Topological Superconductivity

Pierre-Antoine Graham and Jean-Baptiste Bertrand*
(Dated: April 10, 2022)

CONTENTS

LIST OF FIGURES

I. INTRODUCTION

One of the earliest breakthrews allowed by quantum mechanics is the descritpion of metals and insulators with band theory [2] which saw light in 1930. This theory brought a microscopic understanding of the disctinction between these two classes of materials and led to incredible tecnological advances such as the discovery of the transistor by John Bardeen and Walter Brattain in 1947 [3]. Beyond its increadible success, band theory rapidly came in contact with a myriad of intriguing quantum mechanical effects such as the integer quantum Hall effect discovered in 1980 [4]. In 1982, Thouless et al. [5] figured out the topological nature of the effect and, in term, brought topology closer to band theory. Altough the integer quantum Hall effect (QH) requires a strong external magnetic field, it was theorised in 2005 by Charles Kane and Eugene Mele [6] that similar topological properties could be intrinsicaly realised through the quantum spin Hall effect (QSH) [1]. Experiment then showed in 2007 [7] that HgTe/CdTe quantum wells (mercury telluride heterostructure) could produce a QSH effect. The theory and experiment of QSH effect led to a deeper classification of solids with topological band theory [8]. When applied to insulators, the upgraded band theory creates a seperation between the trivial and the *topological insulators* (TI). The latter is generally caracterised by a metallic boundary and an insulating bulk [9] as opposed to trivial insulators which are insulating everywhere. The present review will focus on basic properties of time reversal symetric topological insulators based on the mercury telluride example. Sec.II presents an overview of important ideas from topological band theory. In sec.III, the main properties of the QSH state are given and compared to the QH effect. Finally, a model of HgTe/CdTe quantum is studied in sec. IV.

* Physics Department, Sherbrooke University.

II. ELEMENT OF TOPOLOGICAL BAND THEORY

This sections aims to describe the notion of topological invariant and its consequence on the band structure.

A. Topological equivalence of insulators

In a periodic lattice potential, electrons are described by bloch states $|n, \mathbf{k}\rangle$ where n is a dicrete quantum number and $\mathbf{k}$ is the crystal momentum in the brillouin zone [10]. Each of those states is associated to an energy $E_{n,\mathbf{k}}$. As it varies with $\mathbf{k}$, the energy sweeps a continous range called a band labeled with the number n. Bands are often separated by energy gaps where there are no states. Trivial insulators have a gapped ground state meaning that low energy excitations are forbidden by the presence of the gap. On the contrary, topological insulators have metallic (gapless edge states)[11] and are different from trivial insulators in a fundamental way.

Slow modifications of the hamiltonian of a trivial insulator will change its band structure while leaving it in its ground state. The *deformation* is said to yield equivalent insulators if the gap doesn't close [11]. This equvalence is topological in the same way the continuous disformation from a torus to a coffee mug is. Just like the coffee mug cannot be continuously disformed into a sphere, a trivial insulator cannot be continuously disformed into a topological insulator.

For the coffee mug and the sphere, a number tells if there is topological equivalence or not. This number is called the genus (the number of holes) and it as analogues for topological insulators [12]. The central property of genus is that it only changes when the torus is broken into a sphere in a necessarly discontinuous way. Therefore, the genus is a topological invariant for the sphere and torus. For the QH effect the topological invariant indicating the persistance of the edge states uppon deforming the Hamiltonian is the Chern number. For the QSH effect, the $\mathbb{Z}_2$ invariant is involved [11]. In the following section, the link between time reversal symmetry and this last invariant is detailed. Mixing symmetry, topological invariants and dimensionnality allows for a classification of topological insulators in a form similar to the *periodic table* of elements [13].

B. Time reversal symmetry

In a quantum mechanical setting, an operator U (unitary or anti-unitary) corresponds to a symmetry of a system when $UHU^{-1} = H$ where H is the hamiltonian governing the motion of the system. For a time reversal symetric (TRS) system this means that evolution follows the same equations of motion as a movie of itself set on rewind [10]. The operator sending a system to its time reverse is an anti-unitary operator noted Θ and TRS corresponds to $\Theta H \Theta^{-1} = H$. In the position representation of the spin, momentum and position observables of an electron, the intuitive effects of this time reversal operator are the following [10] :

$$\Theta \mathbf{s} \Theta^{-1} = -\mathbf{s},$$
$$\Theta \mathbf{r} \Theta^{-1} = \mathbf{r},$$
$$\Theta(-i\hbar\nabla)\Theta^{-1} = +i\hbar\nabla$$

meaning it sets electrons spinning backwards with a reversed velocity. The operator satisfying all those porperties is the $\Theta = is_y K$ where K represents complex conjugation and the is_y factor ensures spin is fully reversed [14]. The essential property of the time reversal operator for electrons is that is satsifies $\Theta^2 = -1$. Hence reversing time twice doesn't lead to the original states[13]! This leads to Kramer's theorem which states the following:

$$K$$

Time reversal symmetry (TRS) plays an important role in QSH topological insulators: it protects the gapless edge states [11]. While the QH effect requiers a magnetic field to occur (breaking of TRS), the QSH effect doesn't and it preserves TRS[5].

C. $\mathbb{Z}_2$ invariant

III. HALL EFFECTS

One of the defining properties of topological insulators is the presence of conducting edge states with an insulating bulk. The example considered here in sec. IV is a two dimensionnal material and its edge is therefore one-dimensionnal. The QH and QSH effects both involve one-dimensionnal conduction. In one dimension, electrons can either move forward of backward on the edge of the sample and this restriction is central for Hall effects. [1].

A. QH

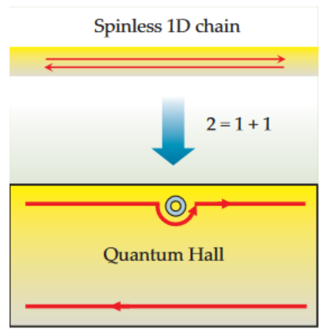

FIG. 1. Schematic representation of the conduction chanels in a spinless quantum Hall system.[1]

B. QSH

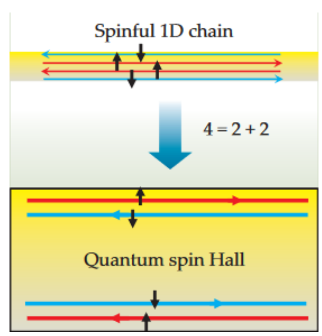

FIG. 2. Schematic representation of the conduction chanels in a spinlful quantum Hall system.

IV. HGTE/CDTE HETEROSTRUCTURE

The first observed topological insulator is a mercury telluride heterostructure consisting of a staking of thin

HgTe layers between $Hg_xCd_{1-x}Te$ [15]. If the thickness of the HgTe layers is right, a spin Hall effect arises. To model this system, the

V. CONCLUSION

1. Opening on other topological systems (topological Superconductivity and charge pumps)

[1] Xiao-Liang Qi and Shou-Cheng Zhang. The quantum spin Hall effect and topological insulators. *Physics Today*, 63(1):33–38, January 2010.

[2] Lillian Hoddeson, Gordon Baym, and Michael Eckert. The development of the quantum-mechanical electron theory of metals: 1928—1933. *Reviews of Modern Physics*, 59(1):287–327, January 1987.

[3] W.F. Brinkman, D.E. Haggan, and W.W. Troutman. A history of the invention of the transistor and where it will lead us. *IEEE Journal of Solid-State Circuits*, 32(12):1858–1865, December 1997. Conference Name: IEEE Journal of Solid-State Circuits.

[4] K. v. Klitzing, G. Dorda, and M. Pepper. New Method for High-Accuracy Determination of the Fine-Structure Constant Based on Quantized Hall Resistance. *Physical Review Letters*, 45(6):494–497, August 1980.

[5] J Cayssol and J N Fuchs. Topological and geometrical aspects of band theory. *Journal of Physics: Materials*, 4(3):034007, July 2021.

[6] C. L. Kane and E. J. Mele. Quantum Spin Hall Effect in Graphene. *Physical Review Letters*, 95(22):226801, November 2005. arXiv: cond-mat/0411737.

[7] Markus Koenig, Steffen Wiedmann, Christoph Bruene, Andreas Roth, Hartmut Buhmann, Laurens W. Molenkamp, Xiao-Liang Qi, and Shou-Cheng Zhang. Quantum Spin Hall Insulator State in HgTe Quantum Wells. *Science*, 318(5851):766–770, November 2007. arXiv: 0710.0582.

[8] Alexey A Soluyanov. TOPOLOGICAL ASPECTS OF BAND THEORY. page 174.

[9] Joel E. Moore. The birth of topological insulators. *Nature*, 464(7286):194–198, March 2010.

[10] R. Shankar. Topological Insulators – A review. *arXiv:1804.06471 [cond-mat]*, April 2018. arXiv: 1804.06471.

[11] C.L. Kane. Topological Band Theory and the z2 Invariant. In *Contemporary Concepts of Condensed Matter Science*, volume 6, pages 3–34. Elsevier, 2013.

[12] Navketan Batra and Goutam Sheet. Physics with Coffee and Doughnuts: Understanding the Physics Behind Topological Insulators Through Su-Schrieffer-Heeger Model. *Resonance*, 25(6):765–786, June 2020.

[13] M. Z. Hasan and C. L. Kane. Topological Insulators. *Reviews of Modern Physics*, 82(4):3045–3067, November 2010. arXiv: 1002.3895.

[14] B. Andrei Bernevig and Taylor L. Hughes. *Topological Insulators and Topological Superconductors*. Princeton University Press, April 2013.

[15] Charles Kane. This newly discovered phase of matter has become one of the hottest topics in condensed-matter physics. It is hard to understand – there is no denying it – but take a deep breath, as Charles Kane and Joel Moore are here to explain what all the fuss is about. *Topological insulators*, page 5, 2011.