第四节　方阵的行列式

1.填空题.

（1）排列 6427531 的逆序数为 15 ，该排列为 奇 排列.

【解题过程】$\tau(6427531)=15$，故 6427531 为奇排列.

（2）$i=$ 8 ，$j=$ 3 时，排列 1274 i 56 j 9 为偶排列.

【解题过程】排列 1274 i 56 j 9 只能为 127435689 或 127485639 这两种排列，只需要计算其逆序数即可. $\tau(127435689)=5, \tau(127485639)=10,$ 即 $i=8,\ j=3$ 时，排列 1274 i 56 j 9 为偶排列.

（3）在 6 阶行列式中，含 $a_{15}a_{23}a_{32}a_{44}a_{51}a_{66}$ 的项的符号为 + ，含 $a_{32}a_{43}a_{14}a_{51}a_{66}a_{25}$ 的项的符号为 + .

【解题过程】 $(-1)^{\tau(532416)}=1$，含 $a_{15}a_{23}a_{32}a_{44}a_{51}a_{66}$ 的项的符号为正；$(-1)^{\tau(341562)+\tau(234165)}=1$，含 $a_{32}a_{43}a_{14}a_{51}a_{66}a_{25}$ 的项的符号为正.

（4）多项式 $f\left(x\right)=\begin{vmatrix} 3x & 2x & 3 & 1 \\ 1 & x & 1 & -2 \\ 3 & 2 & 3x & 1 \\ 1 & 0 & 1 & x \end{vmatrix}$ 中

x^4 的系数为 $\underline{\ 9\ }$，x^3 的系数为 $\underline{\ -6\ }$.

【解题过程】 $a_{11}a_{22}a_{33}a_{44}$ 含有 x^4，x^4 的系数为 $\left(-1\right)^{\tau(1,2,3,4)}3\times1\times3\times1=9$；$a_{12}a_{21}a_{33}a_{44}$ 含有 x^3，x^3 的系数为

$$\left(-1\right)^{\tau(2,1,3,4)}2\times1\times3\times1=-6.$$

（5）$\begin{vmatrix} 0 & 0 & \cdots & 0 & a_{1n} \\ 0 & 0 & \cdots & a_{2,n-1} & a_{2n} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & a_{n-1,2} & \cdots & a_{n-1,n-1} & a_{n-1,n} \\ a_{n1} & a_{n2} & \cdots & a_{n,n-1} & a_{nn} \end{vmatrix}=$

$\underline{\left(-1\right)^{\frac{n(n-1)}{2}}a_{1n}a_{2,n-1}\cdots a_{n-1,2}a_{n1}}$.

【解题过程】

$$\begin{vmatrix} 0 & 0 & \cdots & 0 & a_{1n} \\ 0 & 0 & \cdots & a_{2,n-1} & a_{2n} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & a_{n-1,2} & \cdots & a_{n-1,n-1} & a_{n-1,n} \\ a_{n1} & a_{n2} & \cdots & a_{n,n-1} & a_{nn} \end{vmatrix}$$

$$=\left(-1\right)^{\tau(n,n-1,n-2,\cdots,1)}a_{n1}a_{n-1,2}\cdots a_{2,n-1}a_{1n}$$

$$=\left(-1\right)^{\frac{n(n-1)}{2}}a_{1n}a_{2,n-1}\cdots a_{n-1,2}a_{n1}.$$

（6）$\begin{vmatrix} 2 & 0 & 1 \\ 1 & -4 & -1 \\ -1 & 8 & 3 \end{vmatrix}=\underline{\ -4\ }$，$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}=$

$\underline{(c-a)(c-b)(b-a)}$.

【解题过程】

$$\begin{vmatrix} 2 & 0 & 1 \\ 1 & -4 & -1 \\ -1 & 8 & 3 \end{vmatrix}$$
$$= 2\times(-4)\times3+0\times(-1)\times(-1)+1\times1\times8$$
$$-1\times(-4)\times(-1)-2\times8\times(-1)-0\times1\times3$$
$$=-4;$$

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & b^2-a^2 & c^2-a^2 \end{vmatrix}$$
$$=(b-a)(c^2-a^2)-(c-a)(b^2-a^2)$$
$$=(c-a)(c-b)(b-a).$$

（7）方阵 A 按列分块为 $A=(A_1,A_2,A_3,A_4)$，

且 $\det A=2$，则 $\det(A_1+A_2,2A_1,A_4,A_3)=\underline{4}$.

【解题思路】行列式的性质.

【解题过程】$\det(A_1+A_2,2A_1,A_4,A_3)$

$$=|A_1,2A_1,A_4,A_3|+|A_2,2A_1,A_4,A_3|$$

$$=|A_2,2A_1,A_4,A_3|=2|A_2,A_1,A_4,A_3|$$

$$=2|A_1,A_2,A_3,A_4|=4$$

（8）方阵 A 是奇数阶反对称矩阵，则 $\det A = \underline{0}$。

【解题过程】 设 A 为奇数阶反对称矩阵，则 $A^T = -A$，$\left|A^T\right| = |A| = |-A| = -|A|$，故 $|A| = 0$；则奇数阶反对称矩阵 $\left(a_{ij} = -a_{ji}\right)$ 的行列式为零.

（9）已知四阶行列式 D 的第三行元素分别为：-1，0，2，4；第四行元素对应的余子式依次是 2，10，a，4，则 $a = \underline{\ 9\ }$.

【解题思路】 $A_{ij} = (-1)^{i+j} M_{ij}$;

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nn} \end{vmatrix} = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in};$$

在行列式中，一行的元素与另一行相应元素的代数余子式的乘积之和为零.

【解题过程】

$$D = -1 \times (-1)^{4+1} \times 2 + 0 \times (-1)^{4+2} \times 10 + 2 \times (-1)^{4+3} \times a + 4 \times (-1)^{4+4} \times 4 = 0,$$

得 $a = 9$.

2. 证明

$$\begin{vmatrix} ax+by & ay+bz & az+bx \\ ay+bz & az+bx & ax+by \\ az+bx & ax+by & ay+bz \end{vmatrix} = \left(a^3+b^3\right) \begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix}.$$

【解题思路】

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ b_1+c_1 & b_2+c_2 & \cdots & b_n+c_n \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$=\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ b_1 & b_2 & \cdots & b_n \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}+\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ c_1 & c_2 & \cdots & c_n \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

【解题过程】

$$\text{左边}=\begin{vmatrix} ax+by & ay+bz & az+bx \\ ay+bz & az+bx & ax+by \\ az+bx & ax+by & ay+bz \end{vmatrix}$$

$$=\begin{vmatrix} ax & ay & az \\ ay+bz & az+bx & ax+by \\ az+bx & ax+by & ay+bz \end{vmatrix}$$

$$+\begin{vmatrix} by & bz & bx \\ ay+bz & az+bx & ax+by \\ az+bx & ax+by & ay+bz \end{vmatrix}$$

$$=a\begin{vmatrix} x & y & z \\ ay+bz & az+bx & ax+by \\ az+bx & ax+by & ay+bz \end{vmatrix}$$

$$+b\begin{vmatrix} y & z & x \\ ay+bz & az+bx & ax+by \\ az+bx & ax+by & ay+bz \end{vmatrix}$$

$$=a\begin{vmatrix} x & y & z \\ ay+bz & az+bx & ax+by \\ az & ax & ay \end{vmatrix}$$

$$+b\begin{vmatrix} y & z & x \\ bz & bx & by \\ az+bx & ax+by & ay+bz \end{vmatrix}$$

$$=a^2\begin{vmatrix} x & y & z \\ ay+bz & az+bx & ax+by \\ z & x & y \end{vmatrix}$$

$$+b^2\begin{vmatrix} y & z & x \\ z & x & y \\ az+bx & ax+by & ay+bz \end{vmatrix}$$

$$=a^2\begin{vmatrix} x & y & z \\ ay & az & ax \\ z & x & y \end{vmatrix}+b^2\begin{vmatrix} y & z & x \\ z & x & y \\ bx & by & bz \end{vmatrix}$$

$$=a^3\begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix}+b^3\begin{vmatrix} y & z & x \\ z & x & y \\ x & y & z \end{vmatrix}$$

$$=\left(a^3+b^3\right)\begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix}=\text{右边}$$

即证：

$$\begin{vmatrix} ax+by & ay+bz & az+bx \\ ay+bz & az+bx & ax+by \\ az+bx & ax+by & ay+bz \end{vmatrix}.$$

$$=\left(a^3+b^3\right)\begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix}$$

3. 已知 $D_5=\begin{vmatrix}1&2&3&4&5\\2&2&2&1&1\\3&1&2&4&5\\1&1&1&2&2\\4&3&1&5&0\end{vmatrix}=27$, 计算

（1）$3A_{12}+2A_{22}+2A_{32}+A_{42}+A_{52}$;

【解题过程】 3,2,2,1,1 是第三列的元素，

$A_{12},A_{22},A_{32},A_{42},A_{52}$ 是第 2 列元素的代数余子式

$\because$ 在行列式中，一行的元素与另一行相应元素的代数余子式的乘积之和为零.

$\therefore\ 3A_{12}+2A_{22}+2A_{32}+A_{42}+A_{52}=0$

（2）$A_{41}+A_{42}+A_{43}$ 和 $A_{44}+A_{45}$;

【解题思路】

$$\begin{vmatrix}a_{11}&a_{12}&\cdots&a_{1j}&\cdots&a_{1n}\\ \vdots&\vdots&&\vdots&&\vdots\\ a_{i1}&a_{i2}&\cdots&a_{ij}&\cdots&a_{in}\\ \vdots&\vdots&&\vdots&&\vdots\\ a_{n1}&a_{n2}&\cdots&a_{nj}&\cdots&a_{nn}\end{vmatrix}$$
$=a_{i1}A_{i1}+a_{i2}A_{i2}+\cdots+a_{in}A_{in}$;

在行列式中，一行的元素与另一行相应元素的代数余子式的乘积之和为零

【解题过程】

由行列式的性质：

$$\begin{cases}A_{31}+A_{32}+A_{33}+2\left(A_{34}+A_{35}\right)=0\\2\left(A_{31}+A_{32}+A_{33}\right)+A_{34}+A_{35}=0\end{cases},$$

得 $\begin{cases}A_{31}+A_{32}+A_{33}=0\\A_{34}+A_{35}=0\end{cases}$.

（3）$A_{31}+A_{32}+A_{33}$ 和 $A_{34}+A_{35}$.

【解题过程】由行列式的性质：

$$\begin{cases} A_{41}+A_{42}+A_{43}+2\left(A_{44}+A_{45}\right)=27 \\ 2\left(A_{41}+A_{42}+A_{43}\right)+A_{44}+A_{45}=0 \end{cases},$$

得 $\begin{cases} A_{41}+A_{42}+A_{43}=-9 \\ A_{44}+A_{45}=18 \end{cases}$.

4.若 A 为 n 阶方阵，且满足 $AA^T=E$,若 $|A|<0$,求 $|E+A|$.

【解题过程】

$\because$ A 为 n 阶方阵，且满足 $AA^T=E$

$\therefore \left|AA^T\right|=|A|^2=1$

$\because |A|<0$

$\therefore |A|=-1$

$\because A^T\left(A+E\right)=A^TA+A^T=E+A^T=\left(A+E\right)^T$

$\therefore \left|A^T\left(A+E\right)\right|=-|A+E|=\left|\left(A+E\right)^T\right|=|A+E|$

$\therefore |A+E|=0$.

5. 已知 n 阶方阵 A,B,D 的行列式值分别为 2，3，4，计算 $\det\begin{pmatrix} 0 & A \\ B & D \end{pmatrix}$.

其中 D 是 n 阶方阵。

【解题思路】（Laplace 定理）任意取定行列式的某 k 行，位于这些行上的所有可能的 C_n^k 个 k 阶子式与各自的代数余子式乘积的和，等于原行列式.即对任意固定的 $i_1,\cdots,i_k$ 行，

有

$$\det A=\sum_{1\le j_1<j_2<\cdots<j_k<n}\det A\begin{pmatrix} i_1,\cdots,i_k \\ j_1,\cdots,j_k \end{pmatrix}\det A^{cc}\begin{pmatrix} i_1,\cdots,i_k \\ j_1,\cdots,j_k \end{pmatrix}.$$

【解题过程】 Laplace 定理可知：

$$\begin{aligned}
&\det\begin{pmatrix} 0 & A \\ B & D \end{pmatrix} \\
&=\det A\times(-1)^{1+2+\cdots+n+n+1+(n+2)+\cdots+2n}\det B \\
&=(-1)^{2n^2+n}\det A\det B \\
&=(-1)^n\det A\det B=(-1)^n 6.
\end{aligned}$$

6. 计算下列行列式（1）$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{vmatrix}$；（2）

$$\begin{vmatrix} 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 \end{vmatrix}.$$

【解题思路】 行列式的性质.

【解题过程】

$$(1)\begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{vmatrix}\overset{r_1+r_k(k=2,3,4)}{=}\begin{vmatrix} 10 & 10 & 10 & 10 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{vmatrix}=10\begin{vmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{vmatrix}$$

$$\overset{r_2-2r_1,r_3-3r_2,r_4-4r_1}{=}10\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & -2 & -1 \\ 0 & -3 & -2 & -1 \end{vmatrix}\overset{\text{按第一行展开}}{=}10\begin{vmatrix} 1 & 2 & -1 \\ 1 & -2 & -1 \\ -3 & -2 & -1 \end{vmatrix}$$

$$\overset{r_2-r_1}{=}10\begin{vmatrix} 1 & 2 & -1 \\ 0 & -4 & 0 \\ -3 & -2 & -1 \end{vmatrix}\overset{\text{按第二行展开}}{=}10\times(-4)\times(-1)^{2+2}\begin{vmatrix} 1 & -1 \\ -3 & -1 \end{vmatrix}$$

$$=160.$$

(2) $$\begin{vmatrix}1&-1&1&1\\1&-1&-1&-1\\1&1&-1&-1\\1&1&1&-1\end{vmatrix}\overset{r_2-r_1,r_3-r_1,r_4-r_1}{=}\begin{vmatrix}1&-1&1&1\\0&0&-2&-2\\0&2&-2&-2\\0&2&0&-2\end{vmatrix}$$

$$\overset{\text{按第一行展开}}{=}\begin{vmatrix}0&-2&-2\\2&-2&-2\\2&0&-2\end{vmatrix}\overset{r_2-r_1}{=}\begin{vmatrix}0&-2&-2\\2&0&0\\2&0&-2\end{vmatrix}$$

$$\overset{\text{按第二行展开}}{=}2\times(-1)^{2+1}\begin{vmatrix}-2&-2\\0&-2\end{vmatrix}=-8.$$

7.利用三角形行列式的结果计算下列 n 阶行列式.

(1) $$\begin{vmatrix}x&y&0&\cdots&0&0\\0&x&y&\cdots&0&0\\\vdots&\vdots&\vdots&&\vdots&\vdots\\0&0&0&\cdots&x&y\\y&0&0&\cdots&0&x\end{vmatrix}.$$

【解题过程】

$$\begin{vmatrix}x&y&0&\cdots&0&0\\0&x&y&\cdots&0&0\\\vdots&\vdots&\vdots&&\vdots&\vdots\\0&0&0&\cdots&x&y\\y&0&0&\cdots&0&x\end{vmatrix}.\overset{\text{按第一列展开}}{=}x\cdot(-1)^{1+1}\begin{vmatrix}x&y&\cdots&0&0\\\vdots&\vdots&&\vdots&\vdots\\0&0&\cdots&x&y\\0&0&\cdots&0&x\end{vmatrix}_{n-1}$$

$$+y\cdot(-1)^{n+1}\begin{vmatrix}y&0&0&\cdots&0\\x&y&0&\cdots&0\\\vdots&\vdots&&\vdots&\vdots\\0&0&\cdots&y&0\\0&0&\cdots&x&y\end{vmatrix}_{n-1}$$

$$=x^n+(-1)^{1+n}y^n.$$

(2)设 $\prod_{i=1}^{n}a_i\neq 0$,

$$D_n = \begin{vmatrix} 1+a_1 & 1 & \cdots & 1 \\ 2 & 2+a_2 & \cdots & 2 \\ \vdots & \vdots & & \vdots \\ n & n & \cdots & n+a_n \end{vmatrix}.$$

【解题过程】

$$\begin{vmatrix} 1+a_1 & 1 & \cdots & 1 \\ 2 & 2+a_2 & \cdots & 2 \\ \vdots & \vdots & & \vdots \\ n & n & \cdots & n+a_n \end{vmatrix}$$

$$\overset{r_2-2r_1,r_3-3r_1,\cdots,r_n-nr_1}{=} \begin{vmatrix} 1+a_1 & 1 & \cdots & 1 \\ -2a_1 & a_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ -na_1 & 0 & \cdots & a_n \end{vmatrix}$$

$$\overset{c_1+\frac{ia_1}{a_i}(i=2,3,\cdots,n)}{=} \begin{vmatrix} 1+a_1+\dfrac{2a_1}{a_2}+\cdots+\dfrac{na_1}{a_n} & 1 & \cdots & 1 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_n \end{vmatrix}$$

$$\overset{\text{按照第一行展开}}{=} \left(1+a_1+\frac{2a_1}{a_2}+\cdots+\frac{na_1}{a_n}\right)a_2\cdots a_n$$

$$=\left(1+\sum_{i=1}^{n}\frac{i}{a_i}\right)a_1a_2\cdots a_n.$$

8.设 $D_n=\begin{vmatrix} x & a & a & \cdots & a \\ a & x & a & \cdots & a \\ a & a & x & \cdots & a \\ \vdots & \vdots & \vdots & & \vdots \\ a & a & a & \cdots & x \end{vmatrix}$,

$(n-1)a+x\neq 0$, 求 $A_{n1}+A_{n2}+\cdots+A_{nn}$.

【解题过程】

$$D_n=\begin{vmatrix} x & a & a & \cdots & a \\ a & x & a & \cdots & a \\ a & a & x & \cdots & a \\ \vdots & \vdots & \vdots & & \vdots \\ a & a & a & \cdots & x \end{vmatrix} \overset{r_j+r_k(k=1,2,\cdots,j-1,j+1,\cdots,n)}{=}$$

$$\begin{vmatrix} x & a & a & \cdots & a \\ a & x & a & \cdots & a \\ a & a & x & \cdots & a \\ \vdots & \vdots & \vdots & & \vdots \\ (n-1)a+x & (n-1)a+x & (n-1)a+x & \cdots & (n-1)a+x \end{vmatrix}$$

$$=\left[(n-1)a+x\right]\begin{vmatrix} x & a & a & \cdots & a \\ a & x & a & \cdots & a \\ a & a & x & \cdots & a \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{vmatrix}$$

$$\overset{r_j-ar_i(j=1,2,\cdots,i-1,i+1,\cdots,n)}{=}\left[(n-1)a+x\right]\begin{vmatrix} x-a & 0 & 0 & \cdots & 0 \\ 0 & x-a & 0 & \cdots & 0 \\ 0 & 0 & x-a & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{vmatrix}$$

$$=\left[(n-1)a+x\right](x-a)^{n-1}.$$

故n阶行列式

$$A_{n1}+A_{n2}+\cdots+A_{nn}$$

$$=\begin{vmatrix} x & a & a & \cdots & a \\ a & x & a & \cdots & a \\ a & a & x & \cdots & a \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{vmatrix}=\left(x-a\right)^{n-1}.$$

9.用数学归纳法证明：

$$D_n=\begin{vmatrix} \cos\theta & 1 & 0 & \cdots & 0 & 0 \\ 1 & 2\cos\theta & 1 & \cdots & 0 & 0 \\ 0 & 1 & 2\cos\theta & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2\cos\theta & 1 \\ 0 & 0 & 0 & \cdots & 1 & 2\cos\theta \end{vmatrix}$$

$=\cos n\theta.$

【解题过程】当$n=1$时，$D_1=\cos\theta;$

当$n=2$时，

$$D_2=\begin{vmatrix} \cos\theta & 1 \\ 1 & 2\cos\theta \end{vmatrix}=\cos 2\theta;$$

假设当$n=k-1$时，$D_k=\cos\left(k-1\right)\theta$；

当$n=k$时，

$$\begin{vmatrix} \cos\theta & 1 & 0 & \cdots & 0 & 0 \\ 1 & 2\cos\theta & 1 & \cdots & 0 & 0 \\ 0 & 1 & 2\cos\theta & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2\cos\theta & 1 \\ 0 & 0 & 0 & \cdots & 1 & 2\cos\theta \end{vmatrix}$$

$$\overset{\text{按最后1列展开}}{=} 2\cos\theta\cdot(-1)^{k+k}\begin{vmatrix} cos\theta & 1 & \cdots & 0 & 0 \\ 1 & 2\cos\theta & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 2\cos\theta & 1 \\ 0 & 0 & \cdots & 1 & 2\cos\theta \end{vmatrix}_{k-1}$$

$$+(-1)^{k-1+k}\begin{vmatrix} cos\theta & 1 & \cdots & 0 & 0 \\ 1 & 2\cos\theta & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 2\cos\theta & 1 \\ 0 & 0 & \cdots & 1 & 2\cos\theta \end{vmatrix}_{k-2}$$

$$= 2\cos\theta D_{k-1} - D_{k-2}$$

$$= 2\cos\theta\cos(k-1)\theta - \cos(k-2)\theta$$

$$= 2\cos\theta\cos(k-1)\theta - \cos\theta\cos(k-1)\theta - \sin\theta\sin(k-1)\theta$$

$$= \cos k\theta;$$

由此可知，

$$\begin{vmatrix} \cos\theta & 1 & 0 & \cdots & 0 & 0 \\ 1 & 2\cos\theta & 1 & \cdots & 0 & 0 \\ 0 & 1 & 2\cos\theta & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2\cos\theta & 1 \\ 0 & 0 & 0 & \cdots & 1 & 2\cos\theta \end{vmatrix} = \cos n\theta.$$

10.利用范德蒙行列式的结果计算下列行列式.

（1）$D_{n+1}=\begin{vmatrix} a^n & (a-1)^n & \cdots & (a-n)^n \\ a^{n-1} & (a-1)^{n-1} & \cdots & (a-n)^{n-1} \\ \vdots & \vdots & & \vdots \\ a & a-1 & \cdots & a-n \\ 1 & 1 & \cdots & 1 \end{vmatrix},(a\neq 0,1,2,\cdots,n).$

【解题思路】 范德蒙行列式

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ a_1 & a_2 & a_3 & \cdots & a_n \\ a_1^2 & a_2^2 & a_3^2 & \cdots & a_n^2 \\ \vdots & \vdots & \vdots & & \vdots \\ a_1^{n-1} & a_2^{n-1} & a_3^{n-1} & \cdots & a_n^{n-1} \end{vmatrix}=\prod_{1\le j<i\le n}\left(a_i-a_j\right).$$

【解题过程】

$$D_{n+1}=\begin{vmatrix} a^n & (a-1)^n & \cdots & (a-n)^n \\ a^{n-1} & (a-1)^{n-1} & \cdots & (a-n)^{n-1} \\ \vdots & \vdots & & \vdots \\ a & a-1 & \cdots & a-n \\ 1 & 1 & \cdots & 1 \end{vmatrix}$$

$$=(-1)^{\frac{n(n+1)}{2}}\begin{vmatrix} 1 & 1 & \cdots & 1 \\ a & a-1 & \cdots & a-n \\ \vdots & \vdots & & \vdots \\ a^{n-1} & (a-1)^{n-1} & \cdots & (a-n)^{n-1} \\ a^n & (a-1)^n & \cdots & (a-n)^n \end{vmatrix}$$

$$=(-1)^{\frac{n(n+1)}{2}}\left[(a-1)-a\right]\cdot\left[(a-2)-a\right]\cdot\left[(a-2)-(a-1)\right]$$

$$\cdots\left[(a-n)-a\right]\cdot\left[(a-n)-(a-1)\right]\cdots\left[(a-n)-(a-n+1)\right]$$

$$=(-1)^{\frac{n(n+1)}{2}}\cdot(-1)^{\frac{n(n+1)}{2}}\prod_{i=1}^{n}i!$$

$$=(-1)^{n(n+1)}\prod_{i=1}^{n} i! = \prod_{i=1}^{n} i!.$$

（2）$D_{n+1}=\begin{vmatrix} a_1^n & a_1^{n-1}b_1 & a_1^{n-2}b_1^2 & \cdots & a_1b_1^{n-1} & b_1^n \\ a_2^n & a_2^{n-1}b_2 & a_2^{n-2}b_2^2 & \cdots & a_2b_2^{n-1} & b_2^n \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_{n+1}^n & a_{n+1}^{n-1}b_{n+1} & a_{n+1}^{n-2}b_{n+1} & \cdots & a_{n+1}b_{n+1}^{n-1} & b_{n+1}^n \end{vmatrix}$,

$(a_i \neq 0, i=1,2,\cdots,n)$.

【解题过程】

当$a_i\ (i=1,2,\cdots,n+1)\neq 0$时，将行列式的第$i$行除以$a_i^n\ (i=1,2,\cdots,n+1)$得：

$$\begin{vmatrix} a_1^n & a_1^{n-1}b_1 & a_1^{n-2}b_1^2 & \cdots & a_1b_1^{n-1} & b_1^n \\ a_2^n & a_2^{n-1}b_2 & a_2^{n-2}b_2^2 & \cdots & a_2b_2^{n-1} & b_2^n \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_{n+1}^n & a_{n+1}^{n-1}b_{n+1} & a_{n+1}^{n-2}b_{n+1}^2 & \cdots & a_{n+1}b_{n+1}^{n-1} & b_{n+1}^n \end{vmatrix}$$

$$= a_1^n a_2^n \cdots a_{n+1}^n \begin{vmatrix} 1 & \frac{b_1}{a_1} & \frac{b_1^2}{a_1^2} & \cdots & \frac{b_1^{n-1}}{a_1^{n-1}} & \frac{b_1^n}{a_1^n} \\ 1 & \frac{b_2}{a_2} & \frac{b_2^2}{a_2^2} & \cdots & \frac{b_2^{n-1}}{a_2^{n-1}} & \frac{b_2^n}{a_2^n} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & \frac{b_{n+1}}{a_{n+1}} & \frac{b_{n+1}^2}{a_{n+1}^2} & \cdots & \frac{b_{n+1}^{n-1}}{a_{n+1}^{n-1}} & \frac{b_{n+1}^n}{a_{n+1}^n} \end{vmatrix}$$

令$c_i=\dfrac{b_i}{a_i}\ (i=1,2,\cdots,n+1)$得：

$$\begin{vmatrix} a_1^n & a_1^{n-1}b_1 & a_1^{n-2}b_1^2 & \cdots & a_1b_1^{n-1} & b_1^n \\ a_2^n & a_2^{n-1}b_2 & a_2^{n-2}b_2^2 & \cdots & a_2b_2^{n-1} & b_2^n \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_{n+1}^n & a_{n+1}^{n-1}b_{n+1} & a_{n+1}^{n-2}b_{n+1}^2 & \cdots & a_{n+1}b_{n+1}^{n-1} & b_{n+1}^n \end{vmatrix}$$

$$= a_1^n a_2^n \cdots a_{n+1}^n \begin{vmatrix} 1 & c_1 & c_1^2 & \cdots & c_1^{n-1} & c_1^n \\ 1 & c_2 & c_2^2 & \cdots & c_2^{n-1} & c_2^n \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & c_{n+1} & c_{n+1}^2 & \cdots & c_{n+1}^{n-1} & c_{n+1}^n \end{vmatrix}$$

$$= a_1^n a_2^n \cdots a_{n+1}^n \begin{vmatrix} 1 & 1 & \cdots & 1 \\ c_1 & c_2 & \cdots & c_{n+1} \\ c_1^2 & c_2^2 & \cdots & c_{n+1}^2 \\ \vdots & \vdots & & \vdots \\ c_1^{n-1} & c_2^{n-1} & \cdots & c_{n+1}^{n-1} \\ c_1^n & c_2^n & \cdots & c_{n+1}^n \end{vmatrix}$$

$$= a_1^n a_2^n \cdots a_{n+1}^n \prod_{1\le j<i\le n+1}\left(c_i - c_j\right).$$

当$a_{n+1} = 0$时，

$$\begin{vmatrix} a_1^n & a_1^{n-1}b_1 & a_1^{n-2}b_1^2 & \cdots & a_1b_1^{n-1} & b_1^n \\ a_2^n & a_2^{n-1}b_2 & a_2^{n-2}b_2^2 & \cdots & a_2b_2^{n-1} & b_2^n \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_{n+1}^n & a_{n+1}^{n-1}b_{n+1} & a_{n+1}^{n-2}b_{n+1}^2 & \cdots & a_{n+1}b_{n+1}^{n-1} & b_{n+1}^n \end{vmatrix}$$

$$= \begin{vmatrix} a_1^n & a_1^{n-1}b_1 & a_1^{n-2}b_1^2 & \cdots & a_1b_1^{n-1} & b_1^n \\ a_2^n & a_2^{n-1}b_2 & a_2^{n-2}b_2^2 & \cdots & a_2b_2^{n-1} & b_2^n \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & b_{n+1}^n \end{vmatrix}$$

$$= (-1)^{n+1+n+1} b_{n+1}^n \begin{vmatrix} a_1^n & a_1^{n-1}b_1 & a_1^{n-2}b_1^2 & \cdots & a_1 b_1^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m-1}^n & a_{m-1}^{n-1}b_{m-1} & a_{m-1}^{n-2}b_{m-1}^2 & \cdots & a_{m-1}b_{m-1}^{n-1} \\ a_m^n & a_m^{n-1}b_m & a_m^{n-2}b_m^2 & \cdots & a_m b_m^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ a_n^n & a_n^{n-1}b_n & a_n^{n-2}b_n^2 & \cdots & a_n b_n^{n-1} \end{vmatrix}$$

$$= b_{n+1}^n a_1^n \cdots a_{m-1}^n a_m^n \cdots a_n^n \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ c_1 & c_2 & \cdots & c_{m-1} & c_m & \cdots & c_n \\ c_1^2 & c_2^2 & \cdots & c_{m-1}^2 & c_m^2 & \cdots & c_n^2 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ c_1^{n-1} & c_2^{n-1} & \cdots & c_{m-1}^{n-1} & c_m^{n-1} & \cdots & c_n^{n-1} \end{vmatrix}$$

$$= b_{n+1}^n a_1^n \cdots a_{m-1}^n a_m^n \cdots a_n^n \prod_{1 \le j < i \le n} \left(c_i - c_j\right).$$

11.用递归法计算下列行列式.

(1)已知 $a \neq b$, 计算

$$D_n = \begin{vmatrix} a+b & ab & 0 & \cdots & 0 & 0 & 0 \\ 1 & a+b & ab & \cdots & 0 & 0 & 0 \\ 0 & 1 & a+b & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & a+b & ab \\ 0 & 0 & 0 & \cdots & 0 & 1 & a+b \end{vmatrix}.$$

【解题过程】

$$D_n=\begin{vmatrix} a+b & ab & 0 & \cdots & 0 & 0 & 0 \\ 1 & a+b & ab & \cdots & 0 & 0 & 0 \\ 0 & 1 & a+b & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & a+b & ab \\ 0 & 0 & 0 & \cdots & 0 & 1 & a+b \end{vmatrix}$$

$$\overset{\text{按第一列展开}}{=}\left(a+b\right)D_{n-1}+\left(-1\right)^{2+1}\begin{vmatrix} ab & 0 & \cdots & 0 & 0 & 0 \\ 1 & a+b & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & a+b & ab \\ 0 & 0 & \cdots & 0 & 1 & a+b \end{vmatrix}$$

$$=\left(a+b\right)D_{n-1}-abD_{n-2}$$

$$\because D_n=\left(a+b\right)D_{n-1}-abD_{n-2}$$

$$\begin{aligned}\therefore D_n-aD_{n-1}&=b\left(D_{n-1}-aD_{n-2}\right)\\ &=b^2\left(D_{n-2}-aD_{n-3}\right)\\ &=\cdots\\ &=b^{n-3}\left(D_3-aD_2\right)\\ &=b^n \qquad\qquad (1)\end{aligned}$$

同理可知：$D_n-bD_{n-1}=a^n \qquad\qquad (2)$

$b\times(1)-a\times(2)$得：

$$bD_n-aD_n=b^{n+1}-a^{n+1}$$

当$a\neq b$时，$D_n=\dfrac{b^{n+1}-a^{n+1}}{b-a}$.

（2）$D_n=\begin{vmatrix} 2 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 2 \end{vmatrix}$.

【解题思路】

$$D_n=\begin{vmatrix} a+b & ab & 0 & \cdots & 0 & 0 & 0 \\ 1 & a+b & ab & \cdots & 0 & 0 & 0 \\ 0 & 1 & a+b & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & a+b & ab \\ 0 & 0 & 0 & \cdots & 0 & 1 & a+b \end{vmatrix} \overset{\text{按第一列展开}}{=}$$

$$(a+b)D_{n-1}+(-1)^{2+1}\begin{vmatrix} ab & 0 & \cdots & 0 & 0 & 0 \\ 1 & a+b & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & a+b & ab \\ 0 & 0 & \cdots & 0 & 1 & a+b \end{vmatrix}$$

$$=(a+b)D_{n-1}-abD_{n-2}$$

$$\because D_n=(a+b)D_{n-1}-abD_{n-2}$$

$$\begin{aligned}\therefore D_n-aD_{n-1}&=b\left(D_{n-1}-aD_{n-2}\right)\\&=b^2\left(D_{n-2}-aD_{n-3}\right)\\&=\cdots\\&=b^{n-3}\left(D_3-aD_2\right)\\&=b^n \qquad\qquad (1)\end{aligned}$$

同理可知：$D_n-bD_{n-1}=a^n \qquad\qquad (2)$

$b\times(1)-a\times(2)$得：

$$bD_n-aD_n=b^{n+1}-a^{n+1}$$

当$a=b$时，

$$\because D_n-aD_{n-1}=a^n$$

$$\therefore D_n=a^n+aD_{n-1}=a^n+a\left(aD_{n-2}+a^{n-1}\right)$$

$$=2a^n+a^2D_{n-2}$$

$$=\cdots$$

$$=(n-1)a^n+a^{n-1}D_1$$

$$=(n+1)a^n$$

【解题过程】

$$\because D_n=\begin{vmatrix} a+b & ab & 0 & \cdots & 0 & 0 & 0 \\ 1 & a+b & ab & \cdots & 0 & 0 & 0 \\ 0 & 1 & a+b & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & a+b & ab \\ 0 & 0 & 0 & \cdots & 0 & 1 & a+b \end{vmatrix},$$

$a=b$时,

$$D_n= \ =(n+1)a^n$$

$$\therefore D_n=\begin{vmatrix} 2 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 2 \end{vmatrix}=n+1.$$