The PageRank Algorithm (Chapter 5)

Wojtek Kowalczyk

plus slides

http://www.mmds.org/mmds/v2.1/ch05-linkanalysis2.pptx

Some history

- Internet in early 80’: small, slow, local,
- Internet in early 90’: growing, global, slow
  - newsgroups
  - mailing lists
  - bulletin boards
- “primitive” search engines:
  - Lycos
  - Excite
  - Netscape
  - Yahoo!
  - MSN
  - “hybrid”: combinations of existing engines
- **1997: google.stanford.edu; 15 September 1997: domain google.com registered**

The key to success: the PageRank algorithm

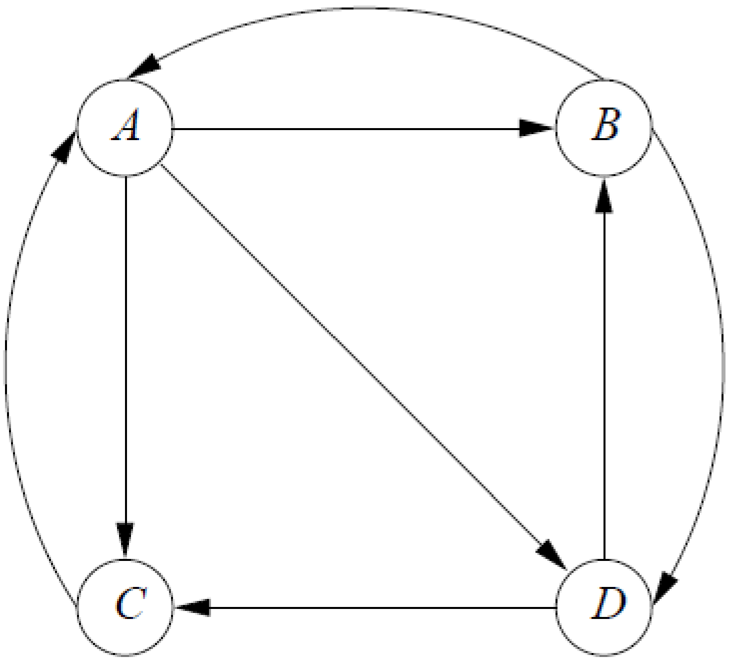
- Invented by Sergiey Brin and Larry Page in 1996 (Stanford University)
- Estimates “page importance” by analyzing **the structure of links between pages**
- Some Rough Statistics (from August 29th, 1996)
  - Total indexable HTML urls: 75 Million
  - Total content downloaded: 207 gigabytes ...
  - BackRub is written in Java and Python and runs on several Sun Ultras and Intel Pentiums running Linux. The primary database is kept on a Sun Ultra II with 28GB of disk. Scott Hassan and Alan Steremberg have provided a great deal of very talented implementation help. **Sergey Brin** has also been very involved and deserves many thanks.
    -**Larry Page** pagecs.stanford.edu
- By the end of 1998, Google had an index of about 60 million pages
- *http://en.wikipedia.org/wiki/History_of_Google*

The PageRank Algorithm (Ch. 5)

- How Google determines the importance of a page?
- A “random surfer” model:
  - visitors start their session at random pages
  - visitors walk along links at random
  - choices are made uniformly (each outgoing link has the same chance)
  - “page importance” = “frequency the surfer visits the page”
- It can be modeled by a Markov Process
  - transition matrix
  - iterative calculation of page probability distribution
  - solution = principal eigen vector of the transition matrix
- Nowadays a much more sophisticated model is used …

The transition matrix

$$\begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}$$



$$M = \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1/2 & 1 & 0 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 1/2 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} A \\ B \\ C \\ D \end{array} \quad \begin{bmatrix} 9/24 \\ 5/24 \\ 5/24 \\ 5/24 \end{bmatrix}$$

Each column X lists probabilities of going to Y

Entries in each column sum up to 1

v=[1/4, 1/4, 1/4, 1/4]'
What is Mv? What is M(Mv)? M(M(v))?…

The transition matrix: iterating Mv

$$
\begin{array}{cc}
 & \begin{array}{cccc} A & B & C & D \end{array} \\
M = & \begin{bmatrix} 0 & 1/2 & 1 & 0 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 1/2 & 0 & 0 \end{bmatrix}
\end{array}
\begin{array}{c} A \\ B \\ C \\ D \end{array}
$$

$$
\begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}
\begin{bmatrix} 9/24 \\ 5/24 \\ 5/24 \\ 5/24 \end{bmatrix}
\begin{bmatrix} 15/48 \\ 11/48 \\ 11/48 \\ 11/48 \end{bmatrix}
\begin{bmatrix} 11/32 \\ 7/32 \\ 7/32 \\ 7/32 \end{bmatrix}
\cdots
\begin{bmatrix} 3/9 \\ 2/9 \\ 2/9 \\ 2/9 \end{bmatrix}
$$

Convergence of the Markov process

if:

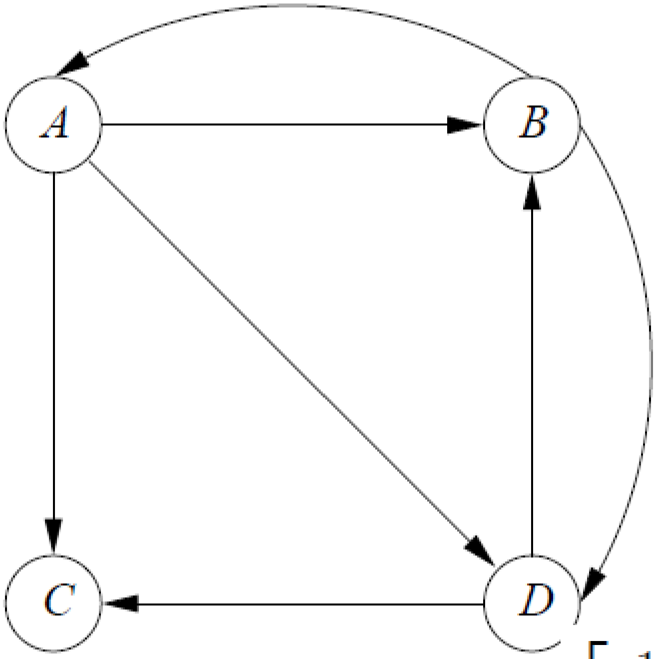
- the graph is strongly connected
  (i.e., there is a path between any two nodes)
- there are no "dead ends" (nodes with no links going out)

then

- the sequence v, Mv, $M(Mv)$, … converges to v' such that $v'=Mv'$
- v' is the principal eigen vector of matrix M (principal = biggest eigen value)

- In practice, 50-70 iterations are sufficient
- … even for huge M (billions x billions , sparse) …
- *For small M, $v=Mv$ can be solved directly… (solving a system of linear eqs)*

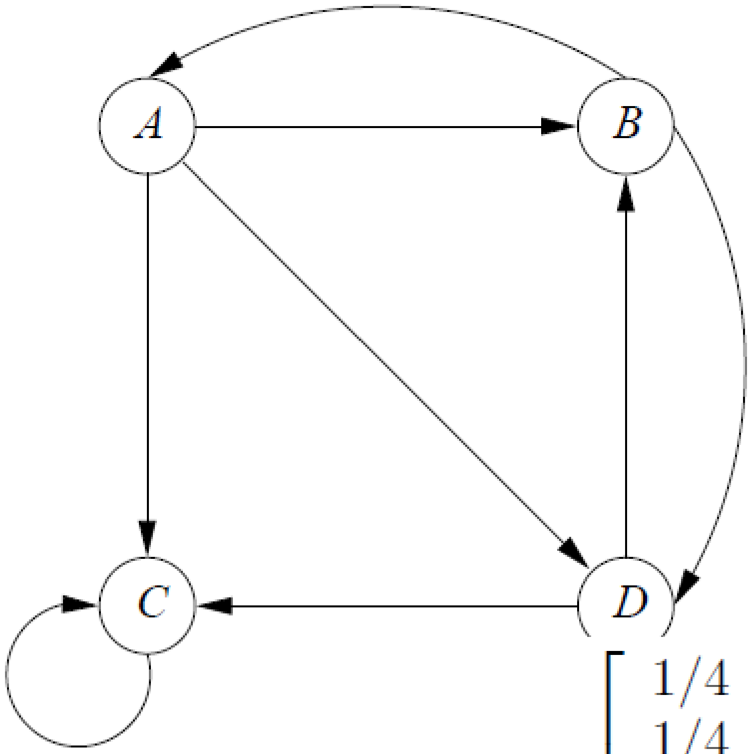
Dead-ends



$$
\begin{array}{cc}
\begin{array}{cccc} A & B & C & D \end{array} & \\
M = \begin{bmatrix} 0 & 1/2 & 0 & 0 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 1/2 & 0 & 0 \end{bmatrix} & \begin{array}{c} A \\ B \\ C \\ D \end{array}
\end{array}
$$

$$
\begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}
\begin{bmatrix} 3/24 \\ 5/24 \\ 5/24 \\ 5/24 \end{bmatrix}
\begin{bmatrix} 5/48 \\ 7/48 \\ 7/48 \\ 7/48 \end{bmatrix}
\begin{bmatrix} 21/288 \\ 31/288 \\ 31/288 \\ 31/288 \end{bmatrix}
\cdots
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

Spider-traps



$$
\begin{array}{cc}
 & \begin{array}{cccc} A & B & C & D \end{array} \\
M = & \begin{bmatrix} 0 & 1/2 & 0 & 0 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 0 & 1 & 1/2 \\ 1/3 & 1/2 & 0 & 0 \end{bmatrix} \begin{array}{c} A \\ B \\ C \\ D \end{array}
\end{array}
$$

$$
\begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}
\begin{bmatrix} 3/24 \\ 5/24 \\ 11/24 \\ 5/24 \end{bmatrix}
\begin{bmatrix} 5/48 \\ 7/48 \\ 29/48 \\ 7/48 \end{bmatrix}
\begin{bmatrix} 21/288 \\ 31/288 \\ 205/288 \\ 31/288 \end{bmatrix}
\cdots
\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}
$$

Teleporting

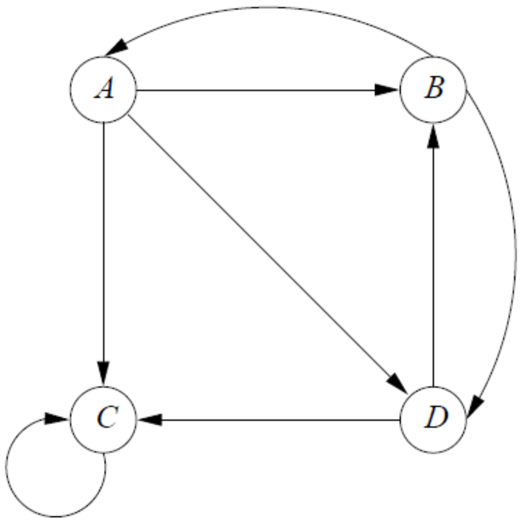
- At each step, decide:
  - with probability β (0< β ≤ 1) continue random walk
  - with probability (1- β) jump to any other node
- The corresponding update rule:

  $$v' = \beta M v + (1-\beta) e/n$$ (e is a vector of n 1's)

- β is usually 0.8 or 0.9
- "adding extra links": all problems solved (*really?*)

Teleporting: example (slide 9 continued…)

β = 0.8 = 4/5



$$\mathbf{v}' = \begin{bmatrix} 0 & 2/5 & 0 & 0 \\ 4/15 & 0 & 0 & 2/5 \\ 4/15 & 0 & 4/5 & 2/5 \\ 4/15 & 2/5 & 0 & 0 \end{bmatrix} \mathbf{v} + \begin{bmatrix} 1/20 \\ 1/20 \\ 1/20 \\ 1/20 \end{bmatrix}$$

$$\begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix} \begin{bmatrix} 9/60 \\ 13/60 \\ 25/60 \\ 13/60 \end{bmatrix} \begin{bmatrix} 41/300 \\ 53/300 \\ 153/300 \\ 53/300 \end{bmatrix} \begin{bmatrix} 543/4500 \\ 707/4500 \\ 2543/4500 \\ 707/4500 \end{bmatrix} \cdots \begin{bmatrix} 15/148 \\ 19/148 \\ 95/148 \\ 19/148 \end{bmatrix}$$

We don’t like it!

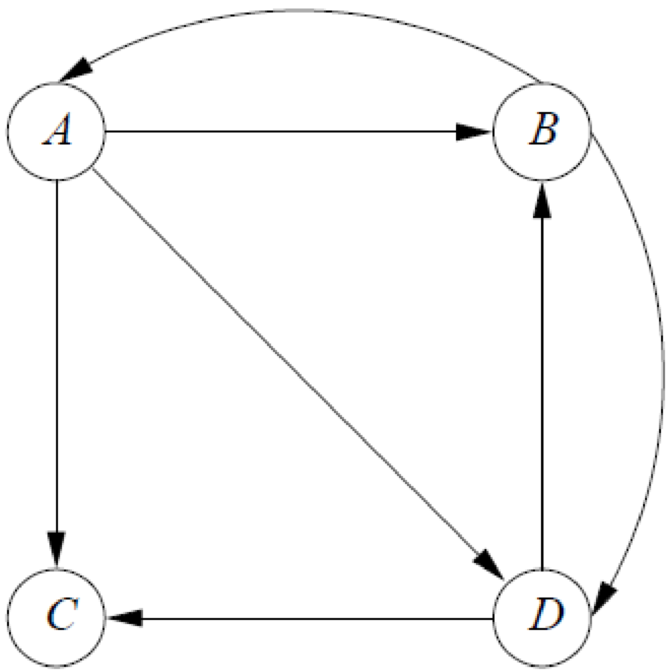
Teleporting: dead ends (slide 8 continued…)

```
Beta=0.8

M=[0,    1/2,  0,   0;
    1/3,   0,    0,  1/2;
    1/3,   0,    0,  1/2;
    1/3, 1/2,   0,   0]

v=[1/4, 1/4, 1/4, 1/4]'


for i=1:20
   v=Beta*M*v +(1-Beta)*ones(size(v))/length(v)
end
```



Teleporting: dead ends (slide 8 continued…)

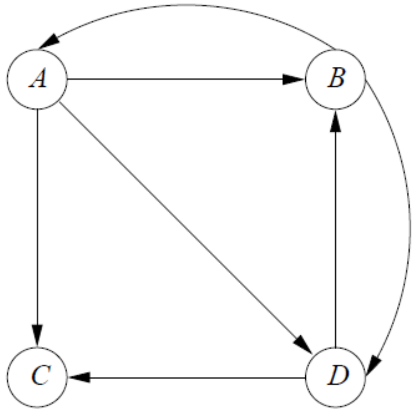
Convergence after 20 iterations:

Iter: 0	1	2	3	10	20
0.2500	0.1500	0.1367	0.1207	0.1018	0.1014
0.2500	0.2167	0.1767	0.1571	0.1290	0.1284
0.2500	0.2167	0.1767	0.1571	0.1290	0.1284
0.2500	0.2167	0.1767	0.1571	0.1290	0.1284

Does it make sense?

- **Why is sum(v) < 1**? Surfers "die" at dead ends? Not exactly…
- Is sum(v) always >0 ? Yes: it is at least (1-Beta)
- What should be the case:
    - p(A)=0.5*p(B)?
    - p(A)= 0.5*p(B) + (1-Beta)/4?
    - **p(A)= Beta*0.5*p(B) + (1-Beta)/4? Check it!**

We don't like it!



Teleporting: dead-ends and spider traps

For graphs with no dead-ends "teleporting" works fine:
the process converges to a vector ***v***, such that ***sum(v)=1***
and can be interpreted as "probabilities".

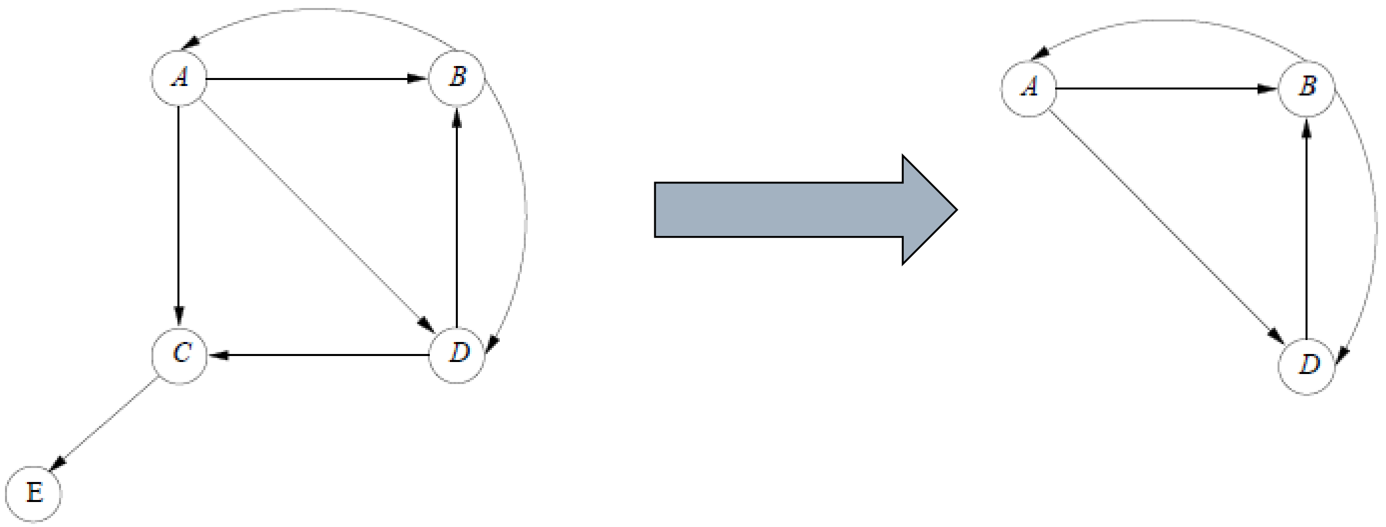
For graphs with dead-ends, the "teleporting" still works,
but the ***sum(v)*** may be smaller than 1 (no probabilistic interpretation).
Still, useful in practice (used in the original "Google" paper).

An alternative algorithm for handling dead-ends (read **Section 5.1.4**!)

- recursively remove dead ends and corresponding links,
- calculate of PageRank for the nodes of the remaining fragment of the graph,
- propagate computed values to the removed nodes.

Another alternative: when reaching a dead end jump to the next node
"uniformly at random with probability 1"

Example from 5.1.4



After removal of nodes E and C, calculate PageRank of A, B, D and set:

PageRank(C) = 1/2*PageRank(A)+1/2*PageRank(D)

PageRank(E) = PageRank(C)

Note that sum of PageRanks is now bigger than 1!

Computing PageRank for HUGE networks

- Imagine a graph with:
  - n=1.000.000.000 (1 billion nodes)
  - on average 10 outgoing links from every node

- Q1: How much RAM and disk space would we need to compute the PageRank for this graph?

  - To keep 1 billion values of PageRank (32bit long floats) we need:
  - 32bits*10^9 =4*10^9 bytes=4GB RAM

- Q2: How much disk space would we need to store the graph structure?
  - 1 billion nodes -> each node represented by a 32bit long integer (2^{32}=4.294.967.296)
  - on average “10 target nodes” => 10 billion addresses => 40GB disk space

Implementation of PageRank

- Key ideas:
  - M is very sparse: say 10 links per page => 10 non-zeros in a column
  - Use “inverted indexing” to represent ***M***: a list of
    <node, outdegree, children of the node>
  - keep on the harddisk ***M*** and the vector ***v_old*** *(of length 1 billion)*
  - keep in RAM only ***v_new***
  - update ***v_new*** in a single scan of ***M*** and ***v_old***
- Some numbers:
  - n=1.000.000.000 (1 billion nodes)
  - RAM needed: 4GB (32bits per node)
  - harddisk: about 40GB (10xRAM)
  - a single scan: about 2-3 minutes
  - 50 iterations => 2-3 hours

Sparse Matrix Encoding

- **Encode sparse matrix using only nonzero entries**
  - Space proportional roughly to number of links
  - Say 10N, or 4*10*1 billion = 40GB
  - **Still won't fit in memory, but will fit on disk**

source node	degree	destination nodes
0	3	1, 5, 7
1	5	17, 64, 113, 117, 245
2	2	13, 23

- **Assume enough RAM to fit v^{new} into memory**
  - Store v^{old} and matrix **M** on disk
- **1 step of power-iteration is:**

$$v' = \beta M v + (1-\beta)e/N$$

Initialize all entries of $v^{new} = (1-\beta) / N$
For each page i (of out-degree d_i):
  Read into memory: i, d_i, $dest_1$, …, $dest_{d_i}$, $v^{old}(i)$
  For $j = 1 \ldots d_i$
    $v^{new}(dest_j) \mathrel{+}= \beta\, v^{old}(i) / d_i$

v^{new}: 0, 1, 2, 3, 4, 5, 6

source	degree	destination
0	3	1, 5, 6
1	4	17, 64, 113, 117
2	2	13, 23

v^{old}: 0, 1, 2, 3, 4, 5, 6

In short: "column-wise" computations

When multiplying **M** by **v** we organize computations **"by columns"**: "a single, linear scan through the harddisk"

- data is already organized in this way,
- we access elements of "**old v**" one by one (a, b, c),
- outdegrees (one per column) are easy to find.

$$\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix} * \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} Aa + Bb + Cc \\ Da + Eb + Fc \\ Ga + Hb + Ic \end{bmatrix}$$

If you like programming: analyze wikipedia links

Go to https://zenodo.org/record/2539424 and fetch the file:

https://zenodo.org/record/2539424/files/enwiki.wikilink_graph.2004-03-01.csv.gz?download=1

Investigate the graph:

- Dead ends
- Distribution of in-degrees
- Distribution of out-degrees of nodes
- Implement the page rank algorithm from slide 19
- Implement direct (sparse) matrix multiplication
- Compare results
- *Is this graph strongly connected?*