Cost Minimization

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Contents

1 Motivation

Profit maximization requires that a firm chooses inputs and outputs to maximize the difference between revenue and costs. One way to think about this is in two steps:

- **Step 1:** Cost Minimization For any given level of output, a firm can choose the combination of inputs that minimizes production costs. We already saw in profit maximization that the firm chooses inputs until the value of the marginal product equals the input's price. This condition essentially implies that the firm is minimizing its cost of production for that output level.
- **Step 2:** Profit Maximization Once the firm knows the minimum cost needed to produce any level of output (its cost function), it can then decide which output level maximizes its overall profit (total revenue minus total cost).

While the previous discussion on profit maximization directly focused on choosing inputs to maximize profit, the next chapter on cost minimization breaks this problem into a more detailed look at the firm's technology and input choices. By understanding how to produce a given output at the lowest cost, we can then combine that insight with revenue considerations to fully maximize profit.

2 Cost Minimization

A firm seeking to minimize its production cost for a given output level y chooses input quantities x_1 and x_2 to minimize the total cost subject to achieving y units of output. Mathematically, the problem is:

$$\begin{aligned} \min_{x_1, x_2} \quad & w_1 x_1 + w_2 x_2 \\ \text{subject to} \quad & f(x_1, x_2) = y, \end{aligned}$$

where w_1 and w_2 are the prices of the two inputs, and $f(x_1, x_2)$ is the production function.

2.1 Cost Function

The cost function, denoted by $c(w_1, w_2, y)$, is defined as the minimum total cost necessary to produce y units of output given the input prices w_1 and w_2. That is,

$$c(w_1, w_2, y) = \min_{x_1, x_2} \{w_1 x_1 + w_2 x_2 \mid f(x_1, x_2) = y\}.$$

2.1.1 Car Manufacturer

Suppose a car manufacturer has a production function for cars given by a simple Cobb-Douglas form:

$$f(x_1, x_2) = \sqrt{x_1 x_2},$$

where:

- x_1 is the number of labor hours,
- x_2 is the number of machine hours.

The firm wants to produce $y = 100$ cars, and we assume the input prices are $w_1 = 4$ (dollars per labor hour) and $w_2 = 9$ (dollars per machine hour). The cost-minimization problem is:

$$\begin{aligned} \min_{x_1, x_2} \quad & 4x_1 + 9x_2 \\ \text{subject to} \quad & \sqrt{x_1 x_2} = 100. \end{aligned}$$

In order to solve it, and as we don't have inequality constraints (we don't need KKT), we follow straightforward:

- **Rewrite the constraint:** Squaring both sides of the production constraint gives (it's equivalent because hours are positive (≥ 0):

$$\sqrt{x_1 x_2} = 100 \longleftrightarrow x_1 x_2 = 10,000.$$

- Solve via Lagrangian Method:

$$\mathcal{L} = 4x_1 + 9x_2 + \lambda(10,000 - x_1 x_2).$$

- **The first-order conditions are:**
  - **With respect to x_1:** $4 - \lambda x_2 = 0 \Rightarrow \lambda = \frac{4}{x_2}$.
  - **With respect to x_2:** $9 - \lambda x_1 = 0 \Rightarrow \lambda = \frac{9}{x_1}$.

  Equating the two expressions for λ: $\frac{4}{x_2} = \frac{9}{x_1} \Rightarrow x_1 = \frac{9}{4}x_2$.

- **Using the Production Constraint:** Substitute $x_1 = \frac{9}{4}x_2$ into $x_1 x_2 = 10,000$:

$$\frac{9}{4}x_2 \cdot x_2 = 10,000 \longleftrightarrow \frac{9}{4}x_2^2 = 10,000 \longleftrightarrow x_2^2 = \frac{10,000 \times 4}{9} = \frac{40,000}{9} \quad \Rightarrow \quad x_2 = \sqrt{\frac{40,000}{9}} \approx 66.67.$$

  Then, $x_1 = \frac{9}{4} \times 66.67 \approx 150$.

- **Calculate the Minimum Cost:** The minimum cost is:

$$c(4, 9, 100) = 4x_1 + 9x_2 = 4(150) + 9(66.67) = 600 + 600 = 1,200.$$

This example shows that to produce 100 cars at the lowest cost given the input prices, the firm should use approximately 150 labor hours and 66.67 machine hours, resulting in a total minimum cost of \$1,200. The cost function $c(4, 9, 100) = 1,200$ tells us the least amount the firm must spend on inputs to achieve the target production level.

2.2 Isocost Lines

Isocost lines represent all combinations of inputs (e.g., x_1 and x_2) that can be purchased for a given total cost C. The equation for an isocost line is

$$w_1 x_1 + w_2 x_2 = C,$$

where w_1 and w_2 are the prices of the two inputs. Rearranging the equation gives:

$$x_2 = \frac{C}{w_2} - \frac{w_1}{w_2}x_1.$$

This is a straight line with a vertical intercept of $\frac{C}{w_2}$ and a slope of $-\frac{w_1}{w_2}$.

2.3 Tangency Condition (Cost Optimal Solution)

The firm's problem is to minimize cost subject to producing a given level of output y, that is:

$$\min_{x_1, x_2} \quad w_1 x_1 + w_2 x_2 \quad \text{subject to} \quad f(x_1, x_2) = y.$$

As recall, we know that an isoquant represents all combinations of inputs that produce the same output y. Its slope is given by the technical rate of substitution (TRS), which is

$$TRS(x_1, x_2) = -\frac{MP_1(x_1, x_2)}{MP_2(x_1, x_2)},$$

where MP_1 and MP_2 are the marginal products of inputs x_1 and x_2. The optimal (cost-minimizing) combination is found where an isocost line is tangent to the isoquant. At the tangency point, the slopes are equal:

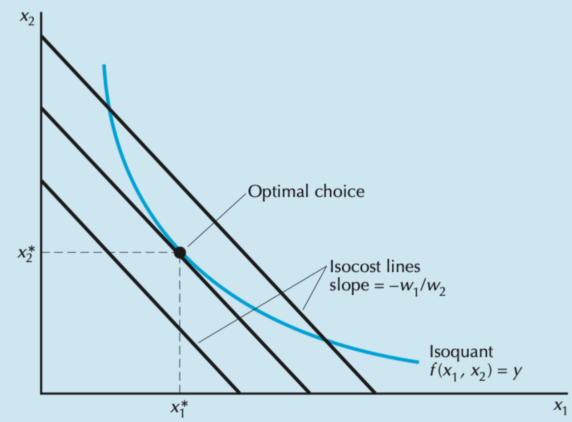
$$-\frac{MP_1(x_1^*, x_2^*)}{MP_2(x_1^*, x_2^*)} = -frac w_1 w_2 \longleftrightarrow \frac{MP_1(x_1^*, x_2^*)}{MP_2(x_1^*, x_2^*)} = \frac{w_1}{w_2}.$$

2.4 Derived Factor Demands

The optimal amounts of inputs that minimize cost for producing a given output y are functions of the input prices w_1, w_2 and output level y. These are called the derived (or conditional) factor demand functions:

$$x_1(w_1, w_2, y) \quad \text{and} \quad x_2(w_1, w_2, y).$$

They show how the firm adjusts its input usage based on the relative input prices and the production target.



Cost minimization. The choice of factors that minimize production costs can be determined by finding the point on the isoquant that has the lowest associated isocost curve.

Figure 1: Optimal Choice - Cost Minimization

2.4.1 Car Manufacturer

Suppose a car manufacturer uses labor (L) and capital (K) with a production function:

$$f(L, K) = 2\sqrt{LK}.$$

- Isocost Line

  If labor costs w_L and capital costs w_K, the isocost line is:

  $$w_L L + w_K K = C \longleftrightarrow K = \frac{C}{w_K} - \frac{w_L}{w_K} L.$$

  For instance, if $w_L = \$20$ per hour, $w_K = \$40$ per hour, and $C = \$1,200$, then:

  $$K = \frac{1,200}{40} - \frac{20}{40} L = 30 - 0.5L.$$

- Marginal Products for Production Function

  The marginal products for the production function are:

  $$MP_L = \frac{\partial f}{\partial L} = 2 \cdot \frac{1}{2} \cdot L^{-1/2} K^{1/2} = \frac{K^{1/2}}{L^{1/2}},$$

  $$MP_K = \frac{\partial f}{\partial K} = 2 \cdot \frac{1}{2} \cdot L^{1/2} K^{-1/2} = \frac{L^{1/2}}{K^{1/2}}.$$

- Slope of the Isoquant

$$TRS = -\frac{MP_L}{MP_K} = -\frac{\frac{K^{1/2}}{L^{1/2}}}{\frac{L^{1/2}}{K^{1/2}}} = -\frac{K}{L}.$$

- Optimal Choice for Cost Minimization

  For optimal cost minimization, set the isoquant slope equal to the isocost slope:

$$-\frac{K}{L} = -\frac{w_L}{w_K} \longleftrightarrow \frac{K}{L} = \frac{w_L}{w_K}.$$

  With $w_L = 20$ and $w_K = 40$, we get:

$$\frac{K}{L} = \frac{20}{40} = 0.5 \longleftrightarrow K = 0.5L.$$

- **Derived Demands**

  Substituting $K = 0.5L$ into the production function:

$$2\sqrt{L\,(0.5L)} = 2\sqrt{0.5\,L^2} = 2\sqrt{0.5}\,L = y \longleftrightarrow L = \frac{y}{\sqrt{2}}.$$

  Using $K = 0.5L$ to find K:

$$K = 0.5\left(\frac{y}{\sqrt{2}}\right) = \frac{y}{2\sqrt{2}}.$$

  For a given output y and input prices $w_L = 20$ and $w_K = 40$, the cost-minimizing input choices are:

$$L^*(20, 40, y) = \frac{y}{\sqrt{2}}, \quad \text{and} \quad K^*(20, 40, y) = \frac{y}{2\sqrt{2}}.$$

2.4.2 Perfect Complements & Substitutes and Cobb-Douglas

- **Perfect Complements:** With $f(x_1, x_2) = \min\{x_1, x_2\}$, to produce y units the firm must use $x_1 = y$ and $x_2 = y$. Thus,

$$c(w_1, w_2, y) = w_1 y + w_2 y = (w_1 + w_2)y.$$

- **Perfect Substitutes:** With $f(x_1, x_2) = x_1 + x_2$, the firm uses the cheaper input. Hence,

$$c(w_1, w_2, y) = \min\{w_1 y,\, w_2 y\} = \min\{w_1, w_2\}\, y.$$

- **Cobb-Douglas:** With $f(x_1, x_2) = x_1^a x_2^b$, using calculus techniques yields the cost function:

$$c(w_1, w_2, y) = K\, w_1^{\frac{a}{a+b}}\, w_2^{\frac{b}{a+b}}\, y^{\frac{1}{a+b}},$$

  where K is a constant that depends on a and b.

2.5 WACM

The Weak Axiom of Cost Minimization (WACM) tells us that if a firm is truly minimizing costs, then the way it chooses inputs under different price regimes must be consistent.

- **Two Price Scenarios:** Imagine the firm produces a fixed output y using inputs x_1 and x_2 under two different sets of input prices - say at time t with prices (w_1^t, w_2^t) and at time s with prices (w_1^s, w_2^s). The firm's cost-minimizing input choices at these times are (x_1^t, x_2^t) and (x_1^s, x_2^s) respectively.

- **Consistency in Cost Minimization:** If the firm is minimizing cost in each situation, then using the t input bundle when facing t prices should be cheaper than if the firm were to use the s input bundle under the same t prices. That is,

$$w_1^t x_1^t + w_2^t x_2^t \leq w_1^t x_1^s + w_2^t x_2^s.$$

Similarly, when facing the s prices, the cost of using the s bundle must be lower than if the firm were to use the t bundle:

$$w_1^s x_1^s + w_2^s x_2^s \leq w_1^s x_1^t + w_2^s x_2^t.$$

- **What It Means:** These inequalities ensure that the firm's chosen combination of inputs is the most cost-effective for producing the output y given the respective prices. If either inequality were violated, it would mean that the firm could have produced the same output at a lower cost by choosing a different combination of inputs, contradicting the assumption that it is minimizing costs.

2.6 Returns to Scale & Average Cost Function

2.6.1 Returns to Scale

Once again, we differentiate between the 3 types of returns to scale:

- Constant Returns to Scale: Doubling all inputs exactly doubles output. Hence, the cost function is linear in output:

$$c(w_1, w_2, y) = y\, c(w_1, w_2, 1),$$

and the average cost remains constant regardless of output level, where $c(w_1, w_2, 1)$ is the **unit cost function** (cost of one unique output):

$$AC(y) = \frac{c(w_1, w_2, y)}{y} = \frac{y\, c(w_1, w_2, 1)}{y} = c(w_1, w_2, 1)$$

- Increasing Returns to Scale: Mathematically, a technology exhibits increasing returns to scale if, for any $t > 1$,

$$f(tx_1, tx_2) > t\, f(x_1, x_2).$$

This implies that when you double (or multiply) all inputs, the output more than doubles. In terms of the cost function $c(w_1, w_2, y)$, if it costs $c(w_1, w_2, 1)$ to produce one unit of output, then under increasing returns the cost of producing y units is less than $y \cdot c(1)$, meaning that the average cost $AC(y) = c(y)/y$ decreases as y increases.

- Decreasing Returns to Scale: A technology exhibits decreasing returns to scale if, for any $t > 1$,

$$f(tx_1, tx_2) < t\, f(x_1, x_2).$$

Here, doubling all inputs results in less than double the output. This means that the cost function $c(y)$ grows more than proportionally: producing y units costs more than $y \cdot c(1)$. Consequently, the average cost $AC(y) = c(y)/y$ increases as output y increases.

2.7 Long-Run and Short-Run Costs

2.7.1 Short-Run Costs

- In the short run, some factors of production (e.g., a factory building) are fixed.

- The short-run cost function, $c_s(y, \bar{x}_2)$, represents the minimum cost of producing y units when, say, factor 2 is fixed at $\bar{x}_2$. This is given by:

$$c_s(y, \bar{x}_2) = \min_{x_1} \{ w_1 x_1 + w_2 \bar{x}_2 \mid f(x_1, \bar{x}_2) = y \}.$$

- The short-run derived factor demand for the variable input (here x_1) depends on input prices and the level of the fixed input.

2.7.2 Long-Run Costs

- In the long run, all factors of production are variable.
- The long-run cost function, $c(y)$, gives the minimum cost of producing y units when the firm can adjust all inputs:
$$c(y) = \min_{x_1,x_2} \{w_1x_1 + w_2x_2 \mid f(x_1, x_2) = y\}.$$
- The long-run derived factor demands, $x_1(w_1, w_2, y)$ and $x_2(w_1, w_2, y)$, show the optimal input levels when the firm is free to adjust every factor.
- A key insight is that the long-run cost function is the envelope of the short-run cost functions, meaning that if the firm chooses the optimal fixed factor level in the short run (say, $x_2 = x_2(y)$), then the long-run cost is simply:
$$c(y) = c_s(y, x_2(y)).$$