Chapter 4

Applications of the Derivative (I)

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4.1

Maxima and Minima

Absolute Maxima and Minima

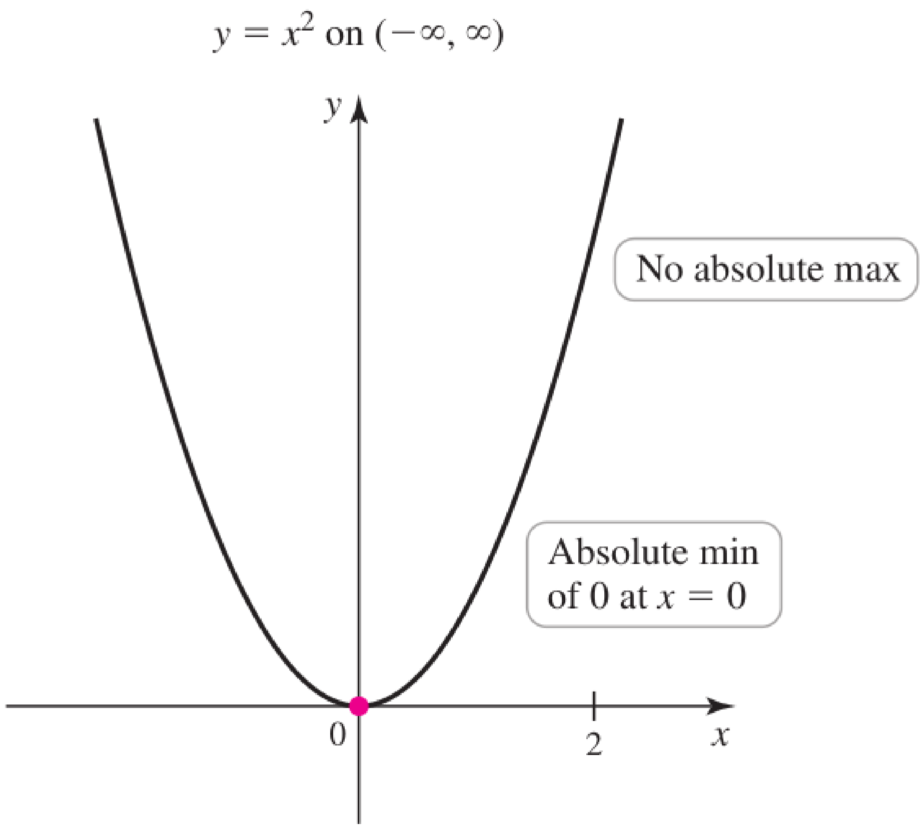
Imagine a long hiking, walk over hills, through valleys, and across plains

DEFINITION Absolute Maximum and Minimum

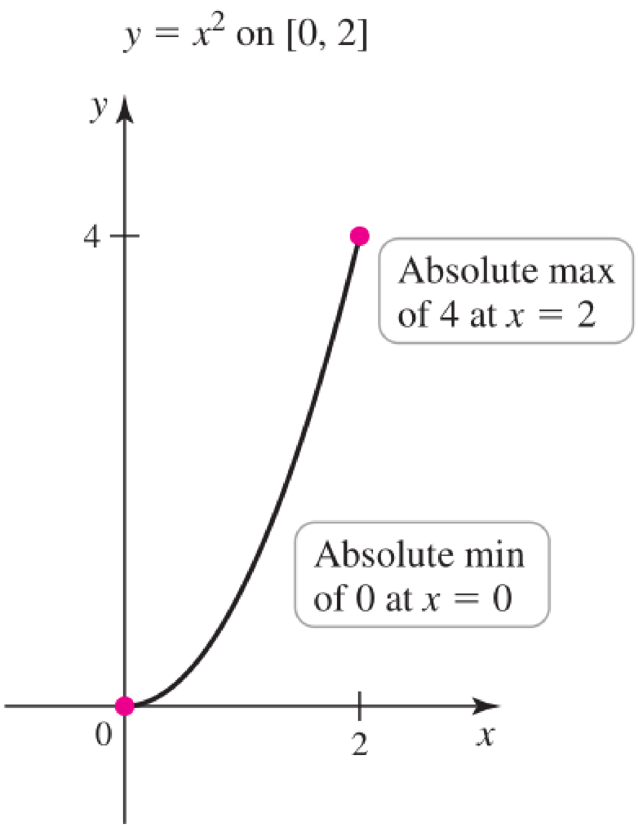
Let f be defined on a set D containing c. If $f(c) \geq f(x)$ for every x in D, then $f(c)$ is an **absolute maximum** value of f on D. If $f(c) \leq f(x)$ for every x in D, then $f(c)$ is an **absolute minimum** value of f on D. An **absolute extreme value** is either an absolute maximum or an absolute minimum value.

The existence and location of absolute extreme values depend on both the *function* and the *interval of interest*.

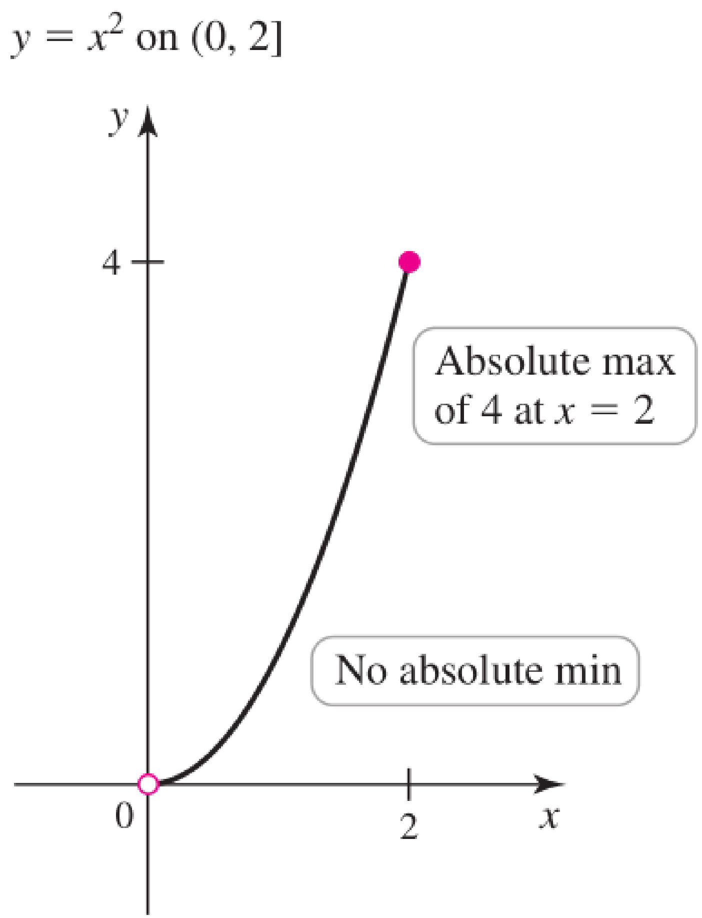
If the interval of interest is not closed, a function might not attain absolute extreme values



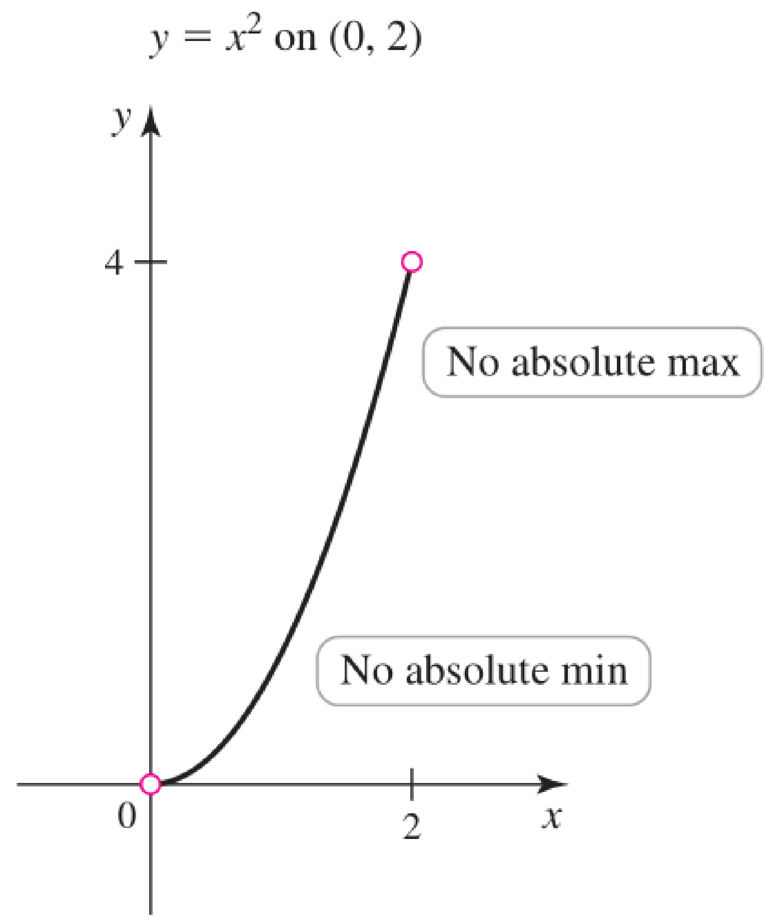
(a) Absolute min only



(b) Absolute max and min

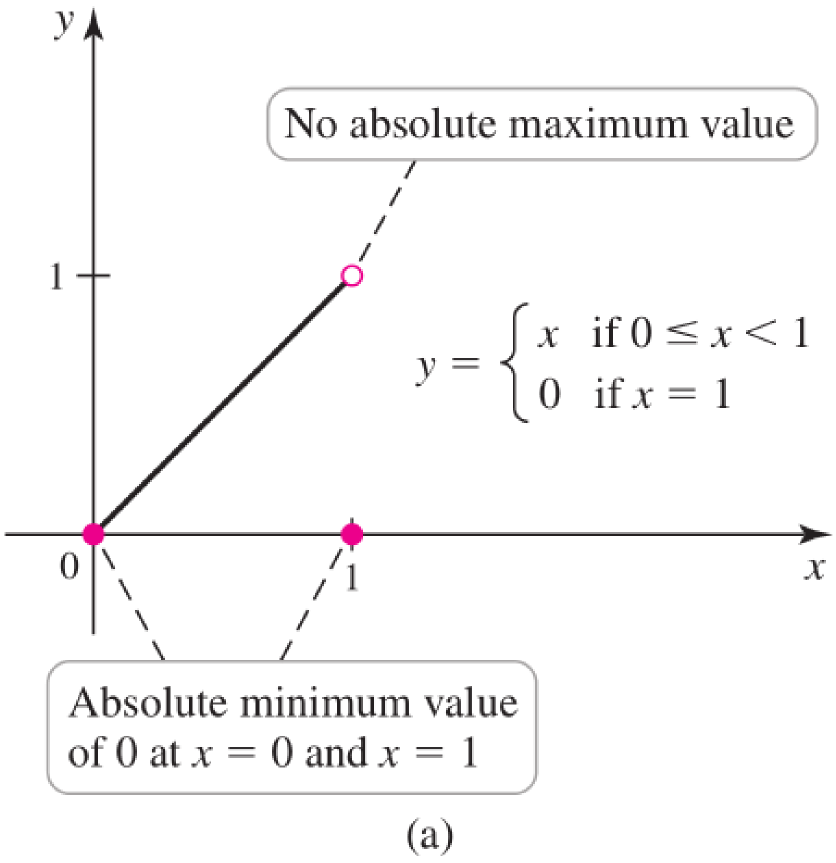


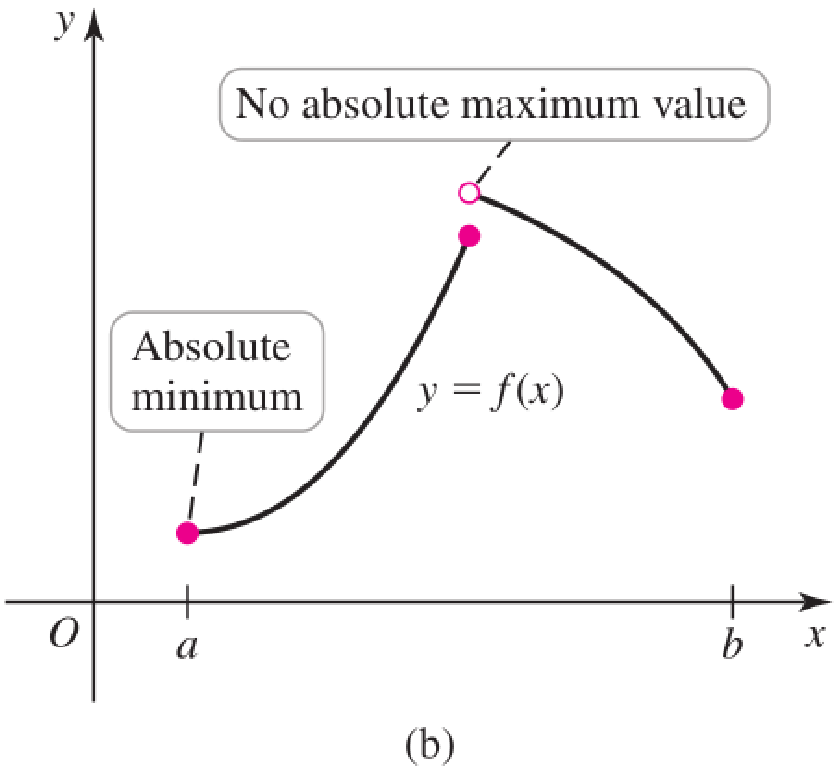
(c) Absolute max only



(d) No absolute max or min

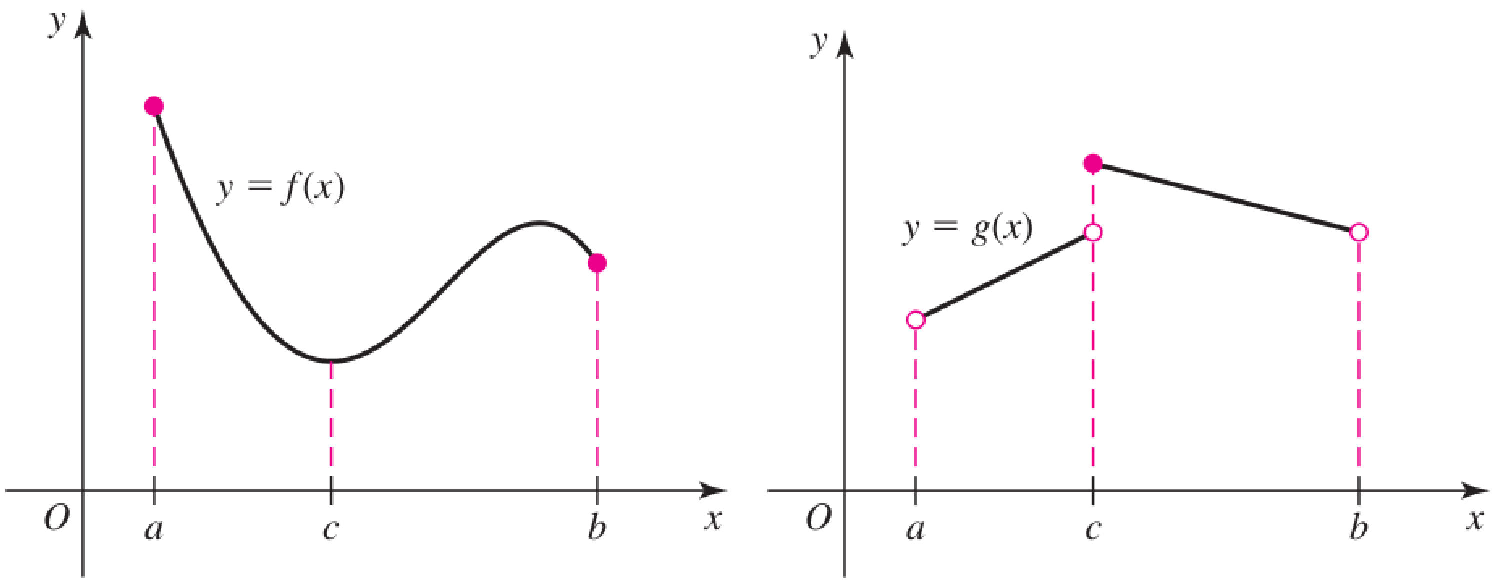
However, defining a function on a closed interval is not enough to guarantee the existence of absolute extreme values





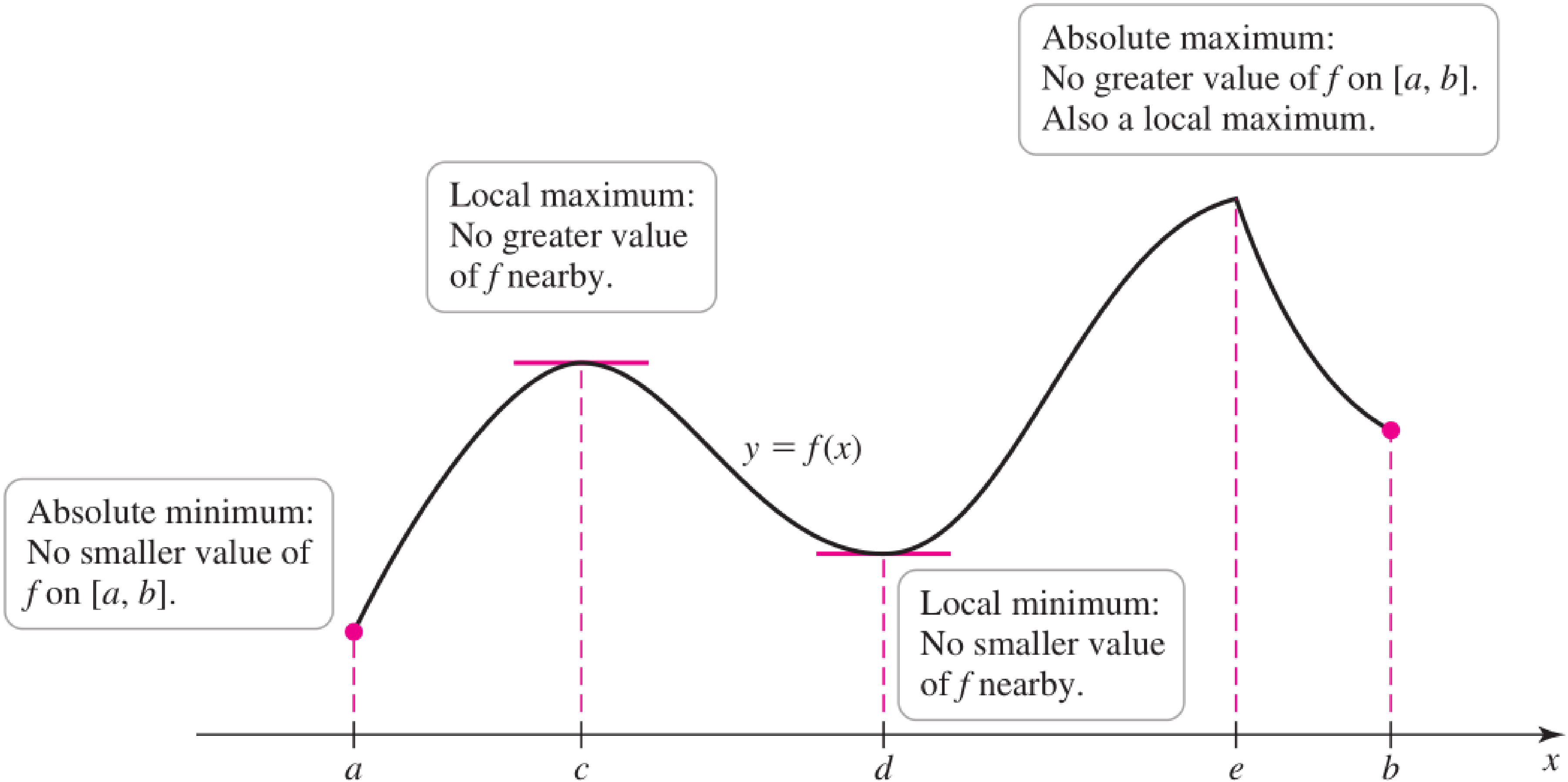
THEOREM 4.1 Extreme Value Theorem
A function that is continuous on a closed interval $[a, b]$ has an absolute maximum value and an absolute minimum value on that interval.

EXAMPLE 1 Locating absolute maximum and minimum values For the functions in Figure 4.4, identify the location of the absolute maximum value and the absolute minimum value on the interval $[a, b]$. Do the functions meet the conditions of the Extreme Value Theorem?



(a) (b)

Local Maxima and Minima



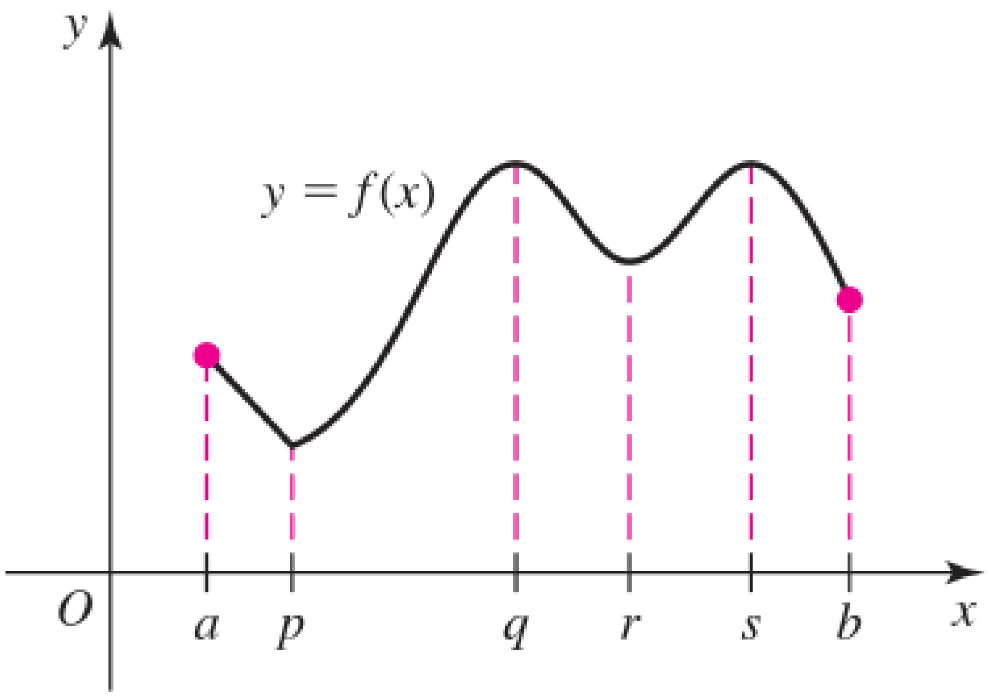
A point at which a function takes on the maximum or minimum value *among values at nearby points* is important.

DEFINITION Local Maximum and Minimum Values

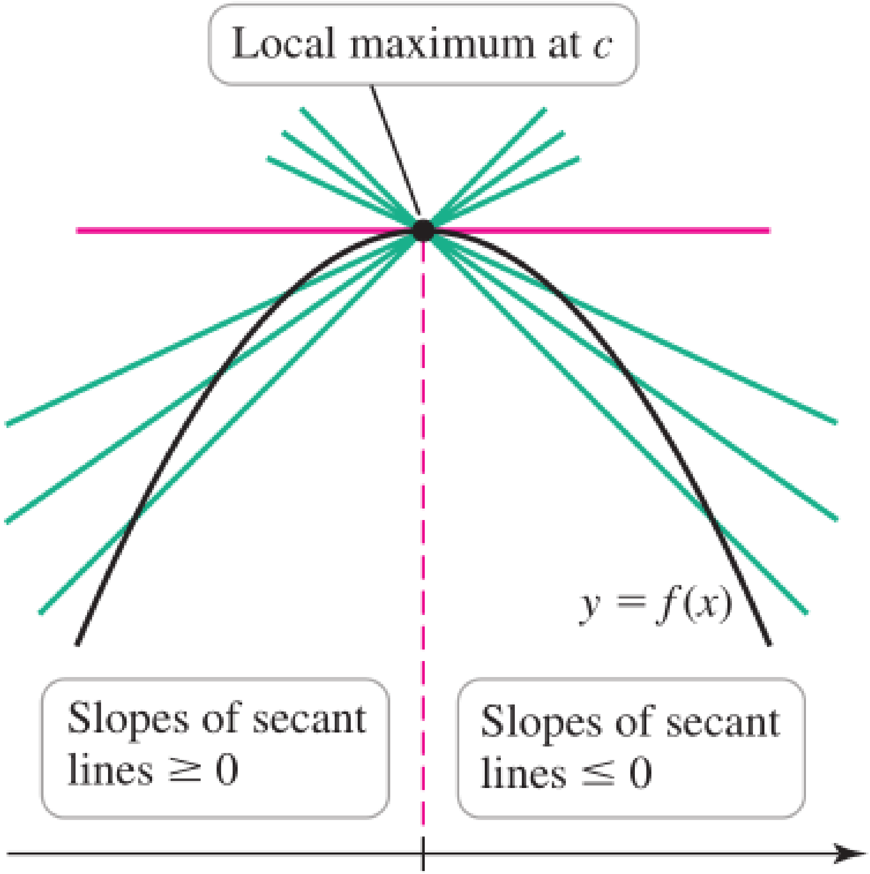
Suppose c is an interior point of some interval I on which f is defined. If $f(c) \geq f(x)$ for all x in I, then $f(c)$ is a **local maximum** value of f. If $f(c) \leq f(x)$ for all x in I, then $f(c)$ is a **local minimum** value of f.

This book adopts the convention that local maximum values and local minimum values occur only at *interior points of the interval(s) of interest*.

EXAMPLE 2 **Locating various maxima and minima** Figure 4.6 shows the graph of a function defined on $[a, b]$. Identify the location of the various maxima and minima using the terms *absolute* and *local*.



Critical Points

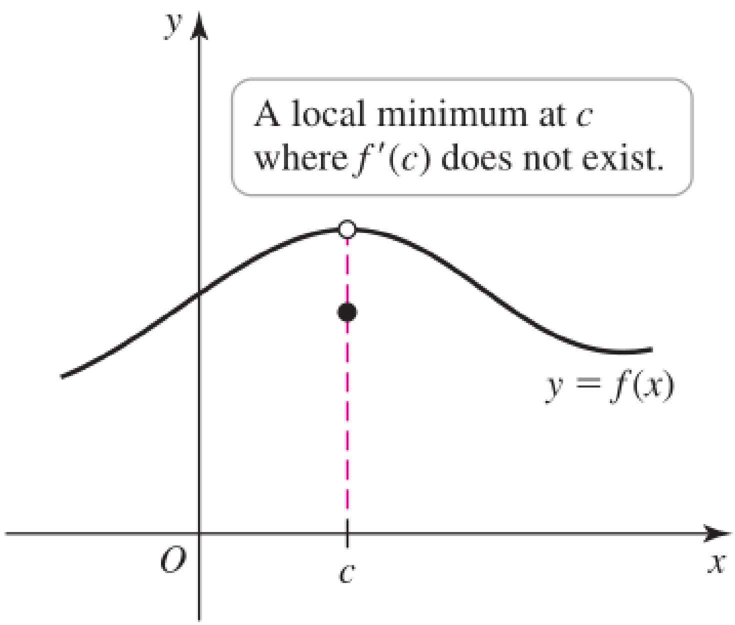


Observations imply that the slope of the tangent line must be both nonnegative and nonpositive, which happens only if $f'(c) = 0$

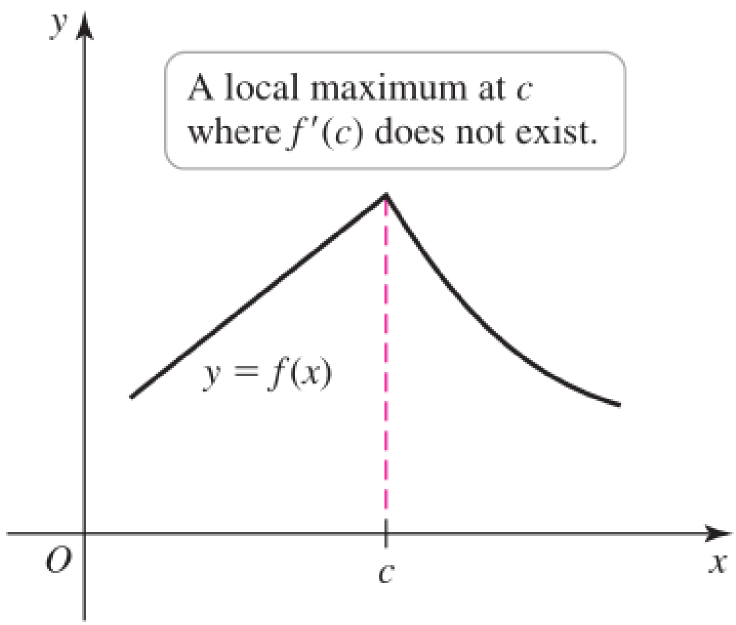
THEOREM 4.2 Local Extreme Value Theorem

If f has a local maximum or minimum value at c and $f'(c)$ exists, then $f'(c) = 0$.

Local extrema can also occur at points c where $f'(c)$ does not exist.



(a)

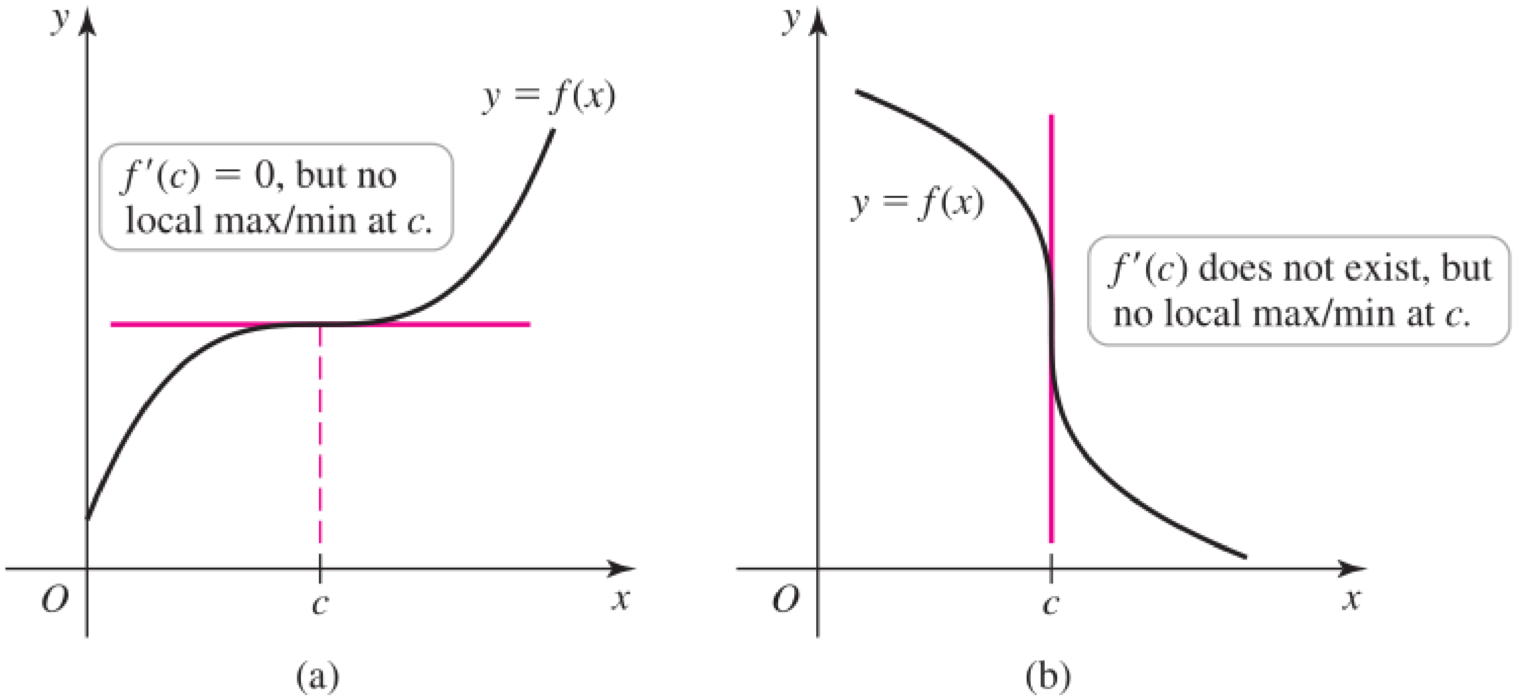


(b)

DEFINITION Critical Point

An interior point c of the domain of f at which $f'(c) = 0$ or $f'(c)$ fails to exist is called a **critical point** of f.

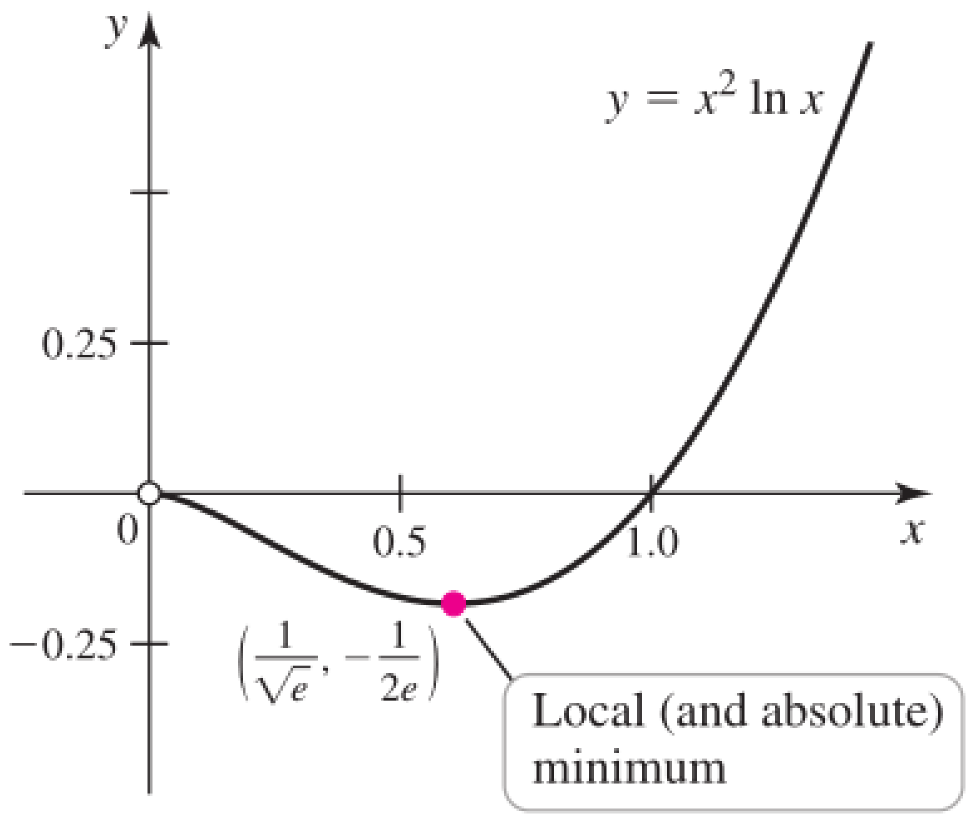
The converse of **Theorem 4.2** is not necessarily true.



Therefore, critical points are candidates for the location of local extreme values.

The determination procedure is discussed in Section 4.2.

EXAMPLE 3 **Locating critical points** Find the critical points of $f(x) = x^2 \ln x$.



Locating Absolute Maxima and Minima

Theorem 4.1 guarantees the existence of absolute extreme values of a continuous function on a closed interval $[a, b]$, but it doesn't say where these values are located.

Two observations

- An absolute extreme value in the interior of an interval is also a local extreme value, and local extreme values occur at the critical points of f.
- Absolute extreme values may also occur at the endpoints of the interval of interest.

PROCEDURE Locating Absolute Extreme Values on a Closed Interval

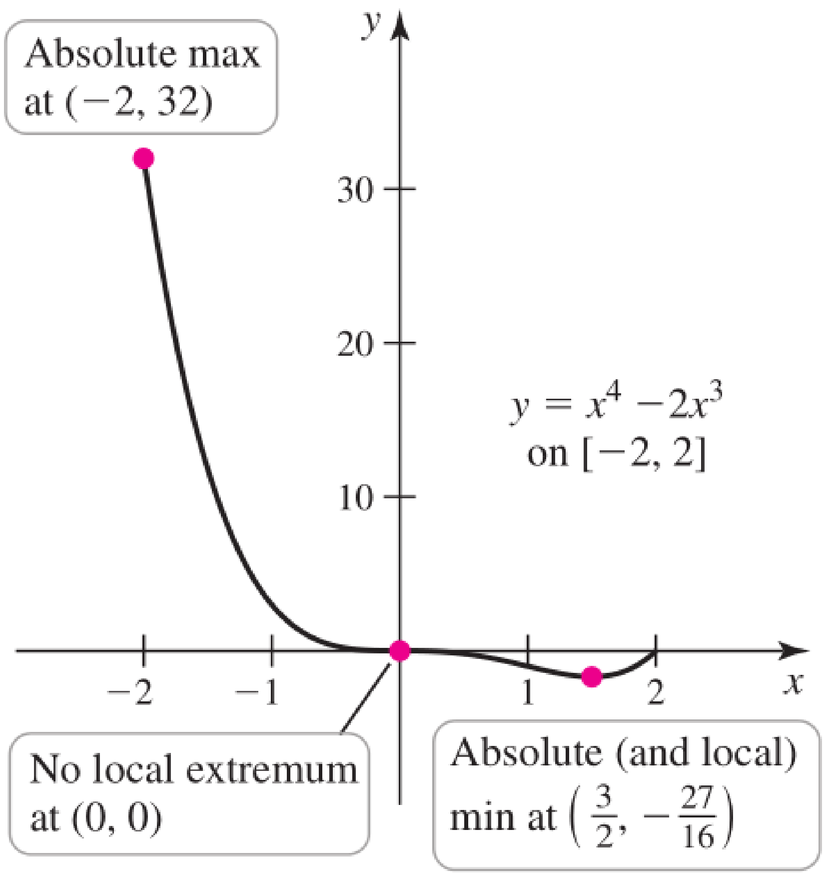
Assume the function f is continuous on the closed interval $[a, b]$.

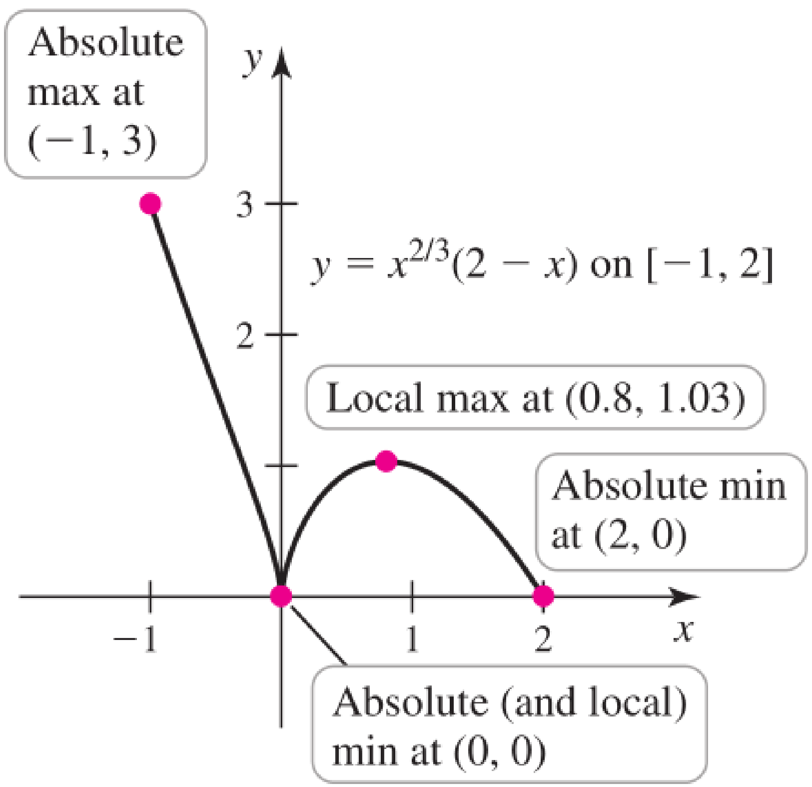
1. Locate the critical points c in (a, b), where $f'(c) = 0$ or $f'(c)$ does not exist. These points are candidates for absolute maxima and minima.
2. Evaluate f at the critical points and at the endpoints of $[a, b]$.
3. Choose the largest and smallest values of f from Step 2 for the absolute maximum and minimum values, respectively.

EXAMPLE 4 **Absolute extreme values** Find the absolute maximum and minimum values of the following functions.

a. $f(x) = x^4 - 2x^3$ on the interval $[-2, 2]$

b. $g(x) = x^{2/3}(2 - x)$ on the interval $[-1, 2]$





EXAMPLE 5 **Trajectory high point** A stone is launched vertically upward from a bridge 80 ft above the ground at a speed of 64 ft/s. Its height above the ground t seconds after the launch is given by

$$f(t) = -16t^2 + 64t + 80, \quad \text{for } 0 \leq t \leq 5.$$

When does the stone reach its maximum height?

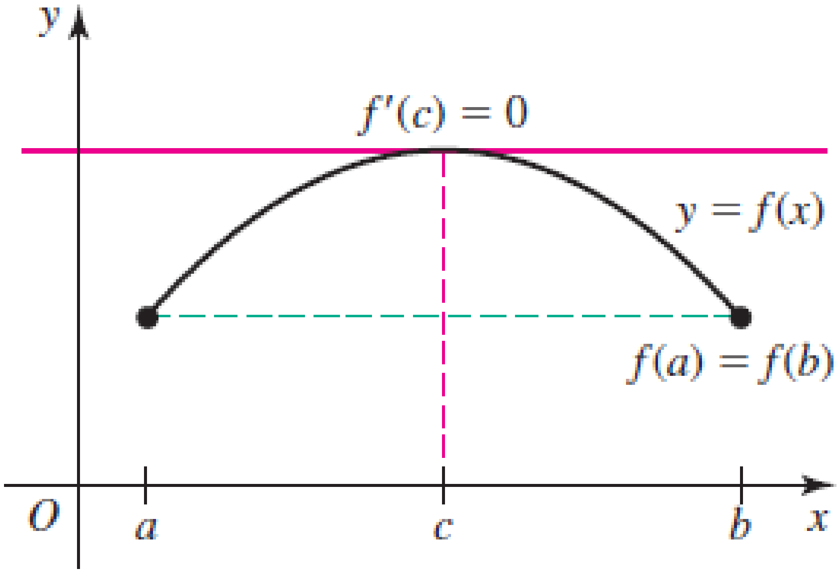
4.2

Mean Value Theorem

Rolle's Theorem

Mean Value Theorem is a cornerstone in the theoretical framework of calculus.

THEOREM 4.8 Rolle's Theorem
Let f be continuous on a closed interval $[a, b]$ and differentiable on (a, b) with $f(a) = f(b)$. There is at least one point c in (a, b) such that $f'(c) = 0$.



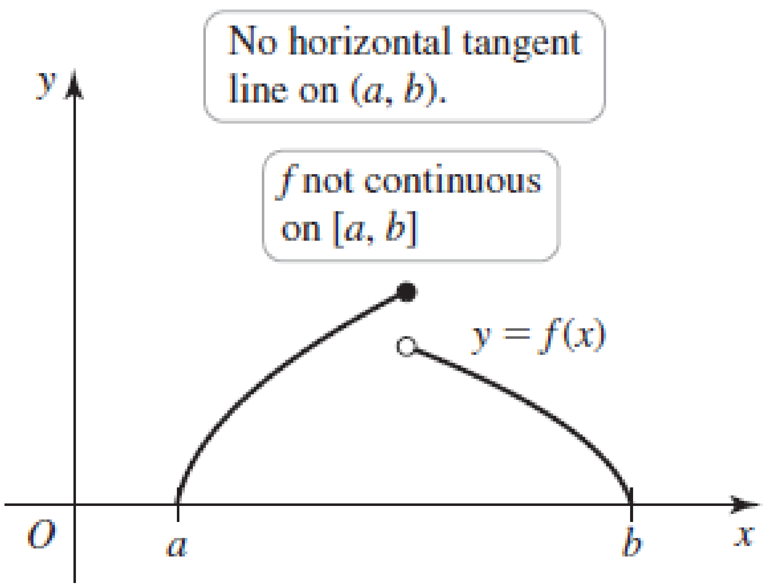
Proof: From Extreme Value Theorem, the function f attains its absolute maximum and minimum values on $[a, b]$.

Case 1: Absolute maximum and minimum values at the endpoints
Because $f(a) = f(b)$, the maximum and minimum values are equal, and it follows that f is a constant function on $[a, b]$.
Therefore, $f'(x) = 0$ for all x in $[a, b]$.

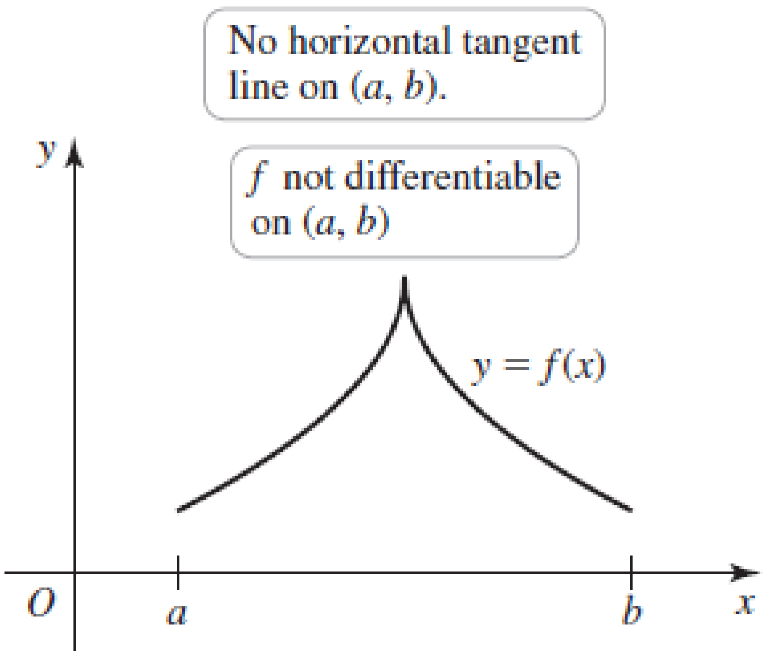
Case 2: At least one absolute extreme value of f does not occur at an endpoint.
Then, f must have either a local maximum or a local minimum at a point c in $[a, b]$.
From Theorem 4.2 that $f'(c) = 0$ where $c \in (a, b)$.

A function that is not continuous on $[a, b]$ may have identical values at both endpoints and still not have a horizontal tangent line at any point on the interval.

A function that is continuous on $[a, b]$ but not differentiable at a point of $[a, b]$ may also fail to have a horizontal tangent line.

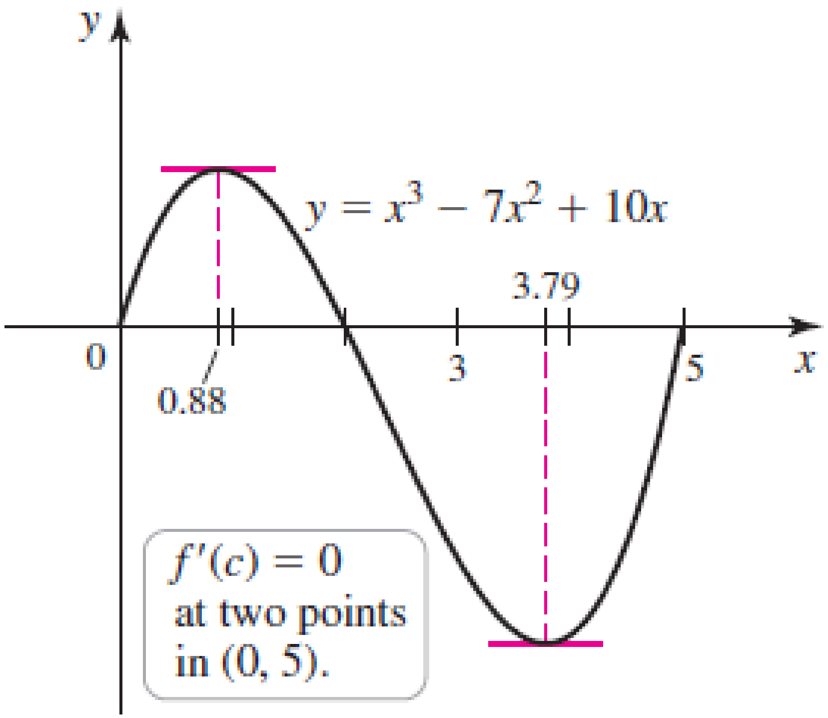


(a)



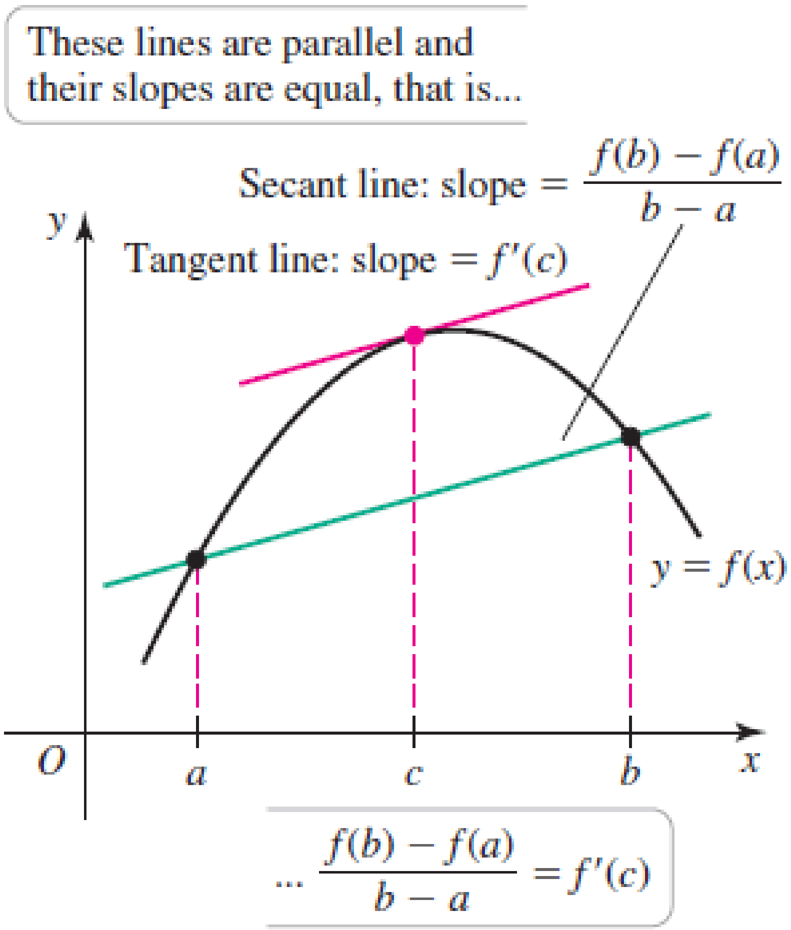
(b)

EXAMPLE 1 **Verifying Rolle's Theorem** Find an interval I on which Rolle's Theorem applies to $f(x) = x^3 - 7x^2 + 10x$. Then find all points c in I at which $f'(c) = 0$.



Mean Value Theorem

The slope of the secant line is the average rate of change of f over $[a, b]$.

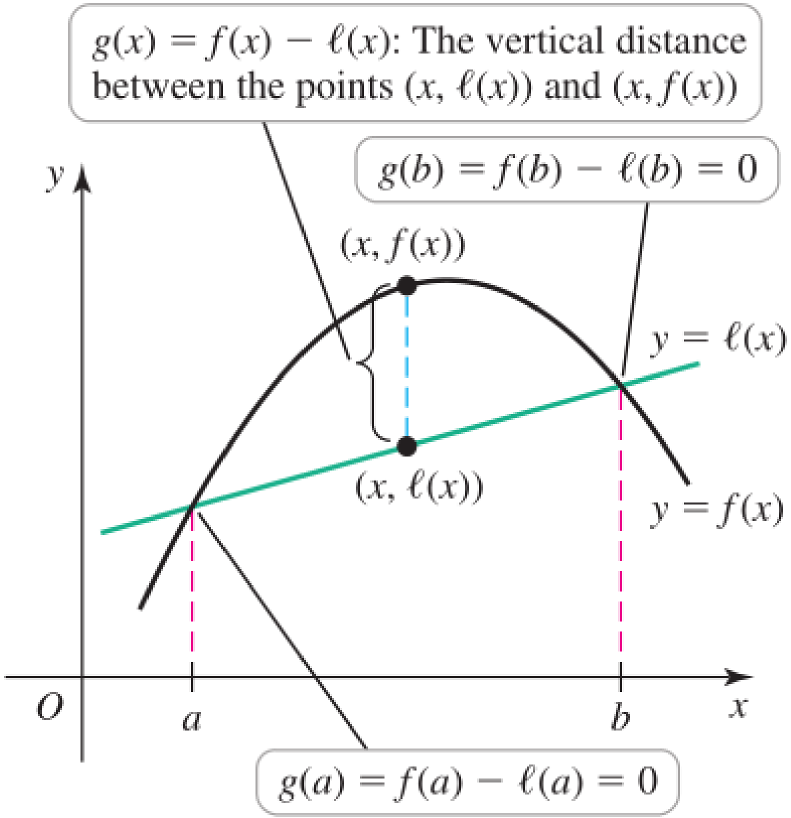


THEOREM 4.9 Mean Value Theorem

If f is continuous on the closed interval $[a, b]$ and differentiable on (a, b), then there is at least one point c in (a, b) such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

Proof: The strategy is to use the function f of the Mean Value Theorem to form a new function g that satisfies Rolle's Theorem.



A new function g that measures the vertical distance between the given function f and the line l:

$$g(x) = f(x) - l(x) \text{ with } g(a) = g(b) = 0.$$

g is also continuous on $[a, b]$ and differentiable in (a, b).
We now have g that satisfies the conditions of Rolle's Theorem.

It guarantees the existence of at least one point c in the interval (a, b) such that $g'(c) = 0$.
It follows that $f'(c) - l'(c) = 0$, i.e., $f'(c) = l'(c)$.

$l'(c)$ is just the slope of the secant line, which is

$$\frac{f(b) - f(a)}{b - a}$$

This completes the proof.

An interpretation of the Mean Value Theorem

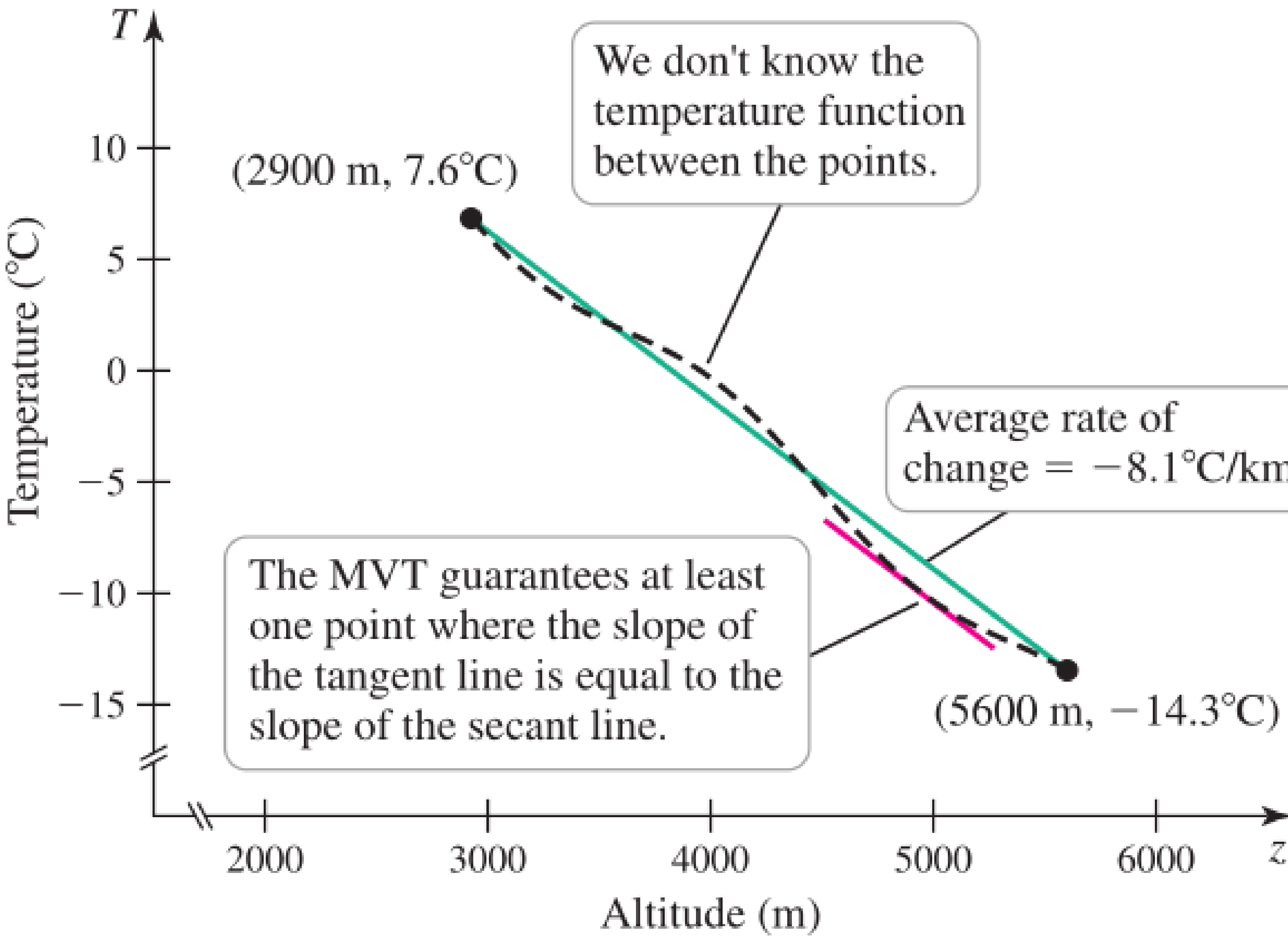
Imagine driving for 2 hours to a town 100 miles away. While your average speed is 100 mi/2 hr = 50 mi/hr.

Your instantaneous speed (measured by the speedometer) almost certainly varies.

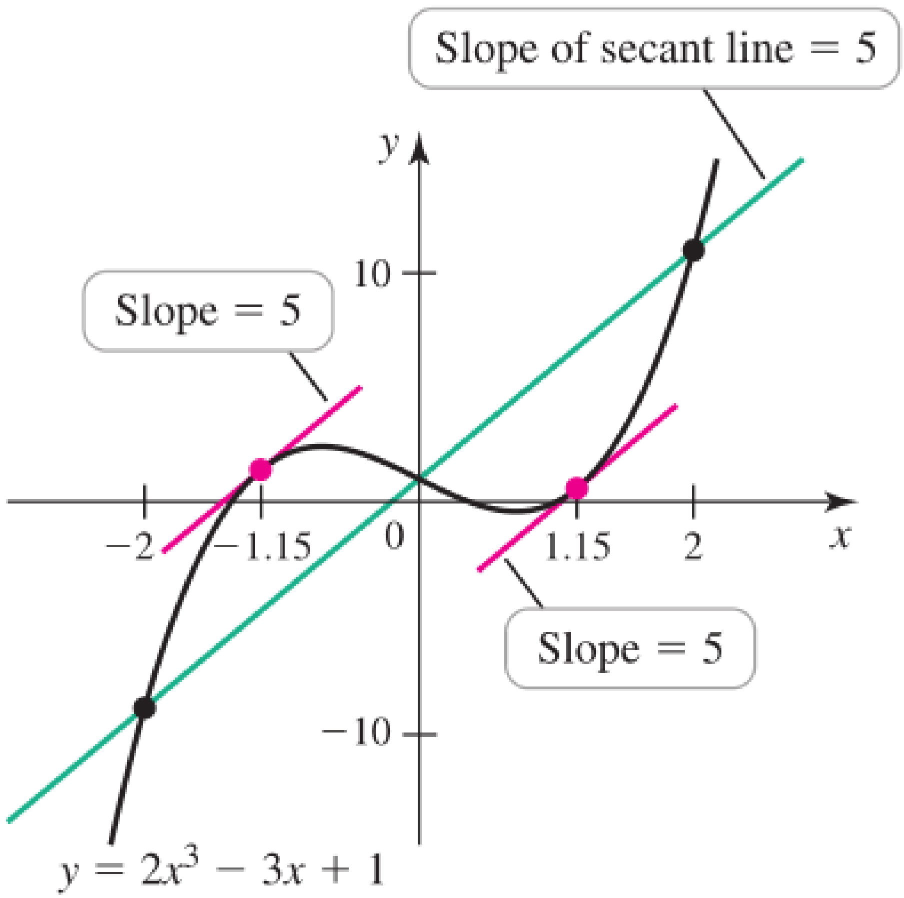
The Mean Value Theorem says that at some point during the trip, your instantaneous speed equals your average speed.

EXAMPLE 2 **Mean Value Theorem in action** The *lapse rate* is the rate at which the temperature T decreases in the atmosphere with respect to increasing altitude z. It is typically reported in units of °C/km and is defined by $\gamma = -dT/dz$. When the lapse rate rises above 7°C/km in a certain layer of the atmosphere, it indicates favorable conditions for thunderstorm and tornado formation, provided other atmospheric conditions are also present.

Suppose the temperature at $z = 2.9$ km is $T = 7.6$°C and the temperature at $z = 5.6$ km is $T = -14.3$°C. Assume also that the temperature function is continuous and differentiable at all altitudes of interest. What can a meteorologist conclude from these data?


T
10
5
0
−5
−10
−15
Temperature (°C)
(2900 m, 7.6°C)
We don't know the temperature function between the points.
Average rate of change = −8.1°C/km
The MVT guarantees at least one point where the slope of the tangent line is equal to the slope of the secant line.
(5600 m, −14.3°C)
2000
3000
4000
5000
6000
z
Altitude (m)

EXAMPLE 3 **Verifying the Mean Value Theorem** Determine whether the function $f(x) = 2x^3 - 3x + 1$ satisfies the conditions of the Mean Value Theorem on the interval $[-2, 2]$. If so, find the point(s) guaranteed to exist by the theorem.



Consequences of the Mean Value Theorem

If $f(x) = C$, then $f'(x) = 0$. How about the converse?

THEOREM 4.10 Zero Derivative Implies Constant Function
If f is differentiable and $f'(x) = 0$ at all points of an interval I, then f is a constant function on I.

And go further

THEOREM 4.11 Functions with Equal Derivatives Differ by a Constant
If two functions have the property that $f'(x) = g'(x)$, for all x of an interval I, then $f(x) - g(x) = C$ on I, where C is a constant; that is, f and g differ by a constant.

4.3

What Derivatives Tell Us

Increasing and Decreasing Functions

DEFINITION Increasing and Decreasing Functions

Suppose a function f is defined on an interval I. We say that f is **increasing** on I if $f(x_2) > f(x_1)$ whenever x_1 and x_2 are in I and $x_2 > x_1$. We say that f is **decreasing** on I if $f(x_2) < f(x_1)$ whenever x_1 and x_2 are in I and $x_2 > x_1$.

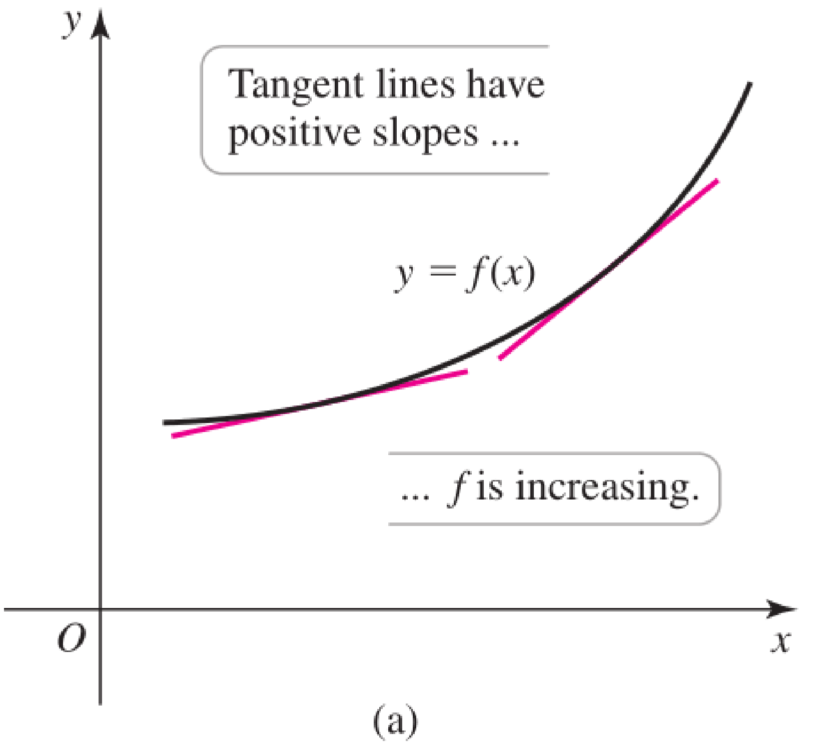
Monotonic: either increasing or decreasing

Nondecreasing: $f(x_2) \geq f(x_1)$ whenever $x_2 > x_1$

Nonincreasing: $f(x_2) \leq f(x_1)$ whenever $x_2 > x_1$

Intervals of Increase and Decrease

How to determine those intervals precisely?



THEOREM 4.12 Intervals of Increase and Decrease
Suppose f is continuous on an interval I and differentiable at all interior points of I. If $f'(x) > 0$ at all interior points of I, then f is increasing on I. If $f'(x) < 0$ at all interior points of I, then f is decreasing on I.

Proof: Let a and b be any two distinct points in the interval I with $b > a$. By Extreme Value Theorem,

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

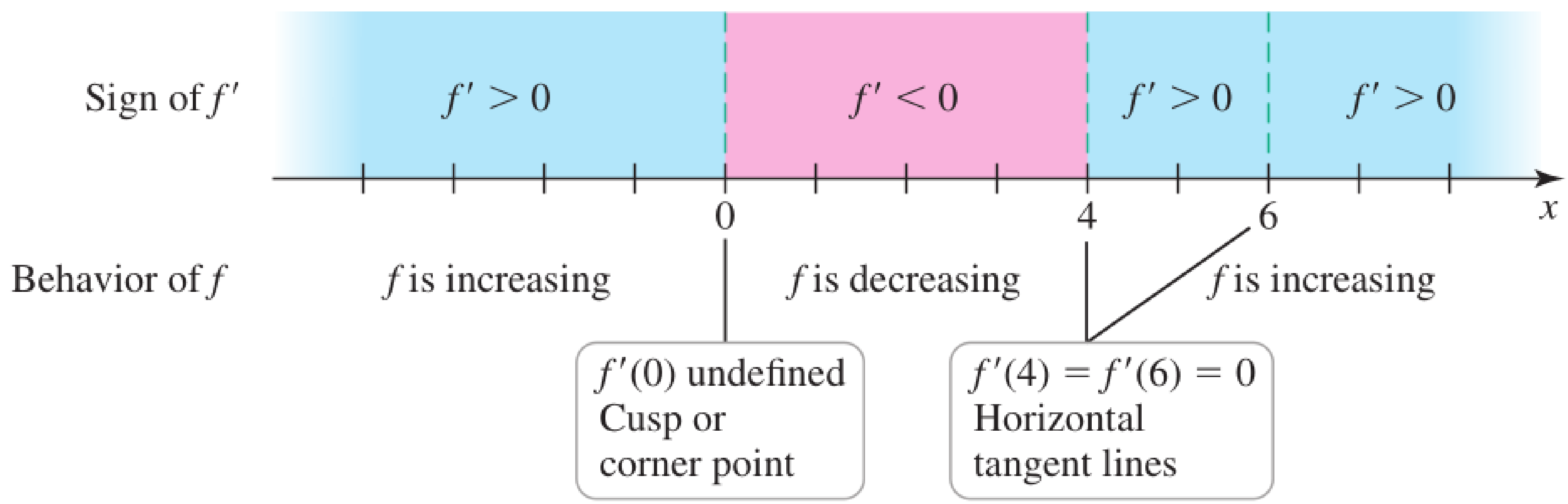
Equivalently,

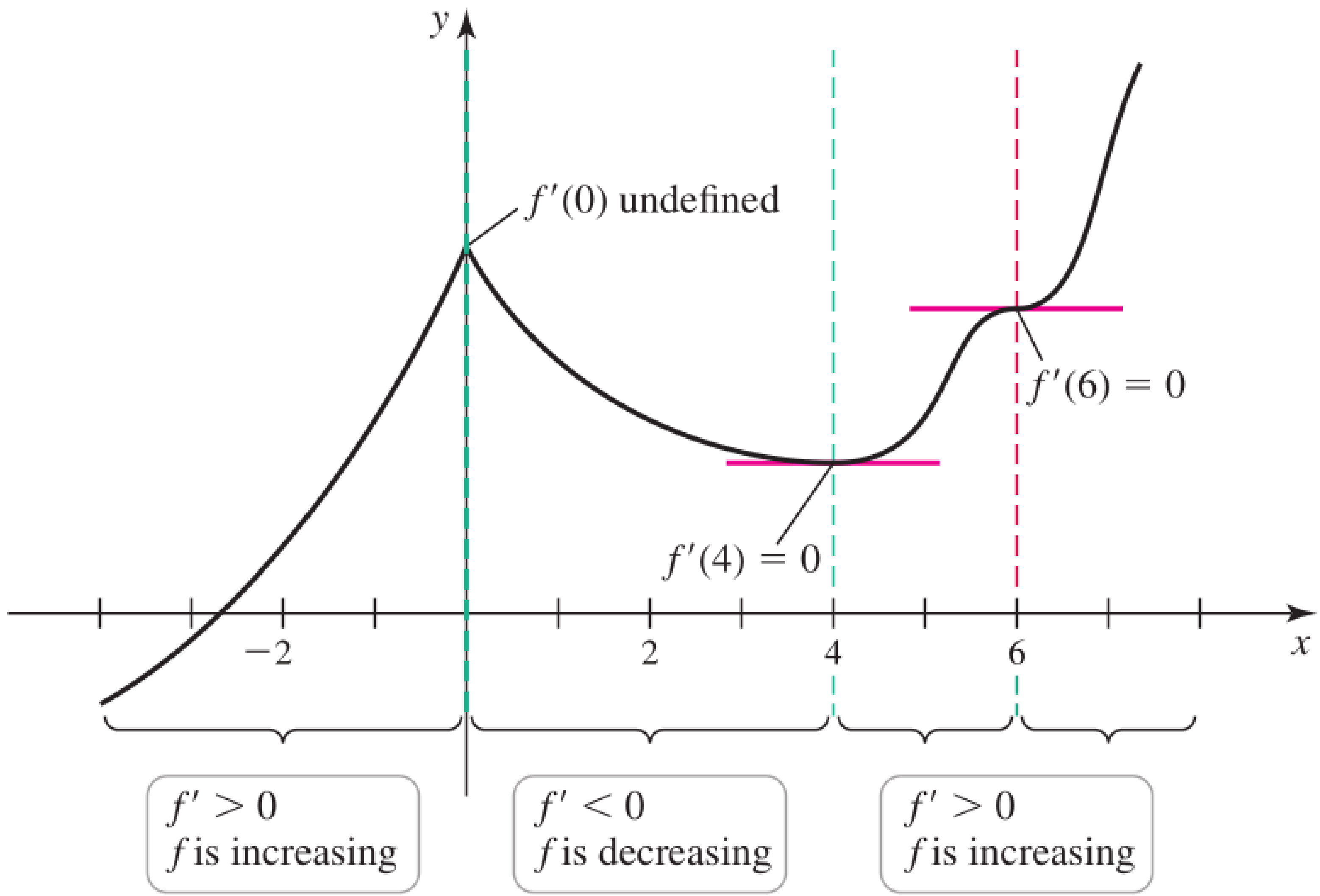
$$f(b) - f(a) = f'(c)(b - a)$$

Notice that $b - a > 0$ by assumption.
So if $f'(c) > 0$, then $f(b) - f(a) > 0$, f is increasing on I.
If $f'(c) < 0$, then $f(b) - f(a) < 0$, f is decreasing on I.

EXAMPLE 1 **Sketching a function** Sketch a graph of a function f that is continuous on $(-\infty, \infty)$ and satisfies the following conditions.

1. $f' > 0$ on $(-\infty, 0)$, $(4, 6)$, and $(6, \infty)$.
2. $f' < 0$ on $(0, 4)$.
3. $f'(0)$ is undefined.
4. $f'(4) = f'(6) = 0$.

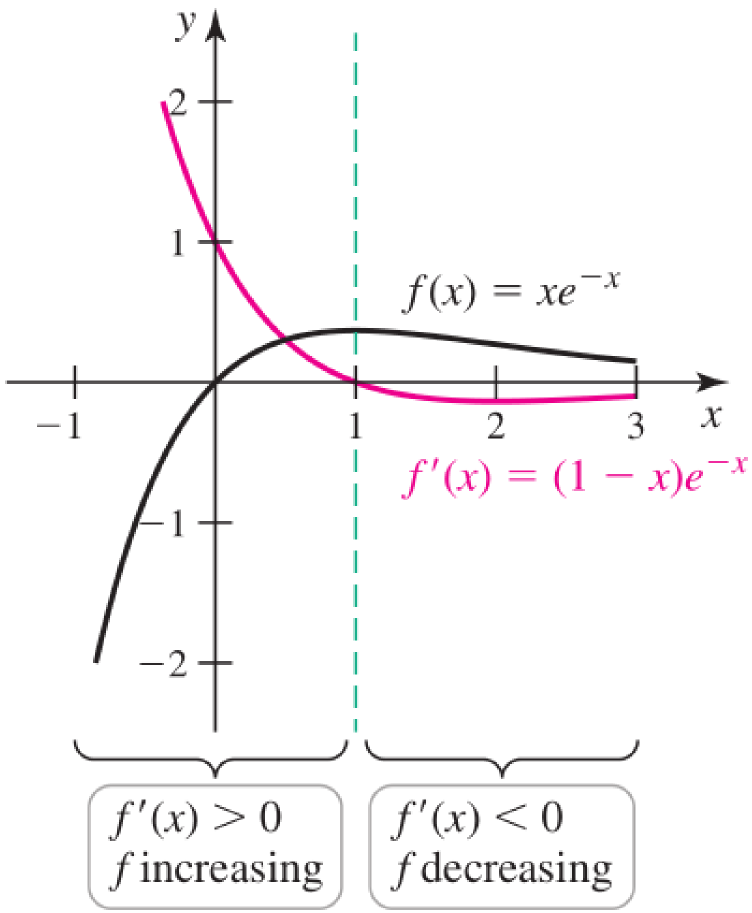


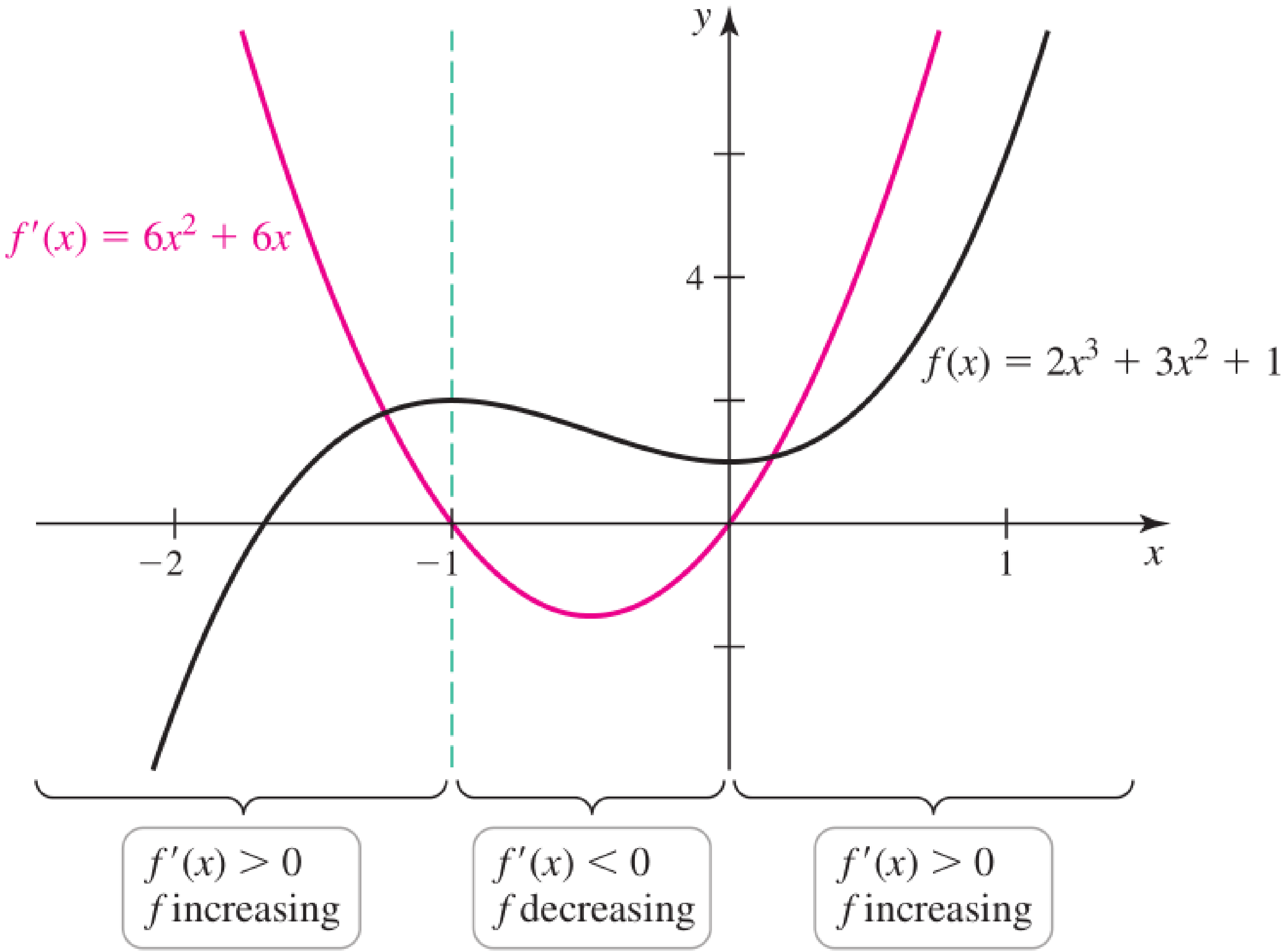


y
x
−2
2
4
6
f′(0) undefined
f′(6) = 0
f′(4) = 0
f′ > 0
f is increasing
f′ < 0
f is decreasing
f′ > 0
f is increasing

EXAMPLE 2 **Intervals of increase and decrease** Find the intervals on which the following functions are increasing and decreasing.

a. $f(x) = xe^{-x}$ **b.** $f(x) = 2x^3 + 3x^2 + 1$

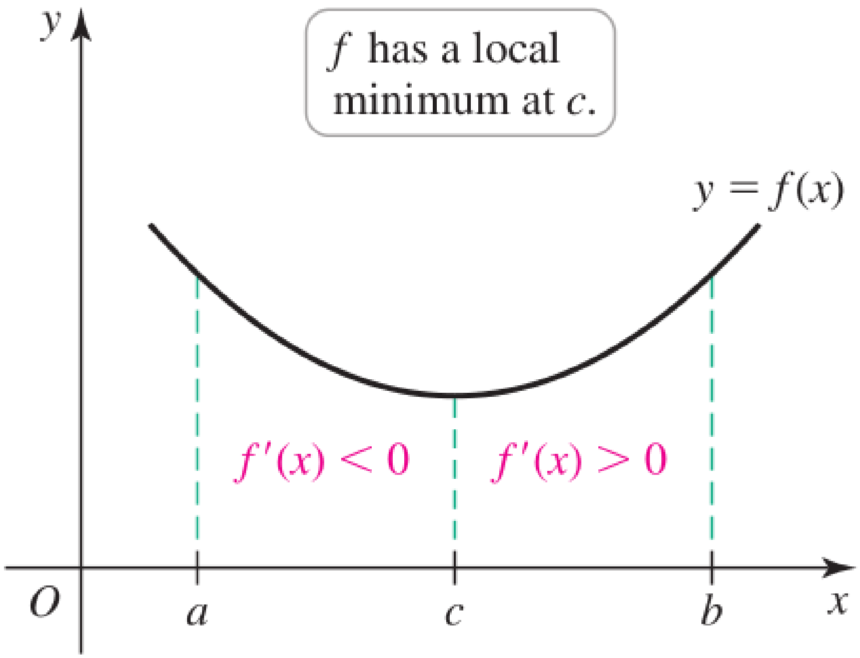




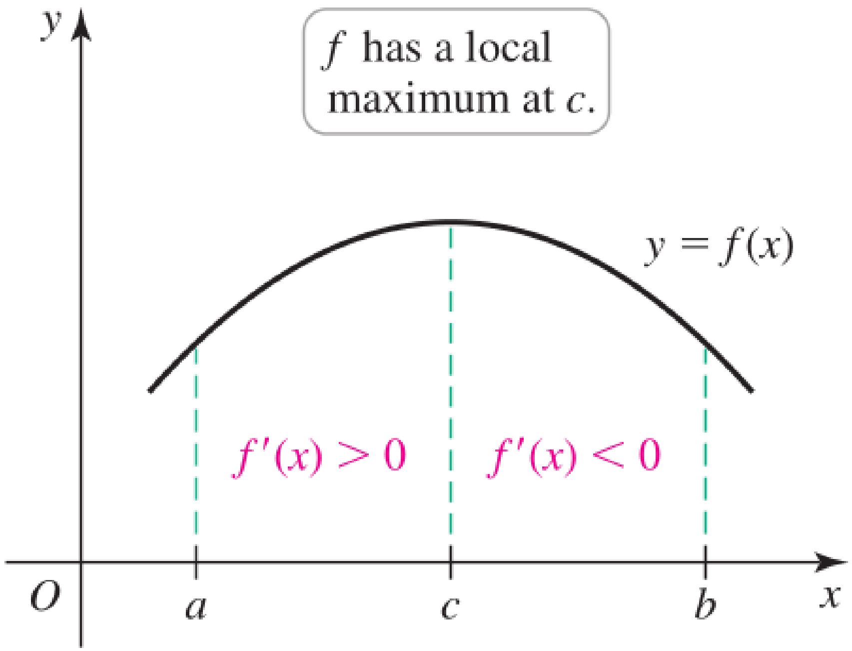
y
$f'(x) = 6x^2 + 6x$
4
$f(x) = 2x^3 + 3x^2 + 1$
-2
-1
1
x
$f'(x) > 0$
f increasing
$f'(x) < 0$
f decreasing
$f'(x) > 0$
f increasing

Identifying Local Maxima and Minima

Suppose $x = c$ is a critical point of f, where $f'(c) = 0$, and f' changes sign at c.



(a)



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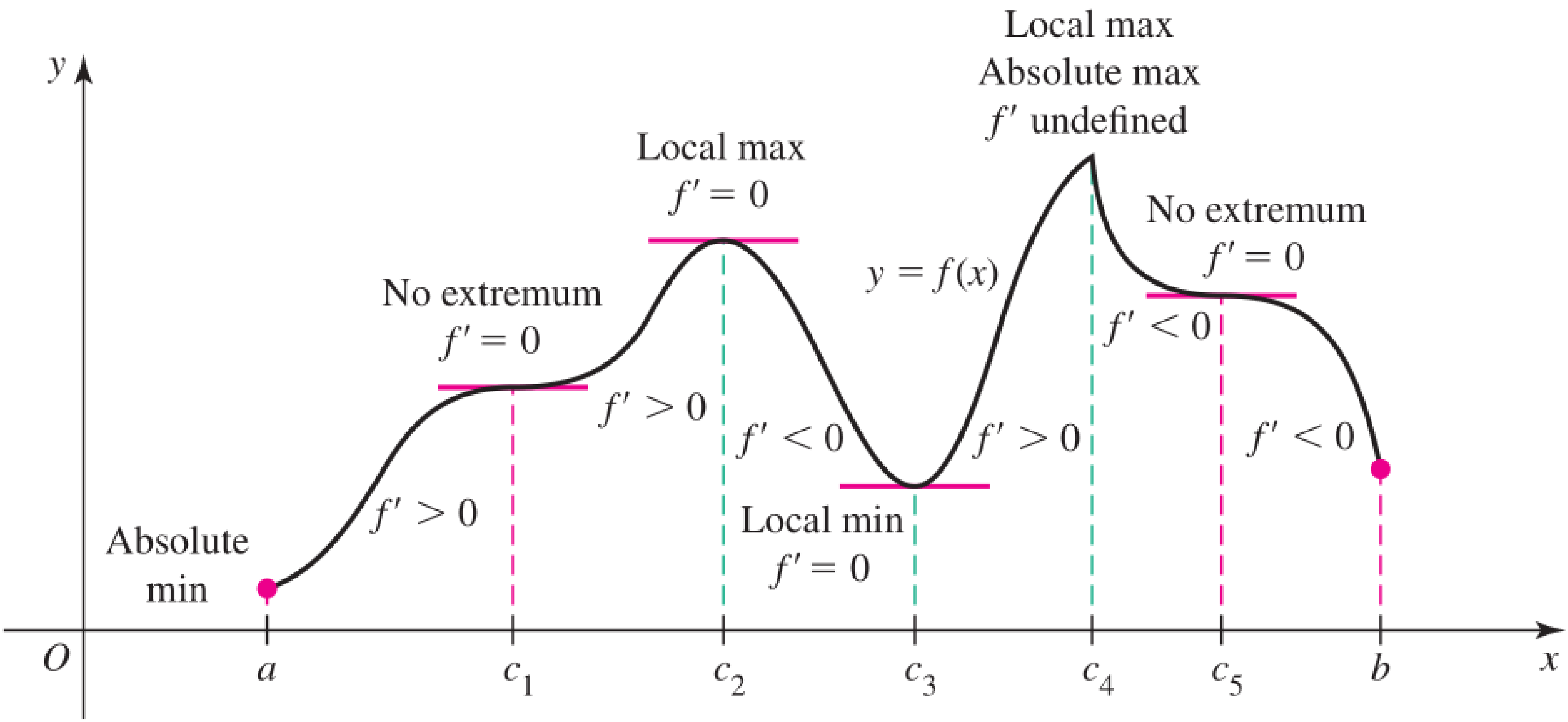


Local max
Absolute max
f' undefined
Local max
$f' = 0$
No extremum
$f' = 0$
No extremum
$f' = 0$
$y = f(x)$
$f' > 0$
$f' < 0$
$f' > 0$
$f' < 0$
$f' < 0$
$f' > 0$
Local min
$f' = 0$
Absolute min
y
O
a
c_1
c_2
c_3
c_4
c_5
b
x

First Derivative test

THEOREM 4.4 First Derivative Test

Suppose that f is continuous on an interval that contains a critical point c and assume f is differentiable on an interval containing c, except perhaps at c itself.

- If f' changes sign from positive to negative as x increases through c, then f has a **local maximum** at c.
- If f' changes sign from negative to positive as x increases through c, then f has a **local minimum** at c.
- If f' is positive on both sides near c or negative on both sides near c, then f has no local extreme value at c.

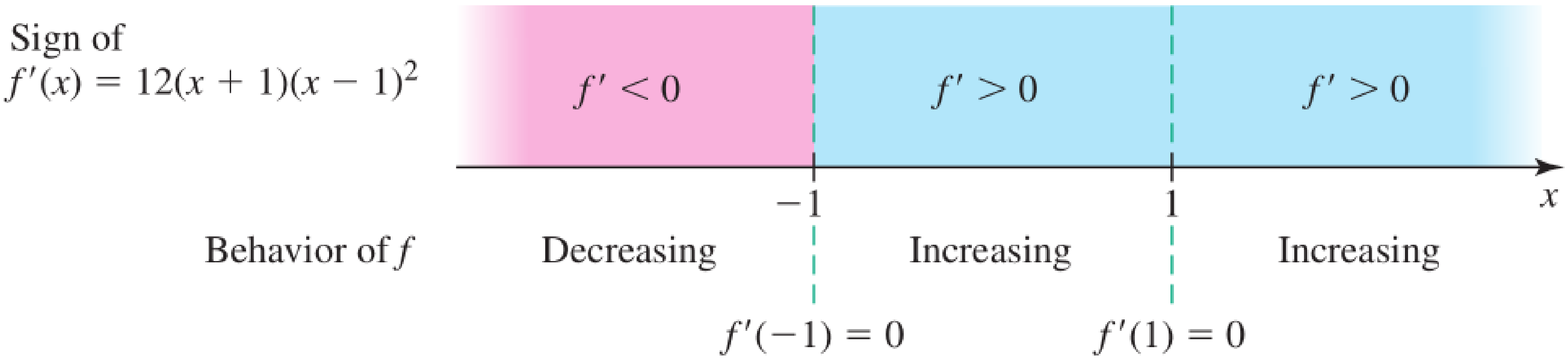
EXAMPLE 3 **Using the First Derivative Test** Consider the function

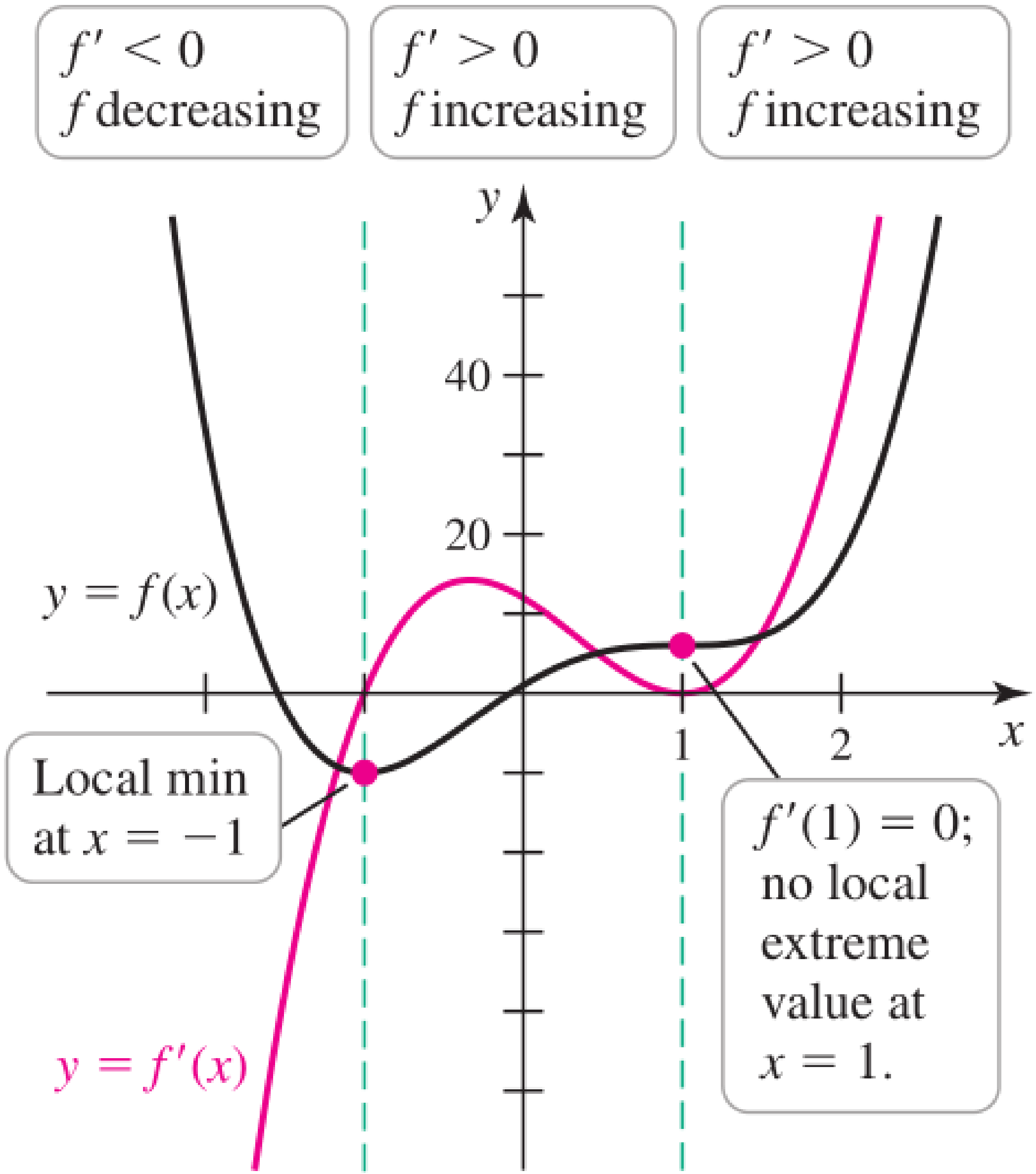
$$f(x) = 3x^4 - 4x^3 - 6x^2 + 12x + 1.$$

a. Find the intervals on which f is increasing and decreasing.

b. Identify the local extrema of f.

$$f'(x) = 12x(x+1)(x-1)^2$$

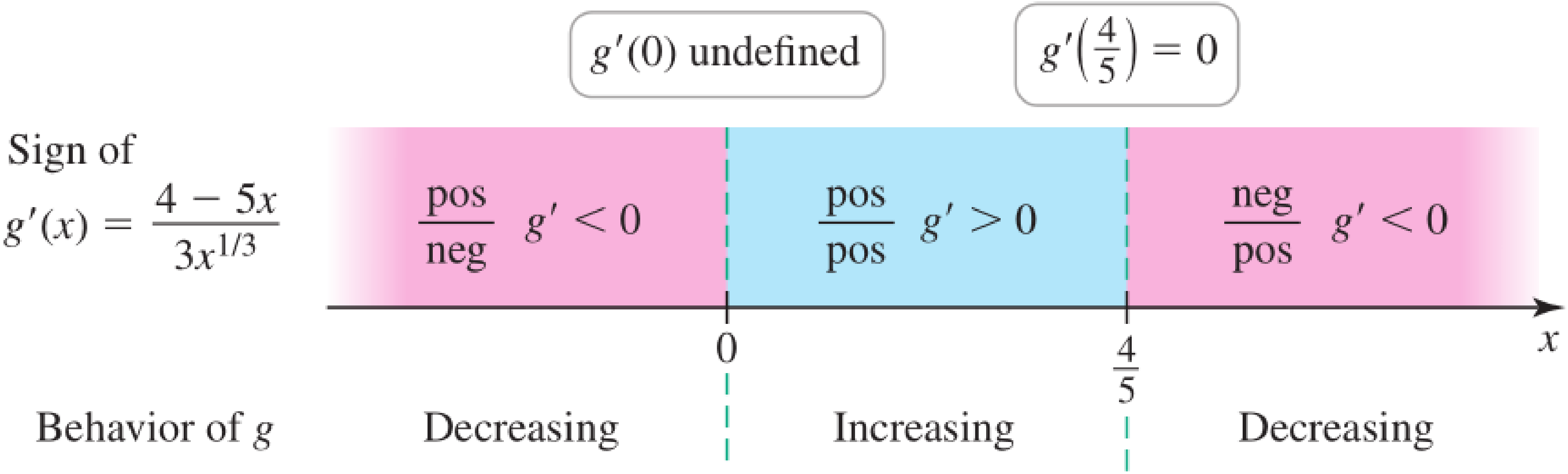


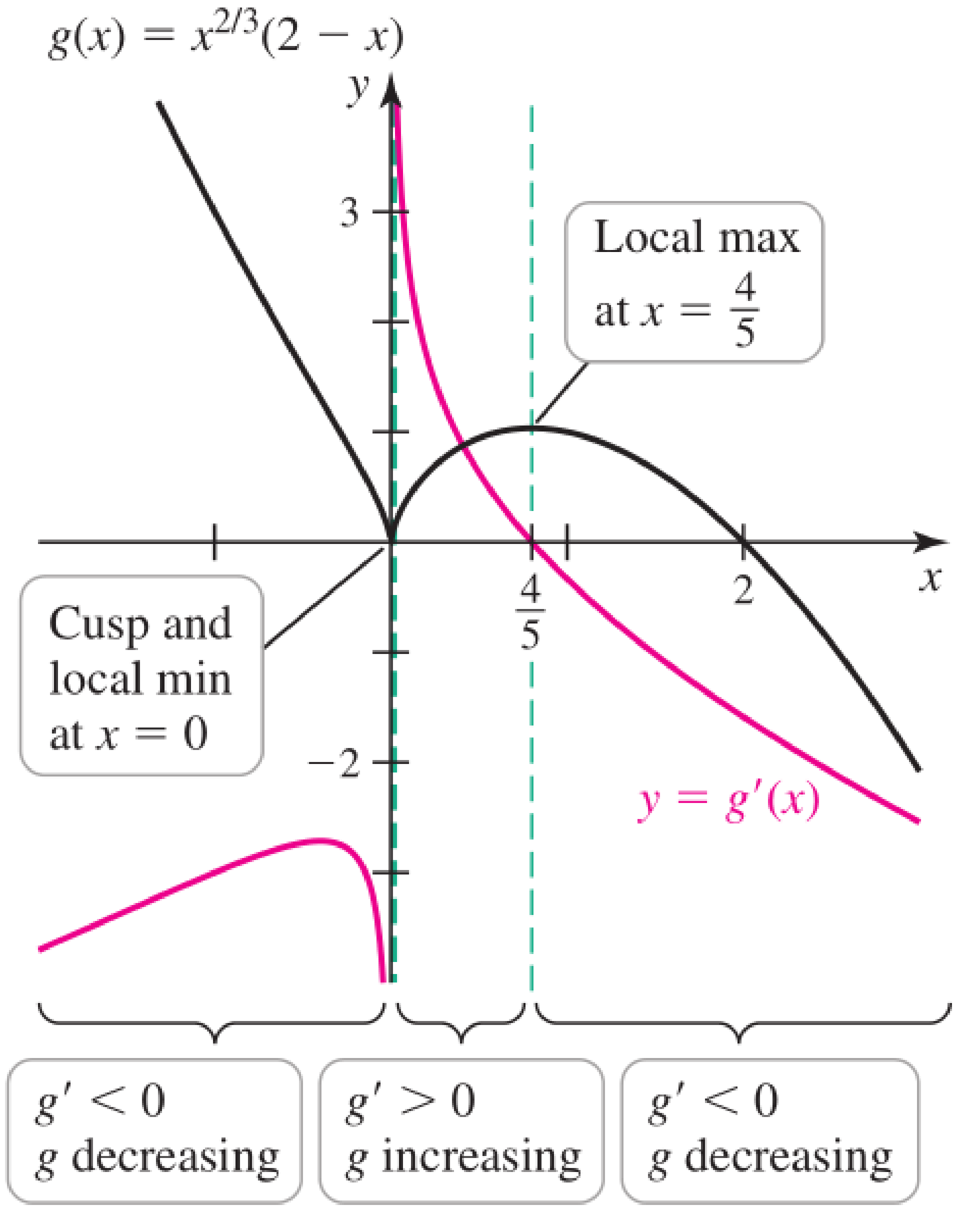


$f' < 0$
f decreasing
$f' > 0$
f increasing
$f' > 0$
f increasing
y
40
20
$y = f(x)$
1
2
x
Local min
at $x = -1$
$f'(1) = 0$;
no local
extreme
value at
$x = 1$.
$y = f'(x)$

EXAMPLE 4 **Extreme values** Find the local extrema of the function $g(x) = x^{2/3}(2 - x)$.

$$g'(x) = \frac{4}{3}x^{-1/3} - \frac{5}{3}x^{2/3} = \frac{4 - 5x}{3x^{1/3}}$$




$g(x) = x^{2/3}(2 - x)$
y
3
Local max
at $x = \frac{4}{5}$
Cusp and
local min
at $x = 0$
−2
4/5
2
x
$y = g'(x)$
$g' < 0$
g decreasing
$g' > 0$
g increasing
$g' < 0$
g decreasing

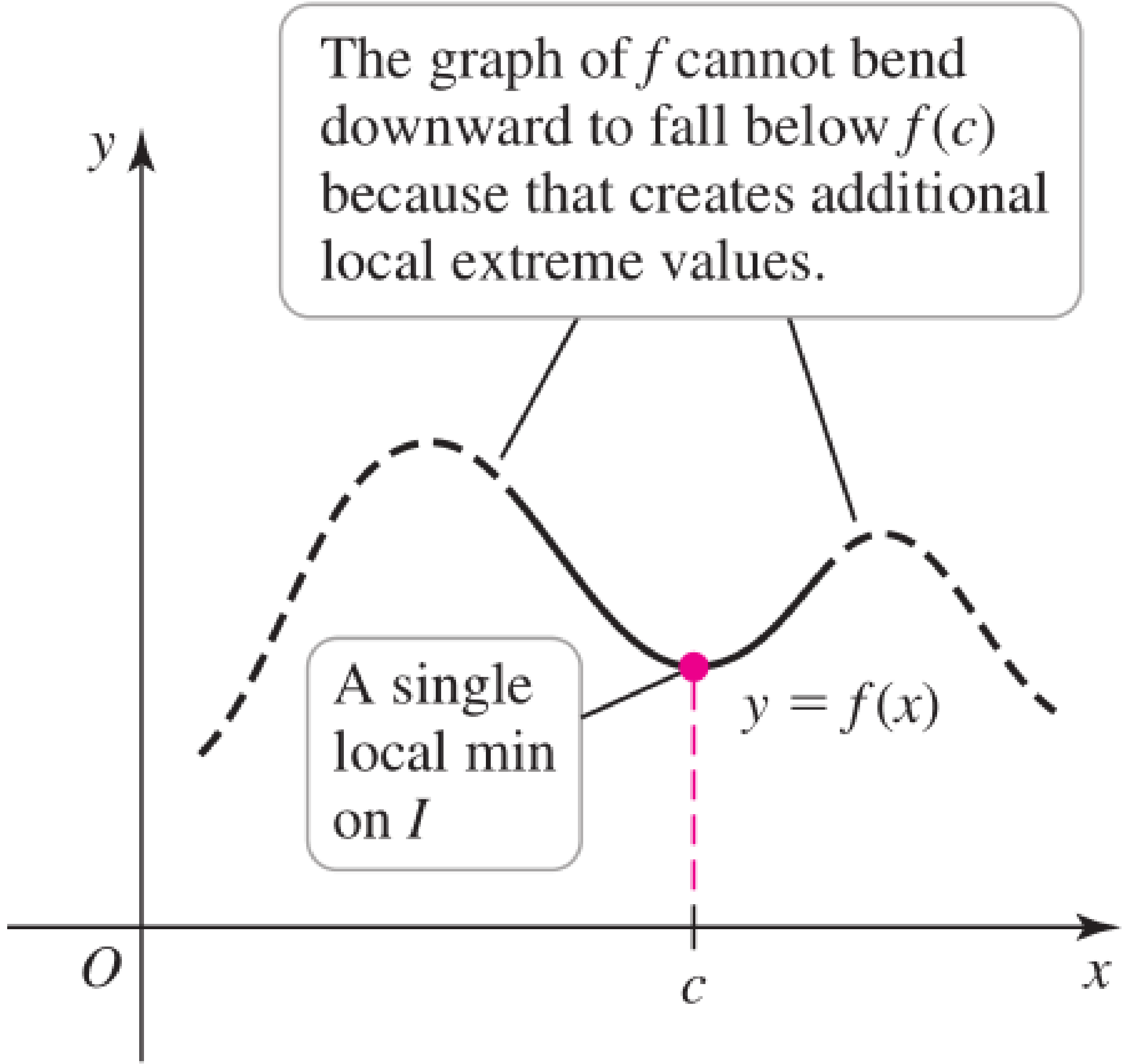
Absolute extreme Values on Any Interval

What about absolute extrema on intervals that are not closed?

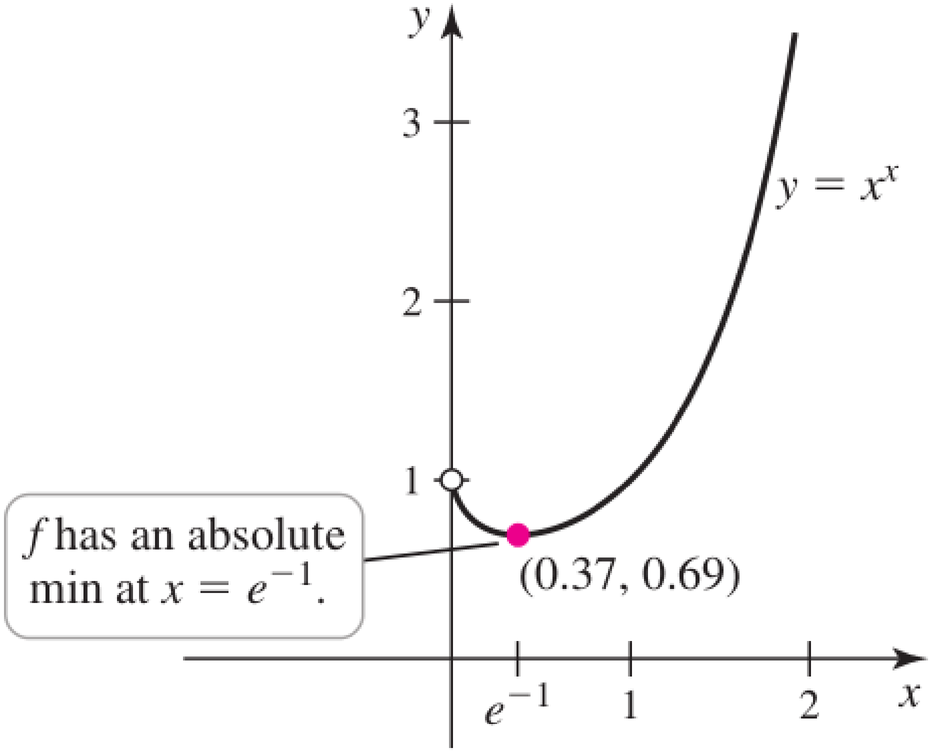
THEOREM 4.5 One Local Extremum Implies Absolute Extremum

Suppose f is continuous on an interval I that contains exactly one local extremum at c.

- If a local maximum occurs at c, then $f(c)$ is the absolute maximum of f on I.
- If a local minimum occurs at c, then $f(c)$ is the absolute minimum of f on I.


The graph of f cannot bend
downward to fall below f(c)
because that creates additional
local extreme values.
y
A single
local min
on I
y = f(x)
O
c
x

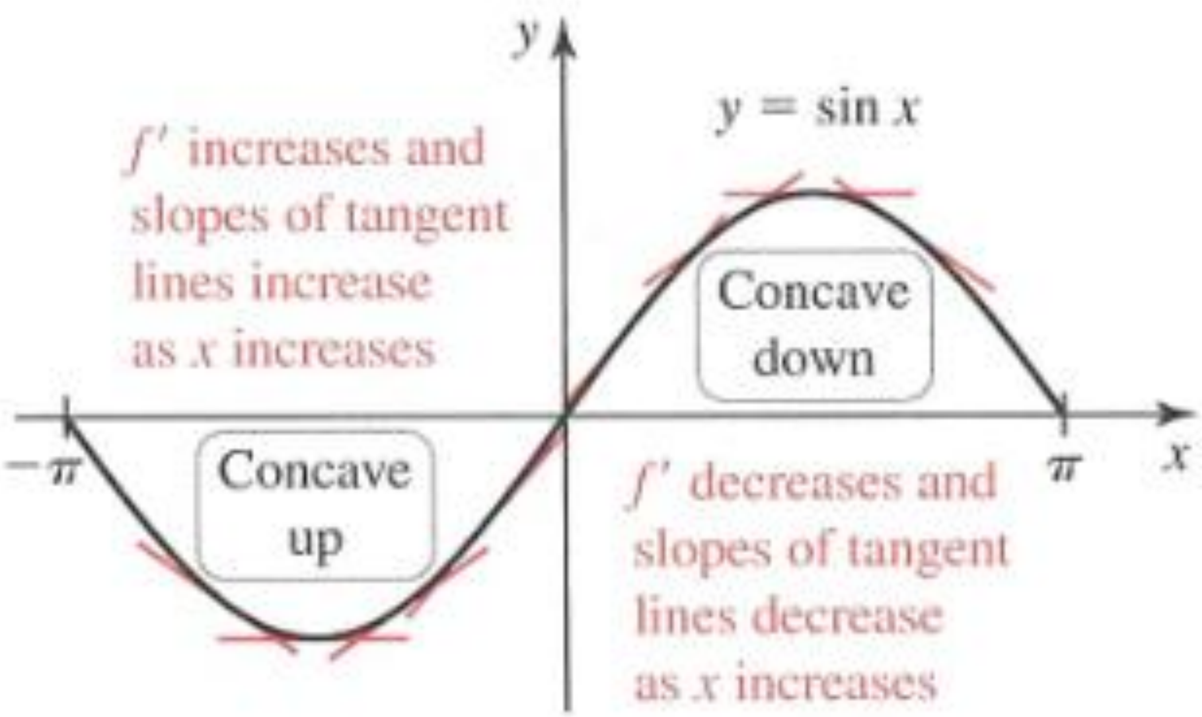
EXAMPLE 5 **Finding an absolute extremum** Verify that $f(x) = x^x$ has an absolute extreme value on its domain.



Concavity and Inflection Points

The second derivative also has geometric meaning

Consider $y = \sin x$



DEFINITION Concavity and Inflection Point

Let f be differentiable on an open interval I. If f' is increasing on I, then f is **concave up** on I. If f' is decreasing on I, then f is **concave down** on I.

If f is continuous at c and f changes concavity at c (from up to down, or vice versa), then f has an **inflection point** at c.

Applying **Theorem 4.3** to f' leads to a test for concavity in terms of the second derivative.

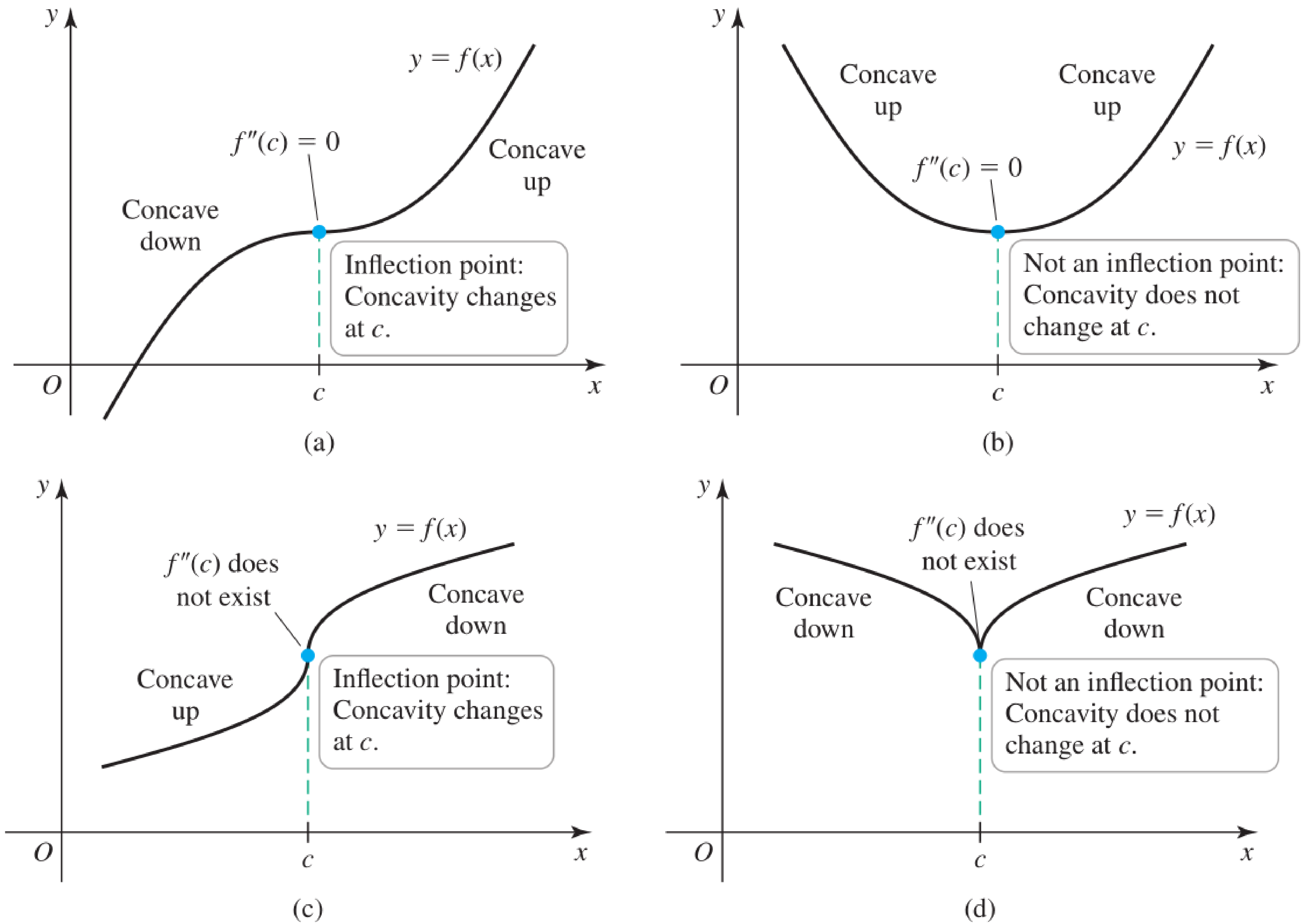
THEOREM 4.6 Test for Concavity

Suppose that f'' exists on an open interval I.

- If $f'' > 0$ on I, then f is concave up on I.
- If $f'' < 0$ on I, then f is concave down on I.
- If c is a point of I at which f'' changes sign at c (from positive to negative, or vice versa), then f has an inflection point at c.

Note

- The fact that $f''(c) = 0$ does not necessarily imply that f has an inflection point at c, e.g., $f(x) = x^4$
- An inflection point may also occur at a point where f'' does not exist, but not vice versa
- In summary, if $f''(c) = 0$ or $f''(c)$ does not exist, then $(c, f(c))$ is a candidate for an inflection point.
- To be certain an inflection point occurs at c, we must show that the concavity of f changes at c.

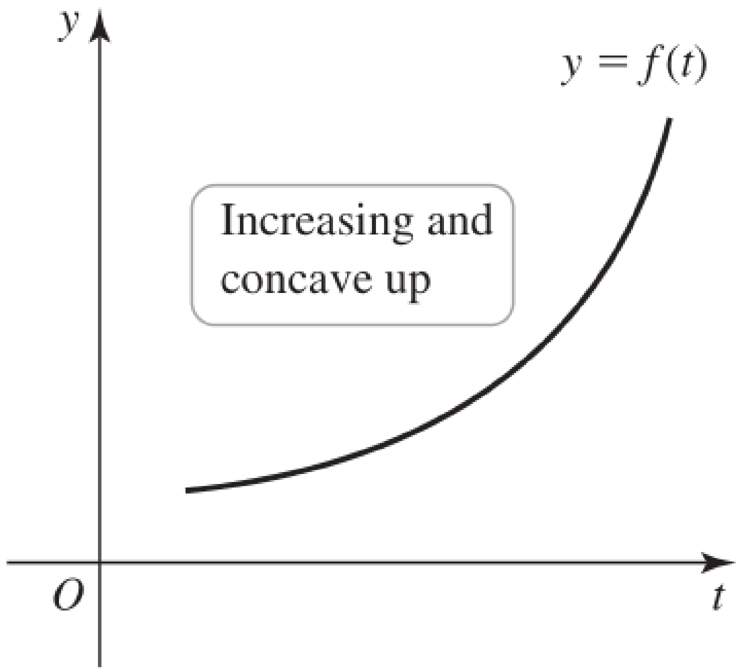

y
y = f(x)
f″(c) = 0
Concave up
Concave down
Inflection point: Concavity changes at c.
O
c
x
(a)
y
Concave up
Concave up
y = f(x)
f″(c) = 0
Not an inflection point: Concavity does not change at c.
O
c
x
(b)
y
y = f(x)
f″(c) does not exist
Concave down
Concave up
Inflection point: Concavity changes at c.
O
c
x
(c)
y
f″(c) does not exist
y = f(x)
Concave down
Concave down
Not an inflection point: Concavity does not change at c.
O
c
x
(d)

EXAMPLE 6 **Interpreting concavity** Sketch a function satisfying each set of conditions on some interval.

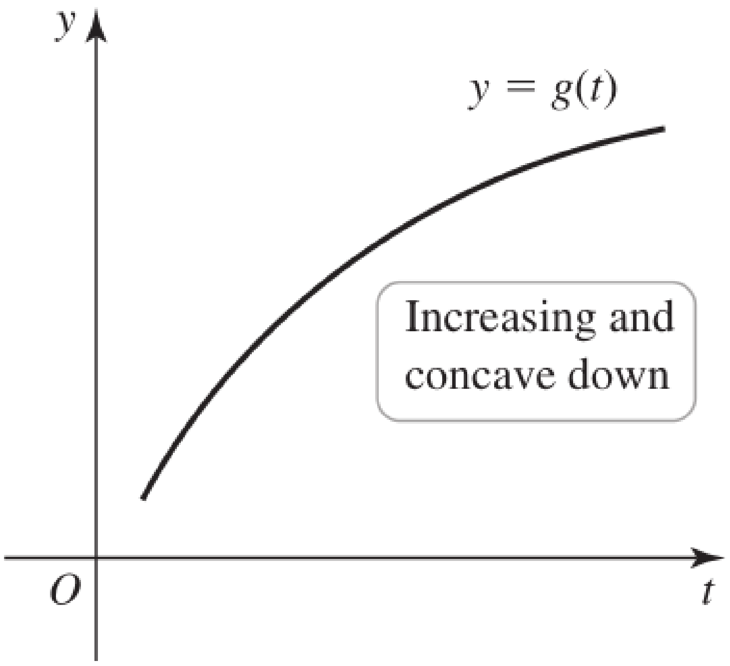
a. $f'(t) > 0$ and $f''(t) > 0$

b. $g'(t) > 0$ and $g''(t) < 0$

c. Which of the functions, f or g, could describe a population that increases and approaches a steady state as $t \to \infty$?



(a)

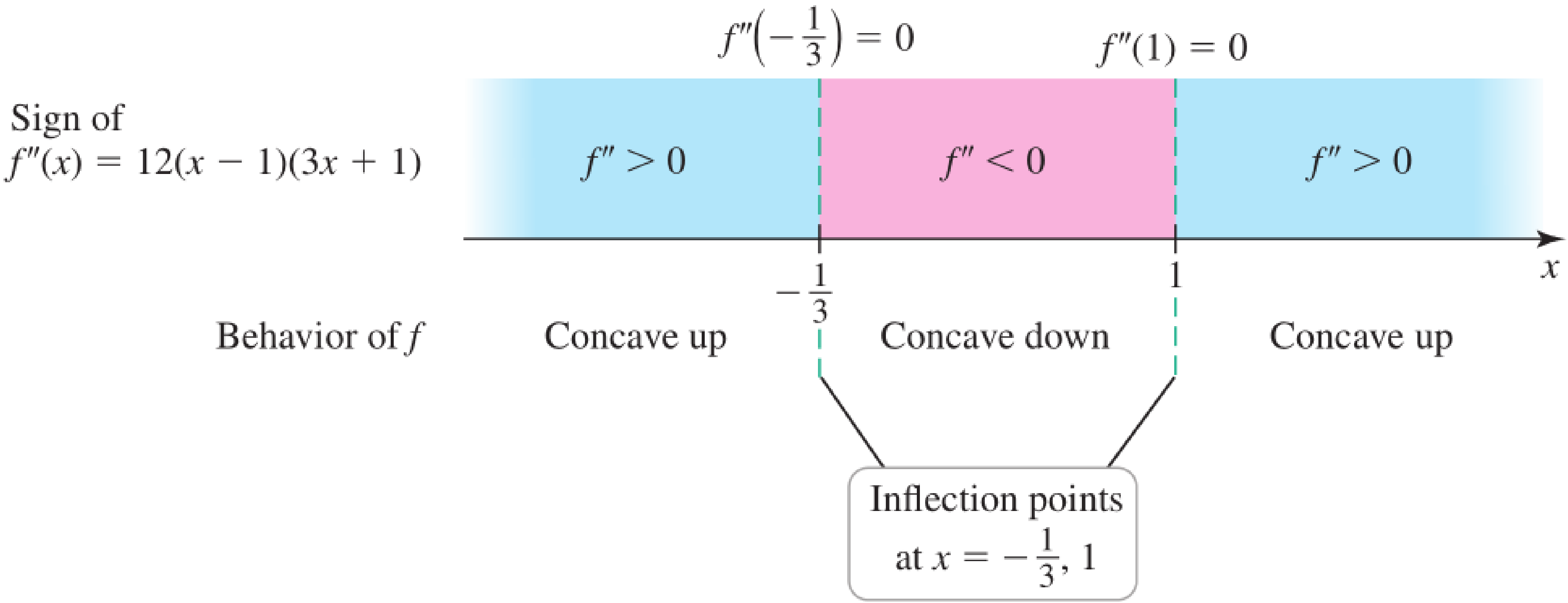


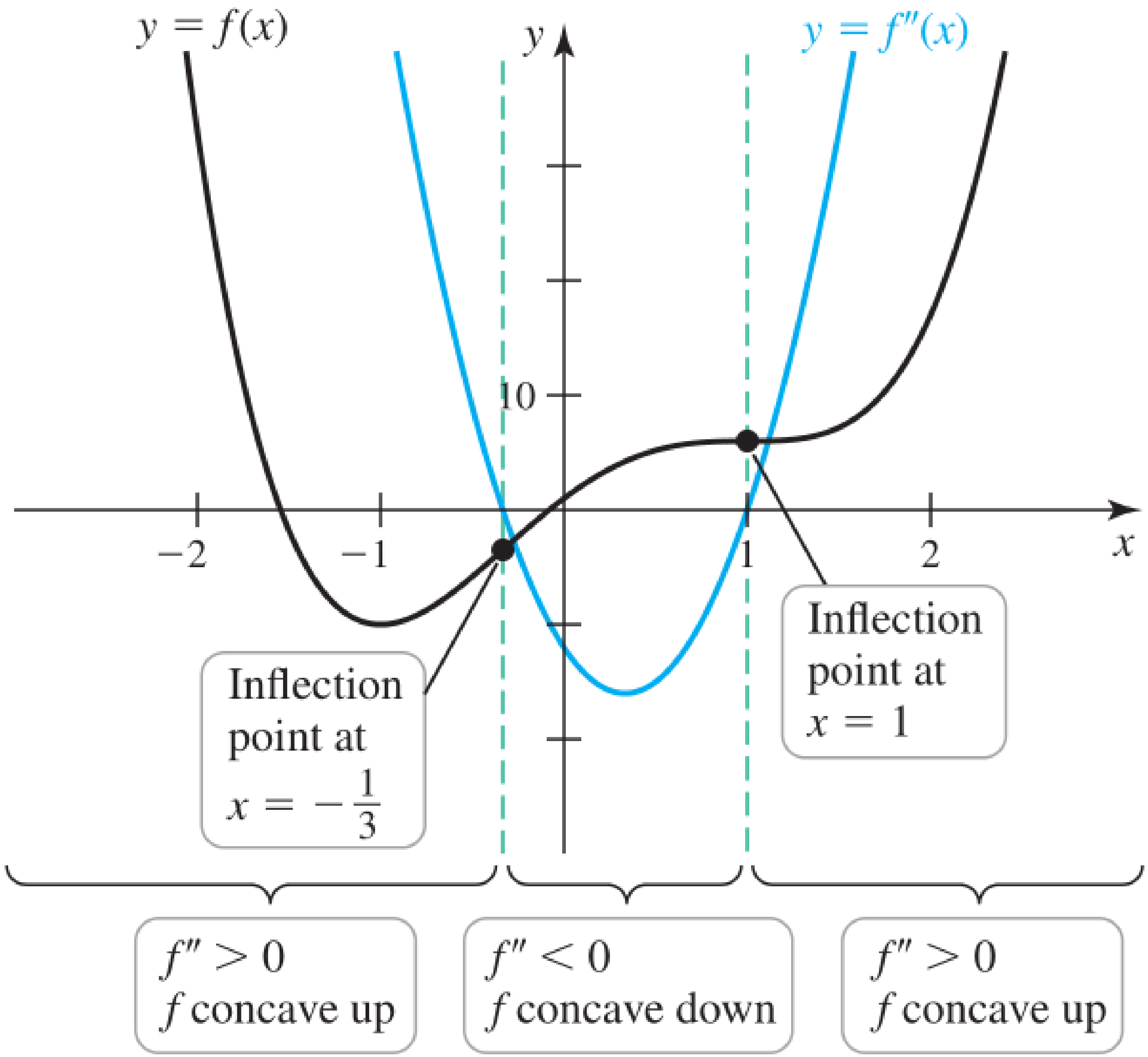
(b)

EXAMPLE 7 **Detecting concavity** Identify the intervals on which the following functions are concave up or concave down. Then locate the inflection points.

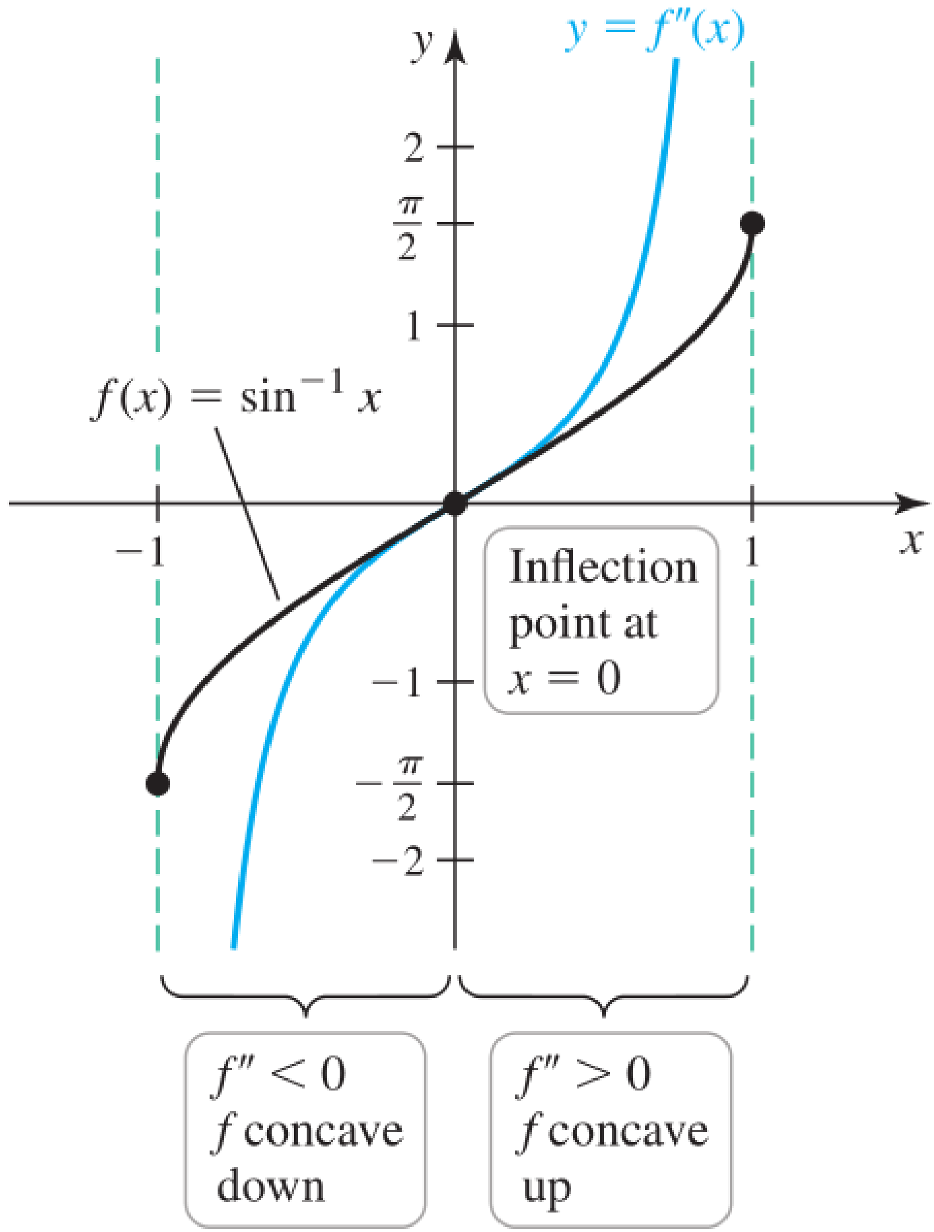
a. $f(x) = 3x^4 - 4x^3 - 6x^2 + 12x + 1$

b. $f(x) = \sin^{-1} x$ on $(-1, 1)$





$y = f(x)$
y
$y = f''(x)$
10
-2
-1
1
2
x
Inflection point at $x = -\frac{1}{3}$
Inflection point at $x = 1$
$f'' > 0$
f concave up
$f'' < 0$
f concave down
$f'' > 0$
f concave up



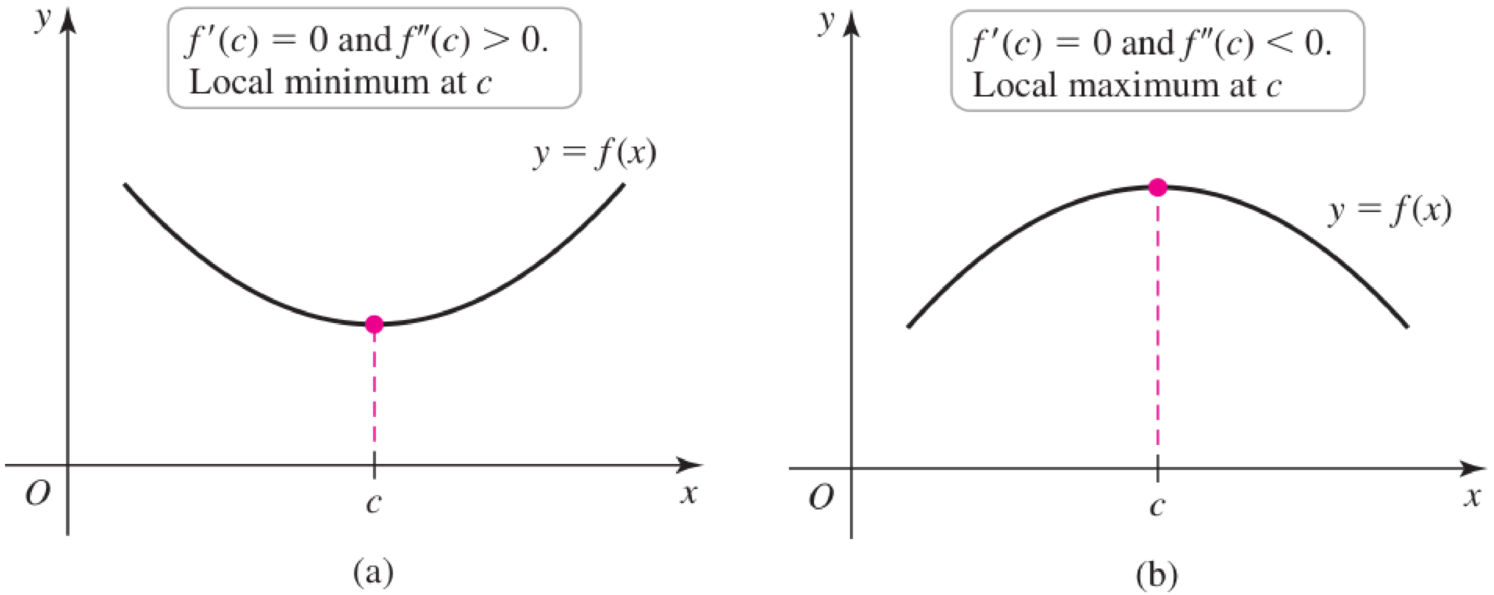
y
$y = f''(x)$
2
$\frac{\pi}{2}$
1
$f(x) = \sin^{-1} x$
−1
1
x
Inflection point at $x = 0$
−1
$-\frac{\pi}{2}$
−2
$f'' < 0$
f concave down
$f'' > 0$
f concave up

Second Derivative Test

THEOREM 4.7 Second Derivative Test for Local Extrema

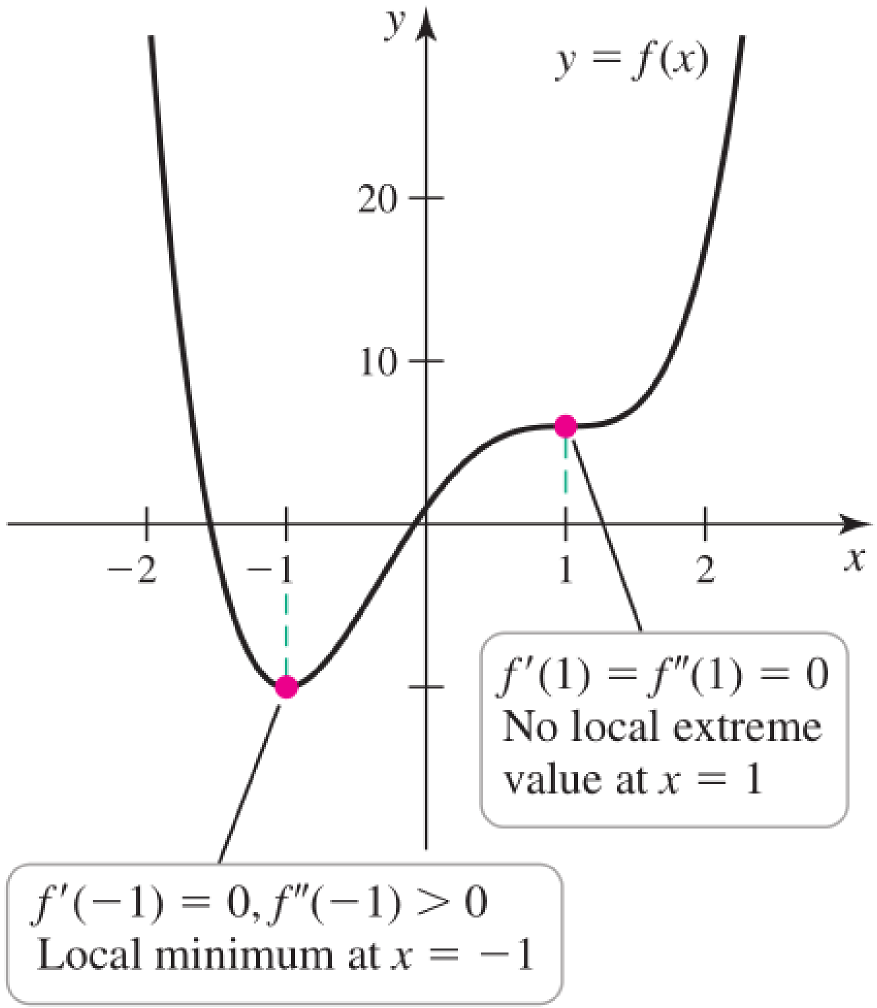
Suppose that f'' is continuous on an open interval containing c with $f'(c) = 0$.

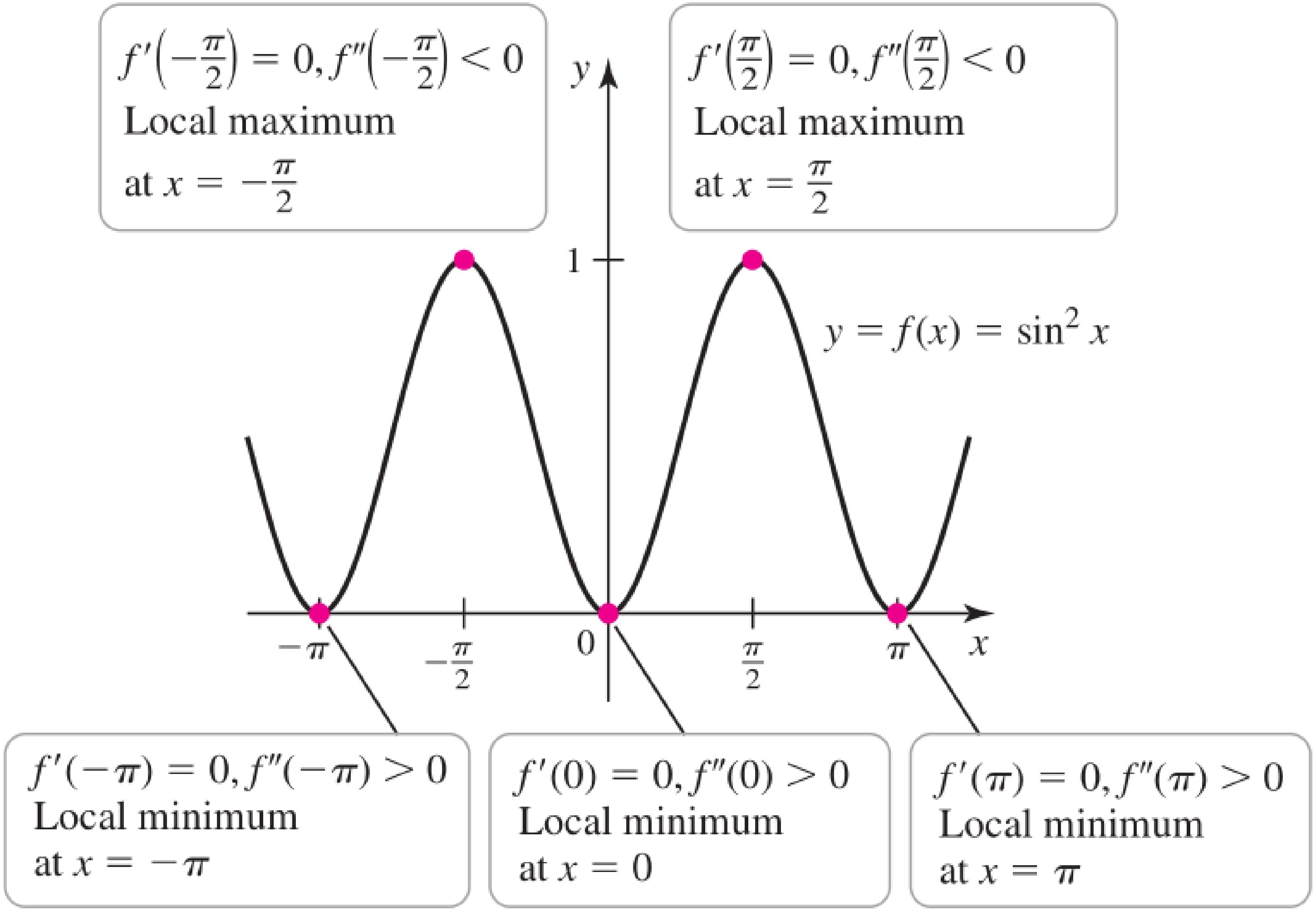
- If $f''(c) > 0$, then f has a local minimum at c (Figure 4.33a).
- If $f''(c) < 0$, then f has a local maximum at c (Figure 4.33b).
- If $f''(c) = 0$, then the test is inconclusive; f may have a local maximum, local minimum, or neither at c.



EXAMPLE 8 **The Second Derivative Test** Use the Second Derivative Test to locate the local extrema of the following functions.

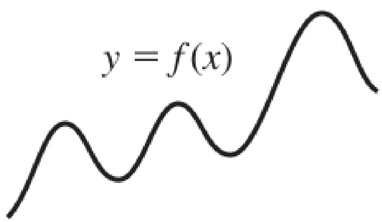
a. $f(x) = 3x^4 - 4x^3 - 6x^2 + 12x + 1$ on $[-2, 2]$ **b.** $f(x) = \sin^2 x$



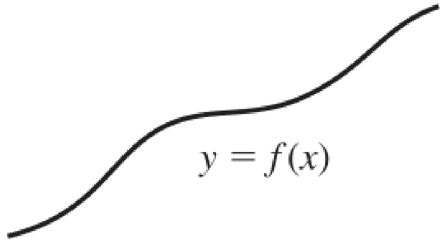


$f'\left(-\frac{\pi}{2}\right) = 0, f''\left(-\frac{\pi}{2}\right) < 0$
Local maximum
at $x = -\frac{\pi}{2}$
$f'\left(\frac{\pi}{2}\right) = 0, f''\left(\frac{\pi}{2}\right) < 0$
Local maximum
at $x = \frac{\pi}{2}$
y
1
$y = f(x) = \sin^2 x$
$-\pi$
$-\frac{\pi}{2}$
0
$\frac{\pi}{2}$
π
x
$f'(-\pi) = 0, f''(-\pi) > 0$
Local minimum
at $x = -\pi$
$f'(0) = 0, f''(0) > 0$
Local minimum
at $x = 0$
$f'(\pi) = 0, f''(\pi) > 0$
Local minimum
at $x = \pi$

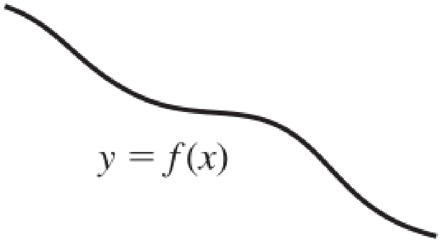
Recap of Derivative Properties



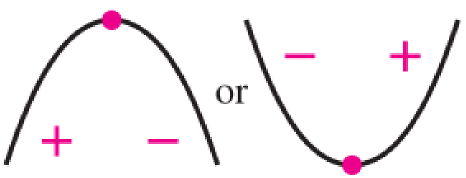
f is differentiable on an interval $\Longrightarrow$ f is a smooth curve



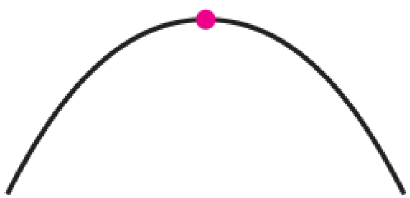
$f' > 0$ on an open interval $\Longrightarrow$ f is increasing on that interval



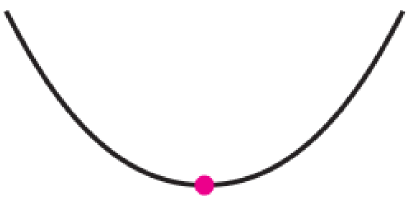
$f' < 0$ on an open interval $\Longrightarrow$ f is decreasing on that interval



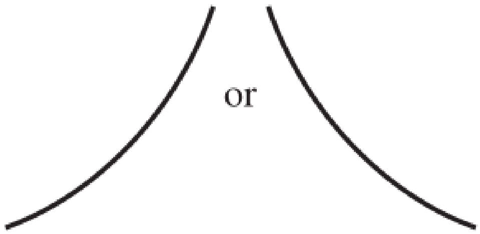
f' changes sign $\Longrightarrow$ f has local maximum or local minimum



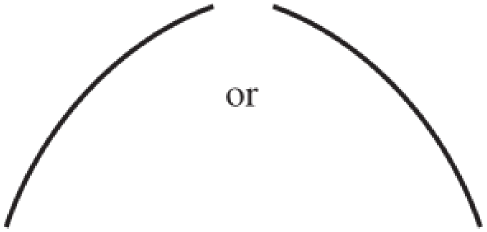
$f' = 0$ and $f'' < 0$ $\Longrightarrow$ f has local maximum



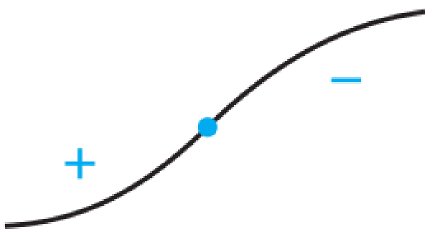
$f' = 0$ and $f'' > 0$ $\Longrightarrow$ f has local minimum



$f'' > 0$ on an open interval $\Longrightarrow$ f is concave up on that interval



$f'' < 0$ on an open interval $\Longrightarrow$ f is concave down on that interval



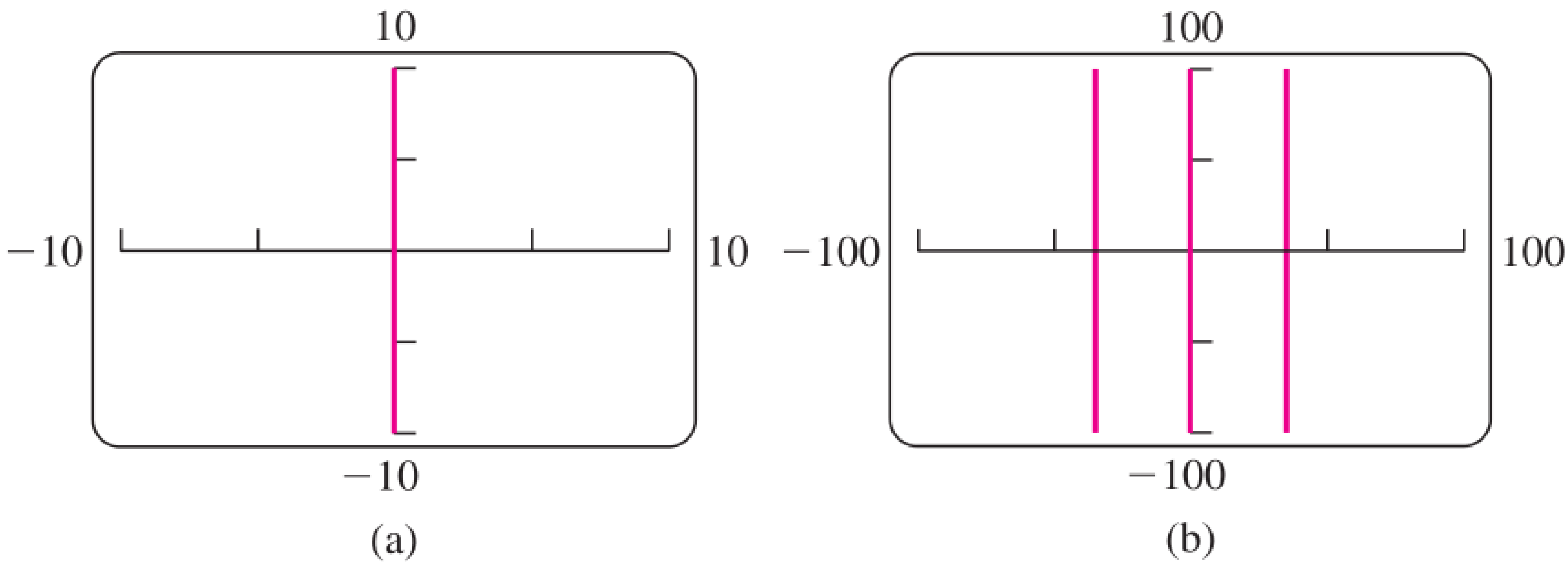
f'' changes sign $\Longrightarrow$ f has inflection point

4.4

Graphing Functions

Calculators and Analysis

Graph the harmless-looking function $f(x) = x^3/3 - 400x$ using a graphing calculator. Just vertical lines



Graphing utilities are valuable for exploring functions

But they should not be relied on exclusively because they cannot explain *why* a graph has its shape

Graphing Guidelines

Graphing Guidelines for $y = f(x)$

1. **Identify the domain or interval of interest.** On what interval(s) should the function be graphed? It may be the domain of the function or some subset of the domain.
2. **Exploit symmetry.** Take advantage of symmetry. For example, is the function *even* ($f(-x) = f(x)$), *odd* ($f(-x) = -f(x)$), or neither?
3. **Find the first and second derivatives.** They are needed to determine extreme values, concavity, inflection points, and intervals of increase and decrease. Computing derivatives—particularly second derivatives—may not be practical, so some functions may need to be graphed without complete derivative information.
4. **Find critical points and possible inflection points.** Determine points at which $f'(x) = 0$ or f' is undefined. Determine points at which $f''(x) = 0$ or f'' is undefined.
5. **Find intervals on which the function is increasing/decreasing and concave up/down.** The first derivative determines the intervals of increase and decrease. The second derivative determines the intervals on which the function is concave up or concave down.

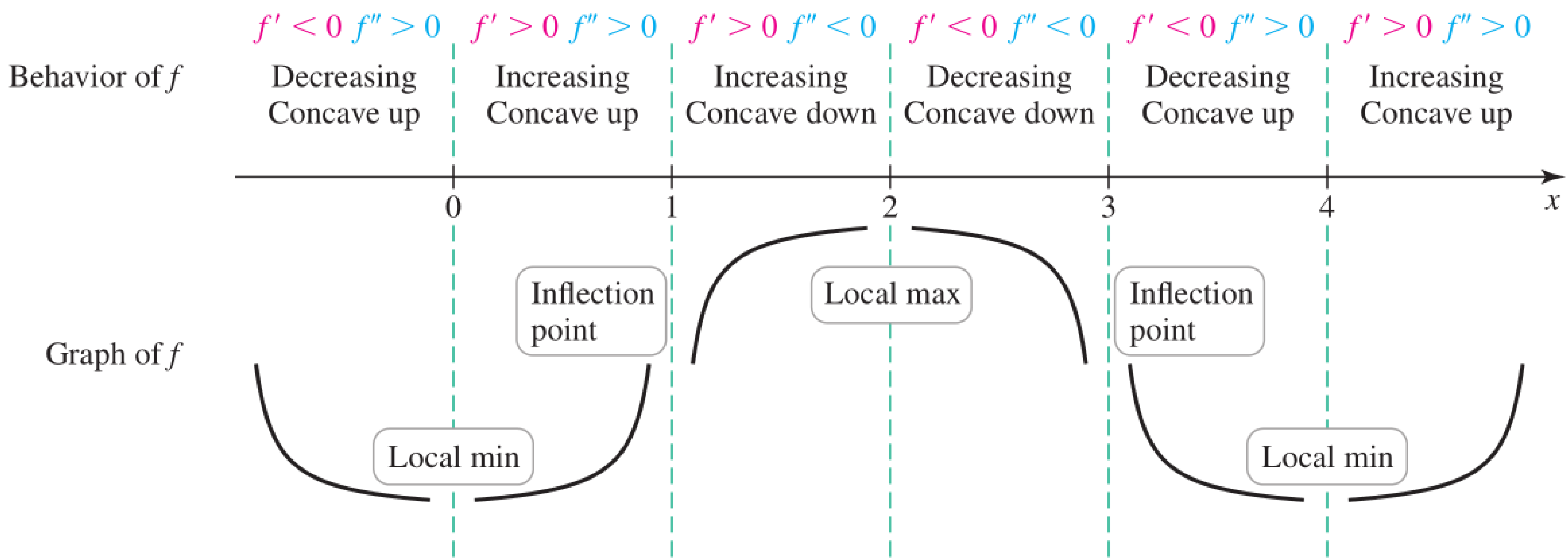
6. **Identify extreme values and inflection points.** Use either the First or Second Derivative Test to classify the critical points. Both x- and y-coordinates of maxima, minima, and inflection points are needed for graphing.
7. **Locate all asymptotes and determine end behavior.** Vertical asymptotes often occur at zeros of denominators. Horizontal asymptotes require examining limits as $x \to \pm\infty$; these limits determine end behavior.
8. **Find the intercepts.** The y-intercept of the graph is found by setting $x = 0$. The x-intercepts are found by setting $y = 0$; they are the real zeros (or roots) of f (those values of x that satisfy $f(x) = 0$).
9. **Choose an appropriate graphing window and plot a graph.** Use the results of the previous steps to graph the function. If you use graphing software, check for consistency with your analytical work. Is your graph *complete*—that is, does it show all the essential details of the function?

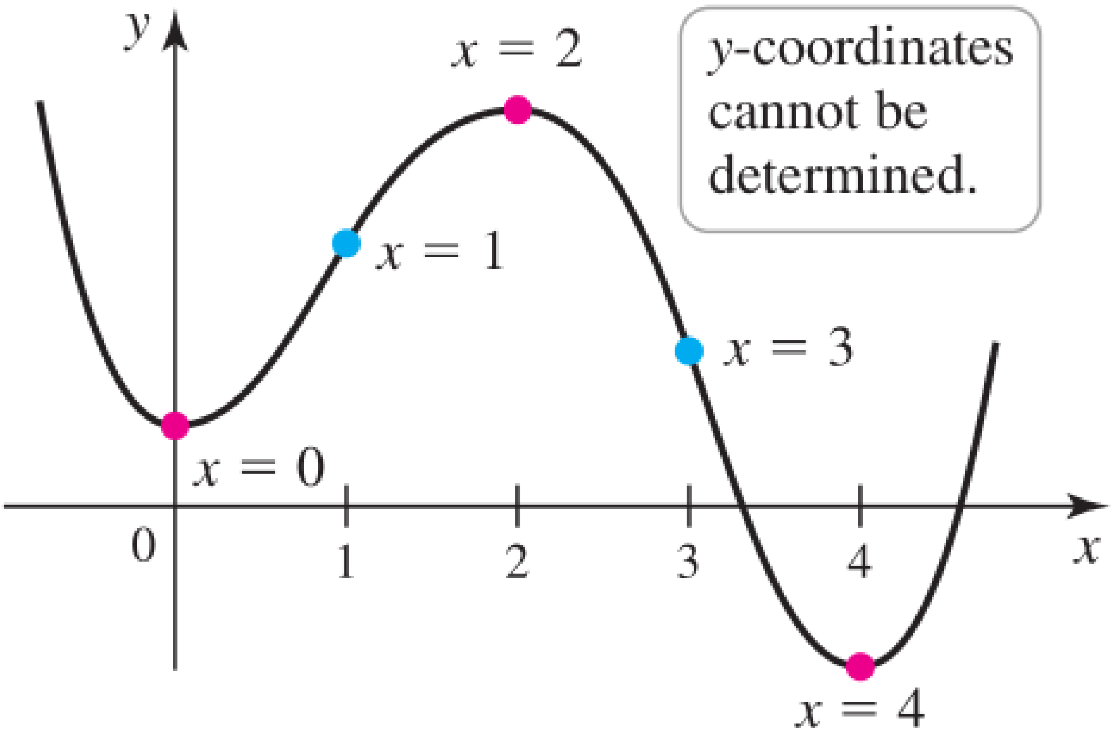
The guidelines need not be followed exactly for every function Some of the steps are best done analytically, while other steps can be done with a graphing utility.

EXAMPLE 1 **A warm-up** Given the following information about the first and second derivatives of a function f that is continuous on $(-\infty, \infty)$, summarize the information using a sign graph, and then sketch a possible graph of f.

$f' < 0, f'' > 0$ on $(-\infty, 0)$ $\quad f' > 0, f'' > 0$ on $(0, 1)$ $\quad f' > 0, f'' < 0$ on $(1, 2)$

$f' < 0, f'' < 0$ on $(2, 3)$ $\quad f' < 0, f'' > 0$ on $(3, 4)$ $\quad f' > 0, f'' > 0$ on $(4, \infty)$




y
x = 2
y-coordinates cannot be determined.
x = 1
x = 3
x = 0
0
1
2
3
4
x
x = 4

EXAMPLE 2 **A deceptive polynomial** Use the graphing guidelines to graph $f(x) = \frac{x^3}{3} - 400x$ on its domain.

SOLUTION

1. **Domain** The domain of any polynomial is $(-\infty, \infty)$.
2. **Symmetry** Because f consists of odd powers of the variable, it is an odd function. Its graph is symmetric about the origin.
3. **Derivatives** The derivatives of f are

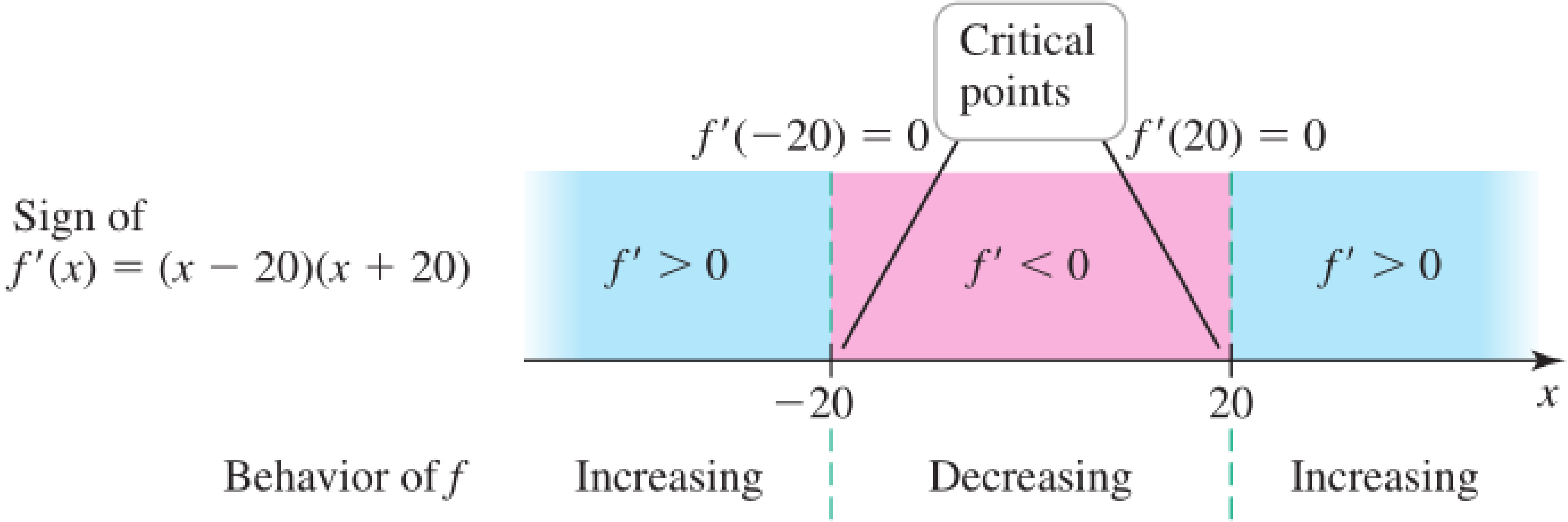
$$f'(x) = x^2 - 400 \quad \text{and} \quad f''(x) = 2x.$$

4. **Critical points and possible inflection points** Solving $f'(x) = 0$, we find that the critical points are $x = \pm 20$. Solving $f''(x) = 0$, we see that a possible inflection point occurs at $x = 0$.

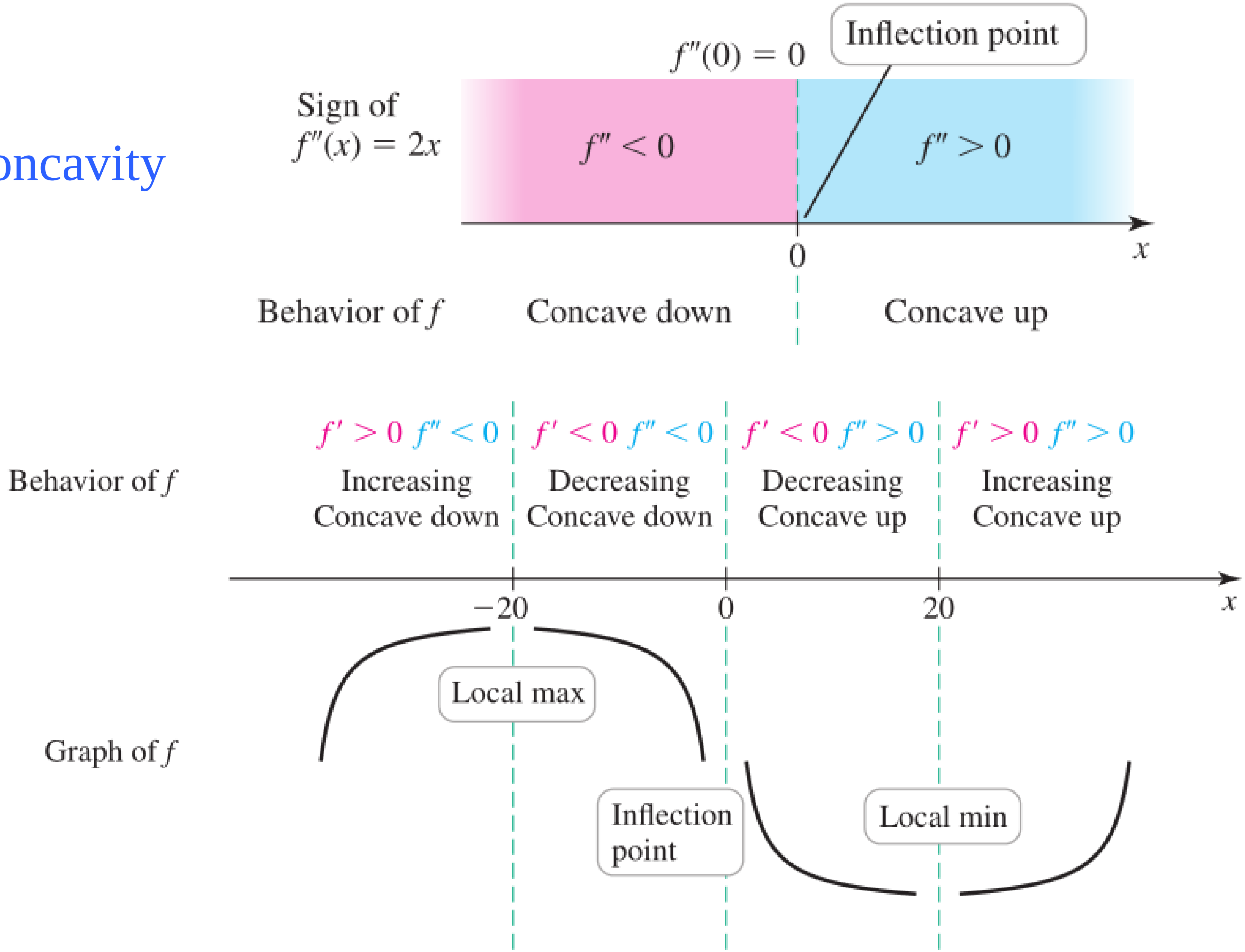
5. Increasing/decreasing and concavity Note that

$$f'(x) = x^2 - 400 = (x - 20)(x + 20).$$

Solving the inequality $f'(x) < 0$, we find that f is decreasing on the interval $(-20, 20)$. Solving the inequality $f'(x) > 0$ reveals that f is increasing on the intervals $(-\infty, -20)$ and $(20, \infty)$ (Figure 4.40). By the First Derivative Test, we have enough information to conclude that f has a local maximum at $x = -20$ and a local minimum at $x = 20$.



Concavity


Inflection point
$f''(0) = 0$
Sign of
$f''(x) = 2x$
$f'' < 0$
$f'' > 0$
0
x
Behavior of f
Concave down
Concave up
$f' > 0$ $f'' < 0$
$f' < 0$ $f'' < 0$
$f' < 0$ $f'' > 0$
$f' > 0$ $f'' > 0$
Behavior of f
Increasing
Concave down
Decreasing
Concave down
Decreasing
Concave up
Increasing
Concave up
-20
0
20
x
Graph of f
Local max
Inflection point
Local min

6. **Extreme values and inflection points** In this case, the Second Derivative Test is easily applied and it confirms what we have already learned. Because $f''(-20) < 0$ and $f''(20) > 0, f$ has a local maximum at $x = -20$ and a local minimum at $x = 20$. The corresponding function values are $f(-20) = 16{,}000/3 = 5333\frac{1}{3}$ and $f(20) = -f(-20) = -5333\frac{1}{3}$. Finally, we see that f'' changes sign at $x = 0$, making $(0, 0)$ an inflection point.

7. **Asymptotes and end behavior** Polynomials have neither vertical nor horizontal asymptotes. Because the highest-power term in the polynomial is x^3 (an odd power) and the leading coefficient is positive, we have the end behavior

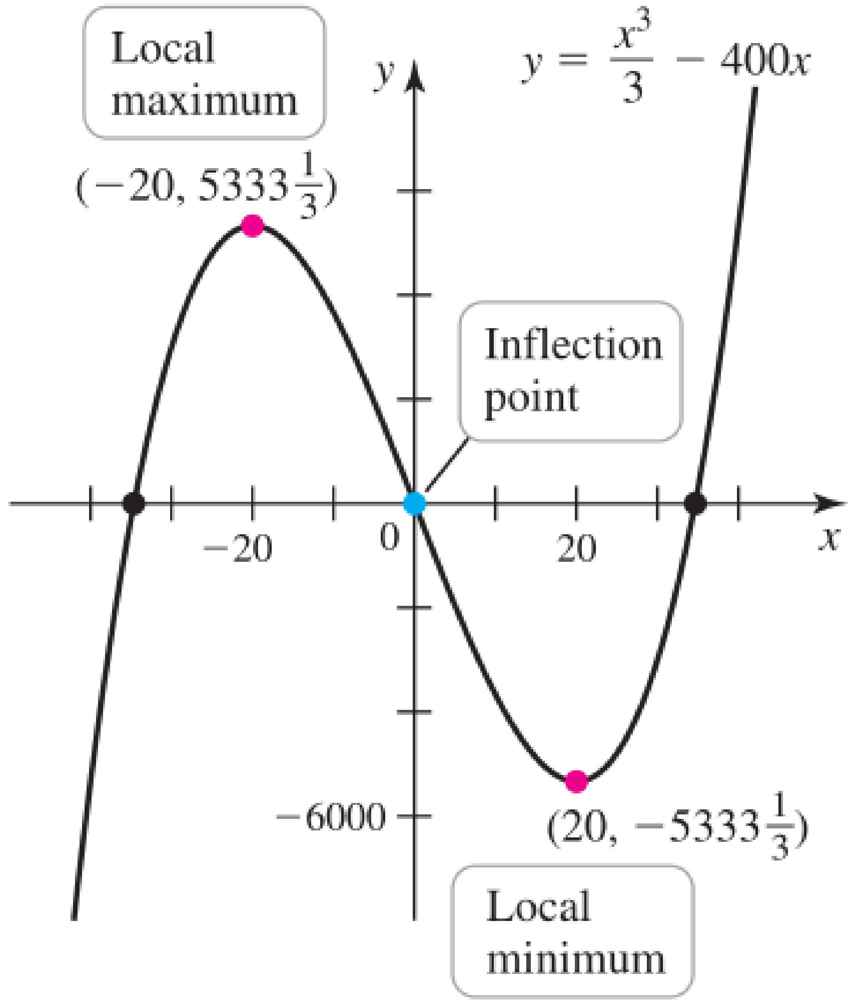
$$\lim_{x \to \infty} f(x) = \infty \quad \text{and} \quad \lim_{x \to -\infty} f(x) = -\infty.$$

8. **Intercepts** The y-intercept is $(0, 0)$. We solve the equation $f(x) = 0$ to find the x-intercepts:

$$\frac{x^3}{3} - 400x = x\left(\frac{x^2}{3} - 400\right) = 0.$$

The roots of this equation are $x = 0$ and $x = \pm\sqrt{1200} \approx \pm 34.6$.

9. Graph the function Using the information found in Steps 1–8, we choose the graphing window $[-40, 40] \times [-6000, 6000]$ and produce the graph shown in Figure 4.43. Notice that the symmetry detected in Step 2 is evident in this graph.



EXAMPLE 3 **The surprises of a rational function** Use the graphing guidelines to graph $f(x) = \dfrac{10x^3}{x^2 - 1}$ on its domain.

SOLUTION

1. **Domain** The zeros of the denominator are $x = \pm 1$, so the domain is $\{x: x \neq \pm 1\}$.
2. **Symmetry** This function consists of an odd function divided by an even function. The product or quotient of an even function and an odd function is odd. Therefore, the graph is symmetric about the origin.
3. **Derivatives** The Quotient Rule is used to find the first and second derivatives:

$$f'(x) = \frac{10x^2(x^2 - 3)}{(x^2 - 1)^2} \quad \text{and} \quad f''(x) = \frac{20x(x^2 + 3)}{(x^2 - 1)^3}.$$

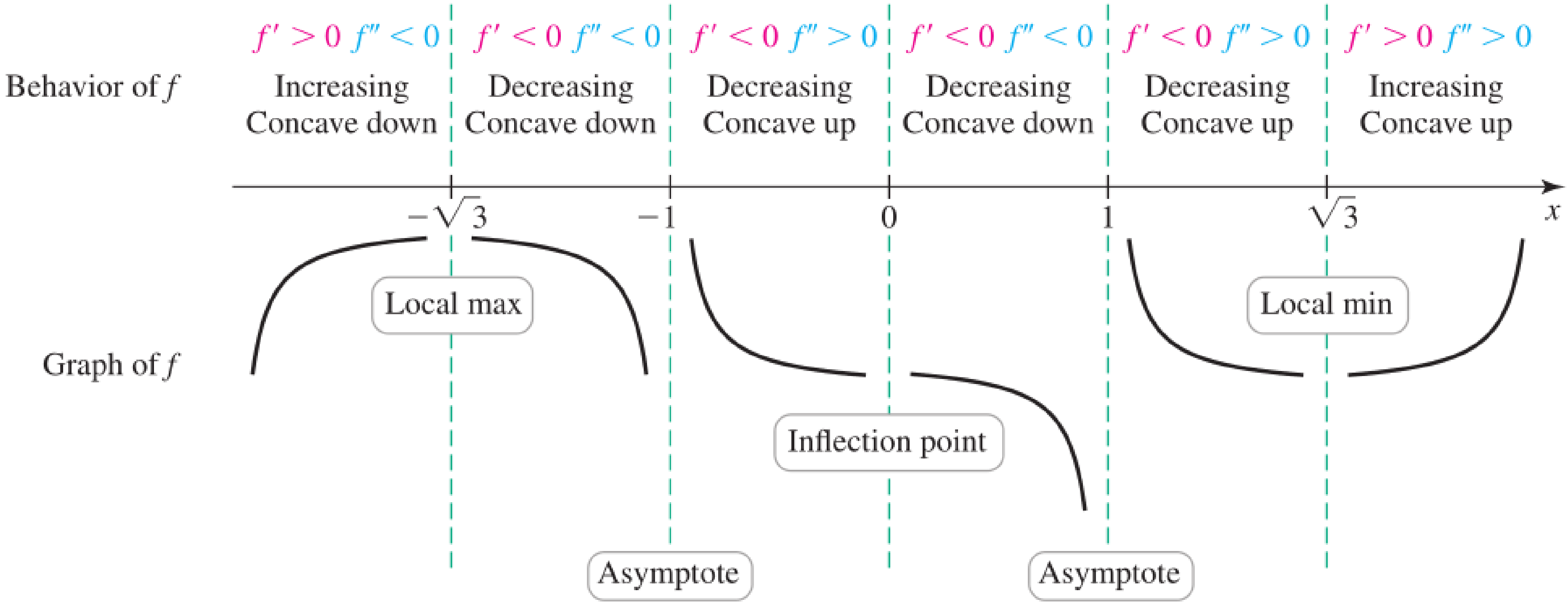
4. **Critical points and possible inflection points** The solutions of $f'(x) = 0$ occur where the numerator equals 0, provided the denominator is nonzero at those points. Solving $10x^2(x^2 - 3) = 0$ gives the critical points $x = 0$ and $x = \pm\sqrt{3}$. The solutions of $f''(x) = 0$ are found by solving $20x(x^2 + 3) = 0$; we see that the only possible inflection point occurs at $x = 0$.

5. **Increasing/decreasing and concavity** To find the sign of f', first note that the denominator of f' is nonnegative, as is the factor $10x^2$ in the numerator. So the sign of f' is determined by the sign of the factor $x^2 - 3$, which is negative on $(-\sqrt{3}, \sqrt{3})$ (excluding $x = \pm 1$) and positive on $(-\infty, -\sqrt{3})$ and $(\sqrt{3}, \infty)$. Therefore, f is decreasing on $(-\sqrt{3}, \sqrt{3})$ (excluding $x = \pm 1$) and increasing on $(-\infty, -\sqrt{3})$ and $(\sqrt{3}, \infty)$.

The sign of f'' is a bit trickier. Because $x^2 + 3$ is positive, the sign of f'' is determined by the sign of $20x$ in the numerator and $(x^2 - 1)^3$ in the denominator. When $20x$ and $(x^2 - 1)^3$ have the same sign, $f''(x) > 0$; when $20x$ and $(x^2 - 1)^3$ have opposite signs, $f''(x) < 0$ (Table 4.1). The results of this analysis are shown in Figure 4.44.

Table 4.1

	$20x$	$x^2 + 3$	$(x^2 - 1)^3$	Sign of f''
$(-\infty, -1)$	$-$	$+$	$+$	$-$
$(-1, 0)$	$-$	$+$	$-$	$+$
$(0, 1)$	$+$	$+$	$-$	$-$
$(1, \infty)$	$+$	$+$	$+$	$+$



Behavior of f
$f' > 0$ $f'' < 0$
Increasing
Concave down
$f' < 0$ $f'' < 0$
Decreasing
Concave down
$f' < 0$ $f'' > 0$
Decreasing
Concave up
$f' < 0$ $f'' < 0$
Decreasing
Concave down
$f' < 0$ $f'' > 0$
Decreasing
Concave up
$f' > 0$ $f'' > 0$
Increasing
Concave up
$-\sqrt{3}$
-1
0
1
$\sqrt{3}$
x
Graph of f
Local max
Local min
Inflection point
Asymptote
Asymptote

6. Extreme values and inflection points The First Derivative Test is easily applied by looking at Figure 4.44. The function is increasing on $(-\infty, -\sqrt{3})$ and decreasing on $(-\sqrt{3}, -1)$; therefore, f has a local maximum at $x = -\sqrt{3}$, where $f(-\sqrt{3}) = -15\sqrt{3}$. Similarly, f has a local minimum at $x = \sqrt{3}$, where $f(\sqrt{3}) = 15\sqrt{3}$. (These results could also be obtained with the Second Derivative Test.) There is no local extreme value at the critical point $x = 0$, only a horizontal tangent line.

Using Table 4.1 from Step 5, we see that f'' changes sign at $x = \pm 1$ and at $x = 0$. The points $x = \pm 1$ are not in the domain of f, so they cannot correspond to inflection points. However, there is an inflection point at $(0, 0)$.

7. Asymptotes and end behavior Recall from Section 2.4 that zeros of the denominator, which in this case are $x = \pm 1$, are candidates for vertical asymptotes. Checking the behavior of f on either side of $x = \pm 1$, we find

$$\lim_{x \to -1^-} f(x) = -\infty, \qquad \lim_{x \to -1^+} f(x) = \infty.$$
$$\lim_{x \to 1^-} f(x) = -\infty, \qquad \lim_{x \to 1^+} f(x) = \infty.$$

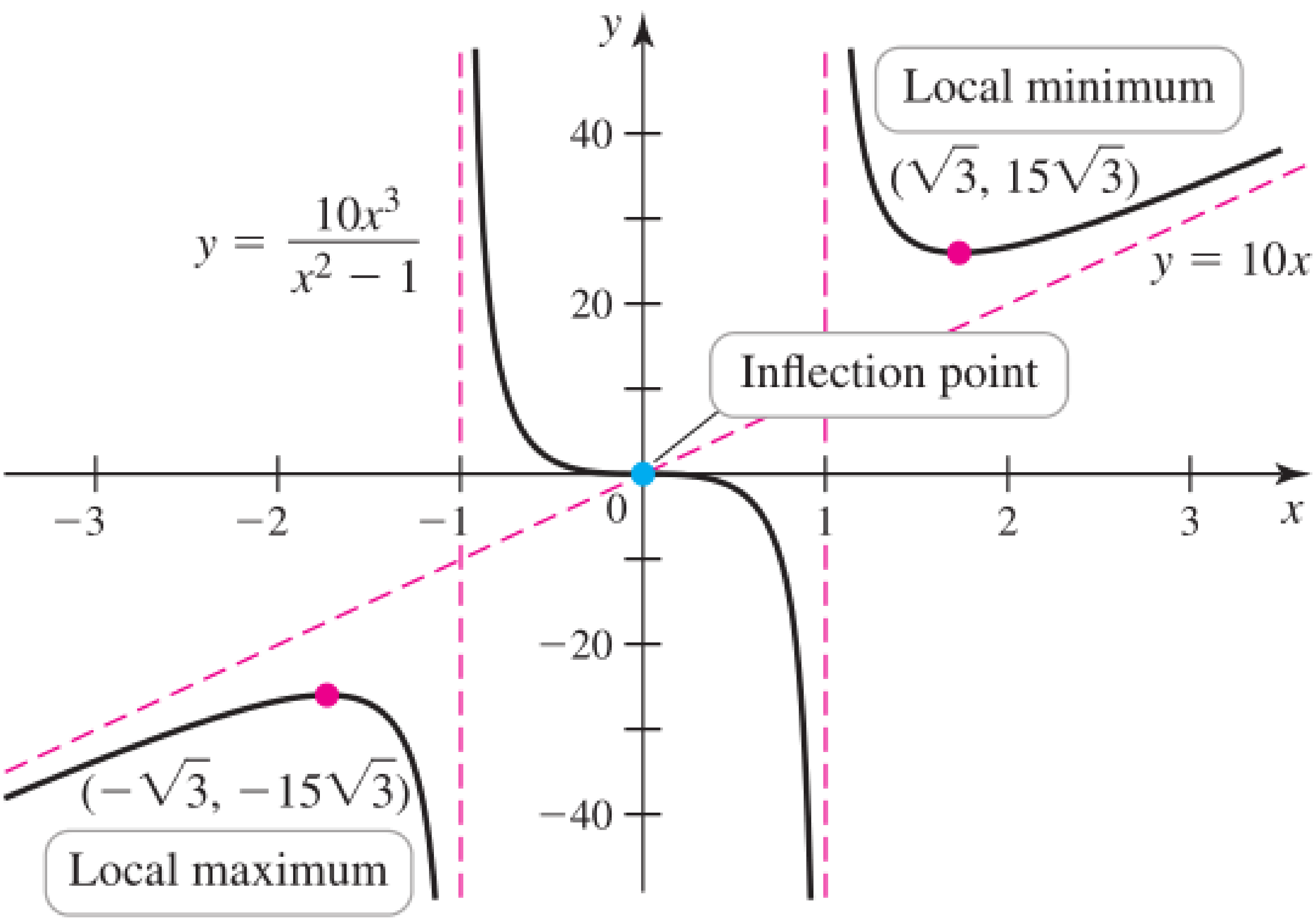
It follows that f has vertical asymptotes at $x = \pm 1$. The degree of the numerator is greater than the degree of the denominator, so there is no horizontal asymptote. Using long division, it can be shown that

$$f(x) = 10x + \frac{10x}{x^2 - 1}.$$

Therefore, as $x \to \pm\infty$, the graph of f approaches the line $y = 10x$. This line is a slant asymptote (Section 2.5).

8. **Intercepts** The zeros of a rational function coincide with the zeros of the numerator, provided that those points are not also zeros of the denominator. In this case, the zeros of f satisfy $10x^3 = 0$, or $x = 0$ (which is not a zero of the denominator). Therefore, $(0, 0)$ is both the x- and y-intercept.

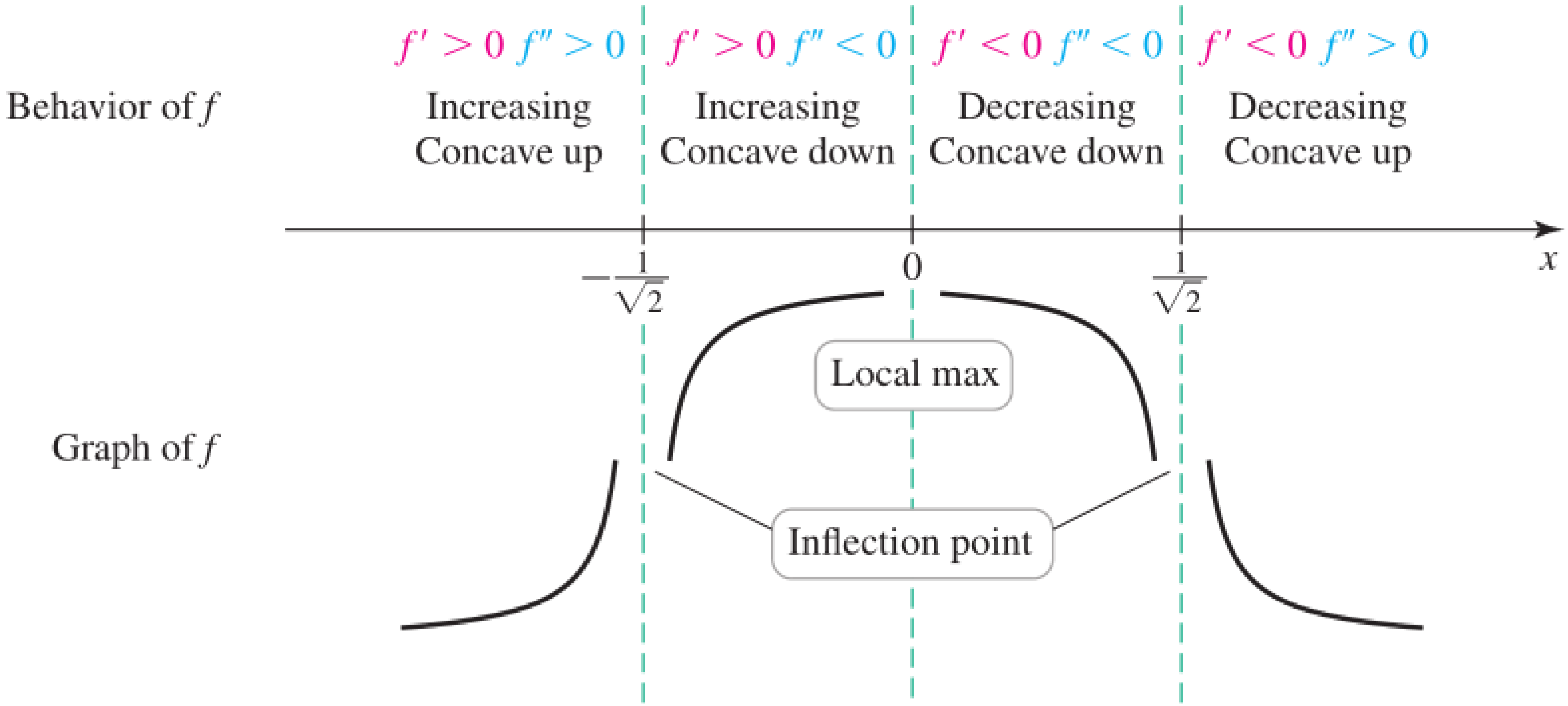
9. **Graphing** We now assemble an accurate graph of f, as shown in Figure 4.45. A window of $[-3, 3] \times [-40, 40]$ gives a complete graph of the function. Notice that the symmetry about the origin deduced in Step 2 is apparent in the graph.



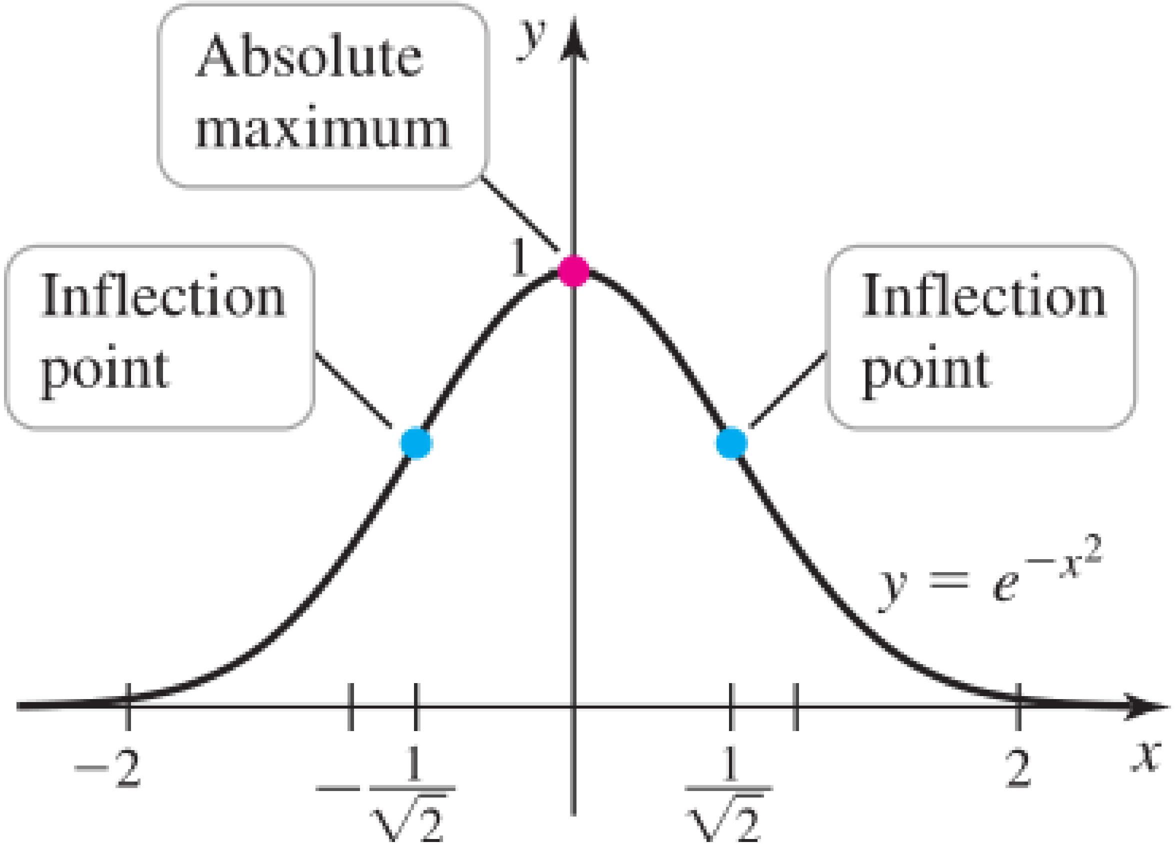
y
x
Local minimum
$(\sqrt{3}, 15\sqrt{3})$
$y = \frac{10x^3}{x^2 - 1}$
$y = 10x$
Inflection point
$(-\sqrt{3}, -15\sqrt{3})$
Local maximum
40
20
-20
-40
-3
-2
-1
0
1
2
3

EXAMPLE 4 **The normal distribution** Analyze the function $f(x) = e^{-x^2}$ and draw its graph.

Domain, Even function, Symmetry



To determine the end behavior, notice that $\lim_{x \to \pm\infty} e^{-x^2} = 0$, so $y = 0$ is a horizontal asymptote of f. Assembling all of these facts, an accurate graph can now be drawn



EXAMPLE 5 **Roots and cusps** Graph $f(x) = \frac{1}{8}x^{2/3}(9x^2 - 8x - 16)$ on its domain.

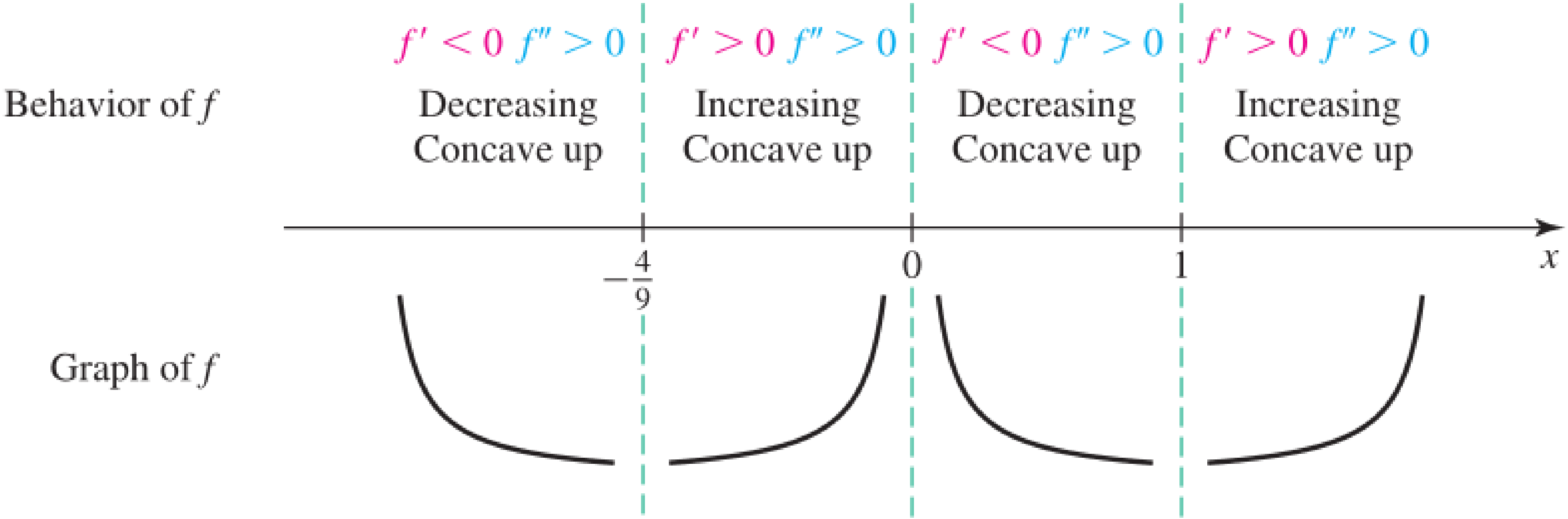
Domain of f is $(-\infty, \infty)$. No special symmetry

$$f'(x) = \frac{(x-1)(9x+4)}{3x^{1/3}}$$

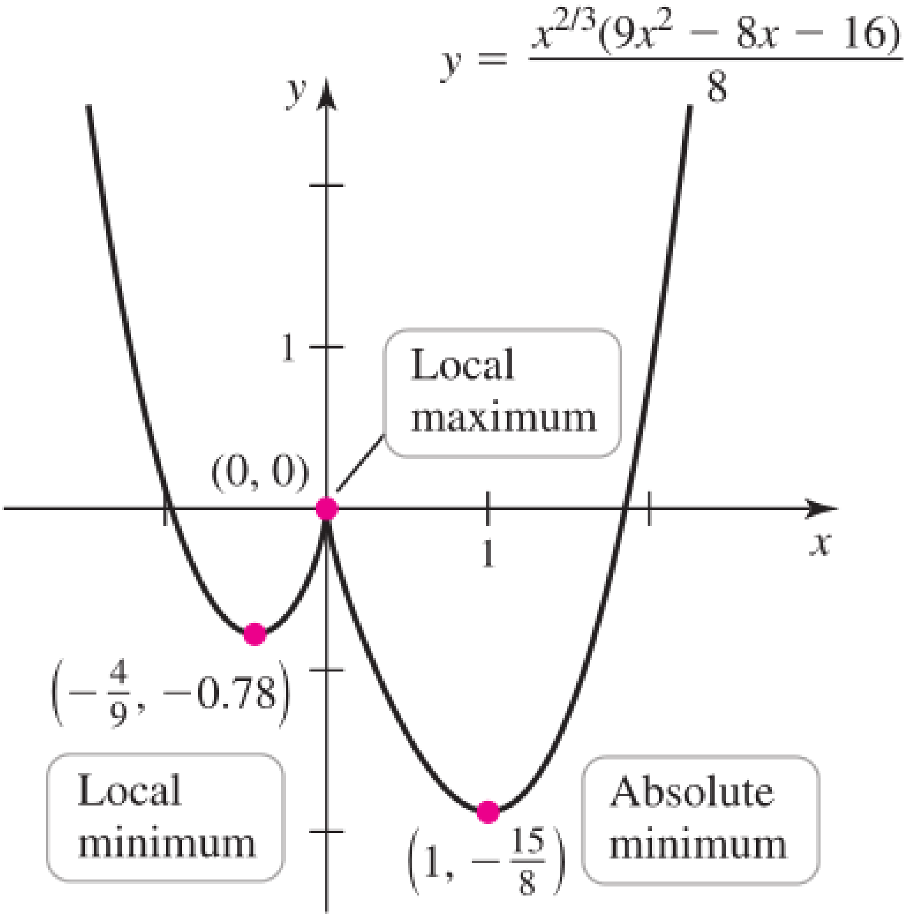
Table 4.2

	$3x^{1/3}$	$9x + 4$	$x - 1$	Sign of f'
$(-\infty, -\frac{4}{9})$	$-$	$-$	$-$	$-$
$(-\frac{4}{9}, 0)$	$-$	$+$	$-$	$+$
$(0, 1)$	$+$	$+$	$-$	$-$
$(1, \infty)$	$+$	$+$	$+$	$+$

$$f''(x) = \frac{45x^2 - 10x + 4}{9x^{4/3}}$$



As $x \rightarrow \pm\infty$, f is dominated by its highest-power term, which is $9x^{8/3}/8$. This term becomes large and positive as $x \rightarrow \pm\infty$; therefore, f has no absolute maximum. Its absolute minimum occurs at $x = 1$ because, comparing the two local minima, $f(1) < f(-\frac{4}{9})$.



Chapter 4

Applications of the Derivative (I)

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