Binary relations

A *binary relation* $\mathcal{R}$ on some set $\mathcal{S}$ is defined as a subset of $\mathcal{S} \times \mathcal{S}$

We write $\mathbf{x}^1 \mathcal{R} \mathbf{x}^2 \Leftrightarrow (\mathbf{x}^1, \mathbf{x}^2) \in \mathcal{R}$.

Properties of relations (cf. Ehrgott05):

$\mathcal{R}$ is *reflexive* $\Leftrightarrow \forall \mathbf{x} \in \mathcal{S} : \mathbf{x} \mathcal{R} \mathbf{x}$

$\mathcal{R}$ is *irreflexive* $\Leftrightarrow \forall \mathbf{x} \in \mathcal{S} : \neg(\mathbf{x} \mathcal{R} \mathbf{x})$.

$\mathcal{R}$ is *symmetric* $\Leftrightarrow \forall \mathbf{x}^1, \mathbf{x}^2 \in \mathcal{S} : \mathbf{x}^1 \mathcal{R} \mathbf{x}^2 \Leftrightarrow \mathbf{x}^2 \mathcal{R} \mathbf{x}^1$

$\mathcal{R}$ is *antisymmetric* $\Leftrightarrow \forall \mathbf{x}^1, \mathbf{x}^2 \in \mathcal{S} : \mathbf{x}^1 \mathcal{R} \mathbf{x}^2 \wedge \mathbf{x}^2 \mathcal{R} \mathbf{x}^1 \Rightarrow \mathbf{x}^1 = \mathbf{x}^2$

$\mathcal{R}$ is *asymmetric* $\Leftrightarrow \forall \mathbf{x}^1, \mathbf{x}^2 \in \mathcal{S} : \mathbf{x}^1 \mathcal{R} \mathbf{x}^2 \Rightarrow \neg(\mathbf{x}^2 \mathcal{R} \mathbf{x}^1)$

$\mathcal{R}$ is *transitive* $\Leftrightarrow \forall \mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3 \in \mathcal{S} : \mathbf{x}^1 \mathcal{R} \mathbf{x}^2 \wedge \mathbf{x}^2 \mathcal{R} \mathbf{x}^3 \Rightarrow \mathbf{x}^1 \mathcal{R} \mathbf{x}^3$

A *preorder* (quasi-order) is a relation that is both transitive and reflexive.

Preorders

Recall: A *preorder* (quasi-order) is a relation that is both transitive and reflexive.

We write $\mathbf{x}^1 \preceq \mathbf{x}^2$ as shorthand for $\mathbf{x}^1 \mathcal{R} \mathbf{x}^2$

We call $(\mathcal{S}, \preceq)$ a *preordered set*.

Given any preorder, we can define the closely related concepts:

$\mathbf{x}^1 \prec \mathbf{x}^2 :\Leftrightarrow \mathbf{x}^1 \preceq \mathbf{x}^2 \wedge \neg(\mathbf{x}^2 \preceq \mathbf{x}^1)$ (strict preference)

$\mathbf{x}^1 \sim \mathbf{x}^2 :\Leftrightarrow \mathbf{x}^1 \preceq \mathbf{x}^2 \wedge \mathbf{x}^2 \preceq \mathbf{x}^1$ (indifference)

A pair of solutions $\mathbf{x}^1, \mathbf{x}^2 \in \mathcal{S}$ is said to be incomparable, iff neither $\mathbf{x}^1 \preceq \mathbf{x}^2$ nor $\mathbf{x}^2 \preceq \mathbf{x}^1$. We write $\mathbf{x}^1 || \mathbf{x}^2$

The relation $\prec$ is irreflexive and transitive, and (as a consequence) asymmetric

The relation $\sim$ is reflexive, transitive and symmetric

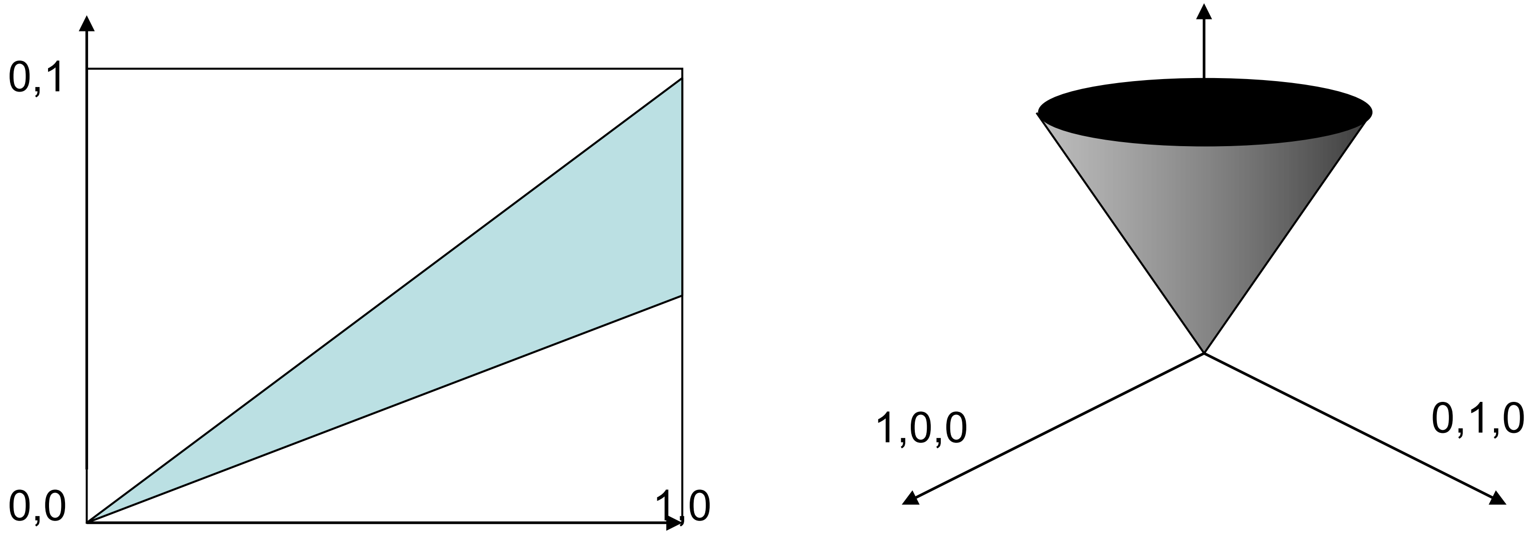
What about the properties of the relation $||$?

Cones

Cones are very useful for geometric interpretation of orders in euclidian space.

A subset $\mathcal{C} \subseteq \mathbb{R}^p$ is called a cone, iff $\alpha\mathbf{d} \in \mathcal{C}$ for all $\mathbf{d} \in \mathcal{C}$ for all $\alpha \in \mathbb{R}, \alpha > 0$.

Remark: Informally, this means that with any point of a set also the ray through this point starting from zero belongs to the cone.



Examples for cones in 2-D (l) and 3-D (r)

Remark: According to the definition above it is possible for $\mathbf{0}$ to belong to a cone or not to belong to it. Verify!

Minkowski* sum and scalar multiplication

Two useful definitions:

The Minkowski sum of two sets $\mathcal{S}^1$ and $\mathcal{S}^2$ is defined as:

$$\mathcal{S}^1 \oplus \mathcal{S}^2 := \{s^1 + s^2 | s^1 \in \mathcal{S}^1, s^2 \in \mathcal{S}^2\}$$

If $\mathcal{S}^1$ is a singleton $\{s\}$, we may write $s \oplus \mathcal{S}^2$ instead of $\{s\} \oplus \mathcal{S}^2$.

The product of a scalar α and a set $\mathcal{S}$ is defined as:

$$\alpha\mathcal{S} := \{\alpha s | s \in \mathcal{S}\}$$

Find examples in $\mathbb{R}$ and $\mathbb{R}^2$!

*Jewish-German mathematician 1864-1909, Goettingen

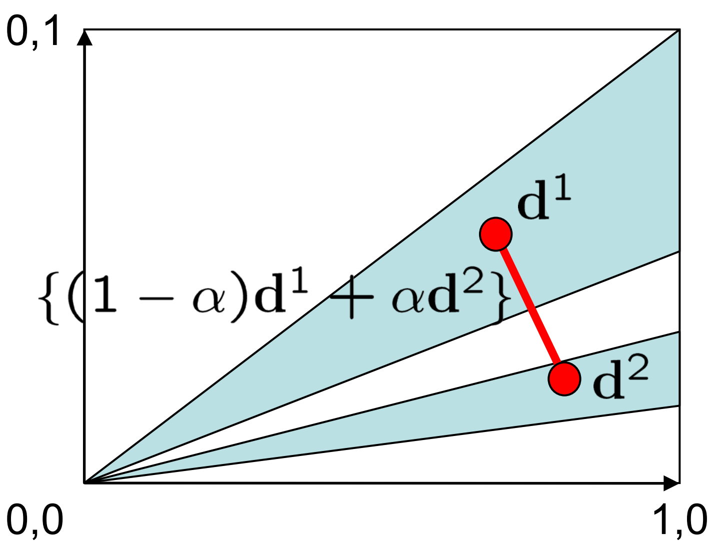
Some properties of cones (2)

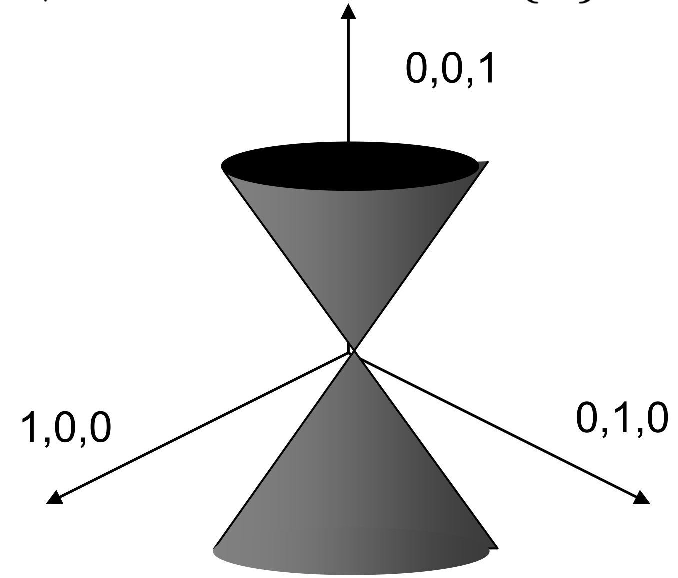
A cone $\mathcal{C} \in \mathbb{R}^m$ is called:

nontrivial or proper, iff $\mathcal{C} \neq \emptyset$.

convex, iff $\alpha \mathbf{d}^1 + (1-\alpha)\mathbf{d}^2 \in \mathcal{C}$ for all $\mathbf{d}^1$ and $\mathbf{d}^2 \in \mathcal{C}$ for all $0 < \alpha < 1$.

pointed, iff for $\mathbf{d} \in \mathcal{C}, \mathbf{d} \neq 0, -\mathbf{d} \notin \mathcal{C}$, i.e. $\mathcal{C} \cap -\mathcal{C} \subseteq \{0\}$





What are the properties of the cones depicted above ?

Polyhedral cones

Def.: Polyhedral cone: A polyhedral cone in $\mathbb{R}^m$is determined by a number of k direction vectors $\mathbf{d_1} \in \mathbb{R}^m$, ... $\mathbf{d}_k \in \mathbb{R}^m$ (cone generators). It is the set that comprises all positive linear combinations of these vectors: $C = \{\mathbf{y} \in \mathbb{R}^m \quad | \text{ exists } \lambda_1 \geq 0, \ldots, \lambda_k \geq 0 : \mathbf{y} = \sum_{i=1}^{k} \lambda_i \mathbf{d}_i\}$

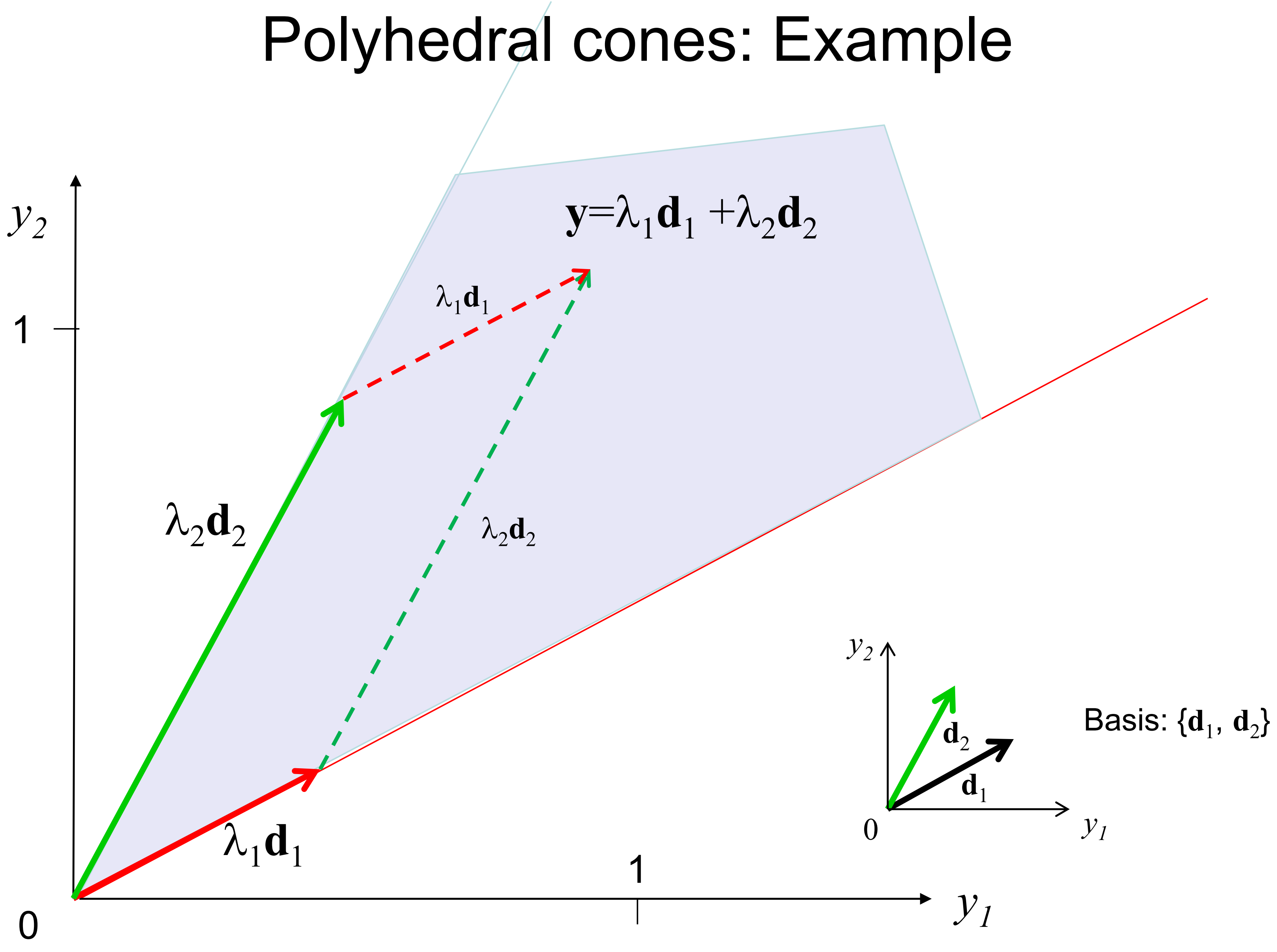
Example: Is $\mathbf{y} \in \mathbf{R}^m$ inside a polyhedral cone given by some linear independent directions $\mathbf{d}_1, \ldots, \mathbf{d}_m$?

Answer: Solve linear equation system:

$$\begin{aligned} y_1 &= \lambda_1 d_{11} + \lambda_2 d_{12} + \ldots + \lambda_m d_{1m} \\ &\vdots \\ y_m &= \lambda_1 d_{m1} + \lambda_2 d_{m2} + \ldots + \lambda_m d_{mm} \end{aligned}$$

If the solution vector $(\lambda_1, \ldots, \lambda_m) \geq 0$ then $\mathbf{y}$ lies inside the cone.

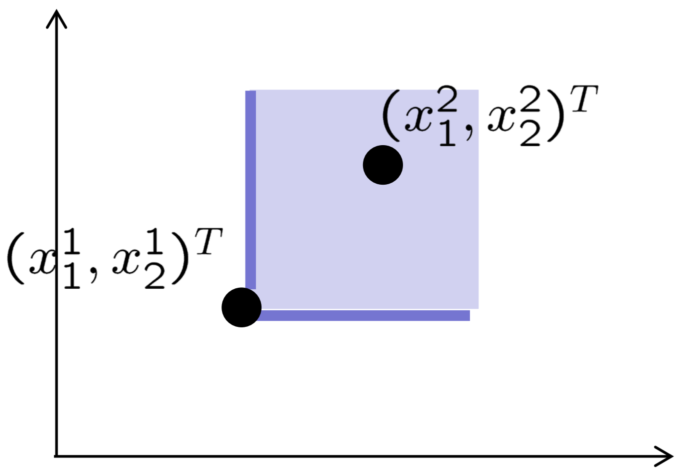
Polyhedral cones: Example

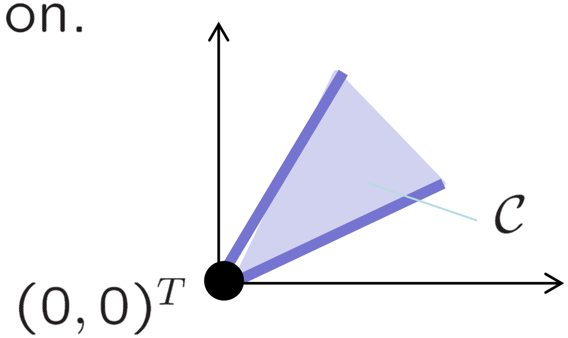


Definition of Pareto optimality via cones

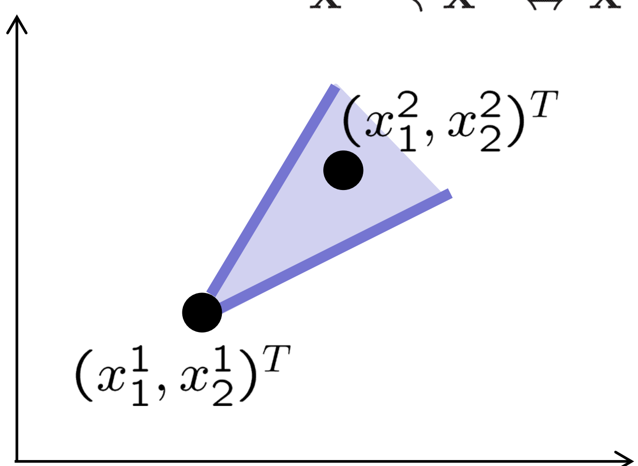
Cones that are not neccassarily orthants can be used to generalize the concept of Pareto domination.

$$\mathbf{x}^1 \prec \mathbf{x}^2 \Leftrightarrow \mathbf{x}^2 \in \mathbf{x}^1 \oplus \mathbb{R}^m_{\succ}$$





$$\mathbf{x}^1 \prec \mathbf{x}^2 \Leftrightarrow \mathbf{x}^2 \in \mathbf{x}^1 \oplus \mathcal{C}$$



Cone orders based on convex cones are the only orders that are invariant to translation and scalar multiplication of the vector components [Noghin, 1991].

Noghin, V. D. (1997). Relative importance of criteria: a quantitative approach. *Journal of Multi-Criteria Decision Analysis*, *6*(6), 355-363.