Multicriteria Optimization and Decision Analysis (MODA)
Exam- MSc LIACS 2021

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1 June 2021

Teachers: Dr. Michael Emmerich (Associate Professor, LIACS, Leiden University)
Examination: Tuesday June 1, 2021, Time 10:15-13:45, 3:00h + 30min (extra time for online submission (+ additional extra time in case special extra time conditions apply))

- **DESCRIPTION:** This is an open-book online exam according to the regulations you received. In particular: You must do the exam on your own and without the help of others. You can use the course materials.
- **GENERAL:** You are allowed to use the lecture notes and slides of the lectures and computer programs (e.g. drawing, plotting).
- **SCORES:** *Maximum score:* 100%. Pass: $\geq$ 55%. To each subquestion, a score is associated as indicated in the headings. (The assignment points will be combined thereafter for computing final grade for MODA).
- **UPLOADING:** Immediately after the end time, submit your answers to Brightspace, one file per tasks (in total 4 pdf files).
  Include your name in the name of the file. For instance, if your name happens to be Albert Einstein use the names
  `Albert_Einstein_Task1.pdf`, `Albert_Einstein_Task2.pdf`,
  `Albert_Einstein_Task3.pdf`, `Albert_Einstein_Task4.pdf`, for your answers. Please do not zip them (it makes it hard to handle for correction).
- **INTEGRITY STATEMENT AND BACKUP EMAIL:** Please also submit the signed integrity statement by email to the course instructor on the same day than the exam. Please also include the four answer pdf in that email

  m.t.m.emmerich@liacs.leidenuniv.nl

  with subject line `MODA2021Exam - YourName - StdNumber`. **Note that this email will only serve as a backup in case something goes wrong with the brightspace submission, and will not be used for the grading and feedback.**. Note: the scanned content of each PDFs file submitted to BrightSpace should be identical to the scanned content of the PDFs sent via email (no new or better-written material is allowed).
- **IN CASE OF TECHNICAL ISSUES:** Contact the examiner as soon as possible by email at m.t.m.emmerich@liacs.leidenuniv.nl, or via the kaltura live room, which will be open throughout the exam.

1 Linear Programming; Karush Kuhn Tucker Conditions; Fritz John Conditions [40%]

Consider this problem:

$$
\begin{aligned}
f_1(x,y) = x + y &\rightarrow \max &(1)\\
g_1(x,y) = 4 - 2x - y &\geq 0 &(2)\\
g_2(x,y) = 2 - \frac{1}{2}x - y &\geq 0 &(3)\\
x &\geq 0 &(4)\\
y &\geq 0 &(5)
\end{aligned}
$$

[Task 1.1 (10%)] Solve the problem graphically. You can use grid paper and send a photo. Indicate the constraint boundaries and infeasible and feasible region, as well as the optimal point(s). (you can make a photo or draw the solution in a drawing program.)

[Task 1.2 (10%)] Formulate the Karush Kuhn Tucker conditions for points $(x, y) \in \mathbb{R}$ for this linear program. and simplify the conditions for the point $(x, y) = (\frac{4}{3}, \frac{4}{3})$. Does this point satisfy the KKT conditions and why? (indicate in your graphical figure the cone of active constraints and the gradient of the objective function).

Next, consider the problem:

$$
\begin{aligned}
f_1(x,y) = x + y &\rightarrow \max &(6)\\
f_2(x,y) = -x - 4y - 4 &\rightarrow \min &(7)\\
g_1(x,y) = 4 - 2x - y &\geq 0 &(8)\\
g_2(x,y) = 2 - \frac{1}{2}x - y &\geq 0 &(9)\\
x &\geq 0 &(10)\\
y &\geq 0 &(11)
\end{aligned}
$$

[Task 1.3 (10%)] Solve the problem graphically by indicating the efficient set. You can use grid paper and send a photo (you can use the same drawing than in Task 1.4) and mark the efficient set. Describe the Pareto front of this problem in the objective space f_1, f_2.

[Task 1.4 (10%)] Formulate the Fritz John conditions for points $(x, y) \in \mathbb{R}$ for this multiobjective linear program!

2 Lagrange Multiplier Rule [20%]

Consider the problem of maximizing the volume for a given surface area. More concretely, let us consider the optimal shape of a cylindrical tin in terms of volume ($\pi r^2 h$) and surface area ($2\pi rh + 2\pi r^2$):

$$\begin{aligned}
\pi r^2 h &\rightarrow \max && (12)\\
2\pi rh + 2\pi r^2 &= C && (13)\\
r &\geq 0 && (14)\\
h &\geq 0 && (15)\\
r, h \in \mathbb{R} & && (16)
\end{aligned}$$

Here C is a constant (the surface area of the cylinder).

> [Task 2.1 (10%)] Formulate the equations of the Lagrange multiplier conditions for this problem. (You can formulate for general r, h. The non-negativity of r, h can be taken care of when candidate solutions have been identified).
>
> [Task 2.2 (5%)] Identify the optimal solution by solving the equations.
>
> [Task 2.3 (5%)] Based don the solution of Task 2.2, provide a expression (parameterized in C) for the efficient set of the problem.

$$\begin{aligned}
\pi r^2 h &\rightarrow \max && (17)\\
2\pi rh + 2\pi r^2 &\rightarrow \min && (18)\\
r &\geq 0 && (19)\\
h &\geq 0 && (20)\\
r, h \in \mathbb{R} & && (21)
\end{aligned}$$

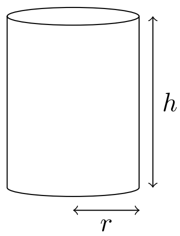


Figure 1: Cylindric tin with radius (r) and height (h).

3 Mathematical Programming Models (20 %)

In the following you are asked use the format of a mathematical program and classify the problem in the classification scheme of the operational research community (choose between: LP, ILP, IP, MILP MINLP, QIP, QP, NLP). If a task belongs to serveral classes, then choose the most specific one.

$$\begin{aligned}
f_i(x_1, \ldots, x_n) &\rightarrow \min, i = 1, \ldots, m && (22)\\
g_j(x_1, \ldots, x_n) &\geq 0, i = 1, \ldots, q && (23)\\
h_k(x_1, \ldots, x_n) &= 0, i = 1, \ldots, \ell && (24)\\
x_1 \in D_1, \ldots, x_n \in D_n & && (25)\\
& && (26)
\end{aligned}$$

[Task 3.1 (10%)] Formulate and classify the problem of finding the maximal number of discs of radius $r = 1.5m$ that fit in a single big disc of radius $10m$. Here by disc we mean a circle plus its interior area. Moreover, note that the ratio between the total area of the big disc and the small disc is given by $\frac{10^2\pi}{1.5^2\pi} = 44.\overline{4}$. Discs are not allowed to overlap, except in their boundary.

[Task 3.2 (10%)] Let a a_{ij}, $i = 1, \ldots, n$, $j = 1, \ldots, n$ denote the adjacency matrix of a graph (network), with $a_{ij} = 1$ if node i and j are connected by a link and $a_{ij} = 0$ if they are disconnected. Elements on the diagonal are set to zero, that is $a_{ii} = 0$, $i = 1, \ldots, n$ and the network is undirected ($a_{ij} = a_{ji}$, for $i, j = 1, \ldots, n$). Formulate the problem of selecting a subset of $k < n$ nodes (k is a given constant), such that the total degree of the selected nodes is maximized. The degree of a node is the number of links attached to it. The total degree of the selected nodes is the sum of the degrees of the selected nodes.

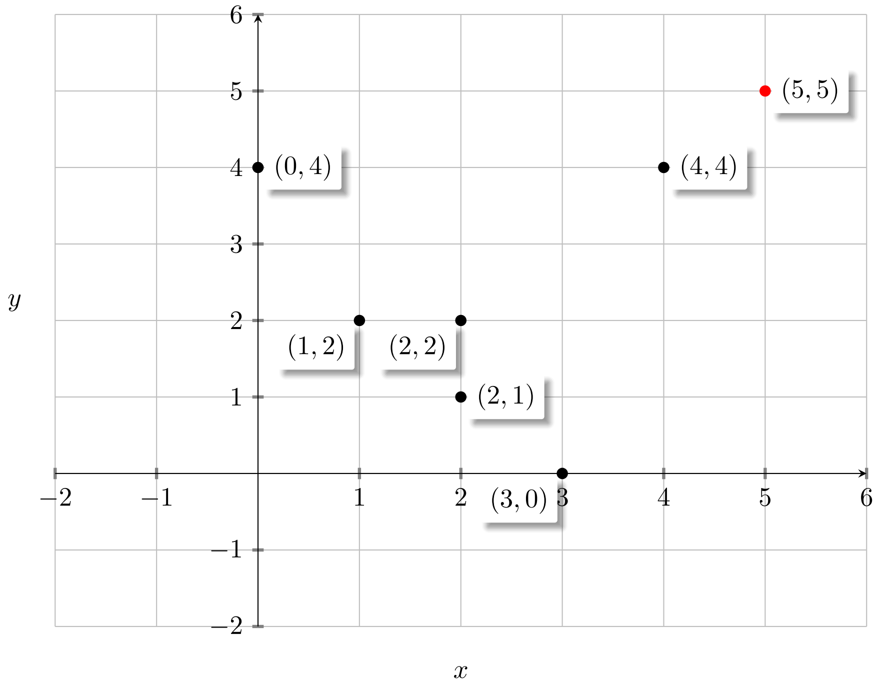
4 Pareto Dominance and Population-based Metaheuristics (20%)

[Task 4.1 (5%)] The Pareto dominance order in the objective space $\mathbb{R}^2$ is a special case of a polyhedral cone-order. What are the generators (vectors) of the ordering cone in the bi-objective case? (chose vectors of length 1); provide a mathematical expression describing the polyhedral cone (set) and draw the cone in a 2-D diagram of the objective case (do also indicate the precise boundaries of the cone).

[Task 4.2 (5%)] Given the set $S = \{(0,4),(1,2),(2,1),(3,0),(4,4),(2,2)\}$. Identify all solutions that are non-dominated. List all solutions that are incomparable to $(2,2)$ under the Pareto dominance order.

[Task 4.3 (5%)] Draw the Hasse diagram of S for Pareto dominance order.

[Task 4.4 (5%)] Compute the ranks of solutions using non-dominated sorting (as used in the first part of the ranking scheme in NSGA-II). What are the hypervolume-contributions (as used in SMS-EMOA) of the solutions in the first (best) ranked subset. Consider reference point $(5,5)$ for computing hypervolume-contributions.



*** END OF EXAM MODA 2021, *Wishing you success!* ***