Chapter 10

Sequences and Infinite Series

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10.1

An Overview

Examples of Sequences

Sequence Example: the *list* of numbers: $\{1,4,7,10,13,16,\cdots\}$
Term: each number in the sequence
Notation: $\{a_1, a_2, \cdots, a_n, \cdots\}$, $\{a_n\}_{n=1}^{\infty}$, or $\{a_n\}$, n is called index

Two ways to define the sequence

Recurrence relation (or an *implicit formula*): specifies the initial term of the sequence, and gives a general rule for computing the next term of the sequence from previous terms, e.g.,

$$a_1 = 1, \text{ and } a_{n+1} = a_n + 3, \text{ for } n = 1,2,3,\cdots$$

Explicit formula: the nth term of the sequence is determined directly from the value of n, e.g., $a_n = 3n - 2$

DEFINITION Sequence

A **sequence** $\{a_n\}$ is an ordered list of numbers of the form

$$\{a_1, a_2, a_3, \ldots, a_n, \ldots\}.$$

A sequence may be generated by a **recurrence relation** of the form $a_{n+1} = f(a_n)$, for $n = 1, 2, 3, \ldots$, where a_1 is given. A sequence may also be defined with an **explicit formula** of the form $a_n = f(n)$, for $n = 1, 2, 3, \ldots$.

EXAMPLE 1 **Explicit formulas** Use the explicit formula for $\{a_n\}_{n=1}^{\infty}$ to write the first four terms of each sequence. Sketch a graph of the sequence.

a. $a_n = \dfrac{1}{2^n}$ **b.** $a_n = \dfrac{(-1)^n n}{n^2 + 1}$

EXAMPLE 2 **Recurrence relations** Use the recurrence relation for $\{a_n\}_{n=1}^{\infty}$ to write the first four terms of the sequences

$$a_{n+1} = 2a_n + 1, a_1 = 1 \quad \text{and} \quad a_{n+1} = 2a_n + 1, a_1 = -1.$$

EXAMPLE 3 **Working with sequences** Consider the following sequences.

a. $\{a_n\} = \{-2, 5, 12, 19, \ldots\}$ **b.** $\{b_n\} = \{3, 6, 12, 24, 48, \ldots\}$

(i) Find the next two terms of the sequence.

(ii) Find a recurrence relation that generates the sequence.

(iii) Find an explicit formula for the *n*th term of the sequence.

Limit of a Sequence

Long-term behavior of a sequence is described by its limit

DEFINITION Limit of a Sequence

If the terms of a sequence $\{a_n\}$ approach a unique number L as n increases—that is, if a_n can be made arbitrarily close to L by taking n sufficiently large—then we say $\lim_{n\to\infty} a_n = L$ exists, and the sequence **converges** to L. If the terms of the sequence do not approach a single number as n increases, the sequence has no limit, and the sequence **diverges**.

EXAMPLE 4 **Limits of sequences** Write the first four terms of each sequence. If you believe the sequence converges, make a conjecture about its limit. If the sequence appears to diverge, explain why.

a. $\left\{\dfrac{(-1)^n}{n^2+1}\right\}_{n=1}^{\infty}$ Explicit formula

b. $\{\cos n\pi\}_{n=1}^{\infty}$ Explicit formula

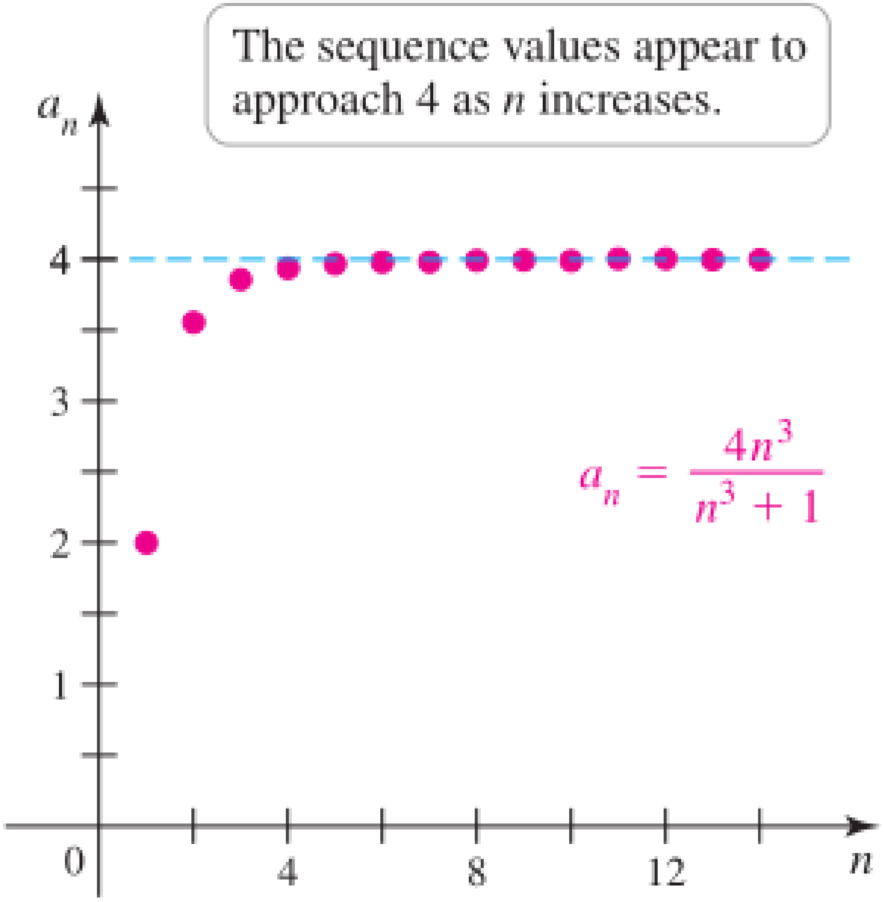
c. $\{a_n\}_{n=1}^{\infty}$, where $a_{n+1} = -2a_n$, $a_1 = 1$ Recurrence relation

EXAMPLE 5 **Limit of a sequence** Enumerate and graph the terms of the following sequence, and make a conjecture about its limit.

$$a_n = \frac{4n^3}{n^3 + 1}, \qquad \text{for } n = 1, 2, 3, \ldots. \quad \text{Explicit formula}$$

Table 1

n	a_n	n	a_n
1	2.000	8	3.992
2	3.556	9	3.995
3	3.857	10	3.996
4	3.938	11	3.997
5	3.968	12	3.998
6	3.982	13	3.998
7	3.988	14	3.999



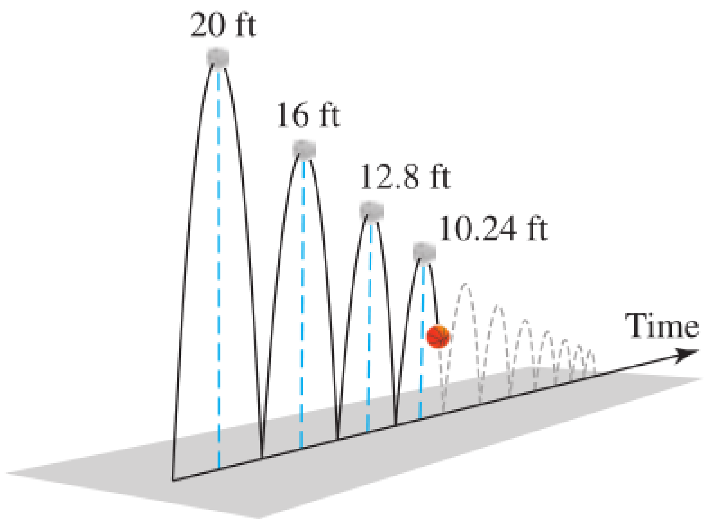
EXAMPLE 6 **A bouncing ball** A basketball tossed straight up in the air reaches a high point and falls to the floor. Each time the ball bounces on the floor it rebounds to 0.8 of its previous height. Let h_n be the high point after the nth bounce, with the initial height being $h_0 = 20$ ft.

a. Find a recurrence relation and an explicit formula for the sequence $\{h_n\}$.

b. What is the high point after the 10th bounce? after the 20th bounce?

c. Speculate on the limit of the sequence $\{h_n\}$.

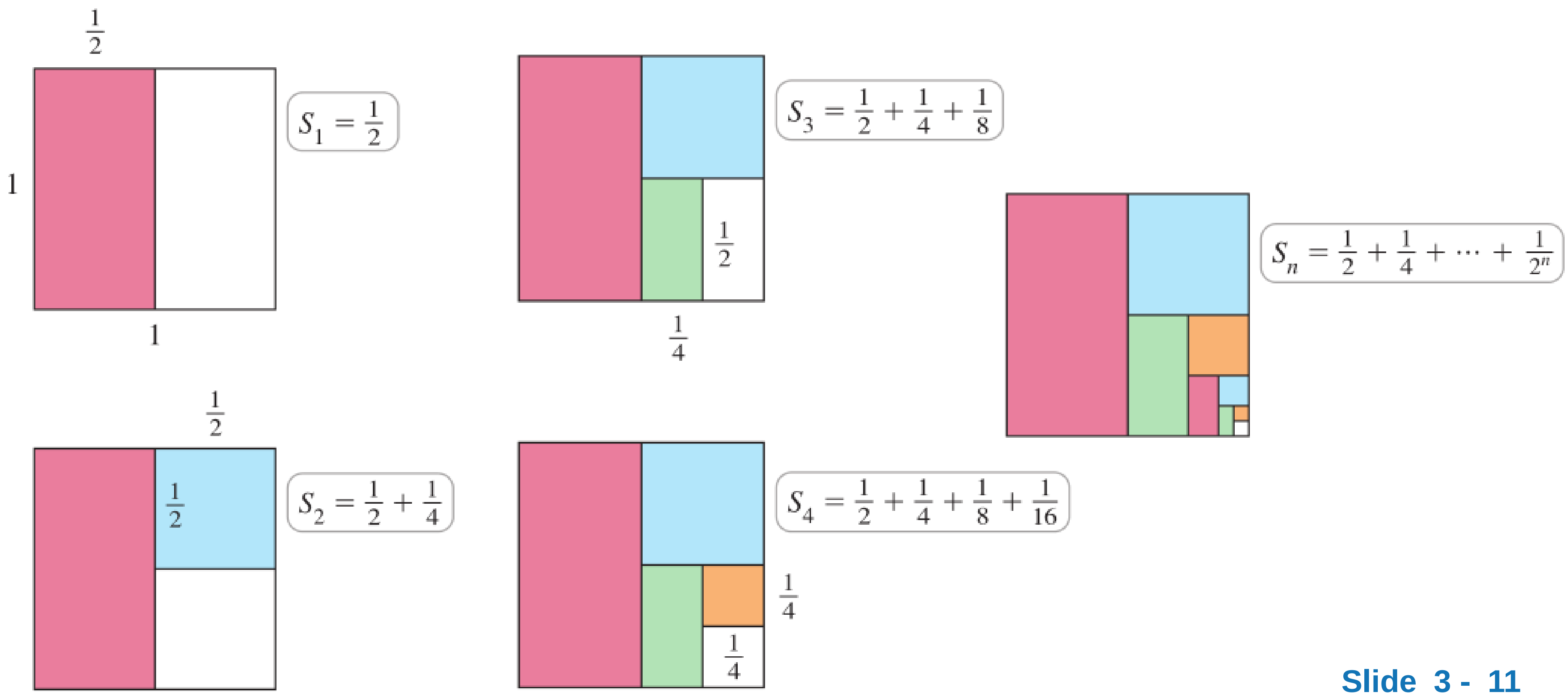
The height of each bounce of the basketball is 0.8 of the height of the previous bounce.



Infinite Series and the Sequence of Partial Sums

An infinite series can be viewed as a *sum* of an infinite set of numbers

Is it possible to sum an infinite set of numbers and produce a finite number?



$$S_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} = \frac{2^n - 1}{2^n}.$$

If this process is continued indefinitely, the area of the colored region S_n approaches the area of the unit square, which is 1

$$\lim_{n\to\infty} S_n = \underbrace{\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots}_{\text{sum continues indefinitely}} = 1.$$

The sequence $\{S_n\}$ generated is extremely important.
It is called a *sequence of partial sums*, and its limit is the value of the infinite series.

EXAMPLE 7 **Working with series** Consider the infinite series

$$0.9 + 0.09 + 0.009 + 0.0009 + \cdots,$$

where each term of the sum is $\frac{1}{10}$ of the previous term.

a. Find the sum of the first one, two, three, and four terms of the series.

b. What value would you assign to the infinite series $0.9 + 0.09 + 0.009 + \cdots$?

DEFINITION **Infinite Series**

Given a sequence $\{a_1, a_2, a_3, \ldots\}$, the sum of its terms

$$a_1 + a_2 + a_3 + \cdots = \sum_{k=1}^{\infty} a_k$$

is called an **infinite series**. The **sequence of partial sums** $\{S_n\}$ associated with this series has the terms

$$\begin{aligned}
S_1 &= a_1 \\
S_2 &= a_1 + a_2 \\
S_3 &= a_1 + a_2 + a_3 \\
&\vdots \\
S_n &= a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k, \quad \text{for } n = 1, 2, 3, \ldots.
\end{aligned}$$

If the sequence of partial sums $\{S_n\}$ has a limit L, the infinite series **converges** to that limit, and we write

$$\sum_{k=1}^{\infty} a_k = \lim_{n \to \infty} \underbrace{\sum_{k=1}^{n} a_k}_{S_n} = \lim_{n \to \infty} S_n = L.$$

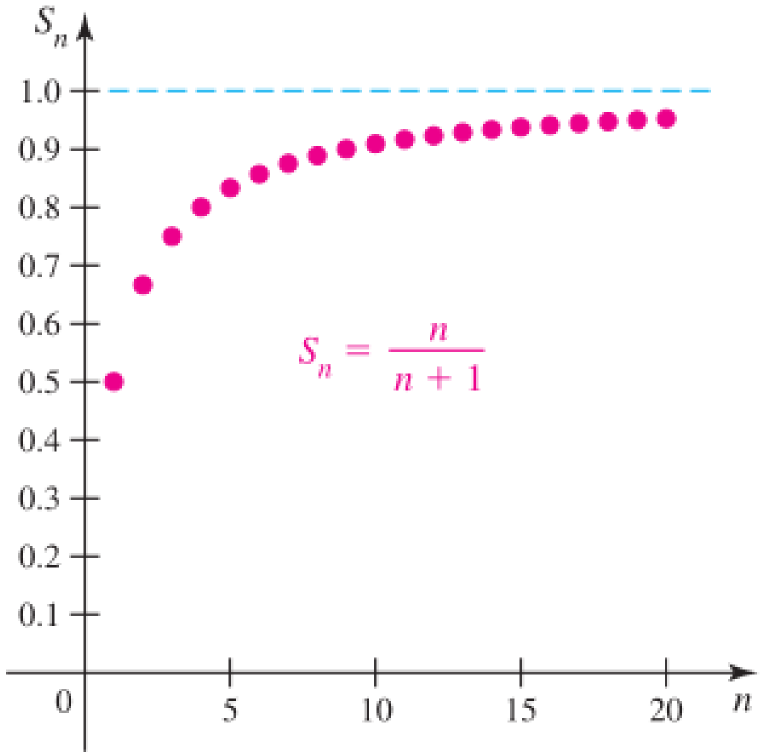
If the sequence of partial sums diverges, the infinite series also **diverges**.

EXAMPLE 8 **Sequence of partial sums** Consider the infinite series

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)}.$$

a. Find the first four terms of the sequence of partial sums.

b. Find an expression for S_n and make a conjecture about the value of the series.



Summary

This section features three key ideas to keep in mind.

- A *sequence* $\{a_1, a_2, \ldots, a_n, \ldots\}$ is an ordered *list* of numbers.
- An *infinite series* $\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots$ is a *sum* of numbers.
- A *sequence of partial sums* $\{S_1, S_2, S_3, \ldots\}$, where $S_n = a_1 + a_2 + \cdots + a_n$, is used to evaluate the series $\sum_{k=1}^{\infty} a_k$.

For sequences, consider the behavior of the individual terms as go out farther and farther in the list, i.e., $\lim_{n\to\infty} a_n$

For infinite series, examine the sequence of partial sums related to the series $\lim_{n\to\infty} \sum_{k=1}^{n} a_k$

The correspondences between sequences / series and functions, and between summation and integration

Table 2

	Sequences / Series	**Functions**
Independent variable	n	x
Dependent variable	a_n	$f(x)$
Domain	Integers e.g., $n = 1, 2, 3, \ldots$	Real numbers e.g., $\{x: x \geq 1\}$
Accumulation	Sums	Integrals
Accumulation over a finite interval	$\sum_{k=1}^{n} a_k$	$\int_1^n f(x)\, dx$
Accumulation over an infinite interval	$\sum_{k=1}^{\infty} a_k$	$\int_1^\infty f(x)\, dx$

10.2

Sequences

Limit of a Sequence and Limit Laws

A fundamental question: the behavior of the terms as we go out farther and farther in the sequence

Limits of sequences are really no different from limits at infinity of functions except that the variable n assumes only integer values as $n \to \infty$.

THEOREM 1 Limits of Sequences from Limits of Functions
Suppose f is a function such that $f(n) = a_n$ for all positive integers n. If $\lim_{x \to \infty} f(x) = L$, then the limit of the sequence $\{a_n\}$ is also L. where L may be $\pm\infty$.

THEOREM 2 Limit Laws for Sequences

Assume that the sequences $\{a_n\}$ and $\{b_n\}$ have limits A and B, respectively. Then

1. $\lim_{n\to\infty} (a_n \pm b_n) = A \pm B$
2. $\lim_{n\to\infty} ca_n = cA$, where c is a real number
3. $\lim_{n\to\infty} a_n b_n = AB$
4. $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{A}{B}$, provided $B \neq 0$.

EXAMPLE 1 **Limits of sequences** Determine the limits of the following sequences.

a. $a_n = \dfrac{3n^3}{n^3 + 1}$ **b.** $b_n = \left(\dfrac{n + 5}{n}\right)^n$ **c.** $c_n = n^{1/n}$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\left(\frac{n + 5}{n}\right)^n = \lim_{n\to\infty}\left(1 + \frac{5}{n}\right)^n$$

Terminology for Sequences

Similar to that used for functions.

DEFINITIONS Terminology for Sequences

$\{a_n\}$ is **increasing** if $a_{n+1} > a_n$; for example, $\{0, 1, 2, 3, \ldots\}$.

$\{a_n\}$ is **nondecreasing** if $a_{n+1} \geq a_n$; for example, $\{1, 1, 2, 2, 3, 3, \ldots\}$.

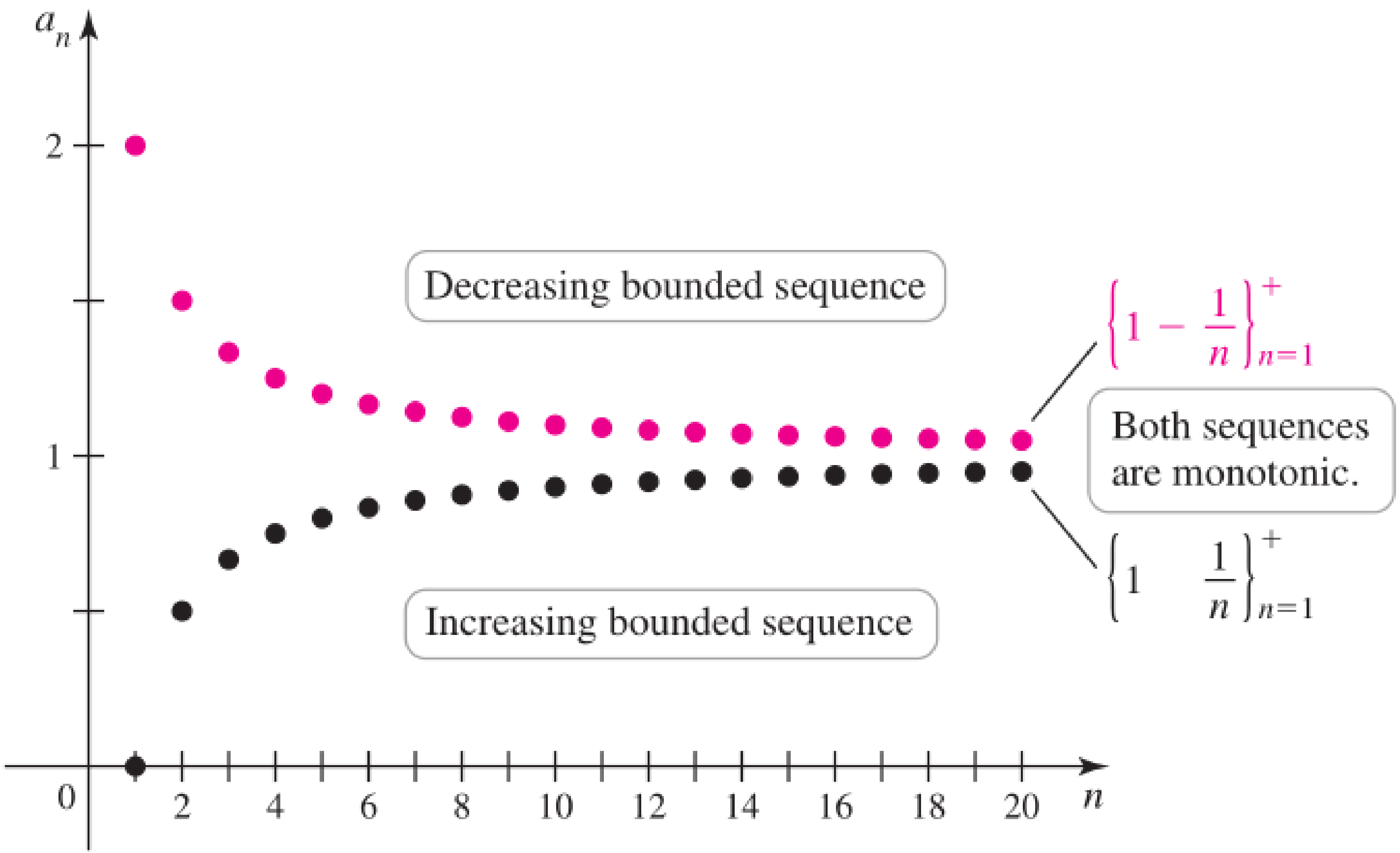
$\{a_n\}$ is **decreasing** if $a_{n+1} < a_n$; for example, $\{2, 1, 0, -1, \ldots\}$.

$\{a_n\}$ is **nonincreasing** if $a_{n+1} \leq a_n$; for example, $\{0, -1, -1, -2, -2, -3, -3, \ldots\}$.

$\{a_n\}$ is **monotonic** if it is either nonincreasing or nondecreasing (it moves in one direction).

$\{a_n\}$ is **bounded** if there is number M such that $|a_n| \leq M$, for all relevant values of n.

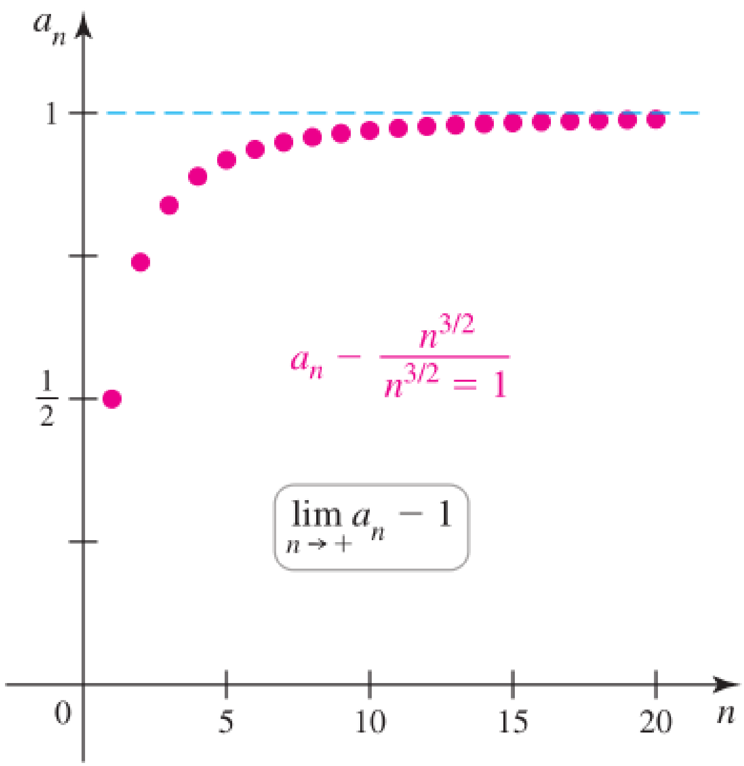
Bounded above & bounded below

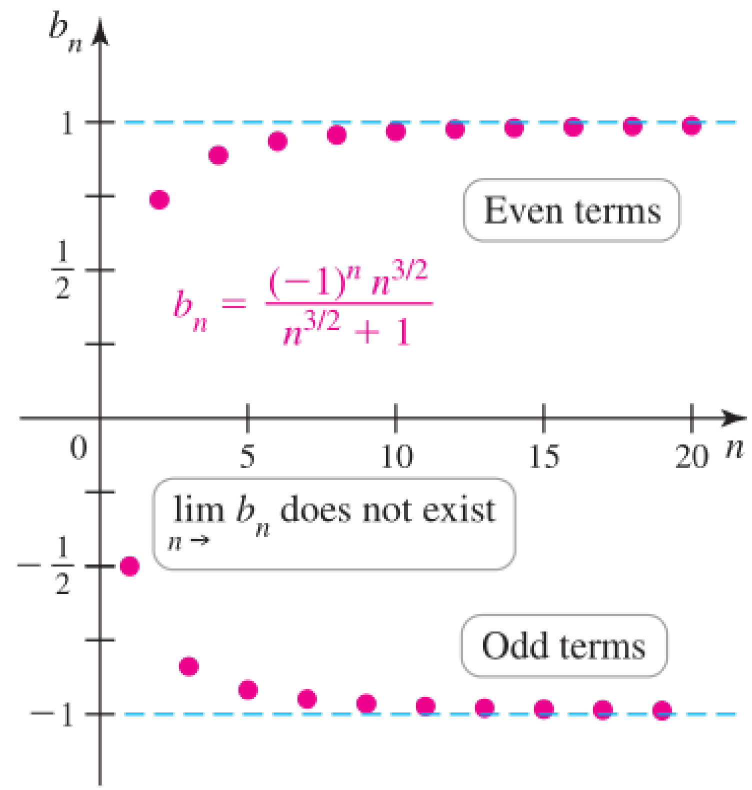


a_n
2
1
0
2
4
6
8
10
12
14
16
18
20
n
Decreasing bounded sequence
Increasing bounded sequence
$\left\{1 - \frac{1}{n}\right\}_{n=1}^{+}$
Both sequences are monotonic.
$\left\{1 \quad \frac{1}{n}\right\}_{n=1}^{+}$

EXAMPLE 2 **Limits of sequences and graphing** Compare and contrast the behavior of $\{a_n\}$ and $\{b_n\}$ as $n \to \neq$.

a. $a_n = \dfrac{n^{3/2}}{n^{3/2} + 1}$ **b.** $b_n = \dfrac{(-1)^n n^{3/2}}{n^{3/2} + 1}$





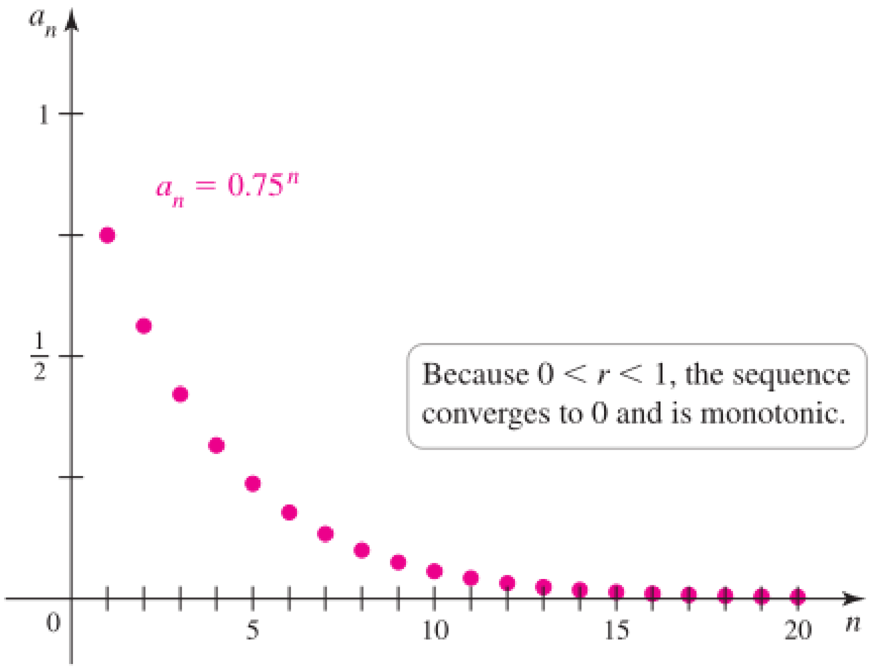
Geometric Sequences

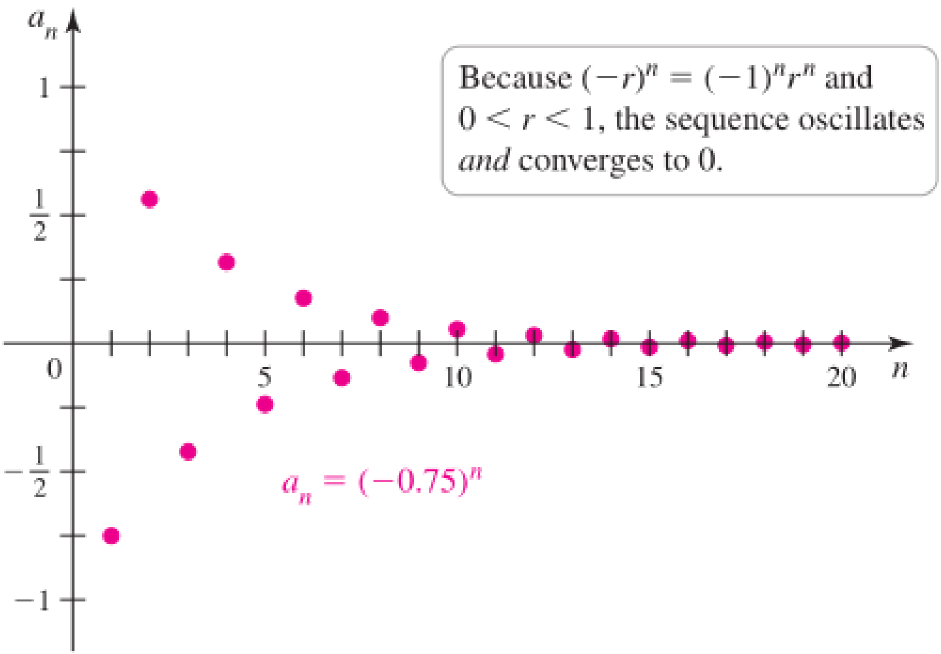
Geometric sequences: each term is obtained by multiplying the previous term by a fixed constant, called the *ratio*.

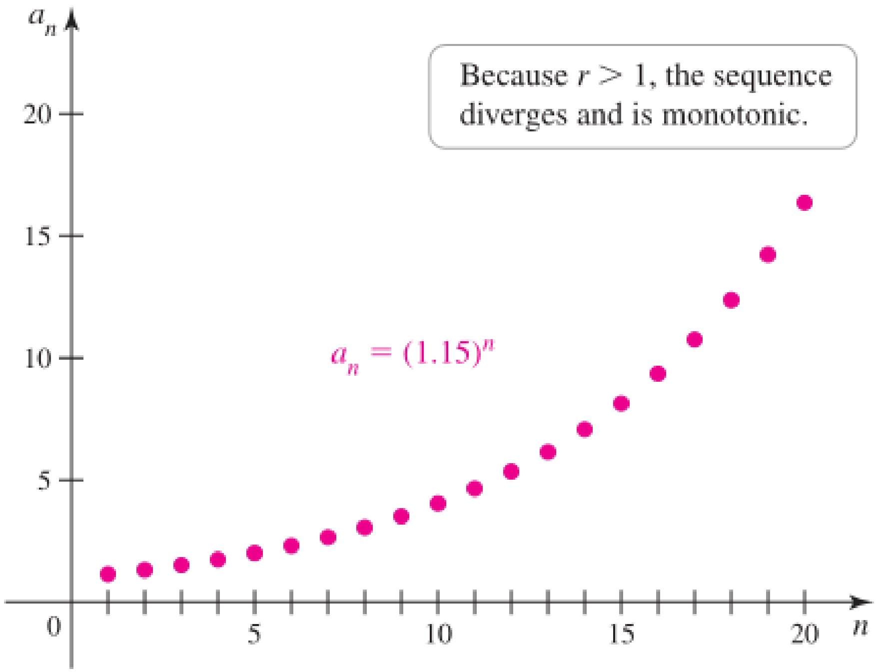
They have the form $\{r^n\}$ or $\{ar^n\}$, where the ratio r and $a \neq 0$ are real numbers

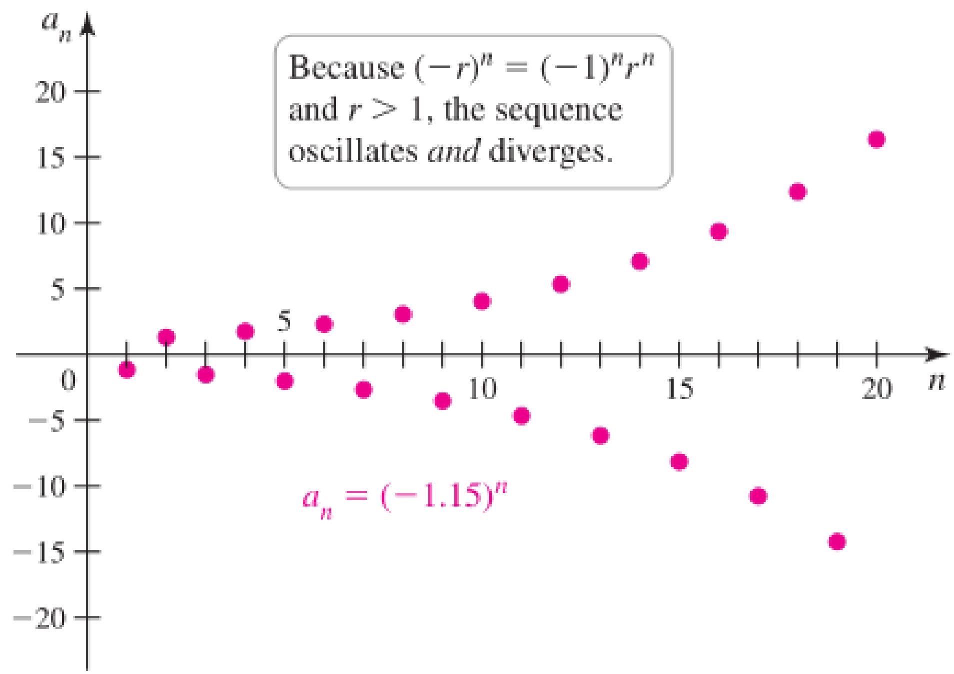
EXAMPLE 3 **Geometric sequences** Graph the following sequences and discuss their behavior.

a. $\{0.75^n\}$ **b.** $\{(-0.75)^n\}$ **c.** $\{1.15^n\}$ **d.** $\{(-1.15)^n\}$






a_n
Because $r > 1$, the sequence diverges and is monotonic.
$a_n = (1.15)^n$
20
15
10
5
0
5
10
15
20
n

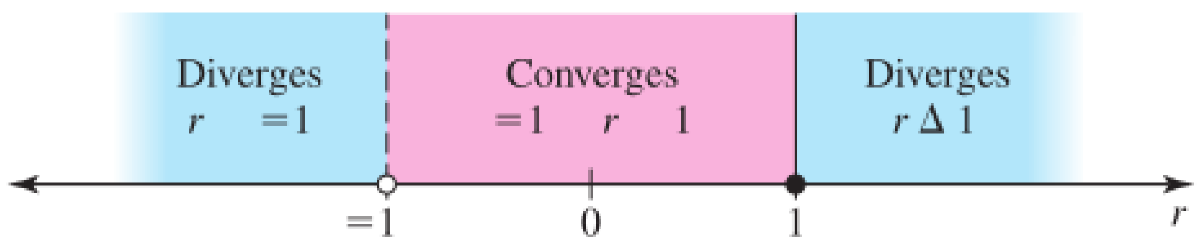

a_n
Because $(-r)^n = (-1)^n r^n$ and $r > 1$, the sequence oscillates *and* diverges.
$a_n = (-1.15)^n$
20
15
10
5
0
−5
−10
−15
−20
5
10
15
20
n

THEOREM 3 Geometric Sequences

Let r be a real number. Then

$$\lim_{n \to \infty} r^n = \begin{cases} 0 & \text{if } |r| < 1 \\ 1 & \text{if } r = 1 \\ \text{does not exist} & \text{if } r \leq -1 \text{ or } r > 1. \end{cases}$$

If $r > 0$, then $\{r^n\}$ is a monotonic sequence. If $r < 0$, then $\{r^n\}$ oscillates.



EXAMPLE 3 **Using Limit Laws** Determine the limits of the following sequences.

a. $a_n = 5(0.6)^n - \dfrac{1}{3^n}$ **b.** $b_n = \dfrac{2n^2 + n}{2^n(3n^2 - 4)}$

A sequence may display any of the following behaviors:

- It may converge to a single value, which is the limit of the sequence.
- Its terms may increase in magnitude without bound (either with one sign or with mixed signs), in which case the sequence diverges.
- Its terms may remain bounded but settle into an oscillating pattern (Chaos) in which the terms approach two or more values; in this case, the sequence diverges.

The Squeeze Theorem

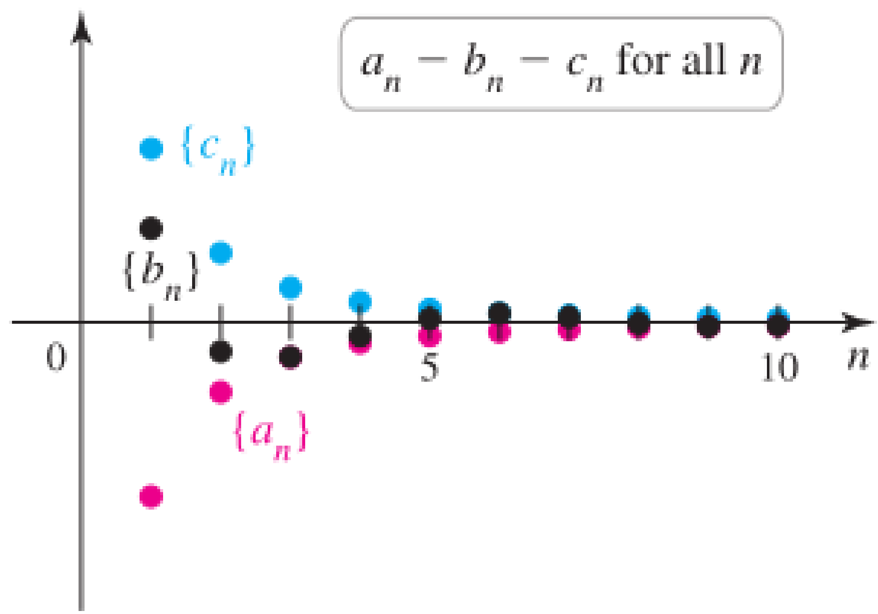
THEOREM 4 Squeeze Theorem for Sequences

Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences with $a_n \leq b_n \leq c_n$ for all integers n greater than some index N. If $\lim_{n \to \neq} a_n = \lim_{n \to \neq} c_n = L$, then $\lim_{n \to \neq} b_n = L$ (Figure 8.19).

EXAMPLE 4 **Squeeze Theorem** Find the limit of the sequence $b_n = \dfrac{\cos n}{n^2 + 1}$.

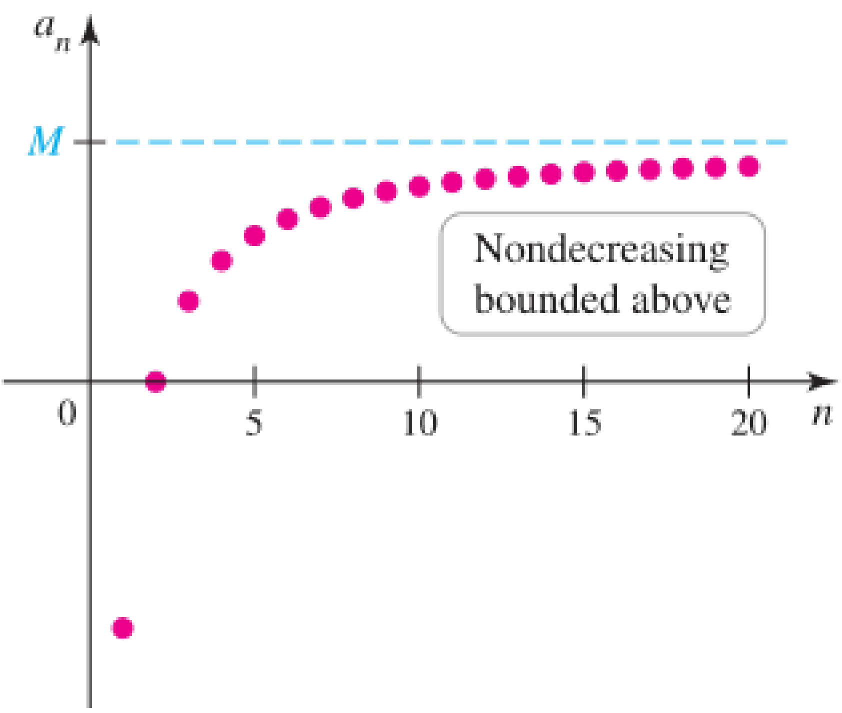
Note that $-1 \le \cos n \le 1$

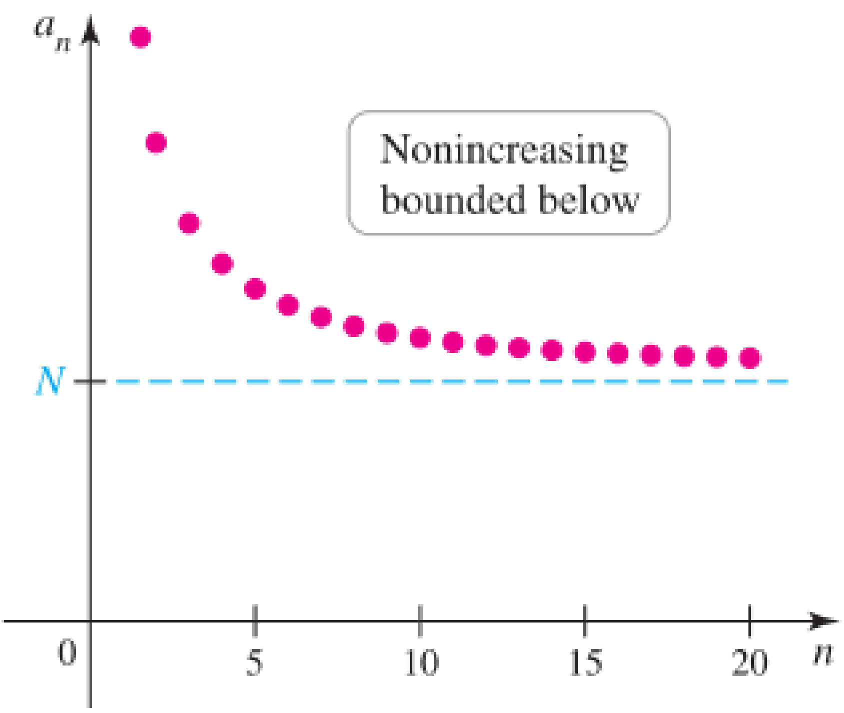
$$\underbrace{-\frac{1}{n^2+1}}_{a_n} \le \underbrace{\frac{\cos n}{n^2+1}}_{b_n} \le \underbrace{\frac{1}{n^2+1}}_{c_n}.$$



Bounded Monotonic Sequence Theorem

THEOREM 5 Bounded Monotonic Sequences
A bounded monotonic sequence converges.





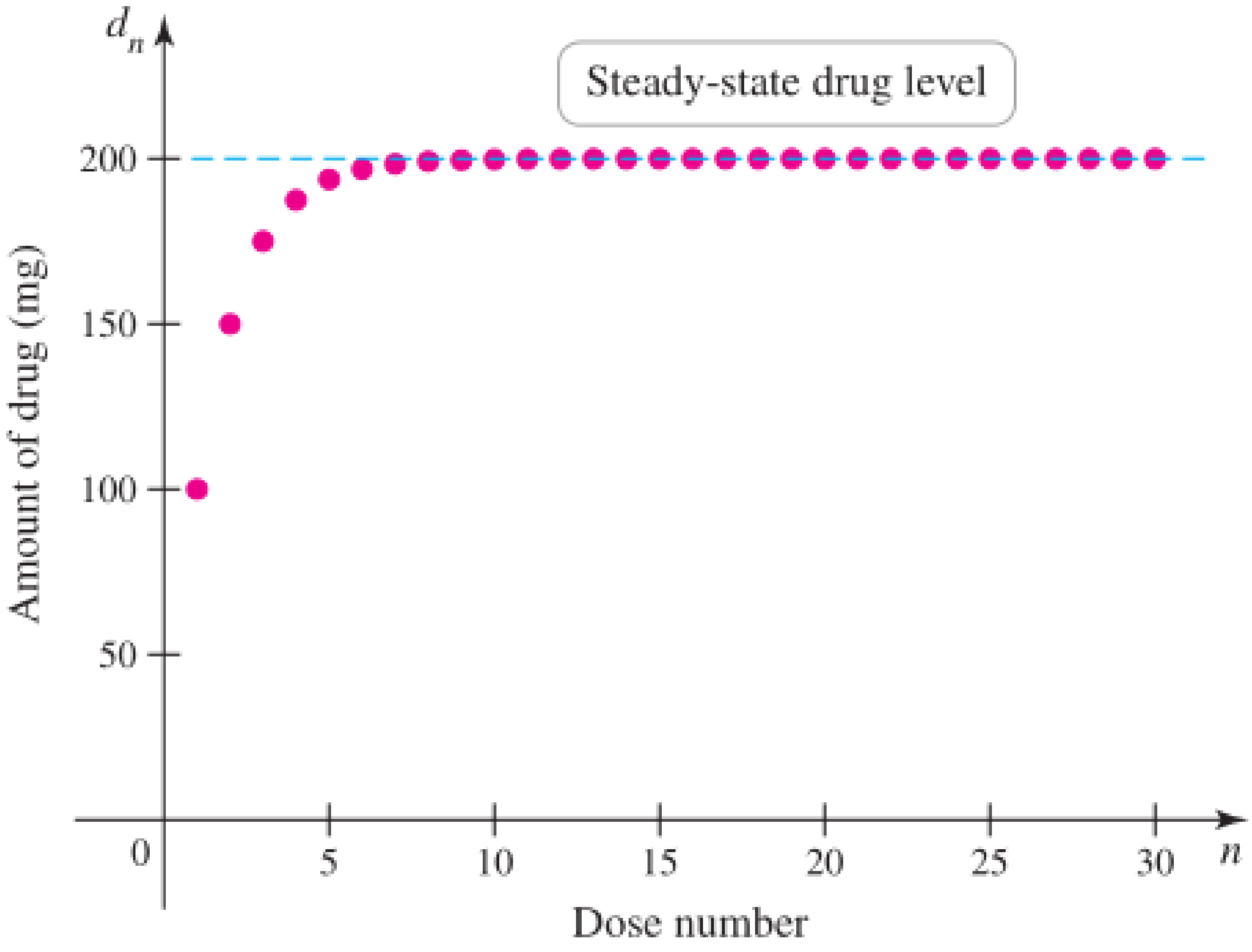
An Application: Recurrence Relations

EXAMPLE 5 **Sequences for drug doses** Suppose your doctor prescribes a 100-mg dose of an antibiotic to be taken every 12 hours. Furthermore, the drug is known to have a half-life of 12 hours; that is, every 12 hours half of the drug in your blood is eliminated.

a. Find the sequence that gives the amount of drug in your blood immediately after each dose.

b. Use a graph to propose the limit of this sequence; that is, in the long run, how much drug do you have in your blood?

c. Find the limit of the sequence directly.


d_n
Steady-state drug level
200
150
100
50
Amount of drug (mg)
0
5
10
15
20
25
30
n
Dose number

Growth Rates of Sequences

Question: Given two *nondecreasing* sequences of *positive* terms $\{a_n\}$ and $\{b_n\}$, which sequence grows faster as $n \to \infty$?

Evaluate $\lim_{n\to\infty} a_n/b_n$

If $\lim_{n\to\infty} a_n/b_n = 0$, then $\{b_n\}$ grows faster than $\{a_n\}$.

If $\lim_{n\to\infty} a_n/b_n = \infty$, then $\{a_n\}$ grows faster than $\{b_n\}$.

Using the results of Section 4.7, we have the following ranking of growth rates of sequences as $n \to \infty$ with positive numbers p, q, r, s, and $b > 1$:

$$\{\ln^q n\} \ll \{n^p\} \ll \{n^p \ln^r n\} \ll \{n^{p+s}\} \ll \{b^n\} \ll \{n^n\}.$$

Another important sequence: factorial sequence $\{n!\}$

$$n^n = \underbrace{n \cdot n \cdot n \cdots n}_{n \text{ factors}}, \qquad \text{whereas}$$

$$n! = \underbrace{n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1}_{n \text{ factors}}.$$

$$b^n = \underbrace{b \cdot b \cdot b \cdots b}_{n \text{ factors}}, \qquad \text{whereas}$$

$$n! = \underbrace{n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1}_{n \text{ factors}}.$$

THEOREM 6 Growth Rates of Sequences

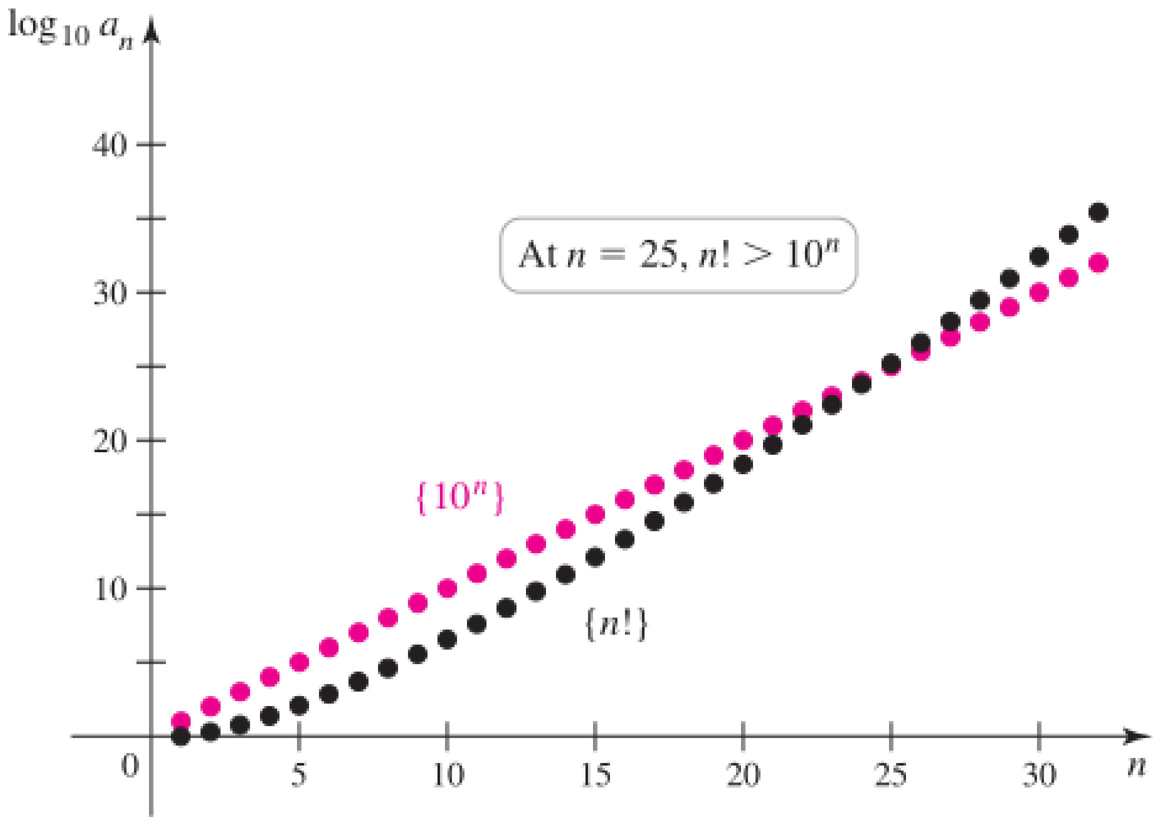
The following sequences are ordered according to increasing growth rates as $n \to \infty$; that is, if $\{a_n\}$ appears before $\{b_n\}$ in the list, then $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\lim_{n\to\infty} \frac{b_n}{a_n} = \infty$:

$$\{\ln^q n\} \ll \{n^p\} \ll \{n^p \ln^r n\} \ll \{n^{p+s}\} \ll \{b^n\} \ll \{n!\} \ll \{n^n\}.$$

The ordering applies for positive real numbers p, q, r, s, and $b > 1$.

EXAMPLE 6 **Convergence and growth rates** Compare growth rates of sequences to determine whether the following sequences converge.

a. $\left\{\dfrac{\ln n^{10}}{0.00001n}\right\}$ **b.** $\left\{\dfrac{n^8 \ln n}{n^{8.001}}\right\}$ **c.** $\left\{\dfrac{n!}{10^n}\right\}$



Formal Definition of a Limit of a Sequence

DEFINITION Limit of a Sequence

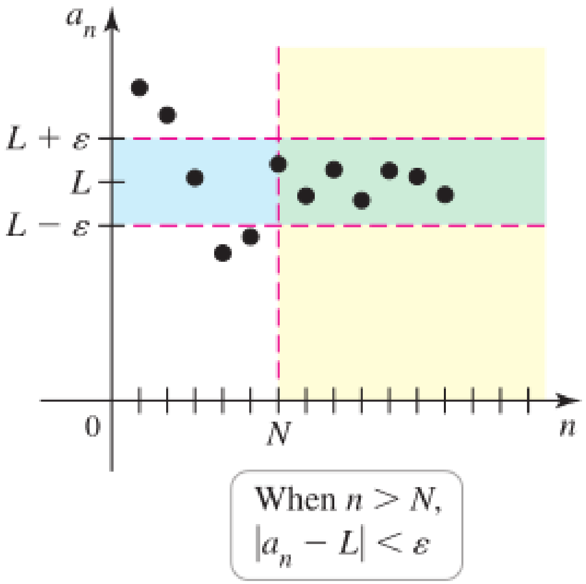
The sequence $\{a_n\}$ converges to L provided the terms of a_n can be made arbitrarily close to L by taking n sufficiently large. More precisely, $\{a_n\}$ has the unique limit L if given any $\varepsilon > 0$, it is possible to find a positive integer N (depending only on ε) such that

$$|a_n - L| < \varepsilon \qquad \text{whenever } n > N.$$

If the **limit of a sequence** is L, we say the sequence **converges** to L, written

$$\lim_{n\to\infty} a_n = L.$$

A sequence that does not converge is said to **diverge**.



EXAMPLE 7 **Limits using the formal definition** Consider the claim that

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{n}{n - 1} = 1.$$

a. Given $\varepsilon = 0.01$, find a value of N that satisfies the conditions of the limit definition.

b. Prove that $\lim_{n \to \infty} a_n = 1$.

10.3

Infinite Series

Geometric Sums and Series

A geometric sum with n terms has the form

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1} = \sum_{k=0}^{n-1} ar^k$$

where $a \neq 0$ and r are real numbers;
r is called the **ratio** of the sum and a is its first term.

A formula for the value of the geometric sum

$$rS_n = ar + ar^2 + \cdots + ar^{n-1} + ar^n$$

$$S_n - rS_n = a - ar^n$$

$$S_n = a\frac{1 - r^n}{1 - r}$$

Then, a short step to *geometric series* $\sum_{k=0}^{\infty} ar^k$

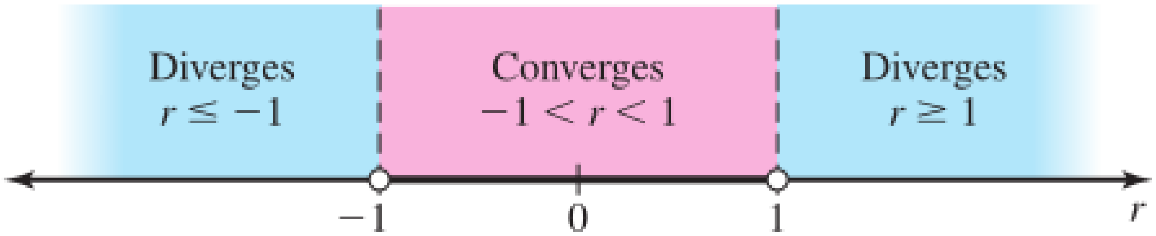
$$\underbrace{\sum_{k=0}^{\infty} ar^k}_{\text{geometric series}} = \lim_{n\to\infty} \underbrace{\sum_{k=0}^{n-1} ar^k}_{\text{geometric sum } S_n} = \lim_{n\to\infty} a\,\frac{1-r^n}{1-r}.$$

To compute this limit, must examine the behavior of r^n as $n \to \infty$

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } |r| < 1 \\ 1 & \text{if } r = 1 \\ \text{does not exist} & \text{if } r \le -1 \text{ or } r > 1. \end{cases}$$

THEOREM 7 Geometric Series

Let $a \neq 0$ and r be real numbers. If $|r| < 1$, then $\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$. If $|r| \geq 1$, then the series diverges.



EXAMPLE 1 **Geometric series** Evaluate the following geometric series or state that the series diverges.

a. $\sum_{k=0}^{\infty} 1.1^k$ **b.** $\sum_{k=0}^{\infty} e^{-k}$ **c.** $\sum_{k=2}^{\infty} 3(-0.75)^k$

EXAMPLE 2 **Decimal expansions as geometric series** Write $1.0\overline{35} = 1.0353535\ldots$ as a geometric series and express its value as a fraction.

$$1.0353535\ldots = 1 + \underbrace{0.035 + 0.00035 + 0.0000035 + \cdots}_{\text{geometric series with } a = 0.035 \text{ and } r = 0.01}.$$

$$1.0353535\ldots = 1 + \frac{a}{1 - r} = 1 + \frac{0.035}{1 - 0.01} = 1 + \frac{35}{990} = \frac{205}{198}.$$

Telescoping Series

EXAMPLE 3 **Telescoping series** Evaluate the following series.

a. $\displaystyle\sum_{k=1}^{\infty}\left(\frac{1}{3^k}-\frac{1}{3^{k+1}}\right)$ **b.** $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k(k+1)}$

The sum telescopes and all the interior terms cancel

The result is a simple formula for the nth term of the sequence of partial sums

10.4

The Divergence and Integral Tests

The Divergence Test

Difficult to obtain the exact value of most convergent series.
Question: *Given an infinite series, does it converge*?
Start from the series with positive terms

THEOREM 8 Divergence Test
If $\sum a_k$ converges, then $\lim_{k \to \infty} a_k = 0$. Equivalently, if $\lim_{k \to \infty} a_k \neq 0$, then the series diverges.

Important note: Theorem 8 *cannot* be used to conclude that a series converges.

Proof

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} (S_n - S_{n-1}) = S - S = 0;$$

EXAMPLE 1 **Using the Divergence Test** Determine whether the following series diverge or state that the Divergence Test is inconclusive.

a. $\displaystyle\sum_{k=0}^{\infty} \frac{k}{k+1}$ **b.** $\displaystyle\sum_{k=1}^{\infty} \frac{1+3^k}{2^k}$ **c.** $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$ **d.** $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$

1(c) diverges, while 1(d) converges.

The Harmonic Series

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

Does it converge? Given $\lim_{k \to \infty} \frac{1}{k} = 0$

Consider the sequence of partial sums

$$S_n = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$$

No simple explicit formula for S_n exists.

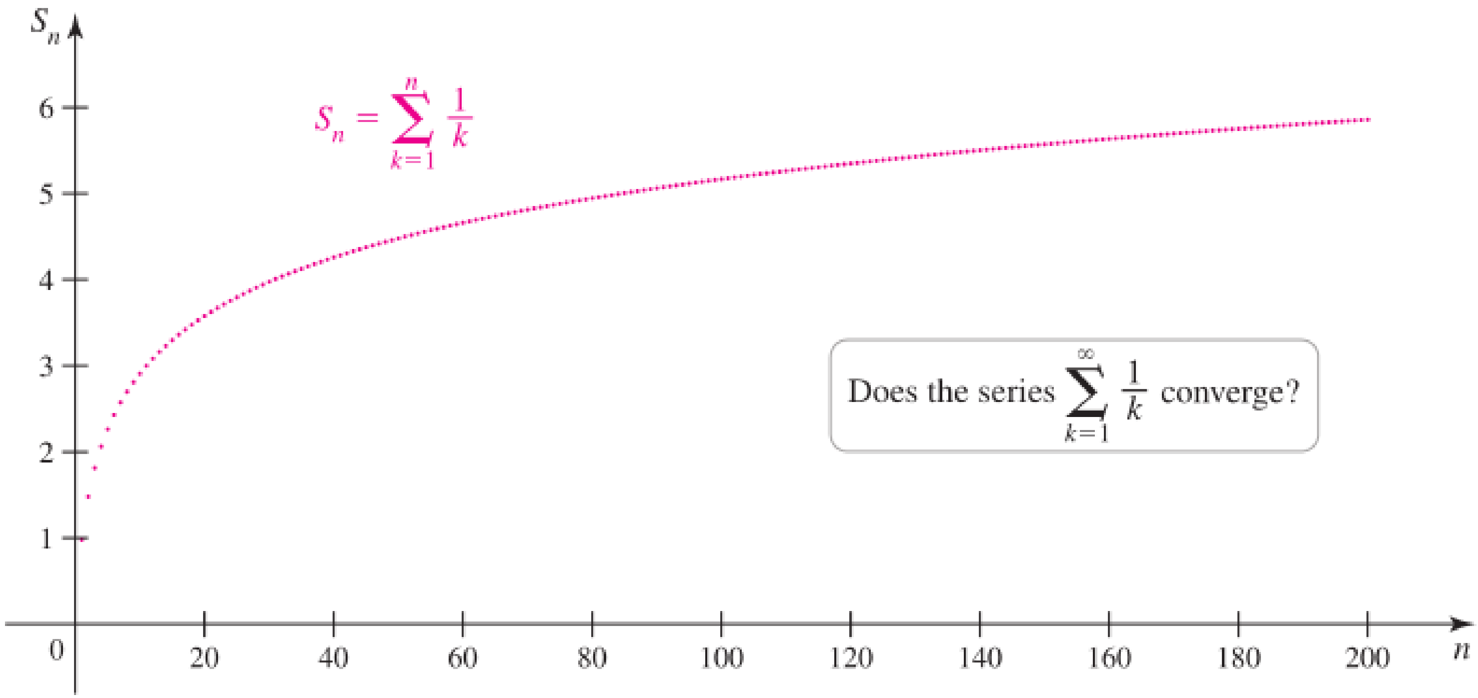
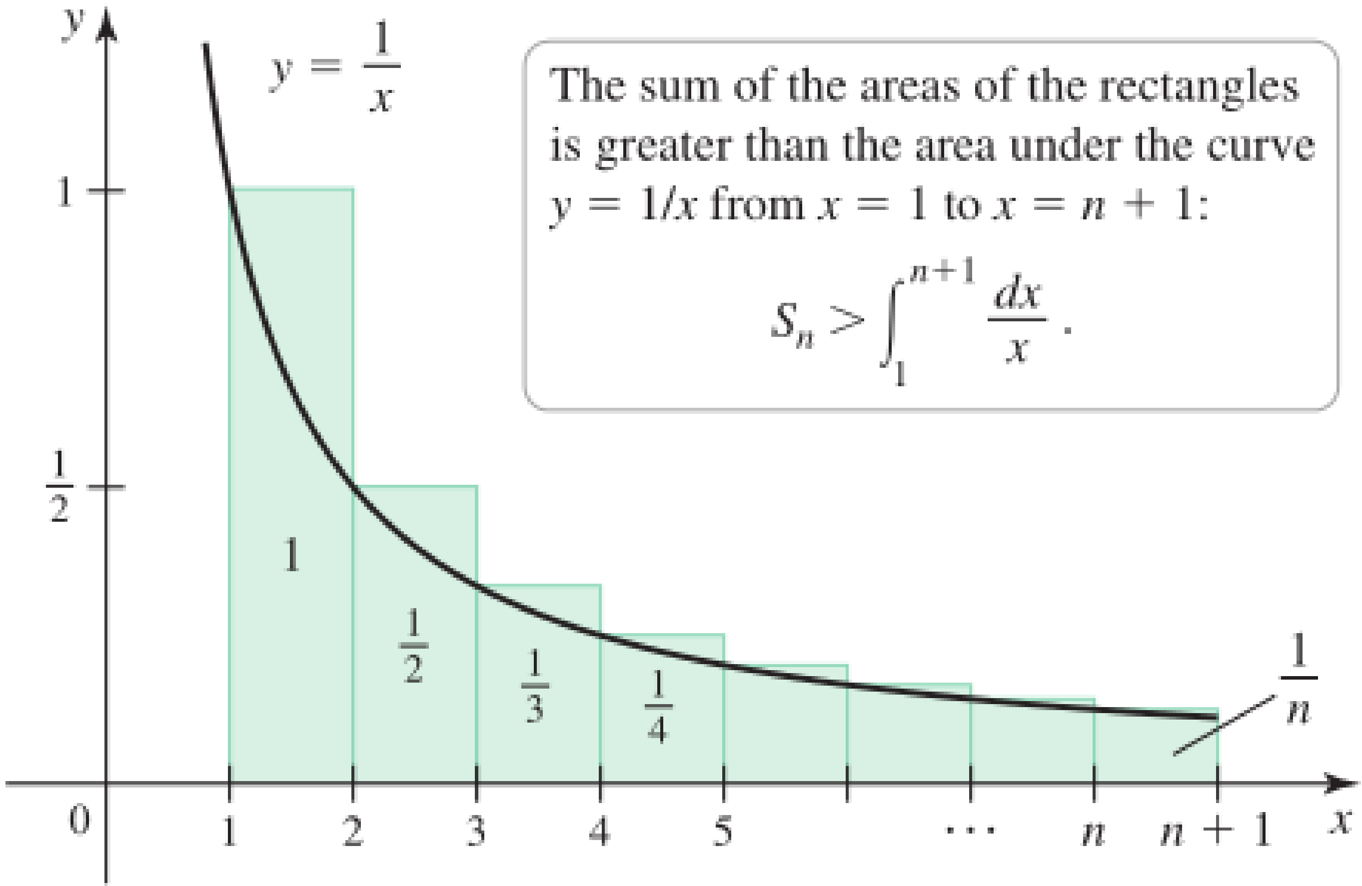


Table 3

n	S_n	n	S_n
10^3	≈ 7.49	10^{10}	≈ 23.60
10^4	≈ 9.79	10^{20}	≈ 46.63
10^5	≈ 12.09	10^{30}	≈ 69.65
10^6	≈ 14.39	10^{40}	≈ 92.68

Computation alone is not sufficient to determine whether a series converges

The nth term of the sequence of partial sums, $\frac{1}{n}$, is represented geometrically by a left Riemann sum of the function $y = \frac{1}{x}$ on the interval $[1, n + 1]$.



And $\int_1^{n+1} \frac{dx}{x} = \ln(n + 1)$ increases without bound.

THEOREM 9 Harmonic Series

The harmonic series $\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$ diverges—even though the terms of the series approach zero.

The Integral Test

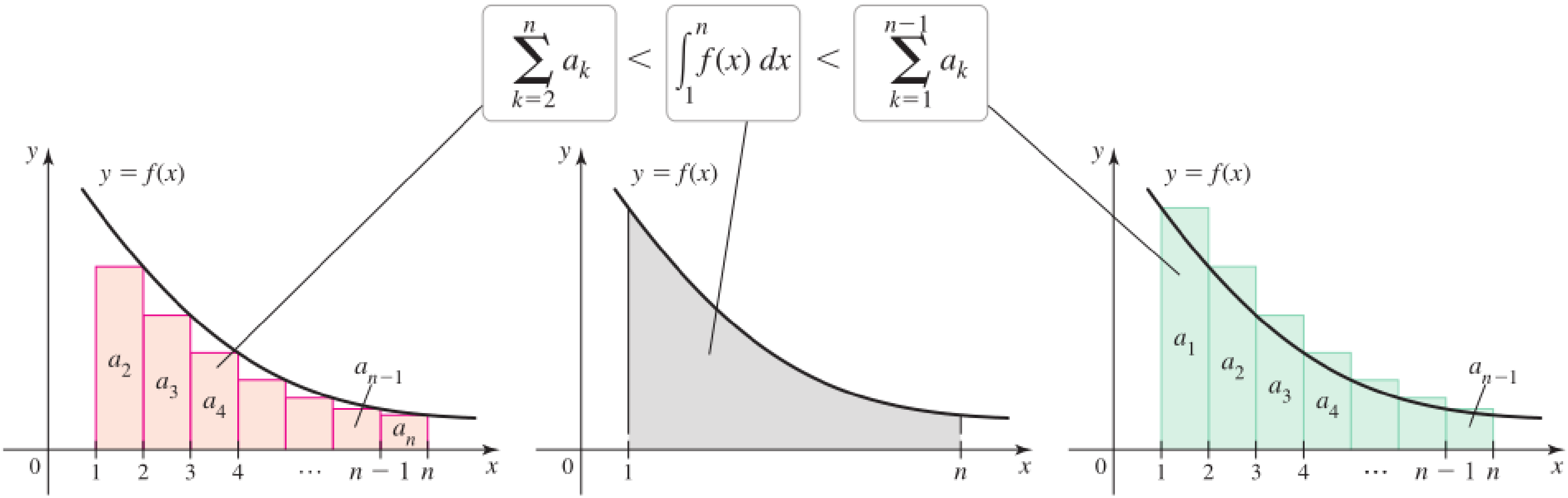
THEOREM 10 Integral Test

Suppose f is a continuous, positive, decreasing function, for $x \geq 1$, and let $a_k = f(k)$, for $k = 1, 2, 3, \ldots$. Then

$$\sum_{k=1}^{\infty} a_k \quad \text{and} \quad \int_1^{\infty} f(x)\,dx$$

either both converge or both diverge. In the case of convergence, the value of the integral is *not* equal to the value of the series.

Proof



Discuss from two directions

(1) Suppose the improper integral $\int_1^\infty f(x)dx$ has a finite value I

$$
\begin{aligned}
\sum_{k=1}^{n} a_k &= a_1 + \sum_{k=2}^{n} a_k && \text{Separate the first term of the series.} \\
&< a_1 + \int_1^n f(x)\,dx && \text{Left inequality in expression (1)} \\
&< a_1 + \int_1^\infty f(x)\,dx && f \text{ is positive, so } \int_1^n f(x)\,dx < \int_1^\infty f(x)\,dx. \\
&= a_1 + I.
\end{aligned}
$$

The terms of the sequence of partial sums $S_n = \sum_{k=1}^{n} a_k$ are bounded above by $a_1 + I$.

Because $\{S_n\}$ is also increasing, the sequence of partial sums converges, , which means the series $\sum_{k=1}^{\infty} a_k$ converges (to a value less than or equal to $a_1 + I$).

(2) Suppose the infinite series $\sum_{k=1}^{\infty} a_k$ converges to a value S.

$$\int_1^n f(x)\,dx < \sum_{k=1}^{n-1} a_k \quad \text{Right inequality in expression (1)}$$

$$< \sum_{k=1}^{\infty} a_k \quad \text{Terms } a_k \text{ are positive.}$$

$$= S. \quad \text{Value of infinite series}$$

The sequence $\{\int_1^n f(x)dx\}$ is increasing and bounded above by a fixed number S.

Therefore, the improper integral $\int_1^{\infty} f(x)dx$ has a finite value (less than or equal to S).

The Integral Test is used to determine *whether* a series converges or diverges.

Powerful, but limited, because it requires evaluating integrals

Adding or subtracting a few terms in the series *or* changing the lower limit of integration to another finite point *does not change* the outcome of the test.

Therefore, the test depends on *neither* the lower index of the series *nor* the lower limit of the integral.

EXAMPLE 2 **Applying the Integral Test** Determine whether the following series converge.

a. $\displaystyle\sum_{k=1}^{\infty} \frac{k}{k^2 + 1}$ **b.** $\displaystyle\sum_{k=3}^{\infty} \frac{1}{\sqrt{2k - 5}}$ **c.** $\displaystyle\sum_{k=0}^{\infty} \frac{1}{k^2 + 4}$

The p-Series

THEOREM 11 Convergence of the p-Series

The **p-series** $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges for $p > 1$ and diverges for $p \leq 1$.

Proof

Apply the Integral Test with the function $f(x) = \frac{1}{x^p}$.

The relevant integral $\int_1^{\infty} \frac{1}{x^p}\,dx$ converges for $p > 1$ and diverges for $p \leq 1$ (Section 8.8).

Therefore, by the Integral Test, the p-series converges $p > 1$ and diverges for $0 < p \leq 1$.

For $p \leq 0$, the series diverges by the Divergence Test.

EXAMPLE 3 **Using the *p*-series test** Determine whether the following series converge or diverge.

a. $\displaystyle\sum_{k=1}^{\infty} k^{-3}$ **b.** $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt[4]{k^3}}$ **c.** $\displaystyle\sum_{k=4}^{\infty} \frac{1}{(k-1)^2}$

Estimating the Value of Infinite Series

The Integral Test can also be used to estimate the value of a convergent series with positive terms.

Remainder R_n: the error in approximating a convergent series by the sum of its first n terms.

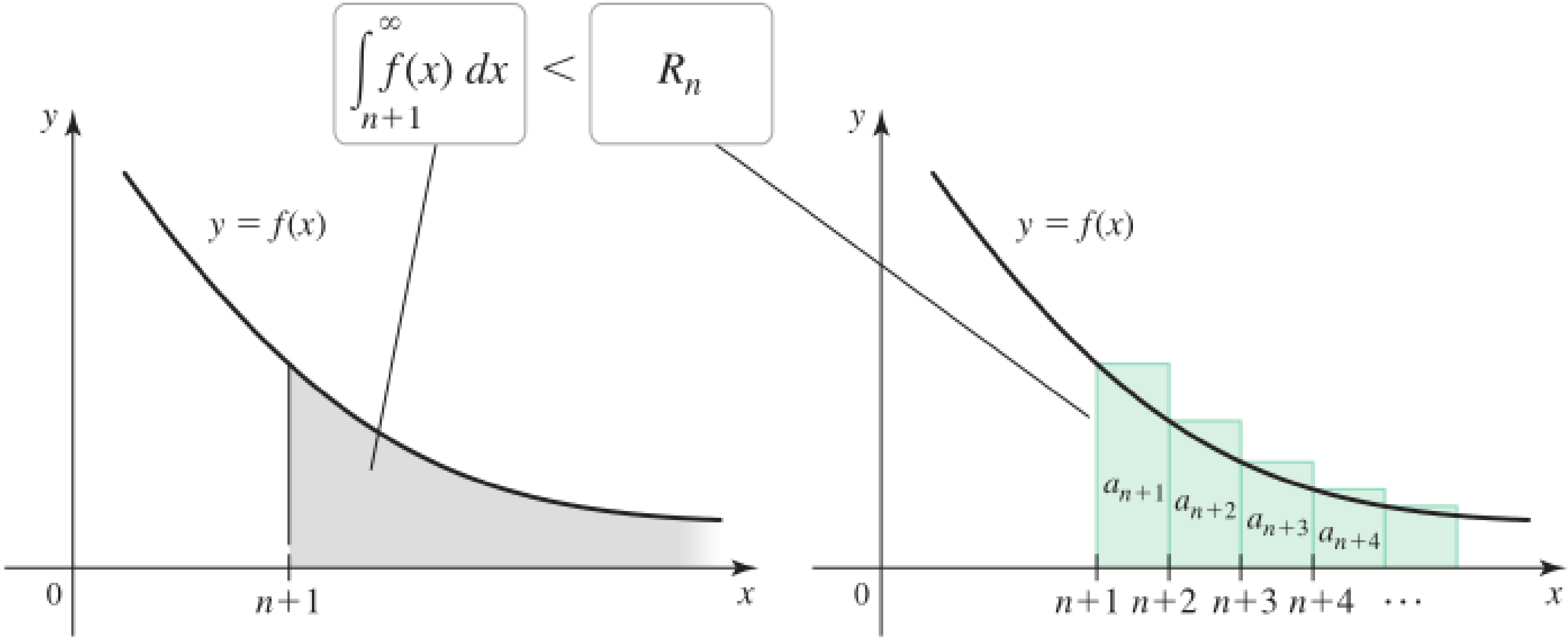
$$R_n = \underbrace{\sum_{k=1}^{\infty} a_k}_{\text{value of series}} - \underbrace{\sum_{k=1}^{n} a_k}_{\text{approximation based on first } n \text{ terms}} = a_{n+1} + a_{n+2} + a_{n+3} + \cdots.$$

The remainder consists of the *tail* of the series—those terms beyond a_n.

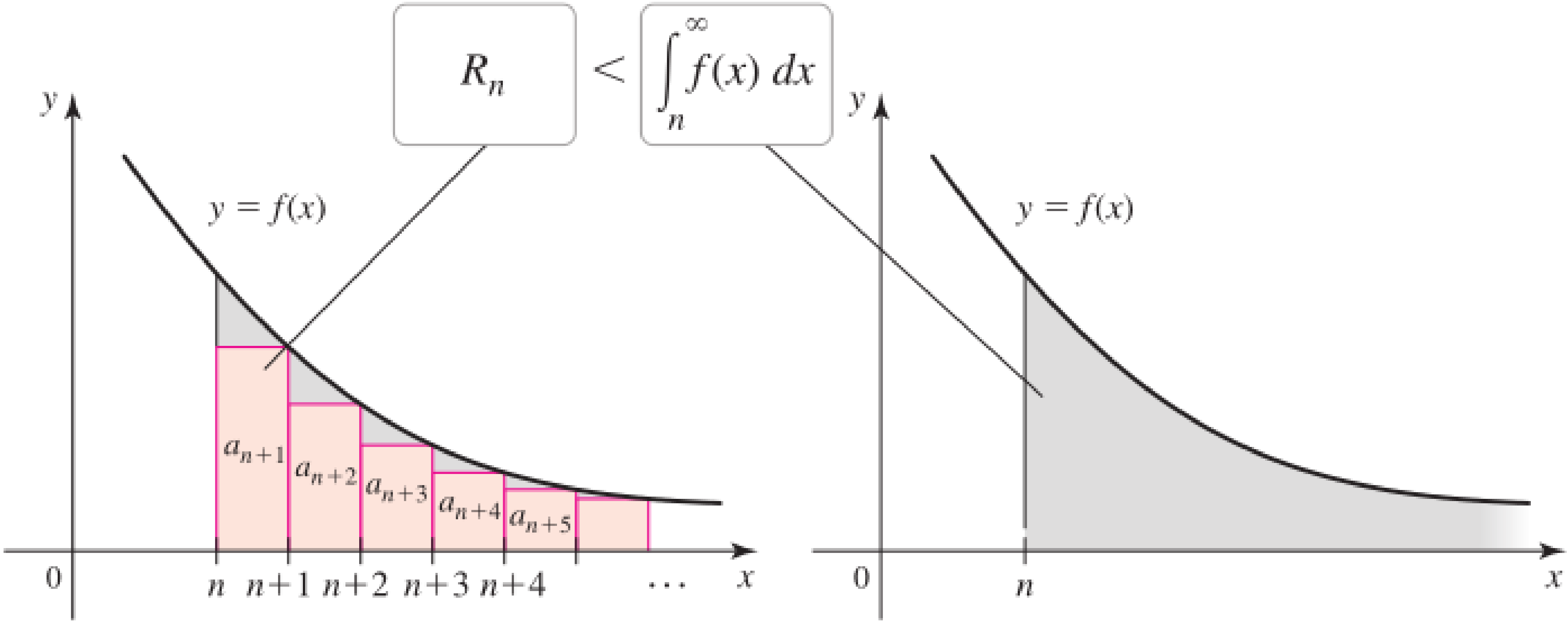
For series with positive terms, the remainder is positive.

Let f be a continuous, positive, decreasing function such that $f(k) = a_k$, for all relevant k, then

$$\int_{n+1}^{\infty} f(x)dx < R_n$$



Similarly, $R_n < \int_n^\infty f(x)dx$



Therefore, the remainder is squeezed between two integrals

$$\int_{n+1}^{\infty} f(x)dx < R_n < \int_n^{\infty} f(x)dx$$

Another equally useful way to express this result.

From $S = \sum_{k=1}^{\infty} a_k = \sum_{k=1}^{n} a_k + R_n$
The series is the sum of the first n terms S_n and the remainder R_n.

Adding S_n to each term of the previous inequality

$$\underbrace{S_n + \int_{n+1}^{\infty} f(x)\,dx}_{L_n} < \underbrace{\sum_{k=1}^{\infty} a_k}_{S_n + R_n = S} < \underbrace{S_n + \int_{n}^{\infty} f(x)\,dx}_{U_n}.$$

These inequalities can be abbreviated as $L_n < S < U_n$
L_n and U_n are lower and upper bounds for S, the exact value of the series

THEOREM 12 Estimating Series with Positive Terms

Let f be a continuous, positive, decreasing function, for $x \geq 1$, and let $a_k = f(k)$, for $k = 1, 2, 3, \ldots$. Let $S = \sum_{k=1}^{\infty} a_k$ be a convergent series and let $S_n = \sum_{k=1}^{n} a_k$ be the sum of the first n terms of the series. The remainder $R_n = S - S_n$ satisfies

$$R_n < \int_n^{\infty} f(x)\,dx.$$

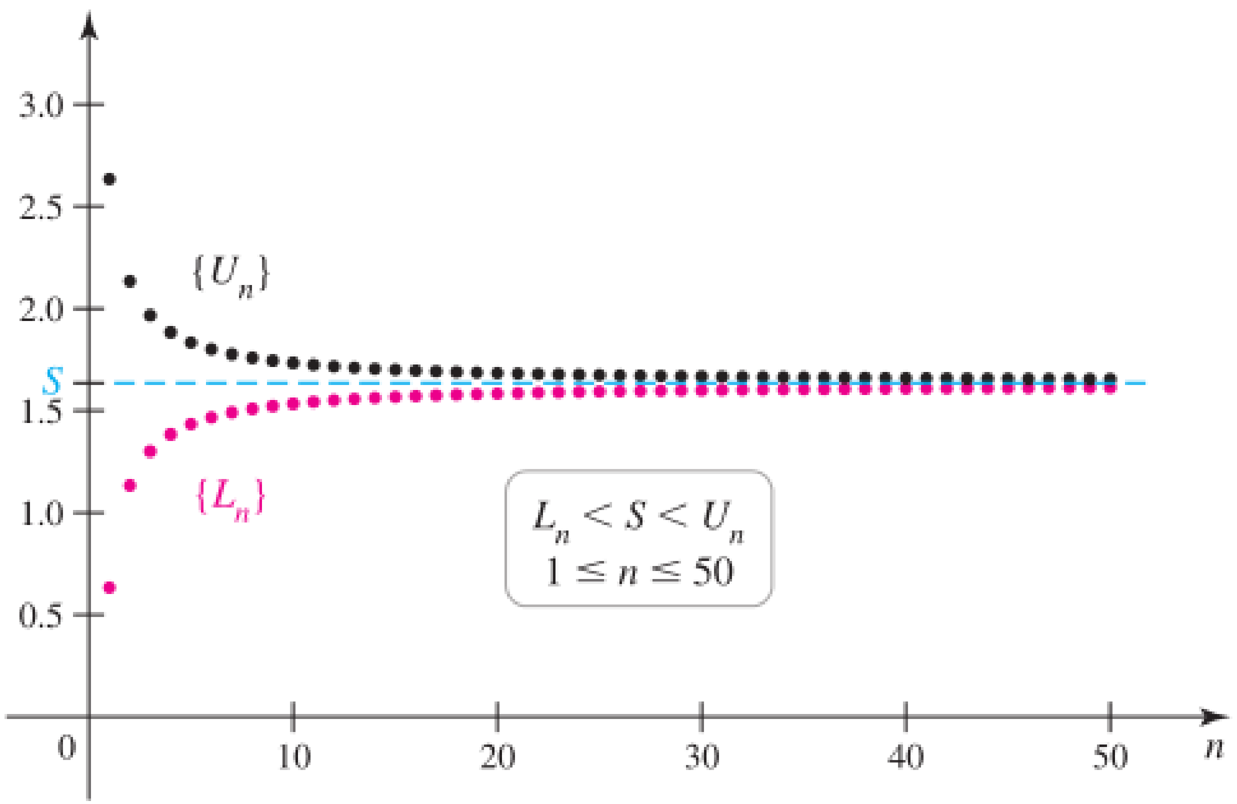
Furthermore, the exact value of the series is bounded as follows:

$$S_n + \int_{n+1}^{\infty} f(x)\,dx < \sum_{k=1}^{\infty} a_k < S_n + \int_n^{\infty} f(x)\,dx.$$

EXAMPLE 4 **Approximating a *p*-series**

a. How many terms of the convergent *p*-series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ must be summed to obtain an approximation that is within 10^{-3} of the exact value of the series?

b. Find an approximation to the series using 50 terms of the series.



Properties of Convergent Series

For a general infinite series $\sum a_k$, whose terms may be positive or negative (or both)

THEOREM 13 Properties of Convergent Series

1. Suppose $\sum a_k$ converges to A and c is a real number. The series $\sum ca_k$ converges, and $\sum ca_k = c\sum a_k = cA$.
2. Suppose $\sum a_k$ converges to A and $\sum b_k$ converges to B. The series $\sum(a_k \pm b_k)$ converges, and $\sum(a_k \pm b_k) = \sum a_k \pm \sum b_k = A \pm B$.
3. If M is a positive integer, then $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=M}^{\infty} a_k$ either both converge or both diverge. In general, *whether* a series converges does not depend on a finite number of terms added to or removed from the series. However, the *value* of a convergent series does change if nonzero terms are added or removed.

Proof

Using properties of finite sums and limits of sequences, e.g., Property 1

$$
\begin{aligned}
\sum_{k=1}^{\infty} ca_k &= \lim_{n\to\infty} \sum_{k=1}^{n} ca_k && \text{Definition of infinite series} \\
&= \lim_{n\to\infty} c \sum_{k=1}^{n} a_k && \text{Property of finite sums} \\
&= c \lim_{n\to\infty} \sum_{k=1}^{n} a_k && \text{Property of limits} \\
&= c \sum_{k=1}^{\infty} a_k && \text{Definition of infinite series} \\
&= cA. && \text{Value of the series}
\end{aligned}
$$

Use caution when applying Theorem 13. For example,

$$\sum_{k=2}^{\infty} \frac{1}{k(k-1)} = \sum_{k=2}^{\infty} \left(\frac{1}{k-1} - \frac{1}{k} \right)$$

A telescoping series (that converges to 1).

An *incorrect* application of Theorem 13 would be to write

$$\sum_{k=2}^{\infty} \left(\frac{1}{k-1} - \frac{1}{k} \right) = \underbrace{\sum_{k=2}^{\infty} \frac{1}{k-1}}_{\text{diverges}} - \underbrace{\sum_{k=2}^{\infty} \frac{1}{k}}_{\text{diverges}} \quad \text{This is } \textit{incorrect}!$$

EXAMPLE 5 **Using properties of series** Evaluate the infinite series

$$S = \sum_{k=1}^{\infty} \left(5\left(\frac{2}{3}\right)^k - \frac{2^{k-1}}{7^k} \right).$$

10.5

Comparison Tests

The Comparison Test

For series with positive terms

THEOREM 16 Comparison Test
Let $\sum a_k$ and $\sum b_k$ be series with positive terms.

1. If $0 < a_k \leq b_k$ and $\sum b_k$ converges, then $\sum a_k$ converges.
2. If $0 < b_k \leq a_k$ and $\sum b_k$ diverges, then $\sum a_k$ diverges.

Idea: For 1, $S_n = \sum_{k=1}^{n} a_k \leq \sum_{k=1}^{n} b_k < \sum_{k=1}^{\infty} b_k = B$
The sequence of partial sums for $\sum a_k$ is increasing (positive terms) and bounded above by B
It implies that $\sum a_k$ converges.

The *key* in using the Comparison Test is finding an appropriate comparison series

EXAMPLE 3 **Using the Comparison Test** Determine whether the following series converge.

a. $\displaystyle\sum_{k=1}^{\infty} \frac{k^3}{2k^4 - 1}$

b. $\displaystyle\sum_{k=2}^{\infty} \frac{\ln k}{k^3}$

The Limit Comparison Test

THEOREM 17 Limit Comparison Test

Let $\sum a_k$ and $\sum b_k$ be series with positive terms and let

$$\lim_{k\to\infty} \frac{a_k}{b_k} = L.$$

1. If $0 < L < \infty$ (that is, L is a finite positive number), then $\sum a_k$ and $\sum b_k$ either both converge or both diverge.
2. If $L = 0$ and $\sum b_k$ converges, then $\sum a_k$ converges.
3. If $L = \infty$ and $\sum b_k$ diverges, then $\sum a_k$ diverges.

Idea: From $\lim_{k\to\infty} \frac{a_k}{b_k} = L$, we have $\left|\frac{a_k}{b_k} - L\right| < \varepsilon = L/2$

Then, $-L/2 < \frac{a_k}{b_k} - L < L/2$, and $\frac{Lb_k}{2} < a_k < \frac{3Lb_k}{2}$

The terms of $\sum a_k$ are sandwiched between the terms of $\sum b_k$.

EXAMPLE 4 **Using the Limit Comparison Test** Determine whether the following series converge.

a. $\sum_{k=1}^{\infty} \frac{5k^4 - 2k^2 + 3}{2k^6 - k + 5}$ **b.** $\sum_{k=1}^{\infty} \frac{\ln k}{k^2}$.

Question b is interesting, because

$$L = \lim_{k \to \infty} \frac{a_k}{b_k} = \lim_{k \to \infty} \frac{\ln k / k^2}{1/k^2} = \lim_{k \to \infty} \ln k = \infty.$$

So, Case (3) of the Limit Comparison Test does not apply here

$$L = \lim_{k \to \infty} \frac{a_k}{b_k} = \lim_{k \to \infty} \frac{\ln k / k^2}{1/k} = \lim_{k \to \infty} \frac{\ln k}{k} = 0.$$

Case (2) of the Limit Comparison Test does not apply either

A series that lies "between" $\sum_{k=1}^{\infty} \frac{1}{k^2}$ and $\sum_{k=1}^{\infty} \frac{1}{k}$ is the convergent p-series $\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$. Try it as a comparison series.

$$L = \lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\ln k / k^2}{1/k^{3/2}} = \lim_{k\to\infty} \frac{\ln k}{\sqrt{k}} = 0.$$

Now, case (2) of the Limit Comparison Test applies.

10.6

Alternating Series

Alternating Series

A series in which the signs strictly alternate, e.g.,

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots.$$

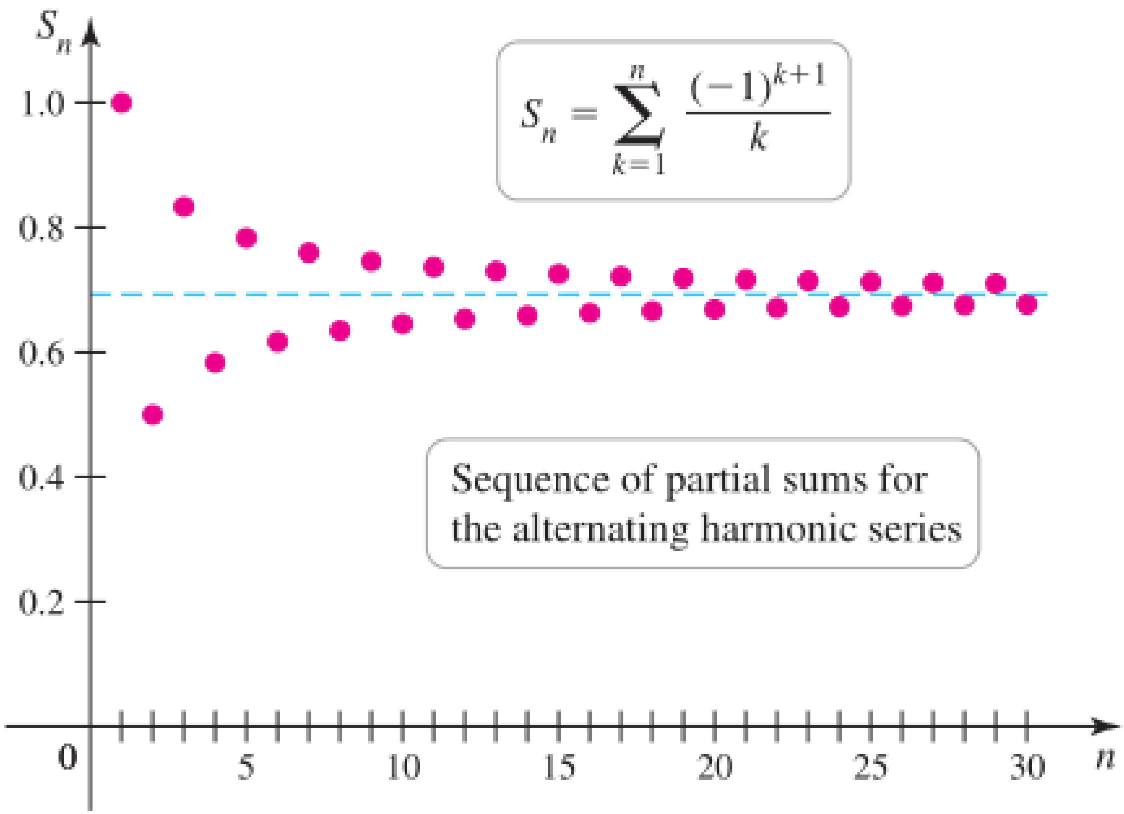
The factor $(-1)^{k+1}$ has the pattern $\{\cdots, 1, -1, 1, -1, \cdots\}$ and provides the alternating signs.

It is called the alternating harmonic series

Alternating Harmonic Series

Recall that this series *without* the alternating signs, $\sum_{k=1}^{\infty} \frac{1}{k}$, is the *divergent* harmonic series.

How is effect of the presence of alternating signs?



$$S_1 = 1$$

$$S_2 = 1 - \frac{1}{2} = \frac{1}{2}$$

$$S_3 = 1 - \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$$

$$S_4 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} = \frac{7}{12}.$$

Two observations:

1. For series with *positive* terms, the sequence of partial sums is necessarily an *increasing* sequence. Because the terms of an *alternating* series alternate in sign, the sequence of partial sums is *not* increasing.

2. For the alternating harmonic series, the odd terms of the sequence of partial sums form a decreasing sequence and the even terms form an increasing sequence.
   As a result, the limit of the sequence of partial sums lies between any two consecutive terms of the sequence.

Alternating Series Test

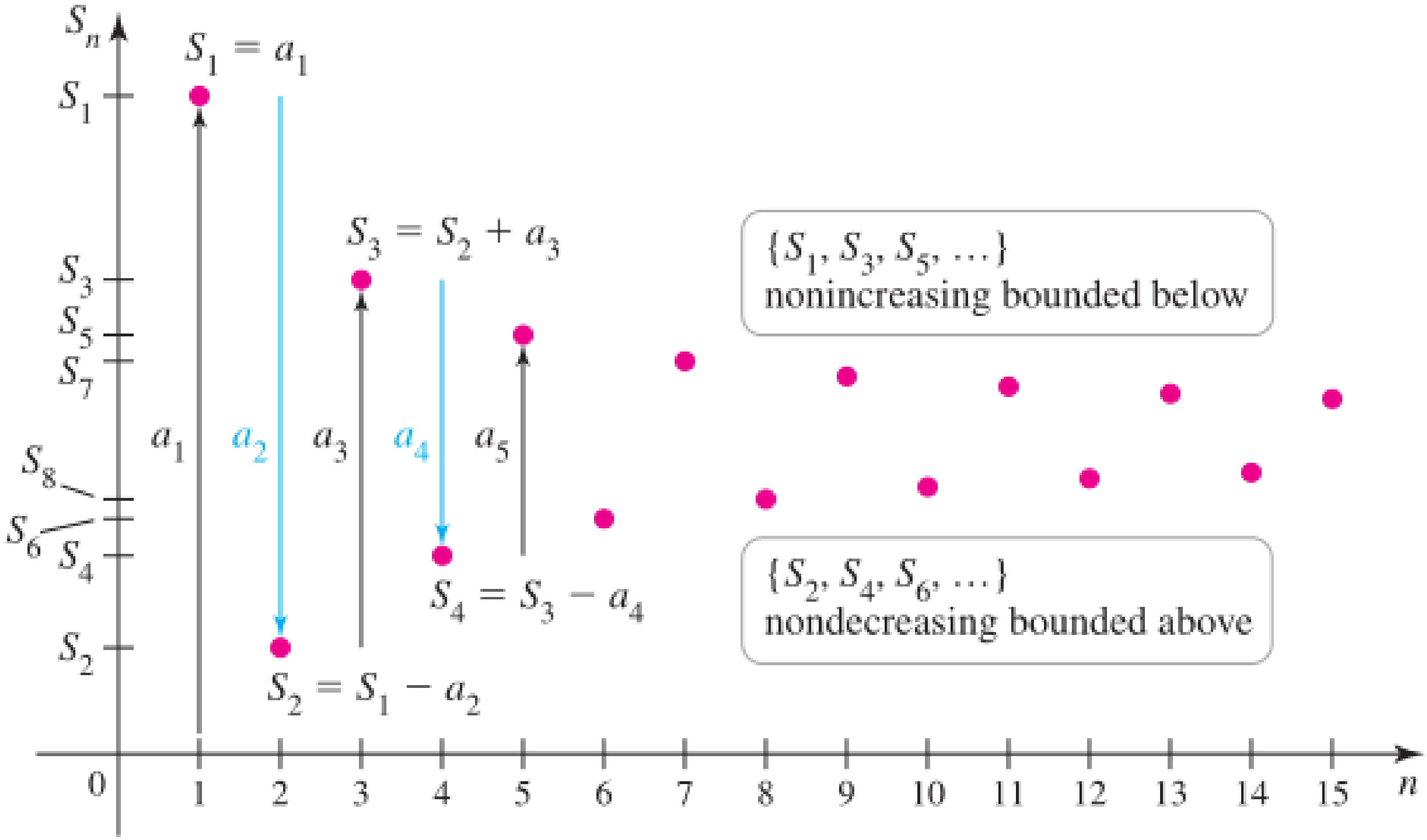
None of the convergence tests for series with positive terms applies to alternating series.

THEOREM 18 Alternating Series Test

The alternating series $\sum(-1)^{k+1}a_k$ converges provided

1. the terms of the series are nonincreasing in magnitude ($0 < a_{k+1} \leq a_k$, for k greater than some index N) and
2. $\lim_{k\to\infty} a_k = 0$.

Proof



$$\underbrace{\lim_{k\to\infty} S_{2k}}_{L} = \underbrace{\lim_{k\to\infty} S_{2k-1}}_{L'} - \underbrace{\lim_{k\to\infty} a_{2k}}_{0},$$

THEOREM 19 Alternating Harmonic Series

The alternating harmonic series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$

converges (even though the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$

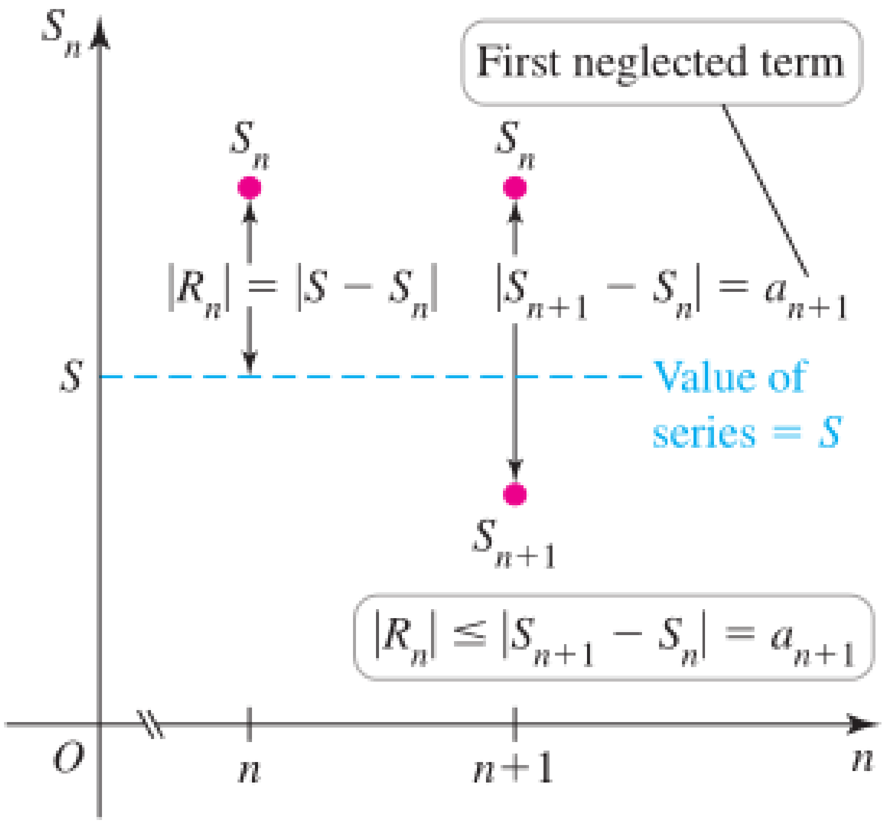
diverges).

EXAMPLE 1 **Alternating Series Test** Determine whether the following series converge or diverge.

a. $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2}$ **b.** $2 - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \cdots$ **c.** $\sum_{k=2}^{\infty} \frac{(-1)^k \ln k}{k}$

Remainders in Alternating Series

The magnitude of the remainder is the absolute error in approximating S by S_n.



When the terms are nonincreasing in magnitude, the value of the series is always trapped between successive terms of the sequence of partial sums.

THEOREM 20 Remainder in Alternating Series

Let $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$ be a convergent alternating series with terms that are nonincreasing in magnitude. Let $R_n = S - S_n$ be the remainder in approximating the value of that series by the sum of its first n terms. Then $|R_n| \leq a_{n+1}$. In other words, the magnitude of the remainder is less than or equal to the magnitude of the first neglected term.

EXAMPLE 2 Remainder in an alternating series

a. In turns out that $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{k=1}^{\neq} \frac{(-1)^{k+1}}{k}$. How many terms of the series are required to approximate $\ln 2$ with an error less than 10^{-6}? The exact value of the series is given but is not needed to answer the question.

b. Consider the series $-1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots = \sum_{k=1}^{\neq} \frac{(-1)^k}{k!}$. Find an upper bound for the magnitude of the error in approximating the value of the series (which is $e^{-1} - 1$) with $n = 9$ terms.

Absolute and Conditional Convergence

Let $\sum a_k$ denote any series.
Removing the alternating signs in a convergent series *may* or *may not* result in a convergent series.

DEFINITION Absolute and Conditional Convergence
If $\sum |a_k|$ converges, then $\sum a_k$ **converges absolutely**. If $\sum |a_k|$ diverges and $\sum a_k$ converges, then $\sum a_k$ **converges conditionally**.

The series $\sum (-1)^{k+1}/k^2$ is an *absolutely convergent* series.
While the convergent alternating harmonic series $\sum (-1)^{k+1}/k$ *converges conditionally*.

THEOREM 10.19 Absolute Convergence Implies Convergence
If $\sum |a_k|$ converges, then $\sum a_k$ converges (absolute convergence implies convergence). Equivalently, if $\sum a_k$ diverges, then $\sum |a_k|$ diverges.

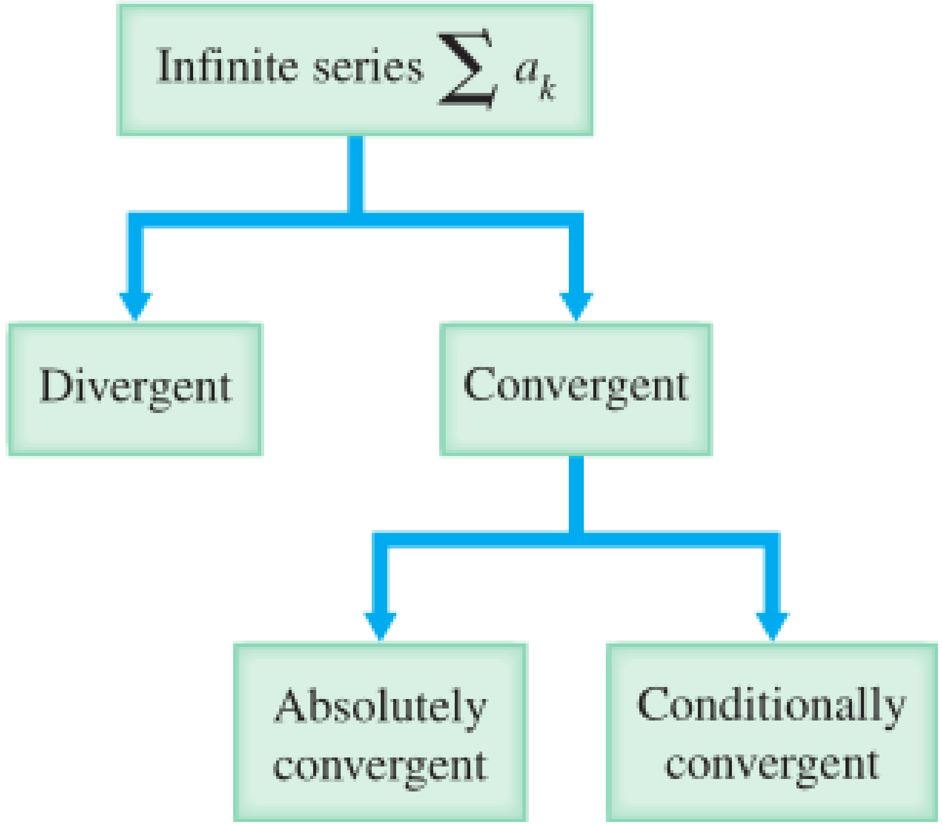
Proof

$$0 \le a_k + |a_k| \le 2|a_k|$$

It follows that $\sum(a_k + |a_k|)$ converges. Note that

$$\sum a_k = \sum(a_k + |a_k| - |a_k|) = \underbrace{\sum(a_k + |a_k|)}_{\text{converges}} - \underbrace{\sum |a_k|}_{\text{converges}}.$$

So, it also converges.



- The distinction between absolute and conditional convergence is relevant only for series of mixed sign.
- To test for absolute convergence, we test the series $\sum|a_k|$, which is a series of positive terms.

EXAMPLE 3 **Absolute and conditional convergence** Determine whether the following series diverge, converge absolutely, or converge conditionally.

a. $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k}}$ **b.** $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k^3}}$ **c.** $\displaystyle\sum_{k=1}^{\infty} \frac{\sin k}{k^2}$ **d.** $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k} k}{k+1}$

10.7

The Ratio and Root Tests

The Ratio Test

THEOREM 10.20 Ratio Test

Let $\sum a_k$ be an infinite series, and let $r = \lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right|$.

1. If $r < 1$, the series converges absolutely, and therefore it converges (by Theorem 10.19).
2. If $r > 1$ (including $r = \infty$), the series diverges.
3. If $r = 1$, the test is inconclusive.

Idea: $|a_{k+1}| \approx r|a_k|$

The tail of the series determines whether the series converges

$$|a_k| + |a_{k+1}| + |a_{k+2}| + \cdots \approx |a_k|(1 + r + r^2 + \cdots)$$

EXAMPLE 1 **Using the Ratio Test** Use the Ratio Test to determine whether the following series converge.

a. $\sum_{k=1}^{\infty} \frac{10^k}{k!}$ b. $\sum_{k=1}^{\infty} \frac{(-1)^k k^k}{k!}$ c. $\sum_{k=1}^{\infty} (-1)^{k+1} e^{-k} (k^2 + 4)$

Inconclusive example: *harmonic series* $\sum_{k=1}^{\infty} \frac{1}{k}$

$$r = \lim_{k \to \infty} \frac{a_{k+1}}{a_k} = \lim_{k \to \infty} \frac{1/(k+1)}{1/k} = \lim_{k \to \infty} \frac{k}{k+1} = 1,$$

The Root Test

THEOREM 10.21 Root Test

Let $\sum a_k$ be an infinite series, and let $\rho = \lim_{k \to \infty} \sqrt[k]{|a_k|}$.

1. If $\rho < 1$, the series converges absolutely, and therefore it converges (by Theorem 10.19).
2. If $\rho > 1$ (including $\rho = \infty$), the series diverges.
3. If $\rho = 1$, the test is inconclusive.

Idea: $\rho \approx \sqrt[k]{|a_k|}$ or $|a_k| \approx \rho^k$

The tail of the series determines whether the series converges

$$|a_k| + |a_{k+1}| + |a_{k+2}| + \cdots \approx \rho^k + \rho^{k+1} + \rho^{k+2} + \cdots$$

The tail is approximately a geometric series with ratio r.

EXAMPLE 2 **Using the Root Test** Use the Root Test to determine whether the following series converge.

a. $\displaystyle\sum_{k=1}^{\infty}\left(\frac{3-4k^2}{7k^2+6}\right)^k$ **b.** $\displaystyle\sum_{k=1}^{\infty}\frac{(-2)^k}{k^{10}}$

10.8

Choosing a Convergence Test

Guidelines for Choosing a Test

1. Begin with the Divergence Test by evaluating $\lim_{k\to\infty} a_k$
2. Is the series a special series?
   - Geometric series: $\sum ar^k$
   - p-series: $\sum \frac{1}{k^p}$
   - Telescoping series
3. If the general kth term looks like an integrable function, then try the Integral Test.
4. If the general kth term involves $k!$, k^k, or a^k, where a is a constant, the Ratio Test is advisable. Series with k in an exponent may yield to the Root Test.
5. If the general kth term is a rational function of k (or a root of a rational function), use the Comparison or the Limit Comparison Test

EXAMPLE 1 **Categorize the series** Discuss the series $\sum_{k=1}^{\infty} \frac{2^k + \cos(\pi k)\sqrt{k}}{3^{k+1}}$.

Table 10.4 Special Series and Convergence Tests

Series or Test	Form of Series	Condition for Convergence	Condition for Divergence	Comments
Geometric series	$\sum_{k=0}^{\infty} ar^k,\ a \neq 0$	$\lvert r \rvert < 1$	$\lvert r \rvert \geq 1$	If $\lvert r \rvert < 1$, then $\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$.
Divergence Test	$\sum_{k=1}^{\infty} a_k$	Does not apply	$\lim_{k\to\infty} a_k \neq 0$	Cannot be used to prove convergence
Integral Test	$\sum_{k=1}^{\infty} a_k$, where $a_k = f(k)$ and f is continuous, positive, and decreasing	$\int_1^{\infty} f(x)\,dx$ converges.	$\int_1^{\infty} f(x)\,dx$ diverges.	The value of the integral is not the value of the series.
p-series	$\sum_{k=1}^{\infty} \frac{1}{k^p}$	$p > 1$	$p \leq 1$	Useful for comparison tests
Ratio Test	$\sum_{k=1}^{\infty} a_k$	$\lim_{k\to\infty} \left\lvert \frac{a_{k+1}}{a_k} \right\rvert < 1$	$\lim_{k\to\infty} \left\lvert \frac{a_{k+1}}{a_k} \right\rvert > 1$	Inconclusive if $\lim_{k\to\infty} \left\lvert \frac{a_{k+1}}{a_k} \right\rvert = 1$

Root Test	$\sum_{k=1}^{\infty} a_k$	$\lim_{k\to\infty} \sqrt[k]{\lvert a_k\rvert} < 1$	$\lim_{k\to\infty} \sqrt[k]{\lvert a_k\rvert} > 1$	Inconclusive if $\lim_{k\to\infty} \sqrt[k]{\lvert a_k\rvert} = 1$
Comparison Test	$\sum_{k=1}^{\infty} a_k$, where $a_k > 0$	$a_k \le b_k$ and $\sum_{k=1}^{\infty} b_k$ converges.	$b_k \le a_k$ and $\sum_{k=1}^{\infty} b_k$ diverges.	$\sum_{k=1}^{\infty} a_k$ is given; you supply $\sum_{k=1}^{\infty} b_k$.
Limit Comparison Test	$\sum_{k=1}^{\infty} a_k$, where $a_k > 0$, $b_k > 0$	$0 \le \lim_{k\to\infty} \frac{a_k}{b_k} < \infty$ and $\sum_{k=1}^{\infty} b_k$ converges.	$\lim_{k\to\infty} \frac{a_k}{b_k} > 0$ and $\sum_{k=1}^{\infty} b_k$ diverges.	$\sum_{k=1}^{\infty} a_k$ is given; you supply $\sum_{k=1}^{\infty} b_k$.
Alternating Series Test	$\sum_{k=1}^{\infty} (-1)^k a_k$, where $a_k > 0$	$\lim_{k\to\infty} a_k = 0$ and $0 < a_{k+1} \le a_k$	$\lim_{k\to\infty} a_k \ne 0$	Remainder R_n satisfies $\lvert R_n\rvert \le a_{n+1}$
Absolute Convergence	$\sum_{k=1}^{\infty} a_k$, a_k arbitrary	$\sum_{k=1}^{\infty} \lvert a_k\rvert$ converges.		Applies to arbitrary series

EXAMPLE 2 **Divergence Test** Does the series $\sum_{k=1}^{\infty}\left(1-\frac{10}{k}\right)^{k}$ converge or diverge?

EXAMPLE 3 **Special series** Does the series $\sum_{k=4}^{\infty}\frac{1}{\sqrt[4]{k^2-6k+9}}$ converge or diverge?

EXAMPLE 4 **More than one test** Does the series $\sum_{k=1}^{\infty} k^2 e^{-2k}$ converge or diverge?

EXAMPLE 5 **Comparison tests** Does the series $\sum_{k=2}^{\infty} \sqrt[3]{\frac{k^2 - 1}{k^4 + 4}}$ converge or diverge?

Chapter 10

Sequences and Infinite Series

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