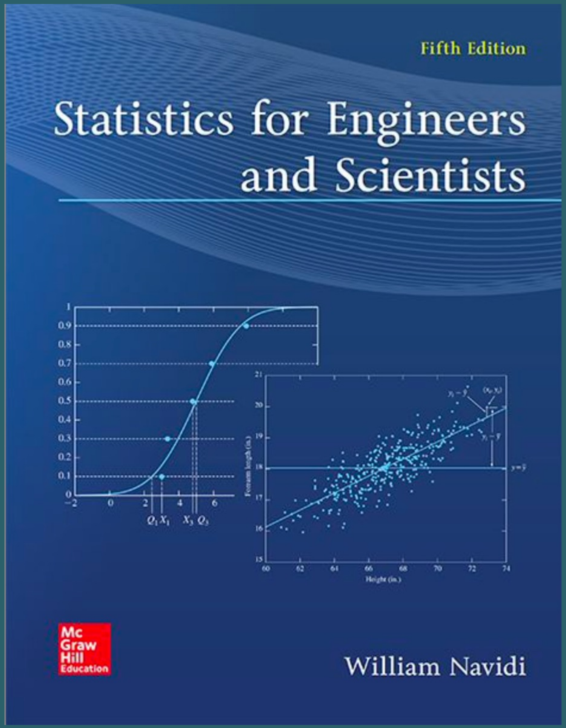

Ch 3: Propagation of Error…

(not all sections are required)

Chapter 3 Overview

3-1 Measurement Error

3-2 Linear Combinations of Measurements

3-3 Uncertainties for Functions of One Measurement

~~*3-4 Uncertainties for Several Measurements*~~

Introduction

- We often add constants to measurements, multiply measurements by constants, or add two or more measurements together.
- We need to understand how uncertainties are affected by these arithmetic operations.
- Since measurements are random variables, and uncertainties are the standard deviations of these random variables, the results used to compute standard deviations of linear combinations of random variables can be applied to compute uncertainties in linear combinations of measurements.

Introduction…

Basic results used to compute uncertainties in linear combinations of independent measurements (p.171):

If X is a measurement and c is a constant, then

$$\sigma_{cX} = |c|\sigma_X \tag{3.3}$$

If $X_1, \ldots, X_n$ are independent measurements and $c_1, \ldots, c_n$ are constants, then

$$\sigma_{c_1X_1+\cdots+c_nX_n} = \sqrt{c_1^2\sigma_{X_1}^2 + \cdots + c_n^2\sigma_{X_n}^2} \tag{3.4}$$

Example 3.4 (p.171)

- The radius of a circle is measured to be 3.0 ± 0.1 cm.
- Estimate the circumference and find the uncertainty in the estimate.

Example 3.4 (p.171) - SOLUTION

- Let R denote the radius of the circle. The measured value of R is 3.0 cm, and the uncertainty is the standard deviation of this measurement, which is $\sigma_R = 0.1$ cm.
- The circumference is given by $C = 2\pi R$. The uncertainty in C is σ_C, the standard deviation of C.
- Since 2π is a constant, we have

$$\begin{aligned} \sigma_C &= |2\pi|\sigma_R && \text{(using Equation 3.3)} \\ &= (6.28)(0.1 \text{ cm}) \\ &= 0.63 \text{ cm} \end{aligned}$$

- The circumference is **18.85 ± 0.63 cm**.

Example 3.5 (p.171)

- An item is formed by placing two components end to end. The lengths of the components are measured independently, by a process that yields a random measurement with uncertainty 0.1 cm.
- The length of the item is estimated by adding the two measured lengths.
- Assume that the measurements are 4.10 cm and 3.70 cm.
- Estimate the length of the item and find the uncertainty in the estimate.

Example 3.5 (p.171) - SOLUTION

- Let X be the measured length of the first component, and let Y be the measured length of the second component.

- The estimated length is $X + Y = 7.80$ cm. The uncertainty is

$$\begin{aligned}\sigma_{X+Y} &= \sqrt{\sigma_X^2 + \sigma_Y^2} \\ &= \sqrt{(0.1)^2 + (0.1)^2} \\ &= 0.14 \text{ cm}\end{aligned}$$

(using Equation 3.4 with $c_1 = c_2 = 1$)

- The estimated length is **7.80 ± 0.14 cm**.

Example 3.6 (p.172)

- A surveyor is measuring the perimeter of a rectangular lot. He measures two adjacent sides to be 50.11 ± 0.05 m and 75.21 ± 0.08 m.
- These measurements are independent.
- Estimate the perimeter of the lot and find the uncertainty in the estimate.

Example 3.6 (p.172) - SOLUTION

- Let X = 50.11 and Y = 75.21 be the two measurements.

- The perimeter is estimated by P = 2X + 2Y = 250.64 m, and the uncertainty in P is

$$\begin{aligned}\sigma_P &= \sigma_{2X+2Y} \qquad \text{(using Equation 3.4)}\\ &= \sqrt{4\sigma_X^2 + 4\sigma_Y^2}\\ &= \sqrt{4(0.05)^2 + 4(0.08)^2}\\ &= 0.19 \text{ m}\end{aligned}$$

- The perimeter is 250.64 ± 0.19 m.

Example 3.7 (p.172)

- In Example 3.6, the surveyor's assistant suggests computing the uncertainty in P by a different method.

- He reasons that since P = X + X + Y + Y, then

$$\begin{aligned}\sigma_P &= \sigma_{X+X+Y+Y} \\ &= \sqrt{\sigma_X^2 + \sigma_X^2 + \sigma_Y^2 + \sigma_Y^2} \\ &= \sqrt{(0.05)^2 + (0.05)^2 + (0.08)^2 + (0.08)^2} \\ &= 0.13 \text{ m}\end{aligned}$$

- This disagrees with the value of 0.19 m calculated in Example 3.6. What went wrong?

Example 3.7 (p.172) - SOLUTION

- What went wrong is that the four terms in the sum for P are not all independent.
- Specifically, X + X is not the sum of independent quantities; neither is Y + Y.
- In order to use Equation (3.4) to compute the uncertainty in P, we **must express P as the sum of independent quantities**, that is, P = 2X + 2Y, as in Example 3.6.

Repeated Measurements

- One of the best ways to reduce uncertainty is to take several independent measurements and average them.

- The measurements in this case are **a simple random sample** from a population, and their average is the sample mean.

If $X_1, \ldots, X_n$ are n *independent* measurements, each with mean μ and uncertainty σ, then the sample mean $\overline{X}$ is a measurement with mean

$$\mu_{\overline{X}} = \mu \tag{3.5}$$

and with uncertainty

$$\sigma_{\overline{X}} = \frac{\sigma}{\sqrt{n}} \tag{3.6}$$

Repeated Measurements…

- If we perform many independent measurements of the same quantity, then the average of these measurements has the same mean as each individual measurement, but

- The standard deviation is reduced by a factor equal to the square root of the sample size.

- In other words, the average of several repeated measurements has the same accuracy as, and is more precise than, any single measurement.

Example 3.8 (p.173)

- The length of a component is to be measured by a process whose uncertainty is 0.05 cm.
- If 25 independent measurements are made and the average of these is used to estimate the length, what will the uncertainty be?
- How much more precise is the average of 25 measurements than a single measurement?

Example 3.8 (p.173) - SOLUTION

- The uncertainty in the average of 25 measurements is

$$0.05/\sqrt{25} = 0.01 \text{ cm}$$

- The uncertainty in a single measurement is 0.05 cm.

- The uncertainty in the average of 25 independent measurements is therefore less than that of a single measurement by a factor of 5, which is the square root of the number of measurements that are averaged.

- Thus, the average of 25 independent measurements is 5 times more precise than a single measurement.

Example 3.9 (p.173)

- The mass of a rock is measured five times on a scale whose uncertainty is unknown.

- The five measurements (in grams) are

  21.10, 21.05, 20.98, 21.12, and 21.05

- Estimate the mass of the rock and find the uncertainty in the estimate.

Example 3.9 (p.173) - SOLUTION

- Let $\bar{X}$ represent the average of the 5 measurements, and let **s** represent the sample standard deviation.
- We compute $\bar{X}$ **= 21.06** g and **s = 0.0543 g**.
- Using Equation (3.6), we would estimate the length of the component to be $\bar{X} \pm \sigma/\sqrt{5}$.
- We do not know σ, which is the uncertainty, or standard deviation, of the measurement process. However, we can approximate σ with s, the sample standard deviation of the five measurements.
- We therefore estimate the mass of the rock to be $21.06 \pm 0.0543/\sqrt{5}$ or **21.06 ± 0.02 g.**

Example 3.10 (p.174)

- In Example 3.6 two adjacent sides of a rectangular lot were measured to be $X = 50.11 \pm 0.05$ m and $Y = 75.21 \pm 0.08$ m.
- Assume that the budget for this project is sufficient to allow 14 more measurements to be made. Each side has already been measured once.
- One engineer suggests allocating the new measurements equally to each side, so that each will be measured 8 times.
- A second engineer suggests using all 14 measurements on the longer side, since that side is measured with greater uncertainty.
- Estimate the uncertainty in the perimeter under each plan. Which plan results in the smaller uncertainty?

Example 3.10 (p.174) - SOLUTION

- Under the first plan, let $\bar{X}$ represent the average of eight measurements of the shorter side, and let $\bar{Y}$ represent the average of eight measurements of the longer side.
- The perimeter will be estimated by $2\bar{X} + 2\bar{Y}$
- The uncertainty in the perimeter under the first plan is therefore

$$\sigma_{2\bar{X}+2\bar{Y}} = \sqrt{4\sigma_{\bar{X}}^2 + 4\sigma_{\bar{Y}}^2} \qquad \text{(using Equation 3.4)}$$

$$= \sqrt{4\left(\frac{\sigma_X}{\sqrt{8}}\right)^2 + 4\left(\frac{\sigma_Y}{\sqrt{8}}\right)^2} \qquad \text{(using Equation 3.6)}$$

$$= \sqrt{\frac{4(0.05)^2}{8} + \frac{4(0.08)^2}{8}}$$

$$= 0.067 \text{ m}$$

Example 3.10 (p.174) - SOLUTION

- Under the second the perimeter will be estimated by $2X + 2\bar{Y}$ where X is a single measurement of the shorter side and $\bar{Y}$ is the average of 15 measurements of the longer side.

- The uncertainty in the perimeter under the second plan is therefore

$$\sigma_{2X+2\bar{Y}} = \sqrt{4\sigma_X^2 + 4\sigma_{\bar{Y}}^2} \qquad \text{(using Equation 3.4)}$$

$$= \sqrt{4\sigma_X^2 + 4\left(\frac{\sigma_Y}{\sqrt{15}}\right)^2} \qquad \text{(using Equation 3.6)}$$

$$= \sqrt{4(0.05)^2 + \frac{4(0.08)^2}{15}}$$

$$= 0.11 \text{ m}$$

- CONCLUSION: The first plan is better

Repeated Measurements with Differing Uncertainties

- Sometimes repeated measurements may have differing uncertainties.
- This can happen, for example, when the measurements are made with different instruments.
- It turns out that the best way to combine the measurements in this case is with a weighted average, rather than with the sample mean.

Example 3.11 (p.175)

- An engineer measures the period of a pendulum (in seconds) to be 2.0 ± 0.2 s.
- Another independent measurement is made with a more precise clock, and the result is 2.2 ± 0.1 s.
- The average of these two measurements is 2.1 s.
- Find the uncertainty in this quantity.

Example 3.11 (p.175) - SOLUTION

- Let X represent the measurement with the less precise clock, so X = 2.0s, with uncertainty $\sigma_X = 0.2$ s.
- Let Y represent the measurement on the more precise clock, so Y = 2.2 s, with uncertainty $\sigma_Y = 0.1$ s.
- The average is (1/2)X + (1/2)Y = 2.10, and the uncertainty in this average is

$$\begin{aligned}\sigma_{\text{avg}} &= \sqrt{\frac{1}{4}\sigma_X^2 + \frac{1}{4}\sigma_Y^2} \\ &= \sqrt{\frac{1}{4}(0.2)^2 + \frac{1}{4}(0.1)^2} \\ &= 0.11 \text{ s}\end{aligned}$$

Example 3.12 (p.175)

- In Example 3.11, another engineer suggests that since Y is a more precise measurement than X, a weighted average in which Y is weighted more heavily than X might be more precise than the unweighted average.
- Specifically, the engineer suggests that by choosing an appropriate constant c between 0 and 1, the weighted average **$cX + (1 - c)Y$** might have a smaller uncertainty than the unweighted average $(1/2)X + (1/2)Y$ considered in Example 3.11.
- Express the uncertainty in the weighted average **$cX + (1 - c)Y$** in terms of c, and find the value of c that minimizes the uncertainty.

Example 3.12 (p.175) - SOLUTION

- The uncertainty in the weighted average is

$$\begin{aligned}\sigma &= \sqrt{c^2\sigma_X^2 + (1-c)^2\sigma_Y^2} \\ &= \sqrt{0.04c^2 + 0.01(1-c)^2} \\ &= \sqrt{0.05c^2 - 0.02c + 0.01}\end{aligned}$$

- We now must find the value of c minimizing σ.
- This is equivalent to finding the value of c minimizing σ^2. We take the derivative of **$\sigma^2 = 0.05c^2 - 0.02c + 0.01$** with respect to c and set it equal to 0:

$$\frac{d\sigma^2}{dc} = 0.10c - 0.02 = 0$$

Example 3.12 (p.175) – SOLUTION…

- Solving for c, we obtain c = 0.2
- The most precise weighted average is therefore **0.2X + 0.8Y = 2.16**.
- The uncertainty in this estimate is

$$\sigma_{\text{best}} = \sqrt{(0.2)^2\sigma_X^2 + (0.8)^2\sigma_Y^2} = \sqrt{(0.2)^2(0.2)^2 + (0.8)^2(0.1)^2} = 0.09 \text{ s}$$

- Note that this is less than the uncertainty of 0.11 s found for the unweighted average used in Example 3.11.

SUMMARY (p.176)

If X and Y are *independent* measurements of the same quantity, with uncertainties σ_X and σ_Y, respectively, then the weighted average of X and Y with the smallest uncertainty is given by $c_{\text{best}}X + (1 - c_{\text{best}})Y$, where

$$c_{\text{best}} = \frac{\sigma_Y^2}{\sigma_X^2 + \sigma_Y^2} \qquad 1 - c_{\text{best}} = \frac{\sigma_X^2}{\sigma_X^2 + \sigma_Y^2} \tag{3.7}$$

End of Section 3-2