PHYS 234 - A10

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Problem 1 a:

$$\Psi(x,0) = A[\psi_1(x) + \psi_2(x)]$$

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} \Psi(x,0)^2 \, \mathrm{d}x \\
&= \int_{-\infty}^{\infty} A^2\psi_1^2(x) + A^2\psi_2^2(x) \, \mathrm{d}x \\
&= \int_{-\infty}^{\infty} A^2\psi_1^2(x) \, \mathrm{d}x + \int_{-\infty}^{\infty} A^2\psi_2^2(x) \, \mathrm{d}x \\
&= A^2 \left[\int_{-\infty}^{\infty} \psi_1^2(x) \, \mathrm{d}x + \int_{-\infty}^{\infty} \psi_2^2(x) \, \mathrm{d}x \right] \\
&\text{(assume } \psi_i(x) \text{ is normalized)} \\
&= 2A^2
\end{aligned}$$

$$\rightarrow A = \frac{1}{\sqrt{2}}$$

Problem 1 b:

$$\begin{aligned}
\Psi(x,0) &= \frac{1}{\sqrt{2}}[\psi_1(x)+\psi_2(x)] \\
\Psi(x,t) &= \frac{1}{\sqrt{2}}\left[\exp\left\{\frac{-i}{\hbar}E_1 t\right\}\psi_1(x)+\exp\left\{\frac{-i}{\hbar}E_2 t\right\}\psi_2(x)\right] \\
&= \frac{1}{\sqrt{2}}\left[\exp\left\{\frac{-i}{\hbar}\frac{\hbar^2\pi^2}{2ma^2}t\right\}\psi_1(x)+\exp\left\{\frac{-i}{\hbar}\frac{2\hbar^2\pi^2}{ma^2}t\right\}\psi_2(x)\right] \\
&= \frac{1}{\sqrt{a}}\left[\exp\left\{\frac{-i}{\hbar}\frac{\hbar^2\pi^2}{2ma^2}t\right\}\sin\frac{\pi x}{a}+\exp\left\{\frac{-i}{\hbar}\frac{2\hbar^2\pi^2}{ma^2}t\right\}\sin\frac{2\pi x}{a}\right]
\end{aligned}$$

$$\begin{aligned}
|\Psi(x,t)|^2 &= \frac{1}{a}\left|\exp\{-i\omega t\}\sin\frac{\pi x}{a}+\exp\{-4i\omega t\}\sin\frac{2\pi x}{a}\right|^2 \\
&= \frac{1}{a}\left[\sin^2\frac{\pi x}{a}+\sin^2\frac{2\pi x}{a}+\exp\{-3i\omega t\}\sin\frac{\pi x}{a}\sin\frac{2\pi x}{a}\right. \\
&\qquad \left.+\exp\{3i\omega t\}\sin\frac{\pi x}{a}\sin\frac{2\pi x}{a}\right] \\
&= \frac{1}{a}\left[\sin^2\frac{\pi x}{a}+\sin^2\frac{2\pi x}{a}+2\sin\frac{\pi x}{a}\sin\frac{2\pi x}{a}\cos 3\omega t\right]
\end{aligned}$$

Problem 1 c:

$$\begin{aligned}
\langle\hat{x}\rangle &= \int_{-\infty}^{\infty}\Psi^*(x,t)\hat{x}\Psi(x,t)\,\mathrm{d}x \\
&= \int_0^a x\,|\Psi(x,t)|^2\,\mathrm{d}x \\
&= \frac{1}{a}\left[\frac{a^2}{2}+\int_{-\infty}^{\infty}2x\sin\frac{\pi x}{a}\sin\frac{2\pi x}{a}\cos 3\omega t\,\mathrm{d}x\right] \\
&= \frac{1}{a}\left[\frac{a^2}{2}-\frac{16a^2\cos 3\omega t}{9\pi^2}\right] \\
&= a\left[\frac{1}{2}-\frac{16\cos 3\omega t}{9\pi^2}\right]
\end{aligned}$$

The angular frequency is 3ω, and the amplitude of the wave is $16a/(9\pi^2)$.

Problem 1 d:

$$\begin{aligned}\mathcal{P}(E_1) &= |\langle E_1 \,|\, \Psi\rangle|^2 \\ &= \frac{1}{2}\end{aligned}$$

$$\begin{aligned}\mathcal{P}(E_2) &= |\langle E_2 \,|\, \Psi\rangle|^2 \\ &= \frac{1}{2}\end{aligned}$$

$$\begin{aligned}\left\langle \hat{H} \right\rangle &= \mathcal{P}(E_1)E_1 + \mathcal{P}(E_2)E_2 \\ &= \frac{1}{2}\left[\frac{\hbar^2\pi^2}{2ma^2} + \frac{4\hbar^2\pi^2}{2ma^2}\right] \\ &= \frac{5\hbar^2\pi^2}{4ma^2}\end{aligned}$$

The expectation of the Hamiltonian is the weighted sum of the individual eigenenergies of the particle.

Problem 2 a:

$$\begin{aligned}\psi_0'(x) &= \frac{1}{\pi^{1/4}\sqrt{x_0'}} \exp\left\{-\frac{x^2}{2x_0'^2}\right\} \\ \psi_1'(x) &= \frac{\sqrt{2}}{\pi^{1/4}\sqrt{x_0'}} \frac{x}{x_0'} \exp\left\{-\frac{x^2}{2x_0'^2}\right\} \\ \psi_2'(x) &= \frac{1}{\pi^{1/4}\sqrt{2x_0'}} \left(2\frac{x^2}{x_0'^2} - 1\right) \exp\left\{-\frac{x^2}{2x_0'^2}\right\}\end{aligned}$$

$$
\begin{aligned}
\psi_0(x)\bigg|_{x_0=2x_0'} &= \frac{1}{\pi^{1/4}\sqrt{2x_0'}}\exp\left\{-\frac{x^2}{2(2x_0')^2}\right\} \\
&= \frac{1}{\pi^{1/4}\sqrt{2x_0'}}\exp\left\{-\frac{x^2}{8x_0'^2}\right\} \\
&= c_0\psi_0'(x) + c_1\psi_1'(x) + c_2\psi_2'(x) \\
\frac{1}{\pi^{1/4}\sqrt{2x_0'}}\kappa^{1/4} &= c_0\frac{1}{\pi^{1/4}\sqrt{x_0'}}\kappa + c_1\frac{\sqrt{2}}{\pi^{1/4}\sqrt{x_0'}}\frac{x}{x_0'}\kappa + c_2\frac{1}{\pi^{1/4}\sqrt{2x_0'}}\left(2\frac{x^2}{x_0'^2}-1\right)\kappa, \qquad \kappa = \exp\left\{-\frac{x^2}{x_0'^2}\right\} \\
&= c_0\frac{\sqrt{2}}{\pi^{1/4}\sqrt{2x_0'}}\kappa + c_1\frac{\sqrt{2}}{\pi^{1/4}\sqrt{x_0'}}\frac{x}{x_0'}\kappa + c_2\frac{1}{\pi^{1/4}\sqrt{2x_0'}}\left(2\frac{x^2}{x_0'^2}-1\right)\kappa \\
&= \frac{1}{\pi^{1/4}\sqrt{2x_0'}}\left[c_0\sqrt{2}\kappa + c_1\sqrt{2}\kappa\frac{x}{x_0'} + c_2\kappa\left(2\frac{x^2}{x_0'^2}-1\right)\right] \\
\kappa^{1/4} &= c_0\sqrt{2}\kappa + c_1\sqrt{2}\kappa\frac{x}{x_0'} + c_2\kappa\left(2\frac{x^2}{x_0'^2}-1\right) \\
&= c_0\sqrt{2}\kappa + c_1\sqrt{2}\kappa\lambda + c_2\kappa\left(2\lambda^2-1\right), \qquad \lambda = x/x_0' \\
\kappa^{-3/4} &= c_0\sqrt{2} + c_1\sqrt{2}\lambda + 2c_2\lambda^2 - c_2 \\
&\ldots \\
&\ldots \\
&\textbf{HNNNNNNG} \\
&\ldots \\
&\ldots
\end{aligned}
$$

Problem 2 b:

The eigenenergies of the state will be E_0, E_1, E_2, found with probabilities:

$$
\mathcal{P}(E_i) = |c_i|^2 \quad \forall i \in \{0, 1, 2\}
$$