Chapter 17

Vector Calculus (II)

Shuwei Chen

swchen@swjtu.edu.cn

17.5

Divergence and Curl

The Divergence

- The circulation form of Green's Theorem relates a line integral over a simple closed oriented curve in the plane to a double integral over the enclosed region.

- The flux form of Green's Theorem relates a line integral over a simple closed oriented curve in the plane to a double integral over the enclosed region.

The Divergence

DEFINITION Divergence of a Vector Field

The **divergence** of a vector field $\mathbf{F} = \langle f, g, h \rangle$ that is differentiable on a region of $\mathbb{R}^3$ is

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}.$$

If $\nabla \cdot \mathbf{F} = 0$, the vector field is **source free**.

The *del operator (operation)* ∇ to define the gradient:

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

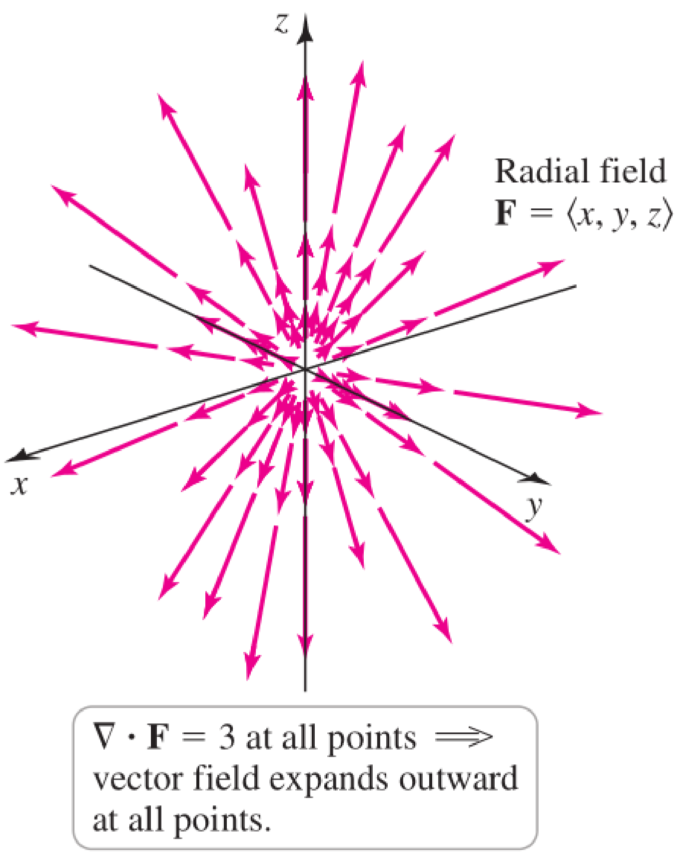
$$\nabla f = \mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}\frac{\partial f}{\partial z} = \langle f_x, f_y, f_z \rangle$$

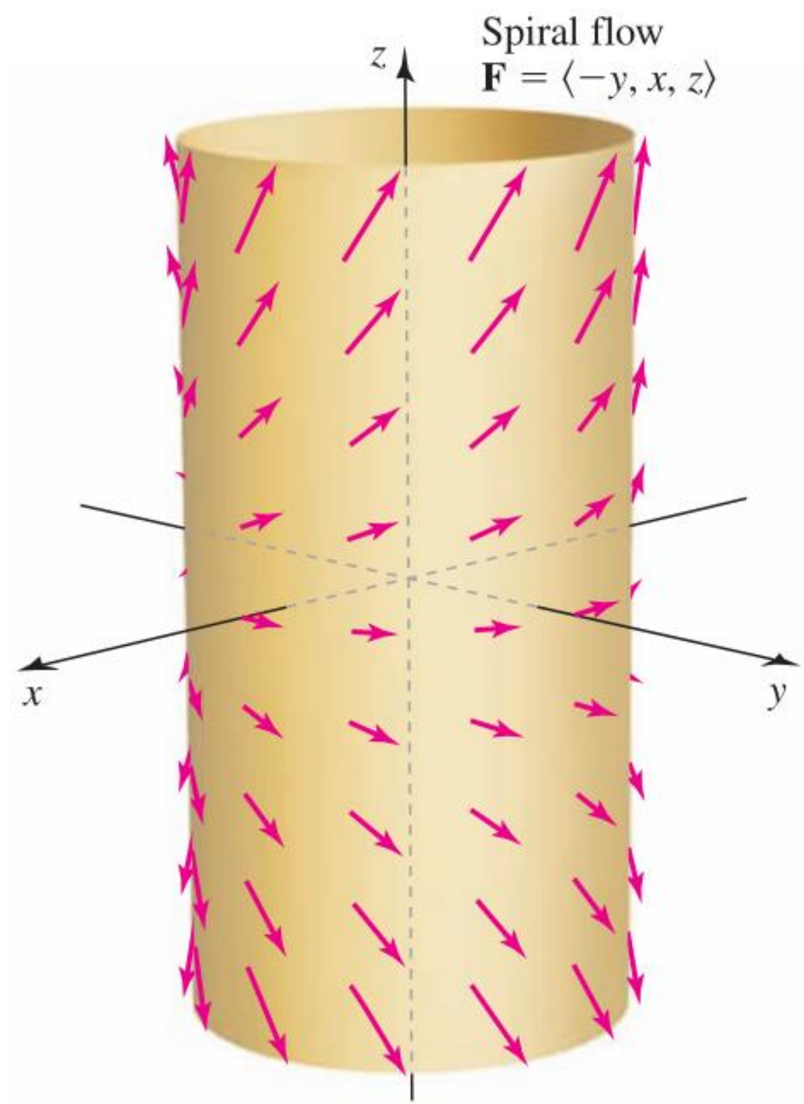
EXAMPLE 1 **Computing the divergence** Compute the divergence of the following vector fields.

a. $\mathbf{F} = \langle x, y, z \rangle$ (a radial field)

b. $\mathbf{F} = \langle -y, x - z, y \rangle$ (a rotation field)

c. $\mathbf{F} = \langle -y, x, z \rangle$ (a spiral flow)





Divergence of a Radial Vector field

EXAMPLE 2 **Divergence of a radial field** Compute the divergence of the radial vector field

$$\mathbf{F} = \frac{\mathbf{r}}{|\mathbf{r}|} = \frac{\langle x, y, z \rangle}{\sqrt{x^2 + y^2 + z^2}}.$$

THEOREM 8 Divergence of Radial Vector Fields

For a real number p, the divergence of the radial vector field

$$\mathbf{F} = \frac{\mathbf{r}}{|\mathbf{r}|^p} = \frac{\langle x, y, z \rangle}{(x^2 + y^2 + z^2)^{p/2}} \quad \text{is} \quad \nabla \cdot \mathbf{F} = \frac{3 - p}{|\mathbf{r}|^p}.$$

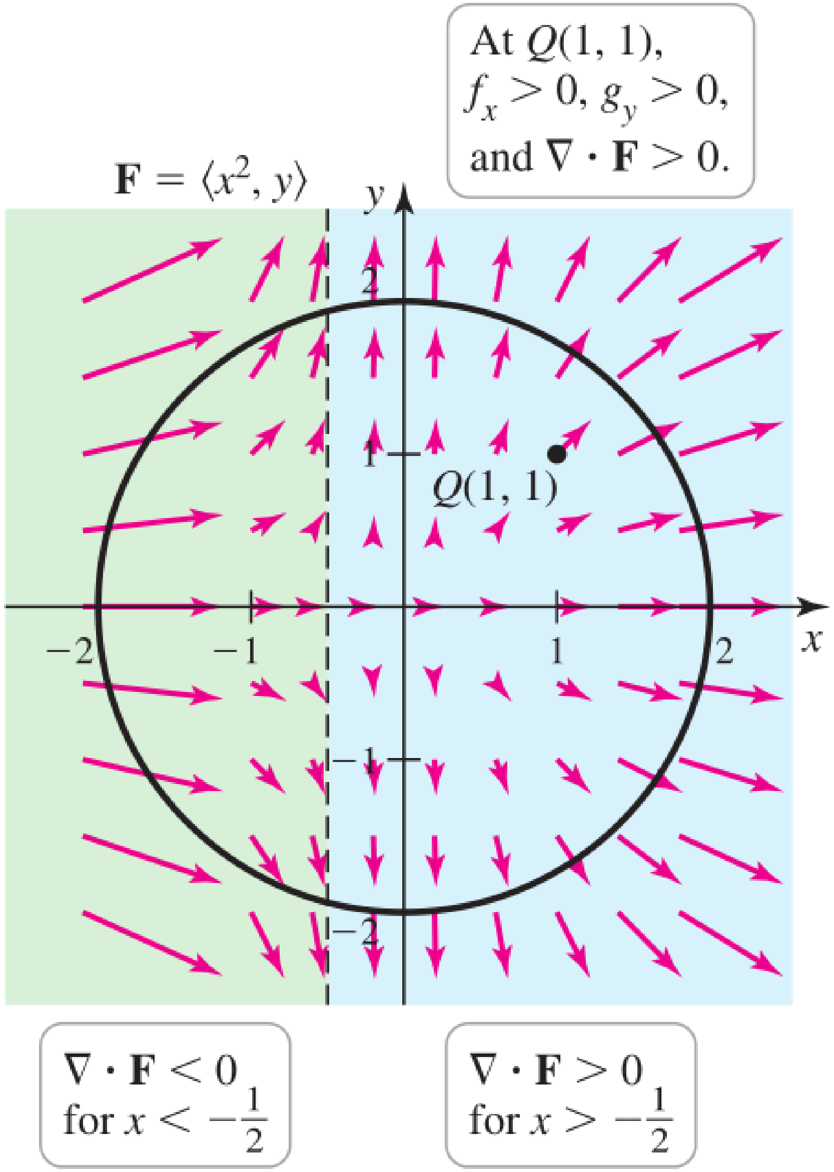
EXAMPLE 3 **Divergence from a graph** To gain some intuition about the divergence, consider the two-dimensional vector field $\mathbf{F} = \langle f, g \rangle = \langle x^2, y \rangle$ and a circle C of radius 2 centered at the origin (Figure 14.39).

a. Without computing it, determine whether the two-dimensional divergence is positive or negative at the point $Q(1, 1)$. Why?

b. Confirm your conjecture in part (a) by computing the two-dimensional divergence at Q.

c. Based on part (b), over what regions within the circle is the divergence positive and over what regions within the circle is the divergence negative?

d. By inspection of the figure, on what part of the circle is the flux across the boundary outward? Is the net flux out of the circle positive or negative?



At $Q(1, 1)$,
$f_x > 0$, $g_y > 0$,
and $\nabla \cdot \mathbf{F} > 0$.
$\mathbf{F} = \langle x^2, y \rangle$
y
2
1
$Q(1, 1)$
-2
-1
1
2
x
-1
-2
$\nabla \cdot \mathbf{F} < 0$
for $x < -\frac{1}{2}$
$\nabla \cdot \mathbf{F} > 0$
for $x > -\frac{1}{2}$

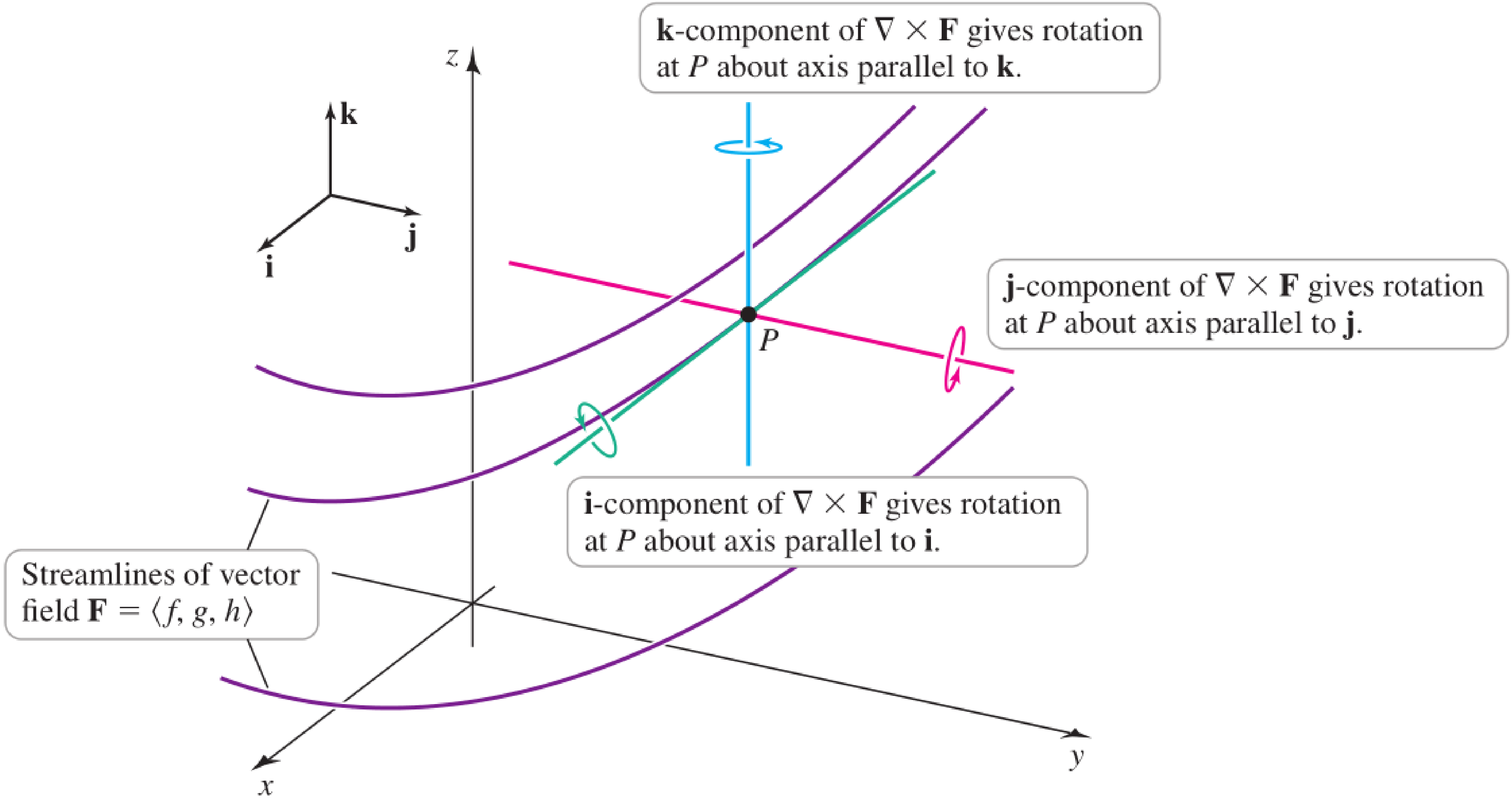
The Curl

The 3-dimensional curl is the cross product:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ f & g & h \end{vmatrix} \begin{array}{l} \leftarrow \text{Unit vectors} \\ \leftarrow \text{Components of } \nabla \\ \leftarrow \text{Components of } \mathbf{F} \end{array}$$

$$= \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\mathbf{i} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)\mathbf{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\mathbf{k}.$$

The **k**-component of the curl $(g_x - f_y)$ is the two-dimensional curl, which gives the rotation in the xy-plane at a point.
Similar for **i**- and **j**-components



k-component of ∇ × F gives rotation at P about axis parallel to k.
j-component of ∇ × F gives rotation at P about axis parallel to j.
i-component of ∇ × F gives rotation at P about axis parallel to i.
Streamlines of vector field F = ⟨f, g, h⟩
z
k
i
j
P
x
y

DEFINITION Curl of a Vector Field

The **curl** of a vector field $\mathbf{F} = \langle f, g, h \rangle$ that is differentiable on a region of $\mathbb{R}^3$ is

$$\begin{aligned}\nabla \times \mathbf{F} &= \operatorname{curl} \mathbf{F} \\ &= \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\mathbf{i} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)\mathbf{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\mathbf{k}.\end{aligned}$$

If $\nabla \times \mathbf{F} = \mathbf{0}$, the vector field is **irrotational**.

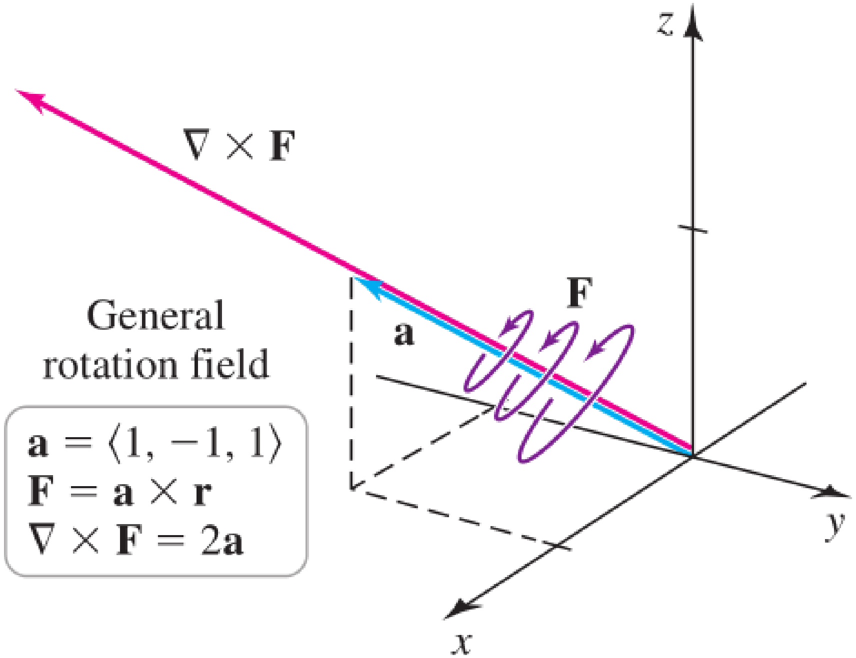
Curl of a General Rotation Vector Field

$$\mathbf{F} = \mathbf{a} \times \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix}$$

$$= (a_2 z - a_3 y)\,\mathbf{i} + (a_3 x - a_1 z)\,\mathbf{j} + (a_1 y - a_2 x)\,\mathbf{k}.$$

- **F** is a general rotation field in three dimensions
- **F** is the superposition of three rotation fields with axes in the **i**-, **j**-, and **k**-directions.
- Result in a single rotation field with an axis in the direction of **a**.
- Degenerate to two-dimensional rotation field $\langle -y, x\rangle$ if $a_1 = a_2 = 0$ and $a_2 = 1$.

- $\nabla \cdot \mathbf{F} = 0$, the divergence of a general rotation field is zero.
- The vector field $\mathbf{F} = \mathbf{a} \times \mathbf{r}$ is indeed a rotation field that circles the vector $\mathbf{a}$ in a counterclockwise direction looking along the length of $\mathbf{a}$ from head to tail.
- $\nabla \times \mathbf{F} = 2\mathbf{a}$, the curl of the general rotation field is in the direction of the axis of rotation $\mathbf{a}$.
- If $\mathbf{F}$ is a velocity field, then $|\mathbf{a}|$ is the constant angular speed of rotation of the field, denoted ω.



General Rotation Vector Field

The **general rotation vector field** is $\mathbf{F} = \mathbf{a} \times \mathbf{r}$, where the nonzero constant vector $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ is the axis of rotation and $\mathbf{r} = \langle x, y, z \rangle$. For all nonzero choices of $\mathbf{a}$, $|\nabla \times \mathbf{F}| = 2|\mathbf{a}|$ and $\nabla \cdot \mathbf{F} = 0$. If $\mathbf{F}$ is a velocity field, then the constant angular speed of the field is

$$\omega = |\mathbf{a}| = \frac{1}{2}|\nabla \times \mathbf{F}|.$$

EXAMPLE 4 **Curl of a rotation field** Compute the curl of the rotational field $\mathbf{F} = \mathbf{a} \times \mathbf{r}$, where $\mathbf{a} = \langle 1, -1, 1 \rangle$ and $\mathbf{r} = \langle x, y, z \rangle$ (Figure 14.41). What is the direction and the magnitude of the curl?

Working with Divergence and Curl

Given a real number c and differentiable vector fields $\mathbf{F}$ and $\mathbf{G}$

Divergence Properties

$$\nabla \cdot (\mathbf{F} + \mathbf{G}) = \nabla \cdot \mathbf{F} + \nabla \cdot \mathbf{G}$$

$$\nabla \cdot (c\mathbf{F}) = c(\nabla \cdot \mathbf{F})$$

Curl Properties

$$\nabla \times (\mathbf{F} + \mathbf{G}) = (\nabla \times \mathbf{F}) + (\nabla \times \mathbf{G})$$

$$\nabla \times (c\mathbf{F}) = c(\nabla \times \mathbf{F})$$

THEOREM 9 Curl of a Conservative Vector Field

Suppose that $\mathbf{F}$ is a conservative vector field on an open region D of $\mathbb{R}^3$. Let $\mathbf{F} = \nabla\varphi$, where φ is a potential function with continuous second partial derivatives on D. Then $\nabla \times \mathbf{F} = \nabla \times \nabla\varphi = \mathbf{0}$: The curl of the gradient is the zero vector and $\mathbf{F}$ is irrotational.

Proof: We must calculate $\nabla \times \nabla\varphi$:

$$\nabla \times \nabla\varphi = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \varphi_x & \varphi_y & \varphi_z \end{vmatrix} = \underbrace{(\varphi_{zy} - \varphi_{yz})}_{0}\mathbf{i} + \underbrace{(\varphi_{xz} - \varphi_{zx})}_{0}\mathbf{j} + \underbrace{(\varphi_{yx} - \varphi_{xy})}_{0}\mathbf{k} = \mathbf{0}.$$

THEOREM 10 Divergence of the Curl

Suppose that $\mathbf{F} = \langle f, g, h \rangle$, where f, g, and h have continuous second partial derivatives. Then $\nabla \cdot (\nabla \times \mathbf{F}) = 0$: The divergence of the curl is zero.

Proof: Again, a calculation is needed:

$$\begin{aligned}
&\nabla \cdot (\nabla \times \mathbf{F}) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) \\
&= \underbrace{(h_{yx} - h_{xy})}_{0} + \underbrace{(g_{xz} - g_{zx})}_{0} + \underbrace{(f_{zy} - f_{yz})}_{0} = 0.
\end{aligned}$$

The gradient, the divergence, and the curl may be combined in many ways.

One important combination is the divergence of the gradient $\nabla \cdot \nabla u$, where u is a scalar-valued function.

Denoted $\nabla^2 u$ and is called the **Laplacian** of u.

$$\nabla \cdot \nabla u = \frac{\partial}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial}{\partial y}\frac{\partial u}{\partial y} + \frac{\partial}{\partial z}\frac{\partial u}{\partial z} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}.$$

THEOREM 11 Product Rule for the Divergence

Let u be a scalar-valued function that is differentiable on a region D and let $\mathbf{F}$ be a vector field that is differentiable on D. Then

$$\nabla \cdot (u\mathbf{F}) = \nabla u \cdot \mathbf{F} + u(\nabla \cdot \mathbf{F}).$$

EXAMPLE 5 **More properties of radial fields** Let $\mathbf{r} = \langle x, y, z \rangle$ and let $\varphi = \dfrac{1}{|\mathbf{r}|} = (x^2 + y^2 + z^2)^{-1/2}$ be a potential function.

a. Find the associated gradient field $\mathbf{F} = \nabla\left(\dfrac{1}{|\mathbf{r}|}\right)$.

b. Compute $\nabla \cdot \mathbf{F}$.

Summary of Properties of Conservative Vector Fields

Properties of a Conservative Vector Field

Let $\mathbf{F}$ be a conservative vector field whose components have continuous second partial derivatives on an open connected region D in $\mathbb{R}^3$. Then $\mathbf{F}$ has the following equivalent properties.

1. There exists a potential function φ such that $\mathbf{F} = \nabla\varphi$ (definition).
2. $\int_C \mathbf{F} \cdot d\mathbf{r} = \varphi(B) - \varphi(A)$ for all points A and B in D and all piecewise-smooth oriented curves C in D from A to B.
3. $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ on all simple piecewise-smooth closed oriented curves C in D.
4. $\nabla \times \mathbf{F} = \mathbf{0}$ at all points of D.

17.6

Surface Integrals

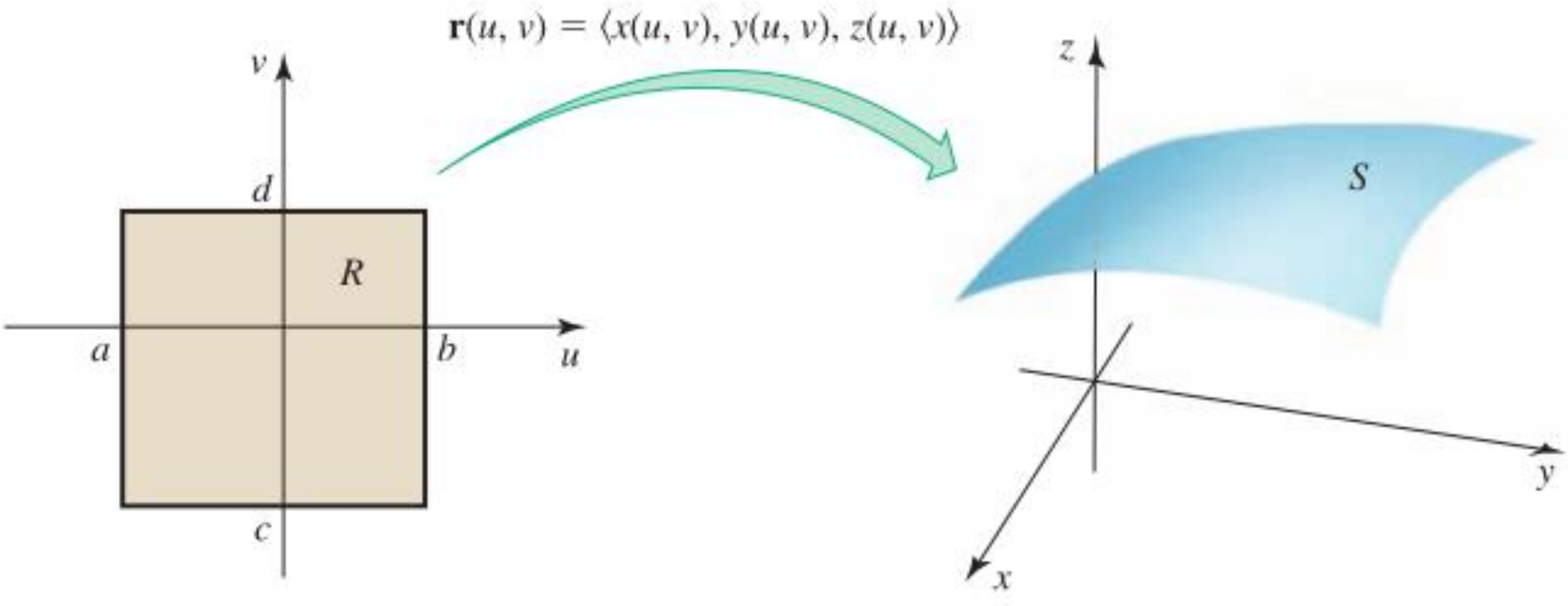
Parallel Concepts	
Curves	**Surfaces**
Arc length	Surface area
Line integrals	Surface integrals
One-parameter description	Two-parameter description

Parameterized Surfaces

Recall: A curve in $\mathbb{R}^2$ is defined parametrically by

$$\mathbf{r}(t) = \langle x(t), y(t) \rangle, \text{ for } a \le t \le b$$

For surface in $\mathbb{R}^3$, we need ***two*** parameters and *three* dependent variables $\quad \mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$



A rectangle in the *uv*-plane is mapped to a surface in *xyz*-space.

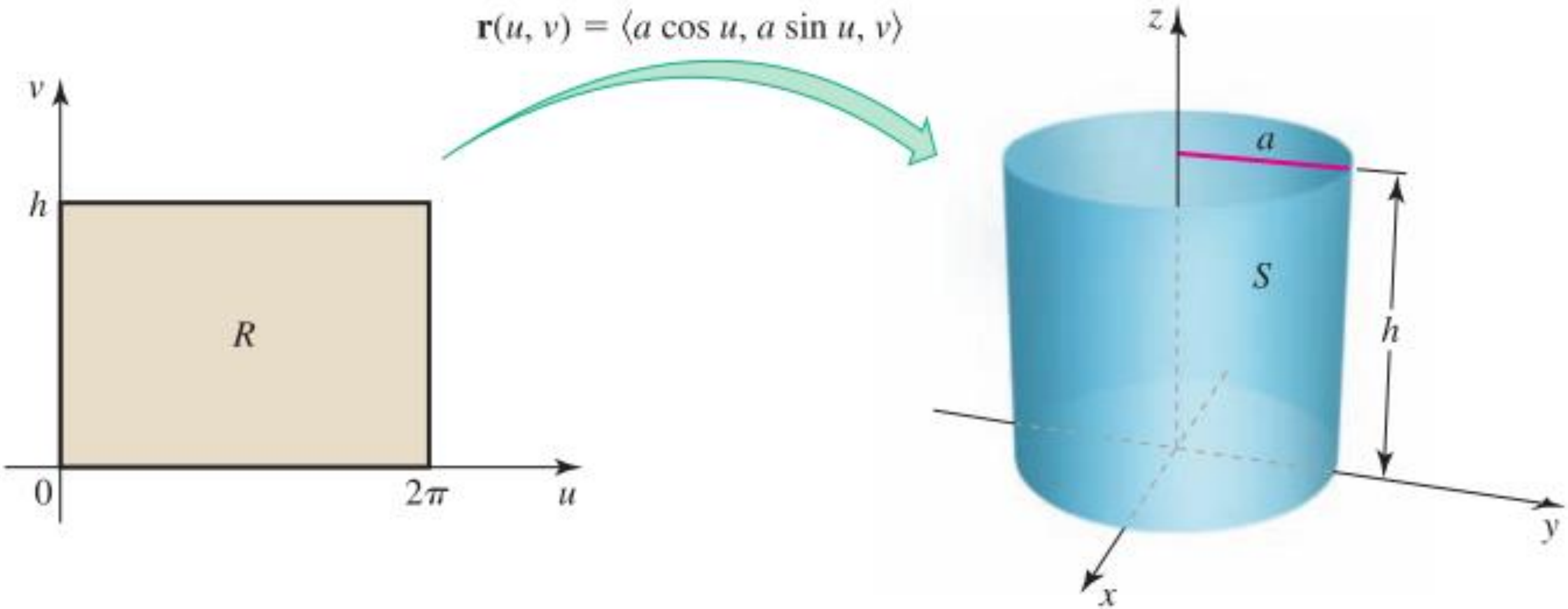
Three Typical Surfaces in Parametric Form: Cylinders

$$\{(x, y, z): x = a\cos\theta\,, y = a\sin\theta\,, 0 \le \theta \le 2\pi, 0 \le z \le h\}$$

Let $u = \theta, v = z$, a parametric description of the cylinder is

$$\mathbf{r}(u, v) = \langle a\cos u\,, a\sin u\,, v\rangle$$

Where $0 \le u \le 2\pi, 0 \le v \le h$



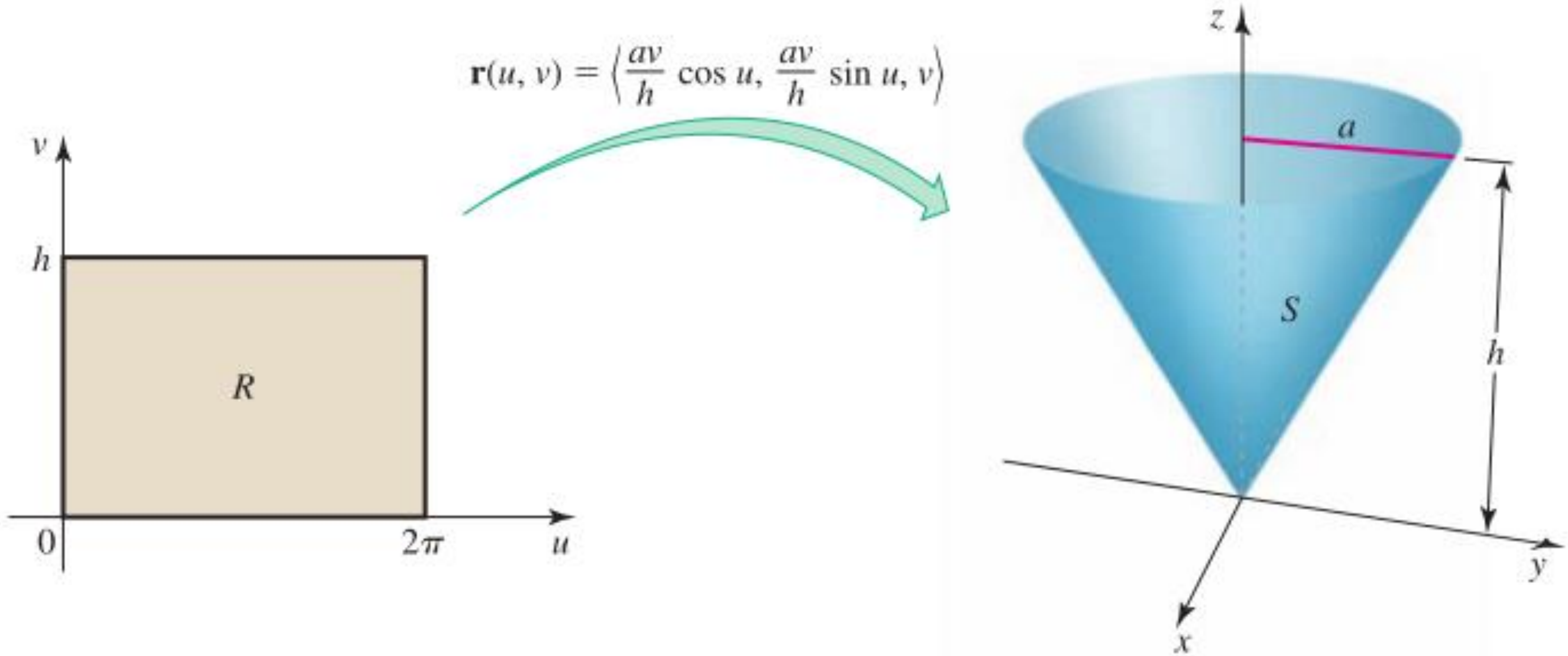
Cones

$$\{(r,\theta,z): 0 \le r \le a, 0 \le \theta \le 2\pi, z = rh/a\}$$

Let $u = \theta, v = z$, parametric description of the conical surface is

$$\mathbf{r}(u,v) = \left\langle \frac{av}{h}\cos u\,, \frac{av}{h}\sin u\,, v \right\rangle$$

Where $0 \le u \le 2\pi, 0 \le v \le h$



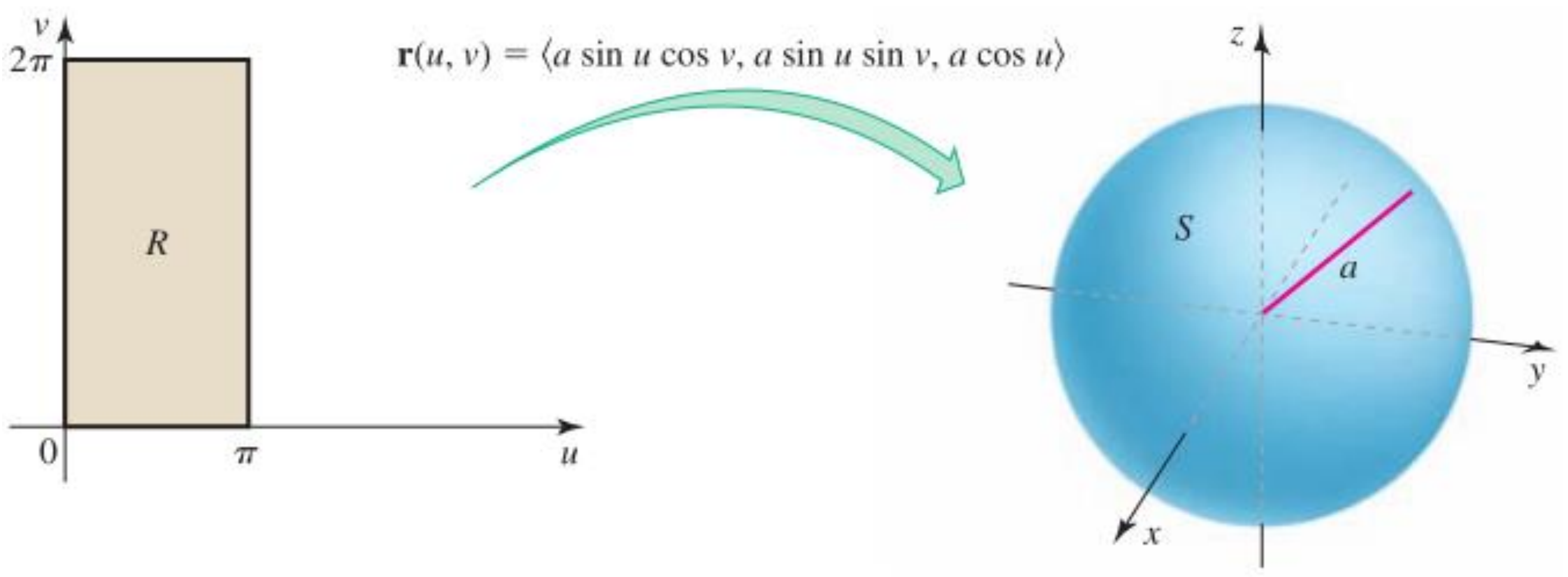
Spheres

$$\{(\rho, \varphi, \theta) : \rho = a, 0 \le \varphi \le \pi, 0 \le \theta \le 2\pi\}$$

Let $u = \varphi, v = \theta$, parametric description of the conical surface is

$$\mathbf{r}(u, v) = \langle a \sin u \cos v\,, a \sin u \sin v\,, a \cos u \rangle$$

Where $0 \le u \le \pi, 0 \le v \le h$



EXAMPLE 1 **Parametric surfaces** Find parametric descriptions for the following surfaces.

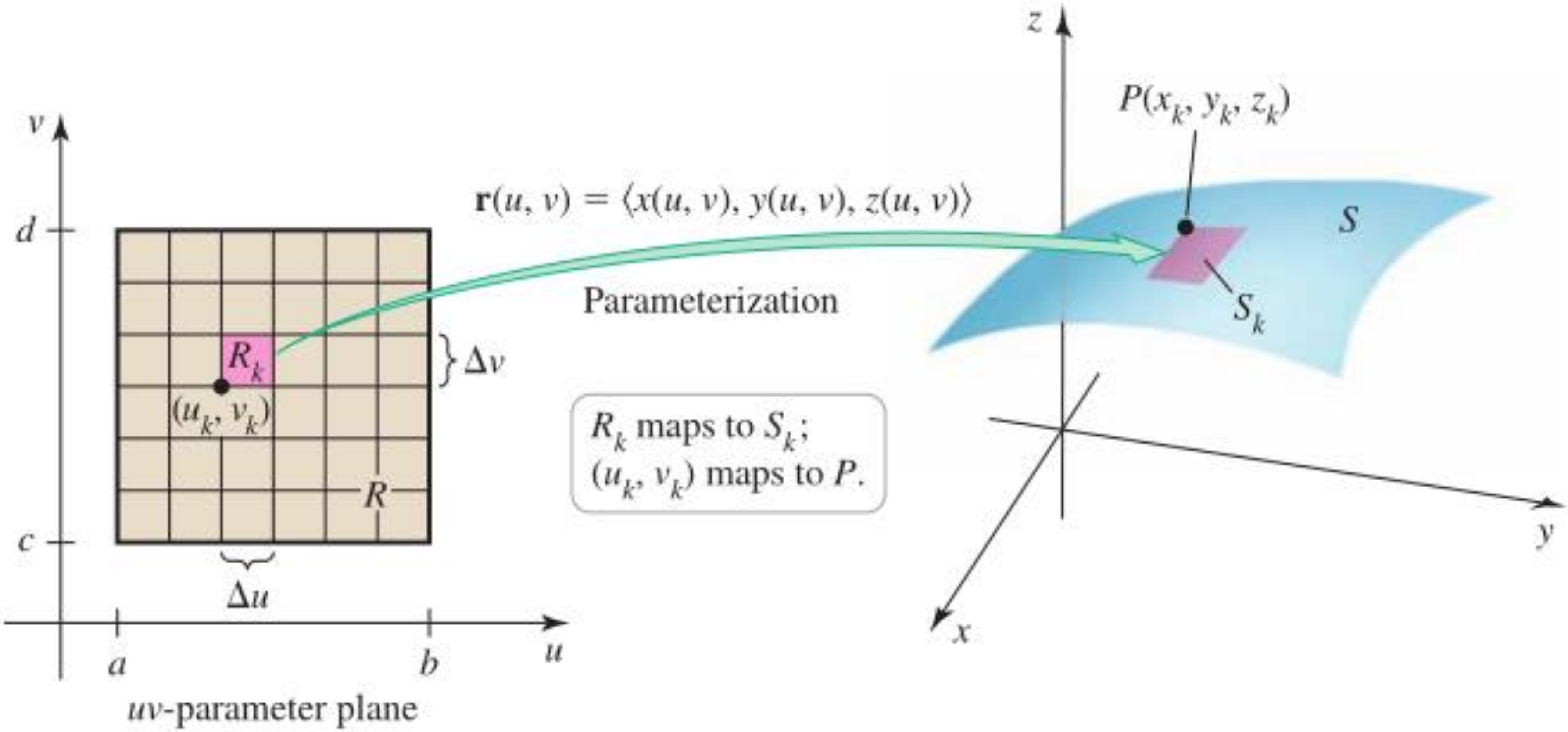
a. The plane $3x - 2y + z = 2$

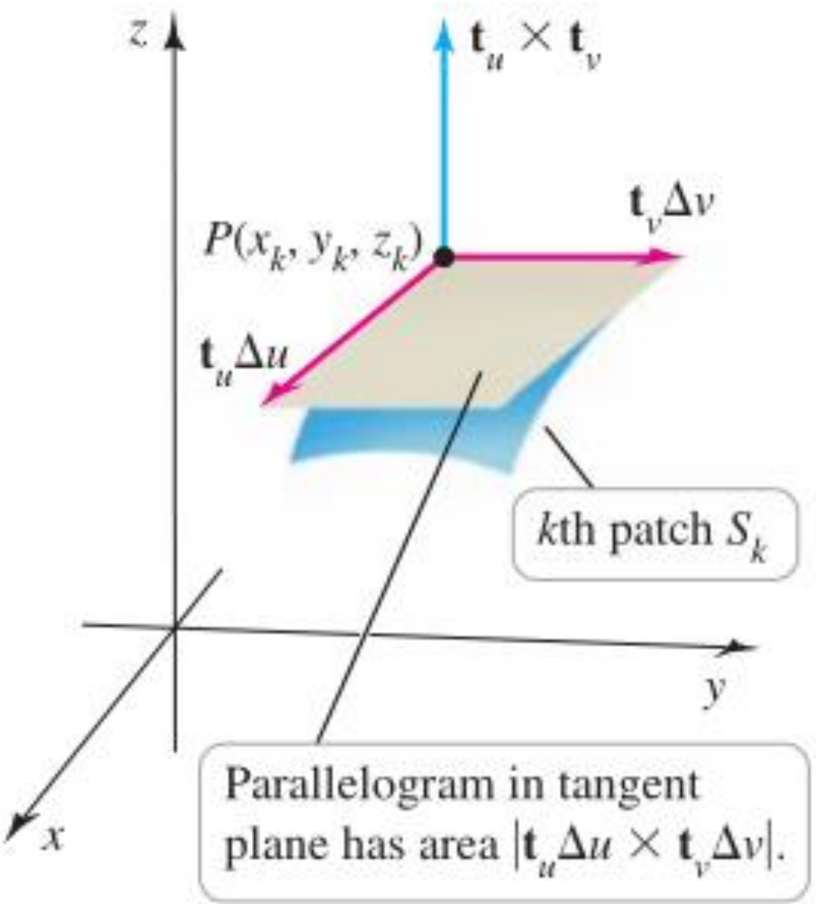
b. The paraboloid $z = x^2 + y^2$, for $0 \leq z \leq 9$

Surface Integrals of Scalar-Valued Functions

A scalar-valued function f defined on a smooth parameterized surface S on the region $R = \{(u, v): a \le u \le b, c \le v \le d\}$:

$$\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$$





- $\mathbf{t}_u$ is a vector tangent to the surface corresponding to a change in u with v constant in the uv-plane.
- $\mathbf{t}_v$ is a vector tangent to the surface corresponding to a change in v with u constant in the uv-plane.

$$\mathbf{t}_u = \frac{\partial \mathbf{r}}{\partial u} = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle$$

$$\mathbf{t}_v = \frac{\partial \mathbf{r}}{\partial v} = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle$$

$$|\mathbf{t}_u \Delta u \times \mathbf{t}_v \Delta v| = |\mathbf{t}_u \times \mathbf{t}_v| \Delta u \Delta v \approx \Delta S_k$$

DEFINITION Surface Integral of Scalar-Valued Functions on Parameterized Surfaces

Let f be a continuous scalar-valued function on a smooth surface S given parametrically by $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$, where u and v vary over $R = \{(u, v): a \le u \le b, c \le v \le d\}$. Assume also that the tangent vectors $\mathbf{t}_u = \frac{\partial \mathbf{r}}{\partial u} = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle$ and $\mathbf{t}_v = \frac{\partial \mathbf{r}}{\partial v} = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle$ are continuous on R and the normal vector $\mathbf{t}_u \times \mathbf{t}_v$ is nonzero on R. Then the **surface integral** of f over S is

$$\iint_S f(x, y, z)\, dS = \iint_R f(x(u, v), y(u, v), z(u, v))\, |\mathbf{t}_u \times \mathbf{t}_v|\, dA.$$

If $f(x, y, z) = 1$, this integral equals the surface area of S.

EXAMPLE 2 **Surface area of a cylinder and sphere** Find the surface area of the following surfaces.

a. A cylinder with radius $a > 0$ and height h (excluding the circular ends)

b. A sphere of radius a

a.

$$\mathbf{t}_u \times \mathbf{t}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix}$$ Definition of cross product

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a \sin u & a \cos u & 0 \\ 0 & 0 & 1 \end{vmatrix}$$ Evaluate derivatives.

$$= \langle a \cos u, a \sin u, 0 \rangle.$$ Compute cross product.

$$\iint_S 1\, dS = \iint_R \underbrace{|\mathbf{t}_u \times \mathbf{t}_v|}_{a}\, dA = \int_0^{2\pi} \int_0^h a\, dv\, du = 2\pi a h,$$

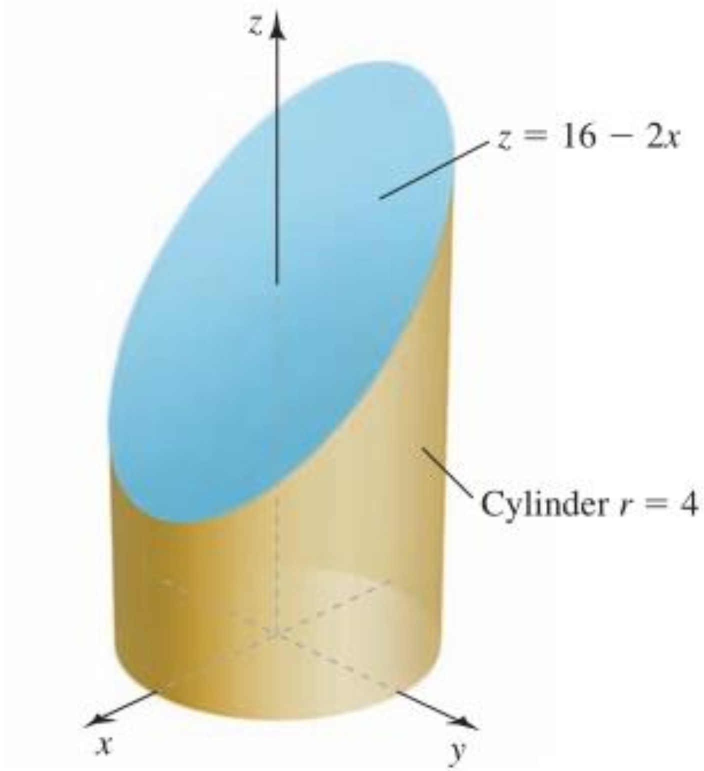
b. Sphere of radius a

$$\mathbf{r}(u, v) = \langle a \sin u \cos v\,, a \sin u \sin v\,, a \cos u \rangle$$

$$\begin{aligned}
\mathbf{t}_u \times \mathbf{t}_v &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a \cos u \cos v & a \cos u \sin v & -a \sin u \\ -a \sin u \sin v & a \sin u \cos v & 0 \end{vmatrix} \\
&= \langle a^2 \sin^2 u \cos v, a^2 \sin^2 u \sin v, a^2 \sin u \cos u \rangle .
\end{aligned}$$

$$\iint_S 1\, dS = \iint_R \underbrace{|\mathbf{t}_u \times \mathbf{t}_v|}_{a^2 \sin u}\, dA = \int_0^{2\pi} \int_0^{\pi} a^2 \sin u \, du \, dv = 4\pi a^2,$$

EXAMPLE 3 **Surface area of a partial cylinder** Find the surface area of the cylinder $\{(r, \theta): r = 4, 0 \le \theta \le 2\pi\}$ between the planes $z = 0$ and $z = 16 - 2x$ (excluding the top and bottom surfaces).



Sliced cylinder is generated by $\mathbf{r}(u, v) = \langle 4\cos u, 4\sin u, v\rangle$, where $0 \le u \le 2\pi$, $0 \le v \le 16 - 8\cos u$.

$$\iint_S 1\, dS = \iint_R \underbrace{|\mathbf{t}_u \times \mathbf{t}_v|}_{4}\, dA$$

$$= \int_0^{2\pi} \int_0^{16 - 8\cos u} 4\, dv\, du$$

EXAMPLE 4 **Average temperature on a sphere** The temperature on the surface of a sphere of radius a varies with latitude according to the function $T(\varphi, \theta) = 10 + 50 \sin \varphi$, for $0 \leq \varphi \leq \pi$ and $0 \leq \theta \leq 2\pi$ (φ and θ are spherical coordinates, so the temperature is $10°$ at the poles, increasing to $60°$ at the equator). Find the average temperature over the sphere.

$$\iint_S (10 + 50 \sin u)\, dS = \iint_R (10 + 50 \sin u) \underbrace{|\mathbf{t}_u \times \mathbf{t}_v|}_{a^2 \sin u} \, dA$$

$$= \int_0^{\pi} \int_0^{2\pi} (10 + 50 \sin u) a^2 \sin u \, dv \, du$$

Surface Integrals on Explicitly Defined Surfaces

For smooth surface S is defined not parametrically, but explicitly, in the form $z = g(x, y)$ over a region R in the xy-plane.

Simply define the parameters to be $u = x$ and $v = y$.

Then, $\mathbf{t}_x = \langle 1,0, z_x \rangle$, $\mathbf{t}_y = \langle 0,1, z_y \rangle$, and the normal vector is

$$\mathbf{t}_x \times \mathbf{t}_y = \langle -z_x, -z_y, 1 \rangle$$

It follows that

$$|\mathbf{t}_x \times \mathbf{t}_y| = |\langle -z_x, -z_y, 1 \rangle| = \sqrt{z_x^2 + z_y^2 + 1}$$

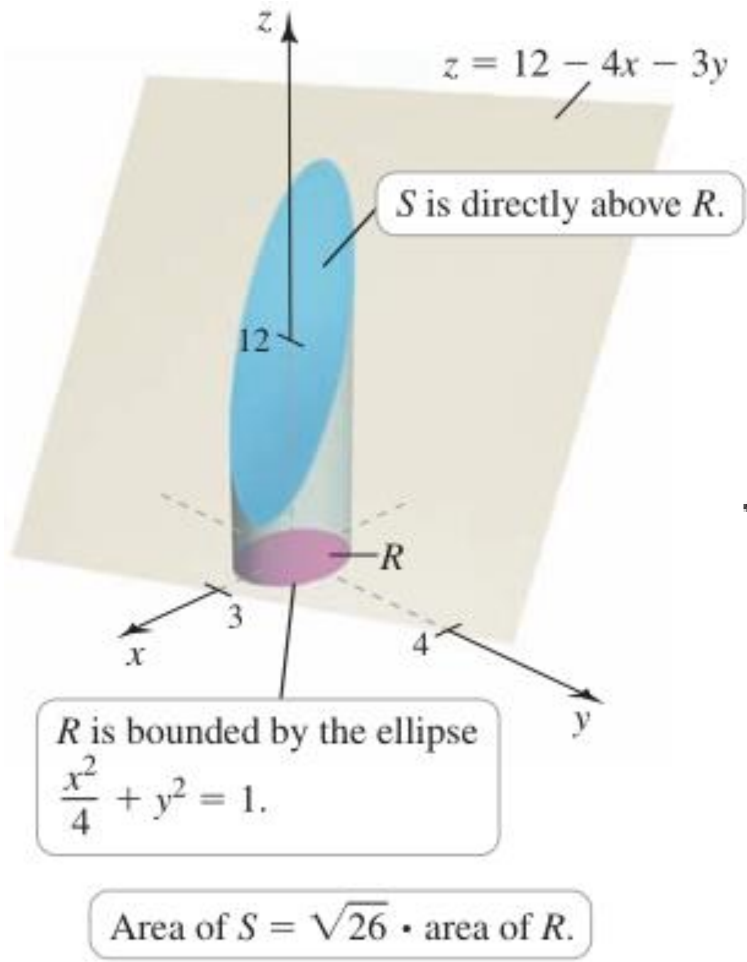
THEOREM 12 Evaluation of Surface Integrals of Scalar-Valued Functions on Explicitly Defined Surfaces

Let f be a continuous function on a smooth surface S given by $z = g(x, y)$, for (x, y) in a region R. The surface integral of f over S is

$$\iint_S f(x, y, z)\, dS = \iint_R f(x, y, g(x, y))\sqrt{z_x^2 + z_y^2 + 1}\, dA.$$

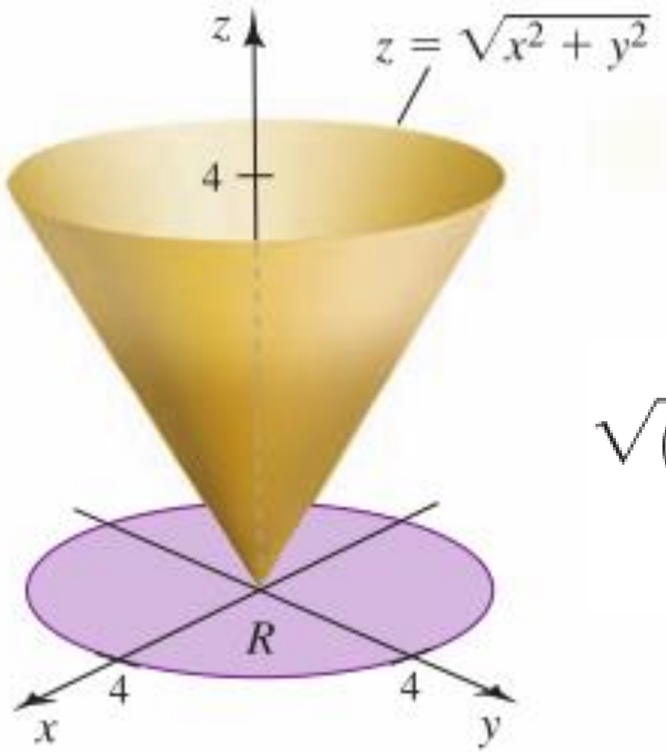
If $f(x, y, z) = 1$, the surface integral equals the area of the surface.

EXAMPLE 5 **Area of a roof over an ellipse** Find the area of the surface S that lies in the plane $z = 12 - 4x - 3y$ directly above the region R bounded by the ellipse $x^2/4 + y^2 = 1$ (Figure 53).



$$\iint_S 1\, dS = \iint_R \underbrace{\sqrt{z_x^2 + z_y^2 + 1}}_{\sqrt{26}}\, dA = \sqrt{26} \iint_R dA.$$

EXAMPLE 6 **Mass of a conical sheet** A thin conical sheet is described by the surface $z = (x^2 + y^2)^{1/2}$, for $0 \le z \le 4$. The density of the sheet in g/cm^2 is $\rho = f(x, y, z) = (8 - z)$ (decreasing from 8 g/cm^2 at the vertex to 4 g/cm^2 at the top of the cone; Figure 54). What is the mass of the cone?



Density function of sheet is $\rho = 8 - z$.

$$\sqrt{z_x^2 + z_y^2 + 1} =$$

$$\sqrt{(x/z)^2 + (y/z)^2 + 1} = \sqrt{\underbrace{\frac{x^2 + y^2}{z^2}}_{1} + 1} = \sqrt{2}.$$

Summary of essential relationships for the explicit and parametric descriptions of cylinders, cones, spheres, and paraboloids.

		Explicit Description $z = g(x, y)$
Surface	**Equation**	**Normal vector; magnitude** $\pm\langle -z_x, -z_y, 1\rangle; \lvert\langle -z_x, -z_y, 1\rangle\rvert$
Cylinder	$x^2 + y^2 = a^2$, $0 \le z \le h$	$\langle x, y, 0\rangle; a$
Cone	$z^2 = x^2 + y^2$, $0 \le z \le h$	$\langle x/z, y/z, -1\rangle; \sqrt{2}$
Sphere	$x^2 + y^2 + z^2 = a^2$	$\langle x/z, y/z, 1\rangle; a/z$
Paraboloid	$z = x^2 + y^2$, $0 \le z \le h$	$\langle 2x, 2y, -1\rangle; \sqrt{1 + 4(x^2 + y^2)}$

Parametric Description	
Equation	**Normal vector; magnitude** $\mathbf{t}_u \times \mathbf{t}_v; \lvert \mathbf{t}_u \times \mathbf{t}_v \rvert$
$\mathbf{r} = \langle a\cos u, a\sin u, v\rangle$, $0 \le u \le 2\pi, 0 \le v \le h$	$\langle a\cos u, a\sin u, 0\rangle; a$
$\mathbf{r} = \langle v\cos u, v\sin u, v\rangle$, $0 \le u \le 2\pi, 0 \le v \le h$	$\langle v\cos u, v\sin u, -v\rangle; \sqrt{2}v$
$\mathbf{r} = \langle a\sin u\cos v,$ $a\sin u\sin v, a\cos u\rangle$, $0 \le u \le \pi, 0 \le v \le 2\pi$	$\langle a^2\sin^2 u\cos v, a^2\sin^2 u\sin v,$ $a^2\sin u\cos u\rangle; a^2\sin u$
$\mathbf{r} = \langle v\cos u, v\sin u, v^2\rangle$, $0 \le u \le 2\pi, 0 \le v \le \sqrt{h}$	$\langle 2v^2\cos u, 2v^2\sin u, -v\rangle; v\sqrt{1 + 4v^2}$

Surface Integrals of Vector Fields

- **Two-sided**, or **orientable**, surfaces
- The normal vectors vary continuously over the surface.
- Möbius strip: a well-known non-orientable surface example.

- At any point of a parameterized orientable surface, there are two **unit normal vectors**.
- Once the direction of the normal vector is determined, the surface becomes oriented.

- Common assumption: a closed orientable surface is oriented so that the normal vectors point in the *outward direction*.
- For other surfaces, the orientation must be specified.
- The sign of $\mathbf{t}_u \times \mathbf{t}_v$ sometimes need to be reversed.

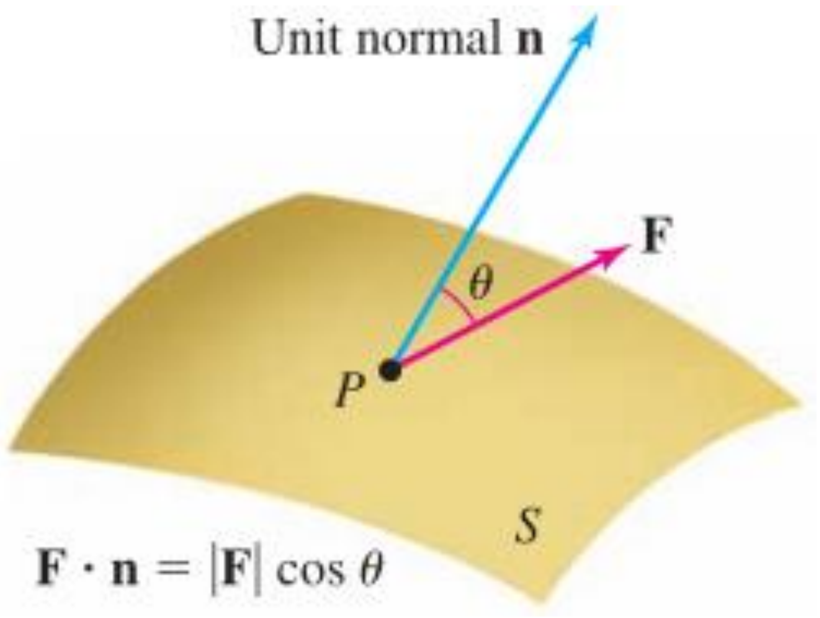
Flux Integrals

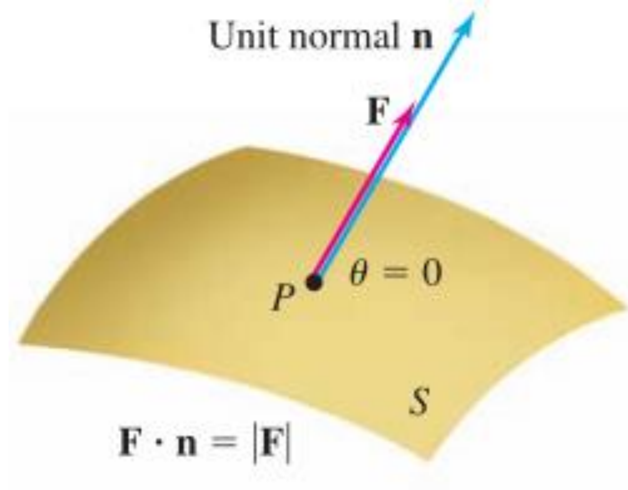
Flux integral: the most common surface integral of a vector field.

The net flux of the vector field across the surface

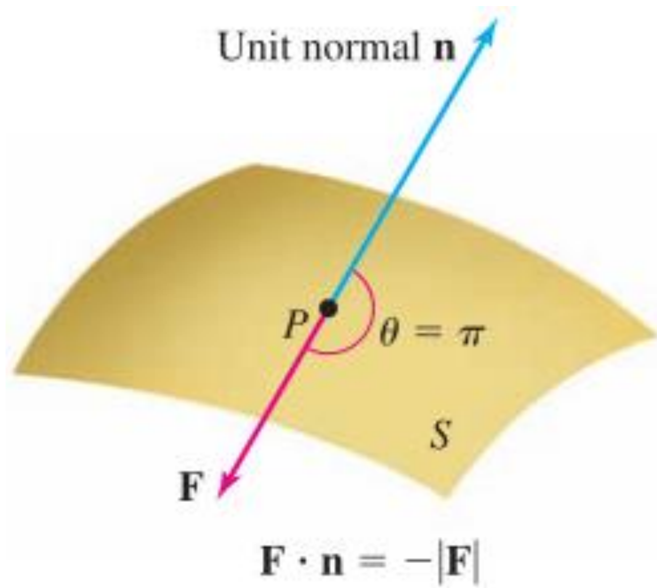
Proportional to the component of **F** in the direction of the unit normal vector **n**, in a small region containing a point P.

$$\mathbf{F} \cdot \mathbf{n} = |\mathbf{F}||\mathbf{n}|\cos\theta = |\mathbf{F}|\cos\theta$$

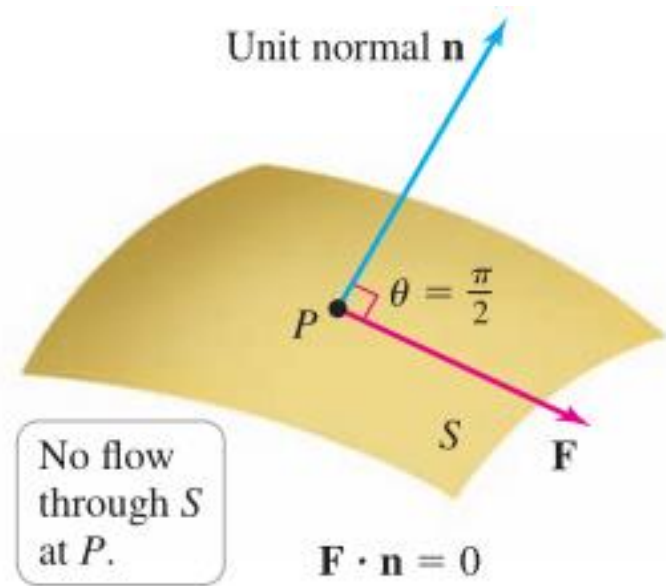



Unit normal n
F
P
θ = 0
S
F · n = |F|

(b)


Unit normal n
P
θ = π
S
F
F · n = −|F|

(c)


Unit normal n
P
θ = π/2
S
F
No flow through S at P.
F · n = 0

(d)

The flux integral adds up the components of **F** normal to the surface at all points of the surface, denoted

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS, \text{ or } \iint_S \mathbf{F}\cdot d\mathbf{S}$$

$$\begin{aligned}
\iint_S \mathbf{F}\cdot\mathbf{n}\,dS &= \iint_R \mathbf{F}\cdot\mathbf{n}\,|\mathbf{t}_u \times \mathbf{t}_v|\,dA && \text{Definition of surface integral}\\
&= \iint_R \mathbf{F}\cdot\underbrace{\frac{\mathbf{t}_u \times \mathbf{t}_v}{|\mathbf{t}_u \times \mathbf{t}_v|}}_{\mathbf{n}}\,|\mathbf{t}_u \times \mathbf{t}_v|\,dA && \text{Substitute for } \mathbf{n}.\\
&= \iint_R \mathbf{F}\cdot(\mathbf{t}_u \times \mathbf{t}_v)\,dA. && \text{Convenient cancellation}
\end{aligned}$$

DEFINITION Surface Integral of a Vector Field

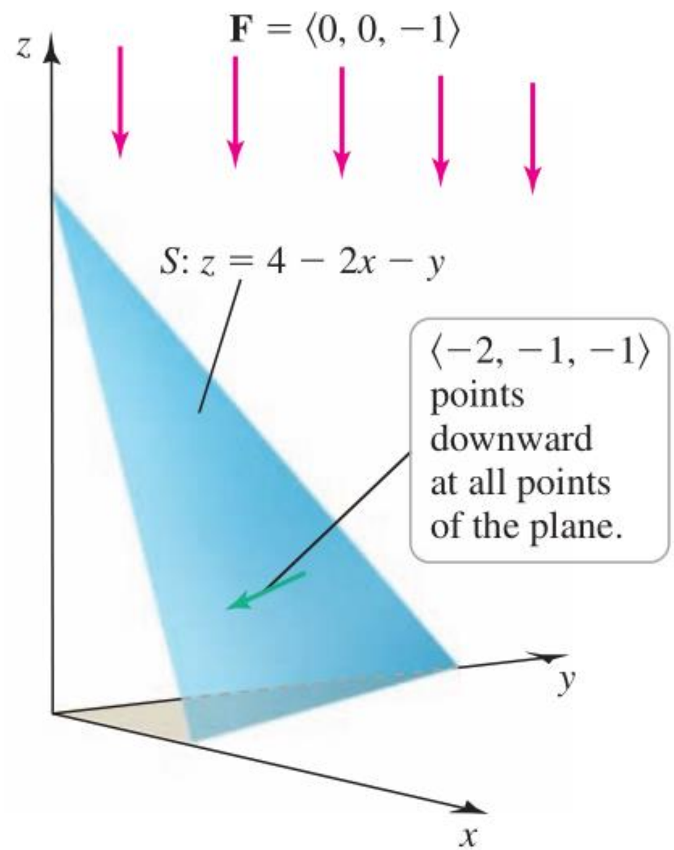
Suppose $\mathbf{F} = \langle f, g, h \rangle$ is a continuous vector field on a region of $\mathbb{R}^3$ containing a smooth oriented surface S. If S is defined parametrically as $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$, for (u, v) in a region R, then

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_R \mathbf{F} \cdot (\mathbf{t}_u \times \mathbf{t}_v)\, dA,$$

where $\mathbf{t}_u = \dfrac{\partial \mathbf{r}}{\partial u} = \left\langle \dfrac{\partial x}{\partial u}, \dfrac{\partial y}{\partial u}, \dfrac{\partial z}{\partial u} \right\rangle$ and $\mathbf{t}_v = \dfrac{\partial \mathbf{r}}{\partial v} = \left\langle \dfrac{\partial x}{\partial v}, \dfrac{\partial y}{\partial v}, \dfrac{\partial z}{\partial v} \right\rangle$ are continuous on R, the normal vector $\mathbf{t}_u \times \mathbf{t}_v$ is nonzero on R, and the direction of the normal vector is consistent with the orientation of S. If S is defined in the form $z = g(x, y)$, for (x, y) in a region R, then

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_R (-f z_x - g z_y + h)\, dA.$$

EXAMPLE 7 **Rain on a roof** Consider the vertical vector field $\mathbf{F} = \langle 0, 0, -1 \rangle$, corresponding to a constant downward flow. Find the flux in the downward direction across the surface S, which is the plane $z = 4 - 2x - y$ in the first octant.



$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \text{area of } R = 4$$

EXAMPLE 8 **Flux of the radial field** Consider the radial vector field $\mathbf{F} = \langle f, g, h \rangle = \langle x, y, z \rangle$. Is the upward flux of the field greater across the hemisphere $x^2 + y^2 + z^2 = 1$, for $z \geq 0$, or across the paraboloid $z = 1 - x^2 - y^2$, for $z \geq 0$? Note that the two surfaces have the same base in the xy-plane and the same high point $(0, 0, 1)$. Use the explicit description for the hemisphere and a parametric description for the paraboloid.

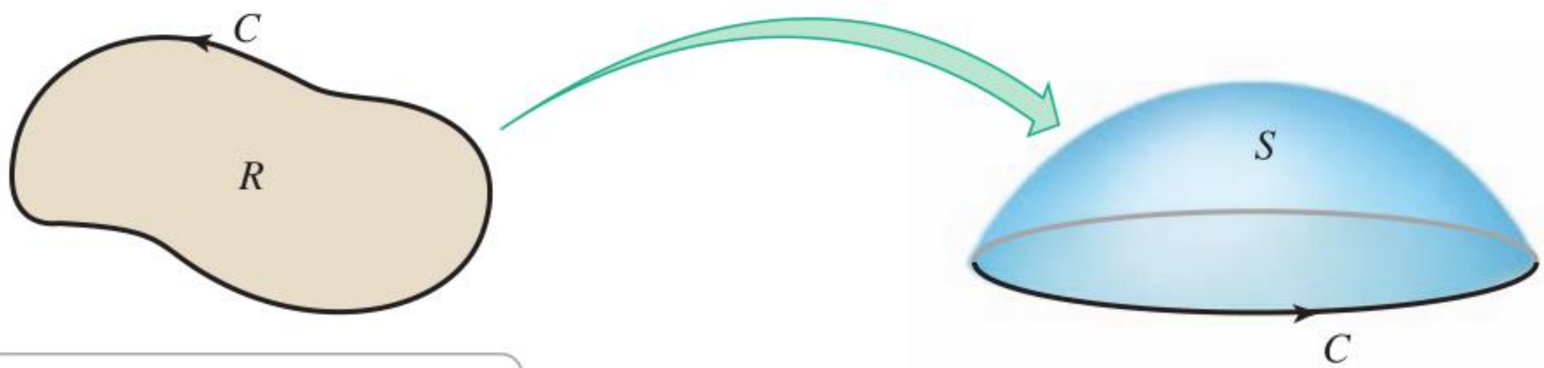
17.7

Stokes' Theorem

Stokes' Theorem

Circulation form of Green's Theorem, the cumulative rotation of the flow within R equals the circulation along the boundary.

$$\underbrace{\oint_C \mathbf{F}\cdot d\mathbf{r}}_{\text{circulation}} = \iint_R \underbrace{(g_x - f_y)}_{\text{curl or rotation}}\, dA.$$



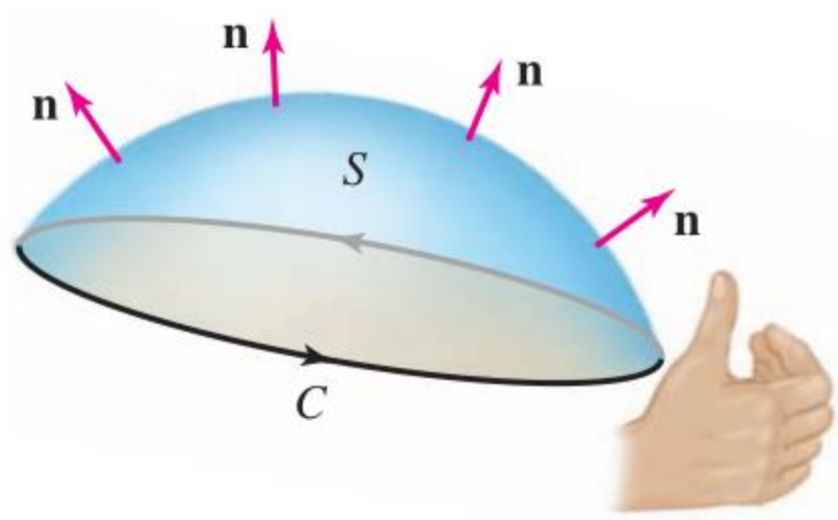
Circulation form of Green's Theorem:

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_R (\nabla \times \mathbf{F})\cdot \mathbf{k}\, dA$$

Stokes' Theorem:

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F})\cdot \mathbf{n}\, dS$$

Stokes' Theorem involves an oriented curve C and an oriented surface S on which two unit normal vectors at every point. The right-hand rule relates the orientations of S and C, and determines the choice of the normal vectors.



THEOREM 13 Stokes' Theorem

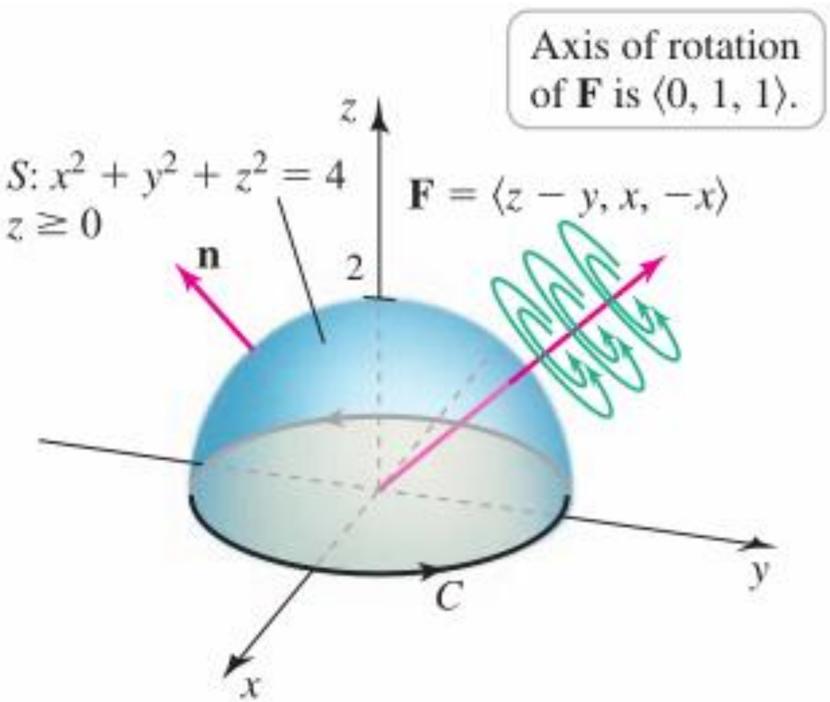
Let S be an oriented surface in $\mathbb{R}^3$ with a piecewise-smooth closed boundary C whose orientation is consistent with that of S. Assume that $\mathbf{F} = \langle f, g, h \rangle$ is a vector field whose components have continuous first partial derivatives on S. Then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS,$$

where $\mathbf{n}$ is the unit vector normal to S determined by the orientation of S.

Remark. If $\mathbf{F}$ is a conservative vector field on a domain D, then $\nabla \times \mathbf{F} = \mathbf{0}$ (Theorem 9), i.e., the circulation integral is zero on all closed curves in D, i.e., no work is done in moving an object on a closed path in a conservative force field.

EXAMPLE 1 **Verifying Stokes' Theorem** Confirm that Stokes' Theorem holds for the vector field $\mathbf{F} = \langle z - y, x, -x \rangle$, where S is the hemisphere $x^2 + y^2 + z^2 = 4$, for $z \geq 0$, and C is the circle $x^2 + y^2 = 4$ oriented counterclockwise.



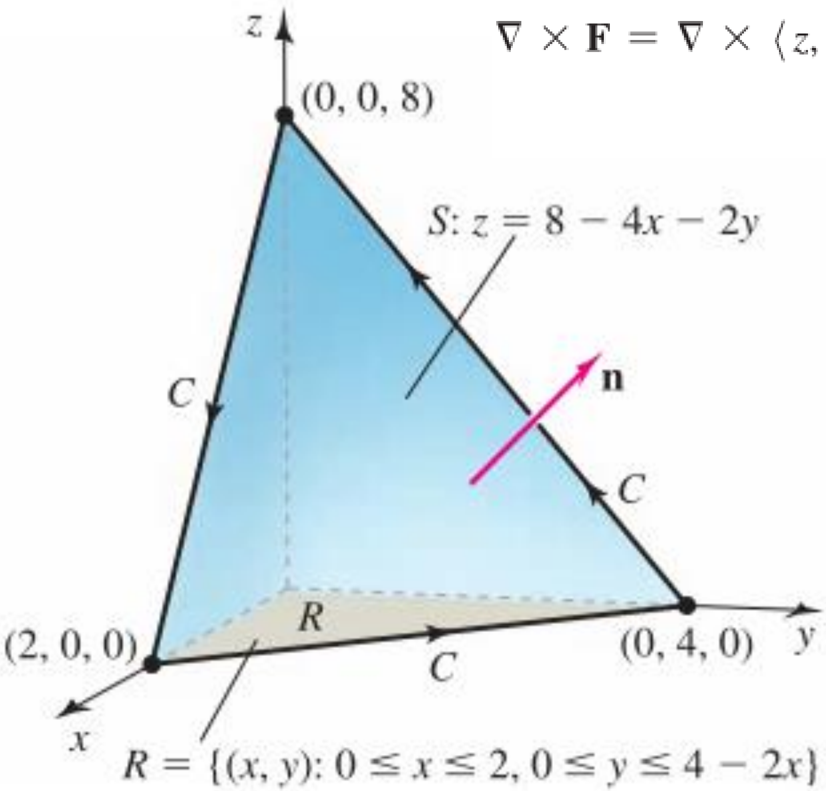
$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F} \cdot \mathbf{r}'(t)\, dt$$

$$= \int_0^{2\pi} \langle \underbrace{z - y}_{-2\sin t}, x, -x \rangle \cdot \langle -2\sin t, 2\cos t, 0 \rangle\, dt$$

$$\iint_S \underbrace{(\nabla \times \mathbf{F})}_{\langle 0,2,2 \rangle} \cdot \mathbf{n}\, dS = \iint_R \langle 0, 2, 2 \rangle \cdot \left\langle \frac{x}{z}, \frac{y}{z}, 1 \right\rangle dA$$

$$= \iint_R \left(\frac{2y}{\sqrt{4 - x^2 - y^2}} + 2 \right) dA$$

EXAMPLE 2 **Using Stokes' Theorem to evaluate a line integral** Evaluate the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = z\,\mathbf{i} - z\,\mathbf{j} + (x^2 - y^2)\,\mathbf{k}$ and C consists of the three line segments that bound the plane $z = 8 - 4x - 2y$ in the first octant, oriented as shown in Figure .62.

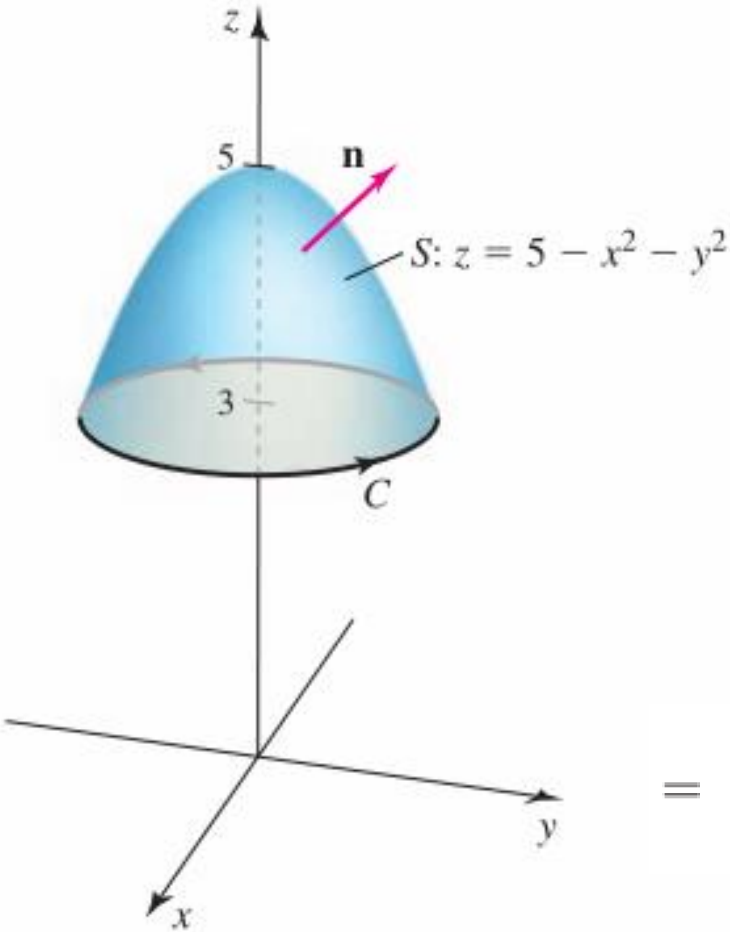


$$\nabla \times \mathbf{F} = \nabla \times \langle z, -z, x^2 - y^2 \rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & -z & x^2 - y^2 \end{vmatrix} = \langle 1 - 2y, 1 - 2x, 0 \rangle$$

Vector normal to the plane $z = 8 - 4x - 2y$ is $\langle -z_x, -z_y, 1 \rangle = \langle 4,2,1 \rangle$ pointing upward

$$\iint_S \underbrace{(\nabla \times \mathbf{F})}_{\langle 1 - 2y,\, 1 - 2x,\, 0 \rangle} \cdot \mathbf{n}\, dS = \iint_R \langle 1 - 2y, 1 - 2x, 0 \rangle \cdot \langle 4, 2, 1 \rangle\, dA$$

EXAMPLE 3 **Using Stokes' Theorem to evaluate a surface integral** Evaluate the integral $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$, where $\mathbf{F} = -xz\,\mathbf{i} + yz\,\mathbf{j} + xye^z\,\mathbf{k}$ and S is the cap of the paraboloid $z = 5 - x^2 - y^2$ above the plane $z = 3$ (Figure 63). Assume $\mathbf{n}$ points in the upward direction on S.



The parametric description of C is

$$\mathbf{r}(t) = \langle \sqrt{2}\cos t, \sqrt{2}\sin t, 3\rangle$$

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \oint_C \mathbf{F} \cdot d\mathbf{r}$$

$$= \int_0^{2\pi} \mathbf{F} \cdot \mathbf{r}'(t)\, dt$$

$$= \int_0^{2\pi} \langle -xz, yz, xye^z\rangle \cdot \langle -\sqrt{2}\sin t, \sqrt{2}\cos t, 0\rangle\, dt$$

Interpreting the Curl

Average circulation of **F** over *S*

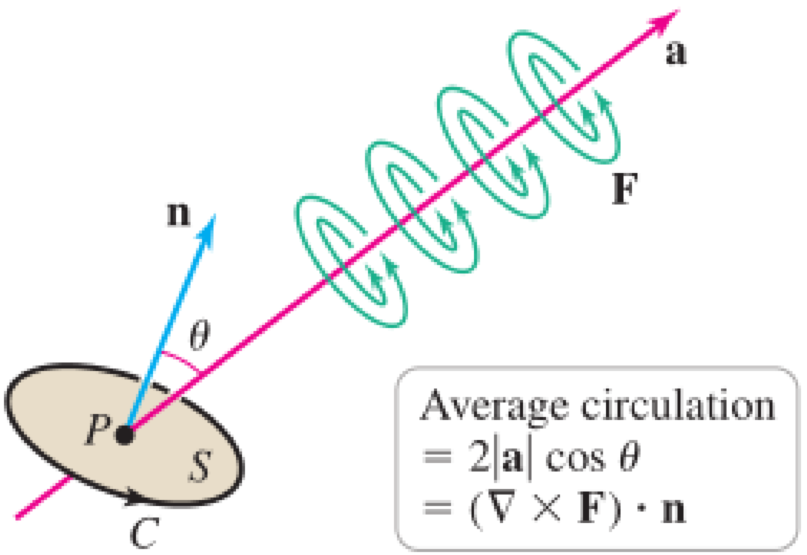
$$\frac{1}{\text{area}(S)}\oint_C \mathbf{F}\cdot d\mathbf{r} = \frac{1}{\text{area}(S)}\iint_S (\nabla \times \mathbf{F})\cdot \mathbf{n}\, dS,$$

$$\frac{1}{\text{area}\,(S)}\iint_S \underbrace{(\nabla \times \mathbf{F})\cdot \mathbf{n}}_{\text{constant}}\, dS$$

$$= \frac{1}{\text{area}\,(S)}(\nabla \times \mathbf{F})\cdot \mathbf{n}\cdot\ \text{area}\,(S)$$

$$= \underbrace{(\nabla \times \mathbf{F})}_{2\mathbf{a}}\cdot \mathbf{n}$$

$$= 2|\mathbf{a}|\cos\theta.$$



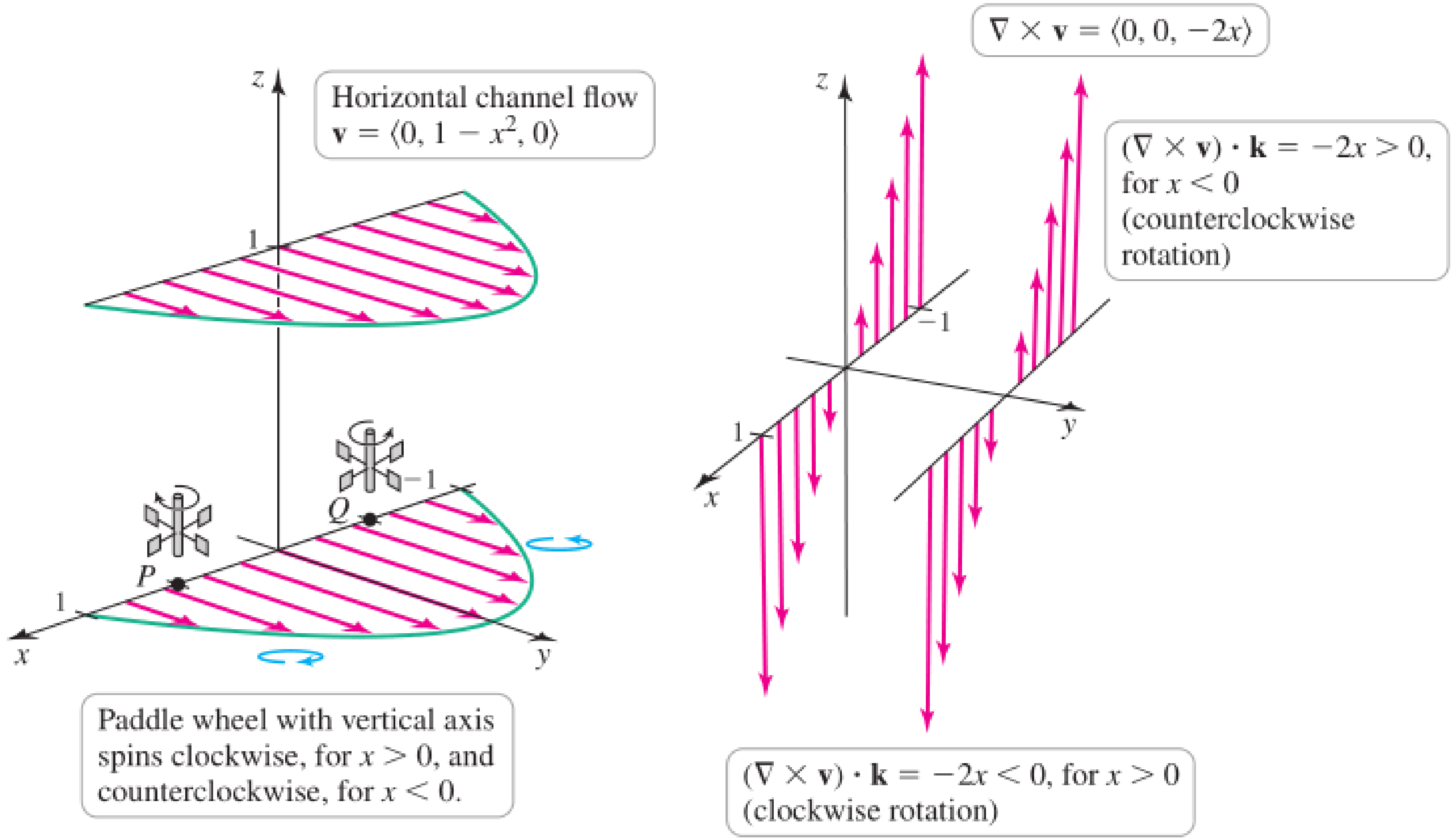
For a general rotation field $\mathbf{F} = \mathbf{a} \times \mathbf{r}$, the curl of $\mathbf{F}$ has the following interpretations, where S is a small disk centered at a point P with a normal vector $\mathbf{n}$.

- The scalar component of $\nabla \times \mathbf{F}$ at P in the direction of $\mathbf{n}$, which is $(\nabla \times \mathbf{F}) \cdot \mathbf{n} = 2|\mathbf{a}|\cos\theta$, is the average circulation of $\mathbf{F}$ on S.
- The direction of $\nabla \times \mathbf{F}$ at P is the direction that maximizes the average circulation of $\mathbf{F}$ on S.

EXAMPLE 4 **Horizontal channel flow** Consider the velocity field $\mathbf{v} = \langle 0, 1 - x^2, 0 \rangle$, for $|x| \leq 1$ and $|z| \leq 1$, which represents a horizontal flow in the y-direction (Figure 14.65a).

a. Suppose you place a paddle wheel at the point $P\left(\frac{1}{2}, 0, 0\right)$. Using physical arguments, in which of the coordinate directions should the axis of the wheel point in order for the wheel to spin? In which direction does it spin? What happens if you place the wheel at $Q\left(-\frac{1}{2}, 0, 0\right)$?

b. Compute and graph the curl of $\mathbf{v}$ and provide an interpretation.

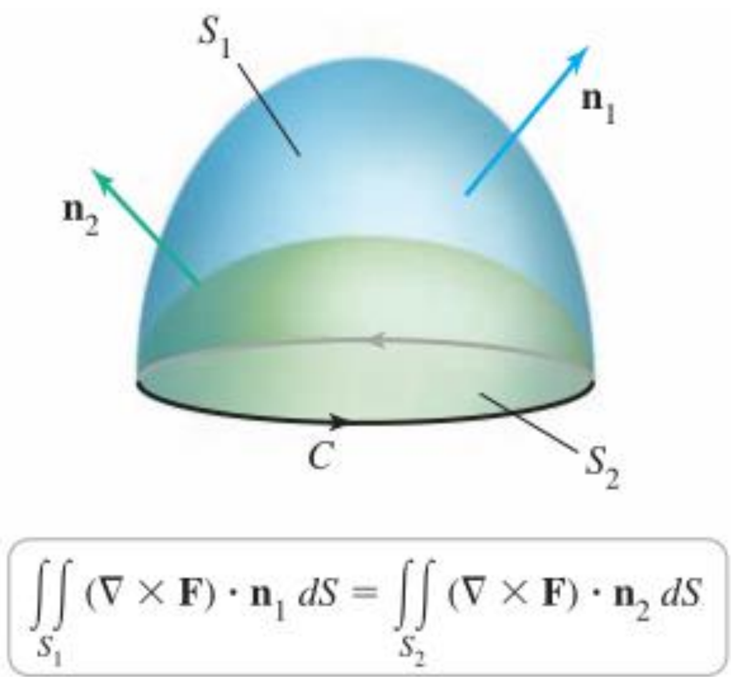


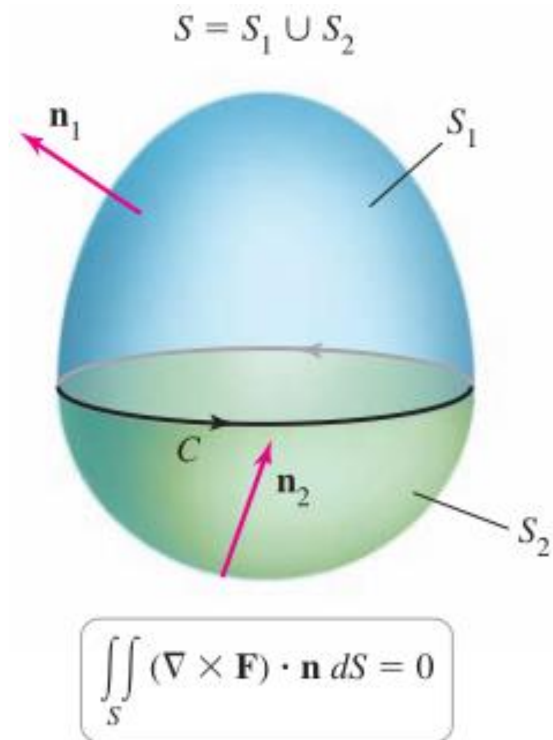
Horizontal channel flow
$\mathbf{v} = \langle 0, 1 - x^2, 0\rangle$
Paddle wheel with vertical axis spins clockwise, for $x > 0$, and counterclockwise, for $x < 0$.
$\nabla \times \mathbf{v} = \langle 0, 0, -2x\rangle$
$(\nabla \times \mathbf{v}) \cdot \mathbf{k} = -2x > 0$, for $x < 0$ (counterclockwise rotation)
$(\nabla \times \mathbf{v}) \cdot \mathbf{k} = -2x < 0$, for $x > 0$ (clockwise rotation)

Two Final Notes on Stokes' Theorem

1. Stokes' Theorem allows a surface integral $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\,dS$ to be evaluated using only the values of the vector field on the boundary C.

   $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\,dS = 0$ over any closed oriented surface S





2. The converse of Theorem 9, i.e., curl $\nabla \times \mathbf{F} = 0$ implies $\mathbf{F}$ is conservative, is true.

THEOREM 14 Curl $\mathbf{F} = \mathbf{0}$ Implies $\mathbf{F}$ Is Conservative
Suppose that $\nabla \times \mathbf{F} = \mathbf{0}$ throughout an open simply connected region D of $\mathbb{R}^3$. Then $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ on all closed simple smooth curves C in D and $\mathbf{F}$ is a conservative vector field on D.

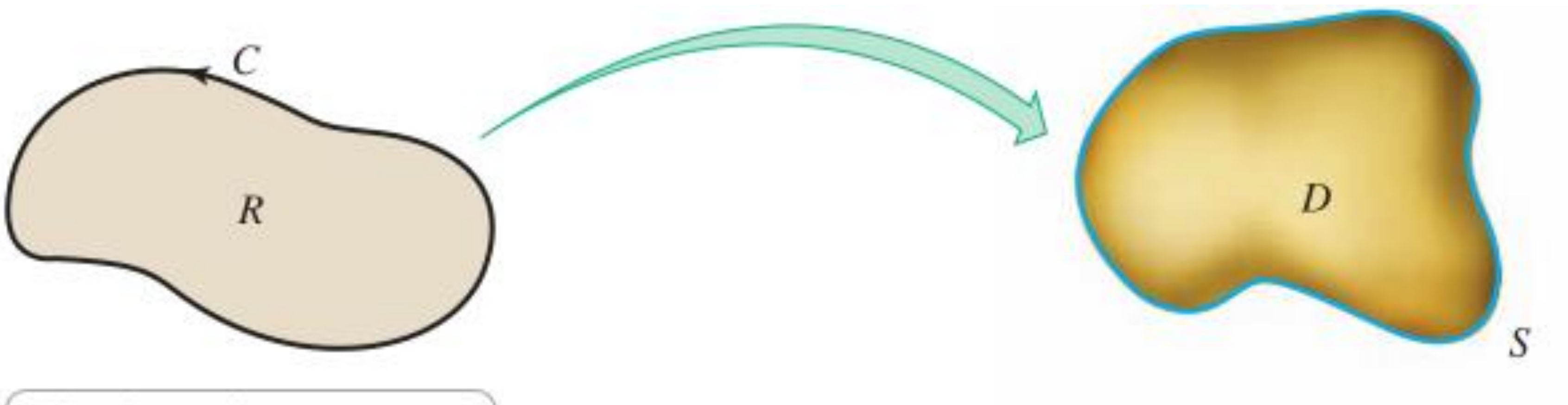
17.8

Divergence Theorem

Divergence Theorem (Gauss' Law)

The Divergence Theorem offers an alternative method for flux calculations.

Three-dimensional version of the flux form of Green's Theorem



Flux form of Green's Theorem

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_R \operatorname{div} \mathbf{F}\, dA$$

Divergence Theorem:

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_D \operatorname{div} \mathbf{F}\, dV$$

THEOREM 15 Divergence Theorem

Let $\mathbf{F}$ be a vector field whose components have continuous first partial derivatives in a connected and simply connected region D in $\mathbb{R}^3$ enclosed by an oriented surface S. Then

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_D \nabla \cdot \mathbf{F}\, dV,$$

where $\mathbf{n}$ is the outward unit normal vector on S.

Circulation form of
Green's Theorem $\longrightarrow$ Stokes' Theorem

Flux form of Green's
Theorem $\longrightarrow$ Divergence Theorem

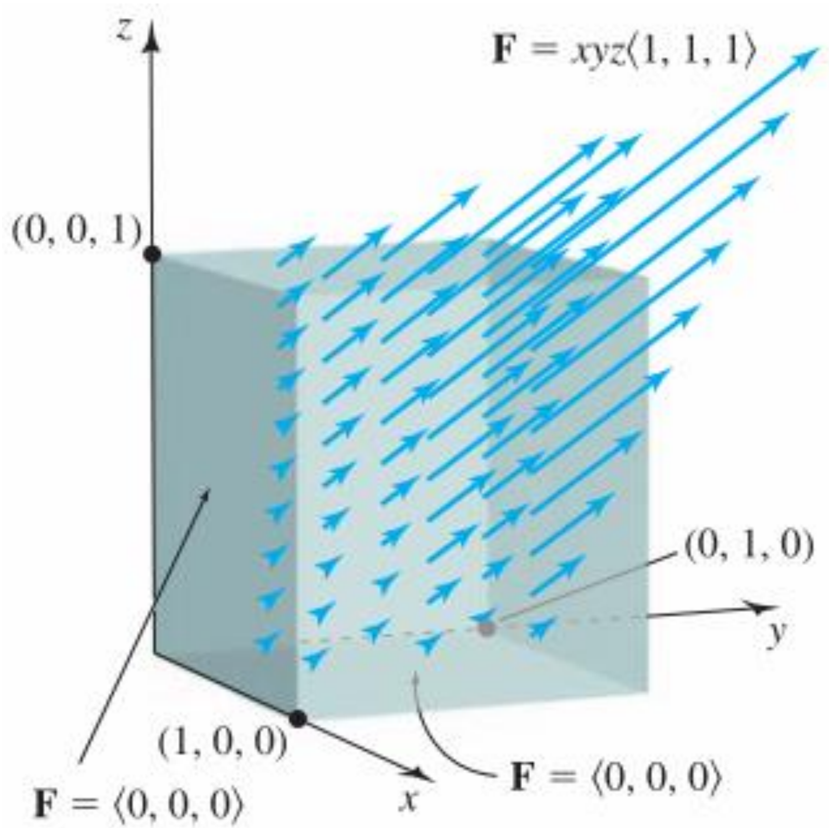
EXAMPLE 1 **Verifying the Divergence Theorem** Consider the radial field $\mathbf{F} = \langle x, y, z \rangle$ and let S be the sphere $x^2 + y^2 + z^2 = a^2$ that encloses the region D. Assume $\mathbf{n}$ is the outward unit normal vector on the sphere. Evaluate both integrals of the Divergence Theorem.

EXAMPLE 2 **Divergence Theorem with a rotation field** Consider the rotation field

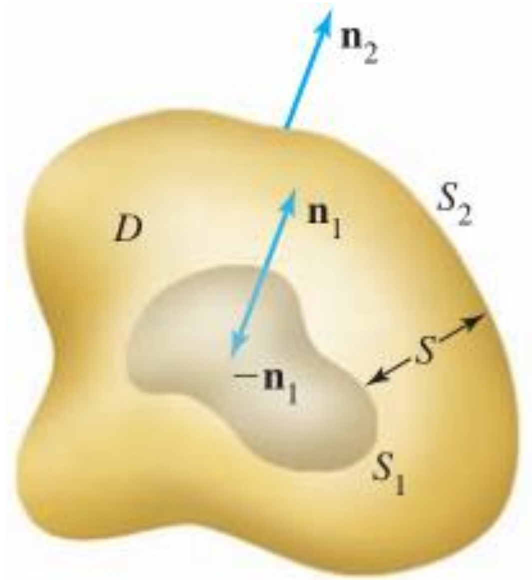
$$\mathbf{F} = \mathbf{a} \times \mathbf{r} = \langle 1, 0, 1 \rangle \times \langle x, y, z \rangle = \langle -y, x - z, y \rangle.$$

Let S be the hemisphere $x^2 + y^2 + z^2 = a^2$, for $z \geq 0$, together with its base in the xy-plane. Find the net outward flux across S.

EXAMPLE 3 **Computing flux with the Divergence Theorem** Find the net outward flux of the field $\mathbf{F} = xyz\langle 1, 1, 1 \rangle$ across the boundaries of the cube $D = \{(x, y, z): 0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1\}$.



Divergence Theorem for Hollow Regions



THEOREM 16 Divergence Theorem for Hollow Regions

Suppose the vector field $\mathbf{F}$ satisfies the conditions of the Divergence Theorem on a region D bounded by two oriented surfaces S_1 and S_2, where S_1 lies within S_2. Let S be the entire boundary of D $(S = S_1 \cup S_2)$ and let $\mathbf{n}_1$ and $\mathbf{n}_2$ be the outward unit normal vectors for S_1 and S_2, respectively. Then

$$\iiint_D \nabla \cdot \mathbf{F}\, dV = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_{S_2} \mathbf{F} \cdot \mathbf{n}_2\, dS - \iint_{S_1} \mathbf{F} \cdot \mathbf{n}_1\, dS.$$

EXAMPLE 4 **Flux for an inverse square field** Consider the inverse square vector field

$$\mathbf{F} = \frac{\mathbf{r}}{|\mathbf{r}|^3} = \frac{\langle x, y, z \rangle}{(x^2 + y^2 + z^2)^{3/2}}.$$

a. Find the net outward flux of $\mathbf{F}$ across the surface of the region $D = \{(x, y, z): a^2 \le x^2 + y^2 + z^2 \le b^2\}$ that lies between concentric spheres with radii a and b.

b. Find the outward flux of $\mathbf{F}$ across any sphere that encloses the origin.

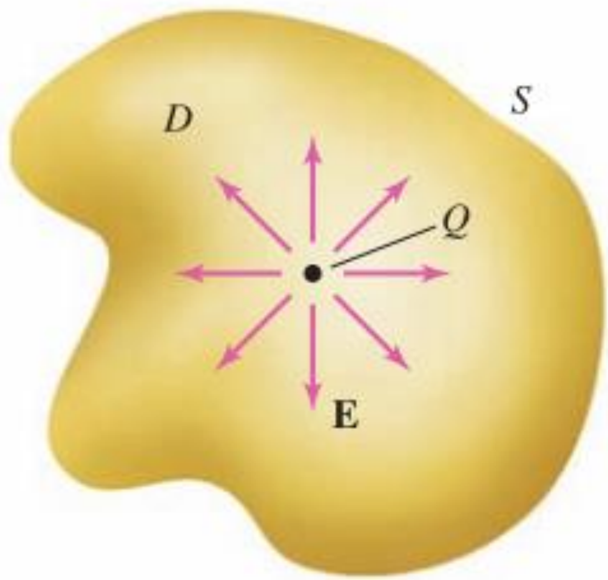
$$\iiint_D \nabla \cdot \mathbf{F}\, dV = 0;$$

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_{S_2} \mathbf{F} \cdot \mathbf{n}_2\, dS - \iint_{S_1} \mathbf{F} \cdot \mathbf{n}_1\, dS = 0.$$

$$\iint_{S_1} \mathbf{F} \cdot \mathbf{n}_1\, dS = \iint_{S_1} \frac{\mathbf{r}}{|\mathbf{r}|^3} \cdot \frac{\mathbf{r}}{|\mathbf{r}|}\, dS$$

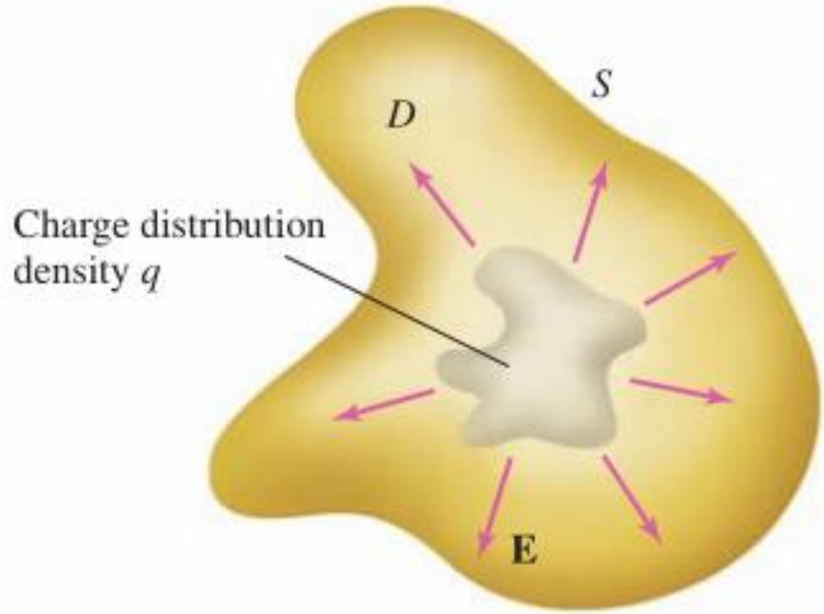
$$= \iint_{S_1} \frac{|\mathbf{r}|^2}{|\mathbf{r}|^4}\, dS$$

Gauss' Law in electric fields



Gauss' Law:
Flux of electric field across S
due to point charge Q

$$= \iint_S \mathbf{E} \cdot \mathbf{n}\, dS = \frac{Q}{\varepsilon_0}$$



Gauss' Law:
Flux of electric field across S
due to charge distribution q

$$= \iint_S \mathbf{E} \cdot \mathbf{n}\, dS = \frac{1}{\varepsilon_0} \iiint_D q\, dV$$

A Final Perspective

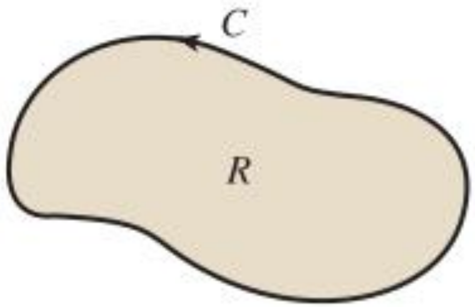
Underlying principle: The cumulative (integrated) effect of the *derivatives* of a function throughout a region is determined by the values of the function on the boundary of that region.

Table 4

Fundamental Theorem of Calculus	$\int_a^b f'(x)\,dx = f(b) - f(a)$	
Fundamental Theorem of Line Integrals	$\int_C \nabla f \cdot d\mathbf{r} = f(B) - f(A)$	

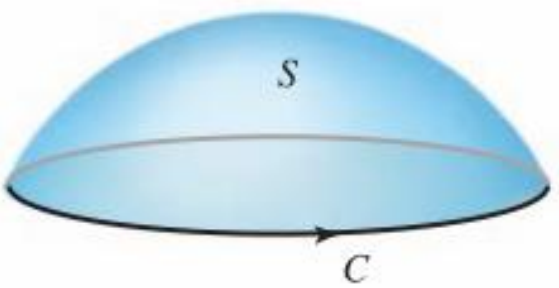
Green's Theorem (Circulation form)

$$\iint_R (g_x - f_y)\, dA = \oint_C f\,dx + g\,dy$$



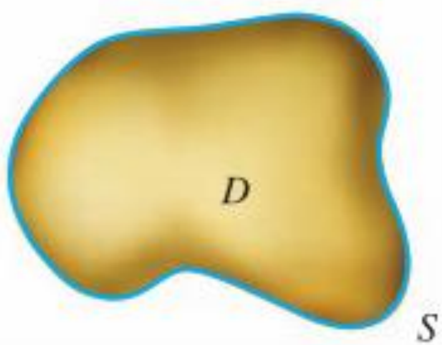
Stokes' Theorem

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \oint_C \mathbf{F} \cdot d\mathbf{r}$$



Divergence Theorem

$$\iiint_D \nabla \cdot \mathbf{F}\, dV = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS$$



Chapter 17

Vector Calculus (II)

Shuwei Chen

swchen@swjtu.edu.cn