Chapter 8

Integration Techniques

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8.1

Basic Approaches

Table 8.1 Basic Integration Formulas

1. $$\int k\,dx = kx + C,\ k \text{ real}$$

2. $$\int x^p\,dx = \frac{x^{p+1}}{p+1} + C,\ p \neq -1 \text{ real}$$

3. $$\int \cos ax\,dx = \frac{1}{a}\sin ax + C$$

4. $$\int \sin ax\,dx = -\frac{1}{a}\cos ax + C$$

5. $$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C$$

6. $$\int \csc^2 ax\,dx = -\frac{1}{a}\cot ax + C$$

7. $$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax + C$$

8. $$\int \csc ax \cot ax\,dx = -\frac{1}{a}\csc ax + C$$

9. $$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$$

10. $$\int \frac{dx}{x} = \ln|x| + C$$

11. $$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C$$

12. $$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

13. $$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C,\ a > 0$$

EXAMPLE 1 **Substitution review** Evaluate $\displaystyle\int_{-1}^{2} \frac{dx}{3 + 2x}$.

EXAMPLE 2 **Subtle substitution** Evaluate $\displaystyle\int \frac{dx}{e^{x} + e^{-x}}$.

EXAMPLE 3 **Split up fractions** Evaluate $\displaystyle\int \frac{\cos x + \sin^{3} x}{\sec x}\, dx$.

EXAMPLE 4 **Division with rational functions** Evaluate $\int \frac{x^2 + 2x - 1}{x + 4}\,dx$.

EXAMPLE 5 **Complete the square** Evaluate $\int \frac{dx}{\sqrt{-x^2 - 8x - 7}}$.

EXAMPLE 6 **Multiply by 1** Evaluate $\int \frac{dx}{1 + \cos x}$.

8.2

Integration by Parts

Integration by Parts for Indefinite Integrals

Substitution Rule arises when we reverse the Chain Rule for derivatives.
Reverse the Product Rule for derivatives lead to ***integration by parts***.

$$\int xe^x dx = ?$$

The Product Rule states that

$$\frac{d}{dx}\big(u(x)v(x)\big) = u'(x)v(x) + u(x)v'(x)$$

Integrating both sides

$$u(x)v(x) = \int (u'(x)v(x) + u(x)v'(x))dx$$

Rearranging this expression

$$\int u(x)\underbrace{v'(x)\,dx}_{dv} = u(x)v(x) - \int v(x)\underbrace{u'(x)\,dx}_{du}$$

Suppressing the independent variable x,

$$\int u\,dv = uv - \int v\,du.$$

Integration by Parts

Suppose that u and v are differentiable functions. Then

$$\int u\,dv = uv - \int v\,du.$$

EXAMPLE 1 **Integration by parts** Evaluate $\int xe^x\,dx$.

Functions in original integral	$u = x$	$dv = e^x\,dx$
Functions in new integral	$du = dx$	$v = e^x$

$$\int \underbrace{x}_{u}\ \underbrace{e^x\,dx}_{dv} = \underbrace{x}_{u}\ \underbrace{e^x}_{v} - \int \underbrace{e^x}_{v}\ \underbrace{dx}_{du}.$$

EXAMPLE 2 **Integration by parts** Evaluate $\int x \sin x \, dx$.

EXAMPLE 3 **Repeated use of integration by parts**

a. Evaluate $\int x^2 e^x \, dx$.

b. How would you evaluate $\int x^n e^x \, dx$, where n is a positive integer?

EXAMPLE 4 **Repeated use of integration by parts** Evaluate $\int e^{2x} \sin x \, dx$.

$u = e^{2x}$	$dv = \sin x \, dx$
$du = 2e^{2x} \, dx$	$v = -\cos x$

$$\int e^{2x} \sin x \, dx = -e^{2x} \cos x + 2 \int e^{2x} \cos x \, dx.$$

$u = e^{2x}$	$dv = \cos x \, dx$
$du = 2e^{2x} \, dx$	$v = \sin x$

$$\int e^{2x} \cos x \, dx = e^{2x} \sin x - 2 \int e^{2x} \sin x \, dx.$$

Integration by Parts for Definite Integrals

Two options:

1. Find an antiderivative using the method for indefinite integrals and then evaluate it at the upper and lower limits of integration.
2. Be incorporated directly into the integration by parts process.

Integration by Parts for Definite Integrals

Let u and v be differentiable. Then

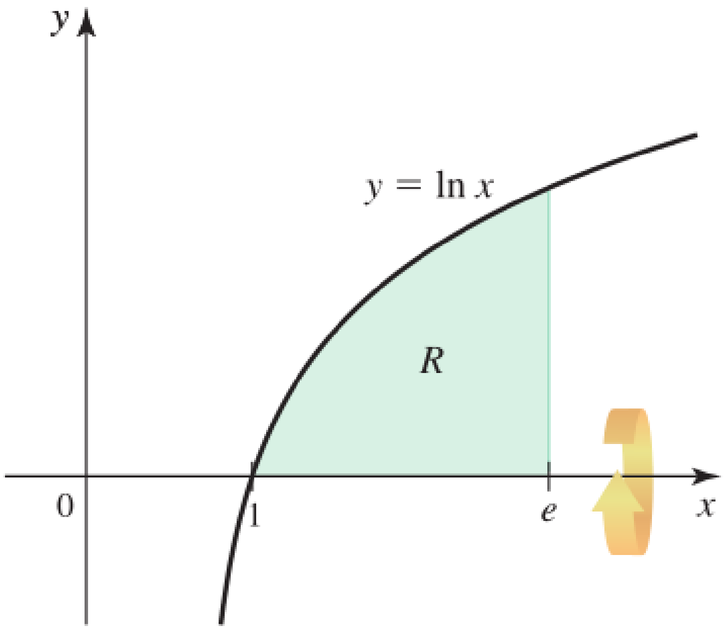
$$\int_a^b u(x)v'(x)\,dx = u(x)v(x)\Big|_a^b - \int_a^b v(x)u'(x)\,dx.$$

EXAMPLE 5 **A definite integral** Evaluate $\int_1^2 \ln x \, dx$.

Integral of ln x

$$\int \ln x \, dx = x \ln x - x + C$$

EXAMPLE 6 **Solids of revolution** Let R be the region bounded by $y = \ln x$, the x-axis, and the line $x = e$ (Figure ´ 1). Find the volume of the solid that is generated when the region R is revolved about the x-axis.



$$V = \int_1^e \pi(\ln x)^2\,dx.$$

When using integration by parts, the acronym **LIPET** may help.

If the integrand is the product of two or more functions, choose u to be the first function type that appears in the list

Logarithmic, Inverse trigonometric, Polynomial, Exponential, Trigonometric.

8.3

Trigonometric Integrals

Integrating Powers of $\sin x$ or $\cos x$

Integrals of the form $\int \sin^m x\, dx$ or $\int \cos^n x\, dx$, where m and n are positive integers.

Two strategies, and both strategies use trigonometric identities to recast the integrand

EXAMPLE 1 **Powers of sine or cosine** Evaluate the following integrals.

a. $\int \cos^5 x\, dx$ **b.** $\int \sin^4 x\, dx$

Integrating Products of Powers of $\sin x$ or $\cos x$

Integrals of the form $\int \sin^m x \cos^n x \, dx$.

If m is an odd, positive integer, split off a factor of $\sin x$ and write the remaining even power of $\sin x$ in terms of cosine functions.

If n is an odd, positive integer, split off a factor of $\cos x$ and write the remaining even power of $\cos x$ in terms of sine functions.

If both m and n are even positive integers,
the half-angle formulas are used to transform the integrand into a polynomial in $\cos 2x$.

EXAMPLE 2 **Products of sine and cosine** Evaluate the following integrals.

a. $\int \sin^4 x \cos^2 x \, dx$ **b.** $\int \sin^3 x \cos^{-2} x \, dx$

Table 8.2

$\int \sin^m x \cos^n x \, dx$	**Strategy**
m odd and positive, n real	Split off $\sin x$, rewrite the resulting even power of $\sin x$ in terms of $\cos x$, and then use $u = \cos x$.
n odd and positive, m real	Split off $\cos x$, rewrite the resulting even power of $\cos x$ in terms of $\sin x$, and then use $u = \sin x$.
m and n both even, nonnegative integers	Use half-angle formulas to transform the integrand into a polynomial in $\cos 2x$ and apply the preceding strategies once again to powers of $\cos 2x$ greater than 1.

Reduction Formulas

A *reduction formula* equates an integral involving a power of a function with another integral in which the power is reduced.

Reduction Formulas

Assume n is a positive integer.

1. $$\int \sin^n x\,dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\,dx$$

2. $$\int \cos^n x\,dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\,dx$$

3. $$\int \tan^n x\,dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx, \quad n \neq 1$$

4. $$\int \sec^n x\,dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx, \quad n \neq 1$$

EXAMPLE 3 **Powers of tan x** Evaluate $\int \tan^4 x \, dx$.

One solution: Apply reduction formula 3 twice.

An alternative solution uses the identity $\tan^2 x = \sec^2 x - 1$.

THEOREM 8.1 Integrals of tan x, cot x, sec x, and csc x

$$\int \tan x\,dx = -\ln|\cos x| + C = \ln|\sec x| + C \qquad \int \cot x\,dx = \ln|\sin x| + C$$

$$\int \sec x\,dx = \ln|\sec x + \tan x| + C \qquad \int \csc x\,dx = -\ln|\csc x + \cot x| + C$$

Integrating Products of Powers of $\tan x$ and $\sec x$

Integrals of the form $\int \tan^m x \sec^n x \, dx$.

If n is an even positive integer, split off a factor of $\sec^2 x$ and write the remaining even power of $\sec x$ in terms of $\tan x$. The substitution $u = \tan x$.

If m is odd and positive, split off a factor of $\sec x \tan x$ (the derivative of $\sec x$). The substitution $u = \sec x$.

If m is even and n is odd,
the integrand is expressed as a polynomial in $\sec x$, each of whose terms is handled by a reduction formula.

EXAMPLE 4 **Products of tan x and sec x** Evaluate the following integrals.

a. $\int \tan^3 x \sec^4 x \, dx$

b. $\int \tan^2 x \sec x \, dx$

Table 8.3

$\int \tan^m x \sec^n x \, dx$	Strategy
n even	Split off $\sec^2 x$, rewrite the remaining even power of $\sec x$ in terms of $\tan x$, and use $u = \tan x$.
m odd	Split off $\sec x \tan x$, rewrite the remaining even power of $\tan x$ in terms of $\sec x$, and use $u = \sec x$.
m even and n odd	Rewrite the even power of $\tan x$ in terms of $\sec x$ to produce a polynomial in $\sec x$; apply reduction formula 4 to each term.

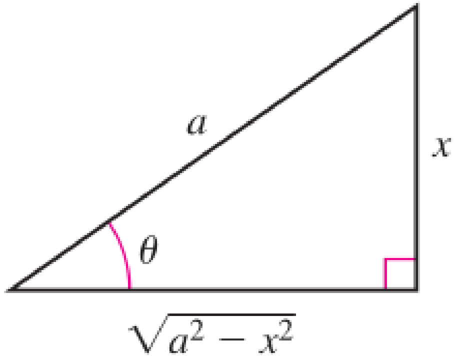
8.4

Trigonometric Substitutions

Integrals Involving $a^2 - x^2$

The goal is to introduce techniques that transform sums of squares $a^2 + x^2$ (and the difference of squares $a^2 - x^2$ and $x^2 - a^2$) into products of squares.

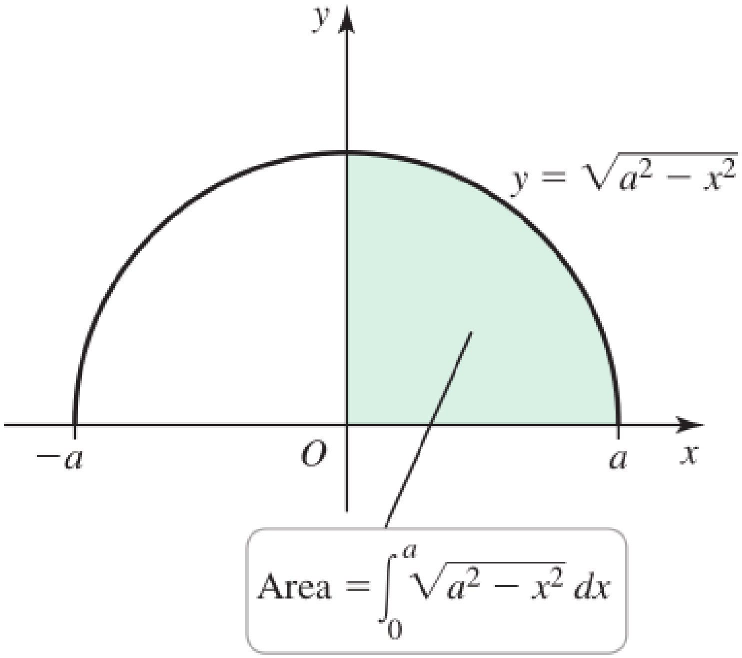
These integrals can be simplified using somewhat unexpected substitutions involving trigonometric functions.



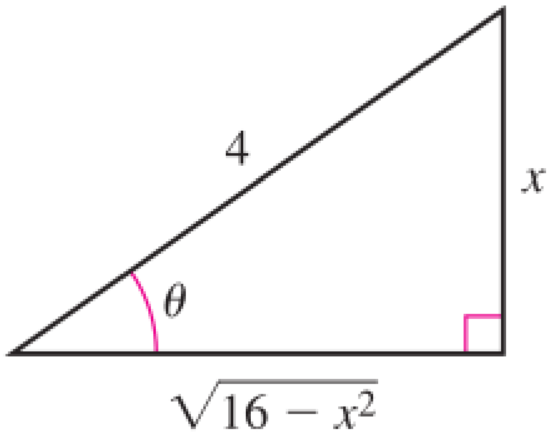
$x = a \sin \theta$

$$\begin{aligned} a^2 - x^2 &= a^2 - (a \sin \theta)^2 && \text{Replace } x \text{ with } a \sin \theta. \\ &= a^2 - a^2 \sin^2 \theta && \text{Simplify.} \\ &= a^2 (1 - \sin^2 \theta) && \text{Factor.} \\ &= a^2 \cos^2 \theta. && 1 - \sin^2 \theta = \cos^2 \theta \end{aligned}$$

EXAMPLE 1 **Area of a circle** Verify that the area of a circle of radius a is πa^2.



EXAMPLE 2 **Sine substitution** Evaluate $\displaystyle\int \frac{dx}{(16 - x^2)^{3/2}}$.



$$\sin\theta = \frac{x}{4}$$
$$\tan\theta = \frac{x}{\sqrt{16 - x^2}}$$

Integrals Involving $a^2 + x^2$ or $x^2 - a^2$

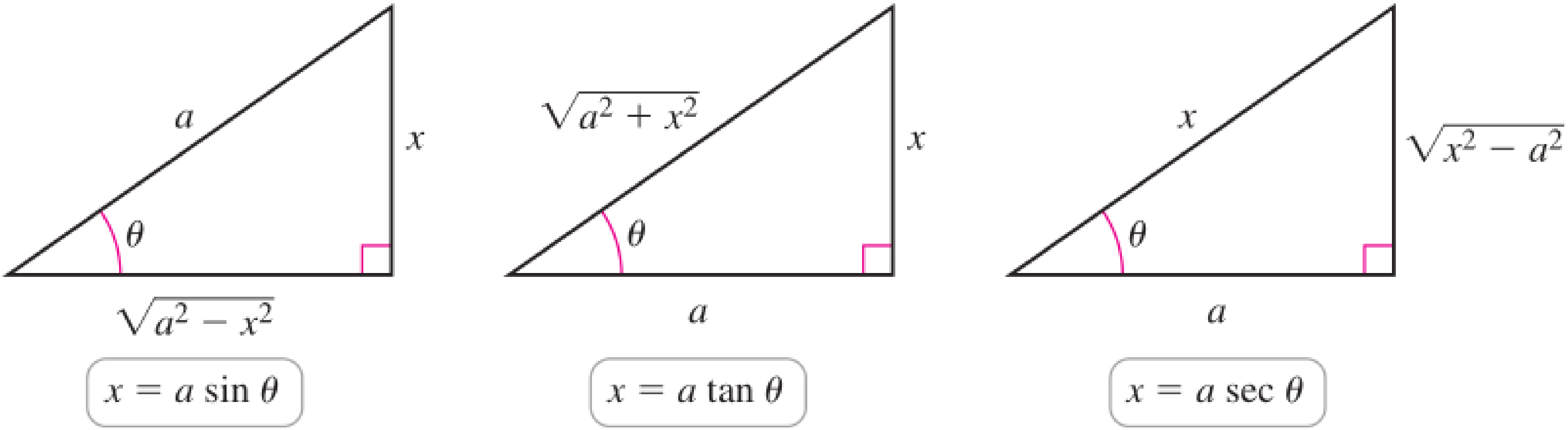
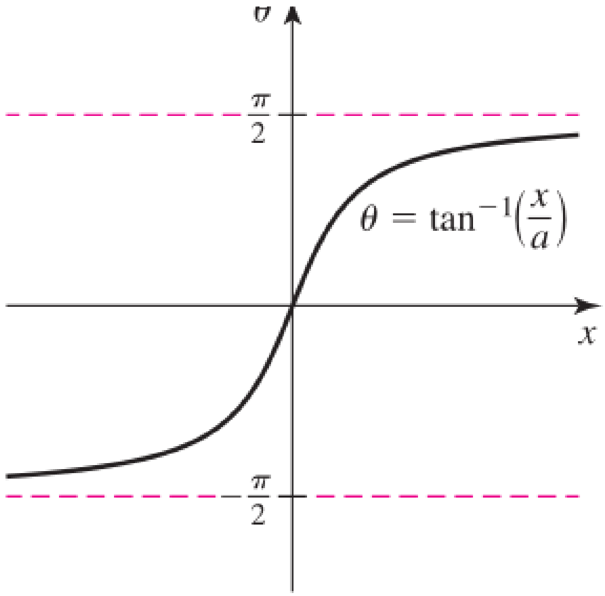


Table 8.4

The Integral Contains . . .	Corresponding Substitution	Useful Identity
$a^2 - x^2$	$x = a\sin\theta,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, for $\lvert x\rvert \le a$	$a^2 - a^2\sin^2\theta = a^2\cos^2\theta$
$a^2 + x^2$	$x = a\tan\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$a^2 + a^2\tan^2\theta = a^2\sec^2\theta$
$x^2 - a^2$	$x = a\sec\theta, \begin{cases} 0 \le \theta < \frac{\pi}{2}, \text{ for } x \ge a \\ \frac{\pi}{2} < \theta \le \pi, \text{ for } x \le -a \end{cases}$	$a^2\sec^2\theta - a^2 = a^2\tan^2\theta$

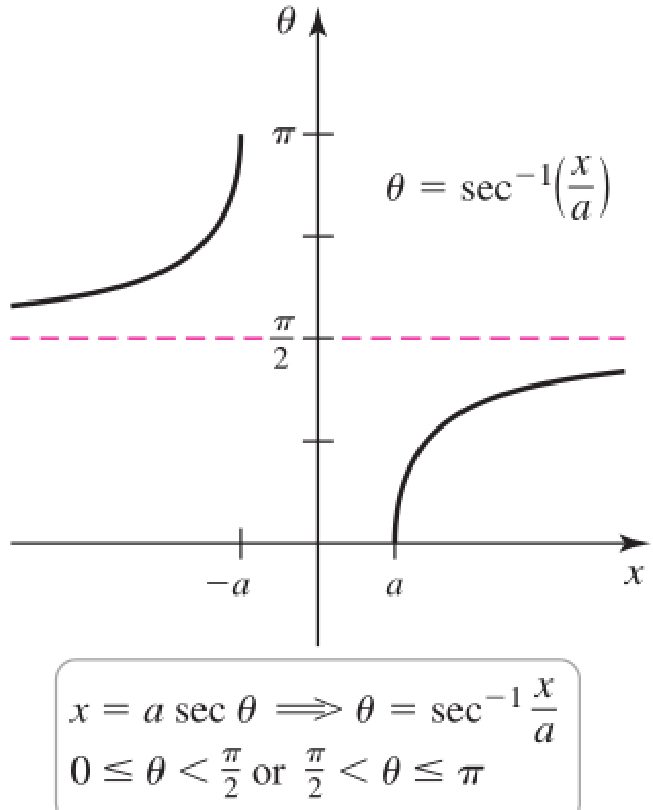
For the tangent substitution $x = a\tan\theta$ to be well defined, the angle θ must be restricted to the interval $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.
On this interval, $\sec\theta > 0$ and with $a > 0$, it is valid

$$\sqrt{a^2 + x^2} = \sqrt{a^2 + (a\tan\theta)^2} = \sqrt{a^2\underbrace{(1 + \tan^2\theta)}_{\sec^2\theta}} = a\sec\theta.$$



$$x = a\tan\theta \implies \theta = \tan^{-1}\left(\frac{x}{a}\right)$$
$$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

For secant substitution $x = a\sec\theta$, $\theta = \sec^{-1}(\frac{x}{a})$ is defined for $x \geq a$, in which case $0 \leq \theta < \pi/2$, *and* for $x \leq -a$, in which case $\frac{\pi}{2} < \theta \leq \pi$. These restrictions on θ must be treated carefully when simplifying integrands with a factor of $\sqrt{x^2 - a^2}$. (next slide)



$\tan\theta$ is positive in the first quadrant but negative in the second

$$\sqrt{x^2 - a^2} = \sqrt{a^2\underbrace{(\sec^2\theta - 1)}_{\tan^2\theta}} = |a\tan\theta| = \begin{cases} a\tan\theta & \text{if } 0 \le \theta < \dfrac{\pi}{2} \\ -a\tan\theta & \text{if } \dfrac{\pi}{2} < \theta \le \pi. \end{cases}$$

When evaluating a definite integral, you should check the limits of integration to see which of these two cases applies.

For indefinite integrals, a piecewise formula is often needed,

EXAMPLE 3 **Arc length of a parabola** Evaluate $\int_0^2 \sqrt{1 + 4x^2}\, dx$, the arc length of the segment of the parabola $y = x^2$ on $[0, 2]$.

$$2\int \sqrt{\left(\tfrac{1}{2}\right)^2 + x^2}\, dx = 2\int \tfrac{1}{2}\sec\theta \underbrace{\tfrac{1}{2}\sec^2\theta\, d\theta}_{dx}$$

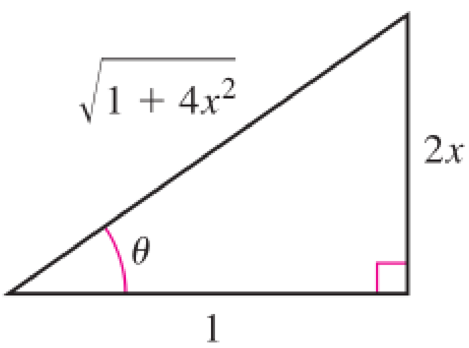
$x = \frac{1}{2}\tan\theta,$
$dx = \frac{1}{2}\sec^2\theta\, d\theta$

$$= \frac{1}{2}\int \sec^3\theta\, d\theta$$

Simplify.

$$= \frac{1}{4}\left(\sec\theta\tan\theta + \ln\left|\sec\theta + \tan\theta\right|\right) + C.$$

Reduction formula 4, Section 7.3



$\tan\theta = 2x$

$\sec\theta = \sqrt{1 + 4x^2}$

$$2\int_0^2 \sqrt{\left(\tfrac{1}{2}\right)^2 + x^2}\, dx = \frac{1}{4}\left(\underbrace{\sqrt{1 + 4x^2}}_{\sec\theta}\,\underbrace{2x}_{\tan\theta} + \ln\left|\underbrace{\sqrt{1 + 4x^2}}_{\sec\theta} + \underbrace{2x}_{\tan\theta}\right|\right)\Bigg|_0^2$$

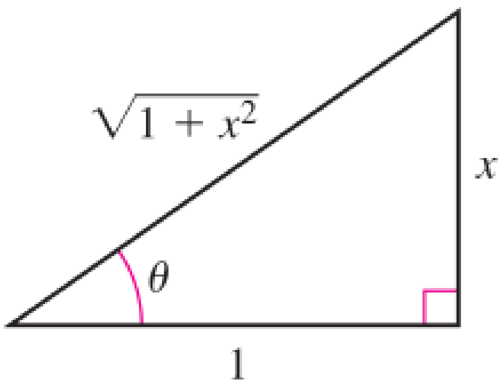
EXAMPLE 4 **Another tangent substitution** Evaluate $\int \frac{dx}{(1+x^2)^2}$.

The factor $1+x^2$ suggests the substitution $x = \tan\theta$.

$$\int \frac{dx}{(1+x^2)^2} = \int \frac{\sec^2\theta}{\sec^4\theta}\,d\theta \qquad x = \tan\theta,\ dx = \sec^2\theta\,d\theta$$

$$= \int \cos^2\theta\,d\theta \qquad \text{Simplify.}$$

$$= \frac{\theta}{2} + \frac{\sin 2\theta}{4} + C. \qquad \text{Integrate } \cos^2\theta = \frac{1+\cos 2\theta}{2}.$$



$$\sin\theta = \frac{x}{\sqrt{1+x^2}}$$
$$\cos\theta = \frac{1}{\sqrt{1+x^2}}$$

$$\int \frac{dx}{(1+x^2)^2} = \frac{\theta}{2} + \frac{\sin 2\theta}{4} + C$$
$$= \frac{1}{2}\tan^{-1}x + \frac{x}{2(1+x^2)} + C.$$

EXAMPLE 5 **Multiple approaches** Evaluate the integral $\int \frac{dx}{\sqrt{x^2 + 4}}$.

Solution 1: The term $x^2 + 4$ suggests the substitution $x = 2\tan\theta$, which implies that $dx = 2\sec^2\theta\, d\theta$

$$\int \frac{dx}{\sqrt{x^2 + 4}} = \int \frac{2\sec^2\theta}{2\sec\theta}\, d\theta = \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C.$$

$$\begin{aligned}
\int \frac{dx}{\sqrt{x^2 + 4}} &= \ln|\sec\theta + \tan\theta| + C \\
&= \ln\left|\frac{1}{2}\sqrt{x^2 + 4} + \frac{x}{2}\right| + C && \text{Substitute for } \sec\theta \text{ and } \tan\theta. \\
&= \ln\left(\frac{1}{2}\left(\sqrt{x^2 + 4} + x\right)\right) + C && \text{Factor; } \sqrt{x^2 + 4} + x > 0. \\
&= \ln\frac{1}{2} + \ln\left(\sqrt{x^2 + 4} + x\right) + C && \ln ab = \ln a + \ln b \\
&= \ln\left(\sqrt{x^2 + 4} + x\right) + C. && \text{Absorb constant in } C.
\end{aligned}$$

Solution 2:
$$\int \frac{dx}{\sqrt{x^2+4}} = \sinh^{-1}\frac{x}{2} + C.$$

$$\sinh^{-1}\frac{x}{2} = \ln\left(\frac{x}{2} + \sqrt{\left(\frac{x}{2}\right)^2 + 1}\right) = \ln\left(\frac{1}{2}(\sqrt{x^2+4} + x)\right),$$

Solution 3: Use the substitution $x = 2\sinh t$, which implies that $dx = 2\cosh t\, dt$ and

$$\sqrt{x^2+4} = \sqrt{4\sinh^2 t + 4} = \sqrt{4(\sinh^2 t + 1)} = 2\sqrt{\cosh^2 t} = 2\cosh t.$$

$$\int \frac{dx}{\sqrt{x^2+4}} = \int \frac{2\cosh t}{2\cosh t}\, dt = \int dt = t + C.$$

$$t = \sinh^{-1}\frac{x}{2},$$

EXAMPLE 6 **A secant substitution** Evaluate $\displaystyle\int_1^4 \frac{\sqrt{x^2+4x-5}}{x+2}\,dx.$

$$\int_1^4 \frac{\sqrt{x^2+4x-5}}{x+2}\,dx = \int_1^4 \frac{\sqrt{(x+2)^2-9}}{x+2}\,dx \qquad \text{Complete the square.}$$

$$= \int_3^6 \frac{\sqrt{u^2-9}}{u}\,du. \qquad u = x+2,\ du = dx \quad \text{Change limits of integration.}$$

$$= \int_0^{\pi/3} \frac{3\tan\theta}{3\sec\theta}\,3\sec\theta\tan\theta\,d\theta \qquad u = 3\sec\theta,\ du = 3\sec\theta\tan\theta\,d\theta$$

$$= 3\int_0^{\pi/3} \tan^2\theta\,d\theta \qquad \text{Simplify.}$$

$$= 3\int_0^{\pi/3} (\sec^2\theta - 1)\,d\theta \qquad \tan^2\theta = \sec^2\theta - 1$$

$$= 3(\tan\theta - \theta)\Big|_0^{\pi/3} \qquad \text{Evaluate integrals.}$$

$$= 3\sqrt{3} - \pi. \qquad \text{Simplify.}$$

8.5

Partial Fractions

Method of Partial Fractions

Feature: Integrands are rational functions

Rational function		Partial fraction decomposition
$\dfrac{3x}{x^2+2x-8}$	$\xrightarrow{\textit{method of partial fractions}}$	$\dfrac{1}{x-2}+\dfrac{2}{x+4}$
Difficult to integrate		**Easy to integrate**
$\displaystyle\int \frac{3x}{x^2+2x-8}\,dx$		$\displaystyle\int\left(\frac{1}{x-2}+\frac{2}{x+4}\right)dx$

Usually combined with standard and trigonometric substitutions

Key idea

Given $f(x) = \frac{3x}{(x-2)(x+4)}$, to write in the form

$$\frac{A}{(x-2)} + \frac{B}{(x+4)}$$

Called partial fraction decomposition

Determine A and B

$$\frac{3x}{(x-2)(x+4)} = \frac{A}{(x-2)} + \frac{B}{(x+4)}$$

Multiplying both sides by $(x-2)(x+4)$

$$3x = A(x+4) + B(x-2)$$

$$3x = (A+B)x + 4A - 2B$$

Solve it to obtain

$$A = 1, B = 2$$

Simple Linear Factors of the form $(x - r)$

PROCEDURE **Partial Fractions with Simple Linear Factors**

Suppose $f(x) = p(x)/q(x)$, where p and q are polynomials with no common factors and with the degree of p less than the degree of q. Assume that q is the product of simple linear factors. The partial fraction decomposition is obtained as follows.

Step 1. **Factor the denominator *q*** in the form $(x - r_1)(x - r_2) \cdots (x - r_n)$, where $r_1, \ldots, r_n$ are real numbers.

Step 2. **Partial fraction decomposition** Form the partial fraction decomposition by writing

$$\frac{p(x)}{q(x)} = \frac{A_1}{(x - r_1)} + \frac{A_2}{(x - r_2)} + \cdots + \frac{A_n}{(x - r_n)}.$$

Step 3. **Clear denominators** Multiply both sides of the equation in Step 2 by $q(x) = (x - r_1)(x - r_2) \cdots (x - r_n)$, which produces conditions for $A_1, \ldots, A_n$.

Step 4. **Solve for coefficients** Equate like powers of x in Step 3 to solve for the undetermined coefficients $A_1, \ldots, A_n$.

EXAMPLE 1 Integrating with partial fractions

a. Find the partial fraction decomposition for $f(x) = \dfrac{3x^2 + 7x - 2}{x^3 - x^2 - 2x}$.

b. Evaluate $\displaystyle\int f(x)\,dx$.

A Shortcut (convenient Values)

The equation holds for all values of x, it must hold for any particular value of x

By choosing values of x judiciously, it is easy to solve for A, B, and C.

For example, setting $x = 0, -1, 2$ in the following equation

$$3x^2 + 7x - 2 = A(x + 1)(x - 2) + Bx(x - 2) + Cx(x + 1).$$

EXAMPLE 2 **Using the shortcut**

a. Find the partial fraction decomposition for $f(x) = \dfrac{3x^2 + 2x + 5}{(x - 1)(x^2 - x - 20)}$.

b. Evaluate $\displaystyle\int_2^4 f(x)\,dx$.

Repeated Linear Factors of the form $(x - r)^m$

PROCEDURE **Partial Fractions for Repeated Linear Factors**

Suppose the repeated linear factor $(x - r)^m$ appears in the denominator of a proper rational function in reduced form. The partial fraction decomposition has a partial fraction for each power of $(x - r)$ up to and including the mth power; that is, the partial fraction decomposition contains the sum

$$\frac{A_1}{(x-r)} + \frac{A_2}{(x-r)^2} + \frac{A_3}{(x-r)^3} + \cdots + \frac{A_m}{(x-r)^m},$$

where $A_1, \ldots, A_m$ are constants to be determined.

EXAMPLE 3 **Integrating with repeated linear factors** Evaluate $\int f(x)\,dx$, where $f(x) = \dfrac{5x^2 - 3x + 2}{x^3 - 2x^2}$.

$$\frac{5x^2 - 3x + 2}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{(x-2)}.$$

$$f(x) = \frac{1}{x} - \frac{1}{x^2} + \frac{4}{x-2}.$$

Irreducible Quadratic Factors of the form $ax^2 + bx + c$

By irreducible, we mean that $ax^2 + bx + c$ cannot be factored over the real numbers. For example,

$$x^9 + 4x^8 + 6x^7 + 34x^6 + 64x^5 - 84x^4 - 287x^3 - 500x^2 - 354x - 180$$

Factors as

$$\underbrace{(x-2)}_{\text{linear factor}}\underbrace{(x+3)^2}_{\text{repeated linear factor}}\underbrace{(x^2-2x+10)}_{\text{irreducible quadratic factor}}\underbrace{(x^2+x+1)^2}_{\text{repeated irreducible quadratic factor}}.$$

Two cases: simple and repeated factors

PROCEDURE **Partial Fractions with Simple Irreducible Quadratic Factors**

Suppose a simple irreducible factor $ax^2 + bx + c$ appears in the denominator of a proper rational function in reduced form. The partial fraction decomposition contains a term of the form

$$\frac{Ax + B}{ax^2 + bx + c},$$

where A and B are unknown coefficients to be determined.

EXAMPLE 4 **Setting up partial fractions** Give the appropriate form of the partial fraction decomposition for the following functions.

a. $\dfrac{x^2 + 1}{x^4 - 4x^3 - 32x^2}$ **b.** $\dfrac{10}{(x - 2)^2(x^2 + 2x + 2)}$

EXAMPLE 5 **Integrating with partial fractions** Evaluate

$$\int \frac{7x^2 - 13x + 13}{(x-2)(x^2-2x+3)}\,dx.$$

$$\frac{7x^2 - 13x + 13}{(x-2)(x^2-2x+3)} = \frac{A}{x-2} + \frac{Bx + C}{x^2 - 2x + 3}.$$

$$\begin{aligned} 7x^2 - 13x + 13 &= A(x^2 - 2x + 3) + (Bx + C)(x - 2) \\ &= (A + B)x^2 + (-2A - 2B + C)x + (3A - 2C). \end{aligned}$$

$$\int \frac{7x^2 - 13x + 13}{(x-2)(x^2-2x+3)}\,dx = \int \frac{5}{x-2}\,dx + \int \frac{2x+1}{x^2-2x+3}\,dx.$$

For the second (more difficult) integral

$$\int \frac{2x+1}{x^2-2x+3}\,dx = \int \frac{2x-2}{x^2-2x+3}\,dx + \int \frac{3}{x^2-2x+3}\,dx.$$

$$\int \frac{7x^2-13x+13}{(x-2)(x^2-2x+3)}\,dx$$

$$= \int \frac{5}{x-2}\,dx + \int \underbrace{\frac{2x-2}{x^2-2x+3}}_{\text{let } u = x^2-2x+3}\,dx + \int \underbrace{\frac{3}{x^2-2x+3}}_{(x-1)^2+2}\,dx$$

$$= 5\ln|x-2| + \ln|x^2-2x+3| + \frac{3}{\sqrt{2}}\tan^{-1}\left(\frac{x-1}{\sqrt{2}}\right) + K \quad \text{Integrate.}$$

$$= \ln|(x-2)^5(x^2-2x+3)| + \frac{3}{\sqrt{2}}\tan^{-1}\left(\frac{x-1}{\sqrt{2}}\right) + K. \quad \text{Property of logarithms}$$

Long Division

If $f(x) = p(x)/q(x)$ is *not* a proper rational function
Divide the denominator into the numerator and express f in two parts: a polynomial + a proper rational function. For example,

$$f(x) = \frac{2x^3 + 11x^2 + 28x + 33}{x^2 - x - 6},$$

$$f(x) = \underbrace{2x + 13}_{\substack{\text{polynomial;} \\ \text{easy to} \\ \text{integrate}}} + \underbrace{\frac{53x + 111}{x^2 - x - 6}}_{\substack{\text{apply partial fraction} \\ \text{decomposition}}}.$$

SUMMARY **Partial Fraction Decompositions**

Let $f(x) = p(x)/q(x)$ be a proper rational function in reduced form. Assume the denominator q has been factored completely over the real numbers and m is a positive integer.

1. **Simple linear factor** A factor $x - r$ in the denominator requires the partial fraction $\dfrac{A}{x - r}$.
2. **Repeated linear factor** A factor $(x - r)^m$ with $m > 1$ in the denominator requires the partial fractions

$$\frac{A_1}{(x - r)} + \frac{A_2}{(x - r)^2} + \frac{A_3}{(x - r)^3} + \cdots + \frac{A_m}{(x - r)^m}.$$

3. **Simple irreducible quadratic factor** An irreducible factor $ax^2 + bx + c$ in the denominator requires the partial fraction

$$\frac{Ax + B}{ax^2 + bx + c}.$$

4. **Repeated irreducible quadratic factor** (See Exercises 83–86.) An irreducible factor $(ax^2 + bx + c)^m$ with $m > 1$ in the denominator requires the partial fractions

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_mx + B_m}{(ax^2 + bx + c)^m}.$$

8.6

Integration Strategies

Problem: How to attack a generic integration problem when the appropriate method is not obvious?

Rewrite the integrand

Split up fractions, use long division or trigonometric identities, complete the square including a quadratic function, perform algebraic manipulations, and cancel common factors.

EXAMPLE 1 **Rewrite the integrand** Evaluate $\displaystyle\int \frac{\sin x + 1}{\cos^2 x}\,dx.$

Try substitution

If an integrand contains the function $f(x)$ and its $f'(x)$, the substitution $u = f(x)$ often works.

EXAMPLE 2 **Substitution after rewriting the integrand** Evaluate $\int \frac{x^2}{\sqrt{4 - x^6}}\, dx$.

Recognize a common pattern

1. Integration by parts when integrating the product of two functions, and sometimes successful for a single function, e.g., $\int \ln x \, dx$ or $\int \sinh^{-1} x \, dx$
2. Powers and products of $\sin x$ and $\cos x$, $\tan x$ and $\sec x$, or $\cot x$ and $\csc x$ are integrated using trigonometric identities, reduction formulas or substitutions.
3. Trigonometric substitution for integrals containing $a^2 - x^2$, $a^2 + x^2$, $x^2 - a^2$
4. Partial fraction decomposition to integrate rational functions. Usually cancel common factors and apply long division first

EXAMPLE 3 **Identify a strategy** Suggest a technique of integration for each of the following integrals based on patterns in the integrand.

a. $\displaystyle\int \frac{4 - 3x^2}{x(x^2 - 4)}\,dx$ **b.** $\displaystyle\int x e^{\sqrt{1+x^2}}\,dx$ **c.** $\displaystyle\int \ln(1 + x^2)\,dx$

Be Persistent

A. More than one technique may work for a integral, try another if one does not work
B. Carry out the first few steps of several techniques to recognize the best approach
C. When encountering dead ends, try a different approach, e.g.,

$$\int \frac{\sin x + 1}{\cos^2 x}\, dx = \int \frac{\sin x + 1}{1 - \sin^2 x}\, dx = \int \frac{\sin x + 1}{(1 + \sin x)(1 - \sin x)}\, dx = \int \frac{1}{1 - \sin x}\, dx.$$

EXAMPLE 4 **Integration by parts?** Evaluate $\int xe^{\sqrt{x}}\,dx$.

$u = e^{\sqrt{x}}$	$dv = x\,dx$
$du = \dfrac{e^{\sqrt{x}}}{2\sqrt{x}}\,dx$	$v = \dfrac{x^2}{2}$

$$\int xe^{\sqrt{x}}\,dx = \frac{x^2e^{\sqrt{x}}}{2} - \frac{1}{4}\int x^{3/2}e^{\sqrt{x}}\,dx.$$

More difficult

Try substitution $u = \sqrt{x}$, $du = \frac{1}{2\sqrt{x}}dx$

Multiple techniques

EXAMPLE 5 **Multiple techniques needed** Find the area of the surface of revolution that results when the curve $f(x) = e^x$ on $[0, \ln 2]$ is revolved about the x-axis.

Surface area $$\int_0^{\ln 2} 2\pi e^x \sqrt{1 + (e^x)^2}\, dx.$$

$$= 2\pi \int_1^2 \sqrt{1 + u^2}\, du$$ $u = e^x$; $du = e^x\, dx$; $x = 0 \Rightarrow u = 1$; $x = \ln 2 \Rightarrow u = 2$

$$= 2\pi \int \sqrt{1 + \tan^2\theta}\, \sec^2\theta\, d\theta$$ Let $u = \tan\theta$; $du = \sec^2\theta\, d\theta$.

$$= 2\pi \int \sec^3\theta\, d\theta.$$ $1 + \tan^2\theta = \sec^2\theta$

$$2\pi \int \sec^3\theta\, d\theta = 2\pi\left(\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\int \sec\theta\, d\theta\right) \qquad \text{Secant reduction formula}$$

$$= \pi(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|) + C \qquad \text{Evaluate } \int \sec\theta\, d\theta.$$

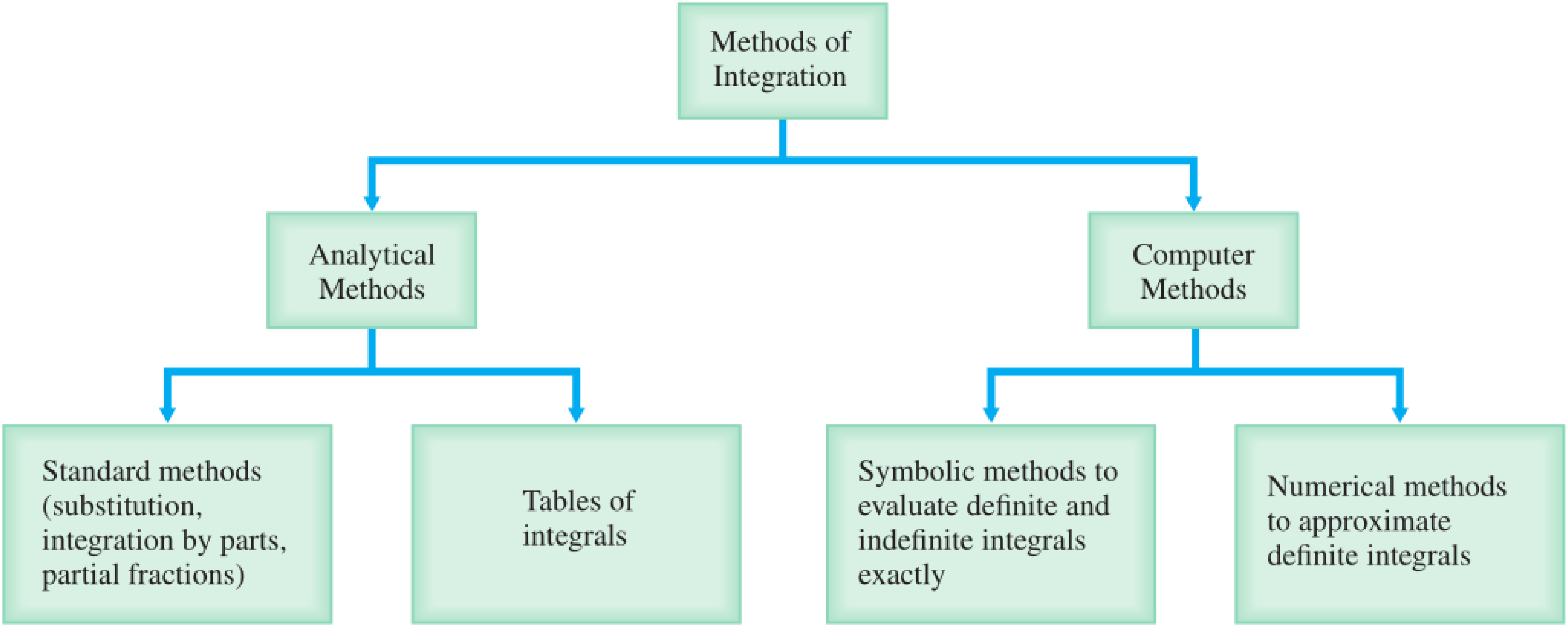
$$= \pi\left(u\sqrt{1+u^2} + \ln\left|u + \sqrt{1+u^2}\right|\right) + C. \qquad \tan\theta = u;\ \sec\theta = \sqrt{1+u^2}$$

$$\text{Area} = \pi\left(u\sqrt{1+u^2} + \ln\left(u + \sqrt{1+u^2}\right)\right)\Big|_1^2$$

$$= \pi\left(2\sqrt{5} - \sqrt{2} + \ln\frac{2+\sqrt{5}}{1+\sqrt{2}}\right) \approx 11.37.$$

8.7

Other Integration Methods


Methods of Integration
Analytical Methods
Computer Methods
Standard methods (substitution, integration by parts, partial fractions)
Tables of integrals
Symbolic methods to evaluate definite and indefinite integrals exactly
Numerical methods to approximate definite integrals

Using Tables of Integrals

EXAMPLE 1 **Using tables of integrals** Evaluate the integral $\displaystyle\int \frac{dx}{x\sqrt{2x-9}}$.

A table of integrals includes the integral

$$\int \frac{dx}{x\sqrt{ax-b}} = \frac{2}{\sqrt{b}}\tan^{-1}\sqrt{\frac{ax-b}{b}} + C, \quad \text{where} \quad b > 0,$$

Or

Letting $u^2 = 2x - 9$, we have
$u\,du = dx$ and $x = \frac{1}{2}(u^2 + 9)$.
Therefore,

$$\int \frac{dx}{x\sqrt{2x-9}} = 2\int \frac{du}{u^2+9}.$$

EXAMPLE 2 **Preliminary work** Evaluate $\int \sqrt{x^2 + 6x}\, dx$.

$$\int \sqrt{x^2 + 6x}\, dx = \int \sqrt{(x+3)^2 - 9}\, dx \qquad \text{Complete the square.}$$

$$= \int \sqrt{u^2 - 9}\, du \qquad u = x + 3,\ du = dx$$

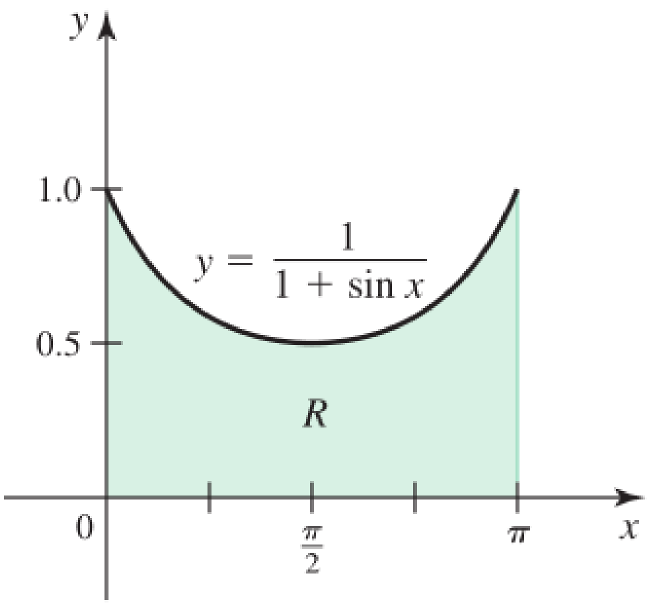
$$= \frac{u}{2}\sqrt{u^2 - 9} - \frac{9}{2}\ln\left|u + \sqrt{u^2 - 9}\right| + C \qquad \text{Table of integrals}$$

$$= \frac{x+3}{2}\sqrt{(x+3)^2 - 9} - \frac{9}{2}\ln\left|x + 3 + \sqrt{(x+3)^2 - 9}\right| + C$$

Replace u with $x + 3$.

$$= \frac{x+3}{2}\sqrt{x^2 + 6x} - \frac{9}{2}\ln\left|x + 3 + \sqrt{x^2 + 6x}\right| + C. \qquad \text{Simplify.}$$

EXAMPLE 3 **Using tables of integrals for area** Find the area of the region bounded by the curve $y = \dfrac{1}{1 + \sin x}$ and the x-axis between $x = 0$ and $x = \pi$.



The area is $\int_0^{\pi} \dfrac{1}{1+\sin x}\, dx$

$$\int \frac{dx}{1 + \sin ax} = -\frac{1}{a}\tan\left(\frac{\pi}{4} - \frac{ax}{2}\right) + C.$$

8.8

Numerical Integration

Absolute and Relative Error

When analytical methods fail, turn to *numerical methods*

DEFINITION Absolute and Relative Error

Suppose c is a computed numerical solution to a problem having an exact solution x. There are two common measures of the error in c as an approximation to x:

$$\textbf{absolute error} = |c - x|$$

and

$$\textbf{relative error} = \frac{|c - x|}{|x|} \quad (\text{if } x \neq 0).$$

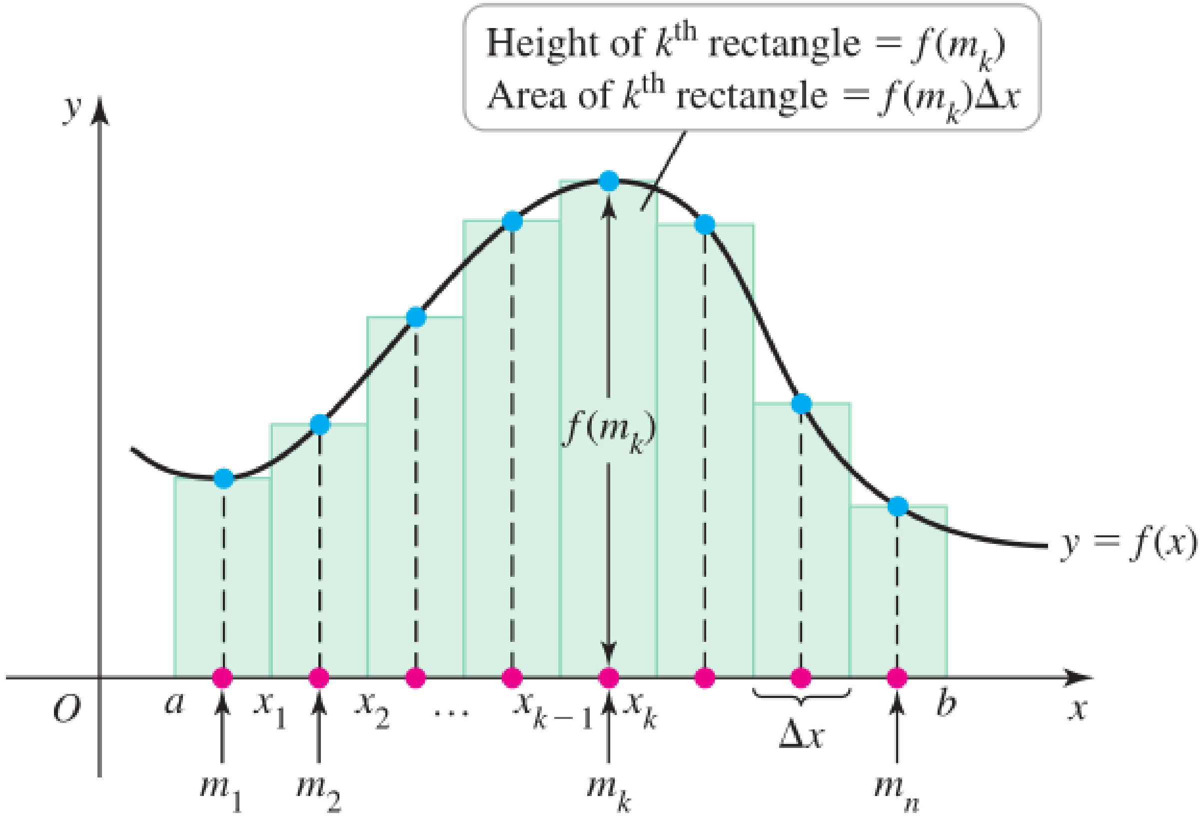
EXAMPLE 1 **Absolute and relative error** The ancient Greeks used $\frac{22}{7}$ to approximate the value of π. Determine the absolute and relative error in this approximation to π.

$$\text{absolute error} = \left| \frac{22}{7} - \pi \right| \approx 0.00126$$

$$\text{relative error} = \frac{|22/7 - \pi|}{|\pi|} \approx 0.000402 \approx 0.04\%.$$

Midpoint Rule

Reimann Sum to approximate the value of $\int_a^b f(x)dx$



Midpoint Rule: $\int_a^b f(x)\,dx \approx f(m_1)\Delta x + f(m_2)\Delta x + \cdots + f(m_n)\Delta x$

DEFINITION Midpoint Rule

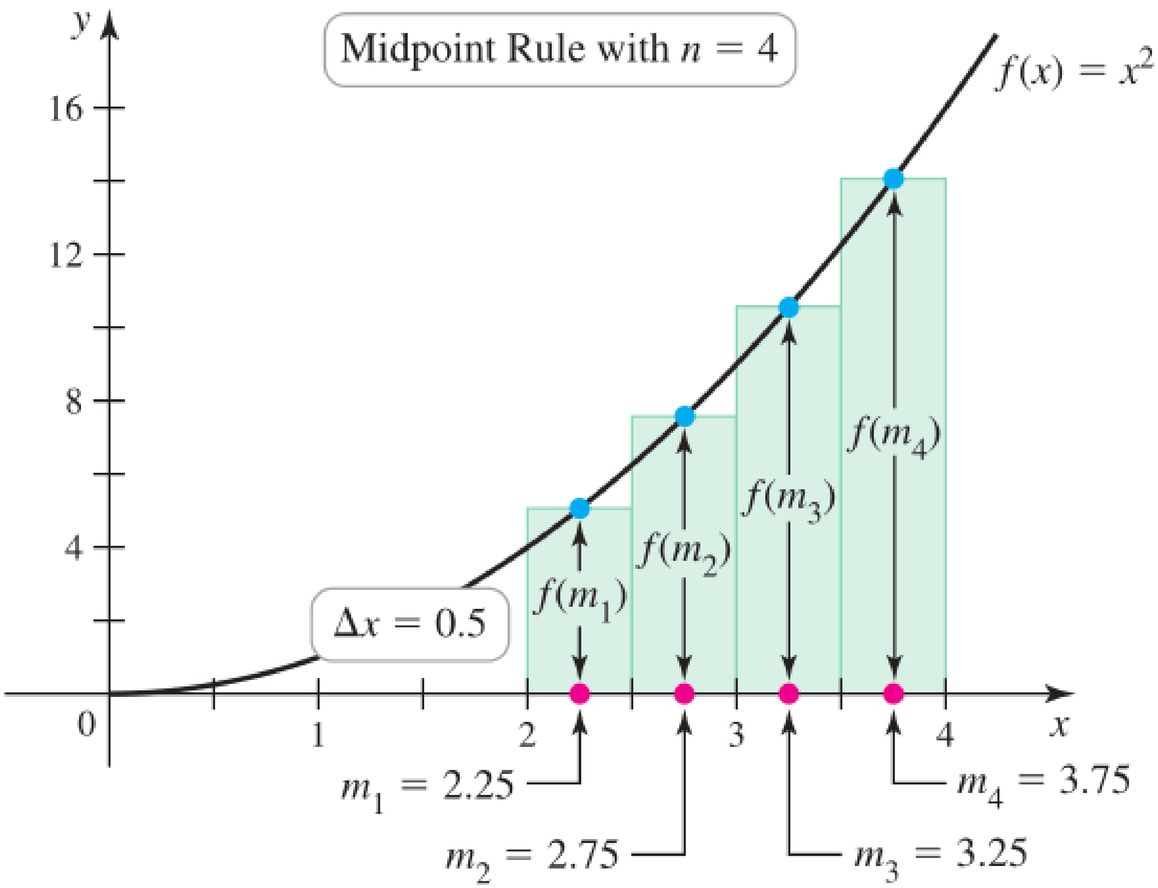
Suppose f is defined and integrable on $[a, b]$. The **Midpoint Rule approximation** to $\int_a^b f(x)\,dx$ using n equally spaced subintervals on $[a, b]$ is

$$M(n) = f(m_1)\Delta x + f(m_2)\Delta x + \cdots + f(m_n)\Delta x$$

$$= \sum_{k=1}^{n} f\left(\frac{x_{k-1} + x_k}{2}\right)\Delta x,$$

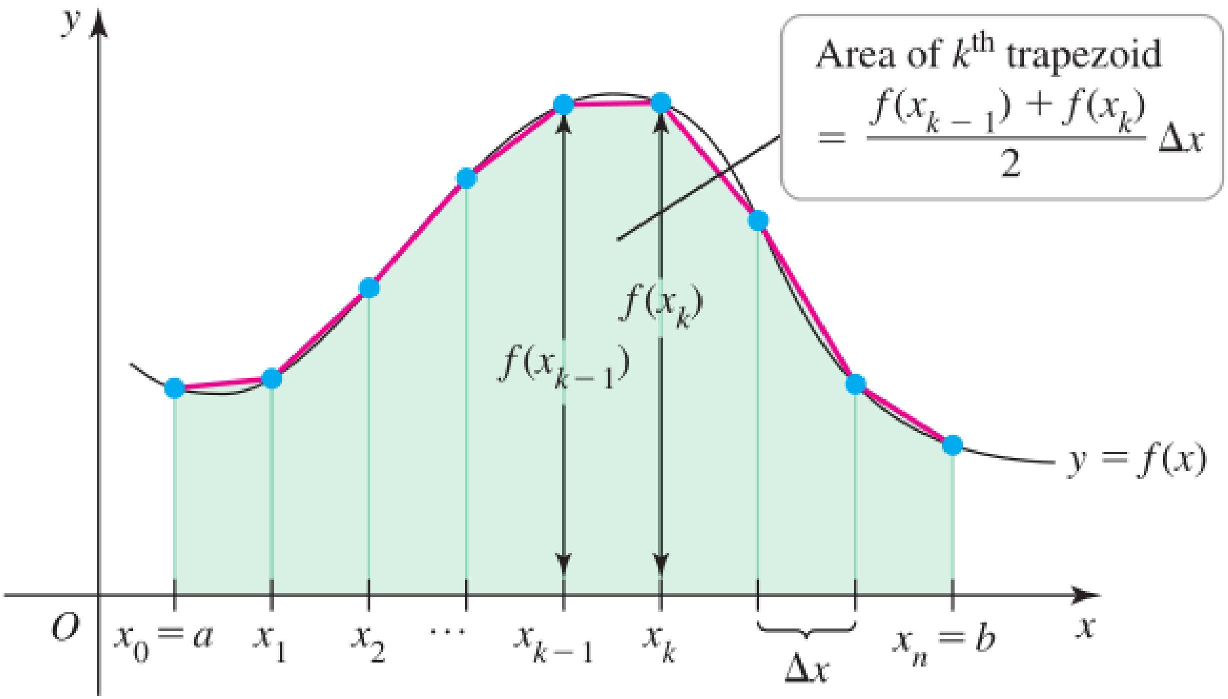
where $\Delta x = (b - a)/n$, $x_0 = a$, $x_k = a + k\Delta x$, and $m_k = (x_{k-1} + x_k)/2$ is the midpoint of $[x_{k-1}, x_k]$, for $k = 1, \ldots, n$.

EXAMPLE 2 **Applying the Midpoint Rule** Approximate $\int_2^4 x^2\,dx$ using the Midpoint Rule with $n = 4$ and $n = 8$ subintervals.

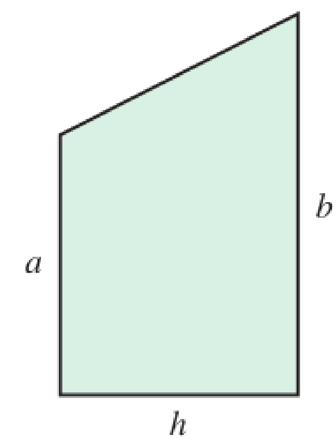


The Trapezoid Rule

Instead of using rectangles, the Trapezoid Rule uses trapezoids



Area of a trapezoid



$$\text{Area} = h\left(\frac{a+b}{2}\right)$$

Trapezoid Rule: $\displaystyle\int_a^b f(x)dx \approx \left(\frac{1}{2}f(x_0) + f(x_1) + \cdots + f(x_{n-1}) + \frac{1}{2}f(x_n)\right)\Delta x$

$$\int_a^b f(x)\,dx \approx T(n)$$

$$= \underbrace{\left(\frac{f(x_0) + f(x_1)}{2}\right)\Delta x}_{\text{area of first trapezoid}} + \underbrace{\left(\frac{f(x_1) + f(x_2)}{2}\right)\Delta x}_{\text{area of second trapezoid}} + \cdots + \underbrace{\left(\frac{f(x_{n-1}) + f(x_n)}{2}\right)\Delta x}_{\text{area of } n\text{th trapezoid}}$$

$$= \left(\frac{f(x_0)}{2} + \underbrace{\frac{f(x_1)}{2} + \frac{f(x_1)}{2}}_{f(x_1)} + \cdots + \underbrace{\frac{f(x_{n-1})}{2} + \frac{f(x_{n-1})}{2}}_{f(x_{n-1})} + \frac{f(x_n)}{2}\right)\Delta x$$

$$= \left(\frac{f(x_0)}{2} + \underbrace{f(x_1) + \cdots + f(x_{n-1})}_{\sum_{k=1}^{n-1} f(x_k)} + \frac{f(x_n)}{2}\right)\Delta x.$$

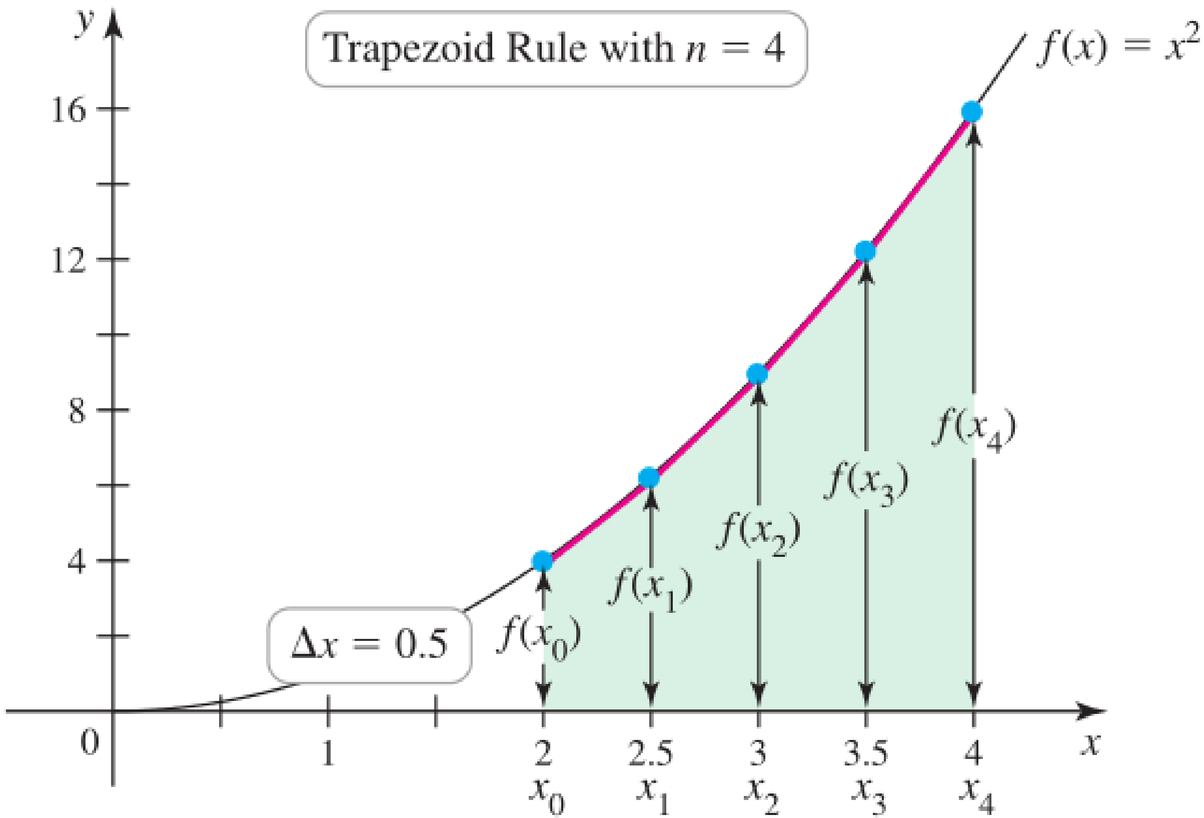
DEFINITION **Trapezoid Rule**

Suppose f is defined and integrable on $[a, b]$. The **Trapezoid Rule approximation** to $\int_a^b f(x)\,dx$ using n equally spaced subintervals on $[a, b]$ is

$$T(n) = \left(\frac{1}{2}f(x_0) + \sum_{k=1}^{n-1} f(x_k) + \frac{1}{2}f(x_n)\right)\Delta x,$$

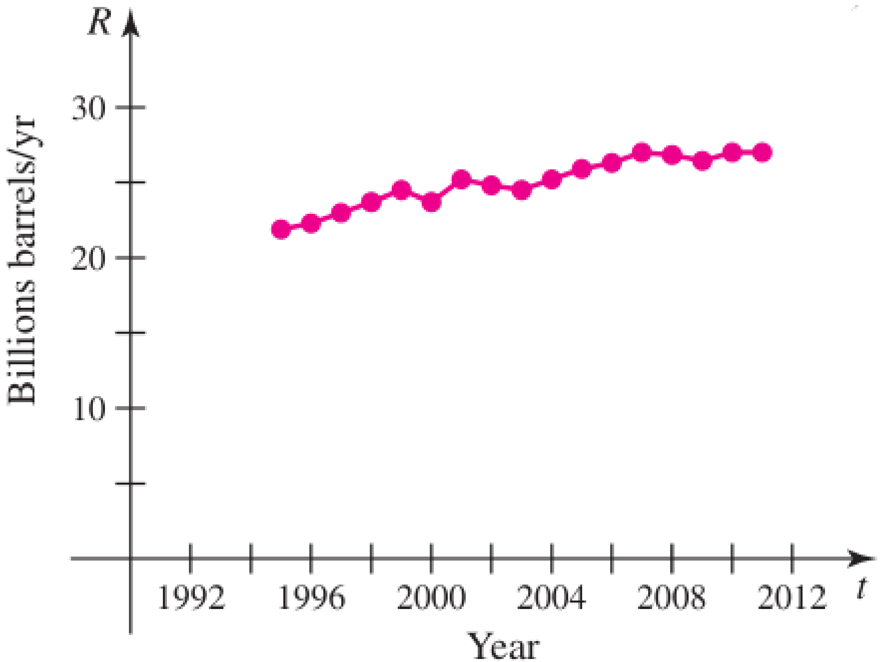
where $\Delta x = (b - a)/n$ and $x_k = a + k\Delta x$, for $k = 0, 1, \ldots, n$.

EXAMPLE 3 **Applying the Trapezoid Rule** Approximate $\int_2^4 x^2\,dx$ using the Trapezoid Rule with $n = 4$ subintervals.



EXAMPLE 5 **World oil production** Table 6 and Figure 16 show data for the rate of world crude oil production (in billions of barrels/yr) over a 16-year period. If the rate of oil production is given by the (assumed to be integrable) function R, then the total amount of oil produced in billions of barrels over the time period $a \leq t \leq b$ is $Q = \int_a^b R(t)\,dt$ (Section 6.1). Use the Midpoint and Trapezoid Rules to approximate the total oil produced between 1995 and 2011.

Year	World Crude Oil Production (billions barrels/yr)
1995	21.9
1996	22.3
1997	23.0
1998	23.7
1999	24.5
2000	23.7
2001	25.2
2002	24.8
2003	24.5
2004	25.2
2005	25.9
2006	26.3
2007	27.0
2008	26.9
2009	26.4
2010	27.0
2011	27.0



Simpson's Rule

An improvement: approximation with curves rather than line segments.

Work with three neighboring points on the curve.
These three points determine a parabola, whose net area is

$$\frac{f(x_0) + 4f(x_1) + f(x_2)}{3}\Delta x$$

$$\int_a^b f(x)\,dx \approx S(n)$$

$$= \left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right)\frac{\Delta x}{3}.$$

n must be an even integer

DEFINITION Simpson's Rule

Suppose f is defined and integrable on $[a, b]$ and $n \geq 2$ is an even integer. The **Simpson's Rule approximation** to $\int_a^b f(x)\,dx$ using n equally spaced subintervals on $[a, b]$ is

$$S(n) = \left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 4f(x_{n-1}) + f(x_n)\right)\frac{\Delta x}{3},$$

where n is an even integer, $\Delta x = (b - a)/n$, and $x_k = a + k\Delta x$, for $k = 0, 1, \ldots, n$.

An easier way:
With the Trapezoid Rule approximations $T(n)$ and $T(2n)$, then the next Simpson's Rule approximation is

$$S(2n) = \frac{4T(2n) - T(n)}{3}$$

EXAMPLE 7 **Errors in the Trapezoid Rule and Simpson's Rule** Given that $\int_0^1 xe^{-x}\,dx = 1 - 2e^{-1}$, find the absolute errors in the Trapezoid Rule and Simpson's Rule approximations to the integral with $n = 8, 16, 32, 64,$ and 128 subintervals.

Table 8

n	$T(n)$	$S(n)$	Error in $T(n)$	Error in $S(n)$
4	0.25904504019141		0.00520	
8	0.26293980164730	0.26423805546593	0.00130	0.00000306
16	0.26391564480235	0.26424092585404	0.000325	0.000000192
32	0.26415974044777	0.26424110566291	0.0000814	0.0000000120
64	0.26422077279247	0.26424111690738	0.0000203	0.000000000750
128	0.26423603140581	0.26424111761026	0.00000509	0.0000000000469

Errors in Numerical Integration

THEOREM 2 Errors in Numerical Integration
Assume that f'' is continuous on the interval $[a, b]$ and that k is a real number such that $|f''(x)| \le k$, for all x in $[a, b]$. The absolute errors in approximating the integral $\int_a^b f(x)\,dx$ by the Midpoint Rule and Trapezoid Rule with n subintervals satisfy the inequalities

$$E_M \le \frac{k(b-a)}{24}(\Delta x)^2 \quad \text{and} \quad E_T \le \frac{k(b-a)}{12}(\Delta x)^2,$$

respectively, where $\Delta x = (b-a)/n$.

Assume that $f^{(4)}$ is continuous on the interval $[a, b]$ and that K is a real number such that $|f^{(4)}(x)| \le K$ on $[a, b]$. The absolute error in approximating the integral $\int_a^b f(x)\,dx$ by Simpson's Rule with n subintervals satisfies the inequality

$$E_S \le \frac{K(b-a)}{180}(\Delta x)^4.$$

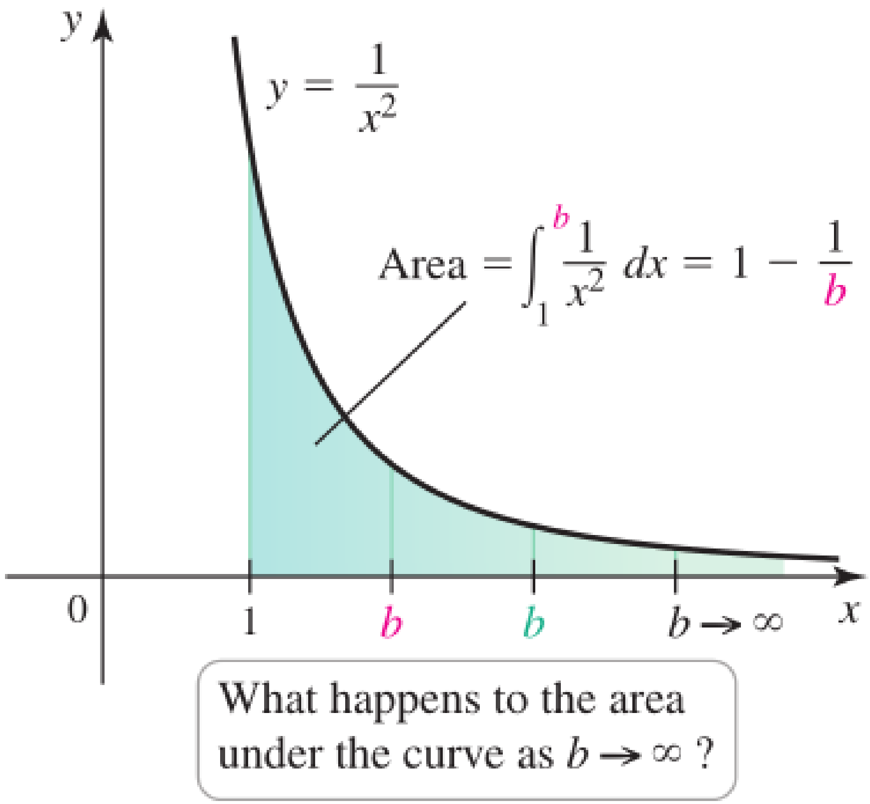
8.9

Improper Integrals

Infinite Intervals

improper integral is used for two cases

- the interval of integration is infinite, or
- the integrand is unbounded on the interval of integration.



Consider the integral $\int_1^b \frac{1}{x^2}\,dx$

$$\int_1^b \frac{1}{x^2}\,dx = -\frac{1}{x}\bigg|_1^b = 1 - \frac{1}{b}.$$

Then let $b \to \infty$

$$\lim_{b\to\infty}\left(1 - \frac{1}{b}\right) = 1.$$

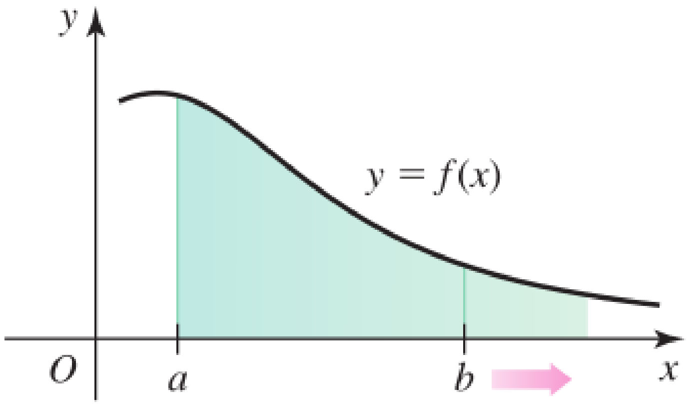
Express this as

$$\int_1^\infty \frac{1}{x^2}\,dx = 1$$

DEFINITION **Improper Integrals over Infinite Intervals**

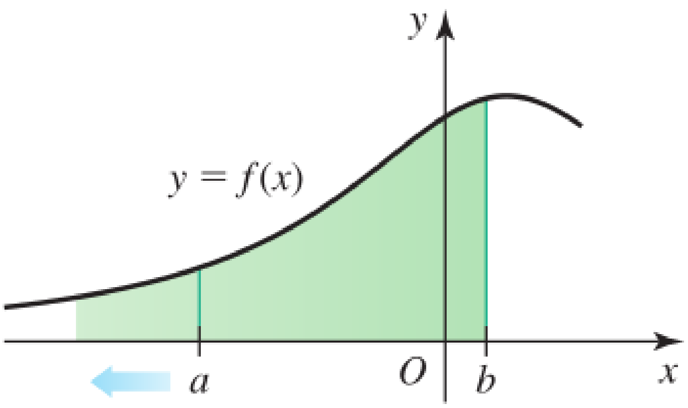
1. If f is continuous on $[a, \infty)$, then

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx.$$



2. If f is continuous on $(-\infty, b]$, then

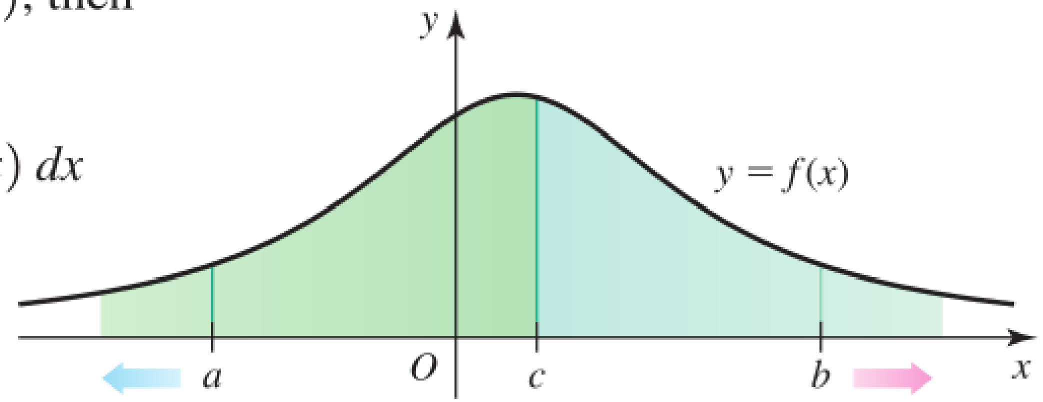
$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx.$$



3. If f is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{a \to -\infty} \int_{a}^{c} f(x)\,dx$$

$$+ \lim_{b \to \infty} \int_{c}^{b} f(x)\,dx,$$

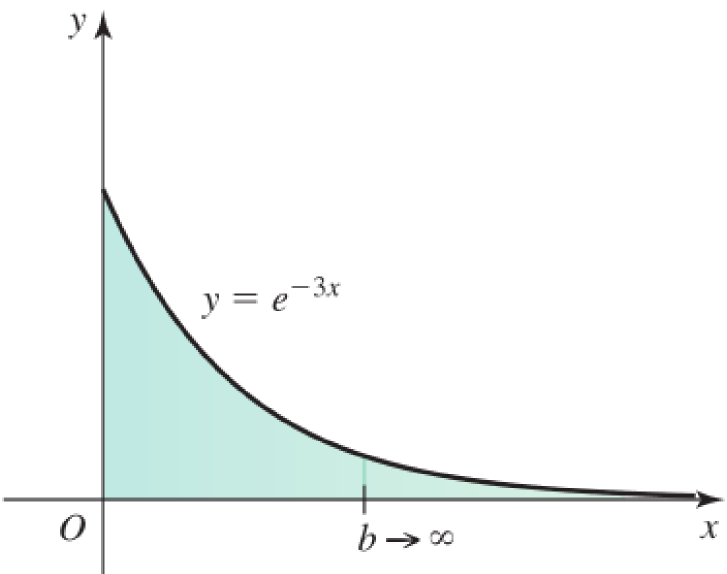


where c is any real number.

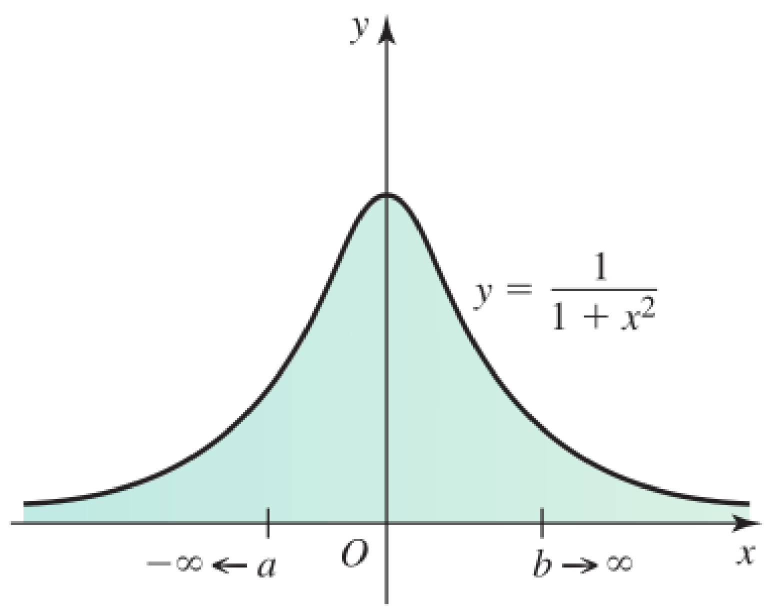
If the limits in cases 1–3 exist, then the improper integrals **converge**; otherwise, they **diverge**.

EXAMPLE 1 **Infinite intervals** Evaluate each integral.

a. $\displaystyle\int_0^\infty e^{-3x}\,dx$ **b.** $\displaystyle\int_{-\infty}^\infty \frac{dx}{1+x^2}$

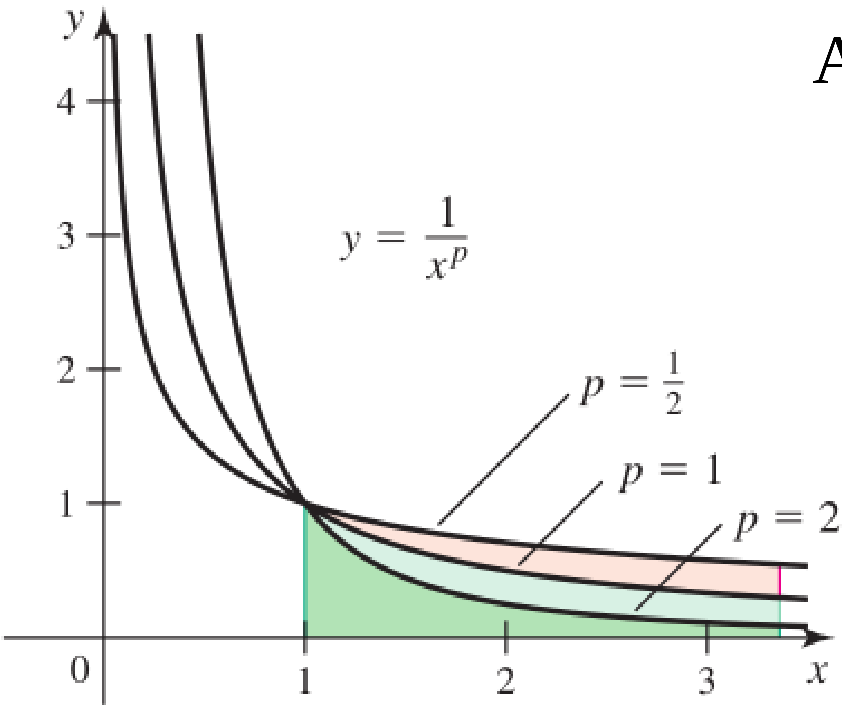


Area of region under the curve $y = e^{-3x}$ on $[0, \infty)$ has finite value $\frac{1}{3}$.



Area of region under the curve $y = \dfrac{1}{1+x^2}$ on $(-\infty, \infty)$ has finite value π.

EXAMPLE 2 **The family $f(x) = 1/x^p$** Consider the family of functions $f(x) = 1/x^p$, where p is a real number. For what values of p does $\int_1^\infty f(x)\,dx$ converge?



$$\int_1^\infty \frac{dx}{x^p} = \frac{1}{p-1}, \text{ if } p > 1.$$

Assuming $p \neq 1$

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{b\to\infty} \int_1^b x^{-p}\,dx$$

$$= \frac{1}{1-p} \lim_{b\to\infty} \left(x^{1-p} \Big|_1^b \right)$$

$$= \frac{1}{1-p} \lim_{b\to\infty} (b^{1-p} - 1).$$

Case 1: If $p > 1$, then $p - 1 > 0$, and $b^{1-p} = 1/b^{p-1}$ approaches 0 as $b \to \infty$. Therefore, the integral converges and its value is

$$\int_1^\infty \frac{1}{x^p}\,dx = \frac{1}{1-p}\lim_{b\to\infty}(\underbrace{b^{1-p}}_{\text{approaches } 0} - 1) = \frac{1}{p-1}.$$

Case 2: If $p < 1$, then $1 - p > 0$, and the integral diverges:

$$\int_1^\infty \frac{1}{x^p}\,dx = \frac{1}{1-p}\lim_{b\to\infty}(\underbrace{b^{1-p}}_{\text{arbitrarily large}} - 1) = \infty.$$

Case 3: If $p = 1$, then $\displaystyle\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty} \ln b = \infty$, so the integral diverges.

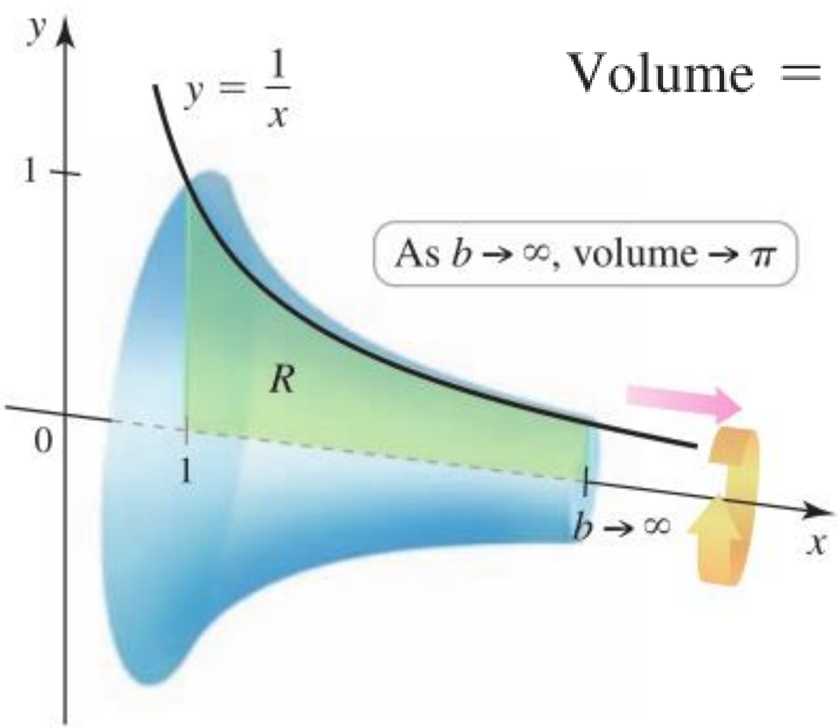
In summary, $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx = \frac{1}{p-1}$ if $p > 1$, and the integral diverges if $p \le 1$.

EXAMPLE 3 **Solids of revolution** Let R be the region bounded by the graph of $y = x^{-1}$ and the x-axis, for $x \geq 1$.

a. What is the volume of the solid generated when R is revolved about the x-axis?

b. What is the surface area of the solid generated when R is revolved about the x-axis?

c. What is the volume of the solid generated when R is revolved about the y-axis?

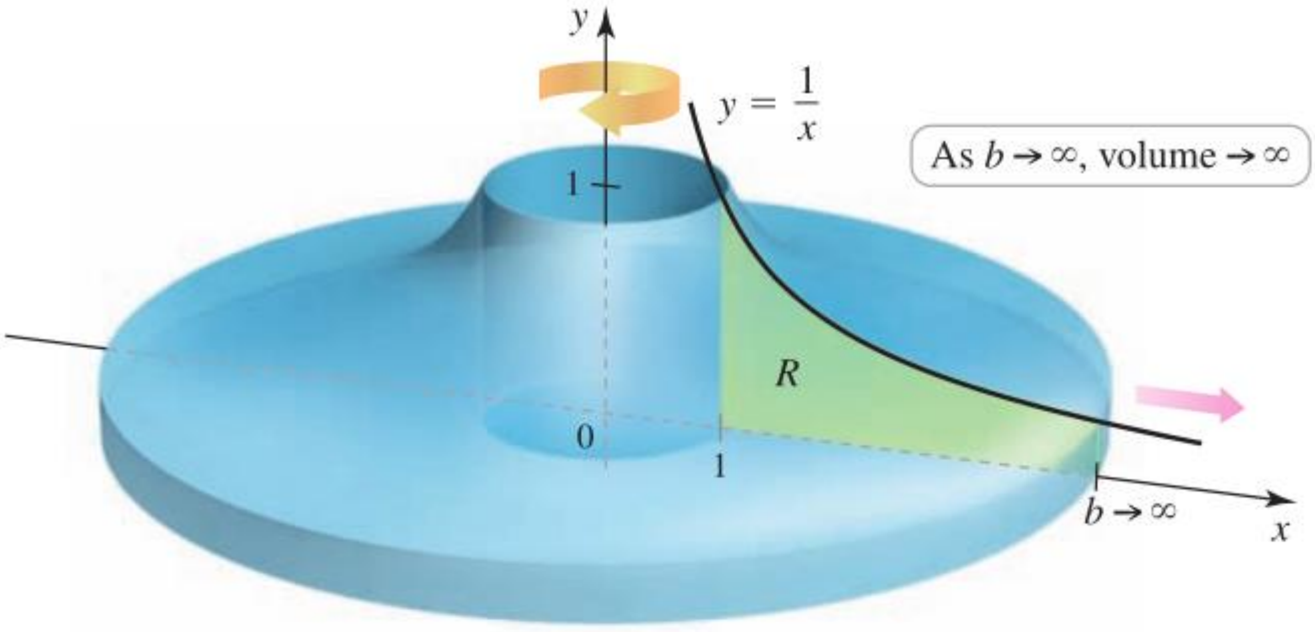


$$\text{Volume} = \int_1^{\infty} \pi (f(x))^2\, dx$$

Disk method

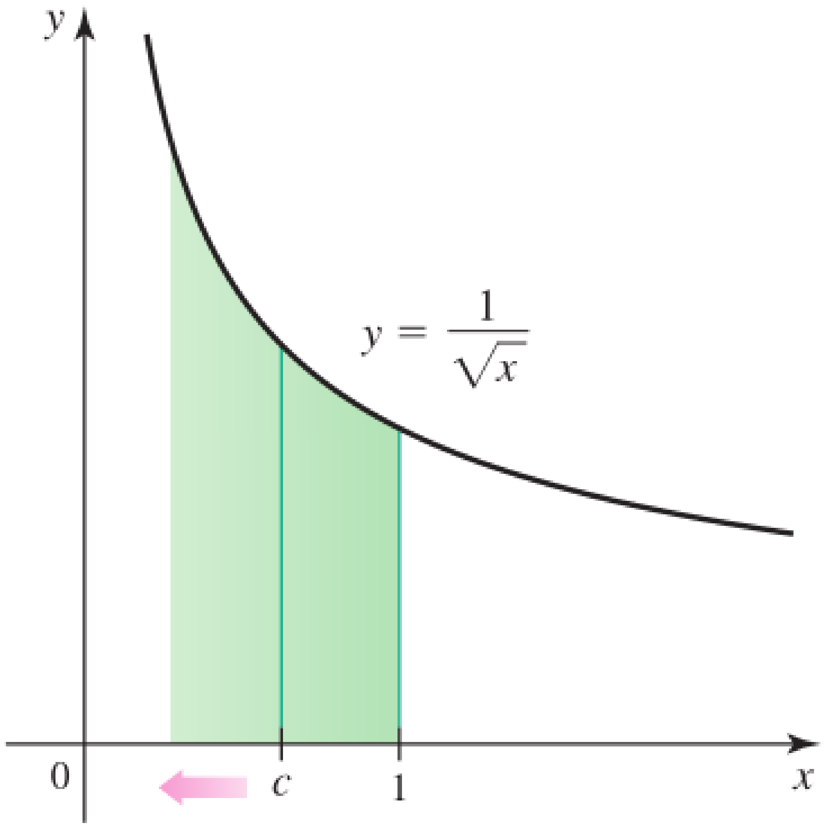
Surface area

$$\int_1^b 2\pi f(x)\sqrt{1 + f'(x)^2}\, dx.$$



$$\text{Volume} = \int_1^{\infty} 2\pi x f(x)\, dx \quad \text{Shell method}$$

Unbounded Integrands



What happens to the area under the curve as $c \to 0^+$?

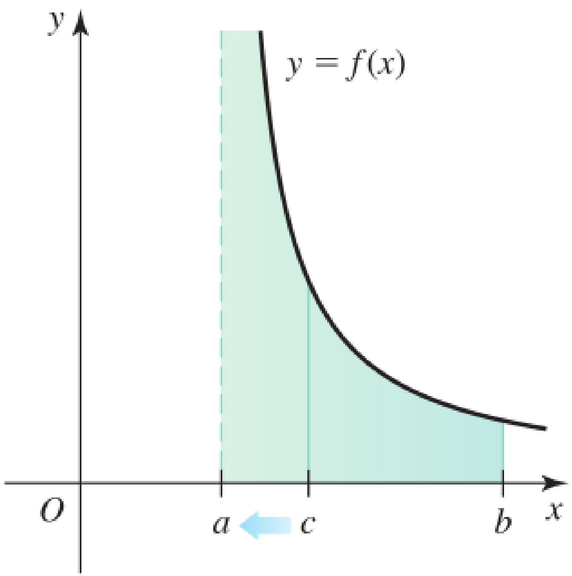
$$\int_c^1 \frac{dx}{\sqrt{x}} = 2\sqrt{x}\Big|_c^1 = 2(1 - \sqrt{c}).$$

$$\lim_{c\to 0^+} \int_c^1 \frac{dx}{\sqrt{x}} = \lim_{c\to 0^+} 2(1 - \sqrt{c}) = 2.$$

DEFINITIONS Improper Integrals with an Unbounded Integrand

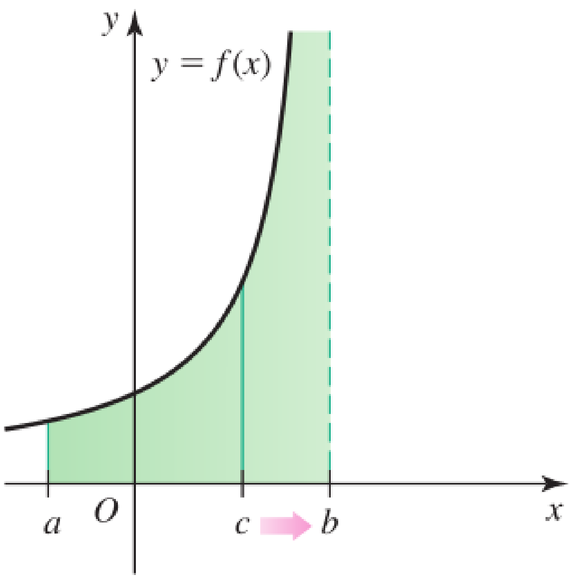
1. Suppose f is continuous on $(a, b]$ with $\lim_{x \to a^+} f(x) = \pm\infty$. Then

$$\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx.$$



2. Suppose f is continuous on $[a, b)$ with $\lim_{x \to b^-} f(x) = \pm\infty$. Then

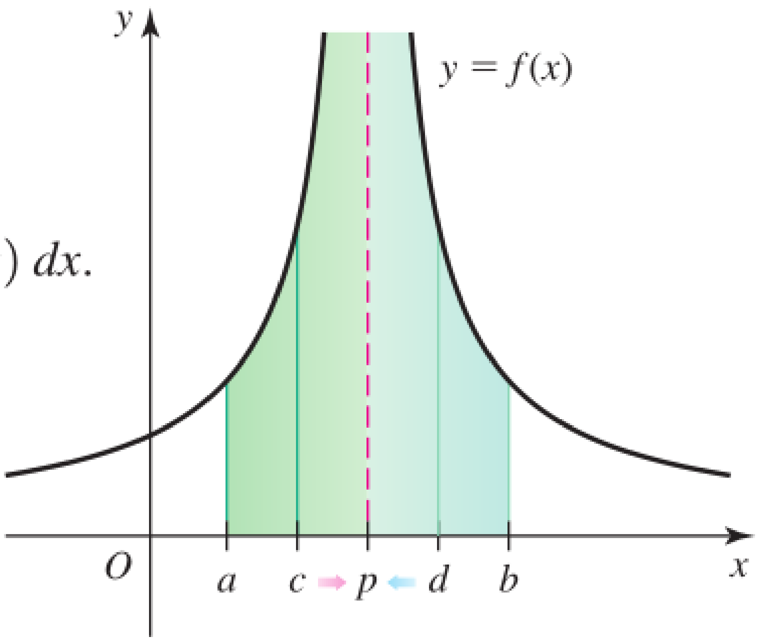
$$\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx.$$



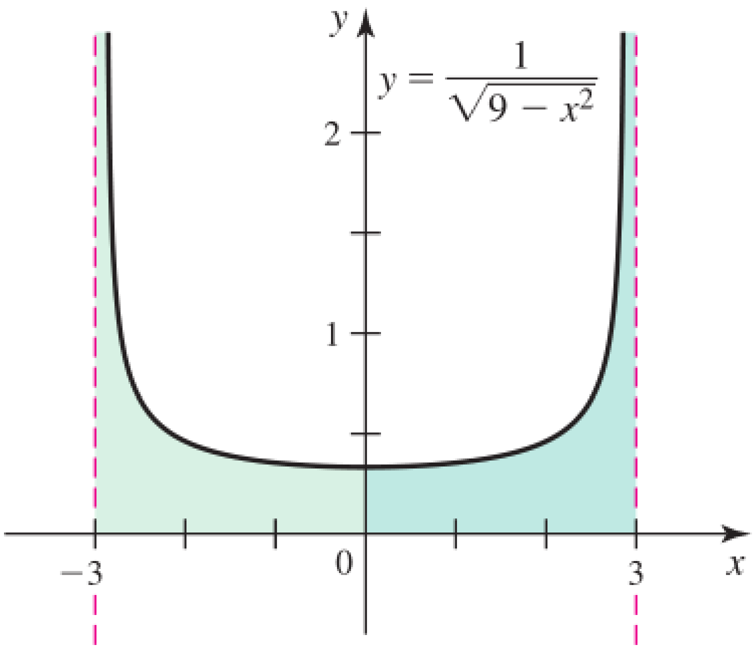
3. Suppose f is continuous on $[a, b]$ except at the interior point p where f is unbounded. Then

$$\int_a^b f(x)\,dx = \lim_{c \to p^-} \int_a^c f(x)\,dx + \lim_{d \to p^+} \int_d^b f(x)\,dx.$$

If the limits in cases 1–3 exist, then the improper integrals **converge**; otherwise, they **diverge**.



EXAMPLE 4 **Infinite integrand** Find the area of the region R between the graph of $f(x) = \dfrac{1}{\sqrt{9 - x^2}}$ and the x-axis on the interval $(-3, 3)$ (if it exists).



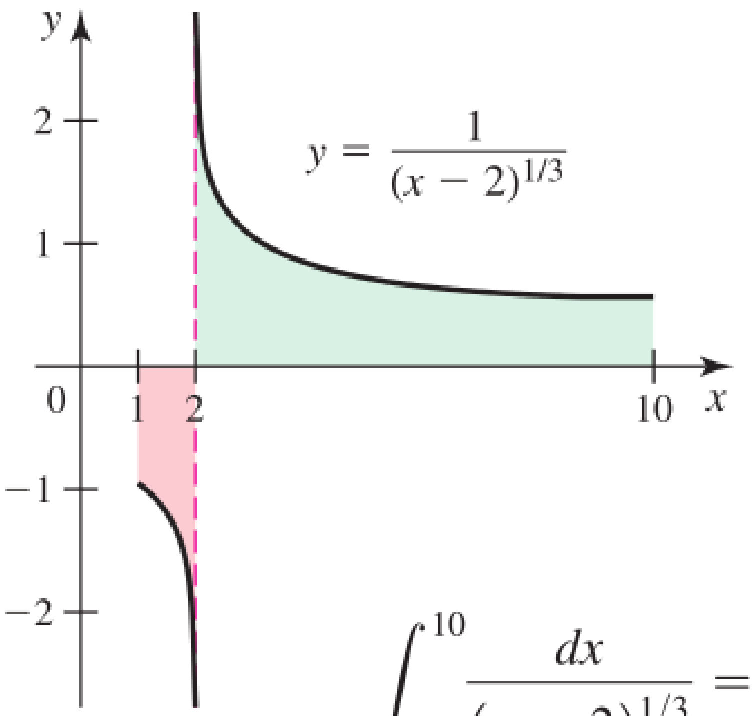
Area of region $= 2\int_0^3 \dfrac{1}{\sqrt{9 - x^2}}\,dx$

$$2\int_0^3 \frac{dx}{\sqrt{9 - x^2}} = 2\lim_{c\to 3^-} \int_0^c \frac{dx}{\sqrt{9 - x^2}}$$

$$= 2\lim_{c\to 3^-} \sin^{-1}\frac{x}{3}\bigg|_0^c$$

$$= 2\lim_{c\to 3^-} \Big(\underbrace{\sin^{-1}\frac{c}{3}}_{\text{approaches } \pi/2} - \underbrace{\sin^{-1} 0}_{\text{equals } 0}\Big).$$

EXAMPLE 5 **Infinite integrand at an interior point** Evaluate $\int_1^{10} \frac{dx}{(x-2)^{1/3}}$.



$$\int_1^{10} \frac{dx}{(x-2)^{1/3}} = \lim_{a \to 2^-} \int_1^{a} \frac{dx}{(x-2)^{1/3}} + \lim_{b \to 2^+} \int_b^{10} \frac{dx}{(x-2)^{1/3}}$$

$$= \lim_{a \to 2^-} \frac{3}{2}(x-2)^{2/3} \Big|_1^a + \lim_{b \to 2^+} \frac{3}{2}(x-2)^{2/3} \Big|_b^{10}$$

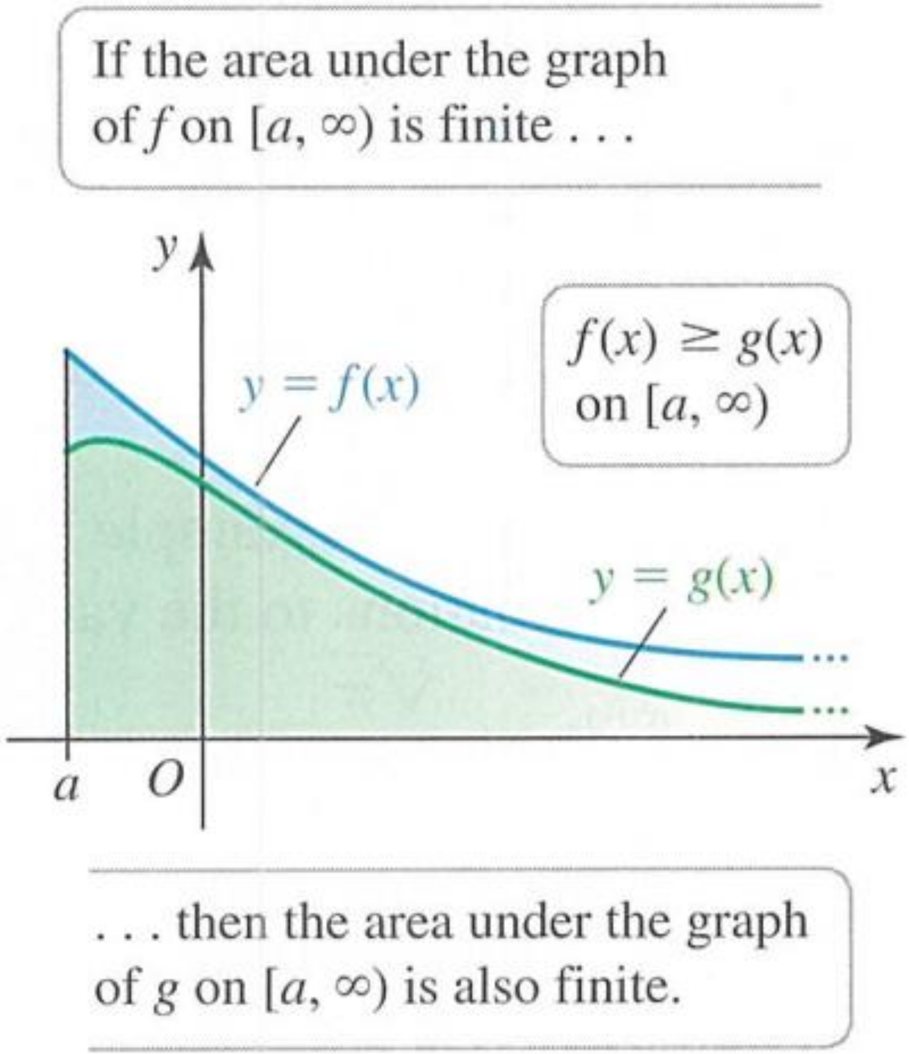
The Comparison Test

Sometimes it is not able to determine whether a given improper integral converges, because it is impossible to compute an antiderivative, e.g., $\int_a^\infty e^{-x^2}\, dx$

THEOREM 8.2 Comparison Test for Improper Integrals

Suppose the functions f and g are continuous on the interval $[a, \infty)$, with $f(x) \geq g(x) \geq 0$, for $x \geq a$.

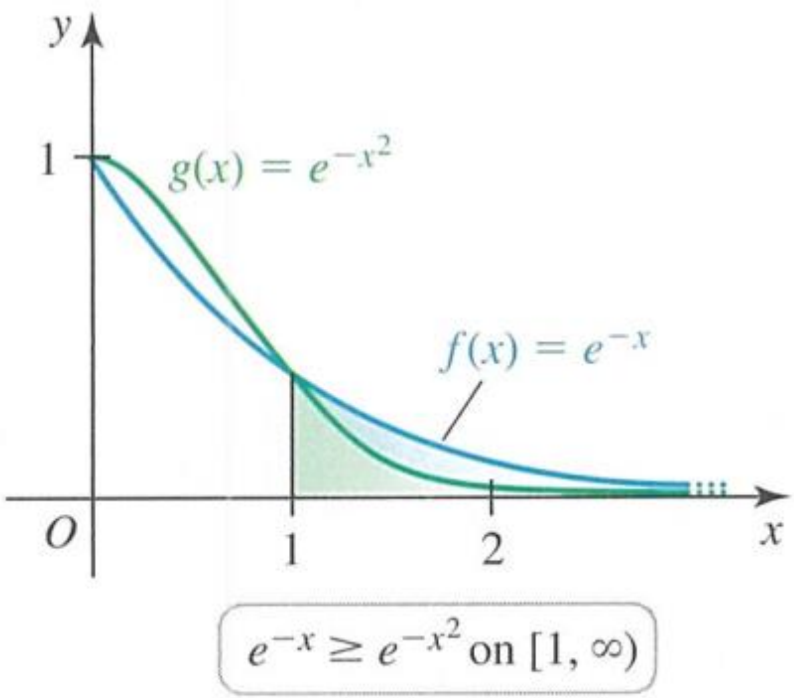
1. If $\int_a^\infty f(x)\, dx$ converges, then $\int_a^\infty g(x)\, dx$ converges.
2. If $\int_a^\infty g(x)\, dx$ diverges, then $\int_a^\infty f(x)\, dx$ diverges.



If the area under the graph of f on $[a, \infty)$ is finite . . .
y
$y = f(x)$
$f(x) \geq g(x)$ on $[a, \infty)$
$y = g(x)$
a
O
x
. . . then the area under the graph of g on $[a, \infty)$ is also finite.

EXAMPLE 7 **Comparison Test** Use Theorem 8.2 to determine whether the following integrals converge.

a. $\int_1^\infty e^{-x^2}\,dx$ **b.** $\int_1^\infty \frac{1}{\sqrt[3]{x^2 - 0.5}}\,dx$



$$\frac{1}{\sqrt[3]{x^2 - 0.5}} > \frac{1}{x^{2/3}} \text{ on } [1, \infty).$$

$$\int_1^\infty \frac{1}{x^p}\,dx \text{ diverges if } p \leq 1.$$

Chapter 8

Integration Techniques

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