Topological Superconductivity

Pierre-Antoine Graham and Jean-Baptiste Bertrand*
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I. INTRODUCTION

One of the earliest breakthrews allowed by quantum mechanics is the descritpion of metals and insulators with band theory [1] which saw light in 1930. This theory brought a microscopic understanding of the disctinction between these two classes of materials and led to incredible tecnological advances such as the discovery of the transistor by John Bardeen and Walter Brattain in 1947 [2]. Beyond its increadible success, band theory rapidly came in contact with a myriad of intriguing quantum mechanical effects such as the integer quantum Hall effect discovered in 1980 [3]. In 1982, Thouless et al. [4] figured out the topological nature of the effect and, in term, brought topology closer to band theory. Altough the integer quantum Hall effect (QH) requires a strong external magnetic field, it was theorised in 2005 by Charles Kane and Eugene Mele [5] that similar topological properties could be intrinsicaly realised through the quantum spin Hall effect (QSH) [6]. Experiment then showed in 2007 [7] that HgTe/CdTe quantum wells (mercury telluride heterostructure) could produce a QSH effect. The theory and experiment of QSH effect led to a deeper classification of solids with topological band theory [8]. When applied to insulators, the upgraded band theory creates a seperation between the trivial and the *topological insulators* (TI). The latter is generally caracterised by a metallic boundary and an insulating bulk [9] as opposed to trivial insulators which are insulating everywhere. The present review will focus on basic properties of time reversal symetric topological insulators based on the mercury telluride example. Sec.II presents an overview of important ideas from topological band theory. In sec.III, the main properties of the QSH state are given and compared to the QH effect. Finally, a model of HgTe/CdTe quantum is studied in sec. IV.

* Physics Department, Sherbrooke University.

II. ELEMENT OF TOPOLOGICAL BAND THEORY

This sections aims to describe the notion of topological invariant and its consequence on the band structure.

A. Topological equivalence of insulators

In a periodic lattice potential, electrons are described by bloch states $|n, \mathbf{k}\rangle$ where n is a dicrete quantum number and $\mathbf{k}$ is the crystal momentum in the brillouin zone [10]. Each of those states is associated to an energy $E_{n,\mathbf{k}}$. As it varies with $\mathbf{k}$, the energy sweeps a continous range called a band labeled with the number n. Bands are often separated by energy gaps where there are no associated states. Trivial insulators have a gapped ground state meaning that low energy excitations are forbidden by the presence of the gap. On the contrary, topological insulators have metallic (gapless) edge states[11] and different from trivial insulators in a fundamental way.

Slow modifications of the hamiltonian of a trivial insulator will change its band structure while leaving it in its ground state. The *deformation* is said to yield equivalent insulators if the gap doesn't close [11]. This equvalence is topological in the same way the continuous disformation from a torus to a coffee mug is. Just like the coffee mug cannot be continuously disformed into a sphere, a trivial insulator cannot be continuously disformed into a topological insulator.

For the coffee mug and the sphere, a number tells if there is topological equivalence or not. This number is called the genus (the number of holes) and it as analogues for topological insulators [12]. The central property of genus is that it only changes when the torus is broken into a sphere in a necessarly discontinuous way. Therefore, the genus is a topological invariant for the sphere and torus. For the QH effect the topological invariant indicating the persistance of the edge states uppon deforming the Hamiltonian is the Chern number. For the QSH effect, a $\mathbb{Z}_2$ invariant is involved [11]. In the following section, the link between time reversal symmetry and this last invariant is detailed. Mixing symmetry, topological invariants and dimensionnality allows for a classification of topological insulators in a form similar to the *periodic table* of elements [13].

B. Time reversal symmetry

In a quantum mechanical setting, an operator U (unitary or anti-unitary) corresponds to a symmetry of a system when $UHU^{-1} = H$ where H is the hamiltonian governing the motion of the system. For a time reversal symetric (TRS) system this means that evolution follows the same equations of motion as a movie of itself set on rewind [14]. The operator sending a system to its time reverse is an anti-unitary operator noted Θ and TRS corresponds to $\Theta H \Theta^{-1} = H$. In the position representation of the spin, momentum and position observables of an electron, the intuitive effects of this time reversal operator are the following [14] :

$$\begin{aligned} \Theta \mathbf{s} \Theta^{-1} &= -\mathbf{s}, \\ \Theta \mathbf{r} \Theta^{-1} &= \mathbf{r}, \\ \Theta(-i\hbar\nabla)\Theta^{-1} &= +i\hbar\nabla \end{aligned}$$

meaning it sets electrons spinning backwards with a reversed velocity. The operator satisfying all those porperties is $\Theta = i s_y K$ where K represents complex conjugation and the $i s_y$ factor ensures spin is fully reversed [15]. The essential property of the time reversal operator for electrons is that is satsifies $\Theta^2 = -1$. Hence reversing time twice for spin 1/2 particles isn't the identity [13]! This leads to Kramer's theorem which is said to be the most important theorem for time reversal invariant two dimensionnal topological insultors [15]. It states the following:

Kramer's Theorem: For a time reversal invariant system containing an odd number of spin 1/2 particles, there are at least two degenerate energy states [15].

To use this theorem here, it must be applied to the periodic bloch states $|n, \mathbf{k}\rangle$ mentionned before. By bloch's theorem [10], these states can be written as the direct product $|u_n(\mathbf{k})\rangle \otimes |\mathbf{k}\rangle$ where $|\mathbf{k}\rangle$ takes care of the tranlsationnal symmetry of the crystal lattice and $|u_n(\mathbf{k})\rangle$ contains the details of the state over a unit cell. The basic idea of Kramers theorem is that provided a state $|u_n(\mathbf{k})\rangle$, the action of the time reversal on it gives a state $\Theta|u_n(\mathbf{k})\rangle$ which has the same eigen energy as the first (if the hamiltonian is time reversal symmetric). Then, if there is no degeneracy the generated state must be related to the first one with a constant c by $\Theta|u_n(\mathbf{k})\rangle = c|u_n(\mathbf{k})\rangle$ or else two physically different states would have the same energy[11]. Applying time reversal again yields

$$\begin{aligned} -|u_n(\mathbf{k})\rangle &= \Theta\{\Theta|u_n(\mathbf{k})\rangle\} \\ &= c^\star \Theta|u_n(\mathbf{k})\rangle \\ &= cc^\star|u_n(\mathbf{k})\rangle \\ &= |c|^2|u_n(\mathbf{k})\rangle \end{aligned}$$

where c gets conjugated by K and the porperty $\Theta^2 = -1$ was used. This series of equalities suggests that, if a state is non degenerate, $|c|^2 = -1$ which is impossible and leads to a contradtiction: the states must be twofold degenerate [11]. This can be made more precise by noting that the time reverse of $|u_n(\mathbf{k})\rangle$ is $|u_{n'}(-\mathbf{k})\rangle$ [15] (a state with reversed crystal momentum possibly living on a different band). Furthermore, the time reversed state also has reversed spin [16].

Special points called time reversal invariant momentas (TRIM) lead to a crucial effect of the theorem. They are points in the Brillouin zone that are sent to points with an equivalent cristal momentum by time reversal ($\mathbf{k} \equiv -\mathbf{k}$) [16]. At TRIM points, the Kramer degeneracy can only be satisfied if the states come from 2 defferent bands having different spins [16]. This forces bands to touch at special points and then split as $\mathbf{k}$ varies trough the effect of spin orbit coupling [11]. In two dimensions, if there are edge states locked inside the gap of the bulk of an insulator, Kramer's theorem forces them to have one of two topologies represented on fig. 1 [11]. Consider the points having cristal momentum norm $k = 0$ (Γ_a) and the point $k = \pi/a$ (Γ_b) (with a being the unit cell size in one direction): they are both TRIM points because $\mathbf{0} = -\mathbf{0}$ and $k = \pi/a$ is equivalent to $k = -\pi/a$. The edge states at the TRIM points must connect two bands by Kramer's theorem. They can either do it in a gapped way (see fig. 1 (A)) or in a metallic gapless way (see fig. 1 (B)).

While the band configuration in fig. 1 (A) can be gapped out by the addition of disorder, configuration (B) always remain gapless and its metallic behavior is robust to deformations of the system (and of its Hamiltonian) [4]. This is where the word *topological* insulator gets its meaning.

Time reversal symmetry (TRS) plays an important role in QSH topological insulators: it protects the gapless edge states [11]. While the QH effect requiers a magnetic field to occur (breaking of TRS), the QSH effect doesn't and it preserves TRS[4].

C. $\mathbb{Z}_2$ invariant

III. HALL EFFECTS

One of the defining properties of topological insulators is the presence of conducting edge states with an insulating bulk. The example considered here in sec. IV

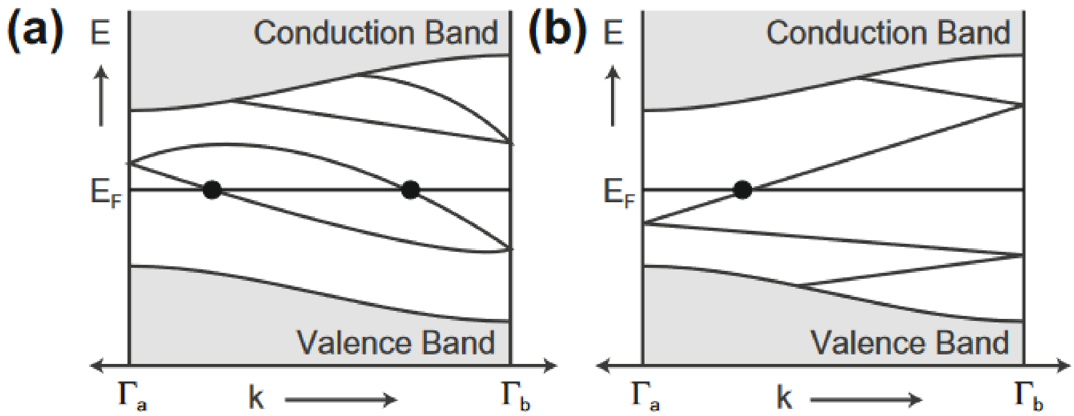

FIG. 1. Schematic representation of the two typical ways Kramer's theorem can be satisfied between TRIM points. (A) Shows a way to connect TRIM points without closing the gap. In this case there is an even amount of intersection points with the Fermi level. (B) Shows second way to connect TRIM points that closes the gap and has an odd number of intersections with the Fermi level [11].

is a two dimensionnal material and its edge is therefore one-dimensionnal. The QH and QSH effects both involve one-dimensionnal conduction. In one dimension, electrons can either move forward of backward on the edge of the sample and this restriction is central for Hall effects. [6].

A. QH

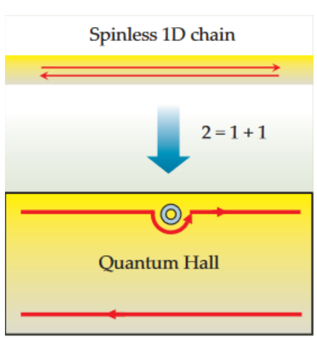

FIG. 2. Schematic representation of the conduction chanels in a spinless quantum Hall system.[6]

B. QSH

IV. HGTE/CDTE HETEROSTRUCTURE

The first observed topological insulator is a mercury telluride heterostructure consisting of a staking of thin HgTe layers between $Hg_xCd_{1-x}Te$ [17]. If the thickness of the HgTe layers is right, a spin Hall effect arises. To model this system, the

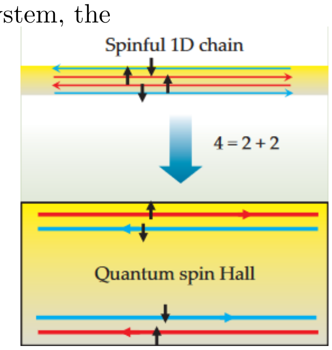

FIG. 3. Schematic representation of the conduction chanels in a spinlful quantum Hall system.

V. CONCLUSION

1. Opening on other topological systems (topological Superconductivity and charge pumps)

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