Integrability of the Toda Lattice

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Introduction

The Toda lattice is an interesting system to physically analyse due to its various unique and somewhat unexpected behavioural traits. The Toda lattice considered in this presentation is a 1-D string of particles considering only the nearest neighbouring interactions. It has been examined both experimentally and theoretically to propagate soliton wave forms without change of shape, this is particularly interesting considering its nonlinear potential. It has been proposed that the Toda lattice may in fact be an integrable system with associated highly stable soliton solutions. The purpose of this presentation is to explore this proposal.

Three-Particle Toda Lattice

For the purpose of this presentation, all written and empirical validation of the integrability of the Toda lattice will be performed considering the three-particle Toda lattice case, for which the Hamiltonian may be written as follows:

$$H = \frac{1}{2}(P_1^2 + P_2^2 + P_3^2) + e^{-(Q2-Q3)} + e^{-(Q2-Q1)} + e^{-(Q1-Q3)} - 3$$

a) Canonical Transformation

In order to bring the Hamiltonian to a more manageable form of:

$$H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{24}\{e^{(2q2+2\sqrt{3}q1)} + e^{(2q2-2\sqrt{3}q1)} + e^{-4q2}\} - \frac{1}{8}$$

A canonical transformation of the form:

$Q_{i=} \sum_{j=1}^{3} A_{ij}\vartheta_j$; $P_{i=} \sum_{j=1}^{3} A_{ij}\mu_j$ is needed.

The matrix A is given by:

$$A = \begin{pmatrix} 6^{-1/2} & 2^{-1/2} & 3^{-1/2} \\ -\frac{2^{-1/2}}{3} & 0 & 3^{-1/2} \\ 6^{-1/2} & -2^{-1/2} & 3^{-1/2} \end{pmatrix}$$

Where the transformation can be proven to be canonical by considering $MJM^T = J$ where J is the unit matrix and M is represented by the following matrix:

$$M = \begin{pmatrix} 6^{-1/2} & 2^{-1/2} & 3^{-1/2} & 0 & 0 & 0 \\ -\frac{2^{1/2}}{3} & 0 & 3^{-1/2} & 0 & 0 & 0 \\ 6^{-1/2} & -2^{-1/2} & 3^{-1/2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 6^{-1/2} & 2^{-1/2} & 3^{-1/2} \\ 0 & 0 & 0 & -\frac{2^{1/2}}{3} & 0 & 3^{-1/2} \\ 0 & 0 & 0 & 6^{-1/2} & -2^{-1/2} & 3^{-1/2} \end{pmatrix}$$

Also let $\vartheta_1 = 2\sqrt{2}q_1$, $\vartheta_2 = 2\sqrt{2}q_2$, $\mu_1 = p_1$, and $\mu_2 = p_2$, to obtain the new Hamiltonian.

It should be noted that the new Hamiltonian is a function of two positions and two momentums rather than three. This is due to the clever choice of canonical transformations which eliminated ϑ_3 from the Hamiltonian, and thus the conjugate momentum μ_3 is known to be a constant of motion and can be ignored in the new Hamiltonian.

b) Fixed Points and Stability

In taking the derivative of the Hamiltonian with respect to q_1, q_2, p_1, and p_2, it can be found that there is only one fixed point for this particular system. It is obvious that this fixed point occurs when (p_1, p_2)=(0,0), and although not as obvious, after minor rearrangement it can be found that the fixed point can only occur when (q_1, q_2) also equals (0,0).

The stability of this point can be inferred using graphical methods, and as can be seen in Fig.1, it is a stable fixed point.

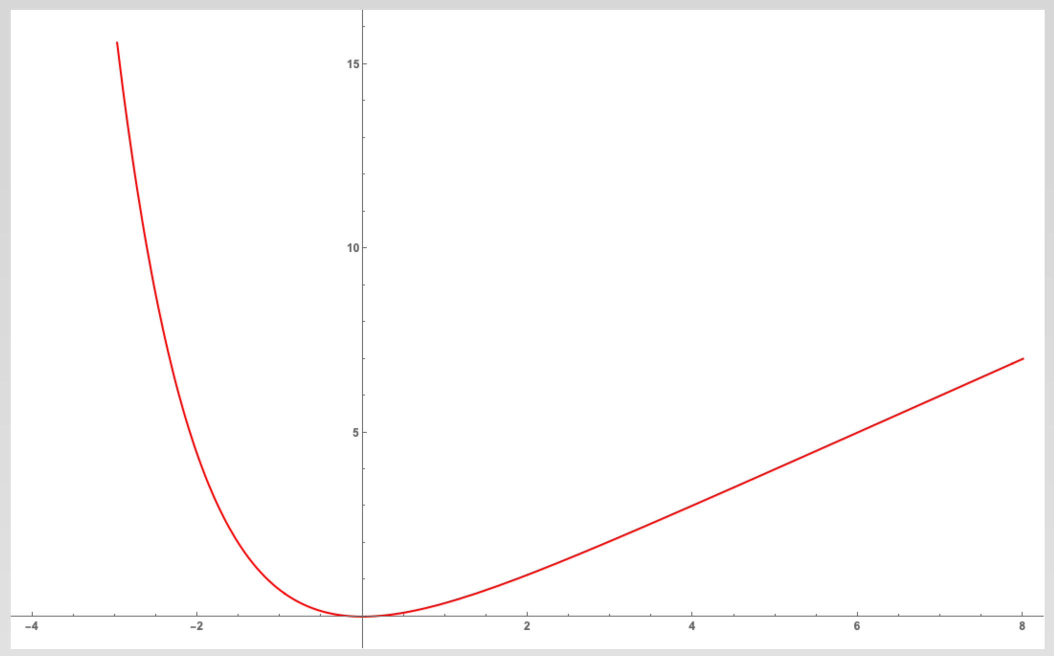

Figure 1: Fixed Point

c) Integrability of the Toda Lattice

There have been various proposed methods to determine the integrability of the Toda lattice. The method analysed in this presentation included plotting a surface of section for the Hamiltonian being considered at various energies to examine whether or not they result in a series of smooth curves when considering different initial conditions for each energy case.

Represented in Fig.2-4 are three unique plots of a surface of section for the Hamiltonian at various energies plotted in the (p_2, q_2) plane. These figures have energies of E=$\frac{1}{12}$, E=1, and E=5 respectively. As the energy increases, the curves cover a larger range of (p_2, q_2) values. However, despite the variation in p_2 and q_2 values, it can be seen below that all of the plots consist of smooth curves. This result indicates that the Toda lattice must indeed be an integrable system for the energies considered in this presentation. This does not guarantee this result will apply for much larger energies.

If a system is completely integrable then there usually exists soliton solutions. Solitons are an important physical anomaly because they are pulse like waves that travel in time without changing their size or shape, even after they interact with one another. It is interesting that the Toda lattice has these soliton solutions, considering its non-linear potential. Therefore the existence of these soliton solutions further validate the integrability of the Toda lattice.

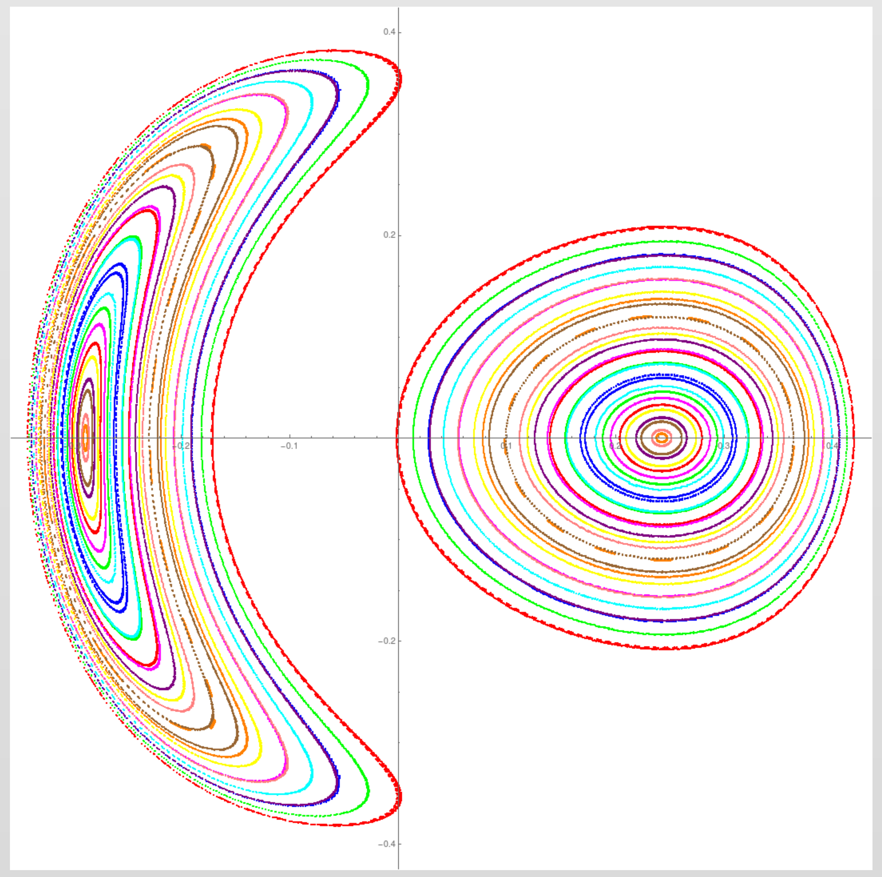

Figure 2: E=1/12

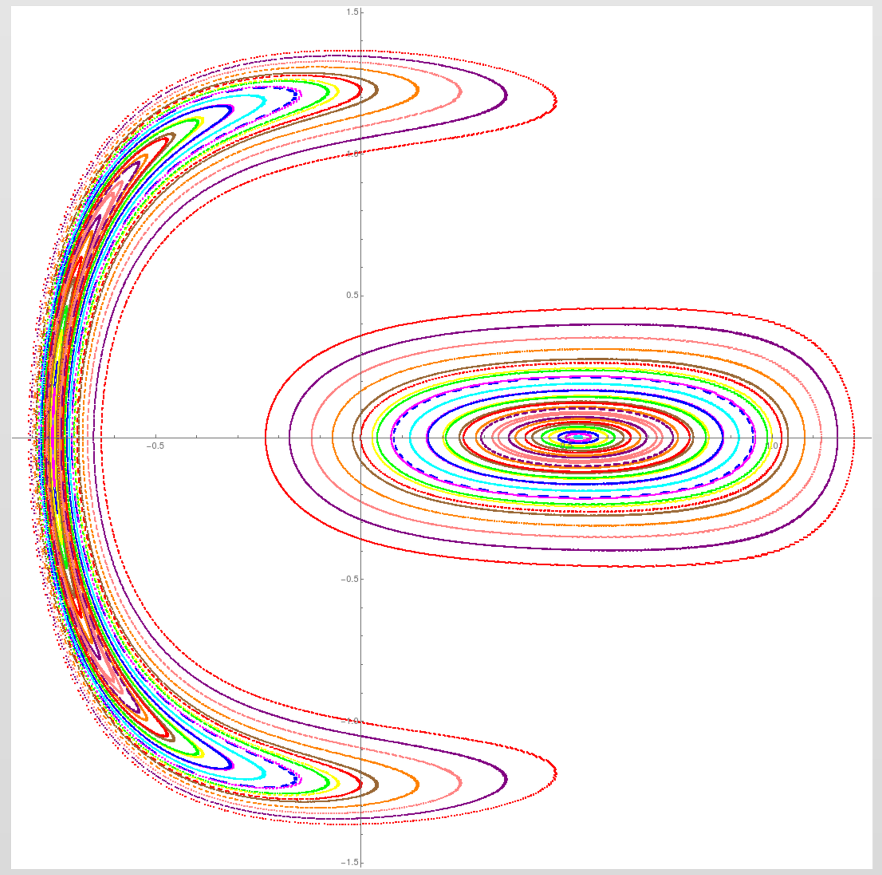

Figure 3: E=1

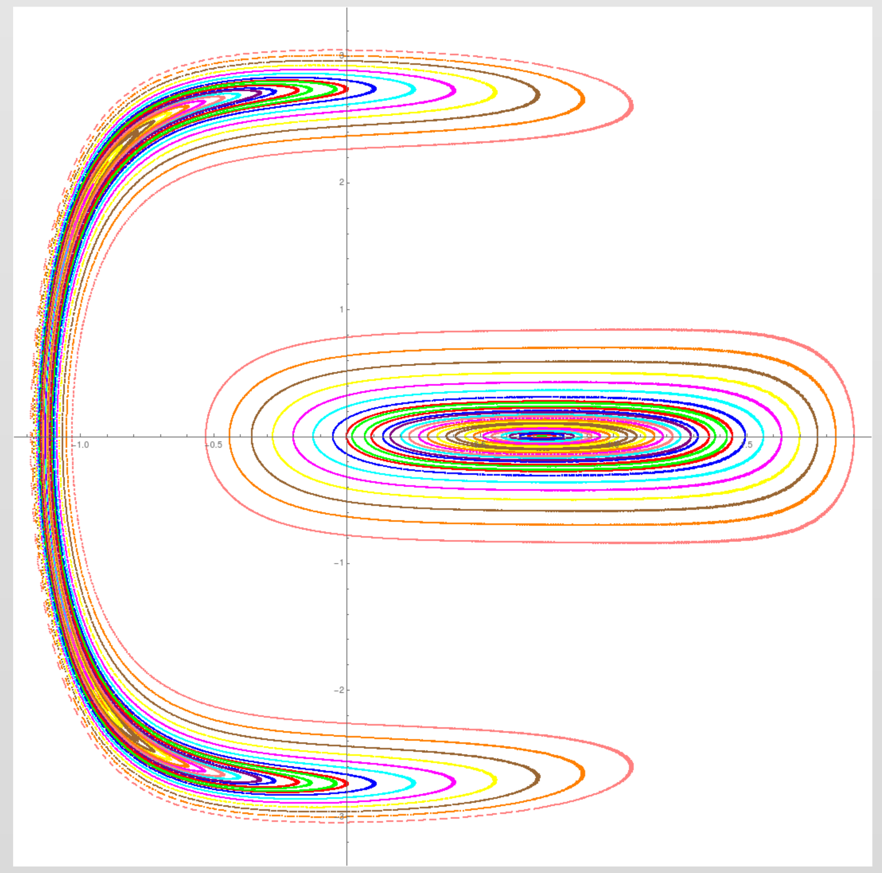

Figure 4: E=5

d) Constant of Motion

If the Hamiltonian considered is integrable, then it must have at least one exact independent constant of motion. This constant of motion was found by Professor M. Henon to be:

$$\emptyset(q_1, p_1, q_2, p_2) = 8p_1(p_1^2 - 3p_2^2) + (p_1 + \sqrt{3}p_2)e^{(2q2-2\sqrt{3}q1)} - 2p_1e^{-4q2} + (p_1 - \sqrt{3}p_2)e^{(2q2+2\sqrt{3}q1)}$$

Which can be proven to be a constant of motion by taking the Poisson bracket of $\{\emptyset, H\}$. The result is expectedly 0.

e) Further Study: The Six-Particle Toda Lattice

Although not explicitly proved in this presentation, the six-particle Toda lattice was another case discussed in the paper by Ford et al. It would be interesting to empirically analyse the integrability of the six-particle Toda lattice as well in order to further prove the integrability of the Toda lattice aside from exclusively the three-particle case.

References:

1. Ford, J., et al. (1973) On the Integrability of the Toda Lattice. *Progress of Theoretical Physics, 50(5),* 1547-1560.
2. Teschl, G. (2009). The Toda Lattice.