DG Lab Project Report – Magnetohydrodynamics

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Lisa implemented the eigenvalues, the Kelvin-Helmholtz instability and experimented with an alternative divergence cleaning approach. Patrick focused on the theoretical background and assisted in the flux implementation and testing. The flux as a whole was a team effort. Christoph implemented the rotor and Orszag-Tang scenarios, required changes in the infrastructure and cleaned up the code.

1 Background and Governing Equations

The magnetohydrodynamics equations (for brevity referred to as the MHD equations) are a set of nonlinear hyperbolic partial differential equations used to describe the movement of an electrically conducting fluid, which is not only influenced by hydrodynamic processes but also electromagnetic processes. This means that the fluid has to conform to the already discussed Euler equations, but also to the Maxwell equations of electrodynamics. An example application for such a fluid would be solar plasma. There are 8 equations, in the conservative formulation the varibales are the density ρ, the momentum components ρu, ρv, ρw, the energy E and the magnetic field components B_x, B_y, B_z.
The equations for ideal MHD are as follows [3][1]:

$$
\begin{aligned}
\delta_t \rho + \triangledown \cdot (\rho \mathbf{u}) &= 0 \\
\delta_t(\rho \mathbf{u}) + \triangledown \cdot (\rho \mathbf{u}\mathbf{u}^T + (p + \frac{|\mathbf{B}|^2}{8\pi})I - \frac{1}{4\pi}\mathbf{B}\mathbf{B}^T) &= 0 \\
\delta_t B + \triangledown \cdot (\mathbf{u}\mathbf{B}^T - \mathbf{B}\mathbf{u}^T) &= 0 \\
\delta_t E + \triangledown \cdot ((E + p + \frac{|\mathbf{B}|^2}{8\pi})\mathbf{u} - \frac{1}{4\pi}\mathbf{B}(\mathbf{u}\cdot\mathbf{B})) &= 0
\end{aligned}
$$

where $\mathbf{u} = (u, v, w)$ and $\mathbf{B} = (B_x, B_y, B_z)$.
A split of these equations into partial derivatives into x, y and z direction can also be found in [1]. The relation between pressure and energy is given by the following equation [3].

$$p = (\gamma - 1)(E - \frac{1}{2}\rho|\mathbf{u}|^2 - \frac{1}{8\pi}|\mathbf{B}|^2)$$

1.1 Eigenvalues

The system of equations has the following eigenvalues for one dimension[3]: u, $u \pm c_s$, $u \pm c_a$ and $u \pm c_f$.
Here

$$
\begin{aligned}
c_a &= |b_x|, \\
c_s &= \sqrt{\frac{1}{2}(a^2 + b^2 - \sqrt{(a^2+b^2)^2 - 4a^2 b_x^2}))} \\
c_f &= \sqrt{\frac{1}{2}(a^2 + b^2 - \sqrt{(a^2+b^2)^2 - 4a^2 b_x^2}))}.
\end{aligned}
$$

The following abbreviations are used: $b_x = \sqrt{\frac{B_x^2}{\rho}}$, $b^2 = \frac{|\mathbf{B}|^2}{\rho}$ and $a = \sqrt{\frac{\gamma p}{\rho}}$.
In particular, the highest eigenvalue is $u + c_f$, since c_f is the speed of the fast magnetosonic waves (c_a is the Alfvén wave speed and c_s is the slow magnetosonic wave speed).

2 Divergence Cleaning

One of the constraints that the Maxwell equations impose on the system is that the divergence of the magnetic field is zero, thus there are no sources or sinks of the magnetic field. This constraint can be violated by numerical simulations because of errors introduced by discretization [3]. Furthermore these errors can increase with time. The divergence cleaning scheme introduced by Dedner [3] ensures the compliance to this constraint by coupling the divergence constraint to the conservation laws by a generalized Lagrange multiplier.

In practice this is done by adding a ninth variable Ψ to the system. This Ψ is now included in the flux of $\mathbf{B}$ by adding ΨI, so the flux becomes $\mathbf{u}\mathbf{B}^T - \mathbf{B}\mathbf{u}^T + \Psi I$, whereas the flux of Ψ is simply the constant factor c_h^2, which can be set to 1 for our simulations.

3 Results

In order to illustrate the obtained results, we chose two well-known examples for the (ideal) MHD equations.

3.1 Rotor

The first one is the rotor problem [2] where an initially rotating high-density fluid is embedded in a low-density environment which is at rest. Due to the rapid rotation torsional Alfvén waves are launched into the atmosphere, slowing the rotation down. The magnetic field additionally compresses the fluid in the rotor into an oblong shape. The simulation was run on a 500×500 grid at order 1, see Figure 1 for a screenshot. Order 1 was chosen so the limiter could be disabled, as the current implementation smears the solution heavily. To utilize higher-orders a more involved limiter (and probably the same holds for the Riemann solver) is required.

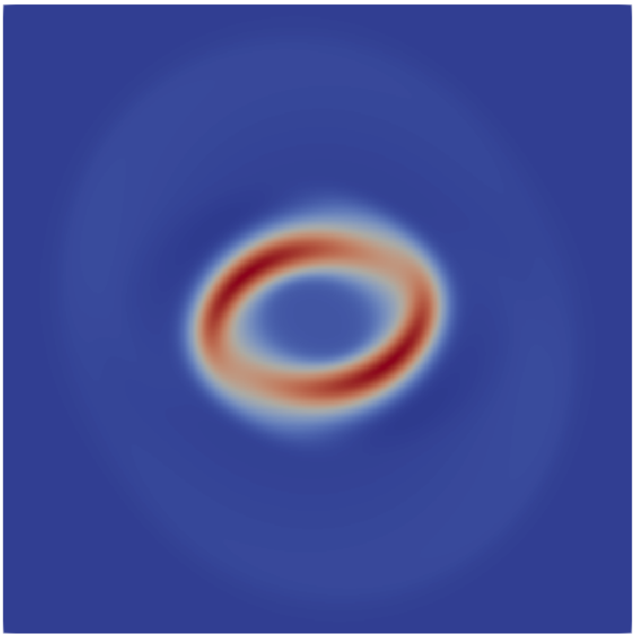

Figure 1: Density of the rotor problem (subsection 3.1) at $t = 0.25$. At this point the Alfvén wave has almost reached the boundary and the rotor is already elongated.

3.2 Orszag-Tang Vortex

The second example is the also well-known Orszag-Tang vortex, which was first investigated in relation to singularities arising from MHD turbulence [5]. We used the initial conditions as given by Dumbser [4], which is also a great source for high-quality figures for both of the presented examples. This simulation was run on an order 1 grid with 300×300 elements until $t = 5.0$. The results are illustrated in Figure 2; up until $t = 3.0$ the results match the ones in [4] reasonably well, at $t = 5.0$ the smearing of the Rusanov flux becomes more apparent.

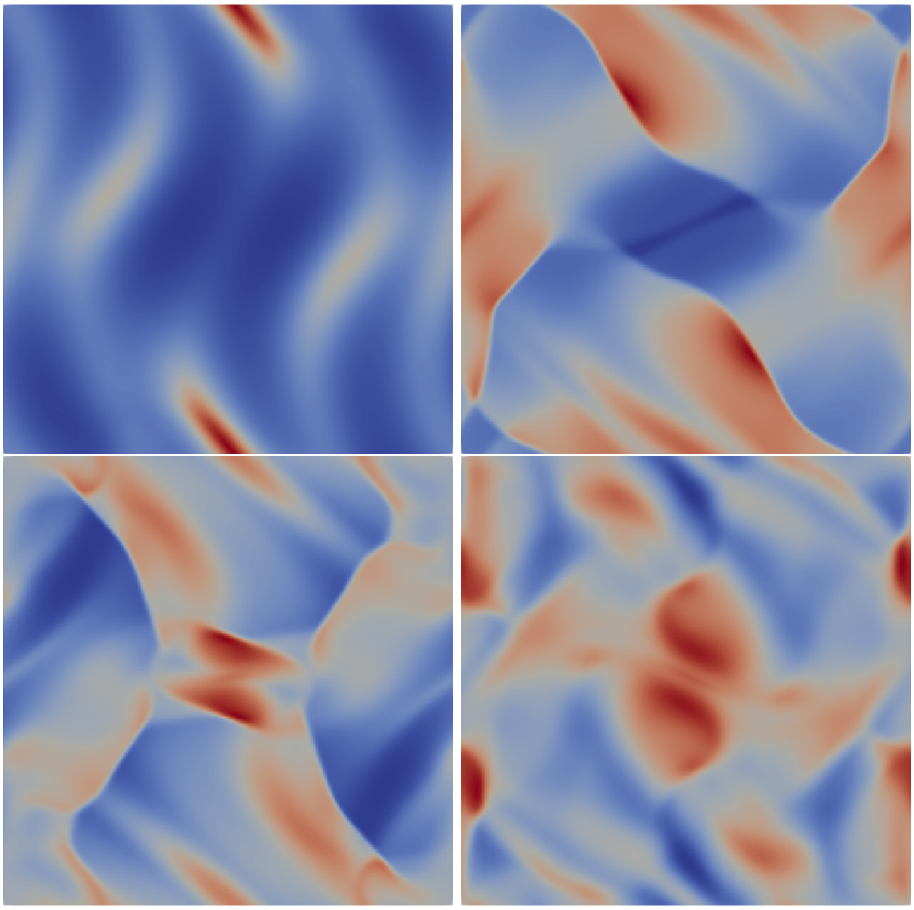

Figure 2: Density of the Orszag-Tang vortex (subsection 3.2) at $t = 0.5$ (top left), $t = 2.0$ (top right), $t = 3.0$ (bottom left) and $t = 5.0$ (bottom right).

References

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