# Chapter 12 Exam

# Online Calculus III

# Fall 2022

Name: KEY

1. Find the domain of $\vec{r}$ and determine the values of $t$ for which the function is continuous.

$$\vec{r}(t) = \sqrt{t}\hat{\imath} + \frac{1}{t-4}\hat{\jmath} + \hat{k}$$

$x(t) = \sqrt{t}$ Dom $(x(t))$ : $[0, \infty)$

$y(t) = \frac{1}{t-4}$ Dom $(y(t))$ : $(-\infty, 4) \cup (4, \infty)$

$z(t) = 1$ Dom $(z(t))$ : $(-\infty, \infty)$

Dom $(\vec{r}(t))$ : $[0, 4) \cup (4, \infty)$

Continuous on $[0, 4) \cup (4, \infty)$

2. Find the unit tangent vector $\vec{T}(t)$ and a set of parametric equations for the tangent line to the space curve at point $P$.

$$r(t) = \langle t, t, \sqrt{4-t^2} \rangle \text{ at } P(1, 1, \sqrt{3}).$$

$$\vec{r}'(t) = \left\langle 1, 1, \frac{-t}{\sqrt{4-t^2}} \right\rangle$$

$$\vec{T}(t) = \frac{\left\langle 1, 1, \frac{-t}{\sqrt{4-t^2}} \right\rangle}{\sqrt{2+\frac{t^2}{4-t^2}}} \qquad P(1,1,\sqrt{3}) \Rightarrow t=1$$

$$\vec{T}(1) = \sqrt{\frac{3}{7}} \left\langle 1, 1, \frac{-1}{\sqrt{3}} \right\rangle$$

Use $\left\langle 1, 1, -\frac{1}{\sqrt{3}} \right\rangle$

$$\begin{cases} x = 1+t \\ y = 1+t \\ z = \sqrt{3} - \frac{1}{\sqrt{3}}t \end{cases}$$

$-\frac{\sqrt{7}}{7} \rangle$

$$\left\langle \frac{\sqrt{21}}{7}, \frac{\sqrt{21}}{7}, \frac{-1}{\sqrt{7}} \right.$$

3. Evaluate the limit.

$$\lim_{t \to 0} \left( \frac{\sin 2t}{t} \hat{\imath} + e^{-t} \hat{\jmath} + e^{t} \hat{k} \right)$$

$$x(t) = \lim_{t \to 0} \frac{\sin 2t}{t} \overset{L'H}{=} \lim_{t \to 0} \frac{2\cos 2t}{1}$$

$$= 2$$

$$y(t) = \lim_{t \to 0} e^{-t} = 1$$

$$z(t) = \lim_{t \to 0} e^{t} = 1$$

$$\lim_{t \to 0} (\vec{r}(t)) = \langle 2, 1, 1 \rangle$$

4. Given

$$\vec{r}(t) = \sin t\, \hat{\boldsymbol{\imath}} + \cos t\, \hat{\boldsymbol{\jmath}} + t\, \hat{\boldsymbol{k}}$$

Find $r'(t)$ and $r''(t)$.

$$\vec{r}\,'(t) = \langle \cos t, -\sin t, 1 \rangle$$

$$\vec{r}\,''(t) = \langle -\sin t, -\cos t, 0 \rangle$$

5. Find the indefinite integral.

$$\int (\sin t\,\hat{\imath} + t\ln t\,\hat{\jmath} + t^2\,\hat{k})\,dt$$

$$x(t) = \int \sin t\,dt = -\cos t + C_1$$

$$y(t) = \int t\ln t\,dt \qquad u = \ln t \quad v = \tfrac{1}{2}t^2$$
$$du = \tfrac{1}{t}dt \quad dv = t\,dt$$

$$= \tfrac{1}{2}t^2\ln t - \tfrac{1}{2}\int t\,dt = \tfrac{1}{2}t^2\ln t - \tfrac{1}{4}t^2 + C_2$$

$$z(t) = \int t^2\,dt = \frac{t^3}{3} + C_3$$

$$\int (\sin t\,\hat{\imath} + t\ln t\,\hat{\jmath} + t^2\hat{k})\,dt =$$

$$\left\langle -\cos t + C_1,\ \tfrac{1}{2}t^2\ln t - \tfrac{1}{4}t^2 + C_2,\ \frac{t^3}{3} + C_3 \right\rangle$$

6. Find $r(t)$ for the given condition.

$$\vec{r}\,'(t) = 2t\,\hat{\imath} + e^{t}\hat{\jmath} + e^{-t}\,\hat{\mathbf{k}}$$

$$\vec{r}(0) = \hat{\imath} + 3\hat{\jmath} - 5\hat{k}$$

$$\vec{r}(t) = \int \vec{r}\,'(t)\,dt = \int \left(2t\hat{i} + e^{t}\hat{j} + e^{-t}\hat{k}\right) dt$$

$$x(t) = \int 2t\,dt = t^2 + C_1$$

$$y(t) = \int e^{t}\,dt = e^{t} + C_2$$

$$z(t) = \int e^{-t}\,dt = -e^{-t} + C_3$$

+7

$$\vec{r}(t) = \langle \underline{t^2 + C_1,\ e^{t} + C_2,\ -e^{-t} + C_3} \rangle \quad +8$$

$$\vec{r}(0) = \langle C_1,\ 1 + C_2,\ -1 + C_3 \rangle$$

$$\vec{r}(0) = \langle 1,\ 3,\ -5 \rangle$$

$$\Rightarrow C_1 = 1,\ C_2 = 2,\ C_3 = -4$$

$$\underline{\vec{r}(t) = \langle t^2 + 1,\ e^{t} + 2,\ -e^{-t} - 4 \rangle} \quad +2$$

7. Evaluate the definite integral.

$$\int_{-1}^{1} \left(t^3\hat{\imath} + \overset{\frac{1}{1+t^2}}{\cancel{\arcsin t}}\,\hat{\jmath} - t^2\hat{k}\right) dt$$

$$x(t) = \int_{-1}^{1} t^3\,dt = \frac{1}{4}t^4\Big]_{-1}^{1}$$

$$= \frac{1}{4}\left(1^4 - (-1)^4\right)$$

$$= \frac{1}{4}(1-1) = 0$$

$$y(t) = \int_{-1}^{1} \frac{1}{1+t^2}\,dt = \arctan t\Big]_{-1}^{1}$$

$$= \arctan(1) - \arctan(-1)$$

$$= \frac{\pi}{4} - \left(-\frac{\pi}{4}\right)$$

$$= \frac{\pi}{2}$$

$$z(t) = \int_{-1}^{1} -t^2\,dt = -\frac{1}{3}t^3\Big]_{-1}^{1} = -\frac{1}{3}\left(1^3 - (-1)^3\right)$$

$$= -\frac{1}{3}(1+1)$$

$$= -\frac{2}{3}$$

$$\int_{-1}^{1} \vec{r}(t)\,dt = \left\langle 0, \frac{\pi}{2}, -\frac{2}{3} \right\rangle$$

8. The position function describes the path of an object moving in space. Find the velocity, speed, and acceleration of the object.

$$\vec{r}(t) = \langle t, -\tan t, e^t \rangle.$$

$$\vec{v}(t) = \langle 1, -\sec^2 t, e^t \rangle$$

$$s(t) = \|\vec{v}(t)\|$$

$$= \sqrt{1 + \sec^4 t + e^{2t}}$$

$$\vec{a}(t) = \langle 0, -2(\sec t)(\sec t \tan t), e^t \rangle$$

9. Given the following position vector for a space curve

$$\vec{r}(t) = t\hat{\imath} + \frac{1}{t}\hat{\jmath}.$$

Find $\vec{T}, \vec{N}, \vec{B}$ at $t = 2$.

$$\vec{r}\,'(t) = \left\langle 1, -\frac{1}{t^2} \right\rangle$$

$$\|\vec{r}\,'(t)\| = \sqrt{1 + \frac{1}{t^4}} = \sqrt{\frac{t^4+1}{t^4}} = \frac{1}{t^2}\sqrt{t^4+1}$$

$$\vec{T}(t) = \frac{t^2}{\sqrt{t^4+1}}\left\langle 1, -\frac{1}{t^2} \right\rangle = \left\langle \frac{t^2}{\sqrt{t^4+1}}, \frac{-1}{\sqrt{t^4+1}} \right\rangle$$

$$\vec{N}(t) = \left\langle \frac{1}{\sqrt{t^4+1}}, \frac{t^2}{\sqrt{t^4+1}} \right\rangle \quad \left(\text{Knowing } T(t)\cdot N(t) = 0\right)$$

$$\vec{T}(2) = \left\langle \frac{4}{\sqrt{17}}, \frac{-1}{\sqrt{17}} \right\rangle \qquad \vec{N}(2) = \left\langle \frac{1}{\sqrt{17}}, \frac{4}{\sqrt{17}} \right\rangle$$

$$\vec{B}(2) = \vec{T}(2) \times \vec{N}(2) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{4}{\sqrt{17}} & \frac{-1}{\sqrt{17}} & 0 \\ \frac{1}{\sqrt{17}} & \frac{4}{\sqrt{17}} & 0 \end{vmatrix}$$

$$= \begin{vmatrix} \frac{-1}{\sqrt{17}} & 0 \\ \frac{4}{\sqrt{17}} & 0 \end{vmatrix}\hat{i} - \begin{vmatrix} \frac{4}{\sqrt{17}} & 0 \\ \frac{1}{\sqrt{17}} & 0 \end{vmatrix}\hat{j} + \begin{vmatrix} \frac{4}{\sqrt{17}} & \frac{1}{\sqrt{17}} \\ \frac{1}{\sqrt{17}} & \frac{4}{\sqrt{17}} \end{vmatrix}\hat{k}$$

$$= \hat{k}$$

10. Find the length of the space curve for the indicated interval $\left[0, \frac{\pi}{2}\right]$.

$$\vec{r}(t) = \langle 8\cos t, 8\sin t, t\rangle.$$

$x(t) = 8\cos t \rightarrow x'(t) = -8\sin t$

$y(t) = 8\sin t \rightarrow y'(t) = 8\cos t$

$z(t) = t \rightarrow z'(t) = 1$

$$\|\vec{r}\,'(t)\| = \sqrt{64\cos^2 t + 64\sin^2 t + 1}$$

$$= \sqrt{64+1} = \sqrt{65}$$

$$L = \int_0^{\pi/2} \|\vec{r}\,'(t)\|\, dt$$

$$= \int_0^{\pi/2} \sqrt{65}\, dt$$

$$= \frac{\pi}{2}\sqrt{65}$$

# Chapter 12 Exam Grade

| Question | Score |
|---|---|
| 1 | /10 |
| 2 | /10 |
| 3 | /10 |
| 4 | /10 |
| 5 | /10 |
| 6 | /10 |
| 7 | /10 |
| 8 | /10 |
| 9 | /10 |
| 10 | /10 |
| **TOTAL** | **/100** |