Multicriteria Optimization and Decision Analysis

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Answer each question on a separate sheet. Put your name, student number and the number of the question you are answering on each and every sheet. Provide full explanations with each of the answers!

Each question is weighted by a number of points, as indicated. The total number of points is 100. Success!

1. [**20 points**] The following table describes diffent options for a building design, based on the criteria Comfort Index (CI), Energy Efficiency (EE), Structural Performance (SP), and Price. Star ratings are between 1 (very low performance) to 5 (excellent performance).

Building	CI Stars	EE Stars	SP Stars	Price 1000 Euro
Cozy	4	1	3	100
EnergyStar	3	4	3	200
Cheap	4	1	3	100
Solid	4	4	5	200

1a. [**5 points**] Draw a Parallel coordinate diagram with all solutions.

1b. [**5 points**] Identify a solution pairs that is indifferent, a solution pair that is incomparable, the efficient set, and the Pareto front?

1c. [**5 points**] Describe the steps to be taken by the user to make a desirability function of the Derringer Suich type.

1d. [**5 points**] A linear weighting utility function should be determined based on pairwise comparisons. Describe the robust ordinal regression linear program that finds the utility function, consistent with the statements: 'EnergyStar is better or equal than Cozy', and 'Solid is better or equal than Cheap'.

2. [**20 points**] Formulate the following problems in the language of mathematical programming, using the following form:
$f(\mathbf{x}) \rightarrow \min$, s.t. $g_i(\mathbf{x}) \geq 0, i = 1, ..., q$, $h_j(\mathbf{x}) = 0, j = 1, ..., s$, $\mathbf{x} \in S$
where S is the decision space, that needs to be specified for the problem. It can be either $\mathbb{R}^n$, $\{0, 1\}^n$, or $\mathbb{Z}^n$ or combinations of these.

2a. [**10 points**] A ship has to be loaded with a subset from a set of

$n = 200$ items. The value of the items is $v_1, v_2, ..., v_n$. The weight of the items is $w_1, w_2, ..., w_n$. The task is to load the ship with at most 50 items and the total weight of the load is not allowed to exceed 3000kg. Items with the index $\{1, ..., 50\}$ cannot be combined with items with the index $\{101, ..., 150\}$. The total value of the loaded items should be maximized.

2b. **[10 points]** A path planning problem for a robot is given by the problem of finding the nodes of a minimal lenght polyline of n vertexes from $\mathbf{v}_{start}$ to $\mathbf{v}_{end}$. The path is described by a sequence of vertices $\left((v_1^1, v_2^1), (v_1^2, v_2^2), ..., (v_1^n, v_2^n)\right)$, where $\mathbf{v}^1 = \mathbf{v}_{start}$ denotes the first vertex, and $\mathbf{v}^n = \mathbf{v}_{end}$ denotes the last vertex.
The robot can move across the plane where $\mathbf{l} \in \mathbb{R}^2$ is the bottom left corner, and $\mathbf{u} \in \mathbb{R}^2$ the upper right corner of the plane. Every point of the polyline must be on a point in the set $\mathbf{c}^1, ..., \mathbf{c}^k$, denoting charging stations. The maximum Euclidean distance between two subsequent point must be below or equal to D_{max}. Note, that the Euclidean distance between two points a and b is given by $d(\mathbf{a}, \mathbf{b}) := \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}$.

If you cannot formulate the full problem, try to formulate as many objectives and constraints as possible.

3. **[15 points]** Let us consider a linear program: $f_1(x_1, x_2) = -x_1 - 2x_2 \to \min$, s.t. $g_1(x_1, x_2) = 4x_1 + x_2 - 4 \leq 0$, $g_2(x_1, x_2) = \frac{1}{2}x_1 + x_2 - 2 \leq 0$, $x_1 \geq 0$, $x_2 \geq 0$.

   3a. **[5 points]** Solve the problem graphically.

   3b. **[5 points]** Formulate the Karush Kuhn Tucker conditions for general nonlinear problems with inequality constraints, and specialize them for the linear program above. What are the active constraints and inactive constraints in the optimal point (minimizer)? Discuss, why the KKT conditions hold for this point (hint: you can give a geometrical reason based on the constraint and objective function gradients).

   3c. **[5 points]** Add a second objective function $f_2(x_1, x_2) = -x_1 \to \min$ and determine the efficient set.

4. **[10 points]** A building with a box shape should be planned with a floor heating. The thermal efficiency is determined by the surface area. The length and width of the building is x_1 and the height is x_2. The surface area is $(x_1)^2 + 4x_2x_1$. The volume is $(x_1)^2x_2$. The surface area is to be minimized. The volume has to be equal to $V = 32m^3$

   4a. **[5 points]** State the equation system of the Lagrange multiplier rule.

   4b. **[5 points]** Find the optimal solution.

5. **[20 points]** A cone order $\mathbf{x}^1 \prec_{\mathcal{C}} \mathbf{x}^2 \Leftrightarrow \mathbf{x}^2 \in \mathbf{x}^1 \oplus \mathcal{C}$ is given by the polyhedral cone $\mathcal{C}$ with basis $\mathbf{u} = (1, -0.5)$ and $\mathbf{v} = (-0.5, 1)$, that is

$\mathcal{C} = \{\mathbf{y} \in \mathbb{R}^2 | \exists \lambda_1 \geq 0, \lambda_2 \geq 0 : \mathbf{y} = \lambda_1 \mathbf{u} + \lambda_2 \mathbf{v}\}$

5a. [**5 points**] Describe graphically the cone and the set of points that is dominated by $\mathbf{p} = (1, 1)$.

5b. [**5 points**] Show that the cone order $\prec_{\mathcal{C}}$ is a partial order.

5c. [**5 points**] Draw the Hasse diagram for the set $\{(1,2), (0,3), (1,3),$ $(2,0),\ (0,0)\}$.

5d. [**5 points**] Recall the weak componentwise order on $\mathbb{R}^2$ is defined as $\mathbf{a} \leq \mathbf{b} \Leftrightarrow a_1 \leq b_1 \wedge a_2 \leq b_2$. Discuss whether the weak componentwise order extends the cone order or the cone order extends the weak componentwise order.

6. [**15 points**] In evolutionary multiobjective optimization it is the goal to find an approximation to the Pareto front.

6a. [**5 points**] Determine the Hypervolume indicator of the population $\{(7,0),\ (6,6),\ (0,8), (9,9), (1,7)\}$ for reference point $(10, 10)$. What are the hypervolume contributions of the points?

6b. [**5 points**] Describe the non-dominated sorting procedure. What is the maximal number of ranks that it can produce for an chain of n points and for an anti-chain of n points?

6c. [**5 points**] Metaheuristics are often used for solving *NP hard* problems, such as integer programming. Describe the difference between *NP complete* and *NP hard* problems. Given $NP \neq P$ can be proven, what would be the difference between problems in P and *NP complete* in terms of asymptotical computational time complexity?

Wishing you success!