# PHYS 234 - A9

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## Problem 1 a:

$$
\begin{aligned}
4\,\text{eV} &= E_{\psi_I} \\
&= E_2 \\
&= \frac{2\hbar^2\pi^2}{ma^2} \\
ma^2 &= \frac{2\hbar^2\pi^2}{4\,\text{eV}} \\
&= 3.43 \times 10^{-48}\,\text{kg} \cdot \text{m}^2
\end{aligned}
$$

$$
\begin{aligned}
E_{\psi_{II}} &= E_3 \\
&= \frac{9\hbar^2\pi^2}{2ma^2} \\
&= 9\,\text{eV}
\end{aligned}
$$

## Problem 1 b:

$$
\begin{aligned}
E_{min} &= E_1 \\
&= \frac{\hbar^2\pi^2}{2ma^2} \\
&= 1\,\text{eV}
\end{aligned}
$$

## Problem 2:

$$\psi_n(x) = A \sin \frac{n\pi x}{L}$$

$$\begin{aligned}
\int_{-\infty}^{\infty} \psi_1^*(x)\psi_3(x)\,\mathrm{d}x &= \int_{-\infty}^{\infty} \psi_1(x)\psi_3(x)\,\mathrm{d}x \\
&= C\int_0^{\infty} \sin(y)\sin(3y)\,\mathrm{d}x \qquad y = \frac{\pi x}{L} \\
&= C\int_0^{\infty} \cos(y+\pi/2)\cos(3y+\pi/2)\,\mathrm{d}x \\
&= C\int_0^{\infty} \frac{\cos(4y+\pi)+\cos(2y)}{2}\,\mathrm{d}x \\
&= C\int_0^{\infty} \cos(2y)-\cos(4y)\,\mathrm{d}x \\
&= C\int_0^{\infty} \cos(2y)\,\mathrm{d}x - C\int_0^{\infty} \cos(4y)\,\mathrm{d}x \\
\text{the integral over any hyperperiod of} & \\
\text{these functions will be 0, and thus:} & \\
&= 0
\end{aligned}$$

## Problem 3 a:

$$\psi(x) = \begin{cases} Nx(x-L) & 0 < x < L \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned}
\int_{-\infty}^{\infty} (Nx(x-L))^2\,\mathrm{d}x &= 1 \\
N^2\left(\frac{L^2x^3}{3} - \frac{Lx^4}{2} + \frac{x^5}{5}\right)\Bigg|_0^L &= 1 \\
\frac{L^5N^2}{30} &= 1 \\
N &= \sqrt{\frac{30}{L^5}}
\end{aligned}$$

## Problem 3 b:

$$\begin{aligned}
\mathcal{P}(E_1) &= |\langle 1 \,|\, \psi \rangle|^2 \\
&= \left| \int_{-\infty}^{\infty} \phi_1^*(x)\psi(x)\,\mathrm{d}x \right|^2 \\
&= \frac{2}{L} \left| \int_0^L \sin\frac{\pi x}{L} Nx(x-L)\,\mathrm{d}x \right|^2 \\
&= \frac{2}{L} \left| -\frac{4L^3 N}{\pi^3} \right|^2 \\
&= \frac{2}{L} \left[ \frac{16L^6 N^2}{\pi^6} \right] \\
&= \frac{32L^5 N^2}{\pi^6} \\
&= \frac{960}{\pi^6}
\end{aligned}$$

## Problem 3 c:

$$\begin{aligned}
\left\langle \hat{H} \right\rangle &= \left\langle \psi \middle| \hat{H} \middle| \psi \right\rangle \\
&= \int_{-\infty}^{\infty} \psi^*(x)\hat{H}\psi(x)\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} \psi^*(x) \left[ \frac{-\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \hat{V} \right] \psi(x)\,\mathrm{d}x \\
&= \int_0^L x(x-L) \left[ \frac{-\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 0 \right] (Nx^2 - NxL)\,\mathrm{d}x \\
&= \frac{5\hbar}{m} \int_0^L x(x-L)\,\mathrm{d}x \\
&= \frac{5\hbar}{L^2 m}
\end{aligned}$$

## Problem 4:

$$\Psi(x,t) = c_1\Psi_1(x,t) + c_2\Psi_2(x,t)$$

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} \Psi^2(x,t)\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} c_1^2\Psi_1^2(x,t)\,\mathrm{d}x + \int_{-\infty}^{\infty} c_2^2\Psi_2^2(x,t)\,\mathrm{d}x + \int_{-\infty}^{\infty} c_1c_2\Psi_1(x,t)\Psi_2(x,t)\,\mathrm{d}x \\
&= c_1^2\int_{-\infty}^{\infty} \Psi_1^2(x,t)\,\mathrm{d}x + c_2^2\int_{-\infty}^{\infty} \Psi_2^2(x,t)\,\mathrm{d}x + 0 \qquad \text{(by orthogonality)} \\
&= c_1^2 + c_2^2 \qquad \text{(by normalization criterion)}
\end{aligned}$$

$$\therefore \quad c_1^2 + c_2^2 = 1$$

## Problem 5:

$$\begin{aligned}
\mathcal{P}(E_n, 3L/4 < x < L) &= \int_{3L/4}^{L} \left|\phi_n(x)\right|^2\,\mathrm{d}x \\
&= \frac{2}{L}\int_{3L/4}^{L} \left|\sin\frac{n\pi x}{L}\right|^2\,\mathrm{d}x \\
&= \frac{\pi n + 2\sin\left(\frac{3\pi n}{2}\right) - 2\sin(2\pi n)}{4\pi n}
\end{aligned}$$

$$\begin{aligned}
\mathcal{P}(E_1) &= \frac{\pi + 2\sin\left(\frac{3\pi}{2}\right) - 2\sin(2\pi)}{4\pi} \\
&= \frac{\pi - 2}{4\pi}
\end{aligned}$$

$$\begin{aligned}
\mathcal{P}(E_2) &= \frac{2\pi + 2\sin\left(\frac{6\pi}{2}\right) - 2\sin(4\pi)}{8\pi} \\
&= \frac{1}{4}
\end{aligned}$$

$$\begin{aligned}
\mathcal{P}(E_3) &= \frac{3\pi + 2\sin\left(\frac{9\pi}{2}\right) - 2\sin(6\pi)}{12\pi} \\
&= \frac{2 + 3\pi}{12\pi}
\end{aligned}$$