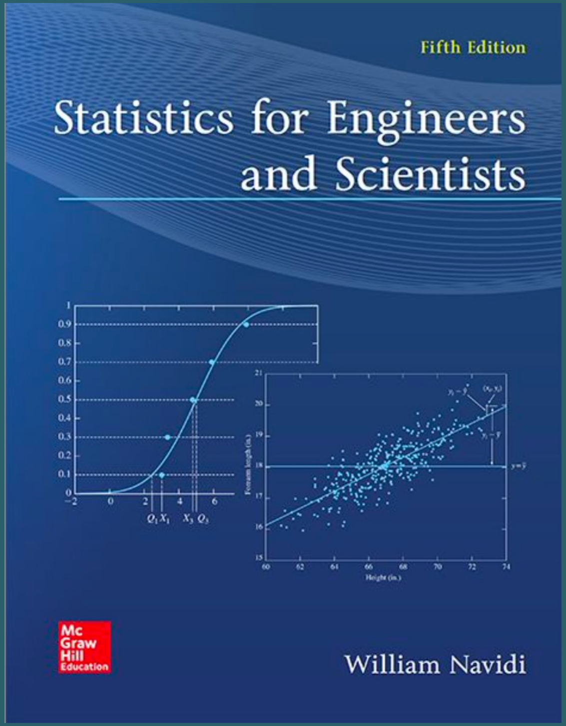

Ch 3: Propagation of Error

(not all sections are required)

Chapter 3 Overview

3-1 **Measurement Error**

3-2 Linear Combinations of Measurements

3-3 Uncertainties for Functions of One Measurement

~~*3-4 Uncertainties for Several Measurements*~~

Introduction

- Measurement is fundamental to scientific work.
- Scientists and engineers often perform calculations with measured quantities.
- Simple examples:
  - Computing the density of an object by dividing a measurement of its mass by a measurement of its volume
  - Computing the area of a rectangle by multiplying measurements of its length and width.

Introduction…

- Any measuring procedure contains error.
- Thus, measured values generally differ from the true values that are being measured.
- When a calculation is performed with measurements, the errors in the measurements produce an error in the calculated value.
- We say that the error is propagated from the measurements to the calculated value.

Measurement Error

Let us use the following example:

- A geologist is weighing a rock on a scale. She weighs the rock five times and obtains the following measurements (in grams):

251.3   252.5   250.8   251.1   250.4

- The measurements are all different, and it is likely that none of them is equal to the true mass of the rock.

- The difference between a measured value and the true value is called the **error** in the measured value.

Measurement Error…

Why does the error occur?

- Any measuring procedure contains many sources of error.
- For example,
  - If the scale was not calibrated properly, this will pull each measurement away from the true value by some fixed amount.
  - Thus, imperfect calibration contributes errors of equal magnitude to each measurement.

Measurement Error…

Why does the error occur?

- For example,
  - Interpolation between graduation marks on the dial could be another source of error.
  - The magnitude of the error due to interpolation is likely to vary from measurement to measurement and is likely to be positive for some measurements and negative for others.
  - It may be reasonable to assume that interpolation errors average out to zero in the long run.

Measurement Error…

We can think of the error in a measurement as being composed of two parts

1. The systematic error (or bias) is the part of the error that is the same for every measurement.
2. The random error that varies from measurement to measurement, and averages out to zero in the long run.

NOTE: Some sources of error contribute both to bias and to random error. For example, consider parallax error.

Measurement Error…

Any measurement can be considered to be the sum of the true value plus contributions from each of the two components of error:

Measured value = true value + bias + random error

Measurement Error…

Measured value = true value + bias + random error

- Since part of the error is random, it is appropriate to use a statistical model to study measurement error.
- We can model each measured value as a random variable, drawn from a population of possible measurements.

Measurement Error…

Measured value = true value + bias + random error

- The mean μ of the population represents that part of the measurement that is the same for every measurement. Therefore, μ is the sum of the true value and the bias.

- The standard deviation σ of the population is the standard deviation of the random error. It represents the variation that is due to the fact that each measurement has a different value for its random error. Therefore, σ represents the size of a typical random error.

Measuring Process

We are interested in 2 aspects of the measuring process

1. **Accuracy** - determined by the bias, which is the difference between the mean measurement μ and the true value being measured.

- ✓ The smaller the bias, the more accurate the measuring process.
- ✓ If the mean μ is equal to the true value, so that the bias is 0, the measuring process is said to be unbiased.

Measuring Process…

We are interested in 2 aspects of the measuring process

2. Precision - the degree to which repeated measurements of the same quantity tend to agree with each other.

- ✓ The precision is determined by the standard deviation σ of the measurement process.
- ✓ The smaller the value of σ, the more precise the measuring process.
- ✓ We will refer to σ as the uncertainty.

INTERIM SUMMARY (p.166)

- A measured value is a random variable with mean μ and standard deviation σ.
- The bias in the measuring process is the difference between the mean measurement and the true value:

$$\text{Bias} = \mu - \text{true value}$$

- The uncertainty in the measuring process is the standard deviation σ.
- The smaller the bias, the more accurate the measuring process.
- The smaller the uncertainty, the more precise the measuring process.

Bias *vs.* Uncertainty

- When reporting a measured value, it is important to report an estimate of the bias and uncertainty along with it, in order to describe the accuracy and precision of the measurement.
- It is in general easier to estimate the uncertainty than the bias

Bias *vs.* Uncertainty…

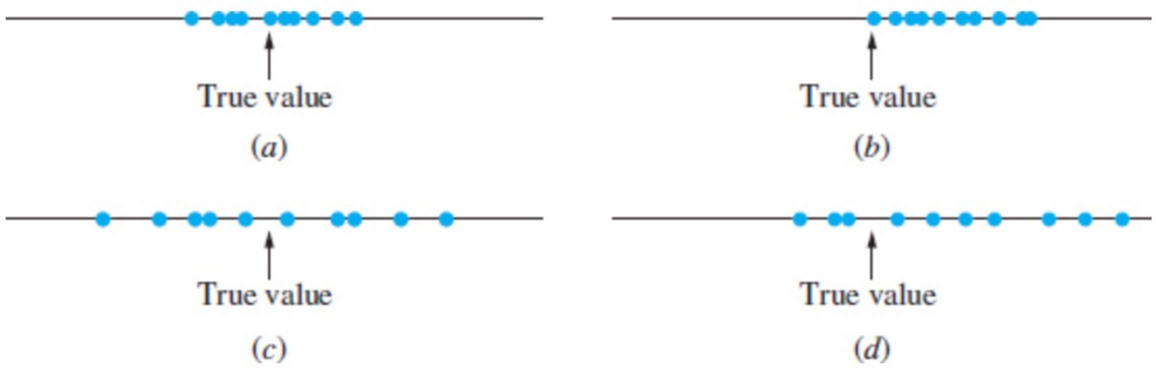

FIGURE 3.1 *(a)* Both bias and uncertainty are small. *(b)* Bias is large; uncertainty is small. *(c)* Bias is small; uncertainty is large. *(d)* Both bias and uncertainty are large.

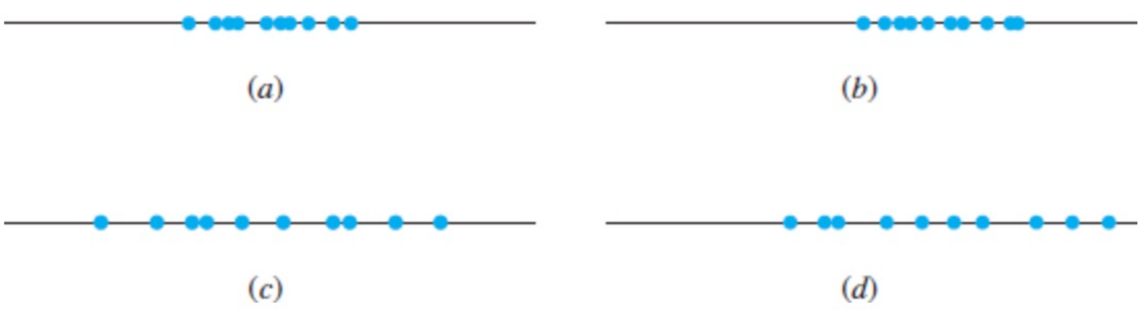

FIGURE 3.2 We can estimate the uncertainty from the set of repeated measurements, but without knowing the true value, we cannot estimate the bias.

Bias *vs.* Uncertainty…

- In real life, of course, we do not know the true value being measured.
- Thus, the plot of measurements shown in Figure 3.1 would look like Figure 3.2
- We can still determine that the sets of measurements in Figure 3.2a and b have smaller uncertainty but without additional information about the true value, we cannot estimate the bias.

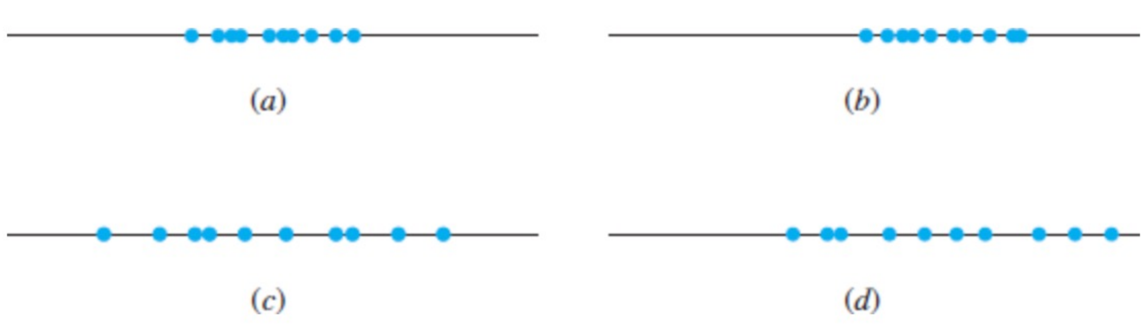

FIGURE 3.2 We can estimate the uncertainty from the set of repeated measurements, but without knowing the true value, we cannot estimate the bias.

Example 3.1 (p.167)

- A laboratory sample of gas is known to have a carbon monoxide (CO) concentration of 50 parts per million (ppm).
- A spectrophotometer is used to take five independent measurements of this concentration.
- The five measurements, in ppm, are **51, 47, 53, 53, and 48.**
- Estimate the bias and the uncertainty in a spectrophotometer measurement.

Example 3.1 (p.167) - SOLUTION

- The five measurements are regarded as a random sample from the population of possible measurements.
- The bias is equal to the mean of this population minus the true value of 50.
- The uncertainty is the standard deviation of the population.
- We do not know the mean and standard deviation of the population, but we can approximate them with the mean and standard deviation of the sample.
- The mean of the five measurements is 50.4. Therefore, we estimate the bias to be $50.4 - 50 = 0.4$ ppm.
- The standard deviation of the five measurements is 2.8 ppm. Therefore, we estimate the uncertainty in each measurement to be 2.8 ppm.

Example 3.2 (p.167)

- A different spectrophotometer is now used to measure the CO concentration in another gas sample.
- The true concentration in this sample is unknown.
- Five measurements are made (in ppm). They are 62, 63, 61, 62, and 59.
- Estimate the uncertainty in a measurement from this spectrophotometer. Can we estimate the bias?

Example 3.2 (p.167) - SOLUTION

- The uncertainty in a single measurement is estimated with the sample standard deviation, which is 1.5 ppm.
- The sample mean is 61.4 ppm, but to estimate the bias, we would have to subtract the true concentration from this.
- Since we do not know the true concentration, we cannot estimate the bias.

INTERIM SUMMARY (p.168)

Let $X_1, \ldots, X_n$ be independent measurements, all made by the same process on the same quantity.

- The sample standard deviation s can be used to estimate the uncertainty.
- Estimates of uncertainty are often crude, especially when based on small samples.
- If the true value is known, the sample mean $\overline{X}$ can be used to estimate the bias: Bias $\approx \overline{X} -$ true value.
- If the true value is unknown, the bias cannot be estimated from repeated measurements.

Calibration

- This calibration procedure follows a chain of comparisons with external standards
- For example, the world's ultimate standard for weight (technically mass) is located in France, near Paris.
- This is the International Prototype Kilogram, a platinum–iridium cylinder whose mass is by definition exactly 1 kg.

Calibration…

- A replica of The Kilogram, located at the National Institute of Standards and Technology in Washington, serves as the standard for measures in the United States.
- Use of this replica, rather than The Kilogram, introduces a bias into every measure of weight in the United States.
- By comparing the U.S. replica to The Kilogram, this bias has been estimated to be -1.9×10^{-8} kg.
- Thus, all weight measurements made at the National Institute of Standards and Technology are adjusted upward by 19 parts in a billion to compensate.

Calibration…

- This adjustment factor could not have been estimated by repeated weighing of the replica; comparison with an external standard was required.
- From here on we will assume, unless otherwise stated, that bias has been reduced to a negligible level.
- We will describe measurements in the form

Measured value ± σ

where σ = the uncertainty in the process that produced the measured value

Example 3.3 (p.169)

- The spectrophotometer in Example 3.1 has been recalibrated, so we may assume that the bias is negligible.
- The spectrophotometer is now used to measure the CO concentration in another gas sample.
- The measurement is 55.1 ppm.
- How should this measurement be expressed?

Example 3.3 (p.169) - SOLUTION

- From the repeated measurements in Example 3.1, the uncertainty in a measurement from this instrument was estimated to be **2.8 ppm**.
- Therefore, we report the CO concentration in this gas sample as **55.1 ± 2.8 ppm**.

End of Section 3-1