# PHYS 234 - A8

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## Problem 1 a:

$$
\begin{aligned}
\Psi(x,t) &= \sin(kx)[i\cos(\omega t/2) + sin(\omega t/2)] \\
\hat{p}\Psi(x,t) &= p\Psi(x,t) \\
-i\hbar\frac{\partial}{\partial x}\Psi(x,t) &= p\Psi(x,t)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial x}\Psi(x,t) &= k\cos(kx)[i\cos(\omega t/2) + \sin(\omega t/2)] \\
-i\hbar k\cos(kx)[i\cos(\omega t/2) + \sin(\omega t/2)] &= p\sin(kx)[i\cos(\omega t/2) + sin(\omega t/2)] \\
-i\hbar k\cos(kx) &= p\sin(kx) \\
p &= -i\hbar k\cot(kx)
\end{aligned}
$$

This particle is **not** in a definite state of momentum, since it is parameterized on $x$ (and furthermore is guaranteed to never be real).

## Problem 1 b:

$$
\begin{aligned}
\Psi(x,t) &= \sin(kx)[i\cos(\omega t/2) + sin(\omega t/2)] \\
\hat{H}\Psi(x,t) &= E\Psi(x,t) \\
i\hbar\frac{\partial}{\partial t}\Psi(x,t) &= E\Psi(x,t) \\
\\
\frac{\partial}{\partial t}\Psi(x,t) &= \frac{\omega sin(kx)}{2}\Bigg[\cos(\omega t/2) - i\sin(\omega t/2)\Bigg] \\
i\hbar\frac{\omega sin(kx)}{2}\Bigg[\cos(\omega t/2) - i\sin(\omega t/2)\Bigg] &= E\sin(kx)[i\cos(\omega t/2) + sin(\omega t/2)] \\
i\hbar\frac{\omega}{2}\Bigg[\cos(\omega t/2) - i\sin(\omega t/2)\Bigg] &= E[i\cos(\omega t/2) + sin(\omega t/2)] \\
i\hbar\frac{\omega}{2}e^{-i\omega t/2} &= Eie^{-i\omega t/2} \\
E &= \frac{\hbar\omega}{2}
\end{aligned}
$$

This particle is in a definite state of energy.

## Problem 2:

$$
\psi_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right); \qquad n \in \mathbb{N}
$$

$$
\begin{aligned}
\langle\hat{x}\rangle &= \int_0^L \psi_n^*(x)x\psi_n(x)\,\mathrm{d}x \\
&= \frac{2}{L}\int_0^L \sin^2\left(\frac{n\pi x}{L}\right)x\,\mathrm{d}x \\
&= \frac{2}{L}\int_0^L 1 - \cos\left(\frac{2n\pi x}{L}\right)x\,\mathrm{d}x \\
&= \frac{2}{L}\left[x - \frac{L\sin(kx)}{k}\right]\Bigg|_0^L \qquad k = \frac{2n\pi}{L} \\
&= \frac{L}{2}
\end{aligned}
$$

$$
\begin{aligned}
\left\langle \hat{x}^2 \right\rangle &= \int_0^L \psi_n^*(x) x^2 \psi_n(x)\,\mathrm{d}x \\
&= \frac{4}{L^2} \int_0^L \sin^4\left(\frac{n\pi x}{L}\right) x\,\mathrm{d}x \\
&\quad \text{(wolfram alpha)} \\
&= \frac{1}{8\pi n L}\left[\sin(2kx) - 8\sin(kx) + 6kx\right]\Bigg|_0^L \\
&= L^2\left(\frac{1}{3} - \frac{1}{2n^2\pi^2}\right)
\end{aligned}
$$

$$
\begin{aligned}
\langle \hat{p} \rangle &= \int_0^L \psi_n^*(x)\left(-i\hbar\frac{\mathrm{d}}{\mathrm{d}x}\right)\psi_n(x)\,\mathrm{d}x \\
&= \frac{-2i\hbar}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right)\frac{\mathrm{d}}{\mathrm{d}x}\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x \\
&= \frac{-i\hbar}{kL}\int_0^L \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x \\
&= \frac{i\hbar}{2k^2L}\cos\left(kx\right)\Bigg|_0^L \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\left\langle \hat{p}^2 \right\rangle &= \int_0^L \psi_n^*(x)\left(-i\hbar\frac{\mathrm{d}}{\mathrm{d}x}\right)^2 \psi_n(x)\,\mathrm{d}x \\
&= \frac{2\hbar^2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right)\frac{\mathrm{d}^2}{\mathrm{d}x^2}\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x \\
&= \frac{-2\hbar^2 n^2\pi^2}{L^3}\int_0^L \sin^2\left(\frac{n\pi x}{L}\right)\mathrm{d}x \\
&= \frac{-2\hbar^2 n^2\pi^2}{L^3}\left[x - \frac{L\sin(kx)}{k}\right]\Bigg|_0^L \\
&= \frac{-\hbar^2 n^2\pi^2}{2L}
\end{aligned}
$$

$$
\begin{aligned}
\sigma_x &= \sqrt{\langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2} \\
&= \sqrt{L^2\left(\frac{1}{3} - \frac{1}{2n^2\pi^2}\right) - \frac{L^2}{4}} \\
&= L\sqrt{\frac{1}{12} - \frac{1}{2n^2\pi^2}}
\end{aligned}
$$

$$
\begin{aligned}
\sigma_p &= \sqrt{\langle \hat{p}^2 \rangle - \langle \hat{p} \rangle^2} \\
&= \sqrt{\frac{-\hbar^2 n^2 \pi^2}{2L}} \\
&= \frac{i\hbar n\pi}{\sqrt{2L}}
\end{aligned}
$$

$$
\begin{aligned}
\sigma_x \sigma_p &= \frac{\hbar}{2} \\
\left(L\sqrt{\frac{1}{12} - \frac{1}{2n^2\pi^2}}\right)\left(\frac{i\hbar n\pi}{\sqrt{2L}}\right) &= \frac{\hbar}{2}
\end{aligned}
$$

## Problem 3 a:

$$
\Psi(x,t) = A\exp\left\{-a(mx^2/\hbar + it)\right\}
$$

$$
\begin{aligned}
\int_{-\infty}^{\infty} \Psi(x,t)\,\mathrm{d}x &= 1 \\
\int_{-\infty}^{\infty} A\exp\left\{-a(mx^2/\hbar + it)\right\}\mathrm{d}x &= 1 \\
&\text{(wolfram alpha)} \\
A &= \sqrt[4]{\frac{2am}{\pi\hbar}}
\end{aligned}
$$

**Problem 3 b:**

$$\Psi(x,t) = A\exp\left\{-a(mx^2/\hbar + it)\right\}$$

$$\begin{aligned}
\hat{H}\Psi(x,t) &= E\Psi(x,t) \\
\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right]\Psi(x,t) &= E\Psi(x,t)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2}{\partial x^2}\left[A\exp\left\{-a(mx^2/\hbar + it)\right\}\right] &= (am/\hbar)^2 \exp\left\{-a(mx^2/\hbar + it)\right\} \\
&= \frac{a^2m^2}{A\hbar}\Psi(x,t)
\end{aligned}$$

$$\begin{aligned}
\left(\frac{a^2m^2}{A\hbar} + V(x)\right)\Psi(x,t) &= E\Psi(x,t) \\
V(x)\Psi(x,t) &= \left(E - \frac{a^2m^2}{A\hbar}\right)\Psi(x,t)
\end{aligned}$$