# Risky Assets

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# 1 Mean-Variance Utility

The mean-variance utility model simplifies decision-making under uncertainty by focusing on two key characteristics of a probability distribution: its mean ($\mu_w$) and variance ($\sigma_w^2$), or alternatively, its standard deviation ($\sigma_w$). Rather than analyzing the entire probability distribution of possible outcomes, this model assumes that preferences can be expressed using just these two parameters.

1. Mean ($\mu_w$): The mean represents the expected value of wealth and is calculated as:

$$\mu_w = \sum_{s=1}^{S} \pi_s w_s,$$

where $w_s$ is the wealth in state $s$ and $\pi_s$ is the probability of that state occurring. This gives the weighted average of all possible outcomes.

2. **Variance ($\sigma_w^2$):** The variance measures the spread of the outcomes around the mean, capturing the risk involved. It is defined as:

$$\sigma_w^2 = \sum_{s=1}^{S} \pi_s (w_s - \mu_w)^2.$$

A larger variance indicates greater uncertainty or risk in the wealth distribution.

3. **Standard Deviation ($\sigma_w$):** The standard deviation is the square root of the variance:

$$\sigma_w = \sqrt{\sigma_w^2}.$$

Both variance and standard deviation quantify the riskiness of the distribution.

The utility of an investor depends on both the mean and the variance of the wealth distribution:

$$u(\mu_w, \sigma_w^2) \quad \text{or equivalently,} \quad u(\mu_w, \sigma_w).$$

In this framework:

- Higher mean ($\mu_w$) increases utility because it represents higher expected wealth.
- Higher variance ($\sigma_w^2$) reduces utility because it represents greater risk, consistent with risk aversion.

The model assumes investors prefer higher expected returns ($\mu_w$ is good) and investors dislike risk ($\sigma_w^2$ or $\sigma_w$ is bad). This model is a simplification of the expected utility model. If the choices can be fully characterized by their mean and variance, the mean-variance utility function ranks choices the same way as the expected utility function. Even when the distribution cannot be fully described by mean and variance, the model often provides a reasonable approximation for decision-making under uncertainty.

## 1.1 Portfolio Problem: Stocks & Bonds

The mean-variance model for a simple portfolio problem analyzes how an investor allocates wealth between a risk-free asset (e.g., Treasury bills) and a risky asset (e.g., stocks) to balance expected return and risk. Let's first define the key components of such model:

1. **Risk-Free Asset:** Always pays a fixed rate of return $r_f$ (return of the risk-free asset), regardless of market conditions.

2. **Risky Asset:** Return depends on the state of the market. Let:
   - $m_s$: Return of the risky asset in state $s$.
   - $\pi_s$: Probability of state $s$.
   - $r_m$: Expected return of the risky asset ($r_m = \sum_{s=1}^{S} \pi_s m_s$).
   - $\sigma_m$: Standard deviation of the risky asset's return, a measure of its risk.

3. **Portfolio Allocation:** Let it be $x$ the fraction of wealth invested in the risky asset and $1 - x$ the fraction of wealth in the risk-free asset.

### 1.1.1 Derivation of the Portfolio's Expected Return

The portfolio's return $r_x$ is a weighted average of the risky and risk-free returns:

$$r_x = \sum_{s-1}^{S}(x \cdot m_s + (1-x)\cdot r_f)\cdot \pi_s = x\sum_{s-1}^{S} m_s\pi_s + (1-x)\cdot r_f\sum_{s-1}^{S}\pi_s \underbrace{=}_{\sum_{s-1}^{S}\pi_s=1} x\cdot r_m + (1-x)\cdot r_f \implies \boxed{\mathrm{r}_x = xr_m + (1-x)r_f}$$

This equation shows that the portfolio's expected return depends on how much wealth ($x$) is allocated to the risky asset.

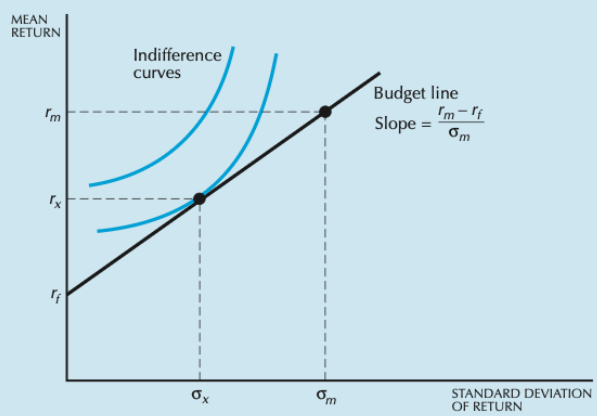
### 1.1.2 Derivation of the Portfolio's Variance

The variance of the portfolio return measures its risk and is derived as follows:

$$\sigma_x^2 = \sum_{s=1}^{S}(xm_s + (1-x)r_f - r_x)^2\pi_s \underbrace{=}_{\bigstar} x^2\sum_{s=1}^{S}(m_s - r_m)^2\pi_s \underbrace{=}_{\bigstar\bigstar} x^2{\sigma_m}^2$$

Where $\bigstar$ is due to the substitution of $r_x = xr_m + (1-x)r_f$ into the variance formula and $\bigstar\bigstar$ by the definition: $\sigma_m^2 = \sum_{s=1}^{S}(m_s - r_m)^2\pi_s$ Thus, the standard deviation (risk) of the portfolio return is:

$$\boxed{\sigma_x = x\sigma_m}$$




**Risk and return.** The budget line measures the cost of achieving a larger expected return in terms of the increased standard deviation of the return. At the optimal choice the indifference curve must be tangent to this budget line.

Figure 1: Portfolio Problem:Risk and Return

### 1.1.3 Price of Risk

Now, we can introduce another definition using the expected return from the risky-asset $r_m$, and the risk-free asset $r_f$. The price of risk measures how much extra return an investor requires per unit of risk. defined as:

$$p = \frac{\text{excess return}}{\text{risk}} = \frac{r_m - r_f}{\sigma_m}$$

For instance, if we consider the values $r_m = 8\%, r_f = 3\%$ and $\sigma_m = 10\%$ then, the price of risk is:

$$\text{Price of Risk} = \frac{8\% - 3\%}{10\%} = 0.5$$

This means the investor earns 0.5% extra return per 1% of additional risk taken.

### 1.1.4 The return of the MRS

In the chapter 5 of Utility, we studied the MRS in the **Simple Two-Good Model** which measures the rate at which a consumer is willing to substitute one good (good 1) for another (good 2) while maintaining the same utility:

$$MRS = -\frac{MU_1}{MU_2}.$$

Here, $MU_1$ is the marginal utility of good 1 and $MU_2$ is the marginal utility of good 2. It describes how much of good 2 the consumer is willing to give up to gain one more unit of good 1 without changing their level of satisfaction.

Now, the MRS can be also defined in the **Portfolio Context under the Mean-Variance Model**, instead the MRS measures the trade-off between risk ($\sigma$) and return ($\mu$) that the investor is willing to make while maintaining the same utility, and is defined as:

$$MRS = -\frac{\Delta U / \Delta\sigma}{\Delta U / \Delta\mu} = \frac{r_m - r_f}{\sigma_m}$$

Here, $\Delta U / \Delta\sigma$ represents how utility changes as risk ($\sigma$) changes and $\Delta U / \Delta\mu$ represents how utility changes as expected return ($\mu$) changes.

The key difference relies on:

- The **two-good model** deals with two physical goods (e.g., apples and oranges), where the MRS reflects substitution between goods.
- The **portfolio model** deals with risk ($\sigma$) and return ($\mu$), where the MRS reflects substitution between risk and return in financial decision-making.

# 2 Measuring Risk, Diversification and Beta Stock

## 2.1 Diversification

Imagine you are considering investing in stocks (Asset A) and bonds (Asset B). Stocks and bonds often have negative correlation—when stock prices fall, bond prices tend to rise, and vice versa. This usually works:

- Stocks (Asset A):
  - In a booming economy, stocks perform well and return \$10 per share.
  - In a recession, stocks perform poorly and lose value, returning - \$5 per share.
- Bonds (Asset B):
  - In a booming economy, bonds return - \$5 per share (interest rates rise, making bonds less valuable).
  - In a recession, bonds perform well and return \$10 per share (investors flock to safe assets like bonds).

These returns are negatively correlated: when stocks are up, bonds are down, and vice versa.

Now, essentially we can follow 2 strategies, either we indivdually select one of the assets or both. If you only invest in stocks or bonds, each asset has an expected return of \$2.50, because we suppose that half of the time it's in booming economiy, and half in recession.

$$\text{Expected Value } (\mu) = 0.5 \cdot 10 + 0.5 \cdot (-5) = 2.50.$$

However, both assets have risk (variance) because their outcomes fluctuate between \$10 and -\$5. If you're risk-averse, you'd value each asset at less than \$2.50 because of this uncertainty.

On the other hand, if you invest equally in both stocks and bonds, your portfolio becomes risk-free:

- **Booming economy:** Stocks return \$10, bonds return -\$5 → Total = \$10 - \$5 = \$5.
- **Recession:** Stocks return -\$5, bonds return \$10 → Total = -\$5 + \$10 = \$5.

No matter the state of the economy, your portfolio's total return is a guaranteed \$5.

This example shows the power of diversification. Assets that move in opposite directions—that are negatively correlated with each other—are very valuable because they reduce overall risk as losses in one are offset by gains in the other. As a result, the combined portfolio is worth \$5, the guaranteed return, which is higher than the individual expected value of either asset (\$2.50). It's important to remember that the amount of risk in an asset depends on its correlation with other assets.

### 2.2 Beta of a Stock

The ($\beta$) of a stock measures how much its price tends to move relative to the overall stock market. If $i$ represents some particular stock, then $\beta_i$ is defined as:

$$\beta_i = \frac{\text{how risky asset } i \text{ is}}{\text{how risky the stock market is}}$$

It shows the stock's sensitivity to market movements:

- $\beta = 1$: The stock is as risky as the market. If the market goes up 10%, the stock tends to go up 10%.
- $\beta > 1$: The stock is riskier than the market. If the market goes up 10%, the stock moves more than 10% (e.g., 15%).
- $\beta < 1$: The stock is less risky than the market. If the market goes up 10%, the stock moves less than 10% (e.g., 5%).

Let's say the *S&P* 500 *Index* represents the stock market, and a tech stock (e.g., TechCorp) has a $\beta = 1.5$. This means TechCorp is 1.5 times as risky as the market.

1. **Market Movement:** If the *S&P* 500 increases by 8% during a period, then TechCorp's price would be expected to increase by:
$$\text{TechCorp Price Change} = 1.5 \times 8\% = 12\%.$$
2. **Market Drop:** If the *S&P* 500 decreases by 6%, then TechCorp's price would be expected to decrease by:
$$\text{TechCorp Price Change} = 1.5 \times (-6\%) = -9\%.$$

Beta helps investors understand the risk of a stock relative to the market. Risk-tolerant investors might favor high-beta stocks for higher potential returns during market booms, while Risk-averse investors might prefer low-beta stocks for more stability during market volatility.

## 3 Equilibrium in Risky Assets Market & Counterparty Risk

### 3.1 Counterparty Risk

Financial institutions often loan money to each other, creating the risk of counterparty risk, where one party may fail to repay its loan. Imagine three banks—A, B, and C—where A owes B \$1 billion, B owes C \$1 billion, and C owes A \$1 billion. If Bank A defaults, Bank B loses \$1 billion and may not be able to pay C, causing C to default as well, pushing A even further in the hole. This chain reaction is called financial contagion or systemic risk, as seen during the 2008 financial crisis.

The solution is a lender of last resort, typically a central bank like the Federal Reserve. The central bank can lend Bank A \$1 billion, enabling A to pay B, B to pay C, and C to pay A, stabilizing the system and allowing

Bank A to repay its emergency loan to the central bank.

**Libertarian Note:** From a libertarian perspective, reliance on a central bank as the "lender of last resort" may be seen as a distortion of free-market principles. It creates moral hazard, encouraging risky behavior by banks since they expect bailouts in times of crisis. Instead of allowing the market to naturally weed out poorly managed institutions, central bank interventions prop up failing entities, potentially leading to inefficiency and undermining accountability. A libertarian advocates for decentralized financial systems or stricter market discipline to prevent such systemic risks without government intervention.

## 3.2 Market Equilibrium: CAPM

### 3.2.1 Total Risk of Assets

We said above that the amount of risk in a given asset $i$ relative to the total risk in the market is denoted by $\beta_i$. This means that to measure the total amount of risk in asset $i$, we have to multiply by the market risk, $\sigma_m$. Thus the total risk in asset $i$ is given by:

$$\text{Total Risk } = \beta_i \sigma_m$$

### 3.2.2 Risk Adjustment

The risk adjustment accounts for the compensation investors demand for bearing risk in a particular asset relative to the market. This adjustment ensures that riskier assets offer higher returns as compensation for their additional risk. In plain english, the risk adjustment is the product of the total risk and the price of the risk. The risk adjustment formula is derived as:

$$\text{Risk Adjustment} = \beta_i \sigma_m \cdot p = \beta_i \sigma_m \cdot \frac{r_m - r_f}{\sigma_m} = \beta_i (r_m - r_f) \implies \text{Risk Adjustment} = \beta_i (r_m - r_f)$$

where:

- $\beta_i$: Measures the risk of asset $i$ relative to the market (how sensitive its return is to market movements). This is the beta of stock $i$.
- $r_m$: The expected return of the market portfolio (a diversified portfolio of all risky assets).
- $r_f$: The risk-free rate, representing a return without any risk, or with very low risk (bonds, treasury bills)
- $(r_m - r_f)$: The market risk premium, which is the extra return investors demand for investing in the market over a risk-free asset.

This formula tells us how much of the market risk premium applies to a given asset based on its $\beta_i$.

### 3.2.3 Risk-Adjusted Return Formula

The risk-adjusted return subtracts the risk adjustment from the asset's expected return to create a standardized return measure across assets. In equilibrium, the risk-adjusted returns of all assets must be equal. For two assets $i$ and $j$, the equilibrium condition is:

$$r_i - \beta_i (r_m - r_f) = r_j - \beta_j (r_m - r_f).$$

This equation ensures that, after accounting for risk differences, no asset offers a higher return than others.

#### 3.2.4 The Capital Asset Pricing Model (CAPM)

The CAPM describes the relationship between an asset's expected return and its risk. It is a cornerstone of modern portfolio theory. The expected return of an asset, $r_i$, is:

$$r_i = r_f + \beta_i(r_m - r_f).$$

Where the components are:

1. $r_f$**:** The risk-free rate, representing the baseline return for risk-free investments (e.g., Treasury bonds).
2. $(r_m - r_f)$**:** The market risk premium, representing the additional return for investing in the market portfolio over the risk-free rate.
3. $\beta_i$**:** The **sensitivity of the asset's returns to the overall market returns**. A higher $\beta_i$ means the asset is riskier relative to the market.

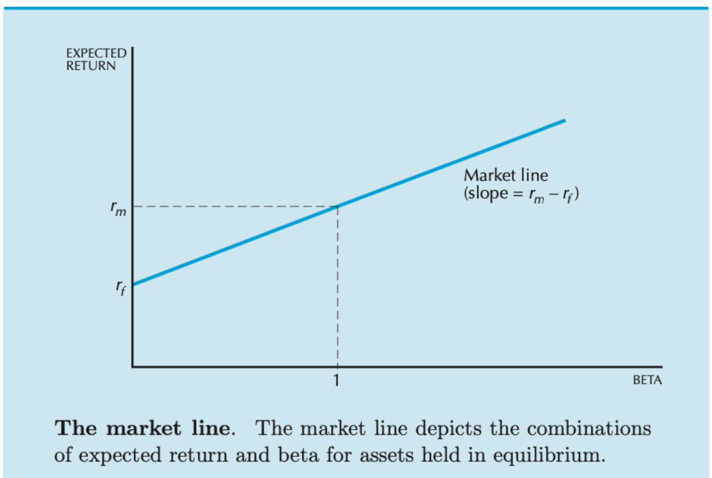



Figure 2: The Market Line

According to our model, all assets that are held in equilibrium have to lie along this line. This line is called **the market line**.

##### 3.2.4.1 Real-Life Example with TSLA, AMAZON, and US Bonds

Let's consider the following assumptions:

- Market Portfolio: S&P 500 (represents overall market returns).
  - Market Expected Return $(r_m) = 10\%$
  - Market Standard Deviation $(\sigma_m) = 15\%$
- Risk-Free Asset: US Bonds
  - Risk-Free Rate $(r_f) = 3\%$
- Individual Assets:
  - Tesla (TSLA): $\beta_{\text{TSLA}} = 1.8$
  - Amazon (AMZN): $\beta_{\text{AMZN}} = 1.2$

1. The **price of risk** represents the return required per unit of market risk ($\sigma_m$):

$$p = \frac{r_m - r_f}{\sigma_m} \frac{10\% - 3\%}{15\%} = \frac{7\%}{15\%} = 0.4667 = 46.67\%$$

   The price of risk is 0.47, meaning investors require an additional 46.67% return per unit of market risk to compensate for taking on that risk. This is a moderate price of risk.

2. The **total risk of each asset** is the product of its beta ($\beta_i$) and the market risk ($\sigma_m$):

   - **Tesla (TSLA):**

$$\text{Total Risk (TSLA)} = \beta_{\text{TSLA}} \cdot \sigma_m = 1.8 \cdot 15\% = 27\%.$$

   - **Amazon (AMZN):**

$$\text{Total Risk (AMZN)} = \beta_{\text{AMZN}} \cdot \sigma_m = 1.2 \cdot 15\% = 18\%.$$

   The interpretation is that **TSLA** has a high total risk (27%), meaning it contributes significant volatility to the portfolio relative to the market and **AMZN** has a moderate total risk (18%), meaning it is less volatile but still riskier than the market.

3. The **Capital Asset Pricing Model (CAPM)** calculates the expected return of an asset:

$$r_i = r_f + \beta_i(r_m - r_f).$$

   - **Tesla (TSLA):**

$$r_{\text{TSLA}} = 3\% + 1.8 \cdot (10\% - 3\%) = 3\% + 1.8 \cdot 7\% = 3\% + 12.6\% = 15.6\%.$$

   - **Amazon (AMZN):**

$$r_{\text{AMZN}} = 3\% + 1.2 \cdot (10\% - 3\%) = 3\% + 1.2 \cdot 7\% = 3\% + 8.4\% = 11.4\%.$$

   **TSLA** has a high expected return (15.6%), reflecting its significant risk ($\beta_{\text{TSLA}} = 1.8$) and **AMZN** has a moderate expected return (11.4%), aligning with its lower risk ($\beta_{\text{AMZN}} = 1.2$).

In conclusion, at 0.47, the price of risk suggests a reasonable compensation for market volatility. **TSLA** contributes substantial risk to the portfolio, while AMZN adds moderate risk. Investors must decide if they can handle **TSLA's** higher volatility in exchange for higher returns. **TSLA** offers a higher return (15.6%) but is significantly riskier, while **AMZN** is less volatile, with a return of 11.4%. Finally, the total risk of your portfolio is not simply the sum of the risks of individual assets (e.g., 27% + 18% in the example). Instead, the portfolio risk depends on how the assets interact with each other. Specifically, it depends on the correlation between the assets (how their returns move together).

**Note:** The market risk represents the overall risk of the market portfolio, typically measured as the standard deviation ($\sigma_m$) of the returns of a well-diversified portfolio that captures the entire market. For example, the S&P 500 is often used as a proxy for the market because it represents a broad range of large companies across various industries. However, if you don't pick a stock from the S&P 500, you would look for the market risk using a benchmark that represents the relevant market for the asset:

- For international stocks: Use indices like the **MSCI World Index**.
- For small-cap stocks: Use benchmarks like the **Russell 2000**.
- For cryptocurrencies: Use a cryptocurrency market index like the **CMBI** (Crypto Market Index).

The key is choosing a benchmark that reflects the broad market relevant to the type of asset you're analyzing.

# 4 Mutual vs Index Funds

The essential difference between an index fund and a mutual fund lies in how they are managed and their investment strategy:

- **Index Fund:**
  - **Passively Managed:** It aims to replicate the performance of a specific market index (e.g., SP 500).
  - **Goal:** Match the index's returns, not beat them.
  - **Lower Fees:** Because it doesn't require active management, costs are typically lower.
- **Mutual Fund:**
  - **Actively Managed (usually):** A professional fund manager selects investments to try to outperform the market or a benchmark.
  - **Goal:** Beat the market or achieve a specific return objective.
  - **Higher Fees:** Active management results in higher management and operational costs.

An index fund mirrors a market index (an index is a statistical measure tracking the performance of a group of assets, such as stocks or bonds, to represent the overall performance of a specific market or sector, like the $S\&P$ 500 Index) and is passive, while a mutual fund actively selects investments to outperform the market. And importantly, you need to remember that, when deciding where to put the money, the comparison has to be made with respect to both risk and return of the investment of the fund.

**Note:** When you invest in the $S\&P$ 500 index fund, it means you're investing in a diversified portfolio based on the $S\&P$ 500 index, where each stock is weighted according to its market capitalization (i.e., the total value of a company's outstanding shares).

**Who Determines the Percentages?** The $S\&P$ 500 is managed by **S&P Dow Jones Indices**, an independent organization. They determine:

- Which companies are included in the index based on specific criteria (e.g., size, profitability, sector, and liquidity).
- The weighting of each stock in the index, which is based on its market capitalization. Larger companies like Apple or Amazon get a bigger percentage than smaller companies.