Chapter 12

Parametric and Polar Curves

Shuwei Chen
swchen@swjtu.edu.cn

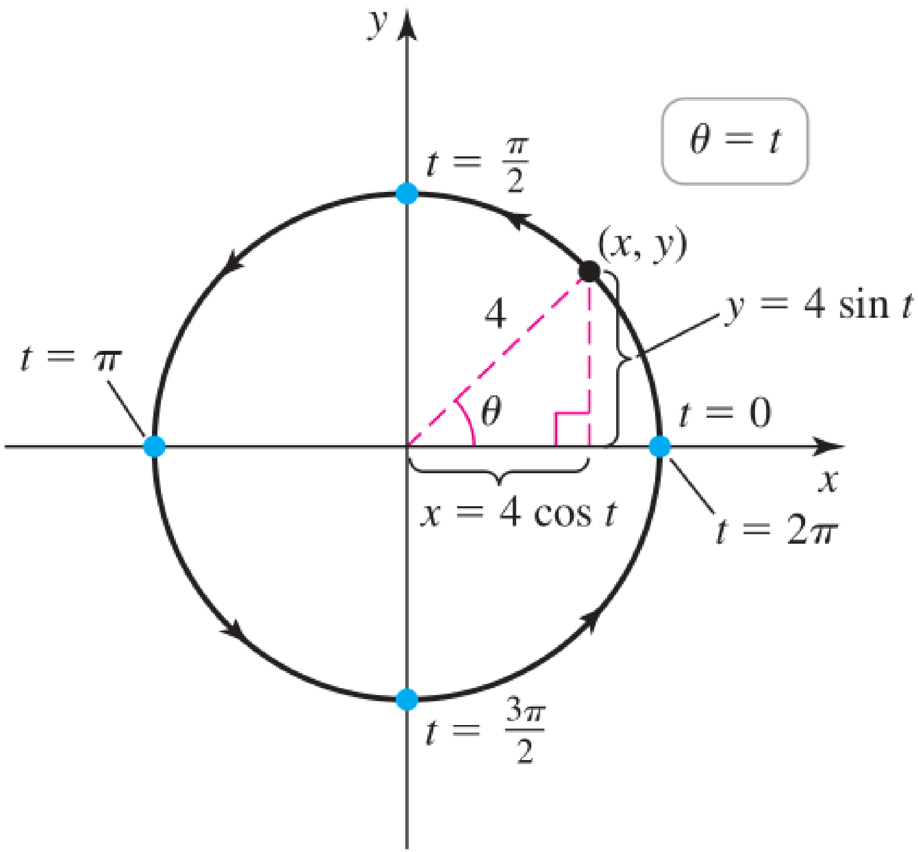
- **Parametric equations**
  - ✓ **to represent curves and trajectories in three-dimensional space (Chapter 13)**
- **Polar coordinate system**
  - ✓ **to work with objects that have circular, cylindrical, or spherical shapes**
- **Conic sections**
  - ✓ **areas of regions bounded by curves, e.g., ellipses, parabolas, and hyperbolas, in polar coordinates**

12.1

Parametric Equations

Basic Ideas

Example. A motor boat travels counterclockwise around a circular course with a radius of 4 miles, completing one lap every 2π hours at a constant speed



Parametric equations

$$x = 4\cos t$$
$$y = 4\sin t$$

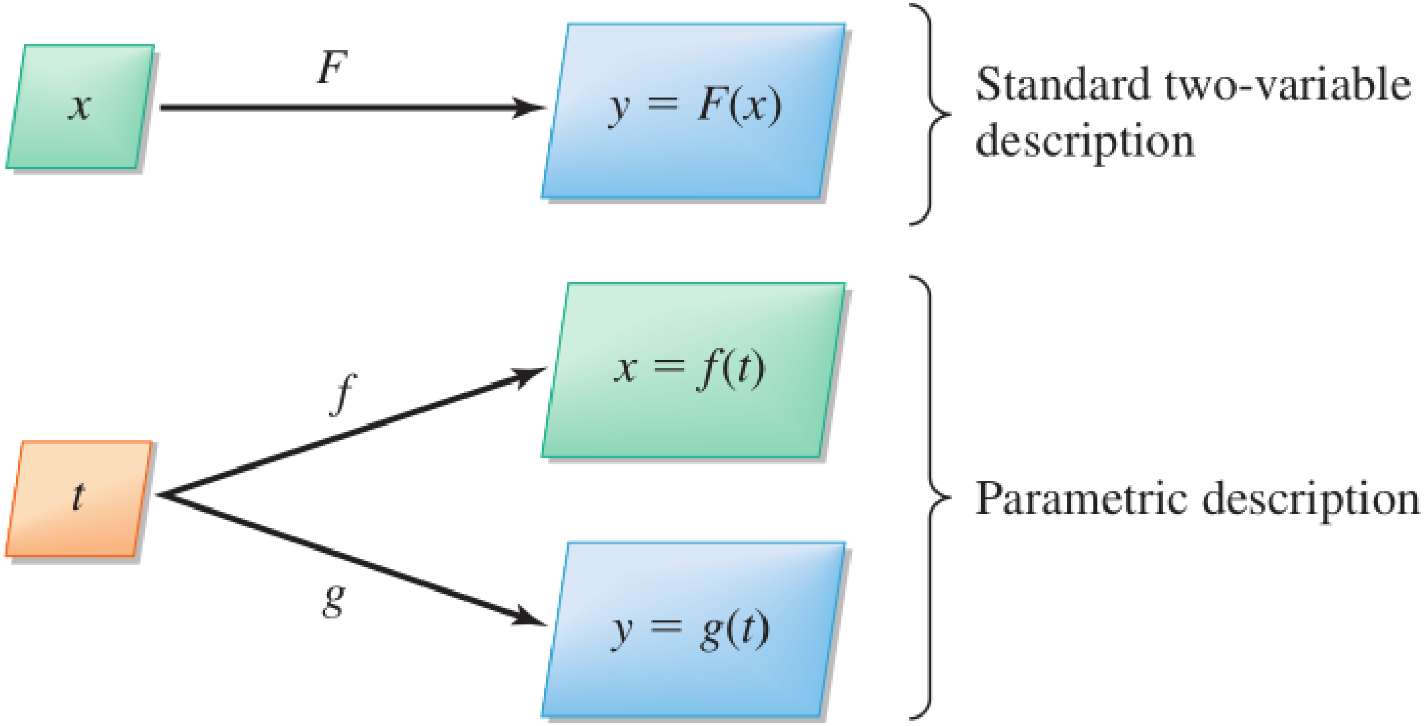
The third variable t is called a parameter

In general, parametric equations

$$x = f(t), y = g(t)$$

Parametric curve consists of the points in the plane

$$(x, y) = \big(f(t), g(t)\big), \text{for } a \leq t \leq b$$



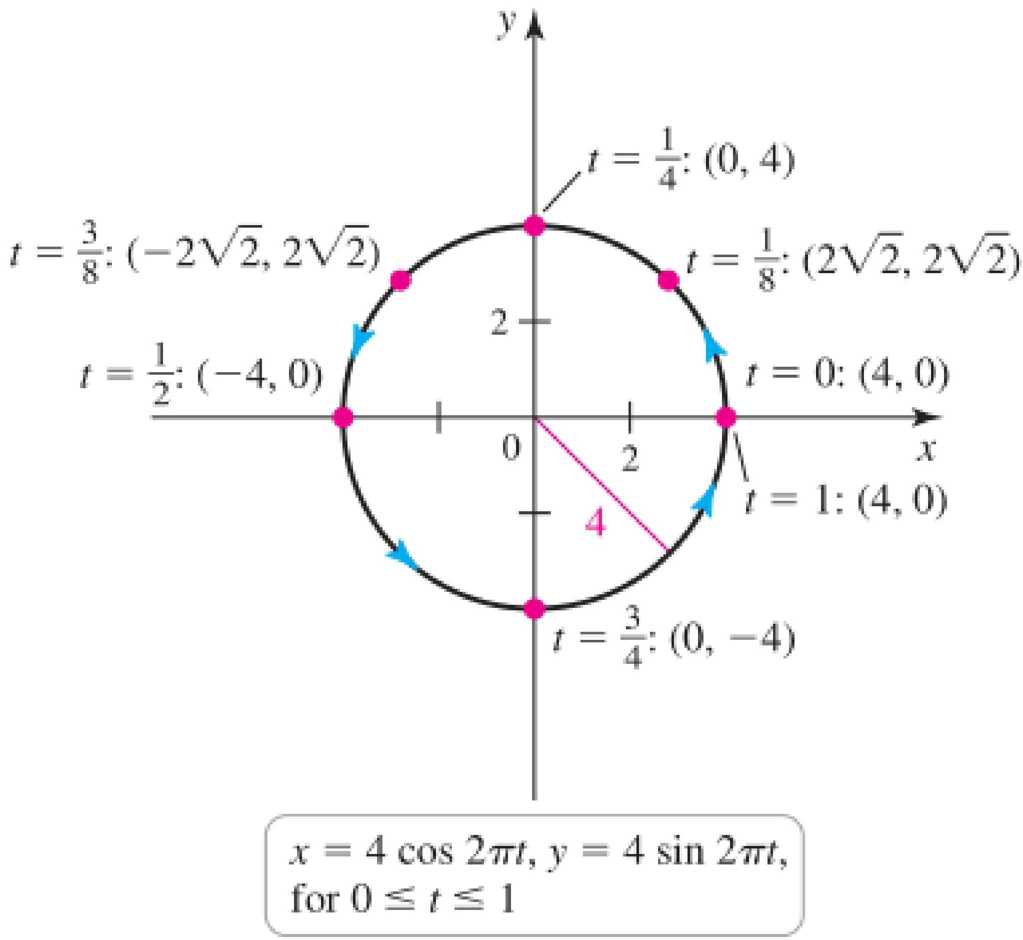
The corresponding curve unfolds in a particular direction as the parameter increases.

DEFINITION Positive Orientation

The direction in which a parametric curve is generated as the parameter increases is called the **positive orientation** of the curve (and is indicated by arrows on the curve).

EXAMPLE 2 **Parametric circle** Graph and analyze the parametric equations

$$x = 4\cos 2\pi t, \quad y = 4\sin 2\pi t, \quad \text{for } 0 \le t \le 1.$$



In general,

$$x = x_0 + a\cos bt\,, y = y_0 + a\sin bt$$

describes all or part of the circle

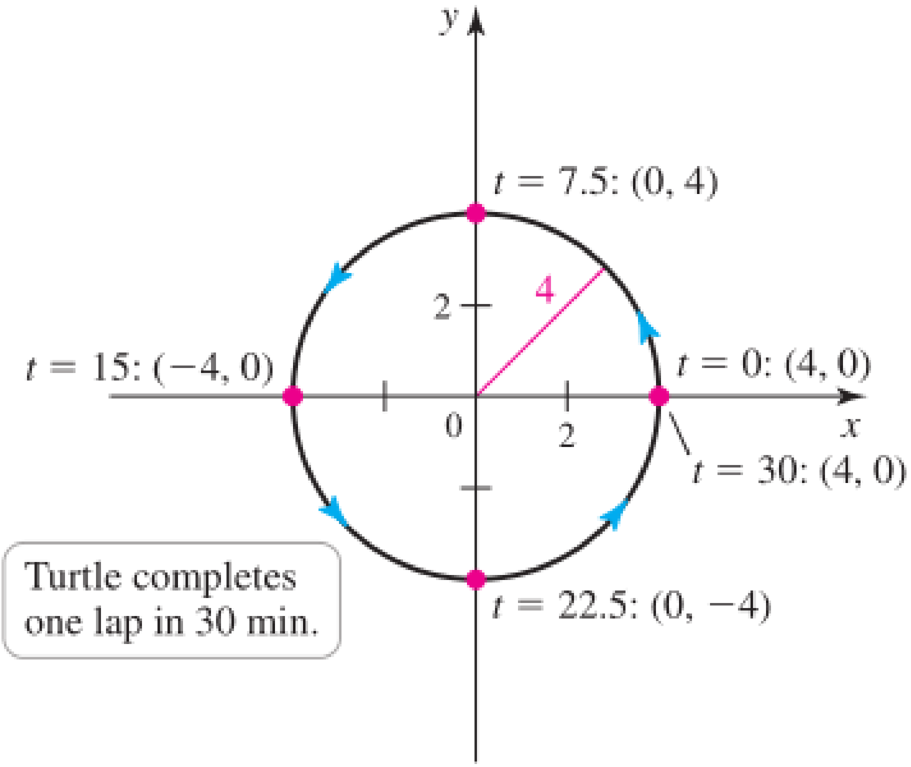
$$(x - x_0)^2 + (y - y_0)^2 = a^2$$

centered at (x_0, y_0) with radius $|a|$.

The circle is traversed once as t varies over any interval of length $2\pi/|b|$.

If $b > 0$ (< 0), then the circle is generated in the counterclockwise (clockwise) direction.

EXAMPLE 3 **Circular path** A turtle walks with constant speed in the counterclockwise direction on a circular track of radius 4 ft centered at the origin. Starting from the point $(4, 0)$, the turtle completes one lap in 30 minutes. Find a parametric description of the path of the turtle at any time $t \geq 0$, where t is measured in minutes.



$x = 4\cos\frac{\pi t}{15},\ y = 4\sin\frac{\pi t}{15},$
for $0 \leq t \leq 30$

SUMMARY Parametric Equation of a Line

The equations

$$x = x_0 + at, y = y_0 + bt, \quad \text{for } -\infty < t < \infty,$$

where x_0, y_0, a, and b are constants with $a \neq 0$, describe a line with slope $\frac{b}{a}$ passing through the point (x_0, y_0). If $a = 0$ and $b \neq 0$, the line is vertical.

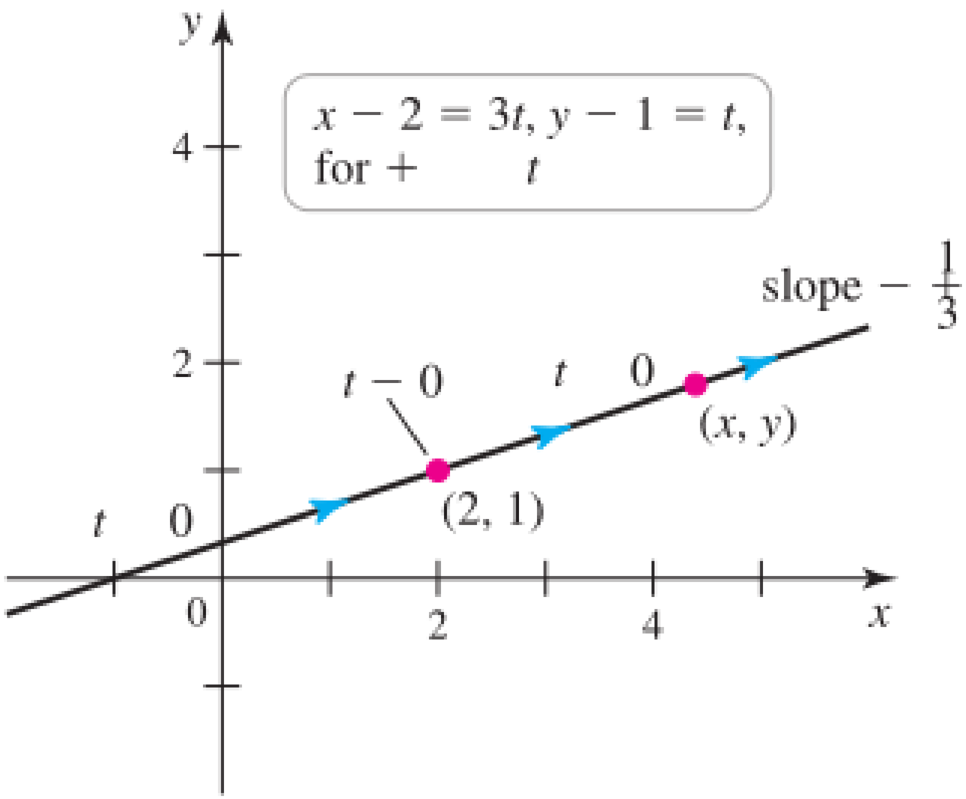
The parametric description of a given line is not unique.

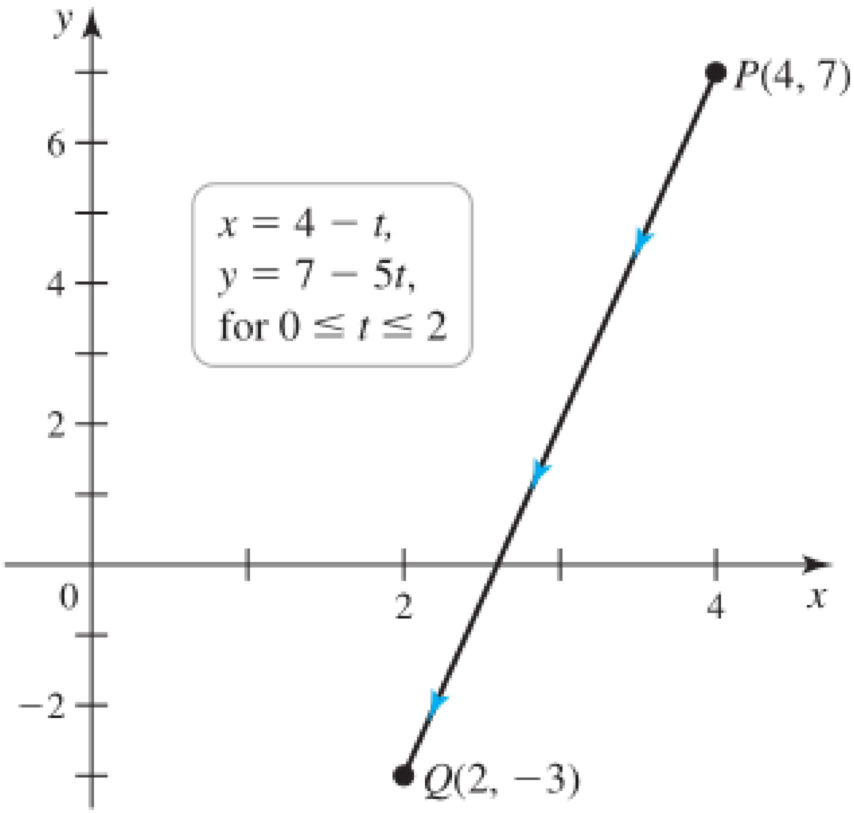
EXAMPLE 4 **Parametric equations of lines**

a. Consider the parametric equations $x = -2 + 3t$, $y = 4 - 6t$, for $-\neq < t < \neq$, which describe a line. Find the slope-intercept form of the line.

b. Find two pairs of parametric equations for the line with slope $\frac{1}{3}$ that passes through the point $(2, 1)$.

c. Find parametric equations for the line segment starting at $P(4, 7)$ and ending at $Q(2, -3)$.


$x - 2 = 3t, y - 1 = t,$
for + t
slope $-\frac{1}{3}$
$t - 0$
t 0
t 0
(2, 1)
(x, y)
y
x
0
2
4

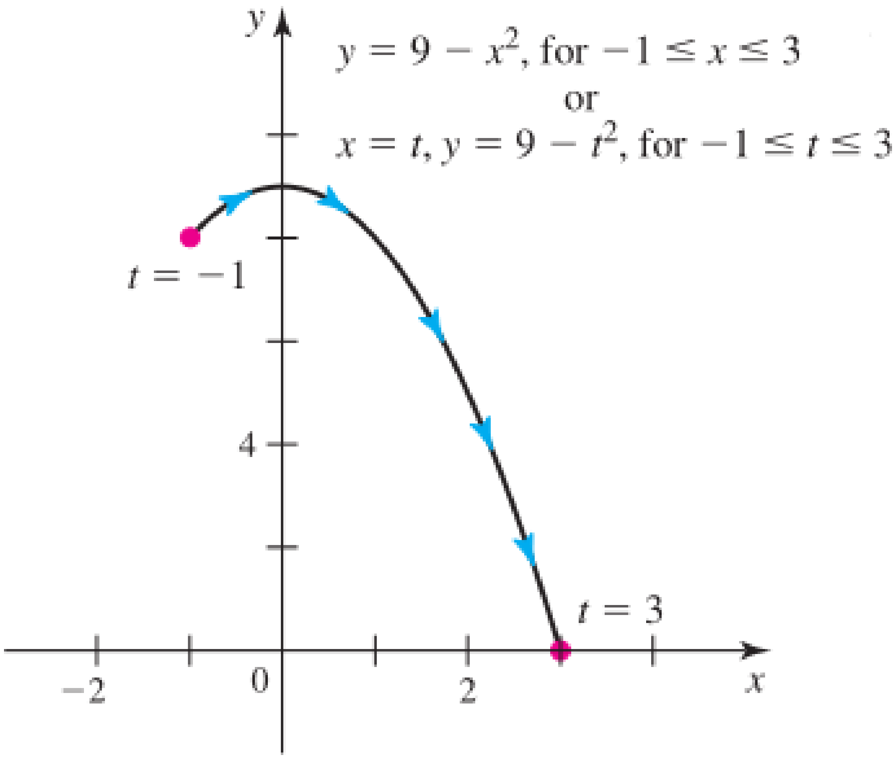

$x = 4 - t,$
$y = 7 - 5t,$
for $0 \le t \le 2$
P(4, 7)
Q(2, −3)
y
x
0
2
4
6
−2

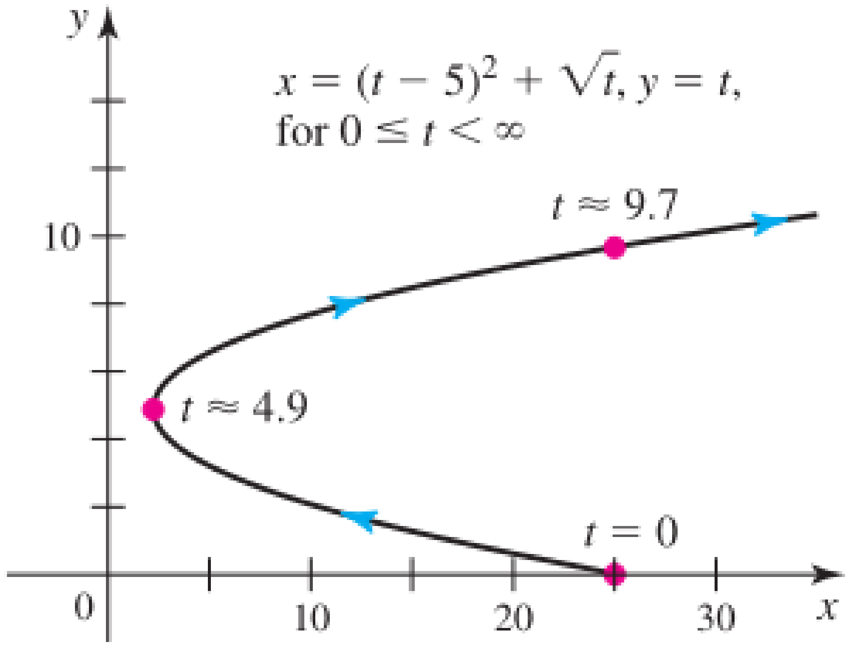
EXAMPLE 5 **Parametric equations of curves** A common task (particularly in upcoming chapters) is to parameterize curves given either by Cartesian equations or by graphs. Find a parametric representation of the following curves.

a. The segment of the parabola $y = 9 - x^2$, for $-1 \le x \le 3$

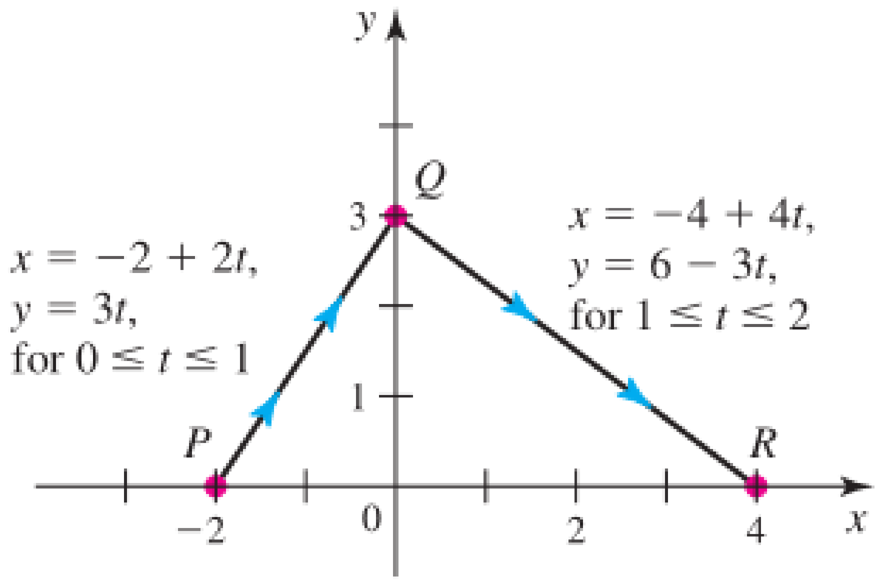
b. The complete curve $x = (y - 5)^2 + \sqrt{y}$

c. The piecewise linear path connecting $P(-2, 0)$ to $Q(0, 3)$ to $R(4, 0)$ (in that order), where the parameter varies over the interval $0 \le t \le 2$





y
$x = (t - 5)^2 + \sqrt{t}, y = t,$
for $0 \le t < \infty$
$t \approx 9.7$
10
$t \approx 4.9$
$t = 0$
0
10
20
30
x



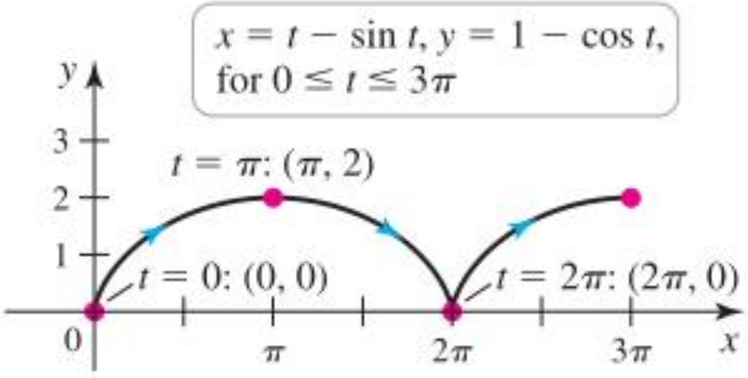
y
Q
3
$x = -4 + 4t,$
$y = 6 - 3t,$
for $1 \le t \le 2$
$x = -2 + 2t,$
$y = 3t,$
for $0 \le t \le 1$
1
P
R
−2
0
2
4
x

EXAMPLE 6 **Rolling wheels** The path of a light on the rim of a wheel rolling on a flat surface (Figure 10.11a) is a **cycloid**, which has the parametric equations

$$x = a(t - \sin t), \quad y = a(1 - \cos t), \quad \text{for } t \geq 0,$$

where $a > 0$. Use a graphing utility to graph the cycloid with $a = 1$. On what interval does the parameter generate the first arch of the cycloid?

(a)



(b)

Derivatives and Parametric Equations

The rate of change of y with respect to x at a point on a parametric curve: dy/dx

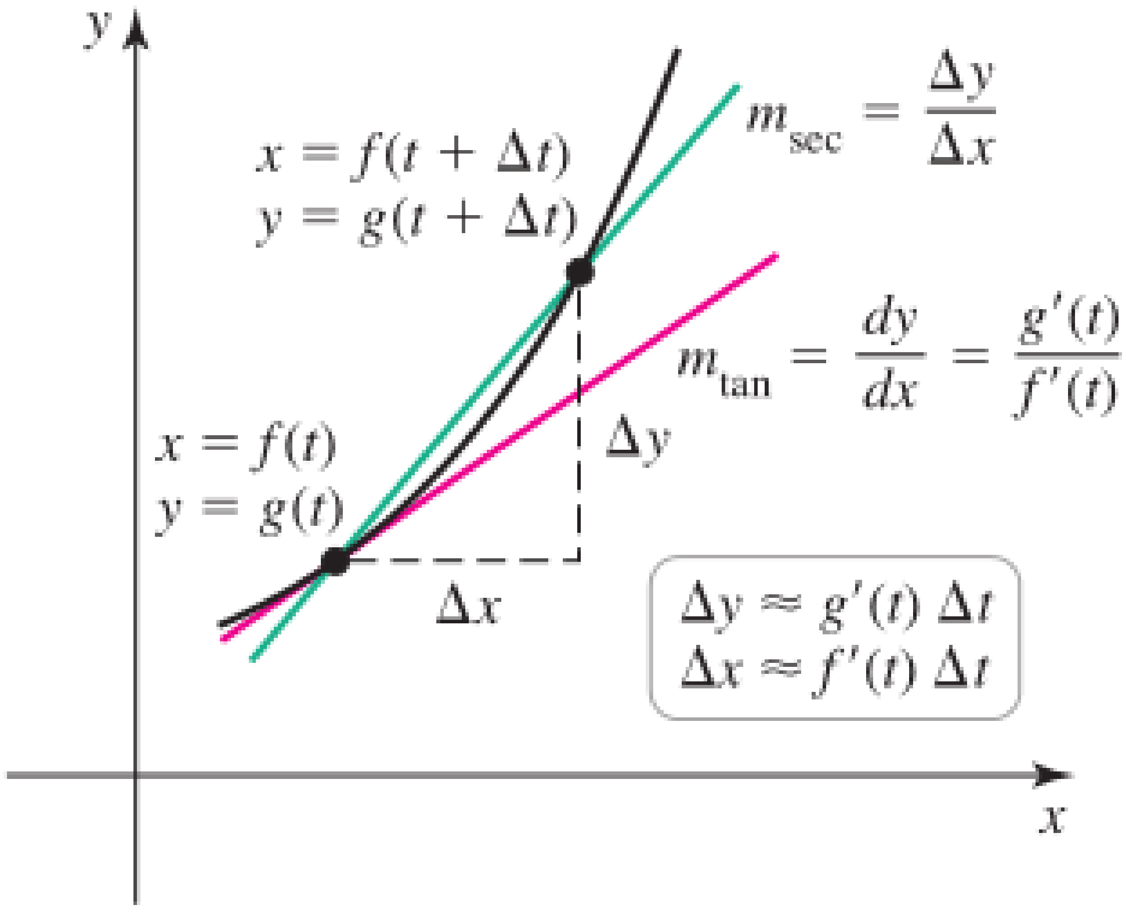
The Chain Rule:

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

THEOREM 1 Derivative for Parametric Curves
Let $x = f(t)$ and $y = g(t)$, where f and g are differentiable on an interval $[a, b]$. Then the slope of the line tangent to the curve at the point corresponding to t is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g'(t)}{f'(t)},$$

provided $f'(t) \neq 0$.

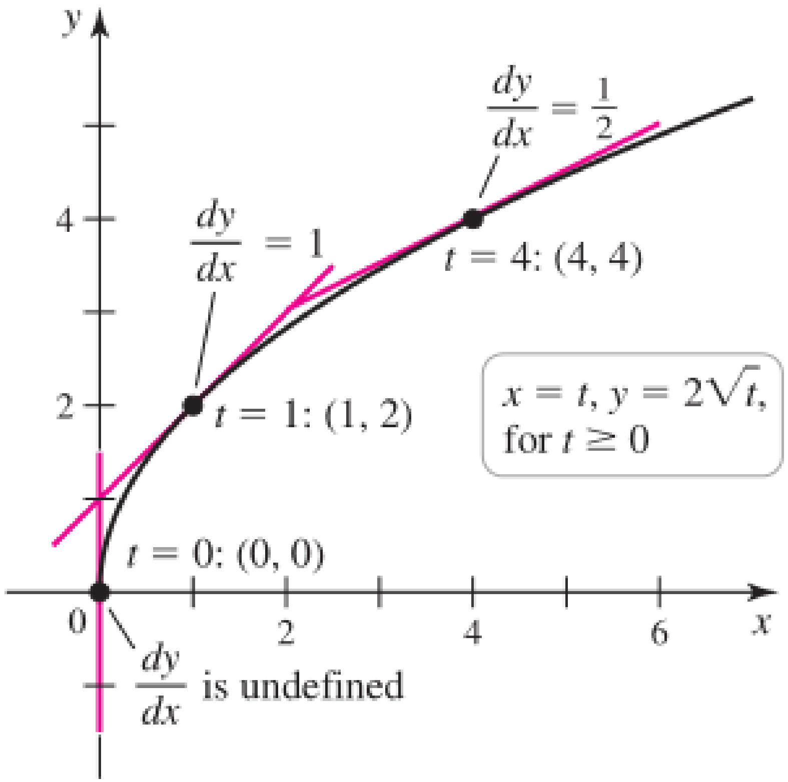


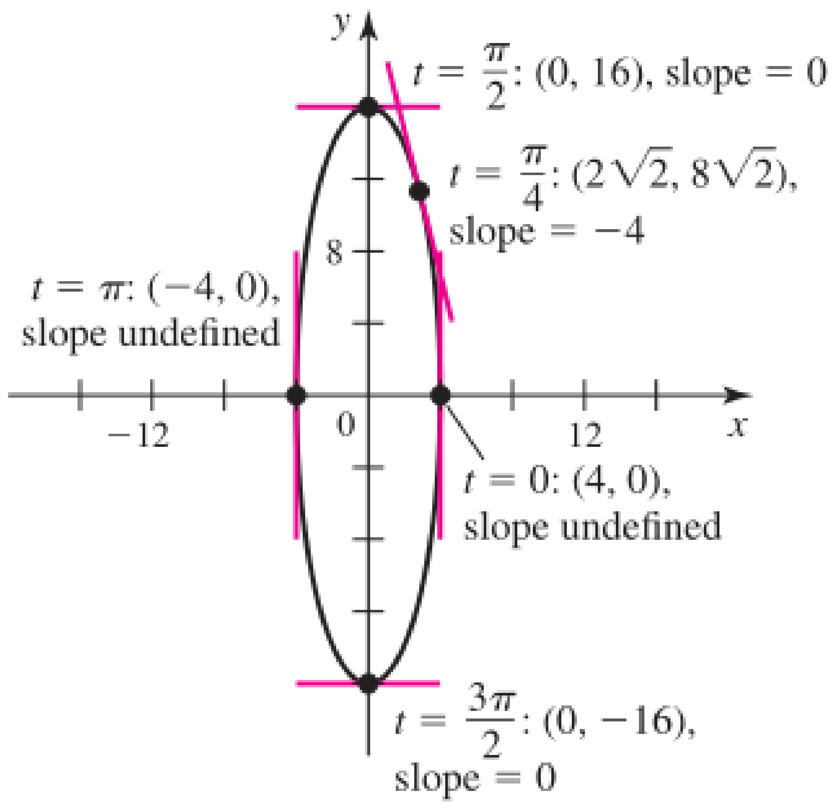
$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{g'(t)\Delta t}{f'(t)\Delta t} = \frac{g'(t)}{f'(t)}$$

EXAMPLE 8 **Slopes of tangent lines** Find $\frac{dy}{dx}$ for the following curves. Interpret the result and determine the points (if any) at which the curve has a horizontal or a vertical tangent line.

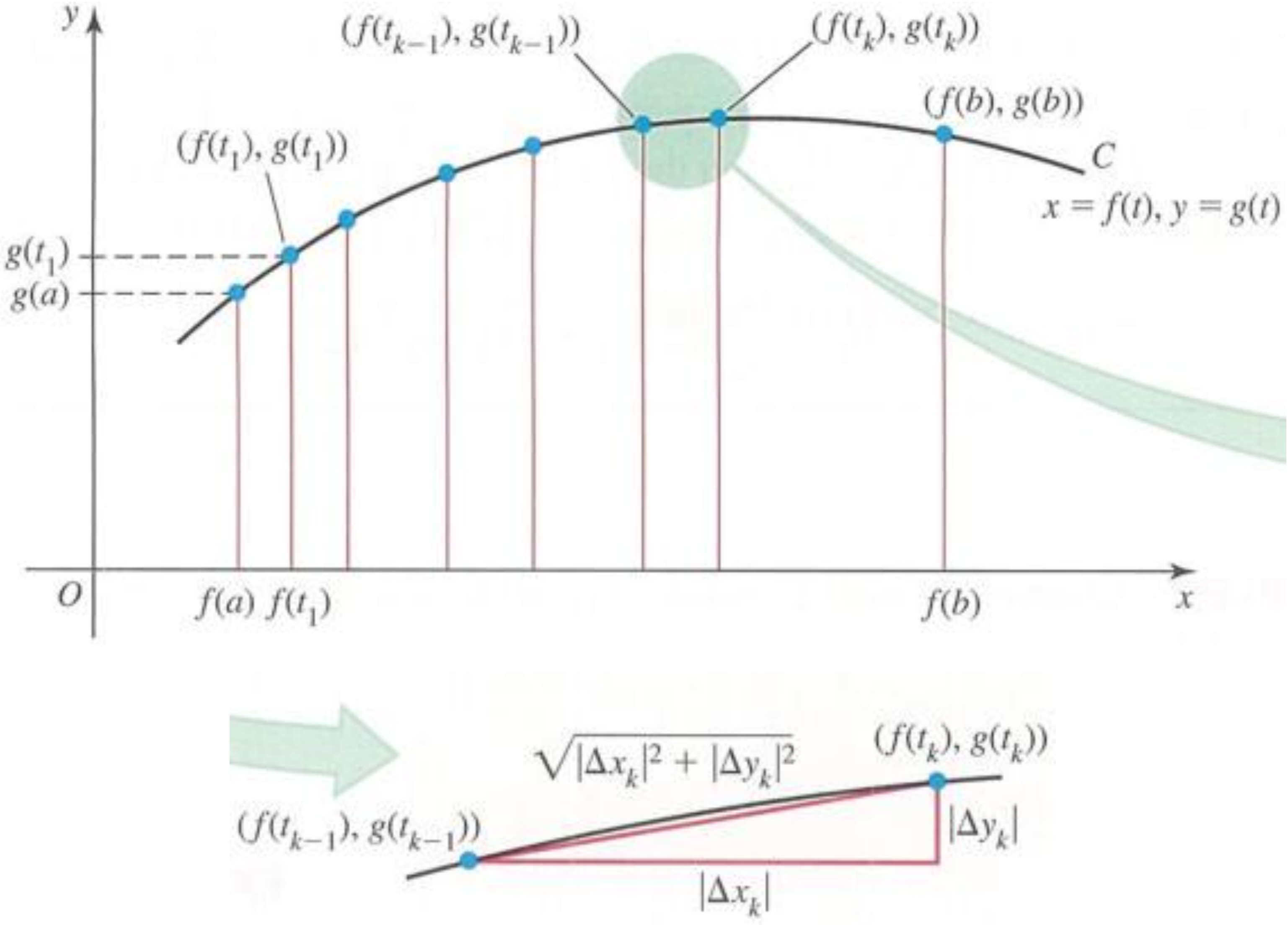
a. $x = f(t) = t, y = g(t) = 2\sqrt{t}$, for $t \geq 0$

b. $x = f(t) = 4\cos t, y = g(t) = 16\sin t$, for $0 \leq t \leq 2\pi$





Arc Length



y
$(f(t_{k-1}), g(t_{k-1}))$
$(f(t_k), g(t_k))$
$(f(b), g(b))$
$(f(t_1), g(t_1))$
C
$x = f(t), y = g(t)$
$g(t_1)$
$g(a)$
O
$f(a)$ $f(t_1)$
$f(b)$
x
$\sqrt{|\Delta x_k|^2 + |\Delta y_k|^2}$
$(f(t_k), g(t_k))$
$(f(t_{k-1}), g(t_{k-1}))$
$|\Delta y_k|$
$|\Delta x_k|$

So, $L \approx \sum_{k=1}^{n} \sqrt{|\Delta x_k|^2 + |\Delta y_k|^2}$

By Mean Value Theorem

$$\frac{\Delta x_k}{\Delta t_k} = \frac{f(t_k) - f(t_{k-1})}{t_k - t_{k-1}} = f'(t_k^*)$$

i.e., $\Delta x_k = f'(t_k^*)\Delta t_k$

Similarly, $\Delta y_k = f'(\hat{t}_k)\Delta t_k$

So, $L \approx \sum_{k=1}^{n} \sqrt{(f'(t_k^*)\Delta t_k)^2 + (f'(\hat{t}_k)\Delta t_k)^2}$

$$= \sum_{k=1}^{n} \sqrt{\left(f'(t_k^*)\right)^2 + \left(f'(\hat{t}_k)\right)^2}\ \Delta t_k$$

Take limit, $L = \lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{\left(f'(t_k^*)\right)^2 + \left(f'(\hat{t}_k)\right)^2}\ \Delta t_k$

$$= \int_a^b \sqrt{\left(f'(t)\right)^2 + \left(f'(t)\right)^2}dt$$

DEFINITION Arc Length for Curves Defined by Parametric Equations

Consider the curve described by the parametric equations $x = f(t), y = g(t)$, where f' and g' are continuous, and the curve is traversed once for $a \le t \le b$. The **arc length** of the curve between $(f(a), g(a))$ and $(f(b), g(b))$ is

$$L = \int_a^b \sqrt{f'(t)^2 + g'(t)^2}\, dt.$$

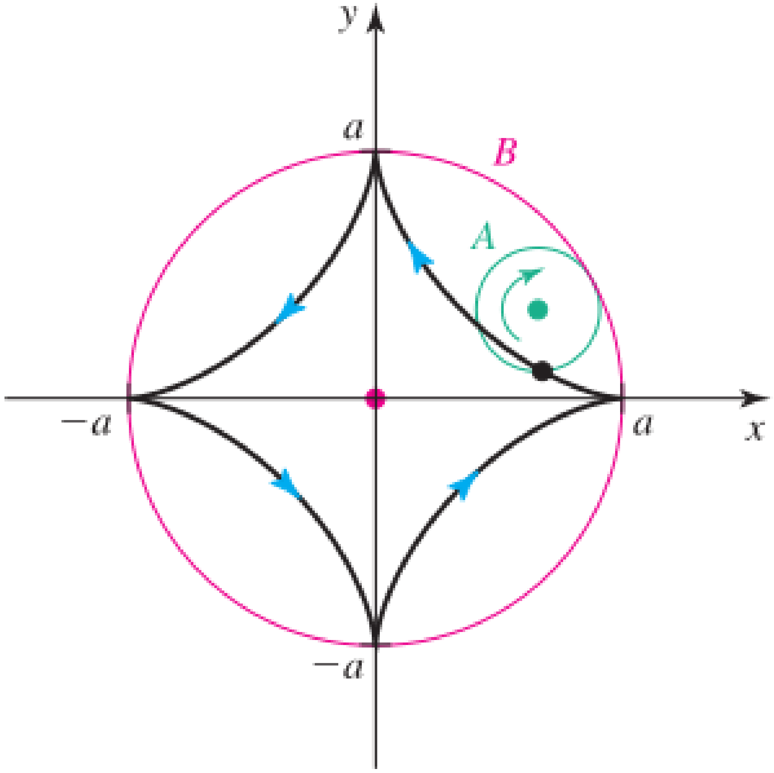
EXAMPLE 7 **Circumference of a circle** Prove that the circumference of a circle of radius $a > 0$ is $2\pi a$.

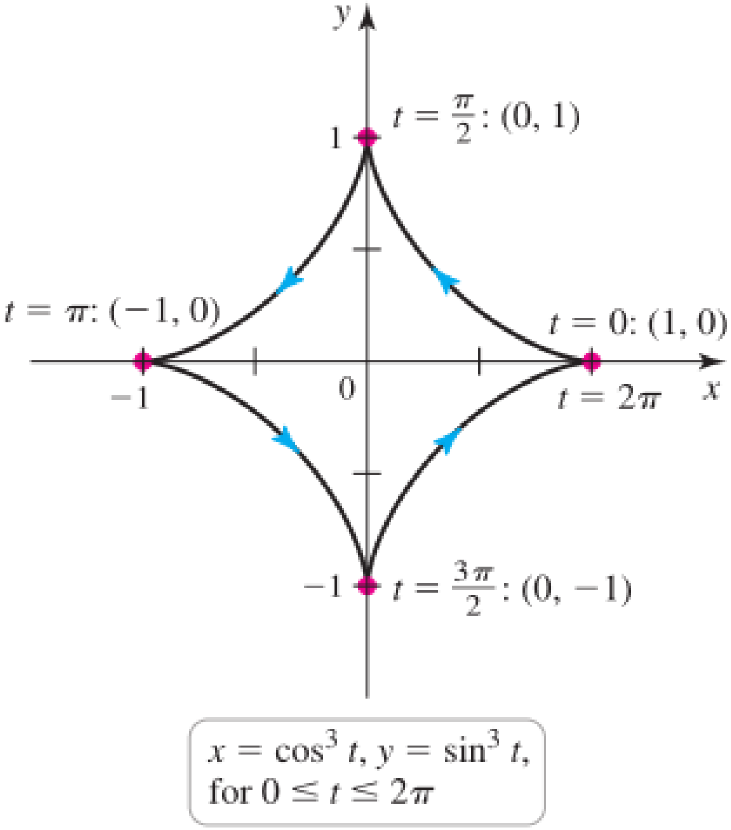
EXAMPLE 7 **More rolling wheels** The path of a point on circle A with radius $a/4$ that rolls on the inside of circle B with radius a (Figure 10.12) is an **astroid** or a **hypocycloid**. Its parametric equations are

$$x = a\cos^3 t, \quad y = a\sin^3 t, \quad \text{for } 0 \le t \le 2\pi.$$

Graph the astroid with $a = 1$ and find its equation in terms of x and y.

b. Find the arc length of the astroid with $a = 1$.



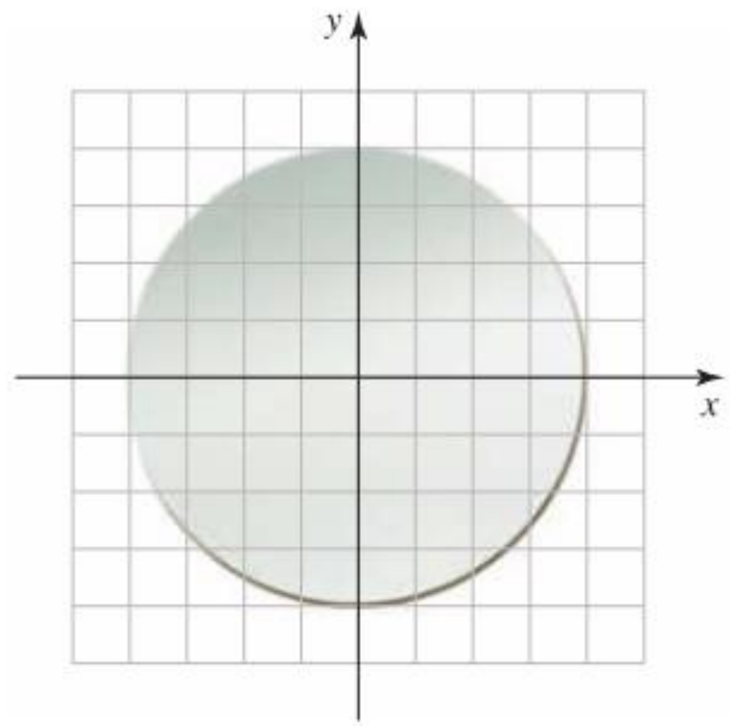


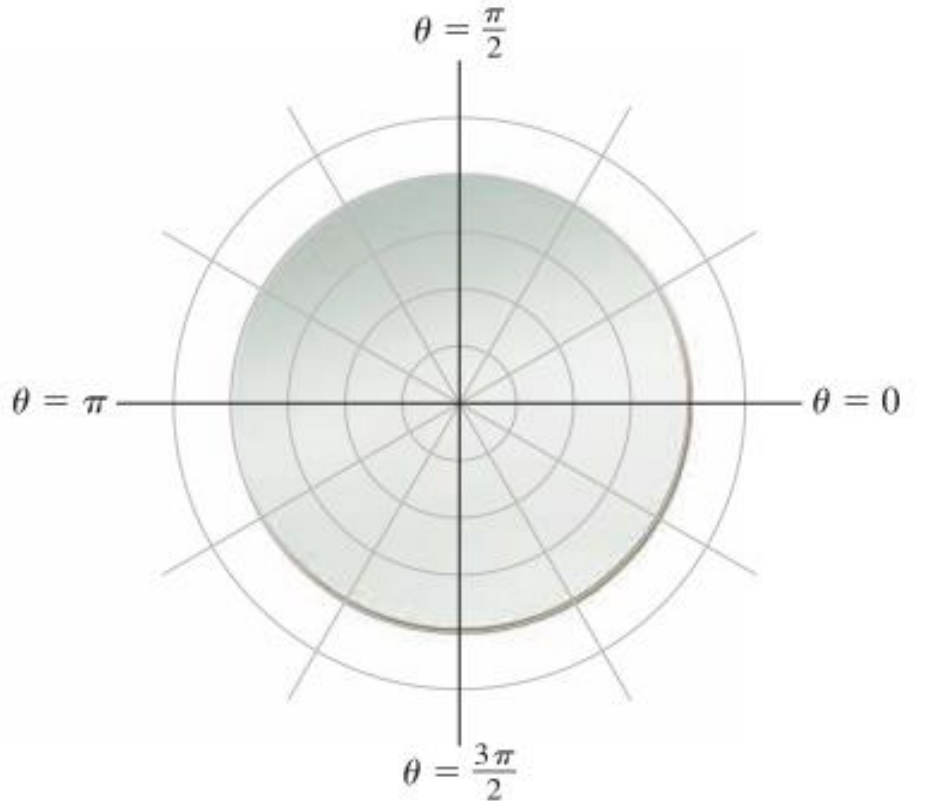
12.2

Polar Coordinates

Defining Polar Coordinates

Polar coordinate system, in which the coordinates are constant on circles and rays, is better suited for studying circular objects





Defining Polar Coordinates

Pole: the origin of the coordinate system

Polar axis: the positive x-axis

Point P in polar coordinates have the form (r, θ)

Radial coordinate r describes the signed (or directed) distance from the origin to P.

Angular coordinate θ describes an angle whose initial side is

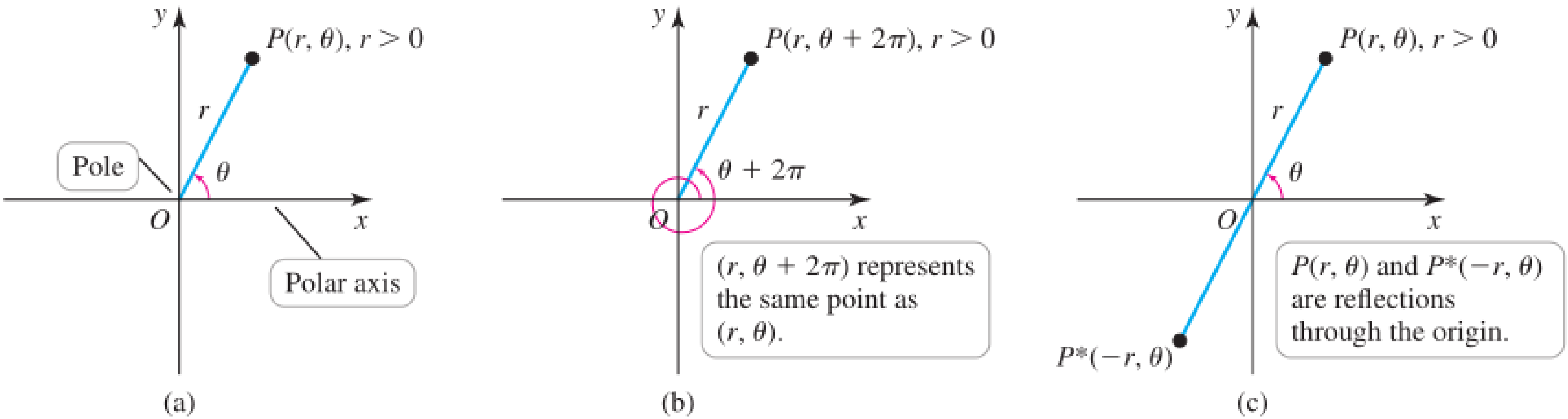
the positive x-axis and whose terminal side lies on the ray passing through the origin and P.

Positive angles are measured counterclockwise from the positive

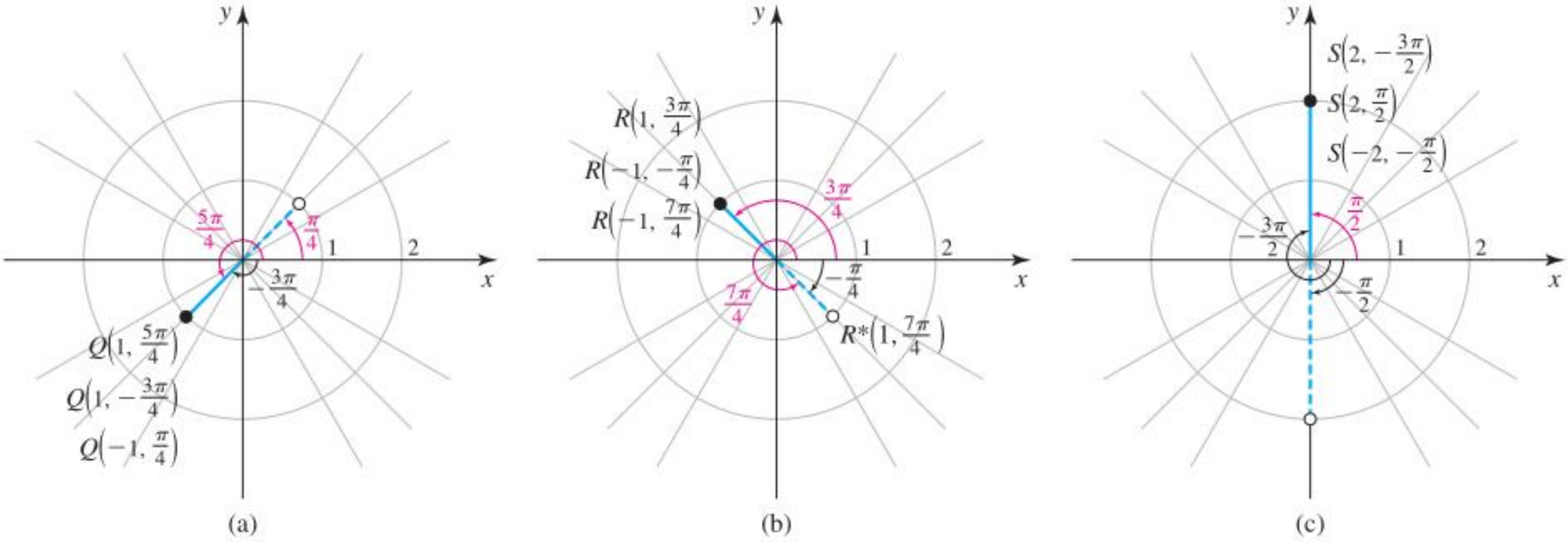
x-axis

Points in polar coordinates have more than one representation for two reasons.

- Angles are determined up to multiples of 2π radians
- The radial coordinate may be negative
- E.g., $(r, \theta\pi)$, $(-r, \theta + \pi)$ and $(-r, \theta - \pi)$ all refer to the same point



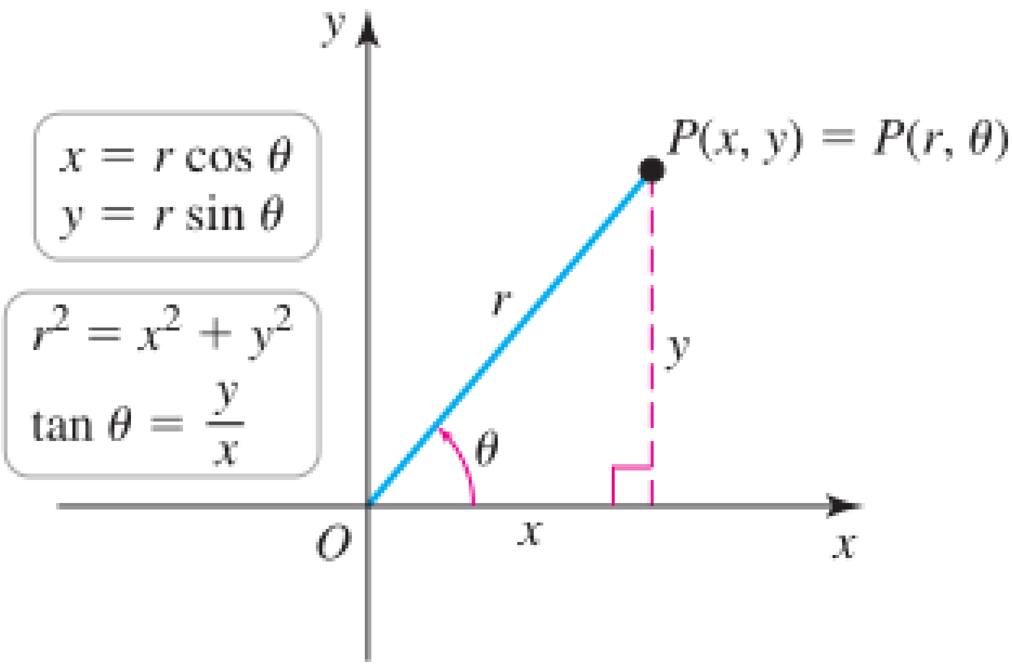
EXAMPLE 1 **Points in polar coordinates** Graph the following points in polar coordinates: $Q\left(1, \frac{5\pi}{4}\right)$, $R\left(-1, \frac{7\pi}{4}\right)$, and $S\left(2, -\frac{3\pi}{2}\right)$. Give two alternative representations for each point.



Converting Between Cartesian and Polar Coordinates

The conversion equations:

$$\cos\theta = \frac{x}{r}, \sin\theta = \frac{y}{r}$$



PROCEDURE **Converting Coordinates**

A point with polar coordinates (r, θ) has Cartesian coordinates (x, y), where

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta.$$

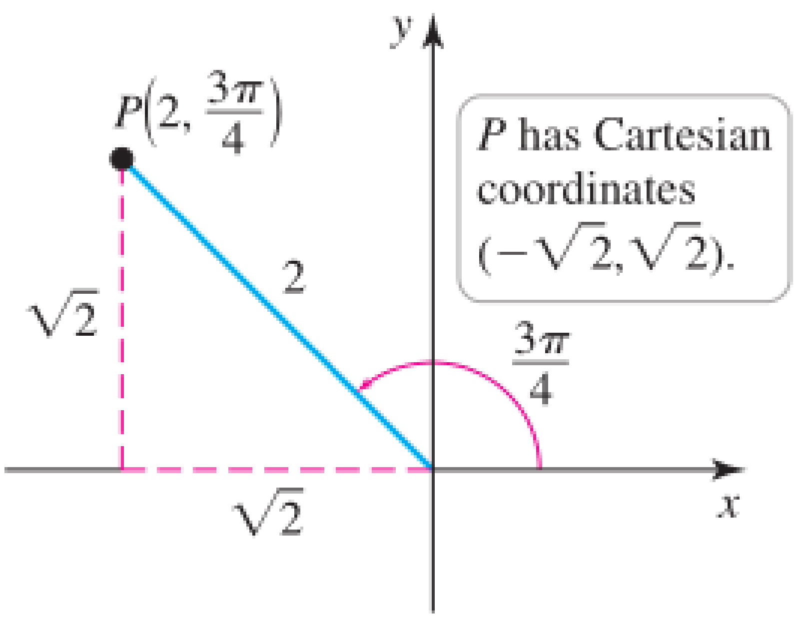
A point with Cartesian coordinates (x, y) has polar coordinates (r, θ), where

$$r^2 = x^2 + y^2 \quad \text{and} \quad \tan\theta = \frac{y}{x}.$$

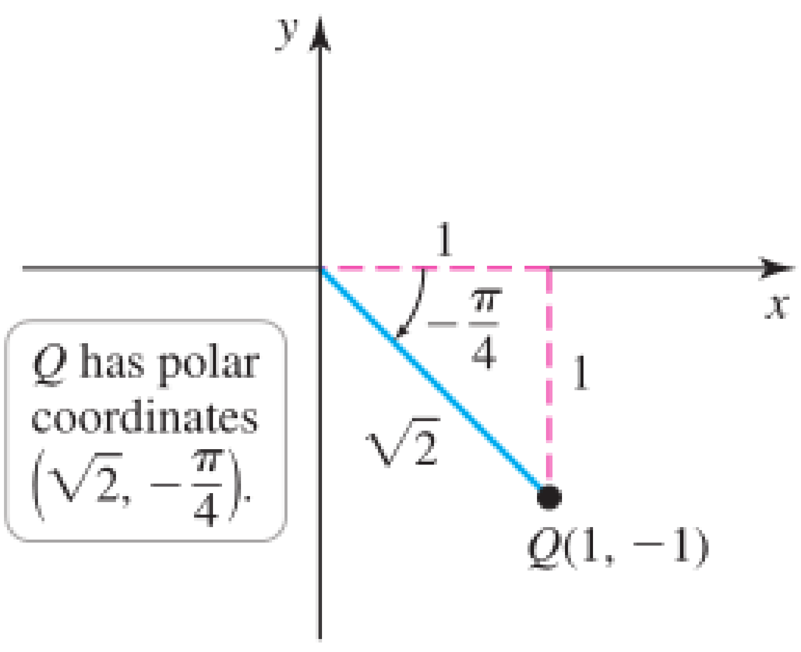
EXAMPLE 2 Converting coordinates

a. Express the point with polar coordinates $P\left(2, \frac{3\pi}{4}\right)$ in Cartesian coordinates.

b. Express the point with Cartesian coordinates $Q(1, -1)$ in polar coordinates.



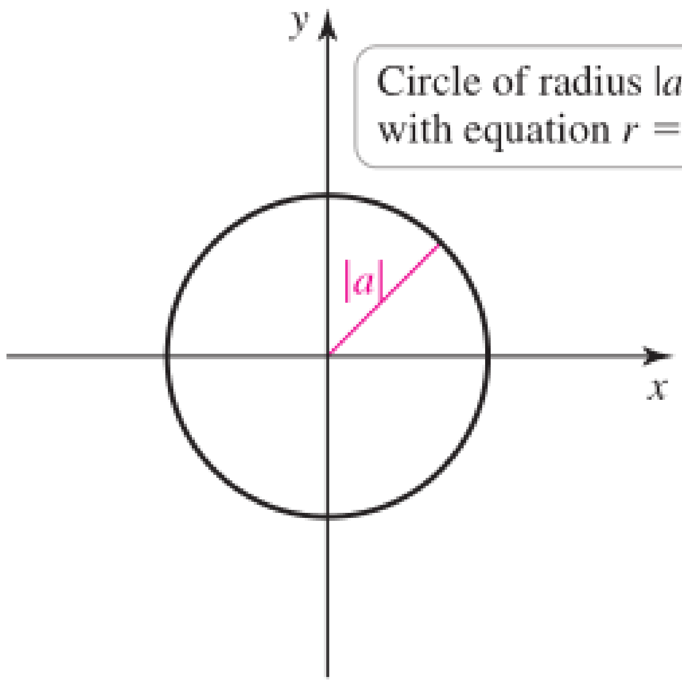
(a)



(b)

Basic Curves in Polar Coordinates

A curve in polar coordinates is the set of points that satisfy an equation in r and θ.



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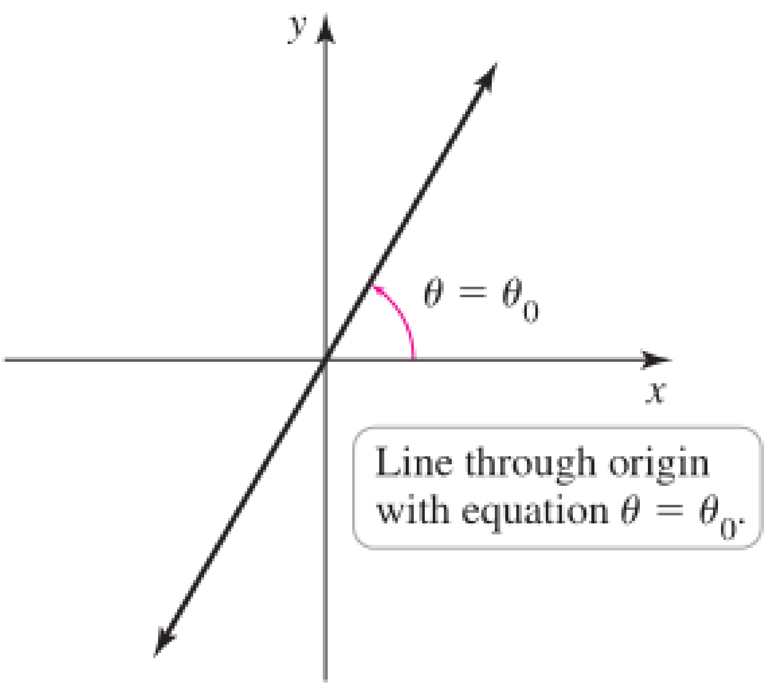

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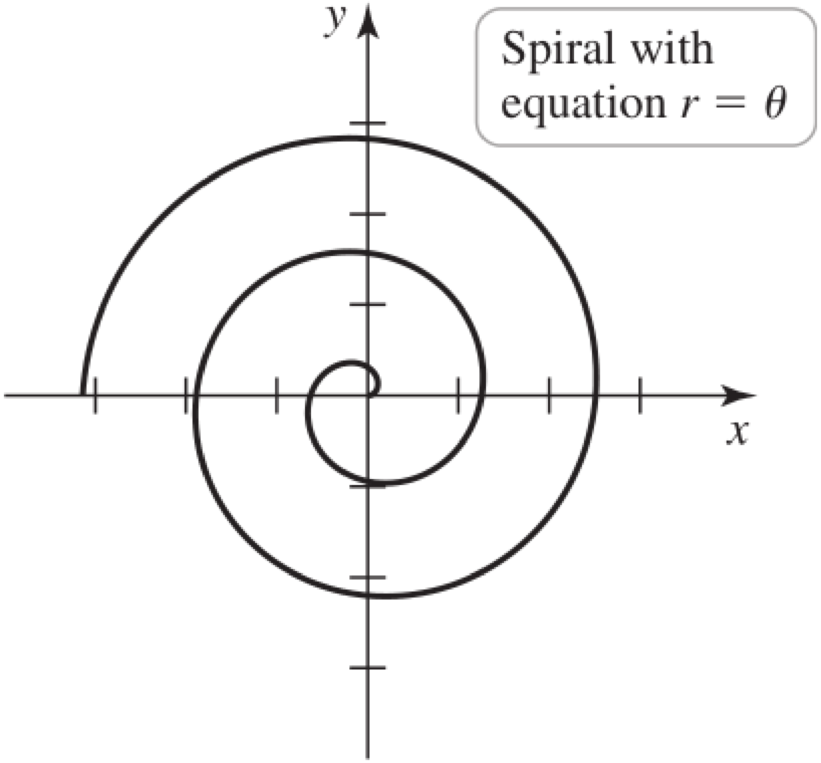
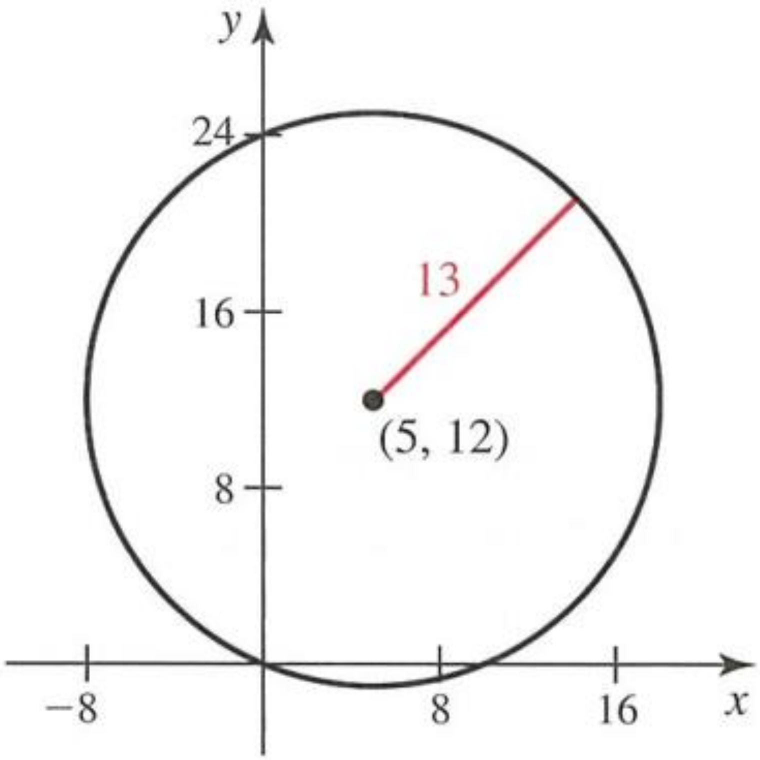

y
x
Spiral with
equation $r = \theta$

EXAMPLE 3 **Polar to Cartesian coordinates** Convert the polar equation $r = 10\cos\theta + 24\sin\theta$ to Cartesian coordinates and describe the corresponding graph.

$$\underbrace{r^2}_{x^2+y^2} = \underbrace{10r\cos\theta}_{10x} + \underbrace{24r\sin\theta}_{24y}$$

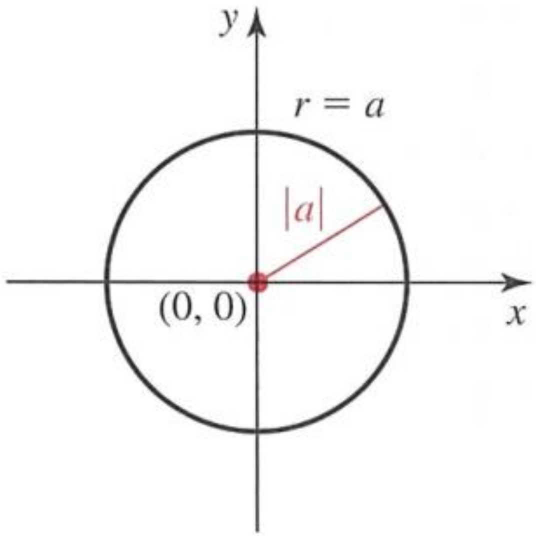
$$(x-5)^2+(y-12)^2=169$$

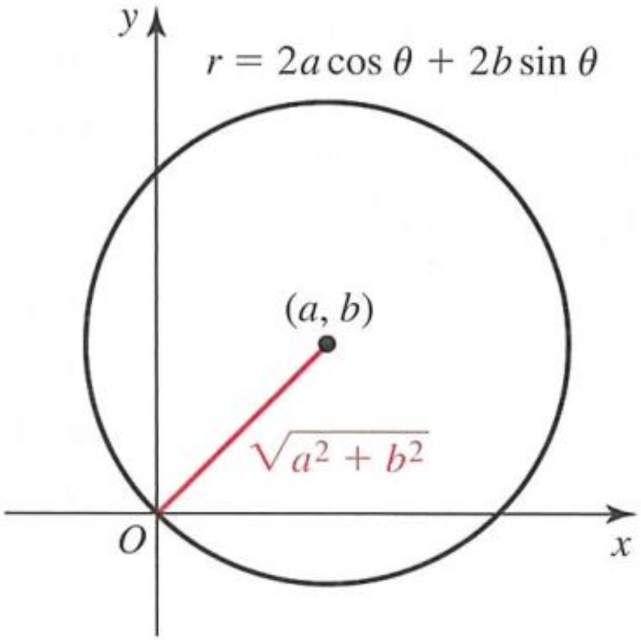


SUMMARY Circles in Polar Coordinates

The equation $r = a$ describes a circle of radius $|a|$ centered at $(0, 0)$.

The equation $r = 2a\cos\theta + 2b\sin\theta$ describes a circle of radius $\sqrt{a^2 + b^2}$ centered at (a, b).





Two important special cases: $b = 0$ or $a = 0$.

Graphing in Polar Coordinates

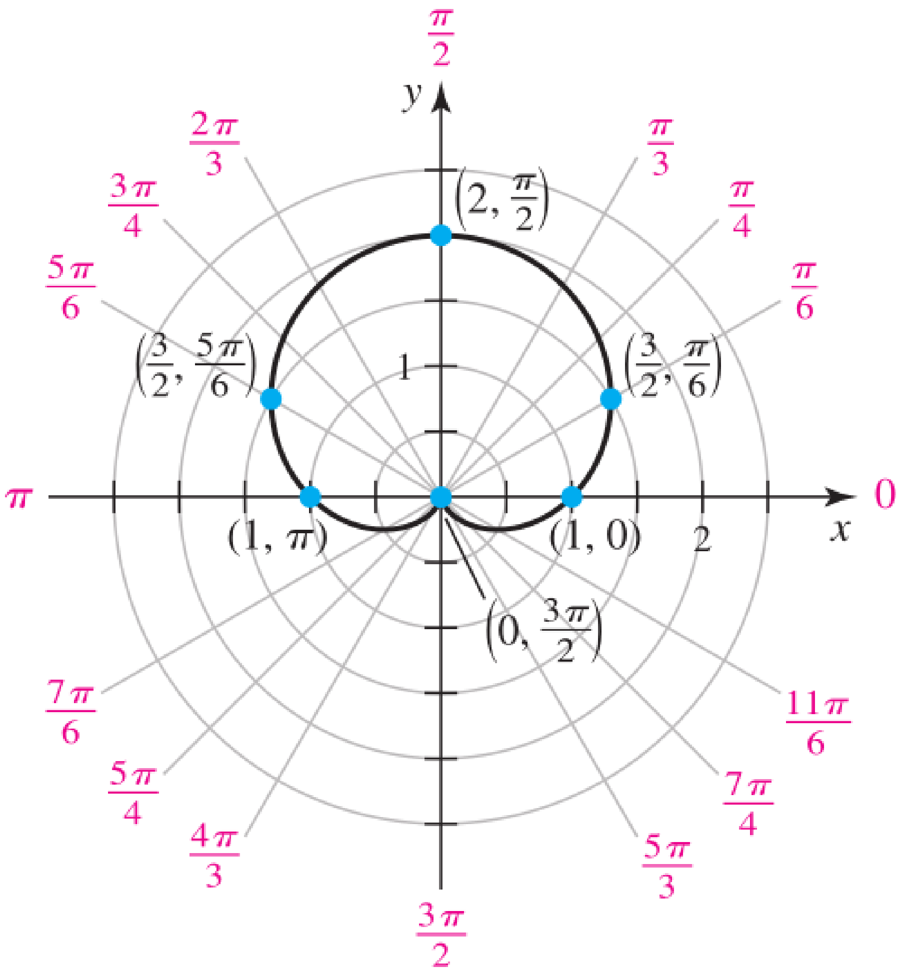
Equations in polar coordinates often describe curves that are difficult to represent in Cartesian coordinates.

The easiest graphing method is to choose several values of θ, calculate the corresponding r-values, and tabulate the coordinates.

EXAMPLE 4 **Plotting a polar curve** Graph the polar equation $r = f(\theta) = 1 + \sin\theta$.

Table 3

θ	$r = 1 + \sin\theta$
0	1
$\pi/6$	$3/2$
$\pi/2$	2
$5\pi/6$	$3/2$
π	1
$7\pi/6$	$1/2$
$3\pi/2$	0
$11\pi/6$	$1/2$
2π	1



Cardioid $r = 1 + \sin\theta$

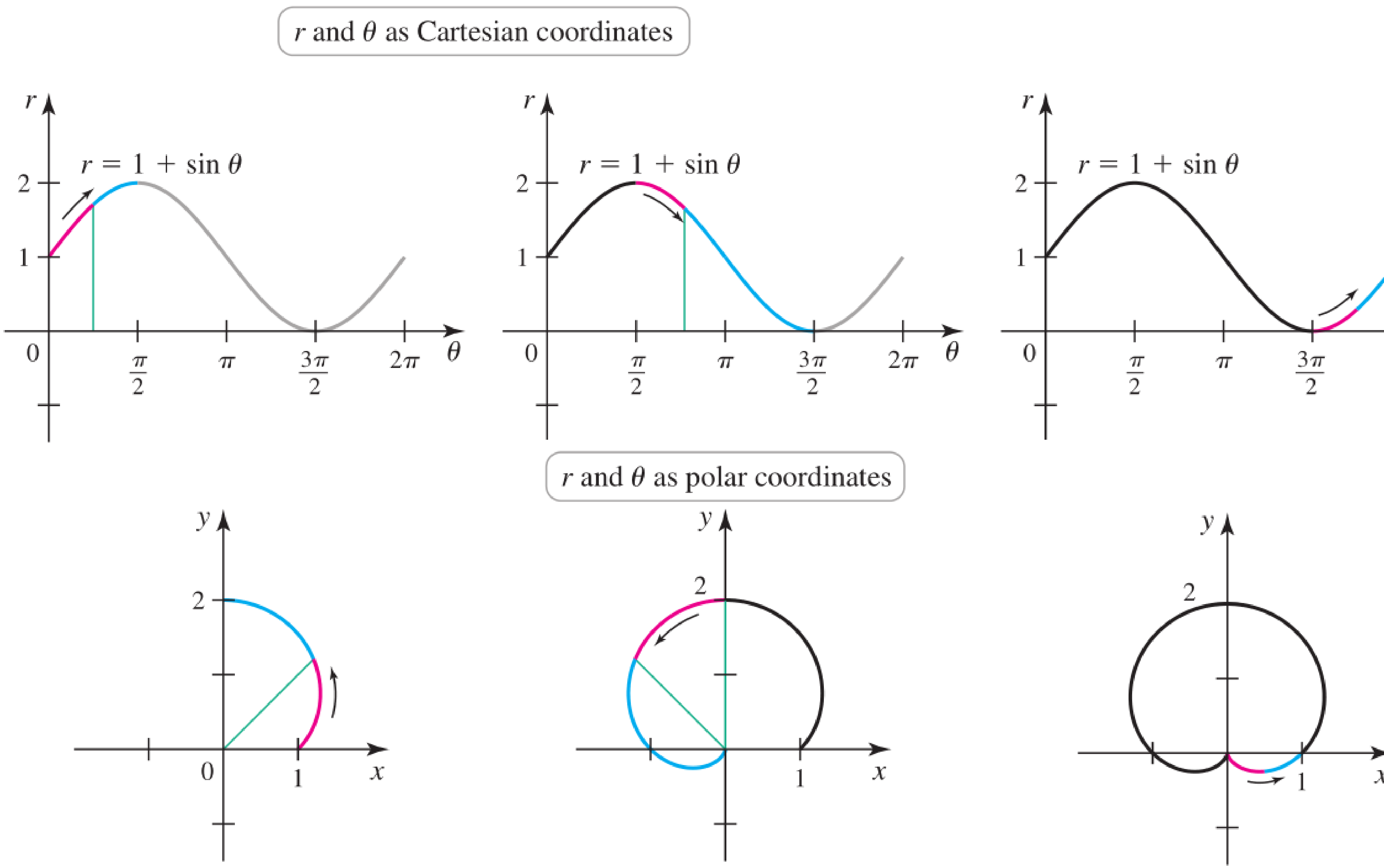
Cardioid, symmetric about the y-axis

Cartesian-to-Polar Method

PROCEDURE **Cartesian-to-Polar Method for Graphing $r = f(\theta)$**

1. Graph $r = f(\theta)$ *as if r and θ were Cartesian coordinates* with θ on the horizontal axis and r on the vertical axis. Be sure to choose an interval for θ on which the entire polar curve is produced.
2. Use the Cartesian graph in Step 1 as a guide to sketch the points (r, θ) on the final *polar* curve.

EXAMPLE 5 **Plotting polar graphs** Use the Cartesian-to-polar method to graph the polar equation $r = 1 + \sin\theta$ (Example 4).



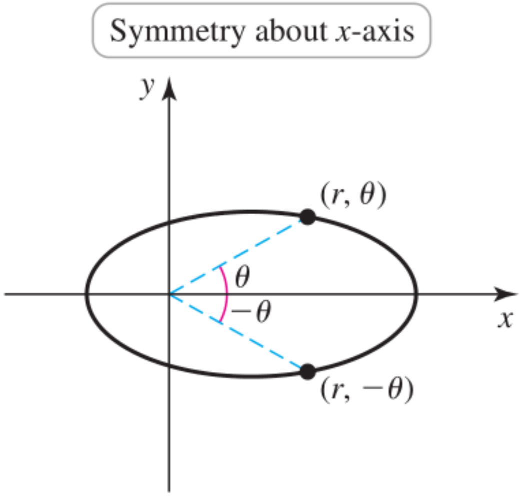
Symmetry

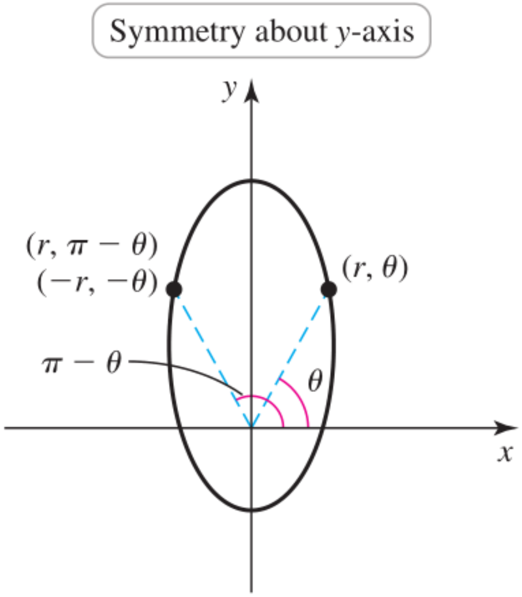
SUMMARY Symmetry in Polar Equations

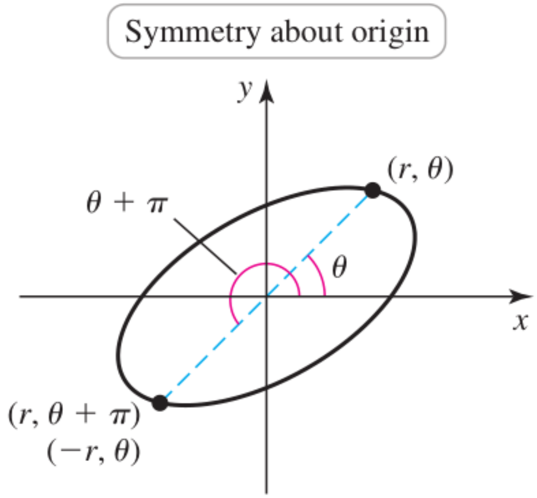
Symmetry about the x-axis occurs if the point (r, θ) is on the graph whenever $(r, -\theta)$ is on the graph.

Symmetry about the y-axis occurs if the point (r, θ) is on the graph whenever $(r, \pi - \theta) = (-r, -\theta)$ is on the graph.

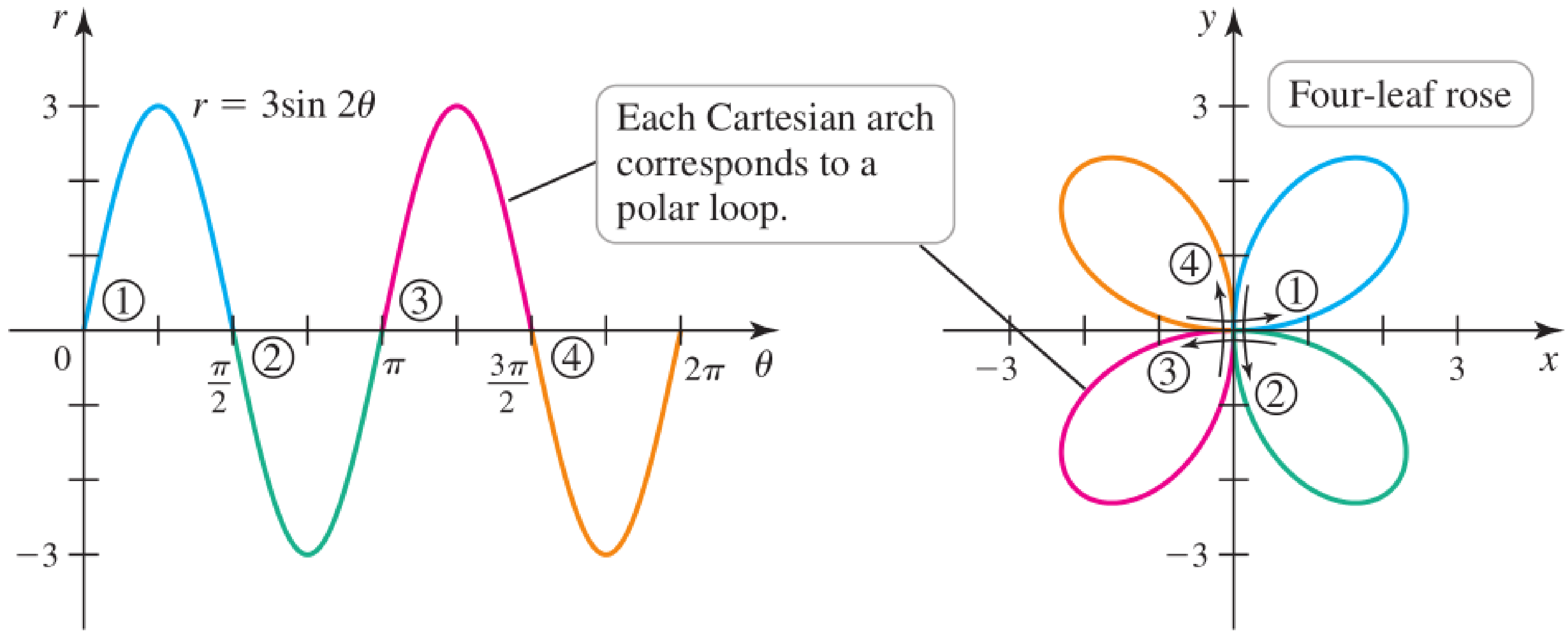
Symmetry about the origin occurs if the point (r, θ) is on the graph whenever $(-r, \theta) = (r, \theta + \pi)$ is on the graph.







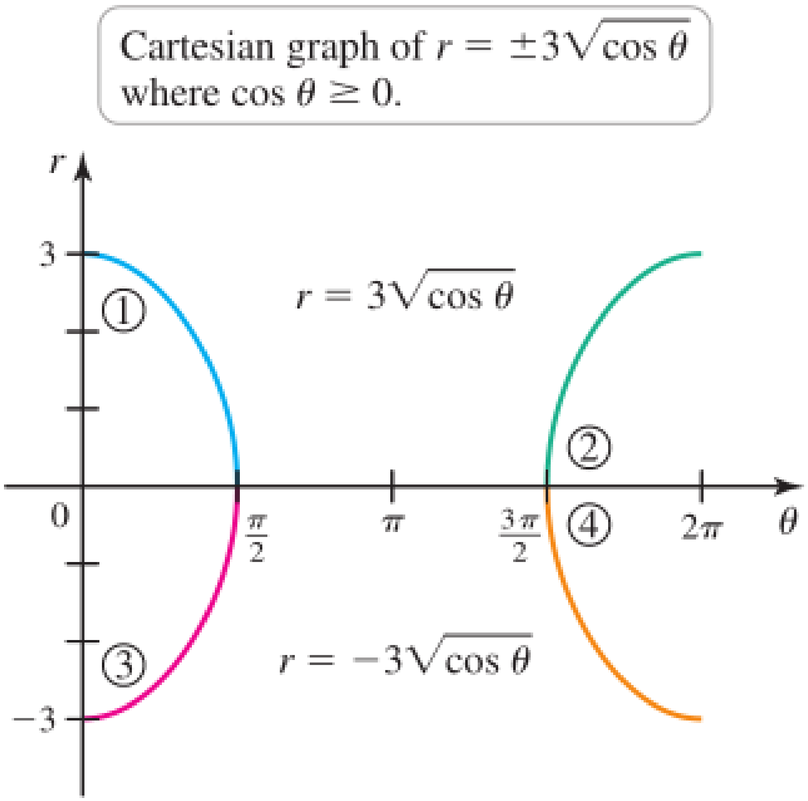
EXAMPLE 6 **Plotting polar graphs** Graph the polar equation $r = 3 \sin 2\theta$.



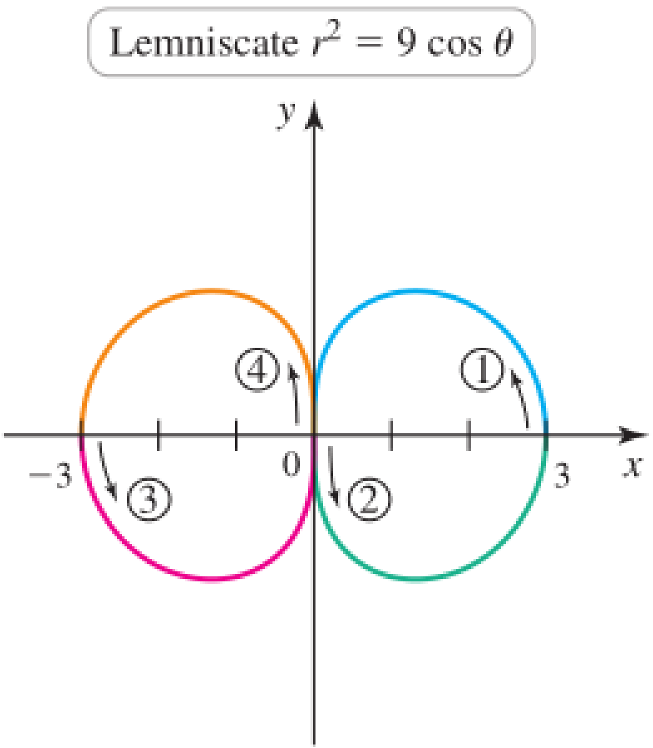
Symmetric about the x -axis, the y-axis, and the origin

Four-leaf rose

EXAMPLE 7 **Plotting polar graphs** Graph the polar equation $r^2 = 9\cos\theta$. Use a graphing utility to check your work.



(a)



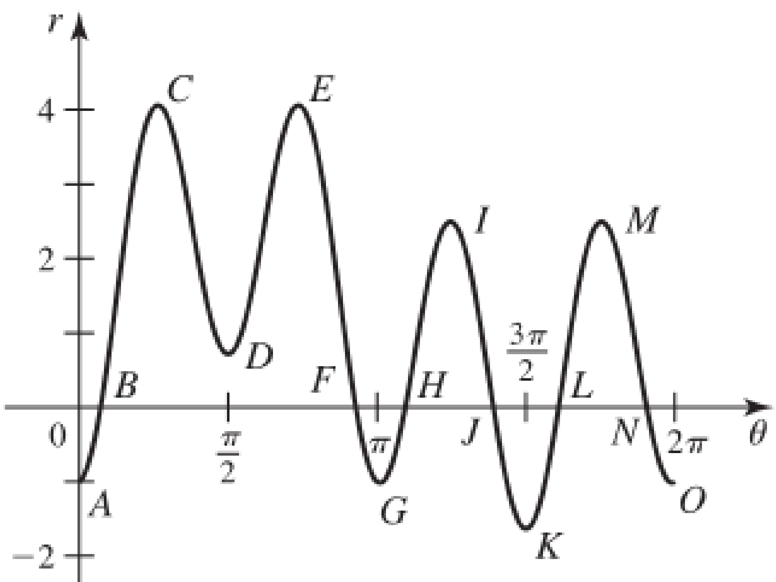
(b)

Lemniscate

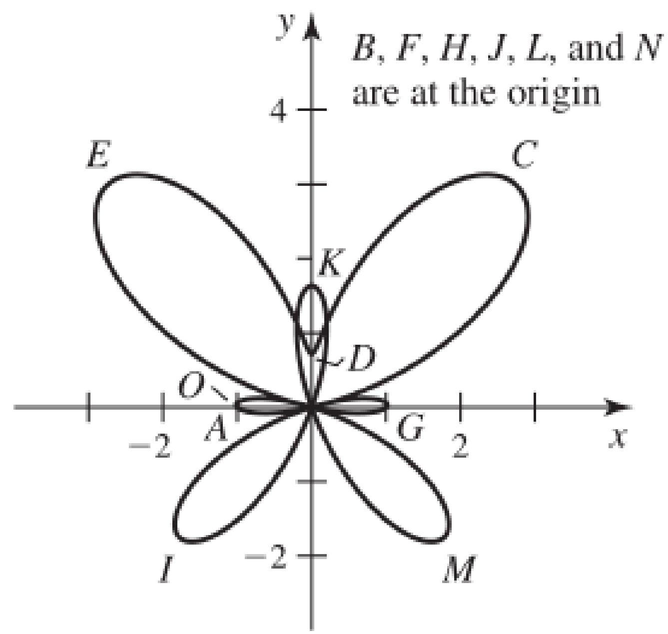
EXAMPLE 8 **Matching polar and Cartesian graphs** The butterfly curve is described by the equation

$$r = e^{\sin\theta} - 2\cos 4\theta, \quad \text{for } 0 \le \theta \le 2\pi,$$

which is plotted in Cartesian and polar coordinates in Figure 30. Follow the Cartesian graph through the points $A, B, C, \ldots, N, O$ and mark the corresponding points on the polar curve



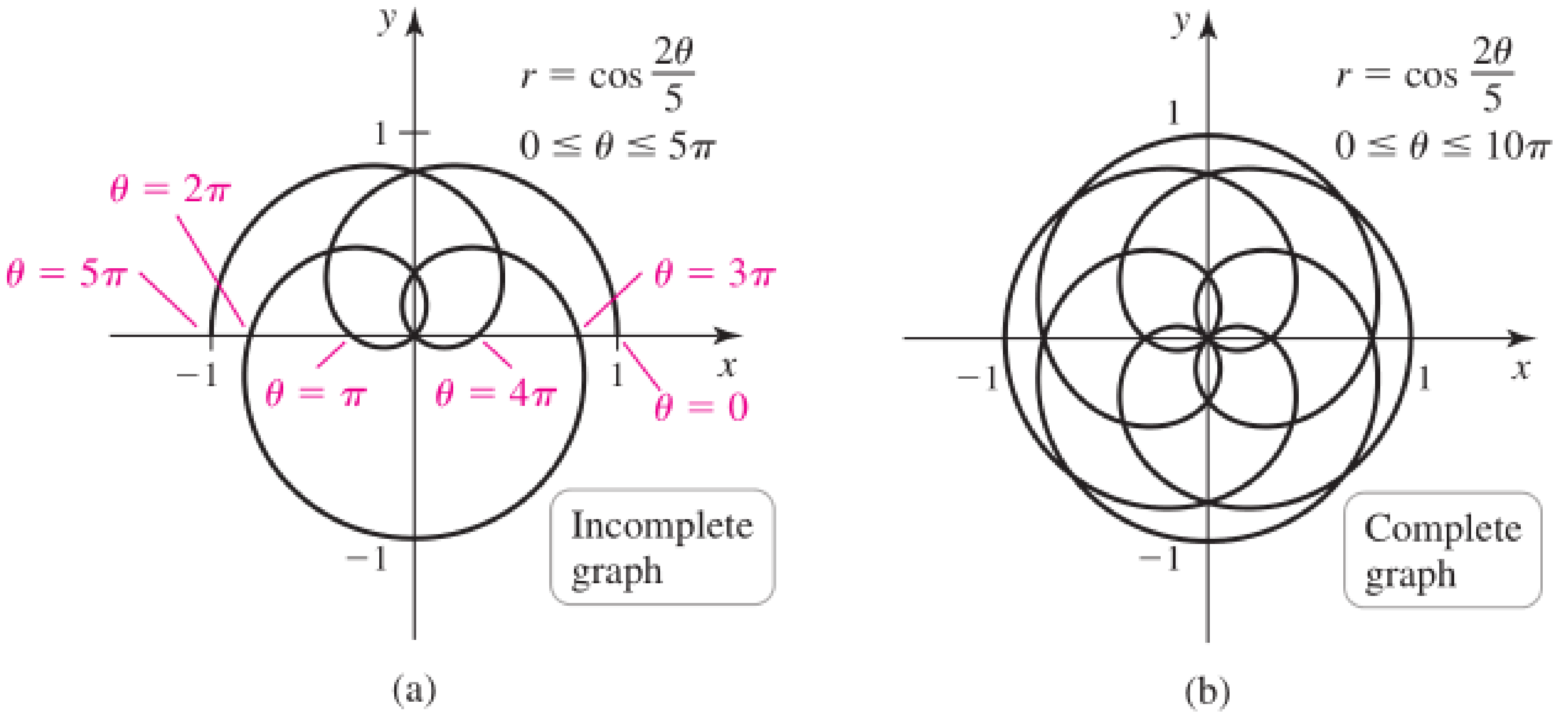
Cartesian graph of $r = e^{\sin\theta} - 2\cos 4\theta$, for $0 \le \theta \le 2\pi$



Polar graph of $r = e^{\sin\theta} - 2\cos 4\theta$, for $0 \le \theta \le 2\pi$

Using Graphing Utilities

EXAMPLE 9 **Plotting complete curves** Consider the closed curve described by $r = \cos(2\theta/5)$. Give an interval in θ that generates the entire curve and then graph the curve.



12.3

Calculus in Polar Coordinates

Slopes of Tangent Lines

For function $y = f(x)$, the slope of its tangent line is $\frac{dy}{dx}$, $f'(x)$

Question: Is the slope of a curve described by the polar equation $r = f(\theta)$ also $\frac{dr}{d\theta} = f'(\theta)$?

Unfortunately not that simple!

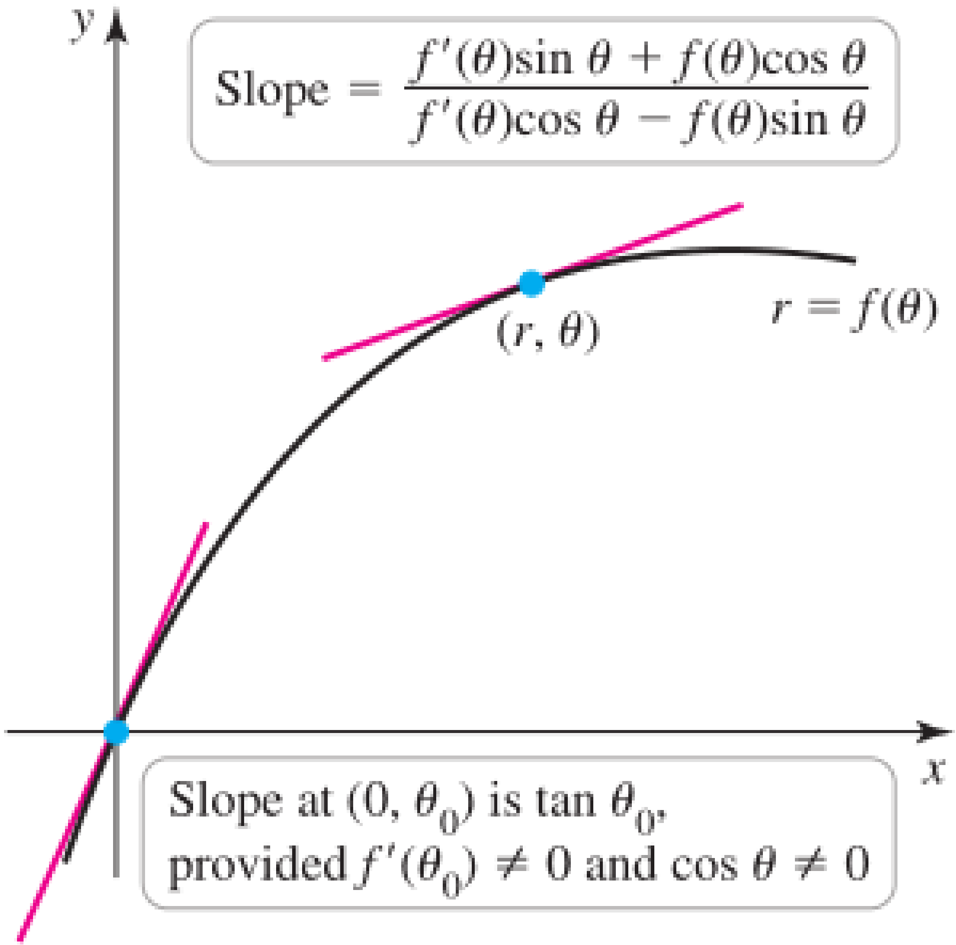
How to solve this problem?

1. Write the polar equation $r = f(\theta)$ in parametric form with θ as a parameter

$$x = r\cos\theta = f(\theta)\cos\theta\,, y = r\sin\theta = f(\theta)\sin\theta$$

2. Consider derivative about parametric equation $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$, applying Product Rule

$$\frac{dy}{dx} = \frac{\overbrace{f'(\theta)\sin\theta + f(\theta)\cos\theta}^{dy/d\theta}}{\underbrace{f'(\theta)\cos\theta - f(\theta)\sin\theta}_{dx/d\theta}}.$$



1. The graph passes through the origin for angle θ_0, $f(\theta_0) = 0$, $f'(\theta_0) \neq 0$, $\cos\theta_0 \neq 0$, then $\frac{dy}{dx} = \frac{\sin\theta_0}{\cos\theta_0} = \tan\theta_0$. The tangent line is simply $\theta = \theta_0$.

2. If $f(\theta_0) = 0$, $f'(\theta_0) \neq 0$, $\cos\theta_0 = 0$, the graph has a vertical tangent line at the origin

THEOREM 2 Slope of a Tangent Line

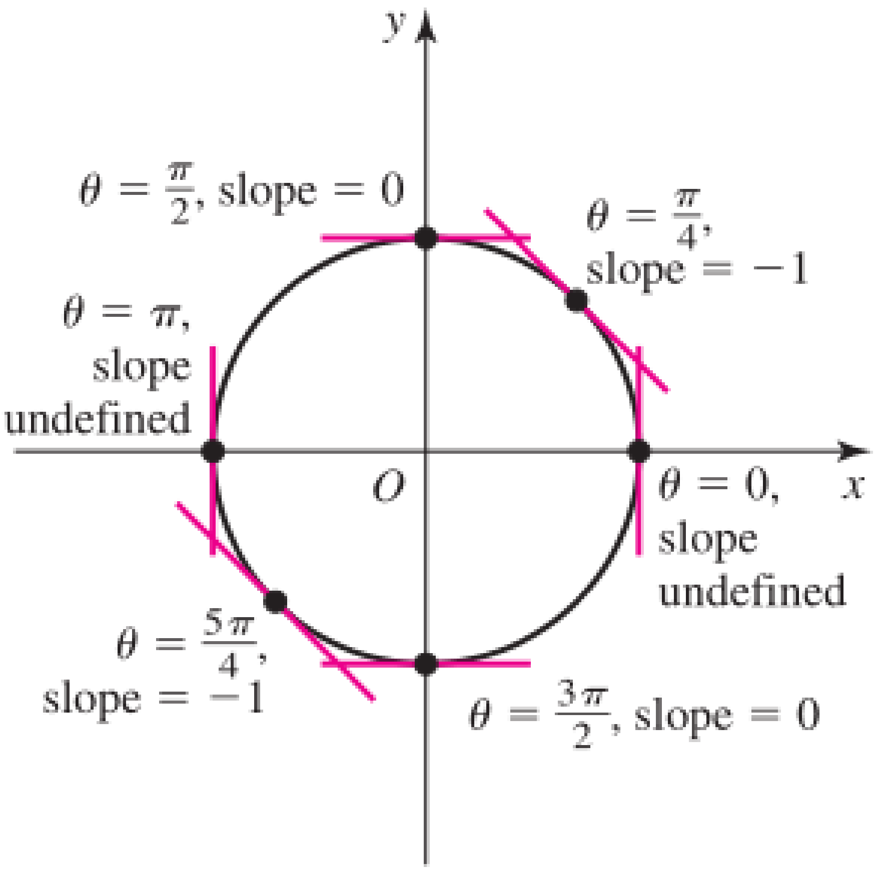
Let f be a differentiable function at θ_0. The slope of the line tangent to the curve $r = f(\theta)$ at the point $(f(\theta_0), \theta_0)$ is

$$\frac{dy}{dx} = \frac{f'(\theta_0)\sin\theta_0 + f(\theta_0)\cos\theta_0}{f'(\theta_0)\cos\theta_0 - f(\theta_0)\sin\theta_0},$$

provided the denominator is nonzero at the point. At angles θ_0 for which $f(\theta_0) = 0$, $f'(\theta_0) \neq 0$, and $\cos\theta_0 \neq 0$, the tangent line is $\theta = \theta_0$ with slope $\tan\theta_0$.

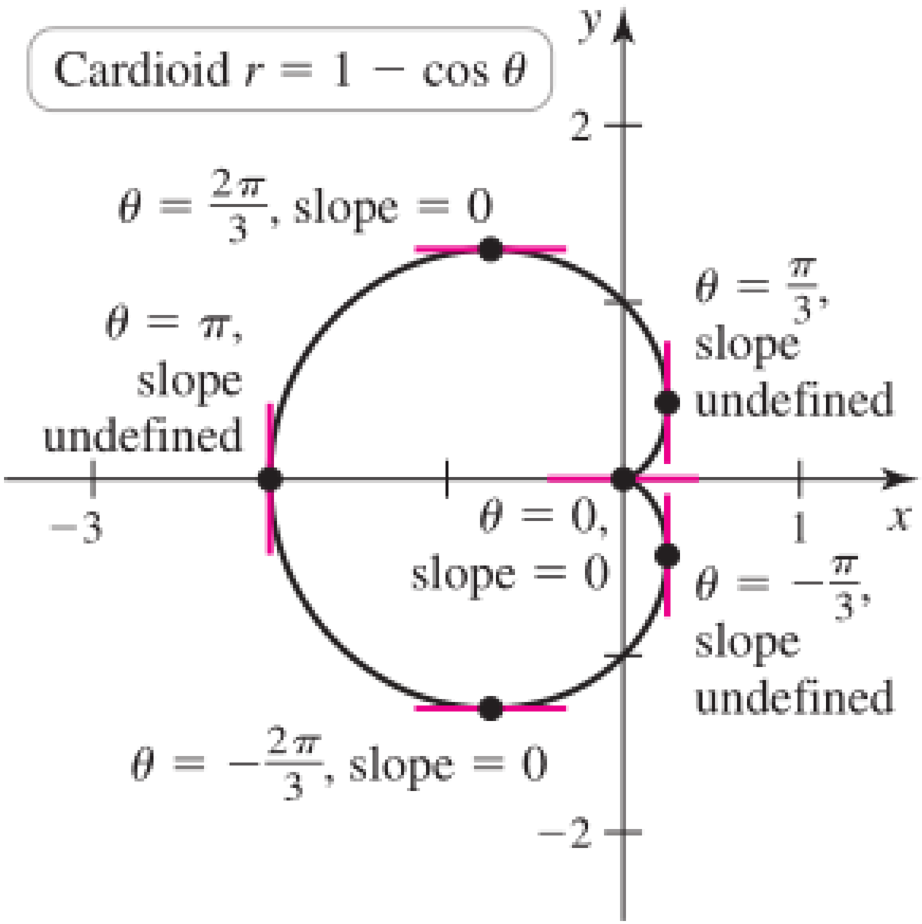
EXAMPLE 1 **Slopes on a circle** Find the slopes of the lines tangent to the circle $r = f(\theta) = 10$.

$$\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta} = -\frac{\cos\theta}{\sin\theta} = -\cot\theta.$$



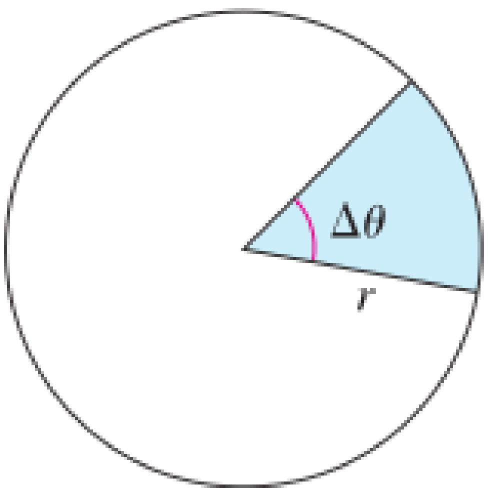
OP, whose slope is $\tan\theta$, is perpendicular to the tangent line at all points P on the circle.

EXAMPLE 2 **Vertical and horizontal tangent lines** Find the points on the interval $-\pi \le \theta \le \pi$ at which the cardioid $r = f(\theta) = 1 - \cos\theta$ has a vertical or horizontal tangent line.



Area of Regions Bounded by Polar Curves

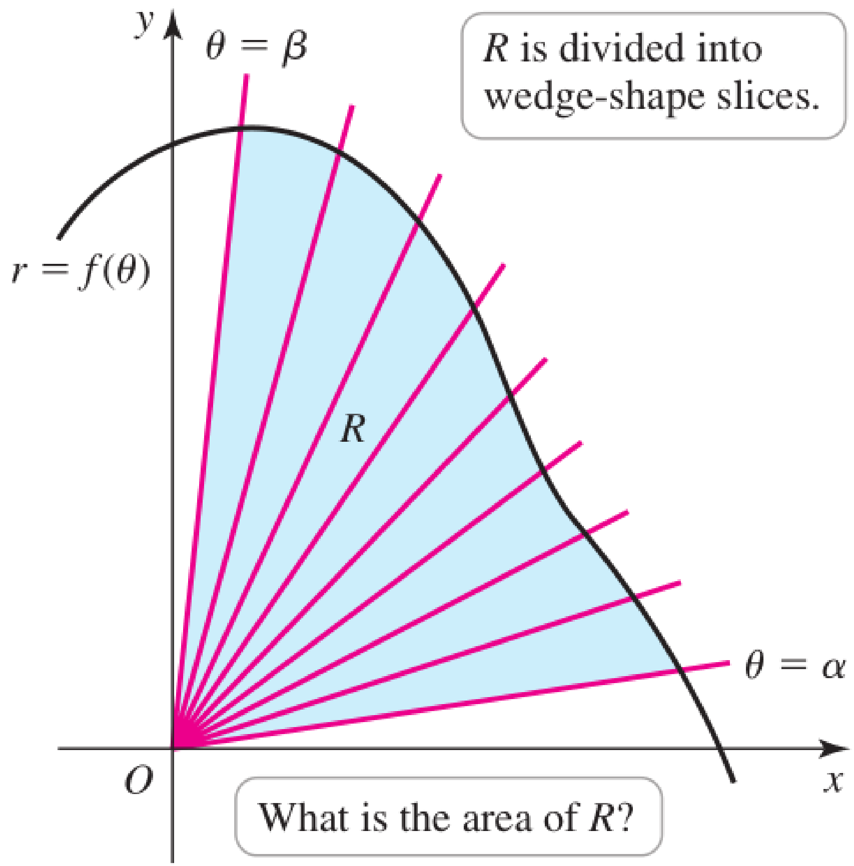
How about the general case?
The area of the region R bounded by the graph of $r = f(\theta)$ between the two rays $\theta = \alpha$ and $\theta = \beta$



Area of circle $= \pi r^2$

Area of $\Delta\theta/(2\pi)$ of a circle

$$= \left(\frac{\Delta\theta}{2\pi}\right)\pi r^2 = \frac{1}{2} r^2 \Delta\theta$$



Slice-and-sum strategy

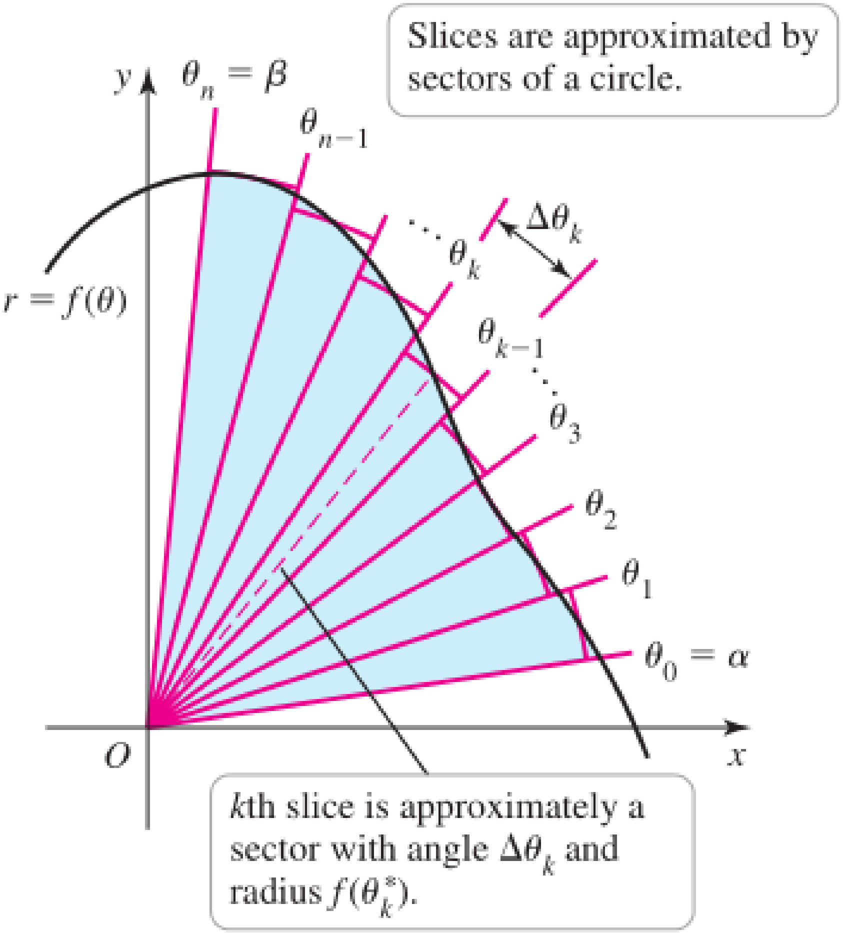
1. **Slice** the region in the radial direction, creating wedge-shaped slices. Interval $[\alpha, \beta]$ is partitioned into n subintervals

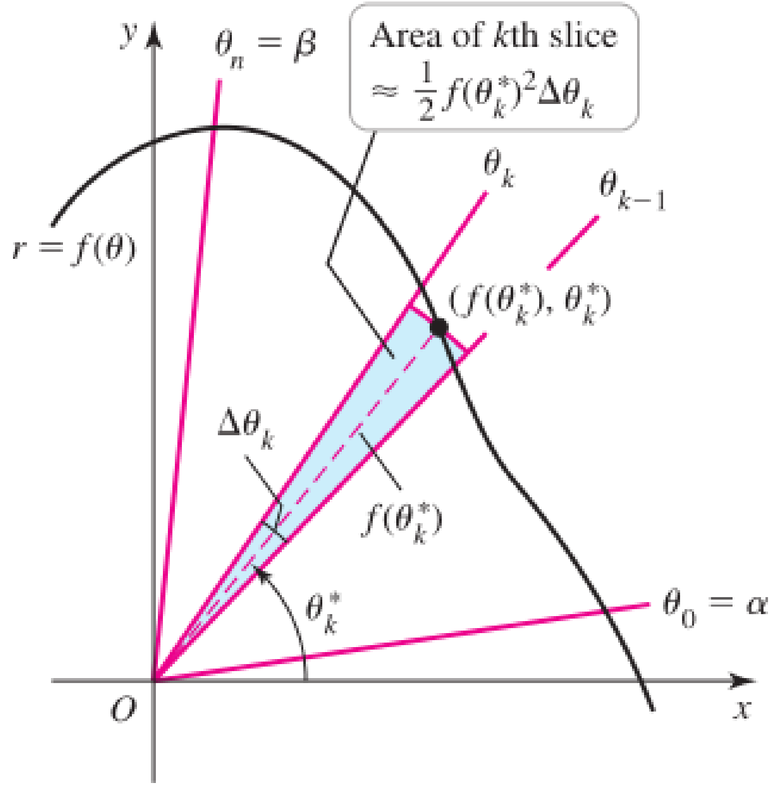
$$\alpha = \theta_0 < \theta_1 < \theta_2 < \cdots < \theta_k < \cdots < \theta_n = \beta$$

2. **Approximation**. Let $\Delta\theta_k = \theta_k - \theta_{k-1}$, $\theta_k^* \in [\theta_{k-1}, \theta_k]$. The kth slice is approximated by the sector of a circle swept out by an angle $\Delta\theta_k^*$ with radius $f(\theta_k^*)$, whose area is $\frac{1}{2} f(\theta_k^*)^2 \Delta\theta_k$

3. **Sum** the areas of these slices to get the approximate area of R

$$\sum_{k=1}^{n} \frac{1}{2} f(\theta_k^*)^2 \Delta\theta_k$$



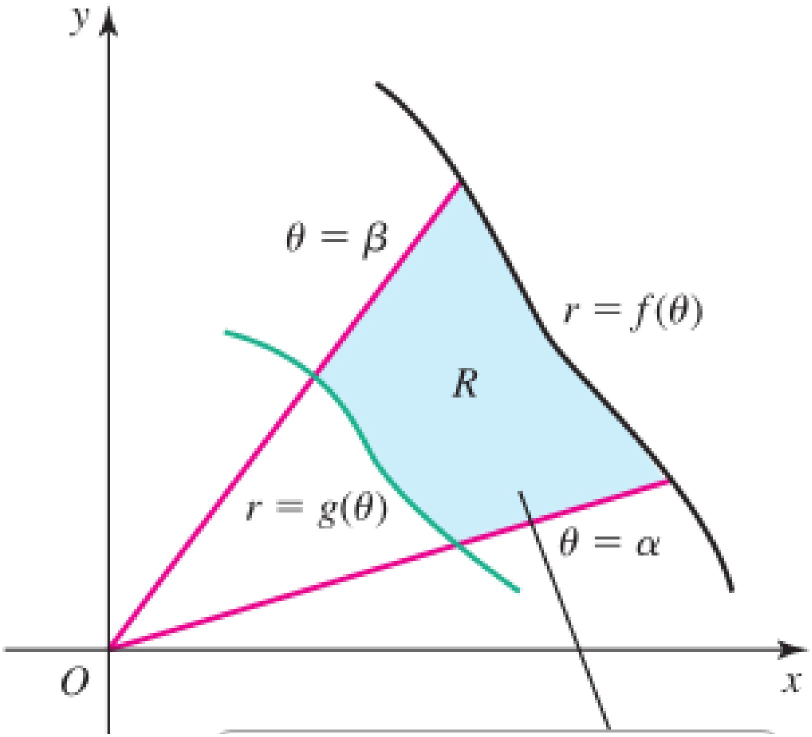


4. **Take limit**. The exact area is

$$\int_{\alpha}^{\beta} \frac{1}{2} f(\theta)^2 \, d\theta = \lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{2} f(\theta_k^*)^2 \Delta\theta_k$$

More generally, the area of a region R bounded by two curves, $r = f(\theta)$ and $r = g(\theta)$ between the rays $\theta = \alpha$ and $\theta = \beta$

Assume that f and g are continuous and $f(\theta) \geq g(\theta) \geq 0$ on $[\alpha, \beta]$

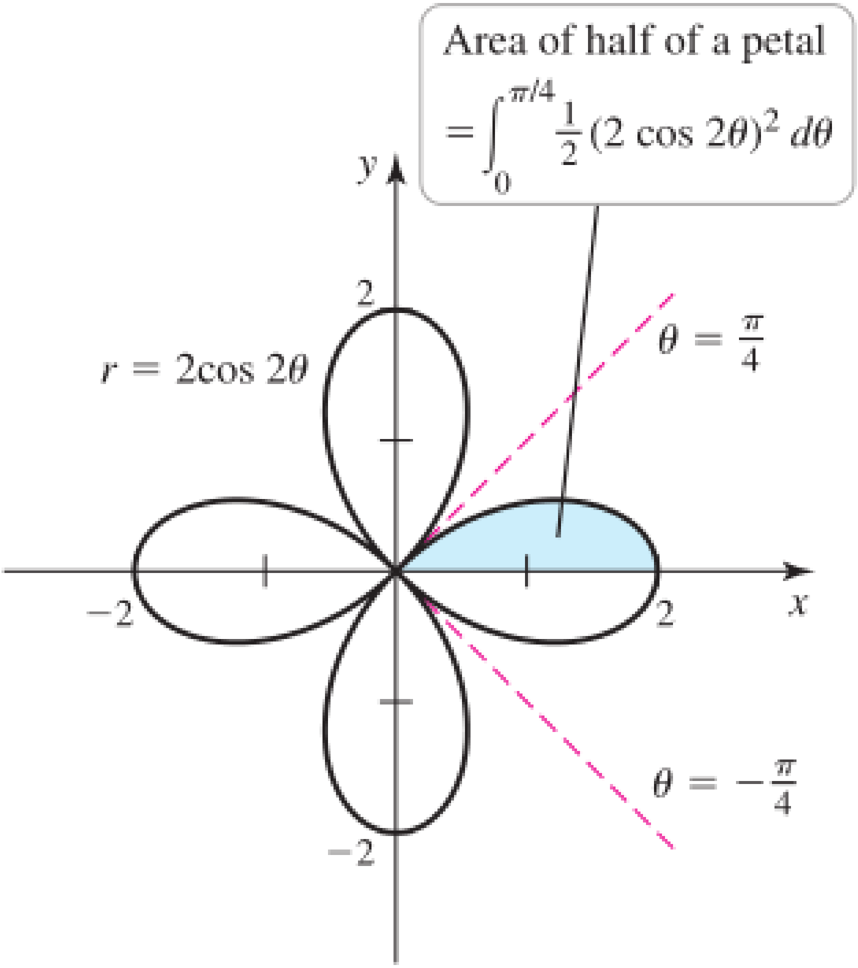


DEFINITION Area of Regions in Polar Coordinates

Let R be the region bounded by the graphs of $r = f(\theta)$ and $r = g(\theta)$, between $\theta = \alpha$ and $\theta = \beta$, where f and g are continuous and $f(\theta) \geq g(\theta) \geq 0$ on $[\alpha, \beta]$. The area of R is

$$\int_{\alpha}^{\beta} \frac{1}{2}\left(f(\theta)^2 - g(\theta)^2\right) d\theta.$$

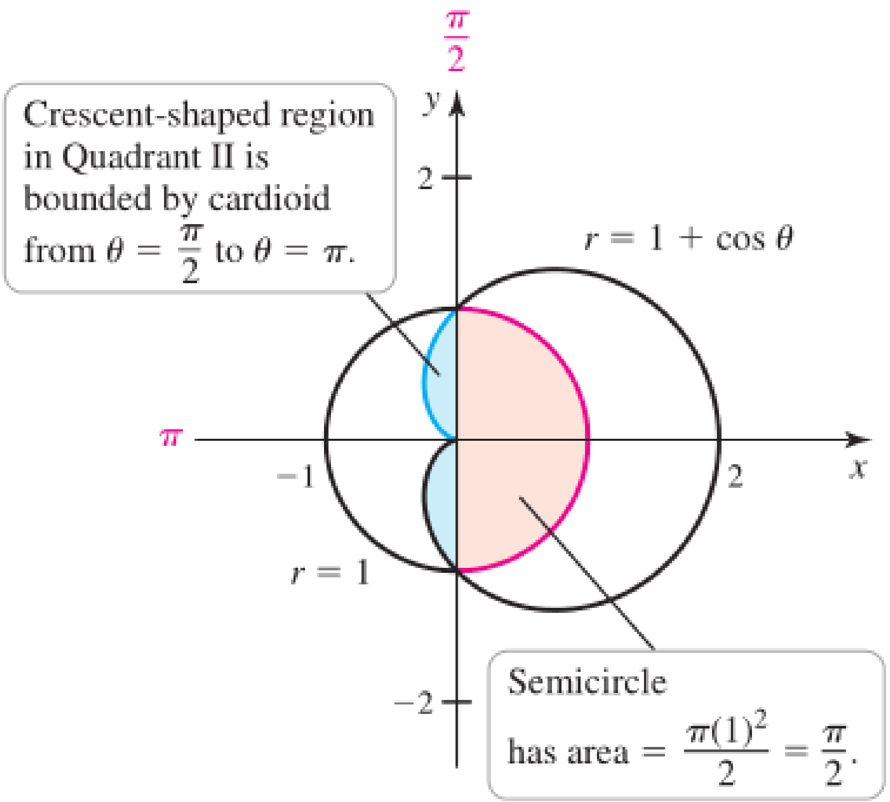
EXAMPLE 3 **Area of a polar region** Find the area of the four-leaf rose $r = f(\theta) = 2\cos 2\theta$.

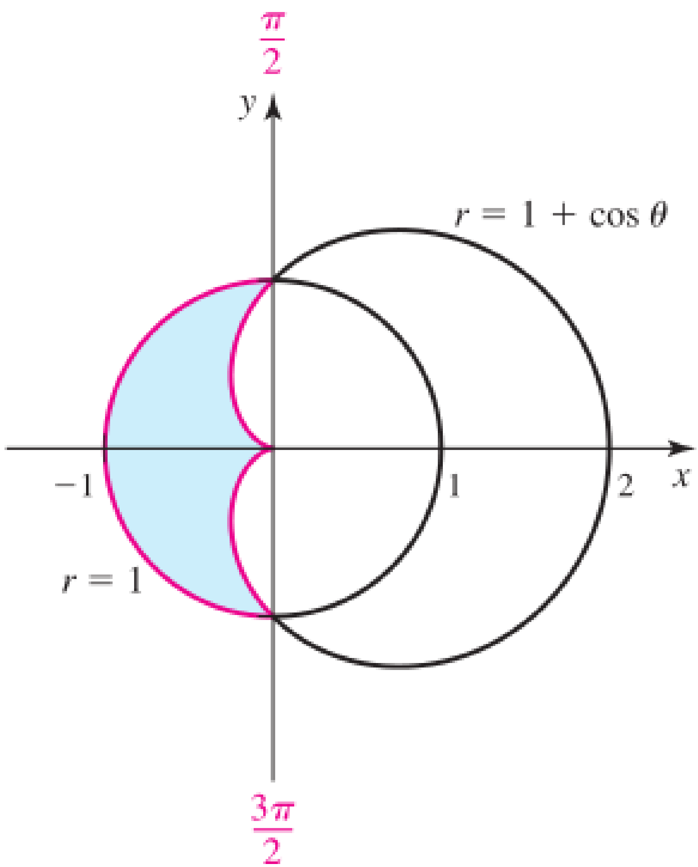


EXAMPLE 4 **Areas of polar regions** Consider the circle $r = 1$ and the cardioid $r = 1 + \cos\theta$ (Figure 38).

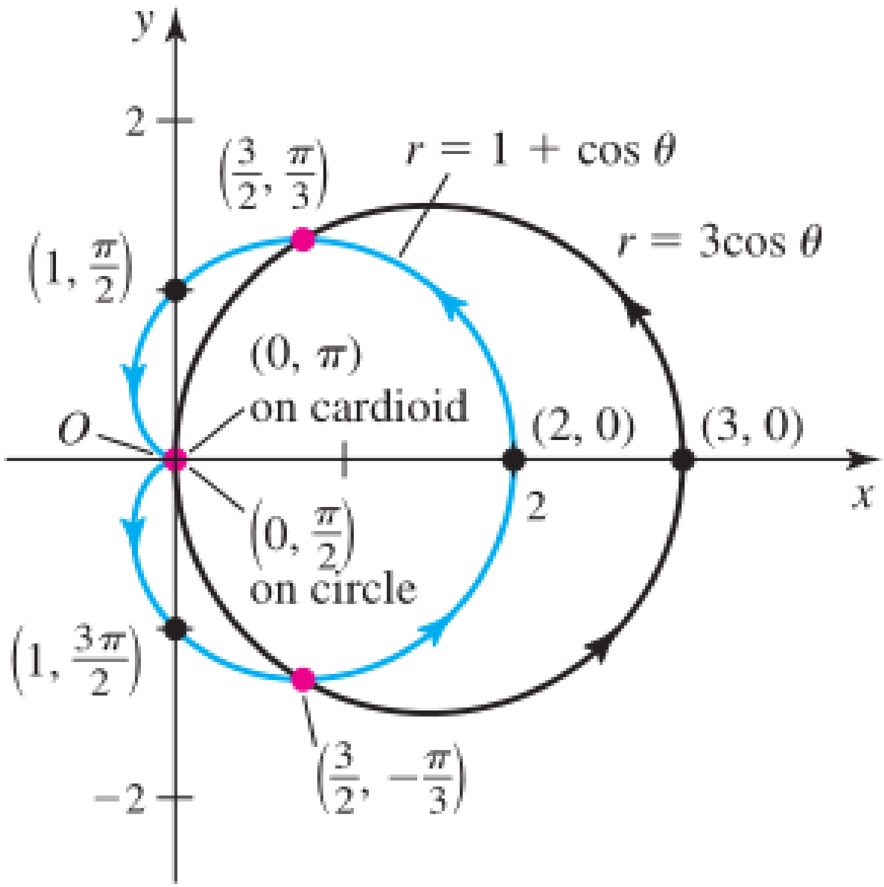
a. Find the area of the region inside the circle and inside the cardioid.

b. Find the area of the region inside the circle and outside the cardioid.



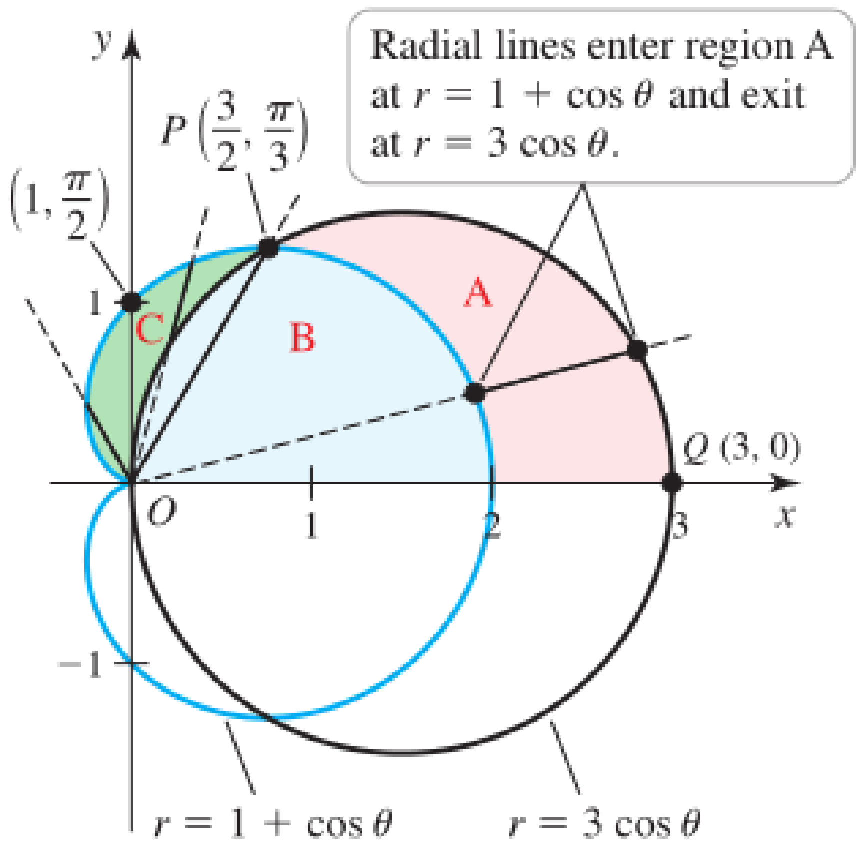


EXAMPLE 5 **Points of intersection** Find the points of intersection of the circle $r = 3\cos\theta$ and the cardioid $r = 1 + \cos\theta$ (Figure 40).



EXAMPLE 6 **Computing areas** Example 5 discussed the points of intersection of the curves $r = 3\cos\theta$ (a circle) and $r = 1 + \cos\theta$ (a cardioid). Use those results to compute the areas of the following non-overlapping regions in Figure 41.

a. region A **b.** region B **c.** region C



Arc Length of a Polar Curve

Question: Given function $r = f(\theta)$, what is the length of the corresponding curve for $\alpha \leq \theta \leq \beta$?

Idea: Express the polar equation as a set of parametric equations in Cartesian coordinates, i.e.,

$$x = r\cos\theta = f(\theta)\cos\theta, \text{ and } y = r\sin\theta = f(\theta)\sin\theta$$

where $\alpha \leq \theta \leq \beta$.

Then the arc length formula is

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$\frac{dx}{d\theta} = f'(\theta)\cos\theta - f(\theta)\sin\theta, \frac{dy}{d\theta} = f'(\theta)\sin\theta + f(\theta)\cos\theta.$$

Arc Length of a Polar Curve

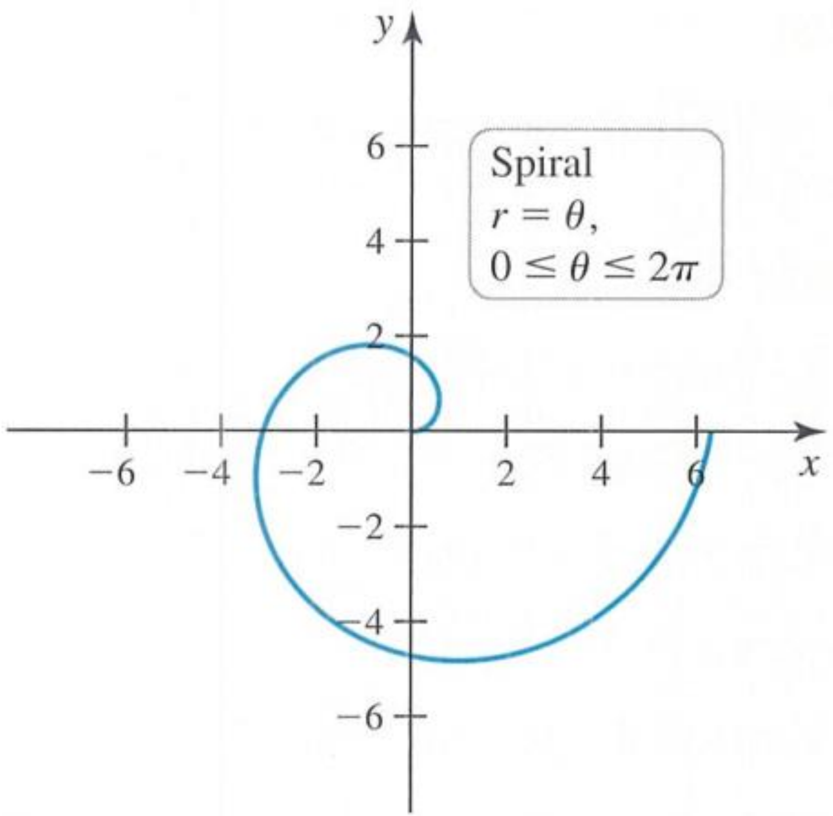
Let f have a continuous derivative on the interval $[\alpha, \beta]$. The **arc length** of the polar curve $r = f(\theta)$ on $[\alpha, \beta]$ is

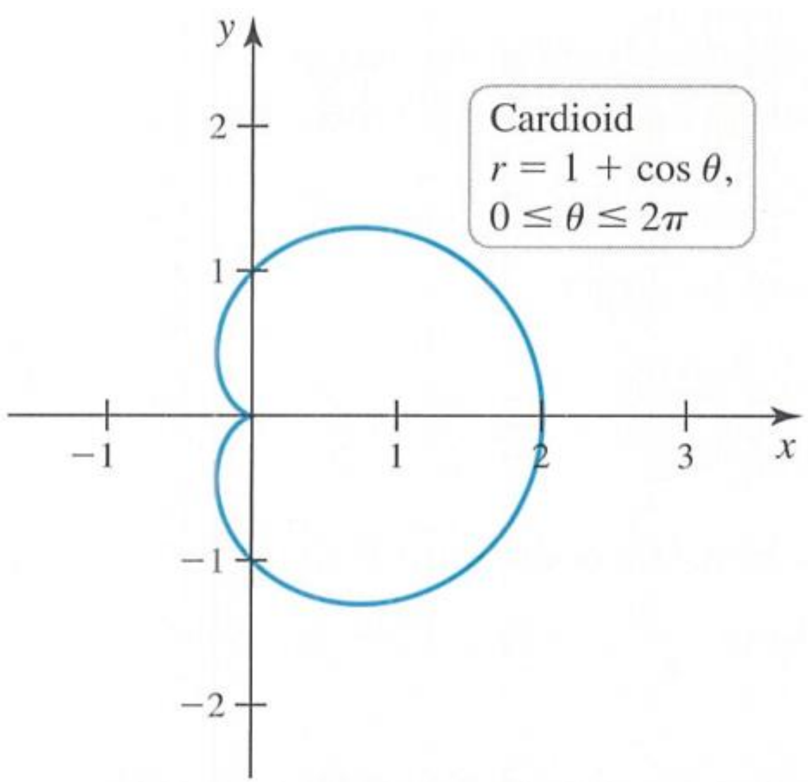
$$L = \int_{\alpha}^{\beta} \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta.$$

EXAMPLE 7 **Arc length of polar curves**

a. Find the arc length of the spiral $r = f(\theta) = \theta$, for $0 \le \theta \le 2\pi$ (**Figure 12.43**).

b. Find the arc length of the cardioid $r = 1 + \cos\theta$ (**Figure 12.44**).



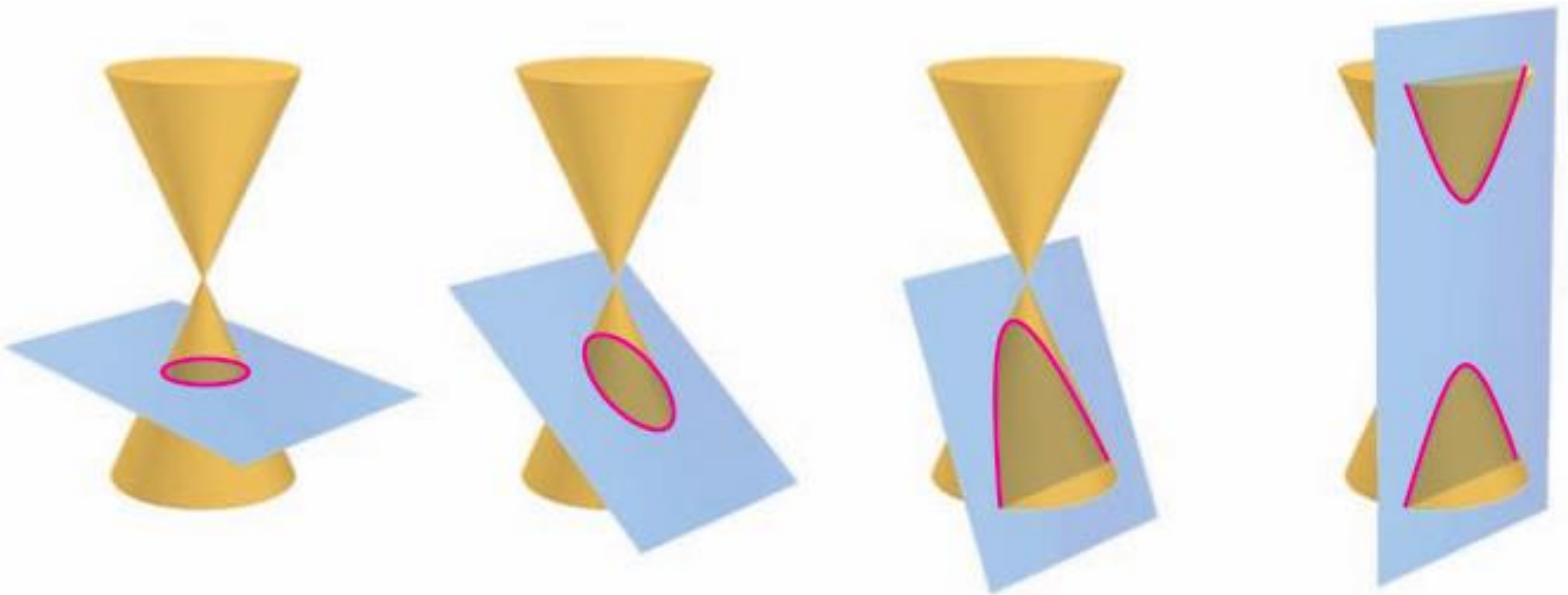


12.4

Conic Sections

Conic sections: Slice a double cone with a plane

Standard ones: *ellipses*, *parabolas*, and *hyperbolas*



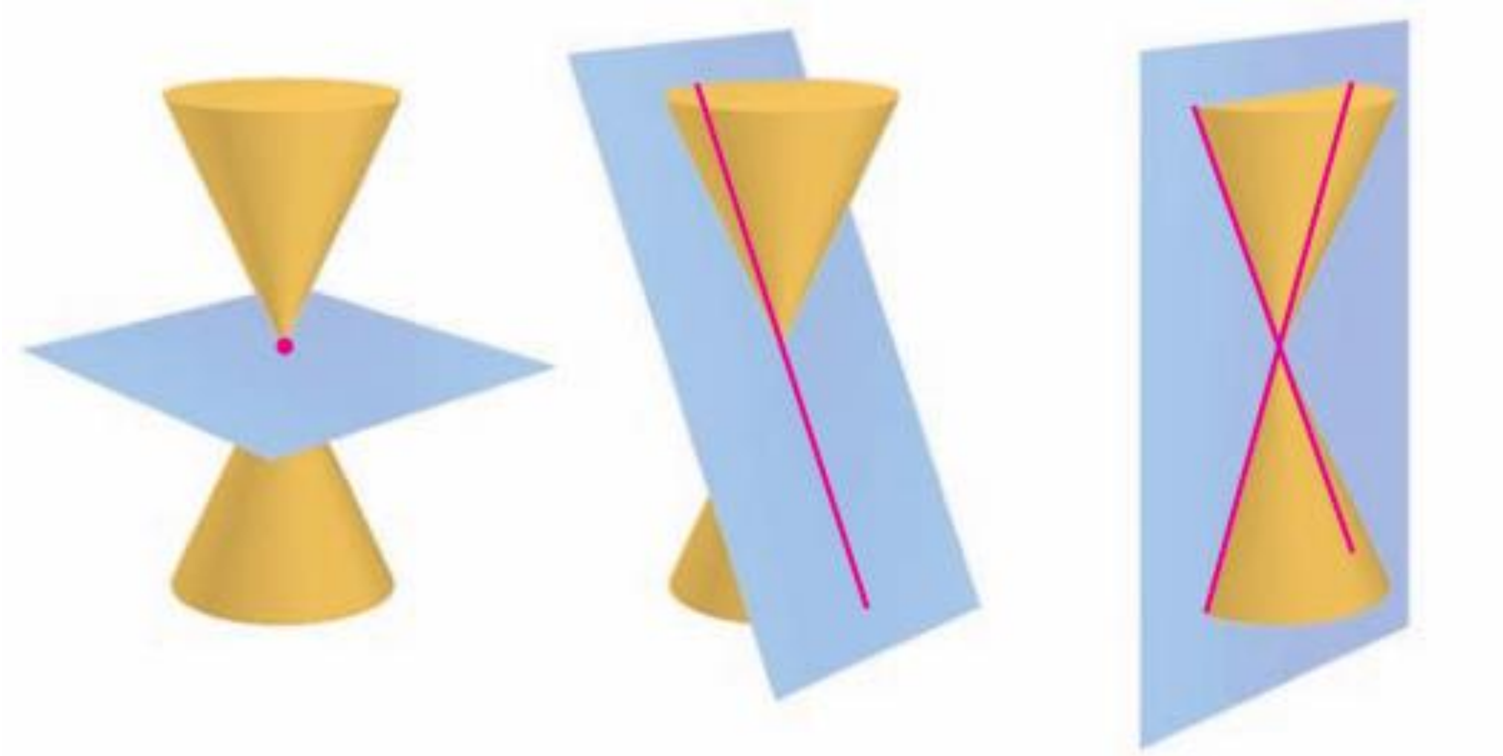
Circle: plane perpendicular to cone axis

Ellipse: plane cuts one half of cone

Parabola: plane parallel to side of cone

Hyperbola: plane cuts both halves of cone

Degenerate conic sections (*lines* and *points*), produced when a plane passes through the vertex of the cone



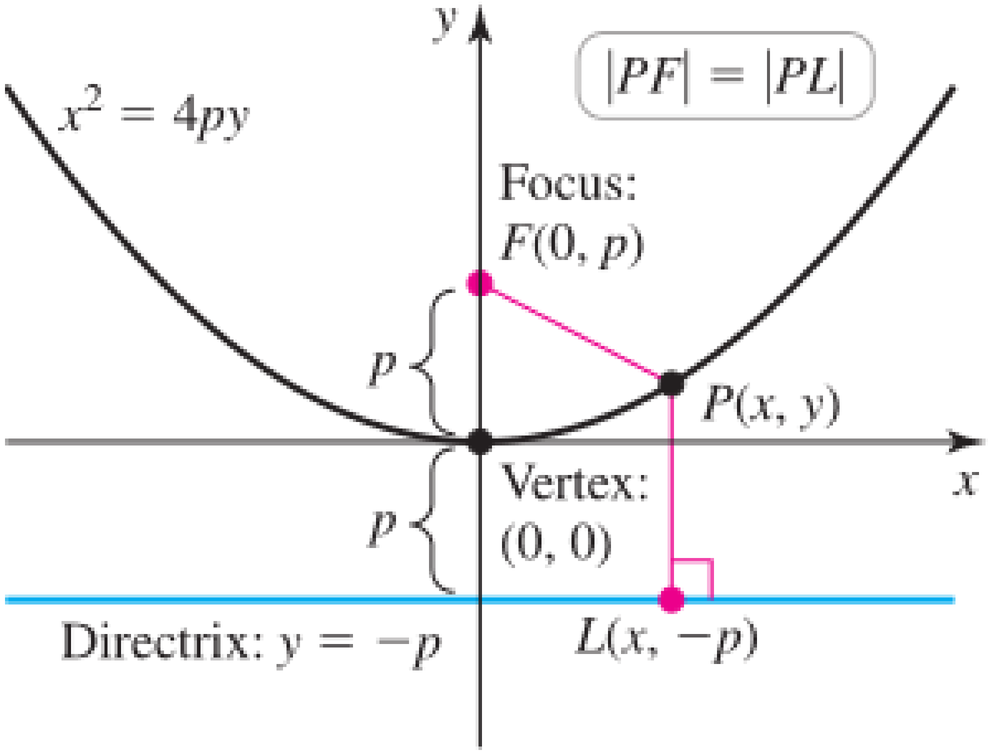
Point: plane through cone vertex only

Single line: plane tangent to cone

Pair of intersecting lines

Parabolas

A **parabola** is the set of points in a plane that are equidistant from a fixed point F (called the **focus**) and a fixed line (called the **directrix**)



$$\underbrace{\sqrt{x^2 + (y-p)^2}}_{|PF|} = \underbrace{y + p}_{|PL|}.$$

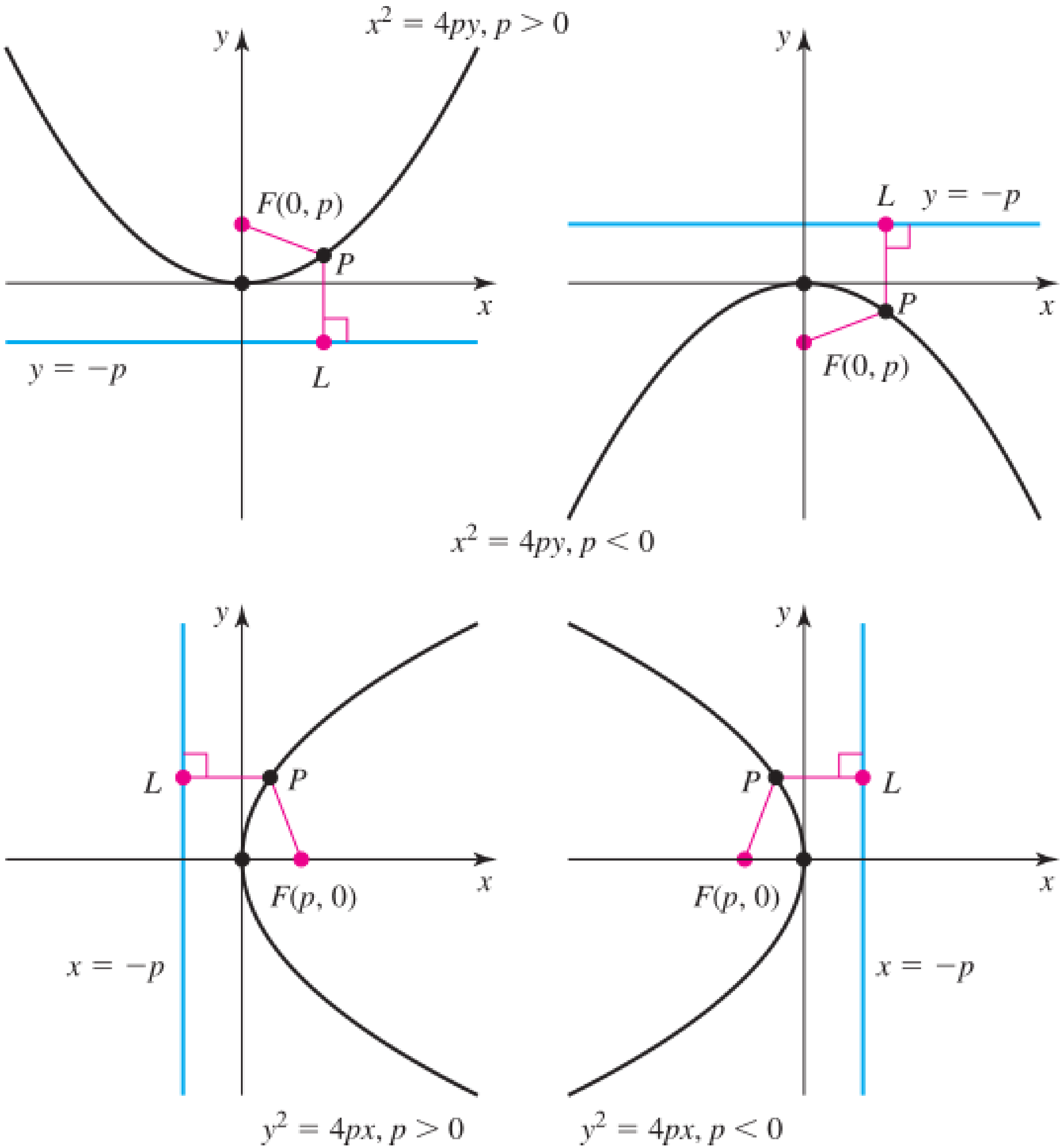
The equation: $x^2 = 4py$

Equations of Four Standard Parabolas

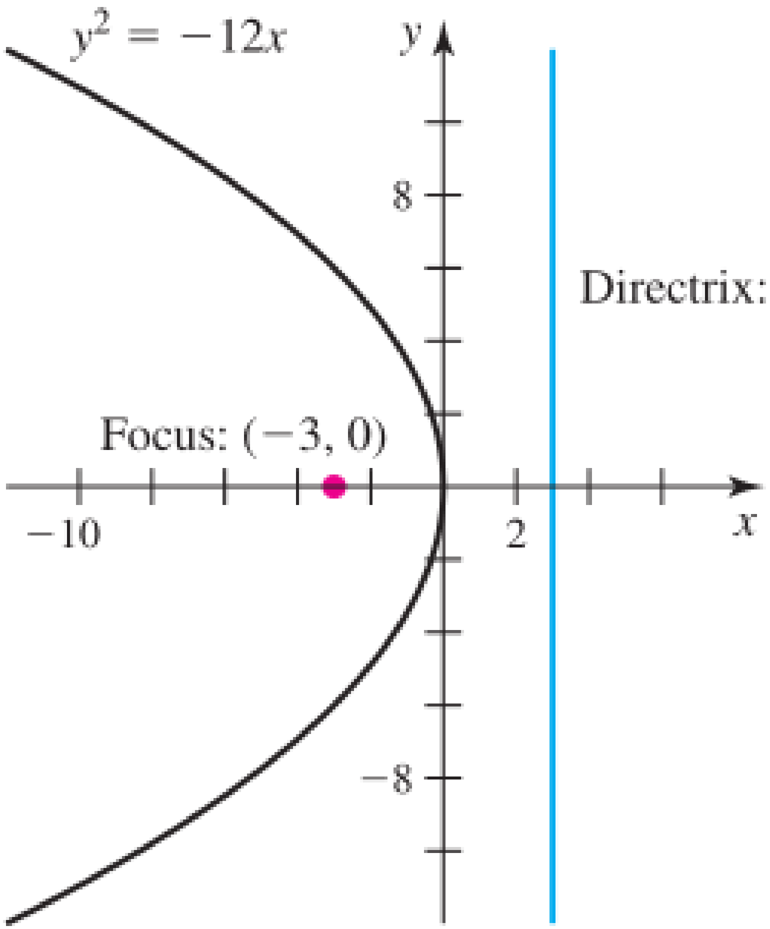
Let p be a real number. The parabola with focus at $(0, p)$ and directrix $y = -p$ is symmetric about the y-axis and has the equation $x^2 = 4py$. If $p > 0$, then the parabola opens *upward*; if $p < 0$, then the parabola opens *downward*.

The parabola with focus at $(p, 0)$ and directrix $x = -p$ is symmetric about the x-axis and has the equation $y^2 = 4px$. If $p > 0$, then the parabola opens *to the right*; if $p < 0$, then the parabola opens *to the left*.

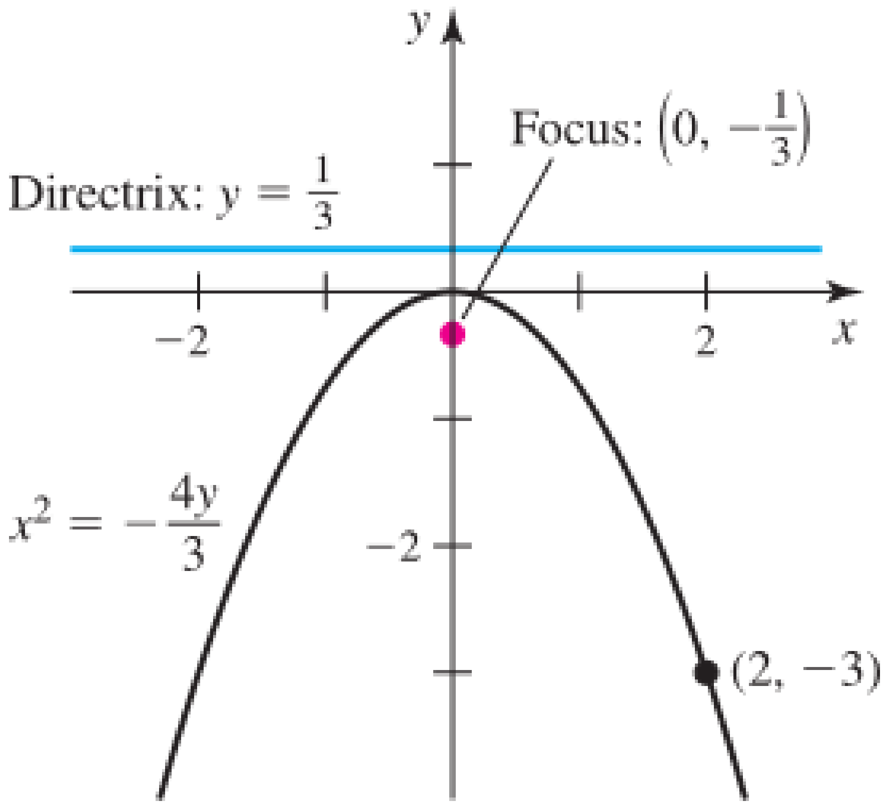
Each of these parabolas has its vertex at the origin (Figure 44).


$x^2 = 4py,\ p > 0$
$F(0, p)$
P
L
$y = -p$
$x^2 = 4py,\ p < 0$
$y^2 = 4px,\ p > 0$
$F(p, 0)$
$x = -p$
$y^2 = 4px,\ p < 0$

EXAMPLE 1 **Graphing parabolas** Find the focus and directrix of the parabola $y^2 = -12x$. Sketch its graph.



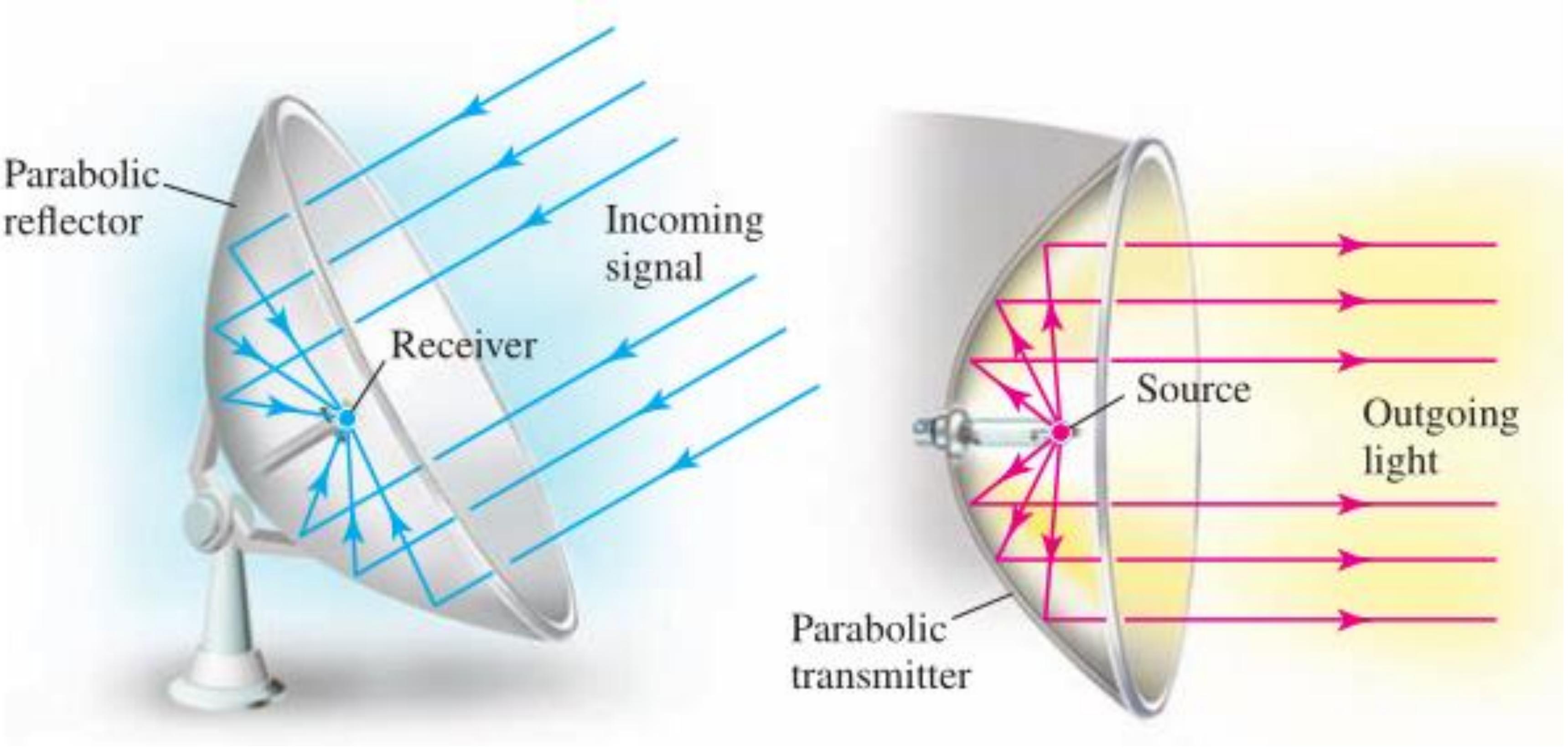
EXAMPLE 2 **Equations of parabolas** Find the equation of the parabola with vertex $(0, 0)$ that opens downward and passes through the point $(2, -3)$.



Reflection Property

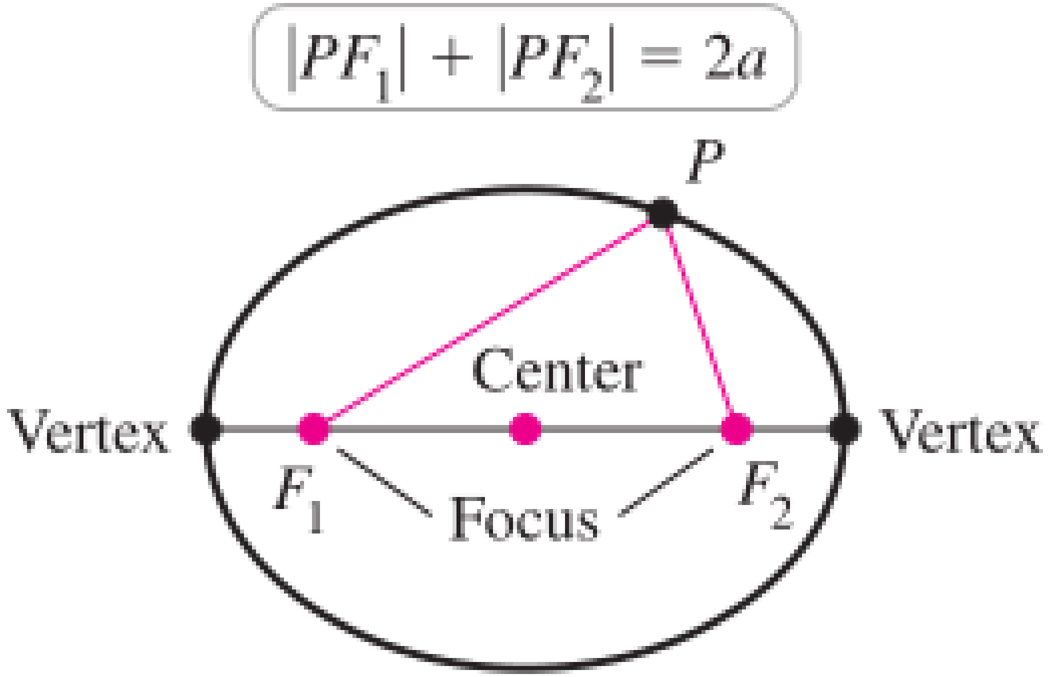
Particle approaching a parabola on any line parallel to the axis of the parabola is reflected on a line that passes through the focus

Useful in the design of reflectors and transmitters

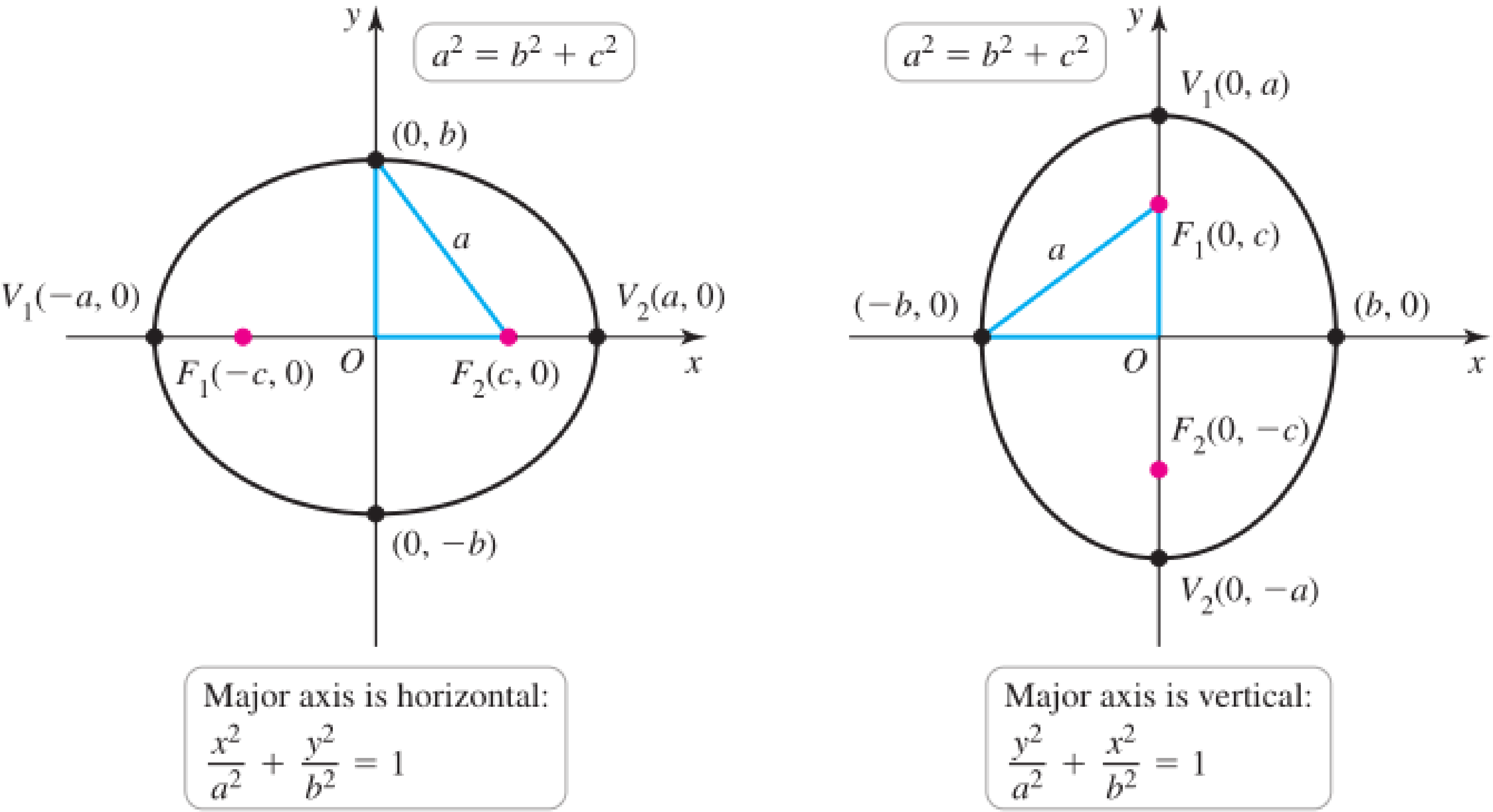


Ellipses

An **ellipse** is the set of points in a plane whose distances from two fixed points have a constant sum that we denote $2a$.
Each of the two fixed points is a **focus** (plural **foci**)



The equation of an ellipse is simplest if the foci are on the x-axis at $(\pm c, 0)$ or on the y-axis at $(0, \pm c)$.



Equations of Standard Ellipses

An ellipse centered at the origin with foci F_1 and F_2 at $(\pm c, 0)$ and vertices V_1 and V_2 at $(\pm a, 0)$ has the equation

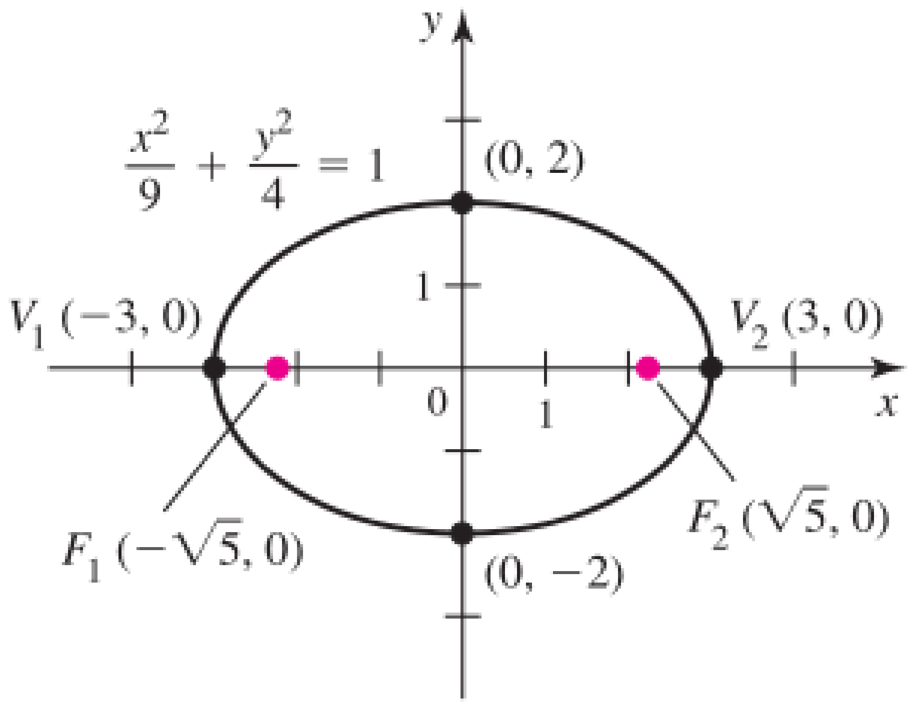
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \text{where } a^2 = b^2 + c^2.$$

An ellipse centered at the origin with foci at $(0, \pm c)$ and vertices at $(0, \pm a)$ has the equation

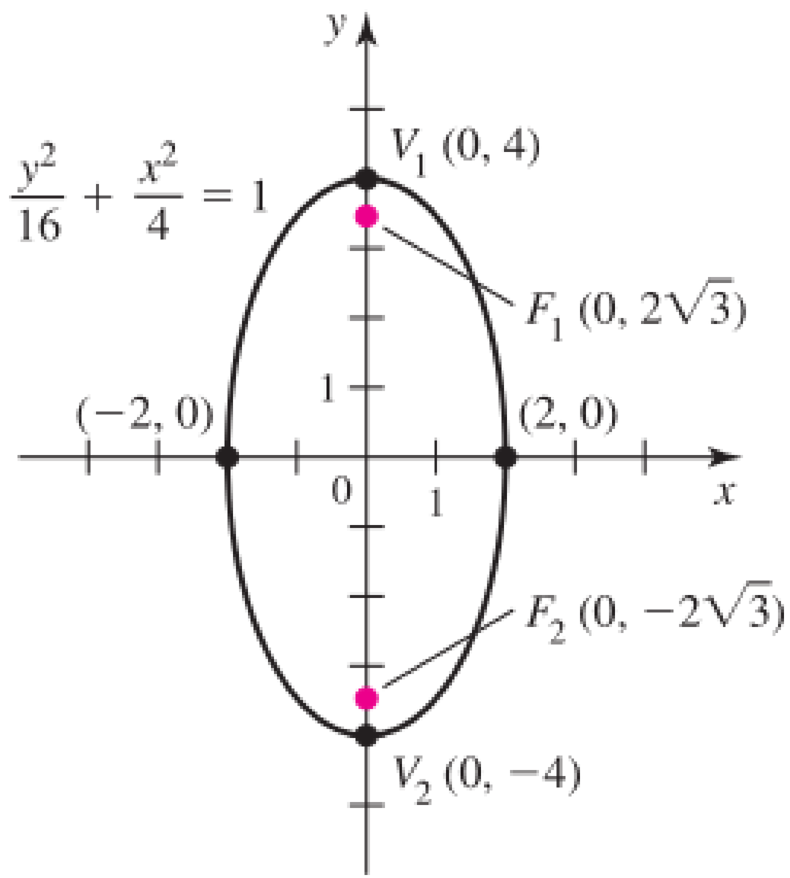
$$\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1, \quad \text{where } a^2 = b^2 + c^2.$$

In both cases, $a > b > 0$ and $a > c > 0$, the length of the long axis (called the **major axis**) is $2a$, and the length of the short axis (called the **minor axis**) is $2b$.

EXAMPLE 3 **Graphing ellipses** Find the vertices, foci, and length of the major and minor axes of the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$. Graph the ellipse.

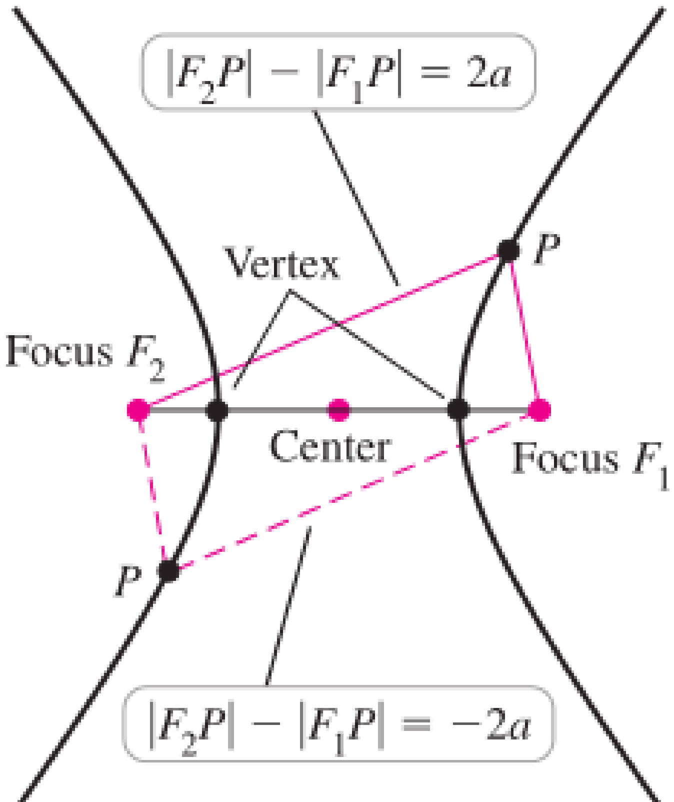


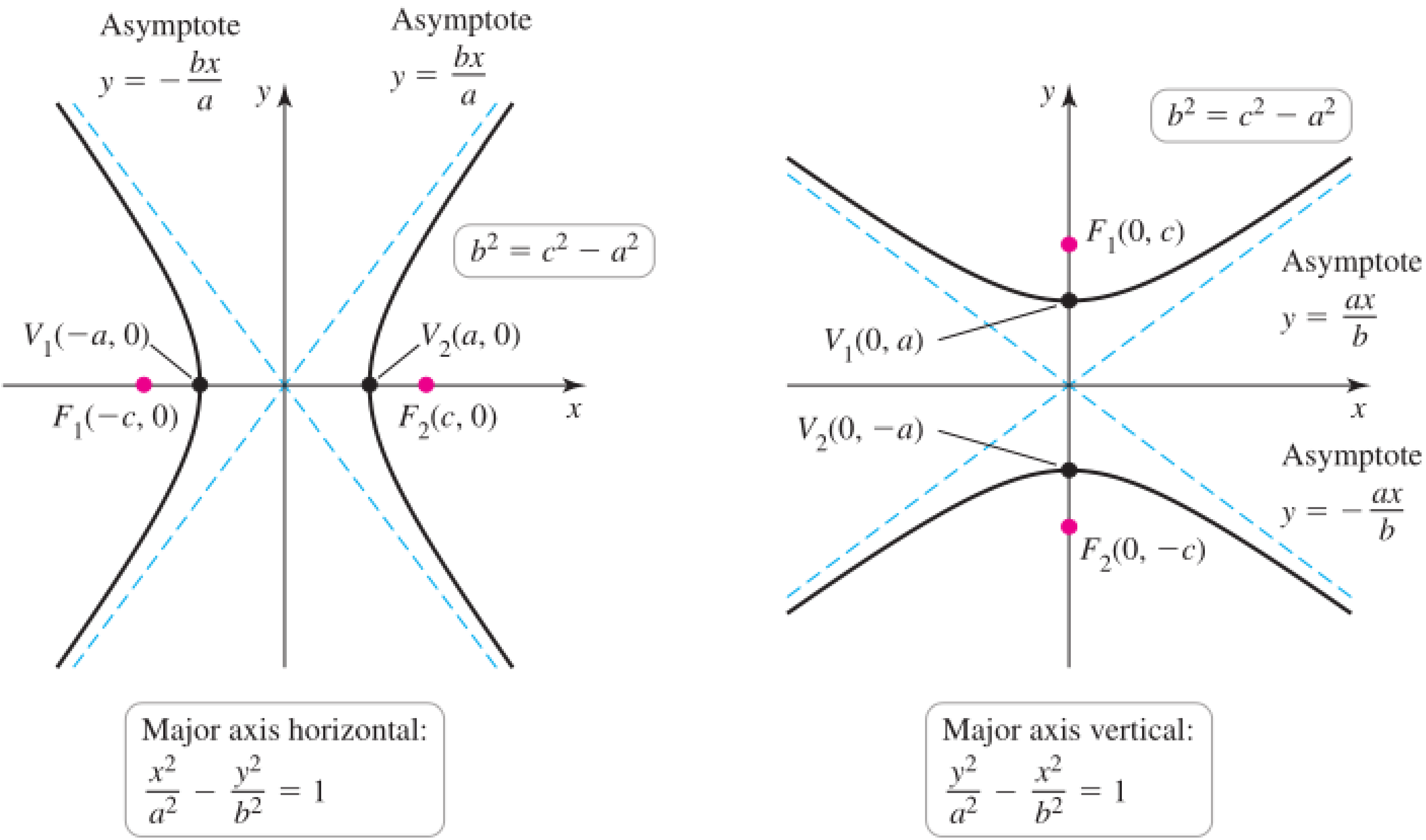
EXAMPLE 4 **Equation of an ellipse** Find the equation of the ellipse centered at the origin with its foci on the y-axis, a major axis of length 8, and a minor axis of length 4. Graph the ellipse.



Hyperbolas

A **hyperbola** is the set of points in a plane whose distances from two fixed points have a constant difference, either $2a$ or $-2a$. The two fixed points are called **foci**.




Asymptote
$y = -\frac{bx}{a}$
Asymptote
$y = \frac{bx}{a}$
y
$b^2 = c^2 - a^2$
$V_1(-a, 0)$
$V_2(a, 0)$
$F_1(-c, 0)$
$F_2(c, 0)$
x
Major axis horizontal:
$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
y
$b^2 = c^2 - a^2$
$F_1(0, c)$
Asymptote
$y = \frac{ax}{b}$
$V_1(0, a)$
x
$V_2(0, -a)$
Asymptote
$y = -\frac{ax}{b}$
$F_2(0, -c)$
Major axis vertical:
$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

Equations of Standard Hyperbolas

A hyperbola centered at the origin with foci F_1 and F_2 at $(\pm c, 0)$ and vertices V_1 and V_2 at $(\pm a, 0)$ has the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \quad \text{where} \quad b^2 = c^2 - a^2.$$

The hyperbola has **asymptotes** $y = \pm bx/a$.

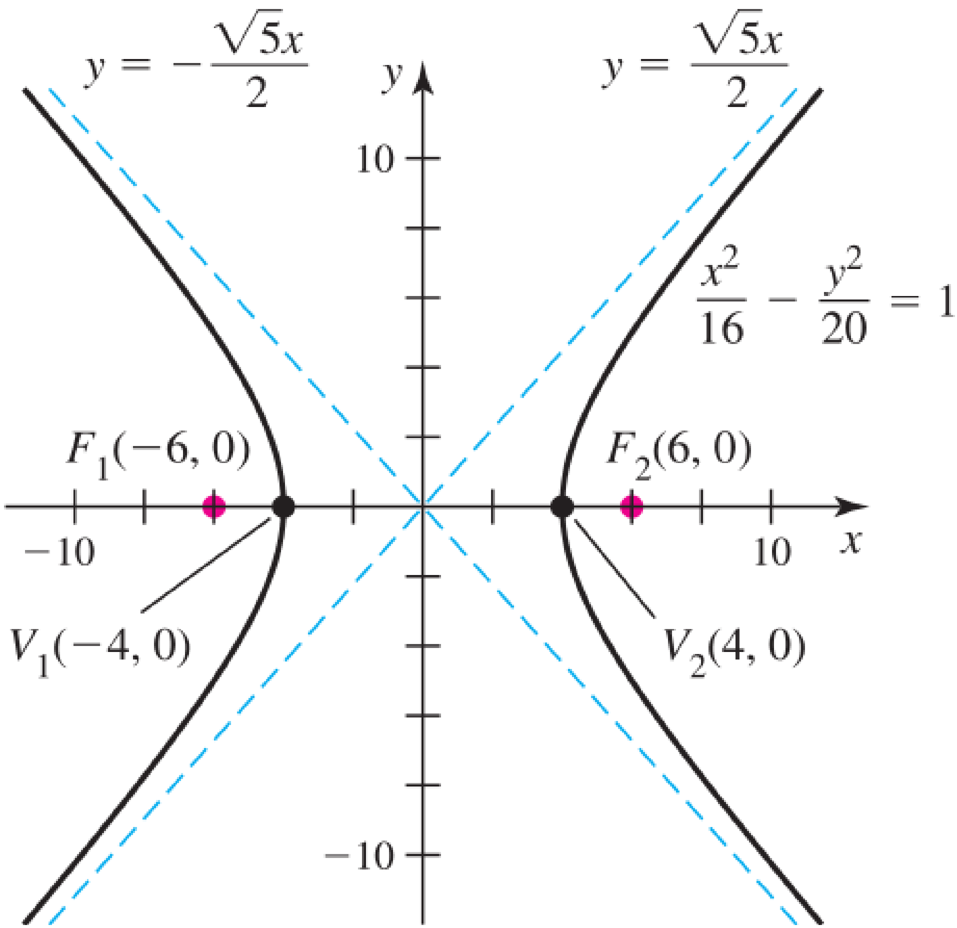
A hyperbola centered at the origin with foci at $(0, \pm c)$ and vertices at $(0, \pm a)$ has the equation

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1, \quad \text{where} \quad b^2 = c^2 - a^2.$$

The hyperbola has **asymptotes** $y = \pm ax/b$.

In both cases, $c > a > 0$ and $c > b > 0$.

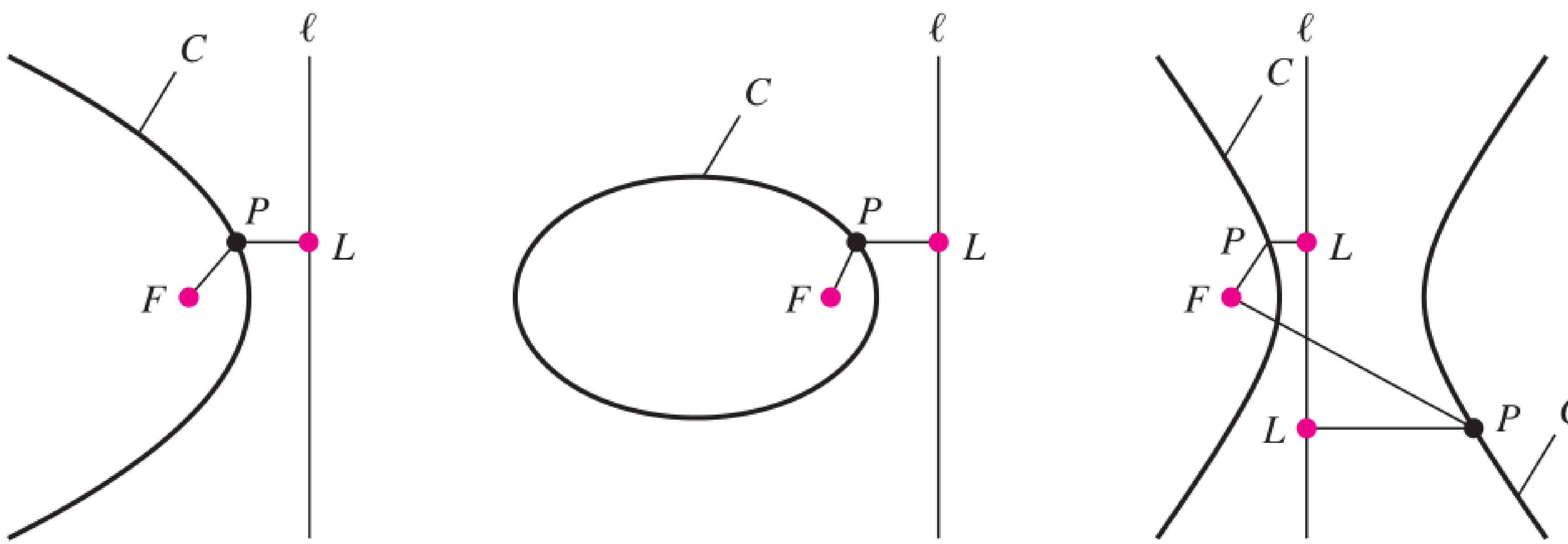
EXAMPLE 5 **Graphing hyperbolas** Find the equation of the hyperbola centered at the origin with vertices V_1 and V_2 at $(\pm 4, 0)$ and foci F_1 and F_2 at $(\pm 6, 0)$. Graph the hyperbola.



Eccentricity and Directrix

Eccentricity-directrix approach, a single unified way to define parabolas, ellipses, and hyperbolas

Directrix is a line l, focus F is a point *not* on l, and eccentricity is a real number $e > 0$.



Parabola: $\dfrac{|PF|}{|PL|} = e = 1$

Ellipse: $\dfrac{|PF|}{|PL|} = e,\ 0 < e < 1$

Hyperbola: $\dfrac{|PF|}{|PL|} = e > 1$

THEOREM 3 Eccentricity-Directrix Theorem

Suppose ℓ is a line, F is a point not on ℓ, and e is a positive real number. Let C be the set of points P in a plane with the property that $\dfrac{|PF|}{|PL|} = e$, where $|PL|$ is the perpendicular distance from P to ℓ.

1. If $e = 1$, C is a **parabola**.

2. If $0 < e < 1$, C is an **ellipse**.

3. If $e > 1$, C is a **hyperbola**.

SUMMARY **Properties of Ellipses and Hyperbolas**

An ellipse or a hyperbola centered at the origin has the following properties.

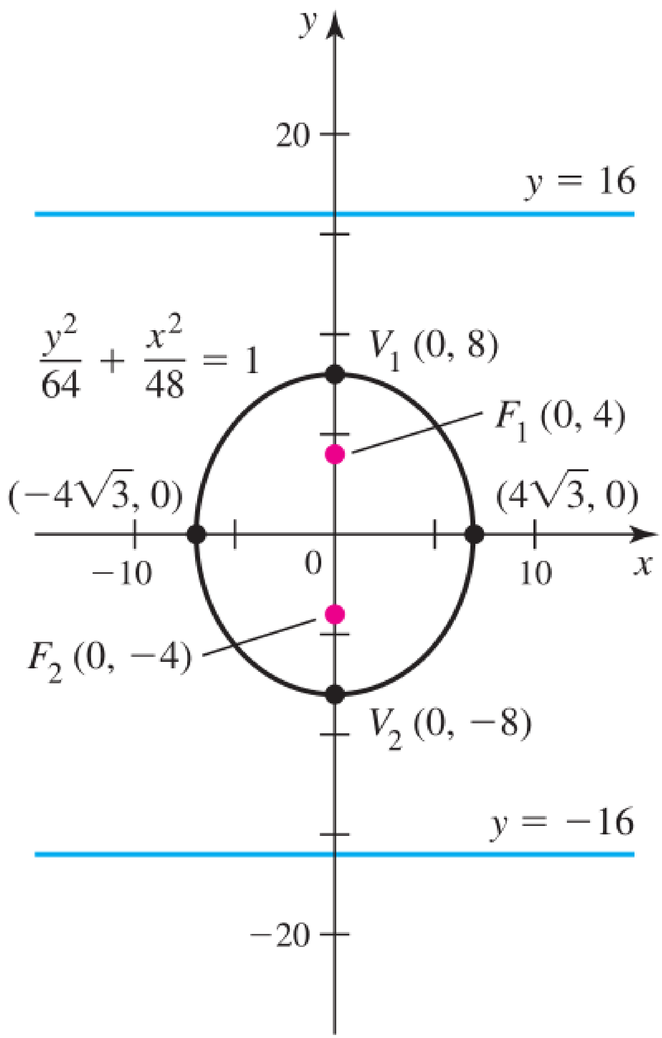
	Foci on x-axis	Foci on y-axis
Major-axis vertices:	$(\pm a, 0)$	$(0, \pm a)$
Minor-axis vertices (for ellipses):	$(0, \pm b)$	$(\pm b, 0)$
Foci:	$(\pm c, 0)$	$(0, \pm c)$
Directrices:	$x = \pm d$	$y = \pm d$
Eccentricity: $0 < e < 1$ for ellipses, $e > 1$ for hyperbolas.		

Given any two of the five parameters a, b, c, d, and e, the other three are found using the relations

$$c = ae, \quad d = \frac{a}{e},$$

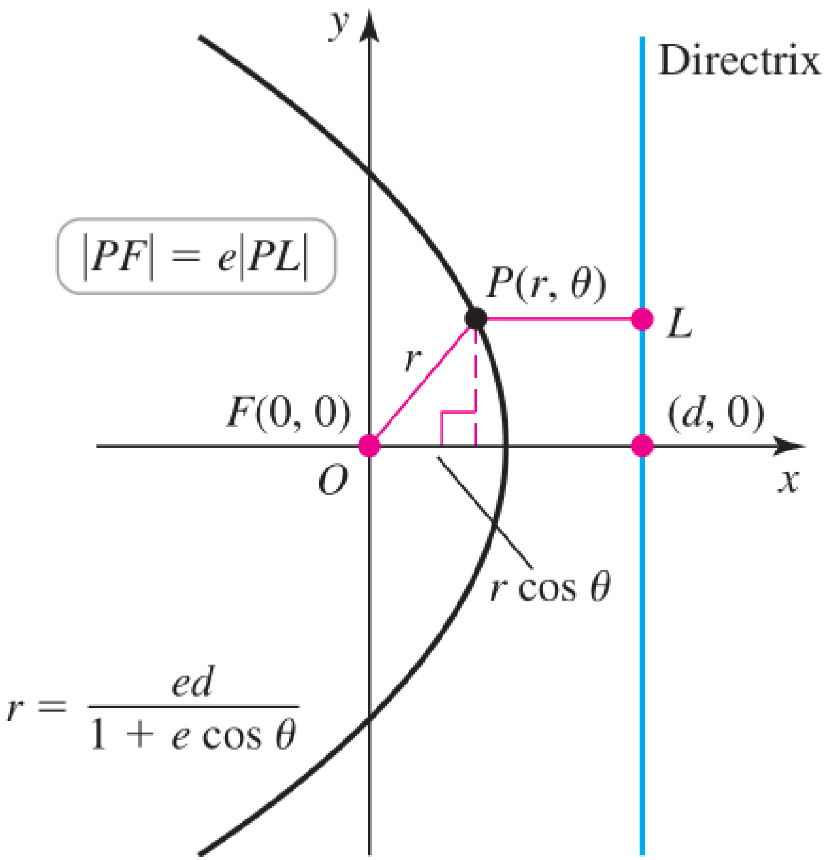
$$b^2 = a^2 - c^2 \quad \text{(for ellipses)}, \qquad b^2 = c^2 - a^2 \quad \text{(for hyperbolas)}.$$

EXAMPLE 6 **Equations of ellipses** Find the equation of the ellipse centered at the origin with foci F_1 and F_2 at $(0, \pm 4)$ and eccentricity $e = \frac{1}{2}$. Give the length of the major and minor axes, the location of the vertices, and the directrices. Graph the ellipse.



Polar Equations of Conic Sections

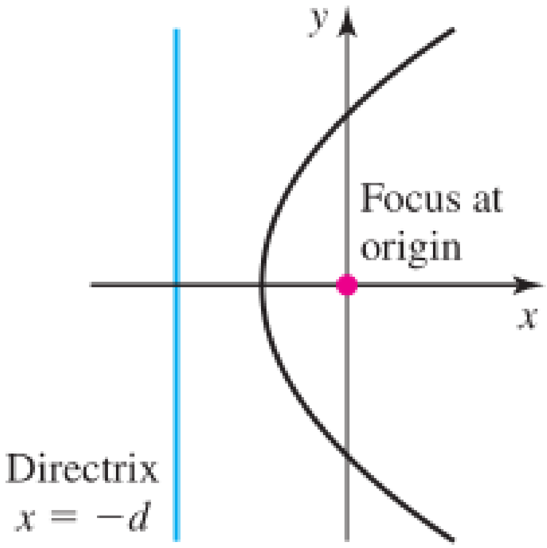
Conic sections have a natural representation in polar coordinates, provided the eccentricity-directrix approach.
A single polar equation covers parabolas, ellipses, and hyperbolas.



Placing one focus F at the origin and taking a directrix perpendicular to the x-axis through $(d, 0)$, $d > 0$.
$|PF| = r$, $|PL| = d - r\cos\theta$
The condition $\frac{|PF|}{|PL|} = e$ implies that $r = e(d - r\cos\theta)$, i.e.,

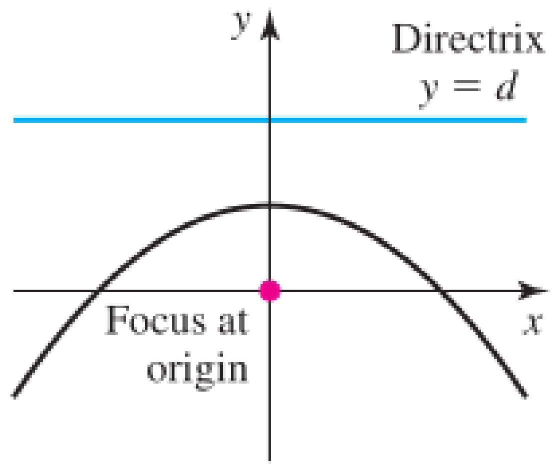
$$r = \frac{ed}{1 + e\cos\theta}$$

$$r = \frac{ed}{1 - e\cos\theta},\ d > 0$$



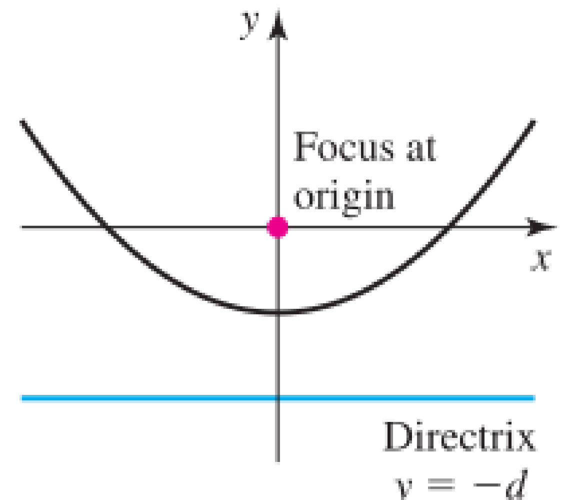
(b)

$$r = \frac{ed}{1 + e\sin\theta},\ d > 0$$



(c)

$$r = \frac{ed}{1 - e\sin\theta},\ d > 0$$



(d)

THEOREM 4 Polar Equations of Conic Sections

Let $d > 0$. The conic section with a focus at the origin and eccentricity e has the polar equation

$$\underbrace{r = \frac{ed}{1 + e\cos\theta}}_{\text{if one directrix is } x = d} \quad \text{or} \quad \underbrace{r = \frac{ed}{1 - e\cos\theta}.}_{\text{if one directrix is } x = -d}$$

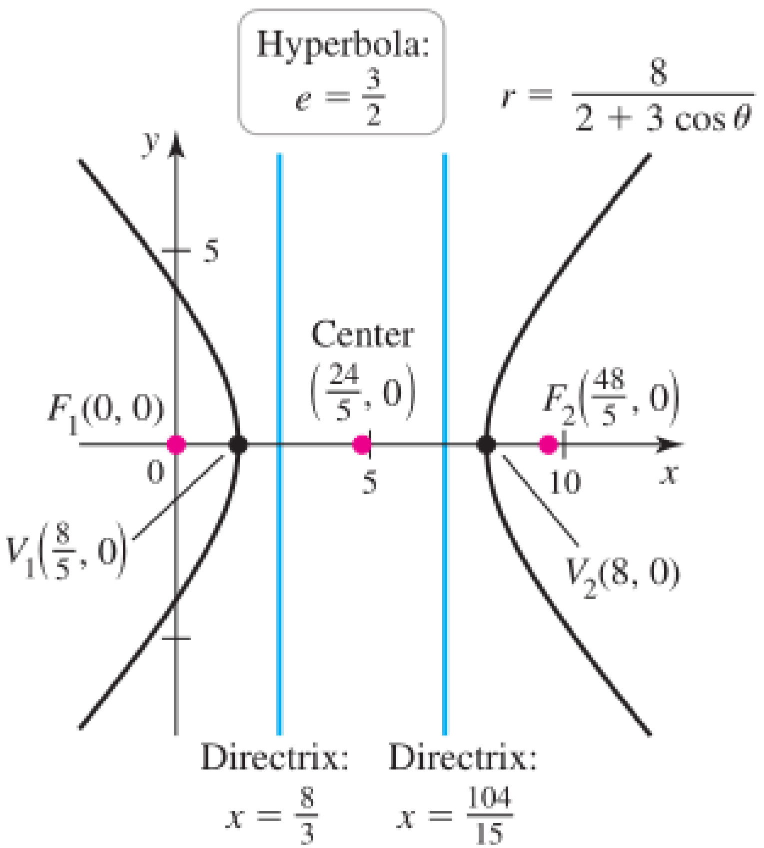
The conic section with a focus at the origin and eccentricity e has the polar equation

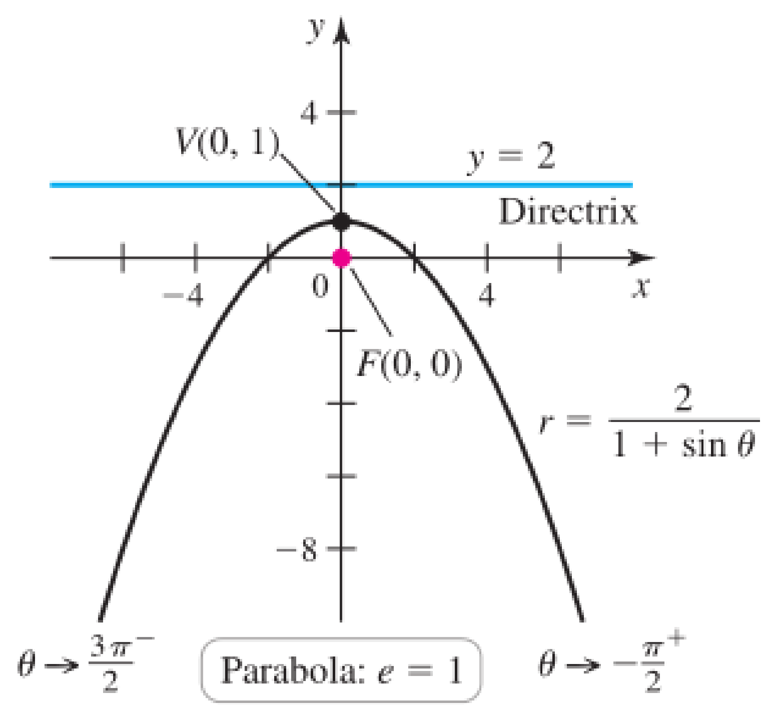
$$\underbrace{r = \frac{ed}{1 + e\sin\theta}}_{\text{if one directrix is } y = d} \quad \text{or} \quad \underbrace{r = \frac{ed}{1 - e\sin\theta}.}_{\text{if one directrix is } y = -d}$$

If $0 < e < 1$, the conic section is an ellipse; if $e = 1$, it is a parabola; and if $e > 1$, it is a hyperbola. The curves are defined over any interval in θ of length 2π.

EXAMPLE 7 **Conic sections in polar coordinates** Find the vertices, foci, and directrices of the following conic sections. Graph each curve and check your work with a graphing utility.

a. $r = \dfrac{8}{2 + 3\cos\theta}$ **b.** $r = \dfrac{2}{1 + \sin\theta}$





Chapter 12

Parametric and Polar Curves

Shuwei Chen
swchen@swjtu.edu.cn