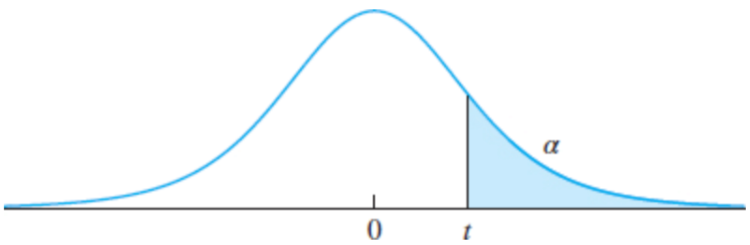
- The test value

$$t = \frac{\overline{D} - \mu_D}{s_D/\sqrt{n}} = \frac{16.7 - 0}{25.4/\sqrt{6}} = 1.610$$

Example: Cholesterol Levels – SOLUTION…

STEP3: Find the P-value

TABLE A.3 Upper percentage points for the Student's *t* distribution



	α						
v	**0.40**	**0.25**	**0.10**	**0.05**	**0.025**	**0.01**	**0.005**
1	0.325	1.000	3.078	6.314	12.706	31.821	63.657
2	0.289	0.816	1.886	2.920	4.303	6.965	9.925
3	0.277	0.765	1.638	2.353	3.182	4.541	5.841
4	0.271	0.741	1.533	2.132	2.776	3.747	4.604
5	0.267	0.727	1.476	2.015	2.571	3.365	4.032
6	0.265	0.718	1.440	1.943	2.447	3.143	3.707
7	0.263	0.711	1.415	1.895	2.365	2.998	3.499

Two-tailed test: 2*0.05 = 0.10 < P-value < 2*0.10 = 0.20
(calculator gives 0.168313)

Example: Cholesterol Levels – SOLUTION…

STEP 4: Decision

P-value > $\alpha = 0.10$; we do not reject H_0

STEP 5: Conclusion

There is not enough evidence to support the claim that the mineral changes a person's cholesterol level.

Power

- A hypothesis test results in a **type II error** if H_0 is not rejected when it is false.
- The power of a test is the probability of rejecting H_0 when it is false.
- Therefore, **Power = 1 – P(type II error)**

- To be useful, a test must have reasonably small probabilities of both type I and type II errors.
- The type I error is kept small by choosing a small value of α as the significance level.
- Then the power of the test is calculated.

Power…

- If the power is large, then the probability of a type II error is small as well, and the test is a useful one.
- Power calculations are generally done before data are collected.
- The purpose of a power calculation is to determine whether or not a hypothesis test, when performed, is likely to reject H_0 in the event that H_0 is false.

Computing Power

Computing the power involves two steps:

1. Compute the rejection region.
2. Compute the probability that the test statistic falls in the rejection region if the alternate hypothesis is true. This is the power.

Example 6.30 (p.491)

Find the power of the 5% level test of $H_0 : \mu \leq 80$ versus $H_1 : \mu > 80$ for the mean yield of the new process under the alternative $\mu = 82$, assuming $n = 50$ and $\sigma = 5$.

$$\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}} =$$

Example 6.30 (p.491) - SOLUTION

We must first find the null distribution.

We know that the statistic $\bar{X}$ has a normal distribution with mean μ and standard deviation $\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}}$ where n is the sample size.

$$\sigma_{\bar{X}} = \frac{5}{\sqrt{50}} = 0.707$$

Example 6.30 (p.491) – SOLUTION…

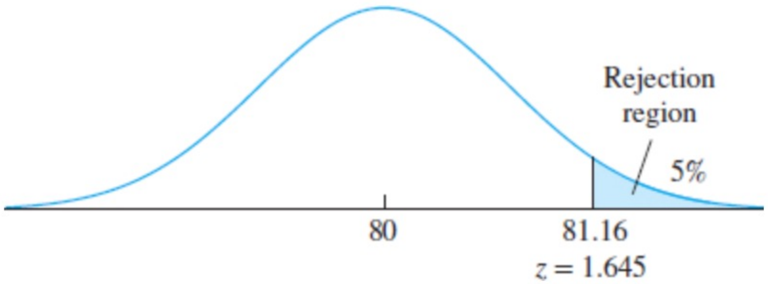


FIGURE 6.27 The hypothesis test will be conducted at a significance level of 5%. The rejection region for this test is the region where the *P*-value will be less than 0.05.

- The critical point has a z-score of 1.645, so its value is $80 + (1.645)(0.707) = 81.16$
- We will reject H_0 if $\bar{X} \geq 81.16$
- This is the rejection region.

Example 6.30 (p.491) – SOLUTION…

- The z-score for the critical point of 81.16 under the alternate hypothesis is

$\boldsymbol{z} = (\bar{X} - \mu)/\sigma_{\bar{X}} = (81.16 - 82)/0.707 = \mathbf{-1.19}$

- The area to the right of z = −1.19 is 1 – 0.1170 = 0.8830.

- This is the power.

Example 6.30 (p.491) – SOLUTION…

Cumulative Standard Normal Distribution

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002
−3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
−3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
−3.1	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007
−3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
−2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
−2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
−2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
−2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
−2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
−2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
−2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
−2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
−2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
−2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
−1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
−1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
−1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
−1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
−1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
−1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
−1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
−1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
−1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
−1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379

Example 6.30 (p.491) – SOLUTION…

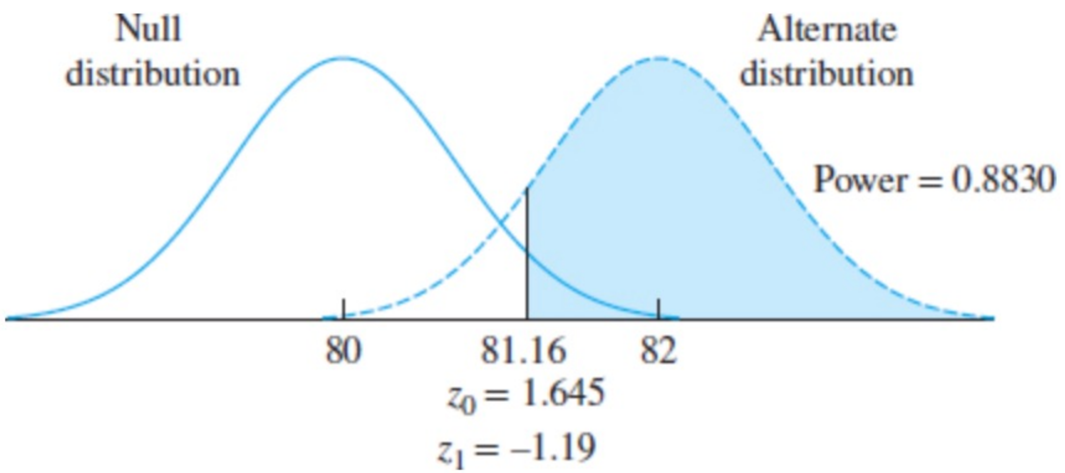
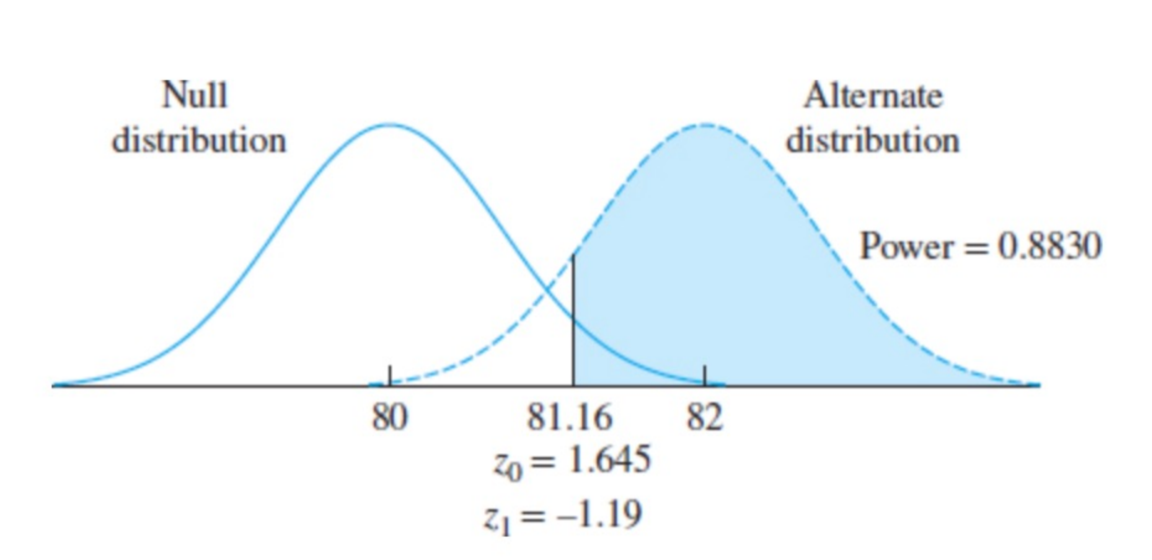


FIGURE 6.29 The rejection region, consisting of the upper 5% of the null distribution, is shaded. The z-score of the critical point is $z_0 = 1.645$ under the null distribution and $z_1 = -1.19$ under the alternate. The power is the area of the rejection region under the alternate distribution, which is 0.8830.

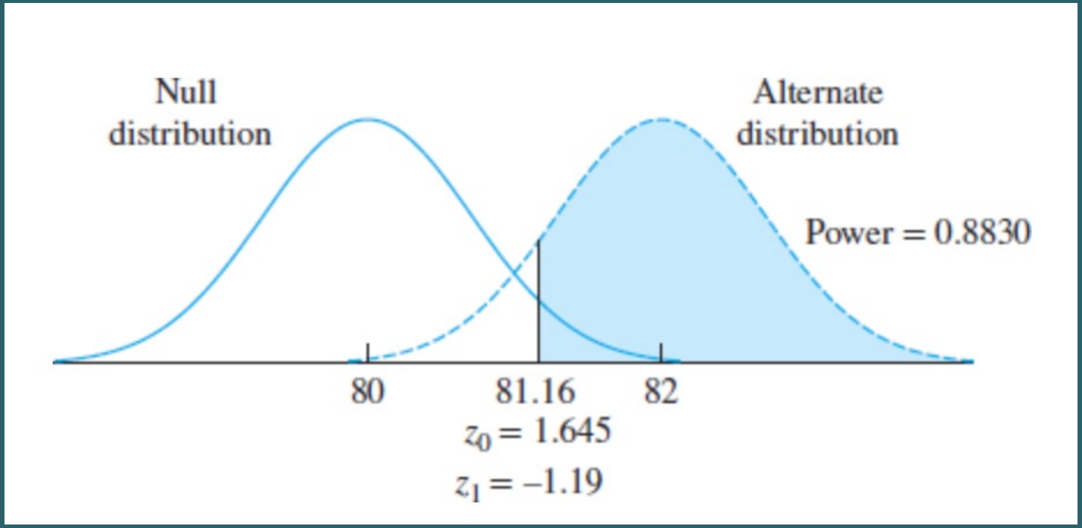
Example 6.30 (p.492) – COMMENTS

- Since the alternate distribution is obtained by shifting the null distribution, the power depends on which alternate value is chosen for μ, and can range from barely greater than the significance level α all the way up to 1.



Example 6.30 (p.492) – COMMENTS…

- If the alternate mean is chosen very close to the null mean, the alternate curve will be almost identical with the null, and the power will be very close to α.
- If the alternate mean is far from the null, almost all the area under the alternate curve will lie in the rejection region, and the power will be close to 1.



Sample size necessary to achieve a desired power

- When power is not large enough, it can be increased by increasing the sample size.

- When planning an experiment, one can determine the sample size necessary to achieve a desired power.

Example 6.31

In testing the hypothesis $H_0 : \mu \leq 80$ versus $H_1 : \mu > 80$ regarding the mean yield of the new process, how many times must the new process be run so that a test conducted at a significance level of 5% will have power 0.90 against the alternative $\mu = 81$, if it is assumed that $\sigma = 5$?

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$$80 + 1.645\left(\frac{5}{\sqrt{n}}\right) = 81 - 1.28\left(\frac{5}{\sqrt{n}}\right)$$

→ n = 214

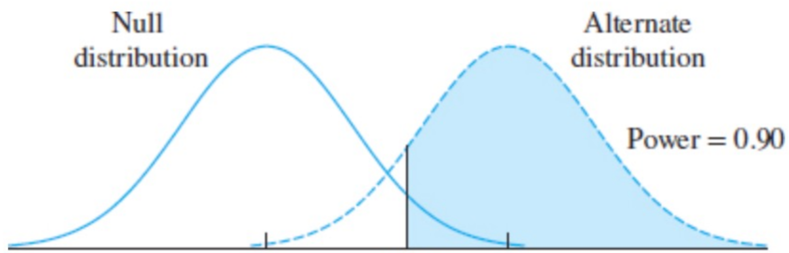

FIGURE 6.30 To achieve a power of 0.90 with a significance level of 0.05, the z-score for the critical point must be $z_0 = 1.645$ under the null distribution and $z_1 = -1.28$ under the alternate distribution.