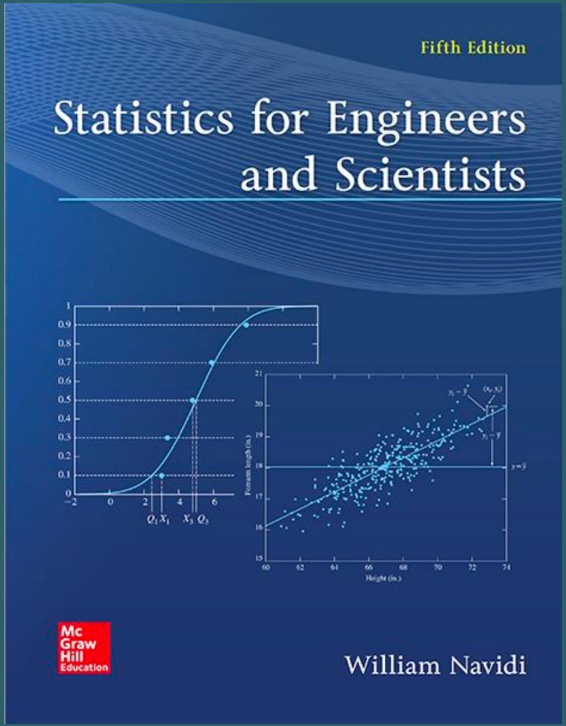


Probability (part 3)

(not all sections are required)

Chapter 2 Overview

2-1 Basic Ideas

2-2 Counting Methods

2-3 Conditional Probability and Independence

2-4 Random Variables

~~*2-5 Linear Functions of Random Variables*~~

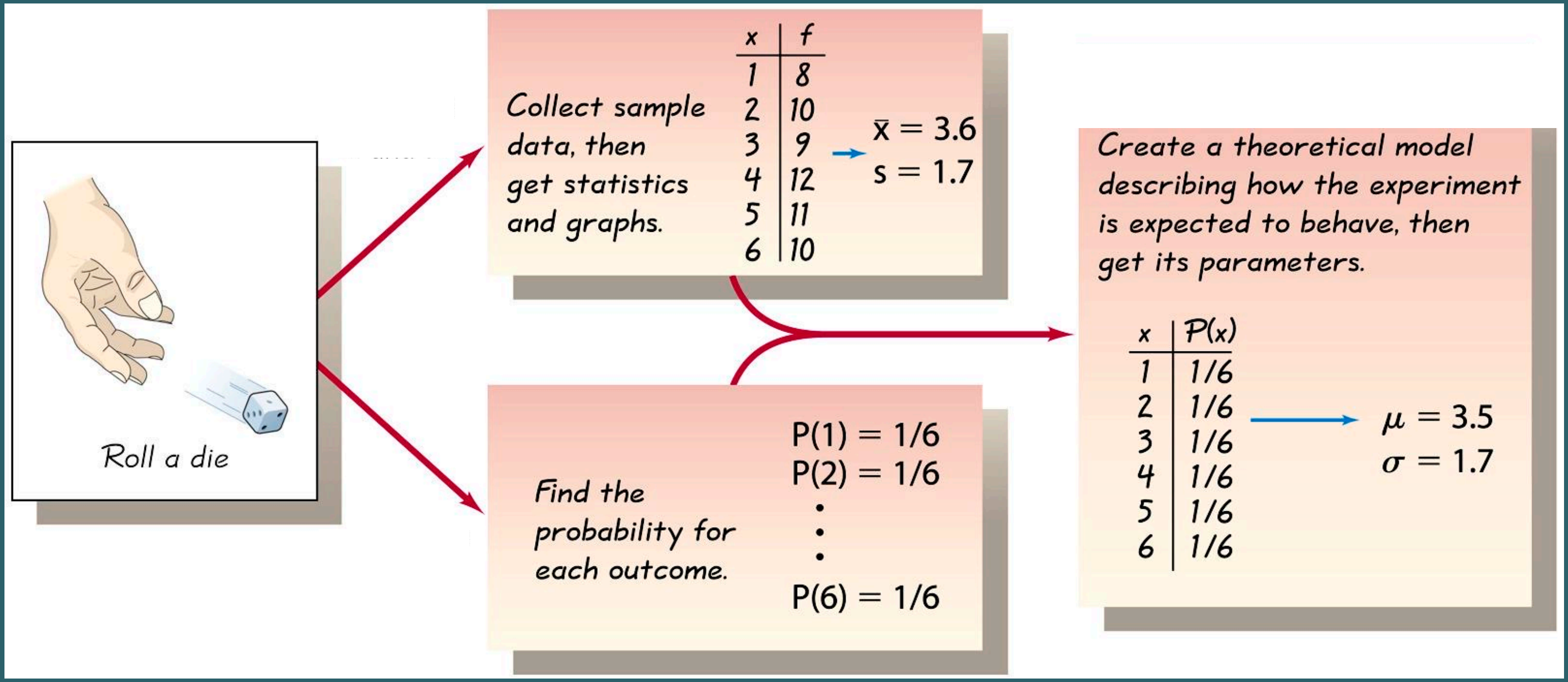
~~*2-6 Jointly Distributed Random Variables*~~

Introduction

- Probability distributions describe what will probably happen instead of what actually did happen.
- They are used to compare theoretical probabilities to actual results is order to determine whether outcomes are unusual.
- The core of inferential statistics is based on some knowledge of probability distributions.

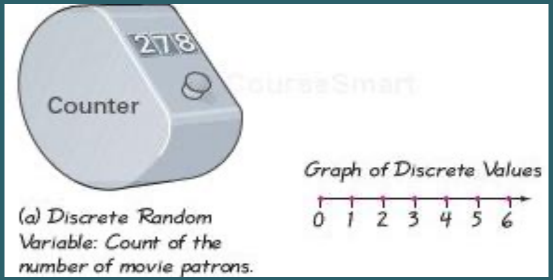
Combining Descriptive Methods and Probabilities

In this chapter we will construct probability distributions by presenting possible outcomes along with the relative frequencies we **expect**.



Recap of Variables

- A **random variable** is a variable whose values are determined by chance (x).
- A **discrete variable** is a variable can assume only a specific number of values; can be counted.



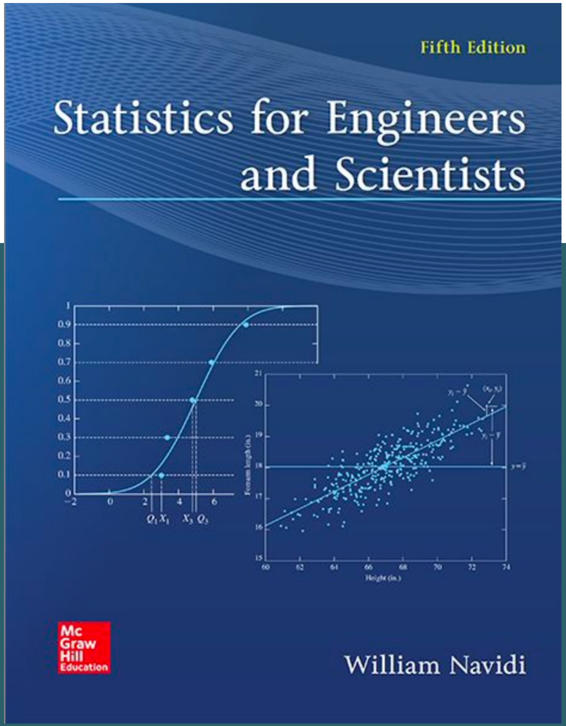
- A **continuous variable** is a variable that can assume all values in the interval between any two given values; can be measured.

Example

State whether the variable is discrete or continuous.

The cost of a Statistics textbook

- ○ **A. Continuous**
- ○ **B. Discrete**



Probability Distributions

A **discrete probability distribution** consists of the values a random variable can assume and the corresponding probabilities of the values.

<u>Two requirements for a probability distribution:</u>

1. The sum of the **probabilities** of all events in a sample space add up to 1.

2. Each probability is between 0 and 1, inclusively.

Example: Tossing 3 coins

When 3 coins are tossed, the sample space is represented as TTT, TTH, THT, HTT, HHT, HTH, THH, HHH.

If X is the random variable for the number of heads, then X assumes the value 0, 1, 2, or 3.

Probabilities of the values of X can be determined as follows:

No heads	One head			Two heads			Three heads
TTT	TTH	THT	HTT	HHT	HTH	THH	HHH
$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
$\frac{1}{8}$		$\frac{3}{8}$			$\frac{3}{8}$		$\frac{1}{8}$

Example: Tossing 3 coins

No heads	One head			Two heads			Three heads
TTT	TTH	THT	HTT	HHT	HTH	THH	HHH
$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
$\frac{1}{8}$	$\frac{3}{8}$			$\frac{3}{8}$			$\frac{1}{8}$

From these values, a probability distribution can be constructed by listing the outcomes and assigning the probability of each outcome.

Number of heads *X*	0	1	2	3
Probability *P(X)*	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

EXAMPLE 5–2 Tossing Coins

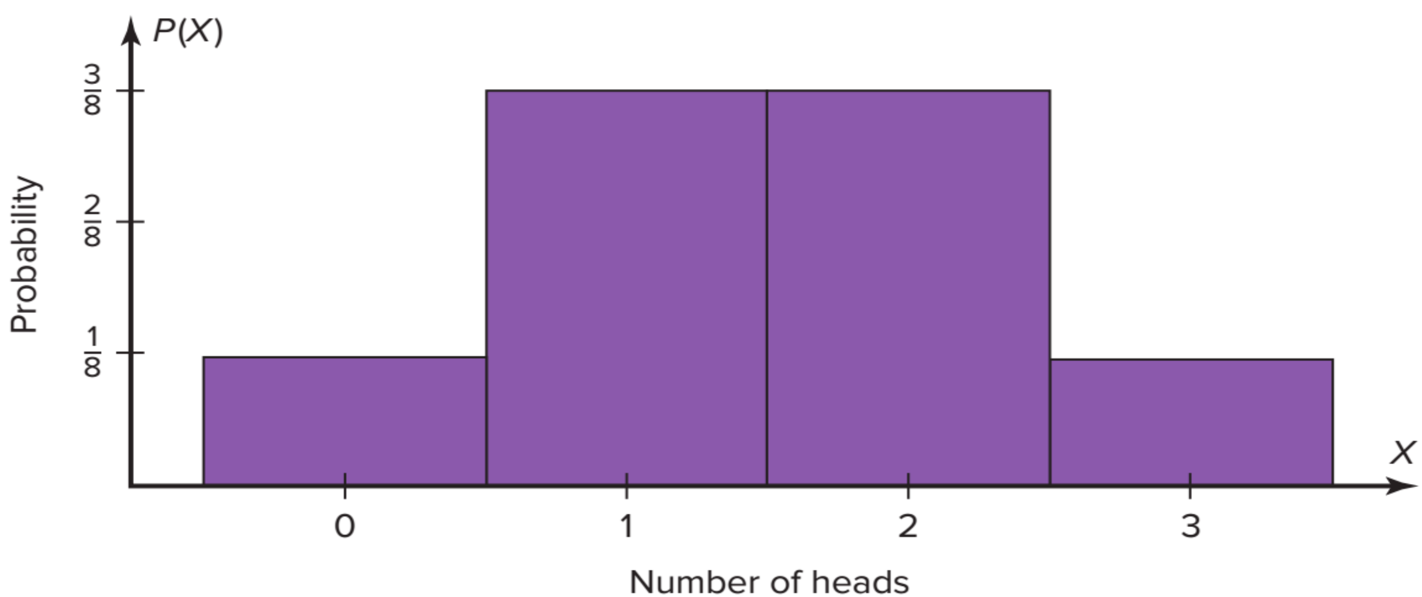
Represent graphically the probability distribution for the sample space for tossing three coins.

Number of heads *X*	0	1	2	3
Probability *P*(*X*)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

SOLUTION

The values that X assumes are located on the x axis, and the values for $P(X)$ are located on the y axis. The graph is shown in Figure 5–1.

FIGURE 5–1 Probability Distribution for Example 5–2



Probability Mass Functions (p.94)

Definition

The **probability mass function** of a discrete random variable X is the function $p(x) = P(X = x)$. The probability mass function is sometimes called the **probability distribution**.

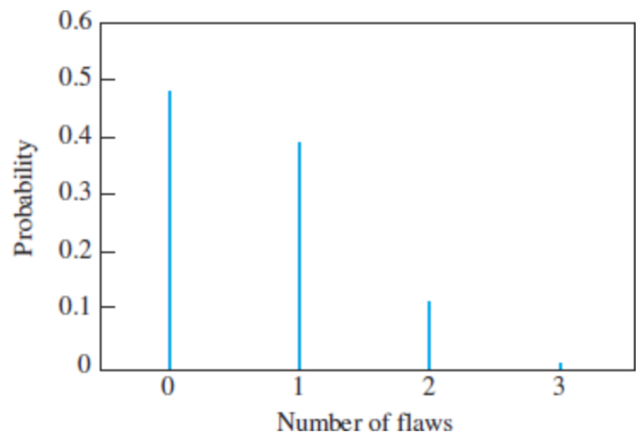


FIGURE 2.8 Probability mass function of X, the number of flaws in a randomly chosen piece of wire.

Example

Construct a probability distribution for a number of girls in a family with three children.

(1) Define Sample Space

In a family with three children, the possible outcomes with regard to gender are 3 boys. Another possibility is two boys, and one girl, where the girl is the youngest, the middle child, or the oldest. Another possibility is one boy and two girls where the boy is the oldest, the middle child, or the youngest. The last possibility is 3 girls.

The following represents the total possibilities:

BBB	BGG
BBG	GBG
BGB	GGB
GBB	GGG

(2) Identify Random Variable

We need to identify our random variable. In this case, we will let x = the number of girls in a family with three children.

BBB	BGG
BBG	GBG
BGB	GGB
GBB	GGG

From our sample space, we can see that the probability that this family would have 0 girls is 1/8 (BBB). The probability that there would be 1 girl is 3/8 (BBG,BGB, GBB). The probability that there would be two girls is 3/8 (BGG, GBG, GGB) and the probability that there would be 3 girls is 1/8 (GGG).

(3) Construct Discrete Probability Distribution

x	0	1	2	3
P(x)	1/8	3/8	3/8	1/8

x = the number of girls in a family with three children

Example: Battery Packages

A convenience store sells AA batteries in 2 per package, 4 per package, 6 per package, and 8 per package.

The store sells 5 two-packs, 10 four-packs, 8 six-packs, and 2 eight-packs over the weekend.

Construct a probability mass function for the variable.

Step 1 Make a frequency distribution for the variable.

Outcome	2	4	6	8
Frequency	5	10	8	2

Example: Battery Packages

Step 2 Find the probability for each outcome. The total of the frequencies is 25. Hence,

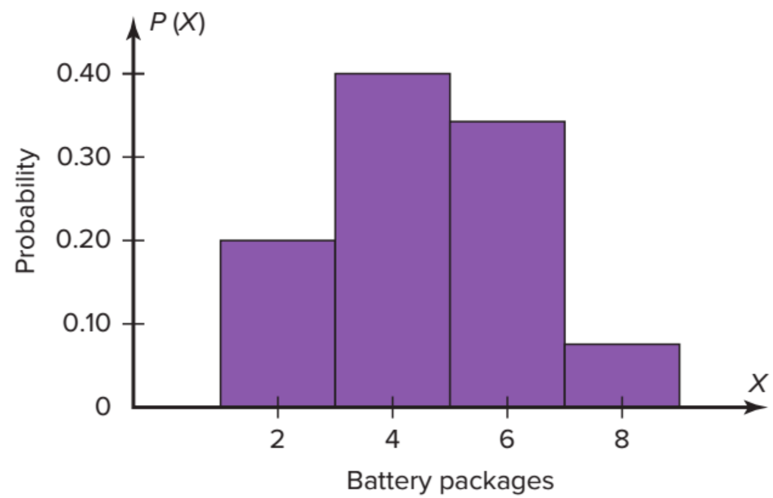
$$P(2) = \frac{5}{25} = 0.20 \qquad P(4) = \frac{10}{25} = 0.40$$

$$P(6) = \frac{8}{25} = 0.32 \qquad P(8) = \frac{2}{25} = 0.08$$

The probability distribution is

Outcome *X*	2	4	6	8
Probability *P*(*X*)	0.20	0.40	0.32	0.08

Step 3 Draw the graph, using the x axis for the outcomes and the y axis for the probabilities.

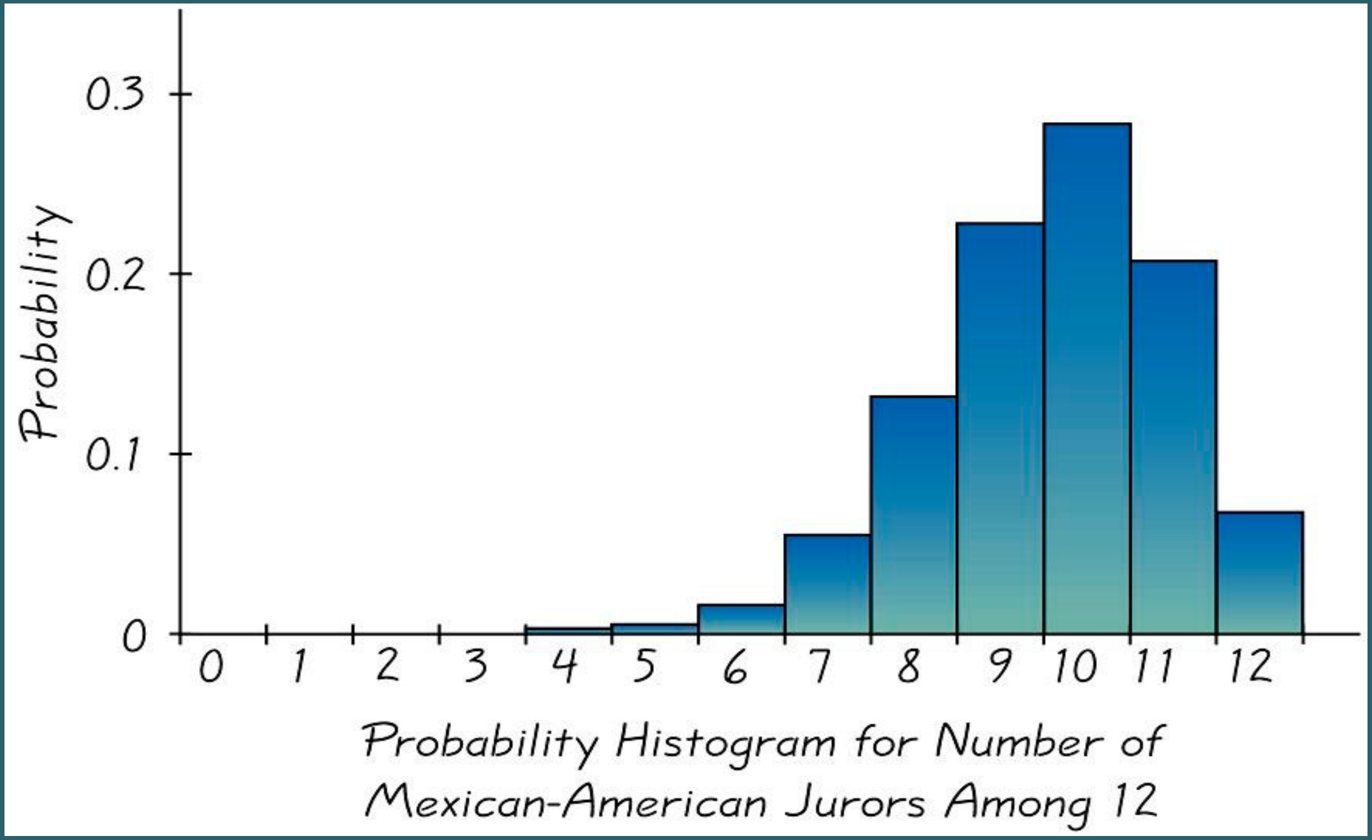


Probability Histogram

- When the possible values of a discrete random variable are evenly spaced, the probability mass function can be represented by a histogram, with rectangles centered at the possible values of the random variable.
- The area of a rectangle centered at a value x is equal to P(X = x). Such a histogram is called a **probability histogram**, because the areas represent probabilities.

Probability Histogram…

The **probability histogram** is very similar to a relative frequency histogram, but the vertical scale shows probabilities.



Example (p.101)

Table 2.3 presents the probability mass function for a random variable X that represents the number of defects in a printed circuit board.

TABLE 2.3

x	$P(X = x)$
0	0.45
1	0.35
2	0.15
3	0.05

Example (p.101)…

Figure 2.10 presents the probability histogram for this random variable.

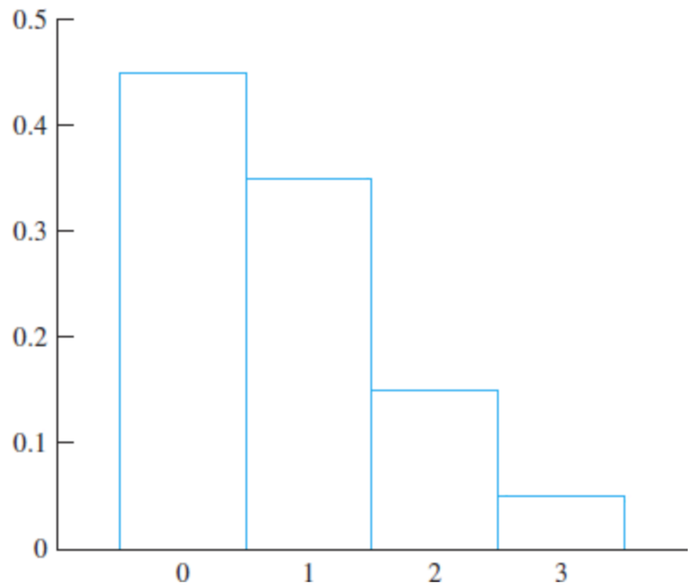


FIGURE 2.10 Probability histogram for X, the number of defects in a printed circuit board.

Mean, Variance and Standard Deviation for Discrete Random Variables

Questions…

- How to calculate **the mean** for a **discrete probability distribution?**
- How to calculate **the variance and/or standard deviation** for a **discrete probability distribution?**

The Mean

We leant that the mean for the sample or population can be computed as:

$$\bar{X} = \frac{X_1 + X_2 + X_3 + \cdots + X_n}{n} = \frac{\sum X}{n}$$

$$\mu = \frac{X_1 + X_2 + X_3 + \cdots + X_N}{N} = \frac{\sum X}{N}$$

Could you use these formulas to compute the average number of girls in a family with 3 children?

Example - Mean

Let x = number of girls in a family with 3 children

x	0	1	2	3
P(x)	1/8	3/8	3/8	1/8

What is the average number of girls in a family with 3 children?

It seems like this average would be 1.5, but it is not possible through experimentation to determine this average.

Formula for the Mean of a Probability Distribution

Such a determination would require infinitely many trials of the experiment. Therefore, it is necessary that this average would be determined theoretically.

The formula for calculating the theoretical mean for a discrete probability distribution shows that we must sum the products of the values of the random variable and the respective probabilities.

$$\mu = \sum X \cdot P(X)$$

Calculation of Mean

We will multiply each value of the random variable by its associated probability and add them all together.

$$\mu = \sum X \cdot P(X) = 0 \cdot \frac{1}{8} + 1 \cdot \frac{3}{8} + 2 \cdot \frac{3}{8} + 3 \cdot \frac{1}{8} = 1.5$$

Expected Value

The **mean** of a probability distribution **is also the expected value of the discrete random variable**. The expected value is expressed as **E(X)**.

We can also interpret this result by saying that if infinitely many families with three children were observed, we would expect an average of 1.5 girls per family.

Expected Value

The **expected value**, or **expectation**, of a discrete random variable of a probability distribution is the theoretical average of the variable.

The expected value is, by definition, the mean of the probability distribution.

$$E(X) = \mu = \sum X \cdot P(X)$$

Textbook (p.97)

Definition

Let X be a discrete random variable with probability mass function $p(x) = P(X = x)$.

The **mean** of X is given by

$$\mu_X = \sum_x x P(X = x) \tag{2.29}$$

where the sum is over all possible values of X.

The mean of X is sometimes called the expectation, or expected value, of X and may also be denoted by $E(X)$ or by μ.

Textbook (p.97)

***E**xample* **2.36**

A certain industrial process is brought down for recalibration whenever the quality of the items produced falls below specifications. Let X represent the number of times the process is recalibrated during a week, and assume that X has the following probability mass function.

x	0	1	2	3	4
$p(x)$	0.35	0.25	0.20	0.15	0.05

Find the mean of X.

Solution

Using Equation (2.29), we compute

$$\mu_X = 0(0.35) + 1(0.25) + 2(0.20) + 3(0.15) + 4(0.05) = 1.30$$

Textbook (p.97)

The population mean has an important physical interpretation. It is the horizontal component of the center of mass of the probability mass function; that is, it is the point on the horizontal axis at which the graph of the probability mass function would balance if supported there. Figure 2.9 illustrates this property for the probability mass function described in Example 2.36, where the population mean is $\mu = 1.30$.

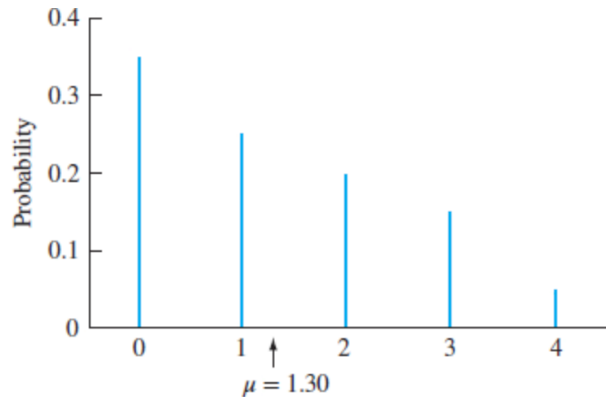


FIGURE 2.9 The graph of a probability mass function will balance if supported at the population mean.

Example: Children in a Family

In families with four children, find the mean number of children who will be girls.

SOLUTION

First, it is necessary to find the sample space. There are 16 outcomes, as shown.

BBBB	BBGG	GGGG
BBBG	BGBG	GGGB
BBGB	GGBB	GGBG
BGBB	GBGB	GBGG
GBBB	BGGB	BGGG
GBBG		

(A tree diagram may help.)

Next, make a probability distribution.

Number of girls *X*	0	1	2	3	4
Probability *P*(*X*)	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

Then multiply X and $P(X)$ for each outcome and find the sum.

$$\mu = \Sigma X \cdot P(X) = 0 \cdot \frac{1}{16} + 1 \cdot \frac{4}{16} + 2 \cdot \frac{6}{16} + 3 \cdot \frac{4}{16} + 4 \cdot \frac{1}{16} = 2$$

Hence, the mean of the number of females is 2.

Variance and Standard Deviation

We also learnt that the population variance and standard deviation can be computed as:

$$\sigma^2 = \frac{\sum (X-\mu)^2}{N}$$

$$\sigma = \sqrt{\frac{\sum (X-\mu)^2}{N}}$$

These formulas cannot be used for a random variable of a probability distribution since **N is infinite**, so the variance and standard deviation must be computed differently.

Variance

Let x = number of girls in a family with 3 children

x	0	1	2	3
P(x)	1/8	3/8	3/8	1/8

What is the variance for the number of girls in a family with 3 children?

Calculation of Variance

The formula for the variance is similar to the formula for the mean, except that the values of the random variable are squared and then the square of the mean is subtracted from the entire sum.

$$\boldsymbol{\sigma^2 = \sum X^2 \cdot P(X) - \mu^2}$$

$$= 0^2 \cdot \frac{1}{8} + 1^2 \cdot \frac{3}{8} + 2^2 \cdot \frac{3}{8} + 3^2 \cdot \frac{1}{8} - 1.5^2 = 0.75$$

Textbook (p.99)

Summary

Let X be a discrete random variable with probability mass function $p(x) = P(X = x)$. Then

- The variance of X is given by

$$\sigma_X^2 = \sum_x (x - \mu_X)^2 P(X = x) \tag{2.30}$$

- An alternate formula for the variance is given by

$$\sigma_X^2 = \sum_x x^2 P(X = x) - \mu_X^2 \tag{2.31}$$

- The variance of X may also be denoted by $V(X)$ or by σ^2.
- The standard deviation is the square root of the variance: $\sigma_X = \sqrt{\sigma_X^2}$.

Example 2.39

A resistor in a certain circuit is specified to have a resistance in the range 99 Ω–101 Ω. An engineer obtains two resistors. The probability that both of them meet the specification is 0.36, the probability that exactly one of them meets the specification is 0.48, and the probability that neither of them meets the specification is 0.16. Let X represent the number of resistors that meet the specification. Find the probability mass function, and the mean, variance, and standard deviation of X.

Solution

The probability mass function is $P(X = 0) = 0.16$, $P(X = 1) = 0.48$, $P(X = 2) = 0.36$, and $P(X = x) = 0$ for $x \neq 0, 1,$ or 2. The mean is

$$\begin{aligned}\mu_X &= (0)(0.16) + (1)(0.48) + (2)(0.36) \\ &= 1.200\end{aligned}$$

The variance is

Page 100

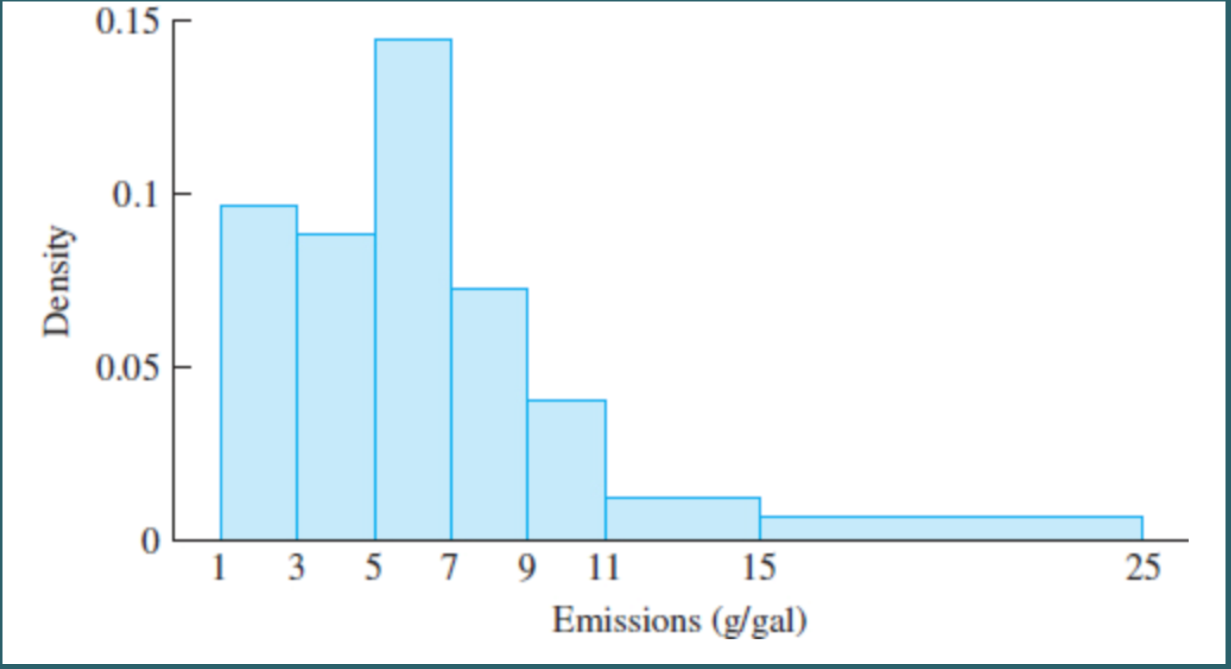
$$\begin{aligned}\sigma_X^2 &= (0 - 1.200)^2(0.16) + (1 - 1.200)^2(0.48) + (2 - 1.200)^2(0.36) \\ &= 0.4800\end{aligned}$$

The standard deviation is $\sigma_X = \sqrt{0.4800} = 0.693$.

Continuous Random Variables

- Figure 1.9 (in Section 1.3) presents a histogram for the emissions, in grams of particulates per gallon of fuel consumed, of a sample of 62 vehicles.

Note that emissions is a continuous variable, because its possible values are not restricted to some discretely spaced set.



Continuous Random Variables…

- The class intervals are chosen so that each interval contains a reasonably large number of vehicles.
- If the sample were larger, we could make the intervals narrower.
- In particular, if we had information on the entire population, containing millions of vehicles, we could make the intervals extremely narrow.
- The histogram would then look quite smooth and could be approximated with a curve, which might look like Figure 2.12.

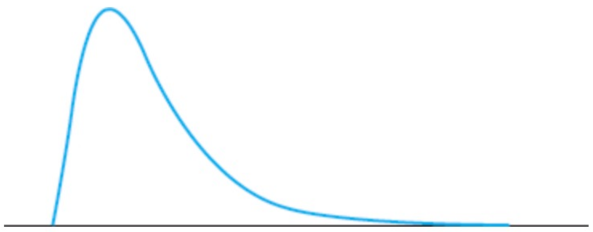


FIGURE 2.12 The histogram for a large continuous population could be drawn with extremely narrow rectangles and might look like this curve.

Definition

A random variable is **continuous** if its probabilities are given by areas under a curve. The curve is called a **probability density function** for the random variable.

The probability density function is sometimes called the **probability distribution**.

Summary

Let X be a continuous random variable with probability density function $f(x)$. Let a and b be any two numbers, with $a < b$. Then

$$P(a \leq X \leq b) = P(a \leq X < b) = P(a < X \leq b) = P(a < X < b) = \int_a^b f(x)\,dx \tag{2.32}$$

In addition,

$$P(X \leq b) = P(X < b) = \int_{-\infty}^{b} f(x)\,dx \tag{2.33}$$

$$P(X \geq a) = P(X > a) = \int_a^{\infty} f(x)\,dx \tag{2.34}$$

Summary

Let X be a continuous random variable with probability density function $f(x)$. Then

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \tag{2.35}$$

Example 2.41

A hole is drilled in a sheet-metal component, and then a shaft is inserted through the hole. The shaft clearance is equal to the difference between the radius of the hole and the radius of the shaft. Let the random variable X denote the clearance, in millimeters. The probability density function of X is

$$f(x) = \begin{cases} 1.25(1 - x^4) & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Components with clearances larger than 0.8 mm must be scrapped. What proportion of components are scrapped?

$$\begin{aligned} P(X > 0.8) &= \int_{0.8}^{\infty} f(x)\,dx \\ &= \int_{0.8}^{1} 1.25(1 - x^4)\,dx \\ &= 1.25\left(x - \frac{x^5}{5}\right)\Bigg|_{0.8}^{1} \\ &= 0.0819 \end{aligned}$$

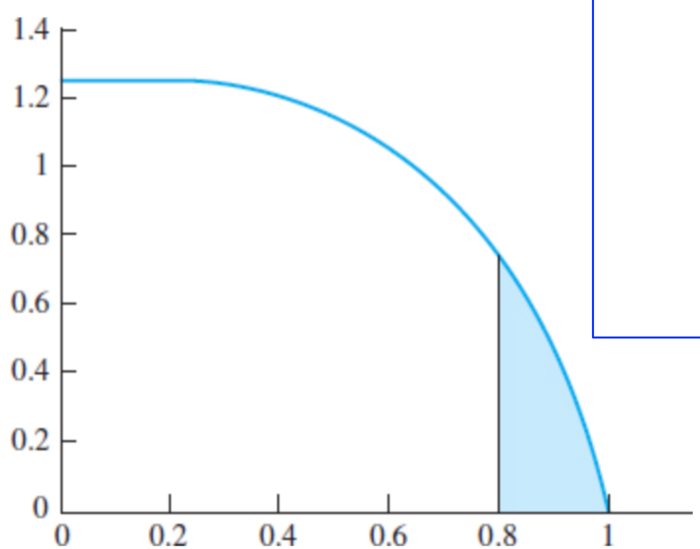


FIGURE 2.13 Graph of the probability density function of X, the clearance of a shaft. The area shaded is equal to $P(X > 0.8)$.

End of Chapter 2