Exercise session 1
MODA 2021/2022

1. Linear Programming

Imagine, you want to produce clay bowls and clay cups. You have 8kg of clay available every week and want to work maximally 40 hours every week.

- For each bowl you need 400g of clay, it takes 1 hour to make it, and you can sell it for 4 Euro.
- For each cup you need 300g of clay, it takes 2 hours to make it, and you can sell it for 5 Euro.

[1.1] Formulate the linear integer programming task of maximizing your weekly profit under the given constraints and solve it graphically. How many bowls and how many cups do you have to make?

1. Linear Programming

[1.1] Solution:

Bowl: 400g of clay, 1 hour, 4 Euro
Cup: 300g of clay, 2 hours, 5 Euro
Max 8kg and max 40 hours

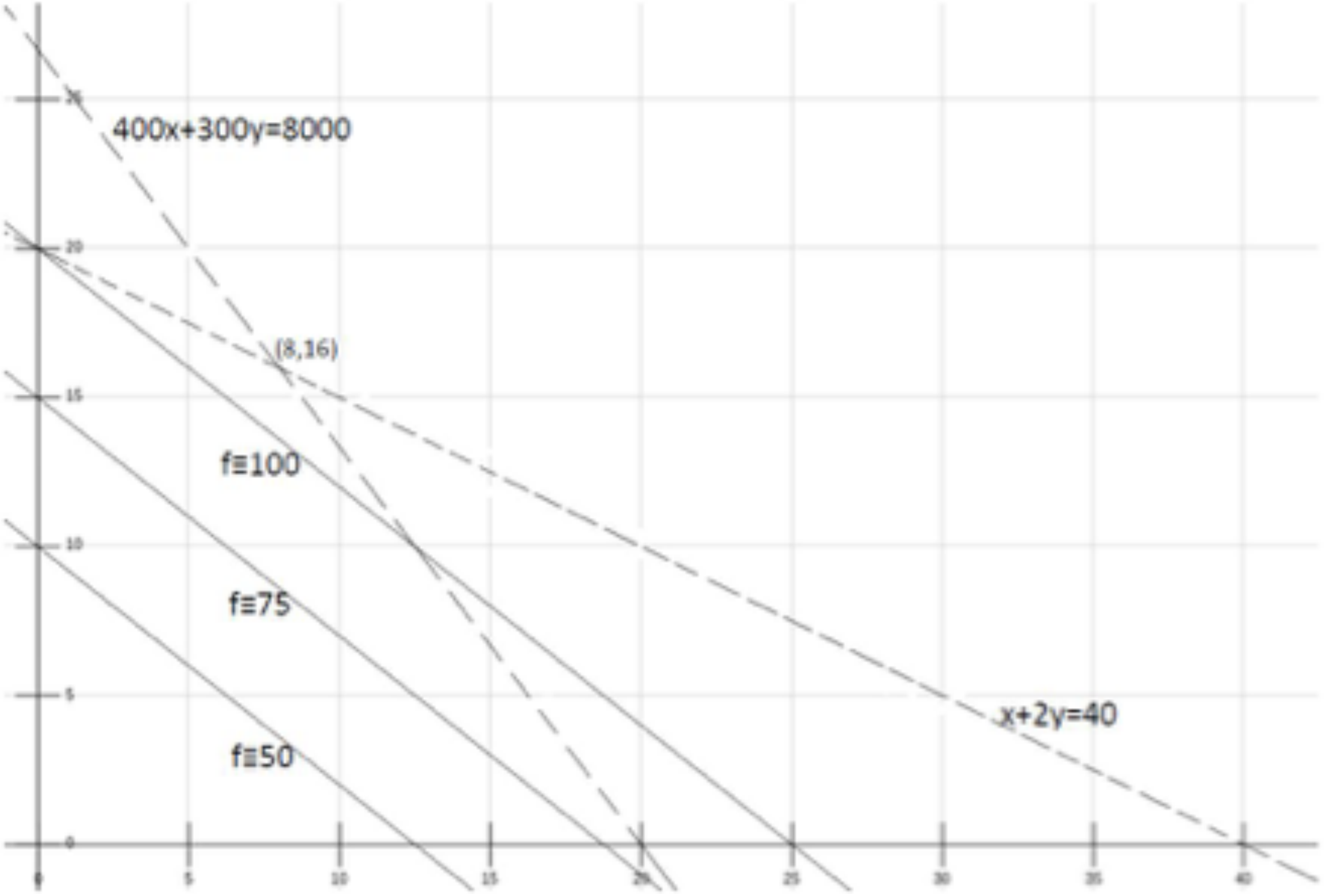
$$f(x,y) = 4x + 5y \rightarrow max$$

subject to

$$g_1(x,y) = 400x + 300y - 8000 \leq 0$$

$$g_2(x,y) = x + 2y - 40 \leq 0$$

$$x, y \in \mathbb{N}$$



1. Linear Programming

[1.2]* What are the active constraints?
Active constraints are the constraints that, if you relax them, it will make your working week more profitable. Is it the time or the amount of available material (clay), or both?

[1.3]* What would be the resulting number of bowls and cups if the time to produce a cup would be 3 hours and all other constants remain the same? (note, that the output must be integer)

1. Linear Programming

[1.4] Show that the feasible region of a linear program is a convex set.

A convex set is a set, for which for every two points in the set, say $\boldsymbol{u} \in X$ and $\mathbf{v} \in X$, also the points of the line segment, that connects these points, is part of X, i.e.,
$\{l\boldsymbol{u} + (1 - l)\boldsymbol{v} \mid l \in [0,1]\} \subseteq X$.

1. Linear Programming

[1.4] Solution:

For a linear programming problem, its constraints can be written as: $Ax \leq b$.
Assume that $\mathbf{u}$ and $\mathbf{v}$ are two points in X. Then $A\mathbf{u} \leq b$ and $A\mathbf{v} \leq b$.

$$A(l\mathbf{u} + (1-l)\mathbf{v}) = lA\mathbf{u} + (1-l)A\mathbf{v} \leq lb + (1-l)b = b$$

$$A(l\mathbf{u} + (1-l)\mathbf{v}) \leq b)$$

Therefore, $l\mathbf{u} + (1-l)\mathbf{v} \in X$.

1. Linear Programming

[1.5] Formulate the Karush Kuhn Tucker conditions for this problem.

1. Linear Programming

[1.5] Solution:

(1) $4 - 400\lambda_1 - \lambda_2 = 0$
(2) $5 - 300\lambda_1 - 2\lambda_2 = 0$
(3) $\lambda_1(400x + 300y - 8000) = 0$
(4) $\lambda_2(x + 2y - 40) = 0$
(5) $\lambda_1, \lambda_2 \geq 0$

$f(x, y) = 4x + 5y \to max$
subject to
$g_1(x, y) = 400x + 300y - 8000 \leq 0$
$g_2(x, y) = x + 2y - 40 \leq 0$
$x, y \in \mathbb{N}$

$\lambda_1 = \frac{3}{500}, \lambda_2 = \frac{8}{5}, x = 8, y = 16$

2. Formulation and Classification of Mathematical Programming Problems

Formulate the following problems in the language of mathematical programming.

The form should be as the following:

$$
\begin{aligned}
f(x_1,\ldots,x_d) &\rightarrow max, \text{ s.t.} \\
g_1(x_1,\ldots,x_d) &\leq c_1 \\
&\vdots \\
g_n(x_n,\ldots,x_d) &\leq c_n \\
h_1(x_1,\ldots,x_d) &= c_{n+1} \\
&\vdots \\
h_q(x_n,\ldots,x_d) &= c_{n+q}
\end{aligned}
$$

Please also specify the domain of each variable c_i: $\{0, 1\}$ / $\mathbb{N}$ / $\mathbb{R}$.

2. Formulation and Classification of Mathematical Programming Problems

[2.1] Formulate the mathematical problem of finding the diameter of the graph, that is the longest path in a network, that touches each node exactly once. Let's assume the graph has N nodes.

2. Formulation and Classification of Mathematical Programming Problems

[2.1] Solution:

Let's represent, for instance, the path $v_1 \to v_2 \to v_4 \to v_3$ through matrix as the following:

$\boldsymbol{v_1}$	$\boldsymbol{v_2}$	$\boldsymbol{v_3}$	$\boldsymbol{v_4}$
1	0	0	0
0	1	0	0
0	0	0	1
0	0	1	0

We will denote the elements of such matrix as $x_{ij} \in \{0, 1\}$, $i, j = 1, \dots, N$.
For each path in the graph, we can construct matrix $\{x_{ij}\}$.

The distance between nodes i and j let's note as d_{ij}.

2. Formulation and Classification of Mathematical Programming Problems

[2.1] Solution:

For each line (vertex) we are looking for a 1-element in the next line. And when we find one, we add the distance between corresponding nodes

v_1	v_2	v_3	v_4
1	0	0	0
0	1	0	0
0	0	0	1
0	0	1	0

$$\sum_{i=1}^{N-1} \sum_{j=1}^{N} \sum_{k=1}^{N} x_{ij} x_{i+1,k} d_{jk} \to max$$

$$\sum_{i=1}^{N} x_{ij} = 1, \quad j = 1, \ldots, N$$

$$\sum_{j=1}^{N} x_{ij} = 1, \quad i = 1, \ldots, N$$

2. Formulation and Classification of Mathematical Programming Problems

[2.2] A clique is a subset of nodes in a graph, where every two nodes in the clique are connected. Formulate the problem of finding the clique of maximal size. A network (or graph) can be represented by an adjacency matrix A. And $a_{ij} = 1$ if and only if node i and node j are connected, and 0 otherwise. The values a_{ij} can be considered as constants.

2. Formulation and Classification of Mathematical Programming Problems

[2.2] Solution:

Let’s represent, for instance, the clique $v_1 v_4 v_5 v_6 v_7$ through array as the following:

$\boldsymbol{v_1}$	$\boldsymbol{v_2}$	$\boldsymbol{v_3}$	$\boldsymbol{v_4}$	$\boldsymbol{v_5}$	$\boldsymbol{v_6}$	$\boldsymbol{v_7}$	$\boldsymbol{v_8}$
1	0	0	1	1	1	1	0

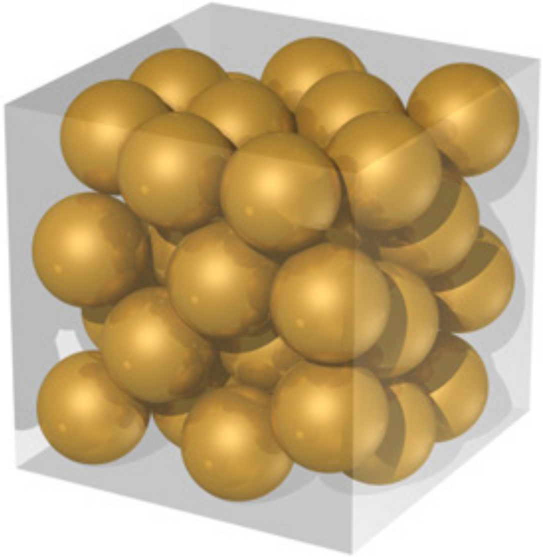
$$\sum_{i=1}^{N} x_i \to max$$

$$\sum_{i=1}^{N}\sum_{j=1}^{N} x_i x_j a_{ij} - \left(\sum_{i=1}^{N} x_i\right)\left(\sum_{i=1}^{N} x_i\right) = 0$$

We will denote the elements of such array as $x_i \in \{0, 1\}$, $i = 1, \dots, N$.
For each clique in the graph, we can construct array $\{x_i\}$.
Matrix $\{a_{ij}\}$ is the adjacency matrix of the graph, we assume that a_{ii} = 1, $i = 1, \dots, N$.

2. Formulation and Classification of Mathematical Programming Problems

[2.3] Formulate the following question as a mathematical programming problem. What is the biggest diameter d, such that k balls (spheres) can fit into a square box of side length 1 meter? The balls are allowed to touch but cannot be squeezed.



2. Formulation and Classification of Mathematical Programming Problems

[2.3] Solution:

Let’s note the coordinates of the center of the ball i as (x_i, y_i, z_i)

d – diameter of the balls

$$d \to max$$

s.t.

$$x_i \leq 1 - \frac{d}{2}; x_i \geq \frac{d}{2};$$

$$y_i \leq 1 - \frac{d}{2}; y_i \geq \frac{d}{2};$$

$$z_i \leq 1 - \frac{d}{2}; z_i \geq \frac{d}{2};$$

$$(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 \geq d^2, i = 1, ..., k, j = 1, ..., k, i \neq j;$$

$$d, x_1, ..., x_k, y_1, ..., y_k, z_1, ..z_k \in [0, 1]$$

2. Formulation and Classification of Mathematical Programming Problems

[2.4]* Now turn the question in 2.3 around. Formulate the problem of fitting as many equal-sized balls as possible of a given diameter d into a square box of side length 1 meter. The Euclidean distance between the centers of the ball can be used to check, whether or not the boxes overlap. The bounding box around the ball can be used to check whether they fit into the box. You are not allowed to squeeze the balls.

3. Order Theory

Let $\overrightarrow{d_1} = (2,1)$ and $\overrightarrow{d_2} = (1,2)$ denote the generators of a 2-D cone. Then the cone is given as the set: $C = \{\lambda_1\overrightarrow{d_1} + \lambda_2\overrightarrow{d_2} \,|\, \lambda_1 \geq 0,\, \lambda_2 \geq 0\}$

[3.1] Visualize the cone C in a 2-D diagram. Is it a pointed cone? Is it convex?

3. Order Theory

[3.1] Solution:

Convex:

(Line connecting two points in the cone is fully contained in the cone)

For two points $u \in C$ and $v \in C$:

$$u = \lambda_{1u} d_1 + \lambda_{2u} d_2, v = \lambda_{1v} d_1 + \lambda_{2v} d_2$$

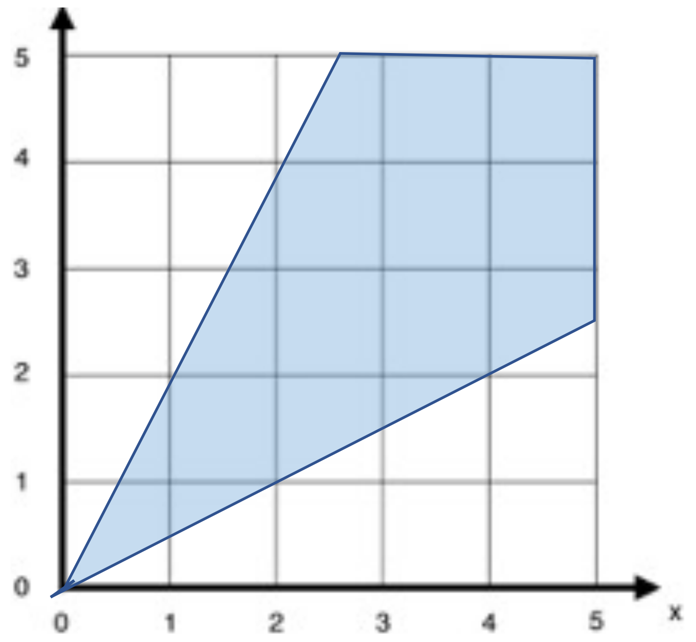
A point x on the line segment from u to v:

$$\exists\ a, b: \ \ a + b = 1, a \geq 0, b \geq 0$$

$$x = au + bv = a\lambda_{1u} d_1 + a\lambda_{2u} d_2 + b\lambda_{1v} d_1 + b\lambda_{2v} d_2$$
$$= (a\lambda_{1u} + b\lambda_{1v}) d_1 + (a\lambda_{2u} + b\lambda_{2v}) d_2$$

and hence for x exists

$$\lambda_{1x} = (a\lambda_{1u} + b\lambda_{1v}) \geq 0, \qquad \lambda_{2x} = (a\lambda_{2u} + b\lambda_{2v}) \geq 0,$$
$$x = v_1 d_1 + v_2 d_2 \Rightarrow x \in C$$



Pointed:

$C \cap -C = \{0\}$ and $C \neq \{0\}$

3. Order Theory

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[3.2] Show that the cone order $\vec{a} \preccurlyeq_C \vec{b} \Leftrightarrow \vec{b} \in \vec{a} \oplus C$ is a partial order.

3. Order Theory

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Solution:

- Show reflexivity ($\forall x, y$: $(x = y) \Rightarrow (x \preccurlyeq_C y)$):

$$(x = y) \Rightarrow (y \in x \oplus C)\ (since\ 0\ is\ part\ of\ C)$$

- Show transitivitiy ($\forall x, y$: $(x \preccurlyeq_C y) \wedge (y \preccurlyeq_C z) \Rightarrow (x \preccurlyeq_C z)$):

$$(\mathrm{y} \in x \oplus C) \wedge (z \in y \oplus C) \Rightarrow (y \oplus C) \subseteq (x \oplus C) \Rightarrow (z \in x \oplus C)$$

- Show antisymmetry $(x \preccurlyeq_C y) \wedge (y \preccurlyeq_C x) \Rightarrow (x = y)$

$$(x \preccurlyeq_C y) \wedge (y \preccurlyeq_C x) \Leftrightarrow (x \in y \oplus C) \wedge (x \in y \oplus -C) \Rightarrow x \in y + \{0\}$$

3. Order Theory

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[3.3] Draw the Hasse diagram for $\{(0, 0), (1, 1), (2, 2), (1, 3), (3, 1)\}$ with respect to $\preccurlyeq_C$.

3. Order Theory

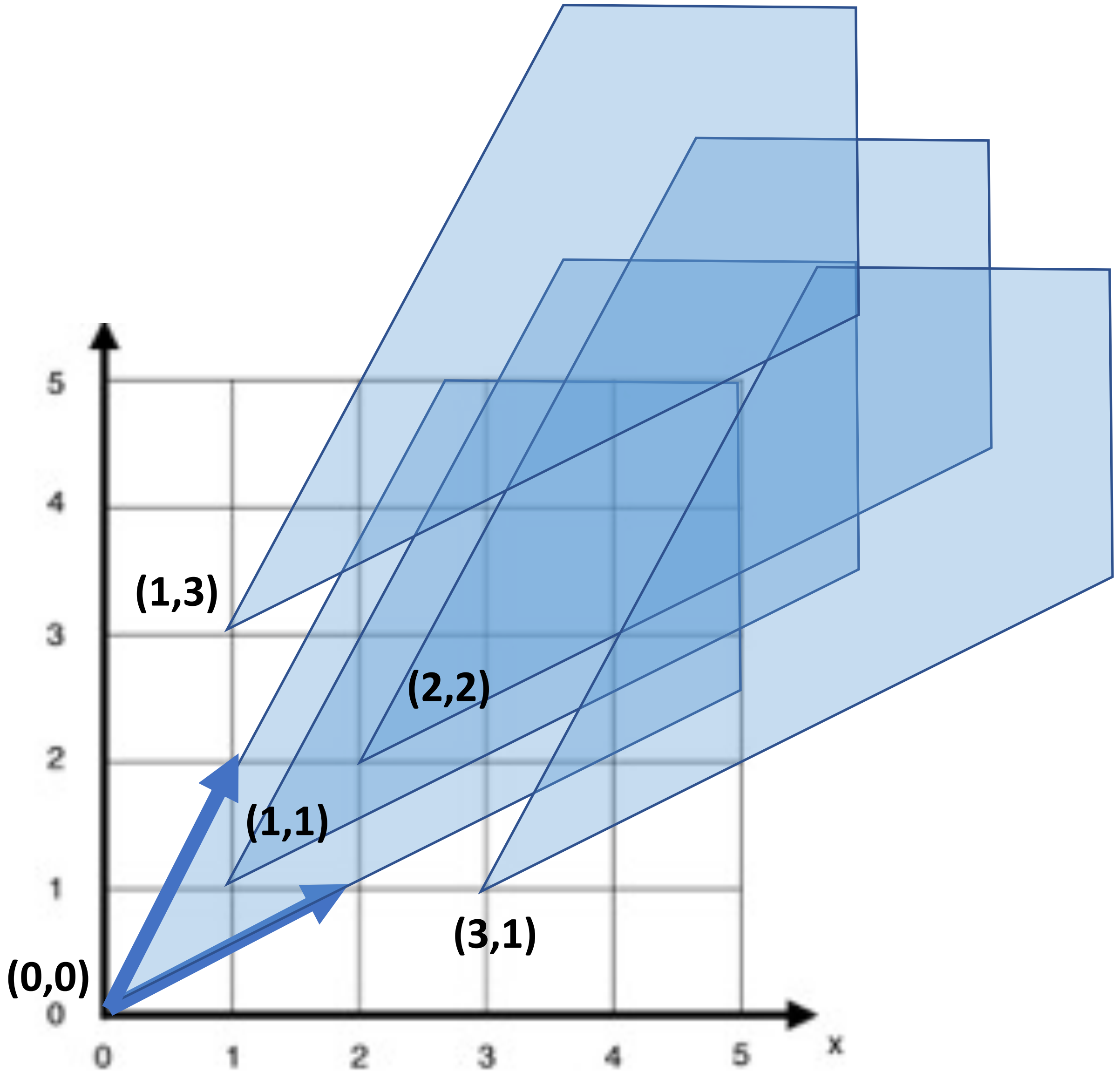
[3.3] Solution:

Hasse diagram - type of mathematical diagram used to represent a finite partially ordered set, in the form of a drawing of its transitive reduction

(2,2)

(1,3) (1,1) (3,1)

(0,0)



3. Order Theory

Let $\overrightarrow{d_1} = (2, 1)$ and $\overrightarrow{d_2} = (1, 2)$ denote the generators of a 2-D cone. Then the cone is given as the set: $C = \left\{\lambda_1 \overrightarrow{d_1} + \lambda_2 \overrightarrow{d_2} \,\middle|\, \lambda_1 \geq 0,\, \lambda_2 \geq 0\right\}$

[3.4] Is the weak componentwise order an extension of $\preccurlyeq_C$ or vice versa?

(Weak componentwise order means that every coordinate not less.
It can be said that $\preccurlyeq_2$ is an order extension of $\preccurlyeq_1$, iff $(x_1 \preccurlyeq_1 x_2) \Rightarrow (x_1 \preccurlyeq_2 x_2)$)

4. Lagrange Multiplier Rule

Let a point in space (e.g., a satellite) have the coordinates $\vec{c} = (c_1, c_2, c_3)^T$.
Moreover, let a point on a sphere (the earth) have the coordinates $\vec{x} = (x_1, x_2, x_3)^T$.
Now the point $\vec{x}^* = (x_1^*, x_2^*, x_3^*)^T$ on the earth that has the closest distance to the satellite is a solution to the following constrained optimization problem.

$$(x_1 - c_1)^2 + (x_2 - c_2)^2 + (x_3 - c_3)^2 \to \min$$

$$x_1{}^2 + x_2^2 + x_3^2 - r^2 = 0$$

Solve the constrained optimization problem using the Lagrange Multiplier Rule.
Consider $\vec{c}$ and r to be constant and $\vec{c}$ to lie outside the sphere.

4. Lagrange Multiplier Rule

Solution:

$$2l_1x_1 - 2l_1c_1 + 2l_2x_1 = 0 \ (A)$$
$$2l_1x_2 - 2l_1c_2 + 2l_2x_2 = 0 \ (B)$$
$$2l_1x_3 - 2l_1c_3 + 2l_2x_3 = 0 \ (C)$$
$${x_1}^2 + x_2^2 + x_3^2 - r^2 = 0 \ (D)$$

$l_1 = 0 \Rightarrow l_2 \neq 0 \Rightarrow x_1 = x_2 = x_3 =$ 0 (this violates equation D)

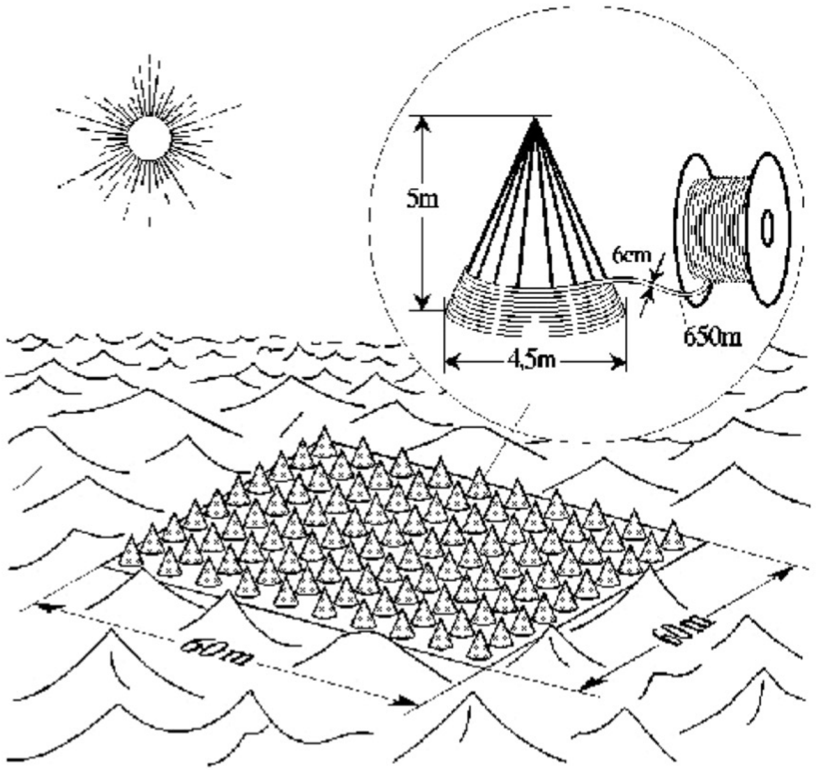
$l_1 = 1$:

$$2x_1 + 2x_1l_2 = 2c_1 \Rightarrow x_1 = c_1/(1 + l_2)$$
$$2x_2 + 2x_2l_2 = 2c_2 \Rightarrow x_2 = c_2/(1 + l_2)$$
$$2x_3 + 2x_3l_2 = c_3 \Rightarrow x_3 = c_3/(1 + l_2)$$

$$\sum(c_i/(1 + l_2))^2 = r^2 \Rightarrow \sqrt{\frac{\sum c_i^2}{r^2}} - 1 = l_2 \ \Rightarrow x_i = rc_i/\sqrt{\sum c_i^2}$$

5. Graphical and analytical solution of Pareto optimization

A heliomite is a cone shaped bio-reactor for hydrogen production. In order to build a large scale hydrogen farm the following trade-off between the cost of the heliomite, determined by its volume, and its visible surface should be maximized for a radius r (in meters) in [1, 5] and a height h (in meters) in [1, 10]. The area of the visible surface to be maximized is given by $f_1 = \pi r\sqrt{r^2 + h^2}$, and the volume to be minimized $f_2 = \frac{1}{3}\pi r^2 h$.



Hydrogen farm in the Sahara consisting of cone-shaped heliomites (https://present5.com/ingo-rechenberg-the-future-becomes-more-biological-a/)

5. Graphical and analytical solution of Pareto optimization

[5.1] Formulate the Fritz John Condition for the problem:

$f_1 = \pi r\sqrt{r^2 + h^2} \ \rightarrow max$, and $f_2 = \frac{1}{3}\pi r^2 h \ \rightarrow min$

5. Graphical and analytical solution of Pareto optimization

[5.1] Formulate the Fritz John Condition for the problem:

$f_1 = \pi r\sqrt{r^2 + h^2} \ \rightarrow max$, and $f_2 = \frac{1}{3}\pi r^2 h \ \rightarrow min,$

$g_1(r, h) = r - 1 \geq 0,$

$g_2(r, h) = 5 - r \geq 0,$

$g_3(r, h) = h - 1 \geq 0,$

$g_4(r, h) = 10 - h \geq 0.$

Solution:

$$\exists \lambda > 0, \nu > 0\text{:}\ \lambda_1 \nabla f_1(r, h) + \lambda_2 \nabla f_2(r, h) - \sum \nu_i \nabla g_i(x) = 0$$

$$\lambda_i\ g_i(r, h) = 0, i = 1, \ldots, 4$$

With

$\nabla\ f_1(r, h) = \left(\frac{\pi\left(2r^2 + h^2\right)}{\sqrt{r^2+h^2}}, \frac{\pi r h}{\sqrt{r^2+h^2}}\right), \nabla f_2(r, h) = \left(\frac{2}{3}\pi r h, \frac{1}{3}\pi r^2\right),$

$\nabla g_1(r, h) = (1{,}0), \nabla g_2(r, h) = (-1{,}0), \nabla g_3(r, h) = (0, 1), \nabla g_4(r, h) = (0, -1)$

5. Graphical and analytical solution of Pareto optimization

[5.1] Formulate the Fritz John Condition for the problem:

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$g_2(r, h) = 5 - r \geq 0,$

$g_3(r, h) = h - 1 \geq 0,$

$g_4(r, h) = 10 - h \geq 0.$

Solution:

$$\lambda_1 \frac{\pi\left(2r^2+h^2\right)}{\sqrt{r^2+h^2}} + \lambda_2 \frac{2}{3}\pi r h - \nu_1 + \nu_2 = 0$$

$$\lambda_1 \frac{\pi r h}{\sqrt{r^2+h^2}} + \lambda_2 \frac{1}{3}\pi r^2 - \nu_3 + \nu_4 = 0$$