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Assignment 3 20396195

1) Base case:
$n=1, \frac{1}{1}+\frac{1}{2^1} \geqslant 1+\frac{1}{2}$
now assume that for some k, $\sum_{i=1}^{2^k} \frac{1}{i} \geqslant 1+\frac{k}{2}$ (induction hypothesis). We need to prove that $k \to k+1$, so $\sum_{i=1}^{2^{k+1}} \frac{1}{i} = \sum_{i=1}^{2^k} \frac{1}{i} + \sum_{i=2^k+1}^{2^{k+1}} \frac{1}{i}$ can be simplified to
$\sum_{i=1}^{2^{k+1}} \frac{1}{i} \geqslant 1+\frac{k}{2}+\sum_{i=2^k+1}^{2^{k+1}} \frac{1}{i}$ (by induction hypothesis)
we can now find a lower bound for the second term $\sum_{i=2^k+1}^{2^{k+1}} \frac{1}{i} \geqslant \frac{2^{k+1}-2^k}{2^{k+1}} = \frac{2^k(2-1)}{2^{k+1}} = \frac{1}{2}$
implying $\sum_{i=1}^{2^{k+1}} \frac{1}{i} \geqslant 1+\frac{k}{2}+\frac{1}{2} = 1+\frac{k+1}{2}$.

2) Given $a_1=2$ and $a_{n+1}=\frac{5a_n-4}{a_n}$
$1 \leqslant a_1 = 2$, now prove that all $a_{n+1} \geqslant a_n$:
$a_{n+1}=\frac{5a_n-4}{a_n}=\frac{5a_n-5+1}{a_n}=\frac{5(a_n-1)+1}{a_n}$
because the base case $a_n=2 \geqslant 1$, the expression $\frac{5(a_n-1)+1}{a_n}$ must always yield a value greater than or equal a_n , thus $\frac{5(a_n-1)+1}{a_n}=a_{n+1} \geqslant a_n$. Now to prove $a_k \leqslant 4$ for $\forall k \in \mathbb{N}$, we can look at $a_{n+1}=\frac{5a_n-4}{a_n}$ as a fixed point iterator, as it maps $[2,4] \to [2,4]$ and $F^*(x)=\frac{d}{dx}\left(\frac{5x-4}{x}\right)=\frac{4}{x^2}$. For values $x \in [2,4]$, $F^*(x) \leqslant 1$ which proves that this fixed point iterator converges to a value in this interval. As I have already proved that $a_{n+1} \geqslant a_n$, this must converge to the upper bound of the interval, and lo-and-behold, $a_n=4, \frac{5(a_n-1)+1}{a_n}=\frac{5(4-1)+1}{4}=\frac{16}{4}=4$. When $a_n=4$, $a_{n+1}=4$. Thus $1 \leqslant a_n \leqslant a_{n+1} \leqslant 4$.

3) Base case: $k=3$,
$x_3=5x_2-6x_1=2^3+3^2$
$x_3=5*7-6*3=8+9$
$x_3=17=17$. The base case is true.
Assume true for $k=1,2,3,\ldots n-1$ (induction hypothesis), so we are required to prove $5x_{n-1}-6x_{n-2}=2^n+3^{n-1}$. We can expand this to $5(2^{n-1}+3^{n-2})-6(2^{n-2}+3^{n-3})=2^n+3^{n-1}$ (by induction hypothesis),

$5*2^{n-1}+5*3^{n-2}-6*2^{n-2}-6*3^{n-3}=2^n+3^{n-1}$ we can now make our exponents equal multiplying them out $10*2^{n-2}+15*3^{n-3}-6*2^{n-2}-6*3^{n-3}=2^n+3^{n-1}$, and then by gathering like terms $4*2^{n-2}+9*3^{n-3}=2^n+3^{n-1}$. The coefficients may now be written as exponents, $2^2*2^{n-2}+3^2*3^{n-3}=2^n+3^{n-1}$ and thus $2^n+3^{n-1}=2^n+3^{n-1}$ shown to be true by the principle of strong induction.

4) Prove $a_n=2a_{n-1}+a_{n-2}=\frac{1}{2}\left((1+\sqrt{2})^n+(1-\sqrt{2})^n\right)$:

Base case $k=2$,

$$2a_1+a_0=\frac{1}{2}\left((1+\sqrt{2})^2+(1-\sqrt{2})^2\right)$$

$$2*1+1=\frac{1}{2}\left((1+\sqrt{2})^2+(1-\sqrt{2})^2\right)$$ (by substitution)

$$3=\frac{1}{2}(1+2\sqrt{2}+2+1-2\sqrt{2}+2)$$

$$3=\frac{1}{2}(6)=3$$, so the base case is true.

Now assume true for $k=1,2,3,\ldots n-1$ (induction hypothesis), so we are required to prove

$$2a_{n-1}+a_{n-2}=\frac{1}{2}\left((1+\sqrt{2})^n+(1-\sqrt{2})^n\right)$$

$$(1+\sqrt{2})^{n-1}+(1-\sqrt{2})^{n-1}+\frac{1}{2}\left((1+\sqrt{2})^{n-2}+(1-\sqrt{2})^{n-2}\right)=\frac{1}{2}\left((1+\sqrt{2})^n+(1-\sqrt{2})^n\right)$$ (by induction hypothesis)

$$(1+\sqrt{2})^{n-1}+(1-\sqrt{2})^{n-1}+\frac{1}{2}\left((1+\sqrt{2})^{n-2}+(1-\sqrt{2})^{n-2}\right)=\frac{1}{2}\left((1+\sqrt{2})^n+(1-\sqrt{2})^n\right)$$

$$(1+\sqrt{2})(1+\sqrt{2})^{n-2}+(1-\sqrt{2})(1-\sqrt{2})^{n-2}+\frac{1}{2}\left((1+\sqrt{2})^{n-2}+(1-\sqrt{2})^{n-2}\right)=\frac{1}{2}\left((1+\sqrt{2})^n+(1-\sqrt{2})^n\right)$$

$$(1+\sqrt{2})^{n-2}\left((1+\sqrt{2})+\frac{1}{2}\right)+(1-\sqrt{2})^{n-2}\left((1-\sqrt{2})+\frac{1}{2}\right)=\frac{1}{2}\left((1+\sqrt{2})^n+(1-\sqrt{2})^n\right)$$

$$(1+\sqrt{2})^{n-2}\left(\frac{3}{2}+\sqrt{2}\right)+(1-\sqrt{2})^{n-2}\left(\frac{3}{2}-\sqrt{2}\right)=\frac{1}{2}\left((1+\sqrt{2})^n+(1-\sqrt{2})^n\right)$$. Notice, now that

$\frac{3}{2}+\sqrt{2}=\frac{(1+\sqrt{2})^2}{2}$, and that $\frac{3}{2}-\sqrt{2}=\frac{(1-\sqrt{2})^2}{2}$, so we can rewrite our previous equation.

$$\frac{(1+\sqrt{2})^{n-2}(1+\sqrt{2})^2}{2}+\frac{(1-\sqrt{2})^{n-2}(1-\sqrt{2})^2}{2}=\frac{1}{2}\left((1+\sqrt{2})^n+(1-\sqrt{2})^n\right)$$

$$\frac{(1+\sqrt{2})^n+(1-\sqrt{2})^n}{2}=\frac{1}{2}\left((1+\sqrt{2})^n+(1-\sqrt{2})^n\right)$$. This this is true by the principle of strong

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induction.