MODA A1.2 Order theory

Q2.1 First let us investigate the K-order in $\mathbb{R}^2$.
We show for this case the equivalence between the K-order $\prec_K$ and the Pareto order $\prec_P$.
There are the following possible cases:

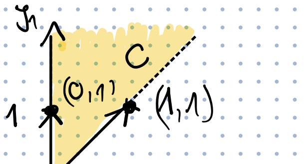
$y_1 \leq x_1, \ y_2 < x_2 \Rightarrow y \prec_K x \ \wedge \ y \prec_P x$

$y_1 < x_1, \ y_2 \leq x_2 \Rightarrow y \prec_K x \ \wedge \ y \prec_P x$

In all other cases it does not hold $y \prec_K x$ nor $y \prec_P x$.
Hence $y \prec_K x$ if and only if $y \prec_P K$

In the case of $\mathbb{R}^3$ to show that $\prec_K$ is a partial order we need to show that $\preceq_K$ is reflexive, antisymmetric and transitive. ($\preceq_K$: $y \preceq_K x :\Longleftrightarrow y \prec_K x$ or $y = x$)
Counterexample to transitivity:
$x = (3,2,2) \preceq_K y = (1,3,3) \preceq_K z = (2,1,4)$
but it does not hold that $x \preceq z$ since $z \prec_K x$.

Q2.2 (a) visualize the cone



(b) Is the cone pointed: Obviously $C \cap -C = \{(0,0)\}$
So the cone is pointed.
Is it convex? >>

If C is convex this means that for every two points in the cone, say $y^{(1)} = u_1 (0,1) + v_1 (1,1)$ and $y^{(2)} = u_2 (0,1) + v_2 (1,1)$ the line segment $\overline{y^{(1)} y^{(2)}}$ is fully contained in the cone.

A point y' is on the line segment $\overline{y^{(1)} y^{(2)}}$ if and only if $\exists \tau \in [0,1]: \quad y'_\tau = \tau y^{(1)} + (1-\tau) y^{(2)}$. Let us see if those points have also coefficients u and v that qualify them to be points in C (that is $u \geq 0$, $v \geq 0$).

$$y'_\tau = \tau \left(u_1 (0,1) + v_1 (1,1)\right) + (1-\tau)\left(u_2 (0,1) + v_2 (1,1)\right)$$

$$= \underbrace{\left(\tau u_1 + (1-\tau) u_2\right)}_{\geq 0} (0,1) + \underbrace{\left(\tau v_1 + (1-\tau) v_2\right)}_{\geq 0} (1,1)$$

The existence of non-negative multipliers shows that y' is indeed part of the cone C for any $\tau \in [0,1]$ □

(c) Partial order is reflexive, antisymmetric, transitive

<u>reflexive</u>: $y^{(1)} = y^{(2)} \Rightarrow y^{(1)} \leq_C y^{(2)}$

$y^{(1)} = y^{(2)} \Leftrightarrow y^{(2)} \in y^{(1)} \oplus C$ or $y^{(2)} - y^{(1)} \in C$

$y^{(1)} - y^{(2)} = (0,0)$ And $(0,0) \in C$.

<u>antisymmetric</u>: $y^{(2)} - y^{(1)} \in C \wedge y^{(1)} - y^{(2)} \in C$

$\boxed{\text{Likewise for } y^{(2)} - y^{(1)} \in C}$

$$\Rightarrow y^{(1)} = y^{(2)}.$$

$$y^{(1)} - y^{(2)} = \underbrace{(u_1 - u_2)}_{c_{0,1}} (0,1) + \underbrace{(v_1 - v_2)}_{c_{1,1}} (1,1)$$

$\boxed{\text{$c_{0,1}$ and $c_{1,1}$ must be non-negative if $y^{(1)} - y^{(2)} \in C$}}$

We can show: $(u_1 - u_2) \geq 0 \wedge (v_1 - v_2) \geq 0 \wedge$

* $(u_2 - u_1) \geq 0 \wedge (v_2 - v_1) \geq 0$

is only satisfied if $u_1 = u_2$ and $v_1 = v_2$ □

Transitivity: $y^{(1)} \preceq_C y^{(2)}$ and $y^{(2)} \preceq_C y^{(3)} \Leftrightarrow$

$$\underbrace{y^{(2)} - y^{(1)} \in C}_{(1)} \quad \text{and} \quad \underbrace{y^{(3)} - y^{(2)} \in C}_{(2)} \Rightarrow \underbrace{y^{(3)} - y^{(1)}}_{(3)} \in C$$

$(1) \Rightarrow \; u_2 - u_1 \geq 0 \;\wedge\; v_2 - v_1 \geq 0$

$(2) \Rightarrow \; u_3 - u_2 \geq 0 \;\wedge\; v_3 - v_2 \geq 0$

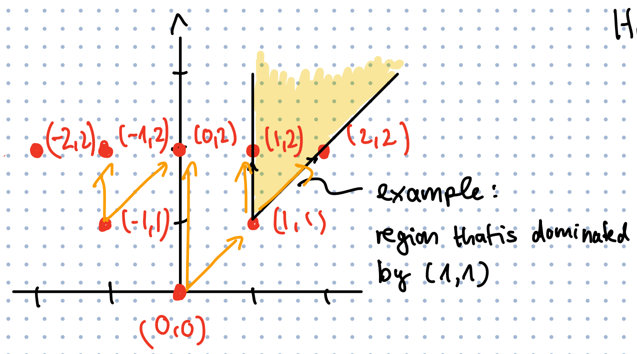
} using the same parameterization as in * above

Now let us investigate whether it follows $u_3 - u_1 \geq 0$

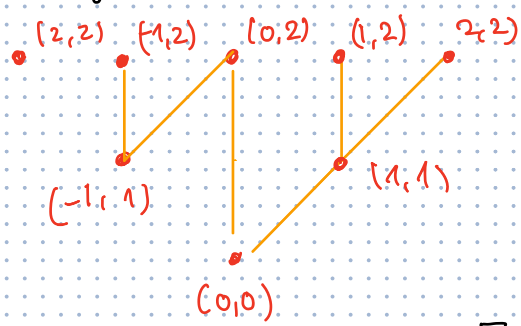
$$u_3 - u_1 = \underbrace{u_3 - u_2}_{\geq 0} + \underbrace{u_2 - u_1}_{\geq 0} \geq 0$$

$$v_3 - v_1 = v_3 - v_2 + v_2 - v_1 \geq 0 \quad \text{(likewise)}$$

(d) Hasse diagram: It is useful to draw first the points and dominance cones in an coordinate system.



Hasse diagram:



$\square$