Unit: Orders and Pareto Dominance

Learning Goals

I. Formal definition of Pareto Dominance orders in Multiobjective Optimization, and relations that are based on it
II. Learning about different types of ordered sets, in particular pre-orders and partial orders, and their fundamental properties.
III. How to compare orders? How to represent them in a diagrams?
IV. What is the geometrical interpretation of Pareto orders and the closely related cone orders?

Pareto* dominance

Given a problem: $f_1(\mathbf{x}) \to$ min, $\ldots, f_n(\mathbf{x}) \to$ min with $\mathbf{x} \in \mathcal{X} \subseteq \mathbb{S}$:

A solution $\mathbf{x}^1$ is said to *dominate* a solution $\mathbf{x}^2$ in the *weak componentwise order*, iff*:

$$\forall i \in \{1, \ldots, n_f\} : f_i(\mathbf{x}^1) \leq f_i(\mathbf{x}^2)$$

A solution $\mathbf{x}^1$ is said to *Pareto dominate* a solution $\mathbf{x}^2$, iff:

$$\forall i \in \{1, \ldots, n_f\} : f_i(\mathbf{x}^1) \leq f_i(\mathbf{x}^2)$$

$$\text{and } \exists i \in \{1, \ldots, n_f\} : f_i(\mathbf{x}^1) < f_i(\mathbf{x}^2)$$

Pareto dominance is a special case of a strict partial order and of a cone order. We will now study what that means ...

*Here "iff" stands for: "if and only if"

*Vilfredo Pareto: Italian economician, 1848-1923

Orders, why?

In multicriteria decision analysis different preference orders are discussed, e.g. lexicographical, Pareto dominance.

Orders are special types of binary relations.

Path in Plitvice Lakes National Park, Croatia,
(c) Michael Emmerich

Used to model:
Hierarchies, Subsumption
Time, Causality
Preferences
Implications

Two ways to define orders:
(1) Axiomatically
(2) Geometrically

Next: Binary relations $\rightarrow$ pre-orders $\rightarrow$ partial orders $\rightarrow$ cone order $\rightarrow$ weak componentwise order $\rightarrow$ Pareto dominance order $\rightarrow$ perfect order $\rightarrow$ total or linear order.

Binary relations

A *binary relation* $\mathcal{R}$ on some set $\mathcal{S}$ is defined as a subset of $\mathcal{S} \times \mathcal{S}$

We write $\mathbf{x}^1 \mathcal{R} \mathbf{x}^2 \Leftrightarrow (\mathbf{x}^1, \mathbf{x}^2) \in \mathcal{R}$.

Properties of relations (cf. Ehrgott05):

- $\mathcal{R}$ is *reflexive* $\Leftrightarrow \forall \mathbf{x} \in \mathcal{S} : \mathbf{x} \mathcal{R} \mathbf{x}$
- $\mathcal{R}$ is *irreflexive* $\Leftrightarrow \forall \mathbf{x} \in \mathcal{S} : \neg(\mathbf{x} \mathcal{R} \mathbf{x})$.
- $\mathcal{R}$ is *symmetric* $\Leftrightarrow \forall \mathbf{x}^1, \mathbf{x}^2 \in \mathcal{S} : \mathbf{x}^1 \mathcal{R} \mathbf{x}^2 \Leftrightarrow \mathbf{x}^2 \mathcal{R} \mathbf{x}^1$
- $\mathcal{R}$ is *antisymmetric* $\Leftrightarrow \forall \mathbf{x}^1, \mathbf{x}^2 \in \mathcal{S} : \mathbf{x}^1 \mathcal{R} \mathbf{x}^2 \wedge \mathbf{x}^2 \mathcal{R} \mathbf{x}^1 \Rightarrow \mathbf{x}^1 = \mathbf{x}^2$
- $\mathcal{R}$ is *asymmetric* $\Leftrightarrow \forall \mathbf{x}^1, \mathbf{x}^2 \in \mathcal{S} : \mathbf{x}^1 \mathcal{R} \mathbf{x}^2 \Rightarrow \neg(\mathbf{x}^2 \mathcal{R} \mathbf{x}^1)$
- $\mathcal{R}$ is *transitive* $\Leftrightarrow \forall \mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3 \in \mathcal{S} : \mathbf{x}^1 \mathcal{R} \mathbf{x}^2 \wedge \mathbf{x}^2 \mathcal{R} \mathbf{x}^3 \Rightarrow \mathbf{x}^1 \mathcal{R} \mathbf{x}^3$

A *preorder* (quasi-order) is a relation that is both transitive and reflexive.

Preorders

Recall: A *preorder* (quasi-order) is a relation that is both transitive and reflexive.

We write $\mathbf{x}^1 \preceq \mathbf{x}^2$ as shorthand for $\mathbf{x}^1 \mathcal{R} \mathbf{x}^2$

We call $(\mathcal{S}, \preceq)$ a *preordered set.*

Given any preorder, we can define the closely related concepts:

$\mathbf{x}^1 \prec \mathbf{x}^2 :\Leftrightarrow \mathbf{x}^1 \preceq \mathbf{x}^2 \wedge \neg(\mathbf{x}^2 \preceq \mathbf{x}^1)$ (strict preference)

$\mathbf{x}^1 \sim \mathbf{x}^2 :\Leftrightarrow \mathbf{x}^1 \preceq \mathbf{x}^2 \wedge \mathbf{x}^2 \preceq \mathbf{x}^1$ (indifference)

A pair of solutions $\mathbf{x}^1, \mathbf{x}^2 \in \mathcal{S}$ is said to be incomparable, iff neither $\mathbf{x}^1 \preceq \mathbf{x}^2$ nor $\mathbf{x}^2 \preceq \mathbf{x}^1$. We write $\mathbf{x}^1 || \mathbf{x}^2$

The relation $\prec$ is irreflexive and transitive, and (as a consequence) asymmetric

The relation $\sim$ is reflexive, transitive and symmetric

What about the properties of the relation $||$?

Minimal elements of a pre-ordered set

Minimal elements of a pre-ordered set are elements that are not preceded by any other element.

x^1 is maximal, iff not exists $x^2 \in S$ such that $x^1 \prec x^2$.

For any finite set (except the empty set $\emptyset$) there exists at least one minimal and one maximal element.

Topological sorting allows to output elements of an order in a sequence, such that minimal elements are always first and maximal elements are last.

To discuss topological sorting algorithm, we will first introduce *covers* relation.

The *covers* relation

Def.: The **covers** relation on a pre-ordered set is defined for any two elements $\mathbf{x}^1$ and $\mathbf{x}^2$ as:

$\mathbf{x}^1$ is covered by $\mathbf{x}^2$, in symbols $\mathbf{x}^1 \lhd \mathbf{x}^2$ iff $\mathbf{x}^1 \prec \mathbf{x}^2$ and $\mathbf{x}^1 \preceq \mathbf{x}^3 \prec \mathbf{x}^2$ implies $\mathbf{x}^1 = \mathbf{x}^3$.

Examples:

In set $\mathbb{N}$ with the common order relations $<$ and $\leq$ we obtain $\mathbf{x}^1 \lhd \mathbf{x}^2$, iff $\mathbf{x}^2 = \mathbf{x}^1 + 1$.

How to specialize this for the subset relation ? Can the covers relation be defined for $(\mathbb{R}, \leq)$?

We can now represent order on finite set as an acyclic graph (V, E) with $V = \mathcal{S}$ and $E \subseteq \mathcal{S} \times \mathcal{S}$, $(\mathbf{x}^1, \mathbf{x}^2) \in E :\Leftrightarrow \mathbf{x}^1 \lhd \mathbf{x}^2$, using, e.g., the transitive reduction algorithm [Aho et al.'72].

How to compute covers relation from order relation on finite set:
Aho, A. V.; Garey, M. R.; Ullman, J. D. (1972), "The transitive reduction of a directed graph", SIAM Journal on Computing 1 (2): 131-137, doi:10.1137/0201008,MR 0306032.

Partial orders, posets

A **partial order** is a preorder that is also antisymmetric.

Recall: $\mathcal{R}$ is *antisymmetric* $\Leftrightarrow \forall \mathbf{x}^1, \mathbf{x}^2 \in \mathcal{S} : \mathbf{x}^1 \mathcal{R} \mathbf{x}^2 \wedge \mathbf{x}^2 \mathcal{R} \mathbf{x}^1 \Rightarrow \mathbf{x}^1 = \mathbf{x}^2$

We inherit the relations of indifference and strict precedence from pre-orders:

$$\mathbf{x}^1 \prec \mathbf{x}^2 :\Leftrightarrow \mathbf{x}^1 \preceq \mathbf{x}^2 \wedge \neg(\mathbf{x}^2 \preceq \mathbf{x}^1) \text{ (strict preference)}$$
$$\mathbf{x}^1 \sim \mathbf{x}^2 :\Leftrightarrow \mathbf{x}^1 \preceq \mathbf{x}^2 \wedge \mathbf{x}^2 \preceq \mathbf{x}^1 \text{ (equivalence)}$$
$$\Rightarrow \text{(antisymmetry) } \mathbf{x}^1 = \mathbf{x}^2$$

We call $(\mathcal{S}, \preceq)$ a partially ordered set *or* **poset**.

Pareto order, invariances

The order defined by the Pareto dominance relation is a special case of a strict partial order.

Additional properties of the Pareto order are:

1. Translation invariance:
   $\forall \mathbf{x} \in \mathbb{R}^m : \mathbf{x}^1 \preceq \mathbf{x}^2 \Rightarrow \mathbf{x}^1 + \mathbf{x} \preceq \mathbf{x}^2 + \mathbf{x}$

2. Scale invariance: $\forall \alpha \in \mathbb{R}^+$: $\mathbf{x}^1 \preceq \mathbf{x}^2 \Rightarrow \alpha\mathbf{x}^1 \preceq \alpha\mathbf{x}^2$.

Besides the Pareto order only cone orders based on convex, pointed, polyhedral cones share this property (Noghin '91). (later more)

Examples: Preordered and partially ordered sets

Examples for preorders that are not partial orders:

Sorting complex numbers by their absolute value*:

$\forall \mathbf{z}^1, \mathbf{z}^2 \in \mathbb{C}$: $\mathbf{z}^1 \preceq \mathbf{z}^2 \Leftrightarrow |\mathbf{z}^1| \leq |\mathbf{z}^2|$.

Order of events by time and place, if there can be many events to a certain time and place.

Examples of partial orders:

Subset relation: $A \preceq B :\Leftrightarrow A \subseteq B$

*$\forall \mathbf{z} \in \mathbb{C}$(Complex numbers): $|\mathbf{z}| = \sqrt{Im(z)^2 + Re(z)^2}$.

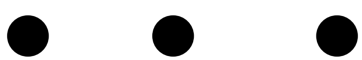
Total (linear) orders and anti-chains

A **total** order (also: linear order, chain) is a partial order that fulfils also the *comparability* or *totality* axiom:

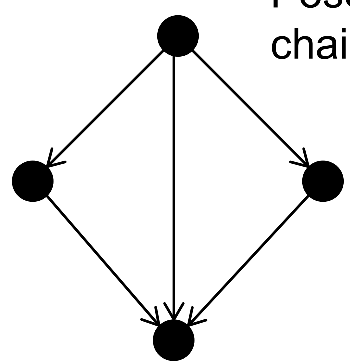
$$\forall \mathbf{x}^1, \mathbf{x}^2 :\in \mathcal{X} : \mathbf{x}^1 \preceq \mathbf{x}^2 \vee \mathbf{x}^2 \preceq \mathbf{x}^1$$

A poset $(\mathcal{S}, \preceq)$ is said to be an **antichain**, iff:

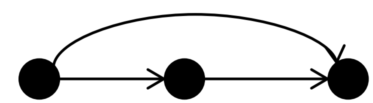
$$\forall \mathbf{x}^1, \mathbf{x}^2 \in \mathcal{S} : \mathbf{x}^1 \neq \mathbf{x}^2 \Rightarrow \mathbf{x}^1 || \mathbf{x}^2$$



antichain



Poset that is neither a chain nor an antichain



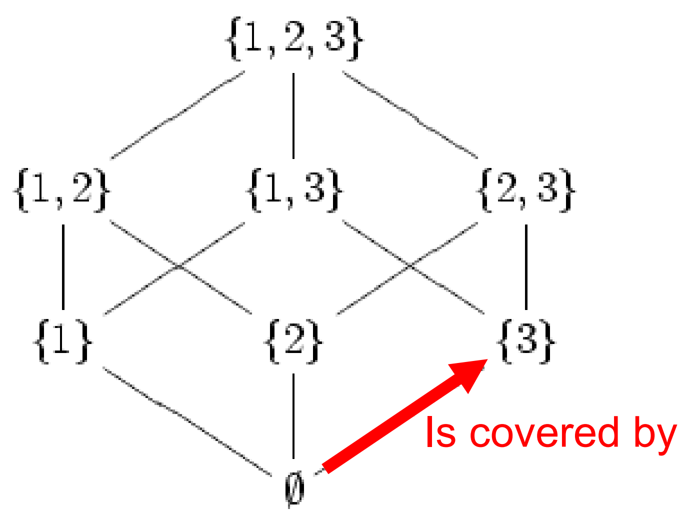
chain
(total order)

What about Pareto fronts? Are they chains, anti-chains or none of these?

Example for a Hasse diagram

Let $\mathcal{A} = \{1, 2, 3\}$ be the poset defined by the subset relation on the set of all subsets of (A), in symbols $\wp(\mathcal{A})$, i.e.:

$$\forall \mathbf{x}^1, \mathbf{x}^2 \in \wp(\mathcal{A}) : \mathbf{x}^1 \preceq \mathbf{x}^2 \Leftrightarrow \mathbf{x}^1 \subseteq \mathbf{x}^2.$$



Remark: $x \preceq x'$, iff x is depicted below x' and there is a path from x to x' that leads upwards.

Drawing the Hasse* diagram

As a reminder: We say $\mathbf{x}^1$ is covered by $\mathbf{x}^2$, in symbols $\mathbf{x}^1 \lhd \mathbf{x}^2$ iff $\mathbf{x}^1 \prec \mathbf{x}^2$ and $\mathbf{x}^1 \preceq \mathbf{x}^3 \prec \mathbf{x}^2$ implies $\mathbf{x}^1 = \mathbf{x}^3$.

(1) To each point $\mathbf{x} \in \mathbf{X}$ assign a point $p(\mathbf{x})$, depicted by a small circle with centre $p(\mathbf{x})$.

(2) For each covering pair $\mathbf{x}^1$ and $\mathbf{x}^2$ draw a line segment $\ell(\mathbf{x}^1, \mathbf{x}^2)$.

(3) Choose the center of circles in a way such that:

(a) if $\mathbf{x}^1 \lhd \mathbf{x}^2$, then $p(\mathbf{x}^1)$ is positioned below $p(\mathbf{x}^2)$.

(b) the circle of $\mathbf{x}^3$ does not intersect the line segment $\ell(\mathbf{x}^1, \mathbf{x}^2)$ if $\mathbf{x}^3 \neq \mathbf{x}^1$ and $\mathbf{x}^3 \neq \mathbf{x}^2$.

* Helmut Hasse, German mathematician, 1898-1979

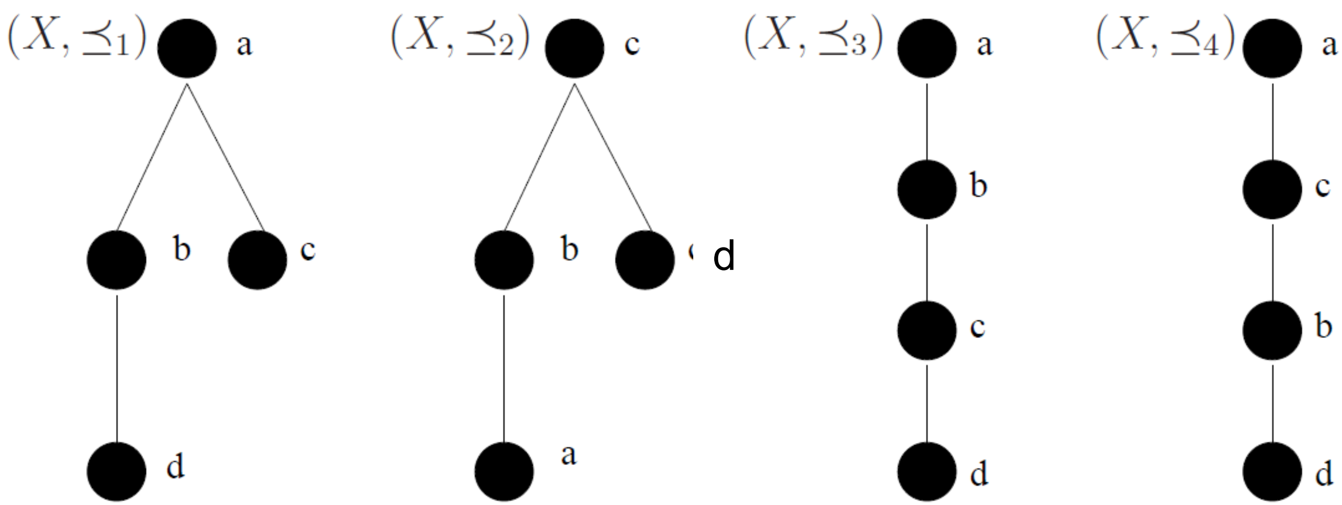
(Linear) extension

Given two preorders $\preccurlyeq_1$ and $\preccurlyeq_2$ on some set X
it can be said that $\preccurlyeq_2$ is an **order extension** of $\preccurlyeq_1$,

iff $x_1 \preccurlyeq_1 x_2 \Rightarrow x_1 \preccurlyeq_2 x_2$.

Moreover, $\preccurlyeq_2$ is a **linear extension** of $\preccurlyeq_1$,

iff $\preccurlyeq_2$ is an order extension of $\preccurlyeq_1$ and $\preccurlyeq_2$ is a total order.



What about these orders on {a,b,c,d}? Identify (linear) extensions! The orders are Represented by their Hasse diagrams.

Orders on the Euclidean space

Several orders on the Euclidean space $\mathbb{R}^m$ can be useful in multi-objective optimization [Ehrgott05 p. 12].

Weak componentwise order:
$\mathbf{y}^1 \leq \mathbf{y}^2 :\Leftrightarrow y_k^1 \leq y_k^2, k = 1, \ldots, n_f$

Pareto order:
$\mathbf{y}^1 \prec \mathbf{y}^2 :\Leftrightarrow y_k^1 \leq y_k^2, k = 1, \ldots, n_f, \mathbf{y}^1 \neq \mathbf{y}^2$

Strict componentwise order
$\mathbf{y}^1 < \mathbf{y}^2 :\Leftrightarrow y_k^1 < y_k^2, k = 1, \ldots, n_f$

Lexikographic order(priority on last component)
$\mathbf{y}^1 \leq_{\mathsf{lex}} \mathbf{y}^2 :\Leftrightarrow \sum_{k=1}^{n_f} \mathrm{I}(y_k^1 < y_k^2) 2^k \leq \sum_{i=1}^{n_f} \mathrm{I}(y_k^2 < y_k^1) 2^k$

$$\mathrm{I} : \{true, false\} \rightarrow \{0, 1\}, \mathrm{I}(true) = 1, \mathrm{I}(false) = 0$$

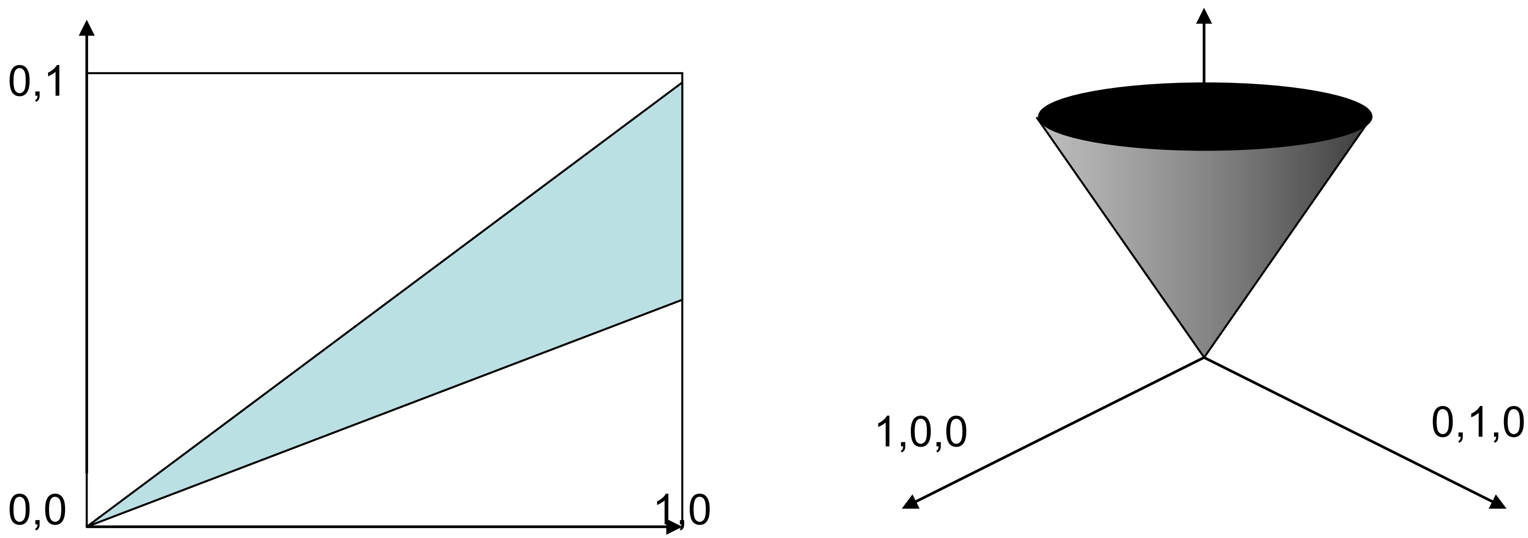
Max order
$\mathbf{y}^1 \leq_{\mathsf{max}} \mathbf{y}^2 :\Leftrightarrow \mathsf{max}_{k \in \{1 \ldots n_f\}} y_k^1 \leq \mathsf{max}_{k \in \{1 \ldots n_f\}} y_k^2$

Cones

Cones are very useful for geometric interpretation of orders in euclidian space.

A subset $\mathcal{C} \subseteq \mathbb{R}^p$ is called a cone, iff $\alpha\mathbf{d} \in \mathcal{C}$ for all $\mathbf{d} \in \mathcal{C}$ for all $\alpha \in \mathbb{R}, \alpha > 0$.

Remark: Informally, this means that with any point of a set also the ray through this point starting from zero belongs to the cone.



Examples for cones in 2-D (l) and 3-D (r)

Remark: According to the definition above it is possible for $\mathbf{0}$ to belong to a cone or not to belong to it. Verify!

Minkowski* sum and scalar multiplication

Two useful definitions:

The Minkowski sum of two sets $\mathcal{S}^1$ and $\mathcal{S}^2$ is defined as:

$$\mathcal{S}^1 \oplus \mathcal{S}^2 := \{s^1 + s^2 | s^1 \in \mathcal{S}^1, s^2 \in \mathcal{S}^2\}$$

If $\mathcal{S}^1$ is a singleton $\{s\}$, we may write $s \oplus \mathcal{S}^2$ instead of $\{s\} \oplus \mathcal{S}^2$.

The product of a scalar α and a set $\mathcal{S}$ is defined as:

$$\alpha\mathcal{S} := \{\alpha s | s \in \mathcal{S}\}$$

Find examples in $\mathbb{R}$ and $\mathbb{R}^2$!

*Jewish-German mathematician 1864-1909, Goettingen

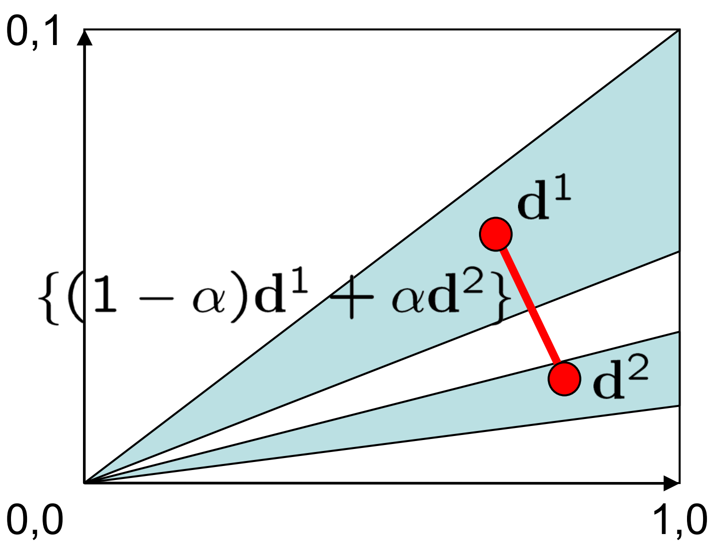
Some properties of cones (2)

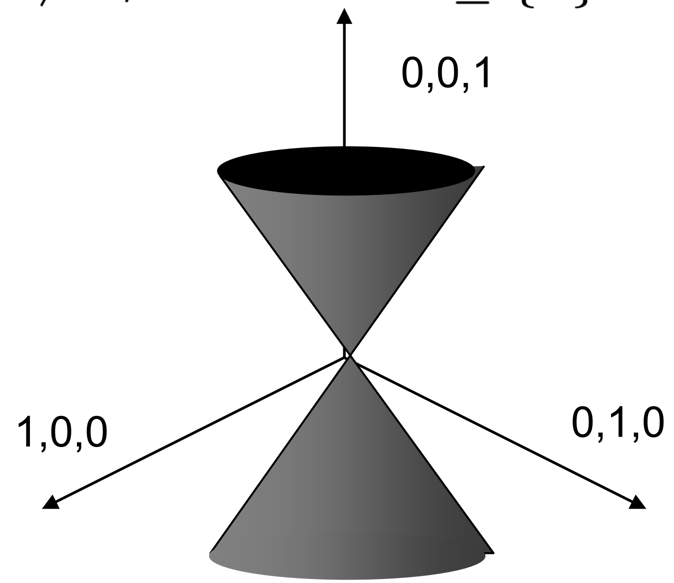
A cone $\mathcal{C} \in \mathbb{R}^m$ is called:

nontrivial or proper, iff $\mathcal{C} \neq \emptyset$.

convex, iff $\alpha \mathbf{d}^1 + (1-\alpha)\mathbf{d}^2 \in \mathcal{C}$ for all $\mathbf{d}^1$ and $\mathbf{d}^2 \in \mathcal{C}$ for all $0 < \alpha < 1$.

pointed, iff for $\mathbf{d} \in \mathcal{C}, \mathbf{d} \neq 0, -\mathbf{d} \notin \mathcal{C}$, i.e. $\mathcal{C} \cap -\mathcal{C} \subseteq \{0\}$





What are the properties of the cones depicted above ?

Polyhedral cones

Def.: Polyhedral cone: A polyhedral cone in $\mathbb{R}^m$is determined by a number of k direction vectors $\mathbf{d_1} \in \mathbb{R}^m$, ... $\mathbf{d}_k \in \mathbb{R}^m$ (cone generators). It is the set that comprises all positive linear combinations of these vectors: $C = \{\mathbf{y} \in \mathbb{R}^m \quad | \text{ exists } \lambda_1 \geq 0, \ldots, \lambda_k \geq 0 : \mathbf{y} = \sum_{i=1}^{k} \lambda_i \mathbf{d}_i\}$

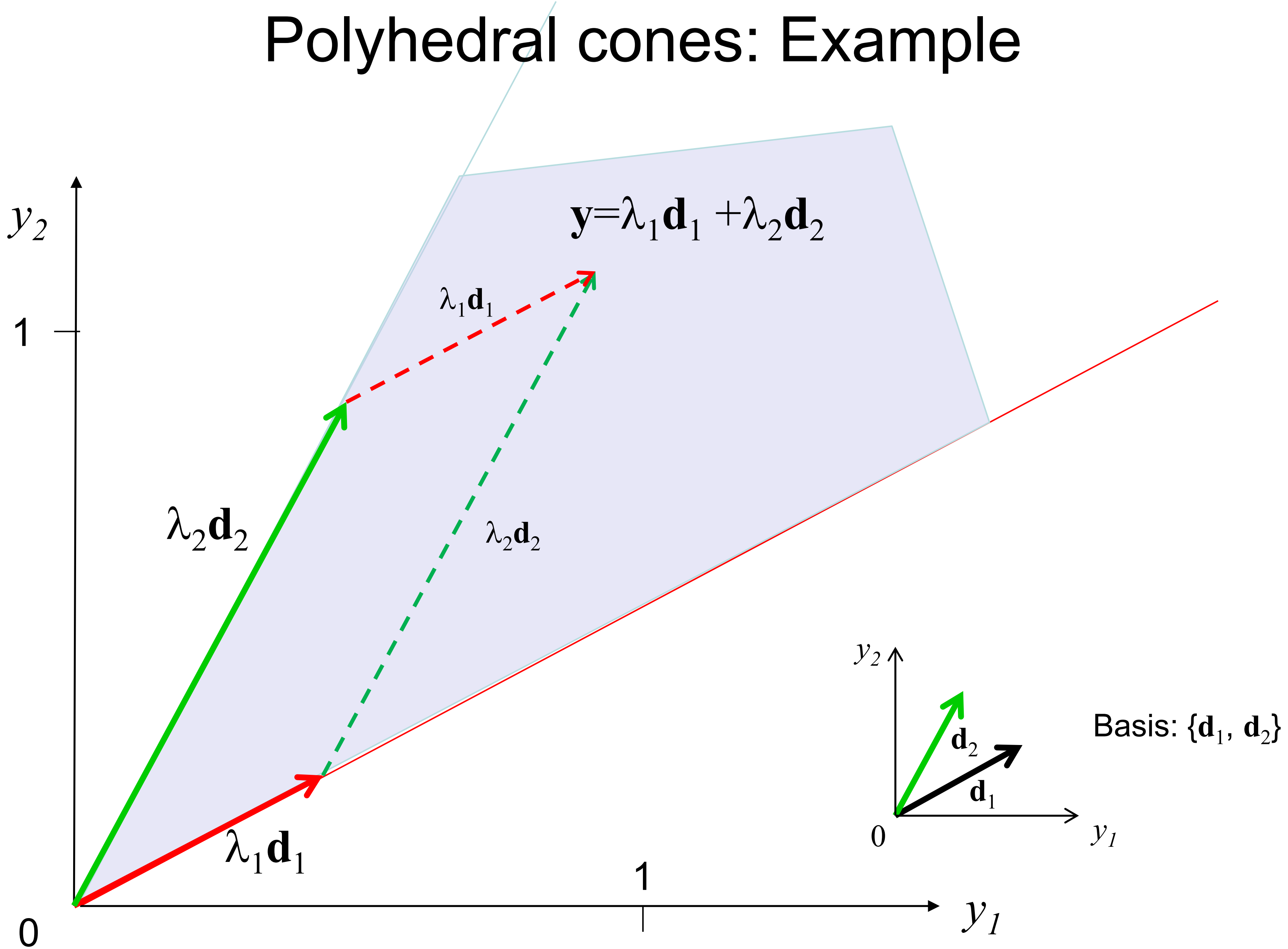
Example: Is $\mathbf{y} \in \mathbf{R}^m$ inside a polyhedral cone given by some linear independent directions $\mathbf{d}_1, \ldots, \mathbf{d}_m$?

Answer: Solve linear equation system:

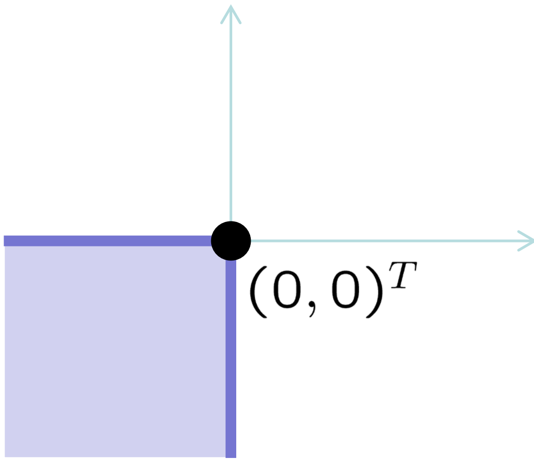
$$\begin{aligned} y_1 &= \lambda_1 d_{11} + \lambda_2 d_{12} + \ldots + \lambda_m d_{1m} \\ &\vdots \\ y_m &= \lambda_1 d_{m1} + \lambda_2 d_{m2} + \ldots + \lambda_m d_{mm} \end{aligned}$$

If the solution vector $(\lambda_1, \ldots, \lambda_m) \geq 0$ then $\mathbf{y}$ lies inside the cone.

Polyhedral cones: Example



The negative orthant



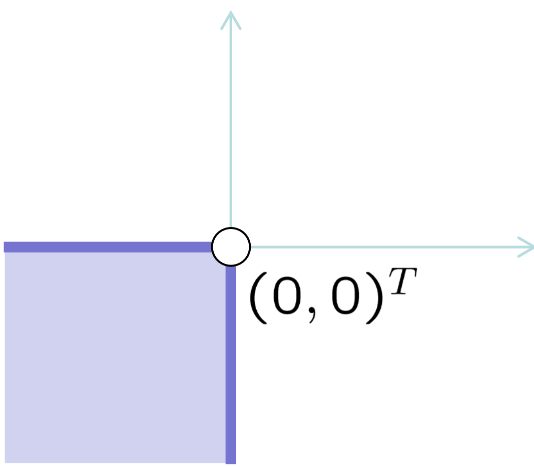
$$\mathbb{R}_{\preceq} = \{\mathbf{x} \in \mathbb{R}^m | x_1 \leq 0, \ldots, x_m \leq 0\}$$

Non-positive orthant

$\mathbb{R}_{\prec} = \mathbb{R}_{\preceq} \setminus \{0\}$ zero-dominated orthant

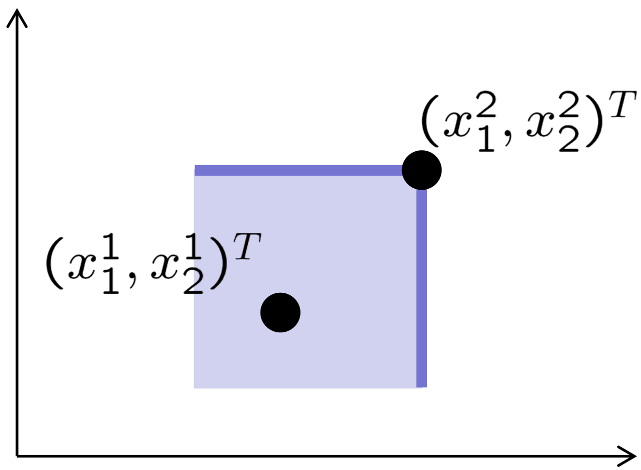
$$\mathbb{R}_{<} = \{\mathbf{x} \in \mathbb{R}^m | x_1 < 0, \ldots, x_m < 0\}$$

negative orthant



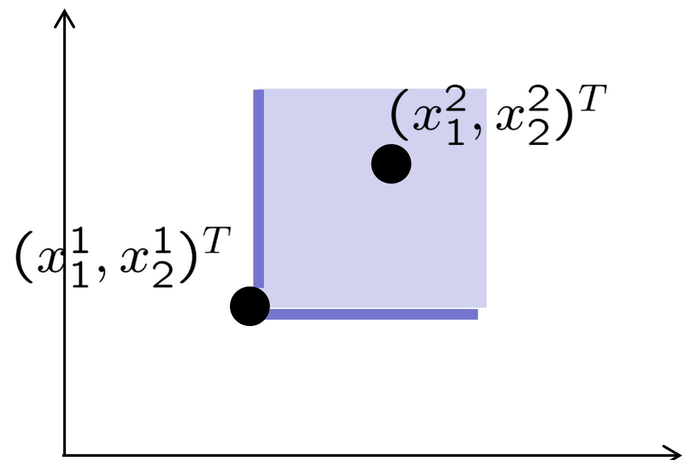
Definition of Pareto optimality via cones

$\mathbf{x}^1 \prec \mathbf{x}^2 \Leftrightarrow \mathbf{x}^1 \in \mathbf{x}^2 \oplus \mathbb{R}^m_{\prec}$



or, equivalently:

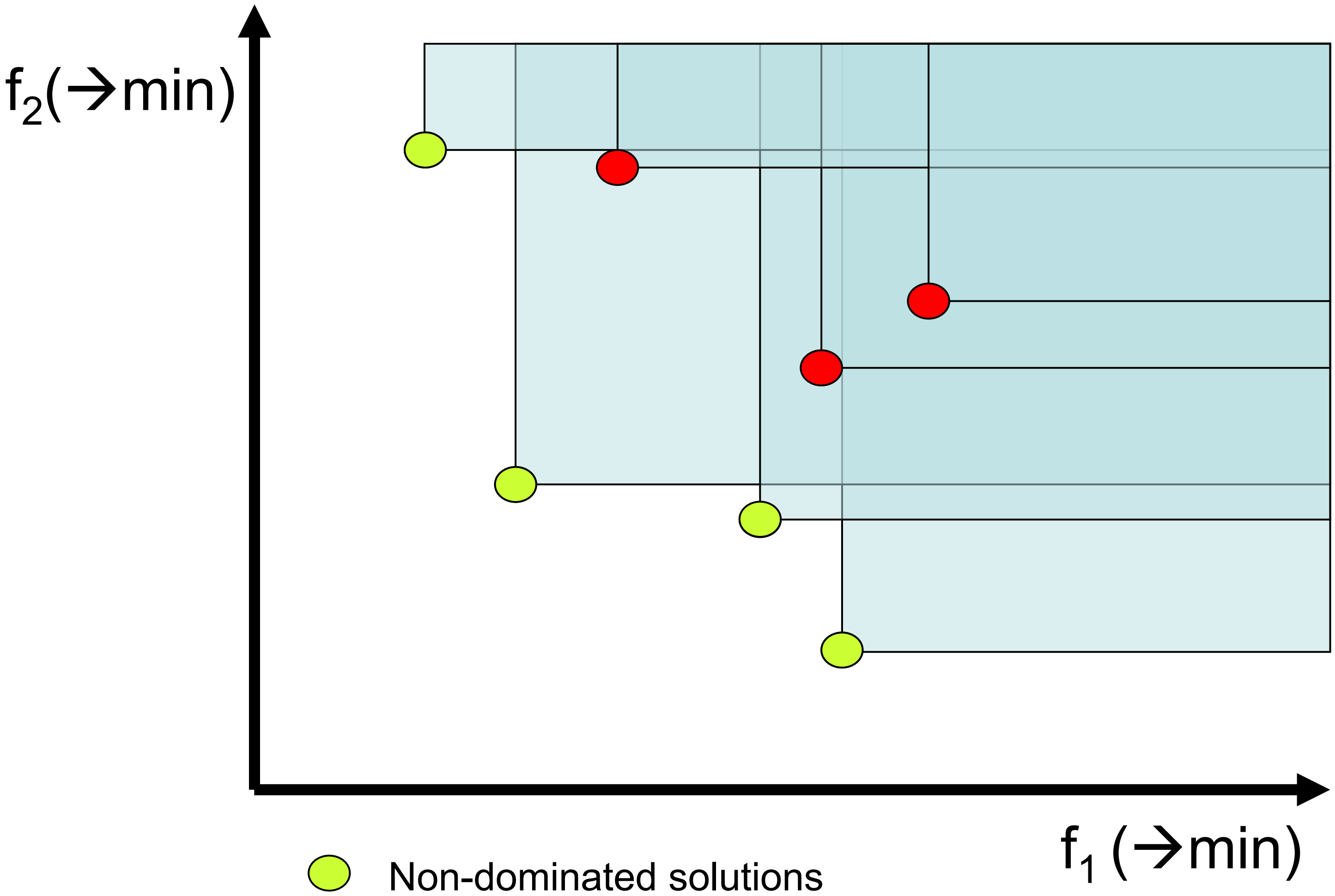
$\mathbf{x}^1 \prec \mathbf{x}^2 \Leftrightarrow \mathbf{x}^2 \in \mathbf{x}^1 \oplus \mathbb{R}^m_{\succ}$ with $\mathbb{R}^m_{\succ} = -\mathbb{R}^m_{\prec}$



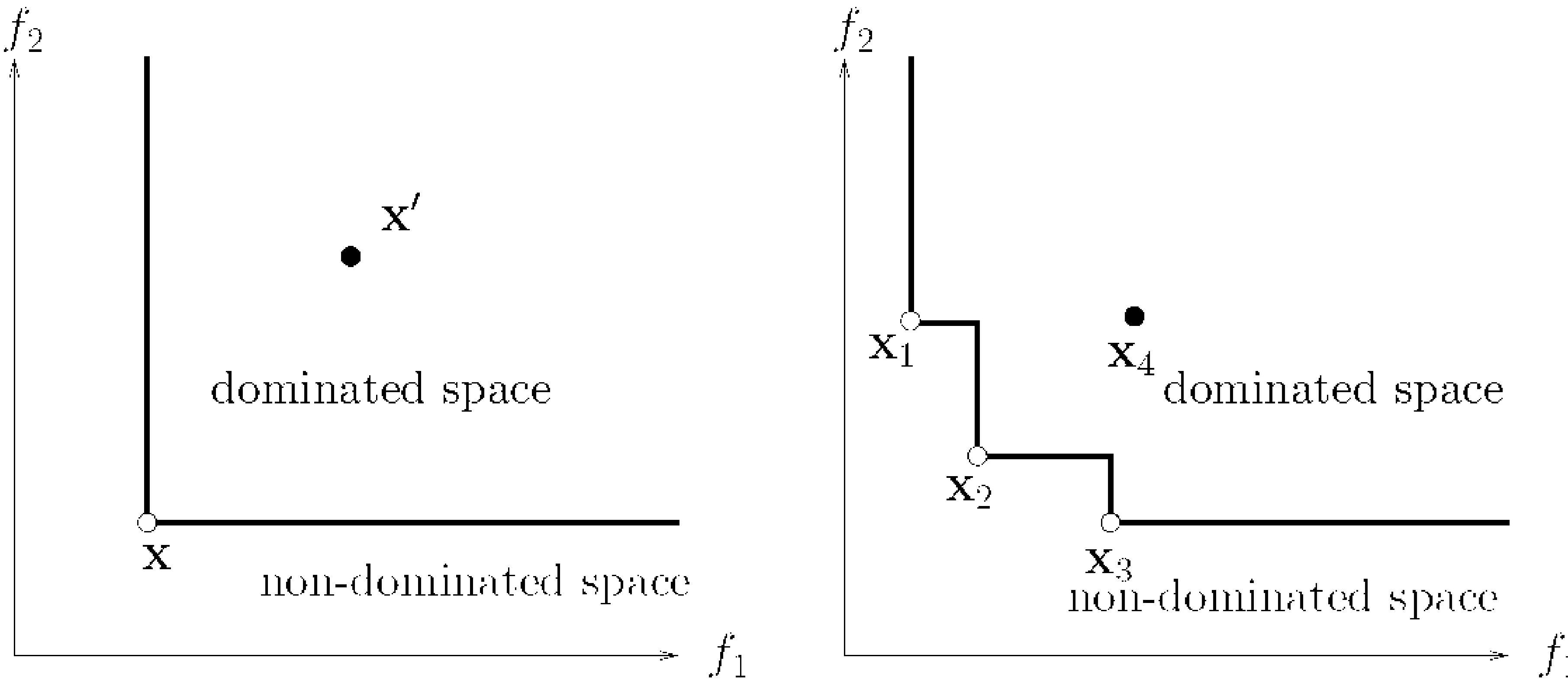
Pareto orders have now a geometrical interpretation!

Cones that are not orthants can be used to generalize the concept of Pareto dominance.

Pareto Optimality and Cones



Attainment curve

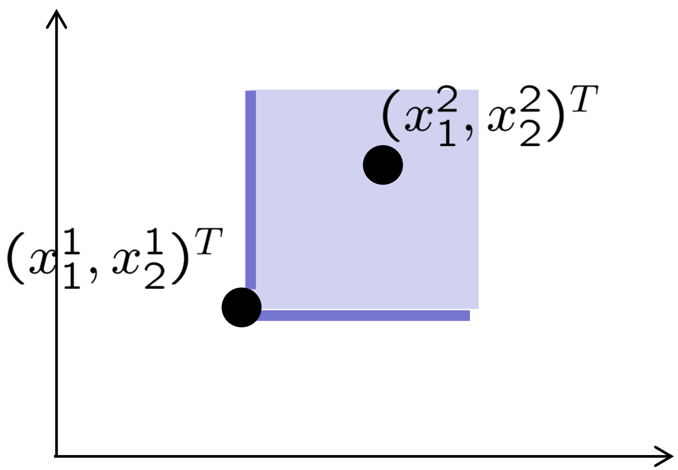


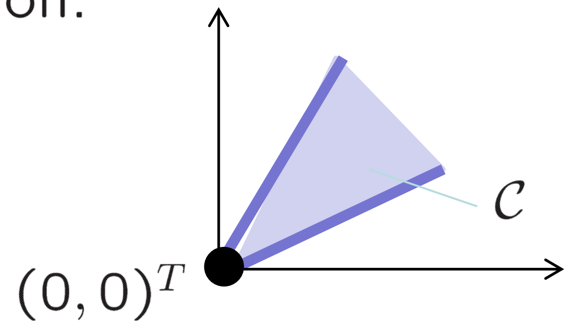
Def. (Attainment curve): For a 2-objective problem, the attainment curve of a set S, $S \subset \mathbb{R}^2$ is defined as the boundary of the dominated space $S \oplus \mathbb{R}^2_{\succsim}$..

Definition of Pareto optimality via cones

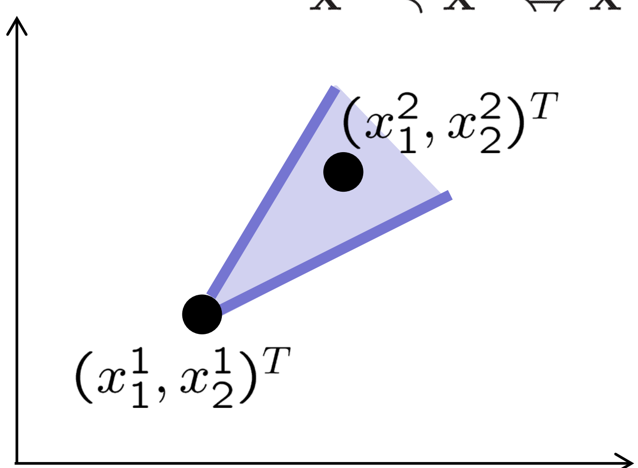
Cones that are not neccassarily orthants can be used to generalize the concept of Pareto domination.

$$\mathbf{x}^1 \prec \mathbf{x}^2 \Leftrightarrow \mathbf{x}^2 \in \mathbf{x}^1 \oplus \mathbb{R}^m_{\succ}$$





$$\mathbf{x}^1 \prec \mathbf{x}^2 \Leftrightarrow \mathbf{x}^2 \in \mathbf{x}^1 \oplus \mathcal{C}$$

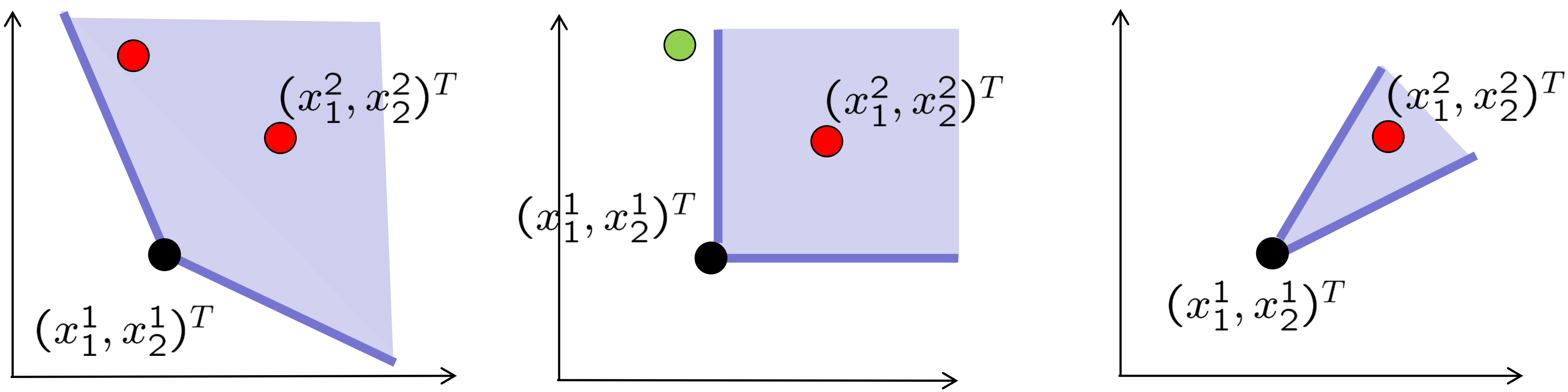


Cone orders based on convex cones are the only orders that are invariant to translation and scalar multiplication of the vector components [Noghin, 1991].

Noghin, V. D. (1997). Relative importance of criteria: a quantitative approach. *Journal of Multi-Criteria Decision Analysis*, *6*(6), 355-363.

Cone-orders and trade-off bounding

If for two cones $\mathcal{C}_1$ and $\mathcal{C}_2$ it holds that $\mathcal{C}_1 \supset \mathcal{C}_2$ then the cone order of $\mathcal{C}_1$ extends the cone order of $\mathcal{C}_2$.

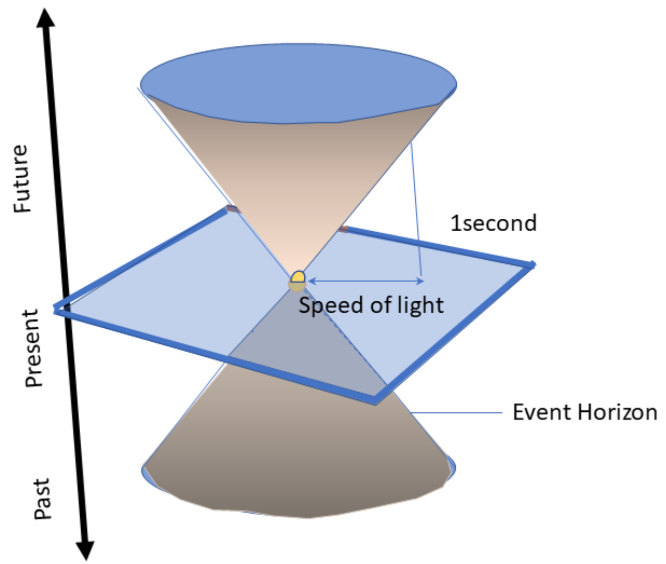


By formulating cones that include Pareto cones we get an order extension to the Pareto order and points with unbalanced trade-off get dominated.

What about cones with opening angle 180◦ ? Are they transitive?

How can we çheck cone dominance?

Example of cone-order: Minkowski's spacetime



Assume a flat world and a maximum speed of a particle of c in any direction of the plane.

Then the possible futures and pasts of this particle are within the following cone:

$$\mathcal{C} = \{\mathcal{D}(t) | t \in \mathbb{R}\}, \quad \mathcal{D}(t) = \{\mathbf{x} \in \mathbb{R}^3 | (x_1)^2 + (x_2)^2 \leq (ct)^2, x_3 = t\}$$

Which are the properties of this cone ?

Unit 2: Take home messages(1/2)

1. A formal definition of Pareto Orders on the criterion space and decision space was given
2. Orders can be introduced as binary relations on a set. The pair (set, order relation) is then called an ordered set. Axioms are used to characterize binary relations, and orders.
3. Preorders, partial orders and linear orders are three imporant classes of orders. Preorders are the most general type of orders.They are introduced (from the left to the right) by requiring additional axioms (antisymmetry, totatlity).
4. Incomparability, strict orders, and indifference are often relations that are defined in conjunction with an preorder.

Unit 2: Take home messages (2/2)

1. Relations between on orders: Order embedding, order isomorphisms, (linear) extension, equality
2. Hasse diagrams are means to exploit the antisymmetry and transitivity of partial orders on finite sets in order to compactly represent them in a diagram
3. Topological sorting can be used to find a linear order that extends an partial order. It is an recursive algorithm. For a given partial order there can be different topological sortings.
4. The Pareto order is a special kind of cone order. Cone-orders are defined on vector spaces by means of a pointed and convex dominance cone.
5. Using the cone order interpretation we can easily check dominance and find Pareto fronts of 2-D finite sets, using a geometrical construction.