PHYS 234 - A6

Alexander Maguire
amaguire@uwaterloo.ca
20396195

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Problem 1 a:

$$\begin{aligned}
\rho &= \sum_i p_i \left|\psi_i\right\rangle \left\langle\psi_i\right| \\
&= 0.2 \left|-\right\rangle_x {}_x\left\langle-\right| + 0.75 \left|+\right\rangle_y {}_y\left\langle+\right| + 0.05 \left|+\right\rangle \left\langle+\right| \\
&= 0.2 \begin{bmatrix} 0.707 \\ -0.707 \end{bmatrix} + 0.75 \begin{bmatrix} 0.707 \\ 0.707i \end{bmatrix} + 0.05 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 0.525 & -0.1 - 0.375i \\ -0.1 + 0.375i & 0.475 \end{bmatrix}
\end{aligned}$$

Problem 1 b:

$$\begin{aligned}
P_{x+} &= 0.2|_x \left\langle + \,|\, - \right\rangle_x|^2 + 0.75|_x \left\langle + \,|\, + \right\rangle_y|^2 + 0.05|_x \left\langle + \,|\, + \right\rangle|^2 \\
&= 0.4
\end{aligned}$$

$$\begin{aligned}
P_{x-} &= 1 - P_{x+} \\
&= 0.6
\end{aligned}$$

Problem 1 c:

$$\begin{aligned}
\langle\psi\rangle &= 0.4\frac{\hbar}{2} - 0.6\frac{\hbar}{2} \\
&= -0.1\hbar
\end{aligned}$$

Problem 1 d:

$$
\begin{aligned}
\langle \mathbf{S_x} \rho \rangle &= Tr\left\{ \frac{\hbar}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0.525 & -0.1 - 0.375i \\ -0.1 + 0.375i & 0.475 \end{bmatrix} \right\} \\
&= \frac{\hbar}{2} Tr\left\{ \begin{bmatrix} -0.1 + 0.375i & 0.475 \\ 0.525 & -0.1 - 0.375i \end{bmatrix} \right\} \\
&= -0.1\hbar
\end{aligned}
$$

Problem 2:

$$
\begin{aligned}
P(x+) &= |_x \langle + \,|\, + \rangle \,|^2 \\
&= 0.5
\end{aligned}
$$

Conditional on the observance of $|+\rangle_x$, we continue:

$$
\begin{aligned}
P(+|x+) &= |\, \langle + \,|\, + \rangle_x \,|^2 \\
&= 0.5 \\
P(-|x+) &= |\, \langle - \,|\, + \rangle_x \,|^2 \\
&= 0.5 \\
P(+) &= P(x+) \times P(+|x+) = 0.25 \\
P(-) &= P(x+) \times P(-|x+) = 0.25
\end{aligned}
$$

Problem 3:

$$\theta = \frac{\pi}{4}$$
$$\phi = \frac{5\pi}{3}$$

$$\begin{aligned}
\left|\psi\right\rangle_n &= \cos\frac{\theta}{2}\left|+\right\rangle + \sin^2\frac{\theta}{2}e^{i\phi}\left|-\right\rangle \\
&= 0.924\left|+\right\rangle + (0.191 - 0.331i)\left|-\right\rangle
\end{aligned}$$

$$\begin{aligned}
P_{y+} &= \left|{}_y\left\langle + \,|\, \psi\right\rangle_n\right|^2 \\
&= 0.194 \\
P_{y-} &= 1 - P_{y+} \\
&= 0.806
\end{aligned}$$

Problem 4 a:

$$\begin{aligned}
AB - BA &= \begin{bmatrix} a_1b_1 & 0 & 0 \\ 0 & 0 & a_2b_2 \\ 0 & a_3b_2 & 0 \end{bmatrix} - \begin{bmatrix} a_1b_1 & 0 & 0 \\ 0 & 0 & a_3b_2 \\ 0 & a_2b_2 & 0 \end{bmatrix} \\
&\neq \hat{0}
\end{aligned}$$

No, these operators do not commute with one another.

Problem 4 b:

Because A is a diagonal basis, it must be expressed in its own basis. Therefore:

$$A_1 = a_1$$
$$A_2 = a_2$$
$$A_3 = a_3$$

$$|a_1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$
$$|a_2\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$
$$|a_3\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Unfortunately B is not so nice, we must compute its eigenvalues:

$$\det B - \lambda I = 0$$
$$\det \begin{bmatrix} b_1 - \lambda & 0 & 0 \\ 0 & -\lambda & b_2 \\ 0 & b_2 & -\lambda \end{bmatrix} = 0$$
$$(b_1 - \lambda)(\lambda^2 - b_2^2) = 0$$
$$\lambda = \{b_1, b_2, -b_2\}$$

$$\begin{bmatrix} b_1 & 0 & 0 \\ 0 & 0 & b_2 \\ 0 & b_2 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = b_1 \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$
$$|b_1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} b_1 & 0 & 0 \\ 0 & 0 & b_2 \\ 0 & b_2 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = b_2 \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$
$$|b_2\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} b_1 & 0 & 0 \\ 0 & 0 & b_2 \\ 0 & b_2 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = -b_2 \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$|-b_2\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Problem 4 c:

$$\begin{aligned} P_{b_1} &= |\, \langle b_1 \,|\, 2 \rangle \,|^2 \\ &= \left| \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right|^2 \\ &= 0 \end{aligned}$$

$$\begin{aligned} P_{b_2} &= |\, \langle b_2 \,|\, 2 \rangle \,|^2 \\ &= \left| \begin{bmatrix} 0 & 0.707 & 0.707 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right|^2 \\ &= 0.5 \end{aligned}$$

$$\begin{aligned} P_{-b_2} &= |\, \langle -b_2 \,|\, 2 \rangle \,|^2 \\ &= \left| \begin{bmatrix} 0 & -0.707 & 0.707 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right|^2 \\ &= 0.5 \end{aligned}$$

$$P_{a_1|b_2} = \left| \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0.707 \\ 0.707 \end{bmatrix} \right|^2$$
$$= 0$$

$$P_{a_2|b_2} = \left| \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0.707 \\ 0.707 \end{bmatrix} \right|^2$$
$$= 0.5$$

$$P_{a_3|b_2} = \left| \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0.707 \\ 0.707 \end{bmatrix} \right|^2$$
$$= 0.5$$

$$P_{a_1|-b_2} = \left| \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -0.707 \\ 0.707 \end{bmatrix} \right|^2$$
$$= 0$$

$$P_{a_2|-b_2} = \left| \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -0.707 \\ 0.707 \end{bmatrix} \right|^2$$
$$= 0.5$$

$$P_{a_3|-b_2} = \left| \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -0.707 \\ 0.707 \end{bmatrix} \right|^2$$
$$= 0.5$$

Problem 4 d:

(a) and (c) are related in that the answers for (c) would be the same if operators A and B commuted.