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Assignment 6 20396195

1) RTS: $an \equiv bn (mod\ mn) \Leftrightarrow a \equiv b (mod\ m)$.
$x \equiv y (mod\ w)$ means that $w | (x-y)$, which implies that $an \equiv bn (mod\ mn)$ is the same as saying $mn | (an-bn)$, or alternatively $mn | n(a-b)$. Because $n|n$ trivially, $mn|n(a-b)$ is true only when $m|(a-b)$. ❑

2) RTS: $n \equiv 3 (mod\ 4)$ implies *n* is not the sum of two squares.
BWOC, assume *n* be the sum of two squares, $n = a^2 + b^2$, $a, b \in \mathbb{Z}$. Thus, $a^2 + b^2 \equiv 3 (mod\ 4)$.
For this to be true, *n* must be odd which means exactly one term in $a^2 + b^2$ be odd.
WLOG, we will say that this term be a^2 . b^2 must thus be even, which means $b \equiv 0 (mod\ 2)$, ie $2k = b$ for some *k*. By squaring this equation, we get $4k^2 = b^2$, which is the same as $b^2 \equiv 0 (mod\ 4)$.
Because $x_1 + x_2 \equiv c_1 + c_2 (mod\ m)$ iff $x_1 \equiv c_1 (mod\ m)$ and $x_2 \equiv c_2 (mod\ m)$, $a^2 \equiv 3 (mod\ 4)$ must hold true. This implies $a^2 = 4k + 3$ for some *k*. Now, every odd number may be written as $4m \pm 1$, implying $(4m+1)^2 = 4k+3$ and $(4m-1)^2 = 4k-3$. Simplifying the first case, we get :

$$16m^2 + 8m + 1 = 4k + 3$$
$$16m^2 + 8m = 4k + 2$$
$$8m^2 + 4m = 2k + 1$$

This is clearly a contradiction. Now, checking the other case:

$$16m - 8m + 1 = 4k + 3$$
$$16m - 8m = 4k + 2$$
$$8m - 4m = 2k + 1$$

Another contradiction, thus *n* must not be the sum of two squares if it is $n \equiv 3 (mod\ 4)$. ❑

3) Required to find all possible pairs of digits (a, b) where $99 | 38a91b$. Because $99 = 9 \cdot 11$, we need only find (a, b) such that $38a91b$ is divisible by 9 and 11.
To check for divisibility by 9, we require that the sum of the digits be divisible by 9:
$9 | 38a91b \equiv 9 | (3+8+a+9+1+b) \equiv 9 | (21+a+b)$.
Divisibility by 11 requires that the alternating series of low-edian-ordered digits be divisible by 11:
$11 | 38a91b \equiv 11 | (b-1+9-a+8-3) \equiv 11 | (13-a+b)$.
These two observations give us the equations $\begin{matrix} 11m = 13 - a + b \\ 9n = 21 + a + b \end{matrix}$. We can fix *m*, *n* so that $0 \leqslant a, b \leqslant 9$, and therefore

$$\begin{matrix} -2 = -a + b \\ 6 = a + b \end{matrix} .$$

Solving this system gives us $b = 4$ and $a = 2$, so our solution becomes the single pair $(2, 4)$.

4) Let n = $n = a_0 + 1000 \cdot a_1 + 1000^2 \cdot a_2 + \cdots + 1000^l \cdot a_l$. RTS $d | n \Leftrightarrow d | (a_0 - a_1 + a_2 + \cdots (-1)^l a_l)$.
We will first solve this for $d = 11$. Because we know that 11 divides any number whose alternating sum of low-edian-ordered digits be divisible by 11, it is trivial to show that the pairs of double zeroes in between each number have no effect. If we were to expand them out, we would get
$a_0 - 0 + 0 - a_1 + 0 - 0 + a_2 + \cdots$. ❑ $for\ d = 11$.

5) a) RTS: An integer is divisible by 8 if and only if the integer formed by its last three digits is divisible by 8.

This is the same as saying an integer *n* is divisible by 8 iff $8|(n \ mod\ 1000)$. 8 evenly divides 1000, and so it must evenly divide all integer multiples of 1000, ie every digit further than the third place. We can define $q=n \ mod\ 1000$ and $p=n-q$. *p* is now an integer multiple of 1000, and *q* is the remainder. We therefore obtain $q\equiv n(mod\ 8)$, that is, that *n* is divisible by 8 iff *q* is divisible 8. ❑

b) Want all possible pairs of digits (a, b) where $72|27b9a4$.

Because $8\cdot 9=72$, the problem then becomes $8|27b9a4$, which from above we know is the same as $8|9a4$, and $9|27b9a4$, which is the same as the sum of digits being divisible by 9, ie $9|(a+b+22)$.

We can then find values for b which satisfy the first equation, namely $b=\{0,4,8\}$. Now that we have b, we can substitute it into the first equation: $9|(a+22), 9|(a+26), 9|(a+30)$.

These are now trivially solvable, providing the solutions: $(9m-22,0),(9m-26,4),(9m-30,8), 0\leqslant a\leqslant 9, m\in\mathbb{N}$.

Finally, enforcing the condition for *a*, we arrive at the solutions $(5,0),(1,4),(6,8)$.