MATH 135
Assignment 2

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1) $\neg\exists x{:}\forall y{:}\ P(x) \lor Q(y)$
$\forall x{:}\neg\forall y{:}\ P(x) \lor Q(y)$
$\forall x{:}\exists y{:}\ \neg(P(x) \lor Q(y))$
$\forall x{:}\exists y{:}\ (\neg P(x)) \land (\neg Q(y))$

2) a) $\forall x{:}\exists y{:}\ x=y^3$, $\mathbb{Z}$ is false:
pick x=2, then $y=\sqrt[3]{2} \notin \mathbb{Z}$

$\forall x{:}\exists y{:}\ x=y^3$, $\mathbb{R}$ is true:
for a positive x, then y is just its cube root. A negative x will have two complex solutions and one real solution, the real solution being the negative cube root of the absolute value.

b) $\forall x{:}\exists y{:}\ (y < x) \land (\forall z{:}\ z < x \rightarrow z \leq y)$, $\mathbb{Z}$ is true:
pick x, pick $y = x - 1$, then for all $z < x$, $z \leq y$

$\forall x{:}\exists y{:}\ (y < x) \land (\forall z{:}\ z < x \rightarrow z \leq y)$, $\mathbb{R}$ is false:
in the universe of $\mathbb{R}$, y and z are bounded by $(-\infty, x)$. Because the top end is open, z can always be picked closer to x than y.

3) $4x^2 + 12x + 11 = 0$
for this to have real roots, its discriminate must be ≥ 0:
$b^2 - 4ac \geq 0$
$12^2 - 4 \cdot 4 \cdot 11 \geq 0$
$144 - 176 \geq 0$
$-32 \geq 0$
This leads to a contradiction, so the polynomial must not have real roots

4) Prove that $\forall n{:}\ 1^2+2^2+3^2 \ldots (2n-1)^2 = \dfrac{n(2n-1)(2n+1)}{3}$, $\mathbb{N}$

$case\ where\ n=1{:}\ 1^2 = \dfrac{1(2-1)(2+1)}{3}$. Now assume that
$1=1$

$1^2+2^2+3^2 \ldots (2k-1)^2 = \dfrac{k(2k-1)(2k+1)}{3}$ is true, then

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$$1^2+2^2+3^2\ldots(2[k+1]-1)^2=1^2+2^2+3^2\ldots(2k-1)^2+(2[k+1]-1)^2$$
$$1^2+2^2+3^2\ldots(2[k+1]-1)^2=\frac{k(2k-1)(2k+1)}{3}+(2[k+1]-1)^2$$
$$(2[k+1]-1)^2=(2k+1)^2=(4k^2+4k+1)$$
$$\frac{k(2k-1)(2k+1)}{3}+(4k^2+4k+1)=\frac{4k^3-k}{3}+\frac{12k^3+12k+3}{3}$$
$$\frac{4k^3-k}{3}+\frac{12k^2+12k+3}{3}=\frac{4k^3+12k^2+11k+3}{3}=\frac{(k+1)(2[k+1]-1)(2[k+1]+1)}{3}$$
$$1^2+2^2+3^2\ldots(2k-1)^2=\frac{k(2k-1)(2k+1)}{3}\Rightarrow 1^2+2^2+3^2\ldots(2[k+1]-1)^2=\frac{(k+1)(2[k+1]-1)(2[k+1]+1)}{3}$$

5) $\sum_{r=1}^{n} r(r!)=(n+1)!-1$

Base case: $1=\sum_{r=1}^{1} 1(1!)=(1+1)!-1=2!-1=1$. Now assume that

$\sum_{r=1}^{k} r(r!)=(k+1)!-1$ is true:

$\sum_{r=1}^{(k+1)} r(r!)=\sum_{r=1}^{k} r(r!)+([k+1]+1)([[k+1]+1]!)$

$\sum_{r=1}^{(k+1)} r(r!)=(k+1)!-1+(k+2)([k+2]!)$

$\sum_{r=1}^{(k+1)} r(r!)=(k+1)!-1+(k+2)^2([k+1]!)\,, thus$

$\sum_{r=1}^{k} r(r!)=(k+1)!-1\Rightarrow\sum_{r=1}^{[k+1]} r(r!)=([k+1]+1)!-1$

QED