Chapter 15

Functions of Several Variables (II)

Shuwei Chen

swchen@swjtu.edu.cn

15.5

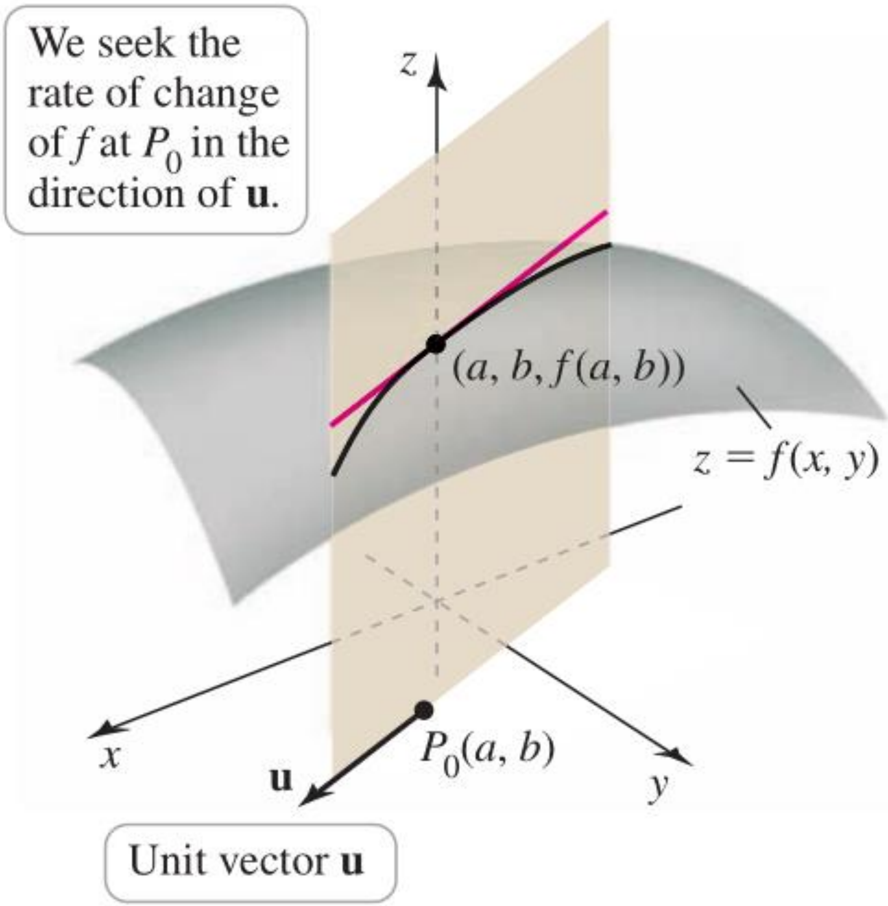
Directional Derivatives and the Gradient

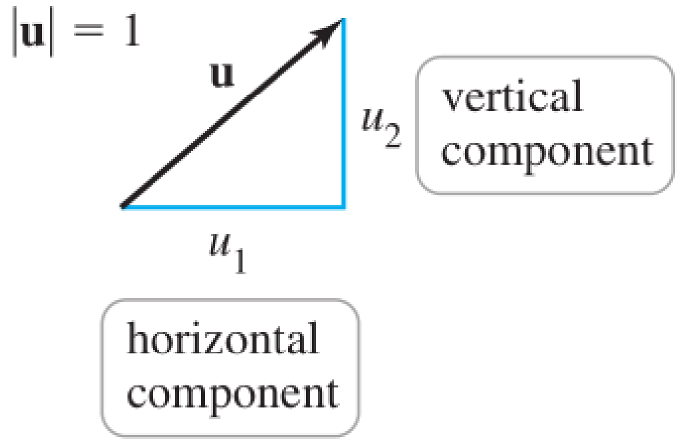
Recall: The partial derivatives f_x and f_y are the rate of change (or slope) of the surface at that point in the directions parallel to the x-axis and y-axis.

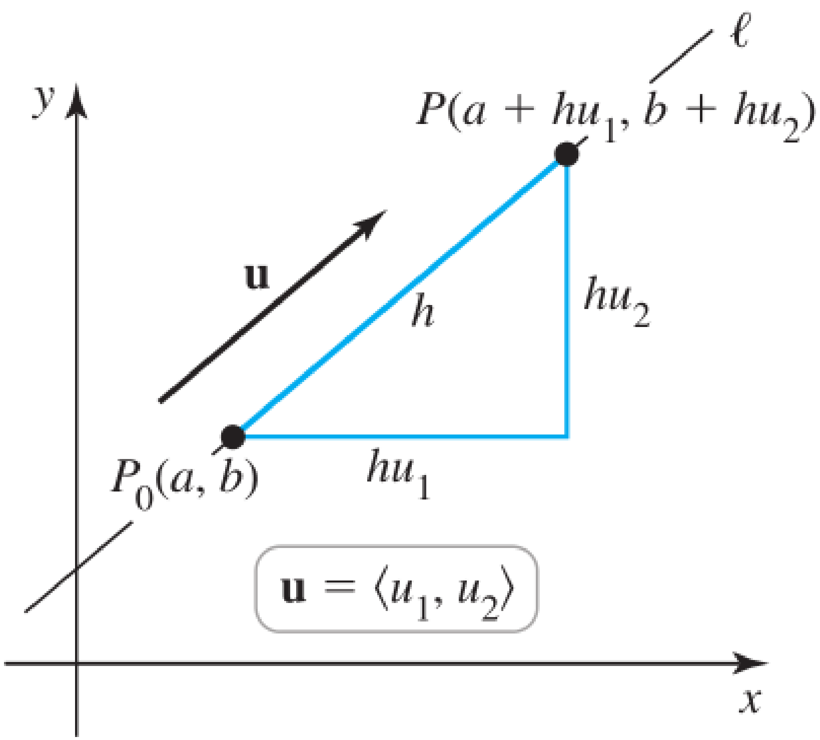
Several questions:

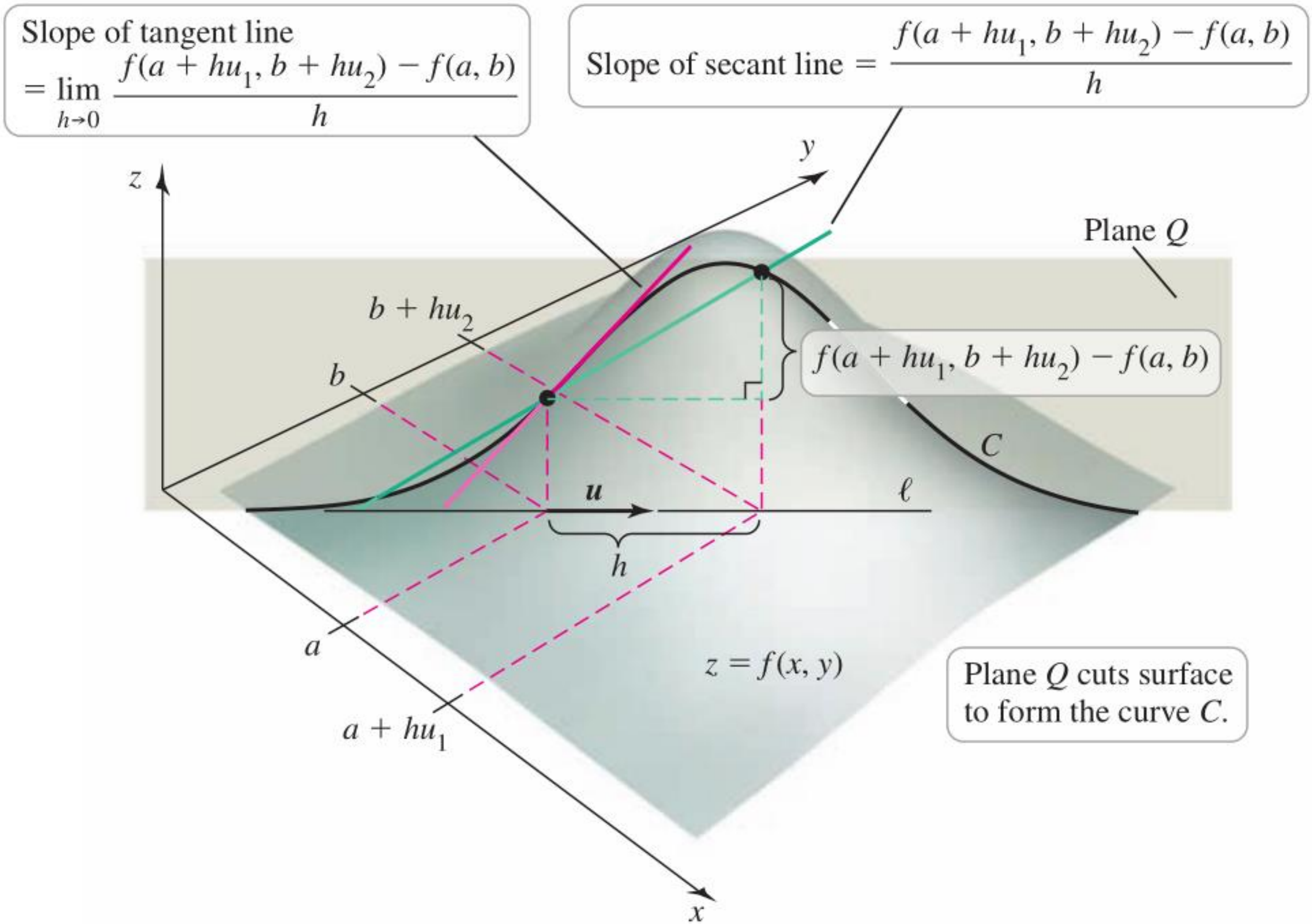
1. What is the rate of change of the function in a direction *other* than a coordinate direction?
2. Suppose you are standing on a surface and you release a ball at your feet and let it roll. In which direction will it roll?.
3. If you are hiking up a mountain, in what direction should you walk after each step if you want to follow the *steepest* path?

Directional Derivatives









Slope of tangent line
$= \lim_{h \to 0} \dfrac{f(a + hu_1, b + hu_2) - f(a, b)}{h}$
Slope of secant line $= \dfrac{f(a + hu_1, b + hu_2) - f(a, b)}{h}$
z
y
Plane Q
$b + hu_2$
b
$f(a + hu_1, b + hu_2) - f(a, b)$
C
$\boldsymbol{u}$
ℓ
h
a
$z = f(x, y)$
$a + hu_1$
x
Plane Q cuts surface to form the curve C.

DEFINITION Directional Derivative

Let f be differentiable at (a, b) and let $\mathbf{u} = \langle u_1, u_2 \rangle$ be a unit vector in the xy-plane. The **directional derivative of f at (a, b) in the direction of u** is

$$D_{\mathbf{u}}f(a, b) = \lim_{h \to 0} \frac{f(a + hu_1, b + hu_2) - f(a, b)}{h},$$

provided the limit exists.

Question: How to evaluate directional derivatives?

Express a directional derivative in terms of partial derivatives.

Key: Define a function that is equal to function f along the line ℓ through (a, b) in the direction of the unit vector $\mathbf{u} = \langle u_1, u_2 \rangle$

The points on ℓ satisfy the parametric equations (s corresponds to arc length)

$$x = a + su_1, y = b + su_2$$

Define the function

$$g(s) = f(\underbrace{a + su_1}_{x}, \underbrace{b + su_2}_{y}),$$

Apply the Chain Rule to find that

$$D_{\mathbf{u}} f(a, b) = g'(0) = \left(\frac{\partial f}{\partial x} \underbrace{\frac{dx}{ds}}_{u_1} + \frac{\partial f}{\partial y} \underbrace{\frac{dy}{ds}}_{u_2} \right) \bigg|_{s=0} \quad \text{Chain Rule}$$

$$= f_x(a, b) u_1 + f_y(a, b) u_2. \quad s = 0 \text{ corresponds to } (a, b).$$

THEOREM 10 Directional Derivative

Let f be differentiable at (a, b) and let $\mathbf{u} = \langle u_1, u_2 \rangle$ be a unit vector in the xy-plane. The **directional derivative of f at (a, b) in the direction of u** is

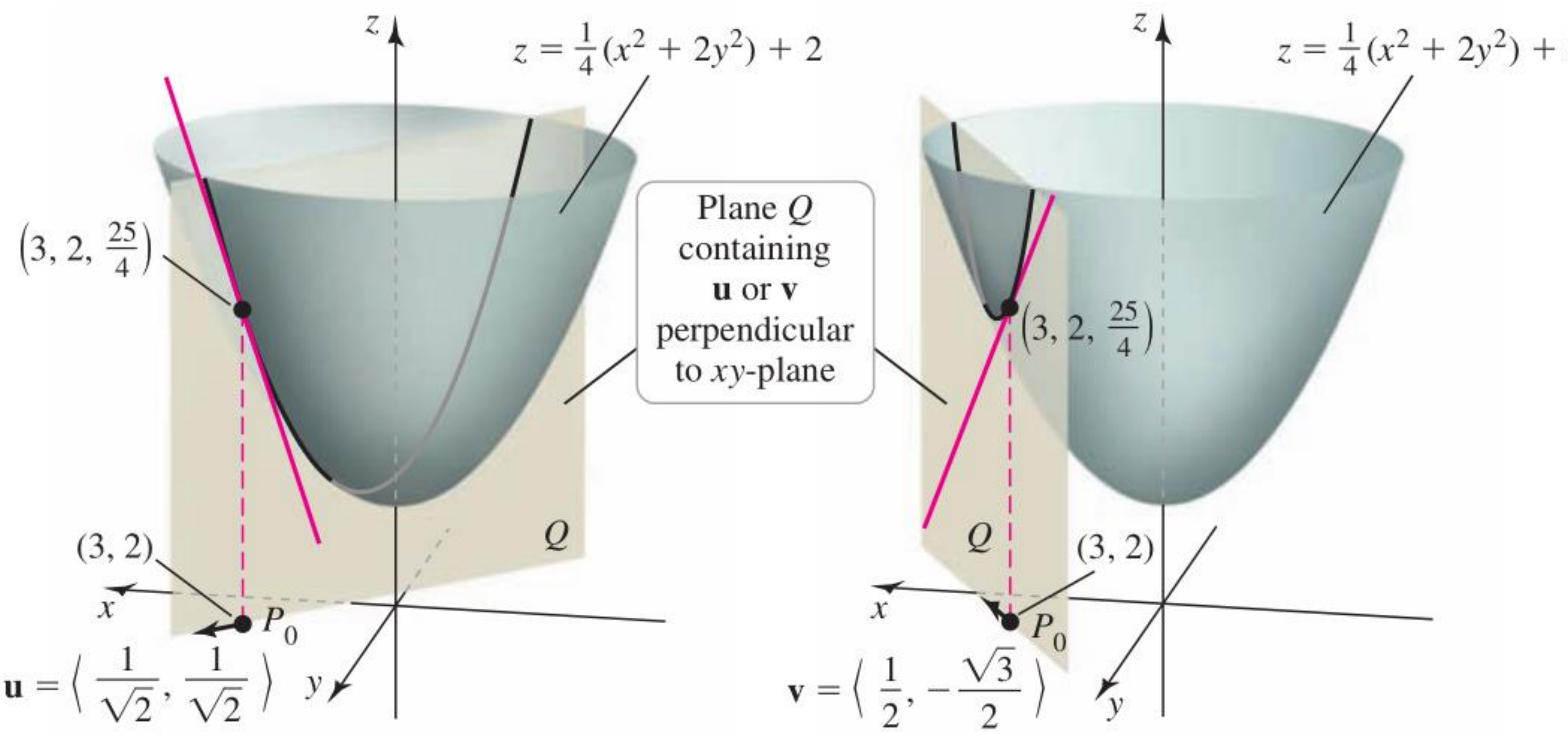
$$D_{\mathbf{u}} f(a, b) = \langle f_x(a, b), f_y(a, b) \rangle \cdot \langle u_1, u_2 \rangle .$$

EXAMPLE 1 **Computing directional derivatives** Consider the paraboloid $z = f(x, y) = \frac{1}{4}(x^2 + 2y^2) + 2$. Let P_0 be the point $(3, 2)$ and consider the unit vectors

$$\mathbf{u} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle \quad \text{and} \quad \mathbf{v} = \left\langle \frac{1}{2}, -\frac{\sqrt{3}}{2} \right\rangle.$$

a. Find the directional derivative of f at P_0 in the directions of $\mathbf{u}$ and $\mathbf{v}$.

b. Graph the surface and interpret the directional derivatives.



Slope of line in Q tangent to the intersection curve C in the direction of $\mathbf{u}$ has slope $D_{\mathbf{u}}f \approx 2.47$.

Slope of line in Q tangent to the intersection curve C in the direction of $\mathbf{v}$ has slope $D_{\mathbf{v}}f \approx -0.98$.

Gradient Vector

DEFINITION Gradient (Two Dimensions)

Let f be differentiable at the point (x, y). The **gradient** of f at (x, y) is the vector-valued function

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = f_x(x, y)\,\mathbf{i} + f_y(x, y)\,\mathbf{j}.$$

With the definition of the gradient, the directional derivative of f at (a, b) in the direction of the unit vector $\mathbf{u}$ can be written

$$D_{\mathbf{u}} f(a, b) = \nabla f(a, b) \cdot \mathbf{u}$$

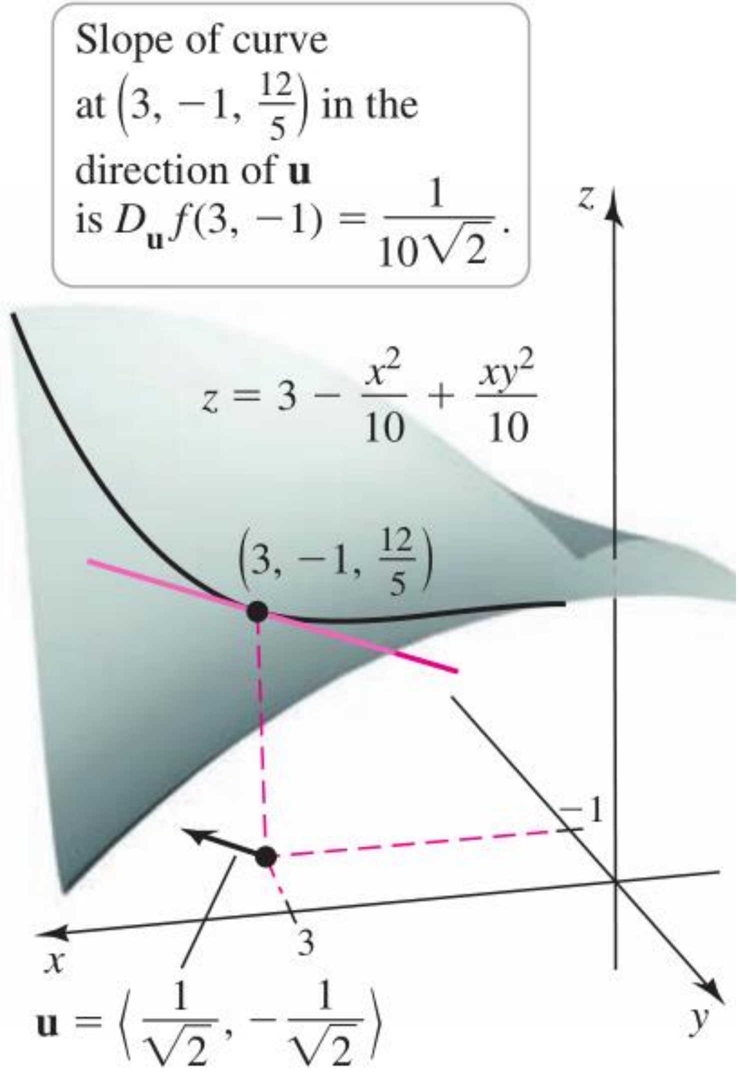
EXAMPLE 2 **Computing gradients** Find $\nabla f(3, 2)$ for $f(x, y) = x^2 + 2xy - y^3$.

EXAMPLE 3 **Computing directional derivatives with gradients** Let $f(x, y) = 3 - \frac{x^2}{10} + \frac{xy^2}{10}$.

a. Compute $\nabla f(3, -1)$.

b. Compute $D_{\mathbf{u}} f(3, -1)$, where $\mathbf{u} = \left\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle$.

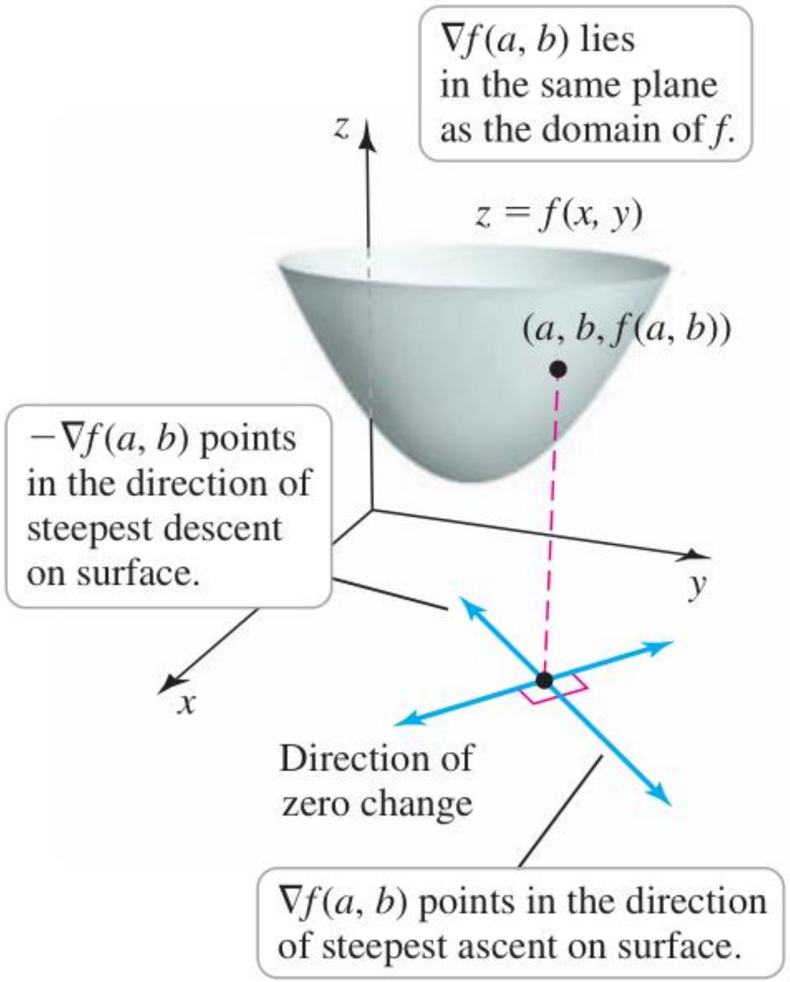
c. Compute the directional derivative of f at $(3, -1)$ in the direction of the vector $\langle 3, 4 \rangle$.


Slope of curve at $\left(3, -1, \frac{12}{5}\right)$ in the direction of $\mathbf{u}$ is $D_{\mathbf{u}}f(3, -1) = \frac{1}{10\sqrt{2}}$.
$z = 3 - \frac{x^2}{10} + \frac{xy^2}{10}$
$\left(3, -1, \frac{12}{5}\right)$
z
x
y
−1
3
$\mathbf{u} = \left\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle$

Interpretations of the Gradient

The directional derivative of f at (a,b) in the direction of the unit vector $\mathbf{u}$ is, where θ is the angle between $\nabla f(a,b)$ and $\mathbf{u}$

$$D_{\mathbf{u}} f(a,b) = \nabla f(a,b) \cdot \mathbf{u}$$
$$= |\nabla f(a,b)||\mathbf{u}| \cos\theta$$
$$= |\nabla f(a,b)| \cos\theta$$



Three cases:

1. When $\theta = 0$.
2. When $\theta = \pi$
3. When $\theta = \pi/2$

THEOREM 11 Directions of Change

Let f be differentiable at (a, b) with $\nabla f(a, b) \neq \mathbf{0}$.

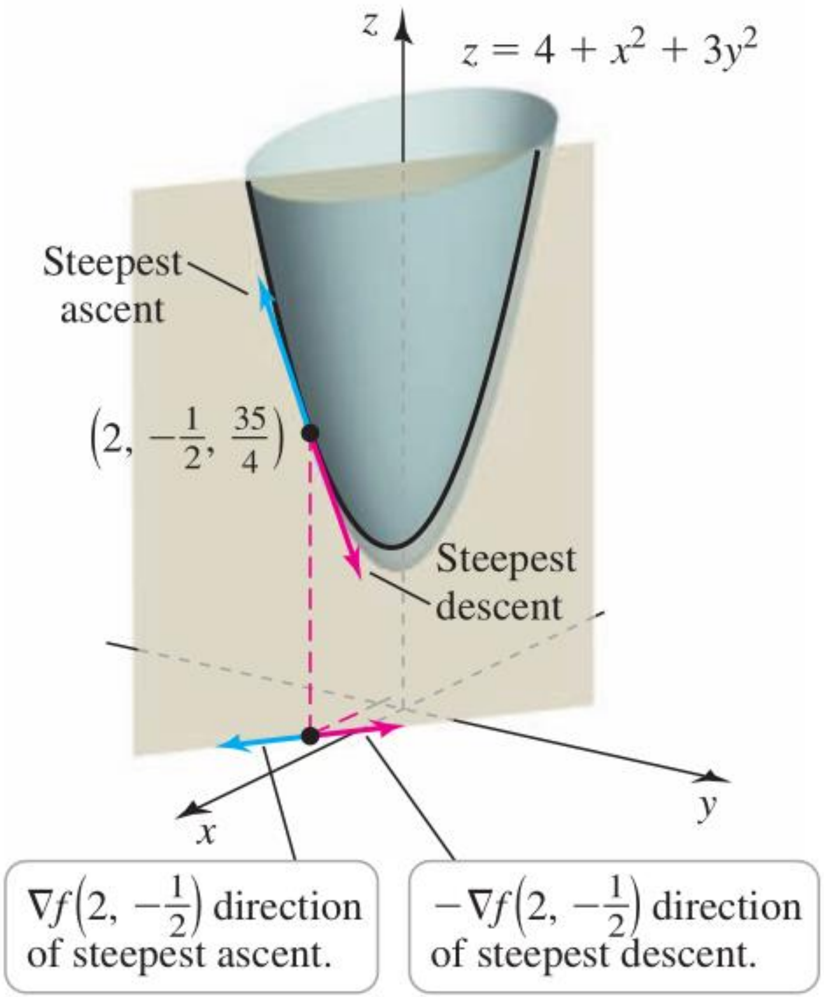
1. f has its maximum rate of increase at (a, b) in the direction of the gradient $\nabla f(a, b)$. The rate of change in this direction is $|\nabla f(a, b)|$.
2. f has its maximum rate of decrease at (a, b) in the direction of $-\nabla f(a, b)$. The rate of change in this direction is $-|\nabla f(a, b)|$.
3. The directional derivative is zero in any direction orthogonal to $\nabla f(a, b)$.

EXAMPLE 4 **Steepest ascent and descent** Consider the bowl-shaped paraboloid $z = f(x, y) = 4 + x^2 + 3y^2$.

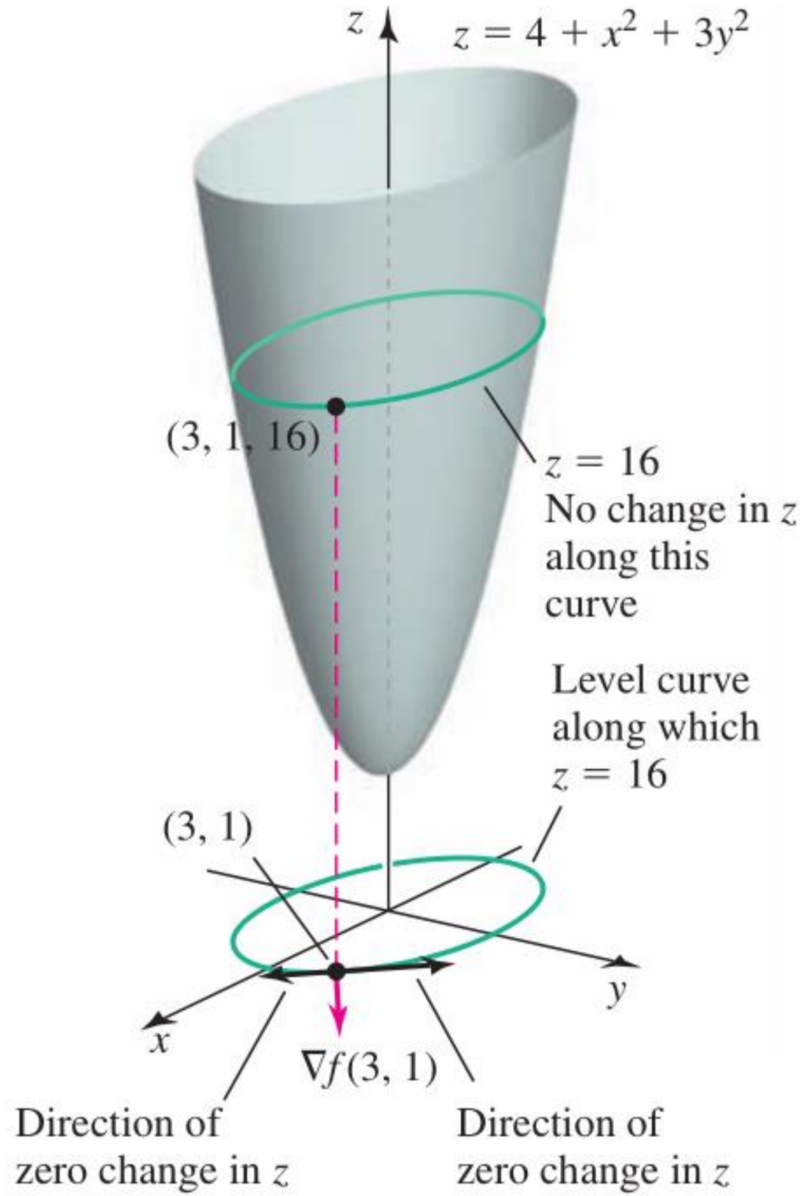
a. If you are located on the paraboloid at the point $\left(2, -\frac{1}{2}, \frac{35}{4}\right)$, in which direction should you move in order to *ascend* on the surface at the maximum rate? What is the rate of change?

b. If you are located at the point $\left(2, -\frac{1}{2}, \frac{35}{4}\right)$, in which direction should you move in order to *descend* on the surface at the maximum rate? What is the rate of change?

c. At the point $(3, 1, 16)$, in what direction(s) is there no change in the function values?



z
$z = 4 + x^2 + 3y^2$
Steepest ascent
$\left(2, -\frac{1}{2}, \frac{35}{4}\right)$
Steepest descent
x
y
$\nabla f\left(2, -\frac{1}{2}\right)$ direction of steepest ascent.
$-\nabla f\left(2, -\frac{1}{2}\right)$ direction of steepest descent.

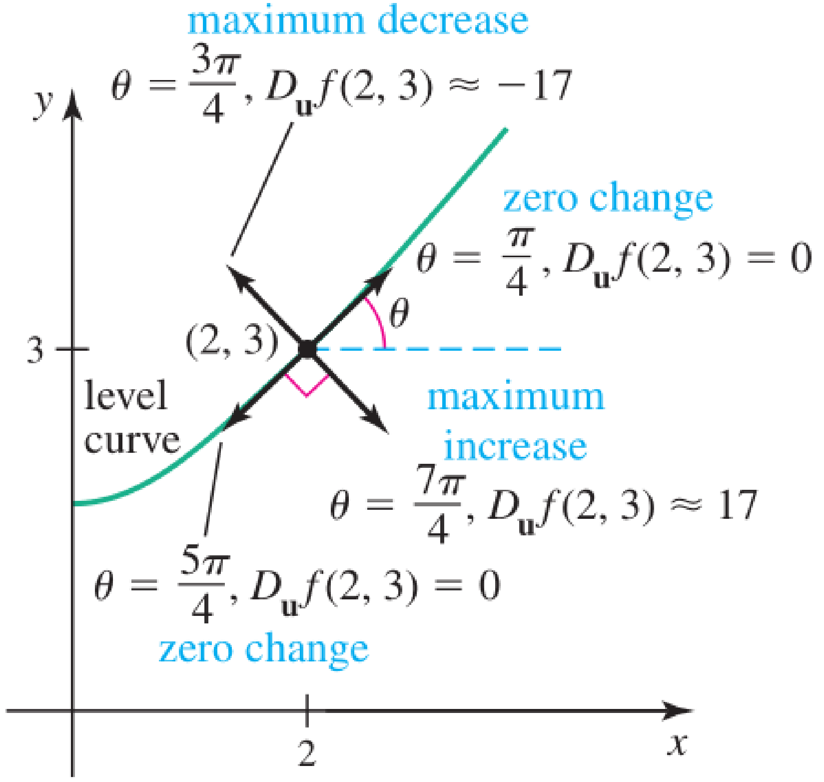


z
$z = 4 + x^2 + 3y^2$
(3, 1, 16)
$z = 16$
No change in z along this curve
Level curve along which $z = 16$
(3, 1)
x
y
$\nabla f(3, 1)$
Direction of zero change in z
Direction of zero change in z

EXAMPLE 5 **Interpreting directional derivatives** Consider the function $f(x, y) = 3x^2 - 2y^2$.

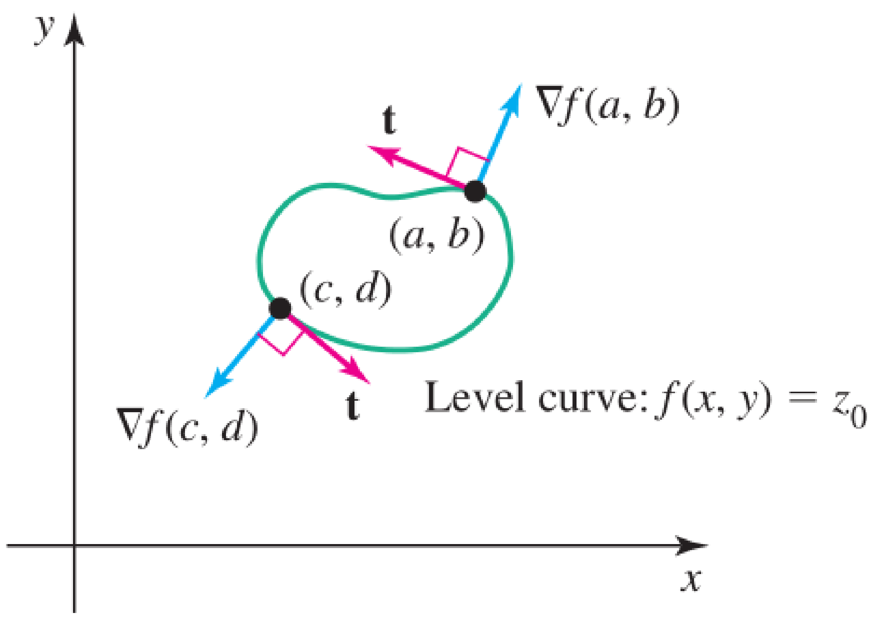
a. Compute $\nabla f(x, y)$ and $\nabla f(2, 3)$.

b. Let $\mathbf{u} = \langle \cos\theta, \sin\theta \rangle$ be a unit vector. At $(2, 3)$, for what values of θ (measured relative to the positive x-axis), with $0 \le \theta < 2\pi$, does the directional derivative have its maximum and minimum values and what are those values?



The Gradient and Level Curves

THEOREM 12 The Gradient and Level Curves
Given a function f differentiable at (a, b), the line tangent to the level curve of f at (a, b) is orthogonal to the gradient $\nabla f(a, b)$, provided $\nabla f(a, b) \neq \mathbf{0}$.



An immediate consequence of Theorem 12 is an alternative equation of the tangent line to the curve $f(a,b) = z_0$.

The tangent line is orthogonal to $\nabla f(a,b)$, i.e.,

$$\nabla f(a,b) \cdot \langle x-a, y-b \rangle = 0$$

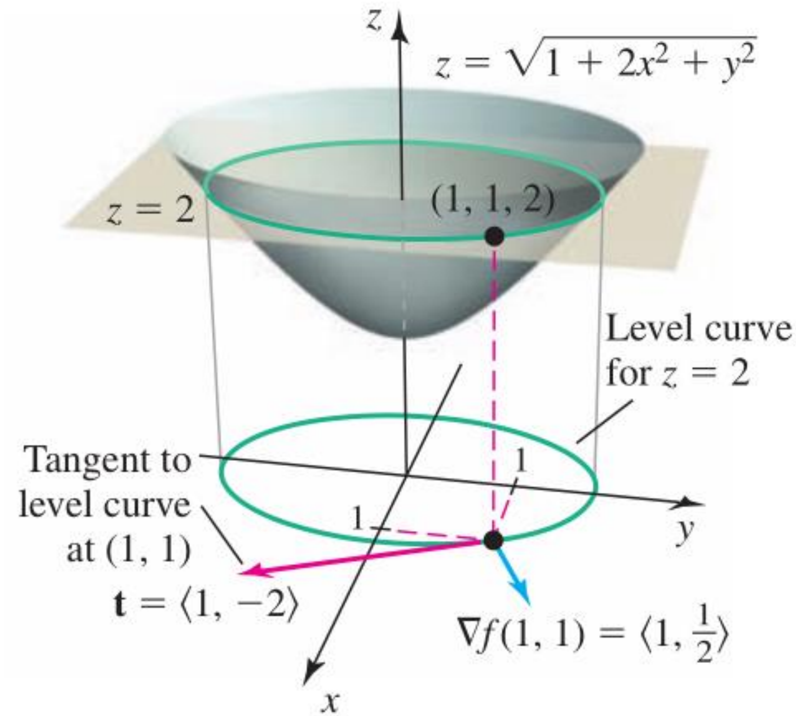
which, when simplified, gives an equation of the line tangent

$$f_x(a,b)(x-a) + f_y(a,b)(y-b) = 0$$

EXAMPLE 6 **Gradients and level curves** Consider the upper sheet $z = f(x, y) = \sqrt{1 + 2x^2 + y^2}$ of a hyperboloid of two sheets.

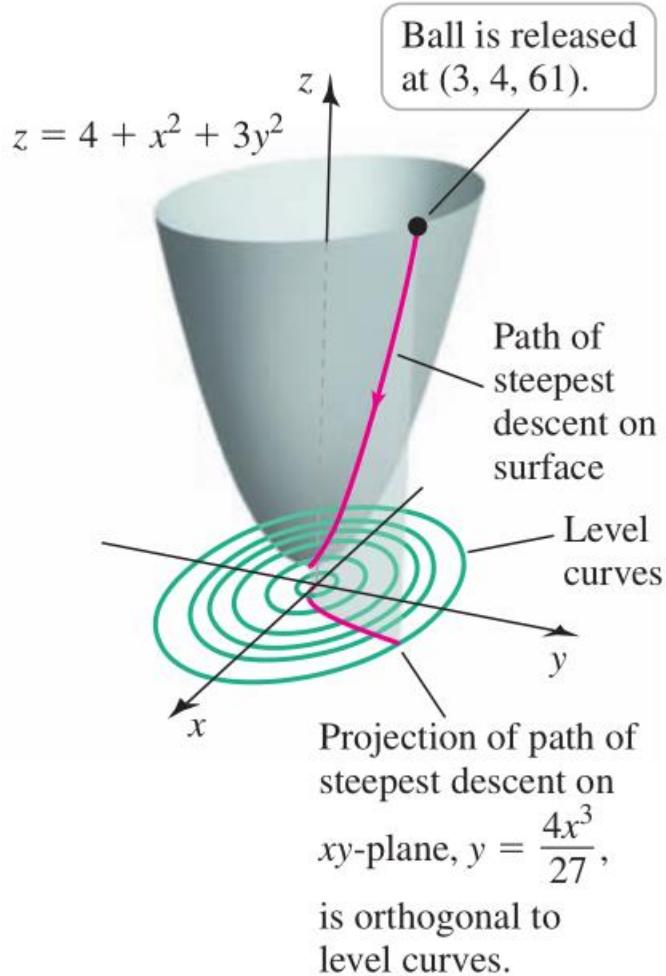
a. Verify that the gradient at $(1, 1)$ is orthogonal to the corresponding level curve at that point.

b. Find an equation of the line tangent to the level curve at $(1, 1)$.



$\mathbf{t} \cdot \nabla f = 0$
∇f is orthogonal to level curves.

EXAMPLE 7 **Path of steepest descent** Consider the paraboloid $z = f(x, y) = 4 + x^2 + 3y^2$ (Figure 73). Beginning at the point $(3, 4, 61)$ on the surface, find the projection in the xy-plane of the path of steepest descent on the surface.

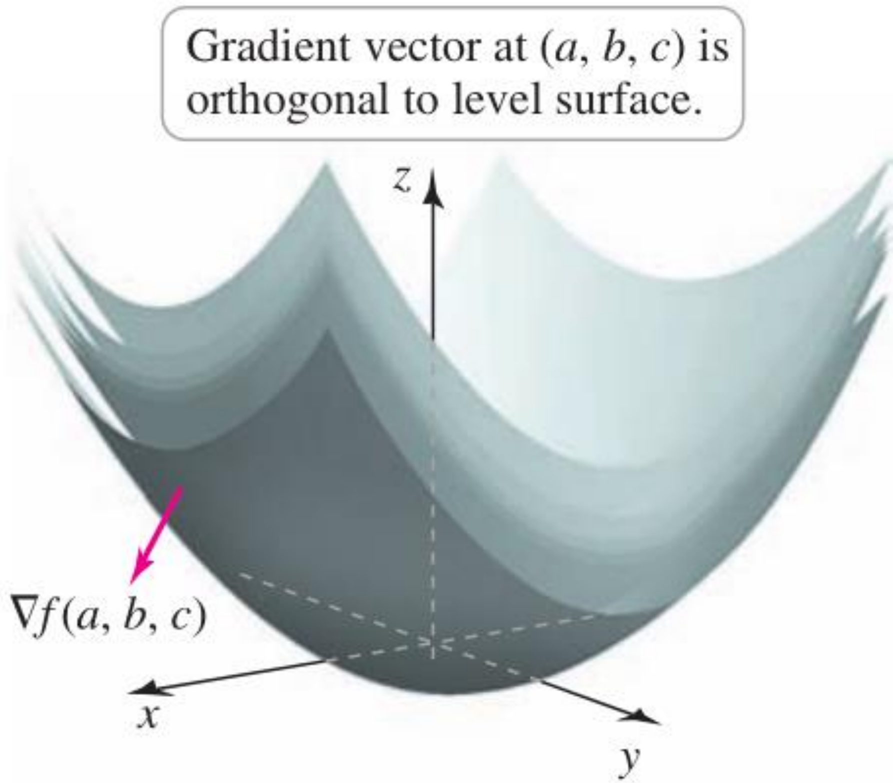


The Gradient in Three Dimensions

In $\mathbb{R}^3$, level curves become level surfaces $f(x, y, z) = C$.
The directional derivative of f in the direction of $\mathbf{u}$ at (a, b, c) is,

$$D_{\mathbf{u}} f(a, b, c) = \nabla f(a, b, c) \cdot \mathbf{u}$$

where $\nabla f(x, y, z) = \langle f_x, f_y, f_z \rangle$ is the gradient in three dimensions



DEFINITION Gradient and Directional Derivative in Three Dimensions

Let f be differentiable at the point (x, y, z). The **gradient** of f at (x, y, z) is the vector-valued function

$$\begin{aligned}\nabla f(x, y, z) &= \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle \\ &= f_x(x, y, z)\,\mathbf{i} + f_y(x, y, z)\,\mathbf{j} + f_z(x, y, z)\,\mathbf{k}.\end{aligned}$$

The **directional derivative** of f in the direction of the unit vector $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ at the point (a, b, c) is $D_{\mathbf{u}} f(a, b, c) = \nabla f(a, b, c) \cdot \mathbf{u}$.

THEOREM 15.13 Directional Derivative and Interpreting the Gradient

Let f be differentiable at (a, b, c) and let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ be a unit vector. The directional derivative of f at (a, b, c) in the direction of $\mathbf{u}$ is

$$\begin{aligned} D_{\mathbf{u}}f(a, b, c) &= \nabla f(a, b, c) \cdot \mathbf{u} \\ &= \langle f_x(a, b, c), f_y(a, b, c), f_z(a, b, c) \rangle \cdot \langle u_1, u_2, u_3 \rangle. \end{aligned}$$

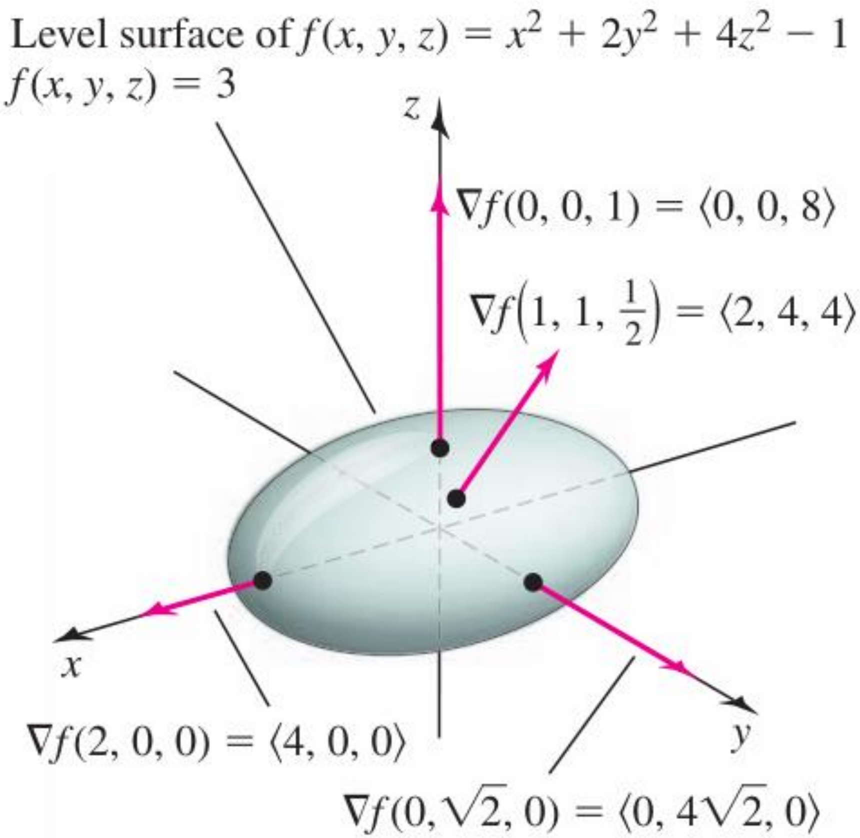
Assuming $\nabla f(a, b, c) \neq \mathbf{0}$, the gradient in three dimensions has the following properties.

1. f has its maximum rate of increase at (a, b, c) in the direction of the gradient $\nabla f(a, b, c)$, and the rate of change in this direction is $|\nabla f(a, b, c)|$.
2. f has its maximum rate of decrease at (a, b, c) in the direction of $-\nabla f(a, b, c)$, and the rate of change in this direction is $-|\nabla f(a, b, c)|$.
3. The directional derivative is zero in any direction orthogonal to $\nabla f(a, b, c)$.

EXAMPLE 8 **Gradients in three dimensions** Consider the function $f(x, y, z) = x^2 + 2y^2 + 4z^2 - 1$ and its level surface $f(x, y, z) = 3$.

a. Find and interpret the gradient at the points $P(2, 0, 0)$, $Q(0, \sqrt{2}, 0)$, $R(0, 0, 1)$, and $S\left(1, 1, \frac{1}{2}\right)$ on the level surface.

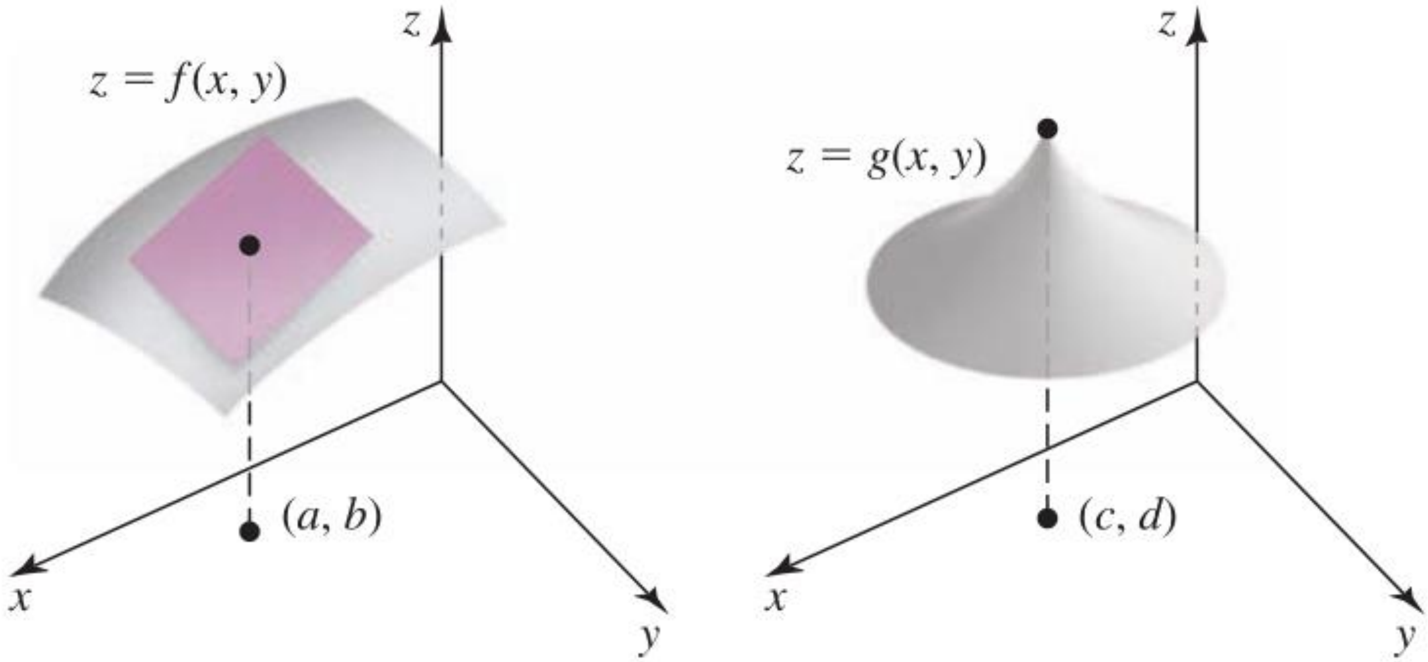
b. What are the actual rates of change of f in the directions of the gradients in part (a)?



15.6

Tangent Planes and Linear Approximation

Tangent lines to tangent planes



f differentiable at $(a, b) \Rightarrow$ tangent plane at $(a, b, f(a, b))$

g not differentiable at $(c, d) \Rightarrow$ no tangent plane at $(c, d, g(c, d))$

Tangent Planes

Recall: a surface in $\mathbb{R}^3$ may be defined in two different ways:

- **Explicitly** in the form $z = f(x, y)$ or
- **Implicitly** in the form $F(x, y, z) = 0$

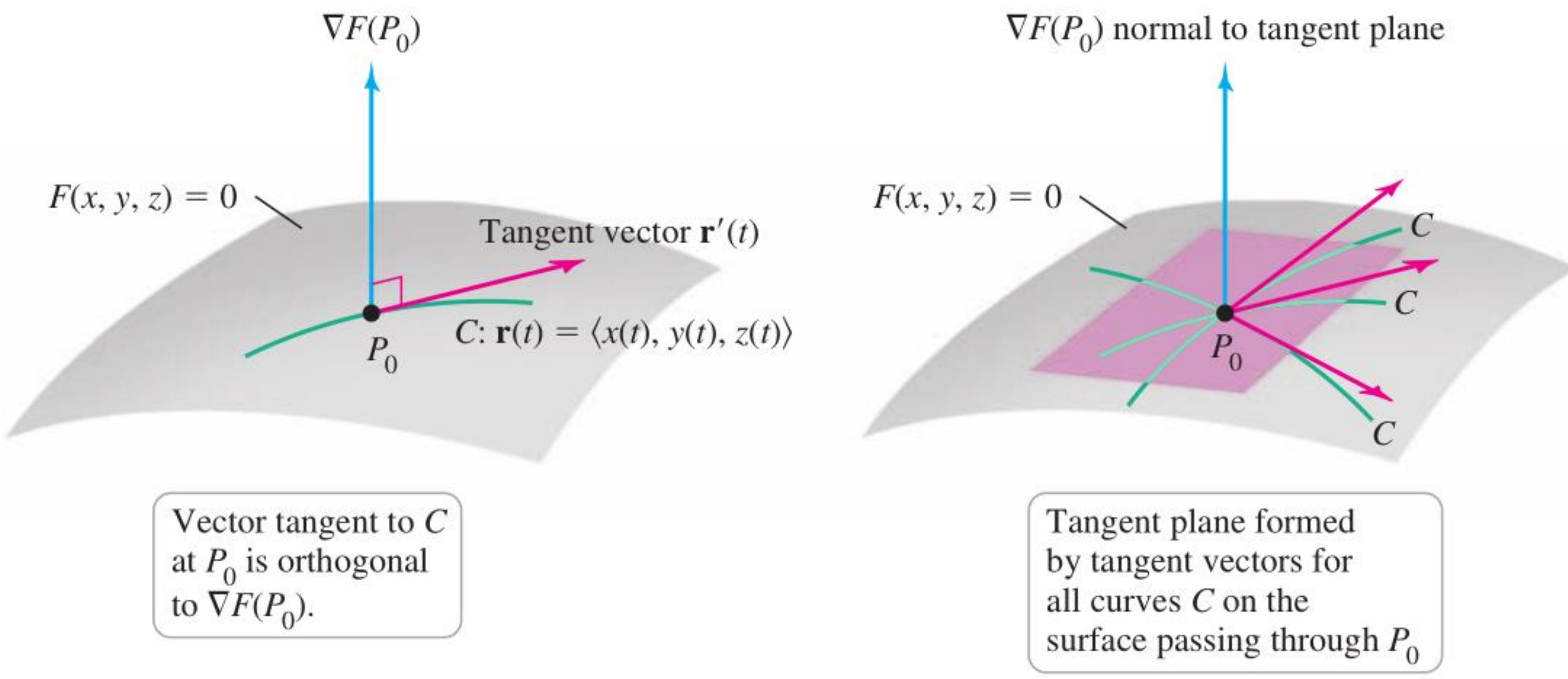
Tangent planes for $F(x, y, z) = 0$

Differentiating both sides of $F(x, y, z) = 0$ with respect to t

$$\begin{aligned} \frac{d}{dt}\left(F(x(t), y(t), z(t))\right) &= \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} \\ &= \underbrace{\left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right\rangle}_{\nabla F(x, y, z)} \cdot \underbrace{\left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle}_{\mathbf{r}'(t)} \\ &= \nabla F(x, y, z) \cdot \mathbf{r}'(t). \end{aligned}$$

Therefore, $\nabla F(x, y, z) \cdot \mathbf{r}'(t) = 0$

The tangent vectors for all these curves (with their tails at P_0) are orthogonal to $\nabla F(a, b, c)$. Therefore, they all lie in the same plane, called the *tangent plane* at P_0.



DEFINITION **Equation of the Tangent Plane for $F(x, y, z) = 0$**

Let F be differentiable at the point $P_0(a, b, c)$ with $\nabla F(a, b, c) \neq \mathbf{0}$. The plane tangent to the surface $F(x, y, z) = 0$ at P_0, called the **tangent plane**, is the plane passing through P_0 orthogonal to $\nabla F(a, b, c)$. An equation of the tangent plane is

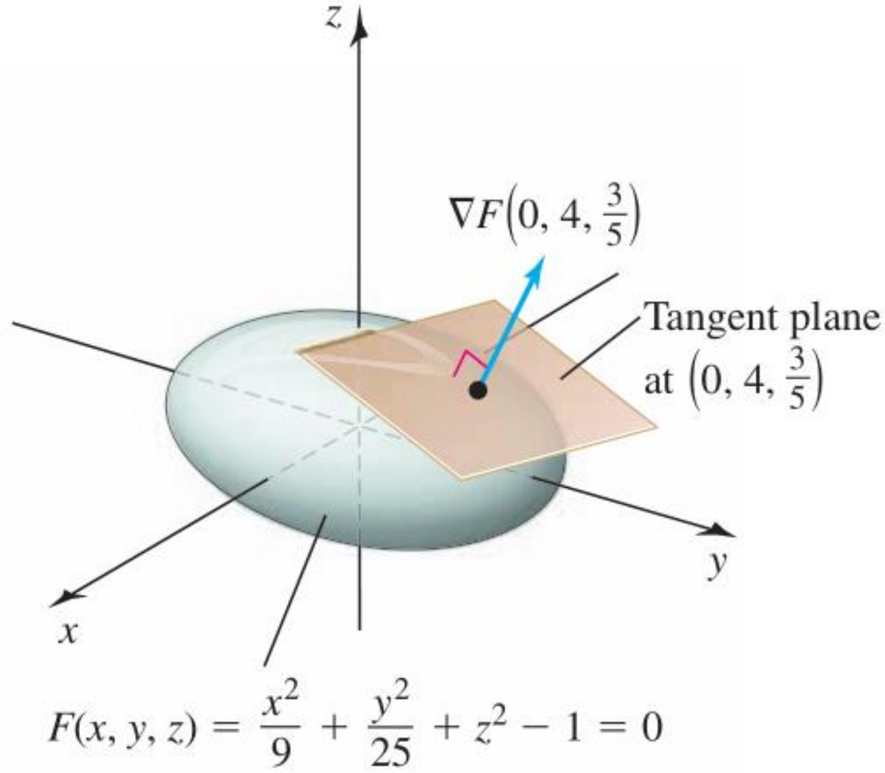
$$F_x(a, b, c)(x - a) + F_y(a, b, c)(y - b) + F_z(a, b, c)(z - c) = 0.$$

EXAMPLE 1 **Equation of a tangent plane** Consider the ellipsoid

$$F(x, y, z) = \frac{x^2}{9} + \frac{y^2}{25} + z^2 - 1 = 0.$$

a. Find the equation of the plane tangent to the ellipsoid at $\left(0, 4, \frac{3}{5}\right)$.

b. At what points on the ellipsoid is the tangent plane horizontal?



Tangent planes for $z = f(x, y)$

Written as $F(x, y, z) = z - f(x, y) = 0$, and the gradient of F at the point $(a, b, f(a, b))$ is

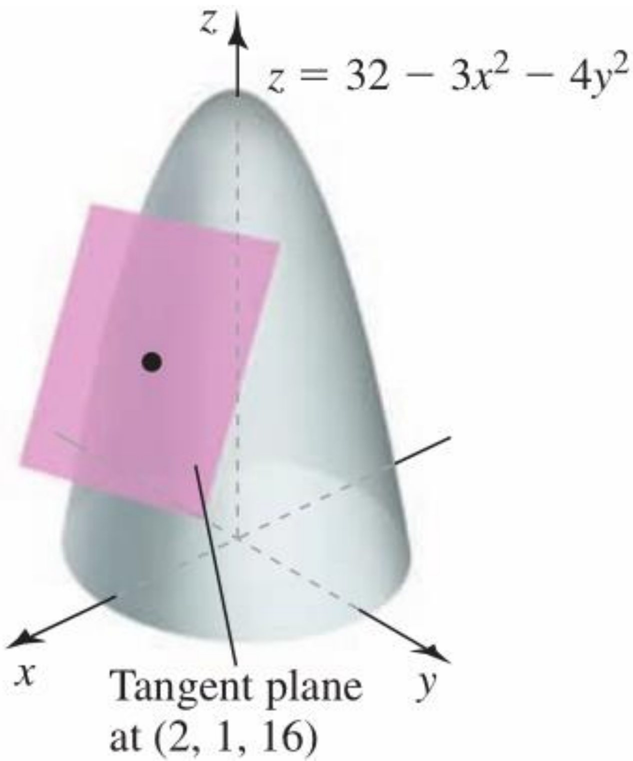
$$\nabla F(a, b, f(a, b))$$
$$= \left\langle F_x(a, b, f(a, b)), F_y(a, b, f(a, b)), F_z(a, b, f(a, b)) \right\rangle$$
$$= \left\langle -f_x(a, b), -f_y(a, b), 1 \right\rangle$$

Tangent Plane for $z = f(x, y)$

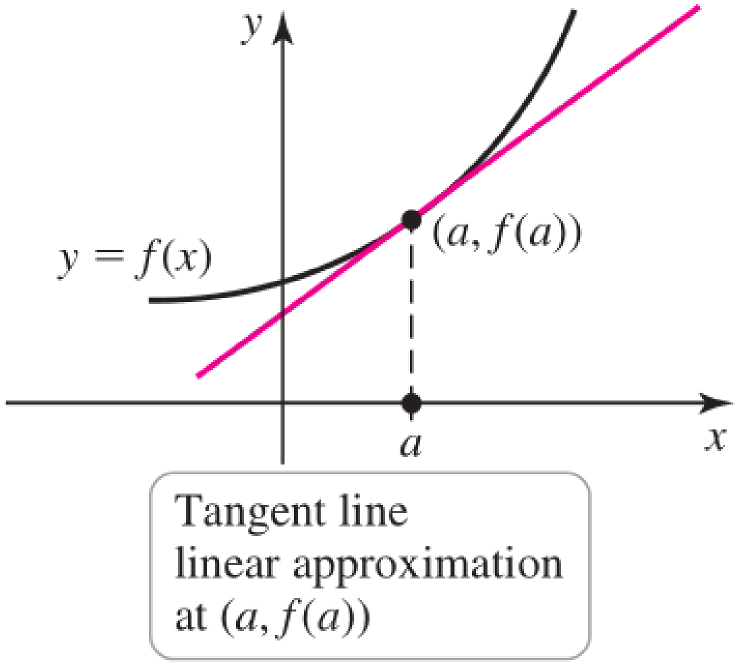
Let f be differentiable at the point (a, b). An equation of the plane tangent to the surface $z = f(x, y)$ at the point $(a, b, f(a, b))$ is

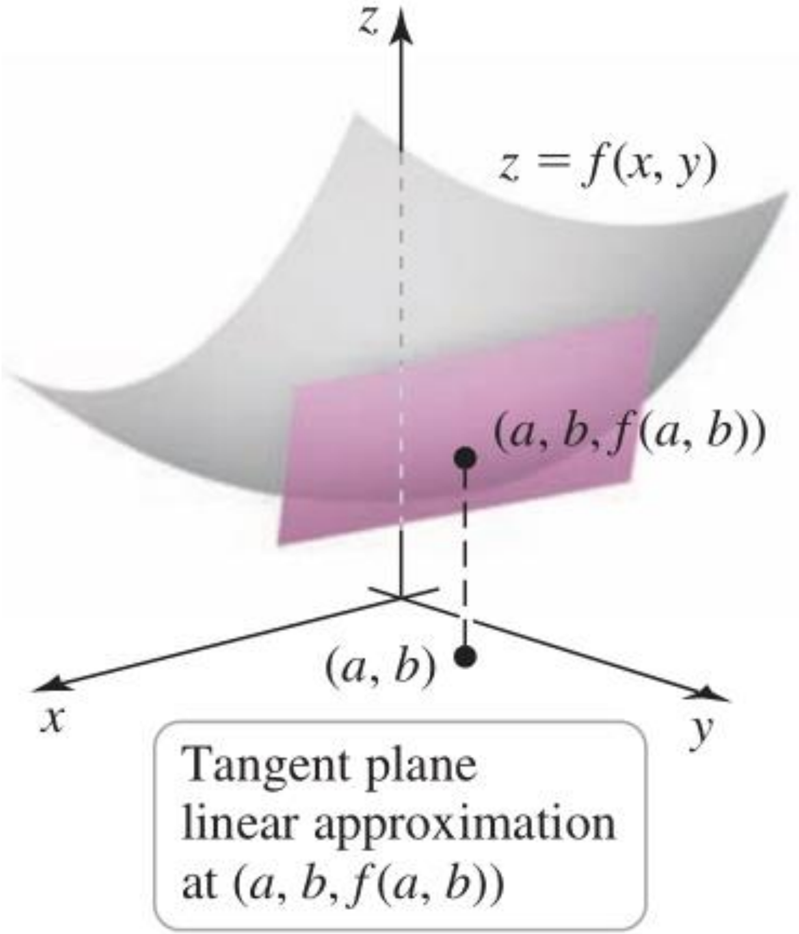
$$z = f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b).$$

EXAMPLE 2 **Tangent plane for $z = f(x, y)$** Find an equation of the plane tangent to the paraboloid $z = f(x, y) = 32 - 3x^2 - 4y^2$ at $(2, 1, 16)$.



Linear Approximation





$$f(x) \approx L(x)$$
$$= f(a) + f'(a)(x - a)$$

DEFINITION Linear Approximation

Let f be differentiable at (a, b). The linear approximation to the surface $z = f(x, y)$ at the point $(a, b, f(a, b))$ is the tangent plane at that point, given by the equation

$$L(x, y) = f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b).$$

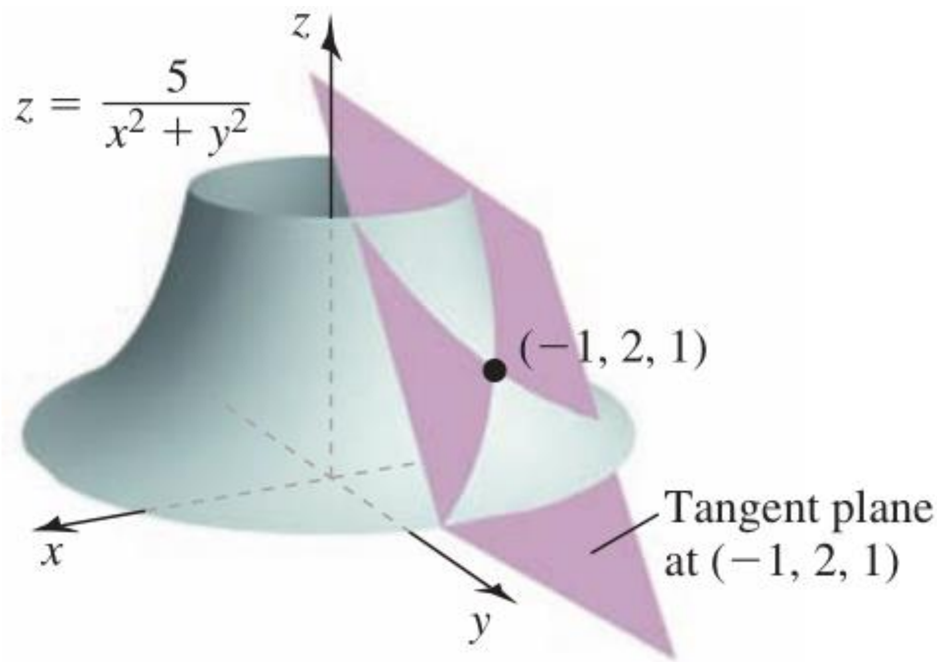
For a function of three variables, the linear approximation to $w = f(x, y, z)$ at the point $(a, b, c, f(a, b, c))$ is given by

$$L(x, y, z) = f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c) + f(a, b, c).$$

EXAMPLE 3 **Linear approximation** Let $f(x, y) = \dfrac{5}{x^2 + y^2}$.

a. Find the linear approximation to the function at the point $(-1, 2, 1)$.

b. Use the linear approximation to estimate the value of $f(-1.05, 2.1)$.

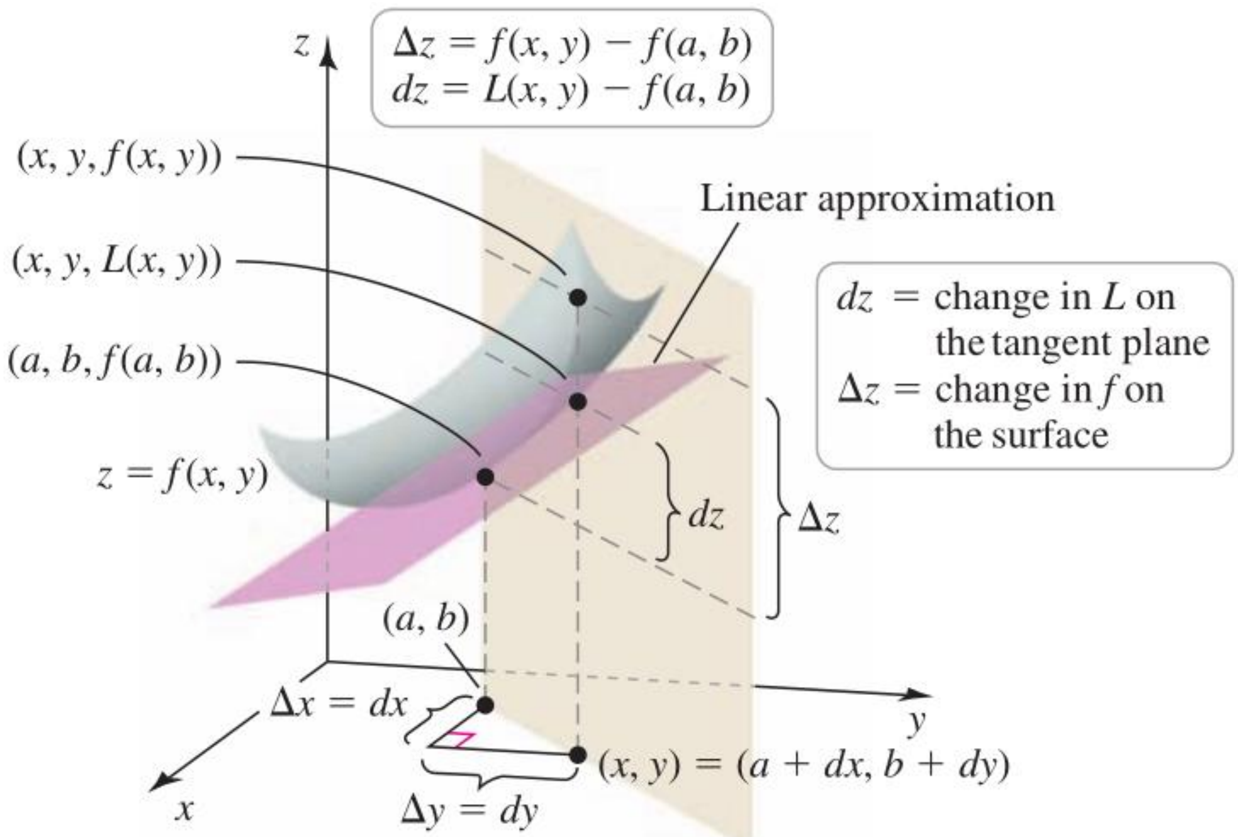


Differentials and Change

For $y = f(x)$, $\Delta y \approx dy = f'(x)dx$

For $z = f(x, y)$,

$$\Delta z \approx L(x, y) - f(a, b) = f_x(a, b)(x - a) + f_y(a, b)(y - b)$$



DEFINITION **The differential dz**

Let f be differentiable at the point (a, b). The change in $z = f(x, y)$ as the independent variables change from (a, b) to $(a + dx, b + dy)$ is denoted Δz and is approximated by the differential dz:

$$\Delta z \approx dz = f_x(a, b)\, dx + f_y(a, b)\, dy.$$

EXAMPLE 5 **Body mass index** The body mass index (BMI) for an adult human is given by the function $B(w, h) = w/h^2$, where w is weight measured in kilograms and h is height measured in meters.

a. Use differentials to approximate the change in the BMI when weight increases from 55 to 56.5 kg and height increases from 1.65 to 1.66 m.

b. Which produces a greater *percentage* change in the BMI, a 1% change in the weight (at a constant height) or a 1% change in the height (at a constant weight)?

15.7

Maximum/Minimum Problems

Local Maximum / Minimum Values

DEFINITION **Local Maximum/Minimum Values**

A function f has a **local maximum value** at (a, b) if $f(x, y) \leq f(a, b)$ for all (x, y) in the domain of f in some open disk centered at (a, b). A function f has a **local minimum value** at (a, b) if $f(x, y) \geq f(a, b)$ for all (x, y) in the domain of f in some open disk centered at (a, b). Local maximum and local minimum values are also called **local extreme values** or **local extrema**.

THEOREM 13 Derivatives and Local Maximum / Minimum Values

If f has a local maximum or minimum value at (a, b) and the partial derivatives f_x and f_y exist at (a, b), then $f_x(a, b) = f_y(a, b) = 0$.

DEFINITION **Critical Point**

An interior point (a, b) in the domain of f is a **critical point** of f if either

1. $f_x(a, b) = f_y(a, b) = 0$, or
2. at least one of the partial derivatives f_x and f_y does not exist at (a, b).

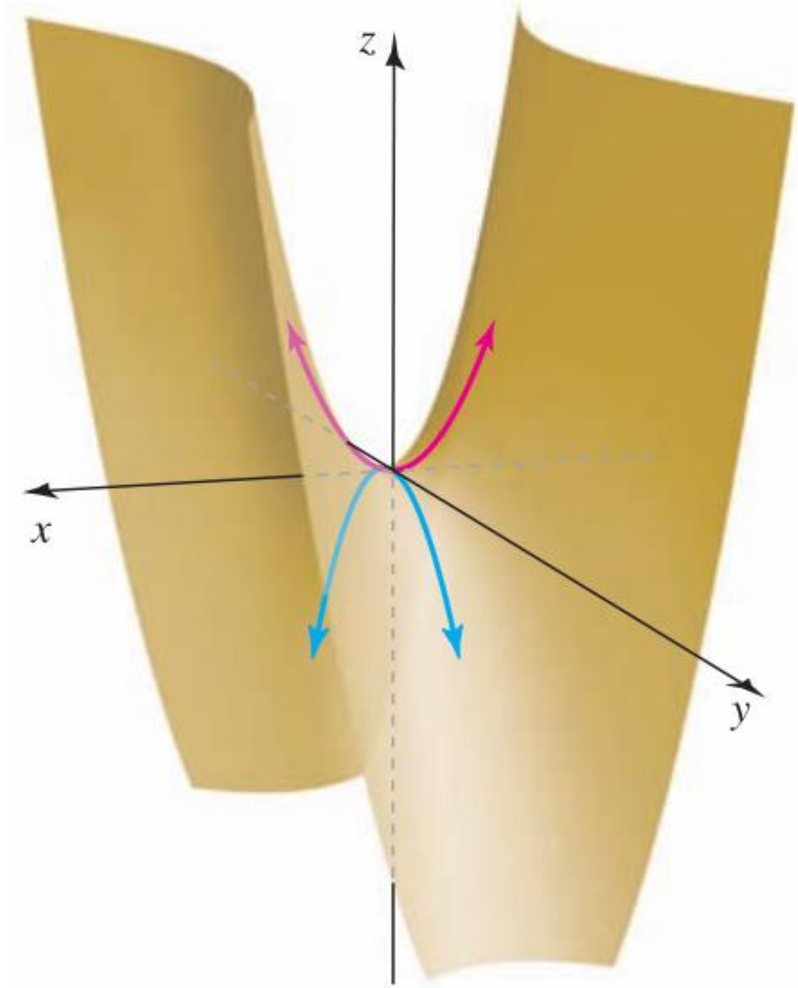
EXAMPLE 1 **Finding critical points** Find the critical points of $f(x, y) = xy(x - 2)(y + 3)$.

Second Derivative Test

DEFINITION Saddle Point

Consider a function f that is differentiable at a critical point (a, b). Then f has a **saddle point** at (a, b) if, in every open disk centered at (a, b), there are points (x, y) for which $f(x, y) > f(a, b)$ and points for which $f(x, y) < f(a, b)$.

If f has a saddle point at (a, b), then from the point $(a, b, f(a, b))$, it is possible to walk uphill in some directions and downhill in other directions.



The hyperbolic paraboloid $z = x^2 - y^2$ has a saddle point at (0, 0).

THEOREM 14 Second Derivative Test

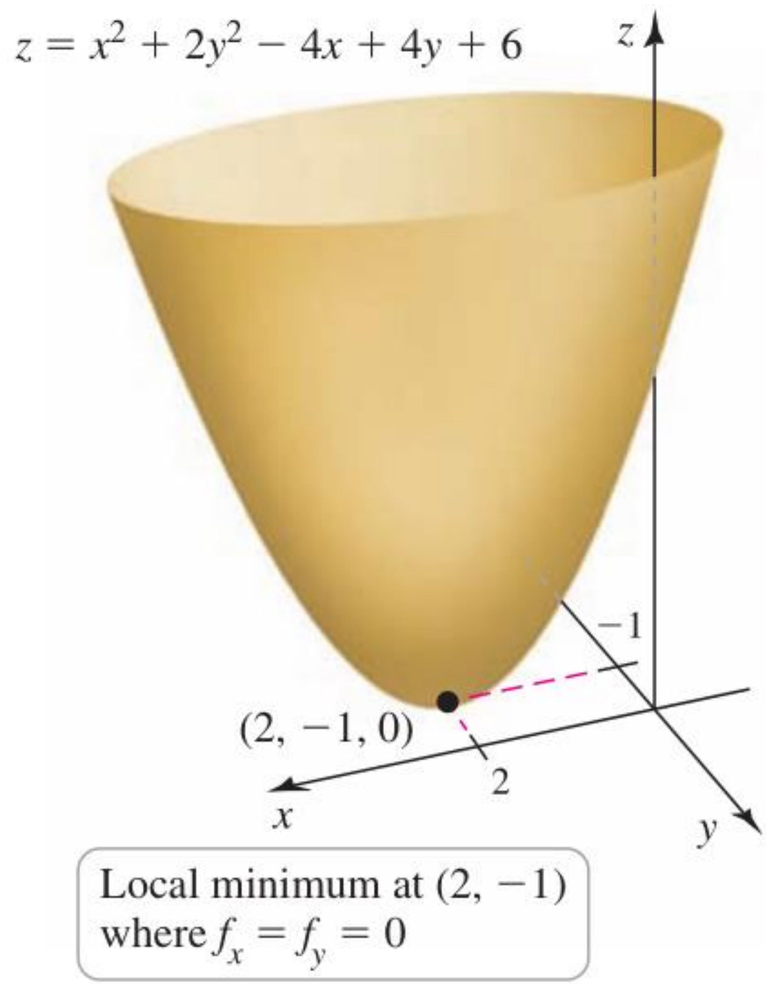
Suppose that the second partial derivatives of f are continuous throughout an open disk centered at the point (a, b), where $f_x(a, b) = f_y(a, b) = 0$. Let $D(x, y) = f_{xx}(x, y) f_{yy}(x, y) - (f_{xy}(x, y))^2$.

1. If $D(a, b) > 0$ and $f_{xx}(a, b) < 0$, then f has a local maximum value at (a, b).
2. If $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then f has a local minimum value at (a, b).
3. If $D(a, b) < 0$, then f has a saddle point at (a, b).
4. If $D(a, b) = 0$, then the test is inconclusive.

A few comments:

- **Discriminant** of f: $D(x,y) = f_{xx}f_{yy} - \left(f_{xy}\right)^2$ can be remembered as the 2×2 determinant of the **Hessian** matrix $\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$ where $f_{xy} = f_{yx}$
- $D(a,b) > 0$ means that the surface has the same general behavior (rises or falls) in all directions near (a,b)
- $D(a,b) = 0$ the test is inconclusive: (a,b) could correspond to a local maximum, a local minimum, or a saddle point

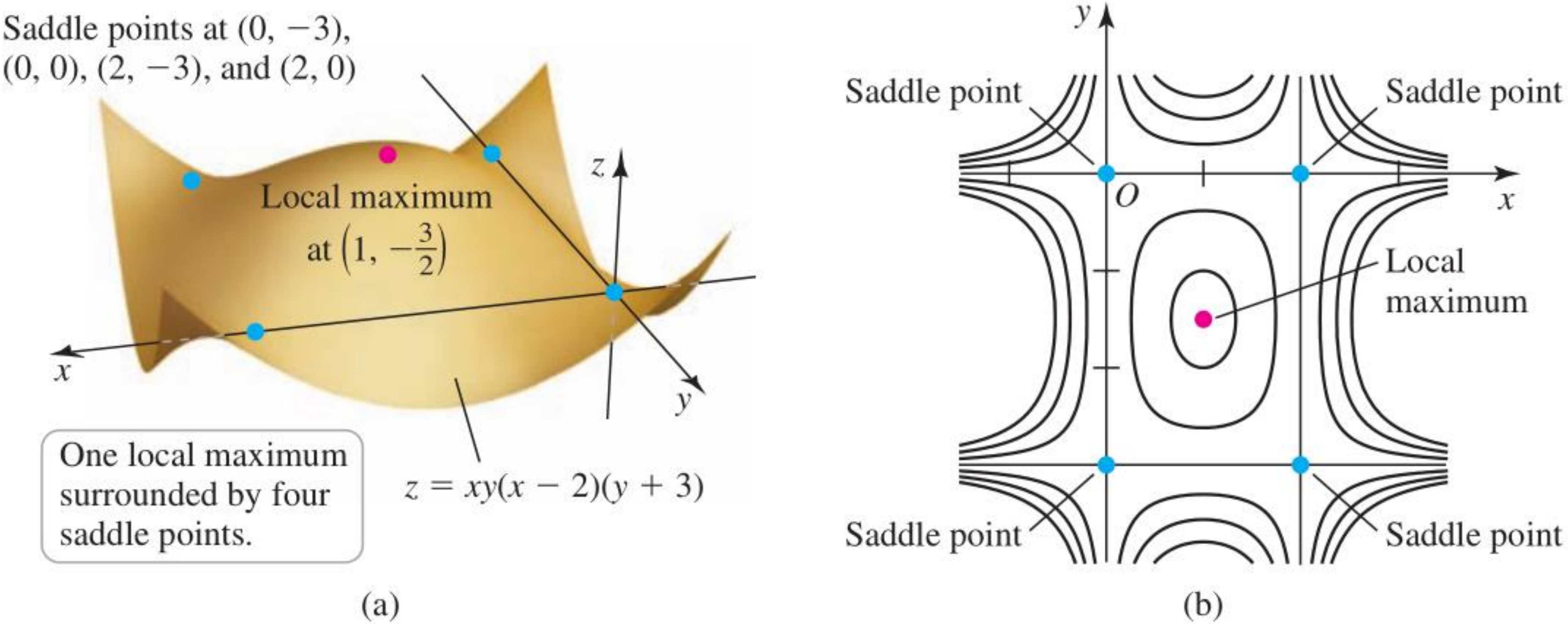
EXAMPLE 2 **Analyzing critical points** Use the Second Derivative Test to classify the critical points of $f(x, y) = x^2 + 2y^2 - 4x + 4y + 6$.



EXAMPLE 3 **Analyzing critical points** Use the Second Derivative Test to classify the critical points of $f(x, y) = xy(x - 2)(y + 3)$.

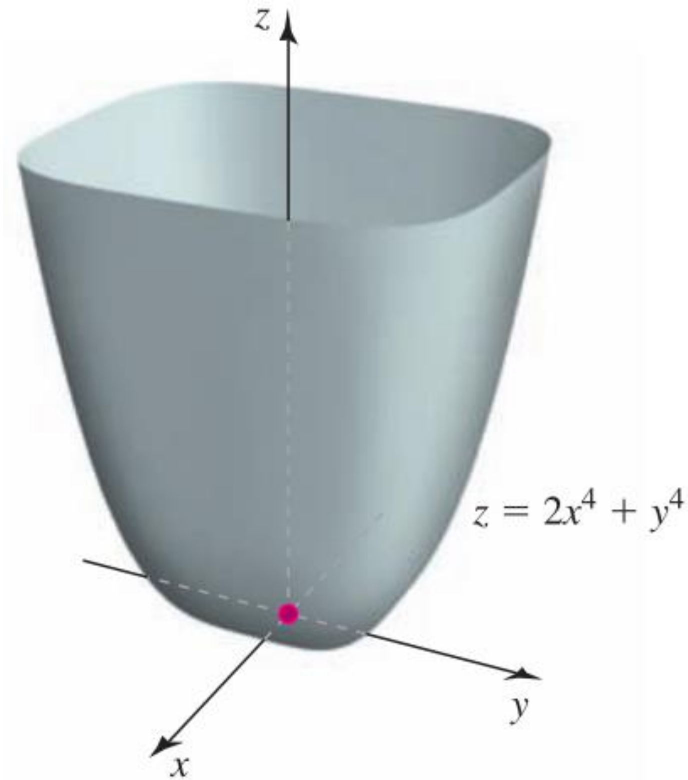
Table 5

(x, y)	$D(x, y)$	f_{xx}	Conclusion
$(0, 0)$	-36	0	Saddle point
$(2, 0)$	-36	0	Saddle point
$\left(1, -\frac{3}{2}\right)$	9	$-\frac{9}{2}$	Local maximum
$(0, -3)$	-36	0	Saddle point
$(2, -3)$	-36	0	Saddle point

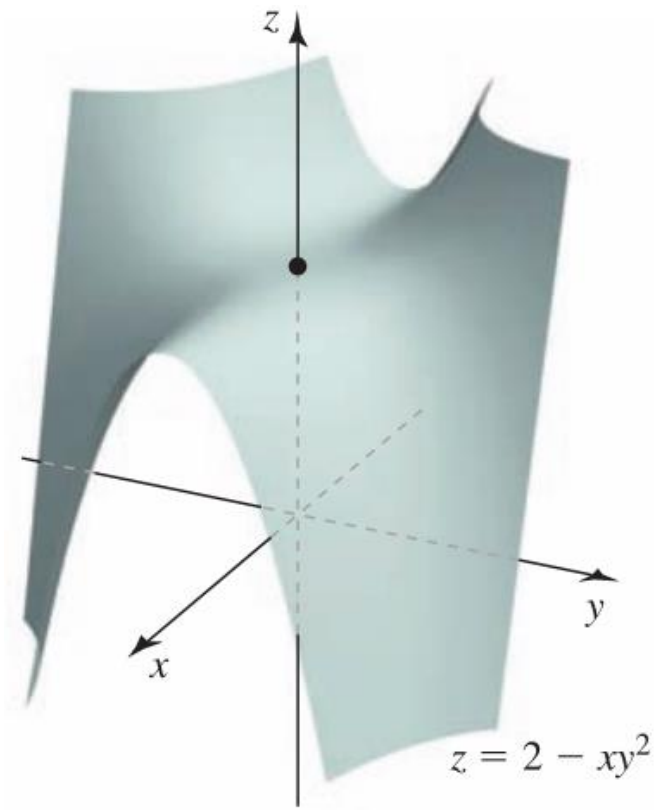

Saddle points at (0, −3), (0, 0), (2, −3), and (2, 0)
Local maximum at $\left(1, -\frac{3}{2}\right)$
One local maximum surrounded by four saddle points.
$z = xy(x - 2)(y + 3)$
x
y
z
(a)
Saddle point
Saddle point
Local maximum
Saddle point
Saddle point
O
x
y
(b)

EXAMPLE 5 **Inconclusive tests** Apply the Second Derivative Test to the following functions and interpret the results.

a. $f(x, y) = 2x^4 + y^4$ **b.** $f(x, y) = 2 - xy^2$



Local minimum at (0, 0), but the Second Derivative Test is inconclusive.



Second derivative test fails to detect saddle point at (0, 0).

Absolute Maximum and Minimum Values

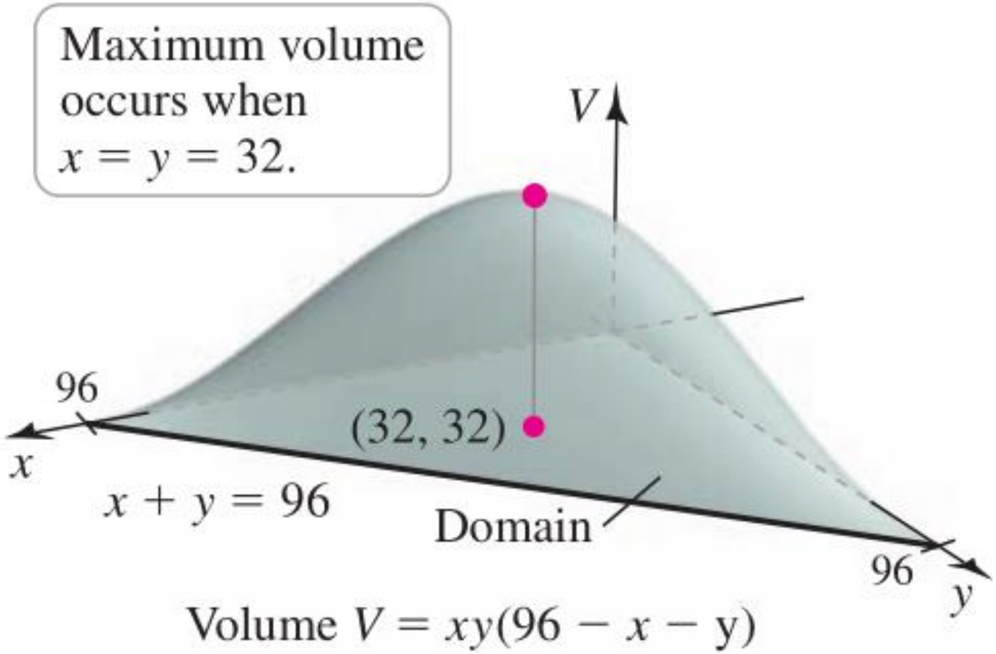
DEFINITION **Absolute Maximum / Minimum Values**

Let f be defined on a set R in $\mathbb{R}^2$ containing the point (a, b). If $f(a, b) \geq f(x, y)$ for every (x, y) in R, then $f(a, b)$ is an **absolute maximum value** of f on R. If $f(a, b) \leq f(x, y)$ for every (x, y) in R, then $f(a, b)$ is an **absolute minimum value** of f on R.

Absolute maximum and minimum values on a closed bounded set R occur in two ways:

- May be local maximum or minimum values at interior points of R, where they are associated with critical points.
- May occur on the boundary of R

EXAMPLE 4 **Shipping regulations** A shipping company handles rectangular boxes provided the sum of the length, width, and height of the box does not exceed 96 in. Find the dimensions of the box that meets this condition and has the largest volume.

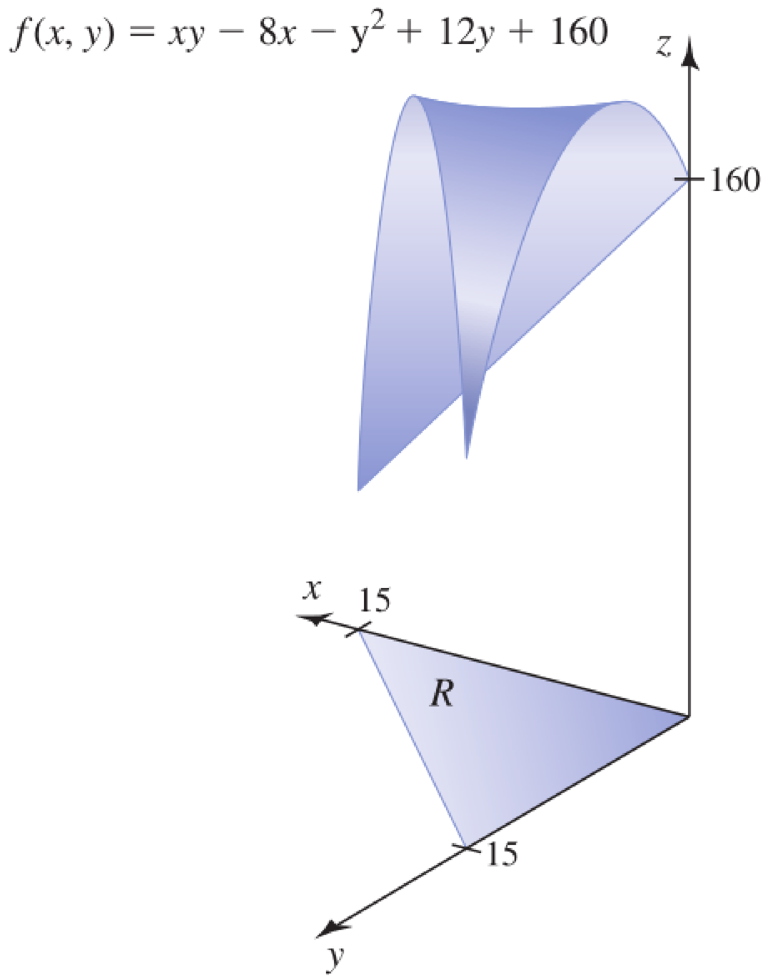


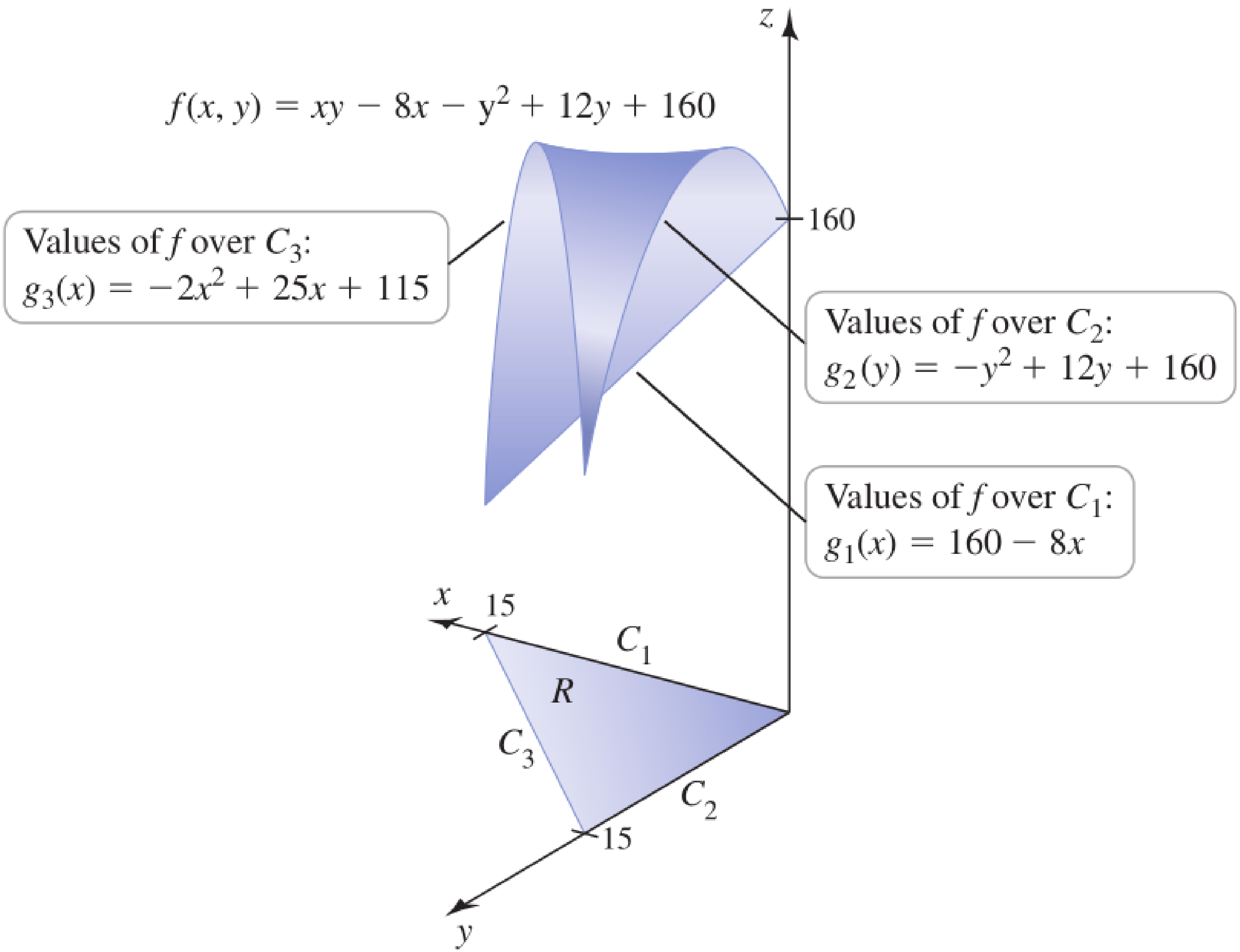
PROCEDURE **Finding Absolute Maximum/Minimum Values on Closed Bounded Sets**

Let f be continuous on a closed bounded set R in $\mathbb{R}^2$. To find the absolute maximum and minimum values of f on R:

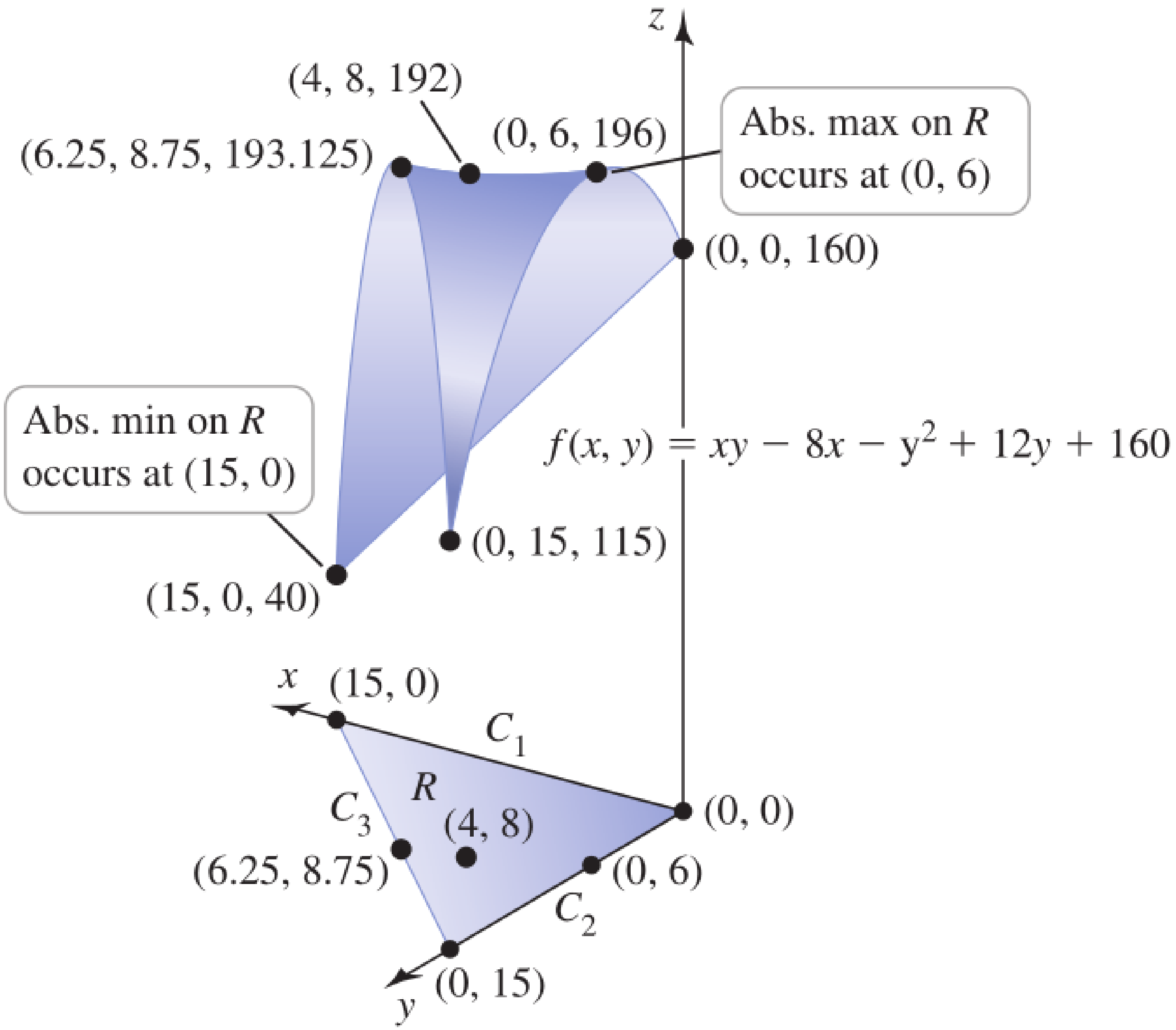
1. Determine the values of f at all critical points in R.
2. Find the maximum and minimum values of f on the boundary of R.
3. The greatest function value found in Steps 1 and 2 is the absolute maximum value of f on R, and the least function value found in Steps 1 and 2 is the absolute minimum value of f on R.

EXAMPLE 6 **Extreme values over a region** Find the absolute maximum and minimum values of $f(x, y) = xy - 8x - y^2 + 12y + 160$ over the triangular region $R = \{(x, y): 0 \leq x \leq 15, 0 \leq y \leq 15 - x\}$.





$f(x, y) = xy - 8x - y^2 + 12y + 160$
Values of f over C_3:
$g_3(x) = -2x^2 + 25x + 115$
Values of f over C_2:
$g_2(y) = -y^2 + 12y + 160$
Values of f over C_1:
$g_1(x) = 160 - 8x$
z
160
x
15
C_1
R
C_3
C_2
15
y


z
(4, 8, 192)
(6.25, 8.75, 193.125)
(0, 6, 196)
Abs. max on R
occurs at (0, 6)
(0, 0, 160)
Abs. min on R
occurs at (15, 0)
$f(x, y) = xy - 8x - y^2 + 12y + 160$
(0, 15, 115)
(15, 0, 40)
x
(15, 0)
C_1
R
C_3
(4, 8)
(0, 0)
(6.25, 8.75)
(0, 6)
C_2
(0, 15)
y

EXAMPLE 7 **Absolute maximum and minimum values** Find the absolute maximum and minimum values of $f(x, y) = \frac{1}{2}(x^3 - x - y^2) + 3$ on the region $R = \{(x, y) : x^2 + y^2 \leq 1\}$ (the closed disk centered at $(0, 0)$ with radius 1).

Critical points:

$$f_x(x, y) = \frac{1}{2}(3x^2 - 1) = 0 \quad \text{and} \quad f_y(x, y) = -y = 0,$$

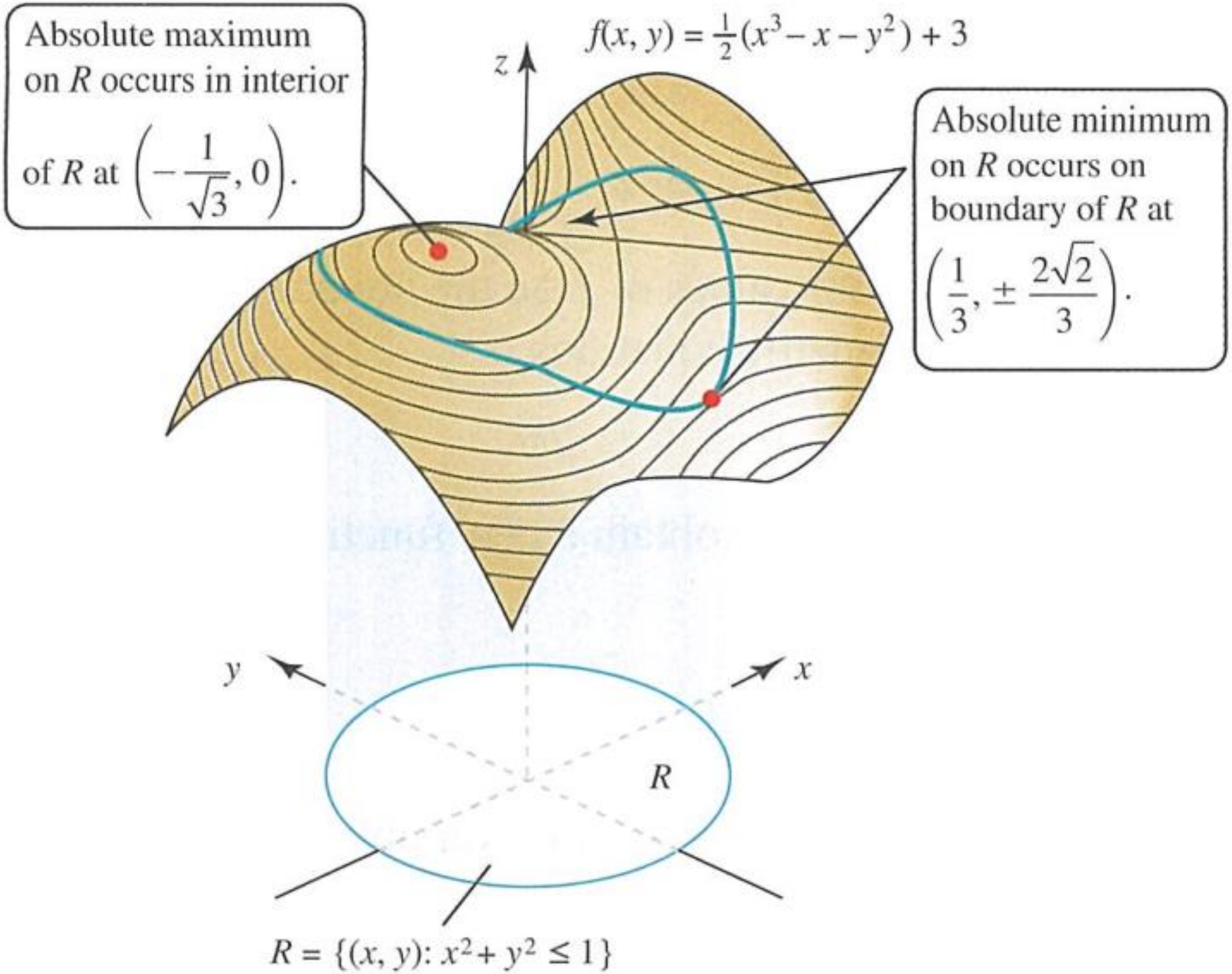
For values on the boundary of R, let

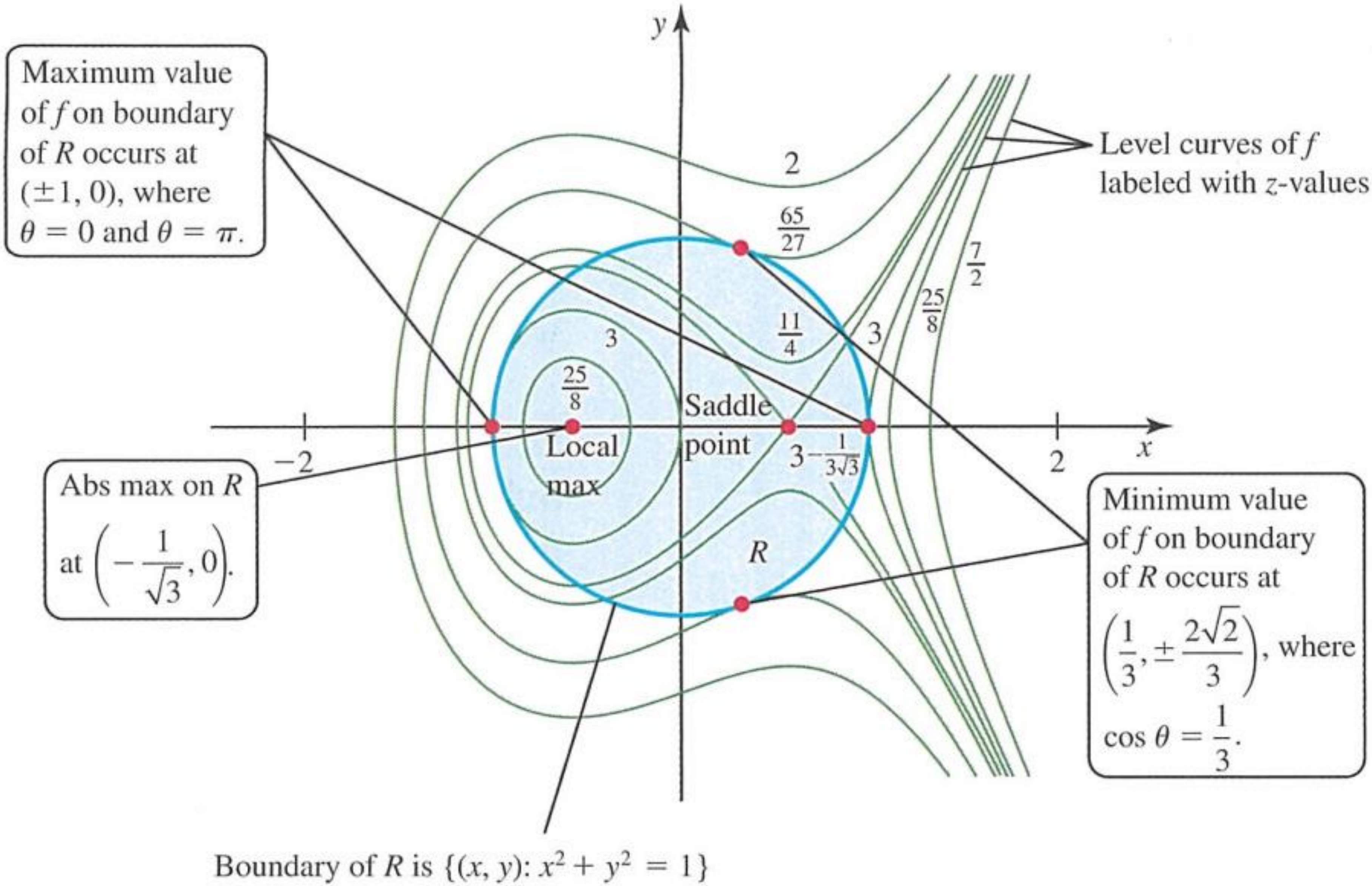
$$x = \cos\theta, y = \sin\theta, 0 \leq \theta \leq 2\pi$$

Obtain a new function $g(\theta)$ that gives the value on the boundary

$$g(\theta) = \frac{1}{2}(\cos^3\theta - \cos\theta - \sin^2\theta) + 3$$

Let $g'(\theta) = 0$


Absolute maximum on R occurs in interior of R at $\left(-\frac{1}{\sqrt{3}}, 0\right)$.
$f(x, y) = \frac{1}{2}(x^3 - x - y^2) + 3$
Absolute minimum on R occurs on boundary of R at $\left(\frac{1}{3}, \pm\frac{2\sqrt{2}}{3}\right)$.
z
y
x
R
$R = \{(x, y): x^2 + y^2 \leq 1\}$

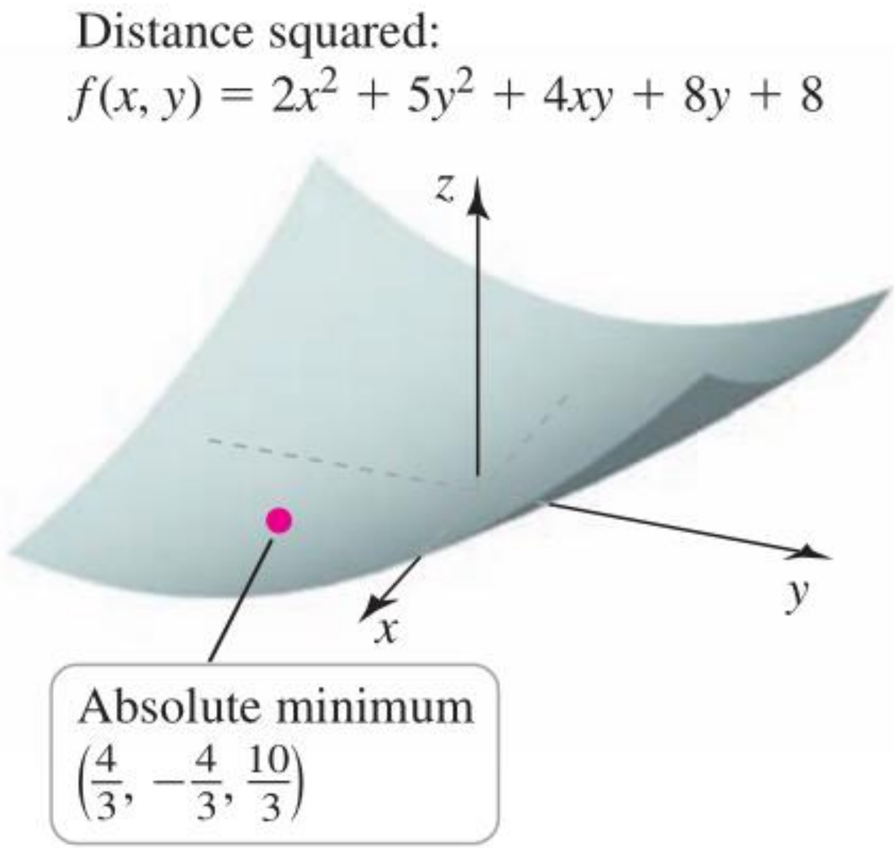


Maximum value of f on boundary of R occurs at $(\pm 1, 0)$, where $\theta = 0$ and $\theta = \pi$.
Level curves of f labeled with z-values
2
$\frac{65}{27}$
$\frac{7}{2}$
$\frac{25}{8}$
3
$\frac{11}{4}$
3
$\frac{25}{8}$
Saddle point
Local max
$3 - \frac{1}{3\sqrt{3}}$
-2
2
x
y
R
Abs max on R at $\left(-\frac{1}{\sqrt{3}}, 0\right)$.
Minimum value of f on boundary of R occurs at $\left(\frac{1}{3}, \pm\frac{2\sqrt{2}}{3}\right)$, where $\cos\theta = \frac{1}{3}$.
Boundary of R is $\{(x, y): x^2 + y^2 = 1\}$

Open and/or Unbounded Regions

EXAMPLE 8 **Absolute extreme values on an open region** Find the absolute maximum and minimum values of $f(x, y) = 4 - x^2 - y^2$ on the open disk $R = \{(x, y): x^2 + y^2 < 1\}$ (if they exist).

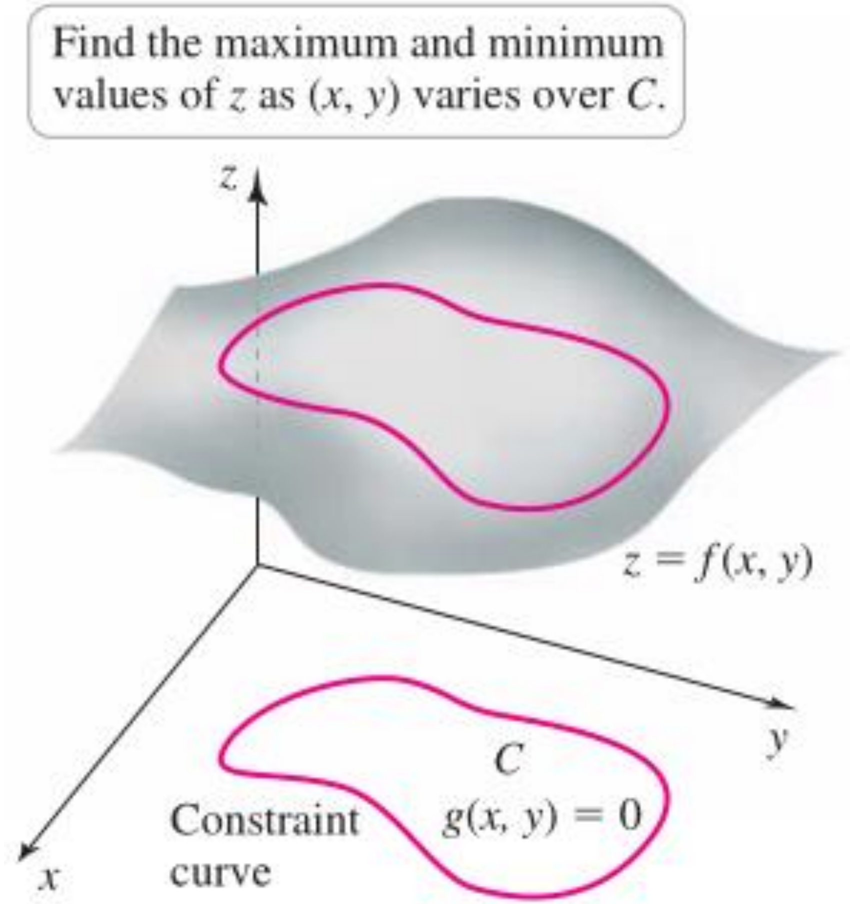
EXAMPLE 9 **Absolute extreme values on an open region** Find the point(s) on the plane $x + 2y + z = 2$ closest to the point $P(2, 0, 4)$.



15.8

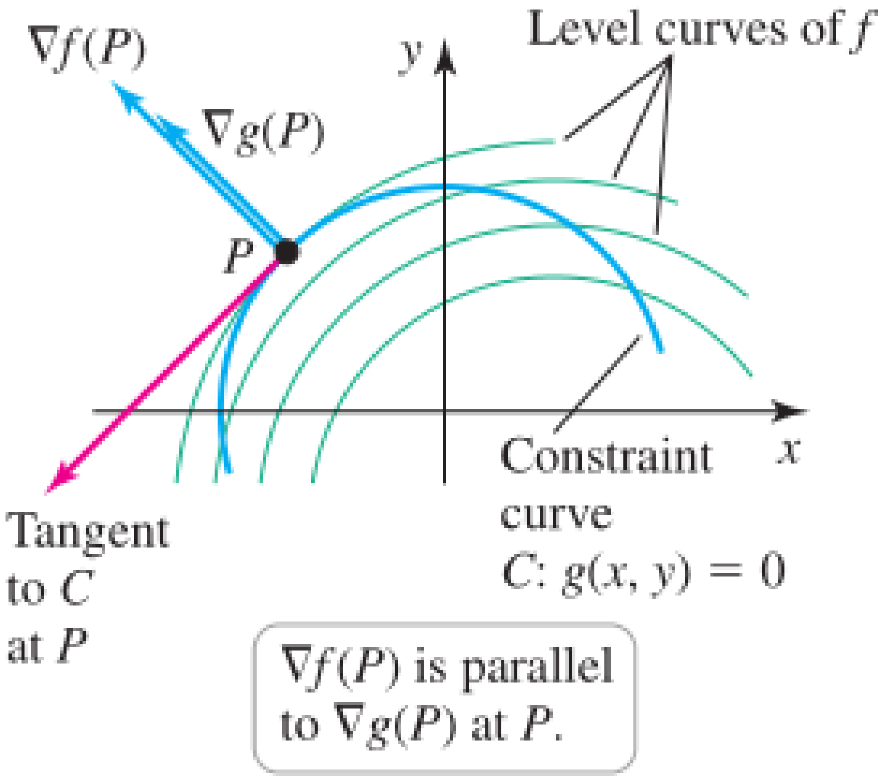
Lagrange Multipliers

The Basic Idea



A typical constrained optimization problem with two independent variables:

- Objective function $f(x, y)$
- Restriction that x and y must lie on a constraint curve C given by $g(x, y) = 0$



- The line tangent to the level curve at P is tangent to the constraint curve C at P.
- The gradients $\nabla f(P)$ and $\nabla g(P)$ are parallel
- These properties characterize the point P at which f has an extreme value on the constraint curve

Lagrange Multipliers with Two Independent Variables

THEOREM 15 Parallel Gradients (Ball Park Theorem)
Let f be a differentiable function in a region of $\mathbb{R}^2$ that contains the smooth curve C given by $g(x, y) = 0$. Assume that f has a local extreme value on C at a point $P(a, b)$. Then $\nabla f(a, b)$ is orthogonal to the line tangent to C at P. Assuming $\nabla g(a, b) \neq \mathbf{0}$, it follows that there is a real number λ (called a **Lagrange multiplier**) such that $\nabla f(a, b) = \lambda \nabla g(a, b)$.

PROCEDURE **Lagrange Multipliers: Absolute Extrema on Closed and Bounded Constraint Curves**

Let the objective function f and the constraint function g be differentiable on a region of $\mathbb{R}^2$ with $\nabla g(x, y) \neq \mathbf{0}$ on the curve $g(x, y) = 0$. To locate the absolute maximum and minimum values of f subject to the constraint $g(x, y) = 0$, carry out the following steps.

1. Find the values of x, y, and λ (if they exist) that satisfy the equations

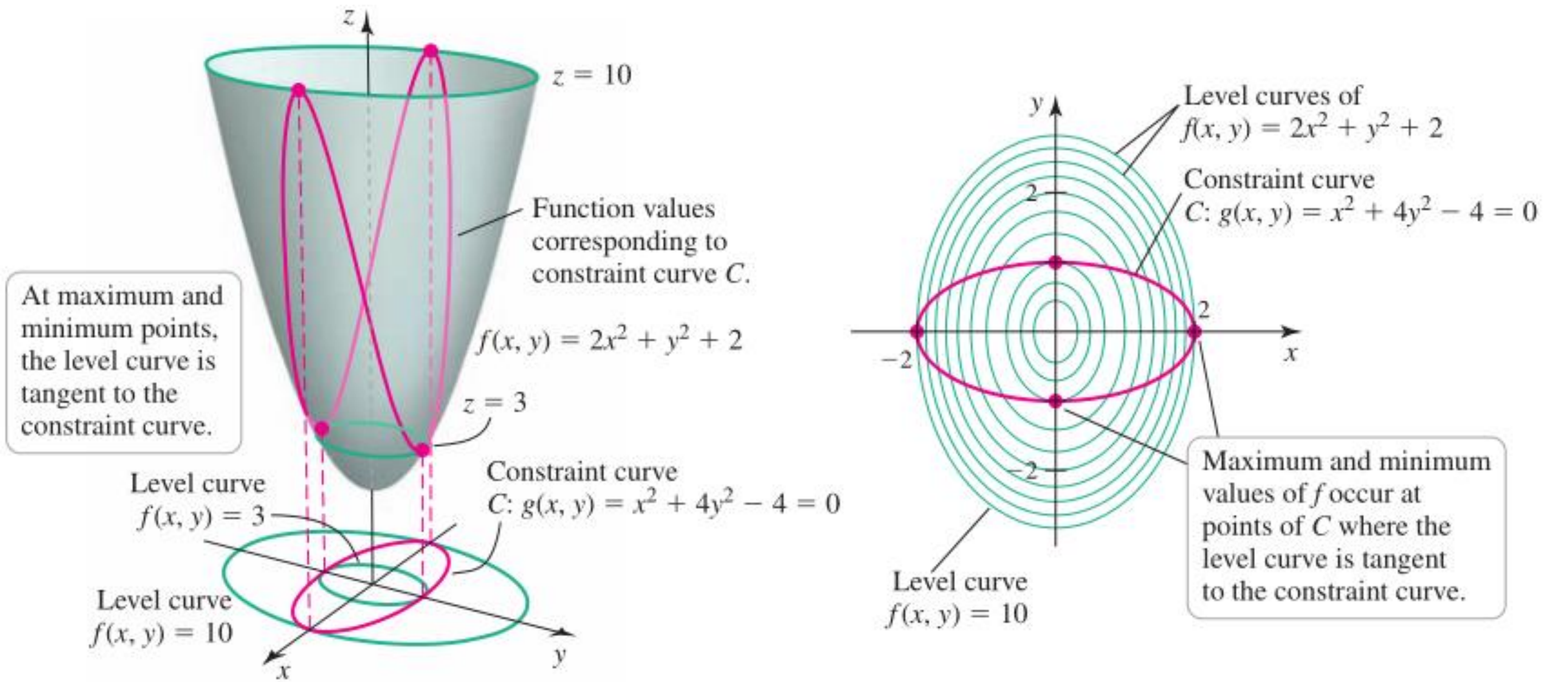
$$\nabla f(x, y) = \lambda \nabla g(x, y) \quad \text{and} \quad g(x, y) = 0.$$

2. Evaluate f at the values (x, y) found in Step 1 and at the endpoints of the constraint curve (if they exist). Select the largest and smallest corresponding function values. These values are the absolute maximum and minimum values of f subject to the constraint.

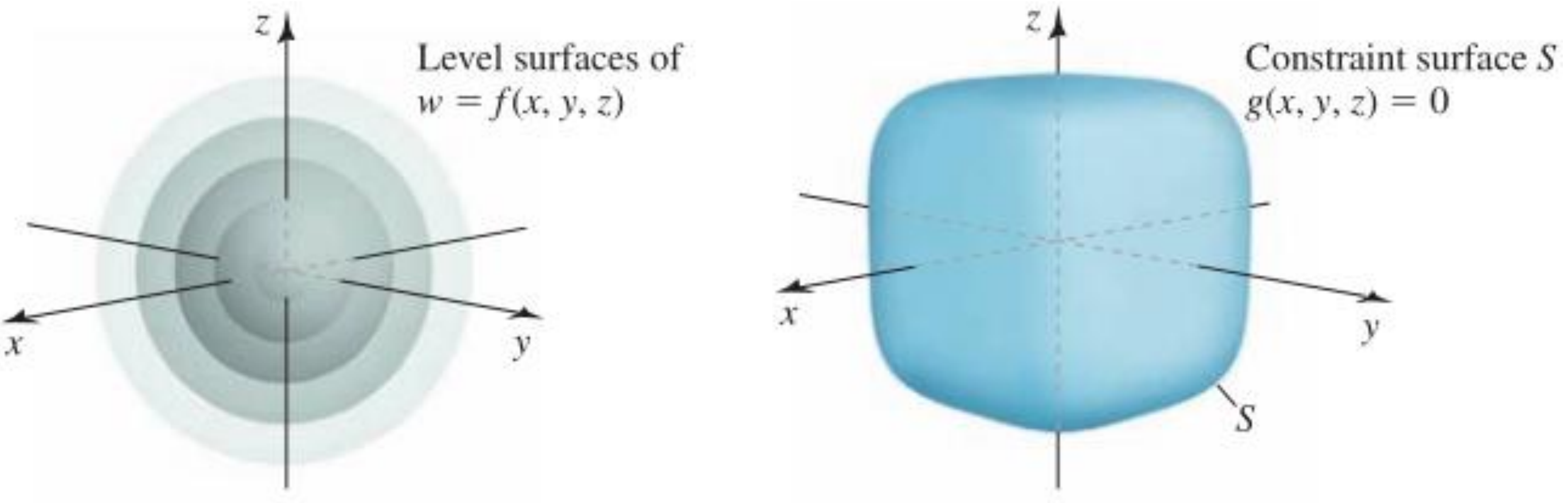
Note $\nabla f = \lambda \nabla g$ is satisfied provided $f_x = \lambda g_x$ and $f_y = \lambda g_y$. Therefore, the crux of the method is solving the three equations

$$f_x = \lambda g_x,\ f_y = \lambda g_y, \text{ and } g(x, y) = 0$$

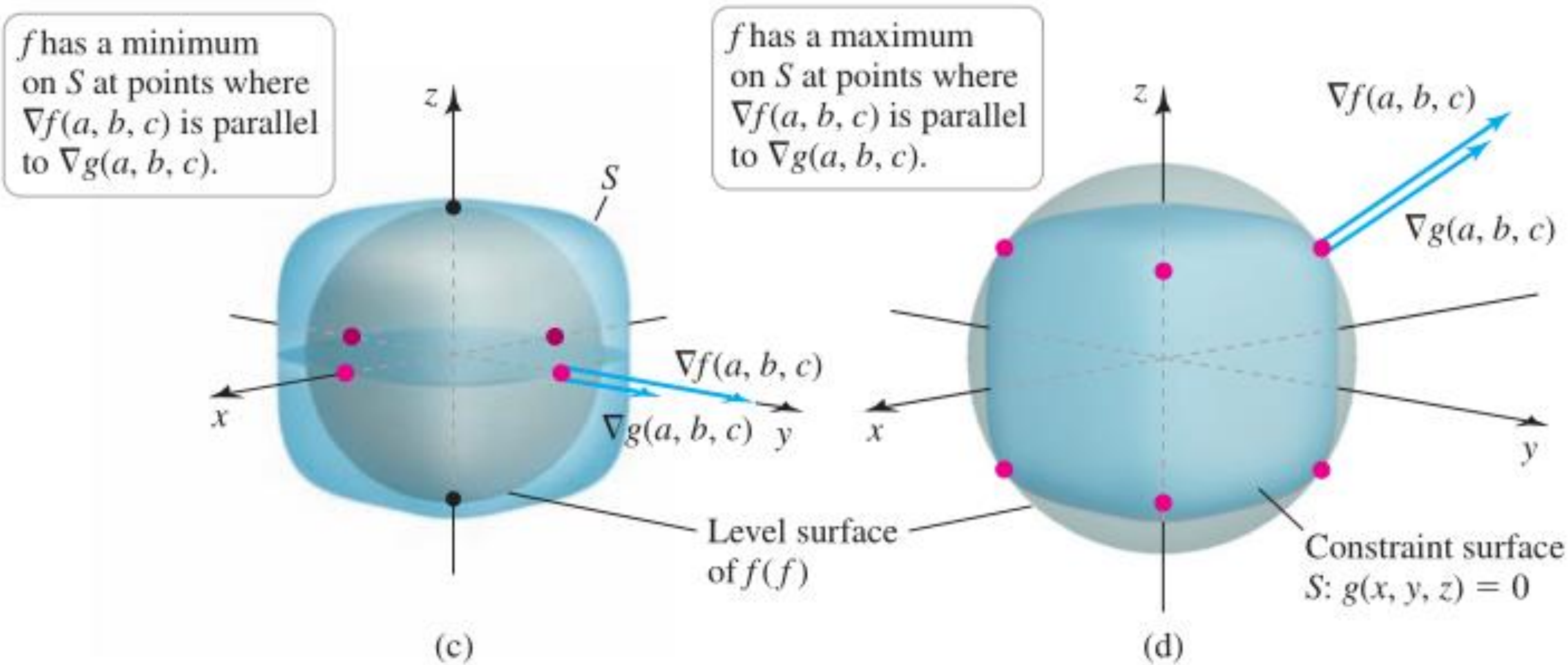
EXAMPLE 1 **Lagrange multipliers with two variables** Find the maximum and minimum values of the objective function $f(x, y) = 2x^2 + y^2 + 2$, where x and y lie on the ellipse C given by $g(x, y) = x^2 + 4y^2 - 4 = 0$.



Lagrange Multipliers with Three Independent Variables



(a) (b)



f has a minimum on S at points where ∇f(a, b, c) is parallel to ∇g(a, b, c).
z
S
∇f(a, b, c)
∇g(a, b, c)
x
y
(c)
f has a maximum on S at points where ∇f(a, b, c) is parallel to ∇g(a, b, c).
z
∇f(a, b, c)
∇g(a, b, c)
x
y
Level surface of f (f)
Constraint surface S: g(x, y, z) = 0
(d)

PROCEDURE **Method of Lagrange Multipliers in Three Variables**

Let f and g be differentiable on a region of $\mathbb{R}^3$ with $\nabla g(x, y, z) \neq \mathbf{0}$ on the surface $g(x, y, z) = 0$. To locate the maximum and minimum values of f subject to the constraint $g(x, y, z) = 0$, carry out the following steps.

1. Find the values of x, y, z, and λ that satisfy the equations

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) \quad \text{and} \quad g(x, y, z) = 0.$$

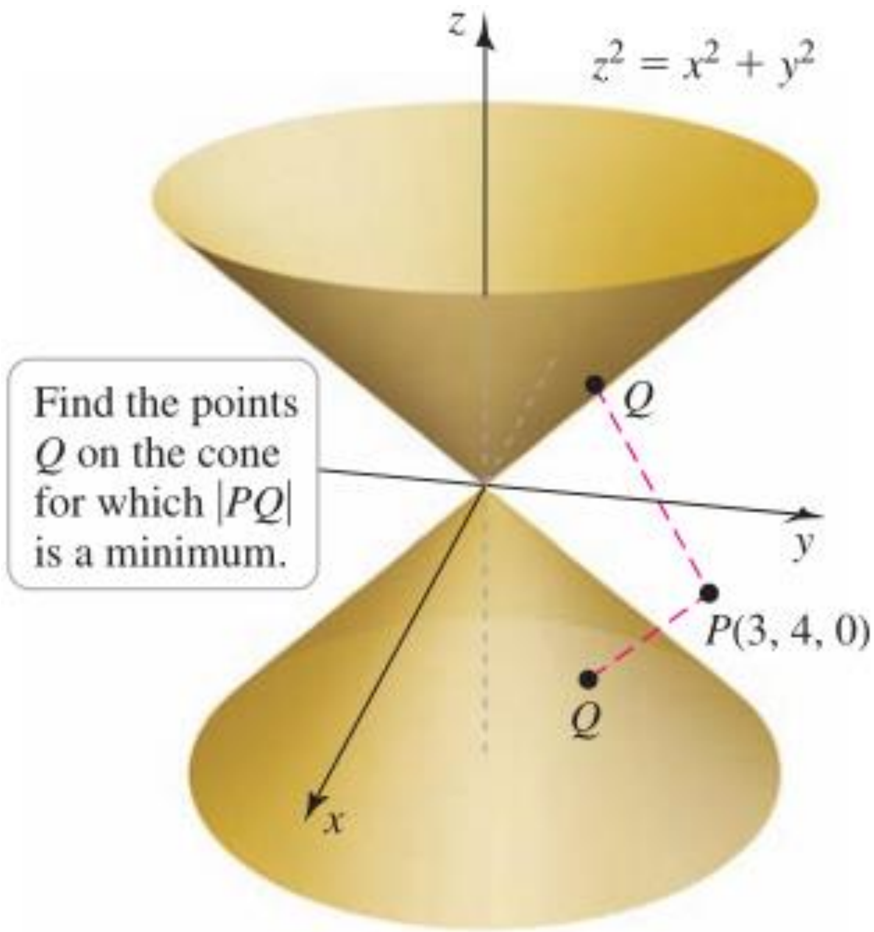
2. Among the points (x, y, z) found in Step 1, select the largest and smallest corresponding function values. These values are the maximum and minimum values of f subject to the constraint.

Now there are four equations to be solved for x, y, z, and λ

$f_x(x, y, z) = \lambda g_x(x, y, z)$, $f_y(x, y, z) = \lambda g_y(x, y, z)$,

$f_z(x, y, z) = \lambda g_z(x, y, z)$, and $g(x, y, z) = 0$

EXAMPLE 2 **A geometry problem** Find the least distance between the point $P(3, 4, 0)$ and the surface of the cone $z^2 = x^2 + y^2$.



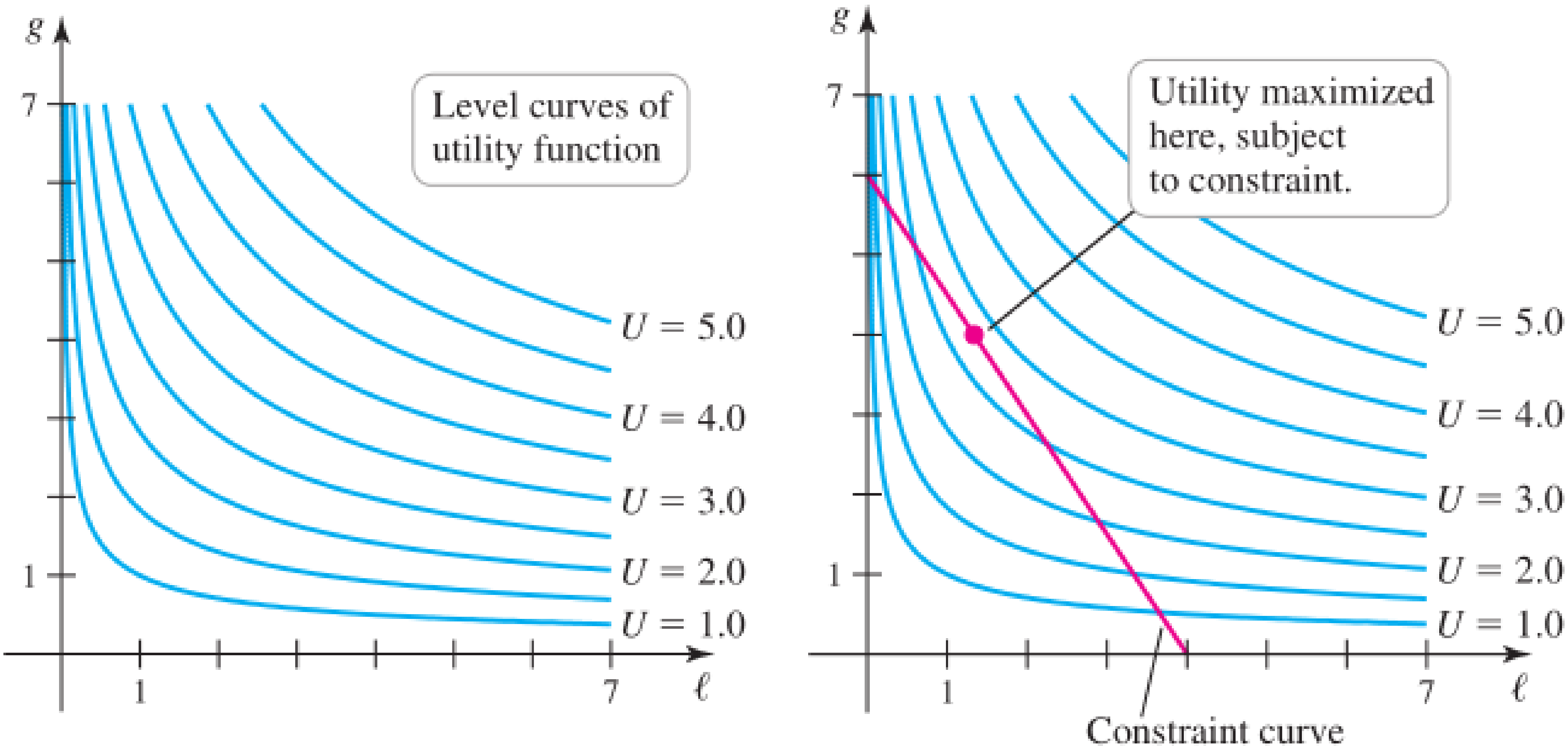
Economic Models

Consumer behavior model

A *utility function* for $U = f(\ell, g)$, measures consumer preferences for various combinations of leisure time ℓ and consumable goods g.

Some assumptions:

1. Utility increases if any variable increases (essentially, *more is better*).
2. Various combinations of leisure time and consumable goods have the same utility; that is, giving up some leisure time for additional consumable goods results in the same utility.



A single level curve shows the combinations of ℓ and g that have the same utility.

So, economists call the level curves *indifference curves*

EXAMPLE 3 **Constrained optimization of utility** Find the maximum value of the utility function $U = f(\ell, g) = \ell^{1/3}g^{2/3}$, subject to the constraint $G(\ell, g) = 3\ell + 2g - 12 = 0$, where $\ell \geq 0$ and $g \geq 0$.

Chapter 15

Functions of Several Variables (II)

Shuwei Chen

swchen@swjtu.edu.cn