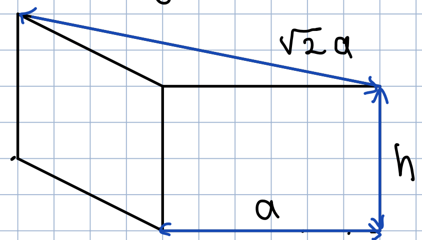
The analysis of the half-square grounded prism.



$S = 2ah + \sqrt{2}\,ah + a^2$ (Surface)

$V = \frac{1}{2}a^2 h$

1) Find the prism with minimal surface area for a given volume V_ε. Use the Lagrange multiplier rule.

$f(a,h) = 2ah + \sqrt{2}\,ah + a^2$

$g(a,h) = \frac{1}{2}a^2 h - V_\varepsilon = 0 \qquad (1)$

$L_{\lambda_1,\lambda_2}(a,h) = \lambda_1 (2ah + \sqrt{2}\,ah + a^2) + \lambda_2 \left(\frac{1}{2}a^2 h - V_\varepsilon\right)$

In the literature the case $\lambda_1 \neq 1$ is often neglected, as it only seldomly leads to a solution. It is however required to investigate it.

This term is like a penalty function. If the goal is minimization it is added and if it is maximization it is substracted.

$$\frac{\partial L_{\lambda_1,\lambda_2}}{\partial a}(a,h) = \lambda_1 (2h + \sqrt{2}\,h + 2a) + \lambda_2 ah = 0 \qquad (2)$$

$$\frac{\partial L_{\lambda_1,\lambda_2}}{\partial h}(a,h) = \lambda_1 (2a + \sqrt{2}\,a) + \frac{1}{2}\lambda_2 a^2 = 0 \qquad (3)$$

Solution $\lambda_1 = 0$: $\frac{1}{2}a^2 h - V_\varepsilon = 0 \Rightarrow h = 2\frac{V_\varepsilon}{a^2} > 0,\ a > 0$

$\Rightarrow$ in (2) $\frac{1}{2}\lambda_2 a^2 = 0 \Rightarrow \lambda_2 = 0$ (this is not allowed because $\lambda \succ (0,0)$ in the Lagrange theorem, and $\lambda_1 = 0$ implies that λ_2 should be positive. So $\lambda_1 = 0$ yields no solution.

Solutions $\lambda_1 = 1$: Here, there are different solution paths; we start with Equation (1) to eliminate h.

from (1) it follows: $h = 2V_\varepsilon / a^2$

from (3) $(2+\sqrt{2})a = -\frac{1}{2}\lambda_2 a^2 \Rightarrow \lambda_2 = -\frac{4+2\sqrt{2}}{a}$

insert in (2) $4V_\varepsilon / a^2 + 2\sqrt{2}\, V_\varepsilon / a^2 + 2a = 2\left(\frac{4+2\sqrt{2}}{a}\right) a V_\varepsilon / a^2$

$$\cdot a^2 \quad \Leftrightarrow: \quad (4+2\sqrt{2})V_\varepsilon + 2a^3 = (8+4\sqrt{2})V_\varepsilon$$

$$\Leftrightarrow: \quad a = \sqrt[3]{(4+2\sqrt{2}-2-\sqrt{2})V_\varepsilon}$$

$$\Leftrightarrow: \quad a = \sqrt[3]{(2+\sqrt{2})V_\varepsilon} =: a^*(V_\varepsilon)$$

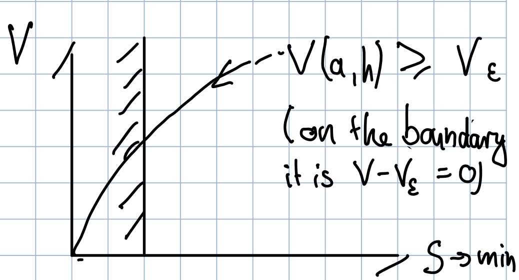
$$h = 2V_\varepsilon \cdot \frac{1}{\left(\sqrt[3]{(2+\sqrt{2})V_\varepsilon}\right)^2} =: h^*(V_\varepsilon)$$

Part 2) How can this result be used to express the efficient set?

Idea: Use the ε-constraint method with equality constraint.

$$S(a,h) \to \min$$

s.t. $V(a,h) \overset{!}{=} V_\varepsilon$, $V_\varepsilon \in (0,\infty)$



Now, we can represent the efficient set as a parametrized curve, with parameter $V_\varepsilon \in (0,\infty)$:

$$X_E = \{ (a^*(V_\varepsilon), h^*(V_\varepsilon)) \mid V_\varepsilon \in (0,\infty) \}$$