Slutsky Equation

Pablo Mollá Chárlez

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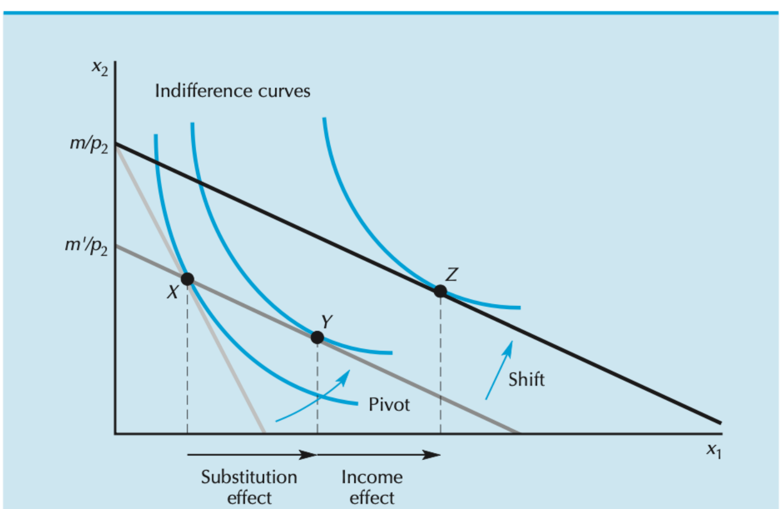
1 Substitution, Income and Total Effect

When the price of a good changes, two key effects arise:

- **Substitution effect:** A change in relative prices reallocates demand toward the cheaper good. For example, if good 1 becomes cheaper, less of good 2 is needed to purchase good 1.
- **Income effect:** A price change alters the consumer's purchasing power. If good 1 becomes cheaper, purchasing power increases, allowing the consumer to afford more goods.

Mathematically, we define the following terms:

1. **Price Change and Income Adjustment:** Let $\Delta p_1 = p_1' - p_1$ represent the price change for good 1 and let $\Delta m = m' - m$ represent the income adjustment necessary to afford the original bundle after the price change. This adjustment is given by:
$$\Delta m = x_1 \cdot \Delta p_1,$$
2. **Substitution effect:** The movement from point X (original choice) to point Y (adjusted for relative prices while holding purchasing power constant) is the substitution effect. It is defined as:
$$\Delta x_1^s = x_1(p_1', m') - x_1(p_1, m),$$
where $m' = m + \Delta m$ is the adjusted income. In the figure 1, the substitution effect leads to a modification in the original budget line corresponding to the equation: $p_1' x_1 + p_2 x_2 = m'$
3. **Income effect:** The movement from point Y (adjusted bundle) to point Z (final choice with new purchasing power) is the income effect. It is defined as:
$$\Delta x_1^i = x_1(p_1', m) - x_1(p_1', m').$$
In the figure 1, the income effect leads to the final budget line corresponding to the equation: $p_1' x_1 + p_2 x_2 = m$
4. The total change in demand for good 1, Δx_1, is the sum of the substitution effect and income effect:
$$\Delta x_1 = \Delta x_1^s + \Delta x_1^i.$$



Substitution effect and income effect. The pivot gives the substitution effect, and the shift gives the income effect.

Figure 1: Substitution and income effect

1.1 Milk Scenario

Let's consider the demand function of the good milk x_1 define as $x_1(m, p_1) = 10 + \frac{m}{10p_1}$ where p_1 denotes the **price** of the good and m the **income** of the consumer. Besides, we know that the income per week is $m = 120\$$ and the price per quart is 3$.

1. **What is the demand per week?**

   To determine the demand per week, we just need to replace the given variables (price and income) into the demand function. Therefore,

$$x_1(120, 3) = 10 + \frac{120}{10 \cdot 3} = 14 \text{ quarts/week}$$

   The initial and optimal bundle that the consumer can buy corresponds to $x_1 = 14$ quarts of milk per week. This is also the original demand of good x_1.

2. **What is the substitution effect if $p_1' = 2\$$?**

   Remark: The substitution effect can be calculated with an utility function, or like in this case with a demand function. Now, let's determine what would be the new bundle of milk that the consumer could get if the price of milk decreased to 2$. First, we need to determine the change of income, which is defined as:

$$\Delta m = \Delta p_1 \cdot x_1 = (p_1' - p_1) \cdot x_1 = (2 - 3) \cdot 14 = -14\$$$

   This change of income corresponds to the amount of dollars that the consumer is saving per week, and with such value, we can compute the new income to keep the same purchasing power:

$$m' = m + \Delta m = 120 - 14 = 106\$$$

   Remark: The new income keeps the same purchasing power because:

   - **Original Expenditure on milk:** $14 \cdot 3 = 42$ quarts/week $\implies 120 - 42 = 78\$$ to spend on the other good which we unknow.
   - **New Expenditure on milk:** $14 \cdot 2 = 28$ quarts/week $\implies 106 - 28 = 78\$$ to spend on the other good which we unknow.

   Now, the new demand of milk with the new price p_1', can be computed as:

$$x_1'(m', p_1') = 10 + \frac{106}{10 \cdot 2} = 15,3 \text{ quarts/week}$$

   Therefore, the substitution effect is:

$$x_1^s = x_1'(106, 2) - x_1(120, 3) = 15,3 - 14 = 1,3 > 0.$$

   The consumer increases his demand for milk by $1,3$ quarts/week by susbtituting milk for other goods.

3. **What is the income effect?**

   The income effect is computed as:

$$x_1^i = x_1'(120, 2) - x_1(106, 2) = 16 - 15,3 = 0,7 > 0.$$

4. **What is the total effect?**

   The total effect is computed as:

$$\Delta x_1 = x_1^s + x_1^i = 1,3 + 0,7 = 2 > 0.$$

   Then, the total effect is 2 quarts/week, meaning that the demand increases by 2 units as a result of the price change.

1.2 Candy Scenario

Let's consider a consumer who wants to buy candies and some other good with price $p_2 = 1$. The **price** for each candy is $p_1 = 0,5\$$, his **income** is $m = 10\$$ and the **first bundle** he wants to buy is $x_1 = 20$. Besides, the consumer provides his utility function defined as follows: $U(x_1, x_2) = x_1 + x_2$ (perfect substitutes), where good 2 is unknown. So, the budget line constraint would be $0,5 \cdot x_1 + x_2 = 10$.

1. **What is the substitution effect if $p_1' = 0,4\$$?**

   Now, let's determine what would be the new bundle of candies that the consumer could get if the price of candies decreased to $0,4\$$. First, we need to determine the change of income, which is defined as:

   $$\Delta m = \Delta p_1 \cdot x_1 = (p_1' - p_1) \cdot x_1 = (0,4 - 0,5) \cdot 20 = -2\$$$

   This change of income corresponds to the amount of dollars that the consumer is saving per purchase, and with such value, we can compute the new income to keep the same purchasing power:

   $$m' = m + \Delta m = 10 - 2 = 8\$$$

   As we are in a perfect substitutes preference scenario, we know that the cheapest good (good 1 because $p_1 < p_2$) will be consumed completely, leaving behind good 2, therefore the optimal consumption bundle would be:

   $$0,5 \cdot x_1 + x_2 = 10 \longleftrightarrow 0,5 \cdot 20 + x_2 = 10 \longleftrightarrow (x_1, x_2) = (20, 0) \implies U(20, 0) = 20$$

   Now, the new demand of candies with the new price p_1' (perfect substitute preference where $p_1' < p_1 < p_2$), can be computed as:

   $$p_1' \cdot x_1 + x_2 = m' \longleftrightarrow 0,4 \cdot x_1 + (0) = 8 \longleftrightarrow (x_1, x_2) = (20, 0)$$

   Therefore, the substitution effect is:

   $$x_1^s = x_1'(8, 0.4) - x_1(10, 0.5) = 20 - 20 = 0.$$

   The consumer's demand for good 1 does not change ($x_1 = 20$) when the price decreases but purchasing power is adjusted to keep the original bundle affordable. This indicates no reallocation of demand between goods due to the price change.

2. **What is the income effect?**

   We need to determine the demand of x_1 under the original income m and with the new price decrease p_1', and as we know it's a perfect substitute preference where $p_1' < p_2$:

   $$p_1' \cdot x_1 + x_2 = m \longleftrightarrow 0,4 \cdot x_1 + (0) = 10 \longleftrightarrow (x_1, x_2) = (25, 0)$$

   The income effect is computed as:

   $$x_1^i = x_1'(10, 0.4) - x_1(8, 0.4) = 25 - 20 = 5 > 0.$$

3. **What is the total effect?**

   The total effect is computed as:

   $$\Delta x_1 = x_1^s + x_1^i = 0 + 5 = 5 > 0.$$

   Then, the total effect is 5 candies, meaning that the demand increases by 5 units as a result of the price change.

2 Sign of Substitution and income effect

2.1 Sign of substitution effect

The sign of the substitution effect is always known:

- For a price decrease, the substitution effect is non-negative (positive or zero) (demand increases).
- For a price increase, the substitution effect is negative (demand decreases).

$\Longrightarrow$ This reflects how consumers substitute toward the relatively cheaper good.

2.2 Sign of income effect

The sign of the income effect depends on whether the good is normal or inferior:

- **Normal Good:** income effect is positive when purchasing power increases, demand increases.
- **Inferior Good:** income effect is negative when purchasing power increases, demand decreases.

Notice the relationship between Purchasing Power and Prices:

- A price decrease increases real income (since the same nominal income can buy more goods).
- A price increase decreases real income (since the same nominal income can buy fewer goods).

3 Total Effect: Cases for Normal and Inferior Goods

The total change in demand (Δx_1) due to a price change is composed of the substitution effect (SE) and the income effect (IE):

$$\Delta x_1 = x_1(p_1', m) - x_1(p_1, m) = \Delta x_1^s + \Delta x_1^i.$$

3.1 Case 1: Normal Good

Substitution and income effects work in the same direction:

- A price increase ($p_1 \to p_1'$) reduces demand due to the substitution effect ($\Delta x_1^s < 0$).
- A price increase also reduces demand because purchasing power decreases, and for a normal good, lower income reduces demand ($\Delta x_1^i < 0$).
- Total effect:

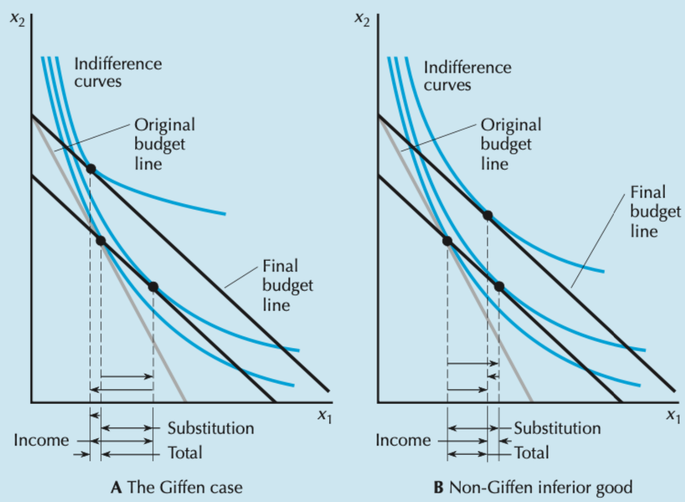
$$\Delta x_1 < 0, \quad \Delta x_1^s < 0, \quad \Delta x_1^i < 0.$$

3.2 Case 2: Inferior Good

Substitution and income effects work in opposite directions:

- A price increase reduces demand due to the substitution effect ($\Delta x_1^s < 0$).
- However, the income effect increases demand because the reduced purchasing power causes the consumer to buy more of the inferior good ($\Delta x_1^i > 0$).
- If the income effect is strong enough ($|\Delta x_1^i| > |\Delta x_1^s|$), the total effect can be positive, as in the Giffen case:

$$\Delta x_1 > 0, \quad \Delta x_1^s < 0, \quad \Delta x_1^i > 0.$$



Inferior goods. Panel A shows a good that is inferior enough to cause the Giffen case. Panel B shows a good that is inferior, but the effect is not strong enough to create a Giffen good.

Figure 2: Inferior Goods

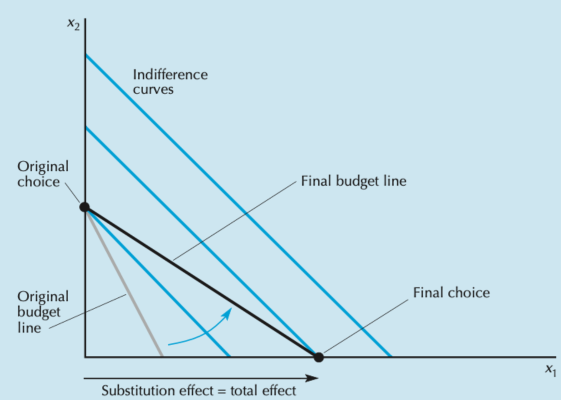
4 Slutsky Equation and Preferences

4.1 Perfect Substitutes

For perfect substitutes, the consumer is indifferent between two goods and chooses only the cheaper one. When the budget line tilts due to a price change:

- The demand bundle jumps from one axis (e.g., vertical, only good 2) to the other axis (e.g., horizontal, only good 1).
- There is no gradual shifting or adjustment because the consumer switches entirely to the cheaper good.
- **Result:** The entire change in demand is due to the substitution effect, and the income effect is zero.

$$\Delta x_1 = \Delta x_1^s, \quad \Delta x_1^i = 0.$$



Perfect substitutes. Slutsky decomposition with perfect substitutes.

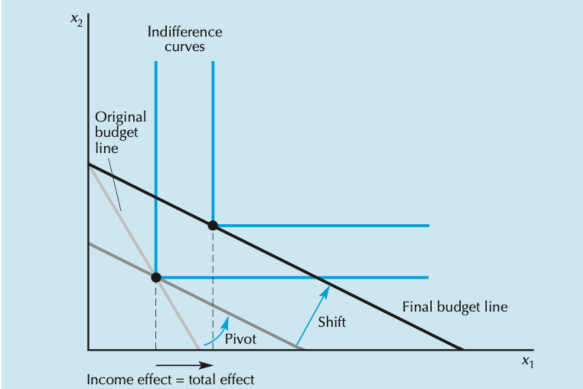
Figure 3: Perfect Substitutes

4.2 Perfect Complements

For perfect complements, goods are consumed in fixed proportions (e.g., one unit of good 1 requires one unit of good 2). When the budget line tilts (pivots) due to a price change:

- The consumer's optimal choice remains at the same proportion, constrained by the fixed ratio.
- The substitution effect is zero because the consumer cannot substitute between the goods.
- **Result:** The entire change in demand is due to the income effect, as a price change impacts the consumer's purchasing power.

$$\Delta x_1 = \Delta x_1^i, \quad \Delta x_1^s = 0.$$



Perfect complements. Slutsky decomposition with perfect complements.

Figure 4: Perfect Complements

4.3 Quasilinear Preferences

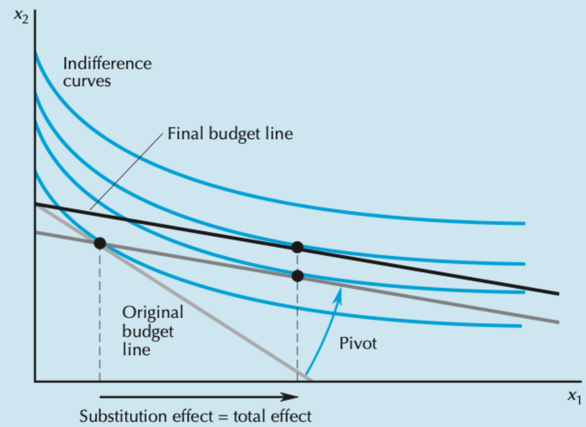
With quasilinear preferences, demand for good 1 does not depend on income:

$$\Delta x_1 = \Delta x_1^s, \quad \Delta x_1^i = 0.$$

For quasilinear preferences, utility takes the form $U(x_1, x_2) = f(x_1) + x_2$, meaning that demand for good 1 is independent of income.

- When income changes, the consumption of good 1 remains constant, while changes occur only for good 2.
- **Result:** The entire change in demand for good 1 is due to the substitution effect, and the income effect is zero.

Preference Type	$SubstitutionEffect$	**Income Effect**	**Explanation**
Perfect Substitutes	Entirely present	Zero	Consumer fully switches to the cheaper good.
Perfect Complements	Zero	Entirely present	Fixed proportions mean no substitution is possible.
Quasilinear Preferences	Entirely present	Zero	Demand for good 1 does not depend on income.



Quasilinear preferences. In the case of quasilinear preferences, the entire change in demand is due to the substitution effect.

Figure 5: Quasilinear Preferences

5 Law of Demand

The law of demand states the following:

The demand for a normal good must decrease when its price increases.

Let's prove the previous statement throughly. Let's recall before proceeding that a normal good is a good such that when the income increases the demand for the good increases as well.

Proof: The law of demand is equivalent to saying that demand for a normal good must increase when its price decreases. If price for a normal good decreases, then after section 3 (Total Effect: Normal & Inferior Goods), the substitution effect Δx_1^s is positive (because it would be negative when the price increases, remember the candy/milk examples). Analogously, the **income effect** Δx_1^i is positive as well, as increased purchasing power leads to higher consumption of x_1. Reason the positivity of the substitution effect due the fact that the consumer substitutes toward the cheaper good x_1, and he positivity of **income effect** because the increase in purchasing power leas to higher demand for the normal good x_1. Therefore, the total effec defined as follows is positive:

$$\Delta x_1 = \underbrace{\Delta x_1^s}_{>0} + \underbrace{\Delta x_1^i}_{>0} > 0.$$

We conclude that as the **total effect** is positive, the demand for good x_1 increases when its price decreases.

6 Holding Income vs. Slutsky vs. Hicks substitution effect

6.1 Holding Income Constant

This refers to the total effect of a price change on demand when nominal income is held constant. No adjustments are made to account for changes in purchasing power or utility. This is the real-world response to a price change because consumers typically face a fixed income. The change in demand reflects both the substitution and income effects combined:

$$\Delta x_1 = \Delta x_1^s + \Delta x_1^i.$$

6.2 Slutsky substitution effect

This isolates the substitution effect by keeping purchasing power constant. The consumer's income is adjusted (m') so they can still afford their original bundle after the price change:

$$m' = m + x_1 \cdot \Delta p_1.$$

This adjustment ensures the consumer can afford their original level of utility in terms of their initial consumption bundle.

6.3 Hicks substitution effect

This isolates the substitution effect by keeping utility constant. It measures substitution behavior while holding utility fixed. Unlike Slutsky, this does not guarantee that the original bundle is affordable; it only ensures the same level of satisfaction.

Aspect	**Holding Income Constant**	**Slutsky substitution effect**	**Hicks substitution effect**
What Is Held Constant?	Income remains fixed	Purchasing power (afford original bundle)	Utility (same indifference curve)
Includes income effect?	Yes	No	No
Purpose	Captures total effect of price change	Isolates substitution due to price change	Isolates substitution due to price change
Adjustment	No adjustment	Adjust income to keep original bundle affordable	Adjust income to keep utility constant
Real-World Application	Observed consumer behavior	Explains substitution effects (theoretical)	Used in welfare and utility analysis (theoretical)

6.3.1 Cobb-Douglas Preferences

Let's use the candy example alongside Cobb-Douglas preferences to clearly illustrate the differences between holding income constant, Slutsky substitution effect, and magentaHicks substitution effect. The utility Function is $U(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$, where $\alpha = 0.5$. The budget constraint is given by the following equation

$$p_1 x_1 + p_2 x_2 = m,$$

and the initial values $p_1 = 0.50$, $p_2 = 1$, $m = 10$ and the new price of the good 1 is $p_1' = 0.40$. Using the Cobb-Douglas optimal choice bundles (Chapter 5) and the given data ($\alpha = 0.5$, $m = 10$, $p_1 = 0.50$, $p_2 = 1$):

$$x_1 = \frac{\alpha \cdot m}{p_1} = \frac{0.5 \cdot 10}{0.50} = 10, \quad x_2 = \frac{(1-\alpha) \cdot m}{p_2} = \frac{0.5 \cdot 10}{1} = 5.$$

So, the initial bundle is:

$$(x_1, x_2) = (10, 5).$$

6.3.2 Total Effect (Holding Income Constant)

Here, we hold $m = 10$ constant and compute the new demand after the price change ($p_1' = 0.40$):

$$x_1' = \frac{0.5 \cdot 10}{0.40} = 12.5, \quad x_2' = \frac{0.5 \cdot 10}{1} = 5.$$

The new bundle is $(x_1', x_2') = (12.5, 5)$ and the total effect:

$$\Delta x_1 = x_1' - x_1 = 12.5 - 10 = 2.5.$$

The total effect reflects the combined impact of the substitution effect and income effect.

6.3.3 Slutsky Substitution Effect

To compute the Slutsky substitution effect, we adjust income (m') to maintain purchasing power constant (so the consumer can still afford the original bundle).

1. Compute adjusted income:

$$m' = m + x_1 \cdot \Delta p_1, \quad \Delta p_1 = p_1' - p_1 = 0.40 - 0.50 = -0.10.$$

$$m' = 10 + 10 \cdot (-0.10) = 10 - 1 = 9.$$

2. With $m' = 9$ and $p_1' = 0.40$, compute the new demand:

$$x_1^s = \frac{0.5 \cdot 9}{0.40} = 11.25, \quad x_2^s = \frac{0.5 \cdot 9}{1} = 4.5.$$

   The substitution bundle is $(x_1^s, x_2^s) = (11.25, 4.5)$ and Slutsky substitution effect:

$$\Delta x_1^s = x_1^s - x_1 = 11.25 - 10 = 1.25.$$

6.3.4 Hicks Substitution Effect

To compute the Hicks substitution effect, we adjust income so the consumer remains on the same indifference curve as the original bundle. First, we need to compute the original utility:

$$U(x_1, x_2) = x_1^{0.5} x_2^{0.5} \implies U(10, 5) = 10^{0.5} \cdot 5^{0.5} = \sqrt{50}.$$

Then, we solve for the new bundle on the same indifference curve using $U = \sqrt{50}$ and the new price $p_1' = 0.40$:

$$U(x_1, x_2) = x_1^{0.5} x_2^{0.5} = \sqrt{50} \quad \text{and} \quad 0.4 \cdot x_1 + x_2 = 10 \longleftrightarrow x_2 = 10 - 0.4x_1$$

Substitute x_2 into $U = \sqrt{50}$ and squaring both sides:

$$x_1^{0.5} (10 - 0.40x_1)^{0.5} = \sqrt{50} \quad \longleftrightarrow \quad x_1(10 - 0.4x_1) = 50 \quad \longleftrightarrow \quad 10x_1 - 0.40x_1^2 = 50 \quad \longleftrightarrow \quad x_1^2 - 25x_1 + 125 = 0.$$

Applying the quadratic formula, we obtain:

$$x_1 = \frac{25 \pm \sqrt{125}}{2} = \frac{25 \pm 5\sqrt{5}}{2} \quad \longleftrightarrow \quad x_{11} = \frac{25 + 5\sqrt{5}}{2} \approx 18.09 \quad \text{and} \quad x_{12} = \frac{25 - 5\sqrt{5}}{2} \approx 6.91.$$

The Hicks substitution effect:

- **Case 1:** $x_1^H = 18.09$

$$\Delta x_1^H = x_1^H - x_1 = 18.09 - 10 = 8.09.$$

  The Hicks substitution effect is positive (+8.09), indicating that the consumer substitutes significantly toward good 1 (candy) when prices change and utility is kept constant.

- **Case 2:** $x_1^H = 6.91$

$$\Delta x_1^H = x_1^H - x_1 = 6.91 - 10 = -3.09.$$

  TheHicks substitution effect is negative (−3.09), indicating that the consumer substitutes away from good 1 (candy) when prices change and utility is kept constant.

Effect	Δx_1	**Explanation**
Holding Income Constant	+2.5	Total effect: observed demand change without adjustments.
Slutsky Substitution Effect	+1.25	Change in demand due to relative price change, holding purchasing power constant.
Hicks Substitution Effect (18.09)	+8.09	Substitution toward good 1, keeping utility constant (solution 1).
Hicks Substitution Effect (6.91)	-3.09	Substitution away from good 1, keeping utility constant (solution 2).