MATH 2233 Differential Equations

Chapter 1 Introduction

Section 1.1 Background

Goal of this section

1. understand what is a differential equation.
2. understand the classification of differential equations.

1. Definition of Differential Equations

Example 1. ***(Types of equations)***

1. *Find* x *in* $x^2 + 2x + 1 = 0$.

2. *Find* $f(t)$ *in* $f(t)e^t + \sin(t) = \cos(t)$.

3. *Find* $y(t)$ *in* $y''(t) + 3y'(t) = e^t$.

Definition. A **differential equation** is

-
-

Remark. If a differential equation involves the derivative of one variable with respect to another, then the former is called a ______________________, and the latter an ______________________.

For example, in the equation

$$\frac{d^2x}{dt^2} + a\frac{dx}{dt} + kx = 0,$$

- x:
- t:
- a and k:

2. Classification of Differential Equations

2.1. By Type

- If a differential equation involves only *ordinary derivatives* of the unknown function, it is called an ____

  e.g.

- If a differential equation involves *partial derivatives*, it is called a ________________________________

  e.g.

2.2 By Order

The _____________ of a differential equation is the order of the highest-order derivatives of the unknown in the equation.

Example 2. *Determine the type and the order of the following differential equations.*

- $y'' + 4y' = e^x$

- $y'' + 4(y')^3 + 5y = e^x$

- $u_t - 2u_{xx} = 0$

- $t^2y''' - t^3y'' + ty^4 = sin(t)$

2.3. By Linearity

An n-th order ordinary differential equation $F(x, y, y', \cdots, y^{(n)}) = 0$ is called ______________, if

More explicitly, an ODE is **linear**, if it can be written as

If a differential equation is not linear, it is called ______________.

Remark: There are two special cases that we are going to discuss throughout this semester

- linear first-order equations:

- linear second-order equations:

Example 3. *Determine the type, order, and linearity of the following differential equations.*

- $x^3y''' + xy' - 5y = e^x$

- $\dfrac{dy}{dx} = \dfrac{x}{y+2}$

- $\dfrac{d^2y}{dx^2} = \dfrac{y+2}{\sqrt{x^2+1}}$

- $(1-y)y' + 2y = e^x$

- $y'' + \sin(x)y = 0$

- $y'' + x\sin(y) = 0$

MATH 2233 Differential Equations

Section 1.2 Solutions and Initial Value Problems

Goal of this section

1. understand the explicit and implicit solution of a differential equation.
2. understand the initial value problem for a differential equation.
3. understand the existence and uniqueness of solution.

1. Explicit Solution

The general form of n-th order ODEs with x independent, y dependent, can be expressed as

In many cases, we can isolate the highest-order term and write the equation as

Definition. A function $\phi(x)$ is called an ____________________ of an ODE if the equation becomes an identity when substituting y by $\phi(x)$.

Example 1. *Verify that $\phi(x) = x^2 - x^{-1}$ is an explicit solution to the differential equation*

$$\frac{d^2y}{dx^2} - \frac{2}{x^2}y = 0,$$

but $\psi(x) = x^3$ is not.

Example 2. *Show that for any choice of the constant* c_1 *and* c_2*, the function* $\phi(x) = c_1 e^{-x} + c_2 e^{2x}$ *is an explicit solution to the linear equation*

$$y'' - y' - 2y = 0.$$

Example 3. *Determine for which values of* m *the function* $\phi(x) = x^m$ *is a solution to the differential equation*

$$5x^2 y'' - 11xy' + 3y = 0.$$

2. Implicit Solution

As we will see in following chapters, the methods for solving differential equations do not always yield an explicit solution. We may have to settle for a solution that is defined implicitly.

Example 4. *Show that the relation $y^2 - x^3 + 8 = 0$ implicitly defines a solution to the nonlinear equation*

$$\frac{dy}{dx} = \frac{3x^2}{2y}$$

on the interval $(2, \infty)$.

Definition. A relation $G(x, y)$ is said to be an ____________________ of an ODE on the interval I if it defines one or more explicit solutions on I.

Example 5. *Show that the relation $x + y + e^{xy} = 0$ is an implicit solution to the nonlinear equation*

$$(1 + xe^{xy})\frac{dy}{dx} + 1 + ye^{xy} = 0.$$

Example 6. *Verify that for every constant C the relation $4x^2 - y^2 = C$ is an implicit solution to*

$$y\frac{dy}{dx} - 4x = 0$$

Remark. For brevity, from now on we use the term **solution** to mean either explicit or implicit solution.

3. Initial Value Problem

As indicated in Example 2, a differential equation usually has infinitely many solutions. To uniquely determine a solution, we often impose additional conditions.

Example 7.

- *Find <u>all</u> solutions of the differential equation* $\frac{dy}{dt} = y$.
- *In addition to the differential equation, we also require* $y(0) = 3$. *What can we say about the solution?*

<u>Remark.</u>

- The additional condition $y(0) = 3$ is often called ____________________________, since the independent variable t often represents time in many physical applications.
- A differential equation together with an initial condition is called an ____________________________.

<u>Remark.</u>

- The IVP for a first-order differential equation is
- The IVP for a second-order differential equation is

Example 8. *As shown in Example 2, the function* $\phi(x) = c_1 e^{-x} + c_2 e^{2x}$ *is a solution to*

$$y'' - y' - 2y = 0$$

for any choice of constants c_1 *and* c_2*. Determine* c_1 *and* c_2 *so that the initial conditions*

$$y(0) = 2 \quad \textit{and} \quad y'(0) = -3$$

are satisfied.

4. Existence and Uniqueness of IVP

Theorem (Existence and Uniqueness of Solution)
Consider the initial value problem

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0.$$

If f and $\dfrac{\partial f}{\partial y}$ are continuous functions in some rectangle

$$R = \{(x, y) : a < x < b, c < y < d\}$$

that contains the point (x_0, y_0), then the initial value problem has a unique solution $\phi(x)$ in some interval $x_0 - \delta < x < x_0 + \delta$, where δ is a positive number.

Example 9. *Show that there exists a unique solution to the initial value problem*

$$3\frac{dy}{dx} = x^2 - xy^3, \quad y(1) = 6.$$

MATH 2233 Differential Equations

Section 1.3 Direction Fields

Goal of this section

1. draw a direction field to see the solution of 1st-order equations without solving them
2. use the direction field to analyze properties of the solution
3. introduce an online software to sketch more accurate direction fields

1. Direction Field

Definition. The **direction field** of the first order differential equation

$$\frac{dy}{dx} = f(x,y)$$

is a plot of short line segments drawn at various points in the xy-plane showing the slope of the solution curve.

Example 1. *Sketch the direction field of the differential equation*

$$\frac{dy}{dx} = 1 - \frac{y}{5}$$

2. Use Direction Field to Analyze the Solution

Example 2. *The following differential equation*

$$\frac{dp}{dt} = p(2 - p)$$

is a logistic equation for modeling the growth of population. Here, p (in thousand) is the population at time t. Sketch the direction field of this equation. Then answer the following questions.

1. *If the initial population is 3000, what can you say about the the population in a long time?*
2. *Can a population of 1000 ever decline to 500?*
3. *Can a population of 1000 ever increase to 3000?*

3. Online Direction Field Softwares

Computer softwares can be used to sketch direction fields of more complicated differential equations accurately. For example,

Geogebra: https://www.geogebra.org/m/W7dAdgqc

Example 3. *Use a computer software to sketch the direction field of*

$$\frac{dy}{dx} = x^2 - y.$$

From the direction field, sketch the solutions with initial conditions $y(0) = 0, 1$*, and* -1.