Consumer Surplus

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Contents

1 Discreet Gross and Net Surplus

We will return tothe original setup outlining a consumer choice model involving two goods, where one good is discrete and the preferences are quasilinear.

- **Model Structure: Two-Goods**
  - **Good 1 (x_1):** Discrete, meaning it can only be consumed in whole units (integers like 0, 1, 2, ...).
  - **Good 2 (x_2):** Continuous, allowing consumption in any non-negative quantity.
  - **Price of Good 1:** p_1:
  - **Price of Good 2:** Normalized to 1 for simplicity $p_2 = 1$.
  - **Budget Constraint:** $p_1x_1 + x_2 = m$ where m is the consumer's income.
  - Quasilinear Preferences: $u(x_1, x_2) = v(x_1) + x_2$ where $v(x_1)$ represents the utility derived from consuming x_1 units of Good 1 and x_2 the linear utility of Good 2. Remember as well that $v(x_1)$ is concave, which ensures diminishing marginal utility for Good 1 (as more units of Good 1 are consumed, each additional unit provides less additional utility.).

1.1 Optimal Choice and Decision Rule

The Decision Rule for quasilinear preferences with discrete goods is:

$$x_1 = n \quad \text{if and only if} \quad r_{n+1} \le p_1 \le r_n$$

Where n is the optimal number of units of Good 1 to consume and r_n the reservation price for the nth unit of Good 1, defined as the additional utility from consuming that unit:

$$r_n = v(n) - v(n-1)$$

The consumer will choose to consume n units of Good 1 if the price p_1 lies between the reservation prices r_n and r_{n+1}, but how did we get to this definition of the reservation price? Let's rewind.

1.2 Reservation Price

By using the budget constraint we can isolate x_2 in terms of a discreet quantity of good 1.

$$p_1x_1 + x_2 = m \longleftrightarrow x_2 = m - p_1x_1 \implies x_2(n) = m - p_1 \cdot n$$

The utility when consuming n units of Good 1, is given by:

$$u(n, x_2(n)) = v(n) + x_2(n) = v(n) + (m - p_1 \cdot n)$$

and, the utility when consuming $n-1$ units of Good 1:

$$\begin{aligned} u(n-1, x_2(n-1)) &= v(n-1) + x_2(n-1) \\ &= v(n-1) + (m - p_1(n-1)) \end{aligned}$$

Therefore, the incremental utility (i.e. the change in utility when moving from $n-1$ to n units of Good 1) calculation is:

$$\begin{aligned} \Delta u &= u(n, x_2(n)) - u(n-1, x_2(n-1)) \\ &= [v(n) + (m - p_1n)] - [v(n-1) + (m - p_1(n-1))] \\ &= v(n) - v(n-1) - p_1 \end{aligned}$$

In the context of consumer choice, the reservation price, denoted r_n for the n-th unit of Good 1, is the maximum price the consumer is willing to pay for that unit without reducing their overall utility. Setting the incremental utility to zero (the point of indifference, where utility doesn't change between consuming n or $n-1$ quantities of Good 1) gives:

$$\Delta u = 0 \quad \longleftrightarrow \quad v(n) - v(n-1) - p_1 = 0 \quad \longleftrightarrow \quad p_1 = v(n) - v(n-1)$$

For such price, the utility doesn't change, and we call such price (dependant on the value of n) the reservation price.

$$r_n = v(n) - v(n-1)$$

This equation defines the reservation price as the marginal utility of the n-th unit of Good 1.

1.3 Deriving Gross and Net Consumer's Surplus

From the previous definition of reservation price, we can normalize the first possible quantity ($n = 0$) by setting it's utility to zero ($v(0) = 0$). This makes sense as if $n = 0$, we don't consume the Good 1, and therefore the whole income m represents the utility of Good 2.

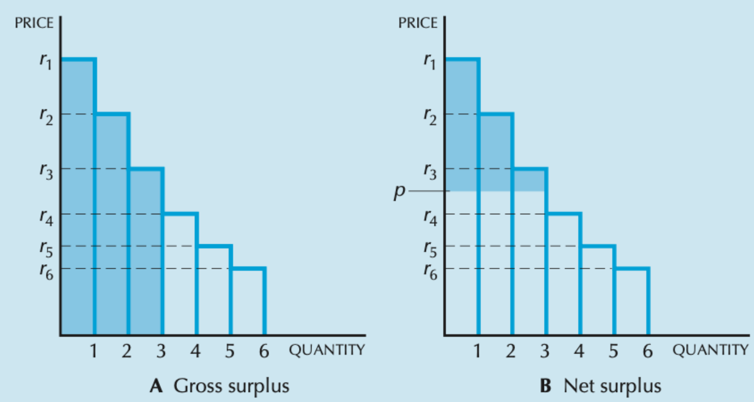
$$u(0, x_2) = v(0) + x_2 \ \wedge \ x_2(0) = m - p_1 \cdot 0 = m \implies u(0, x_2) = v(0) + m$$

By setting $v(0) = 0$, we implicitly imply that the utility of Good 2 is m, as we would obtain $u(0, m) = 0 + m = m$, and it makes sense. Now, we know that the reservation price is defined as $r_n = v(n) - v(n-1)$. We observe that:

$$\begin{aligned}
r_1 &= v(1) - v(0) = v(1) \ \longleftrightarrow \ v(1) = r_1 \\
r_2 &= v(2) - v(1) \ \longleftrightarrow \ r_2 = v(2) - r_1 \longleftrightarrow \ v(2) = r_1 + r_2 \\
r_3 &= v(3) - v(2) \ \longleftrightarrow \ v(3) = r_1 + r_2 + r_3
\end{aligned}$$

We deduce that we have a cumulative utility: $v(n) = r_1 + r_2 + \cdots + r_n = \sum_{i=1}^{n} r_i$

The **interpretation** is that the total utility from consuming n units of Good 1 is the sum of the reservation prices for each unit up to n. This cumulative value $v(n)$ is known as the Gross Consumer's Surplus, and its geometrical interpretation is the area under the demand curve as shown in figure 1. In particular, the gross consumer surplus is represented by the area under the first n bars of the demand curve, where each bar's height is $r_n = v(n) - v(n-1)$, and width is 1.



Reservation prices and consumer's surplus. The gross benefit in panel A is the area under the demand curve. This measures the utility from consuming the x-good. The consumer's surplus is depicted in panel B. It measures the utility from consuming both goods when the first good has to be purchased at a constant price p.

Figure 1: Gross Consumer and Net Consumer Surplus with Discrete Good

The Net Consumer's Surplus is defined as $CS(n) = v(n) - p_1 \cdot n$, and its geometrical interpretation is the area of the gross consumer's surplus minus the **total expenditure** on Good 1.

2 From Consumer's Surplus (Singular) to Consumers' Surplus (Plural)

2.1 Consumer's Surplus (Singular)

Consumer**'s** Surplus refers to the surplus of a single consumer. It measures the individual gain a consumer receives when they purchase a good or service at a price lower than the highest price they are willing to pay. This concept s basically the concept we described in the previous description.

- Gross Consumer's Surplus ($v(n)$): The total utility a consumer derives from consuming n units of a good.
- Net Consumer's Surplus ($CS(n)$): The actual surplus after accounting for the expenditure on the good. It is calculated as:

$$CS(n) = v(n) - p_1 n$$

2.1.1 Example

For instance, let's consider the following scenario. A consumer purchases 3 units of Good 1 and his reservation prices are:

- Reservation Prices: $r_1 = \$50,\ r_2 = \$40,\ r_3 = \$30$
- **Actual Price per Unit** $(p_1) = \$35$

Then, if we want to determine the Gross Consumer's Surplus and Net Consumer's Surplus, we proceed as follows:

1. **Gross Consumer's Surplus ($v(3)$):** $v(3) = r_1 + r_2 + r_3 = \$50 + \$40 + \$30 = \$120$
2. **Total Expenditure on Good 1:** $p_1 n = \$35 \times 3 = \$105$
3. **Net Consumer's Surplus ($CS(3)$):** $CS(3) = v(3) - p_1 n = \$120 - \$105 = \$15$

The gross surplus of \$120 represents the total utility the consumer gains from consuming 3 units of Good 1. After spending \$105 on the good, the consumer retains a net surplus of \$15, reflecting the additional utility beyond the expenditure.

2.2 Consumers' Surplus (Plural)

Consumers' Surplus refers to the aggregate surplus of multiple consumers. It is the sum of individual consumer surpluses across all consumers in the market.

- **Aggregate** Gross Consumers' Surplus: The sum of gross surpluses ($v(n)$) from all consumers.
- **Aggregate** Net Consumers' Surplus: The sum of net surpluses ($CS(n)$) from all consumers, calculated as:

$$\text{Consumers' Surplus} = \sum_{j=1}^{J} CS_j(n_j) = \sum_{j=1}^{J} [v_j(n_j) - p_1 n_j]$$

  where J is the total number of consumers and $CS_j(n_j)$ the net surplus for consumer j consuming n_j units.

2.2.1 Example

We consider a market with 3 Consumers purchasing Good 1, whose **price per unit** is ($p_1 = \$30$). The consumers' reservation prices and consumption are:

- **Consumer A:**
  - **Units Consumed: 2**
  - **Reservation Prices:** $r_{A1} = \$60,\ r_{A2} = \$50$

- **Consumer B:**
  - **Units Consumed: 3**
  - **Reservation Prices:** $r_{B1} = \$55,\ r_{B2} = \$45,\ r_{B3} = \$35$
- **Consumer C:**
  - **Units Consumed: 1**
  - **Reservation Prices:** $r_{C1} = \$40$

Then, for each consumer we obtain the gross surplus, total and the net surplus:

1. **Consumer A:**
   - Gross Surplus ($v_A(2)$): $v_A(2) = r_{A1} + r_{A2} = \$60 + \$50 = \$110$
   - Total Expenditure: $p_1 \times 2 = \$30 \times 2 = \$60$
   - Net Surplus ($CS_A(2)$): $CS_A(2) = v_A(2) - (\$30 \times 2) = \$110 - \$60 = \$50$
2. **Consumer B:**
   - Gross Surplus ($v_B(3)$): $v_B(3) = r_{B1} + r_{B2} + r_{B3} = \$55 + \$45 + \$35 = \$135$
   - Total Expenditure: $p_1 \times 3 = \$30 \times 3 = \$90$
   - Net Surplus ($CS_B(3)$): $CS_B(3) = v_B(3) - (\$30 \times 3) = \$135 - \$90 = \$45$
3. **Consumer C:**
   - Gross Surplus ($v_C(1)$): $v_C(1) = r_{C1} = \$40$
   - Total Expenditure: $p_1 \times 1 = \$30 \times 1 = \$30$
   - Net Surplus ($CS_C(1)$): $CS_C(1) = v_C(1) - (\$30 \times 1) = \$40 - \$30 = \$10$

Finally, the aggregate surpluses are:

- **Aggregate** Consumers' Gross Surplus: $v_A(2) + v_B(3) + v_C(1) = \$110 + \$135 + \$40 = \$285$
- **Aggregate** Net Consumers' Surplus: $CS_A(2) + CS_B(3) + CS_C(1) = \$50 + \$45 + \$10 = \$105$

The **aggregate** gross surplus of \$285 represents the total utility all consumers gain from consuming their respective quantities of Good 1. After accounting for total expenditures of \$180 (\$60 + \$90 + \$30), the **aggregate** net consumers' surplus is \$105, indicating the overall additional utility beyond the expenditures across all consumers.

3 Continuous Gross and Net Surplus

We have seen that the area underneath the demand curve for a discrete good measures the utility of consumption of that good. We can extend this to the case of a good available in continuous quantities by approximating the continuous demand curve by a staircase demand curve. The area under the continuous demand curve is then approximately equal to the area under the staircase demand.

3.1 Gross Consumer Surplus

As before, Gross Consumer's Surplus represents the total utility a consumer derives from consuming a continuous quantity of a good. It is calculated as the area under the continuous demand curve up to the quantity consumed. Mathematically:

$$\text{Gross Surplus} = \int_0^Q D(p)\, dp$$

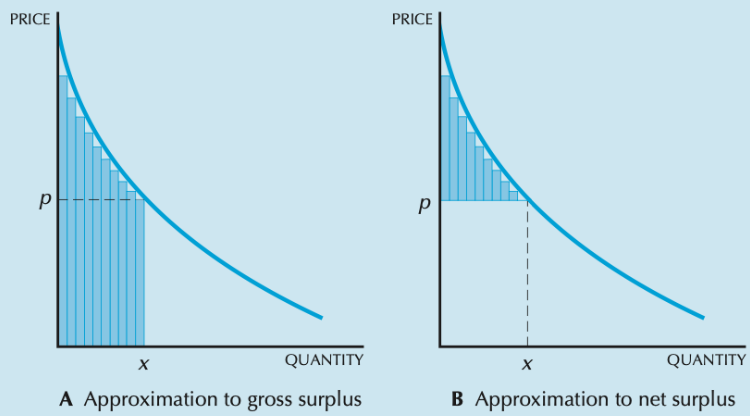
where $D(p)$ is the demand function and Q is the quantity consumed. This area captures the aggregate willingness to pay for each infinitesimal unit of the good, reflecting the total utility from consumption.

3.2 Net Consumer Surplus

Net Consumer's Surplus is the actual surplus after accounting for the total expenditure on the good. It is the Gross Consumer's Surplus minus the total expenditure for the good. Mathematically:

$$\text{Net Surplus} = \int_0^Q D(p)\,dp - P \times Q$$

where P is the market price of the good. This measures the additional utility the consumer gains beyond what they spend, representing their net benefit from the purchase.



Approximating a continuous demand. The consumer's surplus associated with a continuous demand curve can be approximated by the consumer's surplus associated with a discrete approximation to it.

Figure 2: Gross Consumer and Net Consumer Surplus with Continuous Goods

3.3 Change in Gross and Net Consumer Surplus with Price Increase

When the price of a good increases from p to p', the net and gross consumer's surplus — the difference between what consumers are willing to pay and what they actually pay — decreases. This change can be visually represented as a **trapezoidal** area composed of two distinct parts: a rectangle (R) and a small roughly triangular region (T).

- **Rectangle (R)**
  - **Represents:** The loss in consumer surplus due to the price increase applied to the quantity still consumed.
  - **Explanation:** When the price rises from p to p', consumers who continue to purchase the good now pay a higher price for each unit. The rectangle R captures the uniform loss in surplus across the entire original quantity that consumers continue to buy despite the price hike.
  - **Calculation:**

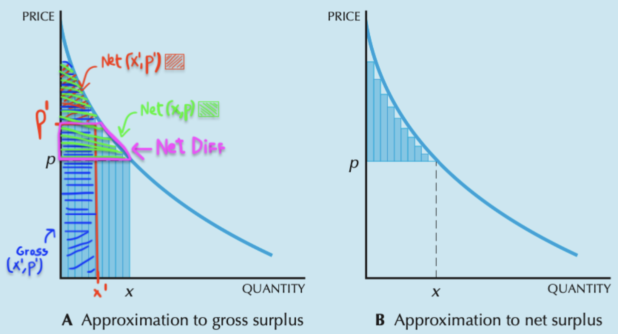
    $$\text{Area of } R = (p' - p) \times Q'$$

    where Q' is the new quantity consumed at price p'.
- **Triangle (T)**
  - **Represents:** The additional loss in consumer surplus due to the reduction in quantity consumed as a result of the price increase.

- **Explanation:** Higher prices typically lead to a decrease in the quantity demanded. Some consumers may reduce their consumption or drop out of the market entirely. The triangle T captures the marginal loss in surplus from the units that are no longer purchased because of the higher price.
- **Calculation:**

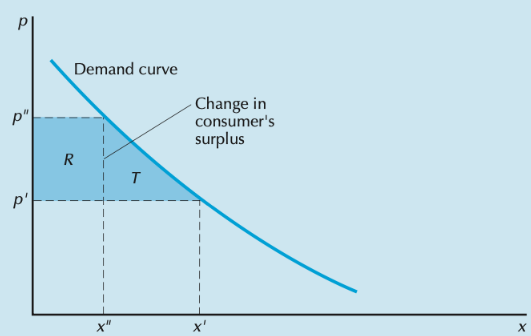
$$\text{Area of } T = \frac{1}{2} \times (p' - p') \times \Delta Q = \frac{1}{2} \times (p' - p') \times (x - x')$$

where ΔQ is the decrease in quantity consumed due to the price increase.



Approximating a continuous demand. The consumer's surplus associated with a continuous demand curve can be approximated by the consumer's surplus associated with a discrete approximation to it.

Figure 3: Net and Gross Change Surplus



Change in consumer's surplus. The change in consumer's surplus is sum of the square area R and the roughly triangular area T, and thus has a roughly trapezoidal shape.

Figure 4: Net and Gross Change Surplus - Another Perspective

For instance, imagine a demand curve where:

Original Price $p = \$30$ **New Price** $p' = \$40$ **Original Quantity** $Q = 10$ units **New Quantity** $Q' = 8$ units **Decrease in Quantity** $\Delta Q = 2$ units

Then, the calculations for the change in net consumer's surplus are:

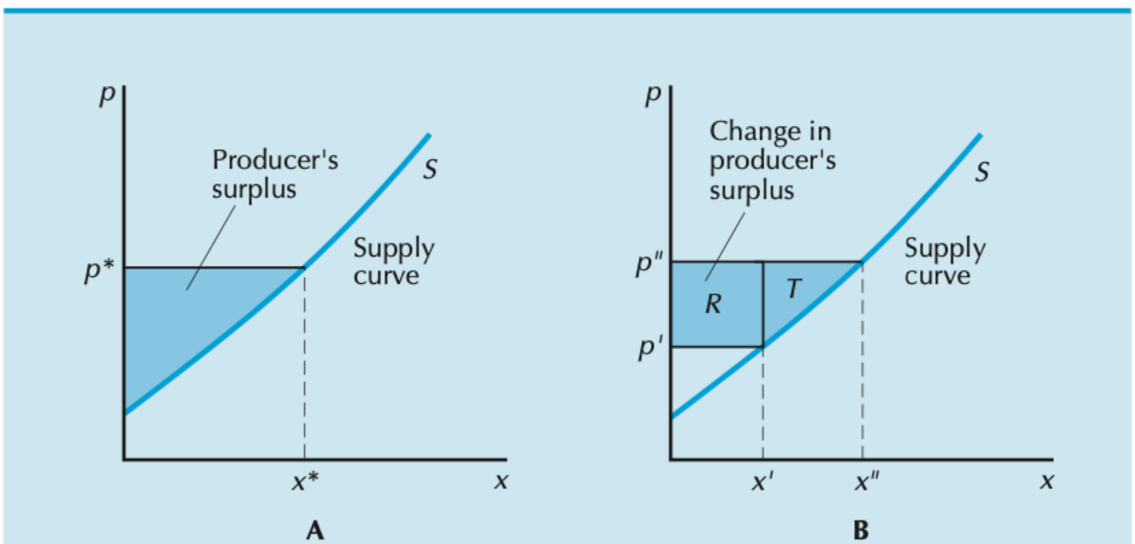
1. **Rectangle R:** $R = (40 - 30) \times 10 = \$100$
2. **Triangle T:** $T = \frac{1}{2} \times (40 - 30) \times 2 = \$10$
3. **Total Change in Consumer's Surplus:** Change $= R + T = \$100 + \$10 = \$110$ decrease

3.4 Change in Gross and Net Supplier Surplus with Price Increase

Important The demand curve measures the amount that will be demanded at each price; the supply curve measures the amount that will be supplied at each price. Just as the area under the demand curve measures the surplus enjoyed by the demanders of a good, the area above the supply curve measures the surplus enjoyed by the suppliers of a good. Remember that the supply curve slopes upward because when the price of a good increases, producing that good becomes more profitable.

When the price of a good increases from p' to p'', the producer's surplus — the benefit producers receive from selling at a higher price than the minimum they are willing to accept — increases. This change, once again, is represented by a trapezoidal area composed of a **rectangle (R)** and a **triangle (T)**.

- The **rectangle** R measures the gain from selling the existing quantity at the higher price, calculated as: $R = (p'' - p') \times Q$, where $Q = x'$ is the original quantity sold.
- The **triangle** T represents the additional gain from selling extra units at the new price, calculated as $T = \frac{1}{2} \times (p'' - p') \times \Delta Q$, where $\Delta Q = x'' - x'$ is the increase in quantity supplied due to the price rise.



Producer's surplus. The net producer's surplus is the triangular area to the left of the supply curve in panel A, and the change in producer's surplus is the trapezoidal area in panel B.

Figure 5: Producer's Surplus and Change Producer's Gross & Net Surplus

For instance, suppose the price of a widget increases from \$20 to \$30. Initially, producers supply 100 widgets. After the price increase, they supply 120 widgets. The change in producer's surplus consists of:

- **Rectangle** R: $(30 - 20) \times 100 = \$1,000$
- **Triangle** T: $\frac{1}{2} \times (30 - 20) \times 20 = \$200$
- **The total increase in producer's surplus:** \$1,200.

4 Benefit-Cost Analysis: Perfect Competition and Imports

Let's consider 3 possible scenarios, under a perfectly competitive market: No Tax Scenario, Tax Scenario and International Trades, as shown in figure 6.

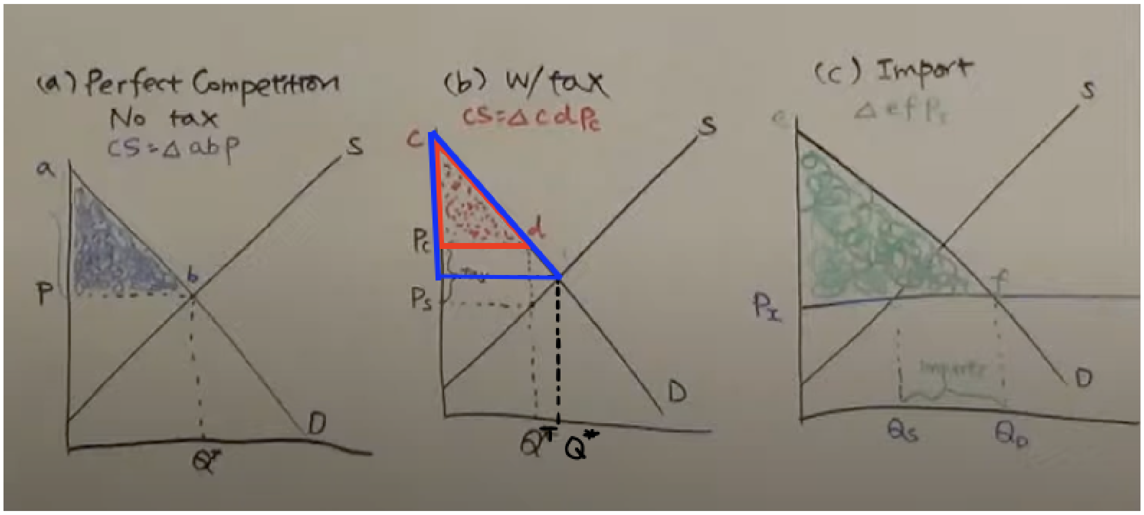


Figure 6: Consumer Surplus vs. Supplier Surplus in Different Scenarios

1. **Perfect Competition without Tax**
   - In this scenario, suppose we have perfect competition so w have the **demand curve D** and the **supply curve S** and at the intersection we find the equilibrium price. Without tax the price **P** paid by the consumer will be equal to the price received by the seller.
   - **Consumer Surplus:** Calculated as the area of triangle ABP.
   - **Formula:** Area $= \frac{1}{2} \times \text{Base} \times \text{Height}$, where Base is from zero to Q^*, and Height is the difference between the demand curve and the price P.
   - **Representation:** The triangle is formed below the demand curve, above the equilibrium price P, and spans from zero to Q^* on the quantity axis.

2. **Perfect Competition with Tax**
   - Now, we suppose the government imposes a tax on such good. The tax is just a wedge between the price paid by the consmer P_C and the price received by the seller P_S. In the figure 6, $Q*$ represents the equilibrium demand which in this case is not achieved due to governmental intervention. When a tax is imposed, the supplier doesn't increase the price; instead, the tax creates a wedge where consumers pay a higher price, and suppliers receive less.

     In practice, suppliers typically do not reduce their prices when taxes are imposed; instead, they try to pass the tax on to consumers by adding it on top of the selling price. This leads to a higher price for consumers while the amount suppliers receive effectively decreases with respecto to what they sell due to the tax. This difference (the tax) is remitted to the government. The decrease in the price received by the supplier may lead them to reduce the quantity supplied because it becomes less profitable to produce the same amount. That's why the demand after tax Q^T is lower than equilibrium demand Q^*.

     Basically, by introducing taxes suppliers earn less due to reduced quantities sold at effectively lower prices to them, impacting their total revenue which could be higher without intervention with a price where the market reaches quilirbrium (such price P^* would be lower than P_C with tax) and consumers face higher prices directly because of the tax and will buy less due to these higher prices. This reduction in consumption hurts consumer surplus.

The less attractive market due to taxes can deter new suppliers from entering the market, affecting long-term supply (which slowly decreases as more suppliers leave the market for such good) and potentially leading to reduced market competition and variety.

- **Consumer Surplus:** Represented by the red triangle area above the new consumer price PC and below the demand curve.

3. **International Trade (Imports)**

- The last scenario contemplates international trades, which as often happens, offer better prices than local/national ones.
  - **Before International Price**: Initially, the market finds a balance where the local demand (consumers) and local supply (producers) curves intersect. This intersection determines two key variables, the Equilibrium Price P^* which is the price at which the quantity supplied perfectly matches the quantity demanded in the market, and the equilibrium quantity Q^* which is the amount of goods that is both supplied and demanded at the equilibrium price. In this setup, everything is stable, with local producers meeting the needs of local consumers at price P^* and quantity Q^*. The equilibrium is not drawn in the international case.
  - **After Introducing International Price** Suppose that in the market enters a new competitor providing the same good with a price P_I which is lower than the local equilibrium price (P^*). This new price shakes up the previous local balance for several reasons:
    - **Consumer Response:** Consumers see that the goods are now available at a lower price (P_I), so they want to buy more. This shifts the demand for goods to a higher quantity (Q_D).
    - **Supplier Response:** Local suppliers find the international price less attractive because it's lower than what they were used to getting (P^*). Consequently, they are willing to supply less than before, reducing their output to Q_S, which is less than Q^*.
  - What is interesting is the Gap created. Now, there is a discrepancy, the new price has increased Demand Q_D (higher demand at the lower price) and decreased Supply Q_S (lower supply because the lower price is less profitable). The difference between Q_D and Q_S represents the gap that must be filled to satisfy consumer demand at this new lower price. This gap is filled by imports. The volume of imports is exactly this difference, making up for the shortfall in local production.

Each scenario visually demonstrates how consumer surplus is impacted by changes in market conditions—such as the absence or presence of taxes, and the influence of international trade prices.