Linear programming formulation of shortest path problem

Consider the shortest path problem with two objectives.

f_1 : Distance to be minimized

f_2: Quality of road to be maxim.

Given : Matrix of distances D

Matrix of quality values Q

City \ t		1	2	3 (j)	4	5
1	1	1	0	0	0	0
2	⋮	0	0	1	0	0
3	i	0	1	0	0	0
4	⋮	0	0	0	0	1
5	C	0	0	0	1	0
time		1	…	t	…	T

FIGURE 1

Let $X = x_{it}, i = 1, \ldots, t$ denote the matrix of selected entries. They indicate a sequence.

In the example of Figure 1: city 1 is visited first, city 3 is visited second, city 2 is visited third, and so on. Notice, that the index i in the column for time step j is 1, if and only if city i is visited in that time step.

We can now state the objective function:

Let us start with discussing the quadratic programming version:

$$f_1^Q(X) = \sum_{t=1}^{T-1} \sum_{i=1}^{C} \sum_{j=1}^{C} d_{ij}\, x_{i,t}\, x_{j,t+1} \rightarrow \min \quad (1)$$

$$f_2^Q(X) = \sum_{t=1}^{T-1} \sum_{i=1}^{C} \sum_{j=1}^{C} q_{ij}\, x_{i,t}\, x_{j,t+1} \rightarrow \max \quad (2)$$

$$\text{s.t.} \quad \sum_{i=1}^{C} x_{i,t} = 1 \quad \forall t = 1, \dots, T \quad (3)$$

(Every time-step exactly 1 city is visited)

$$\sum_{t=1}^{T} x_{i,t} = 1 \quad \forall i = 1, \dots, C \quad (4)$$

(Every city is visited in exactly 1 time step)

Can (1) – (4) be reformulated as a Linear Program?

Let us for this introduce two threshold variables that we will denote with τ_d and τ_q.

We would like to achieve that the total distance is below τ_d and the total quality above τ_q.

$$\min \ \tau_d \ ; \ \max \ \tau_q$$

$$\text{s.t.} \quad \sum_{t=1}^{T-1} \sum_{i=1}^{C} \sum_{j=1}^{C} d_{ij} \, x_{i,t} + \overbrace{L\Big(\big(1 - x_{i,t}\big) + \big(1 - x_{j,t+1}\big)\Big)}^{\text{penalty}(x_{ij}, t)} \leq \tau_d$$

$$\sum_{t=1}^{T-1} \sum_{i=1}^{C} \sum_{j=1}^{C} q_{ij} \, x_{i,t} - \overbrace{L\big(2 - x_{i,t} - x_{j,t+1}\big)}^{\| \ \text{penalty}(x_{ij}, t)} \geq \tau_q$$

$$\text{s.t.} \quad \sum_{i=1}^{C} x_{i,t} = 1 \quad \forall t = 1, \dots, T$$

(Every time-step exactly 1 city is visited)

$$\sum_{t=1}^{T} x_{i,t} = 1 \quad \forall i = 1, \dots, C$$

(Every city is visited in exactly 1 time step)

$X \in \{0,1\}^{C \times T}$, $\tau_d \in \mathbb{R}^+$, $\tau_q \in \mathbb{R}^+$,

L is a sufficiently large constant, e.g. the sum of all d_{ij} or q_{ij}, respectively.

This is a MILP formulation of the shortest path problem. Relaxation bounds to the LP solver can be readily applid to efficiently. □