MATH 2233 Differential Equations
Chapter 8 Series Solutions of Differential Equations

Section 8.1 Taylor Polynomial Approximation

Goal of this section

- Determine the Taylor polynomial approximation of the initial value problems of differential equations.

Taylor Polynomials

The n-th degree Taylor polynomial $p_n(x)$ of a given function $f(x)$ near a particular point $x = x_0$ is

The Taylor polynomial is probably the best approximation of $f(x)$ near $x = x_0$. In fact,

- $p_n(x)$ matches the value of $f(x)$ and its derivatives:

- In particular, when $n = 1$,

Example 1. *Find the first four Taylor polynomials for $f(x) = e^x$ around $x_0 = 0$.*

Example 2. *Determine the fourth-degree Taylor polynomials matching the functions* e^x*,* $\cos(x)$*, and* $\sin(x)$ *at* $x_0 = 2$*.*

Example 3. *Find the fifth-degree Taylor polynomial approximation around $x_0 = 0$ of the initial value problem*

$$y'' = 3y' + x^2y, \qquad y(0) = 10, \quad y'(0) = 5.$$

Example 4. *Determine the Taylor polynomial of degree 3 for the solution of the initial value problem*

$$y' = \frac{1}{x+y+1}, \qquad y(0) = 0.$$

MATH 2233 Differential Equations

Section 8.2 Power Series and Analytic Functions

Goal of this section

- get to know the definition of power series
- find convergence region of power series
- get to know the definition of analytic functions.

Recall

Given a function $y(x)$, the Taylor polynomial approximation is

When $n \to \infty$, it is called Taylor series

Question: Will the series always converge?

Power Series

A **power series** about the point x_0 is

We say that the series **converges** at the point $x = c$ if the infinite series

If the limit does not exist, the power series is called to **diverge** at $x = c$.

Radius of Convergence

Clearly the power series converges at $x = x_0$, In fact, the power series converges *absolutely* inside some interval $(x_0 - \rho, x_0 + \rho)$ centered at x_0, but diverges outside this interval. The positive number ρ is called

Ratio Test

For the power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$, if for n large, the coefficient a_n are nonzero and satisfy

$$\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = L \qquad (0 \leq L \leq \infty),$$

then

Example 1. *Determine the converge set of*

$$\sum_{n=0}^{\infty} \frac{(-2)^n}{n+1}(x-3)^n$$

For each value of x for which the power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converges, we can denote its sum by a function $f(x)$. For example, we have the **geometric series**:

Remark

- If the power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n = 0$ for all x in some open interval, then
- If the power series $f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n$ has a positive convergence radius ρ, then

Example 2. *Using the geometric series find a power series for each of the following functions:*

$$(a)\ \frac{1}{1+x^2} \quad (b)\ \frac{1}{(1-x)^2} \quad (c)\ \arctan(x).$$

Shifting the Summation Index

The index of summation in a power series is a dummy index just like the variable of integration. Thus,

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n =$$

Example 3. *Express the series*

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$$

as a series where the generic term is x^k *instead of* x^{n-2}.

Example 4. *Find the power series expansion* $\sum_{n=0}^{\infty} a_n x^n$ *for* $f(x)+g(x)$ *where*

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n+5} x^n, \quad g(x) = \sum_{n=1}^{\infty} 3^{-n} x^{n-1}.$$

Example 5. *Show that*

$$x^3 \sum_{n=0}^{\infty} n^2(n-2)a_n x^n = \sum_{n=3}^{\infty} (n-3)^2(n-5)a_{n-3}x^n.$$

Analytic Functions

Not all functions can be expressed as power series. Those distinguished functions that can be so represented are called **analytic**.

Definition (Analytic at a point)

A function f is said to be **analytic at** x_0 if

Example 6. *The following functions are typical analytic functions:*

- *polynomial function* $f(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_m x^m$

- *rational function* $f(x) = \dfrac{P(x)}{Q(x)}$*, where* $P(x)$ *and* $Q(x)$ *are polynomials*

- *exponential function*

- *sine cosine functions*

- *logarithm function*

Remark. If $f(x)$ is analytic at x_0, then

MATH 2233 Differential Equations

Section 8.3 Power Series Solutions to Linear Differential Equations

Goal of this section

- determine power series solution to a linear differential equation with polynomial coefficients.

Recall

A second-order linear differential equation has the general form

It can be written in the standard form

Definition: Ordinary and Singular Points

A point x_0 is called an **ordinary point** of the above differential equation if

If x_0 is not an ordinary point, it is called

Example 1. *Determine all singular points of*

$$(x^2 - 7)y'' + 4y' + \sin(x)y = 0$$

Example 2. *Determine all singular points of*

$$xy'' + x(1-x)^{-1}y' + \sin(x)y = 0$$

Example 3. *Find a power series solution about $x = 0$ to the differential equation*

$$y' + 2xy = 0$$

Example 4. *Find a general solution to the differential equation*

$$2y'' + xy' + y = 0.$$

Example 5. *Find the first few terms in a power series expansion about $x = 0$ for a general solution to the differential equation*

$$(1 + x^2)y'' - y' + y = 0.$$