Topological Superconductivity

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CONTENTS

I. INTRODUCTION

One of the earliest breakthrews allowed by quantum mechanics is the descritpion of metals and insulators with band theory [1] which saw light in 1930. This theory brought a microscopic understanding of the disctinction between these two classes of materials and led to incredible tecnological advances such as the discovery of the transistor by John Bardeen and Walter Brattain in 1947 [2]. Beyond its increadible success, band theory rapidly came in contact with a myriad of intriguing quantum mechanical effects such as the integer quantum Hall effect discovered in 1980 [3]. In 1982, Thouless et al. [4] figured out the topological nature of the effect and, in term, brought topology closer to band theory. Altough the integer quantum Hall effect (QH) requires a strong external magnetic field, it was theorised in 2005 by Charles Kane and Eugene Mele [5] that similar topological properties could be intrinsicaly realised through the quantum spin Hall effect (QSH) [6]. Experiment then showed in 2007 [7] that HgTe/CdTe quantum wells (mercury telluride heterostructure) could produce a QSH effect. The theory and experiment of QSH effect led to a deeper classification of solids with topological band theory [8]. When applied to insulators, the upgraded band theory creates a seperation between the trivial and the *topological insulators* (TI). The latter is generally caracterised by a metallic boundary and an insulating bulk [9] as opposed to trivial insulators which are insulating everywhere. The present review will focus on basic properties of time reversal symetric topological insulators based on the mercury telluride example. Sec.II presents an overview of important ideas from topological band theory. In sec.III, the main properties of the QSH state are given and compared to the QH effect. Finally, a model of HgTe/CdTe quantum is studied in sec. IV.

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II. ELEMENT OF TOPOLOGICAL BAND THEORY

This sections aims to describe the notion of topological invariant and its consequence on the band structure.

A. Topological equivalence of insulators

In a periodic lattice potential, electrons are described by bloch states $|n, \mathbf{k}\rangle$ where n is a dicrete quantum number and $\mathbf{k}$ is the crystal momentum in the brillouin zone [10]. Each of those states is associated to an energy $E_{n,\mathbf{k}}$. As it varies with $\mathbf{k}$, the energy sweeps a continous range called a band labeled with the number n. Bands are often separated by energy gaps where there are no associated states. Trivial insulators have a gapped ground state meaning that low energy excitations are forbidden by the presence of the gap. On the contrary, topological insulators have metallic (gapless) edge states[11] and are different from trivial insulators in a fundamental way. By definition, edge states are states that are exponentially localised near the edges of the sample [12]. Usually, the bulk of the crystal is studied under periodic boundary conditions and the edge states are neglected in the analysis. A deep principle called *bulk boundary correspondance* relates the edge states to the spectrum of the bulk system [13].

The adiabatic theorem tells us that sufficiently slow modifications of the hamiltonian of a trivial insulator will change its band structure while leaving it in its ground state[14]. This *deformation* is said to yield equivalent insulators if the gap doesn't close [11]. This equivalence is topological in the same way the continuous disformation from a torus to a coffee mug is. Just like the coffee mug cannot be continuously deformed into a sphere, a trivial insulator cannot be continuously *deformed* into a topological insulator.

A coffee mug and a sphere are considered topologicaly different because they do not have the same number of holes, meaning you can't smoothly deform one into the other. This number (of holes) is called the genus and it as analogues for topological insulators [15]. The central property of genus is that it only changes when the torus is broken into a sphere in a necessarly discontinuous way. For the QHE the relevant topological invariant (under deformation of the Hamiltonian) is the Chern number. For the QSH effect, a $\mathbb{Z}_2$ invariant is involved [11]. In the following section, the link between time reversal symmetry and this last invariant is detailed. Mixing symmetry, topological invariants and dimensionnality allows for a classification of topological insulators in a sort of *periodic table* [16].

B. Time reversal symmetry

In a quantum mechanical setting, an operator U (unitary or anti-unitary) corresponds to a symmetry of a system when $UHU^{-1} = H$ where H is the hamiltonian governing the motion of the system. For a time reversal symetric (TRS) system this means that evolution follows the same equations of motion as a movie of itself set on rewind [17]. The operator sending a system to its time reverse is an anti-unitary operator noted Θ and Hamiltonians of TRS follow the relation $\Theta H \Theta^{-1} = H$. In position representation, the intuitive effects of this time reversal operator on the observables required to describe electrons (namely the spin, momentum and position observables), are the following [17] :

$$\begin{aligned}
\Theta \mathbf{s} \Theta^{-1} &= -\mathbf{s}, \\
\Theta \mathbf{r} \Theta^{-1} &= \mathbf{r}, \\
\Theta(-i\hbar\nabla)\Theta^{-1} &= +i\hbar\nabla
\end{aligned}$$

meaning it sets electrons spinning backwards with a reversed velocity. The operator satisfying all those porperties is $\Theta = is_y K$ where K represents complex conjugation and the is_y factor ensures spin is fully reversed [18]. The essential property of the time reversal operator for electrons is that is satsifies $\Theta^2 = -1$. Hence reversing time twice for spin 1/2 particles isn't the identity [16]! This leads to Kramer's theorem which is said to be the most important theorem for time reversal invariant two dimensionnal topological insultors [18]. It states the following:

> **Kramer's Theorem:**
> For a time reversal invariant system containing an odd number of spin 1/2 particles, there are at least two degenerate energy states [18].

To use this theorem here, it must be applied to the periodic bloch states $|n, \mathbf{k}\rangle$ mentionned before. By bloch's theorem [10], these states can be written as the direct product $|u_n(\mathbf{k})\rangle \otimes |\mathbf{k}\rangle$ where $|\mathbf{k}\rangle$ takes care of the tranlsationnal symmetry of the crystal lattice and $|u_n(\mathbf{k})\rangle$ contains the details of the state over a unit cell. The basic idea of Kramers theoerem is that provided a state $|u_n(\mathbf{k})\rangle$, the action of the time reversal on it gives a state $\Theta|u_n(\mathbf{k})\rangle$ which has the same eigen energy as the first (if the hamiltonian is time reversal symmetric). Then, if there is no degeneracy the generated state must be related to the first one with a constant c by $\Theta|u_n(\mathbf{k})\rangle = c|u_n(\mathbf{k})\rangle$ or else two physically different states would have the same energy[11]. Applying time reversal again yields

$$\begin{aligned}
-|u_n(\mathbf{k})\rangle &= \Theta\{\Theta|u_n(\mathbf{k})\rangle\} \\
&= c^\star \Theta|u_n(\mathbf{k})\rangle \\
&= cc^\star|u_n(\mathbf{k})\rangle \\
&= |c|^2|u_n(\mathbf{k})\rangle
\end{aligned}$$

where c gets conjugated by K and the porperty $\Theta^2 = -1$ was used. This series of equalities suggests that, if a state is non degenerate, $|c|^2 = -1$ which is impossible and leads to a contradtiction: the states must be twofold degenerate [11]. This can be made more precise by noting that the time reverse of $|u_n(\mathbf{k})\rangle$ is $|u_{n'}(-\mathbf{k})\rangle$ [18] (a state with reversed crystal momentum possibly living on a different band). Furthermore, the time reversed state also has reversed spin [19].

Special points called time reversal invariant momentas (TRIM) lead to a crucial effect of the theorem. They are points in the Brillouin zone that are sent to points with an equivalent cristal momentum by time reversal ($\mathbf{k} \equiv -\mathbf{k}$) [19]. At TRIM points, the Kramer degeneracy can only be satisfied if the states come from 2 defferent bands having different spins [19]. This forces bands to touch at special points and then split as $\mathbf{k}$ varies trough the effect of spin orbit coupling [11]. In two dimensions, if there are edge states locked inside the gap of the bulk of an insulator, Kramer's theorem forces them to have one of two topologies represented on fig. 1 [11]. Consider the points having cristal momentum norm $k = 0$ (Γ_a) and the point $k = \pi/a$ (Γ_b) (with a being the unit cell size in one direction): they are both TRIM points because $\mathbf{0} = -\mathbf{0}$ and $k = \pi/a$ is equivalent to $k = -\pi/a$. The edge states at the TRIM points must connect two bands by Kramer's theorem. They can either do it in a gapped way (see fig. 1 (A)) or in a metallic gapless way (see fig. 1 (B)).

While the band configuration in fig. 1 (A) can be gapped out by the addition of disorder (time reversal symmetric impurities [12]), configuration (B) always remain gapless and its metallic behavior is robust to deformations of the system (and of its Hamiltonian) [4]. This is where the word *topological* insulator gets its meaning. To be more precise, disforming the band structure from (A) to (B) would requier closing the bulk gap or breaking TRS which would indicate a topological phase transition [4]. Just like it was mentionned in sec. II A, there is a number distinguishing between configuration (A) and (B) refered to as the $\mathbb{Z}_2$ topological invariant.

Since there are *two* different ways to connect TRIM points, the invariant used to qualify the topology of a 2D TRS topological insulators is an integer ν taking values 0 or 1 [18]. If $\nu = 0$ then the band insulator is trivial and, if $\nu = 1$, it is topological [4]. To give precise meaning to this number, it suffice to use the notion of a *Kramer pair*. Two states related to each other by the application of Θ are called a Kramer pair [12]. In fig. 1, only one member of each Kramer pair is shown so it is important to Keep in mind that all points show effectively count for one pair. The ν invariant corresponds to the parity of the number of kramer pairs at a fixed energy [12]. If the number of pairs is even (A) then $\nu = 0$, the insulator is gapped and trivial. Otherwise the number of pairs is always odd (B) and the insulator is topological.

C. Berry phase

Another important concept of topological band theory is the berry phase. To define it, consider a closed **k**-path in the brillouin zone. Each point of the path corresponds to a state $|u_n(\mathbf{k})\rangle$ [18] of a given band. Normally, trough the passage of time eigenstates of the hamiltonian pick up a dynamical phase depending on their associated energy. In topological systems, a new phase comes into play. Suppose, the state of the system is varied slowly on the **k**-path such that the adiabatic theorem ensures that it stays in a state $|u(\mathbf{k})\rangle$ at each point. After traveling along the entire closed loop, the state has gained a global phase factor. While a part of this phase is dynamical, there is an additionnal geometrical phase γ_n called the berry phase [18]. A *parallel* (pun intended) can seen from the field of three-dimensionnal surfaces [20]. A vector lying on a surface is *parallel transported* on a curve of this surface if it is locally parallel to itself everywhere on the curve. If the curve happens to be a closed loop, the initial vector can be compared to a parallel transported version of itself. The angle difference between these two vectors is called the alholonomy [20] and is the equivalent of the berry phase. With the Gauss-Bonet theorem from differential geometry [20], it is possible to relate the anholonomy to the curvature on the surface enclosed by the loop. Similarly the berry phase associated to a loop C can be computed from the integral of the berry curvature F on the surface S enclosed by this loop in the brillouin zone [11]:

$$\gamma_n(C) = \int_S F \, \mathrm{d}^2\mathbf{k}.$$

Following the analogy with the Gauss-Bonet theorem, if the integral runs over the entire brillouin zone, it yields a topological quantity caracterizing the band n. The sum of these topological quantities for all filled bands is the Chern number [18]. For a three dimensionnal surface, the topological invariant is the genus of the surface like it was mentionned in sec. II A.

III. HALL EFFECTS

One of the defining properties of topological insulators is the presence of conducting edge states with an insulating bulk. The example considered here in sec. IV is a two dimensionnal material and its edge is therefore one-dimensionnal. The QH and QSH effects both involve one-dimensionnal conduction. In one dimension, electrons can only either move forward of backward on the edge of the sample and this restriction is central for Hall effects. [6].

A. QH

1. Chern number

2. Impossiblity of backscattering

3. In which material does it happen

B. QSH

IV. HGTE/CDTE HETEROSTRUCTURE

The first observed topological insulator is a mercury telluride heterostructure consisting of a staking of thin HgTe layers between $\mathrm{Hg_xCd_{1-x}Te}$ [21]. If the thickness of the HgTe layers is right, a spin Hall effect arises. To model this system, the

V. CONCLUSION

1. Opening on other topological systems (topological Superconductivity and charge pumps)

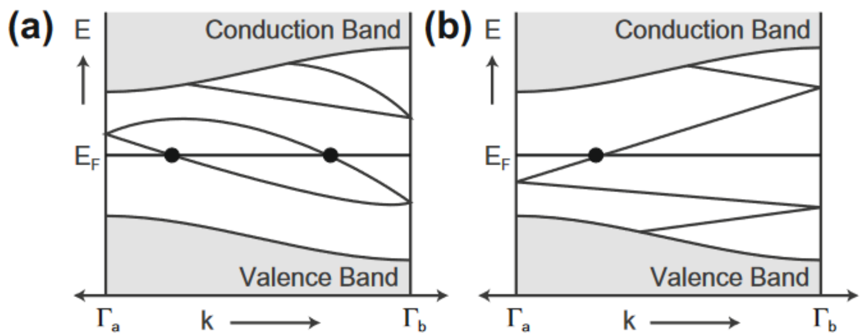

FIG. 1. Schematic representation of the two typical ways Kramer's theorem can be satisfied between TRIM points. (A) Shows a way to connect TRIM points without closing the gap. In this case there is an even amount of intersection points with the Fermi level. (B) Shows second way to connect TRIM points that closes the gap and has an odd number of intersections with the Fermi level [11]. To get a complete picture, on must keep in mind that the energies at $-\mathbf{k}$ are the same as the represented ones along the k values of interest (they are Kramer degenerate).

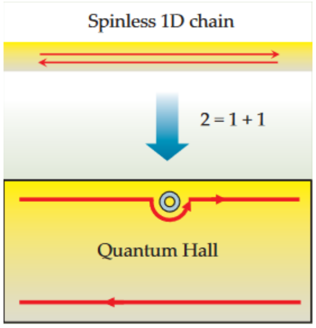

FIG. 2. Schematic representation of the conduction chanels in a spinless quantum Hall system.[6]

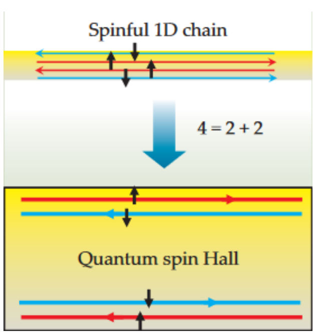

FIG. 3. Schematic representation of the conduction chanels in a spinlful quantum Hall system.

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