1 Linear Programming; Karush Kuhn Tucker Conditions; Fritz John Conditions [40%]

Consider this problem:

$$
\begin{aligned}
f_1(x,y) = x + y &\rightarrow \max && (1)\\
g_1(x,y) = 4 - 2x - y &\geq 0 && (2)\\
g_2(x,y) = 2 - \frac{1}{2}x - y &\geq 0 && (3)\\
x &\geq 0 && (4)\\
y &\geq 0 && (5)
\end{aligned}
$$

[Task 1.1 (10%)] Solve the problem graphically. You can use grid paper and send a photo. Indicate the constraint boundaries and infeasible and feasible region, as well as the optimal point(s). (you can make a photo or draw the solution in a drawing program.)

[Task 1.2 (10%)] Formulate the Karush Kuhn Tucker conditions for points $(x, y) \in \mathbb{R}$ for this linear program. and simplify the conditions for the point $(x, y) = (\frac{4}{3}, \frac{4}{3})$. Does this point satisfy the KKT conditions and why? (indicate in your graphical figure the cone of active constraints and the gradient of the objective function).

Next, consider the problem:

$$
\begin{aligned}
f_1(x,y) = x + y &\rightarrow \max && (6)\\
f_2(x,y) = -x - 4y - 4 &\rightarrow \min && (7)\\
g_1(x,y) = 4 - 2x - y &\geq 0 && (8)\\
g_2(x,y) = 2 - \frac{1}{2}x - y &\geq 0 && (9)\\
x &\geq 0 && (10)\\
y &\geq 0 && (11)
\end{aligned}
$$

[Task 1.3 (10%)] Solve the problem graphically by indicating the efficient set. You can use grid paper and send a photo (you can use the same drawing than in Task 1.4) and mark the efficient set. Describe the Pareto front of this problem in the objective space f_1, f_2.

[Task 1.4 (10%)] Formulate the Fritz John conditions for points $(x, y) \in \mathbb{R}$ for this multiobjective linear program!

2 Lagrange Multiplier Rule [20%]

Consider the problem of maximizing the volume for a given surface area. More concretely, let us consider the optimal shape of a cylindrical tin in terms of volume ($\pi r^2 h$) and surface area ($2\pi rh + 2\pi r^2$):

$$\pi r^2 h \quad \rightarrow \quad \max \tag{12}$$
$$2\pi rh + 2\pi r^2 \quad = \quad C \tag{13}$$
$$r \quad \geq \quad 0 \tag{14}$$
$$h \quad \geq \quad 0 \tag{15}$$
$$r, h \in \mathbb{R} \tag{16}$$

Here C is a constant (the surface area of the cylinder).

[Task 2.1 (10%)] Formulate the equations of the Lagrange multiplier conditions for this problem. (You can formulate for general r, h. The non-negativity of r, h can be taken care of when candidate solutions have been identified).

[Task 2.2 (5%)] Identify the optimal solution by solving the equations.

[Task 2.3 (5%)] Based don the solution of Task 2.2, provide a expression (parameterized in C) for the efficient set of the problem.

$$\pi r^2 h \quad \rightarrow \quad \max \tag{17}$$
$$2\pi rh + 2\pi r^2 \quad \rightarrow \quad \min \tag{18}$$
$$r \quad \geq \quad 0 \tag{19}$$
$$h \quad \geq \quad 0 \tag{20}$$
$$r, h \in \mathbb{R} \tag{21}$$

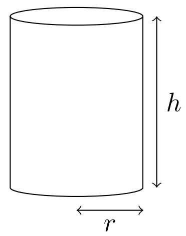


Figure 1: Cylindric tin with radius (r) and height (h).

3 Mathematical Programming Models (20 %)

In the following you are asked use the format of a mathematical program and classify the problem in the classification scheme of the operational research community (choose between: LP, ILP, IP, MILP MINLP, QIP, QP, NLP). If a task belongs to serveral classes, then choose the most specific one.

$$\begin{aligned} f_i(x_1,\ldots,x_n) &\rightarrow \min, i=1,\ldots,m & (22)\\ g_j(x_1,\ldots,x_n) &\geq 0, i=1,\ldots,q & (23)\\ h_k(x_1,\ldots,x_n) &= 0, i=1,\ldots,\ell & (24)\\ x_1\in D_1,\ldots,x_n\in D_n & & (25)\\ & & (26)\end{aligned}$$

[Task 3.1 (10%)] Formulate and classify the problem of finding the maximal number of discs of radius $r=1.5m$ that fit in a single big disc of radius $10m$. Here by disc we mean a circle plus its interior area. Moreover, note that the ratio between the total area of the big disc and the small disc is given by $\frac{10^2\pi}{1.5^2\pi}=44.\bar{4}$. Discs are not allowed to overlap, except in their boundary.

[Task 3.2 (10%)] Let a a_{ij}, $i=1,\ldots,n$, $j=1,\ldots,n$ denote the adjacency matrix of a graph (network), with $a_{ij}=1$ if node i and j are connected by a link and $a_{ij}=0$ if they are disconnected. Elements on the diagonal are set to zero, that is $a_{ii}=0$, $i=1,\ldots,n$ and the network is undirected ($a_{ij}=a_{ji}$, for $i,j=1,\ldots,n$). Formulate the problem of selecting a subset of $k<n$ nodes (k is a given constant), such that the total degree of the selected nodes is maximized. The degree of a node is the number of links attached to it. The total degree of the selected nodes is the sum of the degrees of the selected nodes.

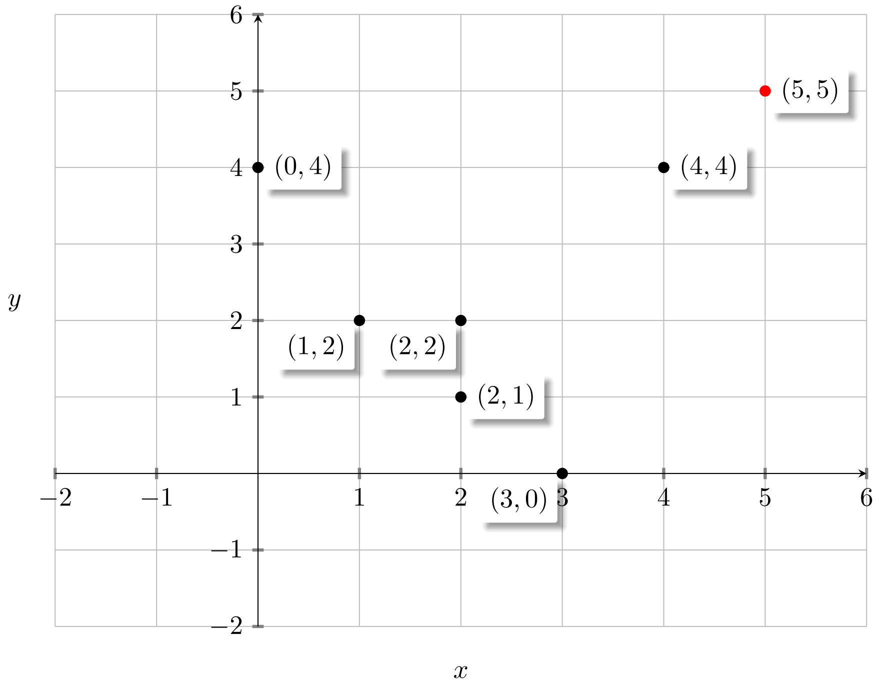
4 Pareto Dominance and Population-based Metaheuristics (20%)

[Task 4.1 (5%)] The Pareto dominance order in the objective space $\mathbb{R}^2$ is a special case of a polyhedral cone-order. What are the generators (vectors) of the ordering cone in the bi-objective case? (chose vectors of length 1); provide a mathematical expression describing the polyhedral cone (set) and draw the cone in a 2-D diagram of the objective case (do also indicate the precise boundaries of the cone).

[Task 4.2 (5%)] Given the set $S = \{(0,4),(1,2),(2,1),(3,0),(4,4),(2,2)\}$. Identify all solutions that are non-dominated. List all solutions that are incomparable to $(2,2)$ under the Pareto dominance order.

[Task 4.3 (5%)] Draw the Hasse diagram of S for Pareto dominance order.

[Task 4.4 (5%)] Compute the ranks of solutions using non-dominated sorting (as used in the first part of the ranking scheme in NSGA-II). What are the hypervolume-contributions (as used in SMS-EMOA) of the solutions in the first (best) ranked subset. Consider reference point $(5,5)$ for computing hypervolume-contributions.



*** END OF EXAM MODA 2021, *Wishing you success!* ***