Cost Curves

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1 Motivation

The cost minimization chapter lays the foundation by showing how a firm chooses its inputs in order to produce a given level of output at the lowest possible cost, thereby defining the cost function $c(y)$. This function encapsulates the firm's best, or most cost-effective, production plan given the input prices.

The cost curves chapter builds on this foundation by graphically representing the cost function. Through constructions like the **total cost curve**, **average cost curve**, and **marginal cost curve**, cost curves illustrate how production costs change with varying output levels. These curves provide essential insights into economies of scale, the firm's efficiency, and optimal output decisions.

In short, cost minimization determines the theoretical minimum cost for each output level, and cost curves visualize these costs, enabling analysis of production efficiency and profit maximization.

2 Average Costs: Variable and Fixed

2.1 Average Cost: Average Fixed Costs + Average Variable Costs

The average cost is the total cost per unit of output, given by

$$AC(y) = \frac{c(y)}{y} = \frac{c_v(y) + F}{y} = AVC(y) + AFC(y),$$

where $c_v(y)$ represents variable costs and F represents fixed costs. As shown, the definition includes 2 concepts:

- **Average Variable Cost (AVC):** This is the variable cost per unit of output, defined as

  $$AVC(y) = \frac{c_v(y)}{y}.$$

  It typically declines initially due to efficiencies, but eventually rises because of diminishing marginal returns or capacity constraints.

- **Average Fixed Cost (AFC):** This is the fixed cost per unit of output, defined as

  $$AFC(y) = \frac{F}{y}.$$

  Since fixed costs remain constant, AFC decreases continuously as output increases, approaching zero as output becomes large.

2.2 Marginal Costs

The marginal cost is the additional cost incurred when producing one extra unit of output. It is calculated as:

$$MC(y) = \frac{c(y + \Delta y) - c(y)}{\Delta y},$$

or, using variable costs (since fixed costs don't change),

$$MC(y) = \frac{c_v(y + \Delta y) - c_v(y)}{\Delta y}.$$

Relationship with Average Variable Cost (AVC):

- At the very first unit, MC equals AVC because there are no changes in fixed costs:

  $$MC(1) = AVC(1).$$

- When AVC is falling (due to economies of scale and improved efficiency), the cost of the additional unit is lower than the current average; hence, $MC < AVC$.
- When AVC starts to rise (as diminishing marginal returns set in), the additional unit costs more than the current average; hence, $MC > AVC$.
- The MC curve always crosses the AVC (and average cost, AC) curve at its minimum point.

There is an important graphical implication: The U-shaped AVC and AC curves reflect that initially, increasing production lowers per-unit costs, but beyond a certain point, the costs begin to rise. The marginal cost curve, which is the rate of change of total costs, intersects these curves at their lowest points, marking the efficient scale of production.

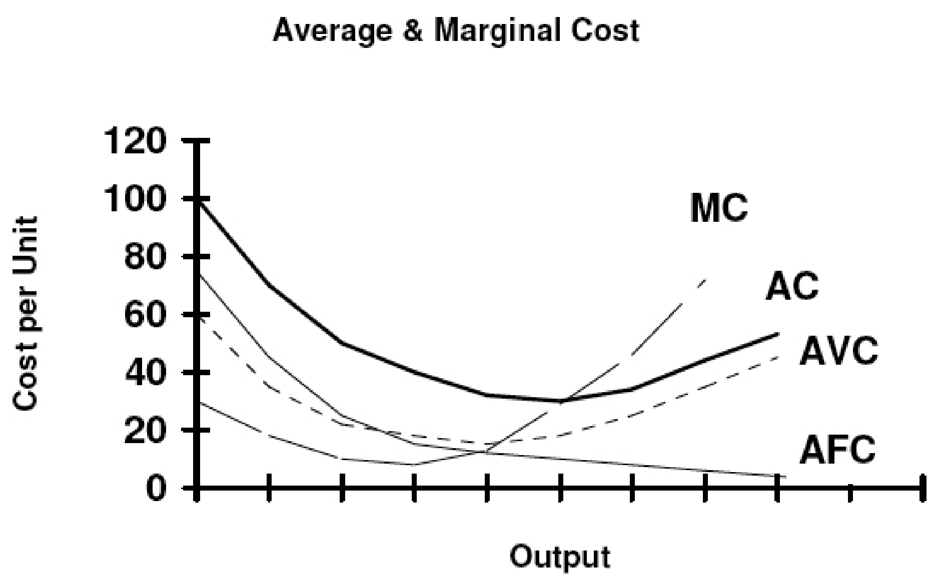


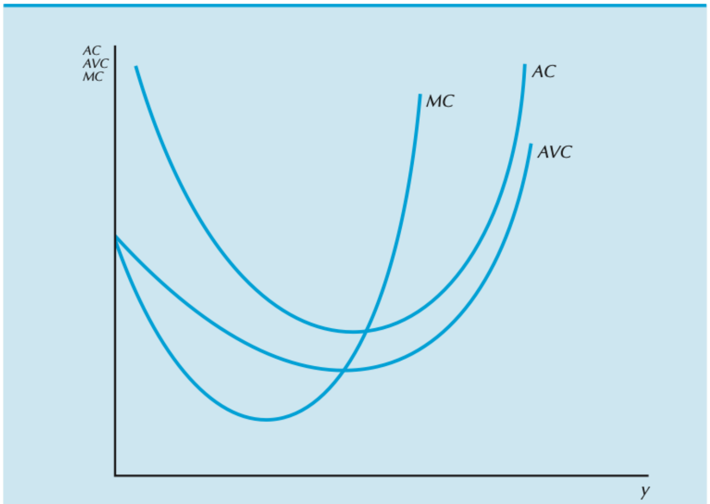
Figure 1: Average Cost Curves

Let's describe the graphical shapes:

- **AFC Curve:** Starts very high (even infinitely high at zero output, because we divide by really small almost non-existent output quantities $y = 0$) and declines steadily as output increases. The curve is downward sloping and convex, asymptotically approaching zero.
- **AVC Curve:** Initially, as output increases, economies of scale and improved efficiency allow the variable costs to be spread over more units, so the average variable cost (AVC) declines. However, once diminishing marginal returns set in, additional inputs yield less extra output, meaning the firm must use a disproportionately larger amount of inputs to increase production further. This causes the AVC to eventually rise, resulting in a U-shaped AVC curve.
- **AC Curve:** The average cost curve is the sum of AFC and AVC. It usually exhibits a U-shape, initially declining (dominated by the rapid fall in AFC) and then rising as the increasing AVC outweighs the falling AFC.
- **MC Curve:** The key idea is that the marginal cost (MC) curve must intersect the average variable cost (AVC) curve at the AVC's minimum and the average total cost (AC) curve at the AC's minimum. In other words:
  - **MC and AVC:** The AVC curve is falling when MC is below it and rising when MC is above it. Therefore, the point where AVC is minimized occurs where $MC = AVC$.

– **MC and AC:** Similarly, the AC curve is falling when $MC < AC$ and rising when $MC > AC$. Thus, the minimum of the AC curve is reached where $MC = AC$.

The intersection of the marginal cost (MC) curve with the average total cost (AC) curve is typically considered more important because it pinpoints the efficient scale of production—the output level at which the firm minimizes its cost per unit in the long run.



Cost curves. The average cost curve (AC), the average variable cost curve (AVC), and the marginal cost curve (MC).

Figure 2: Cost Curves - Another Version

2.2.1 Car Manufacturer

Suppose a car manufacturer has:

- Fixed Costs: $10 million per year (e.g., plant lease, maintenance).
- Variable Costs: These vary with the number of cars produced.

Now, imagine the following scenarios when producing 500 and 1000 cars per year:

	500 cars/year	1000 cars/year
Total Variable Cost	$12 million	$22 million
AVC (per car)	$\frac{\$12 \text{ million}}{500} = \$24{,}000$	$\frac{\$22 \text{ million}}{1{,}000} = \$22{,}000$
AFC (per car)	$\frac{\$10 \text{ million}}{500} = \$20{,}000$	$\frac{\$10 \text{ million}}{1{,}000} = \$10{,}000$
AC (per car)	$44,000	$32,000

Table 1: Cost Breakdown for 500 and 1000 Cars per Year

In this example, as output increases from 500 to $1,000$ cars, the **AFC** decreases (because the same fixed cost is spread over more cars) and the AVC decreases initially due to improved efficiency. The combined effect is a lower average cost per car. However, if production were increased further and capacity constraints kicked in, **AVC** might eventually rise, causing the AC curve to bottom out and then increase—a typical U-shaped pattern.

2.3 Marginal and Variable Costs

As we previously mentioned, the marginal cost (MC) curve shows the additional cost incurred for producing one more unit of output. If we consider only the variable costs $c_v(y)$ we can notice the following for discreet and continuous production:

- **Discreet Production:** If you sum (or integrate) the marginal costs of each additional unit from zero up to y, you obtain the total variable cost of producing y units. Mathematically, for discrete output, we have that $\Delta y = 1$, $c_v(0) = 0$ (as there are no ouputs produced) and:

$$MC(y) = \frac{c_v(y+\Delta y) - c_v(y)}{\Delta y} = c_v(y+1) - c_v(y) \longleftrightarrow MC(y-1) = c_v(y) - c_v(y-1)$$

  Then, we notice that:

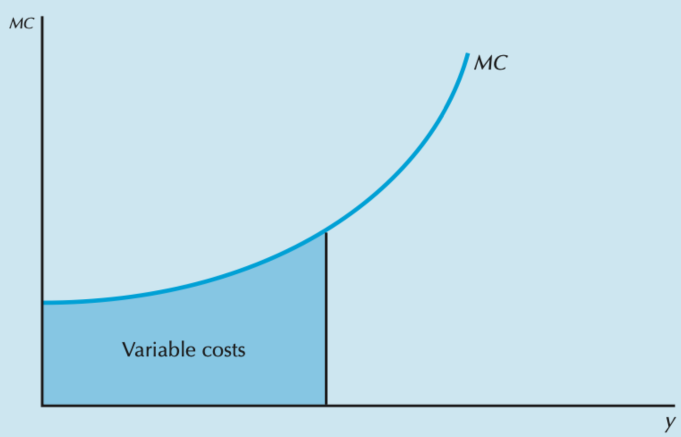
$$c_v(y) = \underbrace{\underbrace{[c_v(y) - c_v(y-1)]}_{MC(y-1)} + \underbrace{[c_v(y-1) - c_v(y-2)]}_{MC(y-2)} + \cdots + \underbrace{[c_v(1) - \overbrace{c_v(0)}^{=0}]}_{MC(0)}}_{\text{Telescoping Sum}} = \sum_{i=0}^{y-1} MC(i),$$

  since $c_v(0) = 0$.

- **For continuous production:** The variable cost is the integral of the MC curve from 0 to y:

$$c_v(y) = \int_0^y MC(u)\,du.$$

  Each unit's marginal cost can be visualized as the height of a small rectangle of width 1. The sum of the areas of these rectangles (i.e., the area under the MC curve) equals the total variable cost.



Marginal cost and variable costs. The area under the marginal cost curve gives the variable costs.

Figure 3: Marginal Cost and Variable Costs

2.3.1 Short Cost Curves 1

Given the cost function

$$c(y) = y^2 + 1,$$

we derive the following cost curves:

- **Variable Costs:** $c_v(y) = y^2$
- **Fixed Costs:** $c_f(y) = 1$
- **Average Variable Costs (AVC):**

$$AVC(y) = \frac{c_v(y)}{y} = \frac{y^2}{y} = y,$$

  which is a straight line with slope 1.
- **Average Fixed Costs (AFC):**

$$AFC(y) = \frac{c_f(y)}{y} = \frac{1}{y},$$

  which declines as output increases.
- **Average Costs (AC):**

$$AC(y) = \frac{c(y)}{y} = \frac{y^2+1}{y}.$$

- **Marginal Costs (MC):** Calculated as the derivative of $c(y)$:

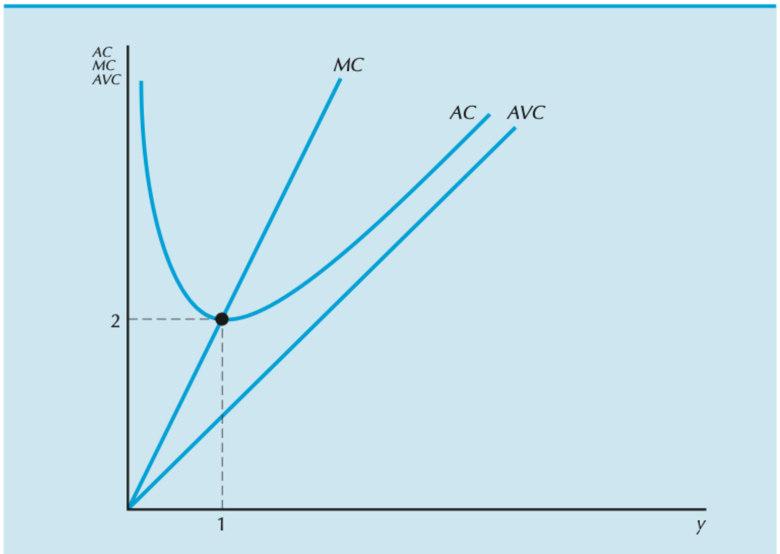
$$MC(y) = \frac{d}{dy}(y^2+1) = 2y,$$

  a straight line with slope 2.

The average cost curve reaches its minimum where $AC(y) = MC(y)$. Setting

$$\frac{y^2+1}{y} = 2y \longleftrightarrow y^2+1 = 2y^2 \Rightarrow y^2 = 1 \Rightarrow y = 1 \text{ } (y = -1 \text{ is discarded as production units } \geq 0),$$

So $y_{\min} = 1$. At $y = 1$, both AC and MC equal 2. This illustrates that the efficient scale of production (the output level minimizing average cost) occurs where the marginal cost curve intersects the average cost curve.



Cost curves. The cost curves for $c(y) = y^2 + 1$.

Figure 4: Short Example

2.3.2 Short Cost Curves 2

Imagine a firm with two production plants which is trying to decide how to allocate production between both plants to minimize total cost.

- **Optimization Setup:** The firm has two cost functions, $c_1(y_1)$ and $c_2(y_2)$, for plants 1 and 2 respectively, and it wants to produce a total of y units. The problem is to choose y_1 and y_2 such that

$$\min_{y_1, y_2} \; c_1(y_1) + c_2(y_2)$$

$$\text{subject to} \quad y_1 + y_2 = y.$$

- **Marginal Cost Equality:** At the optimal allocation, the cost of producing one extra unit in plant 1 must equal the cost of producing one extra unit in plant 2. In other words, if $MC_1(y_1)$ and $MC_2(y_2)$ are the marginal cost curves of plants 1 and 2 respectively, then the optimum is reached when

$$MC_1(y_1) = MC_2(y_2).$$

  If these were not equal, the firm could lower its overall cost by shifting production from the plant with the higher marginal cost to the one with the lower marginal cost.

- **Combined Marginal Cost Curve:** Once the firm determines the optimal output split, the overall cost function $c(y)$ for producing y units reflects this optimal allocation. Graphically, the overall marginal cost curve is the horizontal sum of the individual marginal cost curves. This means the firm always produces the next unit in the plant that offers the lowest marginal cost until the costs are equalized.

In essence, the key insight is that optimal cost minimization across multiple plants requires balancing marginal costs across plants so that no reallocation of production can further reduce total cost.

3 Short and Long Run Costs

3.1 Short-Run Costs

In the short run, some factors (such as plant size) are fixed. The short-run cost function $c_s(y, k)$ represents the minimum cost of producing y units when the fixed factor k is given. As a result, the short-run average cost (SAC) is $AC_s(y, k) = \frac{c_s(y,k)}{y}$ and typically lies higher than the long-run average cost.

3.2 Long-Run Costs

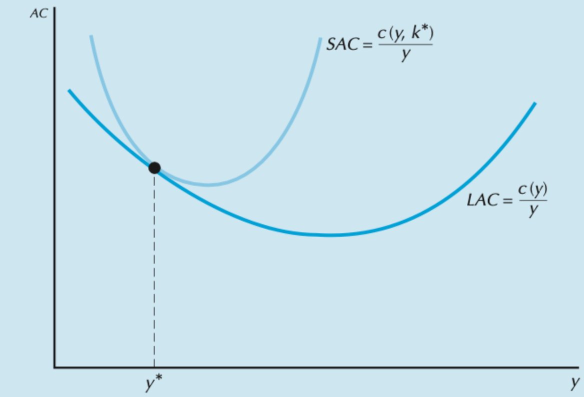
In the long run, all factors of production are variable. The firm can choose the optimal level of the previously fixed factor - say, the optimal plant size $k(y)$ - to minimize costs. Thus, the long-run cost function is defined as

$$c(y) = c_s(y, k(y)).$$

The long-run average cost (LAC) is $AC(y) = \frac{c(y)}{y}$, and it represents the lowest cost per unit that the firm can achieve when it can adjust all inputs.

3.3 Relationship between SAC and LAC

The short-run cost function, constrained by a fixed factor, is always at least as high as the long-run cost function, where the firm can choose the optimal level of that factor. Graphically, the SAC curve lies above the LAC curve, and they are tangent at the output level y^* where the fixed factor is set at its optimal long-run level (i.e., when $k = k(y^*)$). This tangency point indicates the efficient scale of production at which both average and marginal costs are minimized.

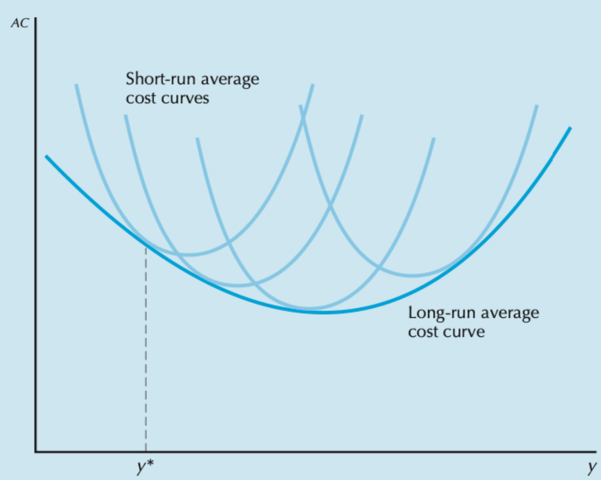


Short-run and long-run average costs. The short-run average cost curve must be tangent to the long-run average cost curve.

Figure 5: SAC and LAC

In essence, while short-run costs are higher due to the inflexibility of fixed inputs, the long-run cost function shows how the firm can achieve lower average costs by optimally adjusting all factors of production.

We can do the same sort of construction for levels of output other than $y*$. Suppose we pick outputs $y_1, y_2, \ldots, y_n$ and accompanying plant sizes $k_1 = k(y_1)$, $k_2 = k(y_2)$, ... , $k_n = k(y_n)$. Then we get a picture like that in Figure **??** and we say that the long-run average cost curve is the lower envelope of the short-run average cost curves.



Short-run and long-run average costs. The long-run average cost curve is the envelope of the short-run average cost curves.

Figure 6: Lower Envelope

3.4 Long-Run Marginal Costs

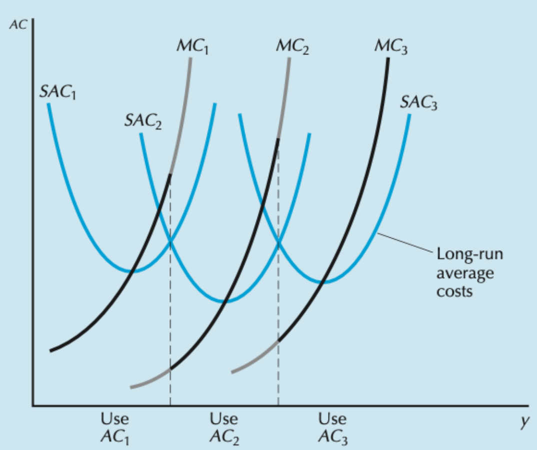
Short-run marginal cost curves (MC_s) are derived from cost functions when some inputs (like plant size) are fixed. In the long run, all inputs can be varied, and the firm chooses the optimal fixed factor for each level of output, resulting in the long-run cost function $c(y)$ and its derivative, the long-run marginal cost ($LRMC$).

The long-run average cost (LAC) curve is the lower envelope of the short-run average cost (SAC) curves, and a similar idea applies to marginal costs.

- **Discrete Case (Limited, Fixed Plant Sizes):**
  - **Short-Run Situation:** A firm may have several discrete plant sizes available, say $k_1, k_2, \ldots, k_n$. For each plant size k_i, there is a short-run cost function $c_s(y, k_i)$ and a corresponding marginal cost curve $MC_s(y, k_i)$.
  - **Long-Run Decision:** For any given output y, the firm chooses the plant size k_i that minimizes cost.Thus, the long-run cost is given by

$$c(y) = \min_i \{c_s(y, k_i)\},$$

    and the long-run marginal cost at y is the marginal cost from the short-run cost curve corresponding to the chosen plant size.
  - **Implication:**
    * The LRMC curve is piecewise defined: for each range of y, it follows the $MC_s(y, k_i)$ of the plant size that minimizes cost in that range.
    * If at some output level the marginal cost in one plant is lower than in another, shifting production from the higher-cost plant to the lower-cost one would reduce total cost. At optimum, the firm uses the plant size with the lowest MC for that y.



Long-run marginal costs. When there are discrete levels of the fixed factor, the firm will choose the amount of the fixed factor to minimize average costs. Thus the long-run marginal cost curve will consist of the various segments of the short-run marginal cost curves associated with each different level of the fixed factor.

Figure 7: LRMC Discreet

- **Continuous Case (Continuously Adjustable Plant Size):**
  - **Short-Run Situation:** The firm's short-run cost function is $c_s(y, k)$ where k is the plant size.
  - **Long-Run Decision:** In the long run, the firm can continuously adjust k to find the optimal plant size $k(y)$ for each output y. The long-run cost function becomes

$$c(y) = c_s(y, k(y)).$$

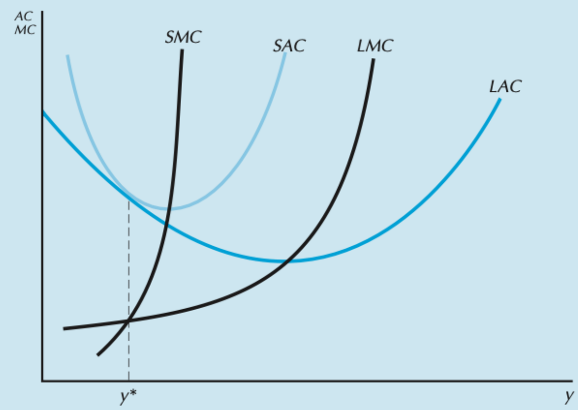
- **Marginal Cost:** The long-run marginal cost is defined as the derivative of $c(y)$:

$$LRMC(y) = c'(y).$$

  By the envelope theorem (or by direct differentiation taking into account that $k(y)$ is optimally chosen), we have

$$LRMC(y) = MC_s(y, k(y)).$$

- **Implication:** The LRMC curve is smooth and represents the incremental cost of producing one more unit when the firm is using the optimally adjusted plant size. At every level of output, the LRMC equals the short-run marginal cost evaluated at the optimal $k(y)$.



Long-run marginal costs. The relationship between the long-run and the short-run marginal costs with continuous levels of the fixed factor.

Figure 8: LRMC Continuous

In both scenarios, the optimal long-run marginal cost at any output level is the lowest possible additional cost to produce one more unit, given that the firm can adjust its plant size optimally.