Course 2023-2024 in Portfolio Allocation and Asset Management
Lecture Notes + Tutorial Exercises

Thierry Roncalli*

*Amundi Asset Management[1]

*University of Paris-Saclay

January 2024

[1]The opinions expressed in this presentation are those of the authors and are not meant to represent the opinions or official positions of Amundi Asset Management.

Part 1. Portfolio Optimization

Part 2. Risk Budgeting

Part 3. Smart Beta, Factor Investing and ARP

Part 4. Equity Portfolio Optimization with ESG Scores (Exercise)

Part 5. Climate Portfolio Construction

Part 6. Equity and Bond Portfolio Optimization with Green Preferences (Exercise)

Part 7. Machine Learning in Asset Management

General information

1. **Overview**
   The objective of this course is to understand the theoretical and practical aspects of asset management
2. **Prerequisites**
   M1 Finance or equivalent
3. **ECTS**
   3
4. **Keywords**
   Finance, Asset Management, Optimization, Statistics
5. **Hours**
   Lectures: 24h, HomeWork: 30h
6. **Evaluation**
   Project + oral examination
7. **Course website**
   `www.thierry-roncalli.com/AssetManagementCourse.html`

Objective of the course

The objective of the course is twofold:

1. having a financial culture on asset management
2. being proficient in quantitative portfolio management

Class schedule

Course sessions

- January 12 (6 hours, AM+PM)
- January 19 (6 hours, AM+PM)
- January 26 (6 hours, AM+PM)
- February 2 (6 hours, AM+PM)

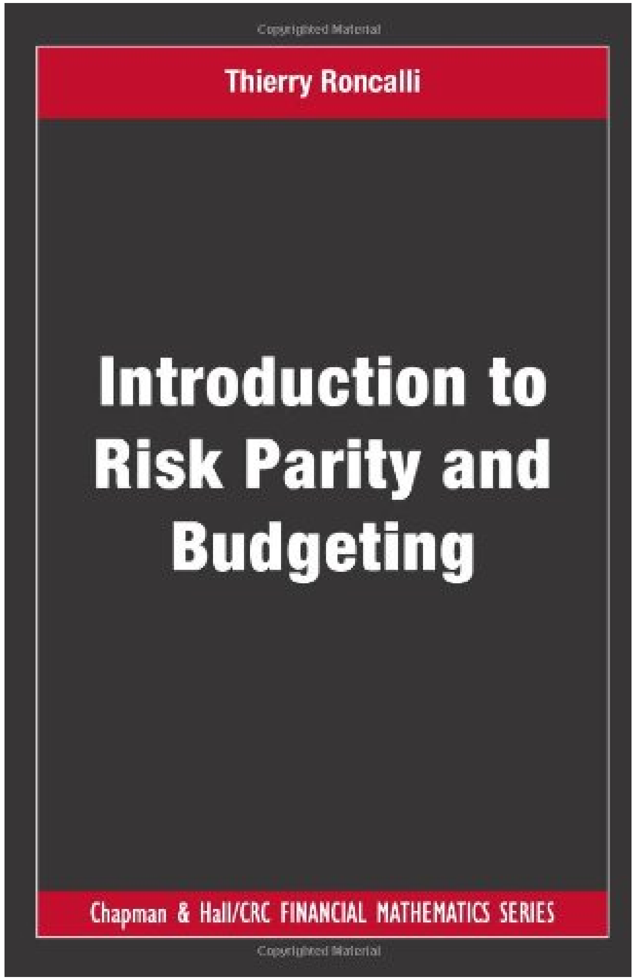
Class times: Fridays 9:00am-12:00pm, 1:00pm–4:00pm, University of Evry

Agenda

- Lecture 1: Portfolio Optimization
- Lecture 2: Risk Budgeting
- Lecture 3: Smart Beta, Factor Investing and Alternative Risk Premia
- Lecture 4: Equity Portfolio Optimization with ESG Scores
- Lecture 5: Climate Portfolio Construction
- Lecture 6: Equity and Bond Portfolio Optimization with Green Preferences
- Lecture 7: Machine Learning in Asset Management

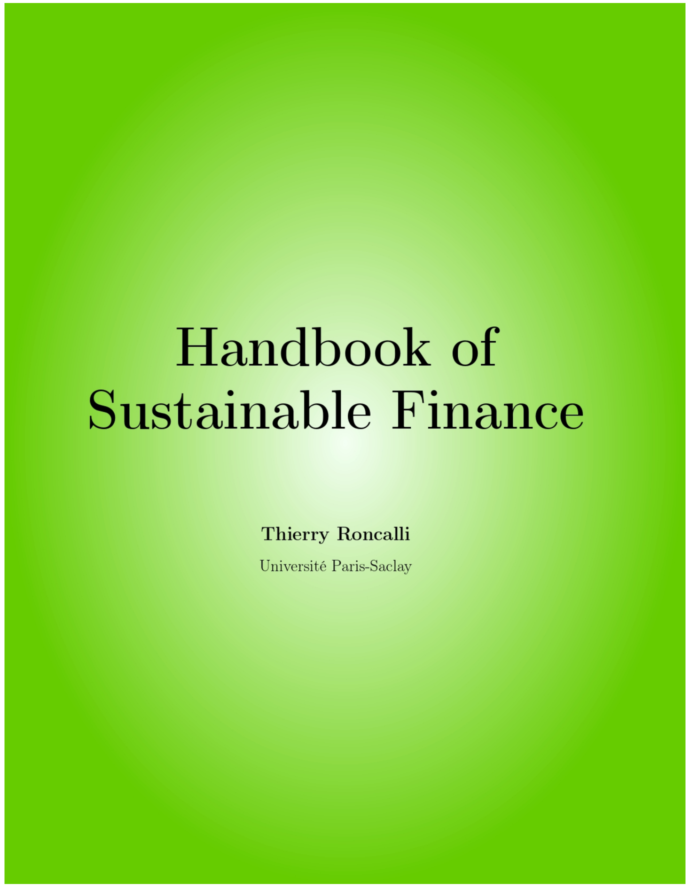
Textbook (Asset Management)

- Roncalli, T. (2013), *Introduction to Risk Parity and Budgeting*, Chapman & Hall/CRC Financial Mathematics Series.



Textbook (Sustainable Finance)

- Roncalli, T. (2024), *Handbook of Sustainable Finance*.



Additional materials

- Slides, tutorial exercises and past exams can be downloaded at the following address:

  `www.thierry-roncalli.com/AssetManagementCourse.html`

- Solutions of exercises can be found in the companion book, which can be downloaded at the following address:

  `http://www.thierry-roncalli.com/RiskParityBook.html`

Course 2023-2024 in Portfolio Allocation and Asset Management
Lecture 1. Portfolio Optimization

Thierry Roncalli*

*Amundi Asset Management[2]

*University of Paris-Saclay

January 2024

[2]The opinions expressed in this presentation are those of the authors and are not meant to represent the opinions or official positions of Amundi Asset Management.

Agenda

- **Lecture 1: Portfolio Optimization**
- Lecture 2: Risk Budgeting
- Lecture 3: Smart Beta, Factor Investing and Alternative Risk Premia
- Lecture 4: Equity Portfolio Optimization with ESG Scores
- Lecture 5: Climate Portfolio Construction
- Lecture 6: Equity and Bond Portfolio Optimization with Green Preferences
- Lecture 7: Machine Learning in Asset Management

Notations

- We consider a universe of n assets
- $x = (x_1, \ldots, x_n)$ is the vector of weights in the portfolio
- The portfolio is fully invested:

$$\sum_{i=1}^{n} x_i = \mathbf{1}_n^\top x = 1$$

- $R = (R_1, \ldots, R_n)$ is the vector of asset returns where R_i is the return of asset i
- The return of the portfolio is equal to:

$$R(x) = \sum_{i=1}^{n} x_i R_i = x^\top R$$

- $\mu = \mathbb{E}[R]$ and $\Sigma = \mathbb{E}\left[(R-\mu)(R-\mu)^\top\right]$ are the vector of expected returns and the covariance matrix of asset returns

Computation of the first two moments

The expected return of the portfolio is:

$$\mu(x) = \mathbb{E}[R(x)] = \mathbb{E}\left[x^\top R\right] = x^\top \mathbb{E}[R] = x^\top \mu$$

whereas its variance is equal to:

$$\begin{aligned}
\sigma^2(x) &= \mathbb{E}\left[(R(x) - \mu(x))(R(x) - \mu(x))^\top\right] \\
&= \mathbb{E}\left[\left(x^\top R - x^\top \mu\right)\left(x^\top R - x^\top \mu\right)^\top\right] \\
&= \mathbb{E}\left[x^\top (R - \mu)(R - \mu)^\top x\right] \\
&= x^\top \mathbb{E}\left[(R - \mu)(R - \mu)^\top\right] x \\
&= x^\top \Sigma x
\end{aligned}$$

Efficient frontier

Two equivalent optimization problems

1. Maximizing the expected return of the portfolio under a volatility constraint (σ-**problem**):

$$\max \mu(x) \quad \text{u.c.} \quad \sigma(x) \leq \sigma^{\star}$$

2. Or minimizing the volatility of the portfolio under a return constraint (μ-**problem**):

$$\min \sigma(x) \quad \text{u.c.} \quad \mu(x) \geq \mu^{\star}$$

Efficient frontier

Example 1

We consider four assets. Their expected returns are equal to 5%, 6%, 8% and 6% while their volatilities are equal to 15%, 20%, 25% and 30%. The correlation matrix of asset returns is given by the following matrix:

$$C = \begin{pmatrix} 1.00 & & & \\ 0.10 & 1.00 & & \\ 0.40 & 0.70 & 1.00 & \\ 0.50 & 0.40 & 0.80 & 1.00 \end{pmatrix}$$

Efficient frontier

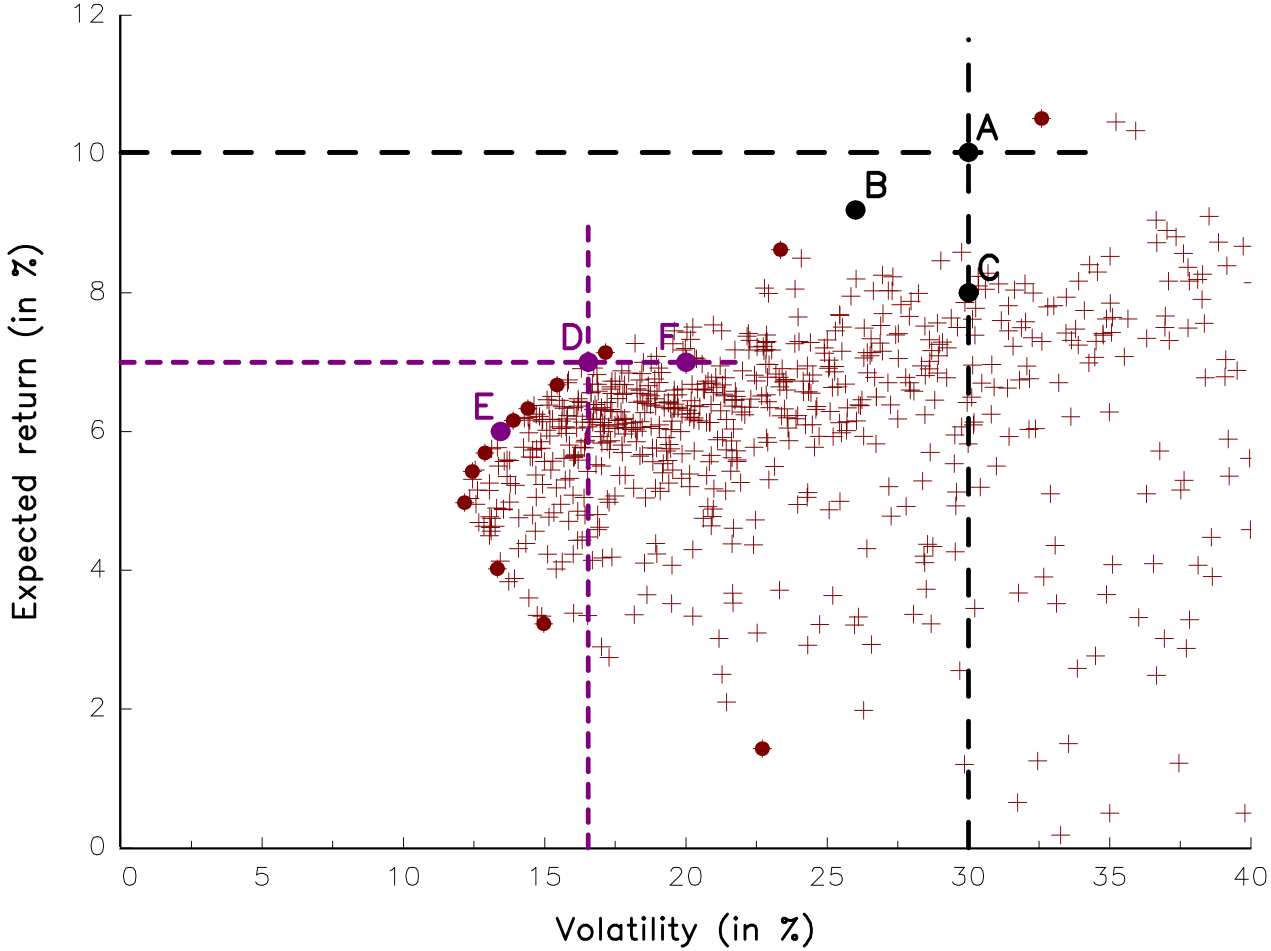


Figure 1: Optimized Markowitz portfolios (1 000 simulations)

Markowitz trick

Markowitz transforms the two original non-linear optimization problems into a quadratic optimization problem:

$$\begin{aligned} x^{\star}(\phi) &= \arg\max x^{\top}\mu - \frac{\phi}{2} x^{\top}\Sigma x \\ & \text{u.c.} \quad \mathbf{1}_n^{\top} x = 1 \end{aligned}$$

where ϕ is a risk-aversion parameter:

- $\phi = 0 \Rightarrow$ we have $\mu(x^{\star}(0)) = \mu^{+}$
- If $\phi = \infty$, the optimization problem becomes:

$$\begin{aligned} x^{\star}(\infty) &= \arg\min \frac{1}{2} x^{\top}\Sigma x \\ & \text{u.c.} \quad \mathbf{1}_n^{\top} x = 1 \end{aligned}$$

$\Rightarrow$ we have $\sigma(x^{\star}(\infty)) = \sigma^{-}$. This is the minimum variance (or MV) portfolio

The γ-problem

The previous problem can also be written as follows:

$$\begin{aligned} x^{\star}(\gamma) &= \arg\min \frac{1}{2} x^{\top} \Sigma x - \gamma x^{\top} \mu \\ \text{u.c.} & \quad \mathbf{1}_n^{\top} x = 1 \end{aligned}$$

with $\gamma = \phi^{-1}$

$\Rightarrow$ This is a standard QP problem

- The minimum variance portfolio corresponds to $\gamma = 0$
- Generally, we use the γ-problem, not the ϕ-problem

Quadratic programming problem

Definition

This is an optimization problem with a quadratic objective function and linear inequality constraints:

$$\begin{aligned} x^{\star} &= \arg\min \frac{1}{2} x^{\top} Q x - x^{\top} R \\ \text{u.c.} &\quad Sx \leq T \end{aligned}$$

where x is a $n \times 1$ vector, Q is a $n \times n$ matrix and R is a $n \times 1$ vector

$\Rightarrow$ $Sx \leq T$ allows specifying linear equality constraints $Ax = B$ ($Ax \geq B$ and $Ax \leq B$) or weight constraints $x^{-} \leq x \leq x^{+}$

Quadratic programming problem

Mathematical softwares consider the following formulation:

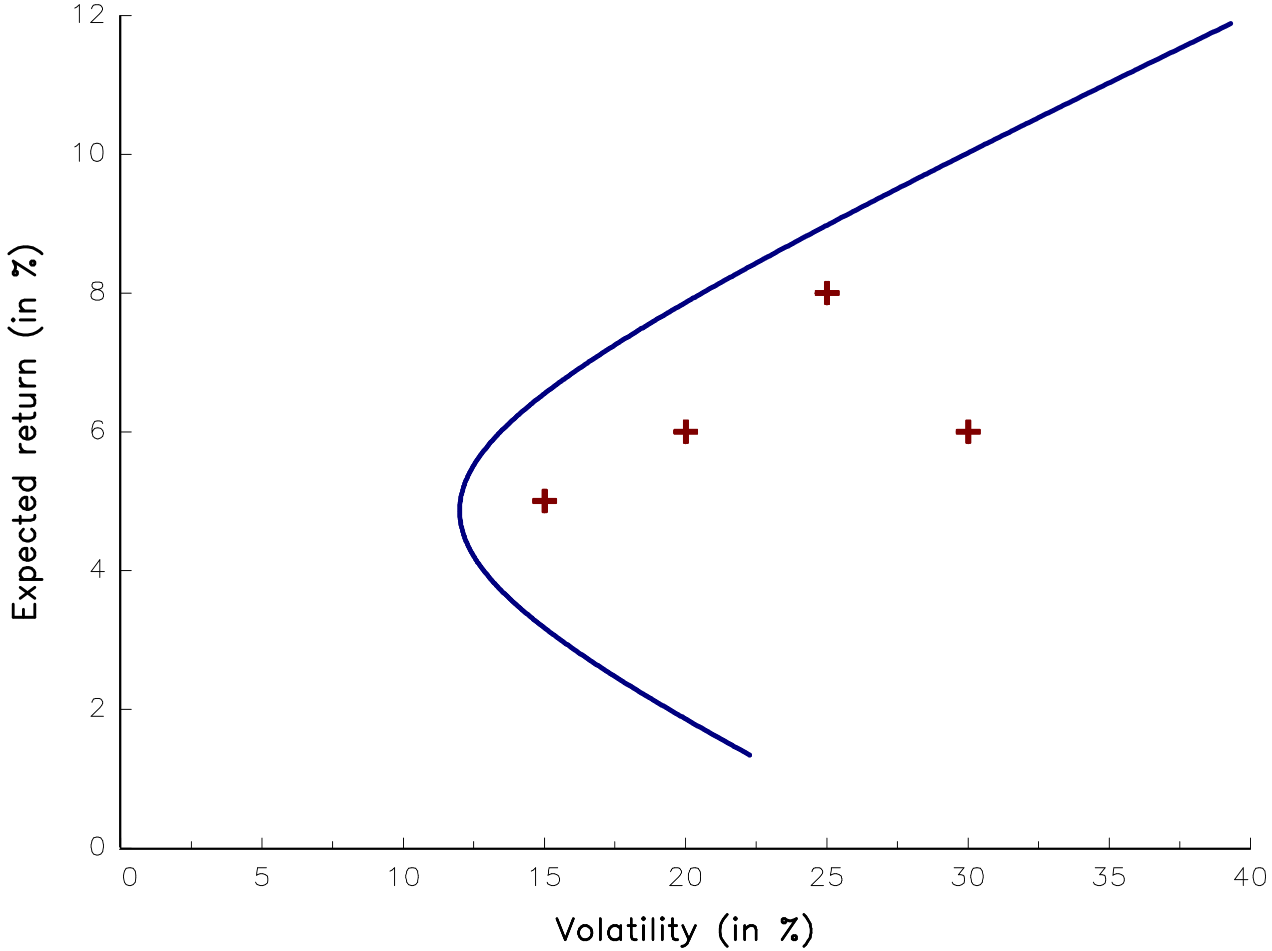
$$
\begin{aligned}
x^{\star} &= \arg\min \frac{1}{2} x^{\top} Q x - x^{\top} R \\
\text{u.c.} & \left\{
\begin{array}{l}
Ax = B \\
Cx \leq D \\
x^{-} \leq x \leq x^{+}
\end{array}
\right.
\end{aligned}
$$

because:

$$
Sx \leq T \Leftrightarrow \begin{bmatrix} -A \\ A \\ C \\ -I_n \\ I_n \end{bmatrix} x \leq \begin{bmatrix} -B \\ B \\ D \\ -x^{-} \\ x^{+} \end{bmatrix}
$$

Efficient frontier

The efficient frontier is the parametric function $(\sigma(x^\star(\phi)), \mu(x^\star(\phi)))$ with $\phi \in \mathbb{R}_+$



Optimized portfolios

Table 1: Solving the ϕ-problem

ϕ	$+\infty$	5.00	2.00	1.00	0.50	0.20
$x_1^\star$	72.74	68.48	62.09	51.44	30.15	−33.75
$x_2^\star$	49.46	35.35	14.17	−21.13	−91.72	−303.49
$x_3^\star$	−20.45	12.61	62.21	144.88	310.22	806.22
$x_4^\star$	−1.75	−16.44	−38.48	−75.20	−148.65	−368.99
$\mu(x^\star)$	4.86	5.57	6.62	8.38	11.90	22.46
$\sigma(x^\star)$	12.00	12.57	15.23	22.27	39.39	94.57

Solving μ- and σ-problems

This is equivalent to finding the optimal value of γ such that:

$$\mu\left(x^{\star}\left(\gamma\right)\right)=\mu^{\star}$$

or:

$$\sigma\left(x^{\star}\left(\gamma\right)\right)=\sigma^{\star}$$

We know that:

- the functions $\mu\left(x^{\star}\left(\gamma\right)\right)$ and $\sigma\left(x^{\star}\left(\gamma\right)\right)$ are increasing with respect to γ
- the functions $\mu\left(x^{\star}\left(\gamma\right)\right)$ and $\sigma\left(x^{\star}\left(\gamma\right)\right)$ are bounded:

$$\begin{aligned}\mu^{-} &\leq \mu\left(x^{\star}\left(\gamma\right)\right)\leq\mu^{+}\\ \sigma^{-} &\leq \sigma\left(x^{\star}\left(\gamma\right)\right)\leq\sigma^{+}\end{aligned}$$

$\Rightarrow$ The optimal value of γ can then be easily computed using the bisection algorithm

Solving μ- and σ-problems

We want to solve $f(\gamma) = c$ where:

- $f(\gamma) = \mu(x^\star(\gamma))$ and $c = \mu^\star$
- or $f(\gamma) = \sigma(x^\star(\gamma))$ and $c = \sigma^\star$

Bisection algorithm

1. We assume that $\gamma^\star \in [\gamma_1, \gamma_2]$
2. If $\gamma_2 - \gamma_1 \leq \varepsilon$, then stop
3. We compute:

$$\bar{\gamma} = \frac{\gamma_1 + \gamma_2}{2}$$

   and $f(\bar{\gamma})$
4. We update γ_1 and γ_2 as follows:
   1. If $f(\bar{\gamma}) < c$, then $\gamma^\star \in [\gamma_c, \gamma_2]$ and $\gamma_1 \leftarrow \gamma_c$
   2. If $f(\bar{\gamma}) > c$, then $\gamma^\star \in [\gamma_1, \gamma_c]$ and $\gamma_2 \leftarrow \gamma_c$
5. Go to Step 2

Solving μ- and σ-problems

Table 2: Solving the unconstrained μ-problem

$\mu^\star$	5.00	6.00	7.00	8.00	9.00
$x_1^\star$	71.92	65.87	59.81	53.76	47.71
$x_2^\star$	46.73	26.67	6.62	−13.44	−33.50
$x_3^\star$	−14.04	32.93	79.91	126.88	173.86
$x_4^\star$	−4.60	−25.47	−46.34	−67.20	−88.07
$\sigma(x^\star)$	12.02	13.44	16.54	20.58	25.10
ϕ	25.79	3.10	1.65	1.12	0.85

Table 3: Solving the unconstrained σ-problem

$\sigma^\star$	15.00	20.00	25.00	30.00	35.00
$x_1^\star$	62.52	54.57	47.84	41.53	35.42
$x_2^\star$	15.58	−10.75	−33.07	−54.00	−74.25
$x_3^\star$	58.92	120.58	172.85	221.88	269.31
$x_4^\star$	−37.01	−64.41	−87.62	−109.40	−130.48
$\mu(x^\star)$	6.55	7.87	8.98	10.02	11.03
ϕ	2.08	1.17	0.86	0.68	0.57

Adding some constraints

We have:

$$\begin{aligned} x^{\star}(\gamma) &= \arg\min \frac{1}{2} x^{\top}\Sigma x - \gamma x^{\top}\mu \\ &\text{u.c.} \quad \left\{ \begin{array}{l} \mathbf{1}_n^{\top} x = 1 \\ x \in \Omega \end{array} \right. \end{aligned}$$

where $x \in \Omega$ corresponds to the set of restrictions

Two classical constraints:

- no short-selling restriction
$$x_i \geq 0$$
- upper bound
$$x_i \leq c$$

Adding some constraints

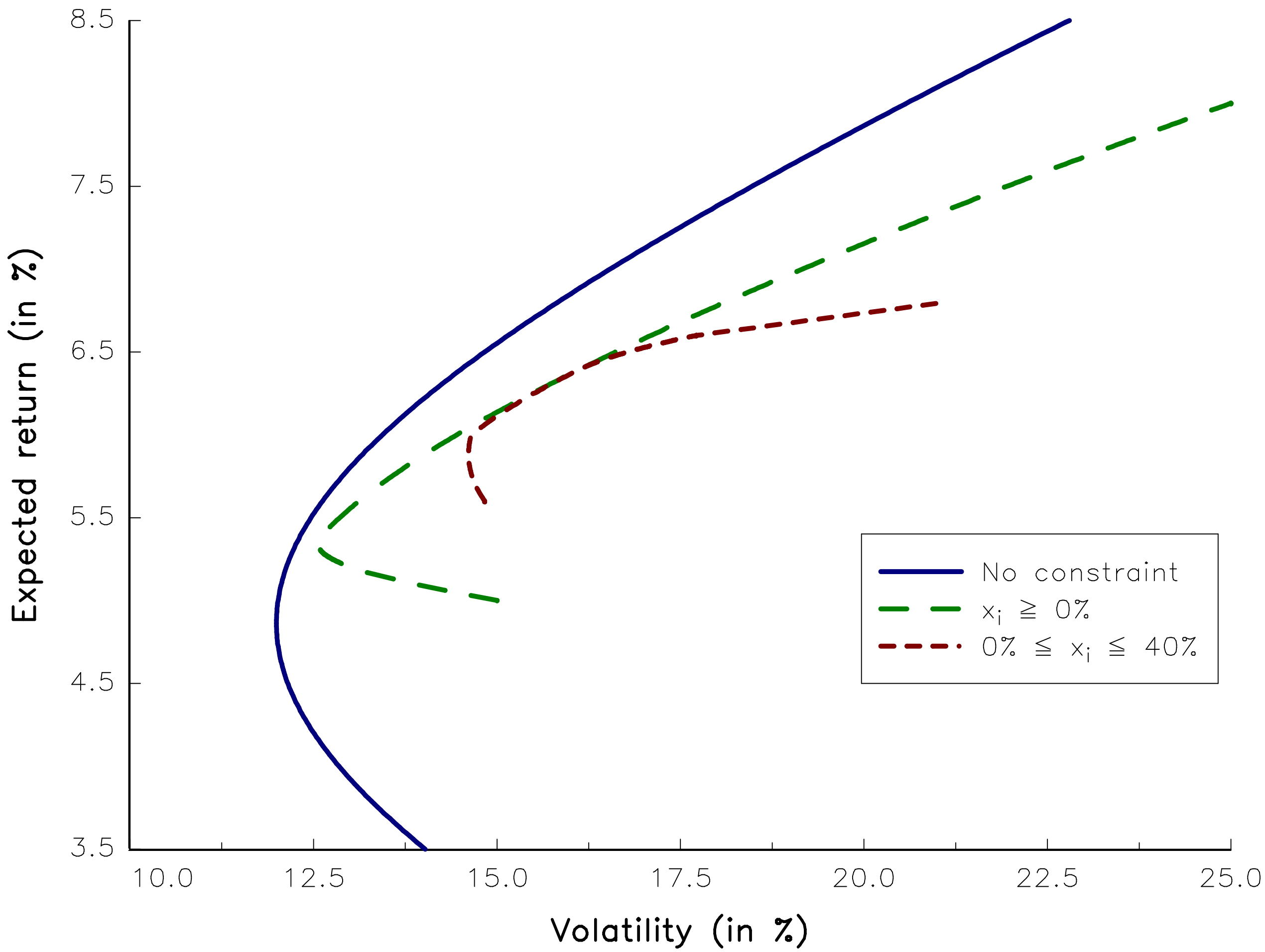


Figure 2: The efficient frontier with some weight constraints

Adding some constraints

Table 4: Solving the σ-problem with weight constraints

	$x_i \in \mathbb{R}$		$x_i \geq 0$		$0 \leq x_i \leq 40\%$	
$\sigma^\star$	15.00	20.00	15.00	20.00	15.00	20.00
$x_1^\star$	62.52	54.57	45.59	24.88	40.00	6.13
$x_2^\star$	15.58	−10.75	24.74	4.96	34.36	40.00
$x_3^\star$	58.92	120.58	29.67	70.15	25.64	40.00
$x_4^\star$	−37.01	−64.41	0.00	0.00	0.00	13.87
$\mu(x^\star)$	6.55	7.87	6.14	7.15	6.11	6.74
ϕ	2.08	1.17	1.61	0.91	1.97	0.28

Analytical solution

The Lagrange function is:

$$\mathcal{L}(x;\lambda_0) = x^\top \mu - \frac{\phi}{2} x^\top \Sigma x + \lambda_0 \left(\mathbf{1}_n^\top x - 1\right)$$

The first-order conditions are:

$$\begin{cases} \partial_x \mathcal{L}(x;\lambda_0) = \mu - \phi \Sigma x + \lambda_0 \mathbf{1}_n = \mathbf{0}_n \\ \partial_{\lambda_0} \mathcal{L}(x;\lambda_0) = \mathbf{1}_n^\top x - 1 = 0 \end{cases}$$

We obtain:

$$x = \phi^{-1}\Sigma^{-1}\left(\mu + \lambda_0 \mathbf{1}_n\right)$$

Because $\mathbf{1}_n^\top x - 1 = 0$, we have:

$$\mathbf{1}_n^\top \phi^{-1}\Sigma^{-1}\mu + \lambda_0 \left(\mathbf{1}_n^\top \phi^{-1}\Sigma^{-1}\mathbf{1}_n\right) = 1$$

It follows that:

$$\lambda_0 = \frac{1 - \mathbf{1}_n^\top \phi^{-1}\Sigma^{-1}\mu}{\mathbf{1}_n^\top \phi^{-1}\Sigma^{-1}\mathbf{1}_n}$$

Analytical solution

The solution is then:

$$x^{\star}(\phi) = \frac{\Sigma^{-1}\mathbf{1}_n}{\mathbf{1}_n^{\top}\Sigma^{-1}\mathbf{1}_n} + \frac{1}{\phi} \cdot \frac{\left(\mathbf{1}_n^{\top}\Sigma^{-1}\mathbf{1}_n\right)\Sigma^{-1}\mu - \left(\mathbf{1}_n^{\top}\Sigma^{-1}\mu\right)\Sigma^{-1}\mathbf{1}_n}{\mathbf{1}_n^{\top}\Sigma^{-1}\mathbf{1}_n}$$

Remark

The global minimum variance portfolio is:

$$x_{\mathrm{mv}} = x^{\star}(\infty) = \frac{\Sigma^{-1}\mathbf{1}_n}{\mathbf{1}_n^{\top}\Sigma^{-1}\mathbf{1}_n}$$

Analytical solution

In the case of no short-selling, the Lagrange function becomes:

$$\mathcal{L}(x; \lambda_0, \lambda) = x^\top \mu - \frac{\phi}{2} x^\top \Sigma x + \lambda_0 \left(\mathbf{1}_n^\top x - 1 \right) + \lambda^\top x$$

where $\lambda = (\lambda_1, \ldots, \lambda_n) \geq \mathbf{0}_n$ is the vector of Lagrange coefficients associated with the constraints $x_i \geq 0$

- The first-order condition is:

$$\mu - \phi \Sigma x + \lambda_0 \mathbf{1} + \lambda = \mathbf{0}_n$$

- The Kuhn-Tucker conditions are:

$$\min(\lambda_i, x_i) = 0$$

The tangency portfolio

Markowitz

There are many optimized portfolios
$\Rightarrow$ there are many optimal portfolios

Tobin

One optimized portfolio dominates all the others if there is a risk-free asset

The tangency portfolio

We consider a combination of the risk-free asset and a portfolio x:

$$R(y) = (1-\alpha)\, r + \alpha R(x)$$

where:

- r is the return of the risk-free asset
- $y = \begin{pmatrix} \alpha x \\ 1-\alpha \end{pmatrix}$ is a vector of dimension $(n+1)$
- $\alpha \geq 0$ is the proportion of the wealth invested in the risky portfolio

It follows that:

$$\mu(y) = (1-\alpha)\, r + \alpha \mu(x) = r + \alpha (\mu(x) - r)$$

and:

$$\sigma^2(y) = \alpha^2 \sigma^2(x)$$

We deduce that:

$$\mu(y) = r + \frac{(\mu(x) - r)}{\sigma(x)} \sigma(y)$$

The tangency portfolio

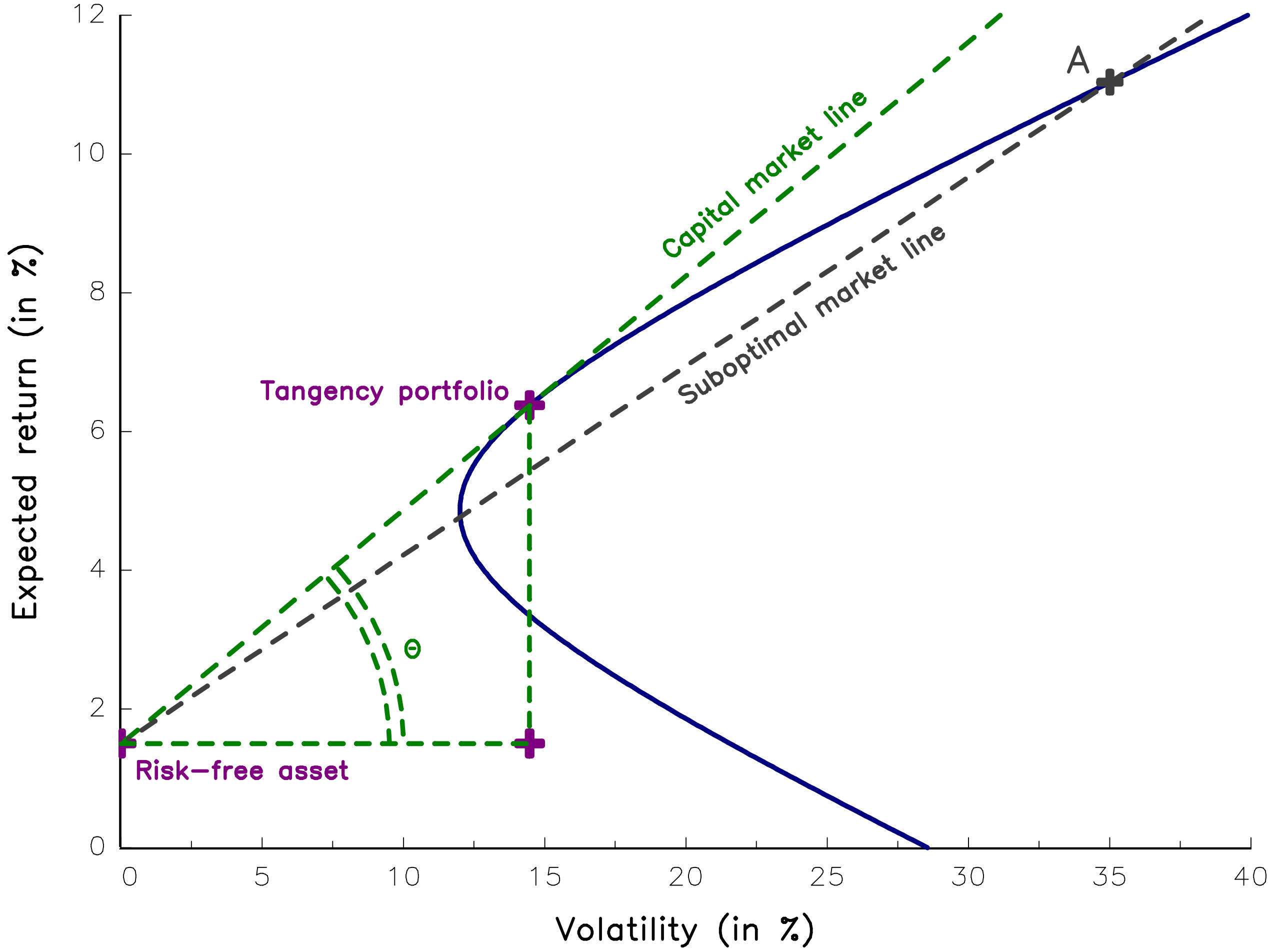


Figure 3: The capital market line ($r = 1.5\%$)

The tangency portfolio

Let $\mathrm{SR}(x \mid r)$ be the Sharpe ratio of portfolio x:

$$\mathrm{SR}(x \mid r) = \frac{\mu(x) - r}{\sigma(x)}$$

We obtain:

$$\frac{\mu(y) - r}{\sigma(y)} = \frac{\mu(x) - r}{\sigma(x)} \Leftrightarrow \mathrm{SR}(y \mid r) = \mathrm{SR}(x \mid r)$$

The tangency portfolio is the one that maximizes the angle θ or equivalently $\tan\theta$:

$$\tan\theta = \mathrm{SR}(x \mid r) = \frac{\mu(x) - r}{\sigma(x)}$$

The tangency portfolio is the risky portfolio corresponding to the maximum Sharpe ratio

The tangency portfolio

Example 2

We consider Example 1 and $r = 1.5\%$

The composition of the tangency portfolio $x^\star$ is:

$$x^\star = \begin{pmatrix} 63.63\% \\ 19.27\% \\ 50.28\% \\ -33.17\% \end{pmatrix}$$

We have:

$$\begin{array}{rcl} \mu\left(x^\star\right) & = & 6.37\% \\ \sigma\left(x^\star\right) & = & 14.43\% \\ \mathrm{SR}\left(x^\star \mid r\right) & = & 0.34 \\ \theta\left(x^\star\right) & = & 18.64 \text{ degrees} \end{array}$$

The tangency portfolio

Let us consider a portfolio x of risky assets and a risk-free asset r. We denote by $\tilde{x}$ the augmented vector of dimension $n+1$ such that:

$$\tilde{x} = \begin{pmatrix} x \\ x_r \end{pmatrix} \quad \text{and} \quad \tilde{\Sigma} = \begin{pmatrix} \Sigma & \mathbf{0}_n \\ \mathbf{0}_n^\top & 0 \end{pmatrix} \quad \text{and} \quad \tilde{\mu} = \begin{pmatrix} \mu \\ r \end{pmatrix}$$

If we include the risk-free asset, the Markowitz γ-problem becomes:

$$\begin{aligned} \tilde{x}^\star(\gamma) &= \arg\min \frac{1}{2}\tilde{x}^\top \tilde{\Sigma}\tilde{x} - \gamma \tilde{x}^\top \tilde{\mu} \\ \text{u.c.} \quad & \mathbf{1}_n^\top \tilde{x} = 1 \end{aligned}$$

Two-fund separation theorem

We can show that (RPB, pages 13-14):

$$\tilde{x}^\star = \underbrace{\alpha \cdot \begin{pmatrix} x_0^\star \\ 0 \end{pmatrix}}_{\text{risky assets}} + \underbrace{(1-\alpha) \cdot \begin{pmatrix} \mathbf{0}_n \\ 1 \end{pmatrix}}_{\text{risk-free asset}}$$

The tangency portfolio

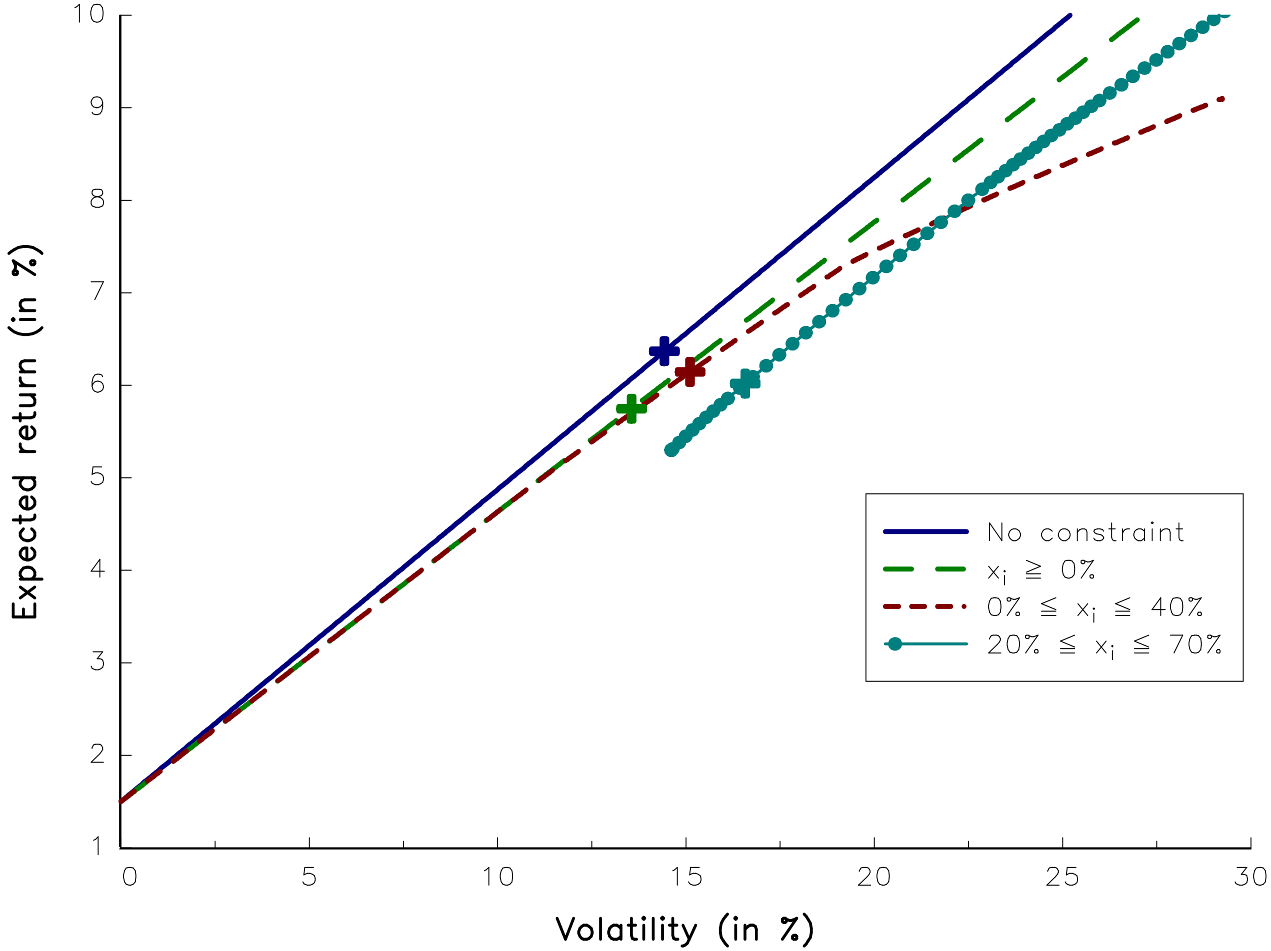


Figure 4: The efficient frontier with a risk-free asset

Market equilibrium and CAPM

- $x^{\star}$ is the tangency portfolio
- On the efficient frontier, we have:

$$\mu(y) = r + \frac{\sigma(y)}{\sigma(x^{\star})}(\mu(x^{\star}) - r)$$

- We consider a portfolio z with a proportion w invested in the asset i and a proportion $(1-w)$ invested in the tangency portfolio $x^{\star}$:

$$\begin{aligned} \mu(z) &= w\mu_i + (1-w)\mu(x^{\star}) \\ \sigma^2(z) &= w^2\sigma_i^2 + (1-w)^2\sigma^2(x^{\star}) + 2w(1-w)\rho(\mathbf{e}_i, x^{\star})\sigma_i\sigma(x^{\star}) \end{aligned}$$

It follows that:

$$\frac{\partial\,\mu(z)}{\partial\,\sigma(z)} = \frac{\mu_i - \mu(x^{\star})}{\left(w\sigma_i^2 + (w-1)\sigma^2(x^{\star}) + (1-2w)\rho(\mathbf{e}_i, x^{\star})\sigma_i\sigma(x^{\star})\right)\sigma^{-1}(z)}$$

Market equilibrium and CAPM

1. When $w = 0$, we have:

$$\frac{\partial \mu(z)}{\partial \sigma(z)} = \frac{\mu_i - \mu(x^\star)}{\left(-\sigma^2(x^\star) + \rho(\mathbf{e}_i, x^\star)\sigma_i\sigma(x^\star)\right)\sigma^{-1}(x^\star)}$$

2. When $w = 0$, the portfolio z is the tangency portfolio $x^\star$ and the previous derivative is equal to the Sharpe ratio $\mathrm{SR}(x^\star \mid r)$

We deduce that:

$$\frac{(\mu_i - \mu(x^\star))\,\sigma(x^\star)}{\rho(\mathbf{e}_i, x^\star)\sigma_i\sigma(x^\star) - \sigma^2(x^\star)} = \frac{\mu(x^\star) - r}{\sigma(x^\star)}$$

which is equivalent to:

$$\pi_i = \mu_i - r = \beta_i(\mu(x^\star) - r)$$

with π_i the risk premium of the asset i and:

$$\beta_i = \frac{\rho(\mathbf{e}_i, x^\star)\sigma_i}{\sigma(x^\star)} = \frac{\mathrm{cov}(R_i, R(x^\star))}{\mathrm{var}(R(x^\star))}$$

Market equilibrium and CAPM

CAPM

The risk premium of the asset i is equal to its beta times the excess return of the tangency portfolio

⇒ We can extend the previous result to the case of a portfolio x (and not only to the asset i):

$$z = wx + (1 - w) x^{\star}$$

In this case, we have:

$$\pi(x) = \mu(x) - r = \beta(x \mid x^{\star})(\mu(x^{\star}) - r)$$

Computation of the beta

The least squares method

- $R_{i,t}$ and $R_t(x)$ be the returns of asset i and portfolio x at time t
- β_i is estimated with the linear regression:

$$R_{i,t} = \alpha_i + \beta_i R_t(x) + \varepsilon_{i,t}$$

- For a portfolio y, we have:

$$R_t(y) = \alpha + \beta R_t(x) + \varepsilon_t$$

Computation of the beta

The covariance method

Another way to compute the beta of portfolio y is to use the following relationship:

$$\beta\left(y \mid x\right) = \frac{\sigma\left(y, x\right)}{\sigma^{2}\left(x\right)} = \frac{y^{\top}\Sigma x}{x^{\top}\Sigma x}$$

We deduce that the expression of the beta of asset i is also:

$$\beta_i = \beta\left(\mathbf{e}_i \mid x\right) = \frac{\mathbf{e}_i^{\top}\Sigma x}{x^{\top}\Sigma x} = \frac{\left(\Sigma x\right)_i}{x^{\top}\Sigma x}$$

The beta of a portfolio is the weighted average of the beta of the assets that compose the portfolio:

$$\beta\left(y \mid x\right) = \frac{y^{\top}\Sigma x}{x^{\top}\Sigma x} = y^{\top}\frac{\Sigma x}{x^{\top}\Sigma x} = \sum_{i=1}^{n} y_i \beta_i$$

Market equilibrium and CAPM

We have $x^{\star} = (63.63\%, 19.27\%, 50.28\%, -33.17\%)$ and $\mu(x^{\star}) = 6.37\%$

Table 5: Computation of the beta and the risk premium (Example 2)

Portfolio y	$\mu(y)$	$\mu(y) - r$	$\beta(y \mid x^{\star})$	$\pi(y \mid x^{\star})$
$\mathbf{e}_1$	5.00	3.50	0.72	3.50
$\mathbf{e}_2$	6.00	4.50	0.92	4.50
$\mathbf{e}_3$	8.00	6.50	1.33	6.50
$\mathbf{e}_4$	6.00	4.50	0.92	4.50
x_{ew}	6.25	4.75	0.98	4.75

Example 2

We consider four assets. Their expected returns are equal to 5%, 6%, 8% and 6% while their volatilities are equal to 15%, 20%, 25% and 30%. The correlation matrix of asset returns is given by the following matrix:

$$C = \begin{pmatrix} 1.00 & & & \\ 0.10 & 1.00 & & \\ 0.40 & 0.70 & 1.00 & \\ 0.50 & 0.40 & 0.80 & 1.00 \end{pmatrix}$$

The risk free rate is equal to $r = 1.5\%$

From active management to passive management

- Active management
- Sharpe (1964)
$$\pi(x) = \beta(x \mid x^\star)\,\pi(x^\star)$$
- Jensen (1969)
$$R_t(x) = \alpha + \beta R_t(b) + \varepsilon_t$$
where $R_t(x)$ is the fund return and $R_t(b)$ is the benchmark return
- Passive management (John McQuown, WFIA, 1971)

Active management = Alpha

Passive management = Beta

Impact of the constraints

If we impose a lower bound $x_i \geq 0$, the tangency portfolio becomes $x^\star = (53.64\%, 32.42\%, 13.93\%, 0.00\%)$ and we have $\mu(x^\star) = 5.74\%$

Table 6: Computation of the beta with a constrained tangency portfolio

Portfolio	$\mu(y) - r$	$\beta(y \mid x^\star)$	$\pi(y \mid x^\star)$
$\mathbf{e}_1$	3.50	0.83	3.50
$\mathbf{e}_2$	4.50	1.06	4.50
$\mathbf{e}_3$	6.50	1.53	6.50
$\mathbf{e}_4$	4.50	1.54	6.53
x_{ew}	4.75	1.24	5.26

$\Rightarrow \mu_4 - r = \beta_4 (\mu(x^\star) - r) + \pi_4^-$ where $\pi_4^- \leq 0$ represents a negative premium due to a lack of arbitrage on the fourth asset

Tracking error

- Portfolio $x = (x_1, \ldots, x_n)$
- Benchmark $b = (b_1, \ldots, b_n)$
- The tracking error between the active portfolio x and its benchmark b is the difference between the return of the portfolio and the return of the benchmark:

$$e = R(x) - R(b) = \sum_{i=1}^{n} x_i R_i - \sum_{i=1}^{n} b_i R_i = x^\top R - b^\top R = (x - b)^\top R$$

- The expected excess return is:

$$\mu(x \mid b) = \mathbb{E}[e] = (x - b)^\top \mu$$

- The volatility of the tracking error is:

$$\sigma(x \mid b) = \sigma(e) = \sqrt{(x - b)^\top \Sigma (x - b)}$$

Markowitz optimization problem

The expected return of the portfolio is replaced by the expected excess return and the volatility of the portfolio is replaced by the volatility of the tracking error

σ-problem

The objective of the investor is to maximize the expected tracking error with a constraint on the tracking error volatility:

$$\begin{array}{rcl} x^{\star} & = & \arg\max \mu\left(x \mid b\right) \\ & \text{u.c.} & \left\{ \begin{array}{l} \mathbf{1}_n^{\top} x = 1 \\ \sigma\left(x \mid b\right) \leq \sigma^{\star} \end{array} \right. \end{array}$$

Equivalent QP problem

We transform the σ-problem into a γ-problem:

$$x^{\star}(\gamma) = \arg\min f(x \mid b)$$

with:

$$\begin{aligned} f(x \mid b) &= \frac{1}{2}(x-b)^{\top}\Sigma(x-b) - \gamma(x-b)^{\top}\mu \\ &= \frac{1}{2}x^{\top}\Sigma x - x^{\top}(\gamma\mu + \Sigma b) + \left(\frac{1}{2}b^{\top}\Sigma b + \gamma b^{\top}\mu\right) \\ &= \frac{1}{2}x^{\top}\Sigma x - x^{\top}(\gamma\mu + \Sigma b) + c \end{aligned}$$

where c is a constant which does not depend on Portfolio x

QP problem with $Q = \Sigma$ and $R = \gamma\mu + \Sigma b$

Remark

The efficient frontier is the parametric curve $(\sigma(x^{\star}(\gamma) \mid b), \mu(x^{\star}(\gamma) \mid b))$ *with* $\gamma \in \mathbb{R}_{+}$

Efficient frontier with a benchmark

Example 3

We consider four assets. Their expected returns are equal to 5%, 6%, 8% and 6% while their volatilities are equal to 15%, 20%, 25% and 30%. The correlation matrix of asset returns is given by the following matrix:

$$C = \begin{pmatrix} 1.00 & & & \\ 0.10 & 1.00 & & \\ 0.40 & 0.70 & 1.00 & \\ 0.50 & 0.40 & 0.80 & 1.00 \end{pmatrix}$$

The benchmark of the portfolio manager is equal to $b = (60\%, 40\%, 20\%, -20\%)$

- 1^{st} case: No constraint
- 2^{nd} case: $x_i^- \leq x_i$ with $x_i^- = -10\%$
- 3^{rd} case: $x_i^- \leq x_i \leq x_i^+$ with $x_1^- = x_2^- = x_3^- = 0\%$, $x_4^- = -20\%$ and $x_i^+ = 50\%$

Efficient frontier with a benchmark

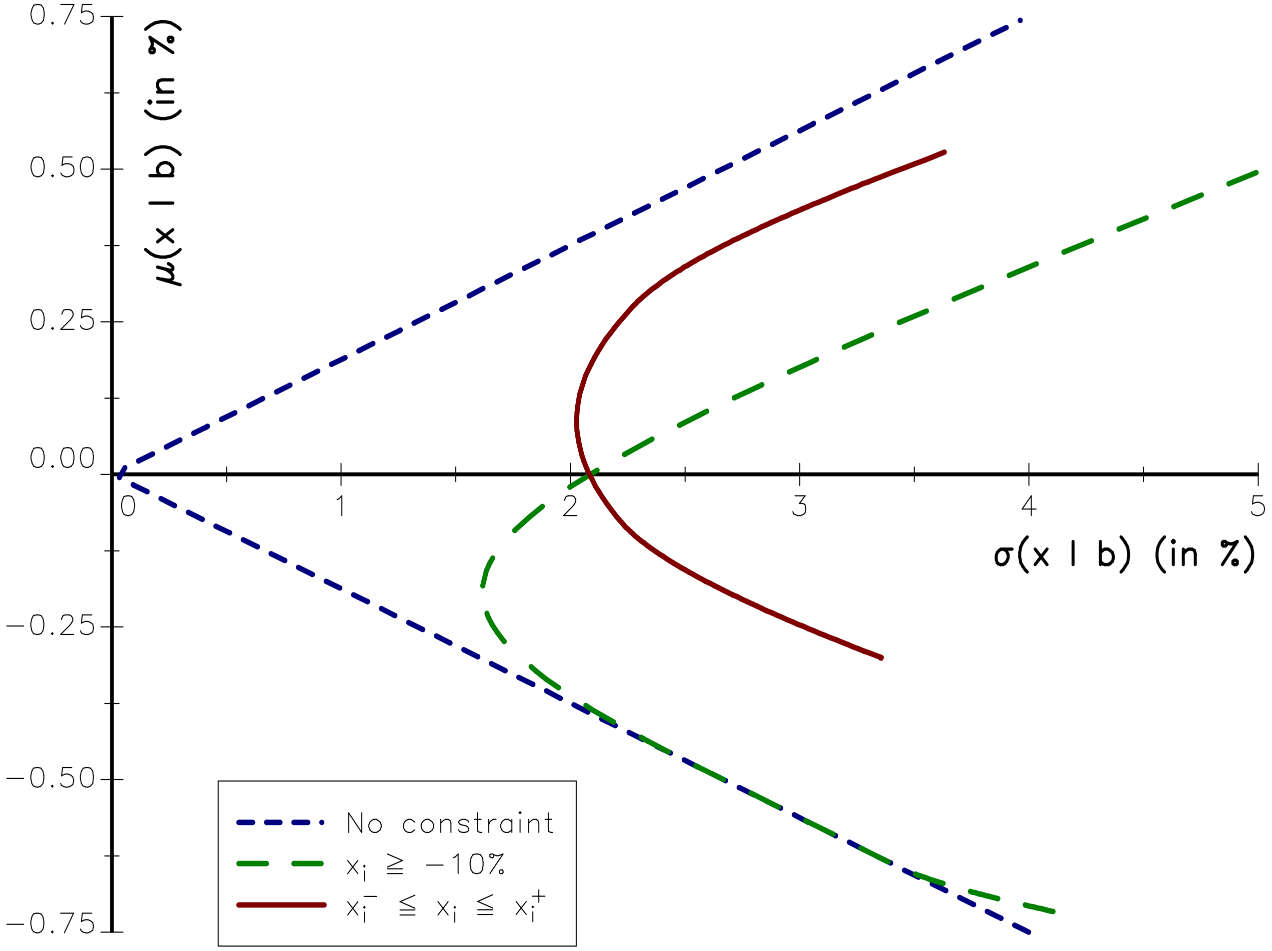


Figure 5: The efficient frontier with a benchmark (Example 3)

Information ratio

Definition

The information ratio is defined as follows:

$$\mathrm{IR}\left(x \mid b\right) = \frac{\mu\left(x \mid b\right)}{\sigma\left(x \mid b\right)} = \frac{\left(x - b\right)^\top \mu}{\sqrt{\left(x - b\right)^\top \Sigma \left(x - b\right)}}$$

Information ratio

If we consider a combination of the benchmark b and the active portfolio x, the composition of the portfolio is:

$$y = (1 - \alpha) b + \alpha x$$

with $\alpha \geq 0$ the proportion of wealth invested in the portfolio x. It follows that:

$$\mu (y \mid b) = (y - b)^\top \mu = \alpha \mu (x \mid b)$$

and:

$$\sigma^2 (y \mid b) = (y - b)^\top \Sigma (y - b) = \alpha^2 \sigma^2 (x \mid b)$$

We deduce that:

$$\mu (y \mid b) = \mathrm{IR} (x \mid b) \cdot \sigma (y \mid b)$$

The efficient frontier is a straight line

Tangency portfolio

If we add some constraints, the portfolio optimization problem becomes:

$$\begin{aligned} x^{\star}(\gamma) &= \arg\min \frac{1}{2} x^{\top} \Sigma x - x^{\top} (\gamma \mu + \Sigma b) \\ & \text{u.c.} \quad \left\{ \begin{array}{l} \mathbf{1}_n^{\top} x = 1 \\ x \in \Omega \end{array} \right. \end{aligned}$$

The efficient frontier is no longer a straight line

Tangency portfolio

One optimized portfolio dominates all the other portfolios. It is the portfolio which belongs to the efficient frontier and the straight line which is tangent to the efficient frontier. It is also the portfolio which maximizes the information ratio

Constrained efficient frontier with a benchmark

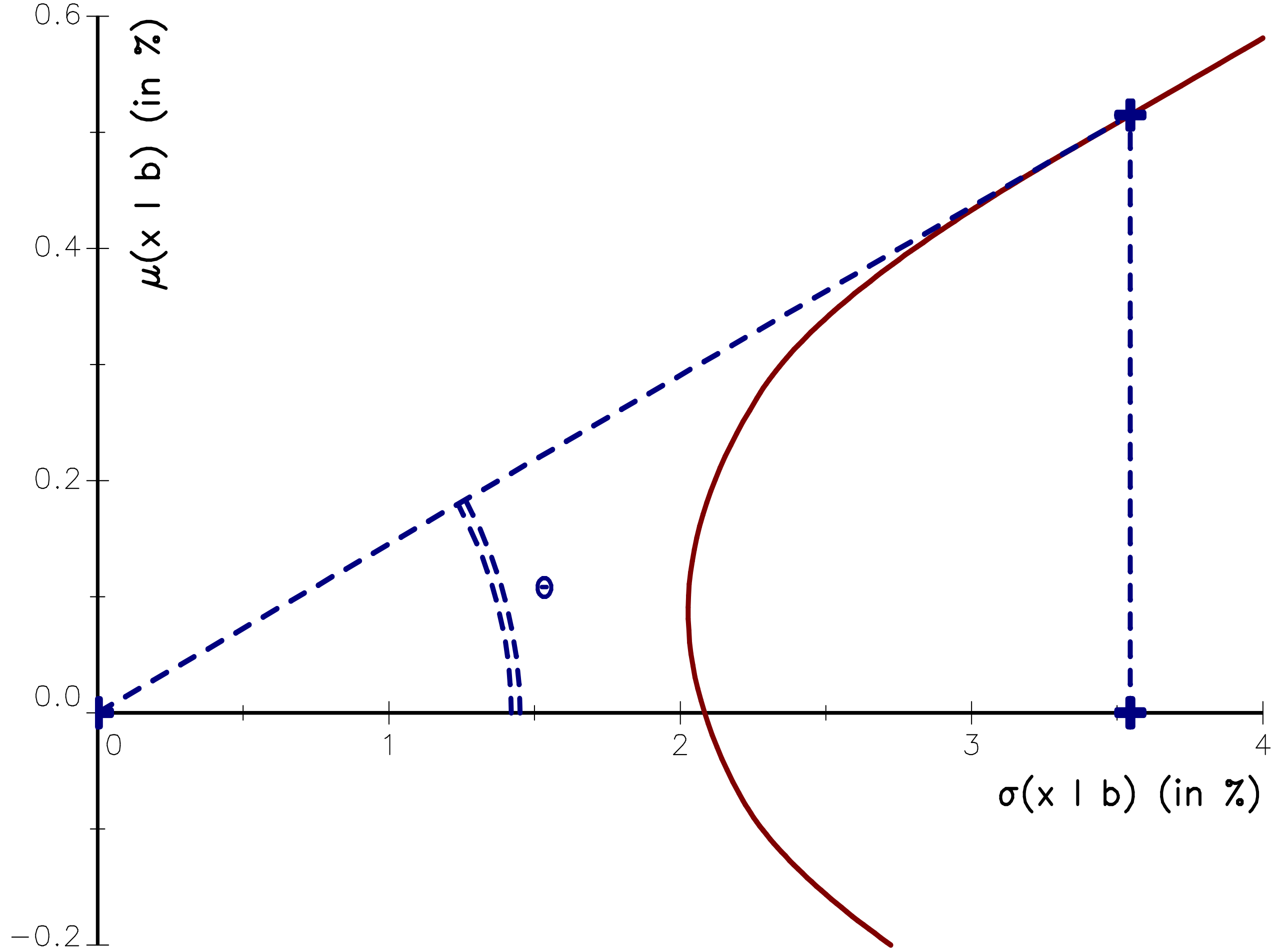


Figure 6: The tangency portfolio with respect to a benchmark (Example 3, 3rd case)

Tangency portfolio

If $x_i^- \leq x_i \leq x_i^+$ with $x_1^- = x_2^- = x_3^- = 0\%$, $x_4^- = -20\%$ and $x_i^+ = 50\%$, the tangency portfolio is equal to:

$$x^\star = \begin{pmatrix} 49.51\% \\ 29.99\% \\ 40.50\% \\ -20.00\% \end{pmatrix}$$

If $r = 1.5\%$, we recall that the MSR (maximum Sharpe ratio) portfolio is equal to:

$$x^\star = \begin{pmatrix} 63.63\% \\ 19.27\% \\ 50.28\% \\ -33.17\% \end{pmatrix}$$

When the benchmark is the risk-free rate

The Markowitz-Tobin-Sharpe approach is obtained when the benchmark is the risk-free asset r. We have:

$$\tilde{x} = \left(\begin{array}{c} x \\ 0 \end{array} \right) \quad \text{and} \quad \tilde{b} = \left(\begin{array}{c} \mathbf{0}_n \\ 1 \end{array} \right)$$

It follows that:

$$\tilde{\Sigma} = \left(\begin{array}{cc} \Sigma & \mathbf{0}_n \\ \mathbf{0}_n^\top & 0 \end{array} \right) \quad \text{and} \quad \tilde{\mu} = \left(\begin{array}{c} \mu \\ r \end{array} \right)$$

When the benchmark is the risk-free rate

The objective function is then defined as follows:

$$\begin{aligned}
f\left(\tilde{x} \mid \tilde{b}\right) &= \frac{1}{2}\left(\tilde{x}-\tilde{b}\right)^\top \Sigma \left(\tilde{x}-\tilde{b}\right) - \gamma \left(\tilde{x}-\tilde{b}\right)^\top \mu \\
&= \frac{1}{2}\tilde{x}^\top \tilde{\Sigma}\tilde{x} - \tilde{x}^\top \left(\gamma\tilde{\mu} + \tilde{\Sigma}\tilde{b}\right) + \left(\frac{1}{2}\tilde{b}^\top \tilde{\Sigma}\tilde{b} + \gamma \tilde{b}^\top \tilde{\mu}\right) \\
&= \frac{1}{2}x^\top \Sigma x - \gamma \left(x^\top \mu - r\right) \\
&= \frac{1}{2}x^\top \Sigma x - \gamma x^\top \left(\mu - r\mathbf{1}_n\right)
\end{aligned}$$

When the benchmark is the risk-free rate

The solution of the QP problem $\tilde{x}^\star(\gamma) = \arg\min f\left(\tilde{x} \mid \tilde{b}\right)$ is related to the solution $x^\star(\gamma)$ of the Markowitz γ-problem in the following way:

$$\tilde{x}^\star(\gamma) = \begin{pmatrix} x^\star(\gamma) \\ 0 \end{pmatrix}$$

We have $\sigma\left(\tilde{x}^\star(\gamma) \mid \tilde{b}\right) = \sigma\left(x^\star(\phi)\right)$

Remark

$\Rightarrow$ *The MSR portfolio is obtained by replacing the vector μ of expected returns by the vector $\mu - r\mathbf{1}_n$ of expected excess returns. We have:*

$$\mathrm{SR}\left(x^\star(\gamma) \mid r\right) = \mathrm{IR}\left(\tilde{x}^\star(\gamma) \mid \tilde{b}\right)$$

Black-Litterman model

Tactical asset allocation (TAA) model

How to incorporate portfolio manager's views in a strategic asset allocation (SAA)?

Two-step approach:

1. Initial allocation $\Rightarrow$ implied risk premia (Sharpe)
2. Portfolio optimization $\Rightarrow$ coherent with the bets of the portfolio manager (Markowitz)

Implied risk premium

$$\begin{aligned} x^{\star} &= \arg\min \frac{1}{2} x^{\top} \Sigma x - \gamma x^{\top} \left(\mu - r \mathbf{1}_n\right) \\ & \text{u.c.} \quad \left\{ \begin{array}{l} \mathbf{1}_n^{\top} x = 1 \\ x \in \Omega \end{array} \right. \end{aligned}$$

If the constraints are satisfied, the first-order condition is:

$$\Sigma x - \gamma \left(\mu - r \mathbf{1}_n\right) = \mathbf{0}_n$$

The solution is:

$$x^{\star} = \gamma \Sigma^{-1} \left(\mu - r \mathbf{1}_n\right)$$

- In the Markowitz model, the unknown variable is the vector x
- If the initial allocation x_0 is given, it must be optimal for the investor, implying that:

$$\tilde{\mu} = r \mathbf{1}_n + \frac{1}{\gamma} \Sigma x_0$$

- $\tilde{\mu}$ is the vector of expected returns which is coherent with x_0

Implied risk premium

We deduce that:

$$\begin{aligned} \tilde{\pi} &= \tilde{\mu} - r \\ &= \frac{1}{\gamma}\Sigma x_0 \end{aligned}$$

The variable $\tilde{\pi}$ is:

- the *risk premium priced* by the portfolio manager
- the '*implied risk premium*' of the portfolio manager
- the '*market risk premium*' when x_0 is the market portfolio

Implied risk aversion

The computation of $\tilde{\mu}$ needs to the value of the parameter γ or the risk aversion $\phi = \gamma^{-1}$

Since we have $\Sigma x_0 - \gamma (\tilde{\mu} - r\mathbf{1}_n) = \mathbf{0}_n$, we deduce that:

$$\begin{aligned}
(*) \quad & \Leftrightarrow & \gamma (\tilde{\mu} - r\mathbf{1}_n) = \Sigma x_0 \\
& \Leftrightarrow & \gamma \left(x_0^\top \tilde{\mu} - r x_0^\top \mathbf{1}_n \right) = x_0^\top \Sigma x_0 \\
& \Leftrightarrow & \gamma \left(x_0^\top \tilde{\mu} - r \right) = x_0^\top \Sigma x_0 \\
& \Leftrightarrow & \gamma = \frac{x_0^\top \Sigma x_0}{x_0^\top \tilde{\mu} - r}
\end{aligned}$$

It follows that

$$\phi = \frac{x_0^\top \tilde{\mu} - r}{x_0^\top \Sigma x_0} = \frac{\mathrm{SR}(x_0 \mid r)}{\sqrt{x_0^\top \Sigma x_0}} = \frac{\mathrm{SR}(x_0 \mid r)}{\sigma(x_0)}$$

where $\mathrm{SR}(x_0 \mid r)$ is the portfolio's expected Sharpe ratio

Implied risk aversion

We have:

$$\tilde{\mu} = r + \mathrm{SR}\left(x_0 \mid r\right) \frac{\Sigma x_0}{\sqrt{x_0^\top \Sigma x_0}}$$

and:

$$\tilde{\pi} = \mathrm{SR}\left(x_0 \mid r\right) \frac{\Sigma x_0}{\sqrt{x_0^\top \Sigma x_0}}$$

Implied risk premium

Example 4

We consider Example 1 and we suppose that the initial allocation x_0 is $(40\%, 30\%, 20\%, 10\%)$

- The volatility of the portfolio is equal to:

$$\sigma(x_0) = 15.35\%$$

- The objective of the portfolio manager is to target a Sharpe ratio equal to 0.25
- We obtain $\phi = 1.63$
- If $r = 3\%$, the implied expected returns are:

$$\tilde{\mu} = \begin{pmatrix} 5.47\% \\ 6.68\% \\ 8.70\% \\ 9.06\% \end{pmatrix}$$

Specification of the bets

Black and Litterman assume that μ is a Gaussian vector with expected returns $\tilde{\mu}$ and covariance matrix Γ:

$$\mu \sim \mathcal{N}\left(\tilde{\mu}, \Gamma\right)$$

The portfolio manager's views are given by this relationship:

$$P\mu = Q + \varepsilon$$

where P is a $(k \times n)$ matrix, Q is a $(k \times 1)$ vector and $\varepsilon \sim \mathcal{N}\left(0, \Omega\right)$ is a Gaussian vector of dimension k

- If the portfolio manager has two views, the matrix P has two rows $\Rightarrow$ k is then the number of views
- Ω is the covariance matrix of $P\mu - Q$, therefore it measures the uncertainty of the views

Absolute views

- We consider the three-asset case:

$$\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix}$$

- The portfolio manager has an absolute view on the expected return of the first asset:

$$\mu_1 = q_1 + \varepsilon_1$$

We have:

$$P = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \; Q = q_1, \; \varepsilon = \varepsilon_1 \text{ and } \Omega = \omega_1^2$$

If $\omega_1 = 0$, the portfolio manager has a very high level of confidence. If $\omega_1 \neq 0$, his view is uncertain

Absolute views

- The portfolio manager has an absolute view on the expected return of the second asset:

$$\mu_2 = q_2 + \varepsilon_2$$

We have:

$$P = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix},\ Q = q_2,\ \varepsilon = \varepsilon_2 \text{ and } \Omega = \omega_2^2$$

- The portfolio manager has two absolute views:

$$\begin{aligned} \mu_1 &= q_1 + \varepsilon_1 \\ \mu_2 &= q_2 + \varepsilon_2 \end{aligned}$$

We have:

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},\ Q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix},\ \varepsilon = \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix} \text{ and } \Omega = \begin{pmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{pmatrix}$$

Relative views

- The portfolio manager thinks that the outperformance of the first asset with respect to the second asset is q:

$$\mu_1 - \mu_2 = q_{1|2} + \varepsilon_{1|2}$$

We have:

$$P = \begin{pmatrix} 1 & -1 & 0 \end{pmatrix}, \; Q = q_{1|2}, \; \varepsilon = \varepsilon_{1|2} \text{ and } \Omega = \omega_{1|2}^2$$

Portfolio optimization

The Markowitz optimization problem becomes:

$$\begin{aligned} x^{\star}(\gamma) &= \arg\min \frac{1}{2} x^{\top} \Sigma x - \gamma x^{\top} \left(\bar{\mu} - r \mathbf{1}_n\right) \\ \text{u.c.} &\quad \mathbf{1}_n^{\top} x = 1 \end{aligned}$$

where $\bar{\mu}$ is the vector of expected returns conditional to the views:

$$\begin{aligned} \bar{\mu} &= \mathbb{E}\left[\mu \mid \text{views}\right] \\ &= \mathbb{E}\left[\mu \mid P\mu = Q + \varepsilon\right] \\ &= \mathbb{E}\left[\mu \mid P\mu - \varepsilon = Q\right] \end{aligned}$$

To compute $\bar{\mu}$, we consider the random vector:

$$\left(\begin{array}{c} \mu \\ \nu = P\mu - \varepsilon \end{array} \right) \sim \mathcal{N}\left(\left(\begin{array}{c} \tilde{\mu} \\ P\tilde{\mu} \end{array} \right), \left(\begin{array}{cc} \Gamma & \Gamma P^{\top} \\ P\Gamma & P\Gamma P^{\top} + \Omega \end{array} \right) \right)$$

Conditional distribution in the case of the normal distribution

Let us consider a Gaussian random vector defined as follows:

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix}, \begin{pmatrix} \Sigma_{x,x} & \Sigma_{x,y} \\ \Sigma_{y,x} & \Sigma_{y,y} \end{pmatrix}\right)$$

We have:

$$Y \mid X = x \sim \mathcal{N}\left(\mu_{y|x}, \Sigma_{y,y|x}\right)$$

where:

$$\mu_{y|x} = \mathbb{E}\left[Y \mid X = x\right] = \mu_y + \Sigma_{y,x}\Sigma_{x,x}^{-1}\left(x - \mu_x\right)$$

and:

$$\Sigma_{y,y|x} = \operatorname{cov}\left(Y \mid X = x\right) = \Sigma_{y,y} - \Sigma_{y,x}\Sigma_{x,x}^{-1}\Sigma_{x,y}$$

Computation of the conditional expectation

We apply the conditional expectation formula:

$$\begin{aligned}\bar{\mu} &= \mathbb{E}\left[\mu \mid \nu = Q\right] \\ &= \mathbb{E}\left[\mu\right] + \operatorname{cov}\left(\mu, v\right) \operatorname{var}\left(v\right)^{-1} \left(Q - \mathbb{E}\left[v\right]\right) \\ &= \tilde{\mu} + \Gamma P^\top \left(P \Gamma P^\top + \Omega\right)^{-1} \left(Q - P\tilde{\mu}\right)\end{aligned}$$

The conditional expectation $\bar{\mu}$ has two components:

1. The first component corresponds to the vector of implied expected returns $\tilde{\mu}$
2. The second component is a correction term which takes into account the *disequilibrium* $(Q - P\tilde{\mu})$ between the manager views and the market views

Computation of the conditional covariance matrix

The condition covariance matrix is equal to:

$$\begin{aligned}\bar{\Sigma} &= \operatorname{var}\left(\mu \mid \nu = Q\right) \\ &= \Gamma - \Gamma P^\top \left(P \Gamma P^\top + \Omega\right)^{-1} P \Gamma\end{aligned}$$

Another expression is:

$$\begin{aligned}\bar{\Sigma} &= \left(I_n + \Gamma P^\top \Omega^{-1} P\right)^{-1} \Gamma \\ &= \left(\Gamma^{-1} + P^\top \Omega^{-1} P\right)^{-1}\end{aligned}$$

The conditional covariance matrix is a weighted average of the covariance matrix Γ and the covariance matrix Ω of the manager views.

Choice of covariance matrices

Choice of Σ

From a theoretical point of view, we have:

$$\Sigma = \bar{\Sigma} = \left(\Gamma^{-1} + P^\top \Omega^{-1} P\right)^{-1}$$

In practice, we use:

$$\Sigma = \hat{\Sigma}$$

Choice of Γ

We assume that:

$$\Gamma = \tau \Sigma$$

We can also target a tracking error volatility and deduce τ

Numerical implementation of the model

The five-step approach to implement the Black-Litterman model is:

1. We estimate the empirical covariance matrix $\hat{\Sigma}$ and set $\Sigma = \hat{\Sigma}$
2. Given the current portfolio, we compute the implied risk aversion $\phi = \gamma^{-1}$ and we deduce the vector $\tilde{\mu}$ of implied expected returns
3. We specify the views by defining the P, Q and Ω matrices
4. Given a matrix Γ, we compute the conditional expectation $\bar{\mu}$
5. We finally perform the portfolio optimization with $\hat{\Sigma}$, $\bar{\mu}$ and γ

Illustration

- We use Example 4 and impose that the optimized weights are positive
- The portfolio manager has an absolute view on the first asset and a relative view on the second and third assets:

$$P = \left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{array} \right), \; Q = \left(\begin{array}{c} q_1 \\ q_{2-3} \end{array} \right) \quad \text{and} \quad \Omega = \left(\begin{array}{cc} \varpi_1^2 & 0 \\ 0 & \varpi_{2-3}^2 \end{array} \right)$$

- $q_1 = 4\%$, $q_{2-3} = -1\%$, $\varpi_1 = 10\%$ and $\varpi_{2-3} = 5\%$

Illustration

- Case #1: $\tau = 1$
- Case #2: $\tau = 1$ and $q_1 = 7\%$
- Case #3: $\tau = 1$ and $\varpi_1 = \varpi_{2-3} = 20\%$
- Case #4: $\tau = 10\%$
- Case #5: $\tau = 1\%$

Illustration

Table 7: Black-Litterman portfolios

	#0	#1	#2	#3	#4	#5
$x_1^\star$	40.00	33.41	51.16	36.41	38.25	39.77
$x_2^\star$	30.00	51.56	39.91	42.97	42.72	32.60
$x_3^\star$	20.00	5.46	0.00	10.85	9.14	17.65
$x_4^\star$	10.00	9.58	8.93	9.77	9.89	9.98
$\sigma\left(x^\star \mid x_0\right)$	0.00	3.65	3.67	2.19	2.18	0.45

Illustration

To calibrate the parameter τ, we could target a tracking error volatility $\sigma^{\star}$:

- If $\sigma^{\star} = 2\%$, the optimized portfolio is between portfolios $\#4$ $(\sigma(x^{\star} \mid x_0) = 2.18\%)$ and $\#5$ $(\sigma(x^{\star} \mid x_0) = 0.45\%)$
- The optimal value of τ is between 10% and 1%
- Using a bisection algorithm, we obtain $\tau = 5.2\%$

The optimal portfolio is:

$$x^{\star} = \begin{pmatrix} 36.80\% \\ 41.83\% \\ 11.58\% \\ 9.79\% \end{pmatrix}$$

Empirical estimator

We have:

$$\hat{\Sigma} = \frac{1}{T} \sum_{t=1}^{T} \left(R_t - \bar{R} \right) \left(R_t - \bar{R} \right)^\top$$

Asynchronous markets

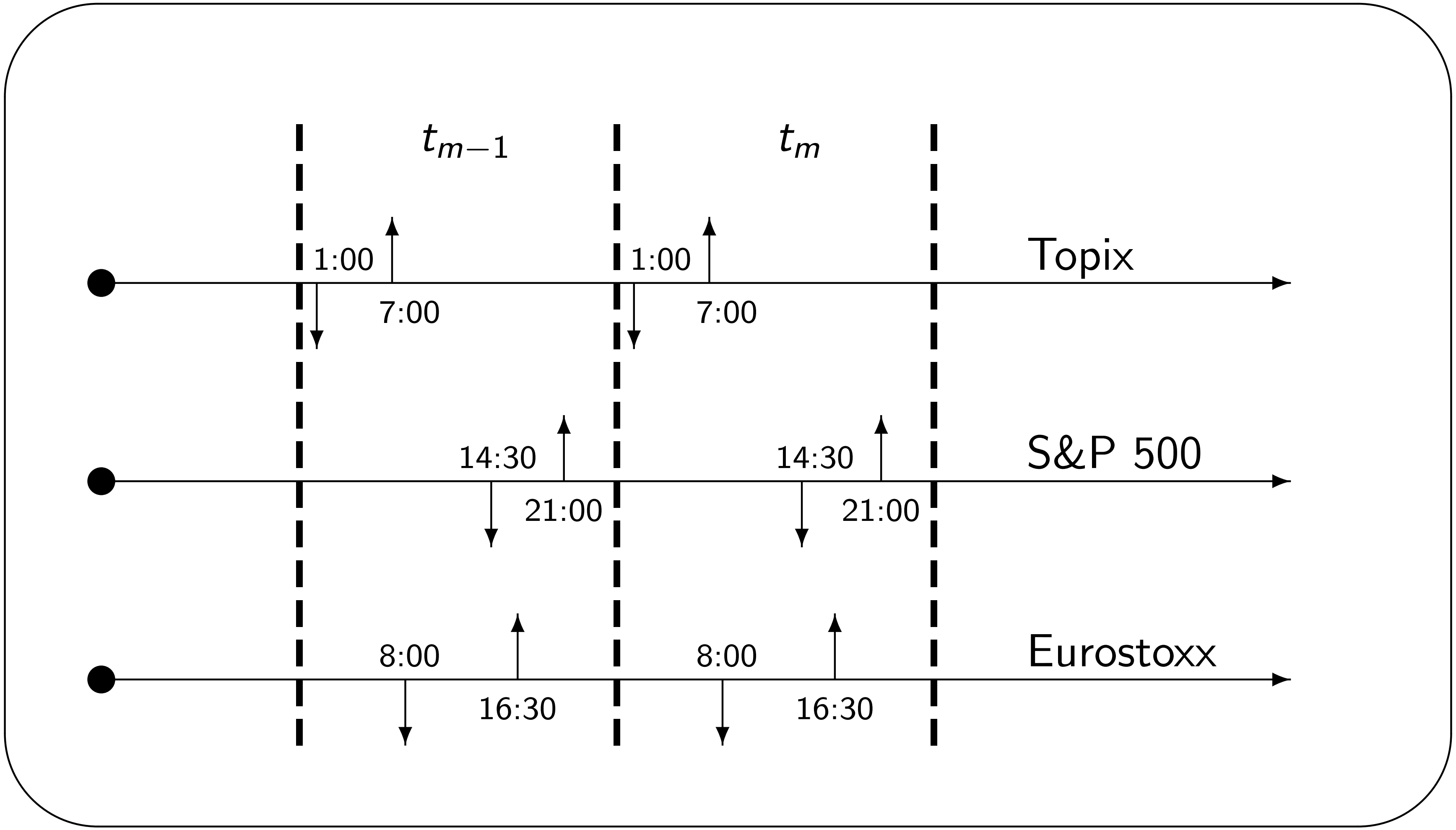


Figure 7: Trading hours of asynchronous markets (UTC time)

Asynchronous markets

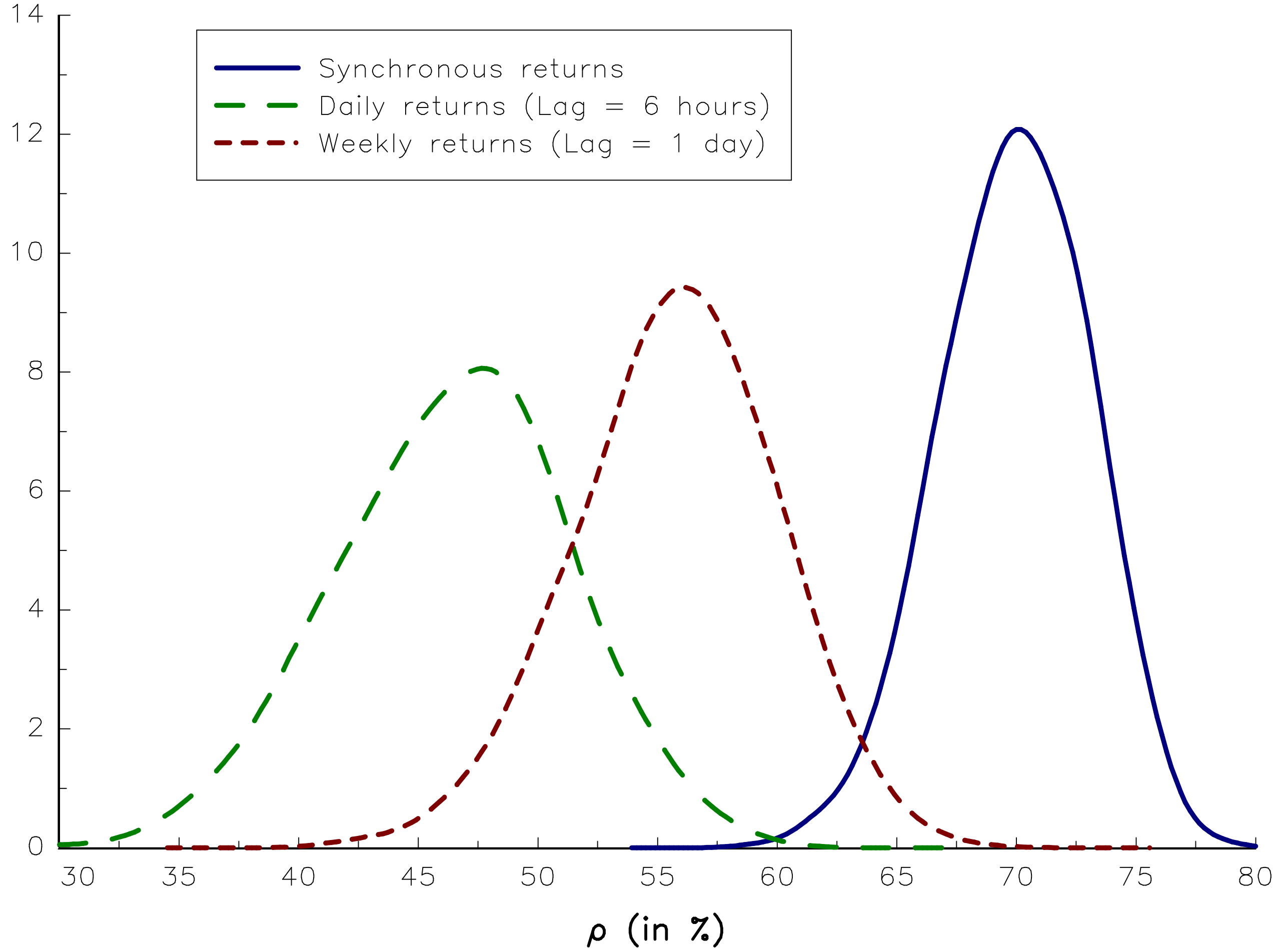


Figure 8: Density of the estimator $\hat{\rho}$ with asynchronous returns ($\rho = 70\%$)

Asynchronous markets

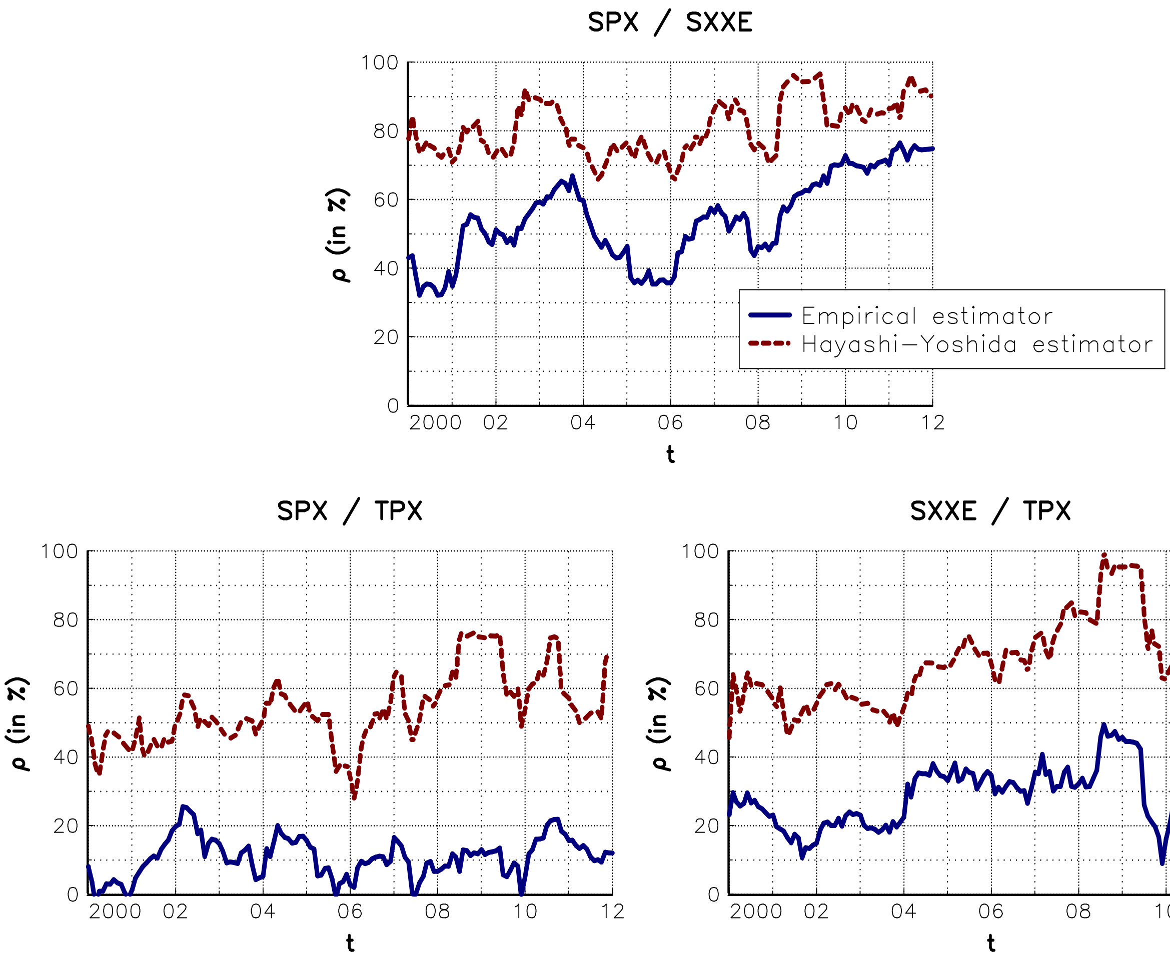


Figure 9: Hayashi-Yoshida estimator

Hayashi-Yoshida estimator

We have:

$$\tilde{\Sigma}_{i,j} = \frac{1}{T}\sum_{t=1}^{T}\left(R_{i,t}-\bar{R}_i\right)\left(R_{j,t}-\bar{R}_j\right)+\frac{1}{T}\sum_{t=1}^{T}\left(R_{i,t}-\bar{R}_i\right)\left(R_{j,t-1}-\bar{R}_j\right)$$

where j is the equity index which has a closing time after the equity index i. In our case, j is necessarily the S&P 500 index whereas i can be the Topix index or the Eurostoxx index. This estimator has two components:

1. The first component is the classical covariance estimator $\hat{\Sigma}_{i,j}$
2. The second component is a correction to take into account the lag between the two closing times

Other statistical methods

- EWMA methods
- GARCH models
- Factor models
  - Uniform correlation
    $$\rho_{i,j} = \rho$$
  - Sector approach (inter-correlation and intra-correlation)
  - Linear factor models:
    $$R_{i,t} = A_i^\top \mathcal{F}_t + \varepsilon_{i,t}$$

Economic/econometric approach

- Market timing (MT)
- Tactical asset allocation (TAA)
- Strategic asset allocation (SAA)

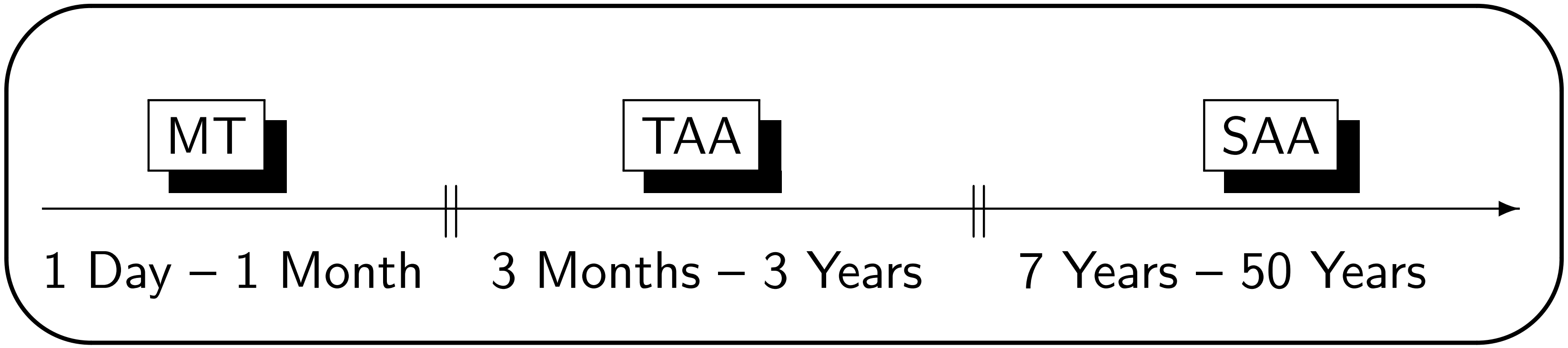


Figure 10: Time horizon of MT, TAA and SAA

Statistical/scoring approach

- Stock picking models: fundamental scoring, value, quality, sector analysis, etc.
- Bond picking models: fundamental scoring, structural model, credit arbitrage model, etc.
- Statistical models: mean-reverting, trend-following, cointegration, etc.
- Machine learning: return forecasting, scoring model, etc.

Stability issues

Example 5

We consider a universe of 3 assets. The parameters are: $\mu_1 = \mu_2 = 8\%$, $\mu_3 = 5\%$, $\sigma_1 = 20\%$, $\sigma_2 = 21\%$, $\sigma_3 = 10\%$ and $\rho_{i,j} = 80\%$. The objective is to maximize the expected return for a 15% volatility target. The optimal portfolio is $(38.3\%, 20.2\%, 41.5\%)$.

Table 8: Sensitivity of the MVO portfolio to input parameters

ρ		70%	90%		90%	
σ_2				18%	18%	
μ_1						9%
x_1	38.3	38.3	44.6	13.7	−8.0	60.6
x_2	20.2	25.9	8.9	56.1	74.1	−5.4
x_3	41.5	35.8	46.5	30.2	34.0	44.8

Solutions

In order to stabilize the optimal portfolio, we have to introduce some regularization techniques:

- Resampling techniques
- Factor analysis
- Shrinkage methods
- Random matrix theory
- Norm penalization
- Etc.

Resampling techniques

- Jacknife
- Cross validation
  - Hold-out
  - K-fold
- Bootstrap
  - Resubstitution
  - Out of the bag
  - .632

Resampling techniques

Example 6

We consider a universe of four assets. The expected returns are $\hat{\mu}_1 = 5\%$, $\hat{\mu}_2 = 9\%$, $\hat{\mu}_3 = 7\%$ and $\hat{\mu}_4 = 6\%$ whereas the volatilities are equal to $\hat{\sigma}_1 = 4\%$, $\hat{\sigma}_2 = 15\%$, $\hat{\sigma}_3 = 5\%$ and $\hat{\sigma}_4 = 10\%$. The correlation matrix is the following:

$$\hat{C} = \begin{pmatrix} 1.00 & & & \\ 0.10 & 1.00 & & \\ 0.40 & 0.20 & 1.00 & \\ -0.10 & -0.10 & -0.20 & 1.00 \end{pmatrix}$$

Resampling techniques

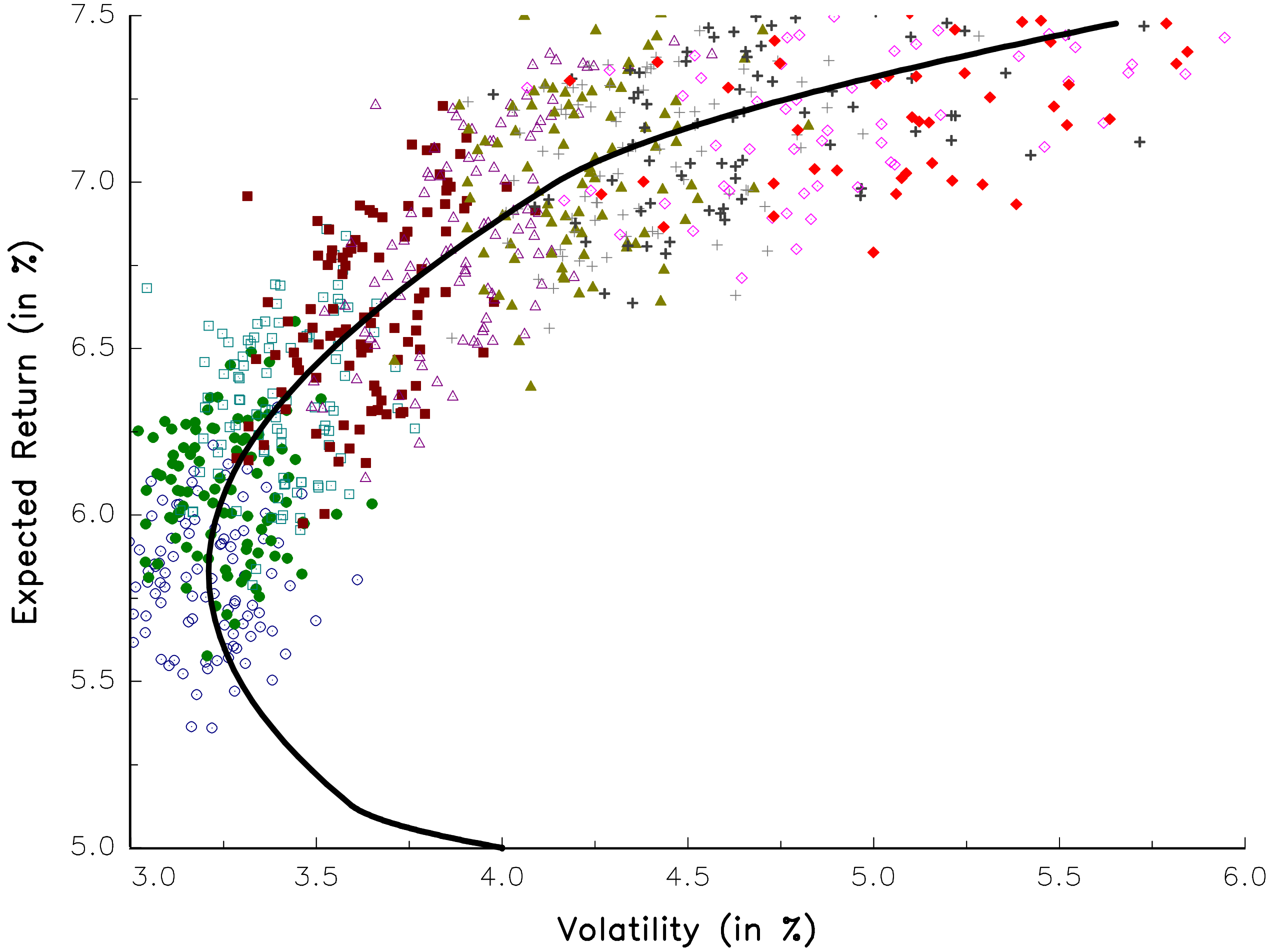


Figure 11: Uncertainty of the efficient frontier

Resampling techniques

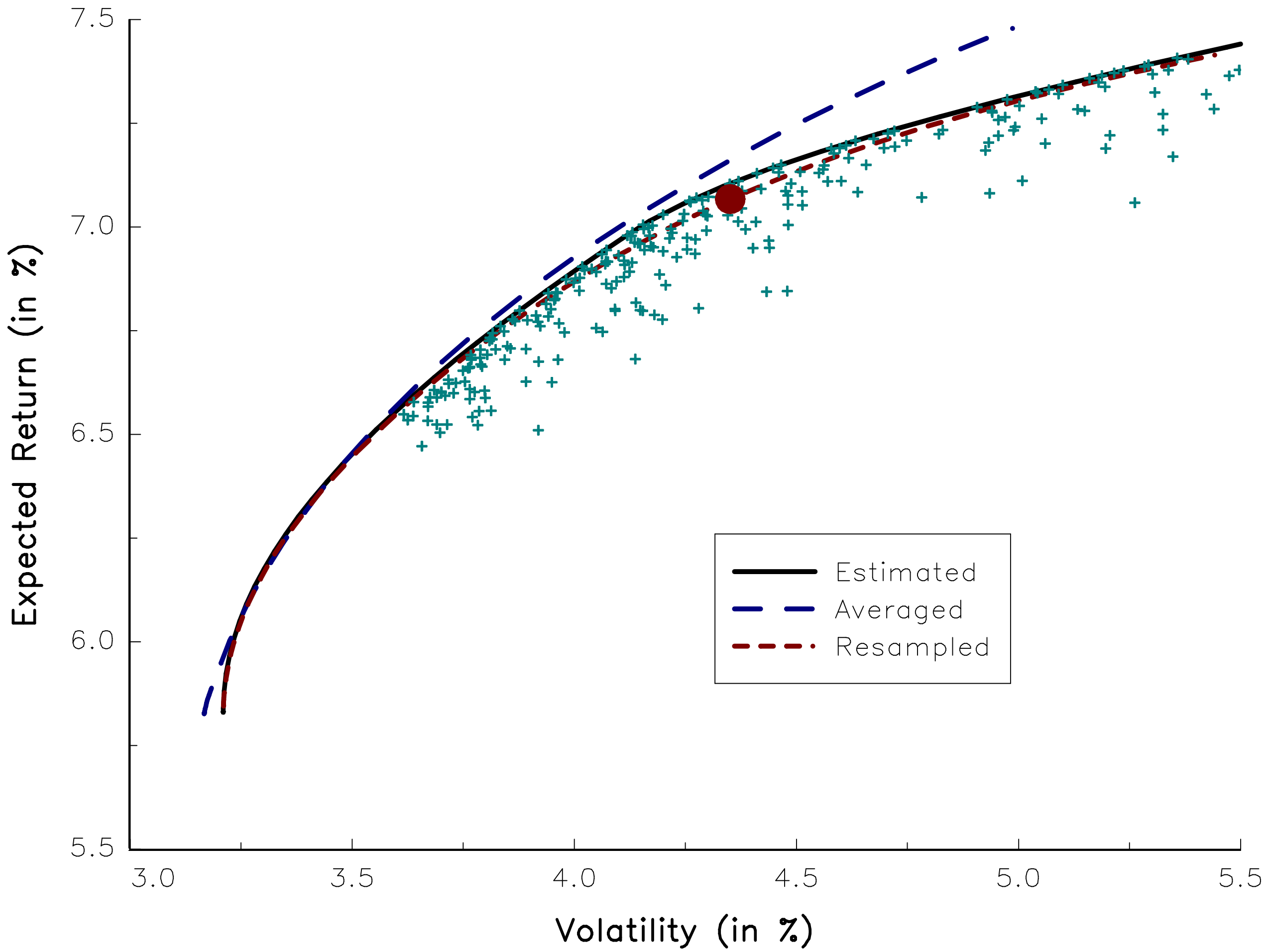


Figure 12: Resampled efficient frontier

Resampling techniques

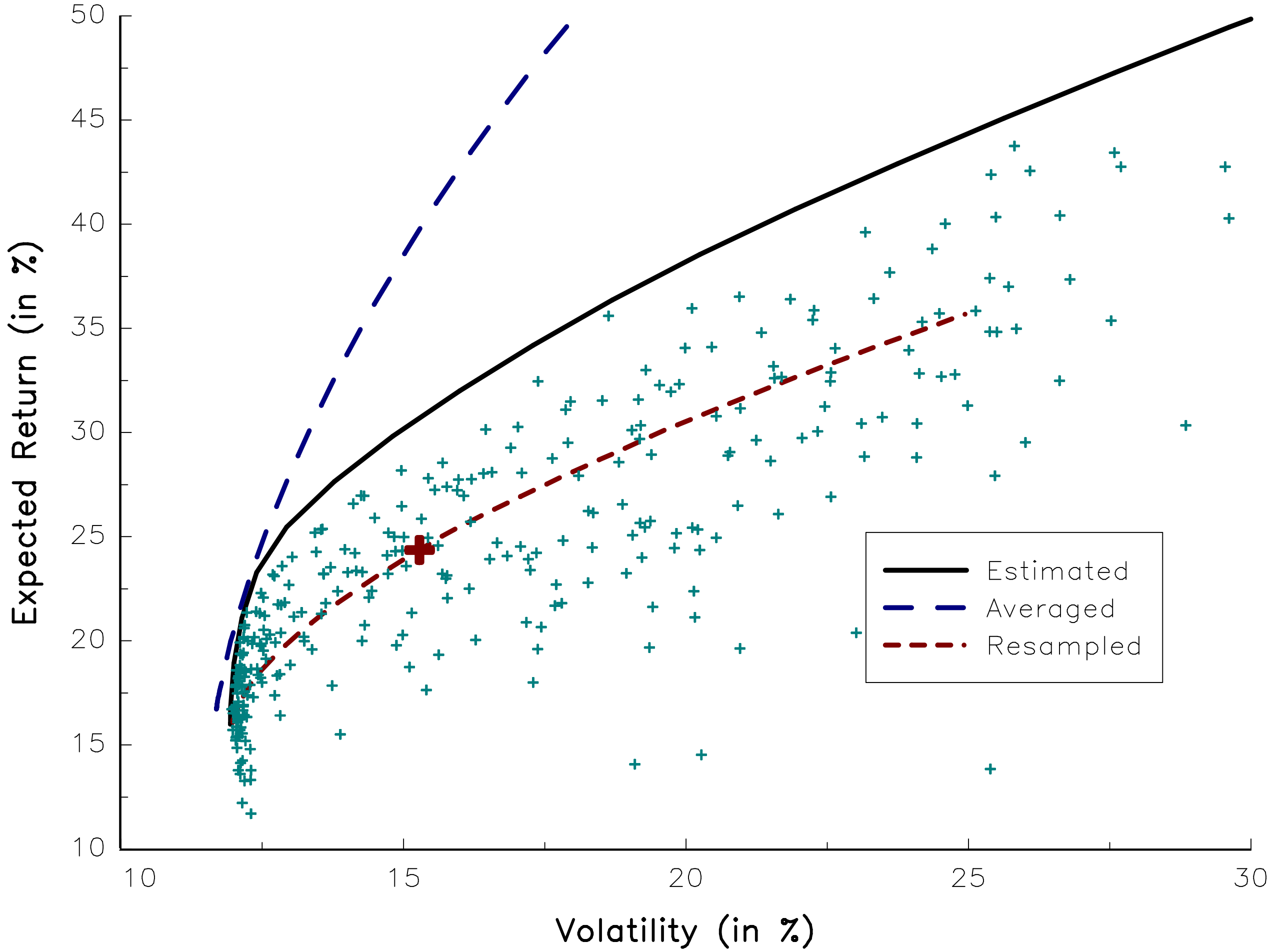


Figure 13: S&P 100 resampled efficient frontier (Bootstrap approach)

Source: Bruder *et al.* (2013)

How to denoise the covariance matrix?

1. Factor analysis by imposing a correlation structure (MSCI Barra)
2. Factor analysis by filtering the correlation structure (APT)
3. Principal component analysis
4. Random matrix theory
5. Shrinkage methods

How to denoise the covariance matrix?

- The eigendecomposition $\hat{\Sigma}$ of is

$$\hat{\Sigma} = V \Lambda V^{\top}$$

where $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ is the diagonal matrix of eigenvalues with $\lambda_1 > \lambda_2 > \ldots > \lambda_n$ and V is an orthonormal matrix

- The endogenous factors are $\mathcal{F}_t = \Lambda^{-1/2} V^{\top} R_t$
- By considering only the m first components, we can build an estimation of Σ with less noise

How to denoise the covariance matrix?

Choice of m

1. We keep factors that explain more than 1/n of asset variance:

$$m = \sup \left\{ i : \lambda_i \geq \left(\lambda_1 + \ldots + \lambda_n\right) / n \right\}$$

2. Laloux *et al.* (1999) propose to use the random matrix theory (RMT)
   1. The maximum eigenvalue of a random matrix M is equal to:

$$\lambda_{\max} = \sigma^2 \left(1 + n/T + 2\sqrt{n/T}\right)$$

      where T is the sample size
   2. We keep the first m factors such that:

$$m = \sup \left\{ i : \lambda_i > \lambda_{\max} \right\}$$

How to denoise the covariance matrix?

Shrinkage methods

- $\hat{\Sigma}$ is an unbiased estimator, but its convergence is very slow
- $\hat{\Phi}$ is a biased estimator that converges more quickly

Ledoit and Wolf (2003) propose to combine $\hat{\Sigma}$ and $\hat{\Phi}$:

$$\hat{\Sigma}_\alpha = \alpha\hat{\Phi} + (1-\alpha)\hat{\Sigma}$$

The value of α is estimated by minimizing a quadratic loss:

$$\alpha^\star = \arg\min \mathbb{E}\left[\left\|\alpha\hat{\Phi} + (1-\alpha)\hat{\Sigma} - \Sigma\right\|^2\right]$$

They find an analytical expression of $\alpha^\star$ when:

- $\hat{\Phi}$ has a constant correlation structure
- $\hat{\Phi}$ corresponds to a factor model or is deduced from PCA

How to denoise the covariance matrix?

Example 7 (equity correlation matrix)

We consider a universe with eight equity indices: S&P 500, Eurostoxx, FTSE 100, Topix, Bovespa, RTS, Nifty and HSI. The study period is January 2005–December 2011 and we use weekly returns.

The empirical correlation matrix is:

$$\hat{C} = \left(\begin{array}{cccc|cccc}
1.00 & & & & & & & \\
0.88 & 1.00 & & & & & & \\
0.88 & 0.94 & 1.00 & & & & & \\
0.64 & 0.68 & 0.65 & 1.00 & & & & \\
\hline
0.77 & 0.76 & 0.78 & 0.61 & 1.00 & & & \\
0.56 & 0.61 & 0.61 & 0.50 & 0.64 & 1.00 & & \\
0.53 & 0.61 & 0.57 & 0.53 & 0.60 & 0.57 & 1.00 & \\
0.64 & 0.68 & 0.67 & 0.68 & 0.68 & 0.60 & 0.66 & 1.00
\end{array}\right)$$

How to denoise the covariance matrix?

- Uniform correlation

$$\hat{\rho} = 66.24\%$$

- One common factor + two specific factors

$$\hat{C} = \left(\begin{array}{cccc|cccc}
1.00 & & & & & & & \\
0.77 & 1.00 & & & & & & \\
0.77 & 0.77 & 1.00 & & & & & \\
0.77 & 0.77 & 0.77 & 1.00 & & & & \\
\hline
0.50 & 0.50 & 0.50 & 0.50 & 1.00 & & & \\
0.50 & 0.50 & 0.50 & 0.50 & 0.59 & 1.00 & & \\
0.50 & 0.50 & 0.50 & 0.50 & 0.59 & 0.59 & 1.00 & \\
0.50 & 0.50 & 0.50 & 0.50 & 0.59 & 0.59 & 0.59 & 1.00
\end{array}\right)$$

How to denoise the covariance matrix?

- Two-linear factor model

$$\hat{C} = \left(\begin{array}{cccc|cccc}
1.00 & & & & & & & \\
0.88 & 1.00 & & & & & & \\
0.88 & 0.94 & 1.00 & & & & & \\
0.63 & 0.67 & 0.66 & 1.00 & & & & \\
\hline
0.73 & 0.78 & 0.78 & 0.63 & 1.00 & & & \\
0.58 & 0.62 & 0.60 & 0.54 & 0.59 & 1.00 & & \\
0.56 & 0.59 & 0.58 & 0.56 & 0.60 & 0.54 & 1.00 & \\
0.64 & 0.68 & 0.66 & 0.65 & 0.69 & 0.62 & 0.67 & 1.00
\end{array}\right)$$

How to denoise the covariance matrix?

- RMT estimation

$$\hat{C} = \left(\begin{array}{cccc|cccc}
1.00 & & & & & & & \\
0.73 & 1.00 & & & & & & \\
0.72 & 0.76 & 1.00 & & & & & \\
0.61 & 0.64 & 0.64 & 1.00 & & & & \\
\hline
0.72 & 0.76 & 0.75 & 0.64 & 1.00 & & & \\
0.71 & 0.75 & 0.74 & 0.63 & 0.74 & 1.00 & & \\
0.63 & 0.66 & 0.65 & 0.56 & 0.66 & 0.65 & 1.00 & \\
0.68 & 0.72 & 0.71 & 0.60 & 0.71 & 0.70 & 0.62 & 1.00
\end{array}\right)$$

How to denoise the covariance matrix?

- Ledoit-Wolf shrinkage estimation (constant correlation matrix)

$$\hat{C} = \left(\begin{array}{cccc|cccc}
1.00 & & & & & & & \\
0.77 & 1.00 & & & & & & \\
0.77 & 0.80 & 1.00 & & & & & \\
0.65 & 0.67 & 0.65 & 1.00 & & & & \\
\hline
0.72 & 0.71 & 0.72 & 0.63 & 1.00 & & & \\
0.61 & 0.64 & 0.63 & 0.58 & 0.65 & 1.00 & & \\
0.60 & 0.64 & 0.62 & 0.60 & 0.63 & 0.62 & 1.00 & \\
0.65 & 0.67 & 0.67 & 0.67 & 0.67 & 0.63 & 0.66 & 1.00
\end{array}\right)$$

- We obtain:

$$\alpha^{\star} = 51.2\%$$

- What does this result become in the case of a multi-asset-class universe?

$$\alpha^{\star} \simeq 0$$

Why standard regularization techniques are not sufficient

Optimized portfolios are solutions of the following quadratic program:

$$
\begin{aligned}
x^{\star}(\gamma) &= \arg\min \frac{1}{2} x^{\top} \Sigma x - \gamma x^{\top} \mu \\
\text{u.c.} & \left\{ \begin{array}{l} \mathbf{1}_n^{\top} x = 1 \\ x \in \mathbb{R}^n \end{array} \right.
\end{aligned}
$$

We have:

$$
x^{\star}(\gamma) = \frac{\Sigma^{-1}\mathbf{1}_n}{\mathbf{1}_n^{\top}\Sigma^{-1}\mathbf{1}_n} + \gamma \cdot \frac{\left(\mathbf{1}_n^{\top}\Sigma^{-1}\mathbf{1}_n\right)\Sigma^{-1}\mu - \left(\mathbf{1}_n^{\top}\Sigma^{-1}\mu\right)\Sigma^{-1}\mathbf{1}_n}{\mathbf{1}_n^{\top}\Sigma^{-1}\mathbf{1}_n}
$$

Why standard regularization techniques are not sufficient

Optimal solutions are of the following form:

$$x^{\star} \propto f\left(\Sigma^{-1}\right)$$

The important quantity is then the precision matrix $\mathcal{I} = \Sigma^{-1}$, not the covariance matrix Σ

Why standard regularization techniques are not sufficient

- For the covariance matrix Σ, we have:

$$\Sigma = V \Lambda V^\top$$

where $V^{-1} = V^\top$ and $\Lambda = (\lambda_1, \ldots, \lambda_n)$ with $\lambda_1 \geq \ldots \geq \lambda_n$ the ordered eigenvalues

- The decomposition for the precisions matrix is

$$\mathcal{I} = U \Delta U^\top$$

- We have:

$$\begin{aligned}\Sigma^{-1} &= \left(V \Lambda V^\top\right)^{-1} \\ &= \left(V^\top\right)^{-1} \Lambda^{-1} V^{-1} \\ &= V \Lambda^{-1} V^\top\end{aligned}$$

- We deduce that $U = V$ and $\delta_i = 1/\lambda_{n-i+1}$

Why standard regularization techniques are not sufficient

Remark

The eigenvectors of the precision matrix are the same as those of the covariance matrix, but the eigenvalues of the precision matrix are the inverse of the eigenvalues of the covariance matrix. This means that the risk factors are the same, but they are in the reverse order

Why standard regularization techniques are not sufficient

Example 8

We consider a universe of 3 assets, where $\mu_1 = \mu_2 = 8\%$, $\mu_3 = 5\%$, $\sigma_1 = 20\%$, $\sigma_2 = 21\%$, $\sigma_3 = 10\%$ and $\rho_{i,j} = 80\%$.

The eigendecomposition of the covariance and precision matrices is:

	Covariance matrix Σ			Information matrix $\mathcal{I}$		
Asset / Factor	1	2	3	1	2	3
1	65.35%	−72.29%	−22.43%	−22.43%	−72.29%	65.35%
2	69.38%	69.06%	−20.43%	−20.43%	69.06%	69.38%
3	30.26%	−2.21%	95.29%	95.29%	−2.21%	30.26%
Eigenvalue	8.31%	0.84%	0.26%	379.97	119.18	12.04
% cumulated	88.29%	97.20%	100.00%	74.33%	97.65%	100.00%

$\Rightarrow$ It means that the first factor of the information matrix corresponds to the last factor of the covariance matrix and that the last factor of the information matrix corresponds to the first factor.

$\Rightarrow$ Optimization on arbitrage risk factors, idiosyncratic risk factors and (certainly) noise factors!

Why standard regularization techniques are not sufficient

Example 9

We consider a universe of 6 assets. The volatilities are respectively equal to 20%, 21%, 17%, 24%, 20% and 16%. For the correlation matrix, we have:

$$\rho = \begin{pmatrix} 1.00 & & & & & \\ 0.40 & 1.00 & & & & \\ 0.40 & 0.40 & 1.00 & & & \\ 0.50 & 0.50 & 0.50 & 1.00 & & \\ 0.50 & 0.50 & 0.50 & 0.60 & 1.00 & \\ 0.50 & 0.50 & 0.50 & 0.60 & 0.60 & 1.00 \end{pmatrix}$$

$\Rightarrow$ We compute the minimum variance (MV) portfolio with a shortsale constraint

Why standard regularization techniques are not sufficient

Table 9: Effect of deleting a PCA factor

$x^\star$	MV	$\lambda_1 = 0$	$\lambda_2 = 0$	$\lambda_3 = 0$	$\lambda_4 = 0$	$\lambda_5 = 0$	$\lambda_6 = 0$
$x_1^\star$	15.29	15.77	20.79	27.98	0.00	13.40	0.00
$x_2^\star$	10.98	16.92	1.46	12.31	0.00	8.86	0.00
$x_3^\star$	34.40	12.68	35.76	28.24	52.73	53.38	2.58
$x_4^\star$	0.00	22.88	0.00	0.00	0.00	0.00	0.00
$x_5^\star$	1.01	17.99	2.42	0.00	15.93	0.00	0.00
$x_6^\star$	38.32	13.76	39.57	31.48	31.34	24.36	97.42

Why standard regularization techniques are not sufficient

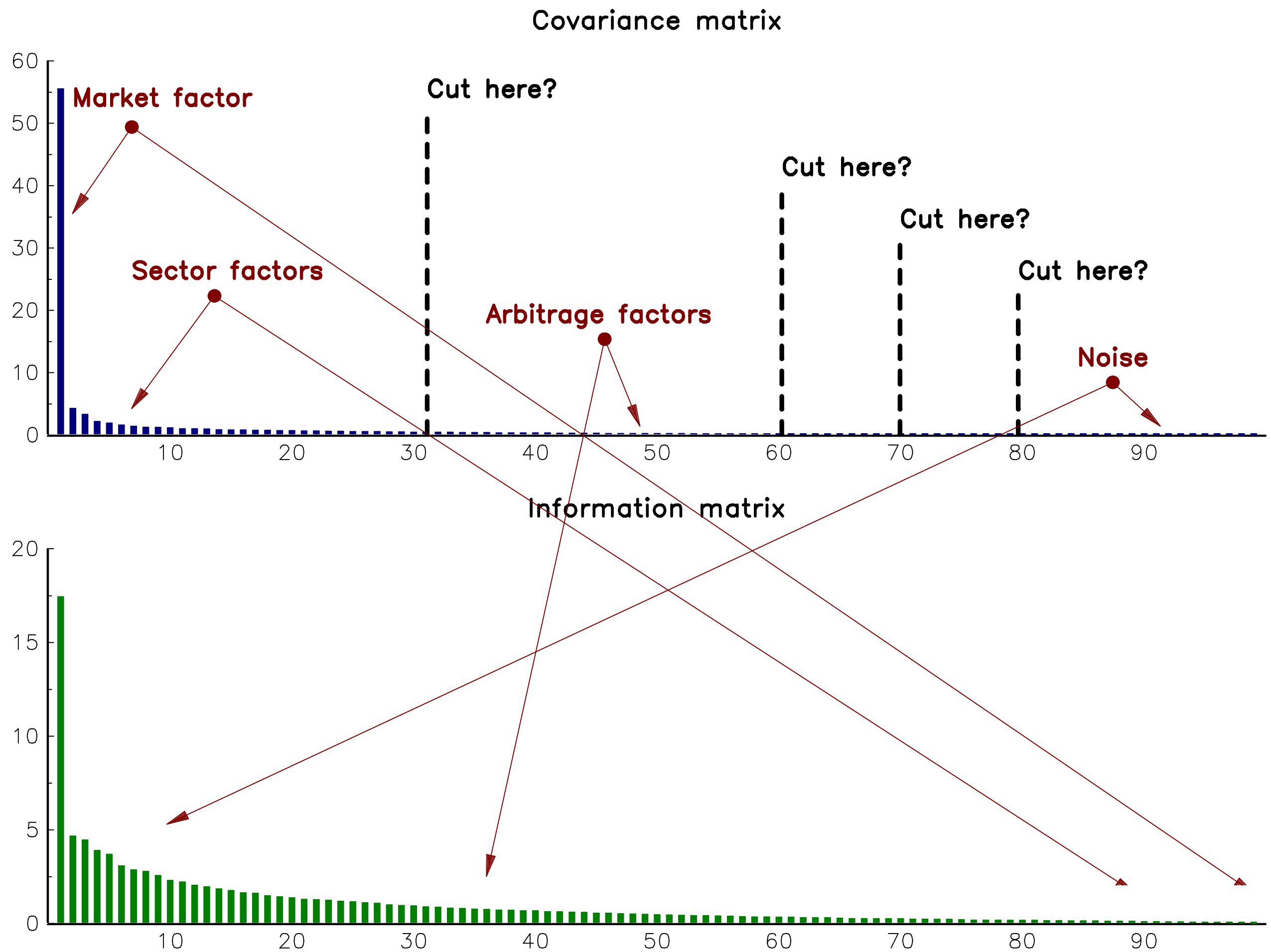


Figure 14: PCA applied to the stocks of the FTSE index (June 2012)

Arbitrage factors, hedging factors or risk factors

We consider the following linear regression model:

$$R_{i,t} = \beta_0 + \beta_i^\top R_t^{(-i)} + \varepsilon_{i,t}$$

- $R_t^{(-i)}$ denotes the vector of asset returns R_t excluding the i^{th} asset
- $\varepsilon_{i,t} \sim \mathcal{N}(0, s_i^2)$
- $\mathcal{R}_i^2$ is the R-squared of the linear regression

Precision matrix

Stevens (1998) shows that the precision matrix is given by:

$$\mathcal{I}_{i,i} = \frac{1}{\hat{\sigma}_i^2 (1 - \mathcal{R}_i^2)} \text{ and } \mathcal{I}_{i,j} = -\frac{\hat{\beta}_{i,j}}{\hat{\sigma}_i^2 (1 - \mathcal{R}_i^2)} = -\frac{\hat{\beta}_{j,i}}{\hat{\sigma}_j^2 \left(1 - \mathcal{R}_j^2\right)}$$

Arbitrage factors, hedging factors or risk factors

Example 10

We consider a universe of four assets. The expected returns are $\hat{\mu}_1 = 7\%$, $\hat{\mu}_2 = 8\%$, $\hat{\mu}_3 = 9\%$ and $\hat{\mu}_4 = 10\%$ whereas the volatilities are equal to $\hat{\sigma}_1 = 15\%$, $\hat{\sigma}_2 = 18\%$, $\hat{\sigma}_3 = 20\%$ and $\hat{\sigma}_4 = 25\%$. The correlation matrix is the following:

$$\hat{C} = \begin{pmatrix} 1.00 & & & \\ 0.50 & 1.00 & & \\ 0.50 & 0.50 & 1.00 & \\ 0.60 & 0.50 & 0.40 & 1.00 \end{pmatrix}$$

We do not impose that the sum of weights are equal to 100%

Arbitrage factors, hedging factors or risk factors

Table 10: Hedging portfolios when $\rho_{3,4} = 40\%$

Asset	$\hat{\beta}_i$				$\mathcal{R}_i^2$	$\hat{s}_i$	$\bar{\mu}_i$	$x^\star$
1		0.139	0.187	0.250	45.83%	11.04%	1.70%	69.80%
2	0.230		0.268	0.191	37.77%	14.20%	2.06%	51.18%
3	0.409	0.354		0.045	33.52%	16.31%	2.85%	53.66%
4	0.750	0.347	0.063		41.50%	19.12%	1.41%	19.28%

Table 11: Hedging portfolios when $\rho_{3,4} = 95\%$

Asset	$\hat{\beta}_i$				$\mathcal{R}_i^2$	$\hat{s}_i$	$\bar{\mu}_i$	$x^\star$
1		0.244	−0.595	0.724	47.41%	10.88%	3.16%	133.45%
2	0.443		0.470	−0.157	33.70%	14.66%	2.23%	52.01%
3	−0.174	0.076		0.795	91.34%	5.89%	1.66%	239.34%
4	0.292	−0.035	1.094		92.38%	6.90%	−1.61%	−168.67%

Arbitrage factors, hedging factors or risk factors

Table 12: Hedging portfolios (in %) at the end of 2006

	SPX	SX5E	TPX	RTY	EM	US HY	EMBI	EUR	JPY	GSCI
SPX		58.6	6.0	150.3	-30.8	-0.5	5.0	-7.3	15.3	-25.5
SX5E	9.0		-1.2	-1.3	35.2	0.8	3.2	-4.5	-5.0	-1.5
TPX	0.4	-0.6		-2.4	38.1	1.1	-3.5	-4.9	-0.8	-0.3
RTY	48.6	-2.7	-10.4		26.2	-0.6	1.9	0.2	-6.4	5.6
EM	-4.1	30.9	69.2	10.9		0.9	4.6	9.1	3.9	33.1
US HY	-5.0	53.5	160.0	-18.8	69.5		95.6	48.4	31.4	-211.7
EMBI	10.8	44.2	-102.1	12.3	73.4	19.4		-5.8	40.5	86.2
EUR	-3.6	-14.7	-33.4	0.3	33.8	2.3	-1.4		56.7	48.2
JPY	6.8	-14.5	-4.8	-8.8	12.7	1.3	8.4	50.4		-33.2
GSCI	-1.1	-0.4	-0.2	0.8	10.7	-0.9	1.8	4.2	-3.3	
$\hat{s}_i$	0.3	0.7	0.9	0.5	0.7	0.1	0.2	0.4	0.4	1.2
$\mathcal{R}_i^2$	83.0	47.7	34.9	82.4	60.9	39.8	51.6	42.3	43.7	12.1

Source: Bruder *et al.* (2013)

Arbitrage factors, hedging factors or risk factors

We finally obtain:

$$x_i^{\star}(\gamma) = \gamma \frac{\mu_i - \hat{\beta}_i^{\top} \mu^{(-i)}}{\hat{s}_i^2}$$

From this equation, we deduce the following conclusions:

1. The better the hedge, the higher the exposure. This is why highly correlated assets produces unstable MVO portfolios
2. The long/short position is defined by the sign of $\mu_i - \hat{\beta}_i^{\top} \mu^{(-i)}$. If the expected return of the asset is lower than the conditional expected return of the hedging portfolio, the weight is negative

Markowitz diversification $\neq$ **Diversification of risk factors**
$=$ **Concentration on arbitrage factors**

QP problem

We use the following formulation of the QP problem:

$$
\begin{aligned}
x^{\star} & = & \arg\min \frac{1}{2} x^{\top} Q x - x^{\top} R \\
& \text{u.c.} & \left\{ \begin{array}{l} Ax = B \\ Cx \leq D \\ x^{-} \leq x \leq x^{+} \end{array} \right.
\end{aligned}
$$

Standard constraints

- γ-problem

$$\arg\min \frac{1}{2} x^\top \Sigma x - \gamma x^\top (\mu - r\mathbf{1}_n) \Rightarrow \left\{ \begin{array}{l} Q = \Sigma \\ R = \gamma\mu \end{array} \right.$$

- Full allocation

$$\mathbf{1}_n^\top x = 1 \Rightarrow \left\{ \begin{array}{l} A = \mathbf{1}_n^\top \\ B = 1 \end{array} \right.$$

- No short selling

$$x_i \geq 0 \Rightarrow x^- = \mathbf{0}_n$$

- Cash neutral (and portfolio optimization with unfunded strategies)

$$\mathbf{1}_n^\top x = 0 \Rightarrow \left\{ \begin{array}{l} A = \mathbf{1}_n^\top \\ B = 0 \end{array} \right.$$

Asset class constraints

Example 11

We consider a multi-asset universe of eight asset classes represented by the following indices:

- four equity indices: S&P 500, Eurostoxx, Topix, MSCI EM
- two bond indices: EGBI, US BIG
- two alternatives indices: GSCI, EPRA

The portfolio manager wants the following exposures:

- at least 50% bonds
- less than 10% commodities
- Emerging market equities cannot represent more than one third of the total exposure on equities

Asset class constraints

The constraints are then expressed as follows:

$$\left\{ \begin{array}{l} x_5 + x_6 \geq 50\% \\ x_7 \leq 10\% \\ x_4 \leq \frac{1}{3}\left(x_1 + x_2 + x_3 + x_4\right) \end{array} \right.$$

The corresponding formulation $Cx \leq D$ of the QP problem is:

$$\left(\begin{array}{rrrrrrrr} 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1/3 & -1/3 & -1/3 & 2/3 & 0 & 0 & 0 & 0 \end{array} \right) \left(\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{array} \right) \leq \left(\begin{array}{r} -0.50 \\ 0.10 \\ 0.00 \end{array} \right)$$

Non-standard constraints (turnover management)

- We want to limit the turnover of the long-only optimized portfolio with respect to a current portfolio x^0:

$$\Omega = \left\{ x \in [0,1]^n : \sum_{i=1}^{n} \left| x_i - x_i^0 \right| \leq \tau^+ \right\}$$

  where τ^+ is the maximum turnover

- Scherer (2007) proposes to introduce some additional variables x_i^- and x_i^+ such that:

$$x_i = x_i^0 + \Delta x_i^+ - \Delta x_i^-$$

  with $\Delta x_i^- \geq 0$ and $\Delta x_i^+ \geq 0$

- Δx_i^+ indicates a positive weight change with respect to the initial weight x_i^0
- Δx_i^- indicates a negative weight change with respect to the initial weight x_i^0

Non-standard constraints (turnover management)

- The expression of the turnover becomes:

$$\sum_{i=1}^{n} \left| x_i - x_i^0 \right| = \sum_{i=1}^{n} \left| \Delta x_i^+ - \Delta x_i^- \right| = \sum_{i=1}^{n} \Delta x_i^+ + \sum_{i=1}^{n} \Delta x_i^-$$

- We obtain the following γ-problem:

$$\begin{aligned} x^\star & = & \arg\min \frac{1}{2} x^\top \Sigma x - \gamma x^\top \mu \\ & & \text{u.c.} \left\{ \begin{array}{l} \sum_{i=1}^n x_i = 1 \\ x_i = x_i^0 + \Delta x_i^+ - \Delta x_i^- \\ \sum_{i=1}^n \Delta x_i^+ + \sum_{i=1}^n \Delta x_i^- \leq \tau^+ \\ 0 \leq x_i \leq 1 \\ 0 \leq \Delta x_i^- \leq 1 \\ 0 \leq \Delta x_i^+ \leq 1 \end{array} \right. \end{aligned}$$

Non-standard constraints (turnover management)

We obtain an augmented QP problem of dimension $3n$ instead of n:

$$\begin{aligned} X^{\star} & = & \arg\min \frac{1}{2} X^{\top} Q X - X^{\top} R \\ & & \text{u.c.} \quad \left\{ \begin{array}{l} AX = B \\ CX \leq D \\ \mathbf{0}_{3n} \leq X \leq \mathbf{1}_{3n} \end{array} \right. \end{aligned}$$

where X is a $3n \times 1$ vector:

$$X = \left(x_1, \ldots, x_n, \Delta x_1^{-}, \ldots, \Delta x_n^{-}, \Delta x_1^{+}, \ldots, \Delta x_n^{+} \right)$$

Non-standard constraints (turnover management)

The augmented QP matrices are:

$$Q_{3n\times 3n} = \begin{pmatrix} \Sigma & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} \\ \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} \\ \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} \end{pmatrix}, \quad R_{3n\times 1} = \begin{pmatrix} \gamma\mu \\ \mathbf{0}_n \\ \mathbf{0}_n \end{pmatrix},$$

$$A_{(n+1)\times 3n} = \begin{pmatrix} \mathbf{1}_n^\top & \mathbf{0}_n^\top & \mathbf{0}_n^\top \\ I_n & I_n & -I_n \end{pmatrix}, \quad B_{(n+1)\times 1} = \begin{pmatrix} 1 \\ x^0 \end{pmatrix},$$

$$C_{1\times 3n} = \begin{pmatrix} \mathbf{0}_n^\top & \mathbf{1}_n^\top & \mathbf{1}_n^\top \end{pmatrix} \quad \text{and} \quad D_{1\times 1} = \tau^+$$

Non-standard constraints (turnover management)

Example 12

We consider four assets. Their expected returns are equal to 5%, 6%, 8% and 6% while their volatilities are equal to 15%, 20%, 25% and 30%. The correlation matrix of asset returns is given by the following matrix:

$$\rho = \begin{pmatrix} 1.00 & & & \\ 0.10 & 1.00 & & \\ 0.40 & 0.70 & 1.00 & \\ 0.50 & 0.40 & 0.80 & 1.00 \end{pmatrix}$$

We impose that the weights are positive

- The optimal portfolio $x^\star$ for a 15% volatility target is $(45.59\%, 24.74\%, 29.67\%, 0.00\%)$
- We assume that the current portfolio x^0 is $(30\%, 45\%, 15\%, 10\%)$
- If we move directly from portfolio x^0 to portfolio $x^\star$, the turnover is equal to 60.53%

Non-standard constraints (turnover management)

Table 13: Limiting the turnover of MVO portfolios

τ^{+}	5.00	10.00	25.00	50.00	75.00	x^{0}
$x_1^{\star}$		35.00	36.40	42.34	45.59	30.00
$x_2^{\star}$		45.00	42.50	30.00	24.74	45.00
$x_3^{\star}$		15.00	21.10	27.66	29.67	15.00
$x_4^{\star}$		5.00	0.00	0.00	0.00	10.00
$\mu\left(x^{\star}\right)$		5.95	6.06	6.13	6.14	6.00
$\sigma\left(x^{\star}\right)$		15.00	15.00	15.00	15.00	15.69
$\tau\left(x^{\star} \mid x^{0}\right)$		10.00	25.00	50.00	60.53	

Non-standard constraints (transaction cost management)

Let c_i^- and c_i^+ be the bid and ask transactions costs. The net expected return is equal to:

$$\mu(x) = \sum_{i=1}^{n} x_i \mu_i - \sum_{i=1}^{n} \Delta x_i^- c_i^- - \sum_{i=1}^{n} \Delta x_i^+ c_i^+$$

The γ-problem becomes:

$$\begin{aligned} x^\star &= \arg\min \frac{1}{2} x^\top \Sigma x - \gamma \left(\sum_{i=1}^{n} x_i \mu_i - \sum_{i=1}^{n} \Delta x_i^- c_i^- - \sum_{i=1}^{n} \Delta x_i^+ c_i^+ \right) \\ &\text{u.c.} \quad \left\{ \begin{array}{l} \sum_{i=1}^{n} \left(x_i + \Delta x_i^- c_i^- + \Delta x_i^+ c_i^+ \right) = 1 \\ x_i = x_i^0 + \Delta x_i^+ - \Delta x_i^- \\ 0 \leq x_i \leq 1 \\ 0 \leq \Delta x_i^- \leq 1 \\ 0 \leq \Delta x_i^+ \leq 1 \end{array} \right. \end{aligned}$$

Non-standard constraints (transaction cost management)

The augmented QP problem becomes:

$$\begin{aligned} X^\star &= \arg\min \frac{1}{2} X^\top Q X - X^\top R \\ \text{u.c.} & \left\{ \begin{array}{l} AX = B \\ \mathbf{0}_{3n} \leq X \leq \mathbf{1}_{3n} \end{array} \right. \end{aligned}$$

where X is a $3n \times 1$ vector:

$$X = \left(x_1, \ldots, x_n, \Delta x_1^-, \ldots, \Delta x_n^-, \Delta x_1^+, \ldots, \Delta x_n^+\right)$$

and:

$$Q_{3n \times 3n} = \left(\begin{array}{ccc} \Sigma & \mathbf{0}_{n \times n} & \mathbf{0}_{n \times n} \\ \mathbf{0}_{n \times n} & \mathbf{0}_{n \times n} & \mathbf{0}_{n \times n} \\ \mathbf{0}_{n \times n} & \mathbf{0}_{n \times n} & \mathbf{0}_{n \times n} \end{array} \right), \quad R_{3n \times 1} = \left(\begin{array}{c} \gamma \mu \\ -c^- \\ -c^+ \end{array} \right),$$

$$A_{(n+1) \times 3n} = \left(\begin{array}{ccc} \mathbf{1}_n^\top & (c^-)^\top & (c^+)^\top \\ I_n & I_n & -I_n \end{array} \right) \quad \text{and} \quad B_{(n+1) \times 1} = \left(\begin{array}{c} 1 \\ x^0 \end{array} \right)$$

Index sampling

Index sampling

The underlying idea is to replicate an index b with n stocks by a portfolio x with n_x stocks and $n_x \ll n$

From a mathematical point of view, index sampling can be written as a portfolio optimization problem with a benchmark:

$$\begin{aligned} x^{\star} & = & \arg\min \frac{1}{2} (x - b)^{\top} \Sigma (x - b) \\ & \text{u.c.} & \left\{ \begin{array}{l} \mathbf{1}_n^{\top} x = 1 \\ x \geq \mathbf{0}_n \\ \sum_{i=1}^{n} \mathbb{1}\{x_i > 0\} \leq n_x \end{array} \right. \end{aligned}$$

where b is the vector of index weights

We obtain a mixed integer non-linear optimization problem

Index sampling

Three stepwise algorithms:

1. The backward elimination algorithm starts with all the stocks, computes the optimized portfolio, deletes the stock which presents the highest tracking error variance, and repeats this process until the number of stocks in the optimized portfolio reaches the target value n_x
2. The forward selection algorithm starts with no stocks in the portfolio, adds the stock which presents the smallest tracking error variance, and repeats this process until the number of stocks in the optimized portfolio reaches the target value n_x
3. The heuristic algorithm is a variant of the backward elimination algorithm, but the elimination process of the heuristic algorithm uses the criterion of the smallest weight

Heuristic algorithm

1. The algorithm is initialized with $\mathcal{N}_{(0)} = \emptyset$ and $x^{\star}_{(0)} = b$.
2. At the iteration k, we define a set $\mathcal{I}_{(k)}$ of stocks having the smallest positive weights in the portfolio $x^{\star}_{(k-1)}$. We then update the set $\mathcal{N}_{(k)}$ with $\mathcal{N}_{(k)} = \mathcal{N}_{(k-1)} \cup \mathcal{I}_{(k)}$ and define the upper bounds $x^{+}_{(k)}$:
$$x^{+}_{(k),i} = \begin{cases} 0 & \text{if} \quad i \in \mathcal{N}_{(k)} \\ 1 & \text{if} \quad i \notin \mathcal{N}_{(k)} \end{cases}$$
3. We solve the QP problem by using the new upper bounds $x^{+}_{(k)}$:
$$\begin{aligned} x^{\star}_{(k)} &= \arg\min \frac{1}{2} \left(x_{(k)} - b\right)^{\top} \Sigma \left(x_{(k)} - b\right) \\ & \text{u.c.} \quad \begin{cases} \mathbf{1}_n^{\top} x_{(k)} = 1 \\ \mathbf{0}_n \leq x_{(k)} \leq x^{+}_{(k)} \end{cases} \end{aligned}$$
4. We iterate steps 2 and 3 until the convergence criterion:
$$\sum_{i=1}^{n} \mathbb{1}\left\{x^{*}_{(k),i} > 0\right\} \leq n_x$$

Complexity of the three numerical algorithms

The number of solved QP problems is respectively equal to:

- $n_b - n_x$ for the heuristic algorithm
- $(n_b - n_x)(n_b + n_x + 1)/2$ for the backward elimination algorithm
- $n_x(2n_b - n_x + 1)/2$ for the forward selection algorithm

		Number of solved QP problems		
n_b	n_x	Heuristic	Backward	Forward
50	10	40	1 220	455
	40	10	455	1 220
500	50	450	123 975	23 775
	450	50	23 775	123 975
1 500	100	1 400	1 120 700	145 050
	1 000	500	625 250	1 000 500

Index sampling (Eurostoxx 50, June 2012)

Table 14: Sampling the SX5E index with the heuristic algorithm

k	Stock	b_i	$\sigma\left(x_{(k)} \mid b\right)$
1	Nokia	0.45	0.18
2	Carrefour	0.60	0.23
3	Repsol	0.71	0.28
4	Unibail-Rodamco	0.99	0.30
5	Muenchener Rueckver	1.34	0.32
6	RWE	1.18	0.36
7	Koninklijke Philips	1.07	0.41
8	Generali	1.06	0.45
9	CRH	0.82	0.51
10	Volkswagen	1.34	0.55
42	LVMH	2.39	3.67
43	Telefonica	3.08	3.81
44	Bayer	3.51	4.33
45	Vinci	1.46	5.02
46	BBVA	2.13	6.53
47	Sanofi	5.38	7.26
48	Allianz	2.67	10.76
49	Total	5.89	12.83
50	Siemens	4.36	30.33

Index sampling (Eurostoxx 50, June 2012)

Table 15: Sampling the SX5E index with the backward elimination algorithm

k	Stock	b_i	$\sigma\left(x_{(k)} \mid b\right)$
1	Iberdrola	1.05	0.11
2	France Telecom	1.48	0.18
3	Carrefour	0.60	0.22
4	Muenchener Rueckver	1.34	0.26
5	Repsol	0.71	0.30
6	BMW	1.37	0.34
7	Generali	1.06	0.37
8	RWE	1.18	0.41
9	Koninklijke Philips	1.07	0.44
10	Air Liquide	2.10	0.48
42	GDF Suez	1.92	3.49
43	Bayer	3.51	3.88
44	BNP Paribas	2.26	4.42
45	Total	5.89	4.99
46	LVMH	2.39	5.74
47	Allianz	2.67	7.15
48	Sanofi	5.38	8.90
49	BBVA	2.13	12.83
50	Siemens	4.36	30.33

Index sampling (Eurostoxx 50, June 2012)

Table 16: Sampling the SX5E index with the forward selection algorithm

k	Stock	b_i	$\sigma\left(x_{(k)} \mid b\right)$
1	Siemens	4.36	12.83
2	Banco Santander	3.65	8.86
3	Bayer	3.51	6.92
4	Eni	3.32	5.98
5	Allianz	2.67	5.11
6	LVMH	2.39	4.55
7	France Telecom	1.48	3.93
8	Carrefour	0.60	3.62
9	BMW	1.37	3.35
41	Société Générale	1.07	0.50
42	CRH	0.82	0.45
43	Air Liquide	2.10	0.41
44	RWE	1.18	0.37
45	Nokia	0.45	0.33
46	Unibail-Rodamco	0.99	0.28
47	Repsol	0.71	0.24
48	Essilor	1.17	0.18
49	Muenchener Rueckver	1.34	0.11
50	Iberdrola	1.05	0.00

Index sampling

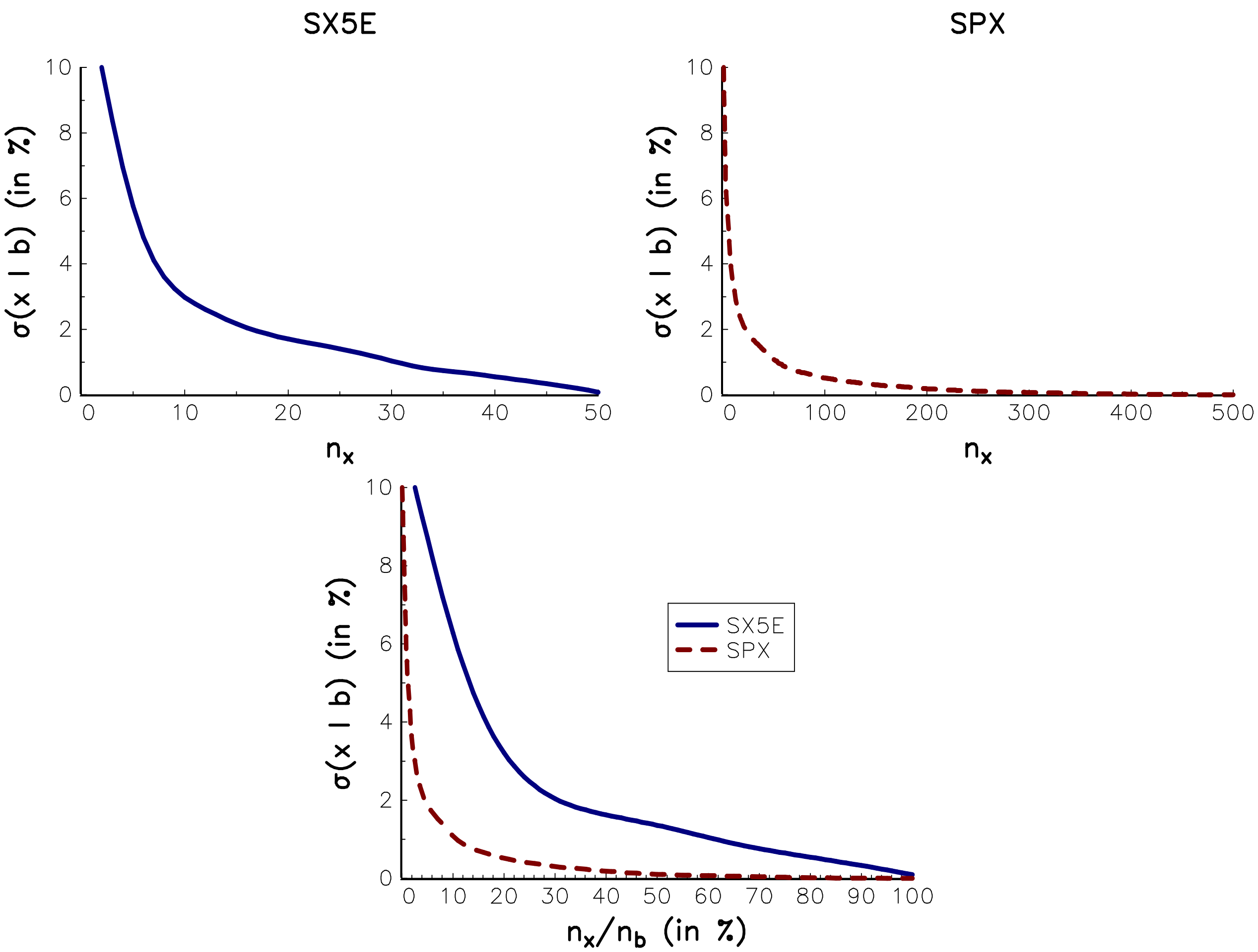


Figure 15: Sampling the SX5E and SPX indices (June 2012)

The impact of weight constraints

We specify the optimization problem as follows:

$$\min \frac{1}{2} x^\top \Sigma x$$
$$\text{u.c.} \left\{ \begin{array}{l} \mathbf{1}_n^\top x = 1 \\ \mu^\top x \geq \mu^\star \\ x \in \mathcal{C} \end{array} \right.$$

where $\mathcal{C}$ is the set of weights constraints. We define:

- the unconstrained portfolio $x^\star$ or $x^\star(\mu, \Sigma)$:

$$\mathcal{C} = \mathbb{R}^n$$

- the constrained portfolio $\tilde{x}$:

$$\mathcal{C}\left(x^-, x^+\right) = \left\{x \in \mathbb{R}^n : x_i^- \leq x_i \leq x_i^+\right\}$$

The impact of weight constraints

Theorem

Jagannathan and Ma (2003) show that the constrained portfolio is the solution of the unconstrained problem:

$$\tilde{x} = x^{\star}\left(\tilde{\mu}, \tilde{\Sigma}\right)$$

with:

$$\left\{ \begin{array}{l} \tilde{\mu} = \mu \\ \tilde{\Sigma} = \Sigma + \left(\lambda^{+} - \lambda^{-}\right) \mathbf{1}_n^{\top} + \mathbf{1}_n \left(\lambda^{+} - \lambda^{-}\right)^{\top} \end{array} \right.$$

where λ^{-} and λ^{+} are the Lagrange coefficients vectors associated to the lower and upper bounds.

$\Rightarrow$ Introducing weights constraints is equivalent to introduce a shrinkage method or to introduce some relative views (similar to the Black-Litterman approach).

The impact of weight constraints

Proof (step 1)

Without weight constraints, the expression of the Lagrangian is:

$$\mathcal{L}(x;\lambda_0,\lambda_1) = \frac{1}{2}x^\top \Sigma x - \lambda_0 \left(\mathbf{1}_n^\top x - 1\right) - \lambda_1 \left(\mu^\top x - \mu^\star\right)$$

with $\lambda_0 \geq 0$ and $\lambda_1 \geq 0$. The first-order conditions are:

$$\left\{\begin{array}{l} \Sigma x - \lambda_0 \mathbf{1}_n - \lambda_1 \mu = \mathbf{0}_n \\ \mathbf{1}_n^\top x - 1 = 0 \\ \mu^\top x - \mu^\star = 0 \end{array}\right.$$

We deduce that the solution $x^\star$ depends on the vector of expected return μ and the covariance matrix Σ and we note $x^\star = x^\star(\mu,\Sigma)$

The impact of weight constraints

Proof (step 2)

If we impose now the weight constraints $\mathcal{C}\left(x^{-},x^{+}\right)$, we have:

$$\begin{aligned}\mathcal{L}\left(x;\lambda_0,\lambda_1,\lambda^{-},\lambda^{+}\right) &= \frac{1}{2}x^{\top}\Sigma x-\lambda_0\left(\mathbf{1}_n^{\top}x-1\right)-\lambda_1\left(\mu^{\top}x-\mu^{\star}\right)- \\ & \quad \lambda^{-\top}\left(x-x^{-}\right)-\lambda^{+\top}\left(x^{+}-x\right)\end{aligned}$$

with $\lambda_0\geq 0$, $\lambda_1\geq 0$, $\lambda_i^{-}\geq 0$ and $\lambda_i^{+}\geq 0$. In this case, the Kuhn-Tucker conditions are:

$$\left\{\begin{array}{l}\Sigma x-\lambda_0\mathbf{1}_n-\lambda_1\mu-\lambda^{-}+\lambda^{+}=\mathbf{0}_n \\ \mathbf{1}_n^{\top}x-1=0 \\ \mu^{\top}x-\mu^{\star}=0 \\ \min\left(\lambda_i^{-},x_i-x_i^{-}\right)=0 \\ \min\left(\lambda_i^{+},x_i^{+}-x_i\right)=0\end{array}\right.$$

The impact of weight constraints

Proof (step 3)

Given a constrained portfolio $\tilde{x}$, it is possible to find a covariance matrix $\tilde{\Sigma}$ such that $\tilde{x}$ is the solution of unconstrained mean-variance portfolio. Let $\mathcal{E} = \left\{ \tilde{\Sigma} > 0 : \tilde{x} = x^{\star} \left(\mu, \tilde{\Sigma} \right) \right\}$ denote the corresponding set:

$$\mathcal{E} = \left\{ \tilde{\Sigma} > 0 : \tilde{\Sigma} \tilde{x} - \lambda_0 \mathbf{1}_n - \lambda_1 \mu = \mathbf{0}_n \right\}$$

Of course, the set $\mathcal{E}$ contains several solutions. From a financial point of view, we are interested in covariance matrices $\tilde{\Sigma}$ that are close to Σ. Jagannathan and Ma note that the matrix $\tilde{\Sigma}$ defined by:

$$\tilde{\Sigma} = \Sigma + \left(\lambda^{+} - \lambda^{-} \right) \mathbf{1}_n^{\top} + \mathbf{1}_n \left(\lambda^{+} - \lambda^{-} \right)^{\top}$$

is a solution of $\mathcal{E}$

The impact of weight constraints

Proof (step 4)

Indeed, we have:

$$\begin{aligned}
\tilde{\Sigma}\tilde{x} &= \Sigma\tilde{x} + \left(\lambda^{+} - \lambda^{-}\right)\mathbf{1}_n^\top \tilde{x} + \mathbf{1}_n \left(\lambda^{+} - \lambda^{-}\right)^\top \tilde{x} \\
&= \Sigma\tilde{x} + \left(\lambda^{+} - \lambda^{-}\right) + \mathbf{1}_n \left(\lambda^{+} - \lambda^{-}\right)^\top \tilde{x} \\
&= \lambda_0 \mathbf{1}_n + \lambda_1 \mu + \mathbf{1}_n \left(\lambda_0 \mathbf{1}_n + \lambda_1 \mu - \Sigma\tilde{x}\right)^\top \tilde{x} \\
&= \lambda_0 \mathbf{1}_n + \lambda_1 \mu + \mathbf{1}_n \left(\lambda_0 + \lambda_1 \mu^\star - \tilde{x}^\top \Sigma \tilde{x}\right) \\
&= \left(2\lambda_0 - \tilde{x}^\top \Sigma \tilde{x} + \lambda_1 \mu^\star\right) \mathbf{1}_n + \lambda_1 \mu
\end{aligned}$$

It proves that $\tilde{x}$ is the solution of the unconstrained optimization problem. The Lagrange coefficients $\lambda_0^\star$ and $\lambda_1^\star$ for the unconstrained problem are respectively equal to $2\tilde{\lambda}_0 - \tilde{x}^\top \Sigma \tilde{x} + \tilde{\lambda}_1 \mu^\star$ and $\tilde{\lambda}_1$ where $\tilde{\lambda}_0$ and $\tilde{\lambda}_1$ are the Lagrange coefficient for the constrained problem. Moreover, $\tilde{\Sigma}$ is generally a positive definite matrix

The impact of weight constraints

Example 13

We consider four assets. Their expected returns are equal to 5%, 6%, 8% and 6% while their volatilities are equal to 15%, 20%, 25% and 30%. The correlation matrix of asset returns is given by the following matrix:

$$C = \begin{pmatrix} 1.00 & & & \\ 0.10 & 1.00 & & \\ 0.40 & 0.70 & 1.00 & \\ 0.50 & 0.40 & 0.80 & 1.00 \end{pmatrix}$$

Given these parameters, the global minimum variance portfolio is equal to:

$$x^{\star} = \begin{pmatrix} 72.742\% \\ 49.464\% \\ -20.454\% \\ -1.753\% \end{pmatrix}$$

The impact of weight constraints

Table 17: Minimum variance portfolio when $x_i \geq 10\%$

$x_i^\star$	$\tilde{x}_i$	λ_i^-	λ_i^+	$\tilde{\sigma}_i$	$\tilde{\rho}_{i,j}$			
72.742	56.195	0.000	0.000	15.00	100.00			
49.464	23.805	0.000	0.000	20.00	10.00	100.00		
−20.454	10.000	1.190	0.000	19.67	10.50	58.71	100.00	
−1.753	10.000	1.625	0.000	23.98	17.38	16.16	67.52	100.00

Table 18: Minimum variance portfolio when $10\% \leq x_i \leq 40\%$

$x_i^\star$	$\tilde{x}_i$	λ_i^-	λ_i^+	$\tilde{\sigma}_i$	$\tilde{\rho}_{i,j}$			
72.742	40.000	0.000	0.915	20.20	100.00			
49.464	40.000	0.000	0.000	20.00	30.08	100.00		
−20.454	10.000	0.915	0.000	21.02	35.32	61.48	100.00	
−1.753	10.000	1.050	0.000	26.27	39.86	25.70	73.06	100.00

The impact of weight constraints

Table 19: Mean-variance portfolio when $10\% \leq x_i \leq 40\%$ and $\mu^\star = 6\%$

$x_i^\star$	$\tilde{x}_i$	λ_i^-	λ_i^+	$\tilde{\sigma}_i$	$\tilde{\rho}_{i,j}$			
65.866	40.000	0.000	0.125	15.81	100.00			
26.670	30.000	0.000	0.000	20.00	13.44	100.00		
32.933	20.000	0.000	0.000	25.00	41.11	70.00	100.00	
−25.470	10.000	1.460	0.000	24.66	23.47	19.06	73.65	100.00

Table 20: MSR portfolio when $10\% \leq x_i \leq 40\%$

$x_i^\star$	$\tilde{x}_i$	λ_i^-	λ_i^+	$\tilde{\sigma}_i$	$\tilde{\rho}_{i,j}$			
51.197	40.000	0.000	0.342	17.13	100.00			
50.784	39.377	0.000	0.000	20.00	18.75	100.00		
−21.800	10.000	0.390	0.000	23.39	36.25	66.49	100.00	
19.818	10.623	0.000	0.000	30.00	50.44	40.00	79.96	100.00

Variations on the efficient frontier

Exercise

We consider an investment universe of four assets. We assume that their expected returns are equal to 5%, 6%, 8% and 6%, and their volatilities are equal to 15%, 20%, 25% and 30%. The correlation matrix is:

$$\rho = \begin{pmatrix} 100\% & & & \\ 10\% & 100\% & & \\ 40\% & 70\% & 100\% & \\ 50\% & 40\% & 80\% & 100\% \end{pmatrix}$$

We note x_i the weight of the i^{th} asset in the portfolio. We only impose that the sum of the weights is equal to 100%.

Variations on the efficient frontier

Question 1

Represent the efficient frontier by considering the following values of γ: -1, -0.5, -0.25, 0, 0.25, 0.5, 1 and 2.

Variations on the efficient frontier

We deduce that the covariance matrix is:

$$\Sigma = \begin{pmatrix} 2.250 & 0.300 & 1.500 & 2.250 \\ 0.300 & 4.000 & 3.500 & 2.400 \\ 1.500 & 3.500 & 6.250 & 6.000 \\ 2.250 & 2.400 & 6.000 & 9.000 \end{pmatrix} \times 10^{-2}$$

We then have to solve the γ-formulation of the Markowitz problem:

$$\begin{aligned} x^{\star}(\gamma) &= \arg\min \frac{1}{2} x^{\top} \Sigma x - \gamma x^{\top} \mu \\ \text{u.c.} \quad & \mathbf{1}_n^{\top} x = 1 \end{aligned}$$

We obtain the results[3] given in Table 21. We represent the efficient frontier in Figure 16.

[3] The weights, expected returns and volatilities are expressed in %.

Variations on the efficient frontier

Table 21: Solution of Question 1

γ	-1.00	-0.50	-0.25	0.00	0.25	0.50	1.00	2.00
$x_1^\star$	94.04	83.39	78.07	72.74	67.42	62.09	51.44	30.15
$x_2^\star$	120.05	84.76	67.11	49.46	31.82	14.17	-21.13	-91.72
$x_3^\star$	-185.79	-103.12	-61.79	-20.45	20.88	62.21	144.88	310.22
$x_4^\star$	71.69	34.97	16.61	-1.75	-20.12	-38.48	-75.20	-148.65
$\mu(x^\star)$	1.34	3.10	3.98	4.86	5.74	6.62	8.38	11.90
$\sigma(x^\star)$	22.27	15.23	12.88	12.00	12.88	15.23	22.27	39.39

Variations on the efficient frontier

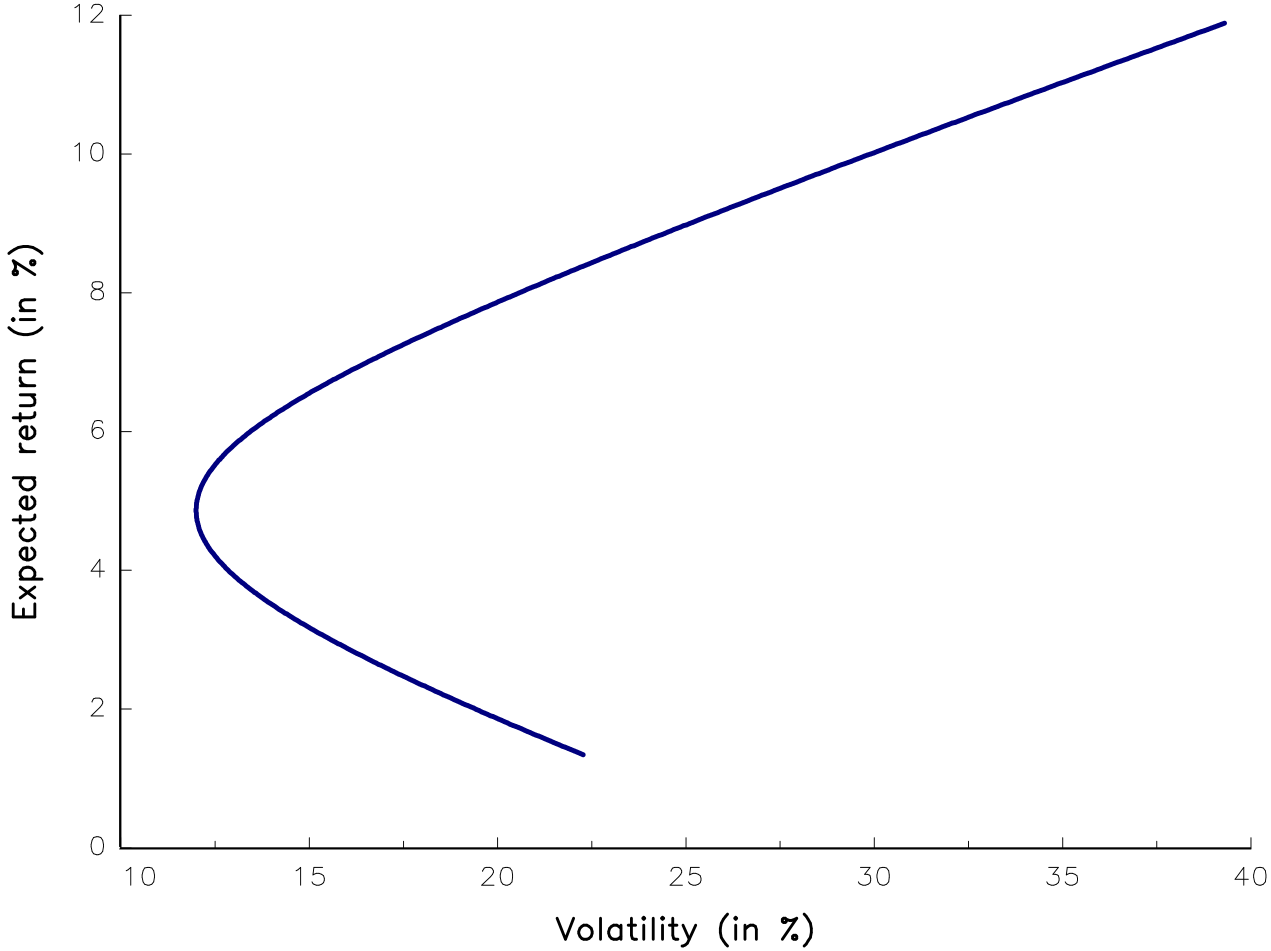


Figure 16: Markowitz efficient frontier

Variations on the efficient frontier

Question 2

Calculate the minimum variance portfolio. What are its expected return and its volatility?

Variations on the efficient frontier

We solve the γ-problem with $\gamma = 0$. The minimum variance portfolio is then $x_1^\star = 72.74\%$, $x_2^\star = 49.46\%$, $x_3^\star = -20.45\%$ and $x_4^\star = -1.75\%$. We deduce that $\mu(x^\star) = 4.86\%$ and $\sigma(x^\star) = 12.00\%$.

Variations on the efficient frontier

Question 3

Calculate the optimal portfolio which has an ex-ante volatility $\sigma^{\star}$ equal to 10%. Same question if $\sigma^{\star} = 15\%$ and $\sigma^{\star} = 20\%$.

Variations on the efficient frontier

There is no solution when the target volatility $\sigma^{\star}$ is equal to 10% because the minimum variance portfolio has a volatility larger than 10%. Finding the optimized portfolio for $\sigma^{\star} = 15\%$ or $\sigma^{\star} = 20\%$ is equivalent to solving a σ-problem. If $\sigma^{\star} = 15\%$ (resp. $\sigma^{\star} = 20\%$), we obtain an implied value of γ equal to 0.48 (resp. 0.85). Results are given in the following Table:

$\sigma^{\star}$	15.00	20.00
$x_1^{\star}$	62.52	54.57
$x_2^{\star}$	15.58	−10.75
$x_3^{\star}$	58.92	120.58
$x_4^{\star}$	−37.01	−64.41
$\mu(x^{\star})$	6.55	7.87
γ	0.48	0.85

Variations on the efficient frontier

Question 4

We note $x^{(1)}$ the minimum variance portfolio and $x^{(2)}$ the optimal portfolio with $\sigma^{\star} = 20\%$. We consider the set of portfolios $x^{(\alpha)}$ defined by the relationship:

$$x^{(\alpha)} = (1 - \alpha)\, x^{(1)} + \alpha x^{(2)}$$

In the previous efficient frontier, place the portfolios $x^{(\alpha)}$ when α is equal to -0.5, -0.25, 0, 0.1, 0.2, 0.5, 0.7 and 1. What do you observe? Comment on this result.

Variations on the efficient frontier

Let $x^{(\alpha)}$ be the portfolio defined by the relationship $x^{(\alpha)} = (1-\alpha)\, x^{(1)} + \alpha x^{(2)}$ where $x^{(1)}$ is the minium variance portfolio and $x^{(2)}$ is the optimized portfolio with a 20% ex-ante volatility. We obtain the following results:

α	$\sigma\left(x^{(\alpha)}\right)$	$\mu\left(x^{(\alpha)}\right)$
-0.50	14.42	3.36
-0.25	12.64	4.11
0.00	12.00	4.86
0.10	12.10	5.16
0.20	12.41	5.46
0.50	14.42	6.36
0.70	16.41	6.97
1.00	20.00	7.87

We have reported these portfolios in Figure 17. We notice that they are located on the efficient frontier. This is perfectly normal because we know that a combination of two optimal portfolios corresponds to another optimal portfolio.

Variations on the efficient frontier

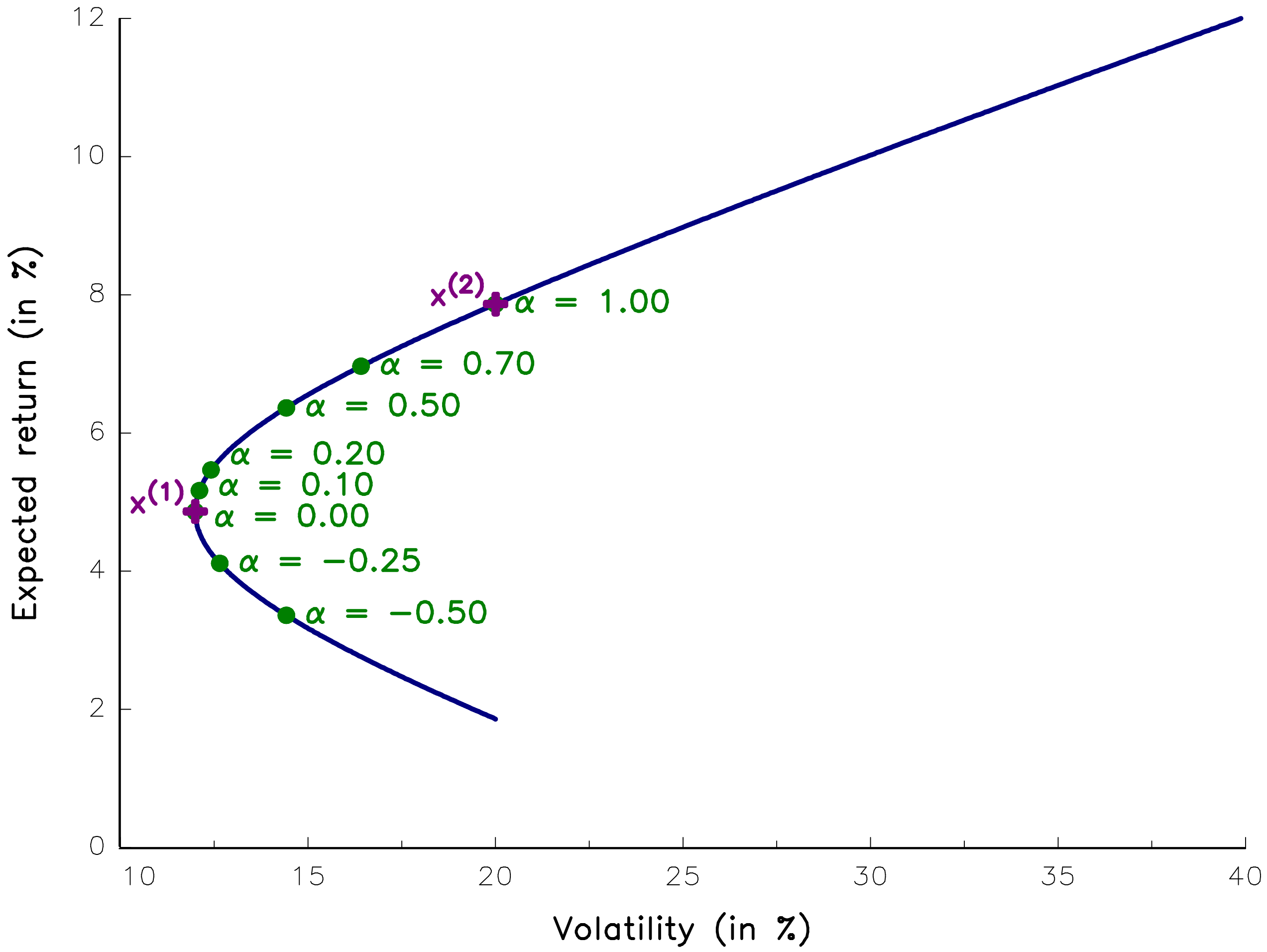


Figure 17: Mean-variance diagram of portfolios $x^{(\alpha)}$

Variations on the efficient frontier

Question 5

Repeat Questions 3 and 4 by considering the constraint $0 \leq x_i \leq 1$.
Explain why we do not retrieve the same observation.

Variations on the efficient frontier

If we consider the constraint $0 \leq x_i \leq 1$, the γ-formulation of the Markowitz problem becomes:

$$\begin{aligned} x^{\star}(\gamma) &= \arg\min \frac{1}{2} x^{\top} \Sigma x - \gamma x^{\top} \mu \\ \text{u.c.} & \left\{ \begin{array}{l} \mathbf{1}_n^{\top} x = 1 \\ \mathbf{0}_n \leq x \leq \mathbf{1}_n \end{array} \right. \end{aligned}$$

Variations on the efficient frontier

We obtain the following results:

$\sigma^{\star}$	MV	12.00	15.00	20.00
$x_1^{\star}$	65.49	✓	45.59	24.88
$x_2^{\star}$	34.51	✓	24.74	4.96
$x_3^{\star}$	0.00	✓	29.67	70.15
$x_4^{\star}$	0.00	✓	0.00	0.00
$\mu(x^{\star})$	5.35	✓	6.14	7.15
$\sigma(x^{\star})$	12.56	✓	15.00	20.00
γ	0.00	✓	0.62	1.10

We observe that we cannot target a volatility $\sigma^{\star} = 10\%$. Moreover, the expected return $\mu(x^{\star})$ of the optimal portfolios are reduced due to the additional constraints.

Variations on the efficient frontier

Question 6

We now include in the investment universe a fifth asset corresponding to the risk-free asset. Its return is equal to 3%.

Variations on the efficient frontier

Question 6.a

Define the expected return vector and the covariance matrix of asset returns.

Variations on the efficient frontier

We have:

$$\mu = \begin{pmatrix} 5.0 \\ 6.0 \\ 8.0 \\ 6.0 \\ 3.0 \end{pmatrix} \times 10^{-2}$$

and:

$$\Sigma = \begin{pmatrix} 2.250 & 0.300 & 1.500 & 2.250 & 0.000 \\ 0.300 & 4.000 & 3.500 & 2.400 & 0.000 \\ 1.500 & 3.500 & 6.250 & 6.000 & 0.000 \\ 2.250 & 2.400 & 6.000 & 9.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \end{pmatrix} \times 10^{-2}$$

Variations on the efficient frontier

Question 6.b

Deduce the efficient frontier by solving directly the quadratic problem.

Variations on the efficient frontier

We solve the γ-problem and obtain the efficient frontier given in Figure 18.

Variations on the efficient frontier

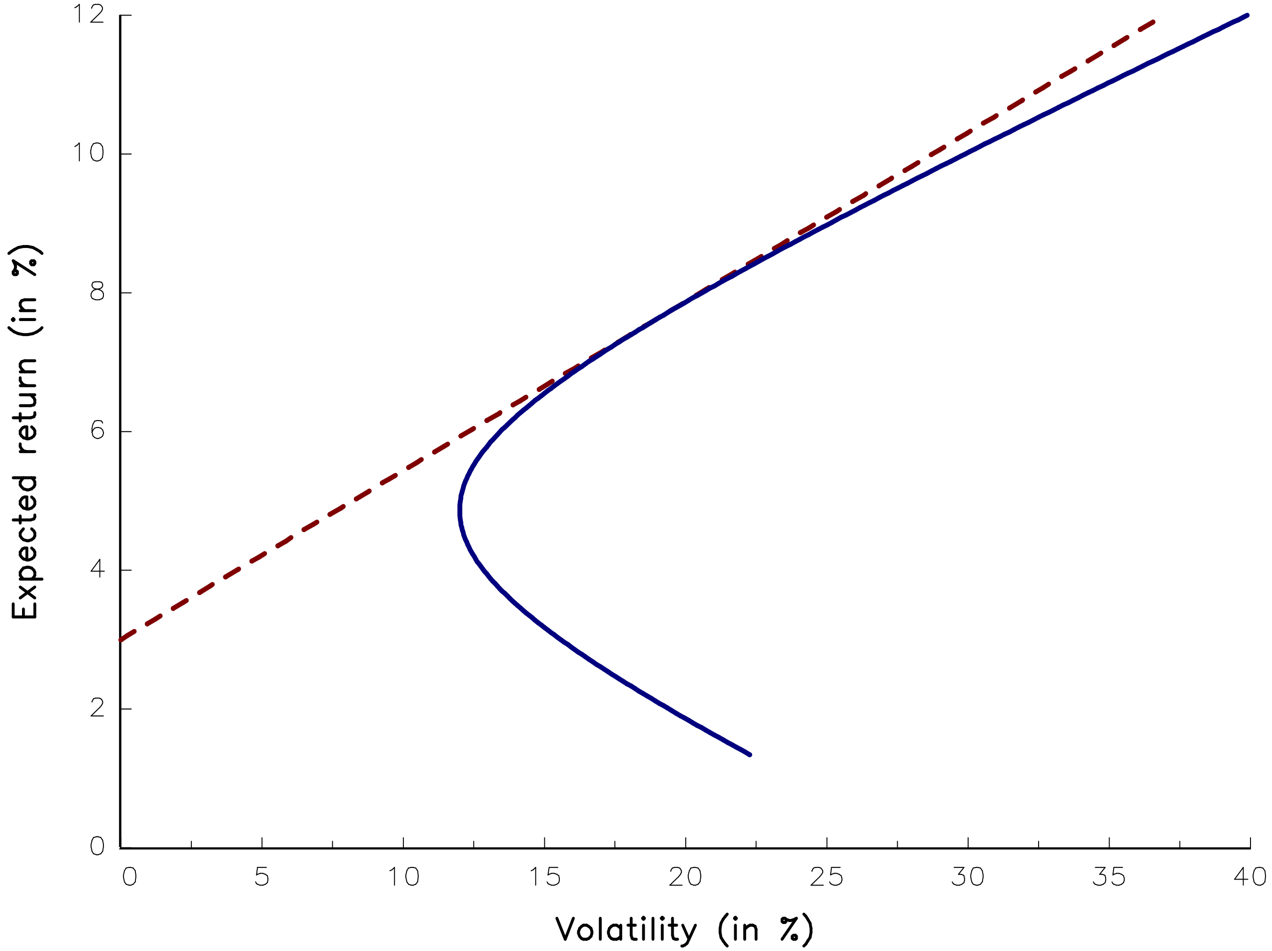


Figure 18: Efficient frontier when the risk-free asset is introduced

Variations on the efficient frontier

Question 6.c

What is the shape of the efficient frontier? Comment on this result.

Variations on the efficient frontier

This efficient frontier is a straight line. This line passes through the risk-free asset and is tangent to the efficient frontier of Figure 16. This question is a direct application of the *Separation Theorem* of Tobin.

Variations on the efficient frontier

Question 6.d

Choose two arbitrary portfolios $x^{(1)}$ and $x^{(2)}$ of this efficient frontier. Deduce the Sharpe ratio of the tangency portfolio.

Variations on the efficient frontier

We consider two optimized portfolios of this efficient frontier. They corresponds to $\gamma = 0.25$ and $\gamma = 0.50$. We obtain the following results:

γ	0.25	0.50
$x_1^\star$	18.23	36.46
$x_2^\star$	-1.63	-3.26
$x_3^\star$	34.71	69.42
$x_4^\star$	-18.93	-37.86
$x_5^\star$	67.62	35.24
$\mu(x^\star)$	4.48	5.97
$\sigma(x^\star)$	6.09	12.18

Variations on the efficient frontier

The first portfolio has an expected return equal to 4.48% and a volatility equal to 6.09%. The weight of the risk-free asset is 67.62%. This explains the low volatility of this portfolio. For the second portfolio, the weight of the risk-free asset is lower and equal to 35.24%. The expected return and the volatility are then equal to 5.97% and 12.18%. We note $x^{(1)}$ and $x^{(2)}$ these two portfolios. By definition, the Sharpe ratio of the market portfolio $x^{\star}$ is the tangency of the line. We deduce that:

$$\begin{aligned}
\mathrm{SR}\left(x^{\star} \mid r\right) &= \frac{\mu\left(x^{(2)}\right)-\mu\left(x^{(1)}\right)}{\sigma\left(x^{(2)}\right)-\sigma\left(x^{(1)}\right)} \\
&= \frac{5.97-4.48}{12.18-6.09} \\
&= 0.2436
\end{aligned}$$

The Sharpe ratio of the market portfolio $x^{\star}$ is then equal to 0.2436.

Variations on the efficient frontier

Question 6.e

Calculate then the composition of the tangency portfolio from $x^{(1)}$ and $x^{(2)}$.

Variations on the efficient frontier

By construction, every portfolio $x^{(\alpha)}$ which belongs to the tangency line is a linear combination of two portfolios $x^{(1)}$ and $x^{(2)}$ of this efficient frontier:

$$x^{(\alpha)} = (1 - \alpha)\, x^{(1)} + \alpha x^{(2)}$$

The market portfolio $x^{\star}$ is the portfolio $x^{(\alpha)}$ which has a zero weight in the risk-free asset. We deduce that the value $\alpha^{\star}$ which corresponds to the market portfolio satisfies the following relationship:

$$(1 - \alpha^{\star})\, x_5^{(1)} + \alpha^{\star} x_5^{(2)} = 0$$

because the risk-free asset is the fifth asset of the portfolio.

Variations on the efficient frontier

It follows that:

$$\begin{aligned}\alpha^\star &= \frac{x_5^{(1)}}{x_5^{(1)} - x_5^{(2)}} \\ &= \frac{67.62}{67.62 - 35.24} \\ &= 2.09\end{aligned}$$

We deduce that the market portfolio is:

$$x^\star = (1 - 2.09) \cdot \begin{pmatrix} 18.23 \\ -1.63 \\ 34.71 \\ -18.93 \\ 67.62 \end{pmatrix} + 2.09 \cdot \begin{pmatrix} 36.46 \\ -3.26 \\ 69.42 \\ -37.86 \\ 35.24 \end{pmatrix} = \begin{pmatrix} 56.30 \\ -5.04 \\ 107.21 \\ -58.46 \\ 0.00 \end{pmatrix}$$

We check that the Sharpe ratio of this portfolio is 0.2436.

Variations on the efficient frontier

Question 7

We consider the general framework with n risky assets whose vector of expected returns is μ and the covariance matrix of asset returns is Σ while the return of the risk-free asset is r. We note $\tilde{x}$ the portfolio invested in the $n+1$ assets. We have:

$$\tilde{x} = \begin{pmatrix} x \\ x_r \end{pmatrix}$$

with x the weight vector of risky assets and x_r the weight of the risk-free asset. We impose the following constraint:

$$\sum_{i=1}^{n} \tilde{x}_i = \sum_{i=1}^{n} x_i = 1$$

Variations on the efficient frontier

Question 7.a

Define $\tilde{\mu}$ and $\tilde{\Sigma}$ the vector of expected returns and the covariance matrix of asset returns associated with the $n+1$ assets.

Variations on the efficient frontier

We have:

$$\tilde{\mu} = \begin{pmatrix} \mu \\ r \end{pmatrix}$$

and:

$$\tilde{\Sigma} = \begin{pmatrix} \Sigma & \mathbf{0}_n \\ \mathbf{0}_n^\top & 0 \end{pmatrix}$$

Variations on the efficient frontier

Question 7.b

By using the Markowitz ϕ-problem, retrieve the *Separation Theorem* of Tobin.

Variations on the efficient frontier

If we include the risk-free asset, the Markowitz ϕ-problem becomes:

$$\begin{aligned} \tilde{x}^\star(\phi) &= \arg\max \tilde{x}^\top \tilde{\mu} - \frac{\phi}{2}\tilde{x}^\top \tilde{\Sigma}\tilde{x} \\ \text{u.c.} \quad & \mathbf{1}_n^\top \tilde{x} = 1 \end{aligned}$$

We note that the objective function can be written as follows:

$$\begin{aligned} f(\tilde{x}) &= \tilde{x}^\top \tilde{\mu} - \frac{\phi}{2}\tilde{x}^\top \tilde{\Sigma}\tilde{x} \\ &= x^\top \mu + x_r r - \frac{\phi}{2} x^\top \Sigma x \\ &= g(x, x_r) \end{aligned}$$

The constraint becomes $\mathbf{1}_n^\top x + x_r = 1$. We deduce that the Lagrange function is:

$$\mathcal{L}(x, x_r; \lambda_0) = x^\top \mu + x_r r - \frac{\phi}{2} x^\top \Sigma x - \lambda_0 \left(\mathbf{1}_n^\top x + x_r - 1\right)$$

Variations on the efficient frontier

The first-order conditions are:

$$\left\{\begin{array}{l}\partial_x \mathcal{L}(x, x_r; \lambda_0) = \mu - \phi\Sigma x - \lambda_0 \mathbf{1}_n = \mathbf{0}_n \\ \partial_{x_r} \mathcal{L}(x, x_r; \lambda_0) = r - \lambda_0 = 0 \\ \partial_{\lambda_0} \mathcal{L}(x, x_r; \lambda_0) = \mathbf{1}_n^\top x + x_r - 1 = 0\end{array}\right.$$

The solution of the optimization problem is then:

$$\left\{\begin{array}{l}x^\star = \phi^{-1}\Sigma^{-1}(\mu - r\mathbf{1}_n) \\ \lambda_0^\star = r \\ x_r^\star = 1 - \phi^{-1}\mathbf{1}_n^\top \Sigma^{-1}(\mu - r\mathbf{1}_n)\end{array}\right.$$

Let $x_0^\star$ be the following portfolio:

$$x_0^\star = \frac{\Sigma^{-1}(\mu - r\mathbf{1}_n)}{\mathbf{1}_n^\top \Sigma^{-1}(\mu - r\mathbf{1}_n)}$$

Variations on the efficient frontier

We can then write the solution of the optimization problem in the following way:

$$\begin{cases} x^{\star} = \alpha x_0^{\star} \\ \lambda_0^{\star} = r \\ x_r^{\star} = 1 - \alpha \\ \alpha = \phi^{-1} \mathbf{1}_n^{\top} \Sigma^{-1} (\mu - r \mathbf{1}_n) \end{cases}$$

The first equation indicates that the relative proportions of risky assets in the optimized portfolio remain constant. If $\phi = \phi_0 = \mathbf{1}_n^{\top} \Sigma^{-1} (\mu - r \mathbf{1}_n)$, then $x^{\star} = x_0^{\star}$ and $x_r^{\star} = 0$. We deduce that $x_0^{\star}$ is the tangency portfolio. If $\phi \neq \phi_0$, $x^{\star}$ is proportional to $x_0^{\star}$ and the wealth invested in the risk-free asset is the complement $(1 - \alpha)$ to obtain a total exposure equal to 100%. We retrieve then the separation theorem:

$$\tilde{x}^{\star} = \underbrace{\alpha \cdot \begin{pmatrix} x_0^{\star} \\ 0 \end{pmatrix}}_{\text{risky assets}} + \underbrace{(1 - \alpha) \cdot \begin{pmatrix} \mathbf{0}_n \\ 1 \end{pmatrix}}_{\text{risk-free asset}}$$

Beta coefficient

Question 1

We consider an investment universe of n assets with:

$$R = \begin{pmatrix} R_1 \\ \vdots \\ R_n \end{pmatrix} \sim \mathcal{N}(\mu, \Sigma)$$

The weights of the market portfolio (or the benchmark) are $b = (b_1, \ldots, b_n)$.

Beta coefficient

Question 1.a

Define the beta β_i of asset i with respect to the market portfolio.

Beta coefficient

The beta of an asset is the ratio between its covariance with the market portfolio return and the variance of the market portfolio return. In the CAPM theory, we have:

$$\mathbb{E}[R_i] = r + \beta_i \left(\mathbb{E}[R(b)] - r\right)$$

where R_i is the return of asset i, $R(b)$ is the return of the market portfolio and r is the risk-free rate. The beta β_i of asset i is:

$$\beta_i = \frac{\operatorname{cov}(R_i, R(b))}{\operatorname{var}(R(b))}$$

Let Σ be the covariance matrix of asset returns. We have $\operatorname{cov}(R, R(b)) = \Sigma b$ and $\operatorname{var}(R(b)) = b^\top \Sigma b$. We deduce that:

$$\beta_i = \frac{(\Sigma b)_i}{b^\top \Sigma b}$$

Beta coefficient

Question 1.b

Let X_1, X_2 and X_3 be three random variables. Show that:

$$\operatorname{cov}\left(c_1 X_1 + c_2 X_2, X_3\right) = c_1 \operatorname{cov}\left(X_1, X_3\right) + c_2 \operatorname{cov}\left(X_2, X_3\right)$$

Beta coefficient

We recall that the mathematical operator $\mathbb{E}$ is bilinear. Let c be the covariance $\operatorname{cov}(c_1 X_1 + c_2 X_2, X_3)$. We then have:

$$\begin{aligned}
c &= \mathbb{E}\left[\left(c_1 X_1 + c_2 X_2 - \mathbb{E}\left[c_1 X_1 + c_2 X_2\right]\right)\left(X_3 - \mathbb{E}\left[X_3\right]\right)\right] \\
&= \mathbb{E}\left[\left(c_1\left(X_1 - \mathbb{E}\left[X_1\right]\right) + c_2\left(X_2 - \mathbb{E}\left[X_2\right]\right)\right)\left(X_3 - \mathbb{E}\left[X_3\right]\right)\right] \\
&= c_1 \mathbb{E}\left[\left(X_1 - \mathbb{E}\left[X_1\right]\right)\left(X_3 - \mathbb{E}\left[X_3\right]\right)\right] + c_2 \mathbb{E}\left[\left(X_2 - \mathbb{E}\left[X_2\right]\right)\left(X_3 - \mathbb{E}\left[X_3\right]\right)\right] \\
&= c_1 \operatorname{cov}\left(X_1, X_3\right) + c_2 \operatorname{cov}\left(X_2, X_3\right)
\end{aligned}$$

Beta coefficient

Question 1.c

We consider the asset portfolio $x = (x_1, \ldots, x_n)$ such that $\sum_{i=1}^{n} x_i = 1$. What is the relationship between the beta $\beta(x \mid b)$ of the portfolio and the betas β_i of the assets?

Beta coefficient

We have:

$$\begin{aligned}
\beta(x \mid b) &= \frac{\operatorname{cov}(R(x), R(b))}{\operatorname{var}(R(b))} = \frac{\operatorname{cov}\left(x^\top R, b^\top R\right)}{\operatorname{var}\left(b^\top R\right)} \\
&= \frac{x^\top \mathbb{E}\left[(R-\mu)(R-\mu)^\top\right] b}{b^\top \mathbb{E}\left[(R-\mu)(R-\mu)^\top\right] b} \\
&= \frac{x^\top \Sigma b}{b^\top \Sigma b} = x^\top \frac{\Sigma b}{b^\top \Sigma b} = x^\top \beta = \sum_{i=1}^n x_i \beta_i
\end{aligned}$$

with $\beta = (\beta_1, \ldots, \beta_n)$. The beta of portfolio x is then the weighted mean of asset betas. Another way to show this result is to exploit the result of Question 1.b. We have:

$$\beta(x \mid b) = \frac{\operatorname{cov}\left(\sum_{i=1}^n x_i R_i, R(b)\right)}{\operatorname{var}(R(b))} = \sum_{i=1}^n x_i \frac{\operatorname{cov}(R_i, R(b))}{\operatorname{var}(R(b))} = \sum_{i=1}^n x_i \beta_i$$

Beta coefficient

Question 1.d

Calculate the beta of the portfolios $x^{(1)}$ and $x^{(2)}$ with the following data:

i	1	2	3	4	5
β_i	0.7	0.9	1.1	1.3	1.5
$x_i^{(1)}$	0.5	0.5			
$x_i^{(2)}$	0.25	0.25	0.5	0.5	-0.5

Beta coefficient

We obtain $\beta\left(x^{(1)} \mid b\right) = 0.80$ and $\beta\left(x^{(2)} \mid b\right) = 0.85$.

Beta coefficient

Question 2

We assume that the market portfolio is the equally weighted portfolio[a].

[a] We have $b_i = n^{-1}$.

Beta coefficient

Question 2.a

Show that $\sum_{i=1}^{n} \beta_i = n$.

Beta coefficient

The weights of the market portfolio are then $b = n^{-1}\mathbf{1}_n$. We have:

$$\beta = \frac{\operatorname{cov}(R, R(b))}{\operatorname{var}(R(b))} = \frac{\Sigma b}{b^\top \Sigma b} = \frac{n^{-1}\Sigma \mathbf{1}_n}{n^{-2}\left(\mathbf{1}_n^\top \Sigma \mathbf{1}_n\right)} = n\frac{\Sigma \mathbf{1}_n}{\left(\mathbf{1}_n^\top \Sigma \mathbf{1}_n\right)}$$

We deduce that:

$$\sum_{i=1}^n \beta_i = \mathbf{1}_n^\top \beta = \mathbf{1}_n^\top n\frac{\Sigma \mathbf{1}_n}{\left(\mathbf{1}_n^\top \Sigma \mathbf{1}_n\right)} = n\frac{\mathbf{1}_n^\top \Sigma \mathbf{1}_n}{\left(\mathbf{1}_n^\top \Sigma \mathbf{1}_n\right)} = n$$

Beta coefficient

Question 2.b

We consider the case $n = 3$. Show that $\beta_1 \geq \beta_2 \geq \beta_3$ implies $\sigma_1 \geq \sigma_2 \geq \sigma_3$ if $\rho_{i,j} = 0$.

Beta coefficient

If $\rho_{i,j} = 0$, we have:

$$\beta_i = n \frac{\sigma_i^2}{\sum_{j=1}^n \sigma_j^2}$$

We deduce that:

$$\begin{aligned}
\beta_1 \geq \beta_2 \geq \beta_3 \quad & \Rightarrow \quad n \frac{\sigma_1^2}{\sum_{j=1}^3 \sigma_j^2} \geq n \frac{\sigma_2^2}{\sum_{j=1}^3 \sigma_j^2} \geq n \frac{\sigma_3^2}{\sum_{j=1}^3 \sigma_j^2} \\
& \Rightarrow \quad \sigma_1^2 \geq \sigma_2^2 \geq \sigma_3^2 \\
& \Rightarrow \quad \sigma_1 \geq \sigma_2 \geq \sigma_3
\end{aligned}$$

Beta coefficient

Question 2.c

What is the result if the correlation is uniform $\rho_{i,j} = \rho$?

Beta coefficient

If $\rho_{i,j} = \rho$, it follows that:

$$\begin{aligned}
\beta_i & \propto & \sigma_i^2 + \sum_{j \neq i} \rho \sigma_i \sigma_j \\
& = & \sigma_i^2 + \rho \sigma_i \sum_{j \neq i} \sigma_j + \rho \sigma_i^2 - \rho \sigma_i^2 \\
& = & (1 - \rho) \sigma_i^2 + \rho \sigma_i \sum_{j=1}^{n} \sigma_j \\
& = & f(\sigma_i)
\end{aligned}$$

with:

$$f(z) = (1 - \rho) z^2 + \rho z \sum_{j=1}^{n} \sigma_j$$

Beta coefficient

The first derivative of $f(z)$ is:

$$f'(z) = 2(1-\rho)z + \rho\sum_{j=1}^{n}\sigma_j$$

If $\rho \geq 0$, then $f(z)$ is an increasing function for $z \geq 0$ because $(1-\rho) \geq 0$ and $\rho\sum_{j=1}^{n}\sigma_j \geq 0$. This explains why the previous result remains valid:

$$\beta_1 \geq \beta_2 \geq \beta_3 \Rightarrow \sigma_1 \geq \sigma_2 \geq \sigma_3 \quad \text{if} \quad \rho_{i,j} = \rho \geq 0$$

If $-(n-1)^{-1} \leq \rho < 0$, then f' is decreasing if $z < -2^{-1}\rho(1-\rho)^{-1}\sum_{j=1}^{n}\sigma_j$ and increasing otherwise. We then have:

$$\beta_1 \geq \beta_2 \geq \beta_3 \nRightarrow \sigma_1 \geq \sigma_2 \geq \sigma_3 \quad \text{if} \quad \rho_{i,j} = \rho < 0$$

In fact, the result remains valid in most cases. To obtain a counter-example, we must have large differences between the volatilities and a correlation close to $-(n-1)^{-1}$. For example, if $\sigma_1 = 5\%$, $\sigma_2 = 6\%$, $\sigma_3 = 80\%$ and $\rho = -49\%$, we have $\beta_1 = -0.100$, $\beta_2 = -0.115$ and $\beta_3 = 3.215$.

Beta coefficient

Question 2.d

Find a general example such that $\beta_1 > \beta_2 > \beta_3$ and $\sigma_1 < \sigma_2 < \sigma_3$.

Beta coefficient

We assume that $\sigma_1 = 15\%$, $\sigma_2 = 20\%$, $\sigma_3 = 22\%$, $\rho_{1,2} = 70\%$, $\rho_{1,3} = 20\%$ and $\rho_{2,3} = -50\%$. It follows that $\beta_1 = 1.231$, $\beta_2 = 0.958$ and $\beta_3 = 0.811$. We thus have found an example such that $\beta_1 > \beta_2 > \beta_3$ and $\sigma_1 < \sigma_2 < \sigma_3$.

Beta coefficient

Question 2.e

Do we have $\sum_{i=1}^{n} \beta_i < n$ or $\sum_{i=1}^{n} \beta_i > n$ if the market portfolio is not equally weighted?

Beta coefficient

There is no reason that we have either $\sum_{i=1}^{n} \beta_i < n$ or $\sum_{i=1}^{n} \beta_i > n$. Let us consider the previous numerical example. If $b = (5\%, 25\%, 70\%)$, we obtain $\sum_{i=1}^{3} \beta_i = 1.808$ whereas if $b = (20\%, 40\%, 40\%)$, we have $\sum_{i=1}^{3} \beta_i = 3.126$.

Beta coefficient

Question 3

We search a market portfolio $b \in \mathbb{R}^n$ such that the betas are the same for all the assets: $\beta_i = \beta_j = \beta$.

Beta coefficient

Question 3.a

Show that there is an obvious solution which satisfies $\beta = 1$.

Beta coefficient

We have:

$$
\begin{aligned}
\sum_{i=1}^{n} b_i \beta_i &= \sum_{i=1}^{n} b_i \frac{(\Sigma b)_i}{b^\top \Sigma b} \\
&= b^\top \frac{\Sigma b}{b^\top \Sigma b} \\
&= 1
\end{aligned}
$$

If $\beta_i = \beta_j = \beta$, then $\beta = 1$ is an obvious solution because the previous relationship is satisfied:

$$
\sum_{i=1}^{n} b_i \beta_i = \sum_{i=1}^{n} b_i = 1
$$

Beta coefficient

Question 3.b

Show that this solution is unique and corresponds to the minimum variance portfolio.

Beta coefficient

If $\beta_i = \beta_j = \beta$, then we have:

$$\sum_{i=1}^{n} b_i \beta = 1 \Leftrightarrow \beta = \frac{1}{\sum_{i=1}^{n} b_i} = 1$$

β can only take one value, the solution is then unique. We know that the marginal volatilities are the same in the case of the minimum variance portfolio x (TR-RPB, page 173):

$$\frac{\partial\,\sigma(x)}{\partial\,x_i} = \frac{\partial\,\sigma(x)}{\partial\,x_j}$$

with $\sigma(x) = \sqrt{x^\top \Sigma x}$ the volatility of the portfolio x.

Beta coefficient

It follows that:

$$\frac{(\Sigma x)_i}{\sqrt{x^\top \Sigma x}} = \frac{(\Sigma x)_j}{\sqrt{x^\top \Sigma x}}$$

By dividing the two terms by $\sqrt{x^\top \Sigma x}$, we obtain:

$$\frac{(\Sigma x)_i}{x^\top \Sigma x} = \frac{(\Sigma x)_j}{x^\top \Sigma x}$$

The asset betas are then the same in the minimum variance portfolio. Because we have:

$$\left\{ \begin{array}{l} \beta_i = \beta_j \\ \sum_{i=1}^n x_i \beta_i = 1 \end{array} \right.$$

we deduce that:

$$\beta_i = 1$$

Beta coefficient

Question 4

We assume that $b \in [0, 1]^n$.

Beta coefficient

Question 4.a

Show that if one asset has a beta greater than one, there exists another asset which has a beta smaller than one.

Beta coefficient

We have:

$$\begin{aligned}
& \sum_{i=1}^{n} b_i \beta_i = 1 \\
\Leftrightarrow \quad & \sum_{i=1}^{n} b_i \beta_i = \sum_{i=1}^{n} b_i \\
\Leftrightarrow \quad & \sum_{i=1}^{n} b_i \beta_i - \sum_{i=1}^{n} b_i = 0 \\
\Leftrightarrow \quad & \sum_{i=1}^{n} b_i \left(\beta_i - 1\right) = 0
\end{aligned}$$

Beta coefficient

We obtain the following system of equations:

$$\left\{ \begin{array}{l} \sum_{i=1}^{n} b_i \left(\beta_i - 1 \right) = 0 \\ b_i \geq 0 \end{array} \right.$$

Let us assume that the asset j has a beta greater than 1. We then have:

$$\left\{ \begin{array}{l} b_j \left(\beta_j - 1 \right) + \sum_{i \neq j} b_i \left(\beta_i - 1 \right) = 0 \\ b_i \geq 0 \end{array} \right.$$

It follows that $b_j \left(\beta_j - 1 \right) > 0$ because $b_j > 0$ (otherwise the beta is zero). We must therefore have $\sum_{i \neq j} x_i \left(\beta_i - 1 \right) < 0$. Because $b_i \geq 0$, it is necessary that at least one asset has a beta smaller than 1.

Beta coefficient

Question 4.b

We consider the case $n = 3$. Find a covariance matrix Σ and a market portfolio b such that one asset has a negative beta.

Beta coefficient

We use standard notations to represent Σ. We seek a portfolio such that $\beta_1 > 0$, $\beta_2 > 0$ and $\beta_3 < 0$. To simplify this problem, we assume that the three assets have the same volatility. We also obtain the following system of inequalities:

$$\left\{ \begin{array}{l} b_1 + b_2\rho_{1,2} + b_3\rho_{1,3} > 0 \\ b_1\rho_{1,2} + b_2 + b_3\rho_{2,3} > 0 \\ b_1\rho_{1,3} + b_2\rho_{2,3} + b_3 < 0 \end{array} \right.$$

It is sufficient that $b_1\rho_{1,3} + b_2\rho_{2,3}$ is negative and b_3 is small. For example, we may consider $b_1 = 50\%$, $b_2 = 45\%$, $b_3 = 5\%$, $\rho_{1,2} = 50\%$, $\rho_{1,3} = 0\%$ and $\rho_{2,3} = -50\%$. We obtain $\beta_1 = 1.10$, $\beta_2 = 1.03$ and $\beta_3 = -0.27$.

Beta coefficient

Question 5

We report the return $R_{i,t}$ and $R_t(b)$ of asset i and market portfolio b at different dates:

t	1	2	3	4	5	6
$R_{i,t}$	−22	−11	−10	−8	13	11
$R_t(b)$	−26	−9	−10	−10	16	14
t	7	8	9	10	11	12
$R_{i,t}$	21	13	−30	−6	−5	−5
$R_t(b)$	14	15	−22	−7	−11	2
t	13	14	15	16	17	18
$R_{i,t}$	19	−17	2	−24	25	−7
$R_t(b)$	15	−15	−1	−23	15	−6

Beta coefficient

Question 5.a

Estimate the beta of the asset.

Beta coefficient

We perform the linear regression $R_{i,t} = \alpha_i + \beta_i R_t(b) + \varepsilon_{i,t}$ and we obtain $\hat{\beta}_i = 1.06$.

Beta coefficient

Question 5.b

What is the proportion of the asset volatility explained by the market?

Beta coefficient

We deduce that the contribution c_i of the market factor is (TR-RPB, page 16):

$$c_i = \frac{\beta_i^2 \operatorname{var}(R(b))}{\operatorname{var}(R_i)} = 90.62\%$$

Black-Litterman model

Exercise

We consider a universe of three assets. Their volatilities are 20%, 20% and 15%. The correlation matrix of asset returns is:

$$\rho = \begin{pmatrix} 1.00 & & \\ 0.50 & 1.00 & \\ 0.20 & 0.60 & 1.00 \end{pmatrix}$$

We would like to implement a trend-following strategy. For that, we estimate the trend of each asset and the volatility of the trend. We obtain the following results:

Asset	1	2	3
$\hat{\mu}$	10%	-5%	15%
$\sigma(\hat{\mu})$	4%	2%	10%

We assume that the neutral portfolio is the equally weighted portfolio.

Black-Litterman model

Question 1

Find the optimal portfolio if the constraint of the tracking error volatility is set to 1%, 2%, 3%, 4% and 5%.

Black-Litterman model

We consider the portfolio optimization problem in the presence of a benchmark (TR-RPB, page 17). We obtain the following results (expressed in %):

$\sigma(x^\star \mid b)$	1.00	2.00	3.00	4.00	5.00
$x_1^\star$	35.15	36.97	38.78	40.60	42.42
$x_2^\star$	26.32	19.30	12.28	5.26	-1.76
$x_3^\star$	38.53	43.74	48.94	54.14	59.34
$\mu(x^\star \mid b)$	1.31	2.63	3.94	5.25	6.56

Black-Litterman model

Question 2

In order to tilt the neutral portfolio, we now consider the Black-Litterman model. The risk-free rate is set to 0.

Black-Litterman model

Question 2.a

Find the implied risk premium of the assets if we target a Sharpe ratio equal to 0.50. What is the value of ϕ?

Black-Litterman model

Let b be the benchmark (that is the equally weighted portfolio). We recall that the implied risk aversion parameter is:

$$\phi = \frac{\mathrm{SR}\left(b \mid r\right)}{\sqrt{b^\top \Sigma b}}$$

and the implied risk premium is:

$$\tilde{\mu} = r + \mathrm{SR}\left(b \mid r\right) \frac{\Sigma b}{\sqrt{b^\top \Sigma b}}$$

We obtain $\phi = 3.4367$ and:

$$\tilde{\mu} = \left(\begin{array}{c} \tilde{\mu}_1 \\ \tilde{\mu}_2 \\ \tilde{\mu}_3 \end{array}\right) = \left(\begin{array}{c} 7.56\% \\ 8.94\% \\ 5.33\% \end{array}\right)$$

Black-Litterman model

Question 2.b

How does one incorporate a trend-following strategy in the Black-Litterman model? Give the P, Q and Ω matrices.

Black-Litterman model

In this case, the views of the portfolio manager corresponds to the trends observed in the market. We then have[4]:

$$\begin{aligned} P &= I_3 \\ Q &= \hat{\mu} \\ \Omega &= \operatorname{diag}\left(\sigma^2\left(\hat{\mu}_1\right), \ldots, \sigma^2\left(\hat{\mu}_n\right)\right) \end{aligned}$$

The views $P\mu = Q + \varepsilon$ become:

$$\mu = \hat{\mu} + \varepsilon$$

with $\varepsilon \sim \mathcal{N}\left(\mathbf{0}_3, \Omega\right)$.

[4] If we suppose that the trends are not correlated.

Black-Litterman model

Question 2.c

Calculate the conditional expectation $\bar{\mu} = \mathbb{E}\left[\mu \mid P\mu = Q + \varepsilon\right]$ if we assume that $\Gamma = \tau\Sigma$ and $\tau = 0.01$.

Black-Litterman model

We have (TR-RPB, page 25):

$$\begin{aligned}\bar{\mu} &= E\left[\mu \mid P\mu = Q + \varepsilon\right] \\ &= \tilde{\mu} + \Gamma P^\top \left(P\Gamma P^\top + \Omega\right)^{-1} \left(Q - P\tilde{\mu}\right) \\ &= \tilde{\mu} + \tau\Sigma \left(\tau\Sigma + \Omega\right)^{-1} \left(\hat{\mu} - \tilde{\mu}\right)\end{aligned}$$

We obtain:

$$\bar{\mu} = \begin{pmatrix} \bar{\mu}_1 \\ \bar{\mu}_2 \\ \bar{\mu}_3 \end{pmatrix} = \begin{pmatrix} 5.16\% \\ 2.38\% \\ 2.47\% \end{pmatrix}$$

Black-Litterman model

Question 2.d

Find the Black-Litterman optimized portfolio.

Black-Litterman model

We optimize the quadratic utility function with $\phi = 3.4367$. The Black-Litterman portfolio is then:

$$x^{\star} = \begin{pmatrix} x_1^{\star} \\ x_2^{\star} \\ x_3^{\star} \end{pmatrix} = \begin{pmatrix} 56.81\% \\ -23.61\% \\ 66.80\% \end{pmatrix}$$

Its volatility tracking error is $\sigma(x^{\star} \mid b) = 8.02\%$ and its alpha is $\mu(x^{\star} \mid b) = 10.21\%$.

Black-Litterman model

Question 3

We would like to compute the Black-Litterman optimized portfolio, corresponding to a 3% tracking error volatility.

Black-Litterman model

Question 3.a

What is the Black-Litterman portfolio when $\tau = 0$ and $\tau = +\infty$?

Black-Litterman model

- If $\tau = 0$, $\bar{\mu} = \tilde{\mu}$. The BL portfolio x is then equal to the neutral portfolio b.
- We also have:

$$
\begin{aligned}
\lim_{\tau \to \infty} \bar{\mu} &= \tilde{\mu} + \lim_{\tau \to \infty} \tau \Sigma^{\top} \left(\tau \Sigma + \Omega \right)^{-1} \left(\hat{\mu} - \tilde{\mu} \right) \\
&= \tilde{\mu} + \left(\hat{\mu} - \tilde{\mu} \right) \\
&= \hat{\mu}
\end{aligned}
$$

In this case, $\bar{\mu}$ is independent from the implied risk premium $\hat{\mu}$ and is exactly equal to the estimated trends $\hat{\mu}$. The BL portfolio x is then the Markowitz optimized portfolio with the given value of ϕ.

Black-Litterman model

Question 3.b

Using the previous results, apply the bisection algorithm and find the Black-Litterman optimized portfolio, which corresponds to a 3% tracking error volatility.

Black-Litterman model

We would like to find the BL portfolio such that $\sigma(x \mid b) = 3\%$. We know that $\sigma(x \mid b) = 0$ if $\tau = 0$. Thanks to Question 2.d, we also know that $\sigma(x \mid b) = 8.02\%$ if $\tau = 1\%$. It implies that the optimal portfolio corresponds to a specific value of τ which is between 0 and 1%. If we apply the bi-section algorithm, we find that:

$$\tau^\star = 0.242\%$$

. The composition of the optimal portfolio is then

$$x^\star = \begin{pmatrix} x_1^\star \\ x_2^\star \\ x_3^\star \end{pmatrix} = \begin{pmatrix} 41.18\% \\ 11.96\% \\ 46.85\% \end{pmatrix}$$

We obtain an alpha equal to 3.88%, which is a little bit smaller than the alpha of 3.94% obtained for the TE portfolio.

Black-Litterman model

Question 3.c

Compare the relationship between $\sigma(x \mid b)$ and $\mu(x \mid b)$ of the Black-Litterman model with the one of the tracking error model. Comment on these results.

Black-Litterman model

We have reported the relationship between $\sigma(x \mid b)$ and $\mu(x \mid b)$ in Figure 19. We notice that the information ratio of BL portfolios is very close to the information ratio of TE portfolios. We may explain that because of the homogeneity of the estimated trends $\hat{\mu}_i$ and the volatilities $\sigma(\hat{\mu}_i)$. If we suppose that $\sigma(\hat{\mu}_1) = 1\%$, $\sigma(\hat{\mu}_2) = 5\%$ and $\sigma(\hat{\mu}_3) = 15\%$, we obtain the relationship #2. In this case, the BL model produces a smaller information ratio than the TE model. We explain this because $\bar{\mu}$ is the right measure of expected return for the BL model whereas it is $\hat{\mu}$ for the TE model. We deduce that the ratios $\bar{\mu}_i / \hat{\mu}_i$ are more volatile for the parameter set #2, in particular when τ is small.

Black-Litterman model

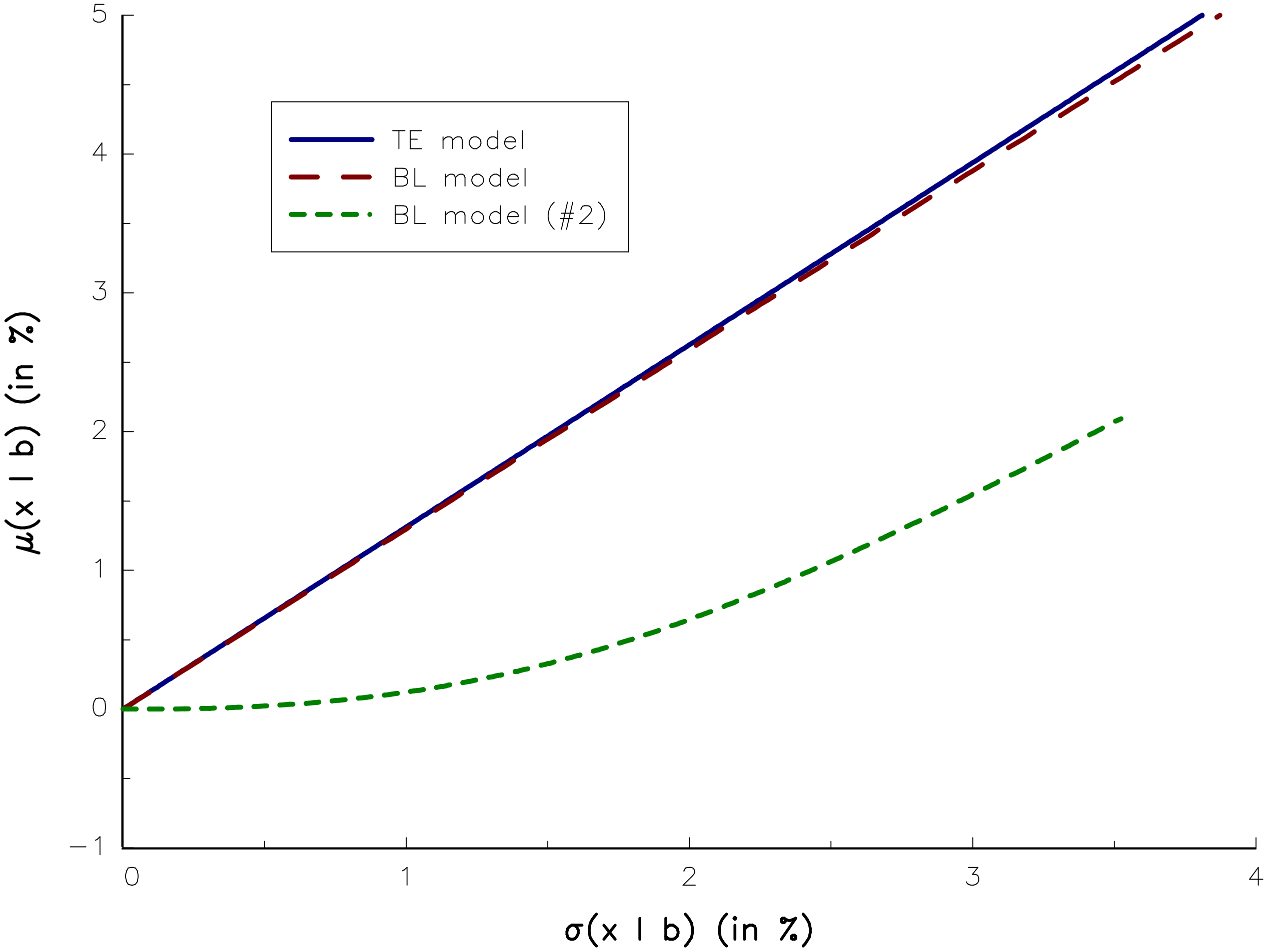


Figure 19: Efficient frontier of TE and BL portfolios

Main references

Roncalli, T. (2013)
Introduction to Risk Parity and Budgeting, Chapman and Hall/CRC Financial Mathematics Series, Chapter 1.

Roncalli, T. (2013)
Introduction to Risk Parity and Budgeting — Companion Book, Chapman and Hall/CRC Financial Mathematics Series, Chapter 1.

References I

- Black, F. and Litterman, R.B. (1992)
  Global Portfolio Optimization, *Financial Analysts Journal*, 48(5), pp. 28-43.
- Bourgeron, T., Lezmi, E., and Roncalli, T. (2018)
  Robust Asset Allocation for Robo-Advisors, *arXiv*, 1902.05710, `https://arxiv.org/abs/1902.07449`.
- Bruder, B., Gaussel, N., Richard, J.C., and Roncalli, T. (2013)
  Regularization of Portfolio Allocation, *SSRN*, `www.ssrn.com/abstract=2767358`.
- Jagannathan, R. and Ma, T. (2003)
  Risk Reduction in Large Portfolios: Why Imposing the Wrong Constraints Helps, *Journal of Finance*, 58(4), pp. 1651-1684.

References II

LALOUX, L., CIZEAU, P., BOUCHAUD, J-P. and POTTERS, M. (1999)
Noise Dressing of Financial Correlation Matrices, *Physical Review Letters*, 83(7), pp. 1467-1470.*

LEDOIT, O. and WOLF, M. (2003)
Improved Estimation of the Covariance Matrix of Stock Returns With an Application to Portfolio Selection, *Journal of Empirical Finance*, 10(5), pp. 603-621.

LEDOIT, O. and WOLF, M. (2004)
Honey, I Shrunk the Sample Covariance Matrix, *Journal of Portfolio Management*, 30(4), pp. 110-119.

MARKOWITZ, H. (1952)
Portfolio Selection, *Journal of Finance*, 7(1), pp. 77-91.

References III

Michaud, R.O. (1989)
The Markowitz Optimization Enigma: Is Optimized Optimal?, *Financial Analysts Journal*, 45(1), pp. 31-42.

Scherer, B. (2007)
Portfolio Construction & Risk Budgeting, Third edition, Risk Books.

Sharpe, W.F. (1964)
Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk, *Journal of Finance*, 19(3), pp. 425-442.

Stevens, G.V.G. (1998)
On the Inverse of the Covariance Matrix in Portfolio analysis, *Journal of Finance*, 53(5), pp. 1821-1827.

Tobin, J. (1958)
Liquidity Preference as Behavior Towards Risk, *Review of Economic Studies*, 25(2), pp. 65-86.

Course 2023-2024 in Portfolio Allocation and Asset Management
Lecture 2. Risk Budgeting

Thierry Roncalli*

*Amundi Asset Management[5]

*University of Paris-Saclay

January 2024

[5] The opinions expressed in this presentation are those of the authors and are not meant to represent the opinions or official positions of Amundi Asset Management.

Agenda

- Lecture 1: Portfolio Optimization
- **Lecture 2: Risk Budgeting**
- Lecture 3: Smart Beta, Factor Investing and Alternative Risk Premia
- Lecture 4: Equity Portfolio Optimization with ESG Scores
- Lecture 5: Climate Portfolio Construction
- Lecture 6: Equity and Bond Portfolio Optimization with Green Preferences
- Lecture 7: Machine Learning in Asset Management

Portfolio optimization & portfolio diversification

Example 1

- We consider an investment universe of 5 assets
- (μ_i, σ_i) are respectively equal to $(8\%, 12\%)$, $(7\%, 10\%)$, $(7.5\%, 11\%)$, $(8.5\%, 13\%)$ and $(8\%, 12\%)$
- The correlation matrix is $\mathcal{C}_5(\rho)$ with $\rho = 60\%$

The optimal portfolio $x^\star$ such that $\sigma(x^\star) = 10\%$ is equal to:

$$x^\star = \begin{pmatrix} 23.97\% \\ 6.42\% \\ 16.91\% \\ 28.73\% \\ 23.97\% \end{pmatrix}$$

Portfolio optimization & portfolio diversification

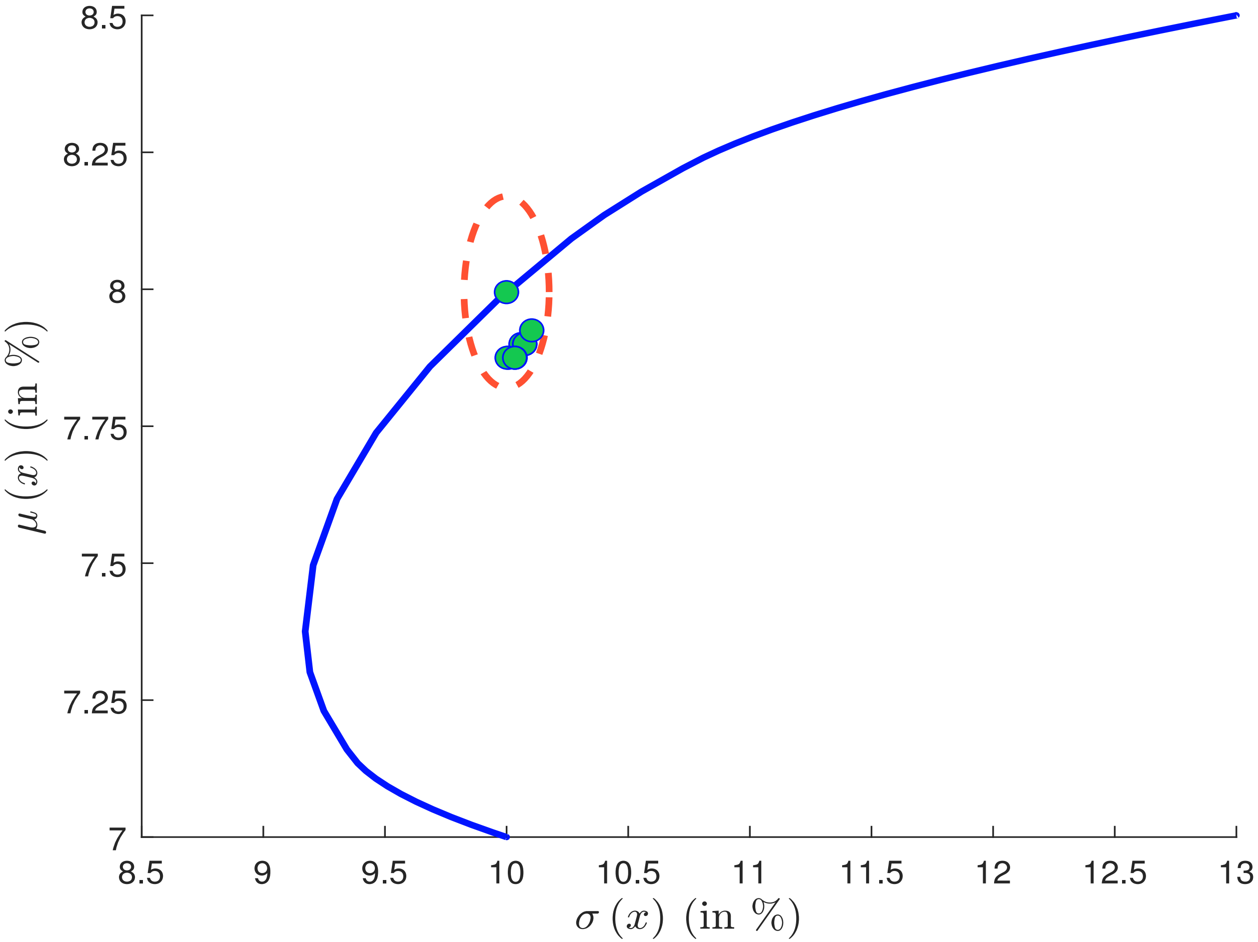


Figure 20: Optimized portfolios versus optimal diversified portfolios

Portfolio optimization & portfolio diversification

Table 22: Some equivalent mean-variance portfolios

x_1	23.97		5	5	35	35	50	5	5	10
x_2	6.42	25		25	10	25	10	30		25
x_3	16.91	5	40		10	5	15		45	10
x_4	28.73	35	20	30	5	35	10	35	20	45
x_5	23.97	35	35	40	40		15	30	30	10
$\mu(x)$	7.99	7.90	7.90	7.90	7.88	7.90	7.88	7.88	7.88	7.93
$\sigma(x)$	10.00	10.07	10.06	10.07	10.01	10.07	10.03	10.00	10.03	10.10

$\Rightarrow$ These portfolios have very different compositions, but lead to very close mean-variance features

Some of these portfolios appear more balanced and more diversified than the optimized portfolio

Other methods to build a portfolio

1. Weight budgeting (WB)
2. Risk budgeting (RB)
3. Performance budgeting (PB)

Ex-ante analysis
$\neq$
Ex-post analysis

Important result

$$RB = PB$$

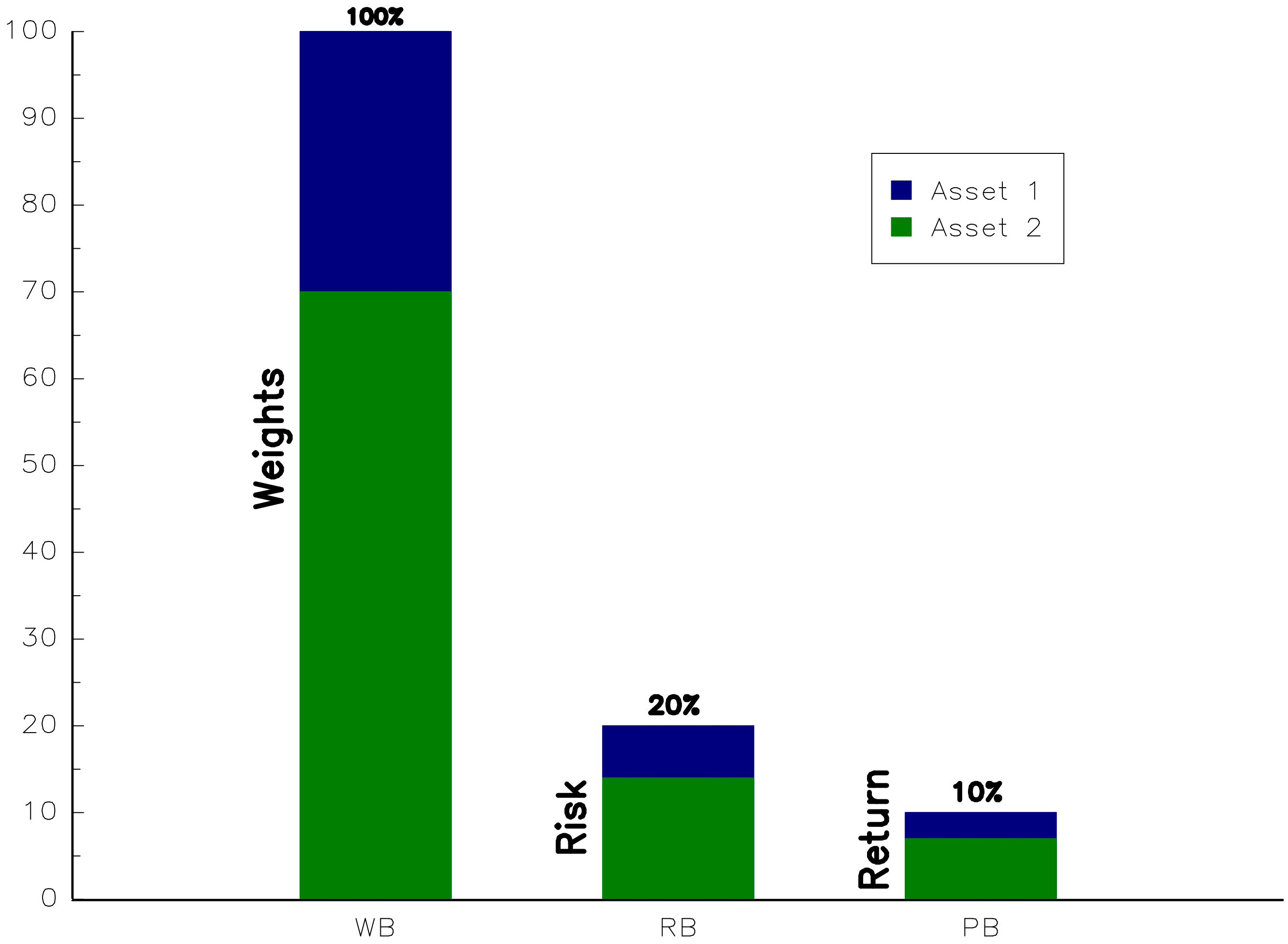


Figure 21: The 30/70 rule

Weight budgeting versus risk budgeting

Let $x = (x_1, \ldots, x_n)$ be the weights of n assets in the portfolio. Let $\mathcal{R}(x_1, \ldots, x_n)$ be a coherent and convex risk measure. We have:

$$\begin{aligned} \mathcal{R}(x_1, \ldots, x_n) &= \sum_{i=1}^{n} x_i \cdot \frac{\partial \, \mathcal{R}(x_1, \ldots, x_n)}{\partial \, x_i} \\ &= \sum_{i=1}^{n} \mathcal{RC}_i(x_1, \ldots, x_n) \end{aligned}$$

Let $b = (b_1, \ldots, b_n)$ be a vector of budgets such that $b_i \geq 0$ and $\sum_{i=1}^{n} b_i = 1$. We consider two allocation schemes:

1. Weight budgeting (WB)

$$x_i = b_i$$

2. Risk budgeting (RB)

$$\mathcal{RC}_i = b_i \cdot \mathcal{R}(x_1, \ldots, x_n)$$

Importance of the coherency and convexity properties

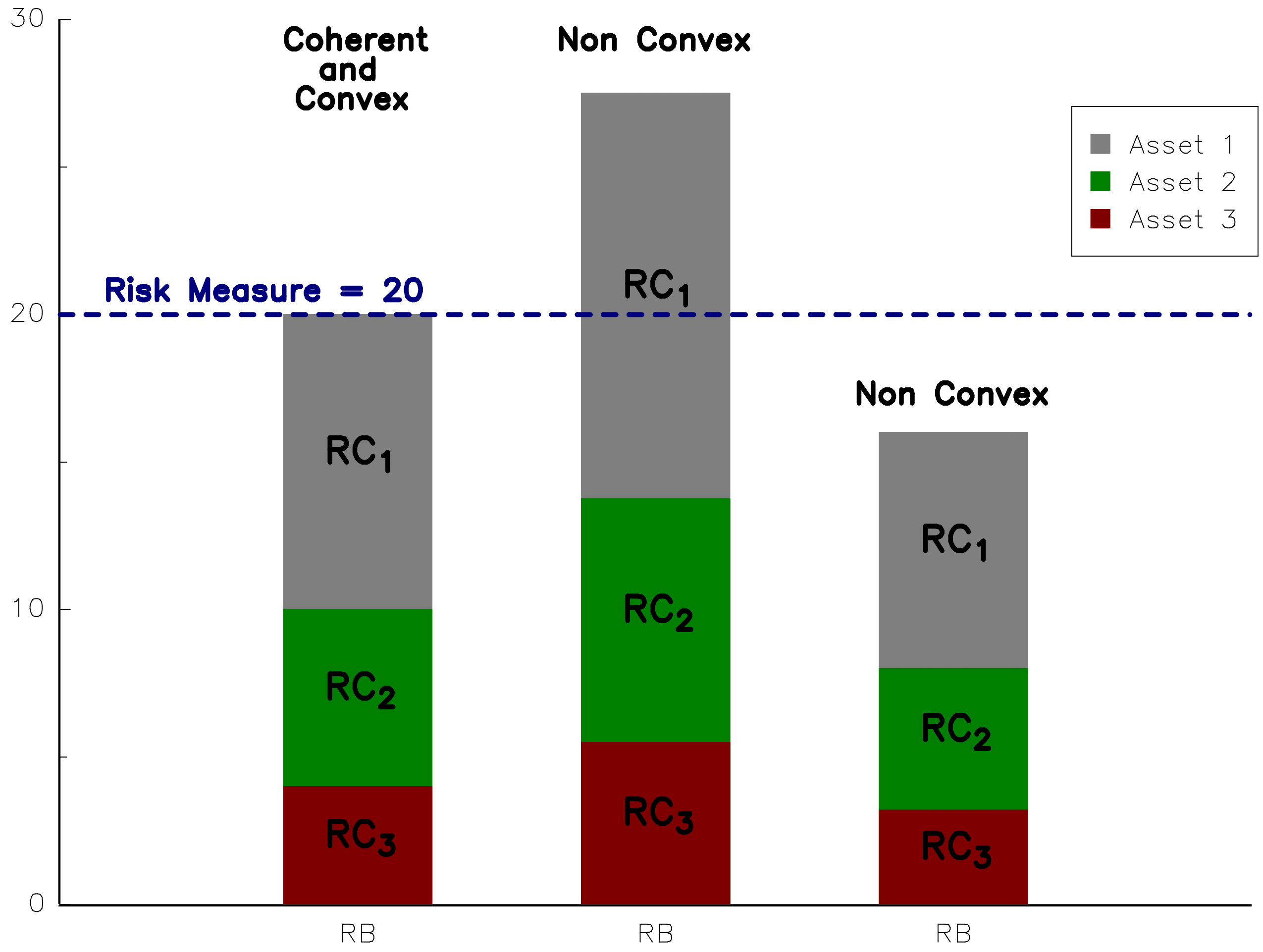


Figure 22: Risk Measure = 20 with a 50/30/20 budget rule

Application to the volatility risk measure

Let Σ be the covariance matrix of the assets returns. We note x the vector of the portfolio's weights:

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

It follows that the portfolio volatility is equal to:

$$\sigma(x) = \sqrt{x^\top \Sigma x}$$

Computation of the marginal volatilities

The vector of marginal volatilities is equal to:

$$\begin{aligned}
\frac{\partial\,\sigma(x)}{\partial\,x} &= \begin{pmatrix} \dfrac{\partial\,\sigma(x)}{\partial\,x_1} \\ \vdots \\ \dfrac{\partial\,\sigma(x)}{\partial\,x_n} \end{pmatrix} \\
&= \frac{\partial}{\partial\,x}\left(x^\top \Sigma x\right)^{1/2} \\
&= \frac{1}{2}\left(x^\top \Sigma x\right)^{1/2-1}(2\Sigma x) \\
&= \frac{\Sigma x}{\sqrt{x^\top \Sigma x}}
\end{aligned}$$

It follows that the marginal volatility of Asset i is given by:

$$\frac{\partial\,\sigma(x)}{\partial\,x_i} = \frac{(\Sigma x)_i}{\sqrt{x^\top \Sigma x}} = \sum_{j=1}^{n} \frac{\rho_{i,j}\sigma_i\sigma_j x_j}{\sigma(x)} = \sigma_i \sum_{j=1}^{n} x_j \frac{\rho_{i,j}\sigma_j}{\sigma(x)}$$

Computation of the risk contributions

We deduce that the risk contribution of the i^{th} asset is then:

$$\begin{aligned}\mathcal{RC}_i &= x_i \cdot \frac{\partial\,\sigma(x)}{\partial\,x_i} \\ &= \frac{x_i \cdot (\Sigma x)_i}{\sqrt{x^\top \Sigma x}} \\ &= \sigma_i x_i \sum_{j=1}^{n} x_j \frac{\rho_{i,j}\sigma_j}{\sigma(x)}\end{aligned}$$

The Euler allocation principle

We verify that the volatility satisfies the full allocation property:

$$\begin{aligned}
\sum_{i=1}^{n} \mathcal{RC}_i &= \sum_{i=1}^{n} \sigma_i x_i \sum_{j=1}^{n} x_j \frac{\rho_{i,j}\sigma_j}{\sigma(x)} = \frac{1}{\sigma(x)} \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j \rho_{i,j} \sigma_i \sigma_j \\
&= \frac{\sigma^2(x)}{\sigma(x)} = \sigma(x)
\end{aligned}$$

An alternative proof uses the definition of the dot product:

$$a \cdot b = \sum_{i=1}^{n} a_i b_i = a^\top b$$

Indeed, we have:

$$\sum_{i=1}^{n} \mathcal{RC}_i = \sum_{i=1}^{n} \frac{x_i \cdot (\Sigma x)_i}{\sqrt{x^\top \Sigma x}} = \frac{1}{\sqrt{x^\top \Sigma x}} \sum_{i=1}^{n} x_i \cdot (\Sigma x)_i = \frac{1}{\sqrt{x^\top \Sigma x}} x^\top \Sigma x = \sigma(x)$$

Definition of the risk contribution

Definition

The marginal risk contribution of Asset i is:

$$\mathcal{MR}_i = \frac{\partial\,\sigma\left(x\right)}{\partial\,x_i} = \frac{\left(\Sigma x\right)_i}{\sqrt{x^\top \Sigma x}}$$

The absolute risk contribution of Asset i is:

$$\mathcal{RC}_i = x_i \frac{\partial\,\sigma\left(x\right)}{\partial\,x_i} = \frac{x_i \cdot \left(\Sigma x\right)_i}{\sqrt{x^\top \Sigma x}}$$

The relative risk contribution of Asset i is:

$$\mathcal{RC}_i^\star = \frac{\mathcal{RC}_i}{\sigma\left(x\right)} = \frac{x_i \cdot \left(\Sigma x\right)_i}{x^\top \Sigma x}$$

The Euler allocation principle

Remark

We have $\sum_{i=1}^{n} \mathcal{RC}_i = \sigma(x)$ *and* $\sum_{i=1}^{n} \mathcal{RC}_i^{\star} = 100\%$.

Application

Example 2

We consider three assets. We assume that their expected returns are equal to zero whereas their volatilities are equal to 30%, 20% and 15%. The correlation of asset returns is given by the following matrix:

$$\rho = \begin{pmatrix} 1.00 & & \\ 0.80 & 1.00 & \\ 0.50 & 0.30 & 1.00 \end{pmatrix}$$

We consider the portfolio x, which is given by:

$$x = \begin{pmatrix} 50\% \\ 20\% \\ 30\% \end{pmatrix}$$

Application

Using the relationship $\Sigma_{i,j} = \rho_{i,j}\sigma_i\sigma_j$, we deduce that the covariance matrix is[6]:

$$\Sigma = \left(\begin{array}{ccc} 9.00 & 4.80 & 2.25 \\ 4.80 & 4.00 & 0.90 \\ 2.25 & 0.90 & 2.25 \end{array} \right) \times 10^{-2}$$

It follows that the variance of the portfolio is:

$$\begin{array}{rcl} \sigma^2(x) & = & 0.50^2 \times 0.09 + 0.20^2 \times 0.04 + 0.30^2 \times 0.0225 + \\ & & 2 \times 0.50 \times 0.20 \times 0.0480 + 2 \times 0.50 \times 0.30 \times 0.0225 + \\ & & 2 \times 0.20 \times 0.30 \times 0.0090 \\ & = & 4.3555\% \end{array}$$

The volatility is then $\sigma(x) = \sqrt{4.3555\%} = 20.8698\%$.

[6] The covariance term between assets 1 and 2 is equal to $\Sigma_{1,2} = 80\% \times 30\% \times 20\%$ or $\Sigma_{1,2} = 4.80\%$

Application

The computation of the marginal volatilities gives:

$$\frac{\Sigma x}{\sqrt{x^\top \Sigma x}} = \frac{1}{20.8698\%} \left(\begin{array}{r} 6.1350\% \\ 3.4700\% \\ 1.9800\% \end{array} \right) = \left(\begin{array}{r} 29.3965\% \\ 16.6269\% \\ 9.4874\% \end{array} \right)$$

Application

Finally, we obtain the risk contributions by multiplying the weights by the marginal volatilities:

$$x \circ \frac{\Sigma x}{\sqrt{x^\top \Sigma x}} = \begin{pmatrix} 50\% \\ 20\% \\ 30\% \end{pmatrix} \circ \begin{pmatrix} 29.3965\% \\ 16.6269\% \\ 9.4874\% \end{pmatrix} = \begin{pmatrix} 14.6982\% \\ 3.3254\% \\ 2.8462\% \end{pmatrix}$$

We verify that the sum of risk contributions is equal to the volatility:

$$\sum_{i=1}^{3} \mathcal{RC}_i = 14.6982\% + 3.3254\% + 2.8462\% = 20.8698\%$$

Application

Table 23: Risk decomposition of the portfolio's volatility (Example 2)

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^{\star}$
1	50.00	29.40	14.70	70.43
2	20.00	16.63	3.33	15.93
3	30.00	9.49	2.85	13.64
$\sigma(x)$			20.87	

The ERC portfolio

Definition

- Let Σ be the covariance matrix of asset returns
- The risk measure corresponds to the volatility risk measure
- The ERC portfolio is the **unique** portfolio x such that the risk contributions are equal:

$$\mathcal{RC}_i = \mathcal{RC}_j \Leftrightarrow \frac{x_i \cdot (\Sigma x)_i}{\sqrt{x^\top \Sigma x}} = \frac{x_j \cdot (\Sigma x)_j}{\sqrt{x^\top \Sigma x}}$$

ERC = Equal Risk Contribution

The concept of risk budgeting

Example 3

- 3 assets
- Volatilities are respectively equal to 20%, 30% and 15%
- Correlations are set to 60% between the 1st asset and the 2nd asset and 10% between the first two assets and the 3rd asset
- Budgets are set to 50%, 25% and 25%
- For the ERC (Equal Risk Contribution) portfolio, all the assets have the same risk budget

Weight budgeting (or traditional approach)

Asset	Weight	Marginal Risk	Risk Contribution Absolute	Risk Contribution Relative
1	**50.00%**	17.99%	9.00%	54.40%
2	**25.00%**	25.17%	6.29%	38.06%
3	**25.00%**	4.99%	1.25%	7.54%
Volatility			16.54%	

Risk budgeting approach

Asset	Weight	Marginal Risk	Risk Contribution Absolute	Risk Contribution Relative
1	41.62%	16.84%	7.01%	**50.00%**
2	15.79%	22.19%	3.51%	**25.00%**
3	42.58%	8.23%	3.51%	**25.00%**
Volatility			14.02%	

ERC approach

Asset	Weight	Marginal Risk	Risk Contribution Absolute	Risk Contribution Relative
1	30.41%	15.15%	4.61%	**33.33%**
2	20.28%	22.73%	4.61%	**33.33%**
3	49.31%	9.35%	4.61%	**33.33%**
Volatility			13.82%	

The concept of risk budgeting

We have:

$$\sigma\left(50\%, 25\%, 25\%\right) = 16.54\%$$

The marginal risk for the first asset is:

$$\frac{\partial\,\sigma\left(x\right)}{\partial\,x_1} = \lim_{\varepsilon \rightarrow 0} \frac{\sigma\left(x_1 + \varepsilon, x_2, x_3\right) - \sigma\left(x_1, x_2, x_3\right)}{\left(x_1 + \varepsilon\right) - x_1}$$

If $\varepsilon = 1\%$, we have:

$$\sigma\left(0.51, 0.25, 0.25\right) = 16.72\%$$

We deduce that:

$$\frac{\partial\,\sigma\left(x\right)}{\partial\,x_1} \simeq \frac{16.72\% - 16.54\%}{1\%} = 18.01\%$$

whereas the true value is equal to:

$$\frac{\partial\,\sigma\left(x\right)}{\partial\,x_1} = 17.99\%$$

The concept of risk budgeting

Example 4

- 3 assets
- Volatilities are respectively 30%, 20% and 15%
- Correlations are set to 80% between the 1^{st} asset and the 2^{nd} asset, 50% between the 1^{st} asset and the 3^{rd} asset and 30% between the 2^{nd} asset and the 3^{rd} asset

Weight budgeting (or traditional) approach

Asset	Weight	Marginal Risk	Risk Contribution Absolute	Risk Contribution Relative
1	**50.00%**	29.40%	14.70%	70.43%
2	**20.00%**	16.63%	3.33%	15.93%
3	**30.00%**	9.49%	2.85%	13.64%
Volatility			20.87%	

Risk budgeting approach

Asset	Weight	Marginal Risk	Risk Contribution Absolute	Risk Contribution Relative
1	31.15%	28.08%	8.74%	**50.00%**
2	21.90%	15.97%	3.50%	**20.00%**
3	46.96%	11.17%	5.25%	**30.00%**
Volatility			17.49%	

ERC approach

Asset	Weight	Marginal Risk	Risk Contribution Absolute	Risk Contribution Relative
1	19.69%	27.31%	5.38%	**33.33%**
2	32.44%	16.57%	5.38%	**33.33%**
3	47.87%	11.23%	5.38%	**33.33%**
Volatility			16.13%	

The concept of risk budgeting

Question

We assume that the portfolio's wealth is set to \$1 000. Calculate the nominal volatility of the previous WB, RB and ERC portfolios.

The concept of risk budgeting

We have:

$$\begin{aligned}\sigma\left(x_{\mathrm{wb}}\right) &= 10^{3} \times 20.87\% = \$208.7 \\ \sigma\left(x_{\mathrm{rb}}\right) &= 10^{3} \times 17.49\% = \$174.9 \\ \sigma\left(x_{\mathrm{erc}}\right) &= 10^{3} \times 16.13\% = \$161.3\end{aligned}$$

The concept of risk budgeting

Question

We increase the exposure of the 3 portfolios by \$10 as follows:

$$\Delta x = \begin{pmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \end{pmatrix} = \begin{pmatrix} \$1 \\ \$5 \\ \$4 \end{pmatrix}$$

Calculate the nominal volatility of these new portfolios.

The concept of risk budgeting

By assuming that $\Delta x \simeq 0$, we have:

$$\begin{array}{rcl}\sigma\left(x_{\mathrm{wb}}+\Delta x\right) & \approx & (\$500+\$1)\times 0.2940 + \\ & & (\$200+\$5)\times 0.1663 + \\ & & (\$300+\$4)\times 0.0949 \\ & \approx & \$210.2\end{array}$$

$\sigma\left(x_{\mathrm{rb}}+\Delta x\right)\approx \$176.4$ and $\sigma\left(x_{\mathrm{erc}}+\Delta x\right)\approx \$162.9$.

Uniform correlation

- We assume a constant correlation matrix $\mathcal{C}_n(\rho)$, meaning that $\rho_{i,j} = \rho$ for all $i \neq j$
- We have:

$$
\begin{aligned}
(\Sigma x)_i &= \sum_{k=1}^{n} \rho_{i,k}\sigma_i\sigma_k x_k \\
&= \sigma_i^2 x_i + \rho\sigma_i \sum_{k\neq i} \sigma_k x_k \\
&= \sigma_i^2 x_i + \rho\sigma_i \sum_{k=1}^{n} \sigma_k x_k - \rho\sigma_i^2 x_i \\
&= (1-\rho)\, x_i\sigma_i^2 + \rho\sigma_i \sum_{k=1}^{n} x_k\sigma_k \\
&= \sigma_i \left((1-\rho)\, x_i\sigma_i + \rho \sum_{k=1}^{n} x_k\sigma_k \right)
\end{aligned}
$$

Uniform correlation

- Since we have:

$$\mathcal{RC}_i = \frac{x_i \left(\Sigma x\right)_i}{\sigma\left(x\right)}$$

we deduce that $\mathcal{RC}_i = \mathcal{RC}_j$ is equivalent to:

$$x_i \sigma_i \left(\left(1 - \rho\right) x_i \sigma_i + \rho \sum_{k=1}^{n} x_k \sigma_k \right) = x_j \sigma_j \left(\left(1 - \rho\right) x_j \sigma_j + \rho \sum_{k=1}^{n} x_k \sigma_k \right)$$

It follows that $x_i \sigma_i = x_j \sigma_j$. Because $\sum_{i=1}^{n} x_i = 1$, we deduce that:

$$x_i = \frac{\sigma_i^{-1}}{\sum_{j=1}^{n} \sigma_j^{-1}}$$

Result

The weight allocated to Asset i is inversely proportional to its volatility and does not depend on the value of the correlation

Minimum uniform correlation

- The global minimum variance portfolio is equal to:

$$x_{\mathrm{gmv}} = \frac{\Sigma^{-1}\mathbf{1}_n}{\mathbf{1}_n^\top \Sigma^{-1}\mathbf{1}_n}$$

- Let $\Sigma = \sigma\sigma^\top \circ \mathcal{C}_n(\rho)$ be the covariance matrix with $\mathcal{C}_n(\rho)$ the constant correlation matrix
- We have:

$$\Sigma^{-1} = \Gamma \circ \mathcal{C}_n^{-1}(\rho)$$

with $\Gamma_{i,j} = \sigma_i^{-1}\sigma_j^{-1}$ and:

$$\mathcal{C}_n^{-1}(\rho) = \frac{\rho\mathbf{1}_n\mathbf{1}_n^\top - ((n-1)\rho + 1) I_n}{(n-1)\rho^2 - (n-2)\rho - 1}$$

Minimum uniform correlation

- We deduce that the expression of the GMV weights are:

$$x_{\mathrm{gmv},i}=\frac{-\left(\left(n-1\right)\rho+1\right)\sigma_i^{-2}+\rho\sum_{j=1}^{n}\left(\sigma_i\sigma_j\right)^{-1}}{\sum_{k=1}^{n}\left(-\left(\left(n-1\right)\rho+1\right)\sigma_k^{-2}+\rho\sum_{j=1}^{n}\left(\sigma_k\sigma_j\right)^{-1}\right)}$$

- The lower bound of $\mathcal{C}_n\left(\rho\right)$ is achieved for $\rho=-\left(n-1\right)^{-1}$
- In this case, the solution becomes:

$$x_{\mathrm{gmv},i}=\frac{\sum_{j=1}^{n}\left(\sigma_i\sigma_j\right)^{-1}}{\sum_{k=1}^{n}\sum_{j=1}^{n}\left(\sigma_k\sigma_j\right)^{-1}}=\frac{\sigma_i^{-1}}{\sum_{k=1}^{n}\sigma_k^{-1}}$$

Result

The ERC portfolio is equal to the GMV portfolio when the correlation is at its lowest possible value:

$$\lim_{\rho\rightarrow-\left(n-1\right)^{-1}}x_{\mathrm{gmv}}=x_{\mathrm{erc}}$$

Uniform volatility

- If all volatilities are equal, i.e. $\sigma_i = \sigma$ for all i, the risk contribution becomes:

$$\mathcal{RC}_i = \frac{\left(\sum_{k=1}^{n} x_i x_k \rho_{i,k}\right) \sigma^2}{\sigma(x)}$$

- The ERC portfolio verifies then:

$$x_i \left(\sum_{k=1}^{n} x_k \rho_{i,k}\right) = x_j \left(\sum_{k=1}^{n} x_k \rho_{j,k}\right)$$

- We deduce that:

$$x_i = \frac{\left(\sum_{k=1}^{n} x_k \rho_{i,k}\right)^{-1}}{\sum_{j=1}^{n} \left(\sum_{k=1}^{n} x_k \rho_{j,k}\right)^{-1}}$$

Uniform volatility

Result

The weight of asset i is inversely proportional to the weighted average of correlations of Asset i

Remark

Contrary to the previous case, this solution is endogenous since x_i is a function of itself directly

General case

- In the general case, we have:

$$\beta_i = \beta(\mathbf{e}_i \mid x) = \frac{\mathbf{e}_i^\top \Sigma x}{x^\top \Sigma x} = \frac{(\Sigma x)_i}{\sigma^2(x)}$$

and:

$$\mathcal{RC}_i = \frac{x_i (\Sigma x)_i}{\sigma(x)} = \sigma(x)\, x_i \beta_i$$

- We deduce that $\mathcal{RC}_i = \mathcal{RC}_j$ is equivalent to:

$$x_i \beta_i = x_j \beta_j$$

- It follows that:

$$x_i = \frac{\beta_i^{-1}}{\sum_{j=1}^n \beta_j^{-1}}$$

General case

- We notice that:

$$\sum_{i=1}^{n} x_i \beta_i = \sum_{i=1}^{n} \frac{\mathcal{RC}_i}{\sigma(x)} = \frac{1}{\sigma(x)} \sum_{i=1}^{n} \mathcal{RC}_i = 1$$

and:

$$\sum_{i=1}^{n} x_i \beta_i = \sum_{i=1}^{n} \left(\frac{1}{\sum_{j=1}^{n} \beta_j^{-1}} \right) = 1$$

It follows that:

$$\frac{1}{\sum_{j=1}^{n} \beta_j^{-1}} = \frac{1}{n}$$

- We finally obtain:

$$x_i = \frac{1}{n \beta_i}$$

General case

Result

The weight of Asset i is proportional to the inverse of its beta:

$$x_i \propto \beta_i^{-1}$$

Remark

This solution is endogenous since x_i is a function of itself because $\beta_i = \beta\left(\mathbf{e}_i \mid x\right)$.

General case

Example 5

We consider an investment universe of four assets with $\sigma_1 = 15\%$, $\sigma_2 = 20\%$, $\sigma_3 = 30\%$ and $\sigma_4 = 10\%$. The correlation of asset returns is given by the following matrix:

$$\rho = \begin{pmatrix} 1.00 & & & \\ 0.50 & 1.00 & & \\ 0.00 & 0.20 & 1.00 & \\ -0.10 & 0.40 & 0.70 & 1.00 \end{pmatrix}$$

General case

Table 24: Composition of the ERC portfolio (Example 5)

Asset	x_i	$\mathcal{MR}_i$	β_i	$\mathcal{RC}_i$	$\mathcal{RC}_i^{\star}$
1	31.34%	8.52%	0.80	2.67%	25.00%
2	17.49%	15.27%	1.43	2.67%	25.00%
3	13.05%	20.46%	1.92	2.67%	25.00%
4	38.12%	7.00%	0.66	2.67%	25.00%
Volatility				10.68%	

We verify that:

$$x_1 = \frac{1}{(4 \times 0.7978)} = 31.34\%$$

Existence and uniqueness

We consider the following optimization problem:

$$\begin{aligned} y^{\star}(c) &= \arg\min \frac{1}{2} y^{\top} \Sigma y \\ \text{u.c.} & \quad \sum_{i=1}^{n} \ln y_i \geq c \end{aligned}$$

The Lagrange function is equal to:

$$\mathcal{L}(y; \lambda_c) = \frac{1}{2} y^{\top} \Sigma y - \lambda_c \left(\sum_{i=1}^{n} \ln y_i - c \right)$$

At the optimum, we have:

$$\frac{\partial \mathcal{L}(y; \lambda_c, \lambda)}{\partial y} = \mathbf{0}_n \Leftrightarrow (\Sigma y)_i - \frac{\lambda_c}{y_i} = 0$$

Existence and uniqueness

It follows that:

$$y_i \left(\Sigma y \right)_i = \lambda_c$$

or equivalently:

$$\mathcal{RC}_i = \mathcal{RC}_j$$

Since we minimize a convex function subject to a lower convex bound, the solution $y^\star(c)$ exists and is unique

Existence and uniqueness

Question

What is the difference between $y^{\star}(c)$ and $y^{\star}(c')$?

Let $y' = \alpha y^{\star}(c)$. The first-order conditions are:

$$y_i^{\star}(c)\left(\Sigma y^{\star}(c)\right)_i = \lambda_c$$

and:

$$y_i'\left(\Sigma y'\right)_i = \alpha^2 \lambda_c = \lambda_{c'}$$

Since $\lambda_c \neq 0$, the Kuhn-Tucker condition becomes:

$$\min\left(\lambda_c, \sum_{i=1}^{n} \ln y_i^{\star}(c) - c\right) = 0 \Leftrightarrow \sum_{i=1}^{n} \ln y_i^{\star}(c) - c = 0$$

Existence and uniqueness

It follows that:

$$\sum_{i=1}^{n} \ln \frac{y_i'(c)}{\alpha} = c$$

or:

$$\sum_{i=1}^{n} \ln y_i'(c) = c + n \ln \alpha = c'$$

We deduce that:

$$\alpha = \exp\left(\frac{c' - c}{n}\right)$$

$y^\star(c')$ is a scaled solution of $y^\star(c)$:

$$y^\star(c') = \exp\left(\frac{c' - c}{n}\right) y^\star(c)$$

Existence and uniqueness

The ERC portfolio is the solution $y^{\star}(c)$ such that $\sum_{i=1}^{n} y_i^{\star}(c) = 1$:

$$x_{\mathrm{erc}} = \frac{y^{\star}(c)}{\sum_{i=1}^{n} y_i^{\star}(c)}$$

and corresponds to the following value of the logarithmic barrier:

$$c_{\mathrm{erc}} = c - n \ln \sum_{i=1}^{n} y_i^{\star}(c)$$

Existence and uniqueness

Theorem

Because of the previous results, x_{erc} exists and is unique. This is the solution of the following optimization problem[a]:

$$\begin{aligned} x_{\mathrm{erc}} & = & \arg\min \frac{1}{2} x^\top \Sigma x \\ & \text{u.c.} & \left\{ \begin{array}{l} \sum_{i=1}^{n} \ln x_i \geq c_{\mathrm{erc}} \\ \mathbf{1}_n^\top x = 1 \\ \mathbf{0}_n \leq x \leq \mathbf{1}_n \end{array} \right. \end{aligned}$$

[a] We can add the last two constraints because they do not change the solution

Location of the ERC portfolio

The global global minimum variance portfolio is defined by:

$$\begin{aligned} x_{\mathrm{gmv}} &= \arg\min \sigma\left(x\right) \\ \text{u.c.} &\quad \mathbf{1}_n^\top x = 1 \end{aligned}$$

We have:

$$\mathcal{L}\left(x;\lambda_0\right) = \sigma\left(x\right) - \lambda_0\left(\mathbf{1}_n^\top x - 1\right)$$

The first-order condition is:

$$\frac{\partial\,\mathcal{L}\left(x;\lambda_0\right)}{\partial\,x} = \mathbf{0}_n \Leftrightarrow \frac{\partial\,\sigma\left(x\right)}{\partial\,x} - \lambda_0\mathbf{1}_n = \mathbf{0}_n$$

Location of the ERC portfolio

Theorem

The global minimum variance portfolio satisfies:

$$\frac{\partial \sigma (x)}{\partial x_i} = \frac{\partial \sigma (x)}{\partial x_j}$$

The marginal volatilities are then the same.

Location of the ERC portfolio

The equally-weighted portfolio is defined by:

$$x_i = \frac{1}{n}$$

We deduce that:

$$x_i = x_j$$

Location of the ERC portfolio

We have:

$$x_i = x_j \tag{EW}$$

$$\frac{\partial \, \sigma(x)}{\partial \, x_i} = \frac{\partial \, \sigma(x)}{\partial \, x_j} \tag{GMV}$$

$$x_i \frac{\partial \, \sigma(x)}{\partial \, x_i} = x_j \frac{\partial \, \sigma(x)}{\partial \, x_j} \tag{ERC}$$

The ERC portfolio is a combination of GMV and EW portfolios

Volatility of the ERC portfolio

We consider the following optimization problem:

$$\begin{aligned} x^{\star}(c) &= \arg\min \frac{1}{2} x^{\top} \Sigma x \\ &\text{u.c.} \quad \begin{cases} \sum_{i=1}^{n} \ln x_i \geq c \\ \mathbf{1}_n^{\top} x = 1 \\ \mathbf{0}_n \leq x \leq \mathbf{1}_n \end{cases} \end{aligned}$$

- We know that there exists a scalar c_{erc} such that:

$$x^{\star}(c_{\mathrm{erc}}) = x_{\mathrm{erc}}$$

- If $c = -\infty$, the logarithmic barrier constraint vanishes and we have:

$$x^{\star}(-\infty) = x_{\mathrm{mv}}$$

where x_{mv} is the long-only minimum variance portfolio

Volatility of the ERC portfolio

- We notice that the function $f(x) = \sum_{i=1}^{n} \ln x_i$ such that $\mathbf{1}_n^\top x = 1$ reaches its maximum when:

$$\frac{1}{x_i} = \lambda_0$$

implying that $x_i = x_j = n^{-1}$. In this case, we have:

$$c_{\max} = \sum_{i=1}^{n} \ln \frac{1}{n} = -n \ln n$$

- If $c = -n \ln n$, we have:

$$x^\star(-n \ln n) = x_{\mathrm{ew}}$$

- Because we have a convex minimization problem and a lower convex bound, we deduce that:

$$c_2 \geq c_1 \Leftrightarrow \sigma(x^\star(c_2)) \geq \sigma(x^\star(c_1))$$

Volatility of the ERC portfolio

Theorem

We obtain the following inequality:

$$\sigma\left(x_{\mathrm{mv}}\right) \leq \sigma\left(x_{\mathrm{erc}}\right) \leq \sigma\left(x_{\mathrm{ew}}\right)$$

The ERC portfolio may be viewed as a portfolio "between" the MV portfolio and the EW portfolio.

Remark

The ERC portfolio is a form of variance-minimizing portfolio subject to a constraint of sufficient diversification in terms of weights

Relationship with naive diversification $(1/n)$

Optimality of the ERC portfolio

Let us consider the tangency (or maximum Sharpe ratio) portfolio defined by:

$$x_{\mathrm{msr}} = \arg\max \frac{\mu(x) - r}{\sigma(x)}$$

where $\mu(x) = x^\top \mu$ and $\sigma(x) = \sqrt{x^\top \Sigma x}$. We recall that the portfolio is MSR if and only if:

$$\frac{\partial_{x_i} \mu(x) - r}{\partial_{x_i} \sigma(x)} = \frac{\mu(x) - r}{\sigma(x)}$$

Therefore, the MSR portfolio x_{msr} verifies the following relationship:

$$\begin{aligned} \mu - r\mathbf{1}_n &= \left(\frac{\mu(x_{\mathrm{msr}}) - r}{\sigma^2(x_{\mathrm{msr}})}\right) \Sigma x_{\mathrm{msr}} \\ &= \mathrm{SR}(x_{\mathrm{msr}} \mid r) \frac{\Sigma x_{\mathrm{msr}}}{\sigma(x_{\mathrm{msr}})} \end{aligned}$$

Optimality of the ERC portfolio

- If we assume a constant correlation matrix, the ERC portfolio is defined by:

$$x_i = \frac{c}{\sigma_i}$$

where $c = \left(\sum_{j=1}^n \sigma_j^{-1}\right)^{-1}$

- We have:

$$(\Sigma x)_i = \sum_{j=1}^n \rho_{i,j}\sigma_i\sigma_j x_j = c\sigma_i \sum_{j=1}^n \rho_{i,j} = c\sigma_i \left(1 + \rho\left(n-1\right)\right)$$

- We deduce that:

$$\frac{\partial\,\sigma\left(x\right)}{\partial\,x_i} = c\frac{\sigma_i\left(\left(1-\rho\right)+\rho n\right)}{\sigma\left(x\right)}$$

Optimality of the ERC portfolio

- The portfolio volatility is equal to:

$$\begin{aligned}
\sigma^2(x) &= \sigma(x)\sum\nolimits_{i=1}^{n} x_i \frac{\partial\,\sigma(x)}{\partial\,x_i} \\
&= \sigma(x)\sum\nolimits_{i=1}^{n} \frac{c}{\sigma_i}\cdot c\frac{\sigma_i\left((1-\rho)+\rho n\right)}{\sigma(x)} \\
&= nc^2\left((1-\rho)+\rho n\right)
\end{aligned}$$

- The ERC portfolio is the MSR portfolio if and only if:

$$\begin{aligned}
\mu_i - r &= \left(\frac{\sum_{j=1}^{n}\left(\mu_j - r\right)x_j}{\sigma^2(x)}\right)(\Sigma x)_i \\
&= \left(\frac{\sum_{j=1}^{n}\left(\mu_j - r\right)c\sigma_j^{-1}}{nc^2\left((1-\rho)+\rho n\right)}\right)c\sigma_i\left(1+\rho(n-1)\right) \\
&= \left(\frac{1}{n}\sum_{j=1}^{n}\frac{\mu_j - r}{\sigma_j}\right)\sigma_i
\end{aligned}$$

Optimality of the ERC portfolio

- We can write this condition as follows:

$$\mu_i = r + \mathrm{SR} \cdot \sigma_i$$

where:

$$\mathrm{SR} = \frac{1}{n} \sum_{j=1}^{n} \frac{\mu_j - r}{\sigma_j}$$

Theorem

The ERC portfolio is the tangency or MSR portfolio if and only if the correlation is uniform and the Sharpe ratio is the same for all the assets

Optimality of the ERC portfolio

Example 6

We consider an investment universe of five assets. The volatilities are respectively equal to 5%, 7%, 9%, 10% and 15%. The risk-free rate is equal to 2%. The correlation is uniform.

Optimality of the ERC portfolio

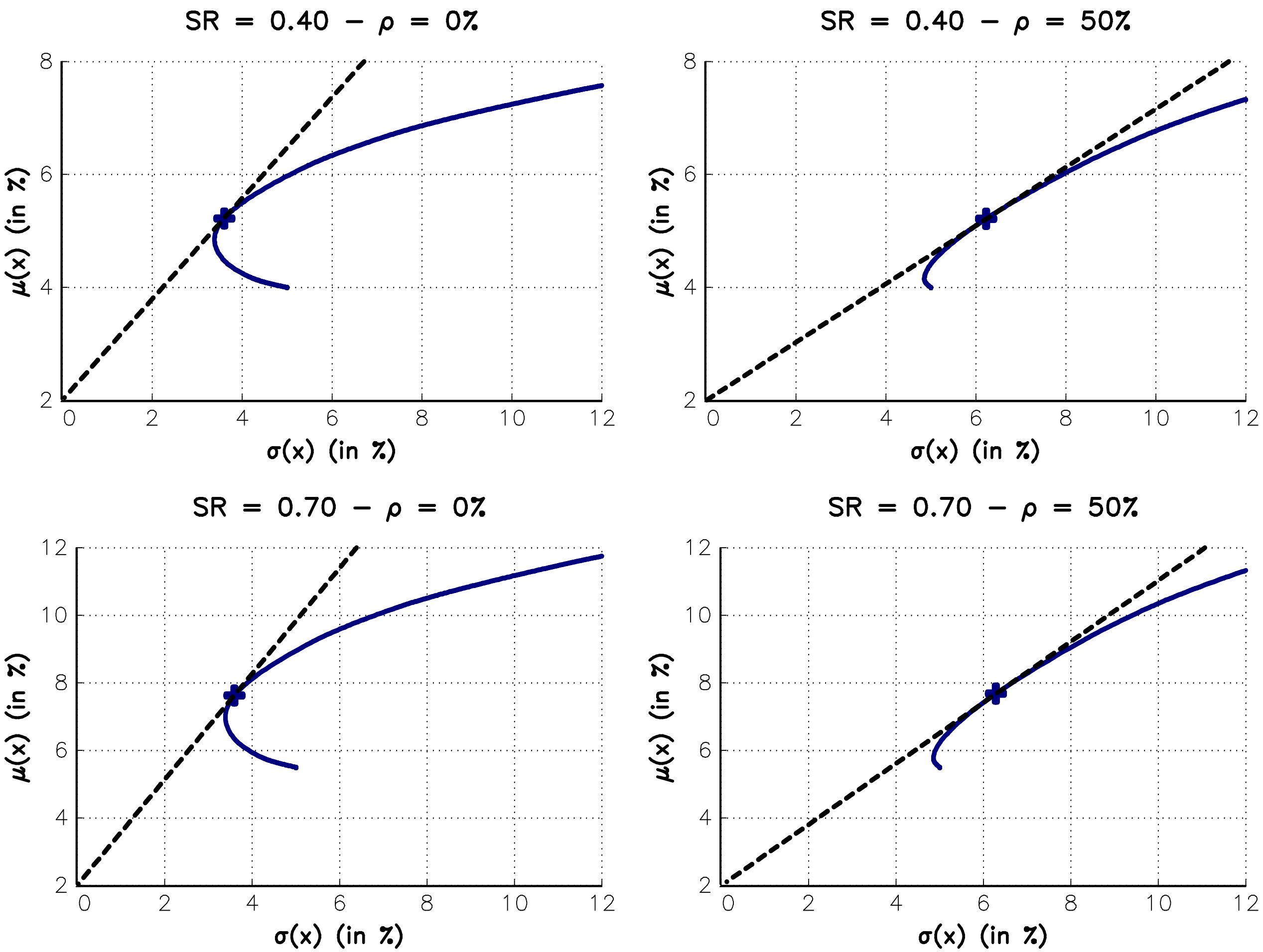


Figure 23: Location of the ERC portfolio in the mean-variance diagram when the Sharpe ratios are the same (Example 6)

Optimality of the ERC portfolio

Example 7

We consider an investment universe of five assets. The volatilities are respectively equal to 5%, 7%, 9%, 10% and 15%. The correlation matrix is equal to:

$$\rho = \begin{pmatrix} 1.00 & & & & \\ 0.50 & 1.00 & & & \\ 0.25 & 0.25 & 1.00 & & \\ 0.00 & 0.00 & 0.00 & 1.00 & \\ -0.25 & -0.25 & -0.25 & 0.00 & 1.00 \end{pmatrix}$$

Optimality of the ERC portfolio

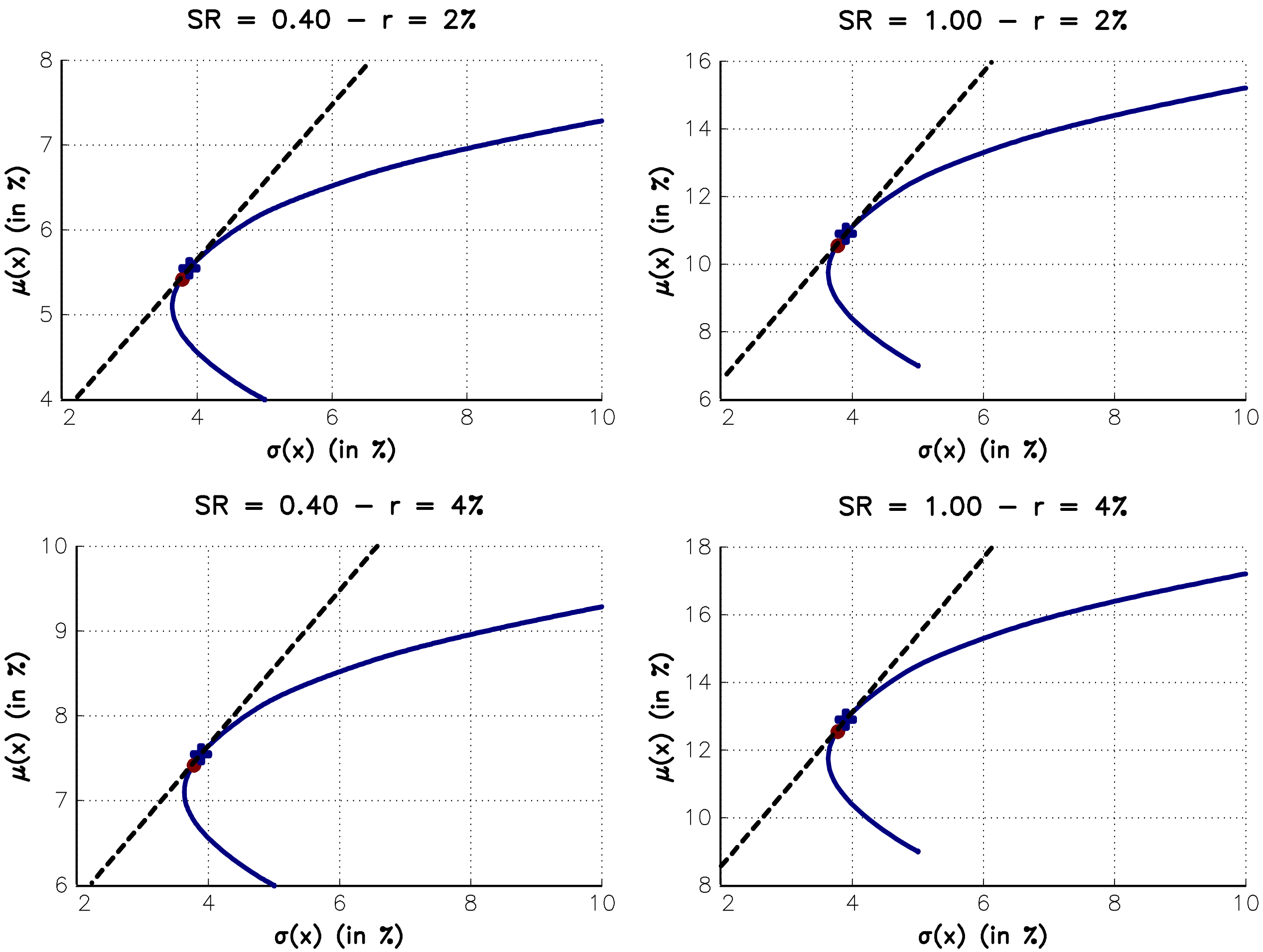


Figure 24: Location of the ERC portfolio in the mean-variance diagram when the Sharpe ratios are the same (Example 7)

The SQP approach

- The ERC portfolio satisfies:

$$x_i \cdot (\Sigma x)_i = x_j \cdot (\Sigma x)_j$$

or:

$$x_i \cdot (\Sigma x)_i = \frac{x^\top \Sigma x}{n}$$

- We deduce that:

$$\begin{aligned} x_{\mathrm{erc}} &= \arg\min f(x) \\ &\text{u.c.} \quad \left\{ \begin{array}{l} \mathbf{1}_n^\top x = 1 \\ \mathbf{0}_n \leq x \leq \mathbf{1}_n \end{array} \right. \end{aligned}$$

and $f(x_{\mathrm{erc}}) = 0$

Remark

The optimization problem is solved using the sequential quadratic programming (or SQP) algorithm

The SQP approach

- We can choose:

$$f(x) = \sum_{i=1}^{n} \left(x_i \cdot (\Sigma x)_i - \frac{1}{n} x^\top \Sigma x \right)^2$$

or:

$$f(x; b) = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(x_i \cdot (\Sigma x)_i - x_j \cdot (\Sigma x)_j \right)^2$$

The Jacobi approach

- We have:

$$\beta_i(x) = \frac{(\Sigma x)_i}{x^\top \Sigma x}$$

- The ERC portfolio satisfies:

$$x_i = \frac{\beta_i^{-1}(x)}{\sum_{j=1}^n \beta_j^{-1}(x)}$$

or:

$$x_i \propto \frac{1}{(\Sigma x)_i}$$

The Jacobi approach

The Jacobi algorithm consists in finding the fixed point by considering the following iterations:

1. We set $k \leftarrow 0$ and we note $x^{(0)}$ the vector of starting values[7]
2. At iteration $k+1$, we compute:

$$y_i^{(k+1)} \propto \frac{1}{\beta_i\left(x^{(k)}\right)} = \frac{1}{\left(\Sigma x^{(k)}\right)_i}$$

   and:

$$x_i^{(k+1)} = \frac{y_i^{(k+1)}}{\sum_{j=1}^n y_j^{(k+1)}}$$

3. We iterate Step 2 until convergence

[7] For instance, we can use the following rule:

$$x_i^{(0)} = \frac{\sigma_i^{-1}}{\sum_{j=1}^n \sigma_j^{-1}}$$

The Newton-Raphson approach

We consider the following optimization problem:

$$x^{\star} = \arg\min f(x)$$

The Newton-Raphson iteration is defined by:

$$x^{(k+1)} = x^{(k)} - \Delta x^{(k)}$$

where $\Delta x^{(k)}$ is the inverse of the Hessian matrix of $f\left(x^{(k)}\right)$ times the gradient vector of $f\left(x^{(k)}\right)$:

$$\Delta x^{(k)} = \left[\partial_x^2 f\left(x^{(k)}\right)\right]^{-1} \partial_x f\left(x^{(k)}\right)$$

The Newton-Raphson approach

- We consider the Lagrange function:

$$f(y) = \frac{1}{2} y^\top \Sigma y - \lambda_c \sum_{i=1}^n \ln y_i$$

- We choose a value of λ_c (e.g. $\lambda_c = 1$)
- We note y^{-m} the vector $n \times 1$ matrix with elements $\left(y_1^{-m}, \ldots, y_n^{-m}\right)$ and $\operatorname{diag}\left(y^{-m}\right)$ the $n \times n$ diagonal matrix with elements $\left(y_1^{-m}, \ldots, y_n^{-m}\right)$:

$$\operatorname{diag}\left(y^{-m}\right) = \begin{pmatrix} y_1^{-m} & 0 & & 0 \\ 0 & y_2^{-m} & & \\ & & \ddots & 0 \\ 0 & & 0 & y_n^{-m} \end{pmatrix}$$

The Newton-Raphson approach

- We apply the Newton-Raphson algorithm with:

$$\partial_y f(y) = \Sigma y - \lambda_c y^{-1}$$

and:

$$\partial_y^2 f(y) = \Sigma + \lambda_c \operatorname{diag}\left(y^{-2}\right)$$

- The solution is given by:

$$x_{\mathrm{erc}} = \frac{y^\star}{\sum_{i=1}^n y_i^\star}$$

The Newton-Raphson approach

- For the starting value $y_i^{(0)}$, we can assume that the correlations are uniform:

$$y_i^{(0)} = \frac{\sigma_i^{-1}}{\sum_{j=1}^n \sigma_j^{-1}}$$

- At the optimum, we recall that $\lambda_c = y_i^\star \cdot (\Sigma y^\star)_i$. We deduce that:

$$\lambda_c = \frac{1}{n} \sum_{i=1}^n y_i^\star \cdot (\Sigma y^\star)_i = \frac{\sigma^2 (y^\star)}{n}$$

Therefore, we can choose:

$$\lambda_c = \frac{\sigma^2 \left(y^{(0)}\right)}{n}$$

The Newton-Raphson approach

- From a numerical point of view, it may be important to control the magnitude order α of $y^{\star}$ (e.g. $\alpha = 10\%$, $\alpha = 1$ or $\alpha = 10$). For instance, we don't want that the magnitude order is 10^{-5} or 10^{5}. In this case, we can use the following rule:

$$\lambda_c = n\alpha^2\sigma^2\left(x_{\mathrm{erc}}\right)$$

- For example, if $n = 10$ and $\alpha = 5$, and we guess that the volatility of the ERC portfolio is around 10%, we set:

$$\lambda_c = 10 \times 5^2 \times 0.10^2 = 2.5$$

The CCD approach

Table 25: Cyclical coordinate descent algorithm

The goal is to find the solution $x^{\star} = \arg\min f(x)$
We initialize the vector $x^{(0)}$
Set $k \leftarrow 0$
repeat
  for $i = 1 : n$ **do**
    $x_i^{(k+1)} = \arg\min_{\varkappa} f\left(x_1^{(k+1)}, \ldots, x_{i-1}^{(k+1)}, \varkappa, x_{i+1}^{(k)}, \ldots, x_n^{(k)}\right)$
  end for
  $k \leftarrow k + 1$
until convergence
return $x^{\star} \leftarrow x^{(k)}$

The CCD approach

We have:

$$\mathcal{L}(y;\lambda_c) = \arg\min \frac{1}{2} y^\top \Sigma y - \lambda_c \sum_{i=1}^{n} \ln y_i$$

The first-order condition is equal to:

$$\frac{\partial\,\mathcal{L}(y;\lambda)}{\partial\, y_i} = (\Sigma y)_i - \frac{\lambda_c}{y_i} = 0$$

or:

$$y_i \cdot (\Sigma y)_i - \lambda_c = 0$$

It follows that:

$$\sigma_i^2 y_i^2 + \left(\sigma_i \sum_{j \neq i} \rho_{i,j} \sigma_j y_j\right) y_i - \lambda_c = 0$$

The CCD approach

We recognize a second-degree equation:

$$\alpha_i y_i^2 + \beta_i y_i + \gamma_i = 0$$

1. The polynomial function is convex because we have $\alpha_i = \sigma_i^2 > 0$
2. The product of the roots is negative:
$$y_i' y_i'' = \frac{\gamma_i}{\alpha_i} = -\frac{\lambda_c}{\sigma_i^2} < 0$$
3. The discriminant is positive:
$$\Delta = \beta_i^2 - 4\alpha_i\gamma_i = \left(\sigma_i \sum_{j \neq i} \rho_{i,j}\sigma_j y_j\right)^2 + 4\sigma_i^2\lambda_c > 0$$

We always have two solutions with opposite signs. We deduce that the solution is the positive root of the second-degree equation:

$$y_i^\star = y_i'' = \frac{-\beta_i + \sqrt{\beta_i^2 - 4\alpha_i\gamma_i}}{2\alpha_i}$$

The CCD approach

The CCD algorithm consists in iterating the following formula:

$$y_i^{(k+1)} = \frac{-\beta_i^{(k+1)} + \sqrt{\left(\beta_i^{(k+1)}\right)^2 - 4\alpha_i^{(k+1)}\gamma_i^{(k+1)}}}{2\alpha_i^{(k+1)}}$$

where:

$$\begin{array}{rcl} \alpha_i^{(k+1)} & = & \sigma_i^2 \\ \beta_i^{(k+1)} & = & \sigma_i \left(\sum_{j<i} \rho_{i,j}\sigma_j y_j^{(k+1)} + \sum_{j>i} \rho_{i,j}\sigma_j y_j^{(k)}\right) \\ \gamma_i^{(k+1)} & = & -\lambda_c \end{array}$$

The ERC portfolio is the scaled solution $y^\star$:

$$x_{\mathrm{erc}} = \frac{y^\star}{\sum_{i=1}^n y_i^\star}$$

Efficiency of the algorithms

$$\text{CCD} \succ \text{NR} \succ \text{SQP} \succ \text{Jacobi}$$

Definition of RB portfolios

Definition

A risk budgeting (RB) portfolio x satisfies the following conditions:

$$\left\{\begin{array}{c} \mathcal{RC}_1 = b_1 \mathcal{R}(x) \\ \vdots \\ \mathcal{RC}_i = b_i \mathcal{R}(x) \\ \vdots \\ \mathcal{RC}_n = b_n \mathcal{R}(x) \end{array}\right.$$

where $\mathcal{R}(x)$ is a coherent and convex risk measure and $b = (b_1, \ldots, b_n)$ is a vector of risk budgets such that $b_i \geq 0$ and $\sum_{i=1}^n b_i = 1$

Definition of RB portfolios

Remark

The ERC portfolio is a particular case of RB portfolios when $\mathcal{R}(x) = \sigma(x)$ and $b_i = \dfrac{1}{n}$

Coherent risk measure

1. Subadditivity
$$\mathcal{R}(x_1 + x_2) \leq \mathcal{R}(x_1) + \mathcal{R}(x_2)$$
2. Homogeneity
$$\mathcal{R}(\lambda x) = \lambda \mathcal{R}(x) \quad \text{if } \lambda \geq 0$$
3. Monotonicity
$$\text{if } x_1 \prec x_2, \text{ then } \mathcal{R}(x_1) \geq \mathcal{R}(x_2)$$
4. Translation invariance
$$\text{if } m \in \mathbb{R}, \text{ then } \mathcal{R}(x + m) = \mathcal{R}(x) - m$$

Convex risk measure

The convexity property is defined as follows:

$$\mathcal{R}\left(\lambda x_1 + (1-\lambda)\, x_2\right) \leq \lambda \mathcal{R}\left(x_1\right) + (1-\lambda)\, \mathcal{R}\left(x_2\right)$$

This condition means that diversification should not increase the risk

Euler allocation principle

This property is necessary for the Euler allocation principle:

$$\mathcal{R}\left(x\right) = \sum_{i=1}^{n} x_i \frac{\partial\, \mathcal{R}\left(x\right)}{\partial\, x_i}$$

Some risk measures

The portfolio loss is $L(x) = -R(x)$ where $R(x)$ is the portfolio return. We consider then different risk measures:

- Volatility of the loss

$$\mathcal{R}(x) = \sigma(L(x)) = \sigma(x)$$

- Standard deviation-based risk measure

$$\mathcal{R}(x) = \mathrm{SD}_c(x) = \mathbb{E}[L(x)] + c \cdot \sigma(L(x)) = -\mu(x) + c \cdot \sigma(x)$$

- Value-at-risk

$$\mathcal{R}(x) = \mathrm{VaR}_\alpha(x) = \inf\{\ell : \Pr\{L(x) \leq \ell\} \geq \alpha\}$$

- Expected shortfall

$$\mathcal{R}(x) = \mathrm{ES}_\alpha(x) = \mathbb{E}[L(x) \mid L(x) \geq \mathrm{VaR}_\alpha(x)] = \frac{1}{1-\alpha}\int_\alpha^1 \mathrm{VaR}_u(x)\,\mathrm{d}u$$

Gaussian risk measures

We assume that the asset returns are normally distributed: $R \sim \mathcal{N}(\mu, \Sigma)$

We have:

$$\begin{aligned}
\sigma(x) &= \sqrt{x^\top \Sigma x} \\
\mathrm{SD}_c(x) &= -x^\top \mu + c \cdot \sqrt{x^\top \Sigma x} \\
\mathrm{VaR}_\alpha(x) &= -x^\top \mu + \Phi^{-1}(\alpha) \sqrt{x^\top \Sigma x} \\
\mathrm{ES}_\alpha(x) &= -x^\top \mu + \frac{\sqrt{x^\top \Sigma x}}{(1-\alpha)} \phi\left(\Phi^{-1}(\alpha)\right)
\end{aligned}$$

Gaussian risk contributions

- Volatility $\sigma(x)$

$$\mathcal{RC}_i = x_i \cdot \frac{(\Sigma x)_i}{\sqrt{x^\top \Sigma x}}$$

- Standard deviation-based risk measure $\mathrm{SD}_c(x)$

$$\mathcal{RC}_i = x_i \cdot \left(-\mu_i + c \frac{(\Sigma x)_i}{\sqrt{x^\top \Sigma x}}\right)$$

- Value-at-risk $\mathrm{VaR}_\alpha(x)$

$$\mathcal{RC}_i = x_i \cdot \left(-\mu_i + \Phi^{-1}(\alpha) \frac{(\Sigma x)_i}{\sqrt{x^\top \Sigma x}}\right)$$

- Expected shortfall $\mathrm{ES}_\alpha(x)$

$$\mathcal{RC}_i = x_i \cdot \left(-\mu_i + \frac{(\Sigma x)_i}{(1-\alpha)\sqrt{x^\top \Sigma x}} \phi\left(\Phi^{-1}(\alpha)\right)\right)$$

Gaussian risk contributions

Example 8

We consider three assets. We assume that their expected returns are equal to zero whereas their volatilities are equal to 30%, 20% and 15%. The correlation of asset returns is given by the following matrix:

$$\rho = \begin{pmatrix} 1.00 & & \\ 0.80 & 1.00 & \\ 0.50 & 0.30 & 1.00 \end{pmatrix}$$

The portfolio is equal to $(50\%, 20\%, 30\%)$.

Gaussian risk contributions

Table 26: Risk decomposition of the portfolio (Example 8)

$\mathcal{R}(x)$	Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
Volatility	1	50.00	29.40	14.70	70.43
	2	20.00	16.63	3.33	15.93
	3	30.00	9.49	2.85	13.64
	$\sigma(x)$			20.87	
Value-at-risk	1	50.00	68.39	34.19	70.43
	2	20.00	38.68	7.74	15.93
	3	30.00	22.07	6.62	13.64
	$\mathrm{VaR}_{99\%}(x)$			48.55	
Expected shortfall	1	50.00	78.35	39.17	70.43
	2	20.00	44.31	8.86	15.93
	3	30.00	25.29	7.59	13.64
	$\mathrm{ES}_{99\%}(x)$			55.62	

Gaussian risk contributions

Example 9

We consider three assets. We assume that their expected returns are equal to 10%, 5% and 8% whereas their volatilities are equal to 30%, 20% and 15%. The correlation of asset returns is given by the following matrix:

$$\rho = \begin{pmatrix} 1.00 & & \\ 0.80 & 1.00 & \\ 0.50 & 0.30 & 1.00 \end{pmatrix}$$

The portfolio is equal to $(50\%, 20\%, 30\%)$.

Gaussian risk contributions

Table 27: Risk decomposition of the portfolio (Example 9)

$\mathcal{R}(x)$	Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^{\star}$
Volatility	1	50.00	29.40	14.70	70.43
	2	20.00	16.63	3.33	15.93
	3	30.00	9.49	2.85	13.64
	$\sigma(x)$			20.87	
Value-at-risk	1	50.00	58.39	29.19	72.71
	2	20.00	33.68	6.74	16.78
	3	30.00	14.07	4.22	10.51
	$\mathrm{VaR}_{99\%}(x)$			40.15	
Expected shortfall	1	50.00	68.35	34.17	72.37
	2	20.00	39.31	7.86	16.65
	3	30.00	17.29	5.19	10.98
	$\mathrm{ES}_{99\%}(x)$			47.22	

Non-Gaussian risk contributions

They are not frequently used in asset management and portfolio allocation, except in the case of skewed assets (Bruder *et al.*, 2016; Lezmi *et al.*, 2018)

Non-parametric risk contributions are given in Chapter 2 in Roncalli (2013)

Gaussian RB portfolios

Example 10

We consider three assets. We assume that their expected returns are equal to 10%, 5% and 8% whereas their volatilities are equal to 30%, 20% and 15%. The correlation of asset returns is given by the following matrix:

$$\rho = \begin{pmatrix} 1.00 & & \\ 0.80 & 1.00 & \\ 0.50 & 0.30 & 1.00 \end{pmatrix}$$

The risk budgets are equal to $(50\%, 20\%, 30\%)$.

Gaussian RB portfolios

Table 28: Risk budgeting portfolios (Example 10)

$\mathcal{R}(x)$	Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^{\star}$
Volatility	1	31.14	28.08	8.74	50.00
	2	21.90	15.97	3.50	20.00
	3	46.96	11.17	5.25	30.00
	$\sigma(x)$			17.49	
Value-at-risk	1	29.18	54.47	15.90	50.00
	2	20.31	31.30	6.36	20.00
	3	50.50	18.89	9.54	30.00
	$\mathrm{VaR}_{99\%}(x)$			31.79	
Expected shortfall	1	29.48	64.02	18.87	50.00
	2	20.54	36.74	7.55	20.00
	3	49.98	22.65	11.32	30.00
	$\mathrm{ES}_{99\%}(x)$			37.74	

Special cases

- The case of uniform correlation[8] $\rho_{i,j} = \rho$
  1. Minimum correlation

$$x_i\left(-\frac{1}{n-1}\right) = \frac{\sigma_i^{-1}}{\sum_{j=1}^n \sigma_j^{-1}}$$

  2. Zero correlation

$$x_i(0) = \frac{\sqrt{b_i}\sigma_i^{-1}}{\sum_{j=1}^n \sqrt{b_j}\sigma_j^{-1}}$$

  3. Maximum correlation

$$x_i(1) = \frac{b_i\sigma_i^{-1}}{\sum_{j=1}^n b_j\sigma_j^{-1}}$$

- The general case

$$x_i = \frac{b_i\beta_i^{-1}}{\sum_{j=1}^n b_j\beta_j^{-1}}$$

where β_i is the beta of Asset i with respect to the RB portfolio

[8] The solution is noted $x_i(\rho)$.

Existence and uniqueness

We have:

$$\frac{\partial \sigma(x)}{\partial x_i} = \frac{x_i \sigma_i^2 + \sigma_i \sum_{j \neq i} x_j \rho_{i,j} \sigma_j}{\sigma(x)}$$

Suppose that the risk budget b_k is equal to zero. This means that:

$$x_k \left(x_k \sigma_k^2 + \sigma_k \sum_{j \neq k} x_j \rho_{k,j} \sigma_j \right) = 0$$

We obtain two solutions:

1. The first one is:
$$x_k' = 0$$
2. The second one verifies:
$$x_k'' = -\frac{\sum_{j \neq k} x_j \rho_{k,j} \sigma_j}{\sigma_k}$$

Existence and uniqueness

- If $\rho_{k,j} \geq 0$ for all j, we have $\sum_{j \neq k} x_j \rho_{k,j} \sigma_j \geq 0$ because $x_j \geq 0$ and $\sigma_j > 0$. This implies that $x''_k \leq 0$ meaning that $x'_k = 0$ is the unique positive solution
- The only way to have $x''_k > 0$ is to have some negative correlations $\rho_{k,j}$. In this case, this implies that:

$$\sum_{j \neq k} x_j \rho_{k,j} \sigma_j < 0$$

- If we consider a universe of three assets, this constraint is verified for $k = 3$ and a covariance matrix such that $\rho_{1,3} < 0$ and $\rho_{2,3} < 0$

Existence and uniqueness

Example 11

We have $\sigma_1 = 20\%$, $\sigma_2 = 10\%$, $\sigma_3 = 5\%$, $\rho_{1,2} = 50\%$, $\rho_{1,3} = -25\%$ and $\rho_{2,3} = -25\%$

If the risk budgets are equal to $(50\%, 50\%, 0\%)$, the two solutions are:

$$(33.33\%, 66.67\%, 0\%)$$

and:

$$(20\%, 40\%, 40\%)$$

Two questions

1. How many solutions do we have in the general case?
2. Which solution is the best?

Existence and uniqueness

Table 29: First solution (Example 11)

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1	33.33	17.32	5.77	50.00
2	66.67	8.66	5.77	50.00
3	0.00	−1.44	0.00	0.00
Volatility			11.55	

Table 30: Second solution (Example 11)

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1	20.00	16.58	3.32	50.00
2	40.00	8.29	3.32	50.00
3	40.00	0.00	0.00	0.00
Volatility			6.63	

Existence and uniqueness

The case with strictly positive risk budgets

- We consider the following optimization problem:

$$\begin{aligned} y^\star &= \arg\min \mathcal{R}(y) \\ \text{u.c.} & \left\{ \begin{array}{l} \sum_{i=1}^n b_i \ln y_i \geq c \\ y \geq \mathbf{0}_n \end{array} \right. \end{aligned}$$

  where c is an arbitrary constant

- The associated Lagrange function is:

$$\mathcal{L}(y;\lambda,\lambda_c) = \mathcal{R}(y) - \lambda^\top y - \lambda_c \left(\sum_{i=1}^n b_i \ln y_i - c \right)$$

  where $\lambda \in \mathbb{R}^n$ and $\lambda_c \in \mathbb{R}$

Existence and uniqueness

The case with strictly positive risk budgets

- The solution $y^\star$ verifies the following first-order condition:

$$\frac{\partial\,\mathcal{L}\left(y;\lambda,\lambda_c\right)}{\partial\,y_i}=\frac{\partial\,\mathcal{R}\left(y\right)}{\partial\,y_i}-\lambda_i-\lambda_c\frac{b_i}{y_i}=0$$

- The Kuhn-Tucker conditions are:

$$\left\{\begin{array}{l}\min\left(\lambda_i,y_i\right)=0\\ \min\left(\lambda_c,\sum_{i=1}^n b_i\ln y_i-c\right)=0\end{array}\right.$$

Existence and uniqueness

The case with strictly positive risk budgets

- Because $\ln y_i$ is not defined for $y_i = 0$, it follows that $y_i > 0$ and $\lambda_i = 0$
- We note that the constraint $\sum_{i=1}^{n} b_i \ln y_i = c$ is necessarily reached (because the solution cannot be $y^{\star} = \mathbf{0}_n$), then $\lambda_c > 0$ and we have:

$$y_i \frac{\partial \mathcal{R}(y)}{\partial y_i} = \lambda_c b_i$$

- We verify that the risk contributions are proportional to the risk budgets:

$$\mathcal{RC}_i = \lambda_c b_i$$

Existence and uniqueness

The case with strictly positive risk budgets

Theorem

The optimization program has a unique solution and the RB portfolio is equal to:

$$x_{\mathrm{rb}} = \frac{y^\star}{\sum_{i=1}^n y_i^\star}$$

Remark

We note that the convexity property of the risk measure is essential to the existence and uniqueness of the RB portfolio. If $\mathcal{R}(x)$ is not convex, the preceding analysis becomes invalid.

Existence and uniqueness

Effect on the solution of setting risk budgets to zero

- Let $\mathcal{N}$ be the set of assets such that $b_i = 0$
- The Lagrange function becomes:

$$\mathcal{L}(y; \lambda, \lambda_c) = \mathcal{R}(y) - \lambda^\top y - \lambda_c \left(\sum_{i \notin \mathcal{N}} b_i \ln y_i - c \right)$$

Existence and uniqueness

Effect on the solution of setting risk budgets to zero

- The solution $y^{\star}$ verifies the following first-order conditions:

$$\frac{\partial\,\mathcal{L}\left(y;\lambda,\lambda_c\right)}{\partial\,y_i}=\left\{\begin{array}{lll}\partial_{y_i}\,\mathcal{R}\left(y\right)-\lambda_i-\lambda_c b_i y_i^{-1}=0 & \text{if} & i\notin\mathcal{N}\\ \partial_{y_i}\,\mathcal{R}\left(y\right)-\lambda_i=0 & \text{if} & i\in\mathcal{N}\end{array}\right.$$

  - If $i\notin\mathcal{N}$, the previous analysis is valid and we verify that risk contributions are proportional to the risk budgets:

$$y_i\frac{\partial\,\mathcal{R}\left(y\right)}{\partial\,y_i}=\lambda_c b_i$$

  - If $i\in\mathcal{N}$, we must distinguish two cases:
    1. If $y_i=0$, it implies that $\lambda_i>0$ and $\partial_{y_i}\,\mathcal{R}\left(y\right)>0$
    2. In the other case, if $y_i>0$, it implies that $\lambda_i=0$ and $\partial_{y_i}\,\mathcal{R}\left(y\right)=0$
  - The solution $y_i=0$ or $y_i>0$ if $i\in\mathcal{N}$ will then depend on the structure of the covariance matrix Σ (in the case of a Gaussian risk measure)

Existence and uniqueness

Effect on the solution of setting risk budgets to zero

Theorem

We conclude that the solution $y^{\star}$ of the optimization problem exists and is unique even if some risk budgets are set to zero. As previously, we deduce the normalized RB portfolio x_{rb} by scaling $y^{\star}$. This solution, noted $\mathcal{S}_1$, satisfies the following relationships:

$$\left\{ \begin{array}{ll} \mathcal{RC}_i = x_i \cdot \partial_{x_i} \mathcal{R}(x) = b_i & \text{if} \quad i \notin \mathcal{N} \\ \left\{ \begin{array}{ll} x_i = 0 \text{ and } \partial_{x_i} \mathcal{R}(x) > 0 & (i) \\ \text{or} & \\ x_i > 0 \text{ and } \partial_{x_i} \mathcal{R}(x) = 0 & (ii) \end{array} \right. & \text{if} \quad i \in \mathcal{N} \end{array} \right.$$

The conditions (i) and (ii) are mutually exclusive for one asset $i \in \mathcal{N}$, but not necessarily for all the assets $i \in \mathcal{N}$.

Existence and uniqueness

Effect on the solution of setting risk budgets to zero

The previous analysis implies that there may be several solutions to the following non-linear system when $b_i = 0$ for $i \in \mathcal{N}$:

$$\left\{\begin{array}{c} \mathcal{RC}_1 = b_1 \mathcal{R}(x) \\ \vdots \\ \mathcal{RC}_i = b_i \mathcal{R}(x) \\ \vdots \\ \mathcal{RC}_n = b_n \mathcal{R}(x) \end{array}\right.$$

- Let $\mathcal{N} = \mathcal{N}_1 \bigsqcup \mathcal{N}_2$ where $\mathcal{N}_1$ is the set of assets verifying the condition (i) and $\mathcal{N}_2$ is the set of assets verifying the condition (ii)
- The number of solutions is equal to 2^m where $m = |\mathcal{N}_2|$ is the cardinality of $\mathcal{N}_2$

Existence and uniqueness

Effect on the solution of setting risk budgets to zero

We note $\mathcal{S}_2$ the solution with $x_i = 0$ for all assets such that $b_i = 0$. Even if $\mathcal{S}_2$ is the solution expected by the investor, the only acceptable solution is $\mathcal{S}_1$. Indeed, if we impose $b_i = \varepsilon_i$ where $\varepsilon_i > 0$ is a small number for $i \in \mathcal{N}$, we obtain:

$$\lim_{\varepsilon_i \to 0} \mathcal{S} = \mathcal{S}_1$$

The solution converges to $\mathcal{S}_1$, and not to $\mathcal{S}_2$ or the other solutions

Existence and uniqueness

Effect on the solution of setting risk budgets to zero

Remark

The non-linear system is not well-defined, whereas the optimization problem is the right approach to define a RB portfolio

Definition

A RB portfolio is a minimum risk portfolio subject to a diversification constraint, which is defined by the logarithmic barrier function

Existence and uniqueness

Example 12

We consider a universe of three assets with $\sigma_1 = 20\%$, $\sigma_2 = 10\%$ and $\sigma_3 = 5\%$. The correlation of asset returns is given by the following matrix:

$$\rho = \begin{pmatrix} 1.00 & & \\ 0.50 & 1.00 & \\ \rho_{1,3} & \rho_{2,3} & 1.00 \end{pmatrix}$$

We would like to build a RB portfolio such that the risk budgets with respect to the volatility risk measure are $(50\%, 50\%, 0\%)$. Moreover, we assume that $\rho_{1,3} = \rho_{2,3}$.

Existence and uniqueness

Table 31: RB solutions when the risk budget b_3 is equal to 0 (Example 12)

$\rho_{1,3} = \rho_{2,3}$	Solution		1	2	3	$\sigma(x)$
-25%	$\mathcal{S}_1$	x_i	20.00%	40.00%	40.00%	6.63%
		$\mathcal{MR}_i$	16.58%	8.29%	0.00%	
		$\mathcal{RC}_i$	50.00%	50.00%	0.00%	
	$\mathcal{S}_2$	x_i	33.33%	66.67%	0.00%	11.55%
		$\mathcal{MR}_i$	17.32%	8.66%	-1.44%	
		$\mathcal{RC}_i$	50.00%	50.00%	0.00%	
	$\mathcal{S}_1'$	x_i	19.23%	38.46%	42.31%	6.38%
		$\mathcal{MR}_i$	16.42%	8.21%	0.15%	
		$\mathcal{RC}_i$	49.50%	49.50%	1.00%	
25%	$\mathcal{S}_1$	x_i	33.33%	66.67%	0.00%	11.55%
		$\mathcal{MR}_i$	17.32%	8.66%	1.44%	
		$\mathcal{RC}_i$	50.00%	50.00%	0.00%	

Existence and uniqueness

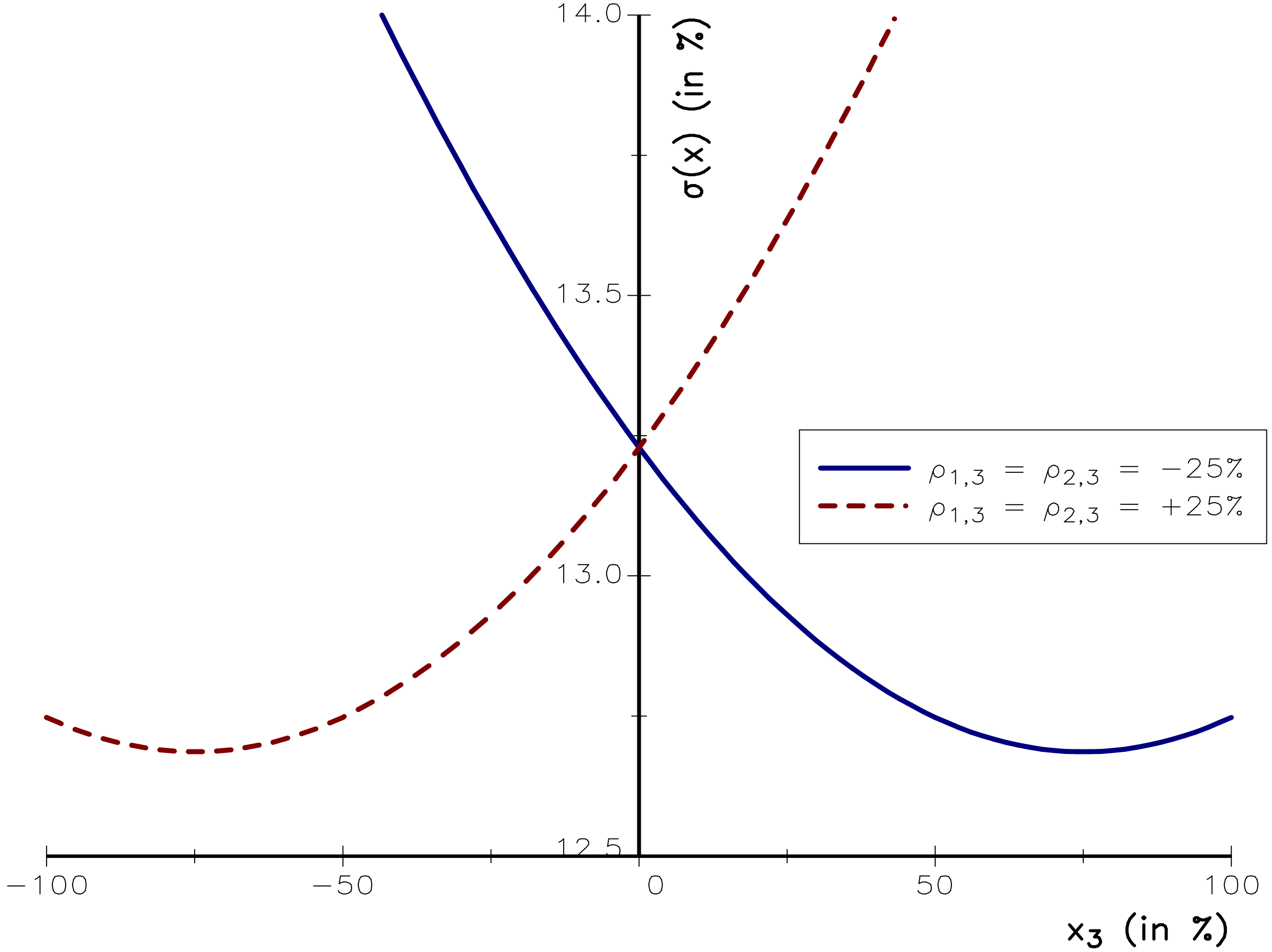


Figure 25: Evolution of the portfolio's volatility with respect to x_3

Location of the RB portfolio

We have:

$$\frac{x_i}{b_i} = \frac{x_j}{b_j} \tag{WB}$$

$$\frac{\partial\,\mathcal{R}(x)}{\partial\,x_i} = \frac{\partial\,\mathcal{R}(x)}{\partial\,x_j} \tag{MR}$$

$$\frac{1}{b_i}\left(x_i\frac{\partial\,\mathcal{R}(x)}{\partial\,x_i}\right) = \frac{1}{b_j}\left(x_j\frac{\partial\,\mathcal{R}(x)}{\partial\,x_j}\right) \tag{ERC}$$

The RB portfolio is a combination of MR (long-only minimum risk) and WB (weight budgeting) portfolios

Risk of the RB portfolio

Theorem

We obtain the following inequality:

$$\mathcal{R}(x_{\mathrm{mr}}) \leq \mathcal{R}(x_{\mathrm{rb}}) \leq \mathcal{R}(x_{\mathrm{wb}})$$

The RB portfolio may be viewed as a portfolio "between" the MR portfolio and the WB portfolio

Diversification index

Definition

The diversification index is equal to:

$$\begin{aligned}\mathcal{D}(x) &= \frac{\mathcal{R}\left(\sum_{i=1}^{n} L_i\right)}{\sum_{i=1}^{n} \mathcal{R}(L_i)} \\ &= \frac{\mathcal{R}(x)}{\sum_{i=1}^{n} x_i \mathcal{R}(\mathbf{e}_i)}\end{aligned}$$

Diversification index

- The diversification index is the ratio between the risk measure of portfolio x and the weighted risk measure of the assets
- If $\mathcal{R}$ is a coherent risk measure, we have $\mathcal{D}(x) \leq 1$
- If $\mathcal{D}(x) = 1$, it implies that the losses are comonotonic
- If $\mathcal{R}$ is the volatility risk measure, we obtain:

$$\mathcal{D}(x) = \frac{\sqrt{x^\top \Sigma x}}{\sum_{i=1}^{n} x_i \sigma_i}$$

It takes the value one if the asset returns are perfectly correlated meaning that the correlation matrix is $\mathcal{C}_n(1)$

Concentration index

- Let $\pi \in \mathbb{R}_+^n$ such that $\mathbf{1}_n^\top \pi = 1 \Rightarrow \pi$ is a probability distribution
- The probability distribution π^+ is perfectly concentrated if there exists one observation i_0 such that $\pi_{i_0}^+ = 1$ and $\pi_i^+ = 0$ if $i \neq i_0$
- When n tends to $+\infty$, the limit distribution is noted π_∞^+
- On the opposite, the probability distribution π^- such that $\pi_i^- = 1/n$ for all $i = 1, \ldots, n$ has no concentration

Concentration index

Definition

A concentration index is a mapping function $\mathcal{C}(\pi)$ such that $\mathcal{C}(\pi)$ increases with concentration and verifies:

$$\mathcal{C}\left(\pi^{-}\right) \leq \mathcal{C}(\pi) \leq \mathcal{C}\left(\pi^{+}\right)$$

- For instance, if π represents the weights of the portfolio, $\mathcal{C}(\pi)$ measures then the weight concentration
- By construction, $\mathcal{C}(\pi)$ reaches the minimum value if the portfolio is equally weighted
- To measure the risk concentration of the portfolio, we define π as the distribution of the risk contributions. In this case, the portfolio corresponding to the lower bound $\mathcal{C}\left(\pi^{-}\right) = 0$ is the ERC portfolio

Herfindahl index

Definition

The Herfindahl index associated with π is defined as:

$$\mathcal{H}(\pi) = \sum_{i=1}^{n} \pi_i^2$$

- This index takes the value 1 for the probability distribution π^+ and $1/n$ for the distribution with uniform probabilities π^-
- To scale the statistics onto $[0, 1]$, we consider the normalized index $\mathcal{H}^\star(\pi)$ defined as follows:

$$\mathcal{H}^\star(\pi) = \frac{n\mathcal{H}(\pi) - 1}{n - 1}$$

Gini index

- The Gini index is based on the Lorenz curve of inequality
- Let X and Y be two random variables. The Lorenz curve $y = \mathbb{L}(x)$ is defined by the following parameterization:

$$\begin{cases} x = \Pr\{X \leq x\} \\ y = \Pr\{Y \leq y \mid X \leq x\} \end{cases}$$

- The Lorenz curve admits two limit cases
  1. If the portfolio is perfectly concentrated, the distribution of the weights corresponds to π_∞^+
  2. On the opposite, the least concentrated portfolio is the equally weighted portfolio and the Lorenz curve is the bisecting line $y = x$

Gini index

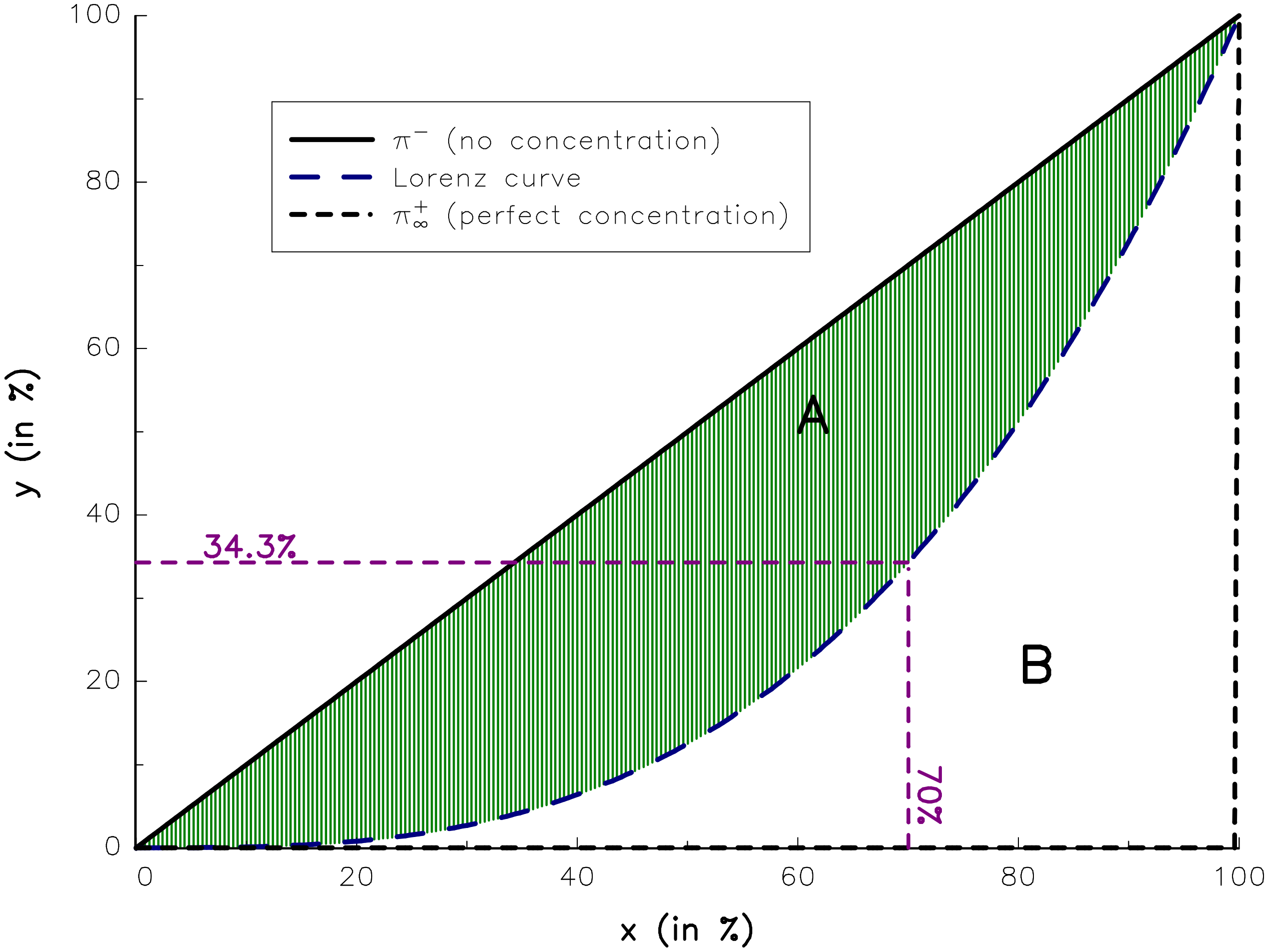


Figure 26: Geometry of the Lorenz curve

Gini index

Definition

The Gini index is then defined as:

$$\mathcal{G}(\pi) = \frac{A}{A+B}$$

with A the area between $\mathbb{L}(\pi^-)$ and $\mathbb{L}(\pi)$, and B the area between $\mathbb{L}(\pi)$ and $\mathbb{L}(\pi_\infty^+)$

Gini index

- By construction, we have $\mathcal{G}(\pi^{-}) = 0$, $\mathcal{G}(\pi_{\infty}^{+}) = 1$ and:

$$\begin{aligned}\mathcal{G}(\pi) &= \frac{(A+B)-B}{A+B} \\ &= 1-\frac{1}{A+B}B \\ &= 1-2\int_0^1 \mathbb{L}(x)\,\mathrm{d}x\end{aligned}$$

In the case when π is a discrete probability distribution, we obtain:

$$\mathcal{G}(\pi) = \frac{2\sum_{i=1}^n i\pi_{i:n}}{n\sum_{i=1}^n \pi_{i:n}} - \frac{n+1}{n}$$

where $\{\pi_{1:n},\ldots,\pi_{n:n}\}$ are the ordered statistics of $\{\pi_1,\ldots,\pi_n\}$.

Shannon entropy

Definition

The Shannon entropy is equal to:

$$\mathcal{I}(\pi) = -\sum_{i=1}^{n} \pi_i \ln \pi_i$$

- The diversity index corresponds to the statistic:

$$\mathcal{I}^\star(\pi) = \exp(\mathcal{I}(\pi))$$

- We have $\mathcal{I}^\star(\pi^-) = n$ and $\mathcal{I}^\star(\pi^+) = 1$

Impact of the reparametrization on the asset universe

- We consider a set of m primary assets $(\mathcal{A}'_1, \ldots, \mathcal{A}'_m)$ with a covariance matrix Ω
- We define n synthetic assets $(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ which are composed of the primary assets
- We denote $W = (w_{i,j})$ the weight matrix such that $w_{i,j}$ is the weight of the primary asset $\mathcal{A}'_j$ in the synthetic asset $\mathcal{A}_i$ (we have $\sum_{j=1}^m w_{i,j} = 1$)
- The covariance matrix of the synthetic assets Σ is equal to $W \Omega W^\top$
- The synthetic assets can be interpreted as portfolios of the primary assets
- For example, $\mathcal{A}'_j$ may represent a stock whereas $\mathcal{A}_i$ may be an index

Impact of the reparametrization on the asset universe

1. We consider a portfolio $x = (x_1, \ldots, x_n)$ defined with respect to the synthetic assets. We have:

$$\mathcal{RC}_i = x_i \cdot \frac{(\Sigma x)_i}{\sqrt{x^\top \Sigma x}}$$

2. We also define the portfolio with respect to the primary assets. In this case, the composition is $y = (y_1, \ldots, y_m)$ where $y_j = \sum_{i=1}^n x_i w_{i,j}$ (or $y = W^\top x$). We have:

$$\mathcal{RC}_j = y_j \cdot \frac{(\Omega y)_j}{\sqrt{y^\top \Omega y}}$$

Impact of the reparametrization on the asset universe

Example 13

We have six primary assets. The volatility of these assets is respectively 20%, 30%, 25%, 15%, 10% and 30%. We assume that the assets are not correlated. We consider two equally weighted synthetic assets with:

$$W = \begin{pmatrix} 1/4 & 1/4 & 1/4 & 1/4 & & \\ & & 1/4 & 1/4 & 1/4 & 1/4 \end{pmatrix}$$

Impact of the reparametrization on the asset universe

Table 32: Risk decomposition of Portfolio #1 with respect to the synthetic assets (Example 13)

Asset i	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
$\mathcal{A}_1$	36.00	9.44	3.40	33.33
$\mathcal{A}_2$	38.00	8.90	3.38	33.17
$\mathcal{A}_3$	26.00	13.13	3.41	33.50

Table 33: Risk decomposition of Portfolio #1 with respect to the primary assets (Example 13)

Asset j	y_j	$\mathcal{MR}_j$	$\mathcal{RC}_j$	$\mathcal{RC}_j^\star$
$\mathcal{A}_1'$	9.00	3.53	0.32	3.12
$\mathcal{A}_2'$	9.00	7.95	0.72	7.02
$\mathcal{A}_3'$	31.50	19.31	6.08	59.69
$\mathcal{A}_4'$	31.50	6.95	2.19	21.49
$\mathcal{A}_5'$	9.50	0.93	0.09	0.87
$\mathcal{A}_6'$	9.50	8.39	0.80	7.82

Impact of the reparametrization on the asset universe

Table 34: Risk decomposition of Portfolio #2 with respect to the synthetic assets (Example 13)

Asset i	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
$\mathcal{A}_1$	48.00	9.84	4.73	49.91
$\mathcal{A}_2$	50.00	9.03	4.51	47.67
$\mathcal{A}_3$	2.00	11.45	0.23	2.42

Table 35: Risk decomposition of Portfolio #2 with respect to the primary assets (Example 13)

Asset j	y_j	$\mathcal{MR}_j$	$\mathcal{RC}_j$	$\mathcal{RC}_j^\star$
$\mathcal{A}'_1$	12.00	5.07	0.61	6.43
$\mathcal{A}'_2$	12.00	11.41	1.37	14.46
$\mathcal{A}'_3$	25.50	16.84	4.29	45.35
$\mathcal{A}'_4$	25.50	6.06	1.55	16.33
$\mathcal{A}'_5$	12.50	1.32	0.17	1.74
$\mathcal{A}'_6$	12.50	11.88	1.49	15.69

Impact of the reparametrization on the asset universe

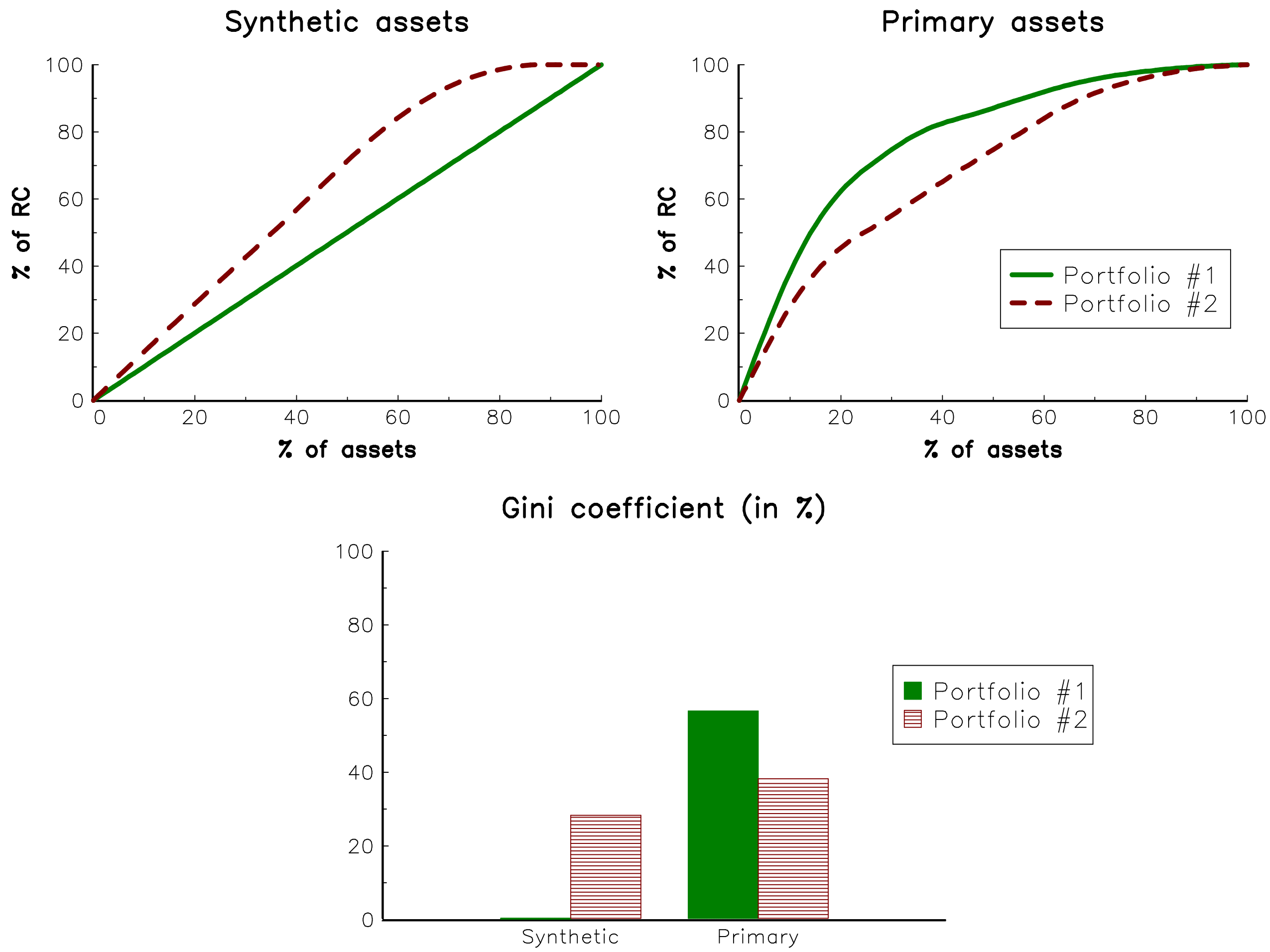


Figure 27: Lorenz curve of risk contributions (Example 13)

Risk decomposition with respect to the risk factors

- We consider a set of n assets $\{\mathcal{A}_1, \ldots, \mathcal{A}_n\}$ and a set of m risk factors $\{\mathcal{F}_1, \ldots, \mathcal{F}_m\}$
- R_t is the $(n \times 1)$ vector of asset returns at time t
- Σ is the covariance matrix of asset returns
- $\mathcal{F}_t$ is the $(m \times 1)$ vector of factor returns at time t
- Ω is the covariance matrix of factor returns

Risk decomposition with respect to the risk factors

Linear factor model

We consider the linear factor model:

$$R_t = A\mathcal{F}_t + \varepsilon_t$$

where $\mathcal{F}_t$ and ε_t are two uncorrelated random vectors, ε_t is a centered random vector $(n \times 1)$ of covariance D and A is the $(n \times m)$ loadings matrix

We have the following relationship:

$$\Sigma = A\Omega A^\top + D$$

Risk decomposition with respect to the risk factors

We decompose the portfolio's asset exposures x by the portfolio's risk factors exposures y in the following way:

$$x = B^{+}y + \tilde{B}^{+}\tilde{y}$$

where:

- B^{+} is the Moore-Penrose inverse of $A^{\top}$
- $\tilde{B}^{+}$ is any $n \times (n - m)$ matrix that spans the left nullspace of B^{+}
- $\tilde{y}$ corresponds to $n - m$ residual (or additional) factors that have no economic interpretation

It follows that:

$$\left\{ \begin{array}{l} y = A^{\top}x \\ \tilde{y} = \tilde{B}x \end{array} \right.$$

where $\tilde{B} = \ker\left(A^{\top}\right)^{\top}$

Risk decomposition with respect to the risk factors

Risk decomposition I

- We can show that the marginal risk of the j^{th} factor exposure is given by:

$$\mathcal{MR}\left(\mathcal{F}_j\right) = \frac{\partial\,\sigma\left(x\right)}{\partial\,y_j} = \frac{\left(A^{+}\Sigma x\right)_j}{\sigma\left(x\right)}$$

whereas its risk contribution is equal to:

$$\mathcal{RC}\left(\mathcal{F}_j\right) = y_j \frac{\partial\,\sigma\left(x\right)}{\partial\,y_j} = \frac{\left(A^{\top} x\right)_j \cdot \left(A^{+}\Sigma x\right)_j}{\sigma\left(x\right)}$$

Risk decomposition with respect to the risk factors

Risk decomposition II

- For the residual factors, we have:

$$\mathcal{MR}\left(\tilde{\mathcal{F}}_j\right) = \frac{\partial\,\sigma\left(x\right)}{\partial\,\tilde{y}_j} = \frac{\left(\tilde{B}\Sigma x\right)_j}{\sigma\left(x\right)}$$

and:

$$\mathcal{RC}\left(\tilde{\mathcal{F}}_j\right) = \tilde{y}_j \frac{\partial\,\sigma\left(x\right)}{\partial\,\tilde{y}_j} = \frac{\left(\tilde{B}x\right)_j \cdot \left(\tilde{B}\Sigma x\right)_j}{\sigma\left(x\right)}$$

Risk decomposition with respect to the risk factors

Remark

We can show that these risk contributions satisfy the allocation principle:

$$\sigma(x) = \sum_{j=1}^{m} \mathcal{RC}(\mathcal{F}_j) + \sum_{j=1}^{n-m} \mathcal{RC}\left(\tilde{\mathcal{F}}_j\right)$$

Risk decomposition with respect to the risk factors

Let $\operatorname{pinv}(C)$ and $\operatorname{null}(C)$ be the Moore-Penrose pseudo-inverse and the orthonormal basis for the right null space of C

1. Computation of A^{+}

$$A^{+} = \operatorname{pinv}(A) = \left(A^{\top}A\right)^{-1} A^{\top}$$

2. Computation of B

$$B = A^{\top}$$

3. Computation of B^{+}

$$B^{+} = \operatorname{pinv}(B) = B^{\top}\left(BB^{\top}\right)^{-1}$$

4. Computation of $\tilde{B}$

$$\tilde{B} = \operatorname{pinv}\left(\operatorname{null}\left({B^{+}}^{\top}\right)\right) \cdot \left(I_n - B^{+}A^{\top}\right)$$

Risk decomposition with respect to the risk factors

Remark

The previous results can be extended to other coherent and convex risk measures (Roncalli and Weisang, 2016)

Risk decomposition with respect to the risk factors

Example 14

We consider an investment universe with four assets and three factors. The loadings matrix A is:

$$A = \begin{pmatrix} 0.9 & 0.0 & 0.5 \\ 1.1 & 0.5 & 0.0 \\ 1.2 & 0.3 & 0.2 \\ 0.8 & 0.1 & 0.7 \end{pmatrix}$$

The three factors are uncorrelated and their volatilities are 20%, 10% and 10%. We assume a diagonal matrix D with specific volatilities 10%, 15%, 10% and 15%.

Risk decomposition with respect to the risk factors

The correlation matrix of asset returns is (in %):

$$\rho = \begin{pmatrix} 100.0 & & & \\ 69.0 & 100.0 & & \\ 79.5 & 76.4 & 100.0 & \\ 66.2 & 57.2 & 66.3 & 100.0 \end{pmatrix}$$

and their volatilities are respectively equal to 21.19%, 27.09%, 26.25% and 23.04%.

Risk decomposition with respect to the risk factors

We obtain that:

$$A^{+}=\begin{pmatrix} 1.260 & -0.383 & 1.037 & -1.196 \\ -3.253 & 2.435 & -1.657 & 2.797 \\ -0.835 & 0.208 & -1.130 & 2.348 \end{pmatrix}$$

and:

$$\tilde{B}=\begin{pmatrix} 0.533 & 0.452 & -0.692 & -0.183 \end{pmatrix}$$

Risk decomposition with respect to the risk factors

Table 36: Risk decomposition of the EW portfolio with respect to the assets (Example 14)

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1	25.00	18.81	4.70	21.97
2	25.00	23.72	5.93	27.71
3	25.00	24.24	6.06	28.32
4	25.00	18.83	4.71	22.00
Volatility			21.40	

Table 37: Risk decomposition of the EW portfolio with respect to the risk factors (Example 14)

Factor	y_j	$\mathcal{MR}_j$	$\mathcal{RC}_j$	$\mathcal{RC}_j^\star$
$\mathcal{F}_1$	100.00	17.22	17.22	80.49
$\mathcal{F}_2$	22.50	9.07	2.04	9.53
$\mathcal{F}_3$	35.00	6.06	2.12	9.91
$\tilde{\mathcal{F}}_1$	2.75	0.52	0.01	0.07
Volatility			21.40	

Risk factor parity (or RFP) portfolios

RFP portfolios are defined by:

$$\mathcal{RC}\left(\mathcal{F}_j\right) = b_j \mathcal{R}\left(x\right)$$

They are computed using the following optimization problem:

$$\begin{aligned}\left(y^{\star},\hat{y}^{\star}\right) &= \arg\min \sum_{j=1}^{m} \left(\mathcal{RC}\left(\mathcal{F}_j\right) - b_j \mathcal{R}\left(x\right)\right)^2 \\ \text{u.c.} &\quad \mathbf{1}_n^\top \left(B^{+} y + \tilde{B}^{+} \tilde{y}\right) = 1\end{aligned}$$

Risk factor parity (or RFP) portfolios

Example 15

We consider an investment universe with four assets and three factors. The loadings matrix A is:

$$A = \begin{pmatrix} 0.9 & 0.0 & 0.5 \\ 1.1 & 0.5 & 0.0 \\ 1.2 & 0.3 & 0.2 \\ 0.8 & 0.1 & 0.7 \end{pmatrix}$$

The three factors are uncorrelated and their volatilities are 20%, 10% and 10%. We assume a diagonal matrix D with specific volatilities 10%, 15%, 10% and 15%. We consider the following factor risk budgets:

$$b = (49\%, 25\%, 25\%)$$

Risk factor parity (or RFP) portfolios

Table 38: Risk decomposition of the RFP portfolio with respect to the risk factors (Example 15)

Factor	y_j	$\mathcal{MR}_j$	$\mathcal{RC}_j$	$\mathcal{RC}_j^\star$
$\mathcal{F}_1$	93.38	11.16	10.42	49.00
$\mathcal{F}_2$	24.02	22.14	5.32	25.00
$\mathcal{F}_3$	39.67	13.41	5.32	25.00
$\tilde{\mathcal{F}}_1$	16.39	1.30	0.21	1.00
Volatility			21.27	

Table 39: Risk decomposition of the RFP portfolio with respect to the assets (Example 15)

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1	15.08	17.44	2.63	12.36
2	38.38	23.94	9.19	43.18
3	0.89	21.82	0.20	0.92
4	45.65	20.29	9.26	43.54
Volatility			21.27	

Minimizing the risk concentration between the risk factors

We now consider the following problem:

$$\mathcal{RC}\left(\mathcal{F}_j\right) \simeq \mathcal{RC}\left(\mathcal{F}_k\right)$$

$\Rightarrow$ The portfolios are computed by minimizing the risk concentration between the risk factors

Remark

We can use the Herfindahl index, the Gini index or the Shanon entropy

Minimizing the risk concentration between the risk factors

Example 16

We consider an investment universe with four assets and three factors. The loadings matrix A is:

$$A = \begin{pmatrix} 0.9 & 0.0 & 0.5 \\ 1.1 & 0.5 & 0.0 \\ 1.2 & 0.3 & 0.2 \\ 0.8 & 0.1 & 0.7 \end{pmatrix}$$

The three factors are uncorrelated and their volatilities are 20%, 10% and 10%. We assume a diagonal matrix D with specific volatilities 10%, 15%, 10% and 15%.

Minimizing the risk concentration between the risk factors

Table 40: Risk decomposition of the balanced RFP portfolio with respect to the risk factors (Example 16)

Factor	y_j	$\mathcal{MR}_j$	$\mathcal{RC}_j$	$\mathcal{RC}_j^\star$
$\mathcal{F}_1$	91.97	7.91	7.28	33.26
$\mathcal{F}_2$	25.78	28.23	7.28	33.26
$\mathcal{F}_3$	42.22	17.24	7.28	33.26
$\tilde{\mathcal{F}}_1$	6.74	0.70	0.05	0.21
Volatility			21.88	

Table 41: Risk decomposition of the balanced RFP portfolio with respect to the assets (Example 16)

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1	0.30	16.11	0.05	0.22
2	39.37	23.13	9.11	41.63
3	0.31	20.93	0.07	0.30
4	60.01	21.09	12.66	57.85
Volatility			21.88	

Minimizing the risk concentration between the risk factors

We have $\mathcal{H}^{\star} = 0$, $\mathcal{G} = 0$ and $\mathcal{I}^{\star} = 3$

Minimizing the risk concentration between the risk factors

Table 42: Balanced RFP portfolios with $x_i \geq 10\%$ (Example 16)

Criterion	$\mathcal{H}(x)$	$\mathcal{G}(x)$	$\mathcal{I}(x)$
x_1	10.00	10.00	10.00
x_2	22.08	18.24	24.91
x_3	10.00	10.00	10.00
x_4	57.92	61.76	55.09
$\mathcal{H}^\star$	0.0436	0.0490	0.0453
$\mathcal{G}$	0.1570	0.1476	0.1639
$\mathcal{I}^\star$	2.8636	2.8416	2.8643

Justification of diversified funds

Investor Profiles

1. Conservative (low risk)
2. Moderate (medium risk)
3. Aggressive (high risk)

Fund Profiles

1. Defensive (20% equities and 80% bonds)
2. Balanced (50% equities and 50% bonds)
3. Dynamic (80% equities and 20% bonds)

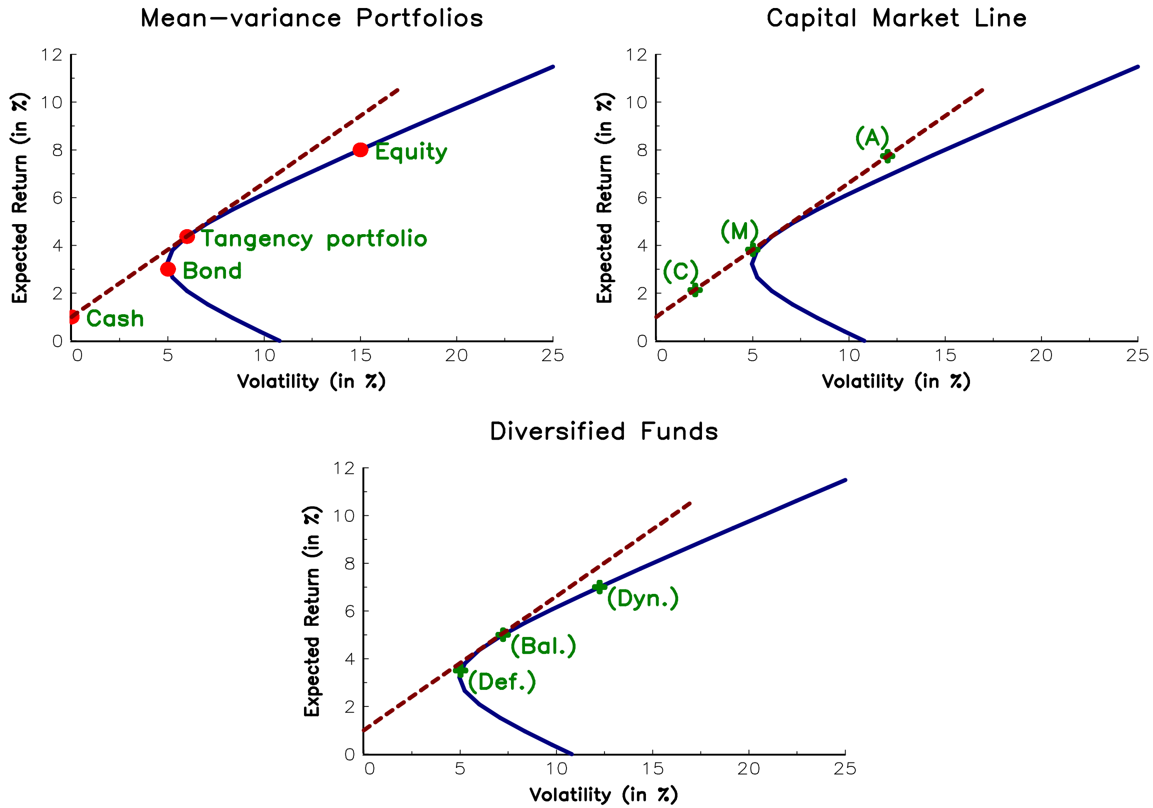


Figure 28: The asset allocation puzzle

What type of diversification is offered by diversified funds?

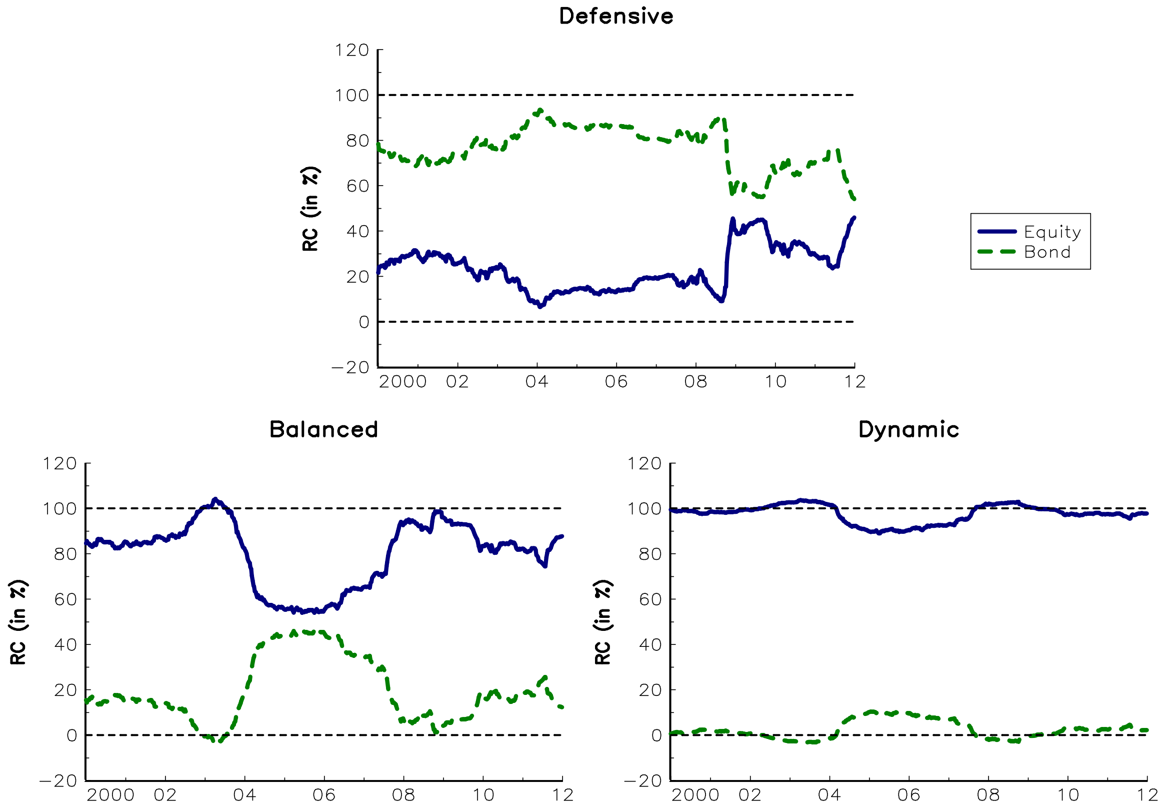


Figure 29: Equity (MSCI World) and bond (WGBI) risk contributions

Diversified funds
=
Marketing idea?

- Contrarian constant-mix strategy
- Deleverage of an equity exposure
- Low risk diversification
- No mapping between fund profiles and investor profiles
- Static weights
- Dynamic risk contributions

Risk-balanced allocation

- Multi-dimensional target volatility strategy
- Trend-following portfolio (if negative correlation between return and risk)
- Dynamic weights
- Static risk contributions (risk budgeting)
- High diversification

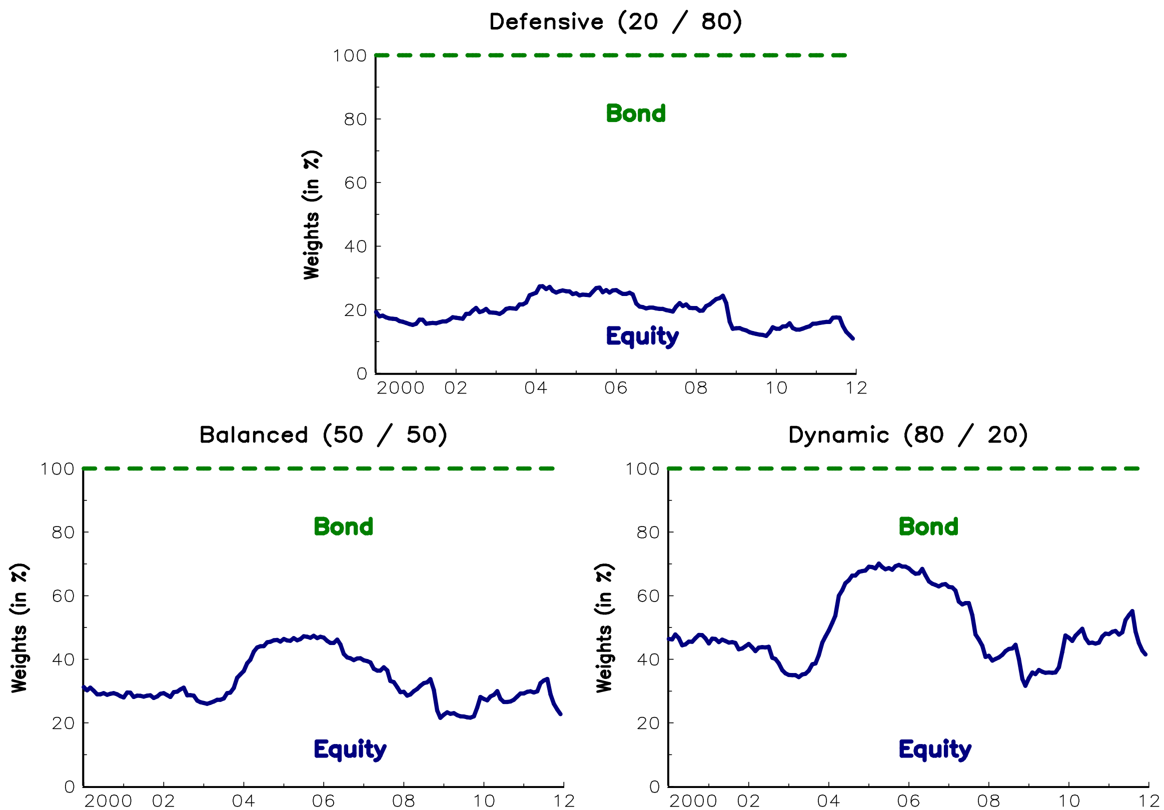


Figure 30: Equity and bond allocation

Characterization of the stock/bond market portfolio

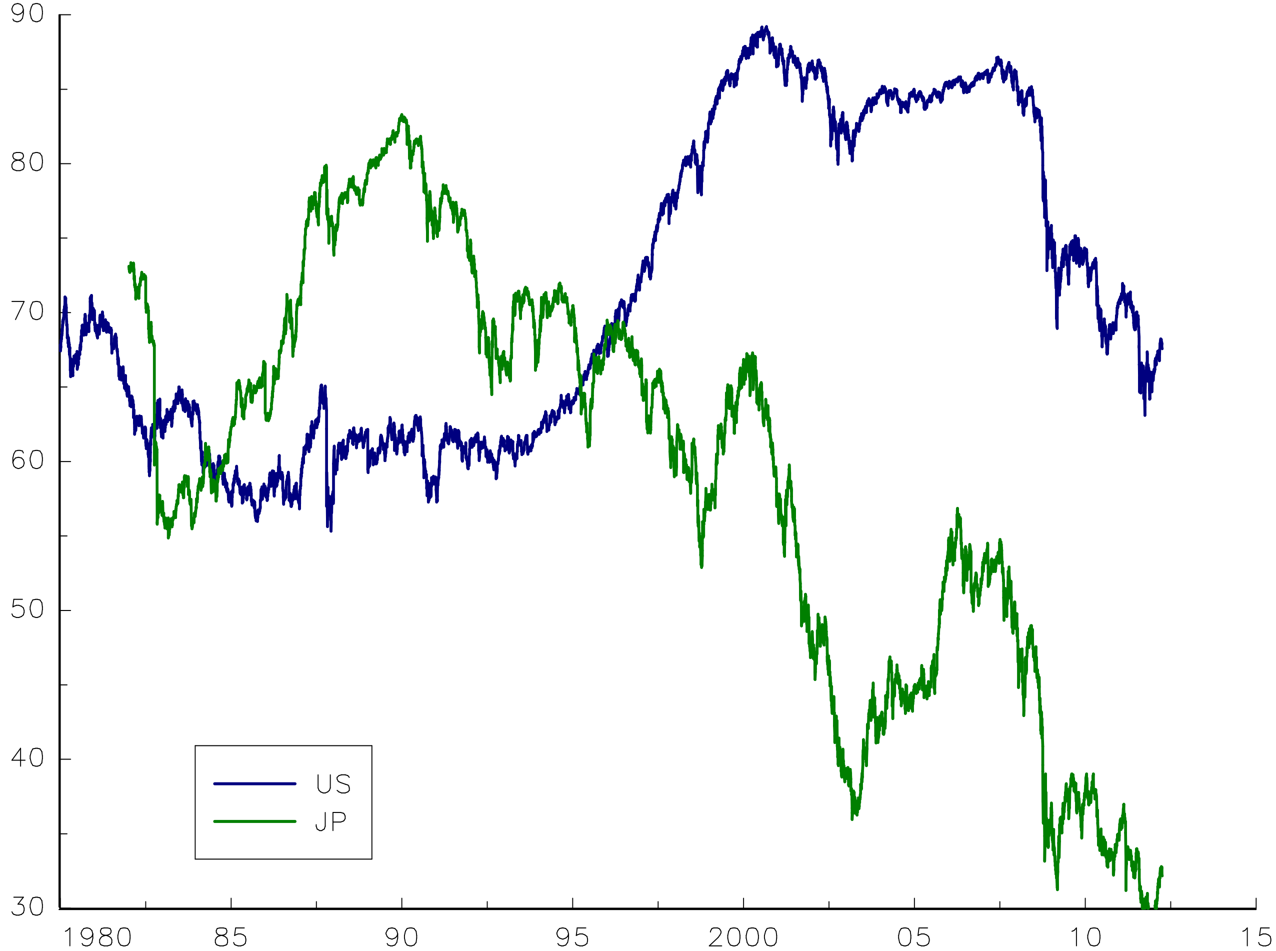


Figure 31: Evolution of the equity weight for United States and Japan

Characterization of the stock/bond market portfolio

Figure 32: Evolution of the equity weight for Germany, France and UK

Link between risk premium and risk contribution

Let π_i and π_M be the risk premium of Asset i and the market risk premium. We have:

$$\begin{aligned}
\pi_i &= \beta_i \cdot \pi_M \\
&= \frac{\operatorname{cov}(R_i, R_M)}{\sigma(R_M)} \cdot \frac{\pi_M}{\sigma(R_M)} \\
&= \frac{\partial\, \sigma(x_M)}{\partial\, x_i} \cdot \mathrm{SR}(x_M)
\end{aligned}$$

The risk premium of Asset i is then proportional to the marginal volatility of Asset i with respect to the market portfolio

Foundation of the risk budgeting approach

For the tangency portfolio, we have:

performance contribution = risk contribution

Link between risk premium and risk contribution

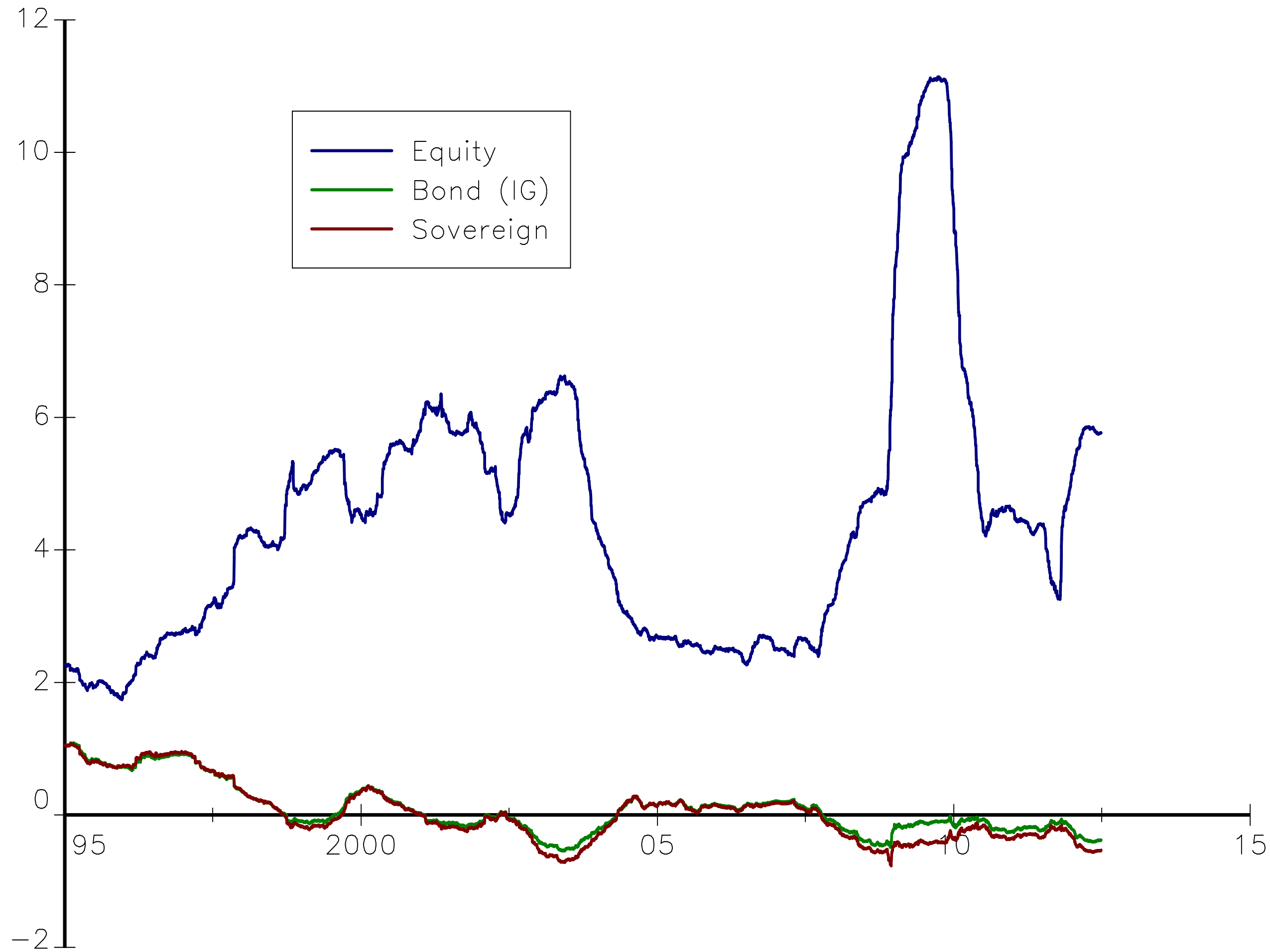


Figure 33: Risk premia (in %) for the US market portfolio ($\mathrm{SR}(x_M) = 25\%$)

Link between risk premium and risk contribution

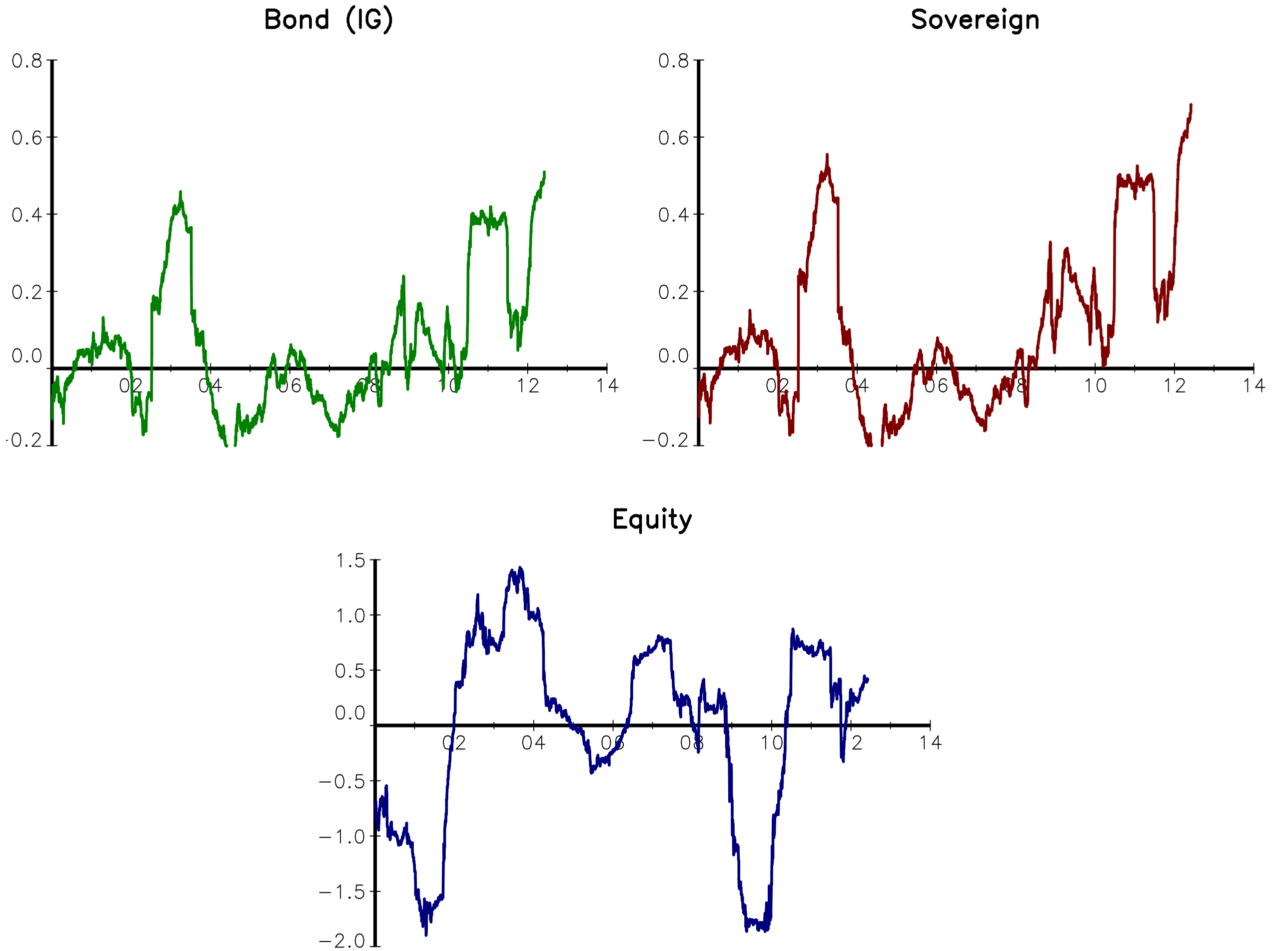


Figure 34: Difference (in %) between EURO and US risk premia $(\mathrm{SR}(x_M) = 25\%)$

Sharpe theory of risk premia

The one-factor risk model

We deduce that:

$$R_i - R_f = \alpha_i + \underbrace{\beta_i \cdot (R_M - R_f)}_{\substack{\text{Systematic} \\ \text{Risk}}} + \underbrace{\varepsilon_i}_{\substack{\text{Specific} \\ \text{Risk}}}$$

We necessarily have:

1. $\alpha_i = 0$
2. $\mathbb{E}[\varepsilon_i] = 0$

$\Rightarrow$ On average, only the systematic risk is rewarded, not the idiosyncratic risk

Sharpe theory of risk premia

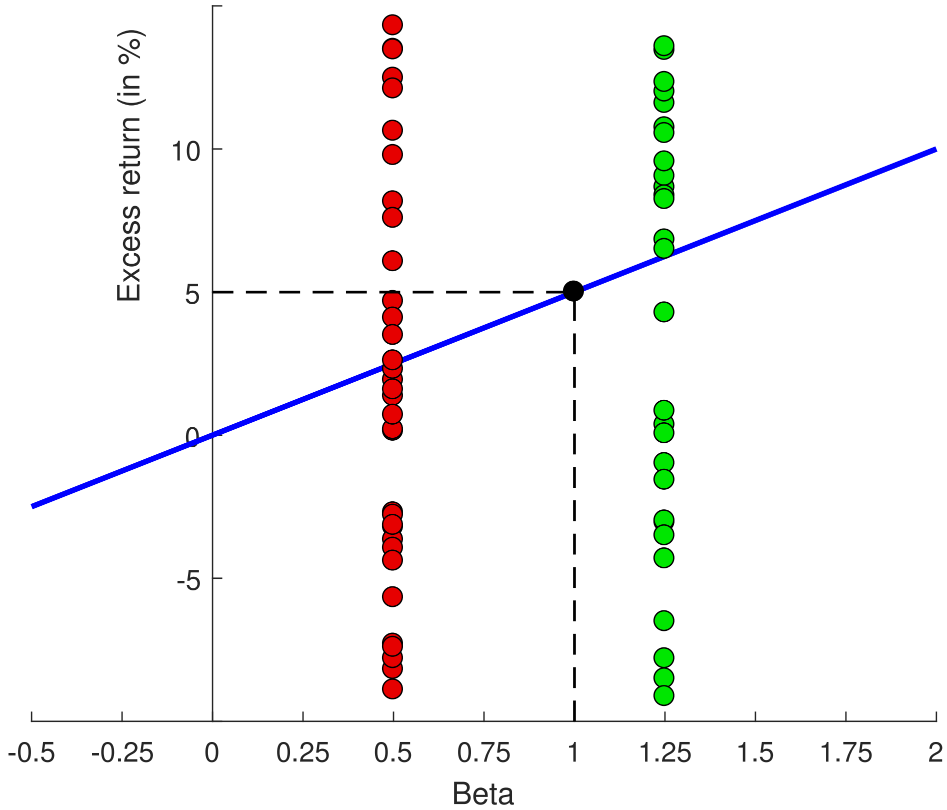


Figure 35: The security market line (SML)

- Risk premium is an increasing function of the systematic risk
- Risk premium may be negative (meaning that some assets can have a return lower than the risk-free asset!)
- More risk $\neq$ more return

Black-Litterman theory of risk premia

In the Black-Litterman model, the expected (or ex-ante/implied) risk premia are equal to:

$$\tilde{\pi} = \tilde{\mu} - r = \mathrm{SR}\left(x \mid r\right) \frac{\Sigma x}{\sqrt{x^\top \Sigma x}}$$

where $\mathrm{SR}\left(x \mid r\right)$ is the expected Sharpe ratio of the portfolio.

Black-Litterman theory of risk premia

Example 17

We consider four assets. Their expected returns are equal to 5%, 6%, 8% and 6% while their volatilities are equal to 15%, 20%, 25% and 30%. The correlation matrix of asset returns is given by the following matrix:

$$C = \begin{pmatrix} 1.00 & & & \\ 0.10 & 1.00 & & \\ 0.40 & 0.70 & 1.00 & \\ 0.50 & 0.40 & 0.80 & 1.00 \end{pmatrix}$$

We also assume that the return of the risk-free asset is equal to 1.5%.

Black-Litterman theory of risk premia

Table 43: Black-Litterman risk premia (Example 17)

Asset	CAPM π_i	CAPM $x_i^\star$	Black-Litterman x_i	Black-Litterman $\tilde{\pi}_i$	Black-Litterman x_i	Black-Litterman $\tilde{\pi}_i$
#1	3.50%	63.63%	25.00%	2.91%	40.00%	3.33%
#2	4.50%	19.27%	25.00%	4.71%	30.00%	4.97%
#3	6.50%	50.28%	25.00%	7.96%	20.00%	7.69%
#4	4.50%	−33.17%	25.00%	9.07%	10.00%	8.18%
$\mu(x)$	6.37%		6.25%		6.00%	
$\sigma(x)$	14.43%		18.27%		15.35%	
$\tilde{\mu}(x)$	6.37%		7.66%		6.68%	

Black-Litterman theory of risk premia

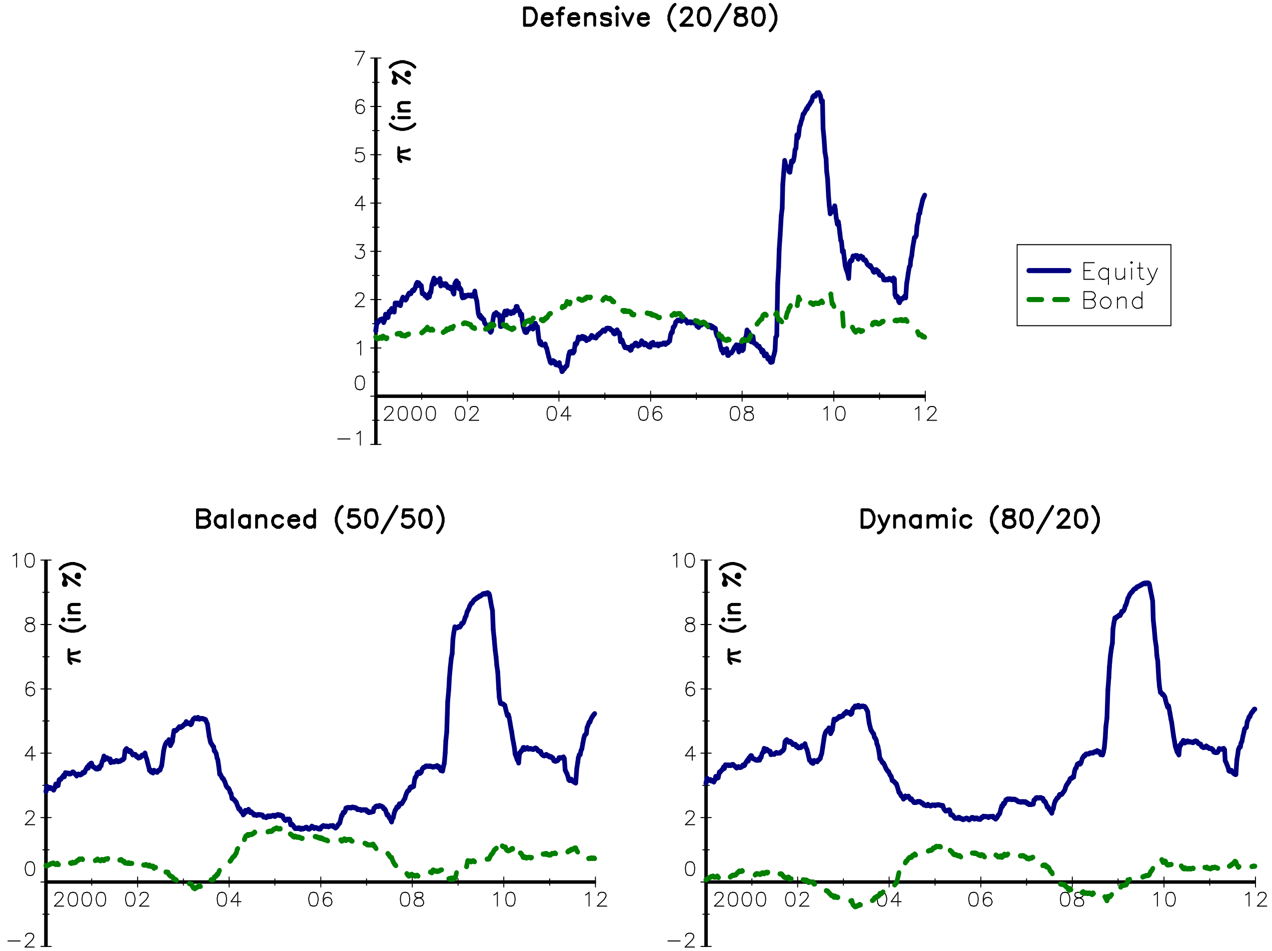


Figure 36: Equity and bond implied risk premia for diversified funds

Performance assets versus hedging assets

- We recall that:
$$\tilde{\pi} = \mathrm{SR}\left(x \mid r\right) \frac{\partial\, \sigma\left(x\right)}{\partial\, x}$$
where $\sigma\left(x\right)$ is the volatility of portfolio x
- We have:
$$\begin{aligned} \frac{\partial\, \sigma\left(x\right)}{\partial\, x_i} &= \frac{\left(\Sigma x\right)_i}{\sigma\left(x\right)} \\ &= \frac{\left(x_i\sigma_i^2 + \sigma_i \sum_{j\neq i} x_j\rho_{i,j}\sigma_j\right)}{\sigma\left(x\right)} \end{aligned}$$
- We deduce that
$$\tilde{\pi}_i = \mathrm{SR}\left(x \mid r\right) \frac{\left(x_i\sigma_i^2 + \sigma_i \sum_{j\neq i} x_j\rho_{i,j}\sigma_j\right)}{\sigma\left(x\right)}$$

Performance assets versus hedging assets

In the two-asset case, we obtain:

$$\tilde{\pi}_1 = c(x)\left(\underbrace{x_1\sigma_1^2}_{\text{variance}} + \underbrace{\rho\sigma_1\sigma_2(1 - x_1)}_{\text{covariance}}\right)$$

and:

$$\tilde{\pi}_2 = c(x)\left(\underbrace{x_2\sigma_2^2}_{\text{variance}} + \underbrace{\rho\sigma_1\sigma_2(1 - x_2)}_{\text{covariance}}\right)$$

where $c(x)$ is equal to $\mathrm{SR}(x \mid r)/\sigma(x)$ and ρ is the cross-correlation between the two asset returns

Performance assets versus hedging assets

In the two-asset case, the implied risk premium becomes:

$$\tilde{\pi}_i = \frac{\mathrm{SR}(x \mid r)}{\sigma(x)} \left(\underbrace{x_i \cdot \sigma_i^2}_{\text{variance}} + \underbrace{(1 - x_i) \cdot \rho \sigma_i \sigma_j}_{\text{covariance}} \right)$$

There are two components in the risk premium:

- a variance risk component, which is an increasing function of the volatility and the weight of the asset
- a (positive or negative) covariance risk component, which depends on the correlation between asset returns

Performance asset versus hedging asset

- When $\tilde{\pi}_i > 0$, the asset i is a performance asset for Portfolio x
- When $\tilde{\pi}_i < 0$, the asset i is a hedging asset for Portfolio x

Performance assets versus hedging assets

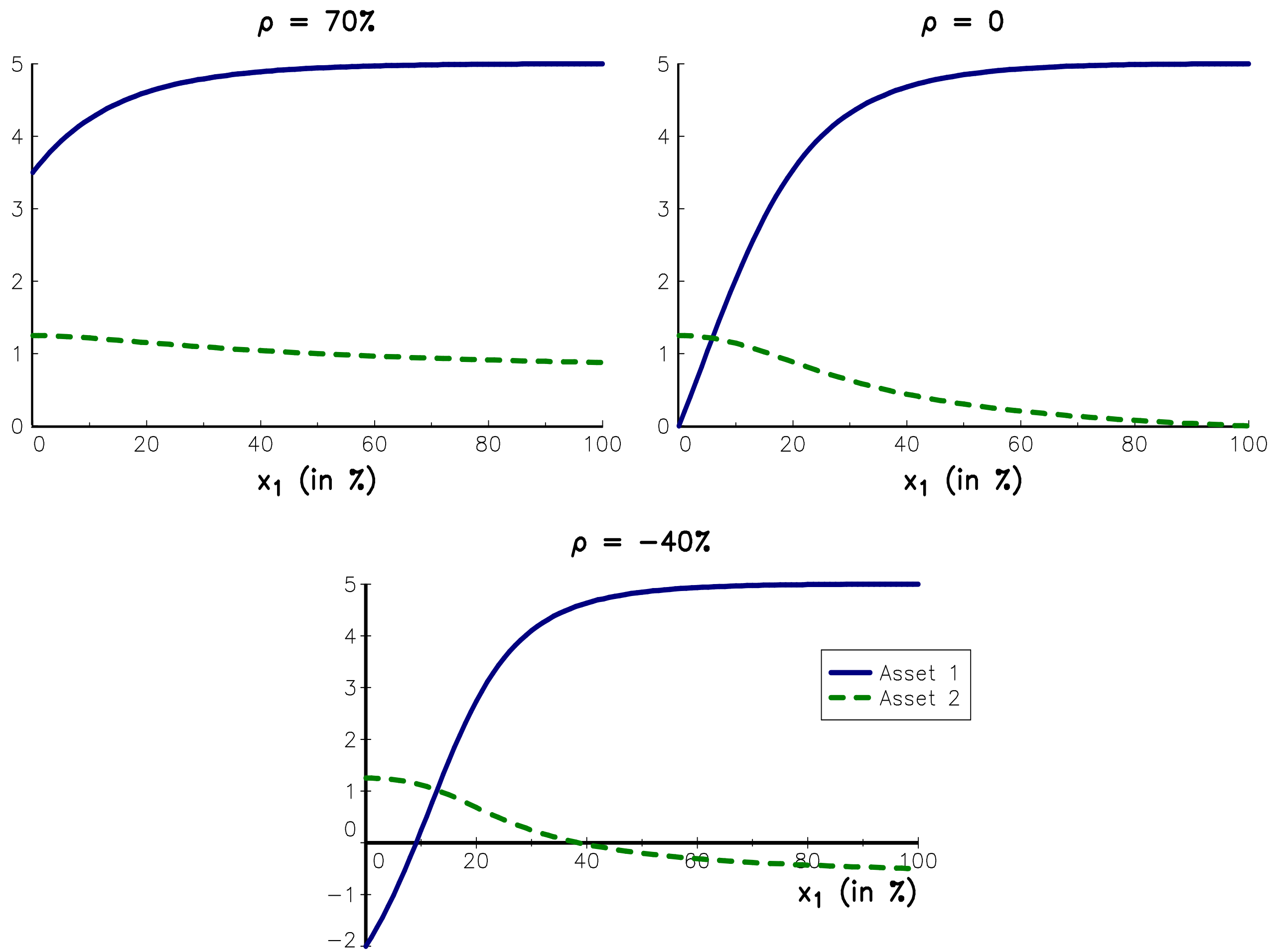


Figure 37: Impact of the correlation on the expected risk premium ($\sigma_1 = 20\%$, $\sigma_2 = 5\%$ and $\mathrm{SR}(x) = 0.25$)

Are bonds performance or hedging assets?

- Stocks are always considered as performance assets, while bonds may be performance or hedging assets, depending on the region and the period[9]
- 1990-2008: (Sovereign) bonds were perceived as performance assets
- The 2008 GFC has strengthened the fly-to-quality characteristic of bonds
- 2013-2017: Bonds are now more and more perceived as hedging assets[10]

Diversified stock-bond portfolios $\Rightarrow$ Deleveraged equity portfolios

[9] For instance bonds were hedging assets in 2008 and performance assets in 2011

[10] This is particular true in the US and Europe, where the implied risk premium is negative. In Japan, the implied risk premium continue to be positive

Diversified versus risk parity funds

Table 44: Statistics of diversified and risk parity portfolios (2000-2012)

Portfolio	$\hat{\mu}_{1Y}$	$\hat{\sigma}_{1Y}$	SR	$\mathcal{MDD}$	γ_1	γ_2
Defensive	5.41	6.89	0.42	−17.23	0.19	2.67
Balanced	3.68	9.64	0.12	−33.18	−0.13	3.87
Dynamic	1.70	14.48	−0.06	−48.90	−0.18	5.96
Risk parity	5.12	7.29	0.36	−21.22	0.08	2.65
Static	4.71	7.64	0.29	−23.96	0.03	2.59
Leveraged RP	6.67	9.26	0.45	−23.74	0.01	0.78

- The 60/40 constant mix strategy is not the right benchmark
- Results depend on the investment universe (number/granularity of asset classes)
- What is the impact of rising interest rates?

Optimality of the RB portfolio

We consider the utility function:

$$\mathcal{U}(x) = (\mu(x) - r) - \phi\mathcal{R}(x)$$

Portfolio x is optimal if the vector of expected risk premia satisfies this relationship:

$$\tilde{\pi} = \phi\frac{\partial\,\mathcal{R}(x)}{\partial\,x}$$

If the RB portfolio is optimal, we deduce that the (excess) performance contribution $\mathcal{PC}_i$ of asset i is proportional to its risk budget:

$$\begin{array}{rcl} \mathcal{PC}_i & = & x_i\tilde{\pi}_i \\ & = & \phi\cdot\mathcal{RC}_i \\ & \propto & b_i \end{array}$$

Optimality of the RB portfolio

In the Black-Litterman approach of risk premia, we have:

$$\tilde{\pi}_i = \tilde{\mu}_i - r = \mathrm{SR}\left(x \mid r\right) \frac{\left(\Sigma x\right)_i}{\sqrt{x^\top \Sigma x}}$$

This implies that the (excess) performance contribution is equal to:

$$\begin{array}{rcl} \mathcal{PC}_i & = & \mathrm{SR}\left(x \mid r\right) \dfrac{x_i \cdot \left(\Sigma x\right)_i}{\sqrt{x^\top \Sigma x}} \\ & = & \mathrm{SR}\left(x \mid r\right) \cdot \mathcal{RC}_i \end{array}$$

where $\mathrm{SR}\left(x \mid r\right)$ is the expected Sharpe ratio of the RB portfolio

Optimality of the RB portfolio

Remark

From an ex-ante point of view, performance budgeting and risk budgeting are equivalent

Optimality of the RB portfolio

Example 18

We consider a universe of four assets. The volatilities are respectively 10%, 20%, 30% and 40%. The correlation of asset returns is given by the following matrix:

$$\rho = \begin{pmatrix} 1.00 & & & \\ 0.80 & 1.00 & & \\ 0.20 & 0.20 & 1.00 & \\ 0.20 & 0.20 & 0.50 & 1.00 \end{pmatrix}$$

The risk-free rate is equal to zero

Optimality of the RB portfolio

Table 45: Implied risk premia when $b = (20\%, 25\%, 40\%, 15\%)$ (Example 18)

Asset	x_i	$\mathcal{MR}_i$	$\tilde{\mu}_i$	$\mathcal{PC}_i$	$\mathcal{PC}_i^{\star}$
1	40.91	7.10	3.55	1.45	20.00
2	25.12	14.46	7.23	1.82	25.00
3	25.26	23.01	11.50	2.91	40.00
4	8.71	25.04	12.52	1.09	15.00
Expected return				7.27	

Table 46: Implied risk premia when $b = (10\%, 10\%, 10\%, 70\%)$ (Example 18)

Asset	x_i	$\mathcal{MR}_i$	$\tilde{\mu}_i$	$\mathcal{PC}_i$	$\mathcal{PC}_i^{\star}$
1	35.88	5.27	2.63	0.94	10.00
2	17.94	10.53	5.27	0.94	10.00
3	10.18	18.56	9.28	0.94	10.00
4	35.99	36.75	18.37	6.61	70.00
Expected return				9.45	

Main result

There is no neutral allocation. Every portfolio corresponds to an active bet.

Variation on the ERC portfolio

Question 1

We note Σ the covariance matrix of asset returns.

Variation on the ERC portfolio

Question 1.a

What is the risk contribution $\mathcal{RC}_i$ of asset i with respect to portfolio x?

Variation on the ERC portfolio

Let $\mathcal{R}(x)$ be a risk measure of the portfolio x. If this risk measure satisfies the Euler principle, we have (TR-RPB, page 78):

$$\mathcal{R}(x) = \sum_{i=1}^{n} x_i \frac{\partial \, \mathcal{R}(x)}{\partial \, x_i}$$

We can then decompose the risk measure as a sum of asset contributions. This is why we define the risk contribution $\mathcal{RC}_i$ of asset i as the product of the weight by the marginal risk:

$$\mathcal{RC}_i = x_i \frac{\partial \, \mathcal{R}(x)}{\partial \, x_i}$$

When the risk measure is the volatility $\sigma(x)$, it follows that:

$$\begin{aligned} \mathcal{RC}_i &= x_i \frac{(\Sigma x)_i}{\sqrt{x^\top \Sigma x}} \\ &= \frac{x_i \left(\sum_{k=1}^{n} \rho_{i,k} \sigma_i \sigma_k x_k \right)}{\sigma(x)} \end{aligned}$$

Variation on the ERC portfolio

Question 1.b

Define the ERC portfolio.

Variation on the ERC portfolio

The ERC portfolio corresponds to the risk budgeting portfolio when the risk measure is the return volatility $\sigma(x)$ and when the risk budgets are the same for all the assets (TR-RPB, page 119). It means that $\mathcal{RC}_i = \mathcal{RC}_j$, that is:

$$x_i \frac{\partial \, \sigma(x)}{\partial \, x_i} = x_j \frac{\partial \, \sigma(x)}{\partial \, x_j}$$

Variation on the ERC portfolio

Question 1.c

Calculate the variance of the risk contributions. Define an optimization program to compute the ERC portfolio. Find an equivalent maximization program based on the $\mathcal{L}^2$ norm.

Variation on the ERC portfolio

We have:

$$\begin{aligned}
\overline{\mathcal{RC}} &= \frac{1}{n}\sum_{i=1}^{n}\mathcal{RC}_i \\
&= \frac{1}{n}\sigma(x)
\end{aligned}$$

It follows that:

$$\begin{aligned}
\operatorname{var}(\mathcal{RC}) &= \frac{1}{n}\sum_{i=1}^{n}\left(\mathcal{RC}_i - \overline{\mathcal{RC}}\right)^2 \\
&= \frac{1}{n}\sum_{i=1}^{n}\left(\mathcal{RC}_i - \frac{1}{n}\sigma(x)\right)^2 \\
&= \frac{1}{n^2\sigma(x)}\sum_{i=1}^{n}\left(nx_i(\Sigma x)_i - \sigma^2(x)\right)^2
\end{aligned}$$

Variation on the ERC portfolio

To compute the ERC portfolio, we may consider the following optimization program:

$$x^{\star} = \arg\min \sum_{i=1}^{n} \left(n x_i \left(\Sigma x \right)_i - \sigma^2 \left(x \right) \right)^2$$

Because we know that the ERC portfolio always exists (TR-RPB, page 108), the objective function at the optimum $x^{\star}$ is necessarily equal to 0. Another equivalent optimization program is to consider the L^2 norm. In this case, we have (TR-RPB, page 102):

$$x^{\star} = \arg\min \sum_{i=1}^{n} \sum_{j=1}^{n} \left(x_i \cdot \left(\Sigma x \right)_i - x_j \cdot \left(\Sigma x \right)_j \right)^2$$

Variation on the ERC portfolio

Question 1.d

Let $\beta_i(x)$ be the beta of asset i with respect to portfolio x. Show that we have the following relationship in the ERC portfolio:

$$x_i \beta_i(x) = x_j \beta_j(x)$$

Propose a numerical algorithm to find the ERC portfolio.

Variation on the ERC portfolio

We have:

$$\begin{aligned}\beta_i(x) &= \frac{(\Sigma x)_i}{x^\top \Sigma x} \\ &= \frac{\mathcal{MR}_i}{\sigma(x)}\end{aligned}$$

We deduce that:

$$\begin{aligned}\mathcal{RC}_i &= x_i \cdot \mathcal{MR}_i \\ &= x_i \beta_i(x)\, \sigma(x)\end{aligned}$$

The relationship $\mathcal{RC}_i = \mathcal{RC}_j$ becomes:

$$x_i \beta_i(x) = x_j \beta_j(x)$$

It means that the weight is inversely proportional to the beta:

$$x_i \propto \frac{1}{\beta_i(x)}$$

Variation on the ERC portfolio

We can use the Jacobi power algorithm (TR-RPB, page 308). Let $x^{(k)}$ be the portfolio at iteration k. We define the portfolio $x^{(k+1)}$ as follows:

$$x^{(k+1)} = \frac{\beta_i^{-1}\left(x^{(k)}\right)}{\sum_{j=1}^{n} \beta_j^{-1}\left(x^{(k)}\right)}$$

Starting from an initial portfolio $x^{(0)}$, the limit portfolio is the ERC portfolio if the algorithm converges:

$$\lim_{k \to \infty} x^{(k)} = x_{\mathrm{erc}}$$

Variation on the ERC portfolio

Question 1.e

We suppose that the volatilities are 15%, 20% and 25% and that the correlation matrix is:

$$\rho = \left(\begin{array}{ccc} 100\% & & \\ 50\% & 100\% & \\ 40\% & 30\% & 100\% \end{array} \right)$$

Compute the ERC portfolio using the beta algorithm.

Variation on the ERC portfolio

Starting from the EW portfolio, we obtain for the first five iterations:

k	0	1	2	3	4	5
$x_1^{(k)}$ (in %)	33.3333	43.1487	40.4122	41.2314	40.9771	41.0617
$x_2^{(k)}$ (in %)	33.3333	32.3615	31.9164	32.3529	32.1104	32.2274
$x_3^{(k)}$ (in %)	33.3333	24.4898	27.6714	26.4157	26.9125	26.7109
$\beta_1\left(x^{(k)}\right)$	0.7326	0.8341	0.8046	0.8147	0.8113	0.8126
$\beta_2\left(x^{(k)}\right)$	0.9767	1.0561	1.0255	1.0397	1.0337	1.0363
$\beta_3\left(x^{(k)}\right)$	1.2907	1.2181	1.2559	1.2405	1.2472	1.2444

Variation on the ERC portfolio

The next iterations give the following results:

k	6	7	8	9	10	11
$x_1^{(k)}$ (in %)	41.0321	41.0430	41.0388	41.0405	41.0398	41.0401
$x_2^{(k)}$ (in %)	32.1746	32.1977	32.1878	32.1920	32.1902	32.1909
$x_3^{(k)}$ (in %)	26.7933	26.7593	26.7734	26.7676	26.7700	26.7690
$\beta_1\left(x^{(k)}\right)$	0.8121	0.8123	0.8122	0.8122	0.8122	0.8122
$\beta_2\left(x^{(k)}\right)$	1.0352	1.0356	1.0354	1.0355	1.0355	1.0355
$\beta_3\left(x^{(k)}\right)$	1.2456	1.2451	1.2453	1.2452	1.2452	1.2452

Variation on the ERC portfolio

Finally, the algorithm converges after 14 iterations with the following stopping criteria:

$$\sup_i \left| x_i^{(k+1)} - x_i^{(k)} \right| \leq 10^{-6}$$

and we obtain the following results:

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1	41.04%	12.12%	4.97%	33.33%
2	32.19%	15.45%	4.97%	33.33%
3	26.77%	18.58%	4.97%	33.33%

Variation on the ERC portfolio

Question 2

We now suppose that the return of asset i satisfies the CAPM model:

$$R_i = \beta_i R_m + \varepsilon_i$$

where R_m is the return of the market portfolio and ε_i is the idiosyncratic risk. We note $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$. We assume that $R_m \perp \varepsilon$, $\mathrm{var}(R_m) = \sigma_m^2$ and $\mathrm{cov}(\varepsilon) = D = \mathrm{diag}(\tilde{\sigma}_1^2, \ldots, \tilde{\sigma}_n^2)$.

Variation on the ERC portfolio

Question 2.a

Calculate the risk contribution $\mathcal{RC}_i$.

Variation on the ERC portfolio

We have:

$$\Sigma = \beta\beta^\top \sigma_m^2 + \operatorname{diag}\left(\tilde{\sigma}_1^2, \ldots, \tilde{\sigma}_n^2\right)$$

We deduce that:

$$\begin{aligned} \mathcal{RC}_i &= \frac{x_i \left(\sum_{k=1}^n \beta_i \beta_k \sigma_m^2 x_k + \tilde{\sigma}_i^2 x_i\right)}{\tilde{\sigma}\left(x\right)} \\ &= \frac{x_i \beta_i B + x_i^2 \tilde{\sigma}_i^2}{\sigma\left(x\right)} \end{aligned}$$

with:

$$B = \sum_{k=1}^n x_k \beta_k \sigma_m^2$$

Variation on the ERC portfolio

Question 2.b

We assume that $\beta_i = \beta_j$. Show that the ERC weight x_i is a decreasing function of the idiosyncratic volatility $\tilde{\sigma}_i$.

Variation on the ERC portfolio

Using Equation 2.a, we deduce that the ERC portfolio satisfies:

$$x_i \beta_i B + x_i^2 \tilde{\sigma}_i^2 = x_j \beta_j B + x_j^2 \tilde{\sigma}_j^2$$

or:

$$\left(x_i \beta_i - x_j \beta_j \right) B = \left(x_j^2 \tilde{\sigma}_j^2 - x_i^2 \tilde{\sigma}_i^2 \right)$$

Variation on the ERC portfolio

If $\beta_i = \beta_j = \beta$, we have:

$$\left(x_i - x_j\right)\beta B = \left(x_j^2\tilde{\sigma}_j^2 - x_i^2\tilde{\sigma}_i^2\right)$$

Because $\beta > 0$, we deduce that:

$$\begin{array}{rcl} x_i > x_j & \Leftrightarrow & x_j^2\tilde{\sigma}_j^2 - x_i^2\tilde{\sigma}_i^2 > 0 \\ & \Leftrightarrow & x_j\tilde{\sigma}_j > x_i\tilde{\sigma}_i \\ & \Leftrightarrow & \tilde{\sigma}_i < \tilde{\sigma}_j \end{array}$$

We conclude that the weight x_i is a decreasing function of the specific volatility $\tilde{\sigma}_i$.

Variation on the ERC portfolio

Question 2.c

We assume that $\tilde{\sigma}_i = \tilde{\sigma}_j$. Show that the ERC weight x_i is a decreasing function of the sensitivity β_i to the common factor.

Variation on the ERC portfolio

If $\tilde{\sigma}_i = \tilde{\sigma}_j = \tilde{\sigma}$, we have:

$$\left(x_i \beta_i - x_j \beta_j\right) B = \left(x_j^2 - x_i^2\right) \tilde{\sigma}^2$$

We deduce that:

$$\begin{aligned} x_i > x_j & \Leftrightarrow \left(x_i \beta_i - x_j \beta_j\right) B < 0 \\ & \Leftrightarrow x_i \beta_i < x_j \beta_j \\ & \Leftrightarrow \beta_i < \beta_j \end{aligned}$$

We conclude that the weight x_i is a decreasing function of the sensitivity β_i.

Variation on the ERC portfolio

Question 2.d

We consider the numerical application: $\beta_1 = 1$, $\beta_2 = 0.9$, $\beta_3 = 0.8$, $\beta_4 = 0.7$, $\tilde{\sigma}_1 = 5\%$, $\tilde{\sigma}_2 = 5\%$, $\tilde{\sigma}_3 = 10\%$, $\tilde{\sigma}_4 = 10\%$, and $\sigma_m = 20\%$. Find the ERC portfolio.

Variation on the ERC portfolio

We obtain the following results:

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1	21.92%	19.73%	4.32%	25.00%
2	24.26%	17.83%	4.32%	25.00%
3	25.43%	17.00%	4.32%	25.00%
4	28.39%	15.23%	4.32%	25.00%

Weight concentration of a portfolio

Question 1

We consider the Lorenz curve defined by:

$$\begin{array}{rcl} [0,1] & \longrightarrow & [0,1] \\ x & \longmapsto & \mathbb{L}(x) \end{array}$$

We assume that $\mathbb{L}$ is an increasing function and $\mathbb{L}(x) > x$.

Weight concentration of a portfolio

Question 1.a

Represent graphically the function $\mathbb{L}$ and define the Gini coefficient $\mathcal{G}$ associated with $\mathbb{L}$.

Weight concentration of a portfolio

We have represented the function $y = \mathcal{L}(x)$ in Figure 38. It verifies $\mathcal{L}(x) \geq x$ and $\mathcal{L}(x) \leq 1$. The Gini coefficient is defined as follows (TR-RPB, page 127):

$$\begin{aligned} G &= \frac{A}{A+B} \\ &= \left(\int_0^1 \mathcal{L}(x)\,\mathrm{d}x - \frac{1}{2}\right) \Big/ \frac{1}{2} \\ &= 2\int_0^1 \mathcal{L}(x)\,\mathrm{d}x - 1 \end{aligned}$$

Weight concentration of a portfolio

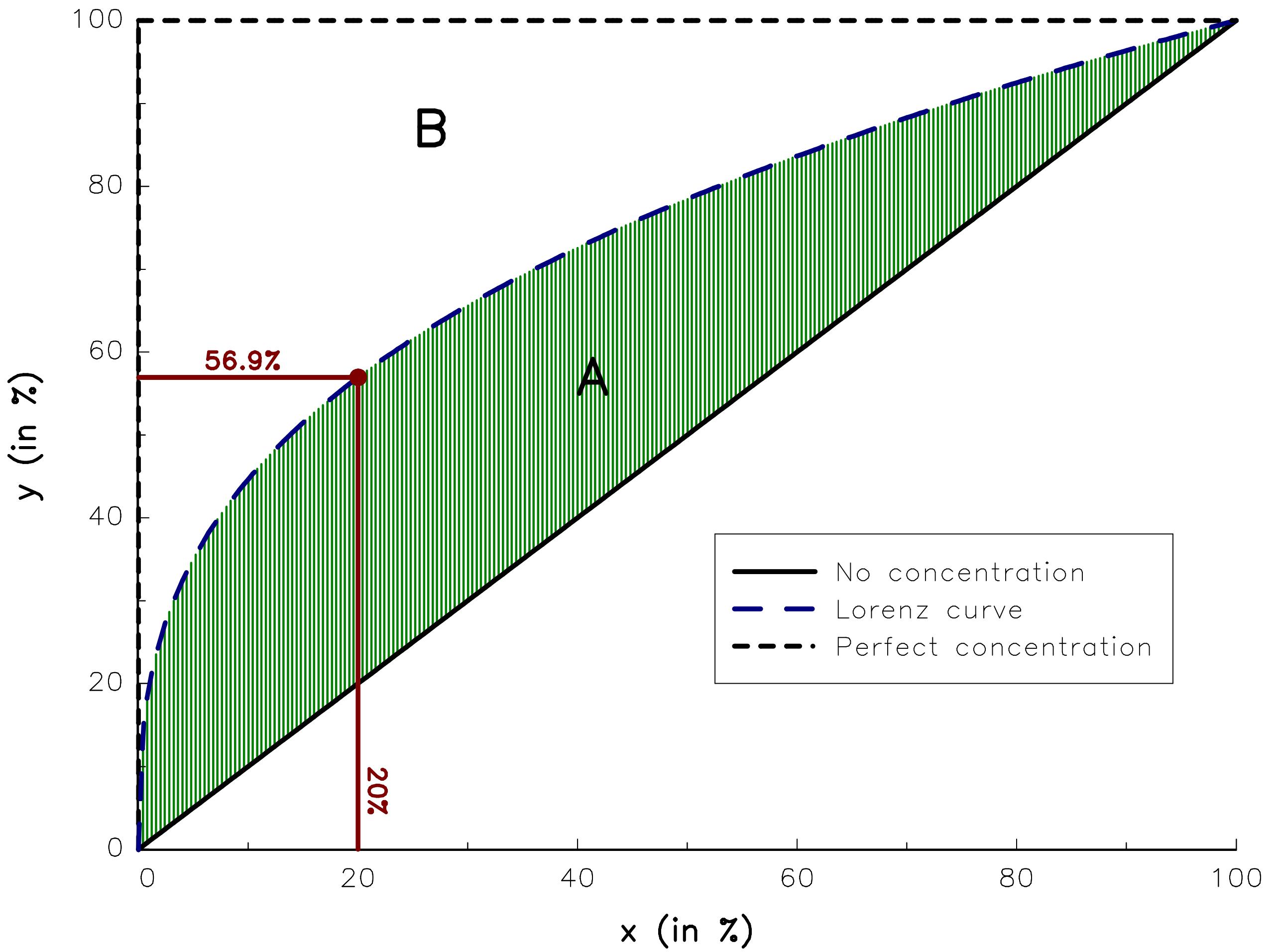


Figure 38: Lorenz curve

Weight concentration of a portfolio

Question 1.b

We set $\mathbb{L}_{\alpha}(x) = x^{\alpha}$ with $\alpha \geq 0$. Is the function $\mathbb{L}_{\alpha}$ a Lorenz curve? Calculate the Gini coefficient $\mathcal{G}(\alpha)$ with respect to α. Deduce $\mathcal{G}(0)$, $\mathcal{G}\left(\frac{1}{2}\right)$ and $\mathcal{G}(1)$.

Weight concentration of a portfolio

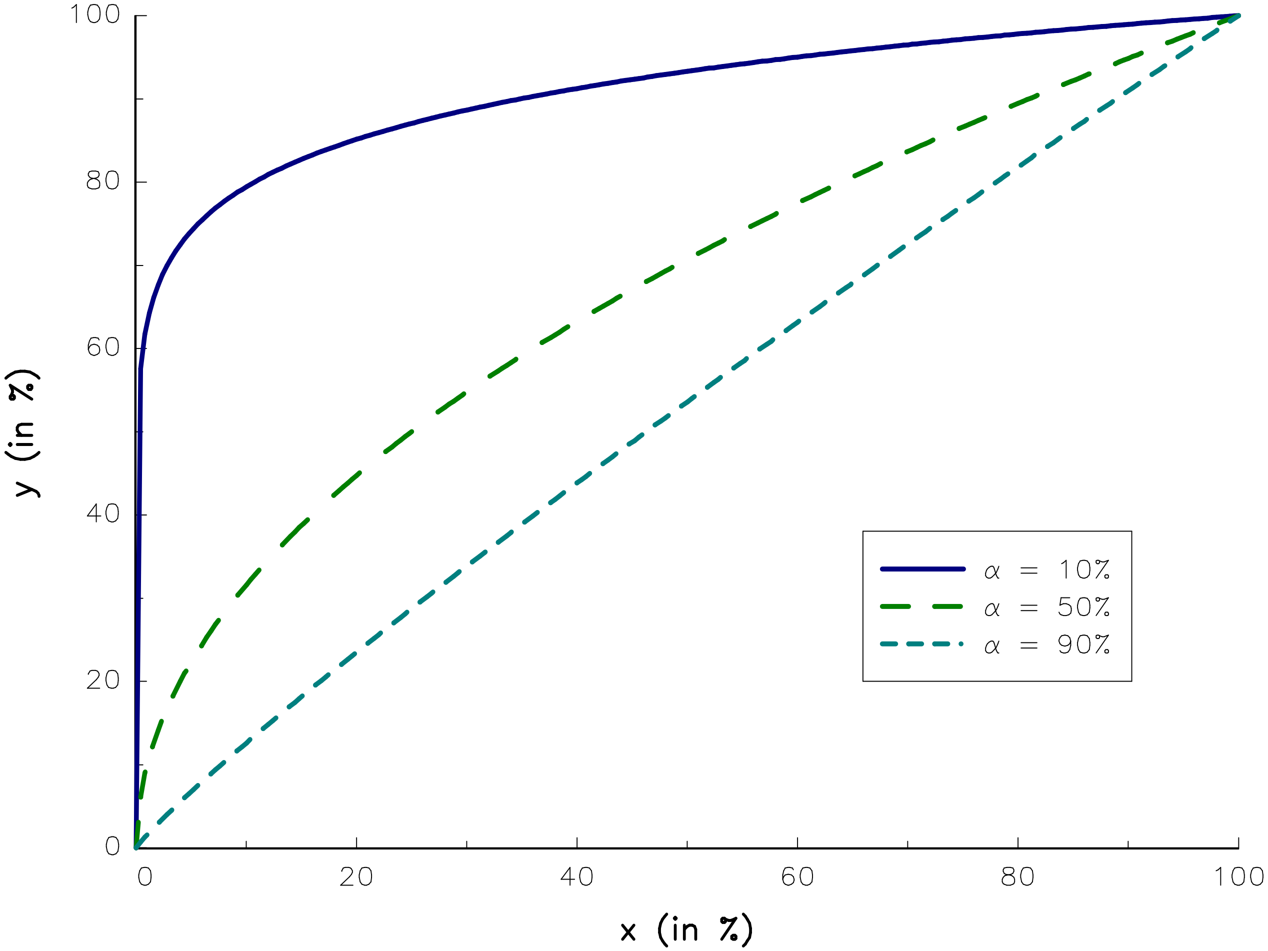


Figure 39: Function $y = x^{\alpha}$

Weight concentration of a portfolio

If $\alpha \geq 0$, the function $\mathcal{L}_{\alpha}(x) = x^{\alpha}$ is increasing. We have $\mathcal{L}_{\alpha}(1) = 1$, $\mathcal{L}_{\alpha}(x) \leq 1$ and $\mathcal{L}_{\alpha}(x) \geq x$. We deduce that $\mathcal{L}_{\alpha}$ is a Lorenz curve. For the Gini index, we have:

$$\begin{aligned}
\mathcal{G}(\alpha) &= 2\int_0^1 x^{\alpha}\,\mathrm{d}x - 1 \\
&= 2\left[\frac{x^{\alpha+1}}{\alpha+1}\right]_0^1 - 1 \\
&= \frac{1-\alpha}{1+\alpha}
\end{aligned}$$

We deduce that $\mathcal{G}(0) = 1$, $\mathcal{G}\left(\frac{1}{2}\right) = {}^1/_3$ et $\mathcal{G}(1) = 0$. $\alpha = 0$ corresponds to the perfect concentration whereas $\alpha = 1$ corresponds to the perfect equality.

Weight concentration of a portfolio

Question 2

Let w be a portfolio of n assets. We suppose that the weights are sorted in a descending order: $w_1 \geq w_2 \geq \ldots \geq w_n$.

Weight concentration of a portfolio

Question 2.a

We define $\mathbb{L}_w(x)$ as follows:

$$\mathbb{L}_w(x) = \sum_{j=1}^{i} w_j \quad \text{if} \quad \frac{i}{n} \leq x < \frac{i+1}{n}$$

with $\mathbb{L}_w(0) = 0$. Is the function $\mathbb{L}_w$ a Lorenz curve ? Calculate the Gini coefficient with respect to the weights w_i. In which cases does $\mathcal{G}$ take the values 0 and 1?

Weight concentration of a portfolio

We have $\mathcal{L}_w(0) = 0$ and $\mathcal{L}_w(1) = \sum_{j=1}^n w_j = 1$. If $x_2 \geq x_1$, we have $\mathcal{L}_w(x_2) \geq \mathcal{L}_w(x_2)$. $\mathcal{L}_w$ is then a Lorenz curve. The Gini coefficient is equal to:

$$\begin{aligned}\mathcal{G} &= 2\int_0^1 \mathcal{L}(x)\,\mathrm{d}x - 1 \\ &= \frac{2}{n}\sum_{i=1}^n \sum_{j=1}^i w_j - 1\end{aligned}$$

If $w_j = n^{-1}$, we have:

$$\begin{aligned}\lim_{n\to\infty} \mathcal{G} &= \lim_{n\to\infty} \frac{2}{n}\sum_{i=1}^n \frac{i}{n} - 1 \\ &= \lim_{n\to\infty} \frac{2}{n} \cdot \frac{n(n+1)}{2n} - 1 \\ &= \lim_{n\to\infty} \frac{1}{n} = 0\end{aligned}$$

Weight concentration of a portfolio

If $w_1 = 1$, we have:

$$\begin{aligned} \lim_{n \to \infty} \mathcal{G} &= \lim_{n \to \infty} 1 - \frac{1}{n} \\ &= 1 \end{aligned}$$

We note that the perfect equality does not correspond to the case $\mathcal{G} = 0$ except in the asymptotic case. This is why we may slightly modify the definition of $\mathcal{L}_w(x)$:

$$\mathcal{L}_w(x) = \begin{cases} \sum_{j=1}^{i} w_j & \text{if} \quad x = n^{-1} i \\ \sum_{j=1}^{i} w_j + w_{i+1}(nx - i) & \text{if} \quad n^{-1} i < x < n^{-1}(i+1) \end{cases}$$

While the previous definition corresponds to a constant piecewise function, this one defines an affine piecewise function. In this case, the computation of the Gini index is done using a trapezoidal integration:

$$\mathcal{G} = \frac{2}{n} \left(\sum_{i=1}^{n-1} \sum_{j=1}^{i} w_j + \frac{1}{2} \right) - 1$$

Weight concentration of a portfolio

Question 2.b

The definition of the Herfindahl index is:

$$\mathcal{H} = \sum_{i=1}^{n} w_i^2$$

In which cases does $\mathcal{H}$ take the value 1? Show that $\mathcal{H}$ reaches its maximum when $w_i = n^{-1}$. What is the interpretation of this result?

Weight concentration of a portfolio

The Herfindahl index is equal to 1 if the portfolio is concentrated in only one asset. We seek to minimize $\mathcal{H} = \sum_{i=1}^{n} w_i^2$ under the constraint $\sum_{i=1}^{n} w_i = 1$. The Lagrange function is then:

$$f(w_1, \ldots, w_n; \lambda) = \sum_{i=1}^{n} w_i^2 - \lambda \left(\sum_{i=1}^{n} w_i - 1 \right)$$

The first-order conditions are $2w_i - \lambda = 0$. We deduce that $w_i = w_j$. $\mathcal{H}$ reaches its minimum when $w_i = n^{-1}$. It corresponds to the equally weighted portfolio. In this case, we have:

$$\mathcal{H} = \frac{1}{n}$$

Weight concentration of a portfolio

Question 2.c

We set $\mathcal{N} = \mathcal{H}^{-1}$. What does the statistic $\mathcal{N}$ mean?

Weight concentration of a portfolio

The statistic $\mathcal{N}$ is the degree of freedom or the equivalent number of equally weighted assets. For instance, if $\mathcal{H} = 0.5$, then $\mathcal{N} = 2$. It is a portfolio equivalent to two equally weighted assets.

Weight concentration of a portfolio

Question 3

We consider an investment universe of five assets. We assume that their asset returns are not correlated. The volatilities are given in the table below:

σ_i	2%	5%	10%	20%	30%
$w_i^{(1)}$		10%	20%	30%	40%
$w_i^{(2)}$	40%	20%		30%	10%
$w_i^{(3)}$	20%	15%	25%	35%	5%

Weight concentration of a portfolio

Question 3.a

Find the minimum variance portfolio $w^{(4)}$.

Weight concentration of a portfolio

The minimum variance portfolio is equal to:

$$w^{(4)} = \begin{pmatrix} 82.342\% \\ 13.175\% \\ 3.294\% \\ 0.823\% \\ 0.366\% \end{pmatrix}$$

Weight concentration of a portfolio

Question 3.b

Calculate the Gini and Herfindahl indices and the statistic $\mathcal{N}$ for the four portfolios $w^{(1)}$, $w^{(2)}$, $w^{(3)}$ and $w^{(4)}$.

Weight concentration of a portfolio

For each portfolio, we sort the weights in descending order. For the portfolio $w^{(1)}$, we have $w_1^{(1)} = 40\%$, $w_2^{(1)} = 30\%$, $w_3^{(1)} = 20\%$, $w_4^{(1)} = 10\%$ and $w_5^{(1)} = 0\%$. It follows that:

$$\begin{aligned}
\mathcal{H}\left(w^{(1)}\right) &= \sum_{i=1}^{5}\left(w_i^{(1)}\right)^2 \\
&= 0.10^2 + 0.20^2 + 0.30^2 + 0.40^2 \\
&= 0.30
\end{aligned}$$

We also have:

$$\begin{aligned}
\mathcal{G}\left(w^{(1)}\right) &= \frac{2}{5}\left(\sum_{i=1}^{4}\sum_{j=1}^{i}\tilde{w}_j^{(1)} + \frac{1}{2}\right) - 1 \\
&= \frac{2}{5}\left(0.40 + 0.70 + 0.90 + 1.00 + \frac{1}{2}\right) - 1 \\
&= 0.40
\end{aligned}$$

Weight concentration of a portfolio

For the portfolios $w^{(2)}$, $w^{(3)}$ and $w^{(4)}$, we obtain $\mathcal{H}\left(w^{(2)}\right) = 0.30$, $\mathcal{H}\left(w^{(3)}\right) = 0.25$, $\mathcal{H}\left(w^{(4)}\right) = 0.70$, $\mathcal{G}\left(w^{(2)}\right) = 0.40$, $\mathcal{G}\left(w^{(3)}\right) = 0.28$ and $\mathcal{G}\left(w^{(4)}\right) = 0.71$. We have $\mathcal{N}\left(w^{(2)}\right) = \mathcal{N}\left(w^{(1)}\right) = 3.33$, $\mathcal{N}\left(w^{(3)}\right) = 4.00$ and $\mathcal{N}\left(w^{(4)}\right) = 1.44$.

Weight concentration of a portfolio

Question 3.c

Comment on these results. What differences do you make between portfolio concentration and portfolio diversification?

Weight concentration of a portfolio

All the statistics show that the least concentrated portfolio is $w^{(3)}$. The most concentrated portfolio is paradoxically the minimum variance portfolio $w^{(4)}$. We generally assimilate variance optimization to diversification optimization. We show in this example that diversifying in the Markowitz sense does not permit to minimize the concentration.

The optimization problem of the ERC portfolio

Question 1

We consider four assets. Their volatilities are equal to 10%, 15%, 20% and 25% whereas the correlation matrix of asset returns is:

$$\rho = \left(\begin{array}{rrrr} 100\% & & & \\ 60\% & 100\% & & \\ 40\% & 40\% & 100\% & \\ 30\% & 30\% & 20\% & 100\% \end{array} \right)$$

The optimization problem of the ERC portfolio

Question 1.a

Find the long-only minimum variance, ERC and equally weighted portfolios.

The optimization problem of the ERC portfolio

The weights of the three portfolios are:

Asset	MV	ERC	EW
1	87.51%	37.01%	25.00%
2	4.05%	24.68%	25.00%
3	4.81%	20.65%	25.00%
4	3.64%	17.66%	25.00%

The optimization problem of the ERC portfolio

Question 1.b

We consider the following portfolio optimization problem:

$$\begin{aligned} x^{\star}(c) &= \arg\min \sqrt{x^{\top}\Sigma x} \qquad (1) \\ \text{u.c.} & \quad \begin{cases} \sum_{i=1}^{n} \ln x_i \geq c \\ \mathbf{1}_n^{\top} x = 1 \\ x \geq \mathbf{0}_n \end{cases} \end{aligned}$$

with Σ the covariance matrix of asset returns. We note λ_c and λ_0 the Lagrange coefficients associated with the constraints $\sum_{i=1}^{n} \ln x_i \geq c$ and $\mathbf{1}_n^{\top} x = 1$. Write the Lagrange function of the optimization problem. Deduce then an equivalent optimization problem that is easier to solve than Problem (1).

The optimization problem of the ERC portfolio

The Lagrange function is:

$$\begin{aligned}
\mathcal{L}(x;\lambda,\lambda_0,\lambda_c) &= \sqrt{x^\top \Sigma x} - \lambda^\top x - \lambda_0\left(\mathbf{1}_n^\top x - 1\right) - \lambda_c\left(\sum_{i=1}^n \ln x_i - c\right) \\
&= \left(\sqrt{x^\top \Sigma x} - \lambda_c \sum_{i=1}^n \ln x_i\right) - \lambda^\top x - \lambda_0\left(\mathbf{1}_n^\top x - 1\right) + \lambda_c c
\end{aligned}$$

We deduce that an equivalent optimization problem is:

$$\begin{aligned}
\tilde{x}^\star(\lambda_c) &= \arg\min \sqrt{\tilde{x}^\top \Sigma \tilde{x}} - \lambda_c \sum_{i=1}^n \ln \tilde{x}_i \\
&\text{u.c.} \quad \left\{\begin{array}{l} \mathbf{1}_n^\top \tilde{x} = 1 \\ \tilde{x} \geq \mathbf{0}_n \end{array}\right.
\end{aligned}$$

The optimization problem of the ERC portfolio

We notice a strong difference between the two problems because they don't use the same control variable. However, the control variable c of the first problem may be deduced from the solution of the second problem:

$$c = \sum_{i=1}^{n} \ln \tilde{x}_i^\star \left(\lambda_c\right)$$

We also know that (TR-RPB, page 131):

$$c_- \leq \sum_{i=1}^{n} \ln x_i \leq c_+$$

where $c_- = \sum_{i=1}^{n} \ln \left(x_{\mathrm{mv}}\right)_i$ and $c_+ = -n \ln n$. It follows that:

$$\left\{ \begin{array}{ll} x^\star \left(c\right) = \tilde{x}^\star \left(0\right) & \text{if } c \leq c_- \\ x^\star \left(c\right) = \tilde{x}^\star \left(\infty\right) & \text{if } c \geq c_+ \end{array} \right.$$

If $c \in \left]c_-, c_+\right[$, there exists a scalar $\lambda_c > 0$ such that:

$$x^\star \left(c\right) = \tilde{x}^\star \left(\lambda_c\right)$$

The optimization problem of the ERC portfolio

Question 1.c

Represent the relationship between λ_c and $\sigma\left(x^\star\left(c\right)\right)$, c and $\sigma\left(x^\star\left(c\right)\right)$ and $\mathcal{I}^\star\left(x^\star\left(c\right)\right)$ and $\sigma\left(x^\star\left(c\right)\right)$ where $\mathcal{I}^\star\left(x\right)$ is the diversity index of the weights.

The optimization problem of the ERC portfolio

For a given value $\lambda_c \in [0, +\infty[$, we solve numerically the second problem and find the optimized portfolio $\tilde{x}^\star(\lambda_c)$. Then, we calculate $c = \sum_{i=1}^n \ln \tilde{x}_i^\star(\lambda_c)$ and deduce that $x^\star(c) = \tilde{x}^\star(\lambda_c)$. We finally obtain $\sigma(x^\star(c)) = \sigma(\tilde{x}^\star(\lambda_c))$ and $\mathcal{I}^\star(x^\star(c)) = \mathcal{I}^\star(\tilde{x}^\star(\lambda_c))$. The relationships between λ_c, c, $\mathcal{I}^\star(x^\star(c))$ and $\sigma(x^\star(c))$ are reported in Figure 40.

The optimization problem of the ERC portfolio

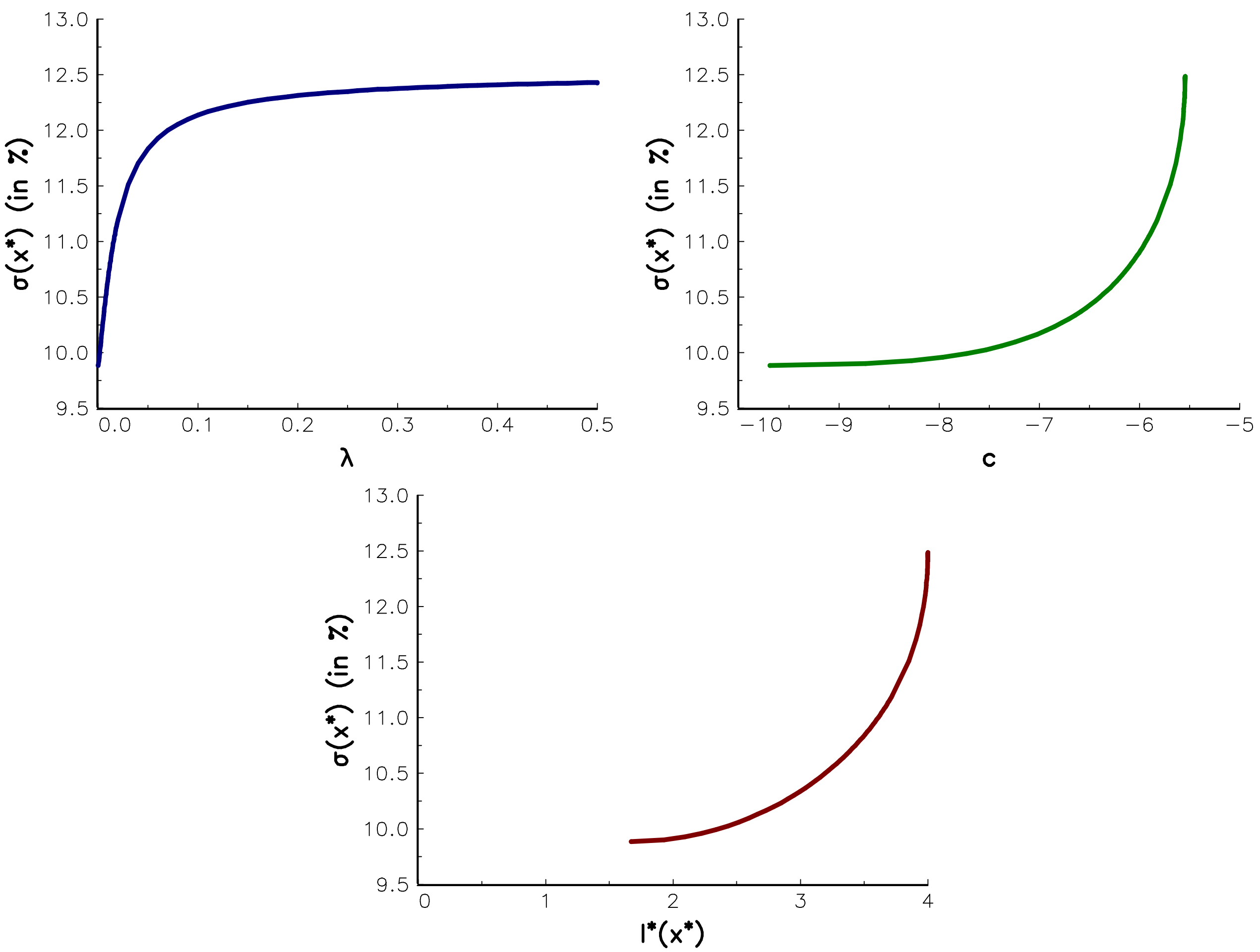


Figure 40: Relationship between λ_c, c, $\mathcal{I}^\star\left(x^\star\left(c\right)\right)$ and $\sigma\left(x^\star\left(c\right)\right)$

The optimization problem of the ERC portfolio

Question 1.d

Represent the relationship between λ_c and $\mathcal{I}^\star(\mathcal{RC})$, c and $\mathcal{I}^\star(\mathcal{RC})$ and $\mathcal{I}^\star(x^\star(c))$ and $\mathcal{I}^\star(\mathcal{RC})$ where $\mathcal{I}^\star(\mathcal{RC})$ is the diversity index of the risk contributions.

The optimization problem of the ERC portfolio

If we consider $\mathcal{I}^{\star}\left(\mathcal{RC}\right)$ in place of $\sigma\left(x^{\star}\left(c\right)\right)$, we obtain Figure 41.

The optimization problem of the ERC portfolio

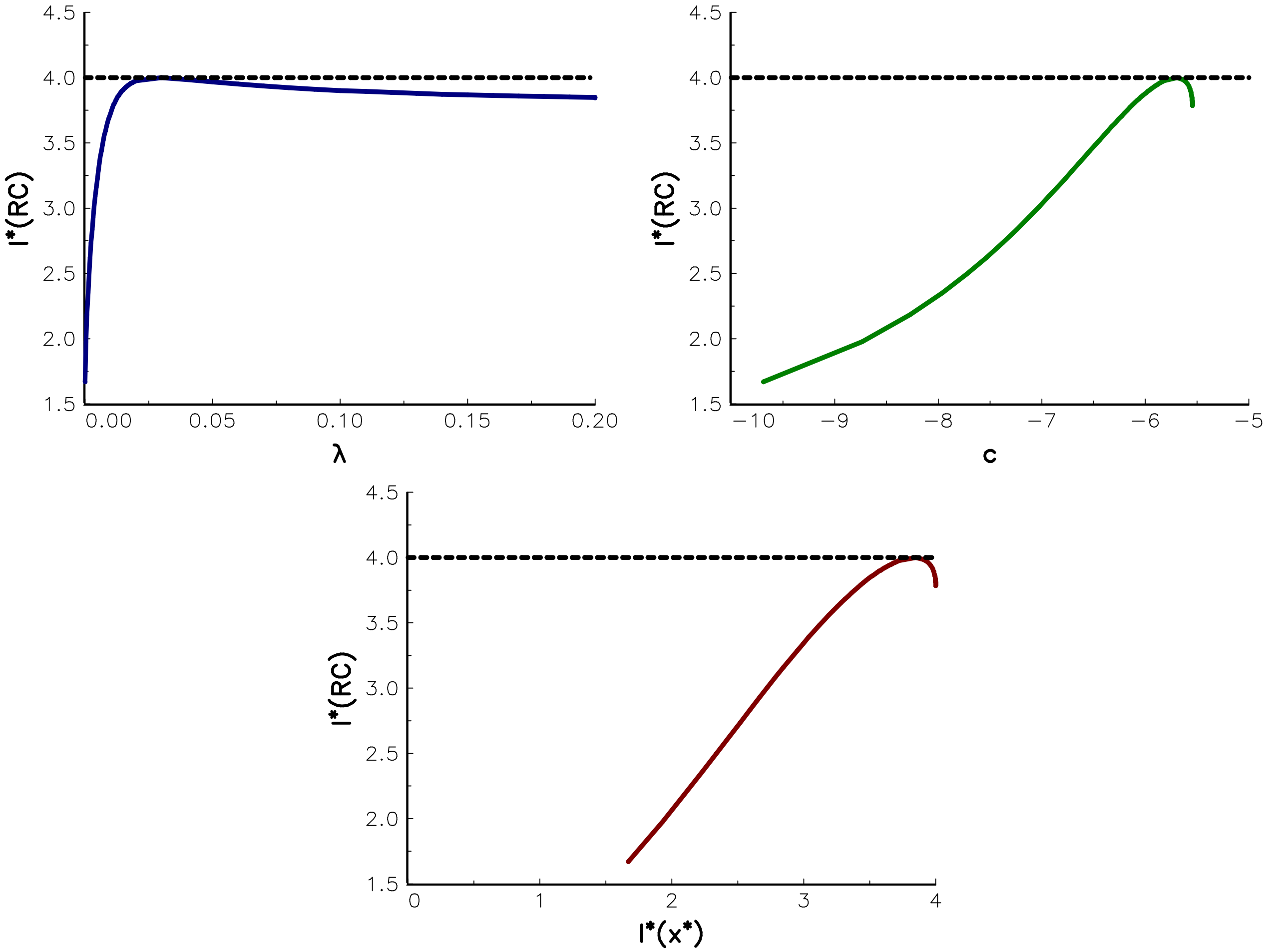


Figure 41: Relationship between λ_c, c, $\mathcal{I}^{\star}\left(x^{\star}\left(c\right)\right)$ and $\mathcal{I}^{\star}\left(\mathcal{RC}\right)$

The optimization problem of the ERC portfolio

Question 1.e

Draw the relationship between $\sigma(x^{\star}(c))$ and $\mathcal{I}^{\star}(\mathcal{RC})$. Identify the ERC portfolio.

The optimization problem of the ERC portfolio

In Figure 42, we have reported the relationship between $\sigma\left(x^{\star}\left(c\right)\right)$ and $\mathcal{I}^{\star}\left(\mathcal{RC}\right)$. The ERC portfolio satisfies the equation $\mathcal{I}^{\star}\left(\mathcal{RC}\right) = n$.

The optimization problem of the ERC portfolio

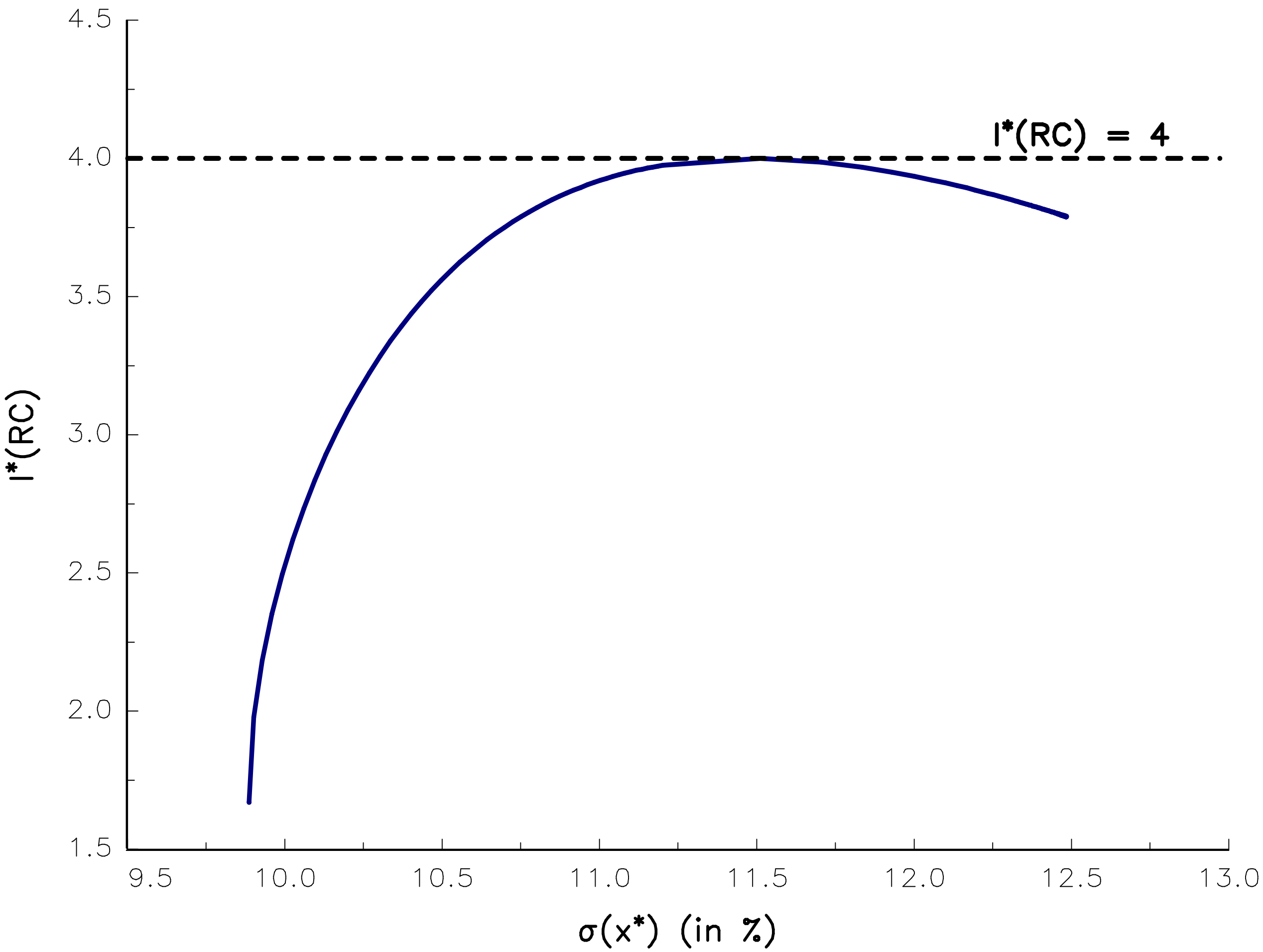


Figure 42: Relationship between $\sigma\left(x^{\star}\left(c\right)\right)$ and $\mathcal{I}^{\star}\left(\mathcal{RC}\right)$

The optimization problem of the ERC portfolio

Question 2

We now consider a slight modification of the previous optimization problem:

$$\begin{aligned} x^{\star}(c) &= \arg\min \sqrt{x^{\top}\Sigma x} \\ &\text{u.c.} \quad \left\{ \begin{array}{l} \sum_{i=1}^{n} \ln x_i \geq c \\ x \geq \mathbf{0}_n \end{array} \right. \end{aligned} \tag{2}$$

The optimization problem of the ERC portfolio

Question 2.a

Why does the optimization problem (1) not define the ERC portfolio?

The optimization problem of the ERC portfolio

Let us consider the optimization problem when we impose the constraint $\mathbf{1}_n^\top x = 1$. The first-order condition is:

$$\frac{\partial \, \sigma \left(x \right)}{\partial \, x_i} - \lambda_i - \lambda_0 - \frac{\lambda_c}{x_i} = 0$$

Because $x_i > 0$, we deduce that $\lambda_i = 0$ and:

$$x_i \frac{\partial \, \sigma \left(x \right)}{\partial \, x_i} = \lambda_0 x_i + \lambda_c$$

If this solution corresponds to the ERC portfolio, we obtain:

$$\mathcal{RC}_i = \mathcal{RC}_j \Leftrightarrow \lambda_0 x_i + \lambda_c = \lambda_0 x_j + \lambda_c$$

If $\lambda_0 \neq 0$, we deduce that:

$$x_i = x_j$$

It corresponds to the EW portfolio meaning that the assumption $\mathcal{RC}_i = \mathcal{RC}_j$ is false.

The optimization problem of the ERC portfolio

Question 2.b

Find the optimized portfolio of the optimization problem (2) when c is equal to -10. Calculate the corresponding risk allocation.

The optimization problem of the ERC portfolio

If c is equal to -10, we obtain the following results:

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1	12.65%	7.75%	0.98%	25.00%
2	8.43%	11.63%	0.98%	25.00%
3	7.06%	13.89%	0.98%	25.00%
4	6.03%	16.25%	0.98%	25.00%
$\sigma(x)$			3.92%	

The optimization problem of the ERC portfolio

Question 2.c

Same question if $c = 0$.

The optimization problem of the ERC portfolio

If c is equal to 0, we obtain the following results:

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1	154.07%	7.75%	11.94%	25.00%
2	102.72%	11.63%	11.94%	25.00%
3	85.97%	13.89%	11.94%	25.00%
4	73.50%	16.25%	11.94%	25.00%
$\sigma(x)$			47.78%	

The optimization problem of the ERC portfolio

Question 2.d

Demonstrate then that the solution to the second optimization problem is:

$$x^{\star}(c) = \exp\left(\frac{c - c_{\mathrm{erc}}}{n}\right) x_{\mathrm{erc}}$$

where $c_{\mathrm{erc}} = \sum_{i=1}^{n} \ln x_{\mathrm{erc},i}$. Comment on this result.

The optimization problem of the ERC portfolio

In this case, the first-order condition is:

$$\frac{\partial\, \sigma(x)}{\partial\, x_i} - \lambda_i - \frac{\lambda_c}{x_i} = 0$$

As previously, $\lambda_i = 0$ because $x_i > 0$ and we obtain:

$$x_i \frac{\partial\, \sigma(x)}{\partial\, x_i} = \lambda_c$$

The solution of the second optimization problem is then a non-normalized ERC portfolio because $\sum_{i=1}^n x_i$ is not necessarily equal to 1. If we note $c_{\mathrm{erc}} = \sum_{i=1}^n \ln\left(x_{\mathrm{erc}}\right)_i$, we deduce that:

$$\begin{aligned} x_{\mathrm{erc}} &= \arg\min \sqrt{x^\top \Sigma x} \\ & \text{u.c.} \quad \left\{ \begin{array}{l} \sum_{i=1}^n \ln x_i \geq c_{\mathrm{erc}} \\ x \geq \mathbf{0}_n \end{array} \right. \end{aligned}$$

The optimization problem of the ERC portfolio

Let $x^{\star}(c)$ be the portfolio defined by:

$$x^{\star}(c) = \exp\left(\frac{c - c_{\mathrm{erc}}}{n}\right) x_{\mathrm{erc}}$$

We have $x^{\star}(c) > \mathbf{0}_n$,

$$\sqrt{x^{\star}(c)^{\top} \Sigma x^{\star}(c)} = \exp\left(\frac{c - c_{\mathrm{erc}}}{n}\right) \sqrt{x_{\mathrm{erc}}^{\top} \Sigma x_{\mathrm{erc}}}$$

and:

$$\begin{aligned}
\sum_{i=1}^{n} \ln x_i^{\star}(c) &= \sum_{i=1}^{n} \ln\left(\exp\left(\frac{c - c_{\mathrm{erc}}}{n}\right) x_{\mathrm{erc}}\right)_i \\
&= c - c_{\mathrm{erc}} + \sum_{i=1}^{n} \ln\left(x_{\mathrm{erc}}\right)_i \\
&= c
\end{aligned}$$

We conclude that $x^{\star}(c)$ is the solution of the optimization problem.

The optimization problem of the ERC portfolio

$x^{\star}(c)$ is then a leveraged ERC portfolio if $c > c_{\mathrm{erc}}$ and a deleveraged ERC portfolio if $c < c_{\mathrm{erc}}$.

In our example, c_{erc} is equal to -5.7046. If $c = -10$, we have:

$$\exp\left(\frac{c - c_{\mathrm{erc}}}{n}\right) = 34.17\%$$

We verify that the solution of Question 2.b is such that $\sum_{i=1}^{n} x_i = 34.17\%$ and $\mathcal{RC}_i^{\star} = \mathcal{RC}_j^{\star}$.

If $c = 0$, we obtain:

$$\exp\left(\frac{c - c_{\mathrm{erc}}}{n}\right) = 416.26\%$$

In this case, the solution is a leveraged ERC portfolio.

The optimization problem of the ERC portfolio

Question 2.e

Show that there exists a scalar c such that the Lagrange coefficient λ_0 of the optimization problem (1) is equal to zero. Deduce then that the volatility of the ERC portfolio is between the volatility of the long-only minimum variance portfolio and the volatility of the equally weighted portfolio:

$$\sigma\left(x_{\mathrm{mv}}\right) \leq \sigma\left(x_{\mathrm{erc}}\right) \leq \sigma\left(x_{\mathrm{ew}}\right)$$

The optimization problem of the ERC portfolio

From the previous question, we know that the ERC optimization portfolio is the solution of the second optimization problem if we use c_{erc} for the control variable. In this case, we have $\sum_{i=1}^{n} x_i^\star \left(c_{\mathrm{erc}}\right) = 1$ meaning that x_{erc} is also the solution of the first optimization problem. We deduce that $\lambda_0 = 0$ if $c = c_{\mathrm{erc}}$. The first optimization problem is a convex problem with a convex inequality constraint. The objective function is then an increasing function of the control variable c:

$$c_1 \leq c_2 \Rightarrow \sigma\left(x^\star\left(c_1\right)\right) \geq \sigma\left(x^\star\left(c_2\right)\right)$$

The optimization problem of the ERC portfolio

We have seen that the minimum variance portfolio corresponds to $c = -\infty$, that the EW portfolio is obtained with $c = -n \ln n$ and that the ERC portfolio is the solution of the optimization problem when c is equal to c_{erc}. Moreover, we have $-\infty \leq c_{\mathrm{erc}} \leq -n \ln n$. We deduce that the volatility of the ERC portfolio is between the volatility of the long-only minimum variance portfolio and the volatility of the equally weighted portfolio:

$$\sigma\left(x_{\mathrm{mv}}\right) \leq \sigma\left(x_{\mathrm{erc}}\right) \leq \sigma\left(x_{\mathrm{ew}}\right)$$

Risk parity funds

Question 1

We consider a universe of three asset classes[a] which are stocks (S), bonds (B) and commodities (C). We have computed the one-year historical covariance matrix of asset returns for different dates and we obtain the following results (all the numbers are expressed in %):

	31/12/1999			31/12/2002			30/12/2005		
σ_i	12.40	5.61	12.72	20.69	7.36	13.59	7.97	7.01	16.93
$\rho_{i,j}$	100.00			100.00			100.00		
	−5.89	100.00		−36.98	100.00		29.25	100.00	
	−4.09	−7.13	100.00	22.74	−13.12	100.00	15.75	15.05	100.00
	31/12/2007			31/12/2008			31/12/2010		
σ_i	12.94	5.50	14.54	33.03	9.73	29.00	16.73	6.88	16.93
$\rho_{i,j}$	100.00	−25.76		100.00			100.00		
	−25.76	100.00		−16.26	100.00		15.31	100.00	
	31.91	6.87	100.00	47.31	9.13	100.00	64.13	15.46	100.00

[a] In fact, we use the MSCI World index, the Citigroup WGBI index and the DJ UBS Commodity index to represent these asset classes.

Risk parity funds

Question 1.a

Compute the weights and the volatility of the risk parity[a] (RP portfolio) portfolios for the different dates.

[a] Here, risk parity refers to the ERC portfolio when we do not take into account the correlations.

Risk parity funds

The RP portfolio is defined as follows:

$$x_i = \frac{\sigma_i^{-1}}{\sum_{j=1}^{n} \sigma_j^{-1}}$$

We obtain the following results:

Date	1999	2002	2005	2007	2008	2010
S	23.89%	18.75%	38.35%	23.57%	18.07%	22.63%
B	52.81%	52.71%	43.60%	55.45%	61.35%	55.02%
C	23.29%	28.54%	18.05%	20.98%	20.58%	22.36%
$\sigma(x)$	4.83%	6.08%	6.26%	5.51%	11.64%	8.38%

Risk parity funds

Question 1.b

Same question by considering the ERC portfolio.

Risk parity funds

In the ERC portfolio, the risk contributions are equal for all the assets:

$$\mathcal{RC}_i = \mathcal{RC}_j$$

with:

$$\mathcal{RC}_i = \frac{x_i \cdot (\Sigma x)_i}{\sqrt{x^\top \Sigma x}} \tag{3}$$

We obtain the following results:

Date	1999	2002	2005	2007	2008	2010
S	23.66%	18.18%	37.85%	23.28%	17.06%	20.33%
B	53.12%	58.64%	43.18%	59.93%	66.39%	59.61%
C	23.22%	23.18%	18.97%	16.79%	16.54%	20.07%
$\sigma(x)$	4.82%	5.70%	6.32%	5.16%	10.77%	7.96%

Risk parity funds

Question 1.c

What do you notice about the volatility of RP and ERC portfolios? Explain these results.

Risk parity funds

We notice that $\sigma(x_{\mathrm{erc}}) \leq \sigma(x_{\mathrm{rp}})$ except for the year 2005. This date corresponds to positive correlations between assets. Moreover, the correlation between stocks and bonds is the highest. Starting from the RP portfolio, it is then possible to approach the ERC portfolio by reducing the weights of stocks and bonds and increasing the weight of commodities. At the end, we find an ERC portfolio that has a slightly higher volatility.

Risk parity funds

Question 1.d

Find the analytical expression of the volatility $\sigma(x)$, the marginal risk $\mathcal{MR}_i$, the risk contribution $\mathcal{RC}_i$ and the normalized risk contribution $\mathcal{RC}_i^\star$ in the case of RP portfolios.

Risk parity funds

The volatility of the RP portfolio is:

$$\begin{aligned}
\sigma(x) &= \frac{1}{\sum_{j=1}^{n}\sigma_j^{-1}}\sqrt{\left(\sigma^{-1}\right)^{\top}\Sigma\sigma^{-1}} \\
&= \frac{1}{\sum_{j=1}^{n}\sigma_j^{-1}}\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{\sigma_i\sigma_j}\rho_{i,j}\sigma_i\sigma_j} \\
&= \frac{1}{\sum_{j=1}^{n}\sigma_j^{-1}}\sqrt{n+2\sum_{i>j}\rho_{i,j}} \\
&= \frac{1}{\sum_{j=1}^{n}\sigma_j^{-1}}\sqrt{n\left(1+(n-1)\bar{\rho}\right)}
\end{aligned}$$

where $\bar{\rho}$ is the average correlation between asset returns.

Risk parity funds

For the marginal risk, we obtain:

$$\begin{aligned}
\mathcal{MR}_i &= \frac{\left(\Sigma\sigma^{-1}\right)_i}{\sigma(x)\sum_{j=1}^n \sigma_j^{-1}} \\
&= \frac{1}{\sqrt{n\left(1+(n-1)\bar{\rho}\right)}}\sum_{j=1}^n \rho_{i,j}\sigma_i\sigma_j\frac{1}{\sigma_j} \\
&= \frac{\sigma_i}{\sqrt{n\left(1+(n-1)\bar{\rho}\right)}}\sum_{j=1}^n \rho_{i,j} \\
&= \frac{\sigma_i\bar{\rho}_i\sqrt{n}}{\sqrt{1+(n-1)\bar{\rho}}}
\end{aligned}$$

where $\bar{\rho}_i$ is the average correlation of asset i with the other assets (including itself).

Risk parity funds

The expression of the risk contribution is then:

$$\begin{aligned}
\mathcal{RC}_i &= \frac{\sigma_i^{-1}}{\sum_{j=1}^n \sigma_j^{-1}} \frac{\sigma_i \bar{\rho}_i \sqrt{n}}{\sqrt{1 + (n-1)\bar{\rho}}} \\
&= \frac{\bar{\rho}_i \sqrt{n}}{\sqrt{1 + (n-1)\bar{\rho}} \sum_{j=1}^n \sigma_j^{-1}}
\end{aligned}$$

We deduce that the normalized risk contribution is:

$$\begin{aligned}
\mathcal{RC}_i^\star &= \frac{\bar{\rho}_i \sqrt{n}}{\sigma(x) \sqrt{1 + (n-1)\bar{\rho}} \sum_{j=1}^n \sigma_j^{-1}} \\
&= \frac{\bar{\rho}_i}{1 + (n-1)\bar{\rho}}
\end{aligned}$$

Risk parity funds

Question 1.e

Compute the normalized risk contributions of the previous RP portfolios. Comment on these results.

Risk parity funds

We obtain the following normalized risk contributions:

Date	1999	2002	2005	2007	2008	2010
S	33.87%	34.96%	34.52%	32.56%	34.45%	36.64%
B	32.73%	20.34%	34.35%	24.88%	24.42%	26.70%
C	33.40%	44.69%	31.14%	42.57%	41.13%	36.67%

We notice that the risk contributions are not exactly equal for all the assets. Generally, the risk contribution of bonds is lower than the risk contribution of equities, which is itself lower than the risk contribution of commodities.

Risk parity funds

Question 2

We consider four parameter sets of risk budgets:

Set	b_1	b_2	b_3
#1	45%	45%	10%
#2	70%	10%	20%
#3	20%	70%	10%
#4	25%	25%	50%

Risk parity funds

Question 2.a

Compute the RB portfolios for the different dates.

Risk parity funds

We obtain the following RB portfolios:

Date	b_i	1999	2002	2005	2007	2008	2010
S	45%	26.83%	22.14%	42.83%	27.20%	20.63%	25.92%
B	45%	59.78%	66.10%	48.77%	66.15%	73.35%	67.03%
C	10%	13.39%	11.76%	8.40%	6.65%	6.02%	7.05%
S	70%	40.39%	29.32%	65.53%	39.37%	33.47%	46.26%
B	10%	37.63%	51.48%	19.55%	47.18%	52.89%	37.76%
C	20%	21.98%	19.20%	14.93%	13.45%	13.64%	15.98%
S	20%	17.55%	16.02%	25.20%	18.78%	12.94%	13.87%
B	70%	69.67%	71.70%	66.18%	74.33%	80.81%	78.58%
C	10%	12.78%	12.28%	8.62%	6.89%	6.24%	7.55%
S	25%	21.69%	15.76%	34.47%	20.55%	14.59%	16.65%
B	25%	48.99%	54.03%	39.38%	55.44%	61.18%	53.98%
C	50%	29.33%	30.21%	26.15%	24.01%	24.22%	29.37%

Risk parity funds

Question 2.b

Compute the implied risk premium $\tilde{\pi}_i$ of the assets for these portfolios if we assume a Sharpe ratio equal to 0.40.

Risk parity funds

To compute the implied risk premium $\tilde{\pi}_i$, we use the following formula (TR-RPB, page 274):

$$\begin{aligned}\tilde{\pi}_i &= \mathrm{SR}\left(x \mid r\right) \cdot \mathcal{MR}_i \\ &= \mathrm{SR}\left(x \mid r\right) \cdot \frac{\left(\Sigma x\right)_i}{\sigma\left(x\right)}\end{aligned}$$

where $\mathrm{SR}\left(x \mid r\right)$ is the Sharpe ratio of the portfolio.

Risk parity funds

We obtain the following results:

Date	b_i	1999	2002	2005	2007	2008	2010
S	45%	3.19%	4.60%	2.49%	3.15%	8.64%	5.20%
B	45%	1.43%	1.54%	2.19%	1.29%	2.43%	2.01%
C	10%	1.42%	1.92%	2.82%	2.86%	6.58%	4.24%
S	70%	4.05%	6.45%	2.86%	4.31%	11.56%	6.32%
B	10%	0.62%	0.52%	1.37%	0.51%	1.04%	1.11%
C	20%	2.13%	2.81%	3.59%	3.61%	8.11%	5.23%
S	20%	2.06%	2.68%	1.91%	1.93%	5.61%	3.91%
B	70%	1.82%	2.10%	2.54%	1.71%	3.14%	2.42%
C	10%	1.42%	1.75%	2.79%	2.64%	5.82%	3.60%
S	25%	2.33%	3.78%	1.98%	2.74%	8.06%	5.13%
B	25%	1.03%	1.10%	1.74%	1.02%	1.92%	1.58%
C	50%	3.45%	3.95%	5.23%	4.69%	9.71%	5.82%

Risk parity funds

Question 2.c

Comment on these results.

Risk parity funds

We have:

$$x_i \tilde{\pi}_i = \mathrm{SR}\left(x \mid r\right) \cdot \mathcal{RC}_i$$

We deduce that:

$$\tilde{\pi}_i \propto \frac{b_i}{x_i}$$

x_i is generally an increasing function of b_i. As a consequence, the relationship between the risk budgets b_i and the risk premiums $\tilde{\pi}_i$ is not necessarily increasing. However, we notice that the bigger the risk budget, the higher the risk premium. This is easily explained. If an investor allocates more risk budget to one asset class than another investor, he thinks that the risk premium of this asset class is higher than the other investor.

Risk parity funds

However, we must be careful. This interpretation is valid if we compare two sets of risk budgets. It is false if we compare the risk budgets among themselves. For instance, if we consider the third parameter set, the risk budget of bonds is 70% whereas the risk budget of stocks is 20%. It does not mean that the risk premium of bonds is higher than the risk premium of equities. In fact, we observe the contrary. If we would like to compare risk budgets among themselves, the right measure is the implied Sharpe ratio, which is equal to:

$$\begin{aligned}\mathrm{SR}_i &= \frac{\tilde{\pi}_i}{\sigma_i} \\ &= \mathrm{SR}\left(x \mid r\right) \cdot \frac{\mathcal{MR}_i}{\sigma_i}\end{aligned}$$

For instance, if we consider the most diversified portfolio, the marginal risk is proportional to the volatility and we retrieve the result that Sharpe ratios are equal if the MDP is optimal.

Main references

Roncalli, T. (2013)
Introduction to Risk Parity and Budgeting, Chapman and Hall/CRC Financial Mathematics Series, Chapter 2.

Roncalli, T. (2013)
Introduction to Risk Parity and Budgeting — Companion Book, Chapman and Hall/CRC Financial Mathematics Series, Chapter 2.

References I

Bruder, B., Kostyuchyk, N., and Roncalli, T. (2016)
Risk Parity Portfolios with Skewness Risk: An Application to Factor Investing and Alternative Risk Premia, *SSRN*, www.ssrn.com/abstract=2813384.

Lezmi, E., Malongo, H., Roncalli, T., and Sobotka, R. (2018)
Portfolio Allocation with Skewness Risk: A Practical Guide, *SSRN*, www.ssrn.com/abstract=3201319.

Maillard, S., Roncalli, T. and Teïletche, J. (2010)
The Properties of Equally Weighted Risk Contribution Portfolios, *Journal of Portfolio Management*, 36(4), pp. 60-70.

Roncalli, T. (2015)
Introducing Expected Returns into Risk Parity Portfolios: A New Framework for Asset Allocation, *Bankers, Markets & Investors*, 138, pp. 18-28.

References II

Roncalli, T., and Weisang, G. (2016)
Risk Parity Portfolios with Risk Factors, *Quantitative Finance*, 16(3), pp. 377-388

Scherer, B. (2007)
Portfolio Construction & Risk Budgeting, Third edition, Risk Books.

Course 2023-2024 in Portfolio Allocation and Asset Management
Lecture 3. Smart Beta, Factor Investing and Alternative Risk Premia

Thierry Roncalli*

*Amundi Asset Management[11]

*University of Paris-Saclay

January 2024

[11] The opinions expressed in this presentation are those of the authors and are not meant to represent the opinions or official positions of Amundi Asset Management.

Agenda

- Lecture 1: Portfolio Optimization
- Lecture 2: Risk Budgeting
- **Lecture 3: Smart Beta, Factor Investing and Alternative Risk Premia**
- Lecture 4: Equity Portfolio Optimization with ESG Scores
- Lecture 5: Climate Portfolio Construction
- Lecture 6: Equity and Bond Portfolio Optimization with Green Preferences
- Lecture 7: Machine Learning in Asset Management

Cap-weighted indexation and modern portfolio theory

Rationale of market-cap indexation

- **Separation Theorem**: there is one unique risky portfolio owned by investors called the tangency portfolio (Tobin, 1958)
- **CAPM**: the tangency portfolio is the Market portfolio, best represented by the capitalization-weighted index (Sharpe, 1964)
- **Performance of active management**: negative alpha in equity mutual funds on average (Jensen, 1968)
- **EMH**: markets are efficient (Fama, 1970)
- **Passive management**: launch of the first index fund (John McQuown, Wells Fargo Investment Advisors, Samsonite Luggage Corporation, 1971)
- **First S&P 500 index fund** by Wells Fargo and American National Bank in Chicago (1973)
- The **first listed ETF** was the SPDRs (Ticker: SPY) in 1993

Index funds

Mutual Fund (MF)

A mutual fund is a **collective investment fund** that are regulated and sold to the general public

Exchange Traded Fund (ETF)

It is a mutual fund which trades **intra-day** on a securities exchange (thanks to market makers)

Exchange Traded Product (ETP)

It is a security that is **derivatively-priced** and that trades intra-day on an exchange. ETPs includes exchange traded funds (ETFs), exchange traded vehicles (ETVs), exchange traded notes (ETNs) and certificates.

Pros of market-cap indexation

- A convenient and **recognized approach** to participate to broad equity markets
- **Management simplicity**: low turnover & transaction costs

Construction of an equity index

- We consider an index universe composed of n stocks
- Let $P_{i,t}$ be the price of the i^{th} stock and $R_{i,t}$ be the corresponding return between times $t-1$ and t:

$$R_{i,t} = \frac{P_{i,t}}{P_{i,t-1}} - 1$$

- The value of the index B_t at time t is defined by:

$$B_t = \varphi \sum_{i=1}^{n} N_i P_{i,t}$$

where φ is a scaling factor and N_i is the total number of shares issued by the company i

Construction of an equity index

- Another expression of B_t is[12]:

$$\begin{aligned} B_t &= \varphi \sum_{i=1}^{n} N_i P_{i,t-1} \left(1 + R_{i,t}\right) \\ &= B_{t-1} \frac{\sum_{i=1}^{n} N_i P_{i,t-1} \left(1 + R_{i,t}\right)}{\sum_{i=1}^{n} N_i P_{i,t-1}} \\ &= B_{t-1} \sum_{i=1}^{n} w_{i,t-1} \left(1 + R_{i,t}\right) \end{aligned}$$

where $w_{i,t-1}$ is the weight of the i^{th} stock in the index:

$$w_{i,t-1} = \frac{N_i P_{i,t-1}}{\sum_{i=1}^{n} N_i P_{i,t-1}}$$

- The computation of the index value B_t can be done at the closing time t and also in an intra-day basis

[12] B_0 can be set to an arbitrary value (e.g. 100, 500, 1000 or 5000)

Construction of an equity index

Remark

The previous computation is purely theoretical because the portfolio corresponds to all the shares outstanding of the n stocks $\Rightarrow$ impossible to hold this portfolio

Remark

Most of equity indices use floating shares[a] instead of shares outstanding

[a] They indicate the number of shares available for trading

Replication of an equity index

- In order to replicate this index, we must build a hedging strategy that consists in investing in stocks
- Let S_t be the value of the strategy (or the index fund):

$$S_t = \sum_{i=1}^{n} n_{i,t} P_{i,t}$$

  where $n_{i,t}$ is the number of stock i held between $t-1$ and t
- The tracking error is the difference between the return of the strategy and the return of the index:

$$e_t \left(S \mid B \right) = R_{S,t} - R_{B,t}$$

Replication of an equity index

The quality of the replication process is measured by the volatility $\sigma\left(e_t\left(S \mid B\right)\right)$ of the tracking error. We may distinguish several cases:

1. Index funds with low tracking error volatility (less than 10 bps) $\Rightarrow$ physical replication or synthetic replication
2. Index funds with moderate tracking error volatility (between 10 bps and 50 bps) $\Rightarrow$ sampling replication
3. Index funds with higher tracking error volatility (larger than 50 bps) $\Rightarrow$ equity universes with liquidity problems and enhanced/tilted index funds

Replication of an equity index

- In a capitalization-weighted index, the weights are given by:

$$w_{i,t} = \frac{C_{i,t}}{\sum_{j=1}^{n} C_{j,t}} = \frac{N_{i,t} P_{i,t}}{\sum_{j=1}^{n} N_{j,t} P_{j,t}}$$

where $N_{i,t}$ and $C_{i,t} = N_{i,t} P_{i,t}$ are the number of shares outstanding and the market capitalization of the i^{th} stock

- If we have a perfect match at time $t-1$:

$$\frac{n_{i,t-1} P_{i,t-1}}{\sum_{i=1}^{n} n_{i,t-1} P_{i,t-1}} = w_{i,t-1}$$

we have a perfect match at time t:

$$n_{i,t} = n_{i,t-1}$$

Replication of an equity index

- We do not need to rebalance the hedging portfolio because of the relationship:

$$n_{i,t} P_{i,t} \propto w_{i,t} P_{i,t}$$

- Therefore, it is not necessarily to adjust the portfolio of the strategy (except if there are subscriptions or redemptions)

A CW index fund remains the most efficient investment in terms of management simplicity, turnover and transaction costs

Cons of market-cap indexation

- Trend-following strategy: momentum bias leads to bubble risk exposure as weight of best performers ever increases
  $\Rightarrow$ Mid 2007, financial stocks represent 40% of the Eurostoxx 50 index
- Growth bias as high valuation multiples stocks weight more than low-multiple stocks with equivalent realized earnings.
  $\Rightarrow$ Mid 2000, the 8 stocks of the technology/telecom sectors represent 35% of the Eurostoxx 50 index
  $\Rightarrow$ $2^1/_2$ years later after the dot.com bubble, these two sectors represent 12%
- Concentrated portfolios
  $\Rightarrow$ The top 100 market caps of the S&P 500 account for around 70%
- Lack of risk diversification and high drawdown risk: no portfolio construction rules leads to concentration issues (e.g. sectors, stocks).

Cons of market-cap indexation

Some illustrations

- Mid 2000: 8 Technology/Telecom stocks represent 35% of the Eurostoxx 50 index
- In 2002: 7.5% of the Eurostoxx 50 index is invested into Nokia with a volatility of 70%
- Dec. 2006: 26.5% of the MSCI World index is invested in financial stocks
- June 2007: 40% of the Eurostoxx 50 is invested into Financials
- January 2013: 20% of the S&P 500 stocks represent 68% of the S&P 500 risk
- Between 2002 and 2012, two stocks contribute on average to more than 20% of the monthly performance of the Eurostoxx 50 index

Cons of market-cap indexation

Table 47: Weight and risk concentration of several equity indices (June 29, 2012)

Ticker	Weights				Risk contributions			
	$\mathcal{G}(x)$	$\mathbb{L}(x)$ 10%	$\mathbb{L}(x)$ 25%	$\mathbb{L}(x)$ 50%	$\mathcal{G}(x)$	$\mathbb{L}(x)$ 10%	$\mathbb{L}(x)$ 25%	$\mathbb{L}(x)$ 50%
SX5P	30.8	24.1	48.1	71.3	26.3	19.0	40.4	68.6
SX5E	31.2	23.0	46.5	72.1	31.2	20.5	44.7	73.3
INDU	33.2	23.0	45.0	73.5	35.8	25.0	49.6	75.9
BEL20	39.1	25.8	49.4	79.1	45.1	25.6	56.8	82.5
DAX	44.0	27.5	56.0	81.8	47.3	27.2	59.8	84.8
CAC	47.4	34.3	58.3	82.4	44.1	31.9	57.3	79.7
AEX	52.2	37.2	61.3	86.0	51.4	35.3	62.0	84.7
HSCEI	54.8	39.7	69.3	85.9	53.8	36.5	67.2	85.9
NKY	60.2	47.9	70.4	87.7	61.4	49.6	70.9	88.1
UKX	60.8	47.5	73.1	88.6	60.4	46.1	72.8	88.7
SXXE	61.7	49.2	73.5	88.7	63.9	51.6	75.3	90.1
SPX	61.8	52.1	72.0	87.8	59.3	48.7	69.9	86.7
MEXBOL	64.6	48.2	75.1	91.8	65.9	45.7	78.6	92.9
IBEX	64.9	51.7	77.3	90.2	68.3	58.2	80.3	91.4
SXXP	65.6	55.0	76.4	90.1	64.2	52.0	75.5	90.0
NDX	66.3	58.6	77.0	89.2	64.6	56.9	74.9	88.6
TWSE	79.7	73.4	86.8	95.2	79.7	72.6	87.3	95.7
TPX	80.8	72.8	88.8	96.3	83.9	77.1	91.0	97.3
KOSPI	86.5	80.6	93.9	98.0	89.3	85.1	95.8	98.8

$\mathcal{G}(x)$ = Gini coefficient, $\mathbb{L}(x)$ = Lorenz curve

Cons of market-cap indexation

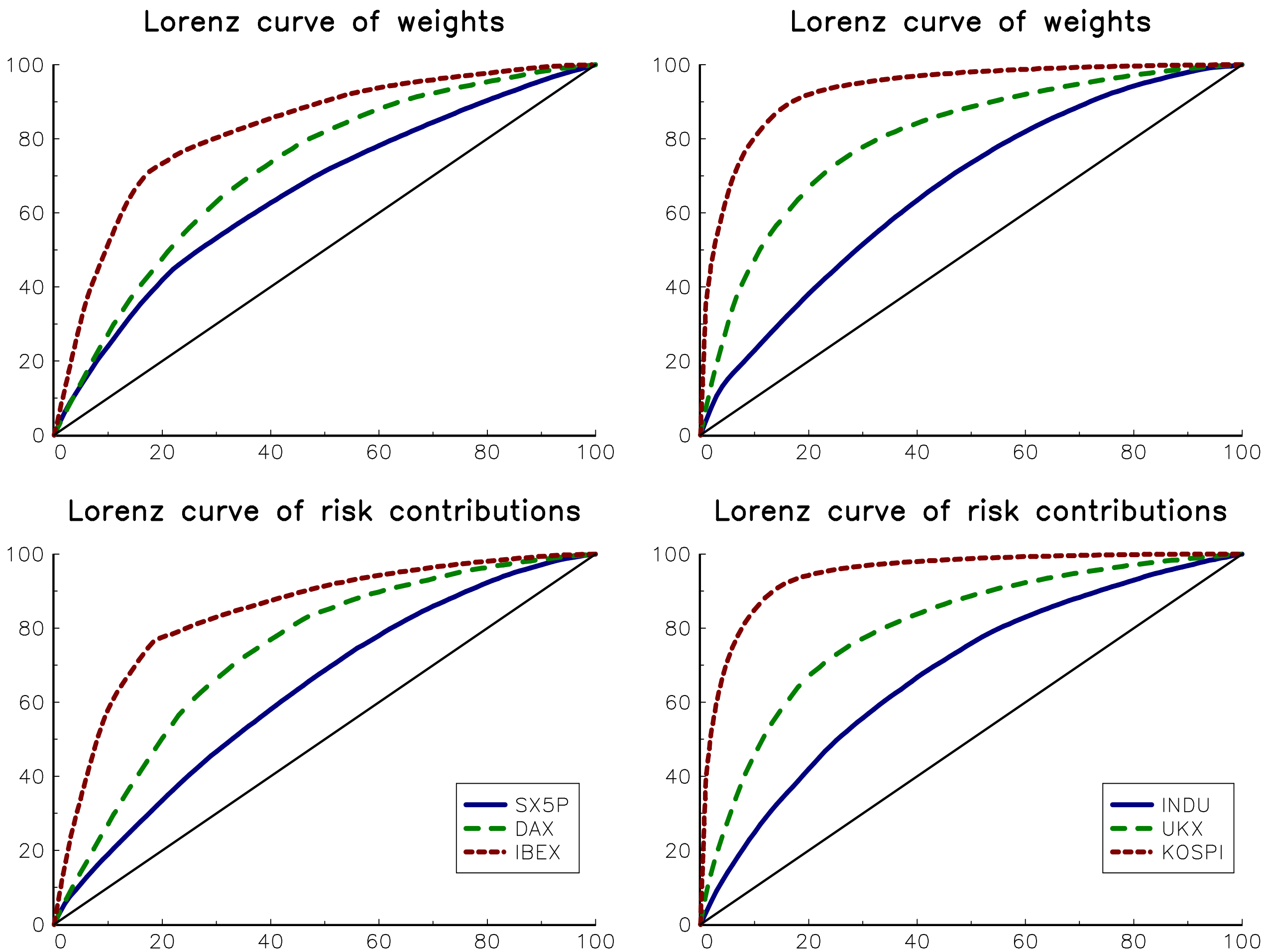


Figure 43: Lorenz curve of several equity indices (June 29, 2012)

Capturing the equity risk premium

	APPLE	EXXON	MSFT	J&J	IBM	PFIZER	CITI	McDO
	Cap-weighted allocation (in %)							
Dec. 1999	1.05	12.40	38.10	7.94	12.20	12.97	11.89	3.46
Dec. 2004	1.74	22.16	19.47	12.61	11.00	13.57	16.76	2.70
Dec. 2008	6.54	35.03	14.92	14.32	9.75	10.30	3.15	5.98
Dec. 2010	18.33	22.84	14.79	10.52	11.29	8.69	8.51	5.02
Dec. 2012	26.07	20.55	11.71	10.12	11.27	9.62	6.04	4.61
Jun. 2013	20.78	19.80	14.35	11.64	11.36	9.51	7.79	4.77
	Implied risk premium (in %)							
Dec. 1999	5.96	2.14	8.51	3.61	5.81	5.91	6.19	2.66
Dec. 2004	3.88	2.66	2.79	2.03	2.32	3.90	3.02	1.86
Dec. 2008	9.83	11.97	10.48	6.24	7.28	8.06	17.15	6.28
Dec. 2010	5.38	3.85	4.42	2.29	3.66	3.76	6.52	2.54
Dec. 2012	5.87	2.85	3.58	1.44	2.80	1.77	5.91	1.88
Jun. 2013	5.59	2.79	3.60	1.55	2.92	1.91	5.24	1.82
	Expected performance contribution (in %)							
Dec. 1999	1.01	4.31	52.63	4.66	11.52	12.43	11.94	1.49
Dec. 2004	2.41	21.04	19.44	9.15	9.12	18.93	18.11	1.79
Dec. 2008	6.60	43.00	16.04	9.17	7.28	8.52	5.55	3.85
Dec. 2010	23.58	21.01	15.62	5.77	9.89	7.81	13.27	3.05
Dec. 2012	42.41	16.23	11.61	4.04	8.73	4.71	9.88	2.40
Jun. 2013	33.96	16.18	15.10	5.28	9.69	5.32	11.93	2.53

Alternative-weighted indexation

Definition

Alternative-weighted indexation aims at building passive indexes where the weights are not based on market capitalization

Alternative-weighted indexation

Three kinds of responses:

1. Fundamental indexation (capturing *alpha*?)
    1. Dividend yield indexation
    2. RAFI indexation
2. Risk-based indexation (capturing *diversification*?)
    1. Equally weighted portfolio
    2. Minimum variance portfolio
    3. Equal risk contribution portfolio
    4. Most diversified portfolio
3. Factor investing (capturing *normal returns or beta*? *abnormal returns or alpha*?)
    1. The market risk factor is not the only systematic risk factor
    2. Other factors: size, value, momentum, low beta, quality, etc.

Alternative-weighted indexation

2008

Smart Beta

=

Fundamental Indexation

+

Risk-Based Indexation

Today

Smart Beta

=

Risk-Based Indexation

+

Factor Investing

Equally-weighted portfolio

- The underlying idea of the equally weighted or '$1/n$' portfolio is to define a portfolio independently from the estimated statistics and properties of stocks
- If we assume that it is impossible to predict return and risk, then attributing an equal weight to all of the portfolio components constitutes a natural choice
- We have:

$$x_i = x_j = \frac{1}{n}$$

Equally-weighted portfolio

The portfolio volatility is equal to:

$$\begin{aligned}\sigma^2(x) &= \sum_{i=1}^n x_i^2\sigma_i^2 + 2\sum_{i>j} x_i x_j \rho_{i,j}\sigma_i\sigma_j \\ &= \frac{1}{n^2}\left(\sum_{i=1}^n \sigma_i^2 + 2\sum_{i>j}\rho_{i,j}\sigma_i\sigma_j\right)\end{aligned}$$

If we assume that $\sigma_i \leq \sigma_{\max}$ and $0 \leq \rho_{i,j} \leq \rho_{\max}$, we obtain:

$$\begin{aligned}\sigma^2(x) &\leq \frac{1}{n^2}\left(\sum_{i=1}^n \sigma_{\max}^2 + 2\sum_{i>j}\rho_{\max}\sigma_{\max}^2\right) \\ &\leq \frac{1}{n^2}\left(n\sigma_{\max}^2 + 2\frac{n(n-1)}{2}\rho_{\max}\sigma_{\max}^2\right) \\ &\leq \left(\frac{1+(n-1)\rho_{\max}}{n}\right)\sigma_{\max}^2\end{aligned}$$

Equally-weighted portfolio

We deduce that:

$$\lim_{n\to\infty} \sigma(x) \leq \sigma_{\max}(x) = \sigma_{\max}\sqrt{\rho_{\max}}$$

Table 48: Value of $\sigma_{\max}(x)$ (in %)

		$\sigma_{\max}$ (in %)					
		5.00	10.00	15.00	20.00	25.00	30.00
$\rho_{\max}$ (in %)	10.00	1.58	3.16	4.74	6.32	7.91	9.49
	20.00	2.24	4.47	6.71	8.94	11.18	13.42
	30.00	2.74	5.48	8.22	10.95	13.69	16.43
	40.00	3.16	6.32	9.49	12.65	15.81	18.97
	50.00	3.54	7.07	10.61	14.14	17.68	21.21
	75.00	4.33	8.66	12.99	17.32	21.65	25.98
	90.00	4.74	9.49	14.23	18.97	23.72	28.46
	99.00	4.97	9.95	14.92	19.90	24.87	29.85

Equally-weighted portfolio

If the volatilities are the same ($\sigma_i = \sigma$) and the correlation matrix is constant ($\rho_{i,j} = \rho$), we deduce that:

$$\sigma(x) = \sigma \sqrt{\frac{1 + (n-1)\rho}{n}}$$

Correlations are more important than volatilities to benefit from diversification (= risk reduction)

Equally-weighted portfolio

Result

The main interest of the EW portfolio is the volatility reduction

It is called "naive diversification"

Equally-weighted portfolio

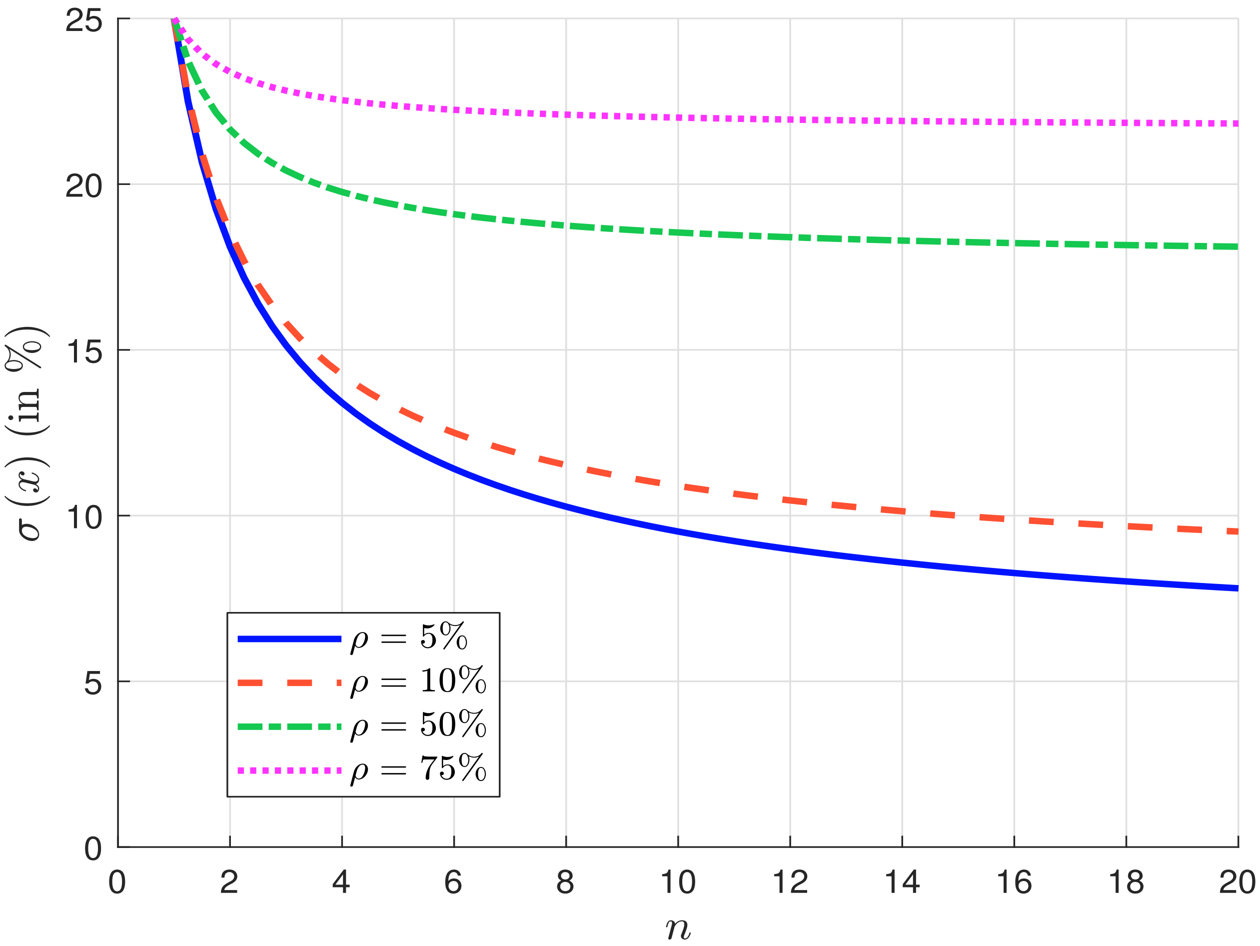


Figure 44: Illustration of the diversification effect ($\sigma = 25\%$)

Equally-weighted portfolio

Another interest of the EW portfolio is its good out-of-sample performance:

> *"We evaluate the out-of-sample performance of the sample-based mean-variance model, and its extensions designed to reduce estimation error, relative to the naive* $1/n$ *portfolio. Of the 14 models we evaluate across seven empirical datasets, none is consistently better than the* $1/n$ *rule in terms of Sharpe ratio, certainty-equivalent return, or turnover, which indicates that, out of sample, the gain from optimal diversification is more than offset by estimation error" (DeMiguel et al., 2009)*

Minimum variance portfolio

The global minimum variance (GMV) portfolio corresponds to the following optimization program:

$$\begin{aligned} x_{\mathrm{gmv}} &= \arg\min \frac{1}{2} x^\top \Sigma x \\ \text{u.c.} &\quad \mathbf{1}_n^\top x = 1 \end{aligned}$$

Minimum variance portfolio

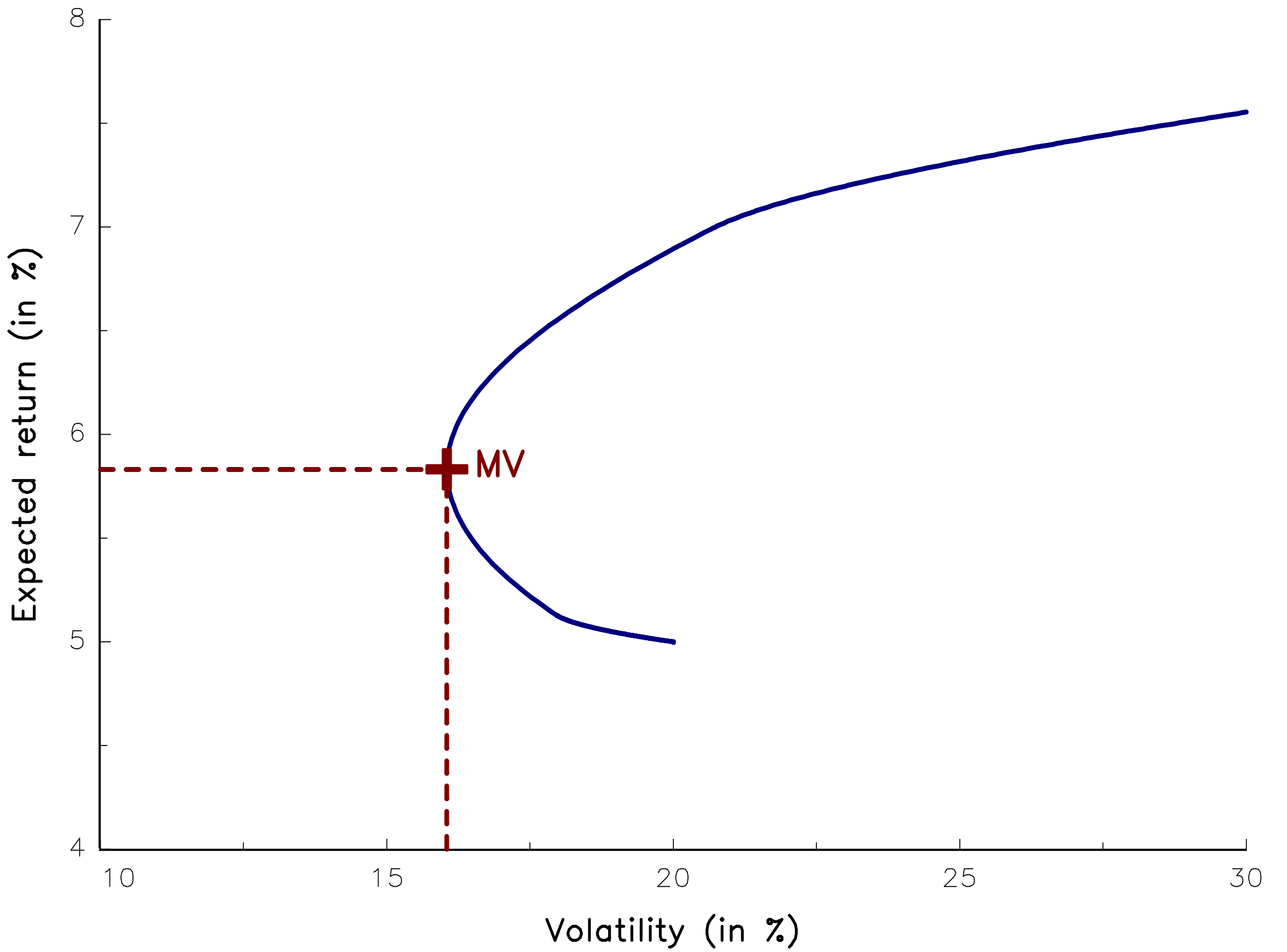


Figure 45: Location of the minimum variance portfolio in the efficient frontier

Minimum variance portfolio

The Lagrange function is equal to:

$$\mathcal{L}(x;\lambda_0) = \frac{1}{2}x^\top \Sigma x - \lambda_0 \left(\mathbf{1}_n^\top x - 1\right)$$

The first-order condition is:

$$\frac{\partial\,\mathcal{L}(x;\lambda_0)}{\partial\, x} = \Sigma x - \lambda_0 \mathbf{1}_n = \mathbf{0}_n$$

We deduce that:

$$x = \lambda_0 \Sigma^{-1} \mathbf{1}_n$$

Since we have $\mathbf{1}_n^\top x = 1$, the Lagrange multiplier satisfies:

$$\mathbf{1}_n^\top \left(\lambda_0 \Sigma^{-1} \mathbf{1}_n\right) = 1$$

or:

$$\lambda_0 = \frac{1}{\mathbf{1}_n^\top \Sigma^{-1} \mathbf{1}_n}$$

Minimum variance portfolio

Theorem

The GMV portfolio is given by the following formula:

$$x_{\mathrm{gmv}} = \frac{\Sigma^{-1}\mathbf{1}_n}{\mathbf{1}_n^\top \Sigma^{-1}\mathbf{1}_n}$$

Minimum variance portfolio

The volatility of the GMV portfolio is equal to:

$$\begin{aligned}
\sigma^2(x_{\mathrm{gmv}}) &= x_{\mathrm{gmv}}^\top \Sigma x_{\mathrm{gmv}} \\
&= \frac{\mathbf{1}_n^\top \Sigma^{-1}}{\mathbf{1}_n^\top \Sigma^{-1} \mathbf{1}_n} \Sigma \frac{\Sigma^{-1}\mathbf{1}_n}{\mathbf{1}_n^\top \Sigma^{-1} \mathbf{1}_n} \\
&= \frac{\mathbf{1}_n^\top \Sigma^{-1} \Sigma \Sigma^{-1} \mathbf{1}_n}{\left(\mathbf{1}_n^\top \Sigma^{-1} \mathbf{1}_n\right)^2} \\
&= \frac{\mathbf{1}_n^\top \Sigma^{-1} \mathbf{1}_n}{\left(\mathbf{1}_n^\top \Sigma^{-1} \mathbf{1}_n\right)^2} \\
&= \frac{1}{\mathbf{1}_n^\top \Sigma^{-1} \mathbf{1}_n}
\end{aligned}$$

Another expression of the GMV portfolio is:

$$x_{\mathrm{gmv}} = \sigma^2(x_{\mathrm{gmv}}) \Sigma^{-1} \mathbf{1}_n$$

Minimum variance portfolio

Example 1

The investment universe is made up of 4 assets. The volatility of these assets is respectively equal to 20%, 18%, 16% and 14%, whereas the correlation matrix is given by:

$$\rho = \begin{pmatrix} 1.00 & & & \\ 0.50 & 1.00 & & \\ 0.40 & 0.20 & 1.00 & \\ 0.10 & 0.40 & 0.70 & 1.00 \end{pmatrix}$$

Minimum variance portfolio

We have:

$$\Sigma = \begin{pmatrix} 400.00 & 180.00 & 128.00 & 28.00 \\ 180.00 & 324.00 & 57.60 & 100.80 \\ 128.00 & 57.60 & 256.00 & 156.80 \\ 28.00 & 100.80 & 156.80 & 196.00 \end{pmatrix} \times 10^4$$

It follows that:

$$\Sigma^{-1} = \begin{pmatrix} 54.35 & -37.35 & -50.55 & 51.89 \\ -37.35 & 62.97 & 41.32 & -60.11 \\ -50.55 & 41.32 & 124.77 & -113.85 \\ 51.89 & -60.11 & -113.85 & 165.60 \end{pmatrix}$$

Minimum variance portfolio

We deduce that:

$$\Sigma^{-1}\mathbf{1}_4 = \begin{pmatrix} 18.34 \\ 6.83 \\ 1.69 \\ 43.53 \end{pmatrix}$$

We also have $\mathbf{1}_4^\top \Sigma^{-1}\mathbf{1}_4 = 70.39$, $\sigma^2(x_{\mathrm{gmv}}) = 1/70.39 = 1.4206\%$ and $\sigma(x_{\mathrm{gmv}}) = \sqrt{1.4206\%} = 11.92\%$. Finally, we obtain:

$$x_{\mathrm{gmv}} = \frac{\Sigma^{-1}\mathbf{1}_4}{\mathbf{1}_4^\top \Sigma^{-1}\mathbf{1}_4} = \begin{pmatrix} 26.05\% \\ 9.71\% \\ 2.41\% \\ 61.84\% \end{pmatrix}$$

We verify that $\sum_{i=1}^4 x_{\mathrm{gmv},i} = 100\%$ and $\sqrt{x_{\mathrm{gmv}}^\top \Sigma x_{\mathrm{gmv}}} = 11.92\%$

Minimum variance portfolio

- If we assume that the correlation matrix is constant – $C = \mathcal{C}_n(\rho)$, the covariance matrix is $\Sigma = \sigma\sigma^\top \circ \mathcal{C}_n(\rho)$ with $\mathcal{C}_n(\rho)$ the constant correlation matrix. We deduce that:

$$\Sigma^{-1} = \Gamma \circ \mathcal{C}_n^{-1}(\rho)$$

with $\Gamma_{i,j} = \sigma_i^{-1}\sigma_j^{-1}$ and:

$$\mathcal{C}_n^{-1}(\rho) = \frac{\rho \mathbf{1}_n \mathbf{1}_n^\top - ((n-1)\rho + 1) I_n}{(n-1)\rho^2 - (n-2)\rho - 1}$$

- By using the trace property $\operatorname{tr}(AB) = \operatorname{tr}(BA)$, we can show that:

$$x_{\mathrm{gmv},i} = \frac{-((n-1)\rho + 1)\sigma_i^{-2} + \rho \sum_{j=1}^n (\sigma_i\sigma_j)^{-1}}{\sum_{k=1}^n \left(-((n-1)\rho + 1)\sigma_k^{-2} + \rho \sum_{j=1}^n (\sigma_k\sigma_j)^{-1}\right)}$$

Minimum variance portfolio

- The denominator is the scaling factor such that $\mathbf{1}_n^\top x_{\mathrm{gmv}} = 1$. We deduce that the optimal weights are given by the following relationship:

$$x_{\mathrm{gmv},i} \propto \frac{\left(\left(n-1\right)\rho+1\right)}{\sigma_i^2} - \frac{\rho}{\sigma_i}\sum_{j=1}^{n}\frac{1}{\sigma_j}$$

Minimum variance portfolio

Here are some special cases:

1. The lower bound of $\mathcal{C}_n(\rho)$ is achieved for $\rho = -(n-1)^{-1}$ and we have:

$$\begin{aligned} x_{\mathrm{gmv},i} &\propto -\frac{\rho}{\sigma_i}\sum_{j=1}^{n}\frac{1}{\sigma_j} \\ &\propto \frac{1}{\sigma_i} \end{aligned}$$

The weights are proportional to the inverse volatilities (GMV = ERC)

2. If the assets are uncorrelated $(\rho = 0)$, we obtain:

$$x_i \propto \frac{1}{\sigma_i^2}$$

The weights are proportional to the inverse variances

Minimum variance portfolio

3. If the assets are perfectly correlated ($\rho = 1$), we have:

$$x_{\mathrm{gmv},i} \propto \frac{1}{\sigma_i}\left(\frac{n}{\sigma_i} - \sum_{j=1}^{n}\frac{1}{\sigma_j}\right)$$

We deduce that:

$$\begin{aligned} x_{\mathrm{gmv},i} \geq 0 \quad & \Leftrightarrow \quad \frac{n}{\sigma_i} - \sum_{j=1}^{n}\frac{1}{\sigma_j} \geq 0 \\ & \Leftrightarrow \quad \sigma_i \leq \left(\frac{1}{n}\sum_{j=1}^{n}\sigma_j^{-1}\right)^{-1} \\ & \Leftrightarrow \quad \sigma_i \leq \bar{H}\left(\sigma_1, \ldots, \sigma_n\right) \end{aligned}$$

where $\bar{H}\left(\sigma_1, \ldots, \sigma_n\right)$ is the harmonic mean of volatilities

Minimum variance portfolio

Example 2

We consider a universe of four assets. Their volatilities are respectively equal to 4%, 6%, 8% and 10%. We assume also that the correlation matrix C is uniform and is equal to $\mathcal{C}_n(\rho)$.

Minimum variance portfolio

Table 49: Global minimum variance portfolios

Asset	ρ = −20%	0%	20%	50%	70%	90%	99%
1	44.35	53.92	65.88	90.65	114.60	149.07	170.07
2	25.25	23.97	22.36	19.04	15.83	11.20	8.38
3	17.32	13.48	8.69	−1.24	−10.84	−24.67	−33.09
4	13.08	8.63	3.07	−8.44	−19.58	−35.61	−45.37
$\sigma(x^{\star})$	1.93	2.94	3.52	3.86	3.62	2.52	0.87

Table 50: Long-only minimum variance portfolios

Asset	ρ = −20%	0%	20%	50%	70%	90%	99%
1	44.35	53.92	65.88	85.71	100.00	100.00	100.00
2	25.25	23.97	22.36	14.29	0.00	0.00	0.00
3	17.32	13.48	8.69	0.00	0.00	0.00	0.00
4	13.08	8.63	3.07	0.00	0.00	0.00	0.00
$\sigma(x^{\star})$	1.93	2.94	3.52	3.93	4.00	4.00	4.00

Minimum variance portfolio

In practice, we impose no short selling constraints

$\Downarrow$

Smart beta products (funds and indices) corresponds to long-only minimum variance portfolios

Minimum variance portfolio

Remark

The minimum variance strategy is related to the low beta effect (Black, 1972; Frazzini and Pedersen, 2014) or the low volatility anomaly (Haugen and Baker, 1991).

Minimum variance portfolio

We consider the single-factor model of the CAPM:

$$R_i = \alpha_i + \beta_i R_m + \varepsilon_i$$

We have:

$$\Sigma = \beta\beta^\top \sigma_m^2 + D$$

where:

- $\beta = (\beta_1, \ldots, \beta_n)$ is the vector of betas
- σ_m^2 is the variance of the market portfolio
- $D = \operatorname{diag}\left(\tilde{\sigma}_1^2, \ldots, \tilde{\sigma}_n^2\right)$ is the diagonal matrix of specific variances

Minimum variance portfolio

Sherman-Morrison-Woodbury formula

Suppose u and v are two $n \times 1$ vectors and A is an invertible $n \times n$ matrix. We can show that:

$$\left(A + uv^\top\right)^{-1} = A^{-1} - \frac{1}{1 + v^\top A^{-1} u} A^{-1} u v^\top A^{-1}$$

Minimum variance portfolio

We have:

$$\Sigma = D + (\sigma_m \beta)(\sigma_m \beta)^\top$$

We apply the Sherman-Morrison-Woodbury with $A = D$ and $u = v = \sigma_m \beta$:

$$\begin{aligned}
\Sigma^{-1} &= \left(D + (\sigma_m \beta)(\sigma_m \beta)^\top\right)^{-1} \\
&= D^{-1} - \frac{1}{1 + (\sigma_m \beta)^\top D^{-1} (\sigma_m \beta)} D^{-1} (\sigma_m \beta)(\sigma_m \beta)^\top D^{-1} \\
&= D^{-1} - \frac{\sigma_m^2}{1 + \sigma_m^2 \left(\beta^\top D^{-1} \beta\right)} \left(D^{-1} \beta\right)\left(D^{-1} \beta\right)^\top
\end{aligned}$$

Minimum variance portfolio

We have:

$$D^{-1}\beta = \tilde{\beta}$$

with $\tilde{\beta}_i = \beta_i / \tilde{\sigma}_i^2$ and:

$$\begin{aligned}
\varphi &= \beta^\top D^{-1}\beta \\
&= \tilde{\beta}^\top \beta \\
&= \sum_{i=1}^{n} \frac{\beta_i^2}{\tilde{\sigma}_i^2}
\end{aligned}$$

We obtain:

$$\Sigma^{-1} = D^{-1} - \frac{\sigma_m^2}{1+\varphi\sigma_m^2}\tilde{\beta}\tilde{\beta}^\top$$

The GMV portfolio is equal to:

$$\begin{aligned}
x_{\mathrm{gmv}} &= \sigma^2\left(x_{\mathrm{gmv}}\right)\Sigma^{-1}\mathbf{1}_n \\
&= \sigma^2\left(x_{\mathrm{gmv}}\right)\left(D^{-1}\mathbf{1}_n - \frac{\sigma_m^2}{1+\varphi\sigma_m^2}\tilde{\beta}\tilde{\beta}^\top\mathbf{1}_n\right)
\end{aligned}$$

Minimum variance portfolio

It follows that:

$$\begin{aligned} x_{\mathrm{gmv},i} &= \sigma^2\left(x_{\mathrm{gmv}}\right)\left(\frac{1}{\tilde{\sigma}_i^2} - \frac{\sigma_m^2\left(\tilde{\beta}^\top \mathbf{1}_n\right)}{1+\varphi\sigma_m^2}\frac{\beta_i}{\tilde{\sigma}_i^2}\right) \\ &= \frac{\sigma^2\left(x_{\mathrm{gmv}}\right)}{\tilde{\sigma}_i^2}\left(1 - \frac{\beta_i}{\beta^\star}\right) \end{aligned}$$

where:

$$\beta^\star = \frac{1+\varphi\sigma_m^2}{\sigma_m^2\left(\tilde{\beta}^\top \mathbf{1}_n\right)}$$

The minimum variance portfolio is positively exposed to stocks with low beta:

$$\left\{\begin{array}{l} \beta_i < \beta^\star \Rightarrow x_{\mathrm{gmv},i} > 0 \\ \beta_i > \beta^\star \Rightarrow x_{\mathrm{gmv},i} < 0 \end{array}\right.$$

Moreover, the absolute weight is a decreasing function of the idiosyncratic volatility: $\tilde{\sigma}_i \searrow \Rightarrow \left|x_{\mathrm{gmv},i}\right| \nearrow$

Minimum variance portfolio

The previous formula has been found by Scherer (2011). Clarke et al. (2011) have extended it to the long-only minimum variance:

$$x_{\mathrm{mv},i} = \frac{\sigma^2\left(x_{\mathrm{gmv}}\right)}{\tilde{\sigma}_i^2}\left(1 - \frac{\beta_i}{\beta^\star}\right)$$

where the threshold $\beta^\star$ is defined as follows:

$$\beta^\star = \frac{1 + \sigma_m^2 \sum_{\beta_i < \beta^\star} \tilde{\beta}_i \beta_i}{\sigma_m^2 \sum_{\beta_i < \beta^\star} \tilde{\beta}_i}$$

In this case, if $\beta_i > \beta^\star$, $x_i^\star = 0$

Minimum variance portfolio

Example 3

We consider an investment universe of five assets. Their beta is respectively equal to 0.9, 0.8, 1.2, 0.7 and 1.3 whereas their specific volatility is 4%, 12%, 5%, 8% and 5%. We also assume that the market portfolio volatility is equal to 25%.

Minimum variance portfolio

- In the case of the GMV portfolio, we have $\varphi = 1879.26$ and $\beta^{\star} = 1.0972$
- In the case of the long-only MV portfolio, we have $\varphi = 121.01$ and $\beta^{\star} = 0.8307$

Table 51: Composition of the MV portfolio

Asset	β_i	$\tilde{\beta}_i$	x_i Unconstrained	x_i Long-only
1	0.90	562.50	147.33	0.00
2	0.80	55.56	24.67	9.45
3	1.20	480.00	-49.19	0.00
4	0.70	109.37	74.20	90.55
5	1.30	520.00	-97.01	0.00
Volatility			11.45	19.19

Minimum variance portfolio

In practice, we use a constrained long-only optimization program:

$$
\begin{aligned}
x^{\star} &= \arg\min \frac{1}{2} x^{\top} \Sigma x \\
\text{u.c.} &\quad \left\{
\begin{array}{l}
\mathbf{1}_n^{\top} x = 1 \\
\mathbf{0}_n \leq x \leq \mathbf{1}_n \\
x \in \mathcal{DC}
\end{array}
\right.
\end{aligned}
$$

$\Rightarrow$ we need to impose some diversification constraints ($x \in \mathcal{DC}$) because Markowitz optimization leads to corner solutions that are not diversified

Minimum variance portfolio

Three main approaches:

1. In order to reduce the concentration of a few number of assets, we can use upper bound on the weights:

$$x_i \leq x_i^+$$

For instance, we can set $x_i \leq 5\%$, meaning that the weight of an asset cannot be larger than 5%. We can also impose lower and upper bounds by sector:

$$s_j^- \leq \sum_{i \in \mathcal{S}_j} x_i \leq s_j^+$$

For instance, if we impose that $3\% \leq \sum_{i \in \mathcal{S}_j} x_i \leq 20\%$, this implied that the weight of each sector must be between 3% and 20%.

Minimum variance portfolio

2. We can impose some constraints with respect to the benchmark composition:

$$\frac{b_i}{m} \leq x_i \leq m \cdot b_i$$

where b_i is the weight of asset i in the benchmark (or index) b. For instance, if $m = 2$, the weight of asset i cannot be lower than 50% of its weight in the benchmark. It cannot also be greater than twice of its weight in the benchmark.

3. The third approach consists of imposing a weight diversification based on the Herfindahl index:

$$\mathcal{H}(x) = \sum_{i=1}^{n} x_i$$

Minimum variance portfolio

- The inverse of the Herfindahl index is called the effective number of bets (ENB):

$$\mathcal{N}(x) = \mathcal{H}^{-1}(x)$$

- $\mathcal{N}(x)$ represents the equivalent number of equally-weighted assets. We can impose a sufficient number of effective bets:

$$\mathcal{N}(x) \geq \mathcal{N}_{\min}$$

- During the period 2000-2020, the ENB of the S&P 500 index is between 90 and 130:

$$90 \leq \mathcal{N}(b) \leq 130$$

- During the same period, the ENB of the S&P 500 minimum variance portfolio is between 15 and 30:

$$15 \leq \mathcal{N}(x) \leq 30$$

- We conclude that the S&P 500 minimum variance portfolio is less diversified than the S&P 500 index

Minimum variance portfolio

We can impose:

$$\mathcal{N}(x) \geq m \cdot \mathcal{N}(b)$$

For instance, if $m = 1.5$, the ENB of the S&P 500 minimum variance portfolio will be 50% larger than the ENB of the S&P 500 index

We notice that:

$$\begin{aligned} \mathcal{N}(x) \geq \mathcal{N}_{\min} & \Leftrightarrow & \mathcal{H}(x) \leq \mathcal{N}_{\min}^{-1} \\ & \Leftrightarrow & x^\top x \leq \mathcal{N}_{\min}^{-1} \end{aligned}$$

The optimization problem becomes:

$$\begin{aligned} x^\star(\lambda) & = & \arg\min \frac{1}{2} x^\top \Sigma x + \lambda \left(x^\top x - \mathcal{N}_{\min}^{-1} \right) \\ & & \text{u.c.} \quad \left\{ \begin{array}{l} \mathbf{1}_n^\top x = 1 \\ \mathbf{0}_n \leq x \leq \mathbf{1}_n \end{array} \right. \end{aligned}$$

Minimum variance portfolio

We can rewrite the objective function as follows:

$$\mathcal{L}(x;\lambda) = \frac{1}{2}x^\top \Sigma x + \lambda x^\top I_n x = \frac{1}{2}x^\top (\Sigma + 2\lambda I_n) x$$

We obtain a standard minimum variance optimization problem where the covariance matrix is shrunk

Remark

The optimal solution is found by applying the bisection algorithm to the QP problem in order to match the constraint:

$$\mathcal{N}(x^\star(\lambda)) = \mathcal{N}_{\min}$$

An alternative approach is to consider the ADMM algorithm (these numerical problems are studied in Lecture 5)

Most diversified portfolio

Definition

Choueifaty and Coignard (2008) introduce the concept of diversification ratio:

$$\mathcal{DR}(x) = \frac{\sum_{i=1}^{n} x_i \sigma_i}{\sigma(x)} = \frac{x^\top \sigma}{\sqrt{x^\top \Sigma x}}$$

$\mathcal{DR}(x)$ is the ratio between the weighted average volatility and the portfolio volatility

- The diversification ratio of a portfolio fully invested in one asset is equal to one:

$$\mathcal{DR}(e_i) = 1$$

- In the general case, it is larger than one:

$$\mathcal{DR}(x) \geq 1$$

Most diversified portfolio

The most diversified portfolio (or MDP) is defined as the portfolio which maximizes the diversification ratio:

$$\begin{aligned} x^{\star} &= \arg\max \ln \mathcal{DR}(x) \\ & \text{u.c.} \quad \left\{ \begin{array}{l} \mathbf{1}_n^\top x = 1 \\ \mathbf{0}_n \leq x \leq \mathbf{1}_n \end{array} \right. \end{aligned}$$

Most diversified portfolio

The associated Lagrange function is equal to:

$$\begin{aligned}\mathcal{L}(x;\lambda_0,\lambda) &= \ln\left(\frac{x^\top\sigma}{\sqrt{x^\top\Sigma x}}\right)+\lambda_0\left(\mathbf{1}_n^\top x-1\right)+\lambda^\top\left(x-\mathbf{0}_n\right) \\ &= \ln\left(x^\top\sigma\right)-\frac{1}{2}\ln\left(x^\top\Sigma x\right)+\lambda_0\left(\mathbf{1}_n^\top x-1\right)+\lambda^\top x\end{aligned}$$

The first-order condition is:

$$\frac{\partial\,\mathcal{L}(x;\lambda_0,\lambda)}{\partial\,x}=\frac{\sigma}{x^\top\sigma}-\frac{\Sigma x}{x^\top\Sigma x}+\lambda_0\mathbf{1}_n+\lambda=\mathbf{0}_n$$

whereas the Kuhn-Tucker conditions are:

$$\min\left(\lambda_i,x_i\right)=0\qquad\text{for } i=1,\ldots,n$$

Most diversified portfolio

The constraint $\mathbf{1}_n^\top x = 1$ can always be matched because:

$$\mathcal{DR}\left(\varphi \cdot x\right) = \mathcal{DR}\left(x\right)$$

We deduce that the MDP $x^\star$ satisfies:

$$\frac{\Sigma x^\star}{x^{\star\top}\Sigma x^\star} = \frac{\sigma}{x^{\star\top}\sigma} + \lambda$$

or:

$$\begin{aligned}
\Sigma x^\star &= \frac{\sigma^2\left(x^\star\right)}{x^{\star\top}\sigma}\sigma + \lambda\sigma^2\left(x^\star\right) \\
&= \frac{\sigma\left(x^\star\right)}{\mathcal{DR}\left(x^\star\right)}\sigma + \lambda\sigma^2\left(x^\star\right)
\end{aligned}$$

If the long-only constraint is not imposed, we have $\lambda = \mathbf{0}_n$

Most diversified portfolio

The correlation between a portfolio x and the MDP $x^{\star}$ is given by:

$$\begin{aligned}
\rho\left(x, x^{\star}\right) &= \frac{x^{\top}\Sigma x^{\star}}{\sigma\left(x\right)\sigma\left(x^{\star}\right)} \\
&= \frac{1}{\sigma\left(x\right)\mathcal{DR}\left(x^{\star}\right)} x^{\top}\sigma + \frac{\sigma\left(x^{\star}\right)}{\sigma\left(x\right)} x^{\top}\lambda \\
&= \frac{\mathcal{DR}\left(x\right)}{\mathcal{DR}\left(x^{\star}\right)} + \frac{\sigma\left(x^{\star}\right)}{\sigma\left(x\right)} x^{\top}\lambda
\end{aligned}$$

Most diversified portfolio

If $x^{\star}$ is the long-only MDP, we obtain (because $\lambda \geq \mathbf{0}_n$ and $x^{\top}\lambda \geq 0$):

$$\rho\left(x, x^{\star}\right) \geq \frac{\mathcal{DR}\left(x\right)}{\mathcal{DR}\left(x^{\star}\right)}$$

whereas we have for the unconstrained MDP:

$$\rho\left(x, x^{\star}\right) = \frac{\mathcal{DR}\left(x\right)}{\mathcal{DR}\left(x^{\star}\right)}$$

The 'core property' of the MDP

"The long-only MDP is the long-only portfolio such that the correlation between any other long-only portfolio and itself is greater than or equal to the ratio of their diversification ratios" (Choueifaty et al., 2013)

Most diversified portfolio

The correlation between Asset i and the MDP is equal to:

$$\begin{aligned}
\rho\left(e_i, x^\star\right) &= \frac{\mathcal{DR}\left(e_i\right)}{\mathcal{DR}\left(x^\star\right)} + \frac{\sigma\left(x^\star\right)}{\sigma\left(e_i\right)} e_i^\top \lambda \\
&= \frac{1}{\mathcal{DR}\left(x^\star\right)} + \frac{\sigma\left(x^\star\right)}{\sigma_i} \lambda_i
\end{aligned}$$

Most diversified portfolio

Because $\lambda_i = 0$ if $x_i^\star > 0$ and $\lambda_i > 0$ if $x_i^\star = 0$, we deduce that:

$$\rho\left(e_i, x^\star\right) = \frac{1}{\mathcal{DR}\left(x^\star\right)} \quad \text{if} \quad x_i^\star > 0$$

and:

$$\rho\left(e_i, x^\star\right) \geq \frac{1}{\mathcal{DR}\left(x^\star\right)} \quad \text{if} \quad x_i^\star = 0$$

Most diversified portfolio

Another diversification concept

"Any stock not held by the MDP is more correlated to the MDP than any of the stocks that belong to it. Furthermore, all stocks belonging to the MDP have the same correlation to it. [...] This property illustrates that all assets in the universe are effectively represented in the MDP, even if the portfolio does not physically hold them. [...] This is consistent with the notion that the most diversified portfolio is the un-diversifiable portfolio" (Choueifaty et al., 2013)

Most diversified portfolio

Remark

In the case when the long-only constraint is omitted, we have $\rho(e_i, x^\star) = \rho(e_j, x^\star)$ meaning that the correlation with the MDP is the same for all the assets

Most diversified portfolio

Example 4

We consider an investment universe of four assets. Their volatilities are equal to 20%, 10%, 20% and 25%. The correlation of asset returns is given by the following matrix:

$$\rho = \begin{pmatrix} 1.00 & & & \\ 0.80 & 1.00 & & \\ 0.40 & 0.30 & 1.00 & \\ 0.50 & 0.10 & -0.10 & 1.00 \end{pmatrix}$$

Most diversified portfolio

Table 52: Composition of the MDP

Asset	Unconstrained		Long-only	
	$x_i^\star$	$\rho(e_i, x^\star)$	$x_i^\star$	$\rho(e_i, x^\star)$
1	-18.15	61.10	0.00	73.20
2	61.21	61.10	41.70	62.40
3	29.89	61.10	30.71	62.40
4	27.05	61.10	27.60	62.40
$\sigma(x^\star)$	9.31		10.74	
$\mathcal{DR}(x^\star)$	1.64		1.60	

Most diversified portfolio

Assumption $\mathcal{H}_0$: all the assets have the same Sharpe ratio

$$\frac{\mu_i - r}{\sigma_i} = s$$

Under $\mathcal{H}_0$, the diversification ratio of portfolio x is proportional to its Sharpe ratio:

$$\begin{aligned}
\mathcal{DR}(x) &= \frac{1}{s} \frac{\sum_{i=1}^{n} x_i (\mu_i - r)}{\sigma(x)} \\
&= \frac{1}{s} \frac{x^\top \mu - r}{\sigma(x)} \\
&= \frac{1}{s} \cdot \mathrm{SR}(x \mid r)
\end{aligned}$$

Most diversified portfolio

Optimality of the MDP

Under $\mathcal{H}_0$, maximizing the diversification ratio is then equivalent to maximizing the Sharpe ratio:

$$\text{MDP} = \text{MSR}$$

Most diversified portfolio

In the CAPM framework, Clarke *et al.* (2013) showed that:

$$x_i^\star = \mathcal{DR}(x^\star) \frac{\sigma_i \sigma(x^\star)}{\tilde{\sigma}_i^2} \left(1 - \frac{\rho_{i,m}}{\rho^\star}\right)$$

where $\sigma_i = \sqrt{\beta_i^2 \sigma_m^2 + \tilde{\sigma}_i^2}$ is the volatility of asset i, $\rho_{i,m} = \beta_i \sigma_m / \sigma_i$ is the correlation between asset i and the market portfolio and $\rho^\star$ is the threshold correlation given by this formula:

$$\rho^\star = \left(1 + \sum_{i=1}^{n} \frac{\rho_{i,m}^2}{1 - \rho_{i,m}^2}\right) \Bigg/ \left(\sum_{i=1}^{n} \frac{\rho_{i,m}}{1 - \rho_{i,m}^2}\right)$$

The weights are then strictly positive if $\rho_{i,m} < \rho^\star$

Most diversified portfolio

The MDP tends to be less concentrated than the MV portfolio because:

$$\begin{aligned} x_{\mathrm{mv},i} &= \frac{1}{\tilde{\sigma}_i^2} \times \cdots \\ x_{\mathrm{mdp},i} &= \frac{\sigma_i}{\tilde{\sigma}_i^2} \times \cdots \approx \frac{1}{\tilde{\sigma}_i} \times \cdots + \cdots \end{aligned}$$

ERC portfolio

In Lecture 2, we have seen that the ERC portfolio corresponds to the portfolio such that the risk contribution from each stock is made equal

The main advantages of the ERC allocation are the following:

1. It defines a portfolio that is well diversified in terms of risk and weights
2. Like the three previous risk-based methods, it does not depend on any expected returns hypothesis
3. It is less sensitive to small changes in the covariance matrix than MV or MDP portfolios (Demey *et al.*, 2010)

ERC portfolio

In the CAPM framework, Clarke *et al.* (2013) showed:

$$x_i^\star = \frac{\sigma^2(x^\star)}{\tilde{\sigma}_i^2}\left(\sqrt{\frac{\beta_i^2}{\beta^{\star 2}} + \frac{\tilde{\sigma}_i^2}{n\sigma^2(x^\star)}} - \frac{\beta_i}{\beta^\star}\right)$$

where:

$$\beta^\star = \frac{2\sigma^2(x^\star)}{\beta(x^\star)\,\sigma_m^2}$$

It follows that:

$$\lim_{n\to\infty} x_{\mathrm{erc}} = x_{\mathrm{ew}}$$

Comparison of the 4 Methods

Equally-weighted (EW)

- Weights are equal
- Easy to understand
- Contrarian strategy with a take-profit scheme
- The least concentrated in terms of weights
- Do not depend on risks

Most Diversified Portfolio (MDP)

- Also known as the Max Sharpe Ratio (MSR) portfolio of EDHEC
- Based on the assumption that sharpe ratio is equal for all stocks
- It is the tangency portfolio if the previous assumption is verified
- Sensitive to the covariance matrix

Minimum variance (MV)

- Low volatility portfolio
- The only optimal portfolio not depending on expected returns assumptions
- Good out of sample performance
- Concentrated portfolios
- Sensitive to the covariance matrix

Equal Risk Contribution (ERC)

- Risk contributions are equal
- Highly diversified portfolios
- Less sensitive to the covariance matrix (than the MV and MDP portfolios)
- Not efficient for universe with a large number of stocks (equivalent to the EW portfolio)

Some properties

In terms of bets

$$\begin{array}{ll} \exists i : w_i = 0 & \text{(MV - MDP)} \\ \forall i : w_i \neq 0 & \text{(EW - ERC)} \end{array}$$

In terms of risk factors

$$\begin{array}{rcll} x_i & = & x_j & \text{(EW)} \\ \dfrac{\partial\, \sigma(x)}{\partial\, x_i} & = & \dfrac{\partial\, \sigma(x)}{\partial\, x_j} & \text{(MV)} \\ x_i \cdot \dfrac{\partial\, \sigma(x)}{\partial\, x_i} & = & x_j \cdot \dfrac{\partial\, \sigma(x)}{\partial\, x_j} & \text{(ERC)} \\ \dfrac{1}{\sigma_i} \cdot \dfrac{\partial\, \sigma(x)}{\partial\, x_i} & = & \dfrac{1}{\sigma_j} \cdot \dfrac{\partial\, \sigma(x)}{\partial\, x_j} & \text{(MDP)} \end{array}$$

Some properties

Proof for the MDP portfolio

For the unconstrained MDP portfolio, we recall that the first-order condition is given by:

$$\frac{\partial \mathcal{L}(x;\lambda_0,\lambda)}{\partial x_i} = \frac{\sigma_i}{x^\top \sigma} - \frac{(\Sigma x)_i}{x^\top \Sigma x} = 0$$

The scaled marginal volatility is then equal to the inverse of the diversification ratio of the MDP:

$$\begin{aligned}
\frac{1}{\sigma_i} \cdot \frac{\partial \sigma(x)}{\partial x_i} &= \frac{1}{\sigma_i} \cdot \frac{(\Sigma x)_i}{\sqrt{x^\top \Sigma x}} \\
&= \frac{\sigma(x)}{\sigma_i} \cdot \frac{(\Sigma x)_i}{x^\top \Sigma x} \\
&= \frac{\sigma(x)}{x^\top \sigma} = \frac{1}{\mathcal{DR}(x)}
\end{aligned}$$

Application to the Eurostoxx 50 index

Table 53: Composition in % (January 2010)

	CW	MV	ERC	MDP	1/n	MV 10%	MDP 10%	MV 5%	MDP 5%
TOTAL	6.1		2.1		2			5.0	
BANCO SANTANDER	5.8		1.3		2				
TELEFONICA SA	5.0	31.2	3.5		2	10.0		5.0	5.0
SANOFI-AVENTIS	3.6	12.1	4.5	15.5	2	10.0	10.0	5.0	5.0
E.ON AG	3.6		2.1		2				1.4
BNP PARIBAS	3.4		1.1		2				
SIEMENS AG	3.2		1.5		2				
BBVA(BILB-VIZ-ARG)	2.9		1.4		2				
BAYER AG	2.9		2.6	3.7	2	2.2	5.0	5.0	5.0
ENI	2.7		2.1		2				
GDF SUEZ	2.5		2.6	4.5	2		5.4	5.0	5.0
BASF SE	2.5		1.5		2				
ALLIANZ SE	2.4		1.4		2				
UNICREDIT SPA	2.3		1.1		2				
SOC GENERALE	2.2		1.2	3.9	2		3.7		5.0
UNILEVER NV	2.2	11.4	3.7	10.8	2	10.0	10.0	5.0	5.0
FRANCE TELECOM	2.1	14.9	4.1	10.2	2	10.0	10.0	5.0	5.0
NOKIA OYJ	2.1		1.8	4.5	2		4.8		5.0
DAIMLER AG	2.1		1.3		2				
DEUTSCHE BANK AG	1.9		1.0		2				
DEUTSCHE TELEKOM	1.9		3.2	2.6	2	5.7	3.7	5.0	5.0
INTESA SANPAOLO	1.9		1.3		2				
AXA	1.8		1.0		2				
ARCELORMITTAL	1.8		1.0		2				
SAP AG	1.8	21.0	3.4	11.2	2	10.0	10.0	5.0	5.0

	CW	MV	ERC	MDP	1/n	MV 10%	MDP 10%	MV 5%	MDP 5%
RWE AG (NEU)	1.7	2.7	2.7		2	7.0		5.0	
ING GROEP NV	1.6		0.8	0.4	2				
DANONE	1.6	1.9	3.4	1.8	2	8.7	3.3	5.0	5.0
IBERDROLA SA	1.6		2.5		2	5.1		5.0	1.2
ENEL	1.6		2.1		2			5.0	2.9
VIVENDI SA	1.6	2.8	3.1	4.5	2	10.0	5.9	5.0	5.0
ANHEUSER-BUSCH INB	1.6	0.2	2.7	10.9	2	2.1	10.0	5.0	5.0
ASSIC GENERALI SPA	1.6		1.8		2				
AIR LIQUIDE(L')	1.4		2.1		2			5.0	
MUENCHENER RUECKVE	1.3		2.1	2.1	2		3.1	5.0	5.0
SCHNEIDER ELECTRIC	1.3		1.5		2				
CARREFOUR	1.3	1.0	2.7	1.3	2	3.7	2.5	5.0	5.0
VINCI	1.3		1.6		2				
LVMH MOET HENNESSY	1.2		1.8		2				
PHILIPS ELEC(KON)	1.2		1.4		2				
L'OREAL	1.1	0.8	2.8		2	5.5		5.0	5.0
CIE DE ST-GOBAIN	1.0		1.1		2				
REPSOL YPF SA	0.9		2.0		2			5.0	
CRH	0.8		1.7	5.1	2		5.2		5.0
CREDIT AGRICOLE SA	0.8		1.1		2				
DEUTSCHE BOERSE AG	0.7		1.5		2				1.9
TELECOM ITALIA SPA	0.7		2.0		2				2.5
ALSTOM	0.6		1.5		2				
AEGON NV	0.4		0.7		2				
VOLKSWAGEN AG	0.2		1.8	7.1	2		7.4		5.0
Total of components	50	11	50	17	50	14	16	20	23

Some examples

To compare the risk-based methods, we report:

- The weights x_i in %
- The relative risk contributions $\mathcal{RC}_i$ in %
- The weight concentration $\mathcal{H}^\star(x)$ in % and the risk concentration $\mathcal{H}^\star(\mathcal{RC})$ in % where $\mathcal{H}^\star$ is the modified Herfindahl index[13]
- The portfolio volatility $\sigma(x)$ in %
- The diversification ratio $\mathcal{DR}(x)$

[13] We have:

$$\mathcal{H}^\star(\pi) = \frac{n\mathcal{H}(\pi) - 1}{n-1} \in [0, 1]$$

Some examples

Example 5

We consider an investment universe with four assets. We assume that the volatility σ_i is the same and equal to 20% for all four assets. The correlation matrix C is equal to:

$$C = \begin{pmatrix} 100\% & & & \\ 80\% & 100\% & & \\ 0\% & 0\% & 100\% & \\ 0\% & 0\% & -50\% & 100\% \end{pmatrix}$$

Some examples

Table 54: Weights and risk contributions (Example 5)

Asset	EW		MV		MDP		ERC	
	x_i	$\mathcal{RC}_i$	x_i	$\mathcal{RC}_i$	x_i	$\mathcal{RC}_i$	x_i	$\mathcal{RC}_i$
1	25.00	4.20	10.87	0.96	10.87	0.96	17.26	2.32
2	25.00	4.20	10.87	0.96	10.87	0.96	17.26	2.32
3	25.00	1.17	39.13	3.46	39.13	3.46	32.74	2.32
4	25.00	1.17	39.13	3.46	39.13	3.46	32.74	2.32
$\mathcal{H}^{\star}(x)$	0.00		10.65		10.65		3.20	
$\sigma(x)$	10.72		8.85		8.85		9.26	
$\mathcal{DR}(x)$	1.87		2.26		2.26		2.16	
$\mathcal{H}^{\star}(\mathcal{RC})$	10.65		10.65		10.65		0.00	

Some examples

Example 6

We modify the previous example by introducing differences in volatilities. They are 10%, 20%, 30% and 40% respectively. The correlation matrix remains the same as in Example 5.

Some examples

Table 55: Weights and risk contributions (Example 6)

Asset	EW		MV		MDP		ERC	
	x_i	$\mathcal{RC}_i$	x_i	$\mathcal{RC}_i$	x_i	$\mathcal{RC}_i$	x_i	$\mathcal{RC}_i$
1	25.00	1.41	74.48	6.43	27.78	1.23	38.36	2.57
2	25.00	3.04	0.00	0.00	13.89	1.23	19.18	2.57
3	25.00	1.63	15.17	1.31	33.33	4.42	24.26	2.57
4	25.00	5.43	10.34	0.89	25.00	4.42	18.20	2.57
$\mathcal{H}^\star(x)$	0.00		45.13		2.68		3.46	
$\sigma(x)$	11.51		8.63		11.30		10.29	
$\mathcal{DR}(x)$	2.17		1.87		2.26		2.16	
$\mathcal{H}^\star(\mathcal{RC})$	10.31		45.13		10.65		0.00	

Some examples

Example 7

We now reverse the volatilities of Example 6. They are now equal to 40%, 30%, 20% and 10%.

Some examples

Table 56: Weights and risk contributions (Example 7)

	EW		MV		MDP		ERC	
Asset	x_i	$\mathcal{RC}_i$	x_i	$\mathcal{RC}_i$	x_i	$\mathcal{RC}_i$	x_i	$\mathcal{RC}_i$
1	25.00	9.32	0.00	0.00	4.18	0.74	7.29	1.96
2	25.00	6.77	4.55	0.29	5.57	0.74	9.72	1.96
3	25.00	1.09	27.27	1.74	30.08	2.66	27.66	1.96
4	25.00	0.00	68.18	4.36	60.17	2.66	55.33	1.96
$\mathcal{H}^{\star}(x)$	0.00		38.84		27.65		19.65	
$\sigma(x)$	17.18		6.40		6.80		7.82	
$\mathcal{DR}(x)$	1.46		2.13		2.26		2.16	
$\mathcal{H}^{\star}(\mathcal{RC})$	27.13		38.84		10.65		0.00	

Some examples

Example 8

We consider an investment universe of four assets. The volatility is respectively equal to 15%, 30%, 45% and 60% whereas the correlation matrix C is equal to:

$$C = \begin{pmatrix} 100\% & & & \\ 10\% & 100\% & & \\ 30\% & 30\% & 100\% & \\ 40\% & 20\% & -50\% & 100\% \end{pmatrix}$$

Some examples

Table 57: Weights and risk contributions (Example 8)

Asset	EW		MV		MDP		ERC	
	x_i	$\mathcal{RC}_i$	x_i	$\mathcal{RC}_i$	x_i	$\mathcal{RC}_i$	x_i	$\mathcal{RC}_i$
1	25.00	2.52	82.61	11.50	0.00	0.00	40.53	4.52
2	25.00	5.19	17.39	2.42	0.00	0.00	22.46	4.52
3	25.00	3.89	0.00	0.00	57.14	12.86	21.12	4.52
4	25.00	9.01	0.00	0.00	42.86	12.86	15.88	4.52
$\mathcal{H}^{\star}(x)$	0.00		61.69		34.69		4.61	
$\sigma(x)$	20.61		13.92		25.71		18.06	
$\mathcal{DR}(x)$	1.82		1.27		2.00		1.76	
$\mathcal{H}^{\star}(\mathcal{RC})$	7.33		61.69		33.33		0.00	

Some examples

Example 9

Now we consider an example with six assets. The volatilities are 25%, 20%, 15%, 18%, 30% and 20% respectively. We use the following correlation matrix:

$$C = \begin{pmatrix} 100\% & & & & & \\ 20\% & 100\% & & & & \\ 60\% & 60\% & 100\% & & & \\ 60\% & 60\% & 60\% & 100\% & & \\ 60\% & 60\% & 60\% & 60\% & 100\% & \\ 60\% & 60\% & 60\% & 60\% & 60\% & 100\% \end{pmatrix}$$

Some examples

Table 58: Weights and risk contributions (Example 9)

Asset	EW x_i	EW $\mathcal{RC}_i$	MV x_i	MV $\mathcal{RC}_i$	MDP x_i	MDP $\mathcal{RC}_i$	ERC x_i	ERC $\mathcal{RC}_i$
1	16.67	3.19	0.00	0.00	44.44	8.61	14.51	2.72
2	16.67	2.42	6.11	0.88	55.56	8.61	18.14	2.72
3	16.67	2.01	65.16	9.33	0.00	0.00	21.84	2.72
4	16.67	2.45	22.62	3.24	0.00	0.00	18.20	2.72
5	16.67	4.32	0.00	0.00	0.00	0.00	10.92	2.72
6	16.67	2.75	6.11	0.88	0.00	0.00	16.38	2.72
$\mathcal{H}^{\star}(x)$	0.00		37.99		40.74		0.83	
$\sigma(x)$	17.14		14.33		17.21		16.31	
$\mathcal{DR}(x)$	1.24		1.14		1.29		1.25	
$\mathcal{H}^{\star}(\mathcal{RC})$	1.36		37.99		40.00		0.00	

Some examples

Example 10

To illustrate how the MV and MDP portfolios are sensitive to specific risks, we consider a universe of n assets with volatility equal to 20%. The structure of the correlation matrix is the following:

$$C = \begin{pmatrix} 100\% & & & & \\ \rho_{1,2} & 100\% & & & \\ 0 & \rho & 100\% & & \\ \vdots & \vdots & \ddots & 100\% & \\ 0 & \rho & \cdots & \rho & 100\% \end{pmatrix}$$

Some examples

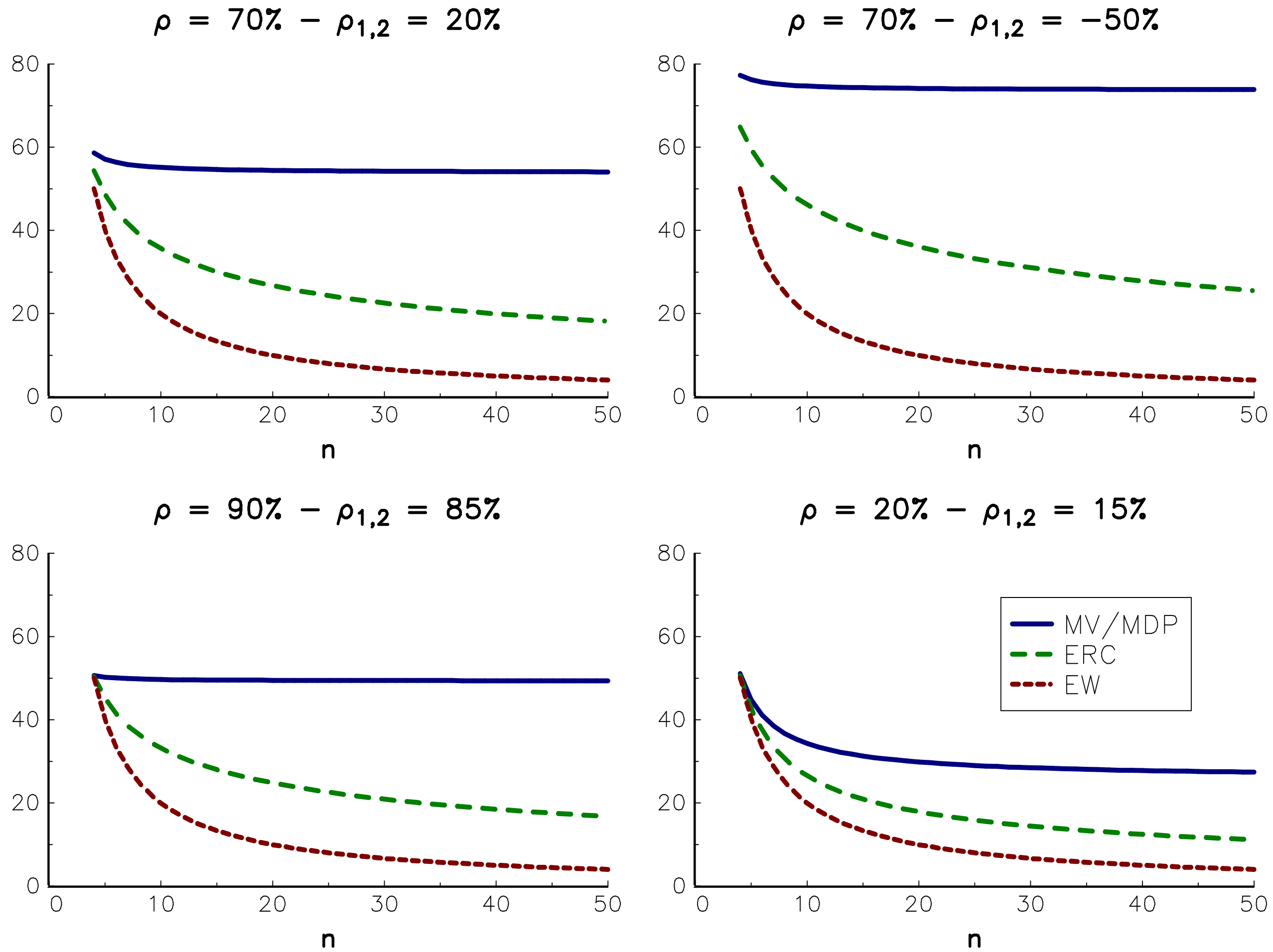


Figure 46: Weight of the first two assets in AW portfolios (Example 10)

Some examples

Example 11

We assume that asset returns follow the one-factor CAPM model. The idiosyncratic volatility $\tilde{\sigma}_i$ is set to 5% for all the assets whereas the volatility of the market portfolio σ_m is equal to 25%.

Some examples

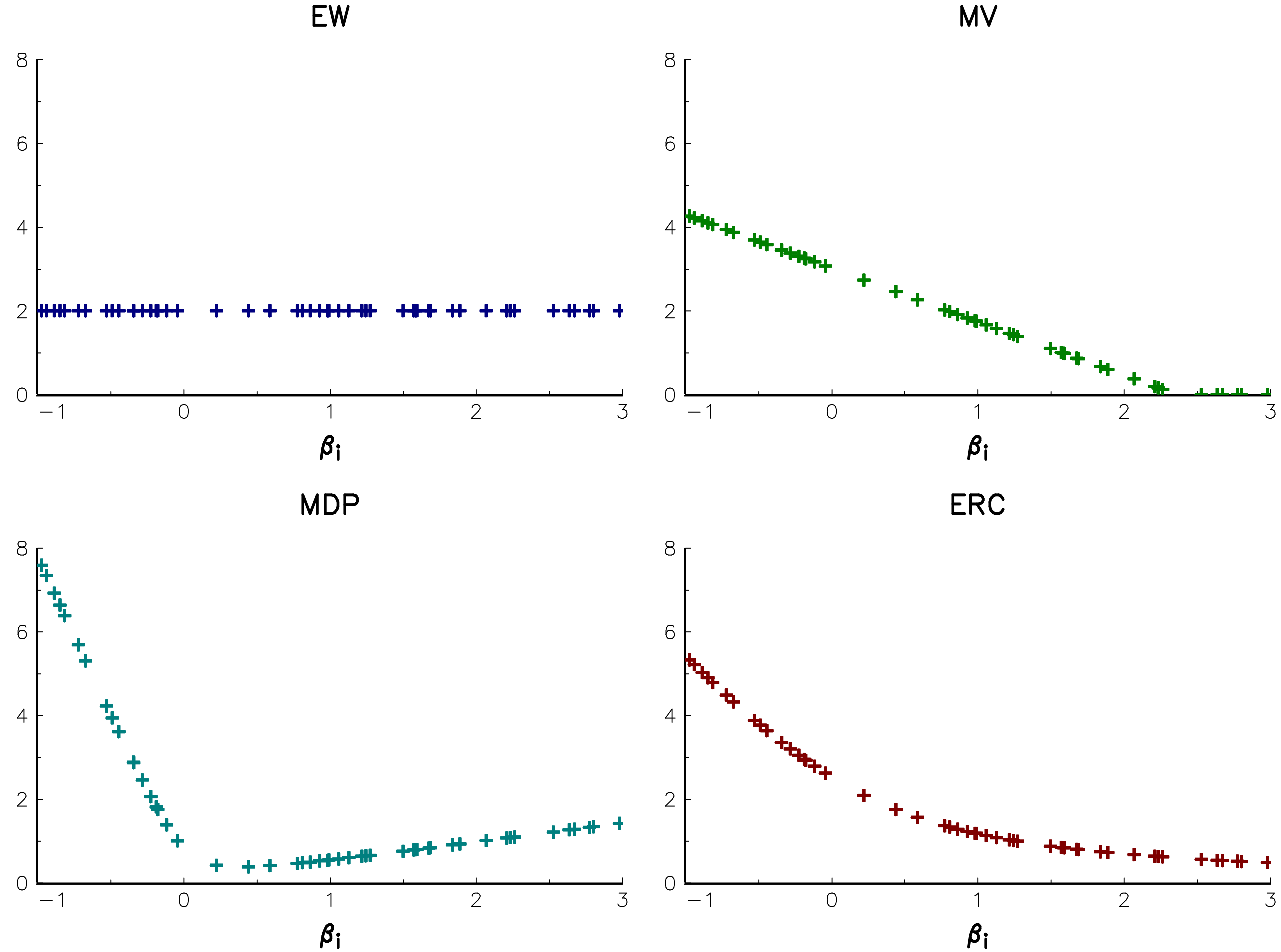


Figure 47: Weight with respect to the asset beta β_i (Example 11)

Smart beta products

- MSCI Equal Weighted Indexes (EW)
  www.msci.com/msci-equal-weighted-indexes
- S&P 500 Equal Weight Index (EW)
  www.spglobal.com/spdji/en/indices/equity/sp-500-equal-weight-index
- FTSE UK Equally Weighted Index Series (EW)
  www.ftserussell.com/products/indices/equally-weighted
- FTSE Global Minimum Variance Index Series (MV)
  www.ftserussell.com/products/indices/min-variance
- MSCI Minimum Volatility Indexes (MV)
  www.msci.com/msci-minimum-volatility-indexes
- S&P 500 Minimum Volatility Index (MV)
  www.spglobal.com/spdji/en/indices/strategy/sp-500-minimum-volatility-index
- FTSE Global Equal Risk Contribution Index Series (ERC)
  www.ftserussell.com/products/indices/erc
- TOBAM MaxDiv Index Series (MDP)
  www.tobam.fr/maximum-diversification-indexes

Smart beta products

Largest ETF issuers in Europe

1. iShares (BlackRock)
2. Xtrackers (DWS)
3. Lyxor ETF
4. UBS ETF
5. Amundi ETF

Largest ETF issuers in US

1. iShares (BlackRock)
2. SPDR (State Street)
3. Vanguard
4. Invesco PowerShares
5. First Trust

- Specialized smart beta ETF issuers: Wisdom Tree (US), Ossiam (Europe), Research affiliates (US), etc.
- Smart beta fund managers in Europe: Amundi, Ossiam, Quoniam, Robeco, Seeyond, Tobam, Unigestion, etc.
- ETFs, mutual funds, mandates

The case of bonds

Two main problems:

1. Benchmarks = debt-weighted indexation (the weights are based on the notional amount of the debt)
2. Fund management driven by the search of yield with little consideration for **credit risk** (carry position $\neq$ arbitrage position)

$\Rightarrow$ Time to rethink bond indexes? (Toloui, 2010)

The case of bonds

Two main problems:

1. Benchmarks = debt-weighted indexation (the weights are based on the notional amount of the debt)
2. Fund management driven by the search of yield with little consideration for **credit risk** (carry position $\neq$ arbitrage position)

$\Rightarrow$ Time to rethink bond indexes? (Toloui, 2010)

Bond indexation

Debt weighting

It is defined by:

$$x_i = \frac{\mathrm{DEBT}_i}{\sum_{i=1}^n \mathrm{DEBT}_i}$$

GDP weighting

It is defined by:

$$x_i = \frac{\mathrm{GDP}_i}{\sum_{i=1}^n \mathrm{GDP}_i}$$

Risk budgeting

It is defined by:

$$b_i = \frac{\mathrm{DEBT}_i}{\sum_{i=1}^n \mathrm{DEBT}_i}$$

or:

$$b_i = \frac{\mathrm{GDP}_i}{\sum_{i=1}^n \mathrm{GDP}_i}$$

$\Rightarrow$ The offering is very small compared to equity indices because of the liquidity issues (see Roncalli (2013), Chapter 4 for more details)

From CAPM to factor investing

How to define risk factors?

Risk factors are common factors that explain the cross-section variance of expected returns

- 1964: Market or MKT (or BETA) factor
- 1972: Low beta or BAB factor
- 1981: Size or SMB factor
- 1985: Value or HML factor
- 1991: Low volatility or VOL factor
- 1993: Momentum or WML factor
- 2000: Quality or QMJ factor

Systematic risk factors $\neq$ Idiosyncratic risk factors

Beta(s) $\neq$ Alpha(s)

Alpha or beta?

At the security level, there is a lot of idiosyncratic risk or alpha[14]:

	Common Risk	Idiosyncratic Risk
GOOGLE	47%	53%
NETFLIX	24%	76%
MASTERCARD	50%	50%
NOKIA	32%	68%
TOTAL	89%	11%
AIRBUS	56%	44%

Carhart's model with 4 factors, 2010-2014
Source: Roncalli (2017)

[14] The linear regression is:

$$R_i = \alpha_i + \sum_{j=1}^{n_{\mathcal{F}}} \beta_i^j \mathcal{F}_j + \varepsilon_i$$

In our case, we measure the alpha as $1 - \mathfrak{R}_i^2$ where:

$$\mathfrak{R}_i^2 = 1 - \frac{\sigma^2(\varepsilon_i)}{\sigma^2(R_i)}$$

The concept of alpha

- Jensen (1968) – **How to measure the performance of active fund managers?**

$$R_t^F = \alpha + \beta R_t^{MKT} + \varepsilon_t$$

Fund	Return	Rank	Beta	Alpha	Rank
A	12%	Best	1.0	-2%	Worst
B	11%	Worst	0.5	4%	Best

Market return = 14%

$$\Rightarrow \bar{\alpha} = -\text{fees}$$

- It is the beginning of passive management:
  - John McQuown (Wells Fargo Bank, 1971)
  - Rex Sinquefield (American National Bank, 1973)

Active management and performance persistence

- Hendricks *et al.* (1993) – **Hot Hands in Mutual Funds**

$$\operatorname{cov}\left(\alpha_t^{\mathsf{Jensen}}, \alpha_{t-1}^{\mathsf{Jensen}}\right) > 0$$

where:

$$\alpha_t^{\mathsf{Jensen}} = R_t^F - \beta^{\mathrm{MKT}} R_t^{MKT}$$

$\Rightarrow$ The persistence of the performance of active management is due to the **persistence of the alpha**

Risk factors and active management

- Grinblatt *et al.* (1995) – **Momentum investors versus Value investors**

  "*77% of mutual funds are momentum investors*"

- Carhart (1997):

$$\left\{\begin{array}{l}\operatorname{cov}\left(\alpha_t^{\text{Jensen}}, \alpha_{t-1}^{\text{Jensen}}\right) > 0 \\ \operatorname{cov}\left(\alpha_t^{\text{Carhart}}, \alpha_{t-1}^{\text{Carhart}}\right) = 0\end{array}\right.$$

  where:

$$\alpha_t^{\text{Carhart}} = R_t^F - \beta^{\text{MKT}} R_t^{\text{MKT}} - \beta^{\text{SMB}} R_t^{\text{SMB}} - \beta^{\text{HML}} R_t^{\text{HML}} - \beta^{\text{WML}} R_t^{\text{WML}}$$

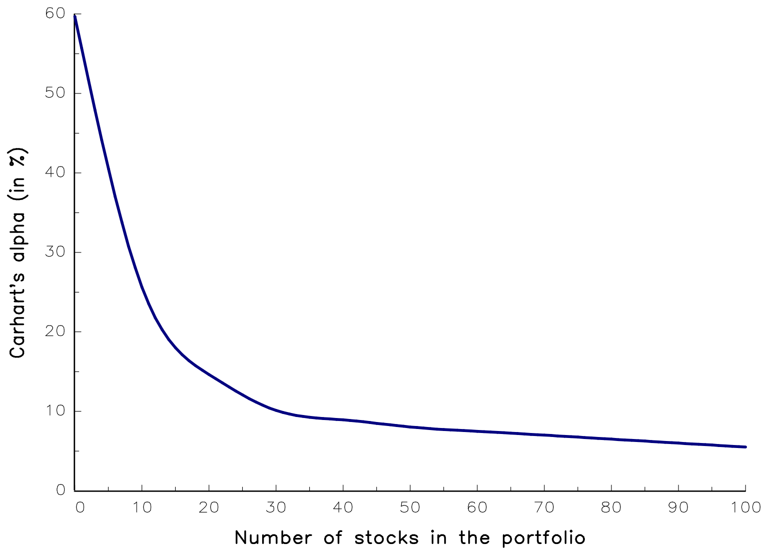
$\Rightarrow$ The (short-term) persistence of the performance of active management is due to the (short-term) **persistence of the performance of risk factors**

Diversification and alpha

David Swensen's rule for effective stock picking

Concentrated portfolio $\Rightarrow$ No more than 20 bets?

Figure 48: Carhart's alpha decreases with the number of holding assets

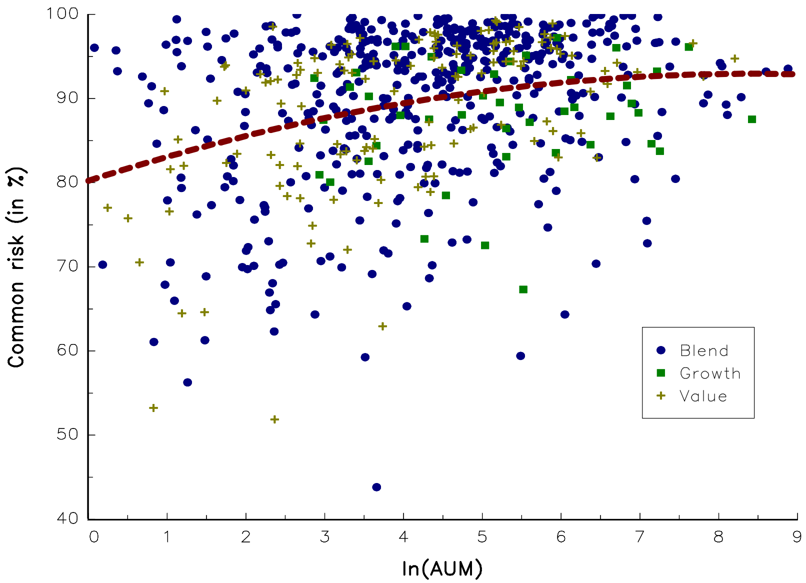


US equity markets, 2000-2014
Source: Roncalli (2017)

"*If you can identify six wonderful businesses, that is all the diversification you need. And you will make a lot of money. And I can guarantee that going into the seventh one instead of putting more money into your first one is going to be a terrible mistake. Very few people have gotten rich on their seventh best idea.*" (Warren Buffett, University of Florida, 1998).

Diversification and alpha

Figure 49: What proportion of return variance is explained by the 4-factor model?



Morningstar database, 880 mutual funds, European equities
Carhart's model with 4 factors, 2010-2014
Source: Roncalli (2017)

How many bets are there in large portfolios of institutional investors?

1986 Less than 10% of institutional portfolio return is explained by security picking and market timing (Brinson *et al.*, 1986)

2009 Professors' Report on the Norwegian GPFG: Risk factors represent 99.1% of the fund return variation (Ang *et al.*, 2009)

Risk factors versus alpha

What lessons can we draw from this?

Idiosyncratic risks and specific bets disappear in (large) diversified portfolios. Performance of institutional investors is then exposed to (common) risk factors.

Alpha is not scalable, but risk factors are scalable

$\Rightarrow$ Risk factors are the only bets that are compatible with diversification

Alpha

- Concentration
- Scarce?

$\neq$

Beta(s)

- Diversification
- Easy access?

Factor investing and active management

Misconception about active management

- Active management $= \boldsymbol{\alpha}$
- Passive management $= \boldsymbol{\beta}$

In this framework, passive management encompasses cap-weighted indexation, risk-based indexation and factor investing because these management styles do not pretend to create alpha

Factor investing and active management

"The question is when is active management good? The answer is never"

Eugene Fama, Morningstar ETF conference, September 2014

"So people say, 'I'm not going to try to beat the market. The market is all-knowing.' But how in the world can the market be all-knowing, if nobody is trying – well, not as many people – are trying to beat it?"

Robert Shiller, CNBC, November 2017

Factor investing and active management

- Discretionary active management $\Rightarrow$ specific/idiosyncratic risks & rule-based management $\Rightarrow$ factor investing and systematic risks?

- Are common risk factors exogenous or endogenous?
- Do risk factors exist without active management?

Risk factors first, active management second

or

Active management first, risk factors second

- Factor investing needs active investing
- Imagine a world without active managers, stock pickers, hedge funds, etc.

$\Rightarrow$ **Should active management be reduced to alpha management?**

Factor investing and active management

- Market risk factor = average of active management
- Low beta/low volatility strategies begin to be implemented in 2003-2004 (after the dot.com crisis)
- Quality strategies begin to be implemented in 2009-2010 (after the GFC crisis)

Alpha strategy $\Rightarrow$ **Risk Factor** (or a beta strategy)

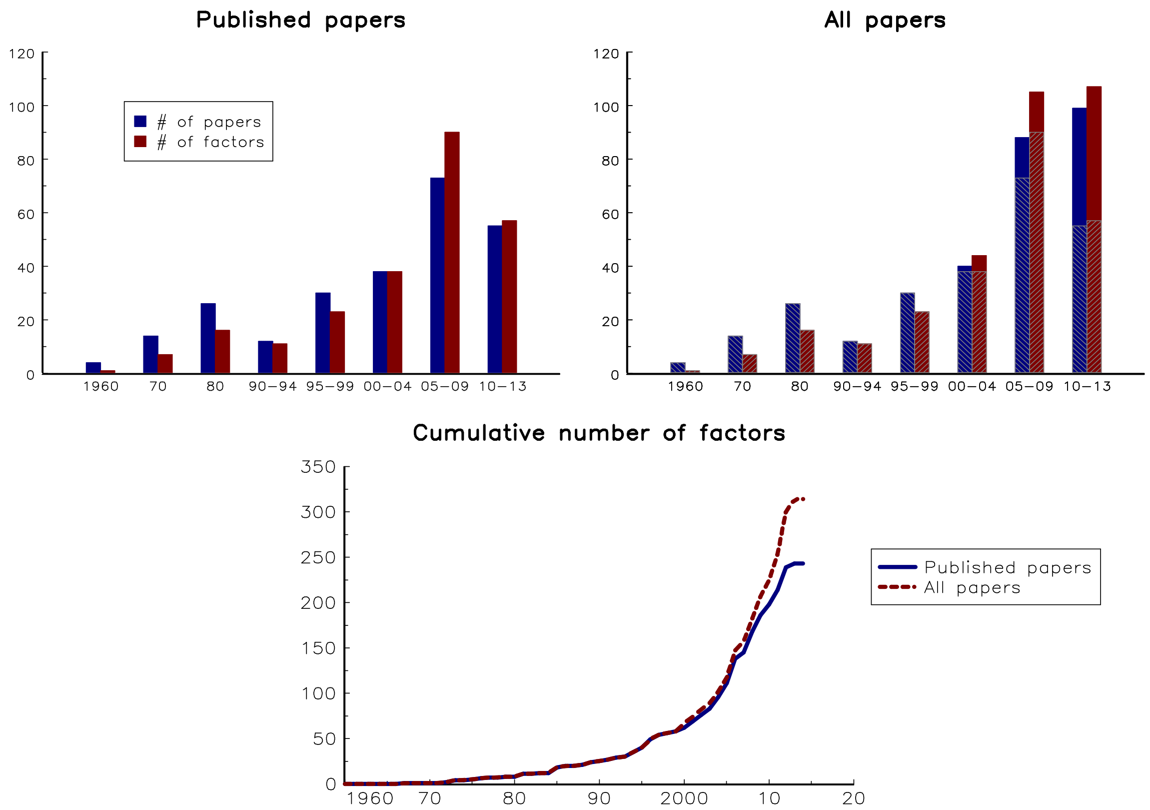
Factor investing and active management

α or β?

"[...] When an alpha strategy is massively invested, it has an enough impact on the structure of asset prices to become a risk factor.
[...] Indeed, an alpha strategy becomes a common market risk factor once it represents a significant part of investment portfolios and explains the cross-section dispersion of asset returns" (Roncalli, 2020)

The factor zoo

Figure 50: Harvey *et al.* (2016)



"*Now we have a zoo of new factors*" (Cochrane, 2011).

Factors, factors everywhere

"Standard predictive regressions fail to reject the hypothesis that the party of the U.S. President, the weather in Manhattan, global warming, El Niño, sunspots, or the conjunctions of the planets, are significantly related to anomaly performance. These results are striking, and quite surprising. In fact, some readers may be inclined to reject some of this paper's conclusions solely on the grounds of plausibility. I urge readers to consider this option carefully, however, as doing do so entails rejecting the standard methodology on which the return predictability literature is built."(Novy-Marx, 2014).

$\Rightarrow$ MKT, SMB, HML, WML, STR, LTR, VOL, IVOL, BAB, QMJ, LIQ, TERM, CARRY, DIV, JAN, CDS, GDP, INF, etc.

The alpha puzzle (Cochrane, 2011)

- Chaos

$$\mathbb{E}\left[R_i\right] - R_f = \boxed{\alpha_i}$$

- Sharpe (1964)

$$\mathbb{E}\left[R_i\right] - R_f = \beta_i^m \left(\mathbb{E}\left[R_m\right] - R_f\right)$$

- Chaos again

$$\mathbb{E}\left[R_i\right] - R_f = \boxed{\alpha_i} + \beta_i^m \left(\mathbb{E}\left[R_m\right] - R_f\right)$$

- Fama and French (1992)

$$\mathbb{E}\left[R_i\right] - R_f = \beta_i^m \left(\mathbb{E}\left[R_m\right] - R_f\right) + \beta_i^{smb}\mathbb{E}\left[R_{smb}\right] + \beta_i^{hml}\mathbb{E}\left[R_{hml}\right]$$

This is not the end of the story...

The alpha puzzle (Cochrane, 2011)

It's just the beginning!

- Chaos again

$$\mathbb{E}[R_i] - R_f = \boxed{\alpha_i} + \beta_i^m (\mathbb{E}[R_m] - R_f) + \beta_i^{smb}\mathbb{E}[R_{smb}] + \beta_i^{hml}\mathbb{E}[R_{hml}]$$

- Carhart (1997)

$$\mathbb{E}[R_i] - R_f = \beta_i^m (\mathbb{E}[R_m] - R_f) + \beta_i^{smb}\mathbb{E}[R_{smb}] + \beta_i^{hml}\mathbb{E}[R_{hml}] + \beta_i^{wml}\mathbb{E}[R_{wml}]$$

- Chaos again

$$\begin{aligned} \mathbb{E}[R_i] - R_f &= \boxed{\alpha_i} + \beta_i^m (\mathbb{E}[R_m] - R_f) + \beta_i^{smb}\mathbb{E}[R_{smb}] + \\ &\quad \beta_i^{hml}\mathbb{E}[R_{hml}] + \beta_i^{wml}\mathbb{E}[R_{wml}] \end{aligned}$$

- Etc.

How can alpha always come back?

The alpha puzzle (Cochrane, 2011)

1. **Because academic backtesting is not the real life**

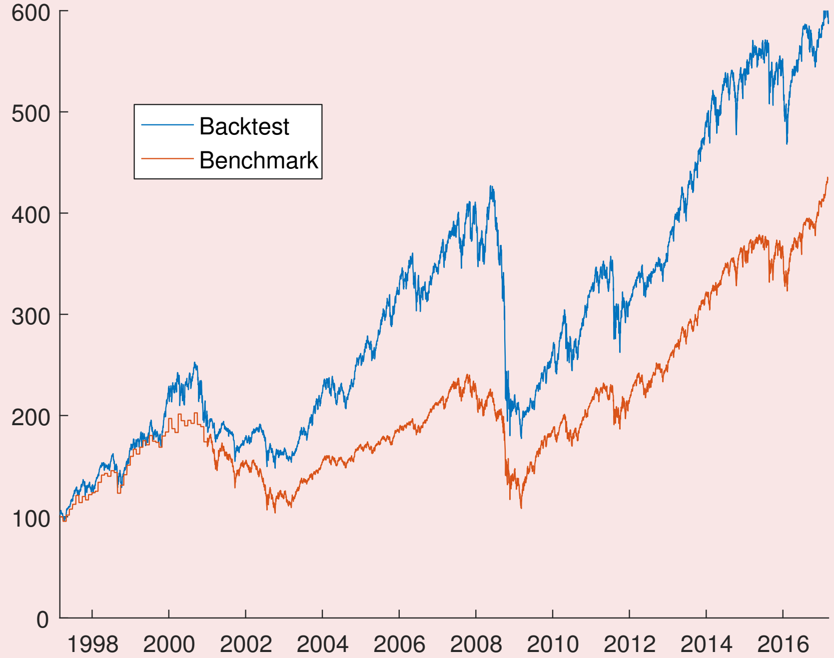
2. **Because risk factors are not independent in practice**

3. **Because the explanatory power of risk factors is time-varying**

4. **Because alpha and beta are highly related (beta strategy = successful alpha strategy)**

The issue of backtesting

Backtesting syndrome



The blue line is above the red line $\Rightarrow$ it's OK!

$\Rightarrow$ Analytical models are important to understand a risk factor

The professional consensus

There is now a consensus among professionals that five factors are sufficient for the equity markets:

1. **Size**

   Small cap stocks $\neq$ Large cap stocks

2. **Value**

   Value stocks $\neq$ Non-value stocks (including growth stocks)

3. **Momentum**

   Past winners $\neq$ Past loosers

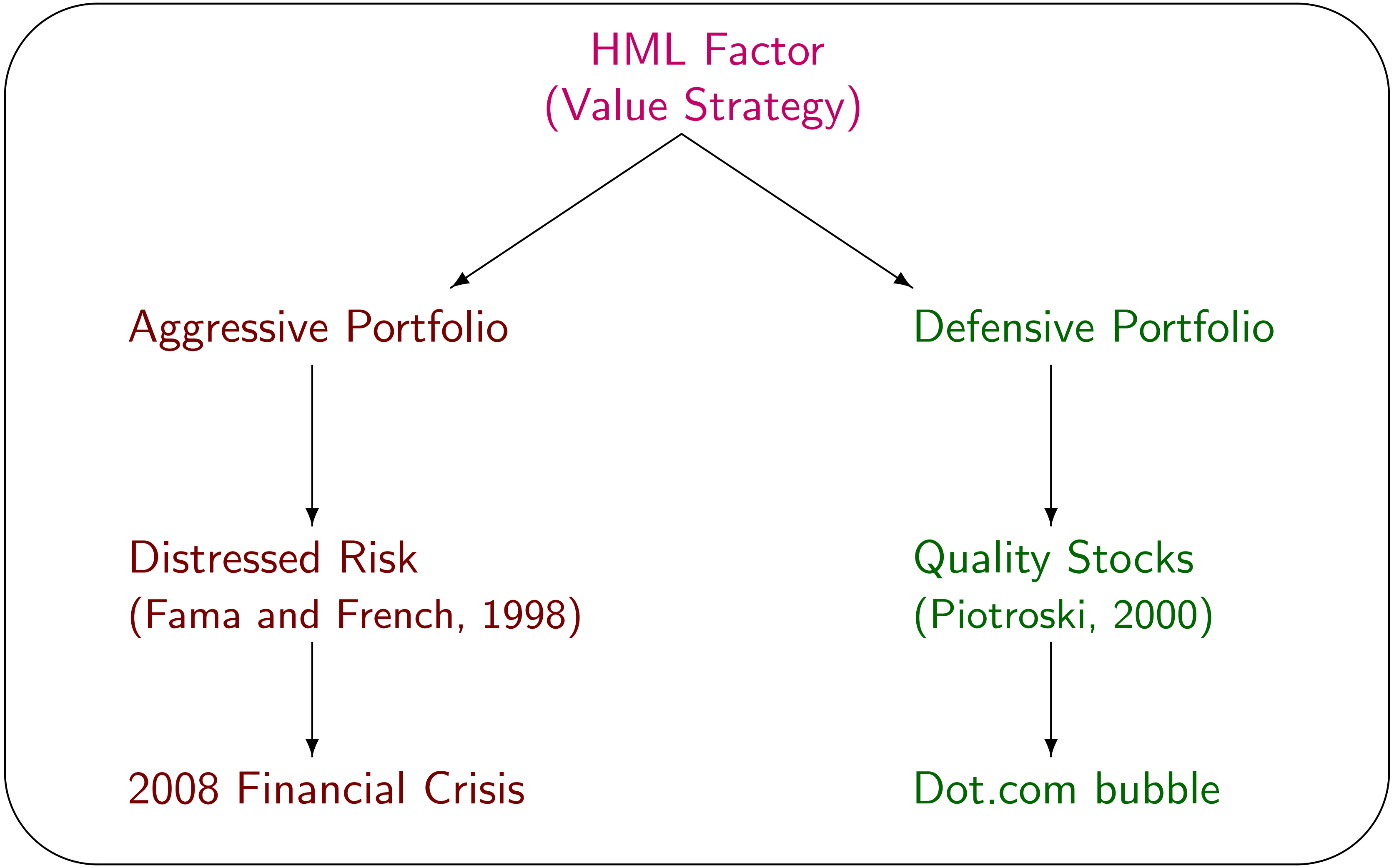
4. **Low-volatility**

   Low-vol (or low-beta) stocks $\neq$ High-vol (or high-beta stocks)

5. **Quality**

   Quality stocks $\neq$ Non-quality stocks (including junk stocks)

The example of the value risk factor



The example of the dividend yield risk factor

- Book-to-price (value risk factor):

$$\mathrm{B2P} = \frac{B}{P}$$

- Dividend yield (carry risk factor):

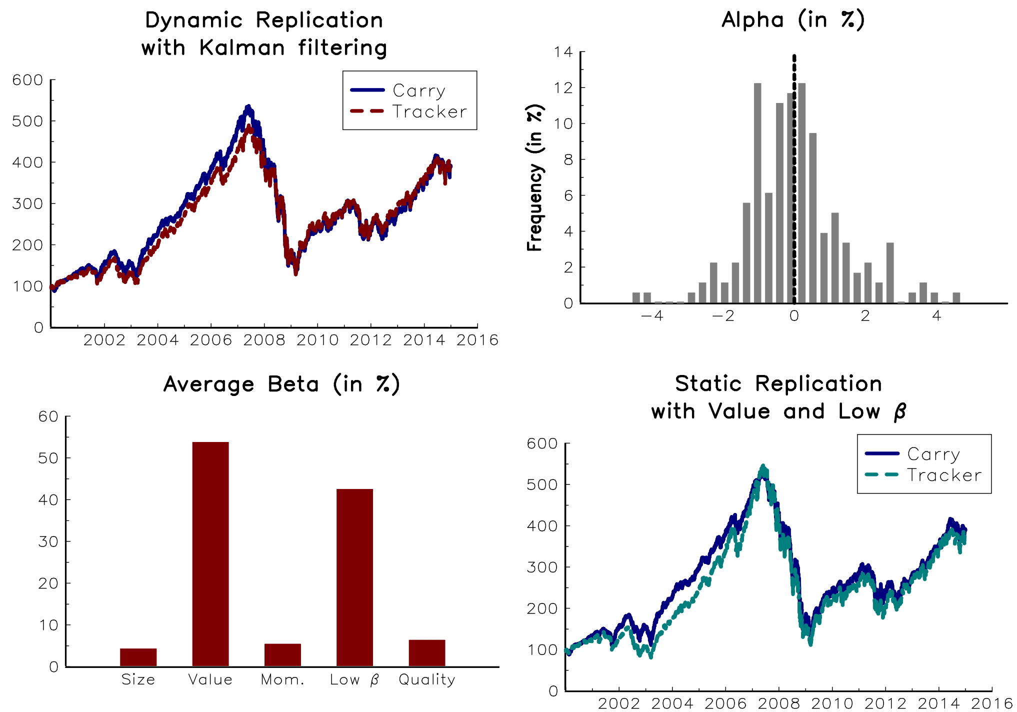
$$\begin{aligned} \mathrm{DY} &= \frac{D}{P} \\ &= \frac{D}{B} \times \frac{B}{P} \\ &= \mathrm{D2B} \times \mathrm{B2P} \end{aligned}$$

- Value component (book and dividend = low-frequency, price = high-frequency)
- Low-volatility component (bond-like stocks)

Risk factors are not orthogonal, they are correlated

The example of the dividend yield risk factor

Figure 51: Value, low beta and carry are not orthogonal risk factors



Source: Richard and Roncalli (2015)

$$\text{Carry} \simeq 60\% \text{ Value} + 40\% \text{ Low-volatility}$$

The example of the dividend yield risk factor

- Why Size + Value + Momentum + Low-volatility + Quality?
- Why not Size + Carry + Momentum + Low-volatility + Quality or Size + Carry + Momentum + Value + Quality?
- Because:

$$\begin{aligned}
\text{Carry} &\simeq 60\%\ \text{Value} + 40\%\ \text{Low-volatility} \\
\text{Value} &\simeq 167\%\ \text{Carry} - 67\%\ \text{Low-volatility} \\
\text{Low-volatility} &\simeq 250\%\ \text{Carry} - 150\%\ \text{Value}
\end{aligned}$$

Question

Why Value + Momentum + Low-volatility + Quality
and not
Size + Value + Momentum + Low-volatility + Quality?

General approach

- We consider a universe $\mathcal{U}$ of stocks (e.g. the MSCI World Index)
- We define a rebalancing period (e.g. every month, every quarter or every year)
- At each rebalancing date t_τ:
  - We define a score $\mathbb{S}_i(t_\tau)$ for each stock i
  - Stocks with high scores are selected to form the long exposure $\mathcal{L}(t_\tau)$ of the risk factor
  - Stocks with low scores are selected to form the short exposure $\mathcal{S}(t_\tau)$ of the risk factor
- We specify a weighting scheme $w_i(t_\tau)$ (e.g. value weighted or equally weighted)

General approach

- The performance of the risk factor between two rebalancing dates corresponds to the performance of the long/short portfolio:

$$\mathcal{F}(t) = \mathcal{F}(t_\tau) \cdot \left(\sum_{i \in \mathcal{L}(t_\tau)} w_i(t_\tau)(1 + R_i(t)) - \sum_{i \in \mathcal{S}(t_\tau)} w_i(t_\tau)(1 + R_i(t)) \right)$$

where $t \in \left]t_\tau, t_{\tau+1}\right]$ and $\mathcal{F}(t_0) = 100$.

- In the case of a long-only risk factor, we only consider the long portfolio:

$$\mathcal{F}(t) = \mathcal{F}(t_\tau) \cdot \left(\sum_{i \in \mathcal{L}(t_\tau)} w_i(t_\tau)(1 + R_i(t)) \right)$$

The Fama-French approach

The SMB and HML factors are defined as follows:

$$\mathrm{SMB}_t = \frac{1}{3}\left(R_t\left(\mathrm{SV}\right) + R_t\left(\mathrm{SN}\right) + R_t\left(\mathrm{SG}\right)\right) - \frac{1}{3}\left(R_t\left(\mathrm{BV}\right) + R_t\left(\mathrm{BN}\right) + R_t\left(\mathrm{BG}\right)\right)$$

and:

$$\mathrm{HML}_t = \frac{1}{2}\left(R_t\left(\mathrm{SV}\right) + R_t\left(\mathrm{BV}\right)\right) - \frac{1}{2}\left(R_t\left(\mathrm{SG}\right) + R_t\left(\mathrm{BG}\right)\right)$$

with the following 6 portfolios[15]:

	Value	Neutral	Growth
Small	SV	SN	SG
Big	BV	BN	BG

[15] We have:

- The scores are the market equity (ME) and the book equity to market equity (BE/ME)
- The size breakpoint is the median market equity (Small = 50% and Big = 50%)
- The value breakpoints are the 30th and 70th percentiles of BE/ME (Value = 30%, Neutral = 40% and Growth = 30%)

The Fama-French approach

Homepage of Kenneth R. French

You can download data at the following webpage:

`https://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html`

- Asia Pacific ex Japan
- Developed
- Developed ex US
- Europe
- Japan
- North American
- US

Quintile portfolios

In this approach, we form five quintile portfolios:

- Q_1 corresponds to the stocks with the highest scores (top 20%)
- Q_2, Q_3 and Q_4 are the second, third and fourth quintile portfolios
- Q_5 corresponds to the stocks with the lowest scores (bottom 20%)

$\Rightarrow$ The long/short risk factor is the performance of $Q_1 - Q_5$, whereas the long-only risk factor is the performance of Q_1

The construction of risk factors

Table 59: An illustrative example

Asset	Score	Rank	Quintile	Selected	L/S	Weight
A_1	1.1	3	Q_2			
A_2	0.5	4	Q_2			
A_3	-1.3	9	Q_5	✓	**Short**	-50%
A_4	1.5	2	Q_1	✓	**Long**	$+50\%$
A_5	-2.8	10	Q_5	✓	**Short**	-50%
A_6	0.3	5	Q_3			
A_7	0.1	6	Q_3			
A_8	2.3	1	Q_1	✓	**Long**	$+50\%$
A_9	-0.7	8	Q_4			
A_{10}	-0.3	7	Q_4			

The scoring system

Variable selection

- Size: market capitalization
- Value: Price to book, price to earnings, price to cash flow, dividend yield, etc.
- Momentum = one-year price return ex 1 month, 13-month price return minus one-month price return, etc.
- Low volatility = one-year rolling volatility, one-year rolling beta, etc.
- Quality: Profitability, leverage, ROE, Debt to Assets, etc.

The scoring system

Variable combination

- Z-score averaging
- Ranking system
- Bottom exclusion
- Etc.

$\Rightarrow$ Finally, we obtain one score for each stock (e.g. the value score, the quality score, etc.)

Single-factor exposure versus multi-factor portfolio

Single-factor

- Trading bet
- Tactical asset allocation (TAA)
- If the investor believe that value stocks will outperform growth stocks in the next six months, he will overweight value stocks or add an exposure on the value risk factor
- Active management

Multi-factor

- Long-term bet
- Strategic asset allocation (SAA)
- The investor believe that a factor investing portfolio allows to better capture the equity risk premium than a CW index
- Factor investing portfolio = diversified portfolio (across risk factors)

Multi-factor portfolio

- Long/short: Market + Size + Value + Momentum + Low-volatility + Quality
- Long-only: Size + Value + Momentum + Low-volatility + Quality (because the market risk factor is replicated by the other risk factors)

Risk factors in sovereign bonds

> *"Market participants have long recognized the importance of identifying the common factors that affect the returns on U.S. government bonds and related securities. To explain the variation in these returns, it is critical to distinguish the systematic risks that have a general impact on the returns of most securities from the specific risks that influence securities individually and hence a negligible effect on a diversified portfolio" (Litterman and Scheinkman, 1991, page 54).*

$\Rightarrow$ The 3-factor model of Litterman and Scheinkman (1991) is based on the PCA analysis:

- the level of the yield curve
- the steepness of the yield curve
- the curvature of the yield curve

Conventional bond model

- Let $B_i(t, D_i)$ be the zero-coupon bond price with maturity D_i:

$$B_i(t, D_i) = e^{-(R(t)+S_i(t))\,D_i}$$

where $R(t)$ is the risk-free interest rate and $S_i(t)$ is the credit spread

- L-CAPM of Acharya and Pedersen (2005):

$$R_i(t) = \underbrace{\underbrace{(R(t) + S_i(t))\;D_i}_{\text{Gross return}} - L_i(t)}_{\text{Net return}}$$

where $L_i(t)$ is the illiquidity cost of Bond i

Conventional bond model

We deduce that:

$$B_i(t, D_i) = e^{-((R(t)+S_i(t))\,D_i - L_i(t))}$$

and:

$$\begin{aligned} \mathrm{d}\ln B_i(t, D_i) &= -D_i\,\mathrm{d}R(t) - D_i\,\mathrm{d}S_i(t) + \mathrm{d}L_i(t) \\ &= -D_i\,\mathrm{d}R(t) - \mathrm{DTS}_i(t)\,\frac{\mathrm{d}S_i(t)}{S_i(t)} + \mathrm{d}L_i(t) \end{aligned}$$

where $\mathrm{DTS}_i(t) = D_i\,S_{i,t}$ is the duration-time-spread factor

Conventional bond model

Liquidity premia (Acharya and Pedersen, 2005)

The illiquidity premium $\mathrm{d}L_{i,t}$ can be decomposed into an illiquidity level component $\mathbb{E}\left[L_{i,t}\right]$ and three illiquidity covariance risks:

1. $\beta\left(L_i, L_M\right)$
   An asset that becomes illiquid when the market becomes illiquid should have a higher risk premium.
2. $\beta\left(R_i, L_M\right)$
   An asset that perform well in times of market illiquidity should have a lower risk premium.
3. $\beta\left(L_i, R_M\right)$
   Investors accept a lower risk premium on assets that are liquid in a bear market.

Conventional bond model

By assuming that:

$$\mathrm{d}L_{i,t} = \alpha_i(t) + \beta(L_i, L_M)\,\mathrm{d}L_M(t)$$

where α_i is the liquidity return that is not explained by the liquidity commonality, we obtain:

$$R_i(t) = \alpha_i(t) - D_i\,\mathrm{d}R(t) - \mathrm{DTS}_i(t)\,\frac{\mathrm{d}S_i(t)}{S_i(t)} + \beta(L_i, L_M)\,\mathrm{d}L_M(t)$$

or:

$$R_i(t) = a(t) - D_i\,\mathrm{d}R(t) - \mathrm{DTS}_i(t)\,\frac{\mathrm{d}S_i(t)}{S_i(t)} + \beta(L_i, L_M)\,\mathrm{d}L_M(t) + u_i(t)$$

Risk factors in corporate bonds

Conventional bond model (or the 'equivalent' CAPM for bonds)

The total return $R_i(t)$ of Bond i at time t is equal to:

$$R_i(t) = a(t) - \mathrm{MD}_i(t)\, R^I(t) - \mathrm{DTS}_i(t)\, R^S(t) + \mathrm{LTP}_i(t)\, R^L(t) + u_i(t)$$

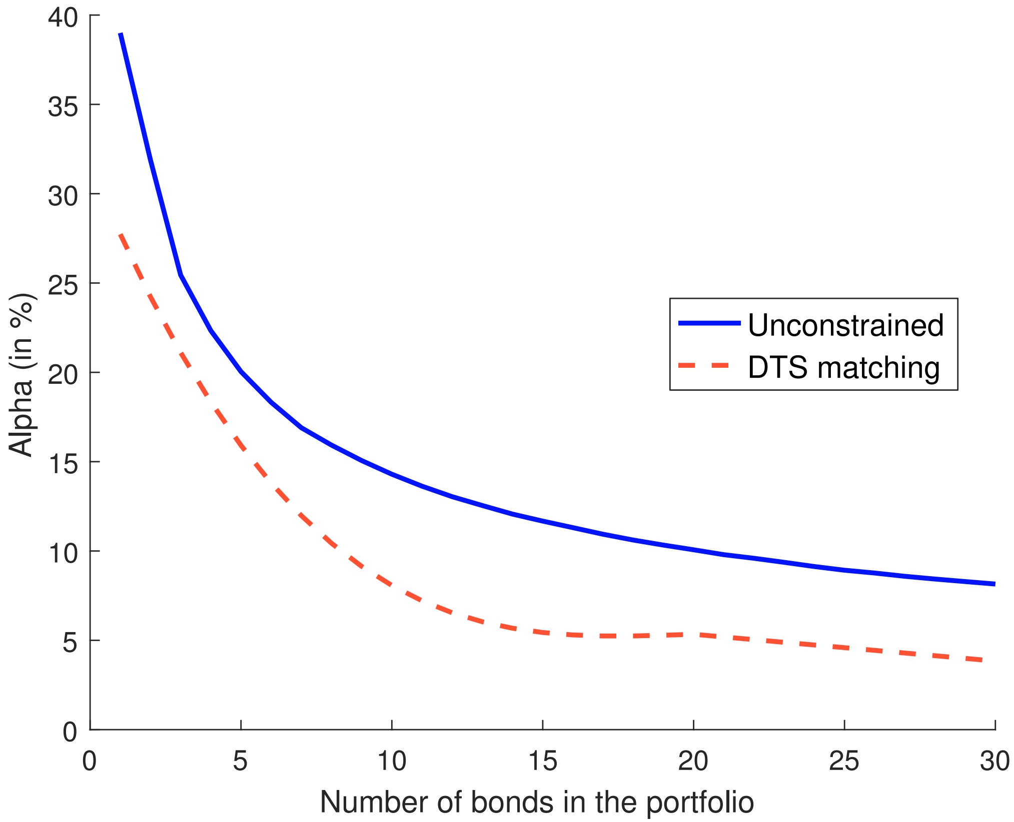
where:

- $a(t)$ is the constant/carry/zero intercept
- $\mathrm{MD}_i(t)$ is the modified duration
- $\mathrm{DTS}_i(t)$ is the duration-times-spread
- $\mathrm{LTP}_i(t)$ is the liquidity-time-price
- $u_i(t)$ is the residual

$\Rightarrow$ $R^I(t)$, $R^S(t)$ and $R^L(t)$ are the return components due to interest rate movements, credit spread variation and liquidity dynamics.

Risk factors in corporate bonds

Figure 52: Conventional alpha decreases with the number of holding assets



EURO IG corporate bonds, 2000-2015
Source: Amundi Research (2017)

- There is less traditional alpha in the bond market than in the stock market

Risk factors in corporate bonds

Since 2015

- Houweling and van Zundert (2017) — HZ
- Bektic, Neugebauer, Wegener and Wenzler (2017) — BNWW
- Israel, Palhares and Richardson (2017) — IPR
- Bektic, Wenzler, Wegener, Schiereck and Spielmann (2019) — BWWSS
- Ben Slimane, De Jong, Dumas, Fredj, Sekine and Srb (2019) — BDDFSS
- Etc.

Risk factors in corporate bonds

Study	HZ	BWWSS	IPR	BNWW
Period	1994-2015	1996-2016 (US) 2000-2016 (EU)	1997-2015	1999-2016
Universe	Bloomberg Barclays US IG & HY	BAML US IG & HY, EU IG	BAML US IG & HY	BAML US IG & HY
Investment		1Y variation in total assets		
Low risk	Short maturity + High rating		Leverage $\times$ Duration $\times$ Profitability	1Y equity beta
Momentum	6M bond return		6M bond return + 6M stock return	1Y stock return
Profitability		Earnings-to-book		
Size	Market value of issuer	Market capitalization		Market capitalization
Value	Comparing OAS to Maturity $\times$ Rating $\times$ 3M OAS variation	Price-to-book	Comparing OAS to Duration $\times$ Rating $\times$ Bond return volatility + Implied default probability	Price-to-book

Risk factors in currency markets

- What are the main risk factors for explaining the cross-section of currency returns?
  1. Momentum (cross-section or time-series)
  2. Carry
  3. Value (short-term, medium-term or long-term)
- The dynamics of some currencies are mainly explained by:
  - Common risk factors (e.g. NZD or CAD)
  - Idiosyncratic risk factors (e.g. IDR or PEN)
- Carry-oriented currency? (e.g. JPY $\neq$ CHF)

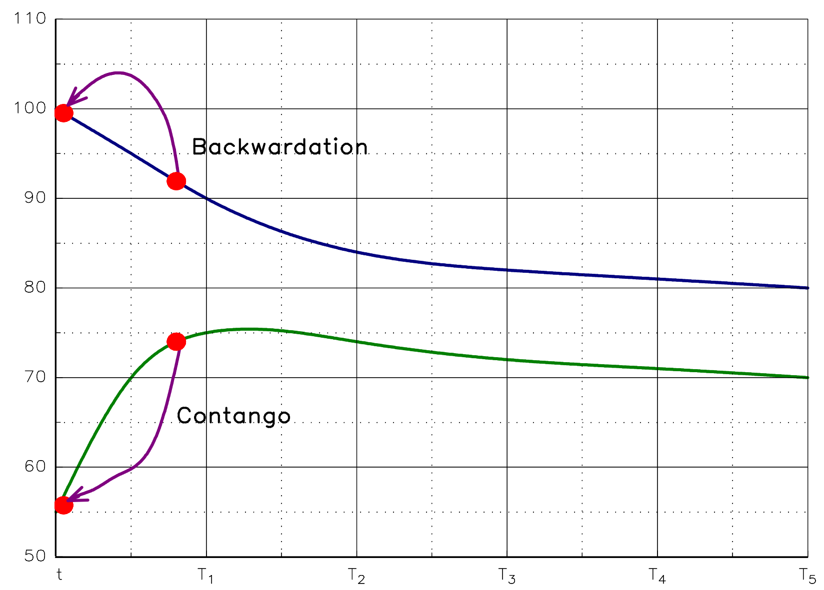
Risk factors in commodities

- Two universal strategies:
  - Contango/backwardation strategy
  - Trend-following strategy

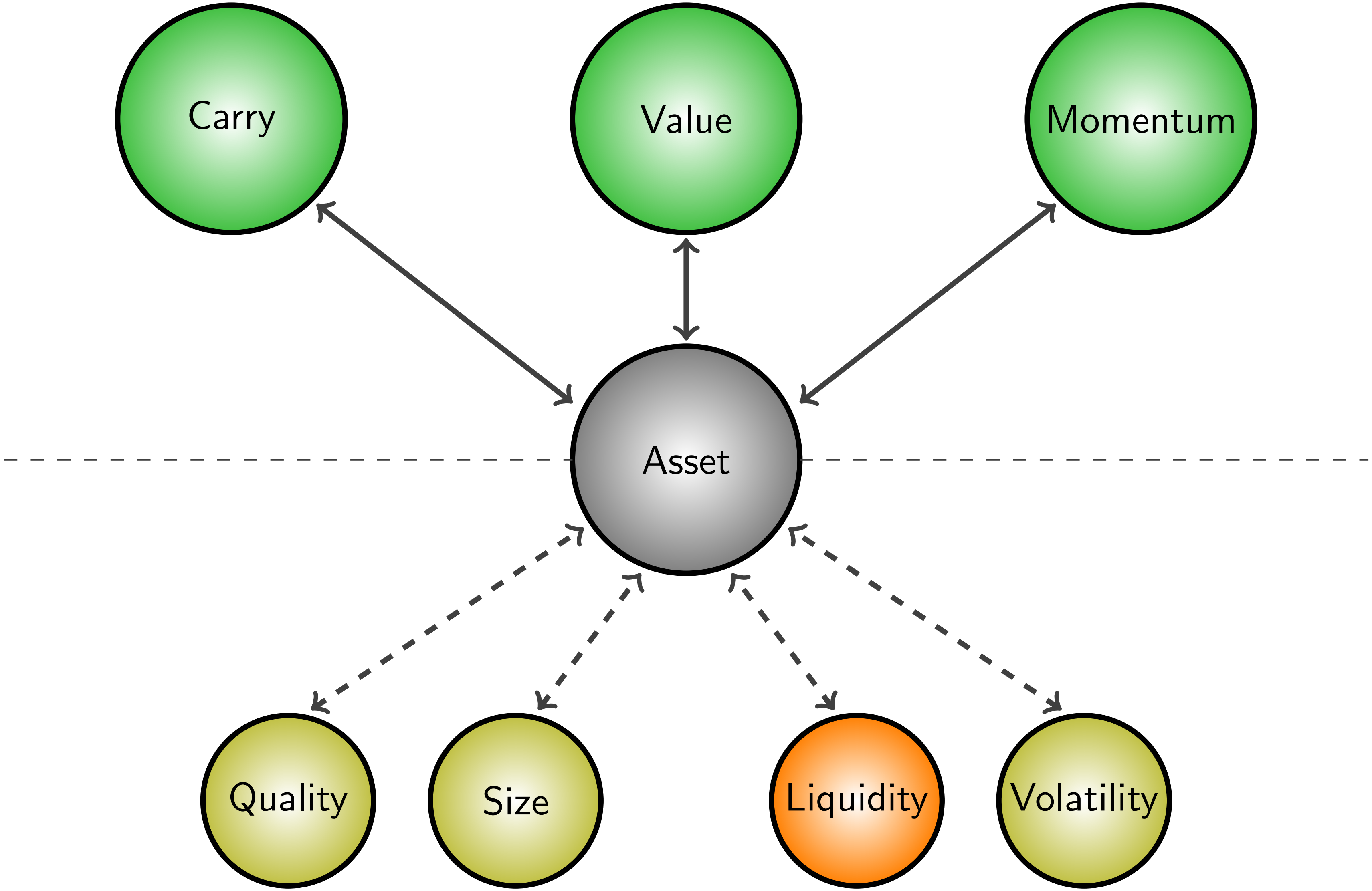
- **CTA = Commodity Trading Advisor**

- Only two risk factors?
  - Carry
  - Momentum

Figure 53: Contango and backwardation movements in commodity futures contracts



Factor analysis of an asset



Factor analysis of an asset

Carry

- Yield
- Income generation
- Risk arbitrage

Value

- Fair price
- Overvalued / undervalued
- Fundamental

Momentum

- Price dynamics
- Trend-following
- Mean-reverting / Reversal

Liquidity

- Tradability property (transaction cost, execution time, scarcity)
- Supply/demande imbalance
- Bad times $\neq$ good times

The concept of alternative risk premia

There are many definitions of ARP:

- ARP $\approx$ factor investing (FI)
  (ARP = long/short portfolios, FI = long portfolios)
- ARP $\approx$ all the other risk premia (RP) than the equity and bond risk premia
- ARP $\approx$ quantitative investment strategies (QIS)

Sell-side

- CIBs & brokers
- ARP = QIS

Buy-side

- Asset managers & asset owners
- ARP = FI (for asset managers)
- ARP = RP (for asset owners)

The concept of alternative risk premia

Alternative Risk Premia

Alternative (or real) assets

- Private equity
- Private debt
- Real estate
- Infrastructure

Traditional financial assets

- Long/short risk factors in equities, rates, credit, currencies & commodities
- Risk premium strategy (e.g. carry, momentum, value, etc.)

The concept of alternative risk premia

- A risk premium is the expected excess return by the investor in order to accept the risk $\Rightarrow$ any (risky) investment strategy has a risk premium!
- Generally, the term "*risk premium*" is associated to asset classes:
  - The equity risk premium
  - The risk premium of high yield bonds
- This means that a risk premium is the expected excess return by the investor in order to accept a future economic risk that cannot be diversifiable
  - For instance, the risk premium of a security does not integrate its specific risk

The concept of alternative risk premia

- What is the relationship between a risk factor and a risk premium?
  - A rewarded risk factor may correspond a to risk premium, while a non-rewarded risk factor is not a risk premium
  - A risk premium can be a risk factor if it helps to explain the cross-section of expected returns
  - The case of cat bonds:

Risk premium	✔
Risk factor	✘

Risk premia & non-diversifiable risk

Consumption-based model (Lucas, 1978; Cochrane, 2001)

A risk premium is a compensation for accepting (systematic) risk in **bad times**.

We have:

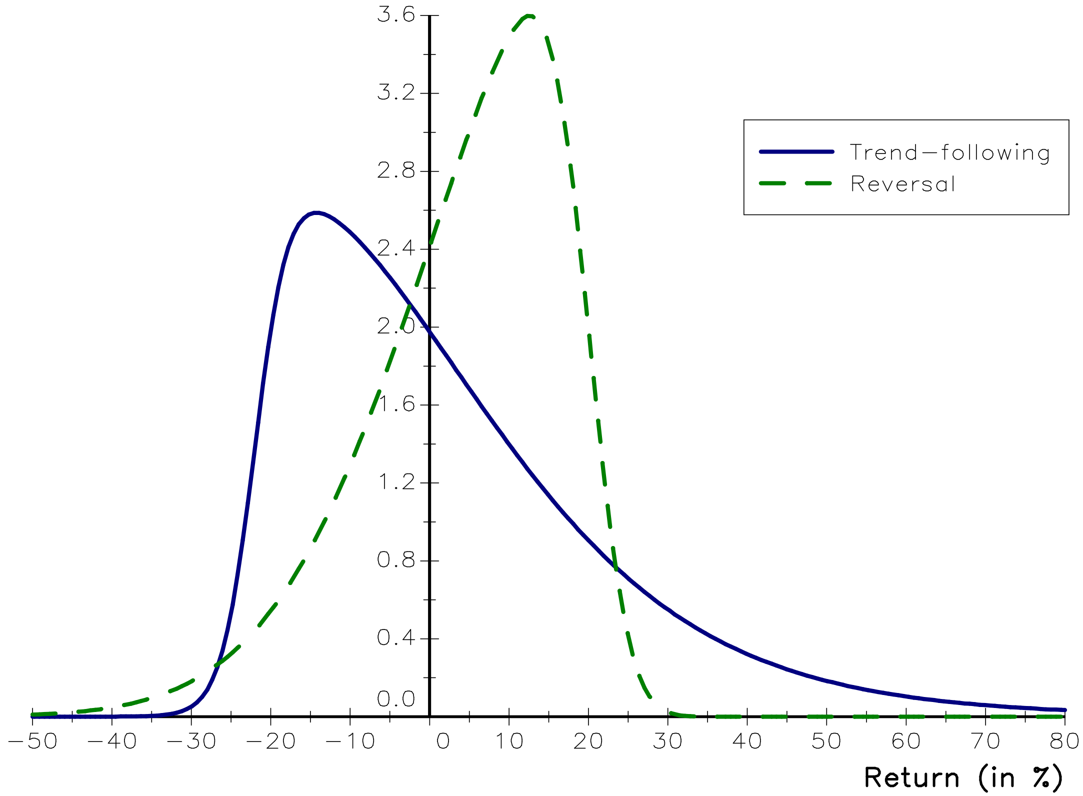
$$\underbrace{\mathbb{E}_t\left[R_{t+1} - R_{f,t}\right]}_{\text{Risk premium}} \propto -\underbrace{\rho\left(u'\left(C_{t+1}\right), R_{t+1}\right)}_{\text{Correlation term}} \times \underbrace{\sigma\left(u'\left(C_{t+1}\right)\right)}_{\text{Smoothing term}} \times \underbrace{\sigma\left(R_{t+1}\right)}_{\text{Volatility term}}$$

where R_{t+1} is the one-period return of the asset, $R_{f,t}$ is the risk-free rate, C_{t+1} is the future consumption and $u\left(C\right)$ is the utility function.

Main results

- Hedging assets help to smooth the consumption $\Rightarrow$ low or negative risk premium
- In bad times, risk premium strategies are correlated and have a negative performance ($\neq$ all-weather strategies)

Risk premia & bad times



The market must reward contrarian and value investors, not momentum investors

Behavioral finance and limits to arbitrage

Bounded rationality

Barberis and Thaler (2003), A Survey of Behavioral Finance.

Decisions of the other economic agents

$\Downarrow$

Feedback effects on our decisions!

Killing Homo Economicus

[...] "conventional economics assumes that people are highly rational, super rational and unemotional. They can calculate like a computer and have no self-control problems" (Richard Thaler, 2009).

"The people I study are humans that are closer to Homer Simpson" (Richard Thaler, 2017).

Behavioral finance and social preferences

- For example, momentum may be a rational behavior if the investor is not informed and his objective is to minimize the loss with respect to the 'average' investor.
- Absolute loss $\neq$ relative loss
- Loss aversion and performance asymmetry
- Imitations between institutional investors $\Rightarrow$ benchmarking
- Home bias

What does the theory become if utility maximization includes the performance of other economic agents?

$\Rightarrow$ **The crowning glory of tracking error and relative performance!**

Behavioral finance and market anomalies

Previously

Positive expected excess returns are explained by:

- risk premia

Today

Positive expected excess returns are explained by:

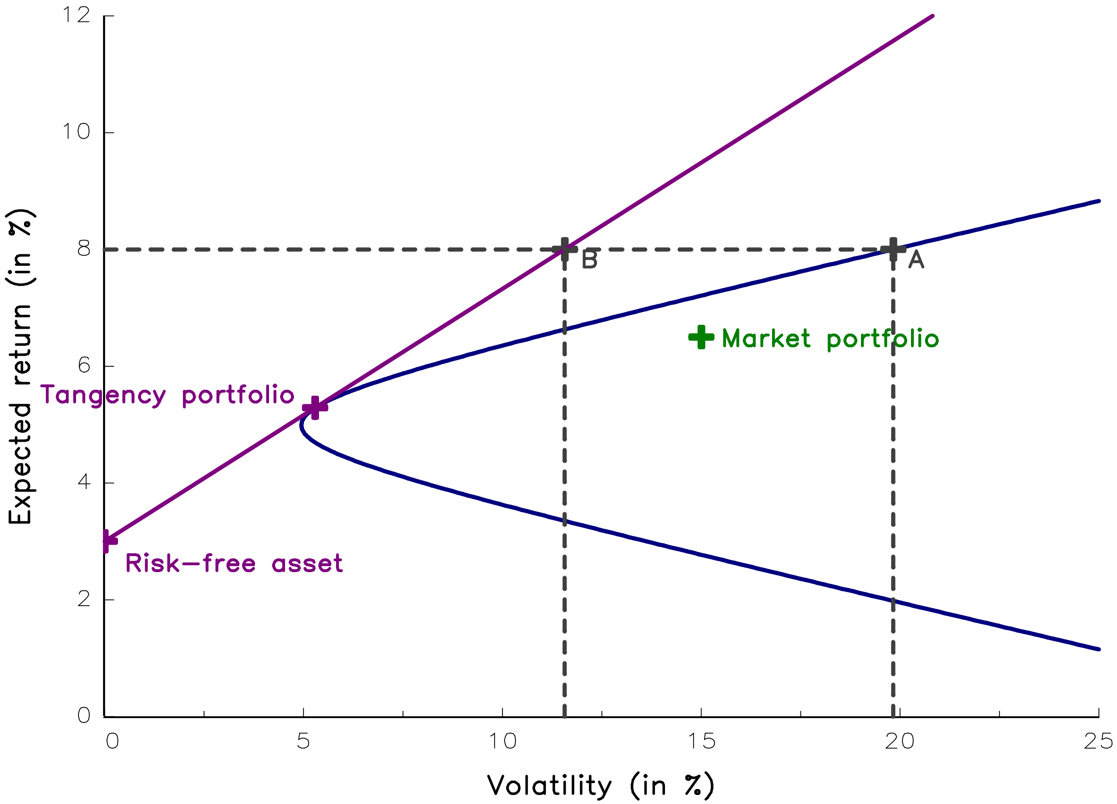
- risk premia
- or market anomalies

Market anomalies correspond to trading strategies that have delivered good performance in the past, but their performance cannot be explained by the existence of a systematic risk (in bad times). Their performance can only be explained by behavioral theories.

$\Rightarrow$ Momentum, low risk and quality risk factors are three market anomalies

The case of low risk assets

Figure 54: What is the impact of borrowing constraints on the market portfolio?



- The investor that targets a 8% expected return must choose Portfolio B
- The demand for high beta assets is higher than this predicted by CAPM
- This effect is called the low beta anomaly

Low risk assets have a higher Sharpe ratio than high risk assets

Skewness risk premia & market anomalies

Characterization of alternative risk premia

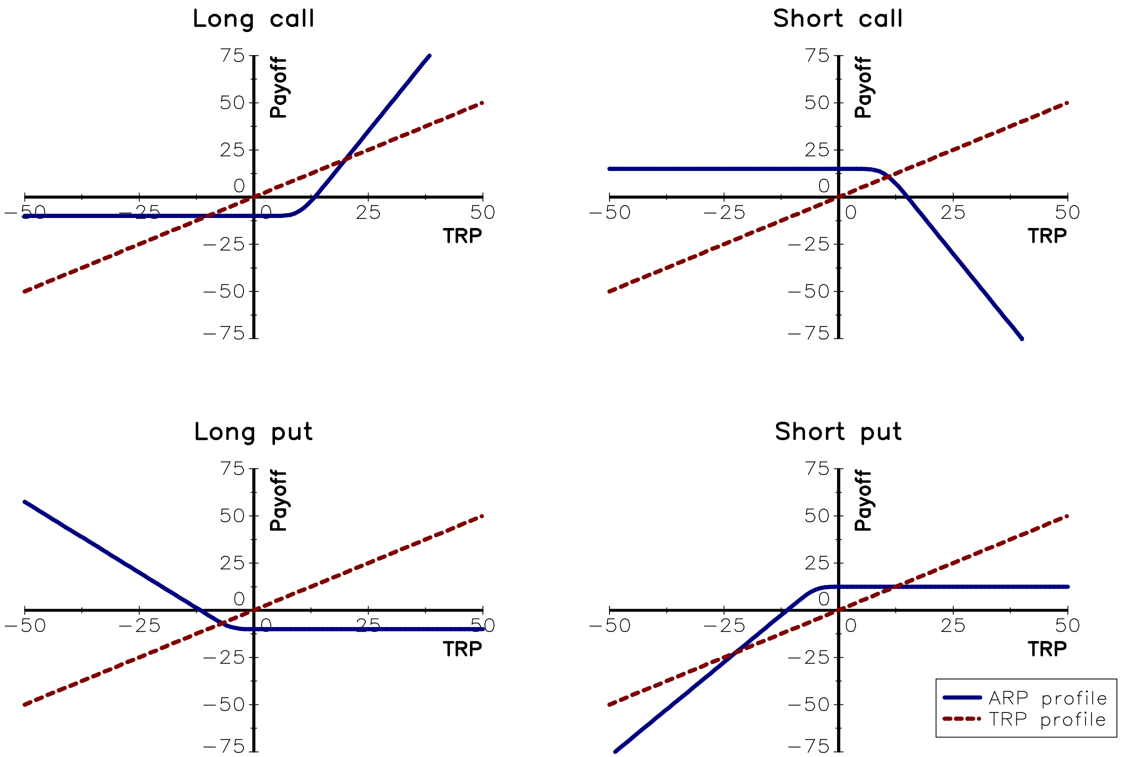
- An alternative risk premium (ARP) is a risk premium, which is not traditional
  - Traditional risk premia (TRP): equities, sovereign/corporate bonds
  - Currencies and some commodities are not TRP
- The drawdown of an ARP must be positively correlated to bad times
  - Risk premia $\neq$ insurance against bad times
  - (SMB, HML) $\neq$ WML
- Risk premia are an increasing function of the volatility and a decreasing function of the skewness

In the market practice, alternative risk premia recover:

1. Skewness risk premia (or pure risk premia), which present high negative skewness and potential large drawdown
2. Markets anomalies

Payoff function of alternative risk premia

Figure 55: Which option profile may be considered as a skewness risk premium?



- ~~Long call~~ (risk adverse)
- ~~Short call~~ (market anomaly)
- ~~Long put~~ (insurance)
- Short put

⇒ SMB, HML, ~~WML~~, ~~BAB~~, ~~QMJ~~

A myriad of alternative risk premia?

Figure 56: Mapping of risk premia strategies (based on existing products)

Strategy	Equities	Rates	Credit	Currencies	Commodities
Carry	Dividend futures High dividend yield	Forward rate bias Term structure slope Cross-term-structure	Forward rate bias	Forward rate bias	Forward rate bias Term structure slope Cross-term-structure
Event	Buyback Merger arbitrage				
Growth	Growth				
Liquidity	Amihud liquidity	Turn-of-the-month	Turn-of-the-month		Turn-of-the-month
Low beta	Low beta Low volatility				
Momentum	Cross-section Time-series	Cross-section Time-series	Time-series	Cross-section Time-series	Cross-section Time-series
Quality	Quality				
Reversal	Time-series Variance	Time-series		Time-series	Time-series
Size	Size				
Value	Value	Value	Value	PPP REER, BEER, FEER NATREX	Value
Volatility	Carry Term structure	Carry		Carry	Carry

Source: Roncalli (2017)

The carry risk premium

Underlying idea

Definition

- The investor takes an investment risk
- This investment risk is rewarded by a high and known yield
- Financial theory predicts a negative mark-to-market return that may reduce or write off the performance
- The investor hopes that the impact of the mark-to-market will be lower than the predicted value

$\Rightarrow$ Carry strategies are highly related to the concept of risk arbitrage[16]

- The carry risk premium is extensively studied by Koijen *et al.* (2018)
- The carry risk premium has a short put option profile

[16] An example is the carry strategy between pure money market instruments and commercial papers = not the same credit risk, not the same maturity risk, but the investor believes that the default will never occur!

The carry risk premium

Not one but several carry strategies

- Equity
  - Carry on dividend futures
  - Carry on stocks with high dividend yields (HDY)
- Rates (sovereign bonds)
  - Carry on the yield curve (term structure & roll-down)
- Credit (corporate bons)
  - Carry on bonds with high spreads
  - High yield strategy
- Currencies
  - Carry on interest rate differentials (uncovered interest rate parity)
- Commodities
  - Carry on contango & backwardation movements
- Volatility
  - Carry on option implied volatilities
  - Short volatility strategy

$\Rightarrow$ Many implementation methods: security-slope, cross-asset, long/short, long-only, basis arbitrage, etc.

The carry risk premium

Analytical model

- Let X_t be the capital allocated at time t to finance a futures position (or an unfunded forward exposure) on asset S_t
- By assuming that the futures price expires at the future spot price ($F_{t+1} = S_{t+1}$), Koijen *et al.* (2018) showed that:

$$\begin{aligned} R_{t+1}(X_t) - R_f &= \frac{F_{t+1} - F_t}{X_t} \\ &= \frac{S_{t+1} - F_t}{X_t} \\ &= \frac{S_t - F_t}{X_t} + \frac{\mathbb{E}_t[S_{t+1}] - S_t}{X_t} + \frac{S_{t+1} - \mathbb{E}_t[S_{t+1}]}{X_t} \end{aligned}$$

The carry risk premium

Analytical model

- At time $t+1$, the excess return of this investment is then equal to:

$$R_{t+1}(X_t) - R_f = \mathcal{C}_t + \frac{\mathbb{E}_t[\Delta S_{t+1}]}{X_t} + \varepsilon_{t+1}$$

where $\varepsilon_{t+1} = (S_{t+1} - \mathbb{E}_t[S_{t+1}])/X_t$ is the unexpected price change and $\mathcal{C}_t$ is the carry:

$$\mathcal{C}_t = \frac{S_t - F_t}{X_t}$$

- It follows that the expected excess return is the sum of the carry and the expected price change:

$$\mathbb{E}_t[R_{t+1}(X_t)] - R_f = \mathcal{C}_t + \frac{\mathbb{E}_t[\Delta S_{t+1}]}{X_t}$$

- The nature of these two components is different:
  1. The carry is an ex-ante observable quantity (known value)
  2. The price change depends on the dynamic model of S_t (unknown value)

The carry risk premium

Analytical model

- If we assume that the spot price does not change (no-arbitrage assumption $\mathcal{H}$), the expected excess return is equal to the carry:

$$\frac{\mathbb{E}_t\left[\Delta S_{t+1}\right]}{X_t} = -\mathcal{C}_t$$

- The carry investor will prefer Asset i to Asset j if the carry of Asset i is higher:

$$\mathcal{C}_{i,t} \geq \mathcal{C}_{j,t} \Longrightarrow A_i \succ A_j$$

- The carry strategy would then be long on high carry assets and short on low carry assets.

Remark

In the case of a fully-collateralized position $X_t = F_t$, the value of the carry becomes:

$$\mathcal{C}_t = \frac{S_t}{F_t} - 1$$

The carry risk premium

Currency carry (or the carry trade strategy)

- Let S_t, i_t and r_t be the spot exchange rate, the domestic interest rate and the foreign interest rate for the period $[t, t+1]$
- The forward exchange rate F_t is equal to:

$$F_t = \frac{1 + i_t}{1 + r_t} S_t$$

- The carry is approximately equal to the interest rate differential:

$$\mathcal{C}_t = \frac{r_t - i_t}{1 + i_t} \simeq r_t - i_t$$

The carry risk premium

Currency carry (or the carry trade strategy)

- The carry strategy is long on currencies with high interest rates and short on currencies with low interest rates
- We can consider the following carry scoring (or ranking) system:

$$\mathcal{C}_t = r_t$$

Uncovered interest rate parity (UIP)

- An interest rate differential of 10% $\Rightarrow$ currency depreciation of 10% per year
- In 10 years, we must observe a depreciation of 65%!

The carry risk premium

Currency carry (or the carry trade strategy)

ARS	Argentine peso	KRW	Korean won
AUD	Australian dollar	LTL	Lithuanian litas
BGN	Bulgarian lev	LVL	Latvian lats
BHD	Bahraini dinar	MXN	Mexican peso
BRL	Brazilian real	MYR	Malaysian ringgit
CAD	Canadian dollar	NOK	Norwegian krone
CHF	Swiss franc	NZD	New Zealand dollar
CLP	Chilean peso	PEN	Peruvian new sol
CNY/RMB	Chinese yuan (Renminbi)	PHP	Philippine peso
COP	Colombian peso	PLN	Polish zloty
CZK	Czech koruna	RON	new Romanian leu
DKK	Danish krone	RUB	Russian rouble
EUR	Euro	SAR	Saudi riyal
GBP	Pound sterling	SEK	Swedish krona
HKD	Hong Kong dollar	SGD	Singapore dollar
HUF	Hungarian forint	THB	Thai baht
IDR	Indonesian rupiah	TRY	Turkish lira
ILS	Israeli new shekel	TWD	new Taiwan dollar
INR	Indian rupee	USD	US dollar
JPY	Japanese yen	ZAR	South African rand

The carry risk premium

Currency carry (or the carry trade strategy)

Baku *et al.* (2019, 2020) consider the most liquid currencies:

- **G10** AUD, CAD, CHF, EUR, GBP, JPY, NOK, NZD, SEK and USD
- **EM** BRL, CLP, CZK, HUF, IDR, ILS, INR, KRW, MXN, PLN, RUB, SGD, TRY, TWD and ZAR
- **G25** G10 + EM

They build currency risk factors using the following characteristics:

- The portfolio is equally-weighted and rebalanced every month
- The portfolio is notional-neutral (number of long exposures = number of short exposures)
- 3/3 for G10, 4/4 for EM and 7/7 for G25
- The long (resp. short) exposures correspond to the highest (resp. lowest) scores

The carry risk premium

Currency carry (or the carry trade strategy)

- Scoring system: $\mathbb{S}_{i,t} = \mathcal{C}_{i,t} = r_{i,t}$
- The carry strategy is long on currencies with high interest rates and short on currencies with low interest rates

Table 60: Risk/return statistics of the carry risk factor (2000-2018)

	G10	EM	G25
Excess return (in %)	3.75	11.21	7.22
Volatility (in %)	9.35	9.12	8.18
Sharpe ratio	0.40	1.23	0.88
Maximum drawdown (in %)	−31.60	−25.27	−17.89

Source: Baku *et al.* (2019, 2020)

The carry risk premium

Currency carry (or the carry trade strategy)

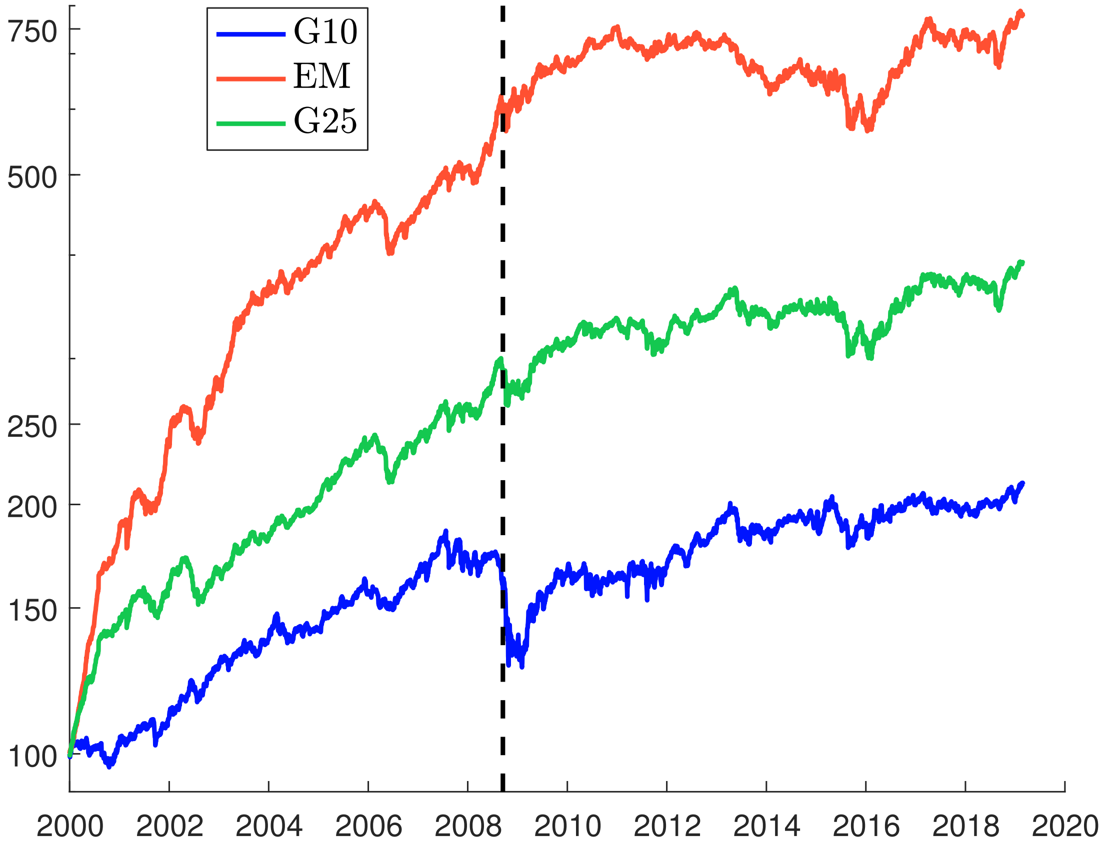


Figure 57: Cumulative performance of the carry risk factor

Source: Baku *et al.* (2019, 2020)

The carry risk premium

Equity carry

- We have:

$$\mathcal{C}_t \simeq \frac{\mathbb{E}_t\left[D_{t+1}\right]}{S_t} - r_t$$

  where $\mathbb{E}_t\left[D_{t+1}\right]$ is the risk-neutral expected dividend for time $t+1$
- If we assume that dividends are constant, the carry is the difference between the dividend yield y_t and the risk-free rate r_t:

$$\mathcal{C}_t = y_t - r_t$$

- The carry strategy is long on stocks with high dividend yields and short on stocks with low dividend yields
- This strategy may be improved by considering forecasts of dividends. In this case, we have:

$$\mathcal{C}_t \simeq \frac{\mathbb{E}_t\left[D_{t+1}\right]}{S_t} - r_t = \frac{D_t + \mathbb{E}_t\left[\Delta D_{t+1}\right]}{S_t} - r_t = y_t + g_t - r_t$$

  where g_t is the expected dividend growth

The carry risk premium

Equity carry

Carry strategy with dividend futures

Another carry strategy concerns dividend futures. The underlying idea is to take a long position on dividend futures where the dividend premium is the highest and a short position on dividend futures where the dividend premium is the lowest. Because dividend futures are on equity indices, the market beta exposure is generally hedged.

Why do we observe a premium on dividend futures?

$\Rightarrow$ Because of the business of structured products and options

The carry risk premium

Bond carry

- The price of a zero-coupon bond with maturity date T is equal to:

$$B_t(T) = e^{-(T-t)R_t(T)}$$

where $R_t(T)$ is the corresponding zero-coupon rate

- Let $F_t(T, m)$ denote the forward interest rate for the period $[T, T+m]$, which is defined as follows:

$$B_t(T+m) = e^{-mF_t(T,m)} B_t(T)$$

We deduce that:

$$F_t(T, m) = -\frac{1}{m} \ln \frac{B_t(T+m)}{B_t(T)}$$

It follows that the instantaneous forward rate is given by this equation:

$$F_t(T) = F_t(T, 0) = \frac{-\partial \ln B_t(T)}{\partial T}$$

The carry risk premium

Bond carry

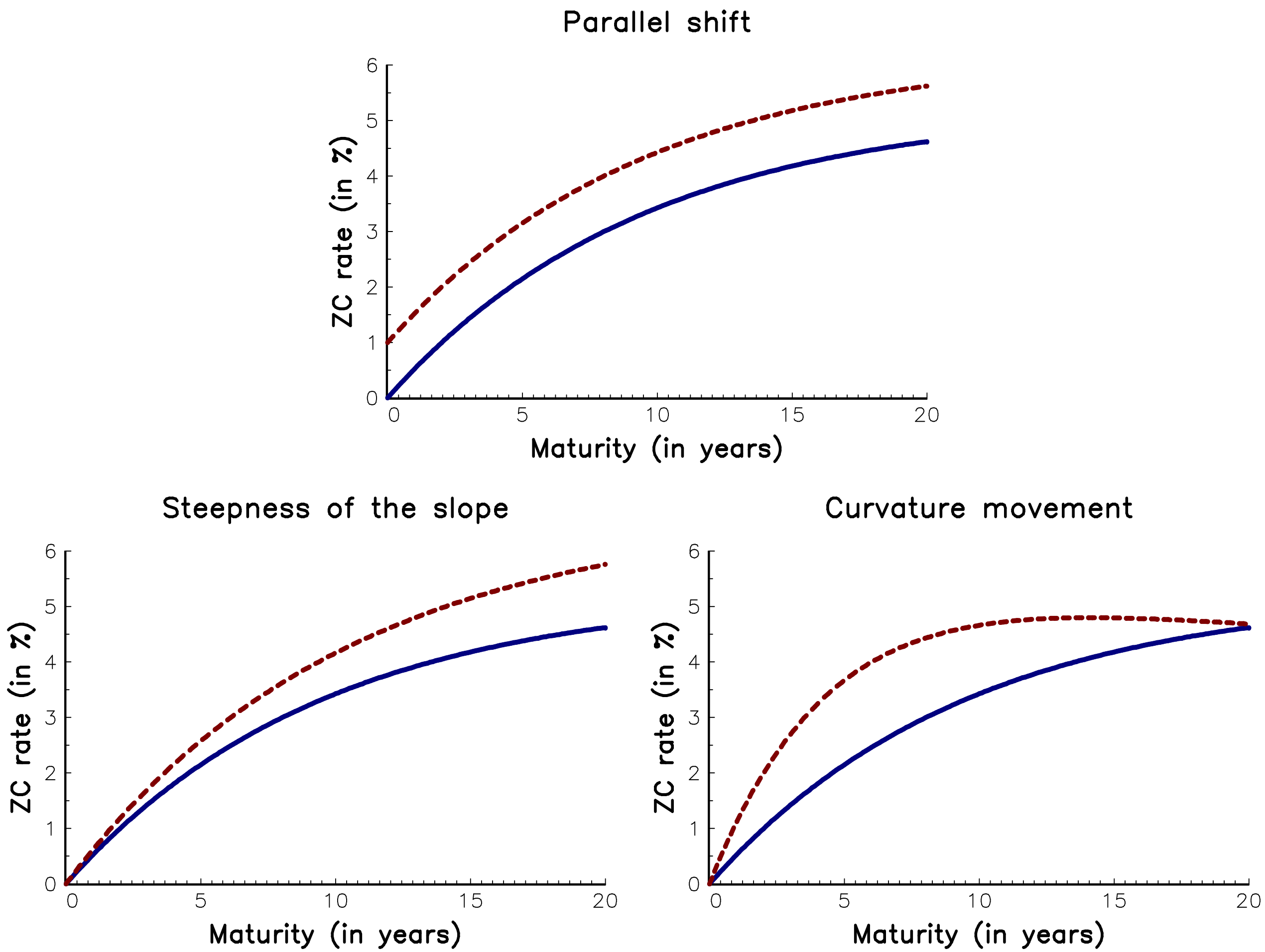


Figure 58: Movements of the yield curve

The carry risk premium

Bond carry

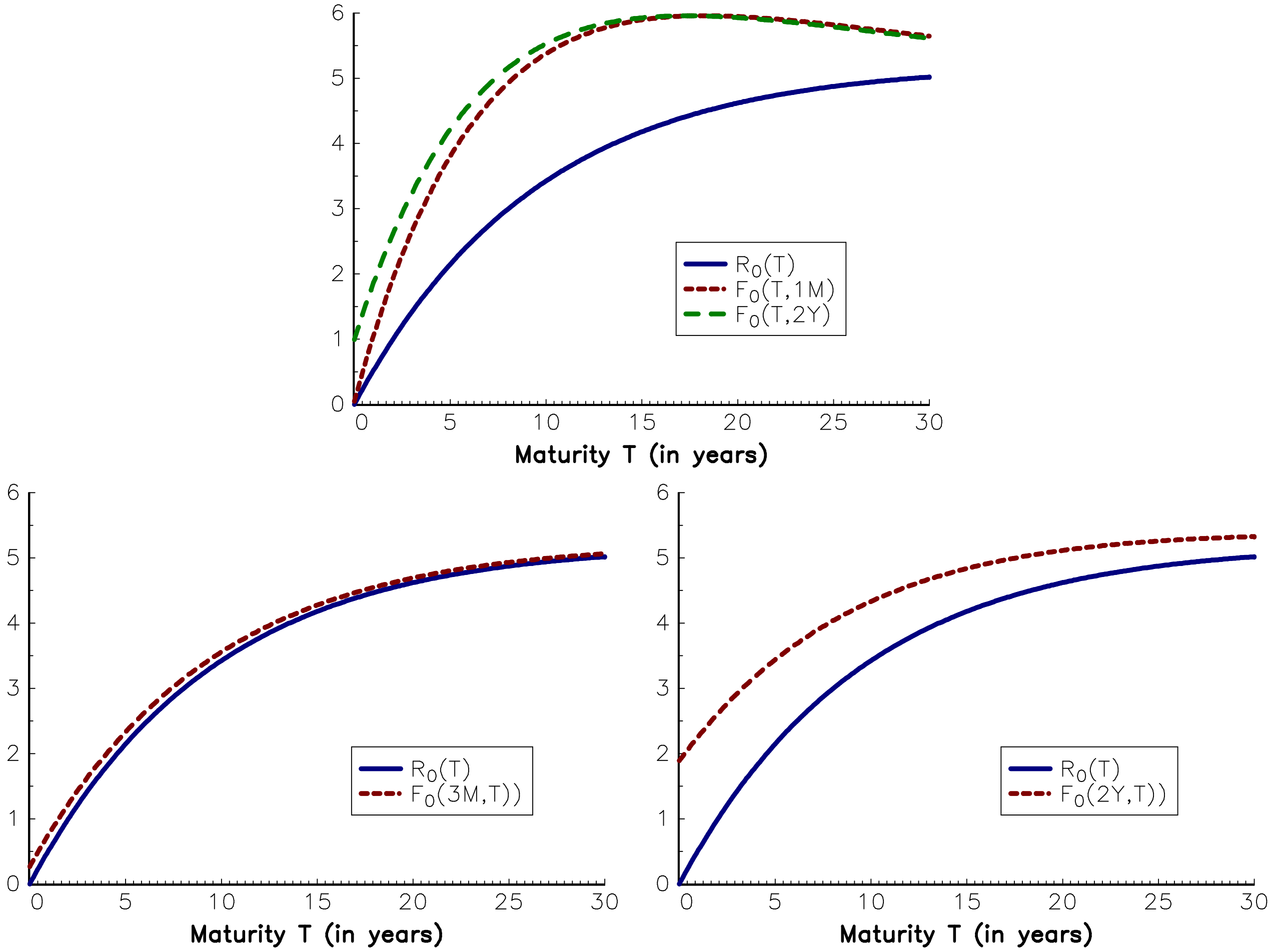


Figure 59: Sport and forward interest rates

The carry risk premium

Bond carry

1. The first carry strategy ("*forward rate bias*") consists in being long the forward contract on the forward rate $F_t(T, m)$ and selling it at time $t + \mathrm{d}t$ with $t + \mathrm{d}t \leq T$
   - Forward rates are generally higher than spot rates
   - Under the hypothesis $(\mathcal{H})$ that the yield curve does not change, rolling forward rate agreements can then capture the term premium and the roll down
   - The carry of this strategy is equal to:

$$\mathcal{C}_t = \underbrace{R_t(T) - r_t}_{\text{term premium}} + \underbrace{\partial_{\bar{T}} \bar{R}_t(\bar{T})}_{\text{roll down}}$$

   where $\bar{R}_t(\bar{T})$ is the zero-coupon rate with a constant time to maturity $\bar{T} = T - t$

The carry risk premium

Bond carry

Implementation

We notice that the difference is higher for long maturities. However, the risk associated with such a strategy is that of a rise in interest rates. This is why this carry strategy is generally implemented by using short-term maturities (less than two years)

The carry risk premium

Bond carry

2. The second carry strategy ("*carry slope*") corresponds to a long position in the bond with maturity T_2 and a short position in the bond with maturity T_1
   - The exposure of the two legs are adjusted in order to obtain a duration-neutral portfolio
   - This strategy is known as the slope carry trade
   - We have:

$$\begin{aligned}\mathcal{C}_t &= \underbrace{\left(R_t\left(T_2\right)-r_t\right)-\frac{D_2\left(T_1\right)}{D_t\left(T_1\right)}\left(R_t\left(T_1\right)-r_t\right)}_{\text{duration neutral slope}}+\\ &\quad \underbrace{\partial_{\bar{T}}\,\bar{R}_t\left(\bar{T}_2\right)-\frac{D_2\left(T_1\right)}{D_t\left(T_1\right)}\partial_{\bar{T}}\,\bar{R}_t\left(\bar{T}_1\right)}_{\text{duration neutral roll down}}\end{aligned}$$

The carry risk premium

Bond carry

Implementation

The classical carry strategy is long 10Y/short 2Y

The carry risk premium

Bond carry

3. The third carry strategy ("*cross-carry slope*") is a variant of the second carry strategy when we consider the yield curves of several countries

Implementation

The portfolio is long on positive or higher slope carry and short on negative or lower slope carry

The carry risk premium

Credit carry

We consider a long position on a corporate bond and a short position on the government bond with the same duration

The carry is equal to:

$$\mathcal{C}_t = \underbrace{\mathcal{S}_t(T)}_{\text{spread}} + \underbrace{\partial_{\bar{T}} \bar{R}^\star_t(\bar{T}) - \partial_{\bar{T}} \bar{R}_t(\bar{T})}_{\text{roll down difference}}$$

where $\mathcal{S}_t(T) = R^\star_t(T) - R_t(T)$ is the credit spread, $R^\star_t(T)$ is the yield-to-maturity of the credit bond and $R^\star_t(T)$ is the yield-to-maturity of the government bond

The carry risk premium

Credit carry

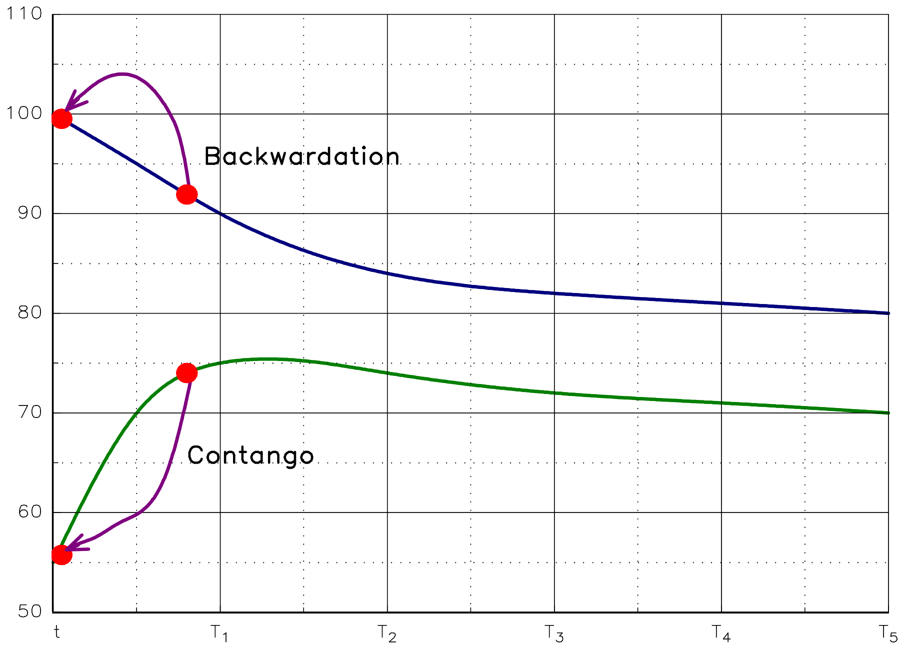
Two implementations

1. The first one is to build a long/short portfolio with corporate bond indices or baskets. The bond universe can be investment grade or high yield. In the case of HY bonds, the short exposure can be an IG bond index
2. The second approach consists in using credit default swaps (CDS). Typically, we sell credit protection on HY credit default indices (e.g. CDX.NA.HY) and buy protection on IG credit default indices (e.g. CDX.NA.IG)

The carry risk premium

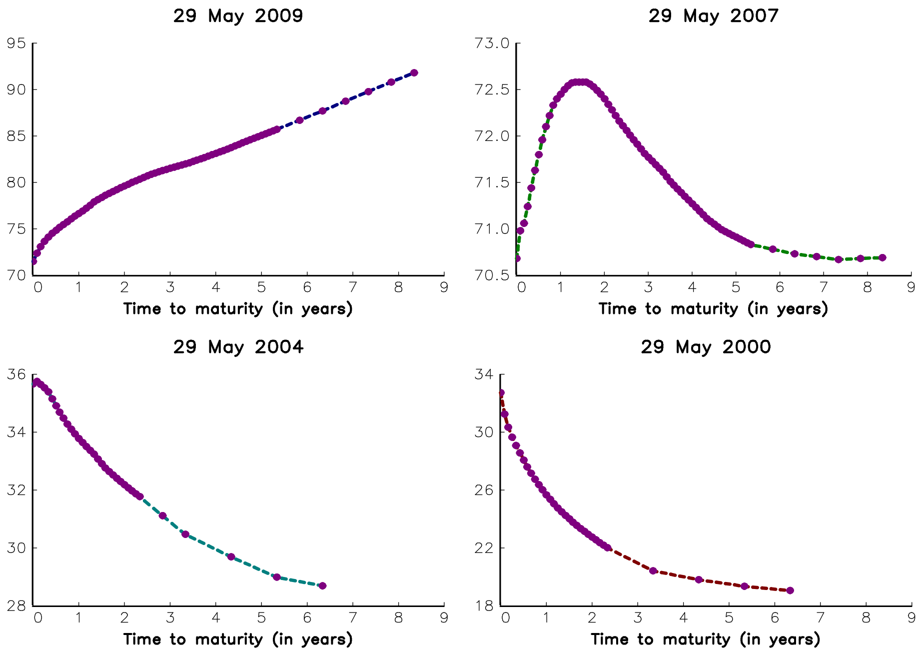
Commodity carry

Figure 60: Contango and backwardation movements in commodity futures contracts



Source: Roncalli (2013)

Figure 61: Term structure of crude oil futures contracts



Source: Roncalli (2013)

The carry risk premium

Volatility carry (or the short volatility strategy)

Volatility carry risk premium

- Long volatility $\Rightarrow$ negative carry ($\neq$ structural exposure)
- Short volatility $\Rightarrow$ positive carry, but the highest skewness risk
- The P&L of selling and delta-hedging an option is equal to:

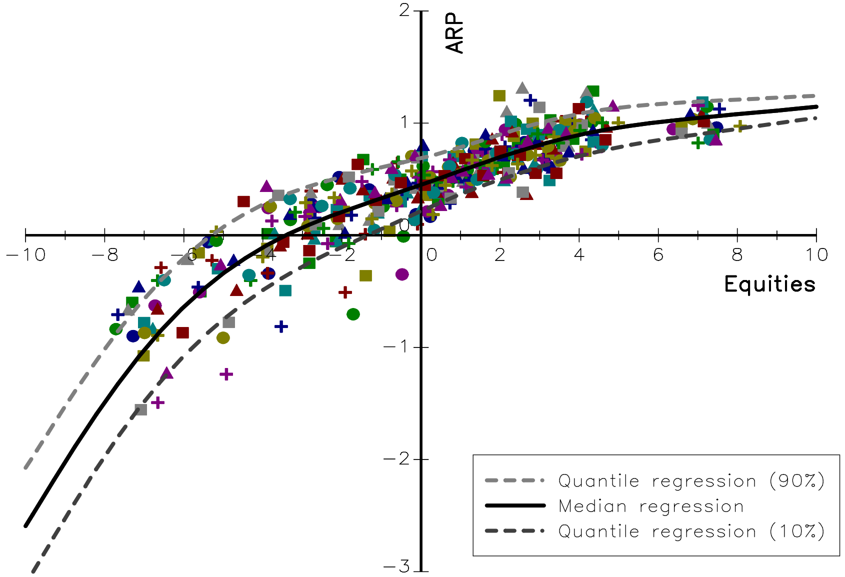
$$\Pi = \frac{1}{2} \int_0^T e^{r(T-t)} S_t^2 \Gamma_t \left(\Sigma_t^2 - \sigma_t^2 \right) \mathrm{d}t$$

  where S_t is the price of the underlying asset, Γ_t is the gamma coefficient, Σ_t is the implied volatility and σ_t is the realized volatility
- $\Sigma_t \geq \sigma_t \Longrightarrow \Pi > 0$
- 3 main reasons:
  1. Asymmetric risk profile between the seller and the buyer
  2. Hedging demand imbalances
  3. Liquidity preferences

The carry risk premium

Volatility carry (or the short volatility strategy)

Figure 62: Non-parametric payoff of the US short volatility strategy



- Income generation
- Short put option profile
- Strategic asset allocation ($\neq$ tactical asset allocation)
- Time horizon is crucial!

It is a skewness risk premium!

Carry strategies exhibit concave payoffs

The value risk premium

Definition

- Let $S_{i,t}$ be the market price of Asset i
- Let $S_i^\star$ be the fundamental price (or the fair value) of Asset i
- The value of Asset i is the relative difference between the two prices:

$$\mathcal{V}_{i,t} = \frac{S_i^\star - S_{i,t}}{S_{i,t}}$$

- The value investor will prefer Asset i to Asset j if the value of Asset i is higher:

$$\mathcal{V}_{i,t} \geq \mathcal{V}_{j,t} \Longrightarrow A_i \succ A_j$$

The value risk premium

The value strategy is an active management bet

- The price of Asset i is undervalued if and only if its value is negative:

$$\mathcal{V}_{i,t} \leq 0 \Leftrightarrow S_i^\star \leq S_{i,t}$$

The value investor should sell securities with negative values

- The price of Asset i is overvalued if and only if its value is positive:

$$\mathcal{V}_{i,t} \geq 0 \Leftrightarrow S_i^\star \geq S_{i,t}$$

The value investor should buy securities with positive values

Remark

While carry is an **objective** measure, value is a **subjective** measure, because the fair value is different from one investor to another (e.g. stock picking = value strategy)

The value risk premium

Computing the fair value

We need a model to estimate the fundamental price $S_i^\star$:

- Stocks: discounted cash flow (DCF) method, fundamental measures (B2P, PE, DY, EBITDA/EV, etc.), machine learning model, etc.
- Sovereign bonds: macroeconomic model, flows model, etc.
- Corporate bonds: Merton model, structural model, econometric model, etc.
- Foreign exchange rates: purchasing power parity (PPP), real effective exchange rate (REER), BEER, FEER, NATREX, etc.
- Commodities: statistical model (5-year average price), etc.

The value risk premium

Equity value

The equity strategy

If we assume that the weight of asset i is proportional to its book-to-price:

$$w_{i,t} \propto \frac{B_{i,t}}{P_{i,t}}$$

We obtain:

$$w_{i,t} = \underbrace{B_{i,t} \bigg/ \sum_{j=1}^{n} B_{j,t}}_{\text{Fundamental component}} \times \underbrace{\sum_{j=1}^{n} P_{j,t} \bigg/ P_{i,t}}_{\text{Reversal component}} \times \underbrace{\text{a cross-effect term}}_{\simeq \text{ constant}}$$

The value risk factor can be decomposed into two main components:

- a fundamental indexation pattern
- a reversal-based pattern

$\Rightarrow$ Reversal strategies $\approx$ value strategies

The value risk premium

Equity value

- In equities, the frequency of the reversal pattern is $\leq$ 1 month or $\geq$ 18 months
- In currencies and commodities, the frequency of the reversal pattern is very short (one or two weeks) or very long ($\geq$ 3 years)

$\Rightarrow$ Value strategy in currencies and commodities?

The value risk premium

The payoff of the equity value risk premium

- We consider two Eurozone Value indices calculated by the same index sponsor
- The index sponsor uses the same stock selection process
- The index sponsor uses two different weighting schemes:
  - The first index considers a capitalization-weighted portfolio
  - The second index considers a minimum variance portfolio

$\Rightarrow$ We recall that the payoff of the low-volatility strategy is long put + short call

The value risk premium

The payoff of the equity value risk premium

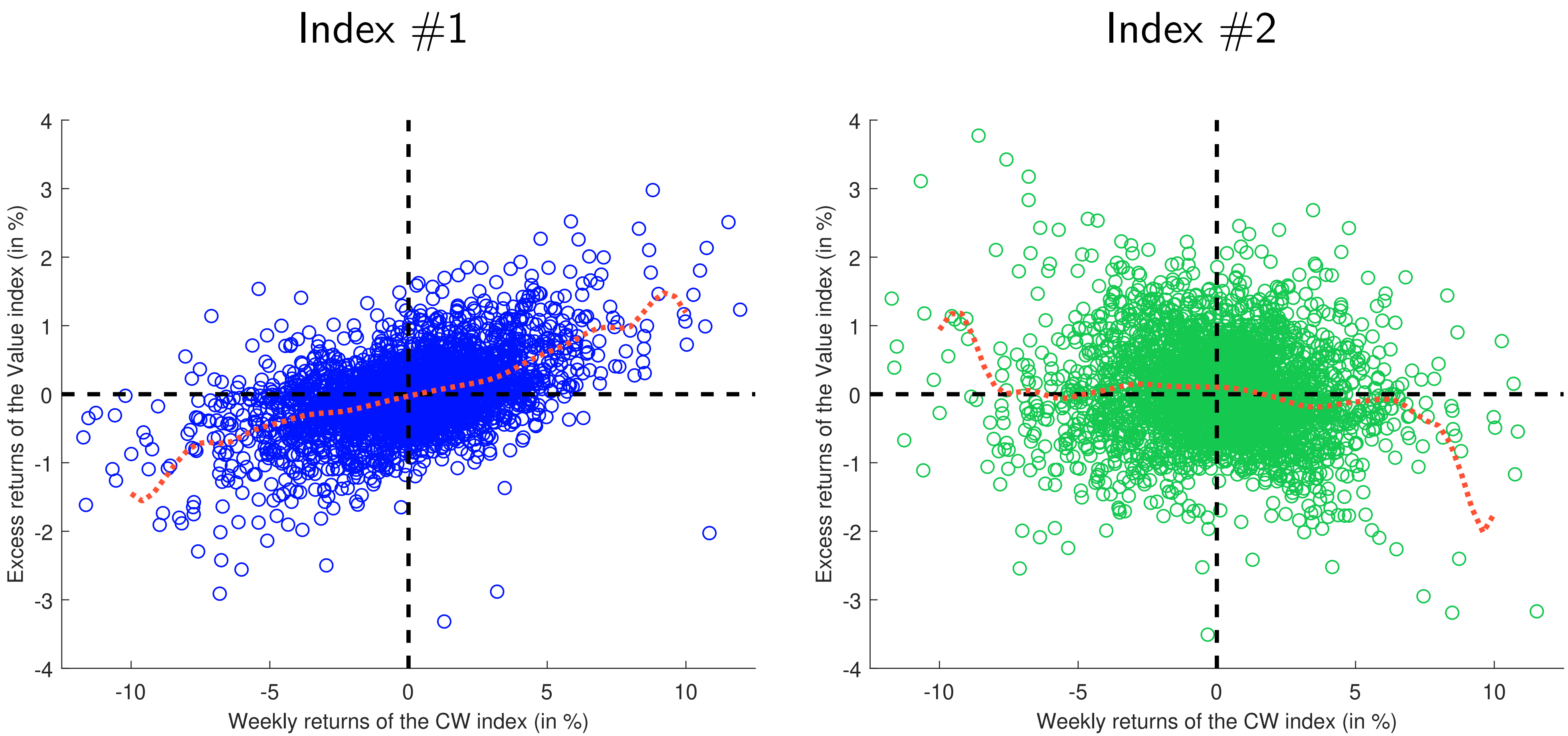


Figure 63: Which Eurozone value index has the right payoff?

The value risk premium

The payoff of the equity value risk premium

Answer

The payoff of the equity value risk premium is:

Short Put + **Long Call**

⇒ It is a skewness risk premium too!

- The design of the strategy is crucial (some weighting schemes may change or destroy the desired payoff!)
- Are the previous results valid for other asset classes, e.g. rates or currencies?

The value risk premium

Misunderstanding of the equity value risk premium

The dot-com crisis (2000-2003)

If we consider the S&P 500 index, we obtain:

- 55% of stocks post a negative performance

$$\approx 75\% \text{ of MC}$$

- 45% of stocks post a positive performance

$$\text{Maximum drawdown} = 49\ \%$$

Small caps stocks $\nearrow$
Value stocks $\nearrow$

The GFC crisis (2008)

If we consider the S&P 500 index, we obtain:

- 95% of stocks post a negative performance

$$\approx 97\% \text{ of MC}$$

- 5% of stocks post a positive performance

$$\text{Maximum drawdown} = 56\ \%$$

Small caps stocks $\searrow$
Value stocks $\searrow$

What is the impact of the liquidity risk premium?

The value risk premium

Extension to other asset classes

- Corporate bonds
  - Houweling and van Zundert (2017)
  - Ben Slimane *et al.* (2019)
  - Roncalli (2020)
- Currencies
  - MacDonald (1995)
  - Menkhoff *et al.* (2016)
  - Baku *et al.* (2019, 2020)

The momentum risk premium

Definition

- Let $S_{i,t}$ be the market price of Asset i
- We assume that:

$$\frac{\mathrm{d}S_{i,t}}{S_{i,t}} = \mu_{i,t}\,\mathrm{d}t + \sigma_{i,t}\,\mathrm{d}W_{i,t}$$

- The momentum of Asset i corresponds to its past trend:

$$\mathcal{M}_{i,t} = \hat{\mu}_{i,t}$$

- The momentum investor will prefer Asset i to Asset j if the momentum of Asset i is higher:

$$\mathcal{M}_{i,t} \geq \mathcal{M}_{j,t} \Longrightarrow A_i \succ A_j$$

The momentum risk premium

Computing the momentum measure

- Past return (e.g. one-month, three-month, one-year, etc.)

$$\mathcal{M}_{i,t} = \frac{S_{i,t} - S_{i,t-h}}{S_{i,t-h}}$$

- Lagged past return[17]
- Econometric and statistical trend estimators (see Bruder *et al.* (2011) for a survey)

[17] For instance, the WML risk factor is generally implemented using the one-month lag of the twelve-month return:

$$\mathcal{M}_{i,t} = \frac{S_{i,t-1\mathrm{M}} - S_{i,t-13\mathrm{M}}}{S_{i,t-13\mathrm{M}}}$$

because the stock market is reversal within a one-month time horizon

The momentum risk premium

Three momentum strategies

1. Cross-section momentum (CSM)

$$\mathcal{M}_{i,t} \geq \mathcal{M}_{j,t} \Longrightarrow A_i \succ A_j$$

2. Time-series momentum (TSM)

$$\mathcal{M}_{i,t} > 0 \Longrightarrow A_i \succ 0 \text{ and } \mathcal{M}_{i,t} < 0 \Longrightarrow A_i \prec 0$$

3. Reversal strategy:

$$\mathcal{M}_{i,t} \geq \mathcal{M}_{j,t} \Longrightarrow A_i \prec A_j$$

Remark

Generally, the momentum risk premium corresponds to the CSM or TSM strategies. When we speak about momentum strategies, we can also include reversal strategies, which are more considered as trading strategies with high turnover ratios and very short holding periods (generally intra-day or daily frequency, less than one week most of the time)

The momentum risk premium

Cross-section versus time-series

Time-series momentum (TSM)

- The portfolio is long (resp. short) on the asset if it has a positive (resp. negative) momentum
- This strategy is also called "trend-following" or "trend-continuation"
- HF: CTA and managed futures
- Between asset classes

Cross-section momentum (CSM)

- The portfolio is long (resp. short) on assets that present a momentum higher (resp. lower) than the others
- This strategy is also called "winners minus losers" (or WML) by Carhart (1997)
- Within an asset class (equities, currencies)

$\Rightarrow$ These two momentum risk premia are very different and not well understood!

The momentum risk premium

Understanding the TSM strategy

Some results (Jusselin *et al.*, 2017)

- EWMA is the optimal trend estimator (Kalman-Bucy filtering)
- Two components
  - a short-term component given by the payoff (dynamics)
  - a long-term component given by the trading impact (performance)
- Main important parameters
  - The Sharpe ratio
  - The duration of the moving average
  - The correlation matrix
  - The term structure of the volatility
- Too much leverage kills momentum (high ruin probability)

The momentum risk premium

Understanding the TSM strategy

Some results (Jusselin *et al.*, 2017)

- The issue of diversification
  - Time-series momentum likes zero-correlated assets (e.g. multi-asset momentum premium)
  - Cross-section momentum likes highly correlated assets (e.g. equity momentum factor)
  - The number of assets decreases the P&L dispersion
  - The symmetry puzzle
  - The n/ρ trade-off
- Short-term versus long-term momentum
  - Short-term momentum is more risky than long-term momentum
  - The Sharpe ratio of long-term momentum is higher
  - The choice of the EWMA duration is more crucial for long-term momentum

The momentum risk premium

Understanding the TSM strategy

Some results (Jusselin *et al.*, 2017)

- The momentum strategy outperforms the buy-and-hold strategy when the Sharpe ratio is lower than 35%
- The specific nature of equities and bonds
  - Performance of equity momentum is explained by leverage patterns
  - Performance of bond momentum is explained by frequency patterns
- A lot of myths about the performance of CTAs (equity contribution, option profile, hedging properties)
- Momentum strategies are not alpha or absolute return strategies, but diversification strategies

The momentum risk premium

Trend-following strategies (or TSM) exhibit a convex payoff

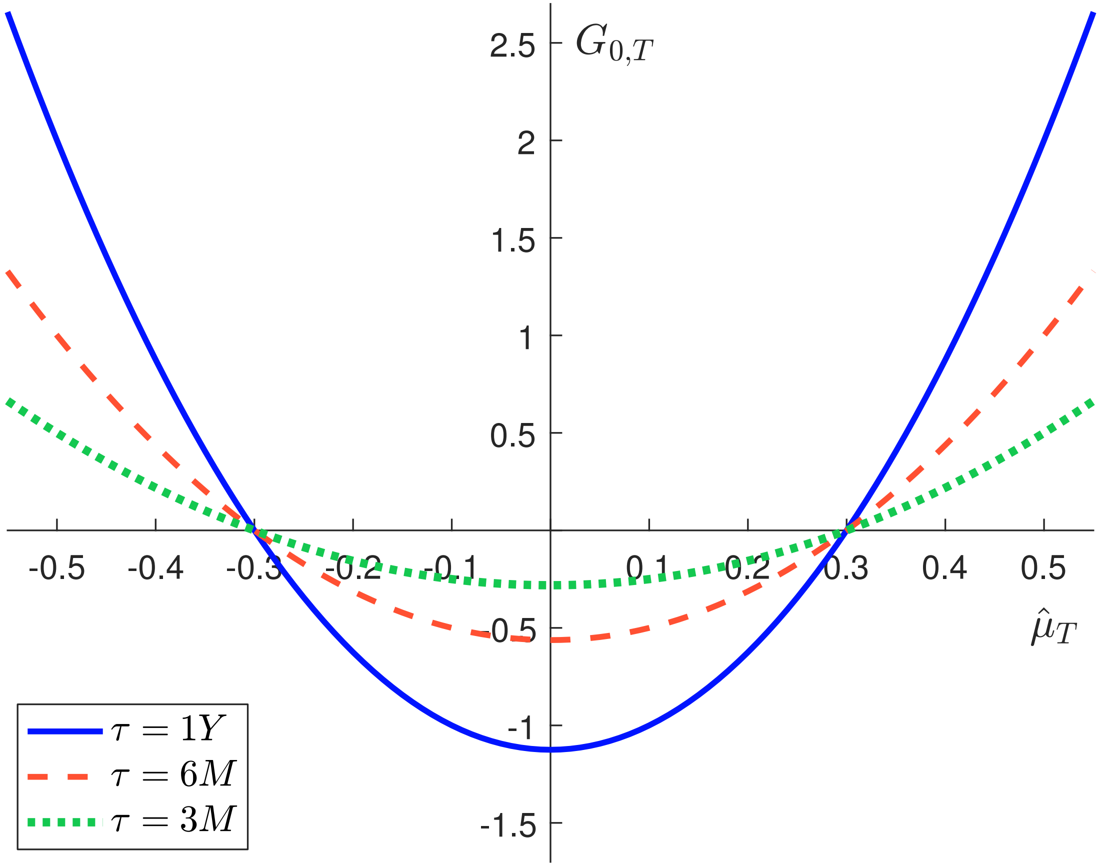


Figure 64: Option profile of the trend-following strategy

- λ is the parameter of the EWMA estimator
- $\tau = 1/\lambda$ is the duration of the EWMA estimator
- Market anomaly: the systematic risk is limited in bad times
- **Trend-following strategies exhibit a convex payoff**

The momentum risk premium

The loss of a trend-following strategy is bounded

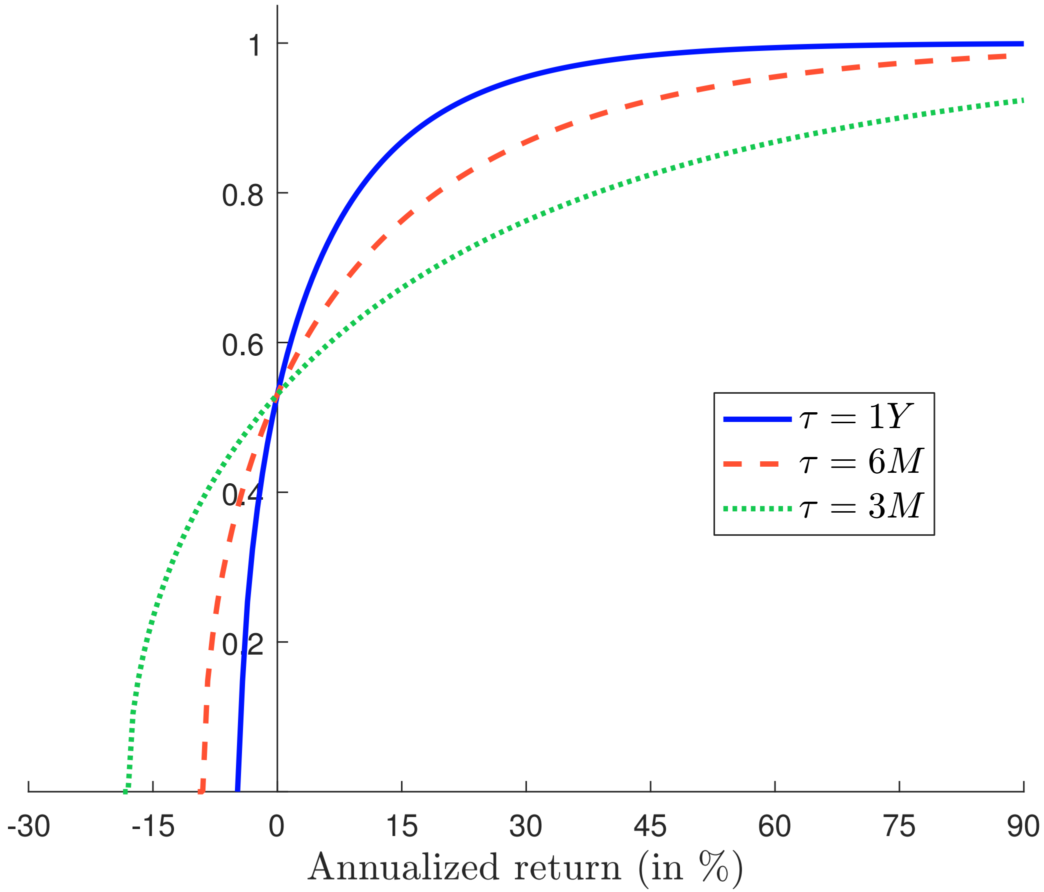


Figure 65: Cumulative distribution function of g_t ($s_t = 0$)

- s_t is the Sharpe ratio
- g_t is the trading impact
- **The loss is bounded**
- The gain may be infinite
- The return variance of short-term momentum strategies is larger than the return variance of long-term momentum strategies
- The skewness is positive

The momentum risk premium

Trend-following strategies exhibit positive skewness

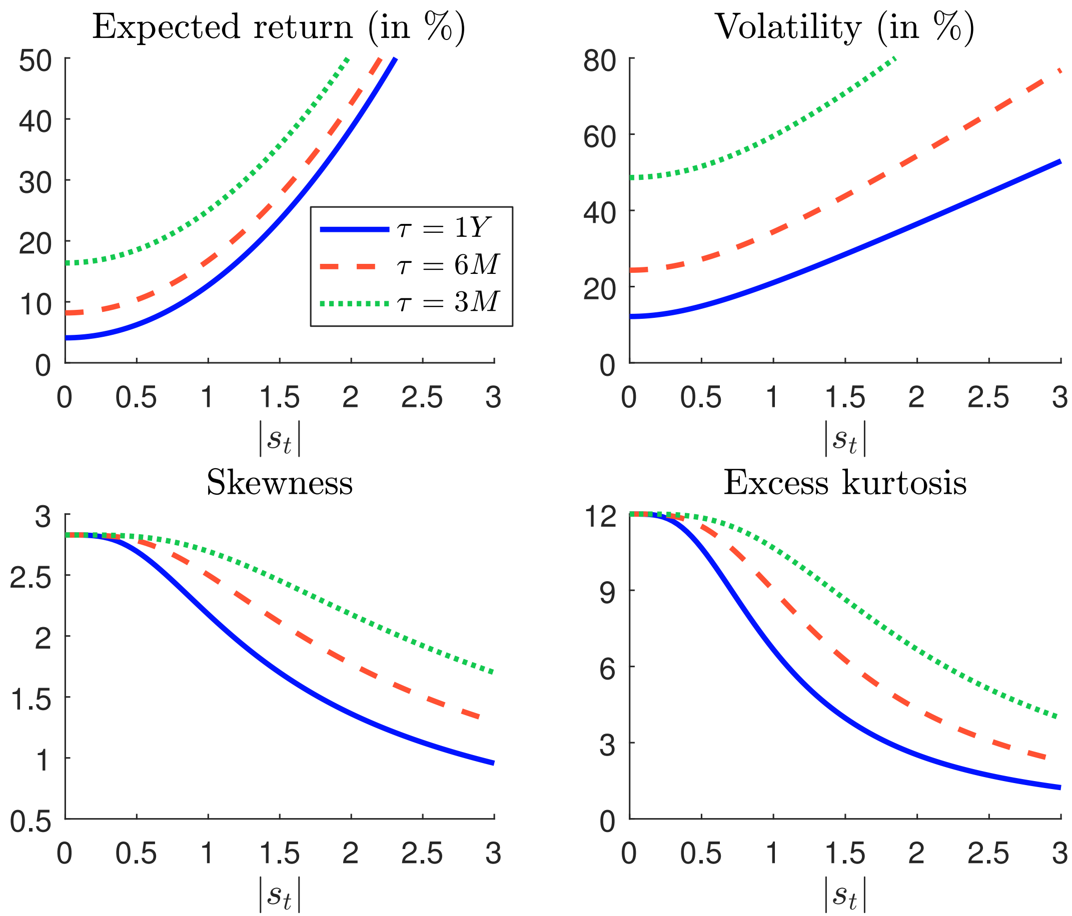


Figure 66: Statistical moments of the momentum strategy

- Short-term trend-following strategies are more risky than long-term trend-following strategies
- The skewness is positive
- It is a market anomaly, not a skewness risk premium

The momentum risk premium

Short-term versus long-term trend-following strategies

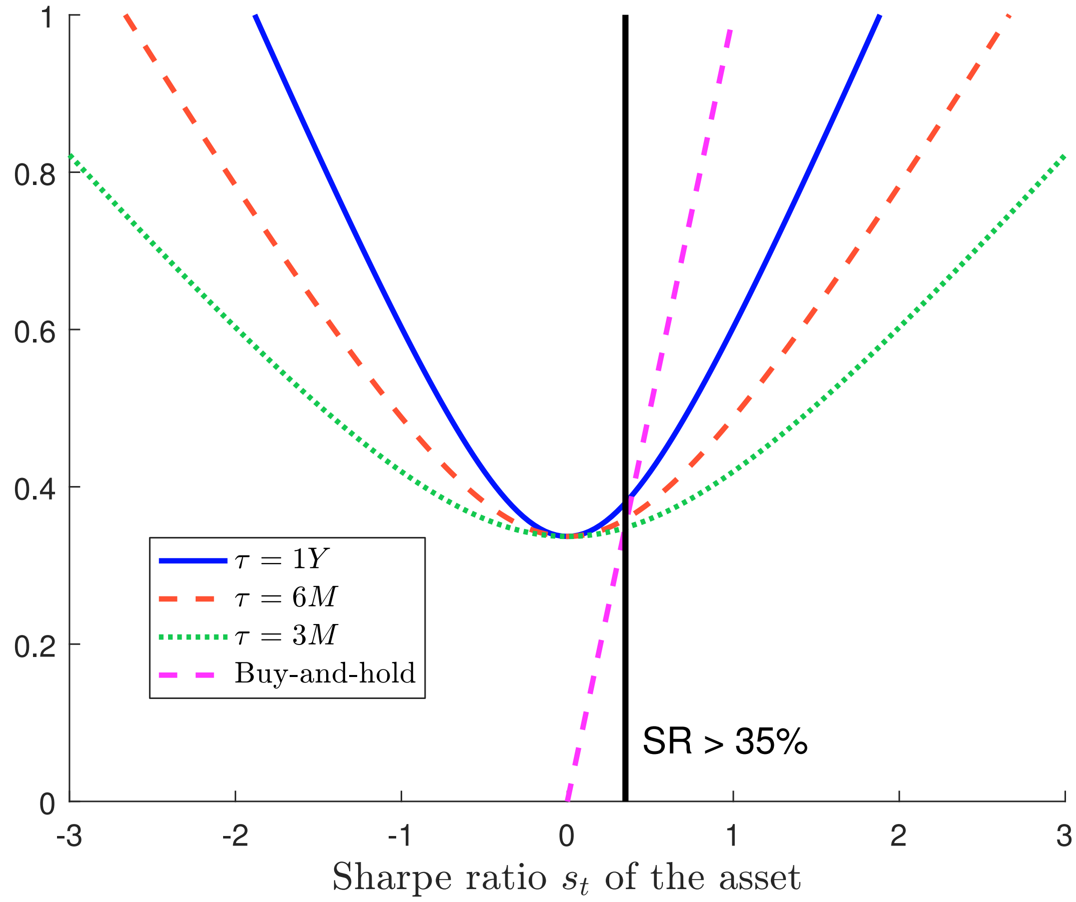


Figure 67: Sharpe ratio of the momentum strategy

- When the Sharpe ratio of the underlying is lower than 35%, the momentum strategy dominates the buy-and-hold strategy
- The Sharpe ratio of long-term momentum strategies is higher than the Sharpe ratio of short-term momentum strategies

The momentum risk premium

Relationship with the Black-Scholes robustness

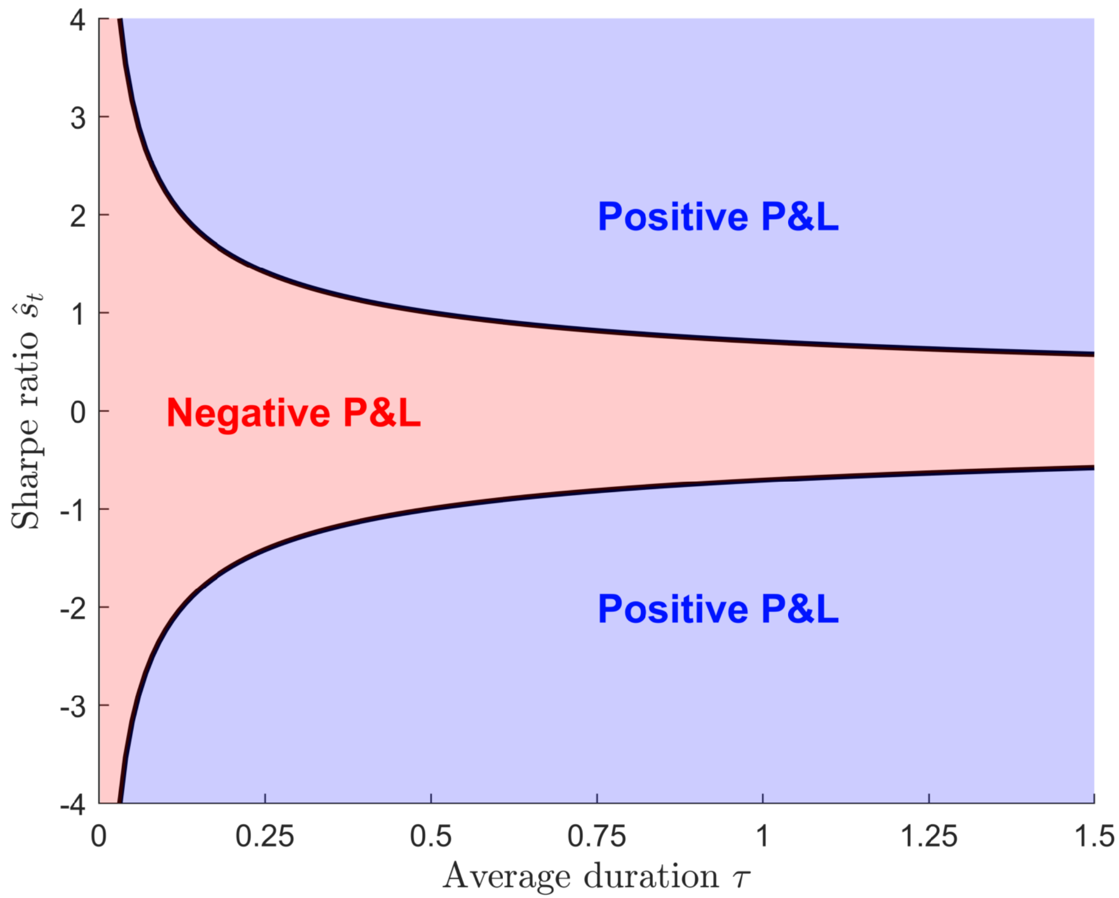


Figure 68: Admissible region for positive P&L

- Delta-hedging: implied volatility vs realized volatility
- Trend-following: duration vs realized Sharpe ratio
- The critical value for the Sharpe ratio is 1.41 for 3M and 0.71 for 1Y

The momentum risk premium

Impact of the correlation on trend-following strategies

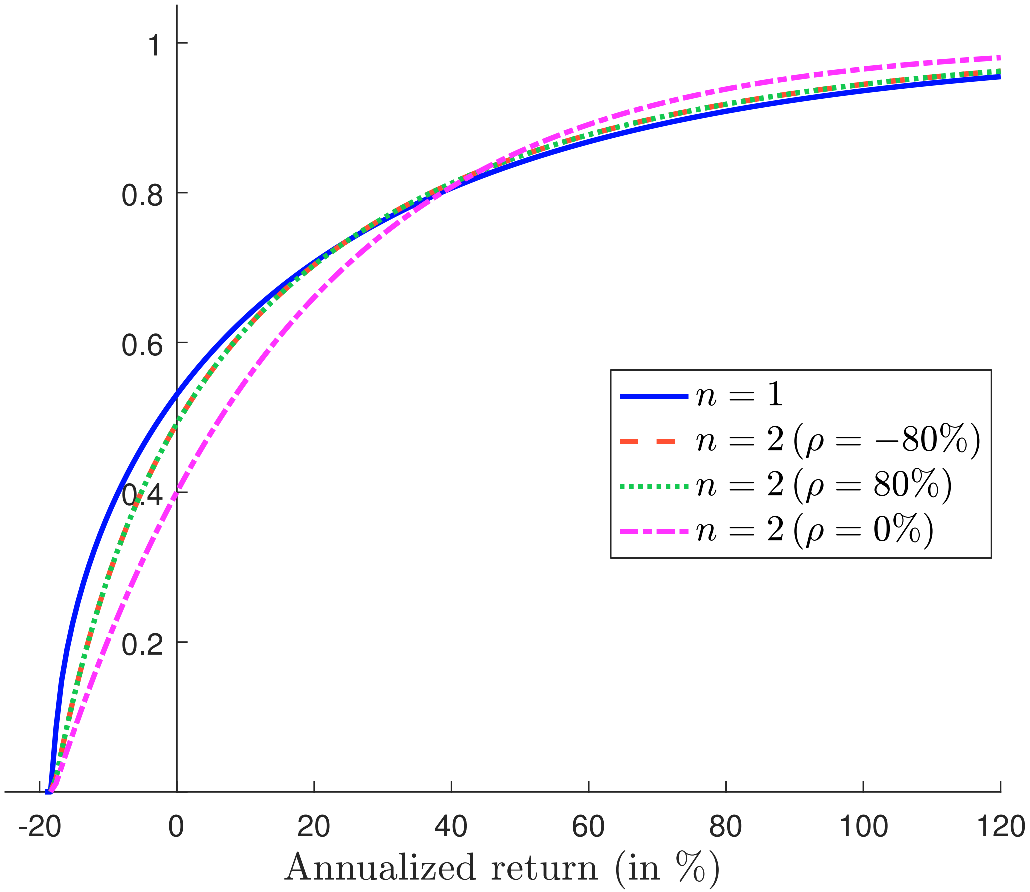


Figure 69: Cumulative distribution function of g_t ($s_t = 0$)

- Sign of correlation does not matter when the Sharpe ratio of assets is zero
- Symmetry puzzle

positive correlation

=

negative correlation

The momentum risk premium

Correlation and diversification

Long-only versus long/short diversification

We consider a portfolio (α_1, α_2) composed of two assets. We have:

$$\sigma(\rho) = \sqrt{\alpha_1^2 \sigma_1^2 + 2\rho\alpha_1\alpha_2\sigma_1\sigma_2 + \alpha_2^2\sigma_2^2}$$

- In the case of a long-only portfolio, the best case for diversification is reached when the correlation is equal to -1:

$$|\alpha_1\sigma_1 - \alpha_2\sigma_2| = \sigma(-1) \leq \sigma(\rho) \leq \sigma(1) = \alpha_1\sigma_1 + \alpha_2\sigma_2$$

- In the case of a long/short portfolio, we generally have $\operatorname{sgn}(\alpha_1\alpha_2) = \operatorname{sgn}(\rho)$. Therefore, the best case for diversification is reached when the correlation is equal to zero: $\sigma(0) \leq \sigma(\rho)$. Indeed, when the correlation is -1, the investor is long on one asset and short on the other asset, implying that this is the same bet.

The momentum risk premium

The number of assets/correlation trade-off

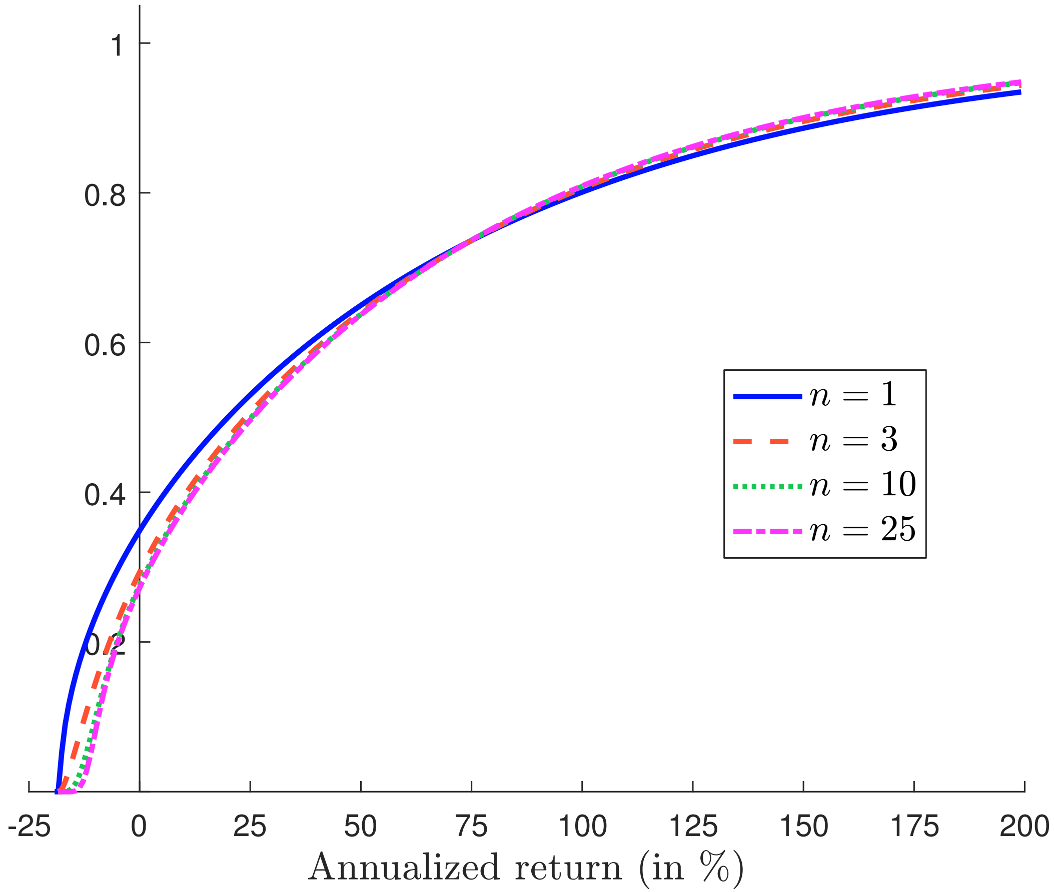


Figure 70: Impact of the number of assets on $\Pr\{g_t \leq g\}$ ($s_t = 2$, $\rho = 80\%$)

- **Correlation is not the friend of time-series momentum**
- A momentum strategy prefers a few number of assets with high Sharpe ratio absolute values than a large number of assets with low Sharpe ratio absolute values

The momentum risk premium

TSM versus CSM

Time-series momentum

- Absolute trends

$$\left\{ \begin{array}{l} \hat{\mu}_{i,t} \geq 0 \Rightarrow e_{i,t} \geq 0 \\ \hat{\mu}_{i,t} < 0 \Rightarrow e_{i,t} < 0 \end{array} \right.$$

- CTA hedge funds
- Alternative risk premia in multi-asset portfolios

Cross-section momentum

- Relative trends

$$\left\{ \begin{array}{l} \hat{\mu}_{i,t} \geq \bar{\mu}_t \Rightarrow e_{i,t} \geq 0 \\ \hat{\mu}_{i,t} < \bar{\mu}_t \Rightarrow e_{i,t} < 0 \end{array} \right.$$

where:

$$\bar{\mu}_t = \frac{1}{n} \sum_{j=1}^{n} \hat{\mu}_{j,t}$$

- Statistical arbitrage / relative value
- Factor investing in equity portfolios

Beta strategy or Alpha strategy?

The momentum risk premium

Performance of cross-section momentum risk premium

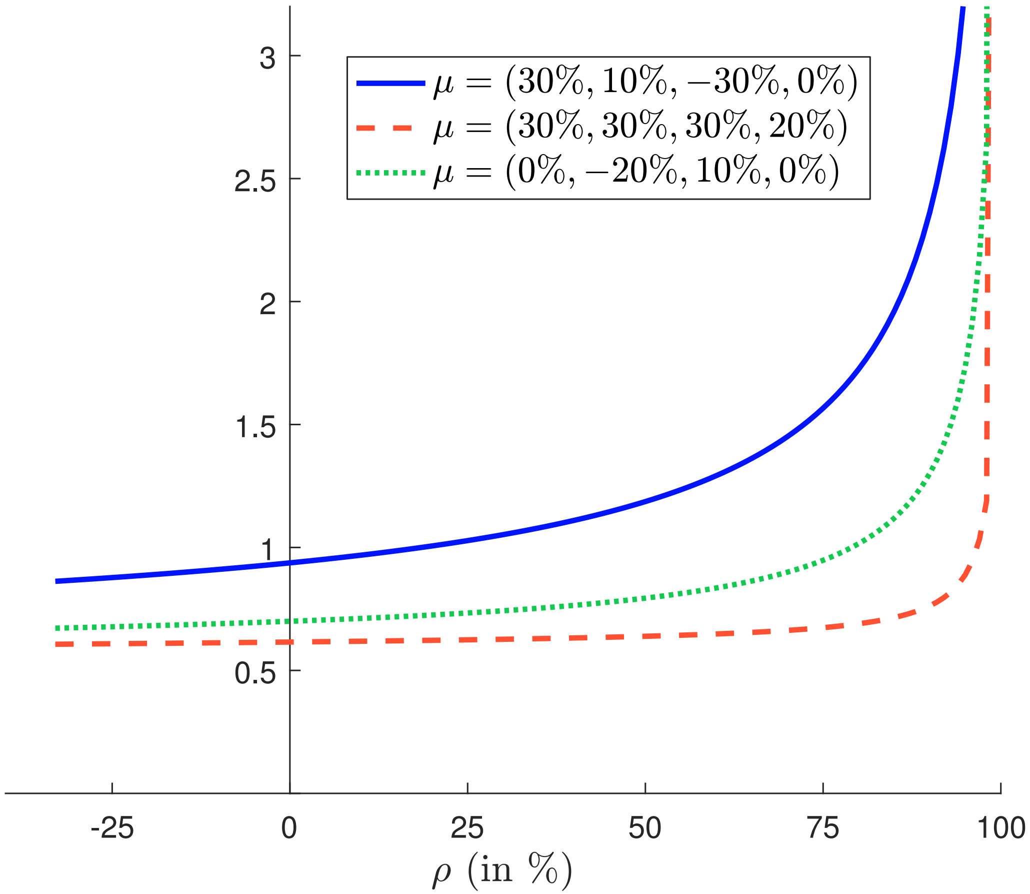


Figure 71: Sharpe ratio of the CSM strategy

- **Correlation is the friend of cross-section momentum!**
- Statistical arbitrage / relative value

The momentum risk premium

Naive replication of the SG CTA Index

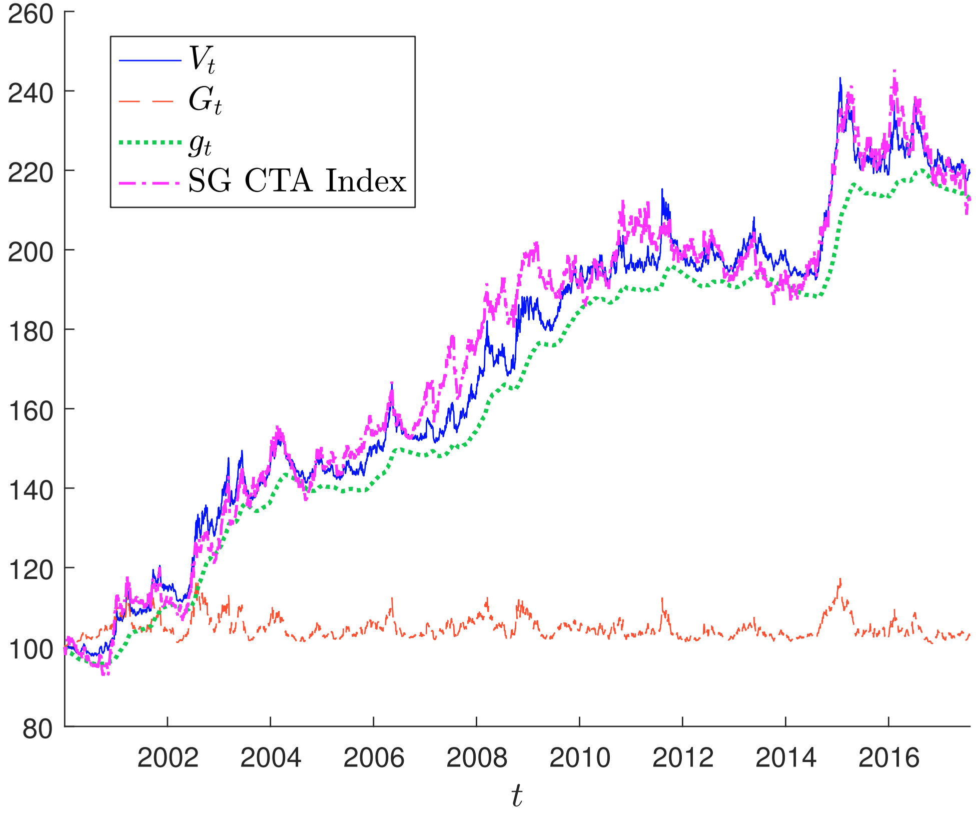


Figure 72: Comparison between the cumulative performance of the naive replication strategy and the SG CTA Index

- The performance of trend-followers comes from the trading impact
- Currencies and commodities are the main contributors!
- Mixing asset classes is the key point in order to capture the diversification premium

The momentum risk premium

Trend-following strategies benefit from traditional risk premia

Table 61: Exposure average of the trend-following strategy (in %)

Asset Class	Average Exposure	Short Exposure	Long Exposure	Short Frequency	Long Frequency
Bond	58%	−100%	122%	29%	**71%**
Equity	52%	−88%	**160%**	44%	56%
Currency	**18%**	−103%	115%	45%	55%
Commodity	**23%**	−108%	113%	41%	59%

- The specific nature of bonds: long exposure frequency $>$ short exposure frequency; long leverage $\approx$ short leverage
- The specific nature of equities: short exposure frequency $\approx$ long exposure frequency; long leverage $>$ short leverage

The momentum risk premium

The myth of short selling

- Equity and bond momentum strategies benefit from the existence of a risk premium
- Currency and commodity momentum strategies benefit from (positive / negative) trend patterns
- Leverage management $\succ$ short management
- The case of equities in the 2008 GFC, the stock-bond correlation and the symmetry puzzle

The good performance of CTAs in 2008 is not explained by their short exposure in equities, but by their long exposure in bonds

The momentum risk premium

The reversal strategy

- The reversal strategy may be defined as the opposite of the momentum strategy (CSM or TSM)
- It is also known as the mean-reverting strategy

How to reconciliate reversal and trend-following strategies?

Because they don't use the same trend windows and holding periods[18]

[18] Generally, reversal strategies use short-term or very long-term trends while trend-following strategies use medium-term trends

The momentum risk premium

The reversal strategy

The mean-reverting (or autocorrelation) strategy

- Let $R_{i,t} = \ln S_{i,t} - \ln S_{i,t-1}$ be the one-period return
- We note $\rho_i(h) = \rho(R_{i,t}, R_{i,t-h})$ the autocorrelation function
- Asset i exhibits a mean-reverting pattern if the short-term autocorrelation $\rho_i(1)$ is negative
- In this case, the short-term reversal is defined by the product of the autocorrelation and the current return:

$$\mathcal{R}_{i,t} = \rho_i(1) \cdot R_{i,t}$$

- The short-term reversal strategy is then defined by the following rule:

$$\mathcal{R}_{i,t} \geq \mathcal{R}_{j,t} \Longrightarrow i \succ j$$

The momentum risk premium

The reversal strategy

First implementation of the autocorrelation strategy

- If $\mathcal{R}_{i,t}$ is positive, meaning that the current return $R_{i,t}$ is negative, we should buy the asset, because a negative return is followed by a positive return on average
- If $\mathcal{R}_{i,t}$ is negative, meaning that the current return $R_{i,t}$ is positive, we should sell the asset, because a positive return is followed by a negative return on average

The momentum risk premium

The reversal strategy

The variance swap strategy

- We assume that the one-period asset return follows an AR(1) process:

$$R_{i,t} = \rho R_{i,t-1} + \varepsilon_t$$

  where $|\rho| < 1$, $\varepsilon_t \sim \mathcal{N}\left(0, \sigma_\varepsilon^2\right)$ and $\operatorname{cov}\left(\varepsilon_t, \varepsilon_{t-j}\right) = 0$ for $j \geq 1$
- Let $\mathrm{RV}\left(h\right)$ be the annualized realized variance of the h-period asset return $R_{i,t}\left(h\right) = \ln S_{i,t} - \ln S_{i,t-h}$
- Hamdan *et al.* (2016) showed that:

$$\mathbb{E}\left[\mathrm{RV}\left(h\right)\right] = \phi\left(h\right)\mathbb{E}\left[\mathrm{RV}\left(1\right)\right]$$

  where:

$$\phi\left(h\right) = 1 + 2\rho\frac{1-\rho^{h-1}}{1-\rho} - 2\sum\nolimits_{j=1}^{h-1}\frac{j}{h}\rho^j$$

The momentum risk premium

The reversal strategy

The variance swap strategy

- We notice that:

$$\lim_{h\to\infty} \mathbb{E}\left[\mathrm{RV}\left(h\right)\right] = \left(1 + \frac{2\rho}{1-\rho}\right) \cdot \mathbb{E}\left[\mathrm{RV}\left(1\right)\right]$$

- When the autocorrelation is negative, this implies that the long-term frequency variance is lower than the short-term frequency variance
- More generally, we have:

$$\left\{\begin{array}{ll} \mathbb{E}\left[\mathrm{RV}\left(h\right)\right] < \mathbb{E}\left[\mathrm{RV}\left(1\right)\right] & \text{if } \rho < 0 \\ \mathbb{E}\left[\mathrm{RV}\left(h\right)\right] \geq \mathbb{E}\left[\mathrm{RV}\left(1\right)\right] & \text{otherwise} \end{array}\right.$$

The momentum risk premium

The reversal strategy

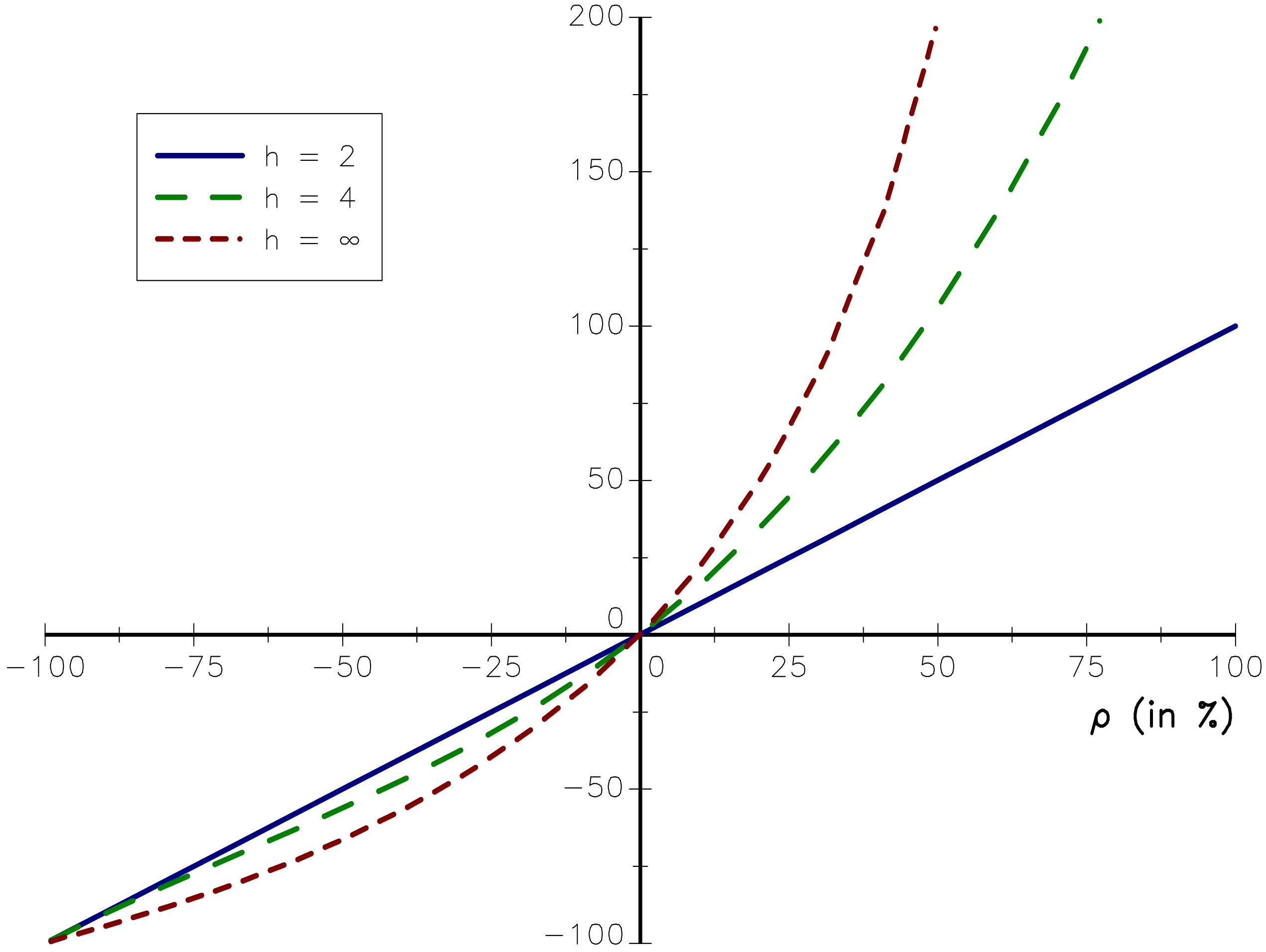


Figure 73: Variance ratio $(\mathrm{RV}(h) - \mathrm{RV}(1)) / \mathrm{RV}(1)$ (in %)

The momentum risk premium

The reversal strategy

Second implementation of the autocorrelation strategy

- The spread between daily/weekly and weekly/monthly variance swaps depends on the autocorrelation of daily returns
- The reversal strategy consists in being long on the daily/weekly variance swaps and short on the weekly/monthly variance swaps

The momentum risk premium

The reversal strategy

The long-term reversal strategy

- The long-term return reversal is defined by the difference between long-run and short-period average prices:

$$\mathcal{R}_{i,t} = \bar{S}_{i,t}^{LT} - \bar{S}_{i,t}^{ST}$$

- Typically, $\bar{S}_{i,t}^{ST}$ is the average price over the last year and $\bar{S}_{i,t}^{LT}$ is the average price over the last five years
- The long-term return reversal strategy follows the same rule as the short-term reversal strategy
- This reversal strategy is equivalent to a value strategy because the long-run average price can be viewed as an estimate of the fundamental price in some asset classes

The momentum risk premium

The reversal strategy

Implementation of the long-term reversal strategy

- If $\mathcal{R}_{i,t}$ is positive, the long-term mean of the asset price is above its short-term mean $\Rightarrow$ we should buy the asset
- If $\mathcal{R}_{i,t}$ is negative, the long-term mean of the asset price is below its short-term mean $\Rightarrow$ we should sell the asset

The liquidity risk premium

What means "*liquidity risk*"?

"[...] there is also broad belief among users of financial liquidity — traders, investors and central bankers — that the principal challenge is not the **average level** *of financial liquidity ... but its* **variability** *and uncertainty " (Persaud, 2003).*

The liquidity risk premium

The liquidity-adjusted CAPM

L-CAPM (Acharya and Pedersen, 2005)

We note L_i the relative (stochastic) illiquidity cost of Asset i. At the equilibrium, we have:

$$\mathbb{E}\left[R_i - L_i\right] - R_f = \tilde{\beta}_i \left(\mathbb{E}\left[R_M - L_M\right] - R_f\right)$$

where:

$$\tilde{\beta}_i = \frac{\operatorname{cov}\left(R_i - L_i, R_M - L_M\right)}{\operatorname{var}\left(R_M - L_M\right)}$$

CAPM in the frictionless economy

$\Downarrow$

CAPM in net returns (including illiquidity costs)

The liquidity risk premium

The liquidity-adjusted CAPM

- The liquidity-adjusted beta can be decomposed into four beta(s):

$$\tilde{\beta}_i = \beta_i + \beta\left(L_{i,}, L_M\right) - \beta\left(R_{i,}, L_M\right) - \beta\left(L_{i,}, R_M\right)$$

  where:

  - $\beta_i = \beta\left(R_{i,} R_M\right)$ is the standard market beta;
  - $\beta\left(L_{i,}, L_M\right)$ is the beta associated to the commonality in liquidity with the market liquidity;
  - $\beta\left(R_{i,}, L_M\right)$ is the beta associated to the return sensitivity to market liquidity;
  - $\beta\left(L_{i,}, R_M\right)$ is the beta associated to the liquidity sensitivity to market returns.

- The risk premium is equal to:

$$\begin{aligned}\pi_i &= \mathbb{E}\left[L_i\right] + \left(\beta_i + \beta\left(L_{i,}, L_M\right)\right)\pi_M - \\ &\quad \left(\tilde{\beta}_i \mathbb{E}\left[L_M\right] + \left(\beta\left(R_{i,}, L_M\right) + \beta\left(L_{i,}, R_M\right)\right)\pi_M\right)\end{aligned}$$

The liquidity risk premium

The liquidity-adjusted CAPM

Acharya and Pedersen (2005)

If assets face some liquidity costs, the relationship between the risk premium and the beta of asset i becomes:

$$\mathbb{E}[R_i] - R_f = \alpha_i + \beta_i (\mathbb{E}[R_M] - R_f)$$

where α_i is a function of the relative liquidity of Asset i with respect to the market portfolio and the liquidity beta(s):

$$\begin{aligned} \alpha_i &= \left(\mathbb{E}[L_i] - \tilde{\beta}_i \mathbb{E}[L_M]\right) + \\ & \beta(L_{i,}, L_M)\pi_M - \beta(R_{i,}, L_M)\pi_M - \beta(L_{i,}, R_M)\pi_M \end{aligned}$$

The liquidity risk premium

Disentangling the liquidity alpha

- We deduce that:

$$\alpha_i \neq \mathbb{E}\left[L_i\right]$$

meaning that the risk premium of an illiquid asset is not the systematic risk premium plus a premium due the illiquidity level:

$$\mathbb{E}\left[R_i\right] - R_f \neq \mathbb{E}\left[L_i\right] + \beta_i \left(\mathbb{E}\left[R_M\right] - R_f\right)$$

- The 4 liquidity premia are highly correlated[19] ($\mathbb{E}\left[L_i\right]$, $\beta\left(L_{i,}, L_M\right)$, $\beta\left(R_{i,}, L_M\right)$ and $\beta\left(L_{i,}, R_M\right)$).
- Acharaya and Pedersen (2005) found that $\mathbb{E}\left[L_i\right]$ represents 75% of α_i on average. The 25% remaining are mainly explained by the liquidity sensitivity to market returns – $\beta\left(L_{i,}, R_M\right)$.

[19] For instance, we have $\rho\left(\beta\left(L_{i,}, L_M\right), \beta\left(R_{i,}, L_M\right)\right) = -57\%$, $\rho\left(\beta\left(L_{i,}, L_M\right), \beta\left(L_{i,}, R_M\right)\right) = -94\%$ and $\rho\left(\beta\left(R_{i,}, L_M\right), \beta\left(L_{i,}, R_M\right)\right) = 73\%$.

The liquidity risk premium

Three liquidity risks

In fact, we have:

$$\alpha_i = \text{illiquidity level} + \text{illiquidity covariance risks}$$

1. $\beta(L_{i,}, L_M)$
   - An asset that becomes illiquid when the market becomes illiquid should have a higher risk premium
   - Substitution effects when the market becomes illiquid
2. $\beta(R_{i,}, L_M)$
   - Assets that perform well in times of market illiquidity should have a lower risk premium
   - Relationship with solvency constraints
3. $\beta(L_{i,}, R_M)$
   - Investors accept a lower risk premium on assets that are liquid in a bear market
   - Selling markets $\neq$ buying markets

The liquidity risk premium

How does market liquidity impact risk premia?

Three main impacts

- Effect on the risk premium
- Effect on the price dynamics
  If liquidity is persistent, negative shock to liquidity implies low current returns and high predicted future returns:
  $$\operatorname{cov}\left(L_{i,t}, R_{i,t}\right) < 0 \text{ and } \partial_{L_{i,t}} \mathbb{E}_t\left[R_{i,t+1}\right] > 0$$
- Effect on portfolio management

- Sovereign bonds
- Corporate bonds
- Stocks
- Small caps
- Private equities

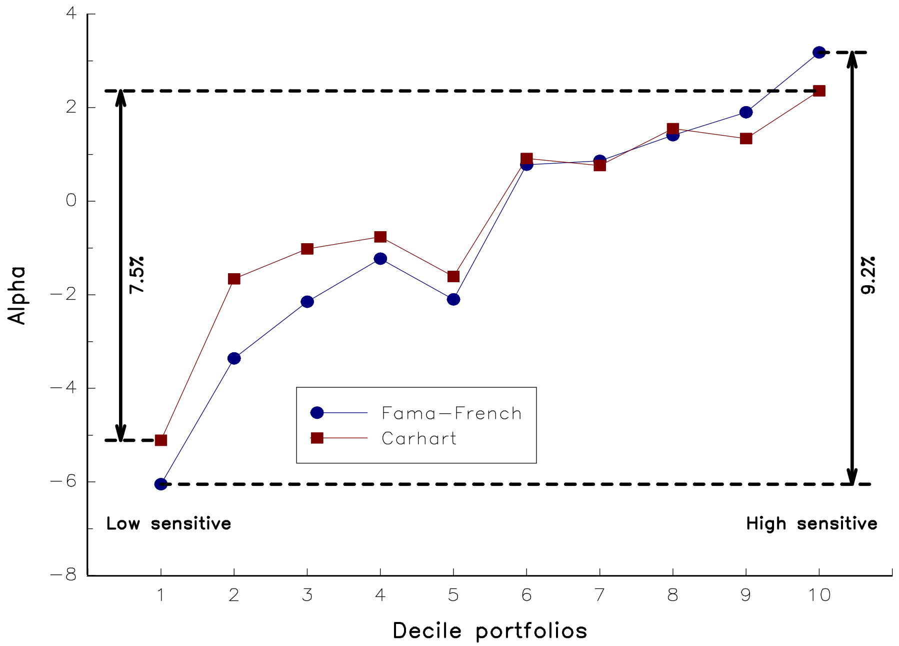
The liquidity risk premium

Application to stocks

Pastor and Stambaugh (2003) include a liquidity premium in the Fama-French-Carhart model:

$$\begin{aligned}\mathbb{E}\left[R_i\right] - R_f &= \beta_i^M \left(\mathbb{E}\left[R_M\right] - R_f\right) + \beta_i^{SMB}\mathbb{E}\left[R_{SMB}\right] + \beta_i^{HML}\mathbb{E}\left[R_{HML}\right] + \\ &\quad \beta_i^{WML}\mathbb{E}\left[R_{WML}\right] + \beta_i^{LIQ}\mathbb{E}\left[R_{LIQ}\right]\end{aligned}$$

where LIQ measures the shock or innovation of the aggregate liquidity.



Alphas of decile portfolios sorted on predicted liquidity beta(s)

Long Q10 / Short Q1:

- 9.2% wrt 3F Fama-French model
- 7.5% wrt 4F Carhart model

The liquidity risk premium

Impact of the liquidity on the stock market

The dot-com crisis (2000-2003)

If we consider the S&P 500 index, we obtain:

- 55% of stocks post a negative performance

$$\approx 75\% \text{ of MC}$$

- 45% of stocks post a positive performance

$$\text{Maximum drawdown} = 49\ \%$$

Small caps stocks $\nearrow$
Value stocks $\nearrow$

The GFC crisis (2008)

If we consider the S&P 500 index, we obtain:

- 95% of stocks post a negative performance

$$\approx 97\% \text{ of MC}$$

- 5% of stocks post a positive performance

$$\text{Maximum drawdown} = 56\ \%$$

Small caps stocks $\searrow$
Value stocks $\searrow$

The liquidity risk premium

The specific status of the stock market

The interconnectedness nature of illiquid assets and liquid assets: the example of the Global Financial Crisis

- Subprime crisis $\Leftrightarrow$ banks (credit risk)
- Banks $\Leftrightarrow$ asset management, e.g. hedge funds (funding & leverage risk)
- Asset management $\Leftrightarrow$ equity market (liquidity risk)
- Equity market $\Leftrightarrow$ banks (asset-price & collateral risk)

The equity market is the ultimate liquidity provider:
GFC $\gg$ internet bubble

Remark

$^1/_3$ *of the losses in the stock market is explained by the liquidity supply*

The liquidity risk premium

Relationship between diversification & liquidity

During good times

- Medium correlation between liquid assets
- Illiquid assets have low impact on liquid assets
- Low substitution effects

Main effect:

$$\mathbb{E}[L_i]$$

During bad times

- High correlation between liquid assets
- Illiquid assets have a high impact on liquid assets
- High substitution effects

Main effects:

$$\beta(L_i, R_M) \text{ and } \beta(R_i, L_M)$$

The skewness puzzle

Skewness aggregation $\neq$ volatility aggregation

When we accumulate long/short skewness risk premia in a portfolio, the volatility of this portfolio decreases dramatically, but its skewness risk generally increases!

- Skewness diversification $\neq$ volatility diversification

$$\begin{array}{rcl} \sigma\left(X_1 + X_2\right) & \leq & \sigma\left(X_1\right) + \sigma\left(X_2\right) \\ \left|\gamma_1\left(X_1 + X_2\right)\right| & \not\leq & \left|\gamma_1\left(X_1\right) + \gamma_1\left(X_2\right)\right| \end{array}$$

Skewness is not a convex risk measure

The skewness puzzle

Example 12

We assume that (X_1, X_2) follows a bivariate log-normal distribution $\mathcal{LN}\left(\mu_1, \sigma_1^2, \mu_2, \sigma_2^2, \rho\right)$. This implies that $\ln X_1 \sim \mathcal{N}\left(\mu_1, \sigma_1^2\right)$, $\ln X_2 \sim \mathcal{N}\left(\mu_2, \sigma_2^2\right)$ and ρ is the correlation between $\ln X_1$ and $\ln X_2$.

The skewness puzzle

We recall that the skewness of X_1 is equal to:

$$\gamma_1(X_1) = \frac{\mu_3(X_1)}{\mu_2^{3/2}(X_1)} = \frac{e^{3\sigma_1^2} - 3e^{\sigma_1^2} + 2}{\left(e^{\sigma_1^2} - 1\right)^{3/2}}$$

whereas the skewness of $X_1 + X_2$ is equal to:

$$\gamma_1(X_1 + X_2) = \frac{\mu_3(X_1 + X_2)}{\mu_2^{3/2}(X_1 + X_2)}$$

where $\mu_n(X)$ is the n^{th} central moment of X

The skewness puzzle

In order to find the skewness of the sum $X_1 + X_2$, we need a preliminary result. By denoting $X = \alpha_1 \ln X_1 + \alpha_2 \ln X_2$, we have[20]:

$$\mathbb{E}\left[e^X\right] = e^{\mu_X + \frac{1}{2}\sigma_X^2}$$

where:

$$\mu_X = \alpha_1 \mu_1 + \alpha_2 \mu_2$$

and:

$$\sigma_X^2 = \alpha_1^2 \sigma_1^2 + \alpha_2^2 \sigma_2^2 + 2\alpha_1 \alpha_2 \rho \sigma_1 \sigma_2$$

It follows that:

$$\mathbb{E}\left[X_1^{\alpha_1} X_2^{\alpha_2}\right] = e^{\alpha_1 \mu_1 + \alpha_2 \mu_2 + \frac{1}{2}\left(\alpha_1^2 \sigma_1^2 + \alpha_2^2 \sigma_2^2 + 2\alpha_1 \alpha_2 \rho \sigma_1 \sigma_2\right)}$$

[20] Because X is a Gaussian random variable

The skewness puzzle

We have:

$$\mu_2 (X_1 + X_2) = \mu_2 (X_1) + \mu_2 (X_2) + 2 \operatorname{cov} (X_1, X_2)$$

where:

$$\mu_2 (X_1) = e^{2\mu_1 + \sigma_1^2} \left(e^{\sigma_1^2} - 1 \right)$$

and:

$$\operatorname{cov} (X_1, X_2) = \left(e^{\rho \sigma_1 \sigma_2} - 1 \right) e^{\mu_1 + \frac{1}{2}\sigma_1^2} e^{\mu_2 + \frac{1}{2}\sigma_2^2}$$

The skewness puzzle

For the third moment of $X_1 + X_2$, we use the following formula:

$$\mu_3 (X_1 + X_2) = \mu_3 (X_1) + \mu_3 (X_2) + 3 (\operatorname{cov} (X_1, X_1, X_2) + \operatorname{cov} (X_1, X_2, X_2))$$

where:

$$\mu_3 (X_1) = e^{2\mu_1 + \frac{3}{2}\sigma_1^2} \left(e^{3\sigma_1^2} - 3e^{\sigma_1^2} + 2 \right)$$

and:

$$\operatorname{cov} (X_1, X_1, X_2) = \left(e^{\rho\sigma_1\sigma_2} - 1 \right) e^{2\mu_1 + \sigma_1^2 + \mu_2 + \frac{\sigma_2^2}{2}} \left(e^{\sigma_1^2 + \rho\sigma_1\sigma_2} + e^{\sigma_2^2} - 2 \right)$$

The skewness puzzle

We deduce that:

$$\gamma_1 \left(X_1 + X_2\right) = \frac{\mu_3 \left(X_1 + X_2\right)}{\mu_2^{3/2} \left(X_1 + X_2\right)}$$

where:

$$\begin{aligned} \mu_2 \left(X_1 + X_2\right) &= e^{2\mu_1 + \sigma_1^2} \left(e^{\sigma_1^2} - 1\right) + e^{2\mu_2 + \sigma_2^2} \left(e^{\sigma_2^2} - 1\right) + \\ & \quad 2 \left(e^{\rho \sigma_1 \sigma_2} - 1\right) e^{\mu_1 + \frac{1}{2}\sigma_1^2} e^{\mu_2 + \frac{1}{2}\sigma_2^2} \end{aligned}$$

and:

$$\begin{aligned} \mu_3 \left(X_1 + X_2\right) &= e^{2\mu_1 + \frac{3}{2}\sigma_1^2} \left(e^{3\sigma_1^2} - 3e^{\sigma_1^2} + 2\right) + e^{2\mu_2 + \frac{3}{2}\sigma_2^2} \left(e^{3\sigma_2^2} - 3e^{\sigma_2^2} + 2\right) + \\ & \quad 3 \left(e^{\rho \sigma_1 \sigma_2} - 1\right) e^{2\mu_1 + \sigma_1^2 + \mu_2 + \frac{\sigma_2^2}{2}} \left(e^{\sigma_1^2 + \rho \sigma_1 \sigma_2} + e^{\sigma_2^2} - 2\right) + \\ & \quad 3 \left(e^{\rho \sigma_1 \sigma_2} - 1\right) e^{\mu_1 + \frac{1}{2}\sigma_1^2 + 2\mu_2 + \sigma_2^2} \left(e^{\sigma_2^2 + \rho \sigma_1 \sigma_2} + e^{\sigma_1^2} - 2\right) \end{aligned}$$

The skewness puzzle

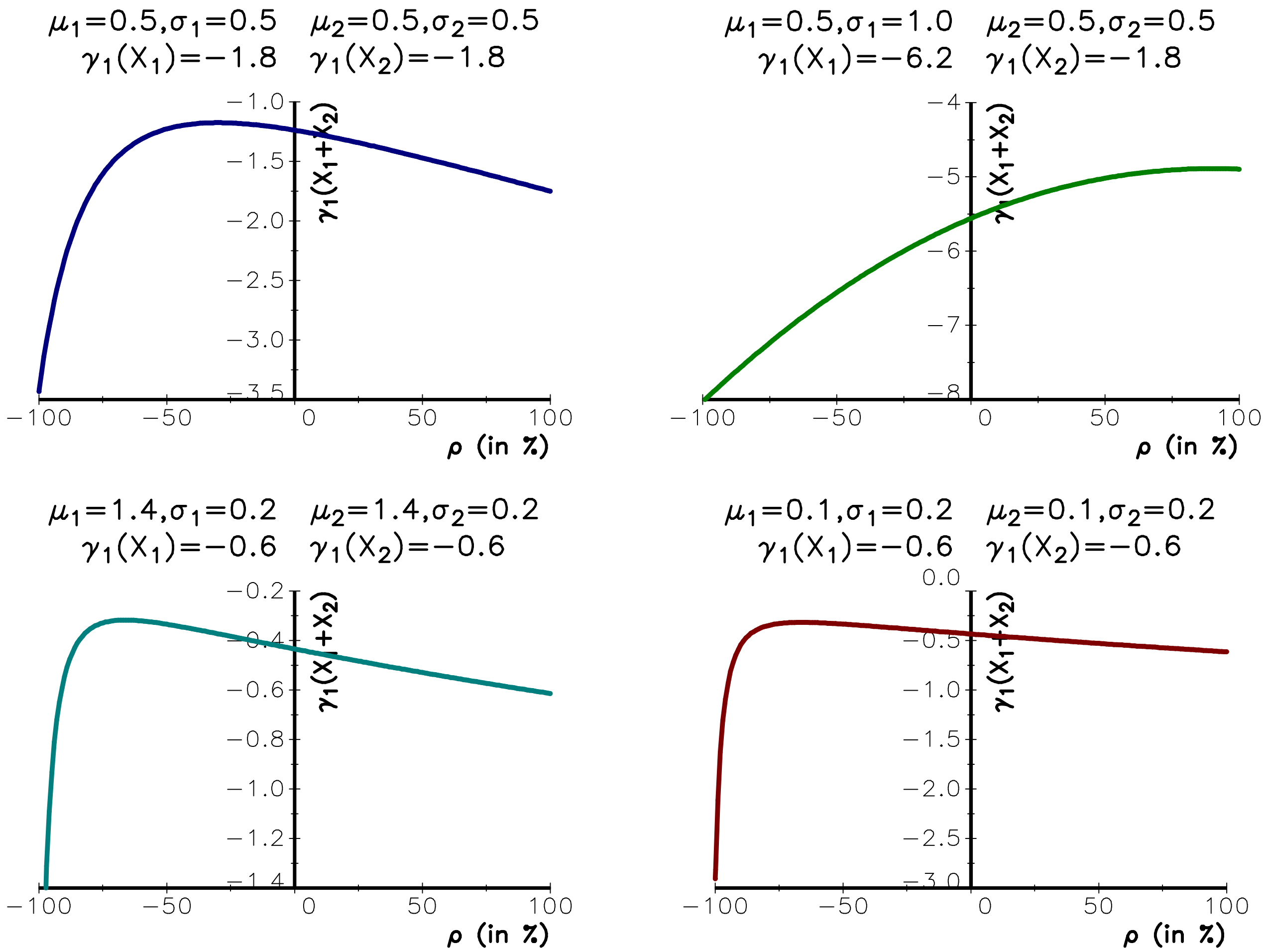


Figure 74: Skewness aggregation of the random vector $(-X_1, -X_3)$

The skewness puzzle

Why?

- Volatility diversification works very well with L/S risk premia:

$$\sigma\left(R\left(x\right)\right) \approx \frac{\bar{\sigma}}{\sqrt{n}}$$

- Drawdown diversification don't work very well because bad times are correlated and are difficult to hedge:

$$\mathrm{DD}\left(x\right) \approx \overline{\mathrm{DD}}$$

The skewness puzzle

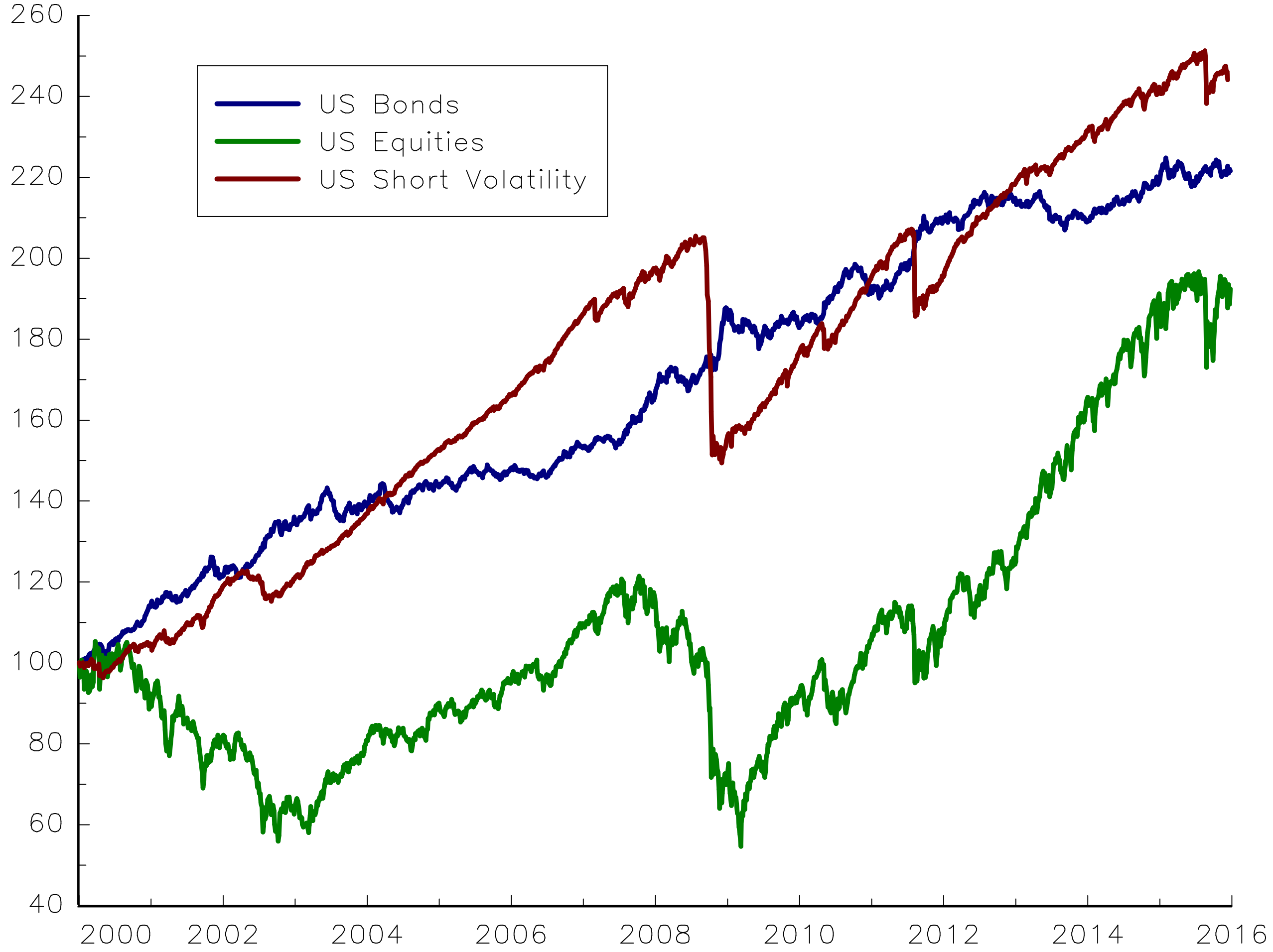


Figure 75: Cumulative performance of US 10Y bonds, US equities and US short volatility

Source: Bruder *et al.* (2016)

The correlation puzzle

We consider the Gaussian random vector (R_1, R_2, R_3), whose volatilities are equal to 25%, 12% and 9.76%. The correlation matrix is given by:

$$C = \begin{pmatrix} 100\% & & \\ -25.00\% & 100\% & \\ 55.31\% & 66.84\% & 100\% \end{pmatrix}$$

Good diversification? (correlation approach)

If R_i represents an asset return (or an excess return), we conclude that (R_1, R_2, R_3) is a well-diversified investment universe

Bad diversification? (payoff approach)

However, we have:

$$R_3 = 0.30 R_1 + 0.70 R_2$$

The correlation puzzle

Fantasies about correlations

- Negative correlations are good for diversification
- Positive correlations are bad for diversification

- If $\rho(R_1, R_2)$ is close to -1, can we hedge Asset 1 with Asset 2?
- If $\rho(R_1, R_2)$ is close to -1, can we diversify Asset 1 with Asset 2?
- If $\rho(R_1, R_2)$ is close to $+1$, can we hedge Asset 1 with a short position on Asset 2?
- If $\rho(R_1, R_2)$ is close to $+1$, can we diversify Asset 1 with a short position on Asset 2?
- Does $\rho(R_1, R_2) = -70\%$ correspond to a better diversification pattern than $\rho(R_1, R_2) = +70\%$?

There is a confusion between diversification and hedging!

The payoff approach

Table 62: Correlation matrix between asset classes (2000-2016)

		Equity				Bond			
		US	Euro	UK	Japan	US	Euro	UK	Japan
Equity	US	100%							
	Euro	78%	100%						
	UK	79%	87%	100%					
	Japan	53%	57%	55%	100%				
Bond	US	−35%	−39%	−32%	−29%	100%			
	Euro	−17%	−16%	−16%	−16%	58%	100%		
	UK	−31%	−37%	−30%	−31%	72%	63%	100%	
	Japan	−17%	−18%	−16%	−33%	37%	31%	36%	100%

Correlation = Pearson correlation = Linear correlation

The payoff approach

Let us consider a Gaussian random vector defined as follows:

$$\begin{pmatrix} Y \\ X \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \mu_y \\ \mu_x \end{pmatrix}, \begin{pmatrix} \Sigma_{yy} & \Sigma_{yx} \\ \Sigma_{xy} & \Sigma_{xx} \end{pmatrix}\right)$$

The conditional distribution of Y given $X = x$ is a MN distribution:

$$\mu_{y|x} = \mathbb{E}\left[Y \mid X = x\right] = \mu_y + \Sigma_{yx}\Sigma_{xx}^{-1}\left(x - \mu_x\right)$$

and:

$$\Sigma_{yy|x} = \sigma^2\left[Y \mid X = x\right] = \Sigma_{yy} - \Sigma_{yx}\Sigma_{xx}^{-1}\Sigma_{xy}$$

We deduce that:

$$\begin{aligned} Y &= \mu_y + \Sigma_{yx}\Sigma_{xx}^{-1}\left(x - \mu_x\right) + u \\ &= \underbrace{\left(\mu_y - \Sigma_{yx}\Sigma_{xx}^{-1}\mu_x\right)}_{\beta_0} + \underbrace{\Sigma_{yx}\Sigma_{xx}^{-1}}_{\beta^\top} x + u \end{aligned}$$

where u is a centered Gaussian random variable with variance $s^2 = \Sigma_{yy|x}$.

The payoff approach

Correlation = linear payoff

It follows that the payoff function is defined by the curve:

$$y = f(x)$$

where:

$$\begin{aligned} f(x) &= \mathbb{E}\left[R_2 \mid R_1 = x\right] \\ &= \left(\mu_2 - \beta_{2|1}\mu_1\right) + \beta_{2|1}x \end{aligned}$$

The payoff approach

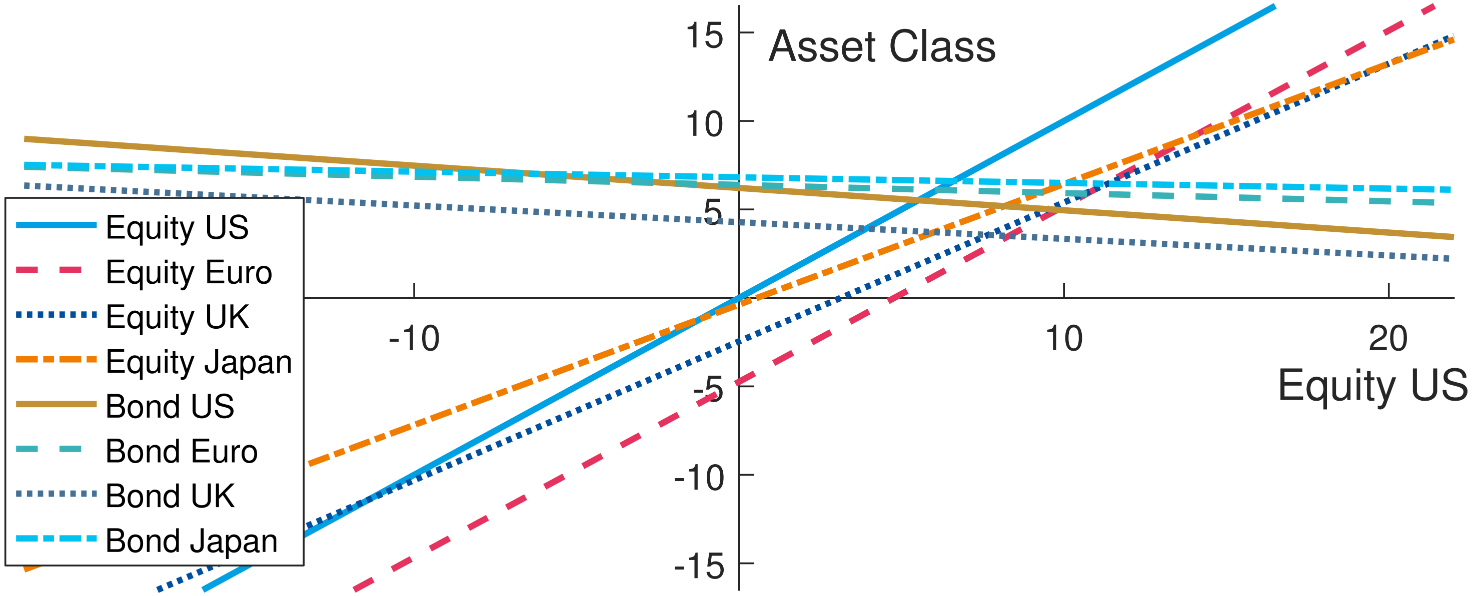


Figure 76: Linear payoff function with respect to the S&P 500 Index

A long-only diversified stock-bond portfolio makes sense!

The payoff approach

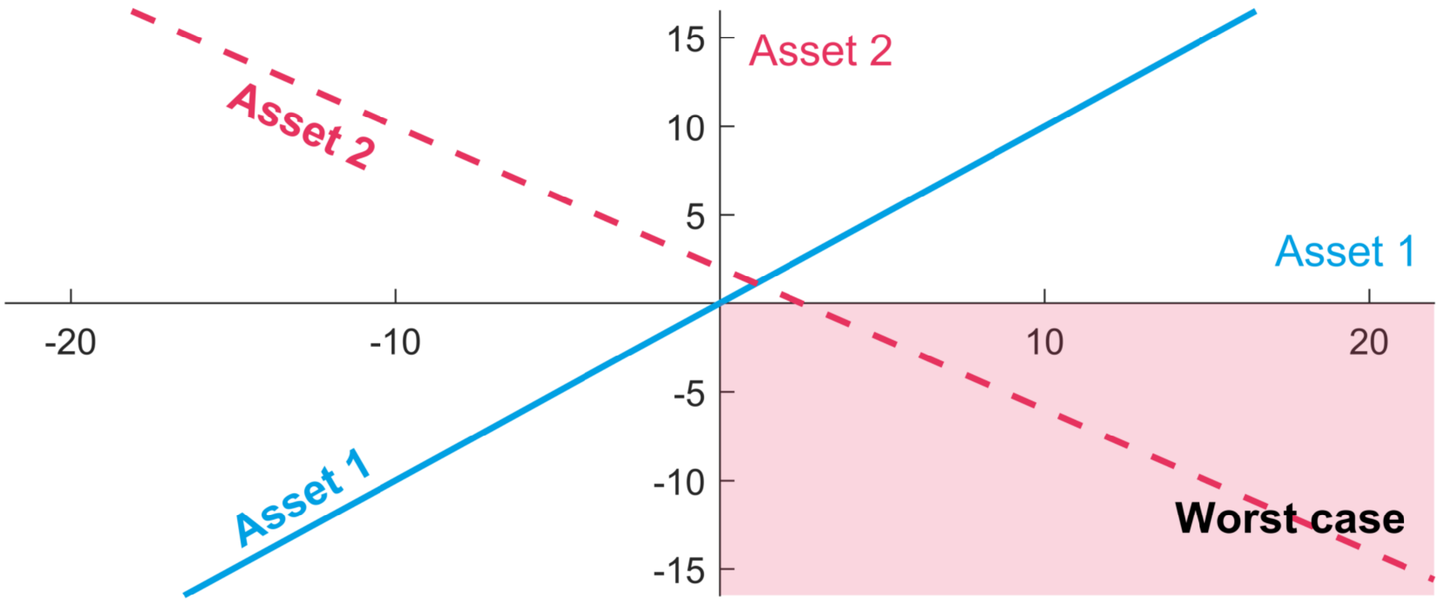


Figure 77: Worst diversification case

What is good diversification? What is bad diversification?

Negative correlation does not necessarily imply good diversification!

The payoff approach

Concave payoff

- Negative skewness
- Positive vega
- Hit ratio $\geq$ 50%
- Gain frequency $>$ loss frequency
- Average gain $<$ average loss
- Positively correlated with bad times

Convex payoff

- Positive skewness
- Negative vega
- Hit ratio $\leq$ 50%
- Gain frequency $<$ loss frequency
- Average gain $>$ average loss
- Negatively correlated with bad times?

Volatility Carry $\neq$ **Time-series Momentum**

The payoff approach

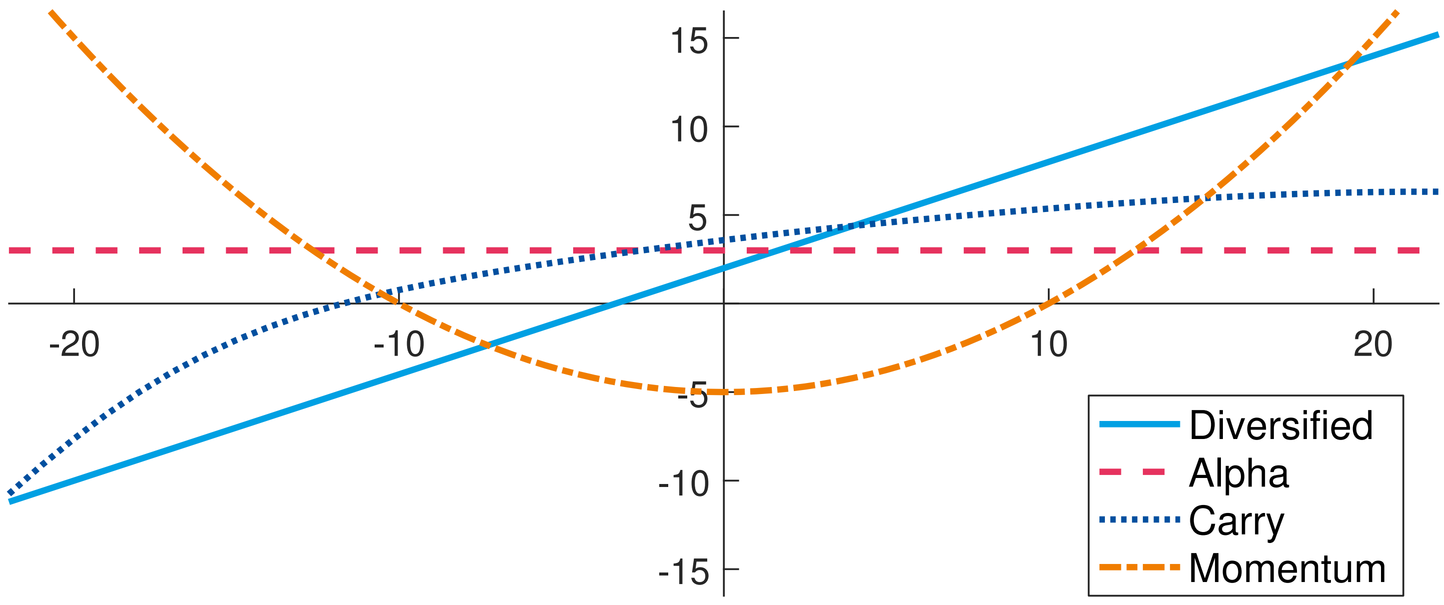


Figure 78: What does portfolio optimization produce with convex and concave strategies?

- Momentum = low allocation during good times and high allocation after bad times
- Carry = high allocation during good times and low allocation after bad times

The payoff approach

The magic formula

Long-run positive correlations, but...

...negative correlations is bad times 🙂

The payoff approach

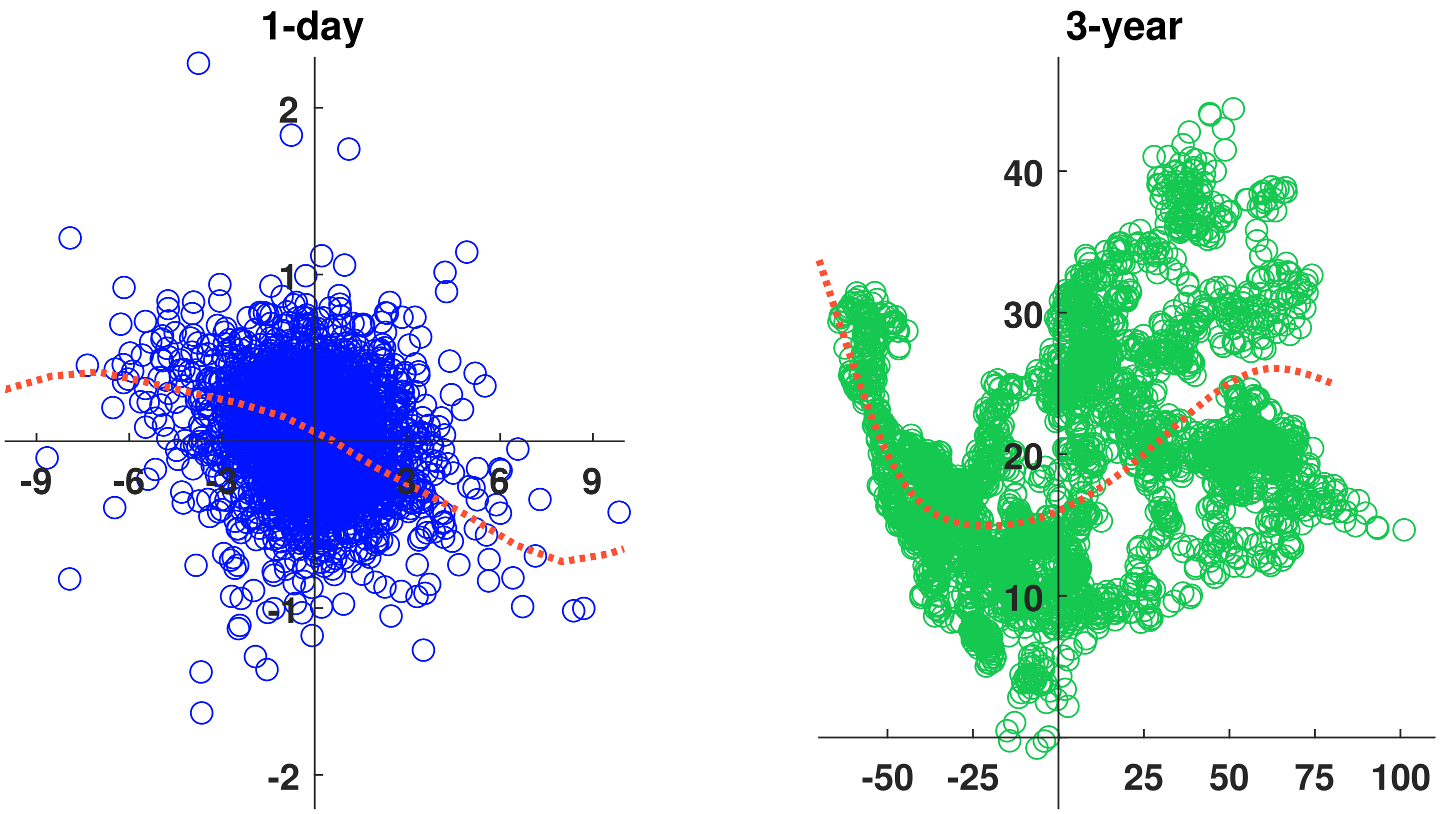


Figure 79: Stock/bond payoff (EUR)

Daily diversification is different than 3-year diversification

Equally-weighted portfolio

Exercise

We note Σ the covariance matrix of n asset returns. In what follows, we consider the equally weighted portfolio based on the universe of these n assets.

Equally-weighted portfolio

Question 1

Let $\Sigma_{i,j} = \rho_{i,j}\sigma_i\sigma_j$ be the elements of the covariance matrix Σ.

Equally-weighted portfolio

Question 1.a

Compute the volatility $\sigma(x)$ of the EW portfolio.

Equally-weighted portfolio

The elements of the covariance matrix are $\Sigma_{i,j} = \rho_{i,j}\sigma_i\sigma_j$. If we consider a portfolio $x = (x_1, \ldots, x_n)$, its volatility is:

$$\begin{aligned}\sigma(x) &= \sqrt{x^\top \Sigma x} \\ &= \sqrt{\sum_{i=1}^n x_i^2\sigma_i^2 + 2\sum_{i>j} x_i x_j \rho_{i,j}\sigma_i\sigma_j}\end{aligned}$$

For the equally weighted portfolio, we have $x_i = n^{-1}$ and:

$$\sigma(x) = \frac{1}{n}\sqrt{\sum_{i=1}^n \sigma_i^2 + 2\sum_{i>j}\rho_{i,j}\sigma_i\sigma_j}$$

Equally-weighted portfolio

Question 1.b

Let $\sigma_0(x)$ and $\sigma_1(x)$ be the volatility of the EW portfolio when the asset returns are respectively independent and perfectly correlated. Calculate $\sigma_0(x)$ and $\sigma_1(x)$.

Equally-weighted portfolio

We have:

$$\sigma_0(x) = \frac{1}{n}\sqrt{\sum_{i=1}^{n} \sigma_i^2}$$

and:

$$\begin{aligned}
\sigma_1(x) &= \frac{1}{n}\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_i \sigma_j} = \frac{1}{n}\sqrt{\sum_{i=1}^{n} \sigma_i \sum_{j=1}^{n} \sigma_j} \\
&= \frac{1}{n}\sqrt{\left(\sum_{i=1}^{n} \sigma_i\right)^2} = \frac{\sum_{i=1}^{n} \sigma_i}{n} \\
&= \bar{\sigma}
\end{aligned}$$

Equally-weighted portfolio

Question 1.c

We assume that the volatilities are the same. Find the expression of the portfolio volatility with respect to the mean correlation $\bar{\rho}$. What is the value of $\sigma(x)$ when $\bar{\rho}$ is equal to zero? What is the value of $\sigma(x)$ when n tends to $+\infty$?

Equally-weighted portfolio

If $\sigma_i = \sigma_j = \sigma$, we obtain:

$$\sigma(x) = \frac{\sigma}{n}\sqrt{n + 2\sum_{i>j}\rho_{i,j}}$$

Let $\bar{\rho}$ be the mean correlation. We have:

$$\bar{\rho} = \frac{2}{n^2 - n}\sum_{i>j}\rho_{i,j}$$

We deduce that:

$$\sum_{i>j}\rho_{i,j} = \frac{n(n-1)}{2}\bar{\rho}$$

Equally-weighted portfolio

We finally obtain:

$$\begin{aligned}\sigma(x) &= \frac{\sigma}{n}\sqrt{n + n(n-1)\bar{\rho}} \\ &= \sigma\sqrt{\frac{1 + (n-1)\bar{\rho}}{n}}\end{aligned}$$

When $\bar{\rho}$ is equal to zero, the volatility $\sigma(x)$ is equal to $\sigma/\sqrt{n}$. When the number of assets tends to $+\infty$, it follows that:

$$\lim_{n\to\infty} \sigma(x) = \sigma\sqrt{\bar{\rho}}$$

Equally-weighted portfolio

Question 1.d

We assume that the correlations are uniform ($\rho_{i,j} = \rho$). Find the expression of the portfolio volatility as a function of $\sigma_0(x)$ and $\sigma_1(x)$. Comment on this result.

Equally-weighted portfolio

If $\rho_{i,j} = \rho$, we obtain:

$$\begin{aligned}
\sigma(x) &= \frac{1}{n}\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\rho_{i,j}\sigma_i\sigma_j} \\
&= \frac{1}{n}\sqrt{\sum_{i=1}^{n}\sigma_i^2 + \rho\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_i\sigma_j - \rho\sum_{i=1}^{n}\sigma_i^2} \\
&= \frac{1}{n}\sqrt{(1-\rho)\sum_{i=1}^{n}\sigma_i^2 + \rho\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_i\sigma_j}
\end{aligned}$$

Equally-weighted portfolio

We have:

$$\sum_{i=1}^{n} \sigma_i^2 = n^2 \sigma_0^2 (x)$$

and:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_i \sigma_j = n^2 \sigma_1^2 (x)$$

It follows that:

$$\sigma (x) = \sqrt{(1 - \rho) \sigma_0^2 (x) + \rho \sigma_1^2 (x)}$$

When the correlation is uniform, the variance $\sigma^2 (x)$ is the weighted average between $\sigma_0^2 (x)$ and $\sigma_1^2 (x)$.

Equally-weighted portfolio

Question 2.a

Compute the normalized risk contributions $\mathcal{RC}_i^{\star}$ of the EW portfolio.

Equally-weighted portfolio

The risk contributions are equal to:

$$\mathcal{RC}_i^\star = \frac{x_i \cdot (\Sigma x)_i}{\sigma^2(x)}$$

In the case of the EW portfolio, we obtain:

$$\begin{aligned}\mathcal{RC}_i^\star &= \frac{\sum_{j=1}^n \rho_{i,j}\sigma_i\sigma_j}{n^2\sigma^2(x)} \\ &= \frac{\sigma_i^2 + \sigma_i \sum_{j\neq i} \rho_{i,j}\sigma_j}{n^2\sigma^2(x)}\end{aligned}$$

Equally-weighted portfolio

Question 2.b

Deduce the risk contributions $\mathcal{RC}_i^\star$ when the asset returns are respectively independent and perfectly correlated[a].

[a] We note them $\mathcal{RC}_{0,i}^\star$ and $\mathcal{RC}_{1,i}^\star$.

Equally-weighted portfolio

If asset returns are independent, we have:

$$\mathcal{RC}^{\star}_{0,i} = \frac{\sigma_i^2}{\sum_{i=1}^{n} \sigma_i^2}$$

In the case of perfect correlation, we obtain:

$$\begin{aligned}
\mathcal{RC}^{\star}_{1,i} &= \frac{\sigma_i^2 + \sigma_i \sum_{j \neq i} \sigma_j}{n^2 \bar{\sigma}^2} \\
&= \frac{\sigma_i \sum_j \sigma_j}{n^2 \bar{\sigma}^2} \\
&= \frac{\sigma_i}{n \bar{\sigma}}
\end{aligned}$$

Equally-weighted portfolio

Question 2.c

Show that the risk contribution $\mathcal{RC}_i$ is proportional to the ratio between the mean correlation of asset i and the mean correlation of the asset universe when the volatilities are the same.

Equally-weighted portfolio

If $\sigma_i = \sigma_j = \sigma$, we obtain:

$$\begin{aligned}
\mathcal{RC}_i^\star &= \frac{\sigma^2 + \sigma^2 \sum_{j \neq i} \rho_{i,j}}{n^2 \sigma^2 (x)} \\
&= \frac{\sigma^2 + (n-1) \sigma^2 \bar{\rho}_i}{n^2 \sigma^2 (x)} \\
&= \frac{1 + (n-1) \bar{\rho}_i}{n (1 + (n-1) \bar{\rho})}
\end{aligned}$$

It follows that:

$$\lim_{n \to \infty} \frac{1 + (n-1) \bar{\rho}_i}{1 + (n-1) \bar{\rho}} = \frac{\bar{\rho}_i}{\bar{\rho}}$$

We deduce that the risk contributions are proportional to the ratio between the mean correlation of asset i and the mean correlation of the asset universe.

Equally-weighted portfolio

Question 2.d

We assume that the correlations are uniform ($\rho_{i,j} = \rho$). Show that the risk contribution $\mathcal{RC}_i$ is a weighted average of $\mathcal{RC}^{\star}_{0,i}$ and $\mathcal{RC}^{\star}_{1,i}$.

Equally-weighted portfolio

We recall that we have:

$$\sigma(x) = \sqrt{(1-\rho)\,\sigma_0^2(x) + \rho\sigma_1^2(x)}$$

It follows that:

$$\begin{aligned}\mathcal{RC}_i &= x_i \cdot \frac{(1-\rho)\,\sigma_0(x)\,\partial_{x_i}\sigma_0(x) + \rho\sigma_1(x)\,\partial_{x_i}\sigma_1(x)}{\sqrt{(1-\rho)\,\sigma_0^2(x) + \rho\sigma_1^2(x)}} \\ &= \frac{(1-\rho)\,\sigma_0(x)\,\mathcal{RC}_{0,i} + \rho\sigma_1(x)\,\mathcal{RC}_{1,i}}{\sqrt{(1-\rho)\,\sigma_0^2(x) + \rho\sigma_1^2(x)}}\end{aligned}$$

We then obtain:

$$\mathcal{RC}_i^\star = \frac{(1-\rho)\,\sigma_0^2(x)}{\sigma^2(x)}\mathcal{RC}_{0,i}^\star + \frac{\rho\sigma_1(x)}{\sigma^2(x)}\mathcal{RC}_{1,i}^\star$$

We verify that the risk contribution $\mathcal{RC}_i$ is a weighted average of $\mathcal{RC}_{0,i}^\star$ and $\mathcal{RC}_{1,i}^\star$.

Equally-weighted portfolio

Question 3

We suppose that the return of asset i satisfies the CAPM model:

$$R_i = \beta_i R_m + \varepsilon_i$$

where R_m is the return of the market portfolio and ε_i is the specific risk. We note $\beta = (\beta_1, \ldots, \beta_n)$ and $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$. We assume that $R_m \perp \varepsilon$, $\operatorname{var}(R_m) = \sigma_m^2$ and $\operatorname{cov}(\varepsilon) = D = \operatorname{diag}\left(\tilde{\sigma}_1^2, \ldots, \tilde{\sigma}_n^2\right)$.

Equally-weighted portfolio

Question 3.a

Calculate the volatility of the EW portfolio.

Equally-weighted portfolio

We have:

$$\Sigma = \beta\beta^\top \sigma_m^2 + D$$

We deduce that:

$$\sigma(x) = \frac{1}{n}\sqrt{\sigma_m^2 \sum_{i=1}^n \sum_{j=1}^n \beta_i \beta_j + \sum_{i=1}^n \tilde{\sigma}_i^2}$$

Equally-weighted portfolio

Question 3.b

Calculate the risk contribution $\mathcal{RC}_i$.

Equally-weighted portfolio

The risk contributions are equal to:

$$\mathcal{RC}_i = \frac{x_i \cdot (\Sigma x)_i}{\sigma(x)}$$

In the case of the EW portfolio, we obtain:

$$\begin{aligned}\mathcal{RC}_i &= \frac{x_i \cdot \left(\sigma_m^2 \beta_i \sum_{j=1}^n x_j \beta_j + x_i \tilde{\sigma}_i^2\right)}{n^2 \sigma(x)} \\ &= \frac{\sigma_m^2 \beta_i \sum_{j=1}^n \beta_j + \tilde{\sigma}_i^2}{n^2 \sigma(x)} \\ &= \frac{n \sigma_m^2 \beta_i \bar{\beta} + \tilde{\sigma}_i^2}{n^2 \sigma(x)}\end{aligned}$$

Equally-weighted portfolio

Question 3.c

Show that $\mathcal{RC}_i$ is approximately proportional to β_i if the number of assets is large. Illustrate this property using a numerical example.

Equally-weighted portfolio

When the number of assets is large and $\beta_i > 0$, we obtain:

$$\mathcal{RC}_i \simeq \frac{\sigma_m^2 \beta_i \bar{\beta}}{n\sigma(x)}$$

because $\bar{\beta} > 0$. We deduce that the risk contributions are approximately proportional to the beta coefficients:

$$\mathcal{RC}_i^\star \simeq \frac{\beta_i}{\sum_{j=1}^n \beta_j}$$

In Figure 80, we compare the exact and approximated values of $\mathcal{RC}_i^\star$. For that, we simulate β_i and $\tilde{\sigma}_i$ with $\beta_i \sim \mathcal{U}_{[0.5,1.5]}$ and $\tilde{\sigma}_i \sim \mathcal{U}_{[0,20\%]}$ whereas σ_m is set to 25%. We notice that the approximated value is very close to the exact value when n increases.

Equally-weighted portfolio

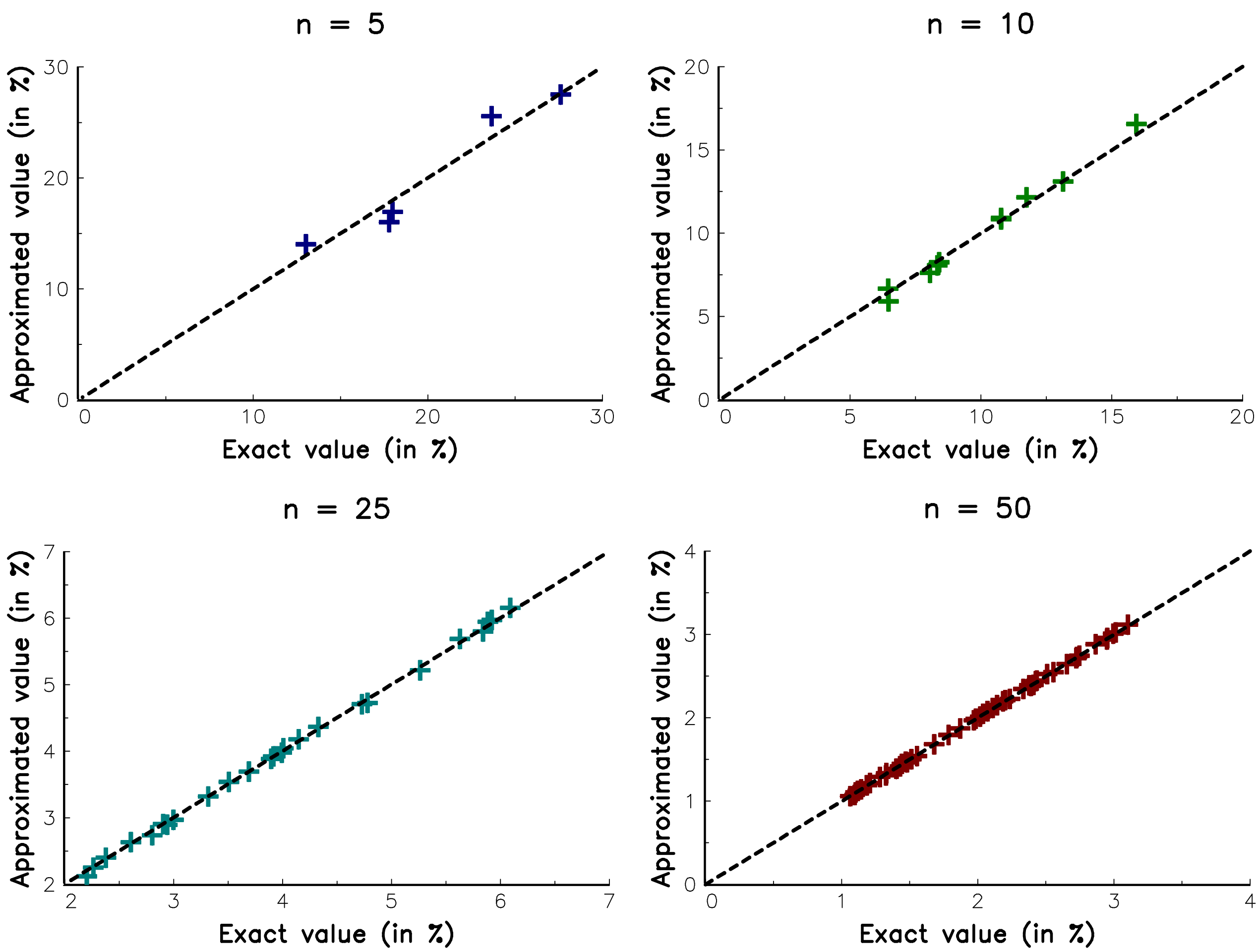


Figure 80: Comparing the exact and approximated values of $\mathcal{RC}_i^\star$

Most diversified portfolio

Exercise

We consider a universe of n assets. We note $\sigma = (\sigma_1, \ldots, \sigma_n)$ the vector of volatilities and Σ the covariance matrix.

Most diversified portfolio

Question 1

In what follows, we consider non-constrained optimized portfolios.

Most diversified portfolio

Question 1.a

Define the diversification ratioDiversification ratio $\mathcal{DR}(x)$ by considering a general risk measure $\mathcal{R}(x)$. How can one interpret this measure from a risk allocation perspective?

Most diversified portfolio

Let $\mathcal{R}(x)$ be the risk measure of the portfolio x. We note $\mathcal{R}_i = \mathcal{R}(\mathbf{e}_i)$ the risk associated to the i^{th} asset. The diversification ratio is the ratio between the weighted mean of the individual risks and the portfolio risk (TR-RPB, page 168):

$$\mathcal{DR}(x) = \frac{\sum_{i=1}^{n} x_i \mathcal{R}_i}{\mathcal{R}(x)}$$

If we assume that the risk measure satisfies the Euler allocation principle, we have:

$$\mathcal{DR}(x) = \frac{\sum_{i=1}^{n} x_i \mathcal{R}_i}{\sum_{i=1}^{n} \mathcal{RC}_i}$$

Most diversified portfolio

Question 1.b

We assume that the weights of the portfolio are positive. Show that $\mathcal{DR}(x) \geq 1$ for all risk measures satisfying the Euler allocation principle. Find an upper bound of $\mathcal{DR}(x)$.

Most diversified portfolio

If $\mathcal{R}(x)$ satisfies the Euler allocation principle, we know that $\mathcal{R}_i \geq \mathcal{MR}_i$ (TR-RPB, page 78). We deduce that:

$$\begin{aligned} \mathcal{DR}(x) &\geq \frac{\sum_{i=1}^{n} x_i \mathcal{R}_i}{\sum_{i=1}^{n} x_i \mathcal{R}_i} \\ &\geq 1 \end{aligned}$$

Let x_{mr} be the portfolio that minimizes the risk measure. We have:

$$\mathcal{DR}(x) \leq \frac{\sup \mathcal{R}_i}{\mathcal{R}(x_{\mathrm{mr}})}$$

Most diversified portfolio

Question 1.c

We now consider the volatility risk measure. Calculate the upper bound of $\mathcal{DR}(x)$.

Most diversified portfolio

If we consider the volatility risk measure, the minimum risk portfolio is the minimum variance portfolio. We have (TR-RPB, page 164):

$$\sigma\left(x_{\mathrm{mv}}\right) = \frac{1}{\sqrt{\mathbf{1}_n^\top \Sigma \mathbf{1}_n}}$$

We deduce that:

$$\mathcal{DR}\left(x\right) \leq \sqrt{\mathbf{1}_n^\top \Sigma^{-1} \mathbf{1}_n} \cdot \sup \sigma_i$$

Most diversified portfolio

Question 1.d

What is the most diversified portfolio (or MDP)? In which case does it correspond to the tangency portfolio? Deduce the analytical expression of the MDP and calculate its volatility.

Most diversified portfolio

The MDP is the portfolio which maximizes the diversification ratio when the risk measure is the volatility (TR-RPB, page 168). We have:

$$\begin{aligned} x^{\star} &= \arg\max \mathcal{DR}(x) \\ \text{u.c.} &\quad \mathbf{1}_n^{\top} x = 1 \end{aligned}$$

Most diversified portfolio

If we consider that the risk premium $\pi_i = \mu_i - r$ of the asset i is proportional to its volatility σ_i, we obtain:

$$\begin{aligned}
\mathrm{SR}(x \mid r) &= \frac{\mu(x) - r}{\sigma(x)} \\
&= \frac{\sum_{i=1}^{n} x_i (\mu_i - r)}{\sigma(x)} \\
&= s \frac{\sum_{i=1}^{n} x_i \sigma_i}{\sigma(x)} \\
&= s \cdot \mathcal{DR}(x)
\end{aligned}$$

where s is the coefficient of proportionality. Maximizing the diversification ratio is equivalent to maximizing the Sharpe ratio.

Most diversified portfolio

We recall that the expression of the tangency portfolio is:

$$x^{\star} = \frac{\Sigma^{-1} (\mu - r \mathbf{1}_n)}{\mathbf{1}_n^{\top} \Sigma^{-1} (\mu - r \mathbf{1}_n)}$$

We deduce that the weights of the MDP are:

$$x^{\star} = \frac{\Sigma^{-1} \sigma}{\mathbf{1}_n^{\top} \Sigma^{-1} \sigma}$$

The volatility of the MDP is then:

$$\begin{aligned} \sigma(x^{\star}) &= \sqrt{\frac{\sigma^{\top} \Sigma^{-1}}{\mathbf{1}_n^{\top} \Sigma^{-1} \sigma} \Sigma \frac{\Sigma^{-1} \sigma}{\mathbf{1}_n^{\top} \Sigma^{-1} \sigma}} \\ &= \frac{\sqrt{\sigma^{\top} \Sigma^{-1} \sigma}}{\mathbf{1}_n^{\top} \Sigma^{-1} \sigma} \end{aligned}$$

Most diversified portfolio

Question 1.e

Demonstrate then that the weights of the MDP are in some sense proportional to $\Sigma^{-1}\sigma$.

Most diversified portfolio

We recall that another expression of the unconstrained tangency portfolio is:

$$x^{\star} = \frac{\sigma^{2}\left(x^{\star}\right)}{\left(\mu\left(x^{\star}\right) - r\right)} \Sigma^{-1}\left(\mu - r\mathbf{1}_{n}\right)$$

We deduce that the MDP is also:

$$x^{\star} = \frac{\sigma^{2}\left(x^{\star}\right)}{\bar{\sigma}\left(x^{\star}\right)} \Sigma^{-1}\sigma$$

where $\bar{\sigma}\left(x^{\star}\right) = x^{\star\top}\sigma$. Nevertheless, this solution is endogenous.

Most diversified portfolio

Question 2

We suppose that the return of asset i satisfies the CAPM:

$$R_i = \beta_i R_m + \varepsilon_i$$

where R_m is the return of the market portfolio and ε_i is the specific risk. We note $\beta = (\beta_1, \ldots, \beta_n)$ and $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$. We assume that $R_m \perp \varepsilon$, $\operatorname{var}(R_m) = \sigma_m^2$ and $\operatorname{cov}(\varepsilon) = D = \operatorname{diag}\left(\tilde{\sigma}_1^2, \ldots, \tilde{\sigma}_n^2\right)$.

Most diversified portfolio

Question 2.a

Compute the correlation $\rho_{i,m}$ between the asset return and the market return. Deduce the relationship between the specific risk $\tilde{\sigma}_i$ and the total risk σ_i of asset i.

Most diversified portfolio

We have:

$$\operatorname{cov}(R_i, R_m) = \beta_i \sigma_m^2$$

We deduce that:

$$\begin{aligned} \rho_{i,m} &= \frac{\operatorname{cov}(R_i, R_m)}{\sigma_i \sigma_m} \\ &= \beta_i \frac{\sigma_m}{\sigma_i} \end{aligned} \tag{4}$$

and:

$$\begin{aligned} \tilde{\sigma}_i &= \sqrt{\sigma_i^2 - \beta_i^2 \sigma_m^2} \\ &= \sigma_i \sqrt{1 - \rho_{i,m}^2} \end{aligned} \tag{5}$$

Most diversified portfolio

Question 2.b

Show that the solution of the MDP may be written as:

$$x_i^\star = \mathcal{DR}\left(x^\star\right) \frac{\sigma_i \sigma\left(x^\star\right)}{\tilde{\sigma}_i^2} \left(1 - \frac{\rho_{i,m}}{\rho^\star}\right) \tag{6}$$

with $\rho^\star$ a scalar to be determined.

Most diversified portfolio

We know that (TR-RPB, page 167):

$$\Sigma^{-1} = D^{-1} - \frac{1}{\sigma_m^{-2} + \tilde{\beta}^\top \beta} \tilde{\beta} \tilde{\beta}^\top$$

where $\tilde{\beta}_i = \beta_i / \tilde{\sigma}_i^2$.

Most diversified portfolio

We deduce that:

$$x^{\star}=\frac{\sigma^{2}\left(x^{\star}\right)}{\bar{\sigma}\left(x^{\star}\right)}\left(D^{-1}\sigma-\frac{1}{\sigma_{m}^{-2}+\tilde{\beta}^{\top}\beta}\tilde{\beta}\tilde{\beta}^{\top}\sigma\right)$$

and:

$$\begin{aligned}
x_{i}^{\star} & = \frac{\sigma^{2}\left(x^{\star}\right)}{\bar{\sigma}\left(x^{\star}\right)}\left(\frac{\sigma_{i}}{\tilde{\sigma}_{i}^{2}}-\frac{\tilde{\beta}^{\top}\sigma}{\sigma_{m}^{-2}+\tilde{\beta}^{\top}\beta}\tilde{\beta}_{i}\right) \\
& = \frac{\sigma_{i}\sigma^{2}\left(x^{\star}\right)}{\bar{\sigma}\left(x^{\star}\right)\tilde{\sigma}_{i}^{2}}\left(1-\frac{\tilde{\beta}^{\top}\sigma}{\sigma_{m}^{-1}+\sigma_{m}\tilde{\beta}^{\top}\beta}\frac{\sigma_{m}\tilde{\sigma}_{i}^{2}\tilde{\beta}_{i}}{\sigma_{i}}\right) \\
& = \frac{\sigma_{i}\sigma^{2}\left(x^{\star}\right)}{\bar{\sigma}\left(x^{\star}\right)\tilde{\sigma}_{i}^{2}}\left(1-\frac{\tilde{\beta}^{\top}\sigma}{\sigma_{m}^{-1}+\sigma_{m}\tilde{\beta}^{\top}\beta}\rho_{i,m}\right) \\
& = \mathcal{DR}\left(x^{\star}\right)\frac{\sigma_{i}\sigma\left(x^{\star}\right)}{\tilde{\sigma}_{i}^{2}}\left(1-\frac{\rho_{i,m}}{\rho^{\star}}\right)
\end{aligned}$$

Most diversified portfolio

Using Equations (4) and (5), $\rho^\star$ is defined as follows:

$$
\begin{aligned}
\rho^\star &= \frac{\sigma_m^{-1} + \sigma_m \tilde{\beta}^\top \beta}{\tilde{\beta}^\top \sigma} \\
&= \left(1 + \sum_{j=1}^{n} \frac{\sigma_m^2 \beta_j^2}{\tilde{\sigma}_j^2}\right) \Bigg/ \left(\sum_{j=1}^{n} \frac{\sigma_m \beta_j \sigma_j}{\tilde{\sigma}_j^2}\right) \\
&= \left(1 + \sum_{j=1}^{n} \frac{\rho_{j,m}^2}{1 - \rho_{j,m}^2}\right) \Bigg/ \left(\sum_{j=1}^{n} \frac{\rho_{j,m}}{1 - \rho_{j,m}^2}\right)
\end{aligned}
$$

Most diversified portfolio

Question 2.c

In which case is the optimal weight $x_i^\star$ positive?

Most diversified portfolio

The optimal weight $x_i^\star$ is positive if:

$$1 - \frac{\rho_{i,m}}{\rho^\star} \geq 0$$

or equivalently:

$$\rho_{i,m} \leq \rho^\star$$

Most diversified portfolio

Question 2.d

Are the weights of the MDP a decreasing or an increasing function of the specific risk $\tilde{\sigma}_i$?

Most diversified portfolio

We recall that:

$$\begin{aligned}\rho_{i,m} &= \beta_i \frac{\sigma_m}{\sigma_i} \\ &= \frac{\beta_i \sigma_m}{\sqrt{\beta_i^2 \sigma_m^2 + \tilde{\sigma}_i^2}}\end{aligned}$$

If $\beta_i < 0$, an increase of the idiosyncratic volatility $\tilde{\sigma}_i$ increases $\rho_{i,m}$ and decreases the ratio $\sigma_i / \tilde{\sigma}_i^2$. We deduce that the weight is a decreasing function of the specific volatility $\tilde{\sigma}_i$. If $\beta_i > 0$, an increase of the idiosyncratic volatility $\tilde{\sigma}_i$ decreases $\rho_{i,m}$ and decreases the ratio $\sigma_i / \tilde{\sigma}_i^2$. We cannot conclude in this case.

Most diversified portfolio

Question 3

In this question, we illustrate that the MDP may be very different than the minimum variance portfolio.

Most diversified portfolio

Question 3.a

In which case does the MDP coincide with the minimum variance portfolio?

Most diversified portfolio

The MDP coincide with the MV portfolio when the volatility is the same for all the assets.

Most diversified portfolio

Question 3.b

We consider the following parameter values:

i	1	2	3	4
β_i	0.80	0.90	1.10	1.20
$\tilde{\sigma}_i$	0.02	0.05	0.15	0.15

with $\sigma_m = 20\%$. Calculate the unconstrained MDP with Formula (6). Compare it with the unconstrained MV portfolio. What is the result if we consider a long-only portfolio?

Most diversified portfolio

The formula cannot be used directly, because it depends on $\sigma(x^\star)$ and $\mathcal{DR}(x^\star)$. However, we notice that:

$$x_i^\star \propto \frac{\sigma_i}{\tilde{\sigma}_i^2}\left(1-\frac{\rho_{i,m}}{\rho^\star}\right)$$

It suffices then to rescale these weights to obtain the solution. Using the numerical values of the parameters, $\rho^\star = 98.92\%$ and we obtain the following results:

	β_i	$\rho_{i,m}$	$x_i \in \mathbb{R}$ MDP	$x_i \in \mathbb{R}$ MV	$x_i \geq 0$ MDP	$x_i \geq 0$ MV
$x_1^\star$	0.80	99.23%	−27.94%	211.18%	0.00%	100.00%
$x_2^\star$	0.90	96.35%	43.69%	−51.98%	25.00%	0.00%
$x_3^\star$	1.10	82.62%	43.86%	−24.84%	39.24%	0.00%
$x_4^\star$	1.20	84.80%	40.39%	−34.37%	35.76%	0.00%
$\sigma(x^\star)$			24.54%	13.42%	23.16%	16.12%

Most diversified portfolio

Question 3.c

We assume that the volatility of the assets is 10%, 10%, 50% and 50% whereas the correlation matrix of asset returns is:

$$\rho = \begin{pmatrix} 1.00 & & & \\ 0.90 & 1.00 & & \\ 0.80 & 0.80 & 1.00 & \\ 0.00 & 0.00 & -0.25 & 1.00 \end{pmatrix}$$

Calculate the (unconstrained and long-only) MDP and MV portfolios.

Most diversified portfolio

The results are:

	$x_i \in \mathbb{R}$		$x_i \geq 0$	
	MDP	MV	MDP	MV
$x_1^\star$	-36.98%	60.76%	0.00%	48.17%
$x_2^\star$	-36.98%	60.76%	0.00%	48.17%
$x_3^\star$	91.72%	-18.54%	50.00%	0.00%
$x_4^\star$	82.25%	-2.98%	50.00%	3.66%
$\sigma(x^\star)$	48.59%	6.43%	30.62%	9.57%

Most diversified portfolio

Question 3.d

Comment on these results.

Most diversified portfolio

These two examples show that the MDP may have a different behavior than the minimum variance portfolio. Contrary to the latter, the most diversified portfolio is not necessarily a low-beta or a low-volatility portfolio.

Computation of risk-based portfolios

Exercise

We consider a universe of five assets. Their expected returns are 6%, 10%, 6%, 8% and 12% whereas their volatilities are equal to 10%, 20%, 15%, 25% and 30%. The correlation matrix of asset returns is defined as follows:

$$\rho = \begin{pmatrix} 100\% & & & & \\ 60\% & 100\% & & & \\ 40\% & 50\% & 100\% & & \\ 30\% & 30\% & 20\% & 100\% & \\ 20\% & 10\% & 10\% & -50\% & 100\% \end{pmatrix}$$

We assume that the risk-free rate is equal to 2%.

Computation of risk-based portfolios

Question 1

We consider unconstrained portfolios. For each portfolio, compute the risk decomposition.

Computation of risk-based portfolios

Question 1.a

Find the tangency portfolio.

Computation of risk-based portfolios

To compute the unconstrained tangency portfolio, we use the analytical formula (TR-RPB, page 14):

$$x^{\star} = \frac{\Sigma^{-1}\left(\mu - r\mathbf{1}_n\right)}{\mathbf{1}_n^{\top}\Sigma^{-1}\left(\mu - r\mathbf{1}_n\right)}$$

We obtain the following results:

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^{\star}$
1	11.11%	6.56%	0.73%	5.96%
2	17.98%	13.12%	2.36%	19.27%
3	2.55%	6.56%	0.17%	1.37%
4	33.96%	9.84%	3.34%	27.31%
5	34.40%	16.40%	5.64%	46.09%

Computation of risk-based portfolios

Question 1.b

Determine the equally weighted portfolio.

Computation of risk-based portfolios

We obtain the following results for the equally weighted portfolio:

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1	20.00%	7.47%	1.49%	13.43%
2	20.00%	15.83%	3.17%	28.48%
3	20.00%	9.98%	2.00%	17.96%
4	20.00%	9.89%	1.98%	17.80%
5	20.00%	12.41%	2.48%	22.33%

Computation of risk-based portfolios

Question 1.c

Compute the minimum variance portfolio.

Computation of risk-based portfolios

For the minimum variance portfolio, we have:

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1	74.80%	9.08%	6.79%	74.80%
2	−15.04%	9.08%	−1.37%	−15.04%
3	21.63%	9.08%	1.96%	21.63%
4	10.24%	9.08%	0.93%	10.24%
5	8.36%	9.08%	0.76%	8.36%

Computation of risk-based portfolios

Question 1.d

Calculate the most diversified portfolio.

Computation of risk-based portfolios

For the most diversified portfolio, we have:

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^{\star}$
1	-14.47%	4.88%	-0.71%	-5.34%
2	4.83%	9.75%	0.47%	3.56%
3	18.94%	7.31%	1.38%	10.47%
4	49.07%	12.19%	5.98%	45.24%
5	41.63%	14.63%	6.09%	46.06%

Computation of risk-based portfolios

Question 1.e

Find the ERC portfolio.

Computation of risk-based portfolios

For the ERC portfolio, we have:

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1	27.20%	7.78%	2.12%	20.00
2	13.95%	15.16%	2.12%	20.00
3	20.86%	10.14%	2.12%	20.00
4	19.83%	10.67%	2.12%	20.00
5	18.16%	11.65%	2.12%	20.00

Computation of risk-based portfolios

Question 1.f

Compare the expected return $\mu(x)$, the volatility $\sigma(x)$ and the Sharpe ratio $\mathrm{SR}(x \mid r)$ of the different portfolios. Calculate then the tracking error volatility $\sigma(x \mid b)$, the beta $\beta(x \mid b)$ and the correlation $\rho(x \mid b)$ if we assume that the benchmark b is the tangency portfolio.

Computation of risk-based portfolios

We recall the definition of the statistics:

$$
\begin{aligned}
\mu(x) &= \mu^\top x \\
\sigma(x) &= \sqrt{x^\top \Sigma x} \\
\mathrm{SR}(x \mid r) &= \frac{\mu(x) - r}{\sigma(x)} \\
\sigma(x \mid b) &= \sqrt{(x-b)^\top \Sigma (x-b)} \\
\beta(x \mid b) &= \frac{x^\top \Sigma b}{b^\top \Sigma b} \\
\rho(x \mid b) &= \frac{x^\top \Sigma b}{\sqrt{x^\top \Sigma x}\sqrt{b^\top \Sigma b}}
\end{aligned}
$$

Computation of risk-based portfolios

We obtain the following results:

Statistic	$x^{\star}$	x_{ew}	x_{mv}	x_{mdp}	x_{erc}
$\mu(x)$	9.46%	8.40%	6.11%	9.67%	8.04%
$\sigma(x)$	12.24%	11.12%	9.08%	13.22%	10.58%
$\mathrm{SR}(x \mid r)$	60.96%	57.57%	45.21%	58.03%	57.15%
$\sigma(x \mid b)$	0.00%	4.05%	8.21%	4.06%	4.35%
$\beta(x \mid b)$	100.00%	85.77%	55.01%	102.82%	81.00%
$\rho(x \mid b)$	100.00%	94.44%	74.17%	95.19%	93.76%

We notice that all the portfolios present similar performance in terms of Sharpe Ratio. The minimum variance portfolio shows the smallest Sharpe ratio, but it also shows the lowest correlation with the tangency portfolio.

Computation of risk-based portfolios

Question 2

Same questions if we impose the long-only portfolio constraint.

Computation of risk-based portfolios

The tangency portfolio, the equally weighted portfolio and the ERC portfolio are already long-only. For the minimum variance portfolio, we obtain:

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^{\star}$
1	65.85%	9.37%	6.17%	65.85%
2	0.00%	13.11%	0.00%	0.00%
3	16.72%	9.37%	1.57%	16.72%
4	9.12%	9.37%	0.85%	9.12%
5	8.32%	9.37%	0.78%	8.32%

Computation of risk-based portfolios

For the most diversified portfolio, we have:

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1	0.00%	5.50%	0.00%	0.00%
2	1.58%	9.78%	0.15%	1.26%
3	16.81%	7.34%	1.23%	10.04%
4	44.13%	12.23%	5.40%	43.93%
5	37.48%	14.68%	5.50%	44.77%

Computation of risk-based portfolios

The results become:

Statistic	$x^{\star}$	x_{ew}	x_{mv}	x_{mdp}	x_{erc}
$\mu(x)$	9.46%	8.40%	6.68%	9.19%	8.04%
$\sigma(x)$	12.24%	11.12%	9.37%	12.29%	10.58%
$\mathrm{SR}(x \mid r)$	60.96%	57.57%	49.99%	58.56%	57.15%
$\sigma(x \mid b)$	0.00%	4.05%	7.04%	3.44%	4.35%
$\beta(x \mid b)$	100.00%	85.77%	62.74%	96.41%	81.00%
$\rho(x \mid b)$	100.00%	94.44%	82.00%	96.06%	93.76%

Building a carry trade exposure

Question 1

We would like to build a carry trade strategy using a *cash neutral* portfolio with equal weights and a notional amount of $100 mn. We use the data given in Table 63. The holding period is equal to three months.

Table 63: Three-month interest rates (March, 15$^{\text{th}}$ 2000)

Currency	AUD	CAD	CHF	EUR	GBP
Interest rate (in %)	5.74	5.37	2.55	3.79	6.21
Currency	JPY	NOK	NZD	SEK	USD
Interest rate (in %)	0.14	5.97	6.24	4.18	6.17

Building a carry trade exposure

Question 1.a

Build the carry trade exposure with two funding currencies and two asset currencies.

Building a carry trade exposure

We rank the currencies according to their interest rate from the lowest to the largest value:

1. JPY	2. CHF	3. EUR	4. SEK	5. CAD
6. AUD	7. NOK	8. USD	9. GBP	10. NZD

We deduce that the carry trade portfolio is:

1. long \$50 mn on NZD
2. long \$50 mn on GBP
3. short \$50 mn on JPY
4. short \$50 mn on CHF

Building a carry trade exposure

Question 1.b

Same question with five funding currencies and two asset currencies.

Building a carry trade exposure

The portfolio becomes:

1. long \$50 mn on NZD and GBP
2. short \$20 mn on JPY, CHF, EUR, SEK and CAD

Building a carry trade exposure

Question 1.c

What is the specificity of the portfolio if we use five funding currencies and five asset currencies.

Building a carry trade exposure

The portfolio is:

1. long \$20 mn on NZD, GBP, USD, NOK and AUD
2. short \$20 mn on JPY, CHF, EUR, SEK and CAD

The asset notional is not equal to the funding notional, because the funding notional is equal to \$100 mn and the asset notional is equal to \$80 mn. Indeed, we don't need to invest the \$20 mn USD exposure since the portfolio currency is the US dollar.

Building a carry trade exposure

Question 1.d

Calculate an approximation of the carry trade P&L if we assume that the spot foreign exchange rates remain constant during the next three months.

Building a carry trade exposure

If we consider the last portfolio, we have:

$$\begin{aligned}
\mathrm{PnL} &\approx 20 \times \frac{1}{4}(6.24\% + 6.21\% + 6.17\% + 5.97\% + 5.74\%) - \\
& \quad 20 \times \frac{1}{4}(0.14\% + 2.55\% + 3.79\% + 4.18\% + 5.37\%) \\
&= \$0.71 \text{ mn}
\end{aligned}$$

If the spot foreign exchange rates remain constant during the next three months, the quarterly P&L is approximated equal to \$710 000.

Building a carry trade exposure

Question 2

We consider the data given in Tables 64 and 65.

Table 64: Three-month interest rates (March, 21$^{\text{th}}$ 2005)

Currency	BRL	CZK	HUF	KRW	MXN
Interest rate (in %)	18.23	2.45	8.95	3.48	8.98
Currency	PLN	SGD	THB	TRY	TWD
Interest rate (in %)	6.63	1.44	2.00	19.80	1.30

Table 65: Annualized volatility of foreign exchange rates (March, 21$^{\text{th}}$ 2005)

Currency	BRL	CZK	HUF	KRW	MXN
Volatility (in %)	11.19	12.57	12.65	6.48	6.80
Currency	PLN	SGD	THB	TRY	TWD
Volatility (in %)	11.27	4.97	4.26	11.61	4.12

Building a carry trade exposure

Question 2.a

Let Σ be the covariance matrix of the currency returns. Which expected returns are used by the carry investor? Write the mean-variance optimization problem if we assume a cash neutral portfolio.

Building a carry trade exposure

Let $\mathcal{C}_i$ and $\mathcal{C} = (\mathcal{C}_1, \ldots, \mathcal{C}_n)$ be the carry of Currency i and the vector of carry values. The carry investor assumes that $\mu_i = \mathcal{C}_i$. We deduce that the mean-variance optimization problem is:

$$\begin{aligned} x^\star(\gamma) &= \arg\min \frac{1}{2} x^\top \Sigma x - \gamma x^\top \mathcal{C} \\ \text{u.c.} &\quad \mathbf{1}_n^\top x = 0 \end{aligned}$$

The constraint $\mathbf{1}_n^\top x = 0$ indicates that the portfolio is cash neutral. If we target a portfolio volatility $\sigma^\star$, we use the bisection algorithm in order to find the optimal value of γ such that:

$$\sigma\left(x^\star(\gamma)\right) = \sigma^\star$$

Building a carry trade exposure

Question 2.b

By assuming a zero correlation between the currencies, calibrate the cash neutral portfolio when the objective function is to target a 3% portfolio volatility.

Building a carry trade exposure

We obtain the following solution:

Currency	BRL	CZK	HUF	KRW	MXN
Weight	15.05%	−1.28%	4.11%	−1.57%	14.30%
Currency	PLN	SGD	THB	TRY	TWD
Weight	2.76%	−13.59%	−14.42%	15.52%	−20.87%

Building a carry trade exposure

Question 2.c

Same question if we use the following correlation matrix:

$$\rho = \begin{pmatrix}
1.00 & & & & & & & & & \\
0.30 & 1.00 & & & & & & & & \\
0.38 & 0.80 & 1.00 & & & & & & & \\
0.00 & 0.04 & 0.08 & 1.00 & & & & & & \\
0.50 & 0.30 & 0.34 & 0.12 & 1.00 & & & & & \\
0.35 & 0.70 & 0.78 & 0.06 & 0.30 & 1.00 & & & & \\
0.33 & 0.49 & 0.56 & 0.29 & 0.27 & 0.53 & 1.00 & & & \\
0.30 & 0.34 & 0.34 & 0.38 & 0.29 & 0.35 & 0.53 & 1.00 & & \\
0.43 & 0.39 & 0.48 & 0.10 & 0.38 & 0.41 & 0.35 & 0.43 & 1.00 & \\
0.03 & 0.07 & 0.06 & 0.63 & 0.09 & 0.07 & 0.30 & 0.40 & 0.20 & 1.00
\end{pmatrix}$$

Building a carry trade exposure

The solution becomes:

Currency	BRL	CZK	HUF	KRW	MXN
Weight	13.69%	−9.45%	4.58%	17.31%	6.56%
Currency	PLN	SGD	THB	TRY	TWD
Weight	2.07%	−17.79%	−20.86%	17.98%	−14.10%

Building a carry trade exposure

Question 2.d

Calculate the carry of this optimized portfolio. For each currency, deduce the maximum value of the devaluation (or revaluation) rate that is compatible with a positive P&L.

Building a carry trade exposure

The carry of the portfolio is equal to:

$$\mathcal{C}(x) = \sum_{i=1}^{n} x_i \cdot \mathcal{C}_i$$

We find $\mathcal{C}(x) = 6.7062\%$ per year. We deduce that the maximum value of the devaluation or revaluation rate D_i that is compatible with a positive P&L is equal to:

$$D_i = \frac{6.7062\%}{4} = 1.6765\%$$

This figure is valid for an exposure of 100%.

Building a carry trade exposure

By considering the weights, we deduce that:

$$D_i = -\frac{\mathcal{C}(x)}{4x_i}$$

Finally, we obtain the following compatible devaluation (negative sign $-$) and revaluation (positive sign $+$) rates:

Currency	BRL	CZK	HUF	KRW	MXN
D_i	-12.25%	$+17.75\%$	-36.64%	-9.69%	-25.55%
Currency	PLN	SGD	THB	TRY	TWD
D_i	-81.08%	$+9.43\%$	$+8.04\%$	-9.32%	$+11.89\%$

Building a carry trade exposure

Question 2.e

Repeat Question 2.b assuming that the volatility target is equal 5%.
Calculate the leverage ratio. Comment on these results.

Building a carry trade exposure

We obtain the following results:

Currency	BRL	CZK	HUF	KRW	MXN
Weight	25.08%	−2.13%	6.84%	−2.62%	23.83%
Currency	PLN	SGD	THB	TRY	TWD
Weight	4.60%	−22.65%	−24.03%	25.86%	−34.78%

The leverage ratio of this portfolio is equal to $\sum_{i=1}^{n} |x_i| = 172.43\%$, whereas it is equal to 103.47% and 124.37% for the portfolios of Questions 2.b and 2.c. This is perfectly normal because the leverage is proportional to the volatility.

Building a carry trade exposure

Question 2.f

Find the analytical solution of the optimal portfolio $x^\star$ when we target a volatility $\sigma^\star$.

Building a carry trade exposure

The Lagrange function is equal to:

$$\mathcal{L}(x;\lambda_0) = \frac{1}{2}x^\top \Sigma x - \gamma x^\top \mathcal{C} + \lambda_0 \left(\mathbf{1}_n^\top x - 0\right)$$

The first-order condition is equal to:

$$\frac{\partial\, \mathcal{L}(x;\lambda_0)}{\partial\, x} = \Sigma x - \gamma \mathcal{C} + \lambda_0 \mathbf{1}_n = \mathbf{0}_n$$

It follows that:

$$x = \Sigma^{-1}\left(\gamma \mathcal{C} - \lambda_0 \mathbf{1}_n\right)$$

Building a carry trade exposure

The cash neutral constraint implies that:

$$\mathbf{1}_n^\top \Sigma^{-1} \left(\gamma \mathcal{C} - \lambda_0 \mathbf{1}_n \right) = 0$$

We deduce that:

$$\lambda_0 = \gamma \frac{\mathbf{1}_n^\top \Sigma^{-1} \mathcal{C}}{\mathbf{1}_n^\top \Sigma^{-1} \mathbf{1}_n}$$

Therefore, the optimal solution is equal to:

$$x^\star = \frac{\gamma \Sigma^{-1}}{\mathbf{1}_n^\top \Sigma^{-1} \mathbf{1}_n} \left(\left(\mathbf{1}_n^\top \Sigma^{-1} \mathbf{1}_n \right) \mathcal{C} - \left(\mathbf{1}_n^\top \Sigma^{-1} \mathcal{C} \right) \mathbf{1}_n \right)$$

Building a carry trade exposure

The volatility of the optimal portfolio is equal:

$$
\begin{aligned}
\sigma^2 \left(x^\star\right) &= x^{\star\top} \Sigma x^\star \\
&= \left(\gamma \mathcal{C}^\top - \lambda_0 \mathbf{1}_n^\top\right) \Sigma^{-1} \Sigma \Sigma^{-1} \left(\gamma \mathcal{C} - \lambda_0 \mathbf{1}_n\right) \\
&= \left(\gamma \mathcal{C}^\top - \lambda_0 \mathbf{1}_n^\top\right) \Sigma^{-1} \left(\gamma \mathcal{C} - \lambda_0 \mathbf{1}_n\right) \\
&= \gamma^2 \mathcal{C}^\top \Sigma^{-1} \mathcal{C} + \lambda_0^2 \mathbf{1}_n^\top \Sigma^{-1} \mathbf{1}_n - 2\gamma\lambda_0 \mathcal{C}^\top \Sigma^{-1} \mathbf{1}_n \\
&= \gamma^2 \left(\mathcal{C}^\top \Sigma^{-1} \mathcal{C} - \frac{\left(\mathbf{1}_n^\top \Sigma^{-1} \mathcal{C}\right)^2}{\mathbf{1}_n^\top \Sigma^{-1} \mathbf{1}_n}\right) \\
&= \frac{\gamma^2}{\mathbf{1}_n^\top \Sigma^{-1} \mathbf{1}_n} \left(\left(\mathbf{1}_n^\top \Sigma^{-1} \mathbf{1}_n\right) \left(\mathcal{C}^\top \Sigma^{-1} \mathcal{C}\right) - \left(\mathbf{1}_n^\top \Sigma^{-1} \mathcal{C}\right)^2\right)
\end{aligned}
$$

Building a carry trade exposure

We deduce that:

$$\gamma = \frac{\sqrt{\mathbf{1}_n^\top \Sigma^{-1} \mathbf{1}_n}}{\sqrt{\left(\mathbf{1}_n^\top \Sigma^{-1} \mathbf{1}_n\right)\left(\mathcal{C}^\top \Sigma^{-1} \mathcal{C}\right) - \left(\mathbf{1}_n^\top \Sigma^{-1} \mathcal{C}\right)^2}} \sigma\left(x^\star\right)$$

Finally, we obtain:

$$\begin{aligned} x^\star &= \sigma\left(x^\star\right) \frac{\Sigma^{-1}\left(\left(\mathbf{1}_n^\top \Sigma^{-1} \mathbf{1}_n\right) \mathcal{C} - \left(\mathbf{1}_n^\top \Sigma^{-1} \mathcal{C}\right) \mathbf{1}_n\right)}{\sqrt{\left(\mathbf{1}_n^\top \Sigma^{-1} \mathbf{1}_n\right)^2 \left(\mathcal{C}^\top \Sigma^{-1} \mathcal{C}\right) - \left(\mathbf{1}_n^\top \Sigma^{-1} \mathbf{1}_n\right)\left(\mathbf{1}_n^\top \Sigma^{-1} \mathcal{C}\right)^2}} \\ &= \sigma^\star \frac{\Sigma^{-1}\left(\left(\mathbf{1}_n^\top \Sigma^{-1} \mathbf{1}_n\right) \mathcal{C} - \left(\mathbf{1}_n^\top \Sigma^{-1} \mathcal{C}\right) \mathbf{1}_n\right)}{\sqrt{\left(\mathbf{1}_n^\top \Sigma^{-1} \mathbf{1}_n\right)^2 \left(\mathcal{C}^\top \Sigma^{-1} \mathcal{C}\right) - \left(\mathbf{1}_n^\top \Sigma^{-1} \mathbf{1}_n\right)\left(\mathbf{1}_n^\top \Sigma^{-1} \mathcal{C}\right)^2}} \end{aligned}$$

Building a carry trade exposure

Question 2.g

We assume that the correlation matrix is the identity matrix I_n. Find the expression of the threshold value $\mathcal{C}^{\star}$ such that all currencies with a carry $\mathcal{C}_i$ larger than $\mathcal{C}^{\star}$ form the long leg of the portfolio.

Building a carry trade exposure

We recall that:

$$x^{\star} \propto \Sigma^{-1}\left(\left(\mathbf{1}_n^{\top}\Sigma^{-1}\mathbf{1}_n\right)\mathcal{C} - \left(\mathbf{1}_n^{\top}\Sigma^{-1}\mathcal{C}\right)\mathbf{1}_n\right)$$

If $\rho = I_n$, we have:

$$\mathbf{1}_n^{\top}\Sigma^{-1}\mathbf{1}_n = \sum_{j=1}^{n}\frac{1}{\sigma_j^2}$$

and:

$$\mathbf{1}_n^{\top}\Sigma^{-1}\mathcal{C} = \sum_{j=1}^{n}\frac{\mathcal{C}_j}{\sigma_j^2}$$

We deduce that:

$$x_i^{\star} \propto \frac{1}{\sigma_i^2}\left(\left(\sum_{j=1}^{n}\frac{1}{\sigma_j^2}\right)\mathcal{C}_i - \left(\sum_{j=1}^{n}\frac{\mathcal{C}_j}{\sigma_j^2}\right)\right)$$

Building a carry trade exposure

The portfolio is long on the currency i if:

$$\mathcal{C}_i \geq \mathcal{C}^\star$$

where:

$$\mathcal{C}^\star = \left(\sum_{j=1}^n \frac{1}{\sigma_j^2}\right)^{-1} \left(\sum_{j=1}^n \frac{\mathcal{C}_j}{\sigma_j^2}\right) = \sum_{j=1}^n \omega_j \mathcal{C}_j$$

and:

$$\omega_j = \frac{\sigma_j^{-2}}{\sum_{k=1}^n \sigma_k^{-2}}$$

$\mathcal{C}^\star$ is the weighted mean of the carry values and the weights are inversely proportional to the variance of the currency returns.

Main references

- Roncalli, T. (2013)
  Introduction to Risk Parity and Budgeting, Chapman and Hall/CRC Financial Mathematics Series, Chapter 3.
- Roncalli, T. (2013)
  Introduction to Risk Parity and Budgeting — Companion Book, Chapman and Hall/CRC Financial Mathematics Series, Chapter 3.
- Roncalli, T. (2017)
  Alternative Risk Premia: What Do We Know?, in Jurczenko, E. (Ed.), *Factor Investing and Alternative Risk Premia*, ISTE Press – Elsevier.

References I

ACHARYA, V.V., and Pedersen, L.H. (2005)
Asset Pricing with Liquidity Risk, *Journal of Financial Economics*, 77(2), pp. 375-410.

BAKU, E., FORTES, R., HERVÉ, K., LEZMI, E., MALONGO, H., RONCALLI, T., and XU, J. (2019)
Factor Investing in Currency Markets: Does it Make Sense?, *Amundi Working Paper*, 89, `www.research-center.amundi.com`.

BAKU, E., FORTES, R., HERVÉ, K., LEZMI, E., MALONGO, H., RONCALLI, T., and XU, J. (2020)
Factor Investing in Currency Markets: Does it Make Sense?, *Journal of Portfolio Management*, 46(2), pp. 141-155.

References II

BARBERIS, N., and THALER, R.H. (2003)
A Survey of Behavioral Finance, in Constantinides, G.M., Harris, M. and Stulz, R.M. (Eds), *Handbook of the Economics of Finance*, 1(B), Elsevier, pp. 1053-1128.

BEKTIC, D., WENZLER, J.S., WEGENER, M., SCHIERECK, D., and SPIELMANN, T. (2019)
Extending Fama-French Factors to Corporate Bond Markets, *Journal of Portfolio Management*, 45(3), pp. 141-158.

BEKTIC, D., NEUGEBAUER, U., WEGENER, M. and WENZLER, J.S. (2017)
Common Equity Factors in Corporate Bond Markets, in Jurczenko, E. (Ed.), *Factor Investing and Alternative Risk Premia*, ISTE Press-Elsevier.

References III

Ben Slimane, M., De Jong, M., Dumas, J-M., Fredj, H., Sekine, T., and Srb, M. (2019)
Traditional and Alternative Factors in Investment Grade Corporate Bond Investing, *Amundi Working Paper*, 78, www.research-center.amundi.com.

Black, F. (1972)
Capital Market Equilibrium with Restricted Borrowing, *Journal of Business*, 45(3), pp. 444-455.

Bruder, B., Dao, T.L., Richard, J.C., and Roncalli, T. (2011)
Trend Filtering Methods for Momentum Strategies, *SSRN*, www.ssrn.com/abstract=2289097.

Carhart, M.M. (1997)
On Persistence in Mutual Fund Performance, *Journal of Finance*, 52(1), pp. 57-82.

References IV

- CAZALET, Z., and RONCALLI, T. (2014)
  Facts and Fantasies About Factor Investing, *SSRN*,
  www.ssrn.com/abstract=2524547.

- CHOUEIFATY, Y., and COIGNARD, Y. (2008)
  Toward Maximum Diversification, *Journal of Portfolio Management*,
  35(1), pp. 40-51.

- CHOUEIFATY, Y., FROIDURE, T., and REYNIER, J. (2013)
  Properties of the Most Diversified Portflio, *Journal of Investment Strategies*, 2(2), pp. 49-70.

- CLARKE, R.G., DE SILVA, H., and THORLEY, S. (2011)
  Minimum Variance Portfolio Composition, *Journal of Portfolio Management*, 37(2), pp. 31-45.

References V

CLARKE, R.G., DE SILVA, H., and THORLEY, S. (2013)
Risk Parity, Maximum Diversification, and Minimum Variance: An Analytic Perspective, *Journal of Portfolio Management*, 39(3), pp. 39-53.

COCHRANE, J.H. (2001)
Asset Pricing, Princeton University Press.

COCHRANE, J.H. (2011)
Presidential Address: Discount Rates, *Journal of Finance*, 66(4), pp. 1047-1108.

DEMIGUEL, V., GARLAPPI, L., and UPPAL, R. (2009)
Optimal Versus Naive Diversification: How Inefficient is the 1/N Portfolio Strategy?, *Review of Financial Studies*, 22(5), pp. 1915-1953.

References VI

FAMA, E.F., and FRENCH, K.R. (1992)
The Cross-Section of Expected Stock Returns, *Journal of Finance*, 47(2), pp. 427-465.

FAMA, E.F., and FRENCH, K.R. (1993)
Common Risk Factors in the Returns on Stocks and Bonds, *Journal of Financial Economics*, 33(1), pp. 3-56.

FRAZZINI, A, and PEDERSEN, L.H. (2014)
Betting Against Beta, *Journal of Financial Economics*, 111(1), pp. 1-25.

GRINBLATT, M., TITMAN, S., and WERMERS, R. (1995)
Momentum Investment Strategies, Portfolio Performance, and Herding: A Study of Mutual Fund Behavior, *American Economic Review*, 85(5), pp. 1088-1105.

References VII

HAMDAN, R., PAVLOWSKY, F., RONCALLI, T., and ZHENG, B. (2016)
A Primer on Alternative Risk Premia, *SSRN*,
www.ssrn.com/abstract=2766850.

HAUGEN, R.A. and BAKER, N.L. (1991)
The Efficient Market Inefficiency of Capitalization-weighted Stock Portfolios, *Journal of Portfolio Management*, 17(3), pp. 35-40.

HARVEY, C.R., LIU, Y., and ZHU, H. (2016)
...and the Cross-Section of Expected Returns, *Review of Financial Studies*, 29(1), pp. 5-68.

HOUWELING, P., and VAN ZUNDERT, J. (2017)
Factor Investing in the Corporate Bond Market, *Financial Analysts Journal*, 73(2), pp. 100-115.

References VIII

ILMANEN, A. (2011)
Expected Returns: An Investor's Guide to Harvesting Market Rewards, Wiley.

ISRAEL, R., PALHARES, D., and RICHARDSON, S.A. (2018)
Common Factors in Corporate Bond and Bond Fund Returns, *Journal of Investment Management*, 16(2), pp. 17-46

JENSEN, M.C. (1968)
The Performance of Mutual Funds in the Period 1945-1964, *Journal of Finance*, 23(2), pp. 389-416.

JURCZENKO, E., MICHEL, T. and TEÏLETCHE, J. (2015)
A Unified Framework for Risk-based Investing, *Journal of Investment Strategies*, 4(4), pp. 1-29.

References IX

Jusselin, P., Lezmi, E., Malongo, H., Masselin, C., Roncalli, T., and Dao, T-L. (2017)
Understanding the Momentum Risk Premium: An In-Depth Journey Through Trend-Following Strategies, *SSRN*, www.ssrn.com/abstract=3042173.

Koijen, R.S.J., Moskowitz, T.J., Pedersen, L.H., and Vrugt, E.B. (2018)
Carry, *Journal of Financial Economics*, 127(2), pp. 197-225.

Lucas, R.E. (1978)
Asset Prices in an Exchange Economy, *Econometrica*, 46(6), pp. 1429-1445.

MacDonald, R. (1995)
Long-Run Exchange Rate Modeling: a Survey of the Recent Evidence, *IMF Staff Papers*, 42(3), pp. 437-489.

References X

- Menkhoff, L., Sarno, L., Schmeling, M., and Schrimpf, A. (2016)
  Currency Value, *Review of Financial Studies*, 30(2), pp. 416-441.
- Roncalli, T. (2017)
  ESG & Factor Investing: A New Stage Has Been Reached, *Amundi ViewPoint*, `www.research-center.amundi.com`.
- Roncalli, T. (2020)
  Handbook of Financial Risk Management, Chapman and Hall/CRC Financial Mathematics Series.
- Scherer, B. (2011)
  A Note on the Returns from Minimum Variance Investing, *Journal of Empirical Finance*, 18(4), pp. 652-660.

Course 2023-2024 in Portfolio Allocation and Asset Management
Lecture 4. Equity Portfolio Optimization with ESG Scores (Exercise)

Thierry Roncalli*

*Amundi Asset Management[21]

*University of Paris-Saclay

January 2024

[21] The opinions expressed in this presentation are those of the authors and are not meant to represent the opinions or official positions of Amundi Asset Management.

Agenda

- Lecture 1: Portfolio Optimization
- Lecture 2: Risk Budgeting
- Lecture 3: Smart Beta, Factor Investing and Alternative Risk Premia
- **Lecture 4: Equity Portfolio Optimization with ESG Scores**
- Lecture 5: Climate Portfolio Construction
- Lecture 6: Equity and Bond Portfolio Optimization with Green Preferences
- Lecture 7: Machine Learning in Asset Management

We consider the CAPM model:

$$R_i - r = \beta_i (R_m - r) + \varepsilon_i$$

where R_i is the return of asset i, R_m is the return of the market portfolio w_m, r is the risk free asset, β_i is the beta of asset i with respect to the market portfolio and ε_i is the idiosyncratic risk of asset i. We have $R_m \perp \varepsilon_i$ and $\varepsilon_i \perp \varepsilon_j$. We note σ_m the volatility of the market portfolio. Let $\tilde{\sigma}_i$, μ_i and $\mathcal{S}_i$ be the idiosyncratic volatility, the expected return and the ESG score of asset i. We use a universe of 6 assets with the following parameter values:

Asset i	1	2	3	4	5	6
β_i	0.10	0.30	0.50	0.90	1.30	2.00
$\tilde{\sigma}_i$ (in %)	17.00	17.00	16.00	10.00	11.00	12.00
μ_i (in %)	1.50	2.50	3.50	5.50	7.50	11.00
$\mathcal{S}_i$	1.10	1.50	2.50	-1.82	-2.35	-2.91

and $\sigma_m = 20\%$. The risk-free return r is set to 1% and the expected return of the market portfolio w_m is equal to $\mu_m = 6\%$.

Question 1

We assume that the CAPM is valid.

Question (a)

Calculate the vector μ of expected returns.

- Using the CAPM, we have:

$$\mu_i = r + \beta_i (\mu_m - r)$$

- For instance, we have:

$$\mu_1 = 1\% + 0.10 \times (6\% - 1\%) = 1.5\%$$

and:

$$\mu_2 = 1\% + 0.30 \times 5\% = 2.5\%$$

- Finally, we obtain $\mu = (1.5\%, 2.5\%, 3.5\%, 5.5\%, 7.5\%, 11\%)$

Question (b)

Compute the covariance matrix Σ. Deduce the volatility σ_i of the asset i and find the correlation matrix $\mathbb{C} = (\rho_{i,j})$ between asset returns.

- We have:

$$\Sigma = \sigma_m^2 \beta \beta^\top + D$$

where:

$$D = \mathrm{diag}\left(\tilde{\sigma}_1^2, \ldots, \tilde{\sigma}_6^2\right)$$

- The numerical value of Σ is:

$$\Sigma = \begin{pmatrix} 293 & & & & & \\ 12 & 325 & & & & \\ 20 & 60 & 356 & & & \\ 36 & 108 & 180 & 424 & & \\ 52 & 156 & 260 & 468 & 797 & \\ 80 & 240 & 400 & 720 & 1\,040 & 1\,744 \end{pmatrix} \times 10^{-4}$$

- We have:

$$\sigma_i = \sqrt{\Sigma_{i,i}}$$

- We obtain:

$$\sigma = (17.12\%, 18.03\%, 18.87\%, 20.59\%, 28.23\%, 41.76\%)$$

- We have:

$$\rho_{i,j} = \frac{\Sigma_{i,j}}{\sigma_i \sigma_j}$$

- We obtain the following correlation matrix expressed in %:

$$\mathbb{C} = \begin{pmatrix} 100.00 & & & & & \\ 3.89 & 100.00 & & & & \\ 6.19 & 17.64 & 100.00 & & & \\ 10.21 & 29.09 & 46.33 & 100.00 & & \\ 10.76 & 30.65 & 48.81 & 80.51 & 100.00 & \\ 11.19 & 31.88 & 50.76 & 83.73 & 88.21 & 100.00 \end{pmatrix}$$

Question (c)

Compute the tangency portfolio w^*. Calculate $\mu(w^*)$ and $\sigma(w^*)$. Deduce the Sharpe ratio and the ESG score of the tangency portfolio.

- We have:

$$w^{*}=\frac{\Sigma^{-1}\left(\mu-r\mathbf{1}\right)}{\mathbf{1}^{\top}\Sigma^{-1}\left(\mu-r\mathbf{1}\right)}=\begin{pmatrix}0.94\%\\2.81\%\\5.28\%\\24.34\%\\29.06\%\\37.57\%\end{pmatrix}$$

- We deduce:

$$\begin{array}{rcl}\mu\left(w^{*}\right) & = & w^{*\top}\mu=7.9201\%\\ \sigma\left(w^{*}\right) & = & \sqrt{w^{*\top}\Sigma w^{*}}=28.3487\%\\ \mathrm{SR}\left(w^{*}\mid r\right) & = & \dfrac{7.9201\%-1\%}{28.3487\%}=0.2441\\ \boldsymbol{\mathcal{S}}\left(w^{*}\right) & = & \displaystyle\sum_{i=1}^{6}w_{i}^{*}\boldsymbol{\mathcal{S}}_{i}=-2.0347\end{array}$$

Question (d)

Compute the beta coefficient $\beta_i(w^*)$ of the six assets with respect to the tangency portfolio w^*, and the implied expected return $\tilde{\mu}_i$:

$$\tilde{\mu}_i = r + \beta_i(w^*)(\mu(w^*) - r)$$

- We have:

$$\beta_i\left(w^*\right)=\frac{\mathbf{e}_i^\top \Sigma w^*}{\sigma^2\left(w^*\right)}$$

- We obtain:

$$\beta\left(w^*\right)=\begin{pmatrix} 0.0723 \\ 0.2168 \\ 0.3613 \\ 0.6503 \\ 0.9393 \\ 1.4451 \end{pmatrix}$$

- The computation of $\tilde{\mu}_i=r+\beta_i\left(w^*\right)\left(\mu\left(w^*\right)-r\right)$ gives:

$$\tilde{\mu}=\begin{pmatrix} 1.50\% \\ 2.50\% \\ 3.50\% \\ 5.50\% \\ 7.50\% \\ 11.00\% \end{pmatrix}$$

Question (e)

Deduce the market portfolio w_m. Comment on these results.

- $\beta_i(w^*) \neq \beta_i(w_m)$ but risk premia are exact
- Let us assume that the allocation of w_m is equal to α of the tangency portfolio w^* and $1-\alpha$ of the risk-free asset. We deduce that:

$$\beta(w_m) = \frac{\Sigma w_m}{\sigma^2(w_m)} = \frac{\alpha \Sigma w^*}{\alpha^2 \sigma^2(w^*)} = \frac{1}{\alpha}\beta(w^*)$$

- We have:

$$\alpha = \frac{\beta_i(w^*)}{\beta_i(w_m)} = 72.25\%$$

- The market portfolio w_m is equal to 72.25% of the tangency portfolio w^* and 27.75% of the risk-free asset

- We have:

$$\mu(w_m) = r + \alpha(\mu(w^*) - r) = 1\% + 72.25\% \times (7.9201\% - 1\%) = 6\%$$

and:

$$\sigma(w_m) = \alpha\,\sigma(w^*) = 72.25\% \times 28.3487\% = 20.48\%$$

- We deduce that:

$$\mathrm{SR}(w_m \mid r) = \frac{6\% - 1\%}{20.48\%} = 0.2441$$

- We do not obtain the true value of the Sharpe ratio:

$$\mathrm{SR}(w_m \mid r) = \frac{6\% - 1\%}{20\%} = 0.25$$

- The tangency portfolio has an idiosyncratic risk:

$$\sqrt{w_m^\top \left(\sigma_m^2 \beta\beta^\top\right) w^\top} = 20\% < \sigma(w_m) = 20.48\%$$

Question 2

We consider long-only portfolios and we also impose a minimum threshold $\mathcal{S}^{\star}$ for the portfolio ESG score:

$$\mathcal{S}(w) = w^{\top}\mathcal{S} \geq \mathcal{S}^{\star}$$

Question (a)

Let γ be the risk tolerance. Write the mean-variance optimization problem.

- We have:

$$
\begin{aligned}
w^{\star} &= \arg\min \frac{1}{2} w^{\top} \Sigma w - \gamma w^{\top} \mu \\
\text{s.t.} &\quad \left\{
\begin{array}{l}
\mathbf{1}_6^{\top} w = 1 \\
w^{\top} \boldsymbol{\mathcal{S}} \geq \boldsymbol{\mathcal{S}}^{\star} \\
\mathbf{0}_6 \leq w \leq \mathbf{1}_6
\end{array}
\right.
\end{aligned}
$$

Question (b)

Find the QP form of the MVO problem.

- The matrix form of the QP problem is:

$$\begin{aligned} w^{\star} &= \arg\min \frac{1}{2} w^{\top} Q w - w^{\top} R \\ & \text{s.t.} \quad \left\{ \begin{array}{l} Aw = B \\ Cw \leq D \\ w^{-} \leq w \leq w^{+} \end{array} \right. \end{aligned}$$

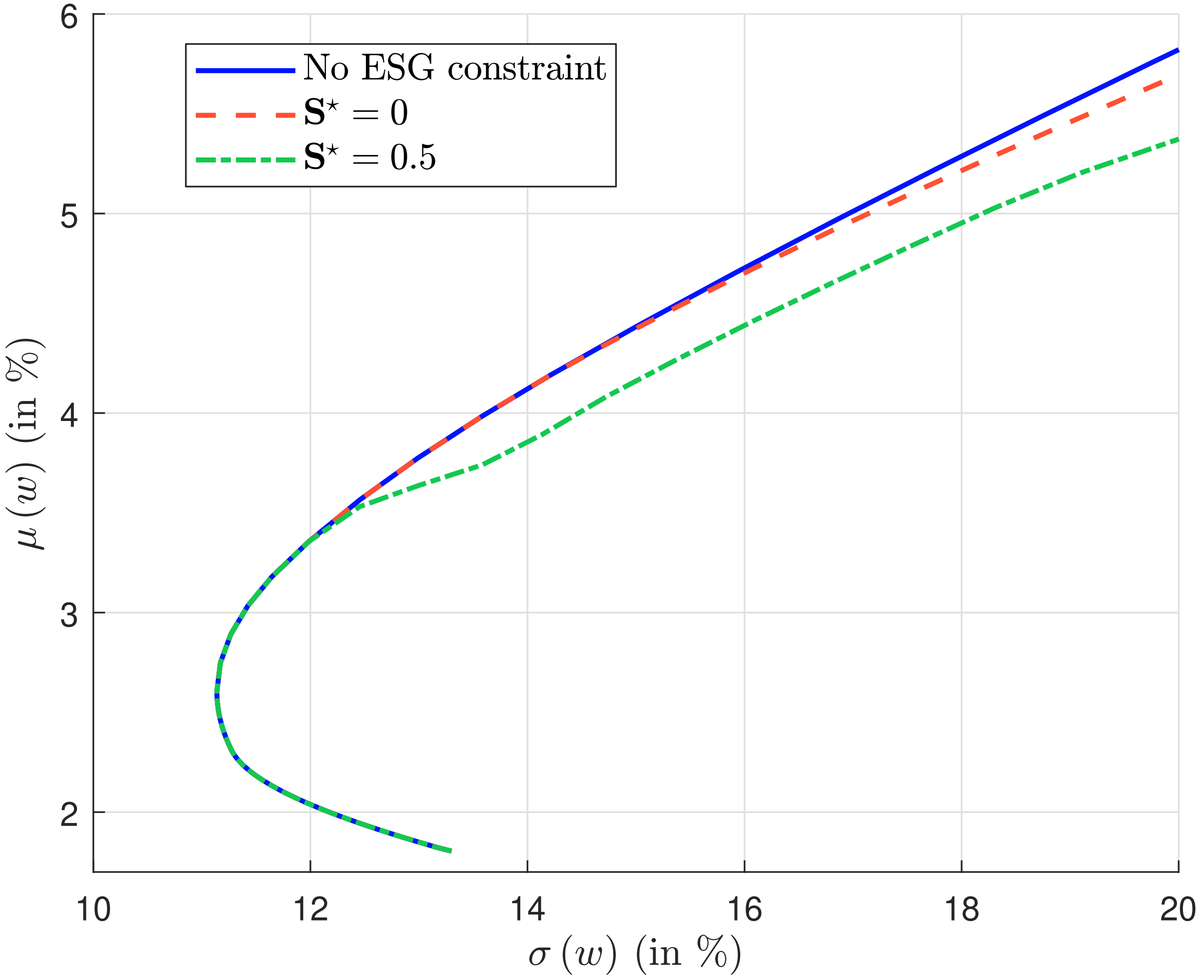
- We deduce that $Q = \Sigma$, $R = \gamma\mu$, $A = \mathbf{1}_6^{\top}$, $B = 1$, $C = -\boldsymbol{\mathcal{S}}^{\top}$, $D = -\mathcal{S}^{\star}$, $w^{-} = \mathbf{0}_6$ and $w^{+} = \mathbf{1}_6$

Question (c)

Compare the efficient frontier when (1) there is no ESG constraint ($\boldsymbol{\mathcal{S}}^{\star} = -\infty$), (2) we impose a positive ESG score ($\boldsymbol{\mathcal{S}}^{\star} = 0$) and (3) the minimum threshold is set to 0.5 ($\boldsymbol{\mathcal{S}}^{\star} = 0.5$). Comment on these results.

- To compute the efficient frontier, we consider several value of $\gamma \in [-1, 2]$
- For each value of γ, we compute the optimal portfolio $w^\star$ and deduce its expected return $\mu(w^\star)$ and its volatility $\sigma(w^\star)$

Figure 81: Impact of the minimum ESG score on the efficient frontier



Question (d)

For each previous cases, find the tangency portfolio w^* and the corresponding risk tolerance γ^*. Compute then $\mu(w^*)$, $\sigma(w^*)$, $\mathrm{SR}(w^* \mid r)$ and $\boldsymbol{\mathcal{S}}(w^*)$. Comment on these results.

- Let $w^{\star}(\gamma)$ be the MVO portfolio when the risk tolerance is equal to γ
- By using a fine grid of γ values, we can find the optimal value γ^{*} by solving numerically the following optimization problem with the brute force algorithm:

$$\gamma^{*} = \arg\max \frac{\mu(w^{\star}(\gamma)) - r}{\sigma(w^{\star}(\gamma))} \qquad \text{for } \gamma \in [0,2]$$

- We deduce the tangency portfolio $w^{*} = w^{\star}(\gamma^{*})$

Table 66: Impact of the minimum ESG score on the efficient frontier

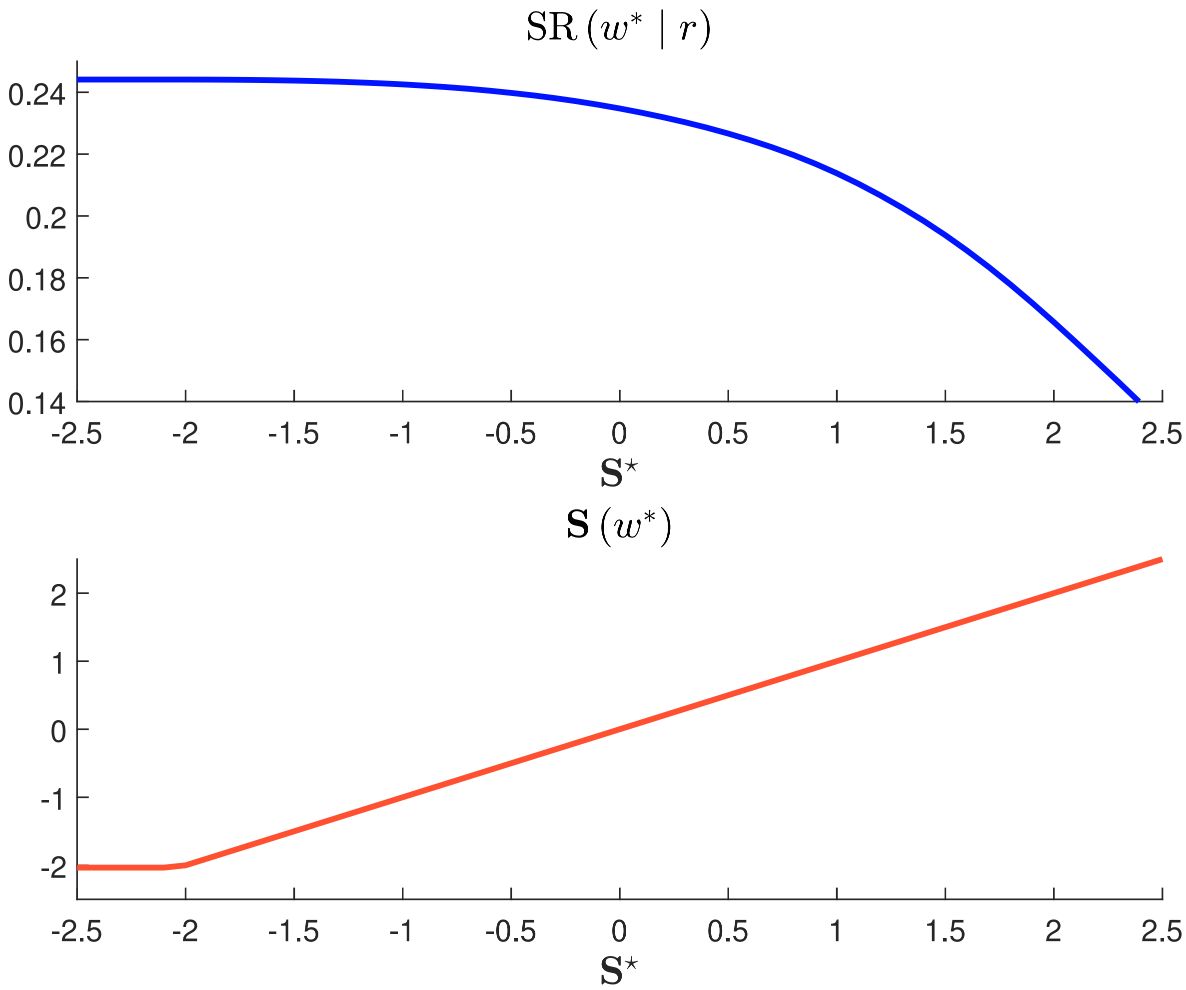
$\mathcal{S}^{\star}$	$-\infty$	0	0.5
γ^{*}	1.1613	0.8500	0.8500
w^{*} (in %)	0.9360	9.7432	9.1481
	2.8079	16.3317	19.0206
	5.2830	31.0176	40.3500
	24.3441	5.1414	0.0000
	29.0609	11.6028	3.8248
	37.5681	26.1633	27.6565
$\mu(w^{*})$ (in %)	7.9201	5.6710	5.3541
$\sigma(w^{*})$ (in%)	28.3487	19.8979	19.2112
$\mathrm{SR}(w^{*} \mid r)$	0.2441	0.2347	0.2266
$\mathcal{S}(w^{*})$	-2.0347	0.0000	0.5000

Question (e)

Draw the relationship between the minimum ESG score $\mathcal{S}^{\star}$ and the Sharpe ratio $\mathrm{SR}\left(w^{*} \mid r\right)$ of the tangency portfolio.

- We perform the same analysis as previously for several values $\boldsymbol{\mathcal{S}}^{\star} \in [-2.5, 2.5]$
- We verify that the Sharpe ratio is a decreasing function of $\boldsymbol{\mathcal{S}}^{\star}$

Figure 82: Relationship between the minimum ESG score $\boldsymbol{\mathcal{S}}^{\star}$ and the Sharpe ratio $\mathrm{SR}\left(w^{*} \mid r\right)$ of the tangency portfolio



Question (f)

We assume that the market portfolio w_m corresponds to the tangency portfolio when $\boldsymbol{\mathcal{S}}^{\star} = 0.5$.

- The market portfolio w_m is then equal to:

$$w_m = \begin{pmatrix} 9.15\% \\ 19.02\% \\ 40.35\% \\ 0.00\% \\ 3.82\% \\ 27.66\% \end{pmatrix}$$

- We deduce that:

$$\begin{aligned} \mu(w_m) &= 5.3541\% \\ \sigma(w_m) &= 19.2112\% \\ \mathrm{SR}(w_m \mid r) &= 0.2266 \\ \mathcal{S}(w_m) &= 0.5 \end{aligned}$$

Question (f).i

Compute the beta coefficient $\beta_i(w_m)$ and the implied expected return $\tilde{\mu}_i(w_m)$ for each asset. Deduce then the alpha return α_i of asset i. Comment on these results.

- We have:

$$\beta_i(w_m) = \frac{\mathbf{e}_i^\top \Sigma w_m}{\sigma^2(w_m)}$$

and:

$$\tilde{\mu}_i(w_m) = r + \beta_i(w_m)(\mu(w_m) - r)$$

- We deduce that the alpha return is equal to:

$$\begin{aligned} \alpha_i &= \mu_i - \tilde{\mu}_i(w_m) \\ &= (\mu_i - r) - \beta_i(w_m)(\mu(w_m) - r) \end{aligned}$$

- We notice that $\alpha_i < 0$ for the first three assets and $\alpha_i > 0$ for the last three assets, implying that:

$$\begin{cases} \boldsymbol{\mathcal{S}}_i > 0 \Rightarrow \alpha_i < 0 \\ \boldsymbol{\mathcal{S}}_i < 0 \Rightarrow \alpha_i > 0 \end{cases}$$

Table 67: Computation of the alpha return due to the ESG constraint

Asset	$\beta_i(w_m)$	$\tilde{\mu}_i(w_m)$ (in %)	$\tilde{\mu}_i(w_m) - r$ (in %)	α_i (in bps)
1	0.1660	1.7228	0.7228	−22.28
2	0.4321	2.8813	1.8813	−38.13
3	0.7518	4.2733	3.2733	−77.33
4	0.8494	4.6984	3.6984	80.16
5	1.2395	6.3967	5.3967	110.33
6	1.9955	9.6885	8.6885	131.15

Question (f).ii

We consider the equally-weighted portfolio w_{ew}. Compute its beta coefficient $\beta(w_{\mathrm{ew}} \mid w_m)$, its implied expected return $\tilde{\mu}(w_{\mathrm{ew}})$ and its alpha return $\alpha(w_{\mathrm{ew}})$. Comment on these results.

- We have:

$$\beta\left(w_{\text{ew}} \mid w_m\right) = \frac{w_{\text{ew}}^\top \Sigma w_m}{\sigma^2\left(w_m\right)} = 0.9057$$

and:

$$\tilde{\mu}\left(w_{\text{ew}}\right) = 1\% + 0.9057 \times (5.3541\% - 1\%) = 4.9435\%$$

- We deduce that:

$$\alpha\left(w_{\text{ew}}\right) = \mu\left(w_{\text{ew}}\right) - \tilde{\mu}\left(w_{\text{ew}}\right) = 5.25\% - 4.9435\% = 30.65 \text{ bps}$$

- We verify that:

$$\alpha\left(w_{\text{ew}}\right) = \sum_{i=1}^{6} w_{\text{ew},i}\alpha_i = \frac{\sum_{i=1}^{6}\alpha_i}{6} = 30.65 \text{ bps}$$

- The equally-weighted portfolio has a positive alpha because:

$$\boldsymbol{\mathcal{S}}\left(w_{\text{ew}}\right) = -0.33 \ll \boldsymbol{\mathcal{S}}\left(w_m\right) = 0.50$$

Question 3

The objective of the investor is twice. He would like to manage the tracking error risk of his portfolio with respect to the benchmark $b = (15\%, 20\%, 19\%, 14\%, 15\%, 17\%)$ and have a better ESG score than the benchmark. Nevertheless, this investor faces a long-only constraint because he cannot leverage his portfolio and he cannot also be short on the assets.

Question (a)

What is the ESG score of the benchmark?

- We have:

$$\mathcal{S}(b) = \sum_{i=1}^{6} b_i \mathcal{S}_i = -0.1620$$

Question (b)

We assume that the investor's portfolio is $w = (10\%, 10\%, 30\%, 20\%, 20\%, 10\%)$. Compute the excess score $\boldsymbol{\mathcal{S}}(w \mid b)$, the expected excess return $\mu(w \mid b)$, the tracking error volatility $\sigma(w \mid b)$ and the information ratio $\mathrm{IR}(w \mid b)$. Comment on these results.

- We have:

$$\left\{\begin{array}{l}\boldsymbol{\mathcal{S}}\left(w \mid b\right) = \left(w - b\right)^{\top} \boldsymbol{\mathcal{S}} = 0.0470 \\ \mu\left(w \mid b\right) = \left(w - b\right)^{\top} \mu = -0.5 \text{ bps} \\ \sigma\left(w \mid b\right) = \sqrt{\left(w - b\right)^{\top} \Sigma \left(w - b\right)} = 2.8423\% \\ \mathrm{IR}\left(w \mid b\right) = \dfrac{\mu\left(w \mid b\right)}{\sigma\left(w \mid b\right)} = -0.0018\end{array}\right.$$

- The portfolio w is not optimal since it improves the ESG score of the benchmark, but its information ratio is negative. Nevertheless, the expected excess return is close to zero (less than -1 bps).

Question (c)

Same question with the portfolio $w = (10\%, 15\%, 30\%, 10\%, 15\%, 20\%)$.

- We have: We have:

$$\left\{\begin{array}{l}\boldsymbol{\mathcal{S}}\left(w \mid b\right)=\left(w-b\right)^{\top}\boldsymbol{\mathcal{S}}=0.1305\\ \mu\left(w \mid b\right)=\left(w-b\right)^{\top}\mu=29.5\text{ bps}\\ \sigma\left(w \mid b\right)=\sqrt{\left(w-b\right)^{\top}\Sigma\left(w-b\right)}=2.4949\%\\ \mathrm{IR}\left(w \mid b\right)=\dfrac{\mu\left(w \mid b\right)}{\sigma\left(w \mid b\right)}=0.1182\end{array}\right.$$

Question (d)

In the sequel, we assume that the investor has no return target. In fact, the objective of the investor is to improve the ESG score of the benchmark and control the tracking error volatility. We note γ the risk tolerance. Give the corresponding esg-variance optimization problem.

- The optimization problem is:

$$
\begin{aligned}
w^{\star} &= \arg\min \frac{1}{2}\sigma^{2}\left(w \mid b\right) - \gamma \boldsymbol{\mathcal{S}}\left(w \mid b\right) \\
\text{s.t.} &\quad \left\{ \begin{array}{l} \mathbf{1}_6^\top w = 1 \\ \mathbf{0}_6 \leq w \leq \mathbf{1}_6 \end{array} \right.
\end{aligned}
$$

Question (e)

Find the matrix form of the corresponding QP problem.

- The objective function is equal to:

$$
\begin{aligned}
(*) &= \frac{1}{2}\sigma^2(w \mid b) - \gamma \boldsymbol{\mathcal{S}}(w \mid b) \\
&= \frac{1}{2}(w-b)^\top \Sigma (w-b) - \gamma (w-b)^\top \boldsymbol{\mathcal{S}} \\
&= \frac{1}{2} w^\top \Sigma w - w^\top (\Sigma b + \gamma \boldsymbol{\mathcal{S}}) + \underbrace{\left(\gamma b^\top \boldsymbol{\mathcal{S}} + \frac{1}{2} b^\top \Sigma b\right)}_{\text{does not depend on } w}
\end{aligned}
$$

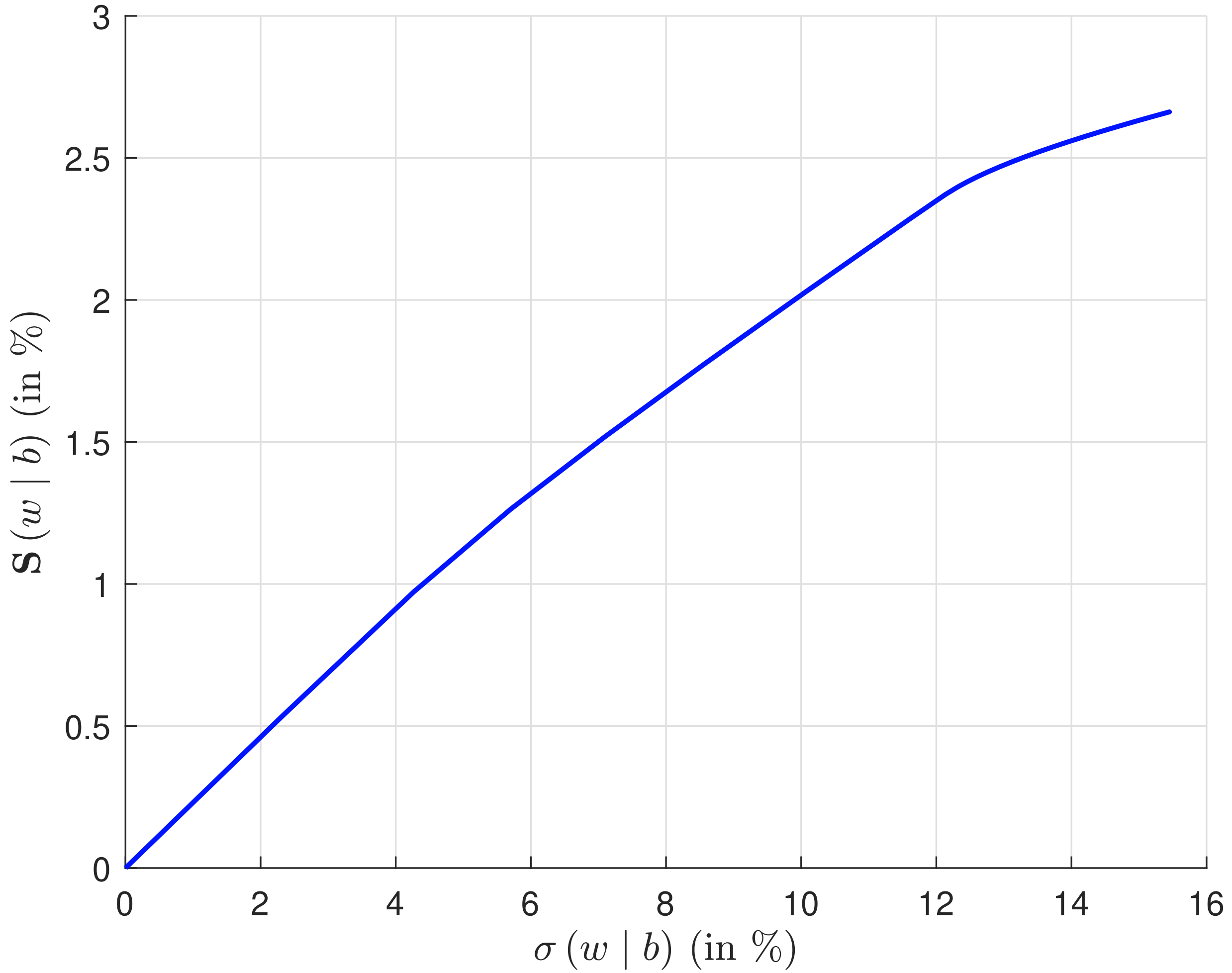
- We deduce that $Q = \Sigma$, $R = \Sigma b + \gamma \boldsymbol{\mathcal{S}}$, $A = \mathbf{1}_6^\top$, $B = 1$, $w^- = \mathbf{0}_6$ and $w^+ = \mathbf{1}_6$

Question (f)

Draw the esg-variance efficient frontier $(\sigma(w^\star \mid b), \boldsymbol{\mathcal{S}}(w^\star \mid b))$ where $w^\star$ is an optimal portfolio.

- We solve the QP problem for several values of $\gamma \in [0, 5\%]$ and obtain Figure 83

Figure 83: Efficient frontier of tracking a benchmark with an ESG score objective



Question (g)

Find the optimal portfolio $w^{\star}$ when we target a given tracking error volatility $\sigma^{\star}$. The values of $\sigma^{\star}$ are 0%, 1%, 2%, 3% and 4%.

- Using the QP numerical algorithm, we compte the optimal value $\sigma\left(w \mid b\right)$ for $\gamma = 0$ and $\gamma = 5\%$
- Then, we apply the bisection algorithm to find the optimal value $\gamma^{\star}$ such that:

$$\sigma\left(w \mid b\right) = \sigma^{\star}$$

Table 68: Solution of the σ-problem

Target $\sigma^\star$	0	1%	2%	3%	4%
$\gamma^\star$ (in bps)	0.000	4.338	8.677	13.015	18.524
$w^\star$ (in %)	15.000	15.175	15.350	15.525	14.921
	20.000	21.446	22.892	24.338	25.385
	19.000	23.084	27.167	31.251	35.589
	14.000	9.588	5.176	0.763	0.000
	15.000	12.656	10.311	7.967	3.555
	17.000	18.052	19.104	20.156	20.550
$\mathcal{S}\left(w^\star \mid b\right)$	0.000	0.230	0.461	0.691	0.915

Question (h)

Find the optimal portfolio $w^\star$ when we target a given excess score $\mathcal{S}^\star$. The values of $\mathcal{S}^\star$ are 0, 0.1, 0.2, 0.3 and 0.4.

- Same method as previously with the following equation:

$$\mathcal{S}\left(w \mid b\right) = \mathcal{S}^{\star}$$

- An alternative approach consists in solving the following optimization problem:

$$\begin{aligned} w^{\star} & = & \arg\min \frac{1}{2}\sigma^{2}\left(w \mid b\right) \\ & & \text{s.t.} \quad \left\{ \begin{array}{l} \mathbf{1}_{6}^{\top} w = 1 \\ \mathcal{S}\left(w \mid b\right) = \mathcal{S}^{\star} \\ \mathbf{0}_{6} \leq w \leq \mathbf{1}_{6} \end{array} \right. \end{aligned}$$

- We have: $Q = \Sigma$, $R = \Sigma b$, $A = \left(\begin{array}{c} \mathbf{1}_{6}^{\top} \\ \mathcal{S}^{\top} \end{array}\right)$, $B = \left(\begin{array}{c} 1 \\ \mathcal{S}^{\star} + \mathcal{S}^{\top} b \end{array}\right)$, $w^{-} = \mathbf{0}_{6}$ and $w^{+} = \mathbf{1}_{6}$

Table 69: Solution of the $\mathcal{S}$-problem

Target $\mathcal{S}^{\star}$	0	0.1	0.2	0.3	0.4
$\gamma^{\star}$ (in bps)	0.000	1.882	3.764	5.646	7.528
$w^{\star}$ (in %)	15.000	15.076	15.152	15.228	15.304
	20.000	20.627	21.255	21.882	22.509
	19.000	20.772	22.544	24.315	26.087
	14.000	12.086	10.171	8.257	6.343
	15.000	13.983	12.966	11.949	10.932
	17.000	17.456	17.913	18.369	18.825
$\sigma(w^{\star} \mid b)$ (in %)	0.000	0.434	0.868	1.301	1.735

Question (i)

We would like to compare the efficient frontier obtained in Question 3(f) with the efficient frontier when we implement a best-in-class selection or a worst-in-class exclusion. The selection strategy consists in investing only in the best three ESG assets, while the exclusion strategy implies no exposure on the worst ESG asset. Draw the three efficient frontiers. Comment on these results.

- For the best-in-class strategy, the optimization problem becomes:

$$
\begin{aligned}
w^{\star} &= \arg\min \frac{1}{2}\sigma^{2}\left(w \mid b\right) - \gamma \boldsymbol{\mathcal{S}}\left(w \mid b\right) \\
& \text{s.t.} \quad \left\{
\begin{array}{l}
\mathbf{1}_6^\top w = 1 \\
w_4 = w_5 = w_6 = 0 \\
\mathbf{0}_6 \leq w \leq \mathbf{1}_6
\end{array}
\right.
\end{aligned}
$$

- The QP form is defined by $Q = \Sigma$, $R = \Sigma b + \gamma \boldsymbol{\mathcal{S}}$, $A = \mathbf{1}_6^\top$, $B = 1$, $w^{-} = \mathbf{0}_6$ and $w^{+} = \left(\begin{array}{c} \mathbf{1}_3 \\ \mathbf{0}_3 \end{array}\right)$

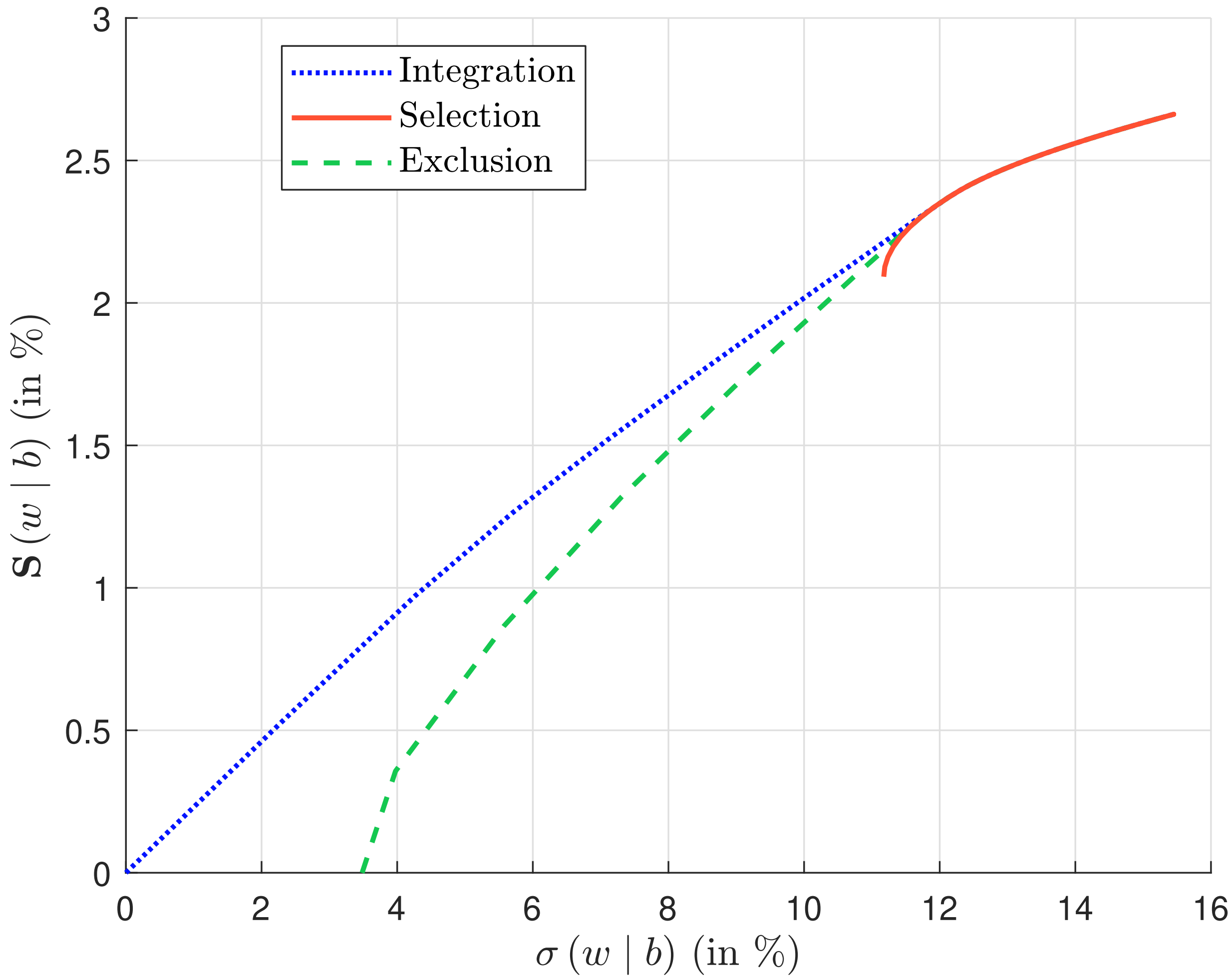
- For the worst-in-class strategy, the optimization problem becomes:

$$
\begin{aligned}
w^{\star} &= \arg\min \frac{1}{2}\sigma^{2}\left(w \mid b\right) - \gamma\boldsymbol{\mathcal{S}}\left(w \mid b\right) \\
\text{s.t.} &\quad \left\{\begin{array}{l}
\mathbf{1}_6^\top w = 1 \\
w_6 = 0 \\
\mathbf{0}_6 \leq w \leq \mathbf{1}_6
\end{array}\right.
\end{aligned}
$$

- The QP form is defined by $Q = \Sigma$, $R = \Sigma b + \gamma\boldsymbol{\mathcal{S}}$, $A = \mathbf{1}_6^\top$, $B = 1$, $w^{-} = \mathbf{0}_6$ and $w^{+} = \left(\begin{array}{c} \mathbf{1}_5 \\ 0 \end{array}\right)$

- The efficient frontiers are reported in Figure 84
- The exclusion strategy has less impact than the selection strategy
- The selection strategy implies a high tracking error risk

Figure 84: Comparison of the efficient frontiers (ESG integration, best-in-class selection and worst-in-class exclusion)



Question (j)

Which minimum tracking error volatility must the investor accept to implement the best-in-class selection strategy? Give the corresponding optimal portfolio.

- We solve the first problem of Question 3(i) with $\gamma = 0$
- We obtain:

$$\sigma(w \mid b) \geq 11.17\%$$

- The lower bound $\sigma(w^\star \mid b) = 11.17\%$ corresponds to the following optimal portfolio:

$$w^\star = \begin{pmatrix} 16.31\% \\ 34.17\% \\ 49.52\% \\ 0\% \\ 0\% \\ 0\% \end{pmatrix}$$

Remark

The impact of ESG scores on optimized portfolios depends on their relationship with expected returns, volatilities, correlations, beta coefficients, etc. In the previous exercise, the results are explained because the best-in-class assets are those with the lowest expected returns and beta coefficients while the worst-in-class assets are those with the highest expected returns and beta coefficients. For instance, we obtain a high tracking error risk for the best-in-class selection strategy, because the best-in-class assets have low volatilities and correlations with respect to worst-in-class assets, implying that it is difficult to replicate these last assets with the other assets.

Course 2023-2024 in Portfolio Allocation and Asset Management
Lecture 5. Climate Portfolio Construction

Thierry Roncalli*

*Amundi Asset Management[22]

*University of Paris-Saclay

January 2024

[22]The opinions expressed in this presentation are those of the authors and are not meant to represent the opinions or official positions of Amundi Asset Management.

Agenda

- Lecture 1: Portfolio Optimization
- Lecture 2: Risk Budgeting
- Lecture 3: Smart Beta, Factor Investing and Alternative Risk Premia
- Lecture 4: Equity Portfolio Optimization with ESG Scores
- **Lecture 5: Climate Portfolio Construction**
- Lecture 6: Equity and Bond Portfolio Optimization with Green Preferences
- Lecture 7: Machine Learning in Asset Management

Quadratic programming

Definition

We have:

$$\begin{aligned} x^\star &= \arg\min \frac{1}{2} x^\top Q x - x^\top R \\ & \text{s.t.} \left\{ \begin{array}{l} Ax = B \\ Cx \leq D \\ x^- \leq x \leq x^+ \end{array} \right. \end{aligned}$$

where x is a $n \times 1$ vector, Q is a $n \times n$ matrix, R is a $n \times 1$ vector, A is a $n_A \times n$ matrix, B is a $n_A \times 1$ vector, C is a $n_C \times n$ matrix, D is a $n_C \times 1$ vector, and x^- and x^+ are two $n \times 1$ vectors

Quadratic form

A quadratic form is a polynomial with terms all of degree two

$$\mathcal{QF}(x_1,\ldots,x_n)=\sum_{i=1}^{n}\sum_{j=1}^{n}a_{i,j}x_ix_j=x^\top Ax$$

Canonical form

$$\mathcal{QF}(x_1,\ldots,x_n)=\frac{1}{2}\left(x^\top Ax+x^\top A^\top x\right)=\frac{1}{2}x^\top\left(A+A^\top\right)x=\frac{1}{2}x^\top Qx$$

Generalized quadratic form

$$\mathcal{QF}(x;Q,R,c)=\frac{1}{2}x^\top Qx-x^\top R+c$$

Quadratic form

Main properties

1. $\varphi \cdot \mathcal{QF}(w; Q, R, c) = \mathcal{QF}(w; \varphi Q, \varphi R, \varphi c)$
2. $\mathcal{QF}(x; Q_1, R_1, c_1) + \mathcal{QF}(x; Q_2, R_2, c_2) =$ $\mathcal{QF}(x; Q_1 + Q_2, R_1 + R_2, c_1 + c_2)$
3. $\mathcal{QF}(x - y; Q, R, c) = \mathcal{QF}\left(x; Q, R + Qy, \frac{1}{2}y^\top Qy + y^\top R + c\right)$
4. $\mathcal{QF}(x - y; Q, R, c) = \mathcal{QF}\left(y; Q, Qx - R, \frac{1}{2}x^\top Qx - x^\top R + c\right)$
5. $\dfrac{1}{2}\sum_{i=1}^n q_i x_i^2 = \mathcal{QF}(x; \mathcal{D}(q), \mathbf{0}_n, 0)$ where $q = (q_1, \ldots, q_n)$ is a $n \times 1$ vector and $\mathcal{D}(q) = \operatorname{diag}(q)$
6. $\dfrac{1}{2}\sum_{i=1}^n q_i (x_i - y_i)^2 = \mathcal{QF}\left(x; \mathcal{D}(q), \mathcal{D}(q)\, y, \frac{1}{2}y^\top \mathcal{D}(q)\, y\right)$
7. $\dfrac{1}{2}\left(\sum_{i=1}^n q_i x_i\right)^2 = \mathcal{QF}(x; \mathcal{T}(q), \mathbf{0}_n, 0)$ where $\mathcal{T}(q) = qq^\top$
8. $\dfrac{1}{2}\left(\sum_{i=1}^n q_i (x_i - y_i)\right)^2 = \mathcal{QF}\left(x; \mathcal{T}(q), \mathcal{T}(q)\, y, \frac{1}{2}y^\top \mathcal{T}(q)\, y\right)$

Quadratic form

Main properties

We note $\omega = (\omega_1, \ldots, \omega_n)$ where $\omega_i = \mathbb{1}\{i \in \Omega\}$

1. $\dfrac{1}{2}\sum_{i\in\Omega} q_i x_i^2 = \mathcal{QF}\left(x; \mathcal{D}(\omega \circ q), \mathbf{0}_n, 0\right)$
2. $\dfrac{1}{2}\sum_{i\in\Omega} q_i (x_i - y_i)^2 =$
$\mathcal{QF}\left(x; \mathcal{D}(\omega \circ q), \mathcal{D}(\omega \circ q) y, \dfrac{1}{2} y^\top \mathcal{D}(\omega \circ q) y\right)$
3. $\dfrac{1}{2}\left(\sum_{i\in\Omega} q_i x_i\right)^2 = \mathcal{QF}\left(x; \mathcal{T}(\omega \circ q), \mathbf{0}_n, 0\right)$
4. $\dfrac{1}{2}\left(\sum_{i\in\Omega} q_i (x_i - y_i)\right)^2 =$
$\mathcal{QF}\left(x; \mathcal{T}(\omega \circ q), \mathcal{T}(\omega \circ q) y, \dfrac{1}{2} y^\top \mathcal{T}(\omega \circ q) y\right)$
5. $\mathcal{D}(\omega \circ q) = \operatorname{diag}(\omega \circ q) = \mathcal{D}(\omega)\mathcal{D}(q)$
6. $\mathcal{T}(\omega \circ q) = (\omega \circ q)(\omega \circ q)^\top = \left(\omega\omega^\top\right) \circ qq^\top = \mathcal{T}(\omega) \circ \mathcal{T}(q)$

Equity portfolio

Basic optimization problems

Mean-variance optimization

The long-only mean-variance optimization problem is given by:

$$\begin{aligned} w^\star &= \arg\min \frac{1}{2} w^\top \Sigma w - \gamma w^\top \mu \\ \text{s.t.} & \left\{ \begin{array}{l} \mathbf{1}_n^\top w = 1 \\ \mathbf{0}_n \leq w \leq \mathbf{1}_n \end{array} \right. \end{aligned}$$

where:

- γ is the risk-tolerance coefficient
- the equality constraint is the budget constraint $\left(\sum_{i=1}^n w_i = 1\right)$
- the bounds correspond to the no short-selling restriction $(w_i \geq 0)$

QP form

$Q = \Sigma$, $R = \gamma\mu$, $A = \mathbf{1}_n^\top$, $B = 1$, $w^- = \mathbf{0}_n$ and $w^+ = \mathbf{1}$

Equity portfolio

Basic optimization problems

Tracking error optimization

The tracking error optimization problem is defined as:

$$\begin{aligned} w^\star &= \arg\min \frac{1}{2} w^\top \Sigma w - w^\top (\gamma\mu + \Sigma b) \\ &\text{s.t.} \left\{ \begin{array}{l} \mathbf{1}_n^\top w = 1 \\ \mathbf{0}_n \leq w \leq \mathbf{1}_n \end{array} \right. \end{aligned}$$

QP form

$Q = \Sigma$, $\boxed{R = \gamma\mu + \Sigma b}$, $A = \mathbf{1}_n^\top$, $B = 1$, $w^- = \mathbf{0}_n$ and $w^+ = \mathbf{1}$

$\Rightarrow$ Portfolio replication: $\boxed{R = \Sigma b}$

Specification of the constraints

Sector weight constraint

- We have

$$s_j^- \leq \sum_{i \in \mathcal{S}ector_j} w_i \leq s_j^+$$

- $\boldsymbol{s}_j$ is the $n \times 1$ sector-mapping vector: $\boldsymbol{s}_{i,j} = \mathbb{1}\left\{i \in \mathcal{S}ector_j\right\}$
- We notice that:

$$\sum_{i \in \mathcal{S}ector_j} w_i = \boldsymbol{s}_j^\top w$$

- We deduce that:

$$s_j^- \leq \sum_{i \in \mathcal{S}ector_j} w_i \leq s_j^+ \Leftrightarrow \left\{ \begin{array}{l} s_j^- \leq \boldsymbol{s}_j^\top w \\ \boldsymbol{s}_j^\top w \leq s_j^+ \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} -\boldsymbol{s}_j^\top w \leq -s_j^- \\ \boldsymbol{s}_j^\top w \leq s_j^+ \end{array} \right.$$

QP form

$$\underbrace{\left(\begin{array}{c} -\boldsymbol{s}_j^\top \\ \boldsymbol{s}_j^\top \end{array} \right)}_{C} w \leq \underbrace{\left(\begin{array}{c} -s_j^- \\ s_j^+ \end{array} \right)}_{D}$$

Specification of the constraints

Score constraint

- General constraint:

$$\sum_{i=1}^{n} w_i \boldsymbol{\mathcal{S}}_i \geq \boldsymbol{\mathcal{S}}^{\star} \Leftrightarrow -\boldsymbol{\mathcal{S}}^{\top} w \leq -\boldsymbol{\mathcal{S}}^{\star}$$

QP form

- $C = -\boldsymbol{\mathcal{S}}^{\top}$
- $D = -\boldsymbol{\mathcal{S}}^{\star}$

Specification of the constraints
Score constraint

- Sector-specific constraint:

$$
\begin{aligned}
\sum_{i \in \mathcal{S}ector_j} w_i \mathcal{S}_i \geq \mathcal{S}_j^\star & \Leftrightarrow \sum_{i=1}^{n} \mathbb{1}\left\{i \in \mathcal{S}ector_j\right\} \cdot w_i \mathcal{S}_i \geq \mathcal{S}_j^\star \\
& \Leftrightarrow \sum_{i=1}^{n} \boldsymbol{s}_{i,j} w_i \mathcal{S}_i \geq \mathcal{S}_j^\star \\
& \Leftrightarrow \sum_{i=1}^{n} w_i \cdot \left(\boldsymbol{s}_{i,j} \mathcal{S}_i\right) \geq \mathcal{S}_j^\star \\
& \Leftrightarrow \left(\boldsymbol{s}_j \circ \mathcal{S}\right)^\top w \geq \mathcal{S}_j^\star
\end{aligned}
$$

QP form

- $C = -\left(\boldsymbol{s}_j \circ \mathcal{S}\right)^\top$
- $D = -\mathcal{S}_j^\star$

Equity portfolios

Example #1

- The capitalization-weighted equity index is composed of 8 stocks
- The weights are equal to 23%, 19%, 17%, 13%, 9%, 8%, 6% and 5%
- The ESG score, carbon intensity and sector of the eight stocks are the following:

Stock	#1	#2	#3	#4	#5	#6	#7	#8
$\mathcal{S}$	−1.20	0.80	2.75	1.60	−2.75	−1.30	0.90	−1.70
$\mathcal{CI}$	125	75	254	822	109	17	341	741
$\mathcal{S}ector$	1	1	2	2	1	2	1	2

Equity portfolios

Example #1 (Cont'd)

- The stock volatilities are equal to 22%, 20%, 25%, 18%, 35%, 23%, 13% and 29%
- The correlation matrix is given by:

$$\mathbb{C} = \begin{pmatrix} 100\% & & & & & & & \\ 80\% & 100\% & & & & & & \\ 70\% & 75\% & 100\% & & & & & \\ 60\% & 65\% & 80\% & 100\% & & & & \\ 70\% & 50\% & 70\% & 85\% & 100\% & & & \\ 50\% & 60\% & 70\% & 80\% & 60\% & 100\% & & \\ 70\% & 50\% & 70\% & 75\% & 80\% & 50\% & 100\% & \\ 60\% & 65\% & 70\% & 75\% & 65\% & 70\% & 80\% & 100\% \end{pmatrix}$$

Equity portfolios
QP problem

- We have:

$$
\begin{aligned}
w^{\star} &= \arg\min \frac{1}{2} w^{\top} Q w - w^{\top} R \\
\text{s.t.} & \left\{
\begin{array}{l}
Aw = B \\
Cw \leq D \\
w^{-} \leq w \leq w^{+}
\end{array}
\right.
\end{aligned}
$$

Equity portfolios
Objective function

- Using $\Sigma_{i,j} = \mathbb{C}_{i,j}\sigma_i\sigma_j$, we obtain:

$$Q = \Sigma = 10^{-4}\times$$

$$\begin{pmatrix}
484.00 & 352.00 & 385.00 & 237.60 & 539.00 & 253.00 & 200.20 & 382.80 \\
352.00 & 400.00 & 375.00 & 234.00 & 350.00 & 276.00 & 130.00 & 377.00 \\
385.00 & 375.00 & 625.00 & 360.00 & 612.50 & 402.50 & 227.50 & 507.50 \\
237.60 & 234.00 & 360.00 & 324.00 & 535.50 & 331.20 & 175.50 & 391.50 \\
539.00 & 350.00 & 612.50 & 535.50 & 1225.00 & 483.00 & 364.00 & 659.75 \\
253.00 & 276.00 & 402.50 & 331.20 & 483.00 & 529.00 & 149.50 & 466.90 \\
200.20 & 130.00 & 227.50 & 175.50 & 364.00 & 149.50 & 169.00 & 301.60 \\
382.80 & 377.00 & 507.50 & 391.50 & 659.75 & 466.90 & 301.60 & 841.00
\end{pmatrix}$$

Equity portfolios

Objective function

- We have:

$$R = \Sigma b = \begin{pmatrix} 3.74 \\ 3.31 \\ 4.39 \\ 3.07 \\ 5.68 \\ 3.40 \\ 2.02 \\ 4.54 \end{pmatrix} \times 10^{-2}$$

Equity portfolios
Constraint specification (bounds)

- The portfolio is long-only

QP form

  - $w^{-} = \mathbf{0}_8$
  - $w^{+} = \mathbf{1}_8$

Equity portfolios

Constraint specification (equality)

- The budget constraint $\sum_{i=1}^{8} w_i = 1 \Rightarrow$ a first linear equation $A_0 w = B_0$

  QP form

  - $A_0 = \mathbf{1}_8^\top$
  - $B_0 = 1$

Equity portfolios

Constraint specification (equality)

- We can impose the sector neutrality of the portfolio meaning that:

$$\sum_{i \in \mathcal{S}ector_j} w_i = \sum_{i \in \mathcal{S}ector_j} b_i$$

The sector neutrality constraint can be written as:

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} w = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$$

QP form

- $A_1 = \boldsymbol{s}_1^\top = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$
- $A_2 = \boldsymbol{s}_2^\top = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$
- $B_1 = \boldsymbol{s}_1^\top b = \sum_{i \in \mathcal{S}ector_1} b_i$
- $B_2 = \boldsymbol{s}_2^\top b = \sum_{i \in \mathcal{S}ector_2} b_i$

Equity portfolios

Constraint specification (inequality)

- We can impose a relative reduction of the benchmark carbon intensity:

$$\mathcal{CI}(w) \leq (1-\mathcal{R})\,\mathcal{CI}(b) \Leftrightarrow C_1 w \leq D_1$$

QP form

- $C_1 = \mathcal{CI}^\top$ (because $\mathcal{CI}(w) = \mathcal{CI}^\top w$)
- $D_1 = (1-\mathcal{R})\,\mathcal{CI}(b)$

- We can impose an absolute increase of the benchmark ESG score:

$$\mathcal{S}(w) \geq \mathcal{S}(b) + \Delta\mathcal{S}^\star$$

Since $\mathcal{S}(w) = \mathcal{S}^\top w$, we deduce that $C_2 w \leq D_2$

QP form

- $C_2 = -\mathcal{S}^\top$
- $D_2 = -\left(\mathcal{S}(b) + \Delta\mathcal{S}^\star\right)$

Equity portfolios

Combination of constraints

Set of constraints	Carbon intensity	ESG score	Sector neutrality	A	B	C	D
#1	✓			A_0	B_0	C_1	D_1
#2		✓		A_0	B_0	C_2	D_2
#3	✓	✓		A_0	B_0	$\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$	$\begin{bmatrix} D_1 \\ D_2 \end{bmatrix}$
#4	✓	✓	✓	$\begin{bmatrix} A_0 \\ A_1 \\ A_2 \end{bmatrix}$	$\begin{bmatrix} B_0 \\ B_1 \\ B_2 \end{bmatrix}$	$\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$	$\begin{bmatrix} D_1 \\ D_2 \end{bmatrix}$

Equity portfolios

Results

Table 70: $\mathcal{R} = 30\%$ and $\Delta \mathcal{S}^{\star} = 0.50$ (Example #1)

		Benchmark	Set #1	Set #2	Set #3	Set #4
Weights (in %)	$w_1^{\star}$	23.00	18.17	25.03	8.64	12.04
	$w_2^{\star}$	19.00	24.25	14.25	29.27	23.76
	$w_3^{\star}$	17.00	16.92	21.95	26.80	30.55
	$w_4^{\star}$	13.00	2.70	27.30	1.48	2.25
	$w_5^{\star}$	9.00	12.31	3.72	10.63	8.51
	$w_6^{\star}$	8.00	11.23	1.34	6.30	10.20
	$w_7^{\star}$	6.00	11.28	1.68	16.87	12.69
	$w_8^{\star}$	5.00	3.15	4.74	0.00	0.00
Statistics	$\sigma(w^{\star} \mid b)$ (in %)	0.00	0.50	1.18	1.90	2.12
	$\mathcal{CI}(w^{\star})$	261.72	183.20	367.25	183.20	183.20
	$\mathcal{R}(w^{\star} \mid b)$ (in %)		30.00	-40.32	30.00	30.00
	$\mathcal{S}(w^{\star})$	0.17	0.05	0.67	0.67	0.67
	$\mathcal{S}(w^{\star}) - \mathcal{S}(b)$		-0.12	0.50	0.50	0.50
	$w^{\star}(\mathcal{S}ector_1)$ (in %)	57.00	66.00	44.67	65.41	57.00
	$w^{\star}(\mathcal{S}ector_2)$ (in %)	43.00	34.00	55.33	34.59	43.00

Equity portfolios

Dealing with constraints on relative weights

- The carbon intensity of the j^{th} sector within the portfolio w is:

$$\mathcal{CI}\left(w;\mathcal{S}ector_j\right) = \sum_{i \in \mathcal{S}ector_j} \tilde{w}_i \mathcal{CI}_i$$

where $\tilde{w}_i$ is the normalized weight in the sector bucket:

$$\tilde{w}_i = \frac{w_i}{\sum_{k \in \mathcal{S}ector_j} w_k}$$

- Another expression of $\mathcal{CI}\left(w;\mathcal{S}ector_j\right)$ is:

$$\mathcal{CI}\left(w;\mathcal{S}ector_j\right) = \frac{\sum_{i \in \mathcal{S}ector_j} w_i \mathcal{CI}_i}{\sum_{i \in \mathcal{S}ector_j} w_i} = \frac{\left(\boldsymbol{s}_j \circ \mathcal{CI}\right)^\top w}{\boldsymbol{s}_j^\top w}$$

Equity portfolios

Dealing with constraints on relative weights

- If we consider the constraint $\mathcal{CI}\left(w;\mathcal{S}ector_j\right) \leq \mathcal{CI}_j^{\star}$, we obtain:

$$
\begin{array}{rcl}
(*) & \Leftrightarrow & \mathcal{CI}\left(w;\mathcal{S}ector_j\right) \leq \mathcal{CI}_j^{\star} \\
 & \Leftrightarrow & \left(\boldsymbol{s}_j \circ \boldsymbol{\mathcal{CI}}\right)^\top w \leq \mathcal{CI}_j^{\star}\left(\boldsymbol{s}_j^\top w\right) \\
 & \Leftrightarrow & \left(\left(\boldsymbol{s}_j \circ \boldsymbol{\mathcal{CI}}\right) - \mathcal{CI}_j^{\star}\boldsymbol{s}_j\right)^\top w \leq 0 \\
 & \Leftrightarrow & \left(\boldsymbol{s}_j \circ \left(\boldsymbol{\mathcal{CI}} - \mathcal{CI}_j^{\star}\right)\right)^\top w \leq 0
\end{array}
$$

QP form

- $C = \left(\boldsymbol{s}_j \circ \left(\boldsymbol{\mathcal{CI}} - \mathcal{CI}_j^{\star}\right)\right)^\top$
- $D = 0$

Equity portfolios

Dealing with constraints on relative weights

Example #2

- Example #1
- We would like to reduce the carbon footprint of the benchmark by 30%
- We impose the sector neutrality

Equity portfolios

Dealing with constraints on relative weights

QP form

- $A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$
- $B = \begin{pmatrix} 100\% \\ 57\% \\ 43\% \end{pmatrix}$
- $C = \begin{pmatrix} 125 & 75 & 254 & 822 & 109 & 17 & 341 & 741 \end{pmatrix}$
- $D = 183.2040$

Equity portfolios

Dealing with constraints on relative weights

- The optimal solution is:

$$w^{\star} = (21.54\%, 18.50\%, 21.15\%, 3.31\%, 10.02\%, 15.26\%, 6.94\%, 3.27\%)$$

- $\sigma(w^{\star} \mid b) = 112$ bps
- $\mathcal{CI}(w^{\star}) = 183.20$ vs. $\mathcal{CI}(b) = 261.72$

BUT

$$\left\{\begin{array}{l} \mathcal{CI}(w^{\star}; \mathcal{S}ector_1) = 132.25 \\ \mathcal{CI}(w^{\star}; \mathcal{S}ector_2) = 250.74 \end{array}\right. \quad \text{versus} \quad \left\{\begin{array}{l} \mathcal{CI}(b; \mathcal{S}ector_1) = 128.54 \\ \mathcal{CI}(b; \mathcal{S}ector_2) = 438.26 \end{array}\right.$$

The global reduction of 30% is explained by:

- an increase of 2.89% of the carbon footprint for the first sector
- a decrease of 42.79% of the carbon footprint for the second sector

Equity portfolios

Dealing with constraints on relative weights

- We impose $\mathcal{R}_1 = 20\%$

QP form

- $C = \begin{pmatrix} \mathcal{CI}^\top \\ \left(\boldsymbol{s}_1 \circ \left(\mathcal{CI} - \left(1 - \mathcal{R}_1\right) \mathcal{CI}\left(b; \mathcal{S}ector_1\right)\right)\right)^\top \end{pmatrix} =$
$$\begin{pmatrix} 125 & 75 & 254 & 822 & 109 & 17 & 341 & 741 \\ 22.1649 & -27.8351 & 0 & 0 & 6.1649 & 0 & 238.1649 & 0 \end{pmatrix}$$
- $D = \begin{pmatrix} 183.2040 \\ 0 \end{pmatrix}$

Equity portfolios

Dealing with constraints on relative weights

- Solving the new QP problem gives the following optimal portfolio:

$$w^{\star} = (22.70\%, 22.67\%, 19.23\%, 5.67\%, 11.39\%, 14.50\%, 0.24\%, 3.61\%)$$

- $\sigma\left(w^{\star} \mid b\right) = 144$ bps
- $\mathcal{CI}\left(w^{\star}\right) = 183.20$
  - $\mathcal{CI}\left(w^{\star}; \mathcal{S}ector_1\right) = 102.84$ (reduction of 20%)
  - $\mathcal{CI}\left(w^{\star}; \mathcal{S}ector_2\right) = 289.74$ (reduction of 33.89%)

Risk measure of a bond portfolio

- We consider a zero-coupon bond, whose price and maturity date are $B(t,T)$ and T:

$$B_t(t,T) = e^{-(r(t)+s(t))(T-t)+L(t)}$$

where $r(t)$, $s(t)$ and $L(t)$ are the interest rate, the credit spread and the liquidity premium

- We deduce that:

$$\begin{aligned} \mathrm{d} \ln B(t,T) &= -(T-t)\,\mathrm{d}r(t) - (T-t)\,\mathrm{d}s(t) + \mathrm{d}L(t) \\ &= -D\,\mathrm{d}r(t) - (D\,s(t))\frac{\mathrm{d}s(t)}{s(t)} + \mathrm{d}L(t) \\ &= -D\,\mathrm{d}r(t) - \mathrm{DTS}(t)\frac{\mathrm{d}s(t)}{s(t)} + \mathrm{d}L(t) \end{aligned}$$

where:

  - $D = T - t$ is the remaining maturity (or duration)
  - $\mathrm{DTS}(t)$ is the duration-times-spread factor

Risk measure of a bond portfolio

- If we assume that $r(t)$, $s(t)$ and $L(t)$ are independent, the risk of the defaultable bond is equal to:

$$\sigma^2 (\mathrm{d} \ln B(t, T)) = D^2 \sigma^2 (\mathrm{d} r(t)) + \mathrm{DTS}(t)^2 \sigma^2 \left(\frac{\mathrm{d} s(t)}{s(t)} \right) + \sigma^2 (\mathrm{d} L(t))$$

- Three risk components

$$\sigma^2 (\mathrm{d} \ln B(t, T)) = D^2 \sigma_r^2 + \mathrm{DTS}(t)^2 \sigma_s^2 + \sigma_L^2$$

$\Longrightarrow$ **The historical volatility of a bond price is not a relevant risk measure**

Bond portfolio optimization

Without a benchmark

- Duration risk:

$$\mathrm{MD}(w) = \sum_{i=1}^{n} w_i \, \mathrm{MD}_i$$

- DTS risk:

$$\mathrm{DTS}(w) = \sum_{i=1}^{n} w_i \, \mathrm{DTS}_i$$

- Clustering approach = generalization of the sector approach, e.g. (EUR, Financials, AAA to A−, 1Y-3Y)
- We have:

$$\mathrm{MD}_j(w) = \sum_{i \in \mathcal{S}ector_j} w_i \, \mathrm{MD}_i$$

and:

$$\mathrm{DTS}_j(w) = \sum_{i \in \mathcal{S}ector_j} w_i \, \mathrm{DTS}_i$$

Bond portfolio optimization

Without a benchmark

Objective function without a benchmark

We have:

$$\begin{aligned} w^{\star} &= \arg\min \frac{\varphi_{\mathrm{MD}}}{2} \sum_{j=1}^{n_{\mathcal{S}ector}} \left(\mathrm{MD}_j\left(w\right) - \mathrm{MD}_j^{\star}\right)^2 + \\ & \frac{\varphi_{\mathrm{DTS}}}{2} \sum_{j=1}^{n_{\mathcal{S}ector}} \left(\mathrm{DTS}_j\left(w\right) - \mathrm{DTS}_j^{\star}\right)^2 - \gamma \sum_{i=1}^{n} w_i \mathcal{C}_i \end{aligned}$$

where:

- $\varphi_{\mathrm{MD}} \geq 0$ and $\varphi_{\mathrm{DTS}} \geq 0$ indicate the relative weight of each risk component
- $\mathcal{C}_i$ is the expected carry of bond i and γ is the risk-tolerance coefficient

Bond portfolio optimization

Without a benchmark

QP form

$$
\begin{array}{rcl}
w^\star & = & \arg\min \mathcal{QF}\left(w; Q, R, c\right) \\
& \text{s.t.} & \left\{ \begin{array}{l} \mathbf{1}_n^\top w = 1 \\ \mathbf{0}_n \leq w \leq \mathbf{1}_n \end{array} \right.
\end{array}
$$

where $\mathcal{QF}\left(w; Q, R, c\right)$ is the quadratic form of the objective function

Bond portfolio optimization

Without a benchmark

We have:

$$\begin{aligned}
\frac{1}{2}\left(\mathrm{MD}_j(w) - \mathrm{MD}_j^{\star}\right)^2 &= \frac{1}{2}\left(\sum\nolimits_{i \in \mathcal{S}ector_j} w_i \,\mathrm{MD}_i - \mathrm{MD}_j^{\star}\right)^2 \\
&= \frac{1}{2}\left(\sum_{i=1}^{n} \boldsymbol{s}_{i,j} w_i \,\mathrm{MD}_i - \mathrm{MD}_j^{\star}\right)^2 \\
&= \frac{1}{2}\left(\sum_{i=1}^{n} \boldsymbol{s}_{i,j} \,\mathrm{MD}_i \, w_i\right)^2 - w^\top \left(\boldsymbol{s}_j \circ \mathrm{MD}\right) \mathrm{MD}_j^{\star} + \frac{1}{2} \mathrm{MD}_j^{\star 2} \\
&= \mathcal{QF}\left(w; \mathcal{T}\left(\boldsymbol{s}_j \circ \mathrm{MD}\right), \left(\boldsymbol{s}_j \circ \mathrm{MD}\right) \mathrm{MD}_j^{\star}, \frac{1}{2} \mathrm{MD}_j^{\star 2}\right)
\end{aligned}$$

where $\mathrm{MD} = (\mathrm{MD}_1, \ldots, \mathrm{MD}_n)$ is the vector of modified durations and $\mathcal{T}(u) = uu^\top$

Bond portfolio optimization

Without a benchmark

We deduce that:

$$\frac{1}{2}\sum_{j=1}^{n_{\mathcal{S}ector}} \left(\mathrm{MD}_j(w) - \mathrm{MD}_j^{\star}\right)^2 = \mathcal{QF}\left(w; Q_{\mathrm{MD}}, R_{\mathrm{MD}}, c_{\mathrm{MD}}\right)$$

where:

$$\left\{\begin{array}{l} Q_{\mathrm{MD}} = \sum_{j=1}^{n_{\mathcal{S}ector}} \mathcal{T}\left(\boldsymbol{s}_j \circ \mathrm{MD}\right) \\ R_{\mathrm{MD}} = \sum_{j=1}^{n_{\mathcal{S}ector}} \left(\boldsymbol{s}_j \circ \mathrm{MD}\right) \mathrm{MD}_j^{\star} \\ c_{\mathrm{MD}} = \frac{1}{2}\sum_{j=1}^{n_{\mathcal{S}ector}} \mathrm{MD}_j^{\star 2} \end{array}\right.$$

Bond portfolio optimization

Without a benchmark

In a similar way, we have:

$$\frac{1}{2}\sum_{j=1}^{n_{\mathcal{S}ector}} \left(\mathrm{DTS}_j(w) - \mathrm{DTS}_j^{\star}\right)^2 = \mathcal{QF}\left(w; Q_{\mathrm{DTS}}, R_{\mathrm{DTS}}, c_{\mathrm{DTS}}\right)$$

where:

$$\begin{cases} Q_{\mathrm{DTS}} = \sum_{j=1}^{n_{\mathcal{S}ector}} \mathcal{T}\left(\boldsymbol{s}_j \circ \mathrm{DTS}\right) \\ R_{\mathrm{MD}} = \sum_{j=1}^{n_{\mathcal{S}ector}} \left(\boldsymbol{s}_j \circ \mathrm{DTS}\right) \mathrm{DTS}_j^{\star} \\ c_{\mathrm{DTS}} = \frac{1}{2}\sum_{j=1}^{n_{\mathcal{S}ector}} \mathrm{DTS}_j^{\star 2} \end{cases}$$

Bond portfolio optimization

Without a benchmark

We have:

$$-\gamma \sum_{i=1}^{n} w_i \mathcal{C}_i = \gamma \mathcal{QF}\left(w; \mathbf{0}_{n,n}, \mathcal{C}, 0\right) = \mathcal{QF}\left(w; \mathbf{0}_{n,n}, \gamma\mathcal{C}, 0\right)$$

where $\mathcal{C} = (\mathcal{C}_1, \ldots, \mathcal{C}_n)$ is the vector of expected carry values

Bond portfolio optimization

Without a benchmark

Quadratic form of the objective function

The function to optimize is:

$$\begin{aligned} \mathcal{QF}\left(w; Q, R, c\right) &= \varphi_{\mathrm{MD}} \mathcal{QF}\left(w; Q_{\mathrm{MD}}, R_{\mathrm{MD}}, c_{\mathrm{MD}}\right) + \\ & \quad \varphi_{\mathrm{DTS}} \mathcal{QF}\left(w; Q_{\mathrm{DTS}}, R_{\mathrm{DTS}}, c_{\mathrm{DTS}}\right) + \\ & \quad \mathcal{QF}\left(w; \mathbf{0}_{n,n}, \gamma \mathcal{C}, 0\right) \end{aligned}$$

where:

$$\left\{ \begin{array}{l} Q = \varphi_{\mathrm{MD}} Q_{\mathrm{MD}} + \varphi_{\mathrm{DTS}} Q_{\mathrm{DTS}} \\ R = \gamma \mathcal{C} + \varphi_{\mathrm{MD}} R_{\mathrm{MD}} + \varphi_{\mathrm{DTS}} R_{\mathrm{DTS}} \\ c = \varphi_{\mathrm{MD}} c_{\mathrm{MD}} + \varphi_{\mathrm{DTS}} c_{\mathrm{DTS}} \end{array} \right.$$

Bond portfolio optimization

With a benchmark

- The MD- and DTS-based tracking error variances are equal to:

$$\mathcal{R}_{\mathrm{MD}}(w \mid b) = \sigma^2_{\mathrm{MD}}(w \mid b) = \sum_{j=1}^{n_{\mathcal{S}ector}} \left(\sum\nolimits_{i \in \mathcal{S}ector_j} (w_i - b_i)\,\mathrm{MD}_i \right)^2$$

and:

$$\mathcal{R}_{\mathrm{DTS}}(w \mid b) = \sigma^2_{\mathrm{DTS}}(w \mid b) = \sum_{j=1}^{n_{\mathcal{S}ector}} \left(\sum\nolimits_{i \in \mathcal{S}ector_j} (w_i - b_i)\,\mathrm{DTS}_i \right)^2$$

This means that $\mathrm{MD}_j^\star = \sum\nolimits_{i \in \mathcal{S}ector_j} b_i\,\mathrm{MD}_i$ and $\mathrm{DTS}_j^\star = \sum\nolimits_{i \in \mathcal{S}ector_j} b_i\,\mathrm{DTS}_i$.

- The active share risk is defined as:

$$\mathcal{R}_{\mathrm{AS}}(w \mid b) = \sigma^2_{\mathrm{AS}}(w \mid b) = \sum_{i=1}^{n} (w_i - b_i)^2$$

Bond portfolio optimization

With a benchmark

Objective function with a benchmark

The optimization problem becomes:

$$\begin{aligned} w^{\star} &= \arg\min \frac{1}{2}\mathcal{R}\left(w \mid b\right) - \gamma \sum_{i=1}^{n} \left(w_i - b_i\right) \mathcal{C}_i \\ & \text{s.t.} \quad \left\{ \begin{array}{l} \mathbf{1}_n^{\top} w = 1 \\ \mathbf{0}_n \leq w \leq \mathbf{1}_n \end{array} \right. \end{aligned}$$

where the synthetic risk measure is equal to:

$$\mathcal{R}\left(w \mid b\right) = \varphi_{\mathrm{AS}} \mathcal{R}_{\mathrm{AS}}\left(w \mid b\right) + \varphi_{\mathrm{MD}} \mathcal{R}_{\mathrm{MD}}\left(w \mid b\right) + \varphi_{\mathrm{DTS}} \mathcal{R}_{\mathrm{DTS}}\left(w \mid b\right)$$

Bond portfolio optimization
With a benchmark

We can show that

$$
\begin{aligned}
w^{\star} & = \arg\min \mathcal{QF}\left(w; Q\left(b\right), R\left(b\right), c\left(b\right)\right) \\
& \text{s.t.} \quad \left\{ \begin{array}{l} \mathbf{1}_n^\top w = 1 \\ \mathbf{0}_n \leq w \leq \mathbf{1}_n \end{array} \right.
\end{aligned}
$$

where:

$$
\left\{ \begin{array}{l}
Q\left(b\right) = \varphi_{\mathrm{AS}} Q_{\mathrm{AS}}\left(b\right) + \varphi_{\mathrm{MD}} Q_{\mathrm{MD}}\left(b\right) + \varphi_{\mathrm{DTS}} Q_{\mathrm{DTS}}\left(b\right) \\
R\left(b\right) = \gamma \mathcal{C} + \varphi_{\mathrm{AS}} R_{\mathrm{AS}}\left(b\right) + \varphi_{\mathrm{MD}} R_{\mathrm{MD}}\left(b\right) + \varphi_{\mathrm{DTS}} R_{\mathrm{DTS}}\left(b\right) \\
c\left(b\right) = \gamma b^\top \mathcal{C} + \varphi_{\mathrm{AS}} c_{\mathrm{AS}}\left(b\right) + \varphi_{\mathrm{MD}} c_{\mathrm{MD}}\left(b\right) + \varphi_{\mathrm{DTS}} c_{\mathrm{DTS}}\left(b\right)
\end{array} \right.
$$

$Q_{\mathrm{AS}}\left(b\right) = I_n$, $R_{\mathrm{AS}}\left(b\right) = b$, $c_{\mathrm{AS}}\left(b\right) = \frac{1}{2} b^\top b$, $Q_{\mathrm{MD}}\left(b\right) = Q_{\mathrm{MD}}$, $R_{\mathrm{MD}}\left(b\right) = Q_{\mathrm{MD}} b = R_{\mathrm{MD}}$, $c_{\mathrm{MD}}\left(b\right) = \frac{1}{2} b^\top Q_{\mathrm{MD}} b = c_{\mathrm{MD}}$, $Q_{\mathrm{DTS}}\left(b\right) = Q_{\mathrm{DTS}}$, $R_{\mathrm{DTS}}\left(b\right) = Q_{\mathrm{DTS}} b = R_{\mathrm{DTS}}$, and $c_{\mathrm{DTS}}\left(b\right) = \frac{1}{2} b^\top Q_{\mathrm{DTS}} b = c_{\mathrm{DTS}}$

Bond portfolio optimization

With a benchmark

Example #3

We consider an investment universe of 9 corporate bonds with the following characteristics[a]:

Issuer	#1	#2	#3	#4	#5	#6	#7	#8	#9
b_i	21	19	16	12	11	8	6	4	3
$\mathcal{CI}_i$	111	52	369	157	18	415	17	253	900
MD_i	3.16	6.48	3.54	9.23	6.40	2.30	8.12	7.96	5.48
DTS_i	107	255	75	996	289	45	620	285	125
$\mathcal{S}ector$	1	1	1	2	2	2	3	3	3

We impose that $0.25 \times b_i \leq w_i \leq 4 \times b_i$. We have $\varphi_{\mathrm{AS}} = 100$, $\varphi_{\mathrm{MD}} = 25$ and $\varphi_{\mathrm{DTS}} = 0.001$.

[a] The units are: b_i in %, $\mathcal{CI}_i$ in $\mathrm{tCO_2e}/\$\ \mathrm{mn}$, MD_i in years and DTS_i in bps

Bond portfolio optimization

With a benchmark

The optimization problem is defined as:

$$
\begin{aligned}
w^{\star}(\mathcal{R}) &= \arg\min \frac{1}{2} w^{\top} Q(b)\, w - w^{\top} R(b) \\
&\text{s.t.} \quad \left\{
\begin{array}{l}
\mathbf{1}_9^{\top} w = 1 \\
\mathcal{CI}^{\top} w \leq (1 - \mathcal{R})\, \mathcal{CI}(b) \\
\dfrac{b}{4} \leq w \leq 4b
\end{array}
\right.
\end{aligned}
$$

where $\mathcal{R}$ is the reduction rate

Bond portfolio optimization

With a benchmark

Since the bonds are ordering by sectors, $Q(b)$ is a block diagonal matrix:

$$Q(b) = \begin{pmatrix} Q_1 & \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} \\ \mathbf{0}_{3\times 3} & Q_2 & \mathbf{0}_{3\times 3} \\ \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} & Q_3 \end{pmatrix} \times 10^3$$

where:

$$Q_1 = \begin{pmatrix} 0.3611 & 0.5392 & 0.2877 \\ 0.5392 & 1.2148 & 0.5926 \\ 0.2877 & 0.5926 & 0.4189 \end{pmatrix}, \qquad Q_2 = \begin{pmatrix} 3.2218 & 1.7646 & 0.5755 \\ 1.7646 & 1.2075 & 0.3810 \\ 0.5755 & 0.3810 & 0.2343 \end{pmatrix}$$

and:

$$Q_3 = \begin{pmatrix} 2.1328 & 1.7926 & 1.1899 \\ 1.7926 & 1.7653 & 1.1261 \\ 1.1899 & 1.1261 & 0.8664 \end{pmatrix}$$

$$R(b) = (2.243, 4.389, 2.400, 6.268, 3.751, 1.297, 2.354, 2.120, 1.424) \times 10^2$$

Bond portfolio optimization

With a benchmark

Table 71: Weights in % of optimized bond portfolios (Example #3)

Portfolio	#1	#2	#3	#4	#5	#6	#7	#8	#9
b	21.00	19.00	16.00	12.00	11.00	8.00	6.00	4.00	3.00
$w^\star$ (10%)	21.92	19.01	15.53	11.72	11.68	7.82	6.68	4.71	0.94
$w^\star$ (30%)	26.29	20.24	10.90	10.24	16.13	3.74	9.21	2.50	0.75
$w^\star$ (50%)	27.48	23.97	4.00	6.94	22.70	2.00	11.15	1.00	0.75

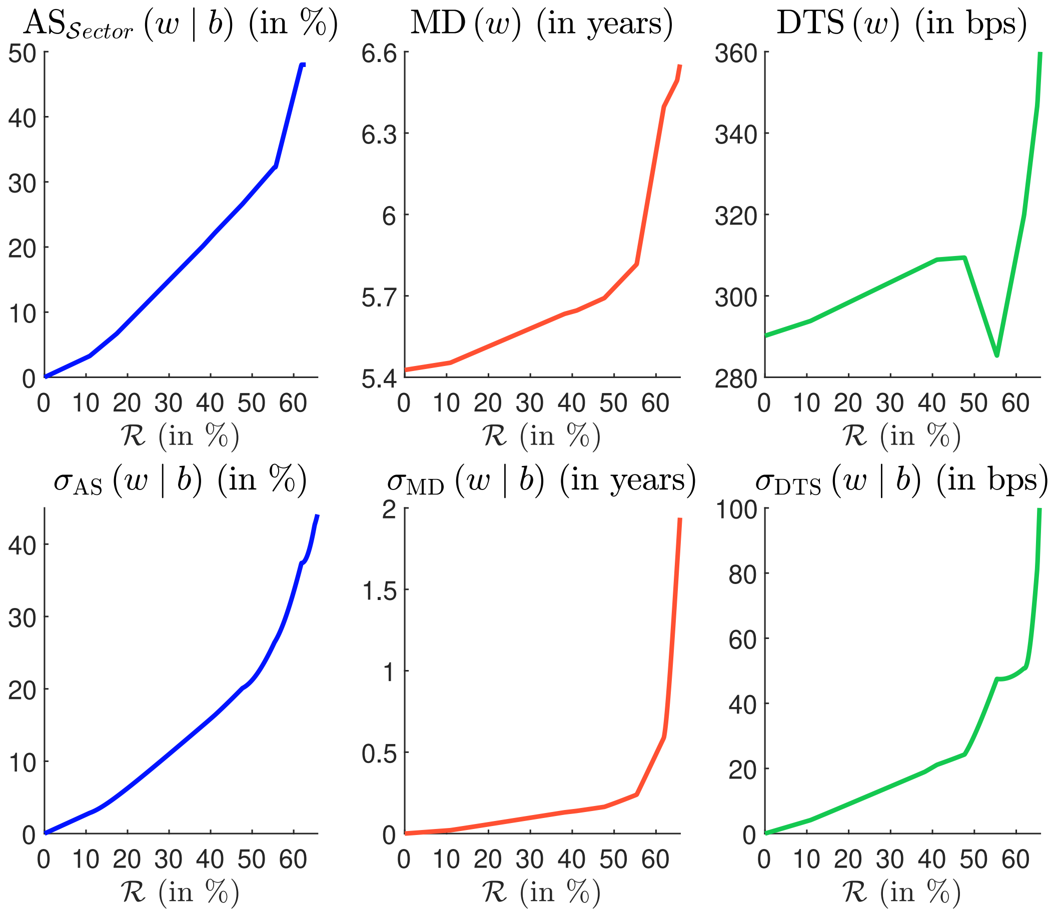
Table 72: Risk statistics of optimized bond portfolios (Example #3)

Portfolio	$\mathrm{AS}_{\mathcal{S}ector}$ (in %)	$\mathrm{MD}(w)$ (in years)	$\mathrm{DTS}(w)$ (in bps)	$\sigma_{\mathrm{AS}}(w \mid b)$ (in %)	$\sigma_{\mathrm{MD}}(w \mid b)$ (in years)	$\sigma_{\mathrm{DTS}}(w \mid b)$ (in bps)	$\mathcal{CI}(w)$ $\mathrm{gCO_2e}/\$$
b	0.00	5.43	290.18	0.00	0.00	0.00	184.39
$w^\star$ (10%)	3.00	5.45	293.53	2.62	0.02	3.80	165.95
$w^\star$ (30%)	14.87	5.58	303.36	10.98	0.10	14.49	129.07
$w^\star$ (50%)	28.31	5.73	302.14	21.21	0.19	30.11	92.19

Bond portfolio optimization

With a benchmark

Figure 85: Relationship between the reduction rate and the tracking risk (Example #3)



Advanced optimization problems

Large bond universe

- QP: $\mathbf{n} \leq \mathbf{5\,000}$ (the dimension of Q is $n \times n$)
- LP: $\mathbf{n} \gg \mathbf{10^6}$

- Some figures as of 31/01/2023
  - MSCI World Index (DM): $n = 1\,508$ stocks
  - MSCI World IMI (DM): $n = 5\,942$ stocks
  - MSCI World AC (DM + EM): $n = 2\,882$ stocks
  - MSCI World AC IMI (DM + EM): $n = 7\,928$ stocks

  - Bloomberg Global Aggregate Total Return Index: $n = 28\,799$ securities
  - ICE BOFA Global Broad Market Index: $n = 33\,575$ securities

- Trick: $\mathcal{L}_2$-norm risk measures $\Rightarrow$ $\mathcal{L}_1$-norm risk measures

Advanced optimization problems

Large bond universe

We replace the synthetic risk measure by:

$$\mathcal{D}(w \mid b) = \varphi'_{\mathrm{AS}} \mathcal{D}_{\mathrm{AS}}(w \mid b) + \varphi'_{\mathrm{MD}} \mathcal{D}_{\mathrm{MD}}(w \mid b) + \varphi'_{\mathrm{DTS}} \mathcal{D}_{\mathrm{DTS}}(w \mid b)$$

where:

$$\begin{aligned}
\mathcal{D}_{\mathrm{AS}}(w \mid b) &= \frac{1}{2} \sum_{i=1}^{n} |w_i - b_i| \\
\mathcal{D}_{\mathrm{MD}}(w \mid b) &= \sum_{j=1}^{n_{\mathcal{S}ector}} \left| \sum\nolimits_{i \in \mathcal{S}ector_j} (w_i - b_i) \, \mathrm{MD}_i \right| \\
\mathcal{D}_{\mathrm{DTS}}(w \mid b) &= \sum_{j=1}^{n_{\mathcal{S}ector}} \left| \sum\nolimits_{i \in \mathcal{S}ector_j} (w_i - b_i) \, \mathrm{DTS}_i \right|
\end{aligned}$$

Advanced optimization problems
Large bond universe

The optimization problem becomes:

$$\begin{aligned} w^{\star} & = & \arg\min \mathcal{D}\left(w \mid b\right) - \gamma \sum_{i=1}^{n} \left(w_i - b_i\right) \mathcal{C}_i \\ & \text{s.t.} & \left\{ \begin{array}{l} \mathbf{1}_n^\top w = 1 \\ \mathbf{0}_n \leq w \leq \mathbf{1}_n \end{array} \right. \end{aligned}$$

Advanced optimization problems

Large bond universe

Absolute value trick

If $c_i \geq 0$, then:

$$\min \sum_{i=1}^{n} c_i \left| f_i \left(x \right) \right| + g \left(x \right) \Leftrightarrow \left\{ \begin{array}{ll} \min & \displaystyle\sum_{i=1}^{n} c_i \tau_i + g \left(x \right) \\ \text{s.t.} & \left\{ \begin{array}{l} \left| f_i \left(x \right) \right| \leq \tau_i \\ \tau_i \geq 0 \end{array} \right. \end{array} \right.$$

The problem becomes linear:

$$\left| f_i \left(x \right) \right| \leq \tau_i \Leftrightarrow -\tau_i \leq f_i \left(x \right) \wedge f_i \left(x \right) \leq \tau_i$$

Advanced optimization problems

Large bond universe

Linear programming

The standard formulation of a linear programming problem is:

$$
\begin{aligned}
x^{\star} & = & \arg\min c^{\top} x \\
& \text{s.t.} & \left\{ \begin{array}{l} Ax = b \\ Cx \leq D \\ x^{-} \leq x \leq x^{+} \end{array} \right.
\end{aligned}
$$

where x is a $n \times 1$ vector, c is a $n \times 1$ vector, A is a $n_A \times n$ matrix, B is a $n_A \times 1$ vector, C is a $n_C \times n$ matrix, D is a $n_C \times 1$ vector, and x^{-} and x^{+} are two $n \times 1$ vectors.

Advanced optimization problems

Large bond universe

We have:

$$
\begin{aligned}
w^{\star} &= \arg\min \frac{1}{2}\varphi'_{\mathrm{AS}}\sum_{i=1}^{n}\tau_{i,w} + \varphi'_{\mathrm{MD}}\sum_{j=1}^{n_{\mathcal{S}ector}}\tau_{j,\mathrm{MD}} + \varphi'_{\mathrm{DTS}}\sum_{j=1}^{n_{\mathcal{S}ector}}\tau_{j,\mathrm{DTS}} - \\
&\qquad \gamma\sum_{i=1}^{n}\left(w_i - b_i\right)\mathcal{C}_i \\
&\text{s.t.} \left\{
\begin{array}{l}
\mathbf{1}_n^{\top} w = 1 \\
\mathbf{0}_n \leq w \leq \mathbf{1}_n \\
\left|w_i - b_i\right| \leq \tau_{i,w} \\
\left|\sum_{i\in\mathcal{S}ector_j}\left(w_i - b_i\right)\mathrm{MD}_i\right| \leq \tau_{j,\mathrm{MD}} \\
\left|\sum_{i\in\mathcal{S}ector_j}\left(w_i - b_i\right)\mathrm{DTS}_i\right| \leq \tau_{j,\mathrm{DTS}} \\
\tau_{i,w} \geq 0, \tau_{j,\mathrm{MD}} \geq 0, \tau_{j,\mathrm{DTS}} \geq 0
\end{array}
\right.
\end{aligned}
$$

Advanced optimization problems
Large bond universe

$$\left|w_i - b_i\right| \leq \tau_{i,w} \Leftrightarrow \left\{ \begin{array}{l} w_i - \tau_{i,w} \leq b_i \\ -w_i - \tau_{i,w} \leq -b_i \end{array} \right.$$

Advanced optimization problems

Large bond universe

$$
\begin{aligned}
(*) \quad & \Leftrightarrow \quad \left| \sum_{i \in \mathcal{S}ector_j} (w_i - b_i) \,\mathrm{MD}_i \right| \leq \tau_{j,\mathrm{MD}} \\
& \Leftrightarrow \quad -\tau_{j,\mathrm{MD}} \leq \sum_{i \in \mathcal{S}ector_j} (w_i - b_i) \,\mathrm{MD}_i \leq \tau_{j,\mathrm{MD}} \\
& \Leftrightarrow \quad -\tau_{j,\mathrm{MD}} + \sum_{i \in \mathcal{S}ector_j} b_i \,\mathrm{MD}_i \leq \sum_{i \in \mathcal{S}ector_j} w_i \,\mathrm{MD}_i \leq \tau_{j,\mathrm{MD}} + \\
& \qquad \sum_{i \in \mathcal{S}ector_j} b_i \,\mathrm{MD}_i \\
& \Leftrightarrow \quad -\tau_{j,\mathrm{MD}} + \mathrm{MD}_j^{\star} \leq \left(\boldsymbol{s}_j \circ \mathrm{MD}\right)^{\top} w \leq \tau_{j,\mathrm{MD}} + \mathrm{MD}_j^{\star} \\
& \Leftrightarrow \quad \left\{ \begin{array}{l} \left(\boldsymbol{s}_j \circ \mathrm{MD}\right)^{\top} w - \tau_{j,\mathrm{MD}} \leq \mathrm{MD}_j^{\star} \\ -\left(\boldsymbol{s}_j \circ \mathrm{MD}\right)^{\top} w - \tau_{j,\mathrm{MD}} \leq -\mathrm{MD}_j^{\star} \end{array} \right.
\end{aligned}
$$

Advanced optimization problems

Large bond universe

$$\left|\sum\nolimits_{i\in\mathcal{S}ector_j}(w_i-b_i)\,\mathrm{DTS}_i\right|\leq\tau_{j,\mathrm{DTS}}\Leftrightarrow\left\{\begin{array}{l}\left(\boldsymbol{s}_j\circ\mathrm{DTS}\right)^\top w-\tau_{j,\mathrm{DTS}}\leq\mathrm{DTS}_j^\star\\-\left(\boldsymbol{s}_j\circ\mathrm{DTS}\right)^\top w-\tau_{j,\mathrm{DTS}}\leq-\mathrm{DTS}_j^\star\end{array}\right.$$

Advanced optimization problems
LP formulation

- x is a vector of dimension $n_x = 2 \times (n + n_{\mathcal{S}ector})$:

$$x = \begin{pmatrix} w \\ \tau_w \\ \tau_{\mathrm{MD}} \\ \tau_{\mathrm{DTS}} \end{pmatrix}$$

Advanced optimization problems

LP formulation

- The vector c is equal to:

$$c = \begin{pmatrix} -\gamma \mathcal{C} \\ \dfrac{1}{2} \varphi'_{\mathrm{AS}} \mathbf{1}_n \\ \varphi'_{\mathrm{MD}} \mathbf{1}_{n_{\mathcal{S}ector}} \\ \varphi'_{\mathrm{DTS}} \mathbf{1}_{n_{\mathcal{S}ector}} \end{pmatrix}$$

Advanced optimization problems

LP formulation

- The linear equality constraint $Ax = B$ is defined by:

$$A = \begin{pmatrix} \mathbf{1}_n^\top & \mathbf{0}_n^\top & \mathbf{0}_{n_{\mathcal{S}ector}}^\top & \mathbf{0}_{n_{\mathcal{S}ector}}^\top \end{pmatrix}$$

and:

$$B = 1$$

Advanced optimization problems

LP formulation

- The linear inequality constraint $Cx \leq D$ is defined by:

$$C = \begin{pmatrix} I_n & -I_n & \mathbf{0}_{n,n_{\mathcal{S}ector}} & \mathbf{0}_{n,n_{\mathcal{S}ector}} \\ -I_n & -I_n & \mathbf{0}_{n,n_{\mathcal{S}ector}} & \mathbf{0}_{n,n_{\mathcal{S}ector}} \\ C_{\mathrm{MD}} & \mathbf{0}_{n_{\mathcal{S}ector},n} & -I_{n_{\mathcal{S}ector}} & \mathbf{0}_{n_{\mathcal{S}ector},n_{\mathcal{S}ector}} \\ -C_{\mathrm{MD}} & \mathbf{0}_{n_{\mathcal{S}ector},n} & -I_{n_{\mathcal{S}ector}} & \mathbf{0}_{n_{\mathcal{S}ector},n_{\mathcal{S}ector}} \\ C_{\mathrm{DTS}} & \mathbf{0}_{n_{\mathcal{S}ector},n} & \mathbf{0}_{n_{\mathcal{S}ector},n_{\mathcal{S}ector}} & -I_{n_{\mathcal{S}ector}} \\ -C_{\mathrm{DTS}} & \mathbf{0}_{n_{\mathcal{S}ector},n} & \mathbf{0}_{n_{\mathcal{S}ector},n_{\mathcal{S}ector}} & -I_{n_{\mathcal{S}ector}} \end{pmatrix}$$

end:

$$D = \begin{pmatrix} b \\ -b \\ \mathrm{MD}^{\star} \\ -\mathrm{MD}^{\star} \\ \mathrm{DTS}^{\star} \\ -\mathrm{DTS}^{\star} \end{pmatrix}$$

Advanced optimization problems
LP formulation

- C_{MD} and C_{DTS} are two $n_{\mathcal{S}ector} \times n$ matrices, whose elements are:

$$\left(C_{\mathrm{MD}}\right)_{j,i} = \boldsymbol{s}_{i,j}\,\mathrm{MD}_i$$

and:

$$\left(C_{\mathrm{DTS}}\right)_{j,i} = \boldsymbol{s}_{i,j}\,\mathrm{DTS}_i$$

- We have:

$$\mathrm{MD}^{\star} = \left(\mathrm{MD}_1^{\star}, \ldots, \mathrm{MD}_{n_{\mathcal{S}ector}}^{\star}\right)$$

and

$$\mathrm{DTS}^{\star} = \left(\mathrm{DTS}_1^{\star}, \ldots, \mathrm{DTS}_{n_{\mathcal{S}ector}}^{\star}\right)$$

Advanced optimization problems

LP formulation

- The bounds are:

$$x^{-} = \mathbf{0}_{n_x}$$

and:

$$x^{+} = \infty \cdot \mathbf{1}_{n_x}$$

Advanced optimization problems

LP formulation

- Additional constraints:

$$\left\{ \begin{array}{l} A'w = B' \\ C'w \leq D' \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} \left(\begin{array}{cc} A' & \mathbf{0}_{n_A,n_x-n} \end{array} \right) x = B' \\ \left(\begin{array}{cc} C' & \mathbf{0}_{n_A,n_x-n} \end{array} \right) x \leq D' \end{array} \right.$$

Advanced optimization problems

Large bond universe

Toy example

We consider a toy example with four corporate bonds:

Issuer	#1	#2	#3	#4
b_i (in %)	35	15	20	30
$\mathcal{CI}_i$ (in $\mathrm{tCO_2e}/\$ \mathrm{mn}$)	117	284	162.5	359
MD_i (in years)	3.0	5.0	2.0	6.0
DTS_i (in bps)	100	150	200	250
$\mathcal{S}ector$	1	1	2	2

We would like to reduce the carbon footprint by 20%, and we set $\varphi'_{\mathrm{AS}} = 100$, $\varphi'_{\mathrm{MD}} = 25$ and $\varphi'_{\mathrm{DTS}} = 1$

Advanced optimization problems

Large bond universe

We have $n = 4$, $n_{\mathcal{S}ector} = 2$ and:

$$x = \left(\underbrace{w_1, w_2, w_3, w_4}_{w}, \underbrace{\tau_{w_1}, \tau_{w_2}, \tau_{w_3}, \tau_{w_4}}_{\tau_w}, \underbrace{\tau_{\mathrm{MD}_1}, \tau_{\mathrm{MD}_2}}_{\tau_{\mathrm{MD}}}, \underbrace{\tau_{\mathrm{DTS}_1}, \tau_{\mathrm{DTS}_2}}_{\tau_{\mathrm{DTS}}}\right)$$

Since the vector $\mathcal{C}$ is equal to $\mathbf{0}_4$, we obtain:

$$c = (0, 0, 0, 0, 50, 50, 50, 50, 25, 25, 1, 1)$$

Advanced optimization problems

Large bond universe

The equality system $Ax = B$ is defined by:

$$A = \left(\begin{array}{cccccccccccc} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

and:

$$B = 1$$

Advanced optimization problems
Large bond universe

The inequality system $Cx \leq D$ is given by:

$$C = \left(\begin{array}{cccc|c|cc|cc}
\multicolumn{4}{c|}{I_4} & -I_4 & \multicolumn{4}{c}{\mathbf{0}_{4,4}} \\ \hline
\multicolumn{4}{c|}{-I_4} & -I_4 & \multicolumn{4}{c}{\mathbf{0}_{4,4}} \\ \hline
3 & 5 & 0 & 0 & & -1 & 0 & 0 & 0 \\
0 & 0 & 2 & 6 & \mathbf{0}_{4,4} & 0 & -1 & 0 & 0 \\
-3 & -5 & 0 & 0 & & -1 & 0 & 0 & 0 \\
0 & 0 & -2 & -6 & & 0 & -1 & 0 & 0 \\ \hline
100 & 150 & 0 & 0 & & 0 & 0 & -1 & 0 \\
0 & 0 & 200 & 250 & \mathbf{0}_{4,4} & 0 & 0 & 0 & -1 \\
-100 & -150 & 0 & 0 & & 0 & 0 & -1 & 0 \\
0 & 0 & -200 & -250 & & 0 & 0 & 0 & -1 \\ \hline
117 & 284 & 162.5 & 359 & \mathbf{0}_{1,4} & 0 & 0 & 0 & 0
\end{array}\right)$$

and:

$$\begin{array}{rcllll}
D & = & (0.35, 0.15, 0.2, 0.3, & -0.35, -0.15, -0.2, -0.3, \ldots & \\
& & 1.8, 2.2, -1.8, -2.2, & 57.5, 115, -57.5, -115, & 179)
\end{array}$$

Advanced optimization problems

Large bond universe

- The last row of $Cx \leq D$ corresponds to the carbon footprint constraint
- We have:

$$\mathcal{CI}(b) = 223.75 \text{ tCO}_2\text{e}/\$ \text{ mn}$$

and:

$$(1 - \mathcal{R})\,\mathcal{CI}(b) = 0.80 \times 223.75 = 179.00 \text{ tCO}_2\text{e}/\$ \text{ mn}$$

Advanced optimization problems

Large bond universe

We solve the LP program, and we obtain the following solution:

$$\begin{aligned} w^{\star} &= (47.34\%, 0\%, 33.3\%, 19.36\%) \\ \tau_{w}^{\star} &= (12.34\%, 15\%, 13.3\%, 10.64\%) \\ \tau_{\mathrm{MD}}^{\star} &= (0.3798, 0.3725) \\ \tau_{\mathrm{DTS}}^{\star} &= (10.1604, 0) \end{aligned}$$

Advanced optimization problems

Large bond universe

- Interpretation of $\tau_w^\star$:

$$w^\star \pm \tau_w^\star = \begin{pmatrix} 47.34\% \\ 0.00\% \\ 33.30\% \\ 19.36\% \end{pmatrix} \begin{pmatrix} - \\ + \\ - \\ + \end{pmatrix} \begin{pmatrix} 12.34\% \\ 15.00\% \\ 13.30\% \\ 10.64\% \end{pmatrix} = \begin{pmatrix} 35\% \\ 15\% \\ 20\% \\ 30\% \end{pmatrix} = b$$

- Interpretation of $\tau_{\mathrm{MD}}^\star$:

$$\begin{pmatrix} \sum_{i \in \mathcal{S}ector_1} w_i^\star \,\mathrm{MD}_i \\ \sum_{i \in \mathcal{S}ector_2} w_i^\star \,\mathrm{MD}_i \end{pmatrix} \pm \tau_{\mathrm{MD}}^\star = \begin{pmatrix} 1.42 \\ 1.83 \end{pmatrix} \begin{pmatrix} + \\ + \end{pmatrix} \begin{pmatrix} 0.38 \\ 0.37 \end{pmatrix} = \begin{pmatrix} 1.80 \\ 2.20 \end{pmatrix} = \begin{pmatrix} \mathrm{MD}_1^\star \\ \mathrm{MD}_2^\star \end{pmatrix}$$

- Interpretation of $\tau_{\mathrm{DTS}}^\star$:

$$\begin{pmatrix} \sum_{i \in \mathcal{S}ector_1} w_i^\star \,\mathrm{DTS}_i \\ \sum_{i \in \mathcal{S}ector_2} w_i^\star \,\mathrm{DTS}_i \end{pmatrix} \pm \tau_{\mathrm{DTS}}^\star = \begin{pmatrix} 47.34 \\ 115.00 \end{pmatrix} \begin{pmatrix} + \\ + \end{pmatrix} \begin{pmatrix} 10.16 \\ 0.00 \end{pmatrix} = \begin{pmatrix} 57.50 \\ 115.00 \end{pmatrix} = \begin{pmatrix} \mathrm{DTS}_1^\star \\ \mathrm{DTS}_2^\star \end{pmatrix}$$

Advanced optimization problems

Large bond universe

Example #4 (Example #3 again)

We consider an investment universe of 9 corporate bonds with the following characteristics[a]:

Issuer	#1	#2	#3	#4	#5	#6	#7	#8	#9
b_i	21	19	16	12	11	8	6	4	3
$\mathcal{CI}_i$	111	52	369	157	18	415	17	253	900
MD_i	3.16	6.48	3.54	9.23	6.40	2.30	8.12	7.96	5.48
DTS_i	107	255	75	996	289	45	620	285	125
$\mathcal{S}ector$	1	1	1	2	2	2	3	3	3

We impose that $0.25 \times b_i \leq w_i \leq 4 \times b_i$ and assume that $\varphi'_{\mathrm{AS}} = \varphi_{\mathrm{AS}} = 100$, $\varphi'_{\mathrm{MD}} = \varphi_{\mathrm{MD}} = 25$ and $\varphi'_{\mathrm{DTS}} = \varphi'_{\mathrm{DTS}} = 0.001$

[a] The units are: b_i in %, $\mathcal{CI}_i$ in $\mathrm{tCO_2e/\$\ mn}$, MD_i in years and DTS_i in bps

Advanced optimization problems

Large bond universe

Table 73: Weights in % of optimized bond portfolios (Example #4)

Portfolio	#1	#2	#3	#4	#5	#6	#7	#8	#9
b	21.00	19.00	16.00	12.00	11.00	8.00	6.00	4.00	3.00
$w^\star$ (10%)	21.70	19.00	16.00	12.00	11.00	8.00	7.46	4.00	0.84
$w^\star$ (30%)	34.44	19.00	4.00	11.65	11.98	6.65	7.52	4.00	0.75
$w^\star$ (50%)	33.69	19.37	4.00	3.91	24.82	2.00	10.46	1.00	0.75

Table 74: Risk statistics of optimized bond portfolios (Example #4)

Portfolio	$\mathrm{AS}_{\mathcal{S}ector}$ (in %)	$\mathrm{MD}(w)$ (in years)	$\mathrm{DTS}(w)$ (in bps)	$\sigma_{\mathrm{AS}}(w \mid b)$ (in %)	$\sigma_{\mathrm{MD}}(w \mid b)$ (in years)	$\sigma_{\mathrm{DTS}}(w \mid b)$ (in bps)	$\mathcal{CI}(w)$ $\mathrm{gCO_2e/\$}$
b	0.00	5.43	290.18	0.00	0.00	0.00	184.39
$w^\star$ (10%)	2.16	5.45	297.28	2.16	0.02	7.10	165.95
$w^\star$ (30%)	15.95	5.43	300.96	15.95	0.00	13.20	129.07
$w^\star$ (50%)	31.34	5.43	268.66	31.34	0.00	65.12	92.19

Equity portfolios

Threshold approach

The optimization problem is:

$$
\begin{aligned}
w^{\star} &= \arg\min \frac{1}{2}\left(w-b\right)^{\top}\Sigma\left(w-b\right) \\
&\text{s.t.} \quad \left\{
\begin{array}{l}
\mathbf{1}_n^{\top} w = 1 \\
w \in \Omega \\
\mathbf{0}_n \leq w \leq \mathbf{1}_n \\
\mathcal{CI}\left(w\right) \leq \left(1-\mathcal{R}\right)\mathcal{CI}\left(b\right)
\end{array}
\right.
\end{aligned}
$$

Equity portfolios

Order-statistic approach

- $\mathcal{CI}_{i:n}$ is the order statistics of $(\mathcal{CI}_1, \ldots, \mathcal{CI}_n)$:

$$\min \mathcal{CI}_i = \mathcal{CI}_{1:n} \leq \mathcal{CI}_{2:n} \leq \cdots \leq \mathcal{CI}_{i:n} \leq \cdots \leq \mathcal{CI}_{n:n} = \max \mathcal{CI}_i$$

- The carbon intensity bound $\mathcal{CI}^{(m,n)}$ is defined as:

$$\mathcal{CI}^{(m,n)} = \mathcal{CI}_{n-m+1:n}$$

  where $\mathcal{CI}_{n-m+1:n}$ is the $(n-m+1)$-th order statistic of $(\mathcal{CI}_1, \ldots, \mathcal{CI}_n)$

- Exclusion process:

$$\mathcal{CI}_i \geq \mathcal{CI}^{(m,n)} \Rightarrow w_i = 0$$

Equity portfolios

Order-statistic approach (Cont'd)

The optimization problem is:

$$\begin{aligned} w^{\star} & = & \arg\min \frac{1}{2}\left(w-b\right)^{\top}\Sigma\left(w-b\right) \\ & \text{s.t.} & \left\{ \begin{array}{l} \mathbf{1}_n^{\top} w = 1 \\ w \in \Omega \\ \mathbf{0}_n \leq w \leq \mathbb{1}\left\{\mathcal{CI} < \mathcal{CI}^{(m,n)}\right\} \end{array} \right. \end{aligned}$$

Equity portfolios

Naive approach

We re-weight the remaining assets:

$$w_i^\star = \frac{\mathbb{1}\left\{\mathcal{CI}_i < \mathcal{CI}^{(m,n)}\right\} \cdot b_i}{\sum_{k=1}^n \mathbb{1}\left\{\mathcal{CI}_k < \mathcal{CI}^{(m,n)}\right\} \cdot b_k}$$

Equity portfolios

Example #5

We consider a capitalization-weighted equity index, which is composed of eight stocks. Their weights are equal to 20%, 19%, 17%, 13%, 12%, 8%, 6% and 5%. The carbon intensities (expressed in $\mathrm{tCO_2e}/\$\ \mathrm{mn}$) are respectively equal to 100.5, 97.2, 250.4, 352.3, 27.1, 54.2, 78.6 and 426.7. To evaluate the risk of the portfolio, we use the market one-factor model: the beta β_i of each stock is equal to 0.30, 1.80, 0.85, 0.83, 1.47, 0.94, 1.67 and 1.08, the idiosyncratic volatilities $\tilde{\sigma}_i$ are respectively equal to 10%, 5%, 6%, 12%, 15%, 4%, 8% and 7%, and the estimated market volatility σ_m is 18%.

Equity portfolios

The covariance matrix is:

$$\Sigma = \beta\beta^\top \sigma_m^2 + D$$

where:

1. β is the vector of beta coefficients
2. σ_m^2 is the variance of the market portfolio
3. $D = diag\left(\tilde{\sigma}_1^2, \ldots, \tilde{\sigma}_n^2\right)$ is the diagonal matrix of idiosyncratic variances

Equity portfolios

Table 75: Optimal decarbonization portfolios (Example #5, threshold approach)

$\mathcal{R}$	0	10	20	30	40	50	$\mathcal{CI}_i$
$w_1^\star$	20.00	20.54	21.14	21.86	22.58	22.96	100.5
$w_2^\star$	19.00	19.33	19.29	18.70	18.11	17.23	97.2
$w_3^\star$	17.00	15.67	12.91	8.06	3.22	0.00	250.4
$w_4^\star$	13.00	12.28	10.95	8.74	6.53	3.36	352.3
$w_5^\star$	12.00	12.26	12.60	13.07	13.53	14.08	27.1
$w_6^\star$	8.00	11.71	16.42	22.57	28.73	34.77	54.2
$w_7^\star$	6.00	6.36	6.69	7.00	7.30	7.59	78.6
$w_8^\star$	5.00	1.86	0.00	0.00	0.00	0.00	426.7
$\sigma(w^\star \mid b)$	0.00	30.01	61.90	104.10	149.65	196.87	
$\mathcal{CI}(w)$	160.57	144.52	128.46	112.40	96.34	80.29	
$\mathcal{R}(w \mid b)$	0.00	10.00	20.00	30.00	40.00	50.00	

The reduction rate and the weights are expressed in % whereas the tracking error volatility is measured in bps

Equity portfolios

Table 76: Optimal decarbonization portfolios (Example #5, order-statistic approach)

m	0	1	2	3	4	5	6	7	$\mathcal{CI}_i$
$w_1^\star$	20.00	20.40	22.35	26.46	0.00	0.00	0.00	0.00	100.5
$w_2^\star$	19.00	19.90	20.07	20.83	7.57	0.00	0.00	0.00	97.2
$w_3^\star$	17.00	17.94	21.41	0.00	0.00	0.00	0.00	0.00	250.4
$w_4^\star$	13.00	13.24	0.00	0.00	0.00	0.00	0.00	0.00	352.3
$w_5^\star$	12.00	12.12	12.32	12.79	13.04	14.26	18.78	100.00	27.1
$w_6^\star$	8.00	10.04	17.14	32.38	74.66	75.12	81.22	0.00	54.2
$w_7^\star$	6.00	6.37	6.70	7.53	4.73	10.62	0.00	0.00	78.6
$w_8^\star$	5.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	426.7
$\sigma(w^\star \mid b)$	0.00	0.37	1.68	2.25	3.98	4.04	4.30	15.41	
$\mathcal{CI}(w)$	160.57	145.12	113.48	73.78	55.08	52.93	49.11	27.10	
$\mathcal{R}(w \mid b)$	0.00	9.62	29.33	54.05	65.70	67.04	69.42	83.12	

The reduction rate, the weights and the tracking error volatility are expressed in %

Equity portfolios

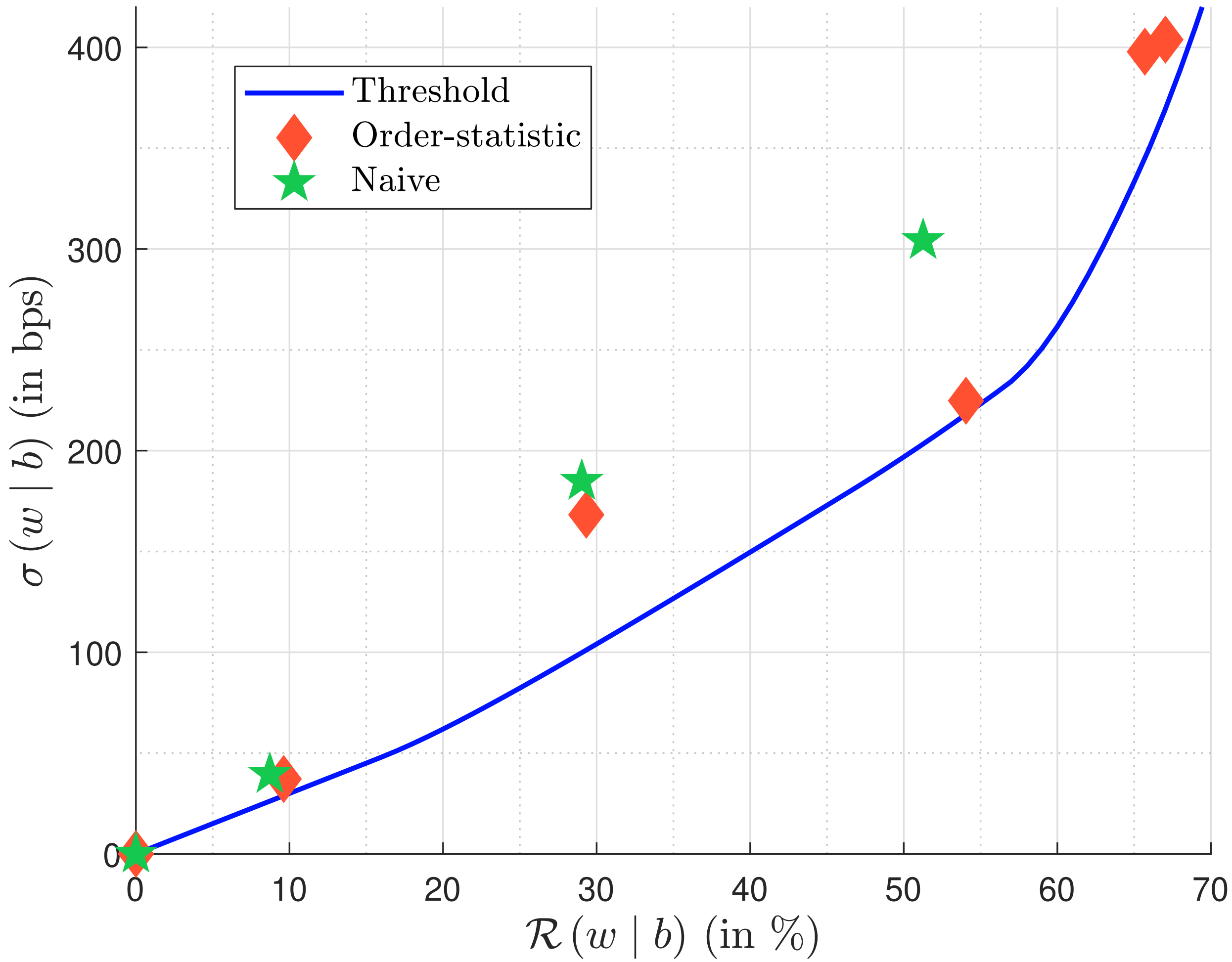
Table 77: Optimal decarbonization portfolios (Example #5, naive approach)

m	0	1	2	3	4	5	6	7	$\mathcal{CI}_i$
$w_1^\star$	20.00	21.05	24.39	30.77	0.00	0.00	0.00	0.00	100.5
$w_2^\star$	19.00	20.00	23.17	29.23	42.22	0.00	0.00	0.00	97.2
$w_3^\star$	17.00	17.89	20.73	0.00	0.00	0.00	0.00	0.00	250.4
$w_4^\star$	13.00	13.68	0.00	0.00	0.00	0.00	0.00	0.00	352.3
$w_5^\star$	12.00	12.63	14.63	18.46	26.67	46.15	60.00	100.00	27.1
$w_6^\star$	8.00	8.42	9.76	12.31	17.78	30.77	40.00	0.00	54.2
$w_7^\star$	6.00	6.32	7.32	9.23	13.33	23.08	0.00	0.00	78.6
$w_8^\star$	5.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	426.7
$\sigma(w^\star \mid b)$	0.00	0.39	1.85	3.04	9.46	8.08	8.65	15.41	
$\mathcal{CI}(w)$	160.57	146.57	113.95	78.26	68.38	47.32	37.94	27.10	
$\mathcal{R}(w \mid b)$	0.00	8.72	29.04	51.26	57.41	70.53	76.37	83.12	

The reduction rate, the weights and the tracking error volatility are expressed in %.

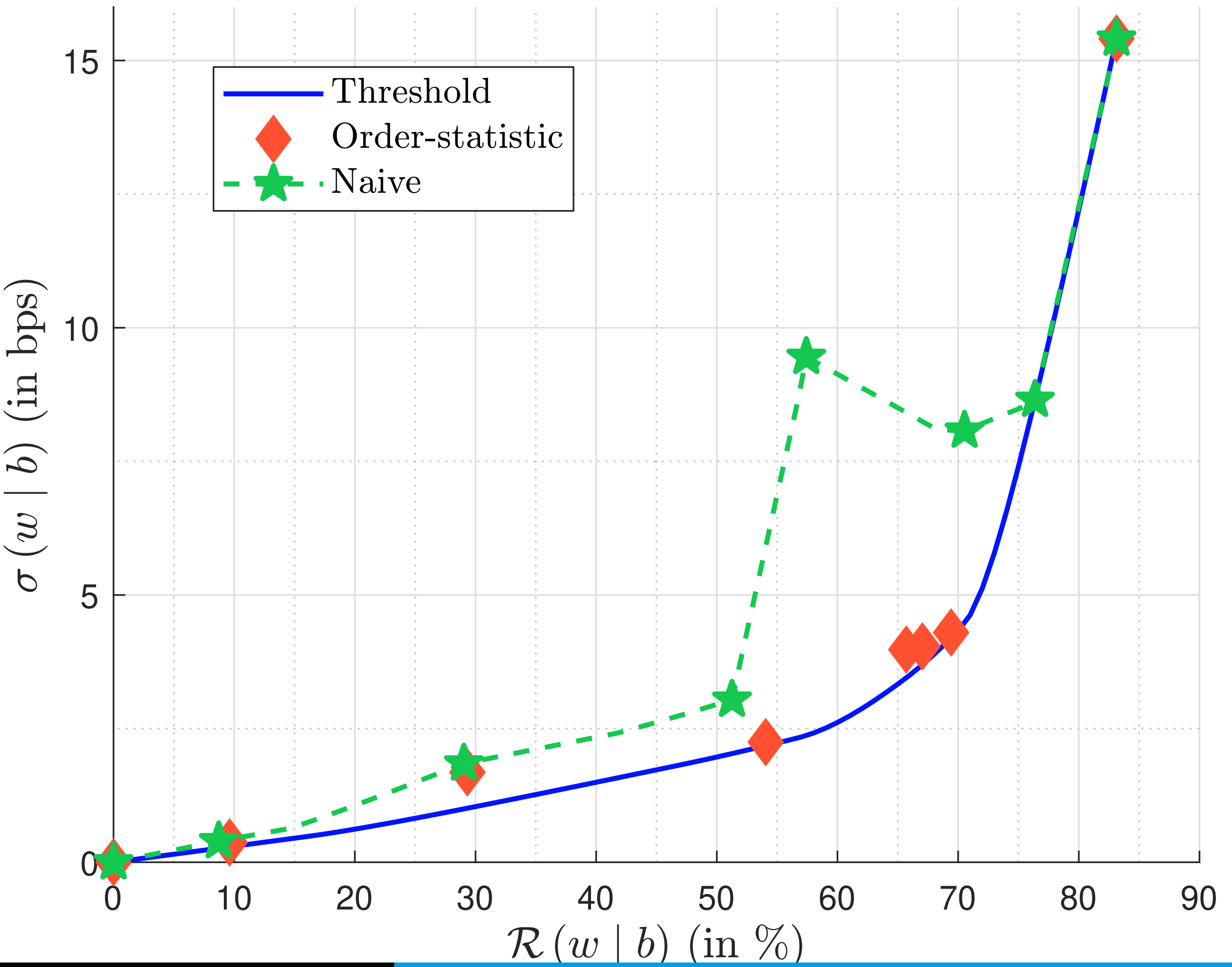
Equity portfolios

Figure 86: Efficient decarbonization frontier (Example #5)



Equity portfolios

Figure 87: Efficient decarbonization frontier of the interpolated naive approach (Example #5)



Bond portfolios

Example #6

We consider a debt-weighted bond index, which is composed of eight bonds. Their weights are equal to 20%, 19%, 17%, 13%, 12%, 8%, 6% and 5%. The carbon intensities (expressed in $\mathrm{tCO_2e/\$\ mn}$) are respectively equal to 100.5, 97.2, 250.4, 352.3, 27.1, 54.2, 78.6 and 426.7. To evaluate the risk of the portfolio, we use the modified duration which is respectively equal to 3.1, 6.6, 7.2, 5, 4.7, 2.1, 8.1 and 2.6 years, and the duration-times-spread factor, which is respectively equal to 100, 155, 575, 436, 159, 145, 804 and 365 bps. There are two sectors. Bonds #1, #3, #4 and #8 belong to $\mathcal{S}ector_1$ while Bonds #2, #5, #6 and #7 belong to $\mathcal{S}ector_2$

Bond portfolios

Table 78: Optimal decarbonization portfolios (Example #6, threshold approach)

$\mathcal{R}$	0	10	20	30	40	50	$\mathcal{CI}_i$
$w_1^\star$	20.00	21.62	23.93	26.72	30.08	33.44	100.5
$w_2^\star$	19.00	18.18	16.98	14.18	7.88	1.58	97.2
$w_3^\star$	17.00	18.92	21.94	22.65	16.82	11.00	250.4
$w_4^\star$	13.00	11.34	5.35	0.00	0.00	0.00	352.3
$w_5^\star$	12.00	13.72	16.14	21.63	33.89	46.14	27.1
$w_6^\star$	8.00	9.60	10.47	10.06	7.21	4.36	54.2
$w_7^\star$	6.00	5.56	5.19	4.75	4.11	3.48	78.6
$w_8^\star$	5.00	1.05	0.00	0.00	0.00	0.00	426.7
$\mathrm{AS}_{\mathcal{S}ector}$	0.00	6.87	15.49	24.07	31.97	47.58	
$\mathrm{MD}(w)$	5.48	5.49	5.45	5.29	4.90	4.51	
$\mathrm{DTS}(w)$	301.05	292.34	282.28	266.12	236.45	206.78	
$\sigma_{\mathrm{AS}}(w \mid b)$	0.00	5.57	12.31	19.82	30.04	43.58	
$\sigma_{\mathrm{MD}}(w \mid b)$	0.00	0.01	0.04	0.17	0.49	0.81	
$\sigma_{\mathrm{DTS}}(w \mid b)$	0.00	8.99	19.29	35.74	65.88	96.01	
$\mathcal{CI}(w)$	160.57	144.52	128.46	112.40	96.34	80.29	
$\mathcal{R}(w \mid b)$	0.00	10.00	20.00	30.00	40.00	50.00	

Bond portfolios

Table 79: Optimal decarbonization portfolios (Example #6, order-statistic approach)

m	0	1	2	3	4	5	6	7	$\mathcal{CI}_i$
$w_1^\star$	20.00	20.83	24.62	64.64	0.00	0.00	0.00	0.00	100.5
$w_2^\star$	19.00	18.60	18.13	21.32	3.32	0.00	0.00	0.00	97.2
$w_3^\star$	17.00	17.79	26.30	0.00	0.00	0.00	0.00	0.00	250.4
$w_4^\star$	13.00	14.53	0.00	0.00	0.00	0.00	0.00	0.00	352.3
$w_5^\star$	12.00	12.89	13.96	6.00	36.57	41.27	41.27	100.00	27.1
$w_6^\star$	8.00	9.74	11.85	0.00	60.11	58.73	58.73	0.00	54.2
$w_7^\star$	6.00	5.62	5.15	8.03	0.00	0.00	0.00	0.00	78.6
$w_8^\star$	5.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	426.7
$\mathrm{AS}_{\mathcal{S}ector}$	0.00	5.78	19.72	49.00	76.68	80.00	80.00	88.00	
$\mathrm{MD}(w)$	5.48	5.52	5.54	4.77	3.27	3.17	3.17	4.70	
$\mathrm{DTS}(w)$	301.05	295.08	284.71	171.82	150.45	150.78	150.78	159.00	
$\sigma_{\mathrm{AS}}(w \mid b)$	0.00	5.73	17.94	50.85	66.96	68.63	68.63	95.33	
$\sigma_{\mathrm{MD}}(w \mid b)$	0.00	0.03	0.04	0.63	2.66	2.64	2.64	3.21	
$\sigma_{\mathrm{DTS}}(w \mid b)$	0.00	6.21	16.87	128.04	197.22	197.29	197.29	199.22	
$\mathcal{CI}(w)$	160.57	147.94	122.46	93.63	45.72	43.02	43.02	27.10	
$\mathcal{R}(w \mid b)$	0.00	7.87	23.74	41.69	71.53	73.21	73.21	83.12	

Sector-specific constraints

Sector scenario

- Decarbonization scenario per sector:

$$\mathcal{CI}\left(w;\mathcal{S}ector_j\right)\leq\left(1-\mathcal{R}_j\right)\mathcal{CI}\left(b;\mathcal{S}ector_j\right)$$

- We have:

$$\left(\boldsymbol{s}_j\circ\left(\mathcal{CI}-\mathcal{CI}_j^{\star}\right)\right)^{\top}w\leq 0$$

where $\mathcal{CI}_j^{\star}=\left(1-\mathcal{R}_j\right)\mathcal{CI}\left(b;\mathcal{S}ector_j\right)$

Sector-specific constraints

Sector scenario

QP form

$$C = \begin{pmatrix} \left(\boldsymbol{s}_1 \circ \left(\mathcal{CI} - \mathcal{CI}_1^\star\right)\right)^\top \\ \vdots \\ \left(\boldsymbol{s}_j \circ \left(\mathcal{CI} - \mathcal{CI}_j^\star\right)\right)^\top \\ \vdots \\ \left(\boldsymbol{s}_{n_{\mathcal{S}ector}} \circ \left(\mathcal{CI} - \mathcal{CI}_{n_{\mathcal{S}ector}}^\star\right)\right)^\top \end{pmatrix}$$

$$D = \begin{pmatrix} \left(1 - \mathcal{R}_1\right) \mathcal{CI}\left(b; \mathcal{S}ector_1\right) \\ \vdots \\ \left(1 - \mathcal{R}_j\right) \mathcal{CI}\left(b; \mathcal{S}ector_j\right) \\ \vdots \\ \left(1 - \mathcal{R}_{n_{\mathcal{S}ector}}\right) \mathcal{CI}\left(b; \mathcal{S}ector_{n_{\mathcal{S}ector}}\right) \end{pmatrix}$$

Sector-specific constraints

Sector scenario

Table 80: Carbon intensity and threshold in $\mathrm{tCO_2e}/\$\ \mathrm{mn}$ per GICS sector (MSCI World, 2030)

Sector	$\mathcal{CI}(b; \mathcal{S}ector_j)$ $\mathcal{SC}_1$	$\mathcal{SC}_{1-2}$	$\mathcal{SC}_{1-3}^{\mathrm{up}}$	$\mathcal{SC}_{1-3}$	$\mathcal{R}_j$ (in %)	$\mathcal{CI}_j^{\star}$ $\mathcal{SC}_1$	$\mathcal{SC}_{1-2}$	$\mathcal{SC}_{1-3}^{\mathrm{up}}$	$\mathcal{SC}_{1-3}$
Communication Services	2	28	134	172	52.4	1	13	64	82
Consumer Discretionary	23	65	206	590	52.4	11	31	98	281
Consumer Staples	28	55	401	929	52.4	13	26	191	442
Energy	632	698	1 006	6 823	56.9	272	301	434	2 941
Financials	13	19	52	244	52.4	6	9	25	116
Health Care	10	22	120	146	52.4	5	10	57	70
Industrials	111	130	298	1 662	18.8	90	106	242	1 350
Information Technology	7	23	112	239	52.4	3	11	53	114
Materials	478	702	1 113	2 957	36.7	303	445	704	1 872
Real Estate	22	101	167	571	36.7	14	64	106	361
Utilities	1 744	1 794	2 053	2 840	56.9	752	773	885	1 224
MSCI World	130	163	310	992	36.6	82	103	196	629

Sector-specific constraints

Sector and weight deviation constraints (equity portfolio)

1. Asset weight deviation constraint:

$$\Omega := \mathcal{C}_1 \left(m_w^-, m_w^+\right) = \left\{w : m_w^- b \leq w \leq m_w^+ b\right\}$$

2. Sector weight deviation constraint:

$$\Omega := \mathcal{C}_2 \left(m_s^-, m_s^+\right) == \left\{\forall j : m_s^- \sum_{i \in \boldsymbol{\mathcal{S}}ector_j} b_i \leq \sum_{i \in \boldsymbol{\mathcal{S}}ector_j} w_i \leq m_s^+ \sum_{i \in \boldsymbol{\mathcal{S}}ector_j} b_i \right\}$$

3. $\mathcal{C}_2 \left(m_s\right) = \mathcal{C}_2 \left(1/m_s, m_s\right)$
4. $\mathcal{C}_3 \left(m_w^-, m_w^+, m_s\right) = \mathcal{C}_1 \left(m_w^-, m_w^+\right) \cap \mathcal{C}_2 \left(m_s\right)$

Sector-specific constraints

Sector and weight deviation constraints (bond portfolio)

1. Modified duration constraint:

$$\Omega := \mathcal{C}'_1 = \{w : \mathrm{MD}(w) = \mathrm{MD}(b)\} = \left\{w : \sum_{i=1}^{n} (x_i - b_i)\, \mathrm{MD}_i = 0\right\}$$

2. DTS constraint

$$\Omega := \mathcal{C}'_2 = \{w : \mathrm{DTS}(w) = \mathrm{DTS}(b)\} = \left\{w : \sum_{i=1}^{n} (x_i - b_i)\, \mathrm{DTS}_i = 0\right\}$$

3. Maturity/rating buckets:

$$\Omega := \left\{w : \sum_{i \in \mathcal{B}ucket_j} (x_i - b_i) = 0\right\}$$

   1. $\mathcal{C}'_3$: $\mathcal{B}ucket_j$ is the j^{th} maturity bucket, e.g., 0–1, 1–3, 3–5, 5–7, 7–10 and 10+
   2. $\mathcal{C}'_4$: $\mathcal{B}ucket_j$ is the j^{th} rating category, e.g., AAA–AA (AAA, AA+, AA and AA−), A (A+, A and A−) and BBB (BBB+, BBB, BBB−)

Sector-specific constraints

HCIS constraint

Two types of sectors:

1. High climate impact sectors (HCIS):

   "**sectors that are key to the low-carbon transition**" (TEG, 2019)

2. Low climate impact sectors (LCIS)

Let $\mathcal{HCIS}(w) = \sum_{i \in \mathrm{HCIS}} w_i$ be the HCIS weight of portfolio w:

$$\mathcal{HCIS}(w) \geq \mathcal{HCIS}(b)$$

Sector-specific constraints

HCIS constraint

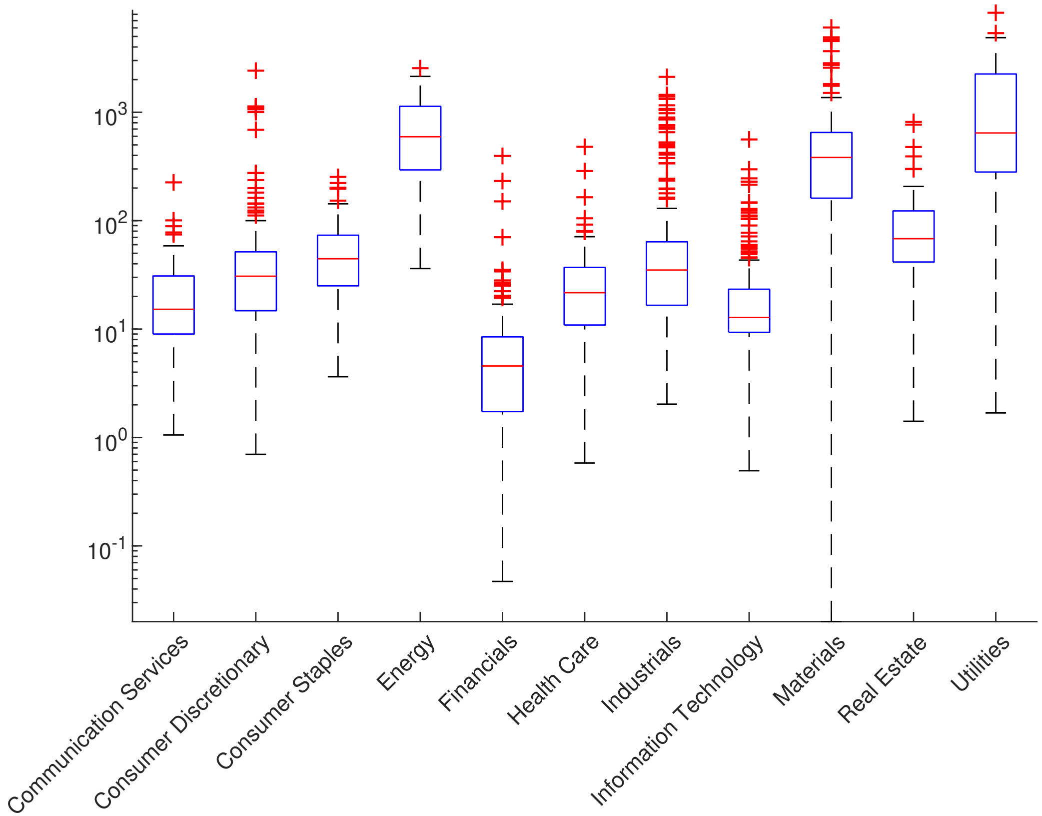
Table 81: Weight and carbon intensity when applying the HCIS filter (MSCI World, June 2022)

Sector	Index	HCIS	$\mathcal{SC}_1$		$\mathcal{SC}_{1-2}$		$\mathcal{SC}_{1-3}^{\mathrm{up}}$		$\mathcal{SC}_{1-3}$	
	b_j	b'_j	$\mathcal{CI}$	$\mathcal{CI}'$	$\mathcal{CI}$	$\mathcal{CI}'$	$\mathcal{CI}$	$\mathcal{CI}'$	$\mathcal{CI}$	$\mathcal{CI}'$
Communication Services	7.58	0.00	2		28		134		172	
Consumer Discretionary	10.56	8.01	23	14	65	31	206	189	590	462
Consumer Staples	7.80	7.80	28	28	55	55	401	401	929	929
Energy	4.99	4.99	632	632	698	698	1 006	1 006	6 823	6 823
Financials	13.56	0.00	13		19		52		244	
Health Care	14.15	9.98	10	13	22	26	120	141	146	177
Industrials	9.90	7.96	111	132	130	151	298	332	1 662	1 921
Information Technology	21.08	10.67	7	12	23	30	112	165	239	390
Materials	4.28	4.28	478	478	702	702	1 113	1 113	2 957	2 957
Real Estate	2.90	2.90	22	22	101	101	167	167	571	571
Utilities	3.21	3.21	1 744	1 744	1 794	1 794	2 053	2 053	2 840	2 840
MSCI World	100.00	59.79	130	210	163	252	310	458	992	1 498

Source: MSCI (2022), Trucost (2022) & Author's calculations

Empirical results (equity portfolios)

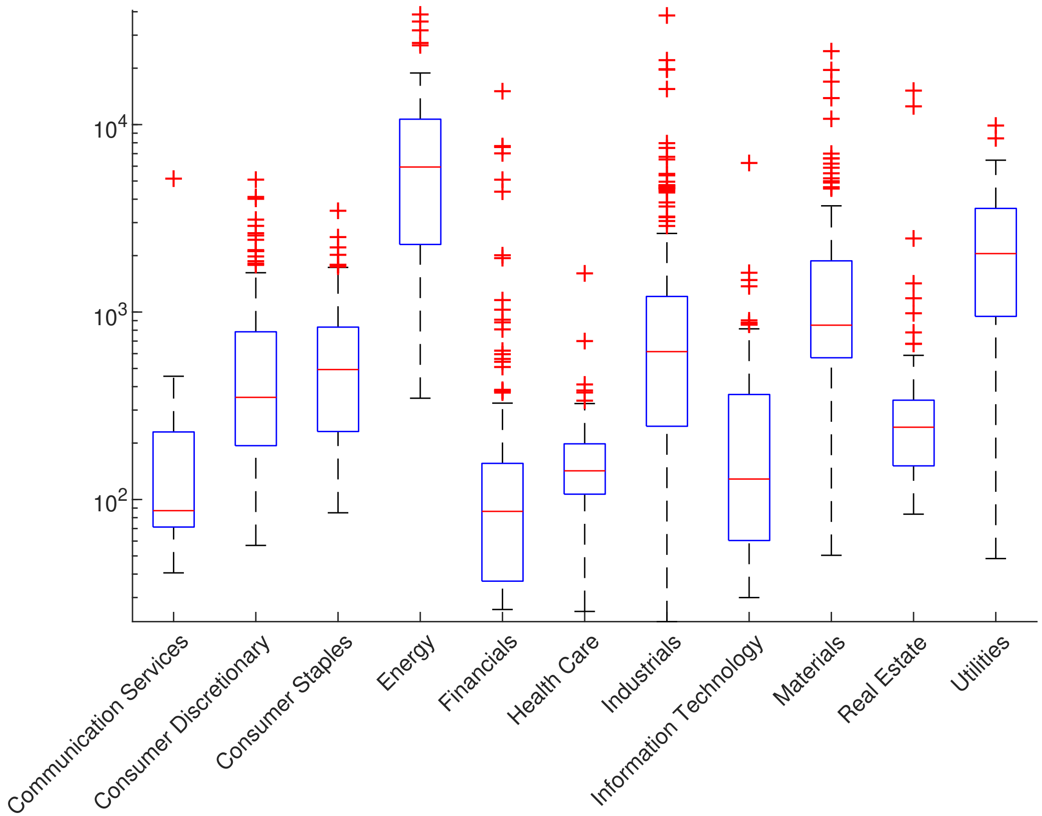
Figure 88: Boxplot of carbon intensity per sector (MSCI World, June 2022, scope $\mathcal{SC}_{1-2}$)



Source: MSCI (2022), Trucost (2022) & Barahhou *et al.* (2022)

Empirical results (equity portfolios)

Figure 89: Boxplot of carbon intensity per sector (MSCI World, June 2022, scope $\mathcal{SC}_{1-3}$)



Source: MSCI (2022), Trucost (2022) & Barahhou *et al.* (2022)

Equity portfolios

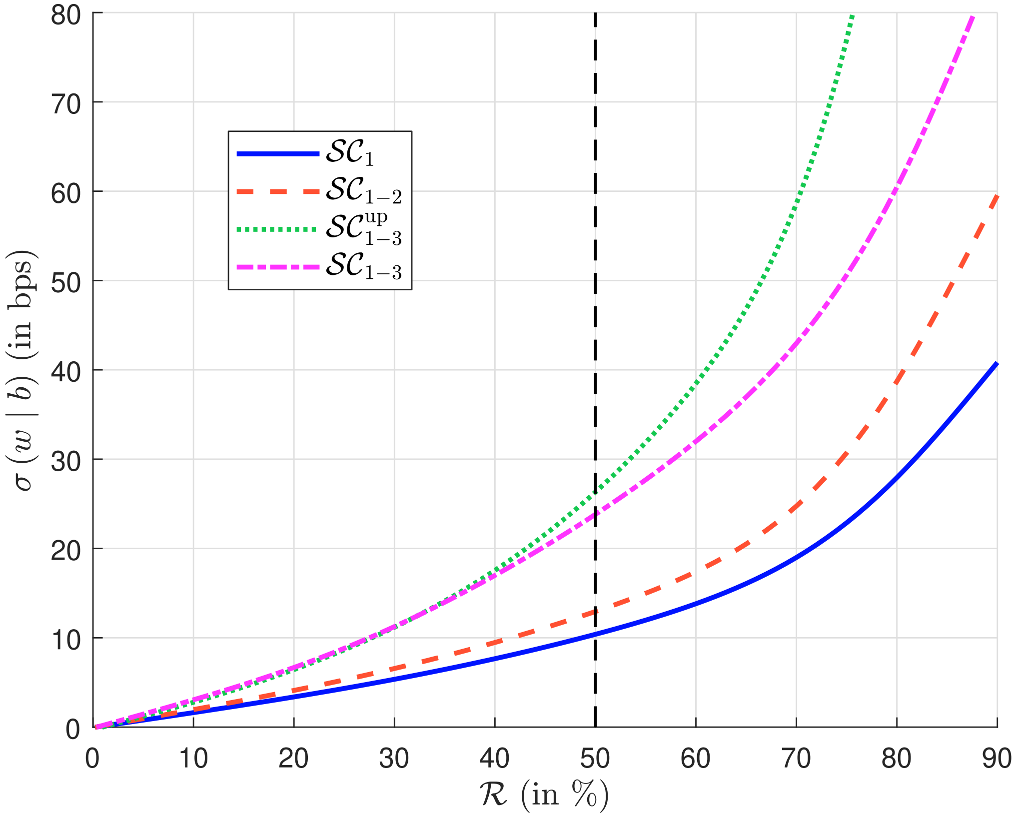
Barahhou *et al.* (2022) consider the basic optimization problem:

$$
\begin{aligned}
w^{\star} &= \arg\min \frac{1}{2}(w-b)^{\top}\Sigma(w-b) \\
&\text{s.t.} \quad \begin{cases} \mathcal{CI}(w) \leq (1-\mathcal{R})\,\mathcal{CI}(b) \\ w \in \Omega_0 \cap \Omega \end{cases}
\end{aligned}
$$

What is the impact of constraints $\Omega_0 \cap \Omega$?

Equity portfolios

Figure 90: Impact of the carbon scope on the tracking error volatility (MSCI World, June 2022, $\mathcal{C}_0$ constraint)



Source: MSCI (2022), Trucost (2022) & Barahhou *et al.* (2022)

Equity portfolios

Table 82: Sector allocation in % (MSCI World, June 2022, scope $\mathcal{SC}_{1-3}$)

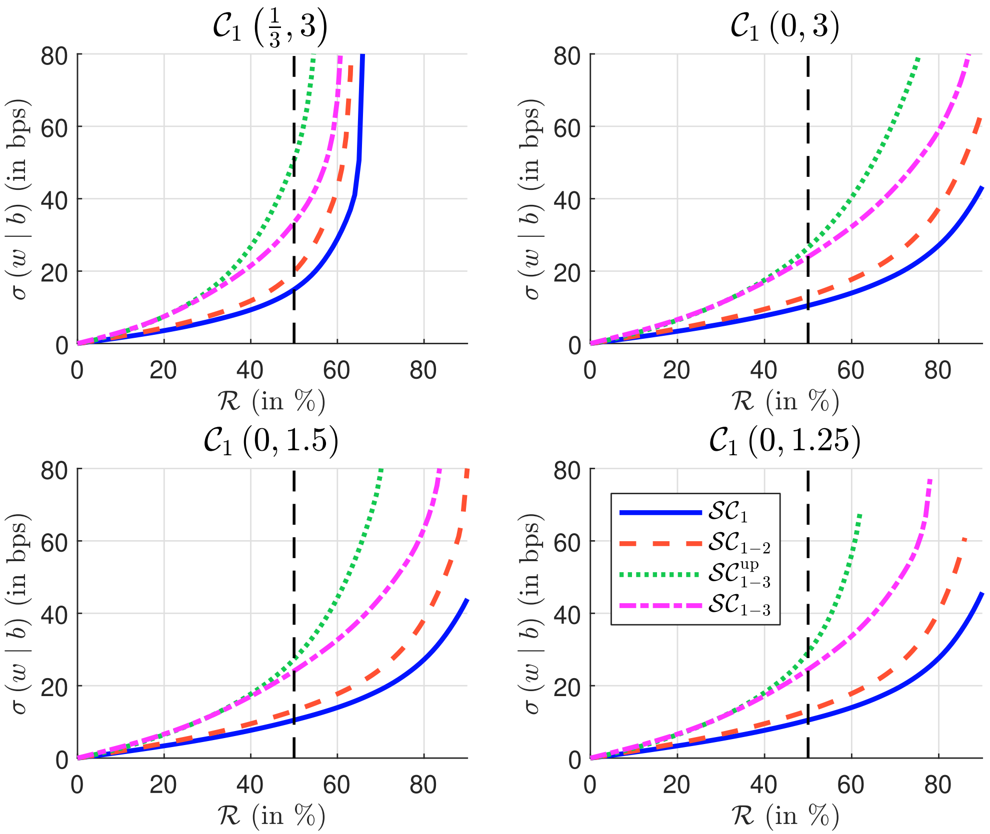
Sector	Index	Reduction rate $\mathcal{R}$ 30%	40%	50%	60%	70%	80%	90%
Communication Services	7.58	7.95	8.15	8.42	8.78	9.34	10.13	12.27
Consumer Discretionary	10.56	10.69	10.69	10.65	10.52	10.23	9.62	6.74
Consumer Staples	7.80	7.80	7.69	7.48	7.11	6.35	5.03	1.77
Energy	**4.99**	4.14	3.65	3.10	2.45	**1.50**	**0.49**	**0.00**
Financials	**13.56**	14.53	15.17	15.94	16.90	**18.39**	**20.55**	**28.62**
Health Care	14.15	14.74	15.09	15.50	16.00	16.78	17.77	17.69
Industrials	9.90	9.28	9.01	8.71	8.36	7.79	7.21	6.03
Information Technology	21.08	21.68	22.03	22.39	22.88	23.51	24.12	24.02
Materials	**4.28**	3.78	3.46	3.06	2.56	**1.85**	**1.14**	**0.24**
Real Estate	2.90	3.12	3.27	3.41	3.57	3.72	3.71	2.51
Utilities	**3.21**	2.28	1.79	1.36	0.90	**0.54**	**0.24**	**0.12**

Source: MSCI (2022), Trucost (2022) & Barahhou *et al.* (2022)

Portfolio decarbonization = strategy **long on Financials** and **short on Energy, Materials and Utilities**

Equity portfolios

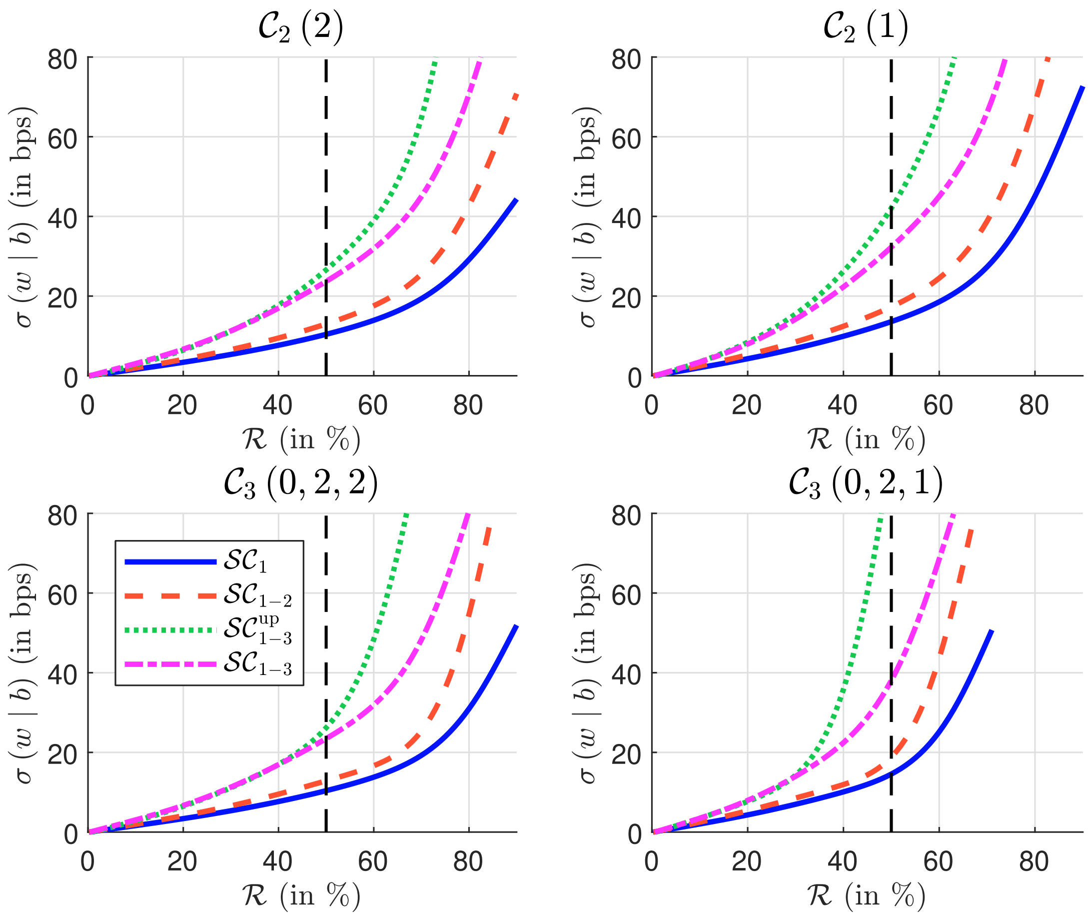
Figure 91: Impact of $\mathcal{C}_1$ constraint on the tracking error volatility (MSCI World, June 2022)



Source: MSCI (2022), Trucost (2022) & Barahhou *et al.* (2022)

Equity portfolios

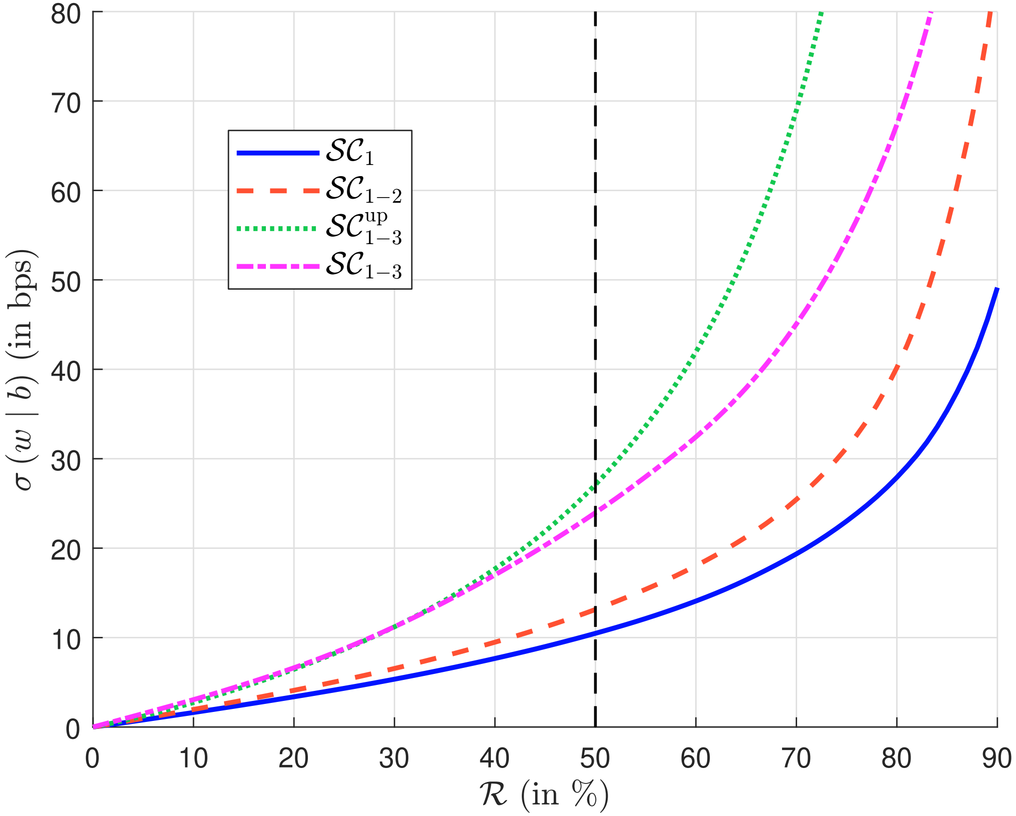
Figure 92: Impact of $\mathcal{C}_2$ and $\mathcal{C}_3$ constraints (MSCI World, June 2022)



Source: MSCI (2022), Trucost (2022) & Barahhou *et al.* (2022)

Equity portfolios

Figure 93: Tracking error volatility with $\mathcal{C}_3\,(0, 10, 2)$ constraint (MSCI World, June 2022)



Source: MSCI (2022), Trucost (2022) & Barahhou *et al.* (2022)

Equity portfolios

First approach

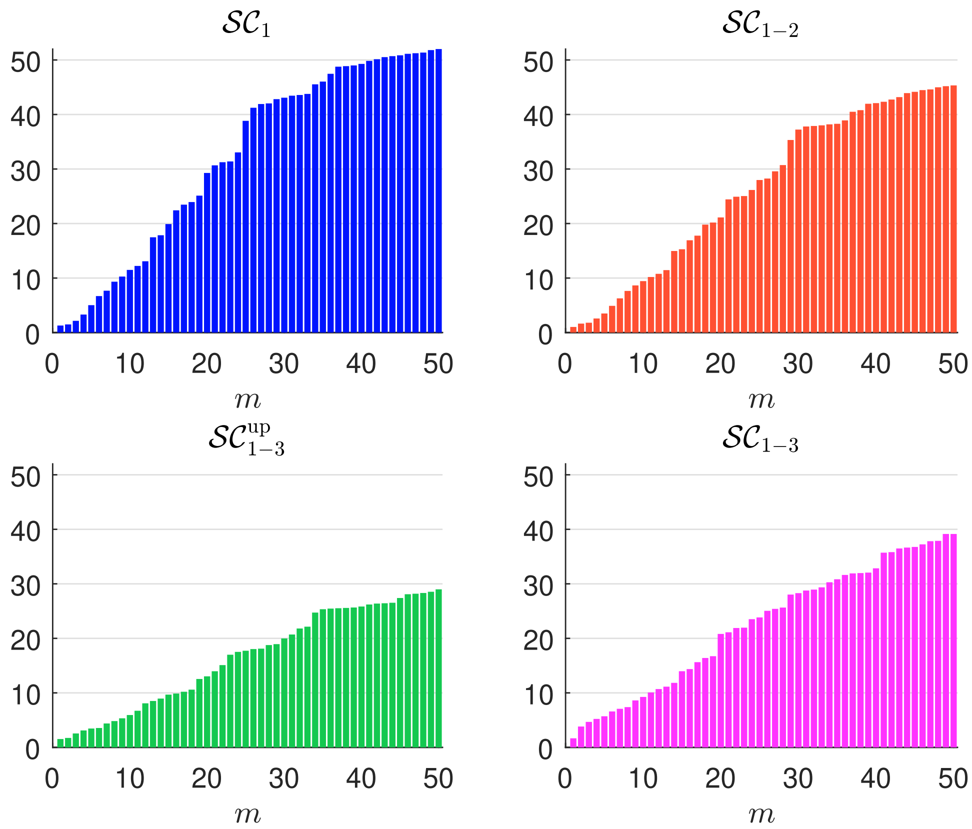
- The carbon footprint contribution of the m worst performing assets is:

$$\mathcal{CFC}^{(m,n)} = \frac{\sum_{i=1}^{n} \mathbb{1}\left\{\boldsymbol{\mathcal{CI}}_i \geq \boldsymbol{\mathcal{CI}}^{(m,n)}\right\} \cdot b_i \boldsymbol{\mathcal{CI}}_i}{CI\left(b\right)}$$

where $\boldsymbol{\mathcal{CI}}^{(m,n)} = \boldsymbol{\mathcal{CI}}_{n-m+1:n}$ is the $(n-m+1)$-th order statistic

Equity portfolios

Figure 94: Carbon footprint contribution $\mathcal{CFC}^{(m,n)}$ in % (MSCI World, June 2022, first approach)



Source: MSCI (2022), Trucost (2022) & Author's calculations

Equity portfolios

Second approach

- Another definition:

$$\mathcal{CFC}^{(m,n)} = \frac{\sum_{i=1}^{n} \mathbb{1}\left\{\mathcal{CIC}_i \geq \mathcal{CIC}^{(m,n)}\right\} \cdot b_i \boldsymbol{\mathcal{CI}}_i}{CI\left(b\right)}$$

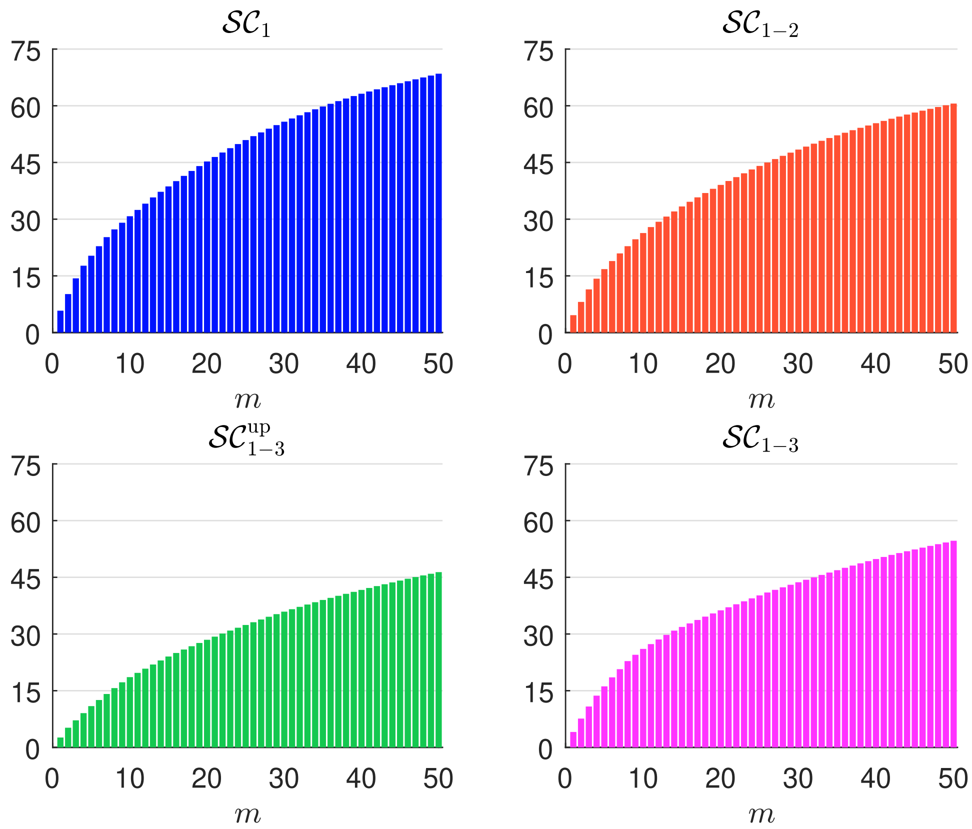
  where $\mathcal{CIC}_i = b_i \boldsymbol{\mathcal{CI}}_i$ and $\mathcal{CIC}^{(m,n)} = \mathcal{CIC}_{n-m+1:n}$

- Weight contribution:

$$\mathcal{WC}^{(m,n)} = \sum_{i=1}^{n} \mathbb{1}\left\{\mathcal{CIC}_i \geq \mathcal{CIC}^{(m,n)}\right\} \cdot b_i$$

Equity portfolios

Figure 95: Carbon footprint contribution $\mathcal{CFC}^{(m,n)}$ in % (MSCI World, June 2022, second approach)



Source: MSCI (2022), Trucost (2022) & Author's calculations

Equity portfolios

Table 83: Carbon footprint contribution $\mathcal{CFC}^{(m,n)}$ in % (MSCI World, June 2022, second approach, $\boldsymbol{\mathcal{SC}}_{1-3}$)

Sector	m = 1	5	10	25	50	75	100	200
Communication Services						0.44	0.44	0.73
Consumer Discretionary				0.78	1.37	2.44	2.93	4.28
Consumer Staples		2.46	2.46	2.46	3.75	4.44	4.92	5.62
Energy		9.61	17.35	23.78	29.56	31.78	33.02	33.89
Financials						0.72	1.53	1.88
Health Care							0.21	0.37
Industrials			2.16	5.59	7.13	8.70	9.48	13.05
Information Technology				0.98	1.58	1.94	2.15	3.30
Materials	4.08	4.08	4.08	5.81	7.31	8.81	9.59	10.75
Real Estate					0.77	0.77	0.77	0.85
Utilities				0.81	3.20	3.89	5.24	7.98
Total	4.08	16.15	26.06	40.21	54.66	63.94	70.29	82.70

Source: MSCI (2022), Trucost (2022) & Author's calculations

Equity portfolios

Table 84: Weight contribution $\mathcal{WC}^{(m,n)}$ in % (MSCI World, June 2022, second approach, $\boldsymbol{\mathcal{SC}}_{1-3}$)

Sector	b_j (in %)	m = 1	5	10	25	50	75	100	200
Communication Services	7.58						0.08	0.08	3.03
Consumer Discretionary	10.56				0.58	1.79	2.44	4.51	5.89
Consumer Staples	7.80		0.70	0.70	0.70	1.90	2.50	2.84	3.84
Energy	4.99		1.71	2.25	2.96	3.62	3.99	4.33	4.65
Financials	13.56						0.74	1.17	2.33
Health Care	14.15							0.95	1.34
Industrials	9.90			0.06	0.32	0.70	0.96	1.20	4.12
Information Technology	21.08				0.16	4.70	8.42	8.78	11.62
Materials	4.28	0.29	0.29	0.29	0.47	0.88	1.10	1.40	1.87
Real Estate	2.90					0.05	0.05	0.05	0.23
Utilities	3.21				0.31	0.86	1.04	1.31	2.33
Total		0.29	2.71	3.30	5.49	14.50	21.32	26.63	41.24

Source: MSCI (2022), Trucost (2022) & Author's calculations

Equity portfolios

- The order-statistic optimization problem is:

$$
\begin{aligned}
w^{\star} & = \arg\min \frac{1}{2} (w - b)^{\top} \Sigma (w - b) \\
& \text{s.t.} \left\{ \begin{array}{l} \mathbf{1}_n^{\top} w = 1 \\ \mathbf{0}_n \leq w \leq w^{(m,n)} \end{array} \right.
\end{aligned}
$$

where the upper bound $w^{(m,n)}$ is equal to $\mathbb{1}\left\{\boldsymbol{\mathcal{CI}} < \boldsymbol{\mathcal{CI}}^{(m,n)}\right\}$ for the first ordering approach and $\mathbb{1}\left\{\mathcal{CIC} < \mathcal{CIC}^{(m,n)}\right\}$ for the second ordering approach

Equity portfolios

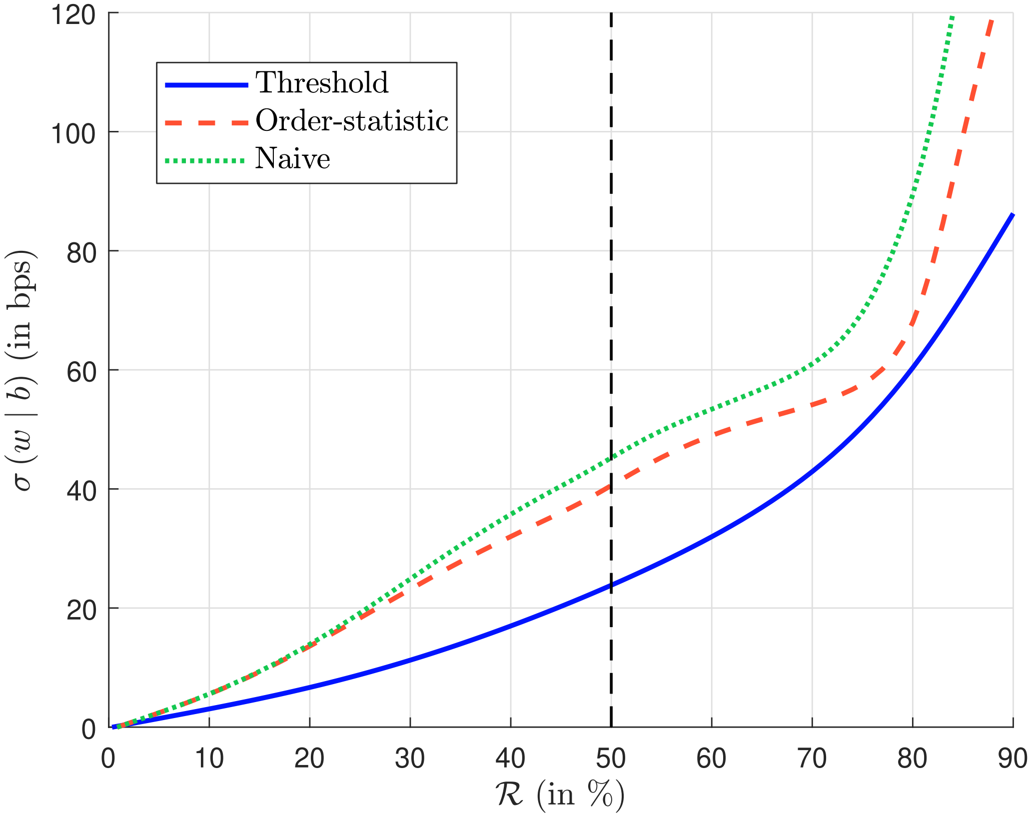
- The naive method is:

$$w_i^\star = \frac{e_i b_i}{\sum_{k=1}^{n} e_k b_k}$$

where e_i is defined as $\mathbb{1}\left\{\boldsymbol{\mathcal{CI}}_i < \boldsymbol{\mathcal{CI}}^{(m,n)}\right\}$ for the first ordering approach and $\mathbb{1}\left\{\mathcal{CIC}_i < \mathcal{CIC}^{(m,n)}\right\}$ for the second ordering approach

Equity portfolios

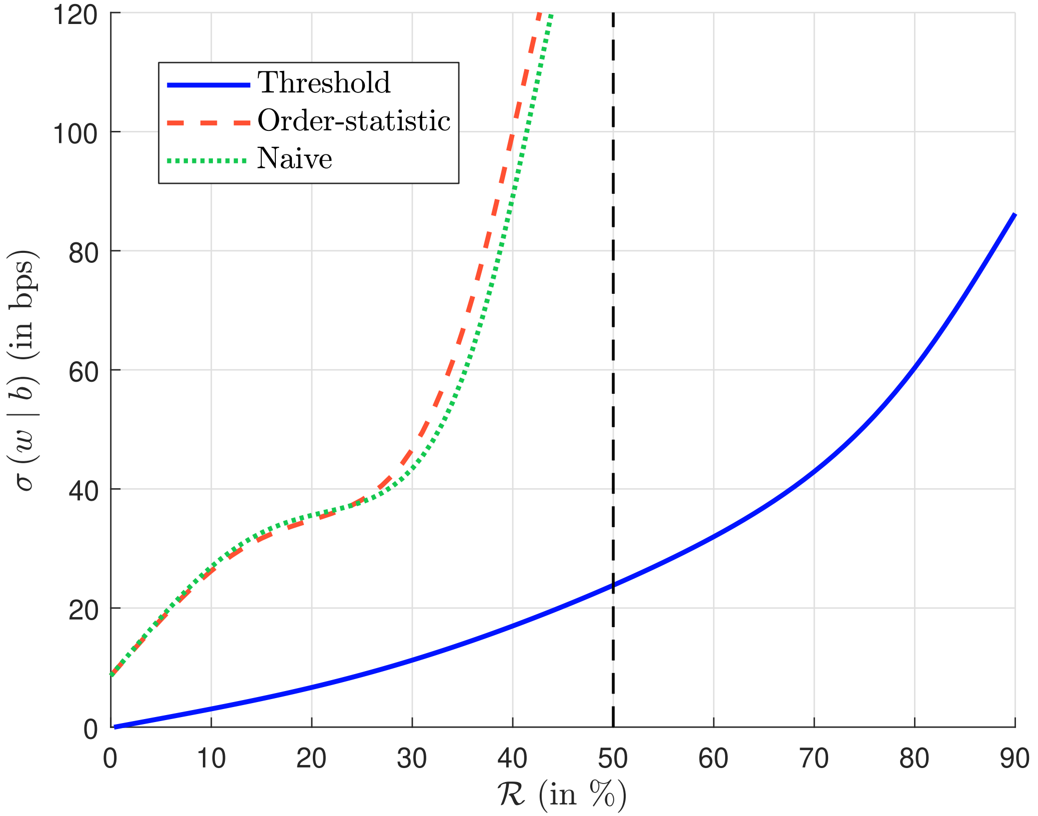
Figure 96: Tracking error volatility (MSCI World, June 2022, $\mathcal{SC}_{1-3}$, first ordering method)



Source: MSCI (2022), Trucost (2022) & Author's calculations

Equity portfolios

Figure 97: Tracking error volatility (MSCI World, June 2022, $\mathcal{SC}_{1-3}$, second ordering method)



Source: MSCI (2022), Trucost (2022) & Author's calculations

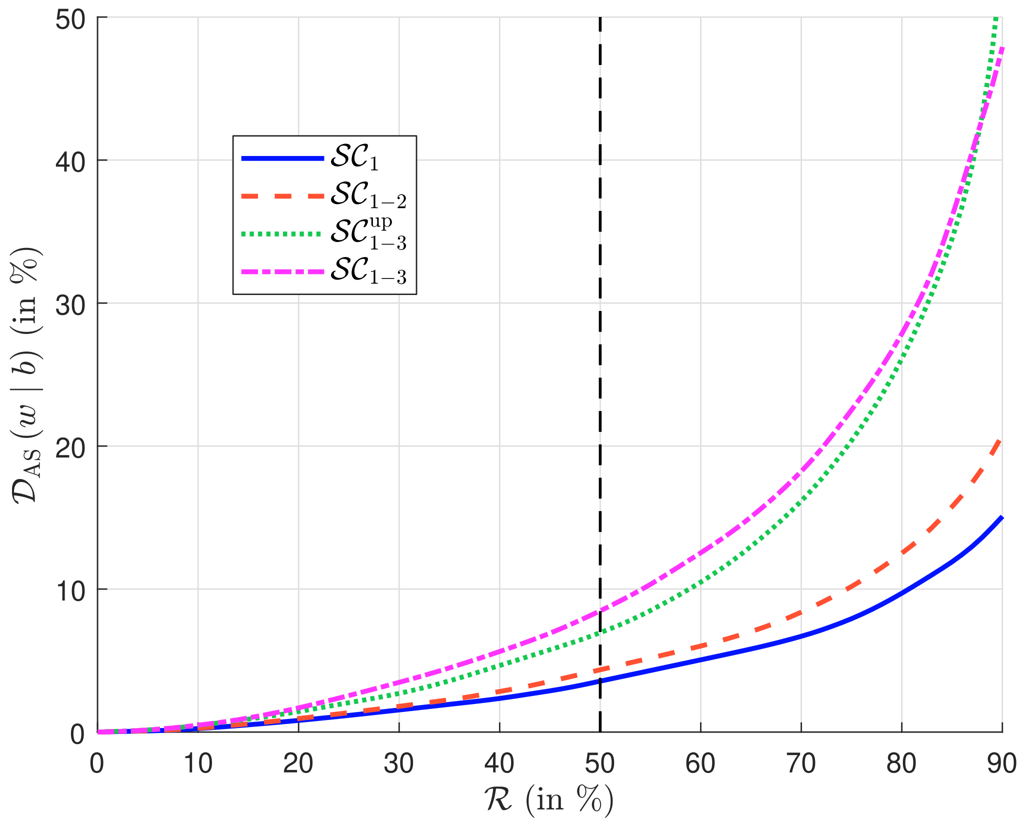
Bond portfolios

The optimization problem is:

$$\begin{aligned} w^{\star} &= \arg\min \frac{1}{2}\sum_{i=1}^{n}\left|w_i - b_i\right| + 50\sum_{j=1}^{n_{\mathcal{S}ector}}\left|\sum_{i\in\mathcal{S}ector_j}\left(w_i - b_i\right)\mathrm{DTS}_i\right| \\ & \text{s.t. } \left\{\begin{array}{l} \mathcal{CI}\left(w\right) \leq \left(1-\mathcal{R}\right)\mathcal{CI}\left(b\right) \\ w \in \mathcal{C}_0 \cap \mathcal{C}'_1 \cap \mathcal{C}'_3 \cap \mathcal{C}'_4 \end{array}\right. \end{aligned}$$

Bond portfolios

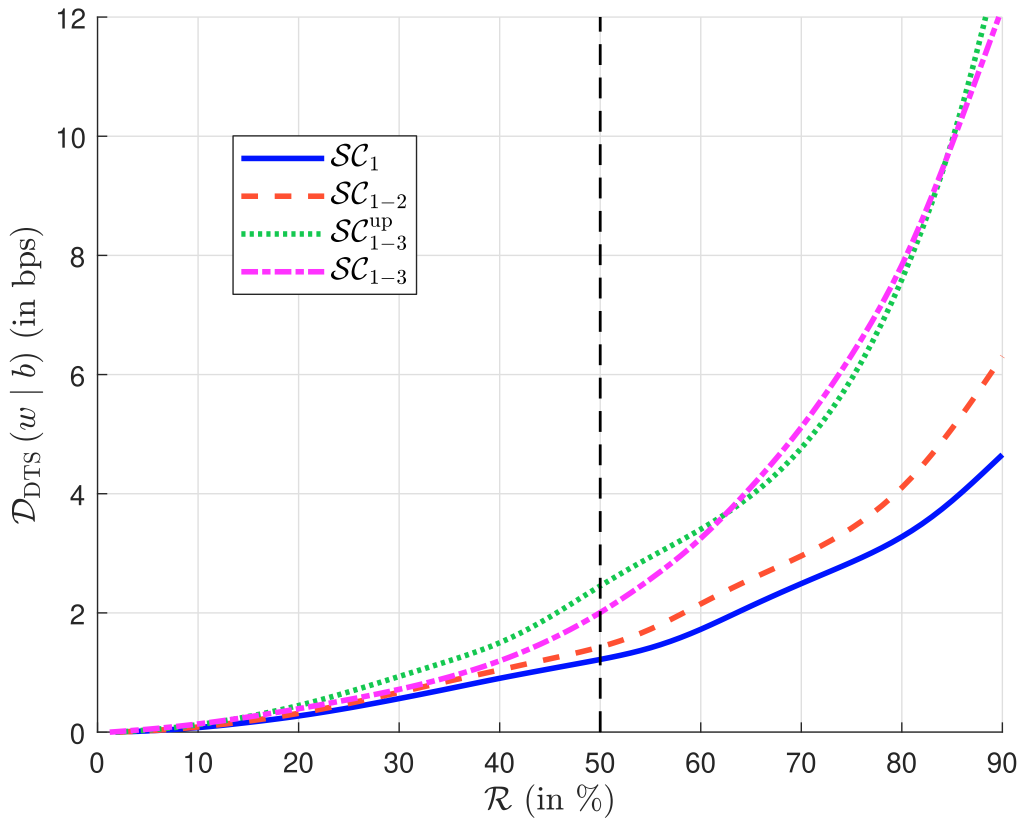
Figure 98: Impact of the carbon scope on the active share in % (ICE Global Corp., June 2022)



Source: ICE (2022), Trucost (2022) & Barahhou *et al.* (2022)

Bond portfolios

Figure 99: Impact of the carbon scope on the DTS risk in bps (ICE Global Corp., June 2022)



Source: ICE (2022), Trucost (2022) & Barahhou *et al.* (2022)

Bond portfolios

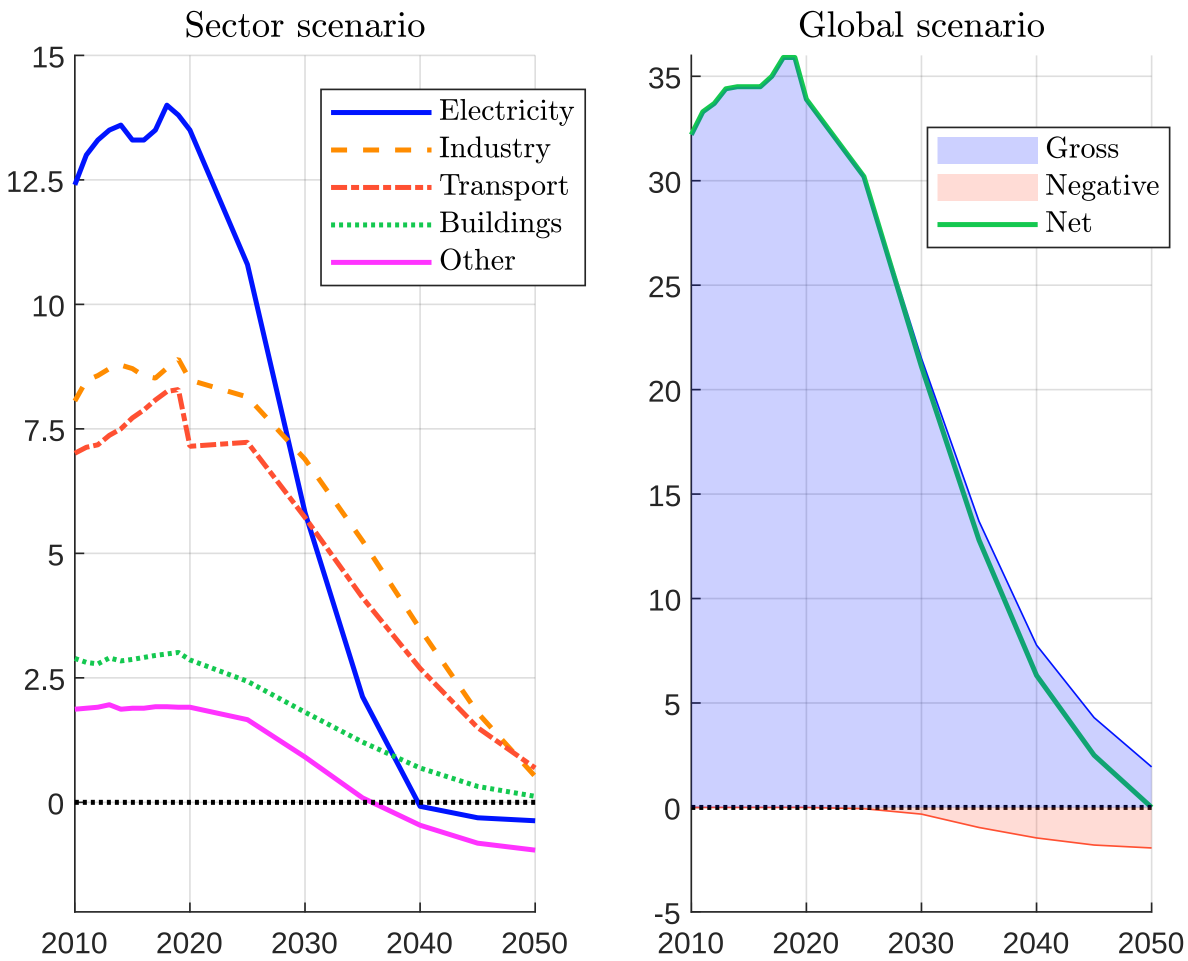
Table 85: Sector allocation in % (ICE Global Corp., June 2022, scope $\mathcal{SC}_{1-3}$)

Sector	Index	Reduction rate $\mathcal{R}$						
		30%	40%	50%	60%	70%	80%	90%
Communication Services	7.34	7.35	7.34	7.37	7.43	7.43	7.31	7.30
Consumer Discretionary	5.97	5.97	5.96	5.94	5.93	5.46	4.48	3.55
Consumer Staples	6.04	6.04	6.04	6.04	6.04	6.02	5.39	4.06
Energy	6.49	5.49	4.42	3.84	3.69	3.23	2.58	2.52
Financials	33.91	34.64	35.66	35.96	36.09	37.36	38.86	39.00
Health Care	7.50	7.50	7.50	7.50	7.50	7.50	7.52	7.48
Industrials	8.92	9.38	9.62	10.19	11.34	12.07	13.55	18.13
Information Technology	5.57	5.57	5.59	5.59	5.60	5.60	5.52	5.27
Materials	3.44	3.43	3.31	3.18	3.12	2.64	2.25	1.86
Real Estate	4.76	4.74	4.74	4.74	4.74	4.66	4.61	3.93
Utilities	10.06	9.89	9.82	9.64	8.52	8.04	7.92	6.88

Source: ICE (2022), Trucost (2022) & Barahhou *et al.* (2022)

Choice of the decarbonization scenario

Figure 100: CO_2 emissions by sector in the IEA NZE scenario (in $\mathrm{GtCO_2e}$)



Source: International Energy Agency (2021).

The carbon emissions/intensity approach

A decarbonization scenario is defined as a function that relates a decarbonization rate to a time index t:

$$\begin{aligned} f \quad : \quad \mathbb{R}^{+} &\longrightarrow [0,1] \\ t &\longmapsto \mathcal{R}(t_0, t) \end{aligned}$$

where t_0 is the base year and $\mathcal{R}\left(t_0, t_0^-\right) = 0$

Two choices

1. Carbon emissions

$$\mathcal{CE}(t) = (1 - \mathcal{R}(t_0, t))\,\mathcal{CE}(t_0)$$

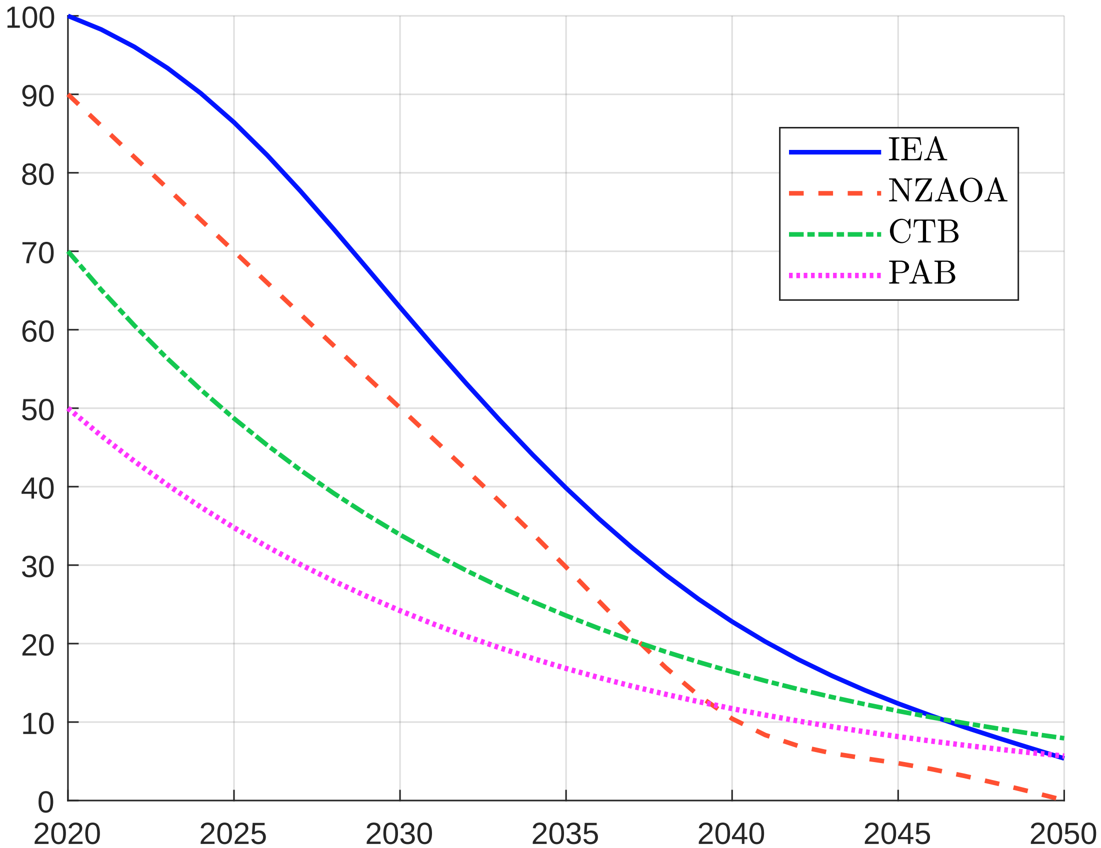
2. Carbon intensity (CTB/PAB)

$$\mathcal{CI}(t) = (1 - \Delta\mathcal{R})^{t - t_0}\left(1 - \mathcal{R}^-\right)\mathcal{CI}(t_0)$$

where $\Delta\mathcal{R} = 7\%$ and $\mathcal{R}^-$ takes the values $30\%/50\%$ (CTB/PAB)

The carbon emissions/intensity approach

Figure 101: IEA, NZAOA, CTB and PAB decarbonization pathways



IEA = International Energy Agency, NZAOA = Net Zero Asset Owners Alliance, CTB = Climate Transition Benchmark, PAB = Paris Aligned Benchmark

The carbon emissions/intensity approach

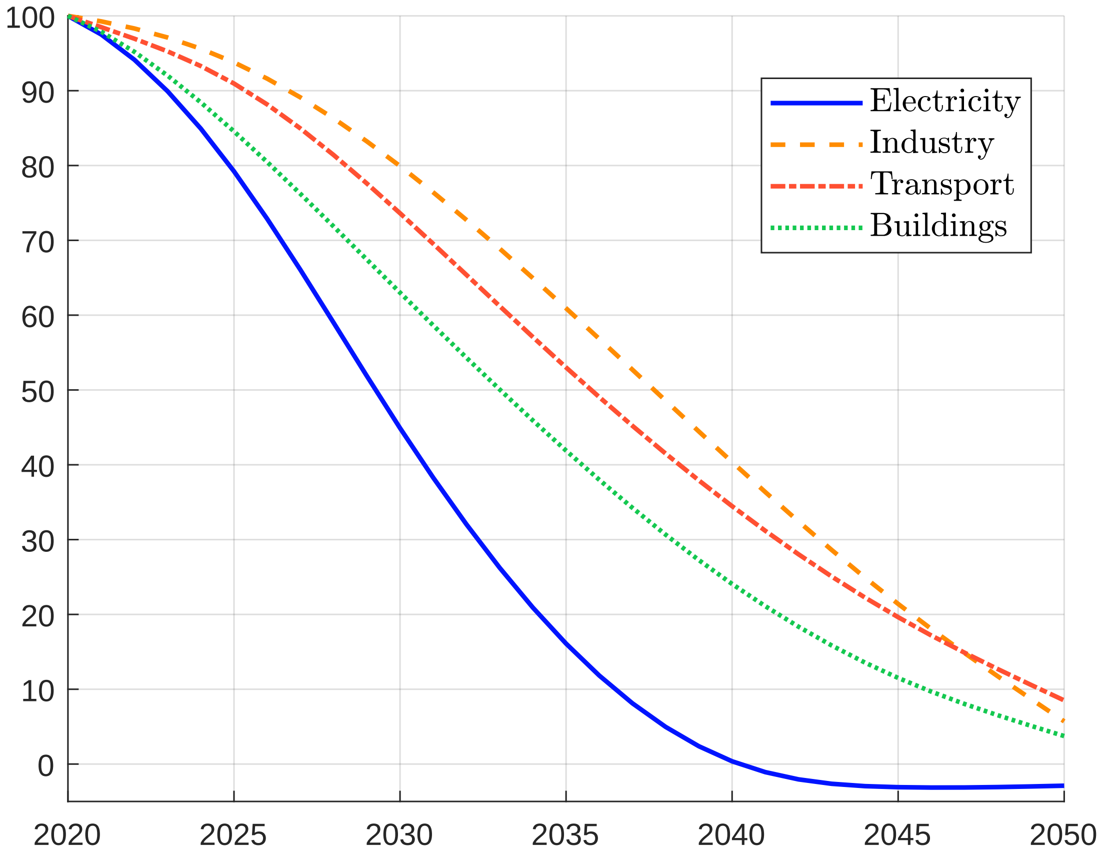
Table 86: IEA, NZAOA, CTB and PAB decarbonization rates (baseline = 2020)

Year $\mathcal{R}^-$ $\Delta\mathcal{R}$	CTB 30% 7%	PAB 50% 7%	NZE IEA Scenario	NZAOA Average Scenario
2020	30.0%	50.0%	0.0%	10.0%
2021	34.9%	53.5%	1.7%	14.0%
2022	39.5%	56.8%	3.9%	18.0%
2023	43.7%	59.8%	6.7%	22.0%
2024	47.6%	62.6%	9.9%	26.0%
2025	51.3%	65.2%	13.6%	30.0%
2026	54.7%	67.7%	17.8%	34.0%
2027	57.9%	69.9%	22.3%	38.0%
2028	60.8%	72.0%	27.2%	42.0%
2029	63.6%	74.0%	32.1%	46.0%
2030	66.1%	75.8%	37.1%	50.0%
2035	76.4%	83.2%	60.2%	70.3%
2040	83.6%	88.3%	77.2%	89.6%
2045	88.6%	91.9%	87.6%	95.2%
2050	92.1%	94.3%	94.6%	100.0%

Source: Ben Slimane *et al.* (2023).

The carbon emissions/intensity approach

Figure 102: Sectoral decarbonization pathways



Electricity $\succ$ Buildings $\succ$ Transport $\succ$ Industry

The carbon budget approach

A NZE scenario is defined by the following constraints:

$$\begin{cases} \mathcal{CB}\left(t_0, 2050\right) \leq \mathcal{CB}^+ \text{ GtCO}_2\text{e} \\ \mathcal{CE}\left(2050\right) \approx 0 \text{ GtCO}_2\text{e} \end{cases}$$

where t_0 is the base date and $\mathcal{CB}^+$ is the maximum carbon budget

IPCC SR15

- $t_0 = 2019$ and $\mathcal{CB}^+ = 580\,\text{GtCO}_2\text{e}$: there is a 50% probability of limiting the global warning to $1.5°\text{C}$
- $t_0 = 2019$ and $\mathcal{CB}^+ = 420\,\text{GtCO}_2\text{e}$: the probability is 66%
- $t_0 = 2019$ and $\mathcal{CB}^+ = 300\,\text{GtCO}_2\text{e}$: the probability is 83%

The carbon budget approach

If we have:

$$\mathcal{CE}(t) = (1 - \Delta\mathcal{R})^{t-t_0}\left(1 - \mathcal{R}^-\right)\mathcal{CE}(t_0)$$

we obtain:

$$\mathcal{CB}(t_0, t) = \left(\frac{(1 - \Delta\mathcal{R})^{t-t_0} - 1}{\ln(1 - \Delta\mathcal{R})}\right)\left(1 - \mathcal{R}^-\right)\mathcal{CE}(t_0)$$

Table 87: Carbon budget $\mathcal{CB}(2020, 2050)$ (in $\mathrm{GtCO_2e}$) when defining the decarbonization pathway of carbon emissions and assuming that $\mathcal{CE}(2020) = 36\,\mathrm{GtCO_2e}$

$\mathcal{R}^-$		0%	10%	20%	30%	50%	75%
	5%	551	496	441	386	276	138
	6%	491	442	393	344	245	123
$\Delta\mathcal{R}$	7%	440	396	352	308	220	110
	8%	396	357	317	277	198	99
	9%	359	323	287	251	180	90
	10%	327	294	262	229	164	82

Dynamic decarbonization and portfolio alignment

We have:

$$\mathcal{CI}(t, w) \leq (1 - \mathcal{R}(t_0, t))\,\mathcal{CI}(t_0, b(t_0))$$

where:

- t_0 is the base year
- $\mathcal{R}(t_0, t)$ is the decarbonization pathway of the NZE scenario
- $\mathcal{CI}(t_0, b(t_0))$ is the carbon intensity of the benchmark at time t_0

Dynamic decarbonization and portfolio alignment

Some properties:

- Decarbonizing the aligned portfolio becomes easier over time as the benchmark decarbonizes itself:

$$\mathcal{CI}\left(t, b\left(t\right)\right) \ll \mathcal{CI}\left(t_0, b\left(t_0\right)\right) \qquad \text{for } t > t_0$$

- Decarbonizing the aligned portfolio becomes more difficult over time as the benchmark carbonizes itself:

$$\mathcal{CI}\left(t, b\left(t\right)\right) \gg \mathcal{CI}\left(t_0, b\left(t_0\right)\right) \qquad \text{for } t > t_0$$

- The aligned portfolio matches the benchmark portfolio if the benchmark is sufficiently decarbonized:

$$\mathcal{CI}\left(t, b\left(t\right)\right) \leq \left(1 - \mathcal{R}\left(t_0, t\right)\right) \mathcal{CI}\left(t_0, b\left(t_0\right)\right)$$

Equity portfolios

The optimization problem becomes:

$$
\begin{aligned}
w^{\star}(t) &= \arg\min \frac{1}{2}(w - b(t))^{\top} \Sigma(t)(w - b(t)) \\
&\text{s.t.} \quad \left\{ \begin{array}{l} \boldsymbol{\mathcal{CI}}(t, w) \leq (1 - \boldsymbol{\mathcal{R}}(t_0, t)) \boldsymbol{\mathcal{CI}}(t_0, b(t_0)) \\ w \in \Omega_0 \cap \Omega \end{array} \right.
\end{aligned}
$$

where:

- $\Omega_0 = \mathcal{C}_0 = \left\{ w : \mathbf{1}_n^{\top} w = 1, \mathbf{0}_n \leq w \leq \mathbf{1}_n \right\}$ defines the long-only constraint
- Ω is the set of additional constraints

Equity portfolios

Example #7

We consider Example #5. We want to align the portfolio with respect to the CTB scenario. To compute the optimal portfolio $w^{\star}(t)$ where $t = t_0 + h$ and $h = 0, 1, 2, \ldots$ years, we assume that the benchmark $b(t)$, the covariance matrix $\Sigma(t)$, and the vector $\mathcal{CI}(t)$ of carbon intensities do not change over time.

Equity portfolios

1. First, we compute the mapping function between the time t and the decarbonization rate $\mathcal{R}(t_0, t)$:

$$\mathcal{R}(t_0, t) = 1 - (1 - 30\%) \times (1 - 7\%)^h$$

   We get $\mathcal{R}(t_0, t_0) = 30\%$, $\mathcal{R}(t_0, t_0 + 1) = 34.90\%$, $\mathcal{R}(t_0, t_0 + 2) = 39.46\%$, and so on

2. Second, we solve the optimization problem for the different values of time t

Equity portfolios

Table 88: Equity portfolio alignment (Example #7)

t	$b(t_0)$	t_0	t_0+1	t_0+2	t_0+3	t_0+4	t_0+5	t_0+10
$w_1^\star$	20.00	21.86	22.21	22.54	22.84	23.02	22.92	8.81
$w_2^\star$	19.00	18.70	18.41	18.15	17.90	17.58	17.04	0.00
$w_3^\star$	17.00	8.06	5.69	3.48	1.43	0.00	0.00	0.00
$w_4^\star$	13.00	8.74	7.66	6.65	5.72	4.56	2.70	0.00
$w_5^\star$	12.00	13.07	13.29	13.51	13.70	13.91	14.18	21.22
$w_6^\star$	8.00	22.57	25.59	28.39	31.00	33.39	35.54	62.31
$w_7^\star$	6.00	7.00	7.15	7.29	7.42	7.53	7.63	7.66
$w_8^\star$	5.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
$\sigma(w^\star \mid b(t))$	0.01	104.10	126.22	147.14	166.79	185.24	203.51	352.42
$\mathcal{CI}(t,w)$	160.57	112.40	104.53	97.22	90.41	84.08	78.20	54.40
$\mathcal{R}(w \mid b(t_0))$	0.00	30.00	34.90	39.46	43.70	47.64	51.30	66.12

The reduction rate and weights are expressed in %, while the tracking error volatility is measured in bps.

Bond portfolios

For bonds, the tracking error volatility is replaced by the active risk function:

$$\mathcal{D}(w \mid b) = \underbrace{\varphi \sum_{s=1}^{n_{\mathcal{S}ector}} \left| \sum_{i \in s} (w_i - b_i) \, \mathrm{DTS}_i \right|}_{\text{DTS component}} + \underbrace{\frac{1}{2} \sum_{i \in b} |w_i - b_i|}_{\text{AS component}} + \underbrace{\mathbb{1}_{\Omega_{\mathrm{MD}}}(w)}_{\text{MD component}}$$

where:

- DTS_i and MD_i are the duration-times-spread and modified duration factors
- $\Omega_{\mathrm{MD}} = \left\{ w : \sum_{i=1}^{n} (w_i - b_i) \, \mathrm{MD}_i = 0 \right\}$
- $\mathbb{1}_{\Omega}(w)$ is the convex indicator function

Bond portfolios

The optimization problem becomes then:

$$
\begin{array}{rcl}
w^{\star}(t) & = & \arg\min \mathcal{D}\left(w \mid b(t)\right) \\
& \text{s.t.} & \left\{ \begin{array}{l} \boldsymbol{\mathcal{CI}}\left(t, w\right) \leq \left(1 - \boldsymbol{\mathcal{R}}\left(t_0, t\right)\right) \boldsymbol{\mathcal{CI}}\left(t_0, b\left(t_0\right)\right) \\ w \in \Omega_0 \cap \Omega \end{array} \right.
\end{array}
$$

Bond portfolios

Example #8

We consider Example #6. We want to align the portfolio with respect to the CTB scenario. To compute the optimal portfolio $w^{\star}(t)$ where $t = t_0 + h$ and $h = 0, 1, 2, \ldots$ years, we assume that the benchmark, the modified duration and the duration-times-spread factors do not change over time.

Bond portfolios

The corresponding LP problem is:

$$\begin{aligned} x^{\star} &= \arg\min c^{\top} x \\ & \text{s.t.} \quad \left\{ \begin{array}{l} Ax = B \\ Cx \leq D \\ x^{-} \leq x \leq x^{+} \end{array} \right. \end{aligned}$$

where:

- $x = (w, \tau_w, \tau_{\mathrm{DTS}})$ is a 18×1 vector
- The 18×1 vector c is equal to $\left(\mathbf{0}_8, \dfrac{1}{2}\mathbf{1}_8, \varphi \mathbf{1}_2\right)$
- The equality constraint includes the convex indicator function $\mathbb{1}_{\Omega_{\mathrm{MD}}}(w)$ and is defined by:

$$Ax = B \Leftrightarrow \left(\begin{array}{ccc} \mathbf{1}_8^{\top} & \mathbf{0}_8^{\top} & \mathbf{0}_2^{\top} \\ \mathrm{MD}^{\top} & \mathbf{0}_8^{\top} & \mathbf{0}_2^{\top} \end{array} \right) x = \left(\begin{array}{c} 1 \\ 5.476 \end{array} \right)$$

Bond portfolios

- The inequality constraints are:

$$Cx \leq D \Leftrightarrow \begin{pmatrix} I_8 & -I_8 & \mathbf{0}_{8,2} \\ -I_8 & -I_8 & \mathbf{0}_{8,2} \\ C_{\mathrm{DTS}} & \mathbf{0}_{2,8} & -I_2 \\ -C_{\mathrm{DTS}} & \mathbf{0}_{2,8} & -I_2 \\ \mathcal{CI}(t)^\top & \mathbf{0}_{1,8} & 0 \end{pmatrix} x \leq \begin{pmatrix} b \\ -b \\ 192.68 \\ 108.37 \\ -192.68 \\ -108.37 \\ 160.574 \times (1 - \mathcal{R}(t_0, t)) \end{pmatrix}$$

where:

$$C_{\mathrm{DTS}} = \begin{pmatrix} 100 & 0 & 575 & 436 & 0 & 0 & 0 & 365 \\ 0 & 155 & 0 & 0 & 159 & 145 & 804 & 0 \end{pmatrix}$$

- Finally, the bounds are $x^- = \mathbf{0}_{18}$ and $x^+ = \infty \cdot \mathbf{1}_{18}$

Bond portfolios

Table 89: Bond portfolio alignment (Example #8)

t	$b(t_0)$	t_0	t_0+1	t_0+2	t_0+3	t_0+4	t_0+5	t_0+10
$w_1^\star$	20.00	20.00	20.00	20.00	13.98	17.64	16.02	5.02
$w_2^\star$	19.00	13.99	17.79	19.00	19.00	19.00	19.00	19.00
$w_3^\star$	17.00	25.43	20.96	17.78	17.00	13.64	11.65	4.61
$w_4^\star$	13.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
$w_5^\star$	12.00	28.97	30.71	35.84	43.52	48.80	53.33	71.37
$w_6^\star$	8.00	8.00	8.00	5.67	6.46	0.92	0.00	0.00
$w_7^\star$	6.00	3.61	2.53	1.70	0.04	0.00	0.00	0.00
$w_8^\star$	5.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
$\mathrm{AS}(w)$	0.00	25.40	22.68	24.62	31.52	36.80	41.33	59.37
$\mathrm{MD}(w)$	5.48	5.48	5.48	5.48	5.48	5.48	5.48	5.48
$\mathrm{DTS}(w)$	301.05	274.61	248.91	230.60	220.10	204.46	197.26	174.46
$\mathcal{D}(w \mid b)$	0.00	0.39	0.49	0.60	0.72	0.85	0.99	1.57
$\mathcal{CI}(w)$	160.57	112.40	104.53	97.22	90.41	84.08	78.20	54.40
$\mathcal{R}(w \mid b)$	0.00	30.00	34.90	39.46	43.70	47.64	51.30	66.12

The reduction rate, weights, and active share metrics are expressed in %, the MD metrics are measured in years, and the DTS metrics are calculated in bps.

Defining a net-zero investment policy

General framework

The set of constraints to be applied must include the transition dimension:

$$\Omega = \Omega_{\text{alignment}} \cap \Omega_{\text{transition}}$$

where:

$$\Omega_{\text{alignment}} = \left\{ w : \mathcal{CI}\left(t, w\right) \leq \left(1 - \mathcal{R}\left(t_0, t\right)\right) \mathcal{CI}\left(t_0, b\left(t_0\right)\right) \right\}$$

and:

$$\Omega_{\text{transition}} = \Omega_{\text{self-decarbonization}} \cap \Omega_{\text{greenness}} \cap \Omega_{\text{exclusion}}$$

Self-decarbonization and endogeneity of the decarbonization pathway

	Bad case	Mixed case	Good case
Effective decarbonization			
at the beginning of the year t	30%	30%	30%
at the end of the year t	25%	33%	36%
Self-decarbonization	0%	3%	6%
Relabancing requirement	10%	2%	0%

We can specify the self-decarbonization constraint as follows:

$$\Omega_{\text{self-decarbonization}} = \left\{ w : \mathcal{CM}\left(t, w\right) \leq \mathcal{CM}^{\star}\left(t\right) \right\}$$

where:

- $\mathcal{CM}\left(t, w\right)$ is the carbon momentum of the portfolio w at time t
- $\mathcal{CM}^{\star}\left(t\right)$ is the self-decarbonization minimum threshold

Green footprint

The greenness constraint can be written as follows:

$$\Omega_{\text{greenness}} = \left\{ w : \mathcal{GI}\left(t, w\right) \geq \mathcal{GI}^{\star}\left(t\right) \right\}$$

where:

- $\mathcal{GI}\left(t, w\right)$ is the green intensity of the portfolio w at time t
- $\mathcal{GI}^{\star}\left(t\right)$ is the minimum threshold

Remark

In general, the absolute measure $\mathcal{GI}^{\star}\left(t\right)$ is expressed as a relative value with respect to the benchmark:

$$\mathcal{GI}^{\star}\left(t\right) = \left(1 + \mathcal{G}\right) \mathcal{GI}\left(t, b\left(t\right)\right)$$

where $\mathcal{G}$ is the minimum growth value. For example, if $\mathcal{G} = 100\%$, we want to improve the green footprint of the benchmark so that the green intensity of the portfolio is at least twice the green intensity of the benchmark

Net-zero exclusion policy

- Net-zero enemies
- Temperature score (Implied Temperature Rating or ITR)
- Barahhou *et al.* (2022) suggest excluding issuers whose carbon momentum is greater than a threshold $\mathcal{CM}^+$:

$$\Omega_{\text{exclusion}} = \left\{ w : \mathcal{CM}_i \geq \mathcal{CM}^+ \Rightarrow w_i = 0 \right\}$$

Equity portfolios

The optimization problem becomes:

$$
\begin{aligned}
w^{\star}(t) &= \arg\min \frac{1}{2}\left(w-b(t)\right)^{\top}\Sigma(t)\left(w-b(t)\right) \\
\text{s.t.} &\quad \left\{
\begin{array}{ll}
\boldsymbol{\mathcal{CI}}(t,w) \leq \left(1-\boldsymbol{\mathcal{R}}(t_0,t)\right)\boldsymbol{\mathcal{CI}}(t_0,b(t_0)) & \leftarrow \text{Alignment} \\
\boldsymbol{\mathcal{CM}}(t,w) \leq \boldsymbol{\mathcal{CM}}^{\star}(t) & \leftarrow \text{Self-decarbonization} \\
\boldsymbol{\mathcal{GI}}(t,w) \geq \left(1+\boldsymbol{\mathcal{G}}\right)\boldsymbol{\mathcal{GI}}(t,b(t)) & \leftarrow \text{Greenness} \\
0 \leq w_i \leq \mathbb{1}\left\{\boldsymbol{\mathcal{CM}}_i(t) \leq \boldsymbol{\mathcal{CM}}^{+}\right\} & \leftarrow \text{Exclusion} \\
w \in \Omega_0 \cap \Omega & \leftarrow \text{Other constraints}
\end{array}
\right.
\end{aligned}
$$

Equity portfolios

We deduce that the quadratic form is $Q = \Sigma(t)$, $R = \Sigma(t)\, b(t)$, $A = \mathbf{1}_n^\top$, $B = 1$, $w^- = \mathbf{0}_n$, $w^+ = \mathbf{1}\left\{\mathcal{CM}(t) \leq \mathcal{CM}^+\right\}$.

- If we assume that the carbon momentum function is a linear function:

$$\mathcal{CM}(t, w) = w^\top \mathcal{CM}(t) = \sum_{i=1}^{n} w_i \mathcal{CM}_i(t)$$

where $\mathcal{CM}(t) = (\mathcal{CM}_1(t), \ldots, \mathcal{CM}_n(t))$ is the carbon momentum vector, we get:

$$Cw \leq D \Leftrightarrow \left(\begin{array}{c} \mathcal{CI}(t)^\top \\ \mathcal{CM}(t)^\top \\ -\mathcal{GI}(t)^\top \end{array}\right) w \leq \left(\begin{array}{c} (1 - \mathcal{R}(t_0, t))\, \mathcal{CI}(t_0, b(t_0)) \\ \mathcal{CM}^\star(t) \\ -(1 + \mathcal{G})\, \mathcal{GI}(t, b(t)) \end{array}\right)$$

Equity portfolios

- If we use an exact calculation of the carbon momentum at the portfolio level, we get:

$$Cw \leq D \Leftrightarrow \left(\begin{array}{c} \mathcal{CI}\left(t\right)^{\top} \\ \zeta^{\top} \\ -\mathcal{GI}\left(t\right)^{\top} \end{array} \right) w \leq \left(\begin{array}{c} \left(1 - \mathcal{R}\left(t_0, t\right)\right) \mathcal{CI}\left(t_0, b\left(t_0\right)\right) \\ 0 \\ -\left(1 + \mathcal{G}\right) \mathcal{GI}\left(t, b\left(t\right)\right) \end{array} \right)$$

where $\zeta = \left(\zeta_1, \ldots, \zeta_n\right)$ and $\zeta_i = \mathcal{CI}_i\left(t\right)\left(\mathcal{CM}_i\left(t\right) - \mathcal{CM}^{\star}\left(t\right)\right)$

Equity portfolios

Example #9

We consider Example #7. The carbon momentum values are equal to -3.1%, -1.2%, -5.8%, -1.4%, $+7.4\%$, -2.6%, $+1.2\%$, and -8.0%. We measure the green intensity by the green revenue share. Its values are equal to 10.2%, 45.3%, 7.5%, 0%, 0%, 35.6%, 17.8% and 3.0%. The net-zero investment policy imposes to follow the CTB decarbonization pathway with a self-decarbonization of 3%, and to improve the green intensity of the benchmark by 100%

Equity portfolios

Table 90: Net-zero equity portfolio (Example #9)

t	$b(t_0)$	t_0	t_0+1	t_0+2	t_0+3	t_0+4	t_0+5	t_0+10
$w_1^\star$	20.00	5.26	3.51	1.49	0.00	0.02		
$w_2^\star$	19.00	20.96	17.27	13.00	8.82	4.16		
$w_3^\star$	17.00	3.35	7.27	11.82	15.02	14.32		
$w_4^\star$	13.00	0.00	0.00	0.00	0.00	0.00	No feasible	
$w_5^\star$	12.00	0.00	0.00	0.00	0.00	0.00	solution	
$w_6^\star$	8.00	60.06	64.69	70.05	75.37	81.51		
$w_7^\star$	6.00	0.00	0.00	0.00	0.00	0.00		
$w_8^\star$	5.00	10.37	7.25	3.64	0.79	0.00		
$\sigma(w^\star \mid b(t))$	0.00	370.16	376.38	398.30	430.94	472.44		
$\mathcal{CI}(t,w)$	160.57	110.85	104.53	97.22	90.41	84.08		
$\mathcal{R}(w \mid b(t_0))$	0.00	30.96	34.90	39.46	43.70	47.64		
$\mathcal{CM}(t,w)$	-1.66	-3.00	-3.00	-3.00	-3.00	-3.00		
$\mathcal{GI}(t,w)$	15.99	31.98	31.98	31.98	31.98	31.98		

The reduction rate, weights, carbon momentum and green intensity are expressed in %, while the tracking error volatility is measured in bps.

Bond portfolios

The optimization problem becomes:

$$
\begin{array}{rcl}
w^{\star}(t) & = & \arg\min \mathcal{D}\left(w \mid b(t)\right) \\
& & \text{s.t.} \left\{
\begin{array}{ll}
\boldsymbol{\mathcal{CI}}(t,w) \leq \left(1 - \boldsymbol{\mathcal{R}}(t_0,t)\right)\boldsymbol{\mathcal{CI}}\left(t_0, b(t_0)\right) & \leftarrow \text{Alignment} \\
\boldsymbol{\mathcal{CM}}(t,w) \leq \boldsymbol{\mathcal{CM}}^{\star}(t) & \leftarrow \text{Self-decarbonization} \\
\boldsymbol{\mathcal{GI}}(t,w) \geq \left(1 + \boldsymbol{\mathcal{G}}\right)\boldsymbol{\mathcal{GI}}\left(t, b(t)\right) & \leftarrow \text{Greenness} \\
0 \leq w_i \leq \mathbb{1}\left\{\boldsymbol{\mathcal{CM}}_i(t) \leq \boldsymbol{\mathcal{CM}}^{+}\right\} & \leftarrow \text{Exclusion} \\
w \in \Omega_0 \cap \Omega & \leftarrow \text{Other constraints}
\end{array}
\right.
\end{array}
$$

Bond portfolios

We get the same LP form except for the set of inequality constraints $Cx \leq D$:

$$\begin{pmatrix} I_n & -I_n & \mathbf{0}_{n,n_{\mathcal{S}ector}} \\ -I_n & -I_n & \mathbf{0}_{n,n_{\mathcal{S}ector}} \\ C_{\mathrm{DTS}} & \mathbf{0}_{n_{\mathcal{S}ector},n} & -I_{n_{\mathcal{S}ector}} \\ -C_{\mathrm{DTS}} & \mathbf{0}_{n_{\mathcal{S}ector},n} & -I_{n_{\mathcal{S}ector}} \\ \mathcal{CI}(t)^\top & \mathbf{0}_{1,n} & \mathbf{0}_{1,n_{\mathcal{S}ector}} \\ \mathcal{CM}(t)^\top & \mathbf{0}_{1,n} & \mathbf{0}_{1,n_{\mathcal{S}ector}} \\ -\mathcal{GI}(t)^\top & \mathbf{0}_{1,n} & \mathbf{0}_{1,n_{\mathcal{S}ector}} \end{pmatrix} x \leq \begin{pmatrix} b \\ -b \\ \mathrm{DTS}^\star \\ -\mathrm{DTS}^\star \\ (1-\mathcal{R}(t_0,t))\,\mathcal{CI}(t_0,b(t_0)) \\ \mathcal{CM}^\star(t) \\ -(1+\mathcal{G})\,\mathcal{GI}(t,b(t)) \end{pmatrix}$$

and the upper bound:

$$x^+ = \left(\mathbb{1}\left\{\mathcal{CM}(t) \leq \mathcal{CM}^+\right\}, \infty \cdot \mathbf{1}_n, \infty \cdot \mathbf{1}_{n_{\mathcal{S}ector}}\right)$$

Bond portfolios

Example #10

We consider Example #8. The carbon momentum values are equal to -3.1%, -1.2%, -5.8%, -1.4%, $+7.4\%$, -2.6%, $+1.2\%$, and -8.0%. We measure the green intensity by the green revenue share. Its values are equal to 10.2%, 45.3%, 7.5%, 0%, 0%, 35.6%, 17.8% and 3.0%. The net-zero investment policy imposes to follow the CTB decarbonization pathway with a self-decarbonization of 2%, and to improve the green intensity of the benchmark by 100%.

Bond portfolios

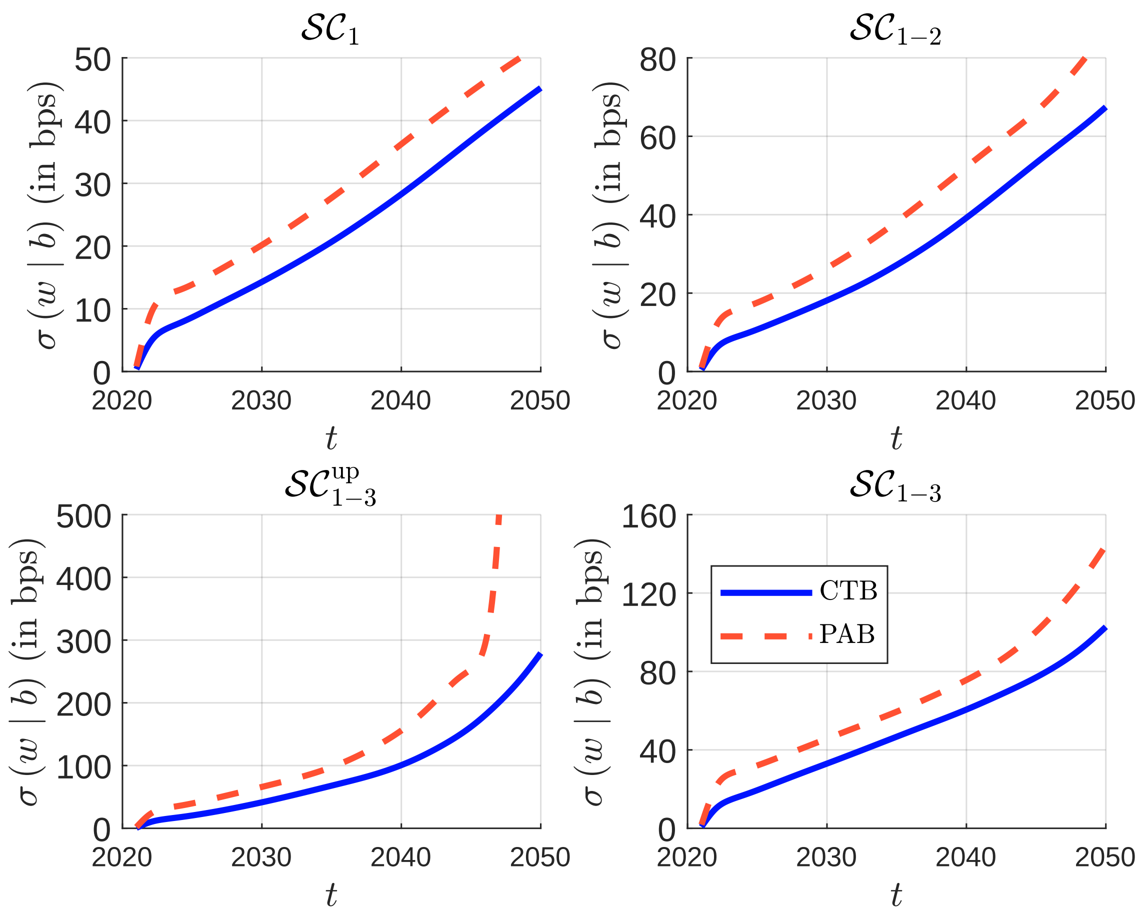
Table 91: Net-zero bond portfolio (Example #10)

t	$b(t_0)$	t_0	t_0+1	t_0+2	t_0+3	t_0+4	t_0+5	t_0+10
$w_1^\star$	20.00	4.28	13.80	20.48	26.34	19.02		
$w_2^\star$	19.00	34.78	38.94	42.72	46.23	49.01		
$w_3^\star$	17.00	21.03	13.86	7.73	2.11	0.00		
$w_4^\star$	13.00	0.00	0.00	0.00	0.00	0.00	No feasible solution	
$w_5^\star$	12.00	0.00	0.00	0.00	0.00	0.00		
$w_6^\star$	8.00	39.91	33.40	29.07	25.32	31.97		
$w_7^\star$	6.00	0.00	0.00	0.00	0.00	0.00		
$w_8^\star$	5.00	0.00	0.00	0.00	0.00	0.00		
$\mathrm{AS}(w)$	0.00	51.72	45.34	45.27	50.89	53.98		
$\mathrm{MD}(w)$	5.48	5.48	5.48	5.48	5.48	5.48		
$\mathrm{DTS}(w)$	301.05	236.99	202.30	173.29	146.83	141.34		
$\mathcal{D}(w \mid b)$	0.00	0.87	0.95	1.09	1.28	1.48		
$\mathcal{CI}(w)$	160.57	112.40	104.53	97.22	90.41	84.08		
$\mathcal{R}(w \mid b)$	0.00	30.00	34.90	39.46	43.70	47.64		
$\mathcal{CM}(t,w)$	−1.66	−2.81	−2.57	−2.35	−2.15	−2.01		
$\mathcal{GI}(t,w)$	15.99	31.98	31.98	32.37	32.80	35.52		

The reduction rate, weights, carbon momentum, green intensity and active share metrics are expressed in %, the MD metrics are measured in years, and the DTS metrics are calculated in bps.

Empirical results

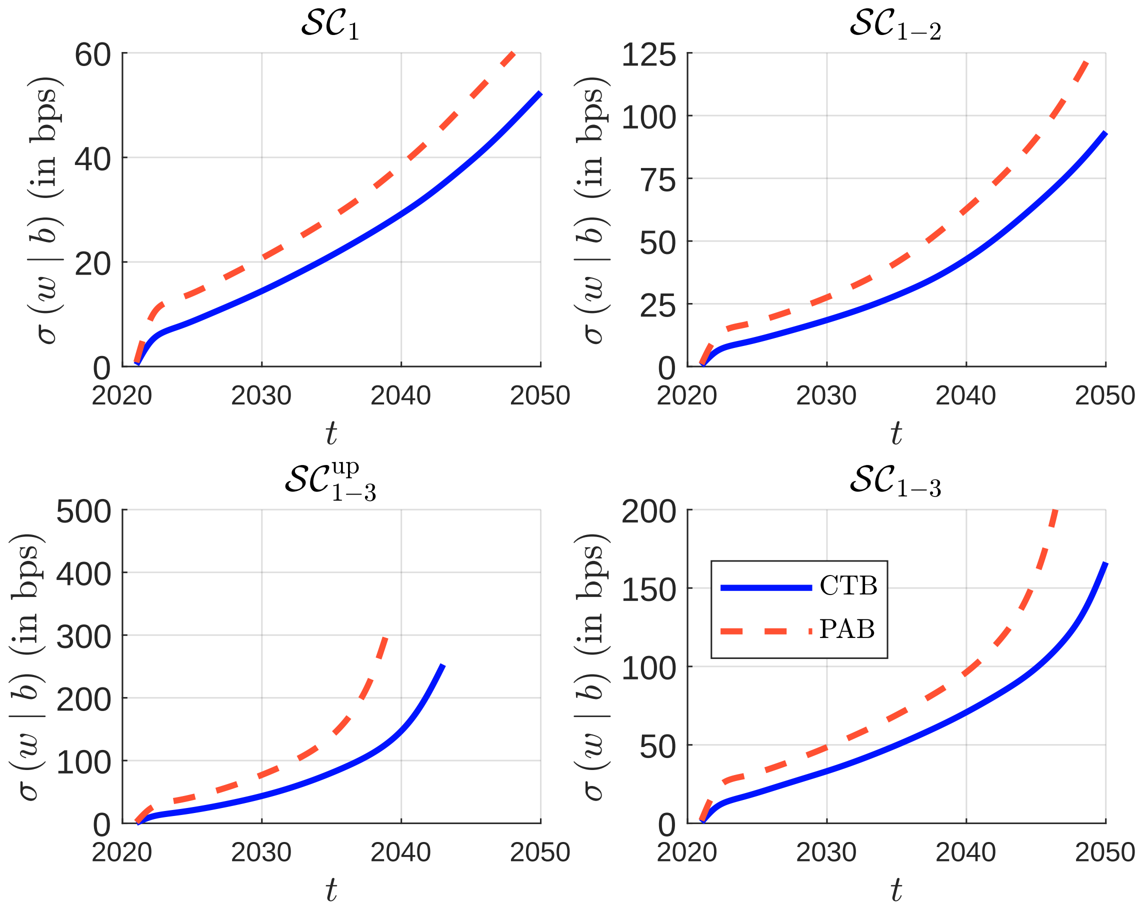
Figure 103: Tracking error volatility of dynamic decarbonized portfolios (MSCI World, June 2022, $\mathcal{C}_0$ constraint)



Source: MSCI (2022), Trucost (2022) & Barahhou *et al.* (2022)

Empirical results

Figure 104: Tracking error volatility of dynamic decarbonized portfolios (MSCI World, June 2022, $\mathcal{C}_3(0, 10, 2)$ constraint)



Source: MSCI (2022), Trucost (2022) & Barahhou *et al.* (2022)

Empirical results

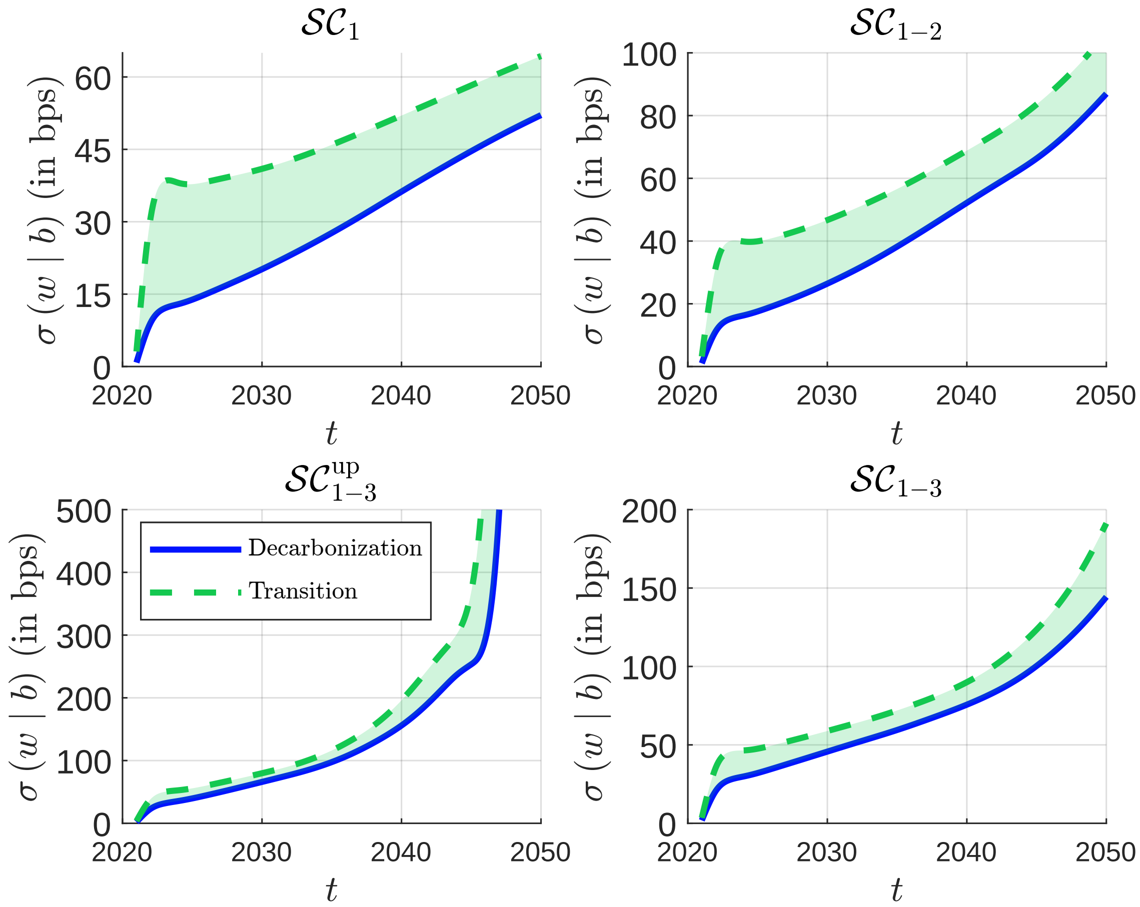
The previous analysis deals only with the decarbonization dimension. Barahhou *et al.* (2022) then introduced the transition dimension and solved the following optimization problem:

$$\begin{aligned} w^{\star}(t) &= \arg\min \frac{1}{2}(w - b(t))^{\top} \Sigma(t)(w - b(t)) \\ & \text{s.t.} \quad \left\{ \begin{array}{l} \mathcal{CI}(t, w) \leq (1 - \mathcal{R}(t_0, t)) \mathcal{CI}(t_0, b(t_0)) \\ \mathcal{CM}(t, w) \leq \mathcal{CM}^{\star}(t) \\ \mathcal{GI}(t, w) \geq (1 + \mathcal{G}) \mathcal{GI}(t, b(t)) \\ w \in \mathcal{C}_0 \cap \mathcal{C}_3(0, 10, 2) \end{array} \right. \end{aligned}$$

where $\mathcal{CM}^{\star}(t) = -5\%$ and $\mathcal{G} = 100\%$

Empirical results

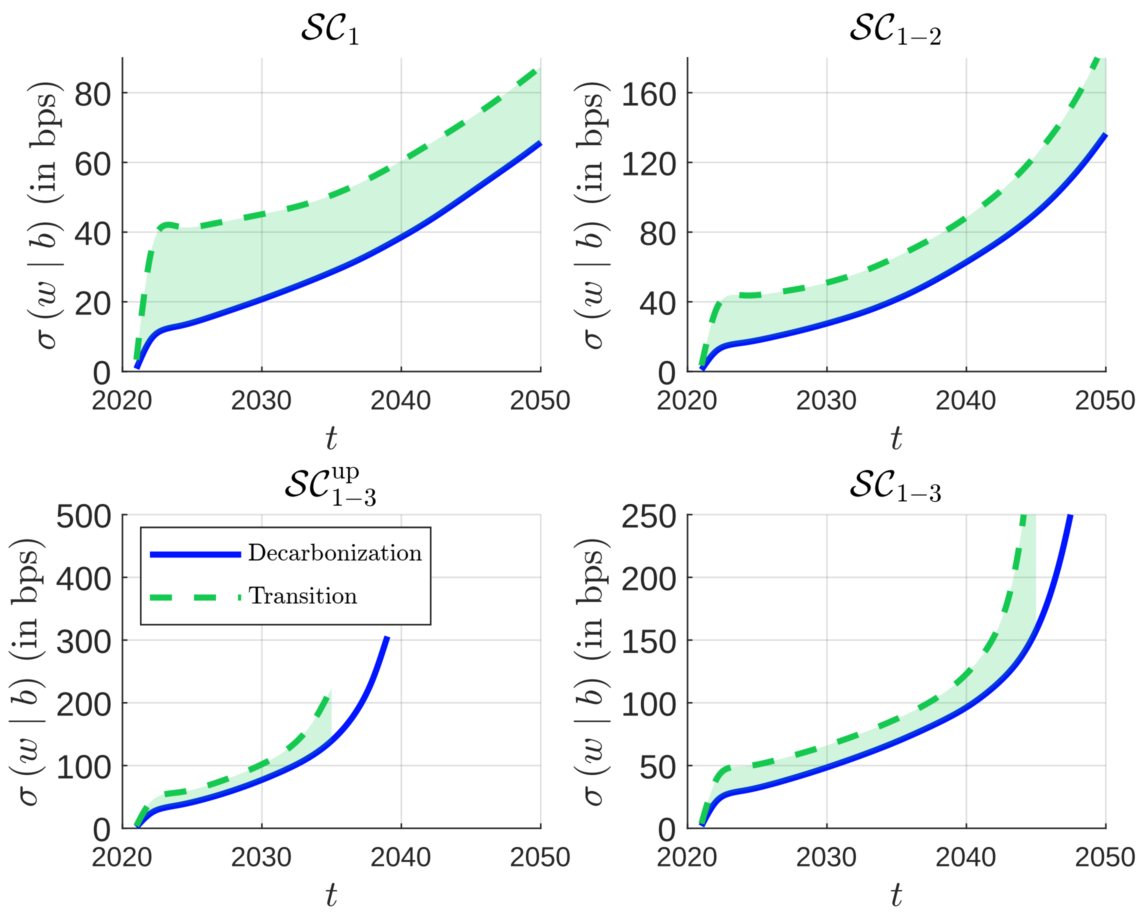
Figure 105: Tracking error volatility of net-zero portfolios (MSCI World, June 2022, $\mathcal{C}_0$ constraint, $\mathcal{G} = 100\%$, $\mathcal{CM}^\star = -5\%$, PAB)



Source: MSCI (2022), Trucost (2022) & Barahhou *et al.* (2022)

Empirical results

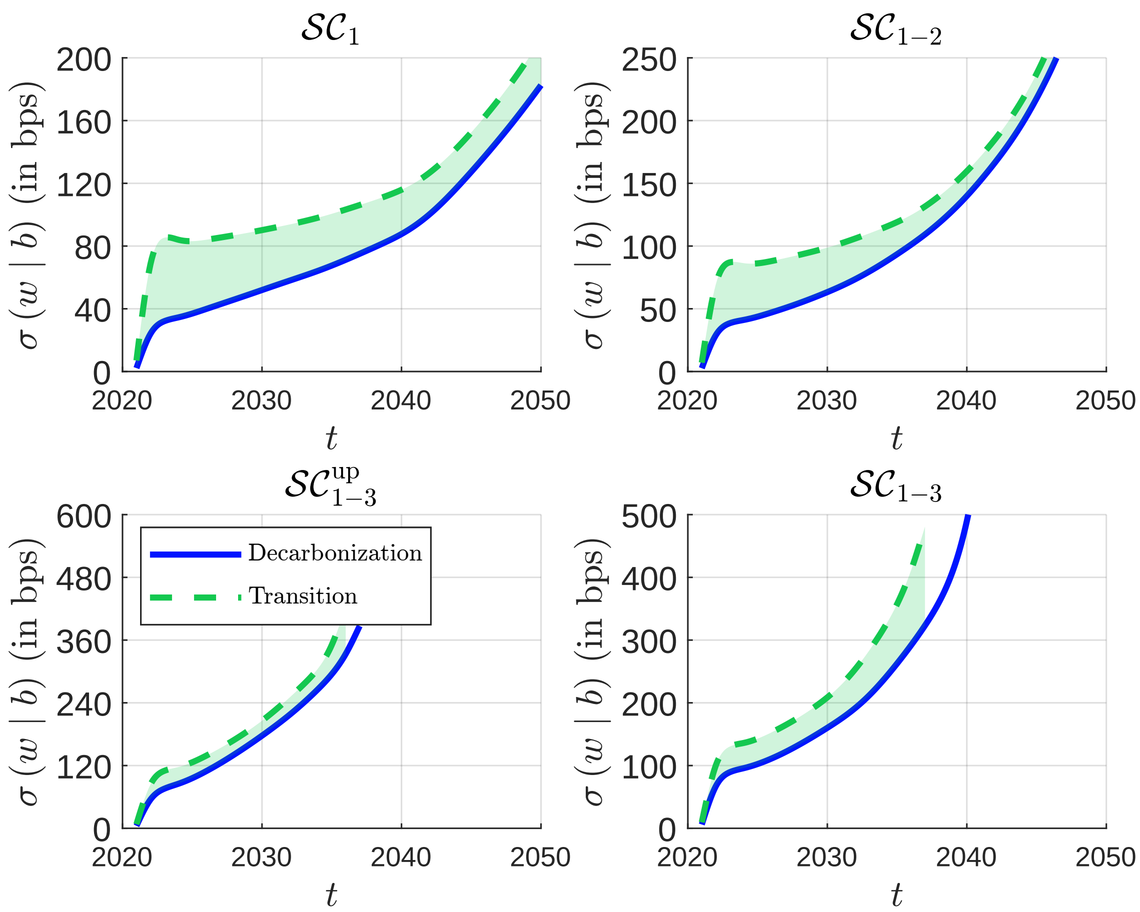
Figure 106: Tracking error volatility of net-zero portfolios (MSCI World, June 2022, $\mathcal{C}_3(0, 10, 2)$ constraint, $\mathcal{G} = 100\%$, $\mathcal{CM}^\star = -5\%$, PAB)



Source: MSCI (2022), Trucost (2022) & Barahhou et al. (2022)

Empirical results

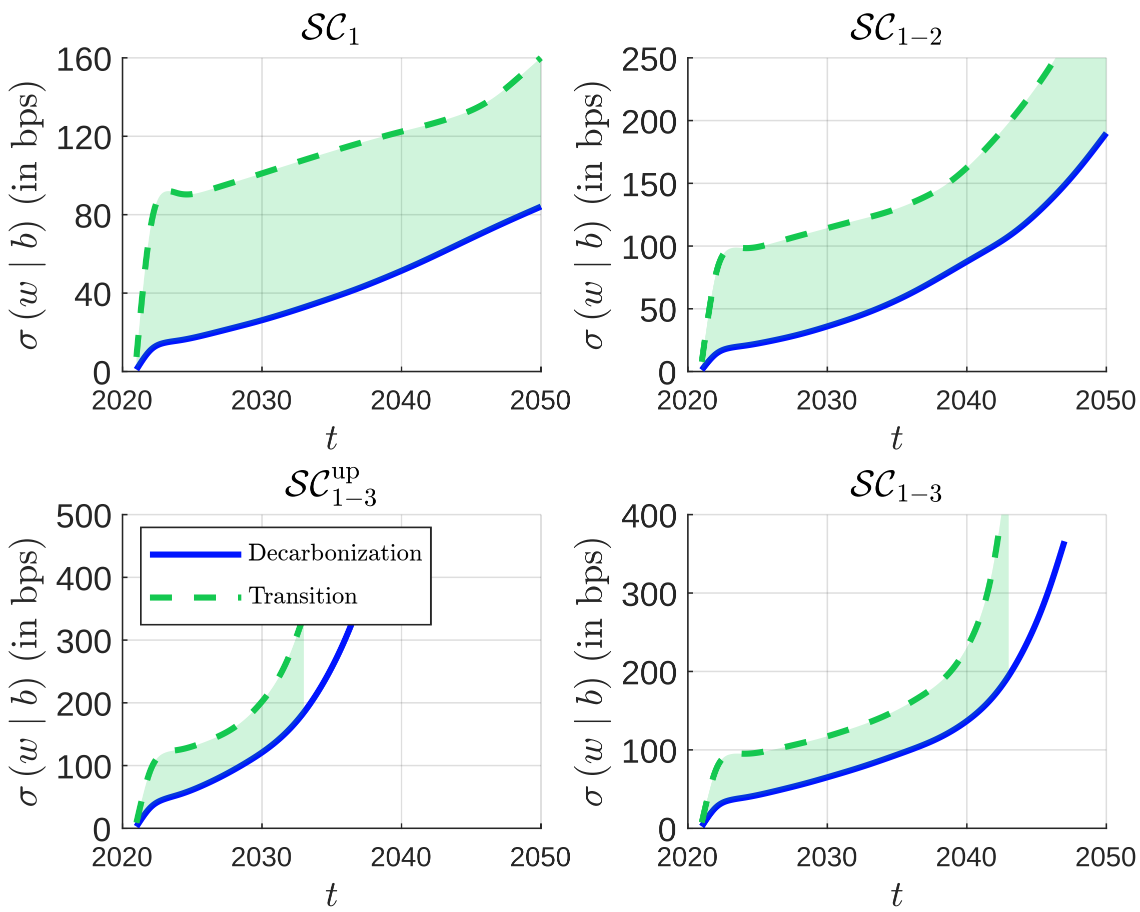
Figure 107: Tracking error volatility of net-zero portfolios (MSCI EMU, June 2022, $\mathcal{C}_3(0, 10, 2)$ constraint, $\mathcal{G} = 100\%$, $\mathcal{CM}^\star = -5\%$, PAB)



Source: MSCI (2022), Trucost (2022) & Barahhou *et al.* (2022)

Empirical results

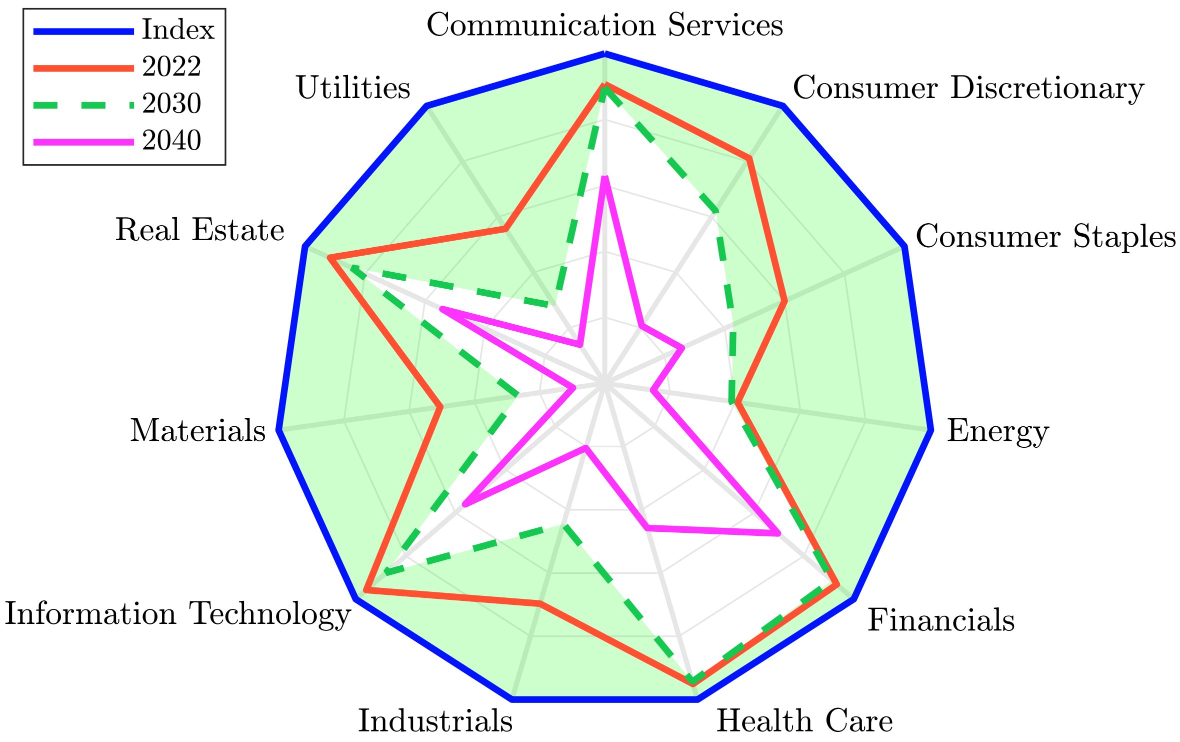
Figure 108: Tracking error volatility of net-zero portfolios (MSCI USA, Jun. 2022, $\mathcal{C}_3(0, 10, 2)$ constraint, $\mathcal{G} = 100\%$, $\mathcal{CM}^{\star} = -5\%$, PAB)



Source: MSCI (2022), Trucost (2022) & Barahhou *et al.* (2022)

Empirical results

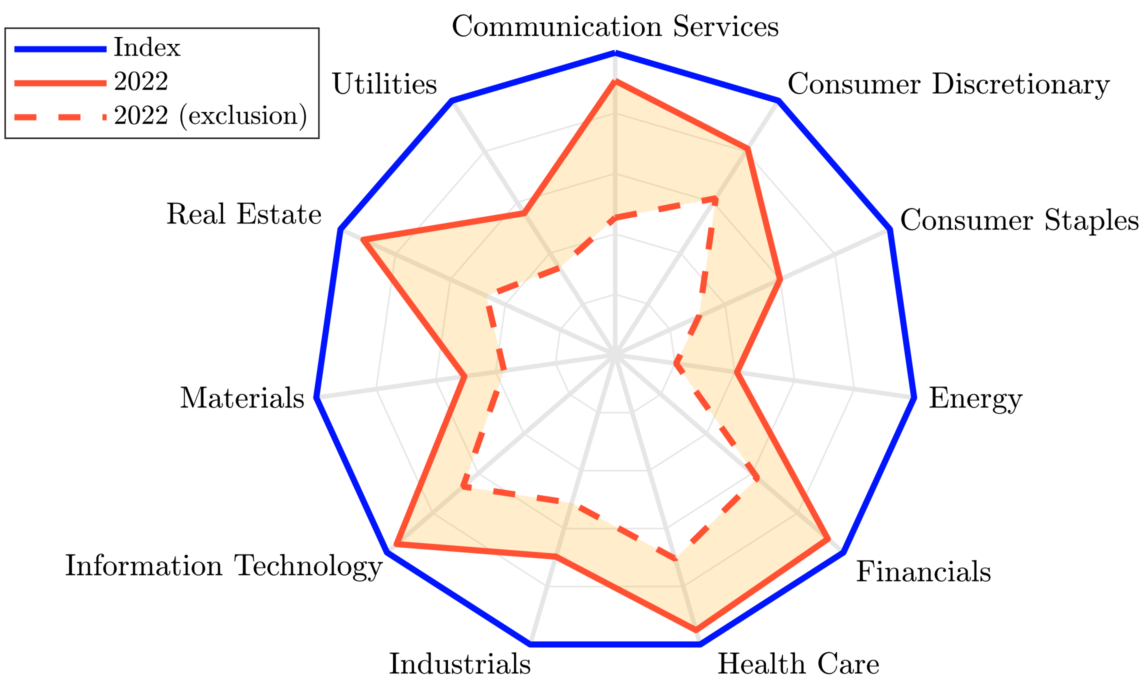
Figure 109: Radar chart of investment universe shrinkage (MSCI World, June 2022, $\mathcal{C}_3(0, 10, 2)$ constraint, $\boldsymbol{\mathcal{G}} = 100\%$, $\boldsymbol{\mathcal{CM}}^\star = -5\%$, PAB, Scope $\boldsymbol{\mathcal{SC}}_{1-3}$)



Source: MSCI (2022), Trucost (2022) & Barahhou *et al.* (2022).

Empirical results

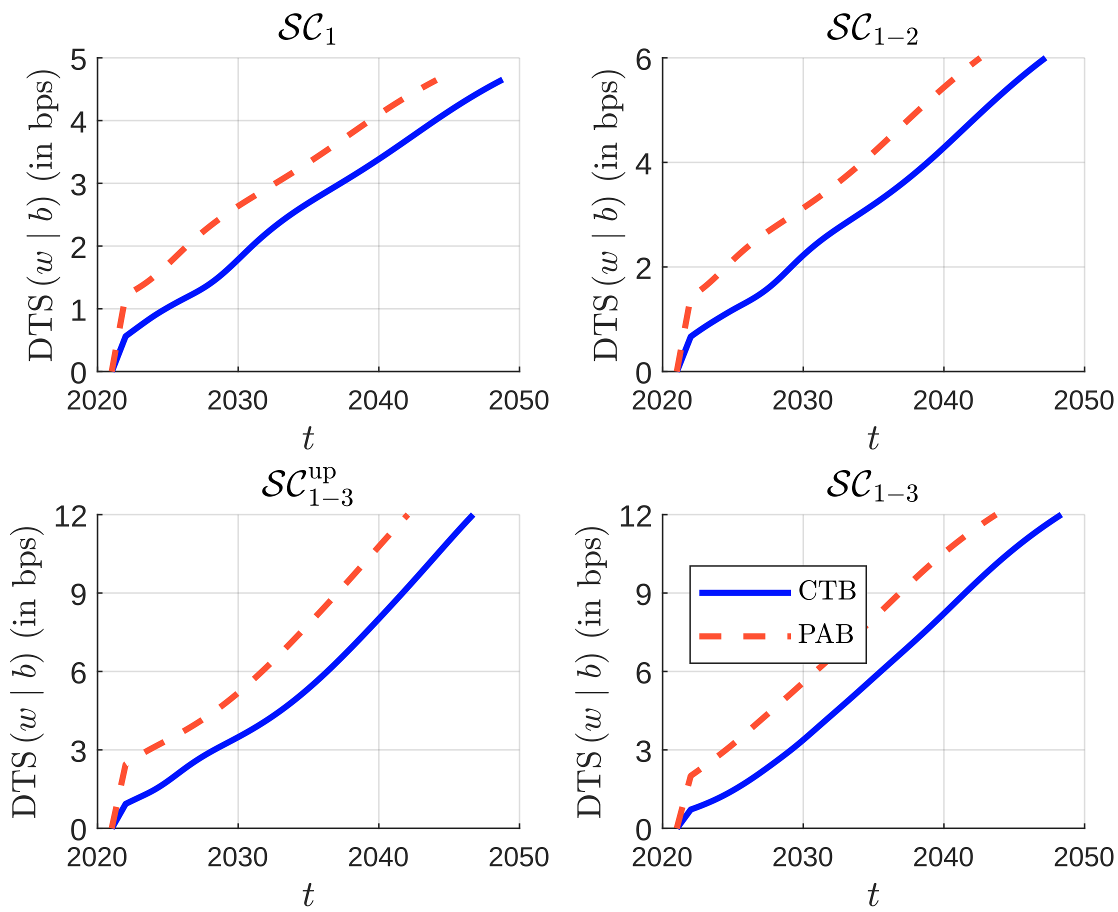
Figure 110: Impact of momentum exclusion on universe shrinkage (MSCI World, June 2022, $\mathcal{C}_3(0,10,2)$ constraint, $\mathcal{G} = 100\%$, $\mathcal{CM}^\star = -5\%$, PAB, Scope $\mathcal{SC}_{1-3}$, $\mathcal{CM}^+ = 0\%$)



Source: MSCI (2022), Trucost (2022) & Barahhou *et al.* (2022).

Empirical results

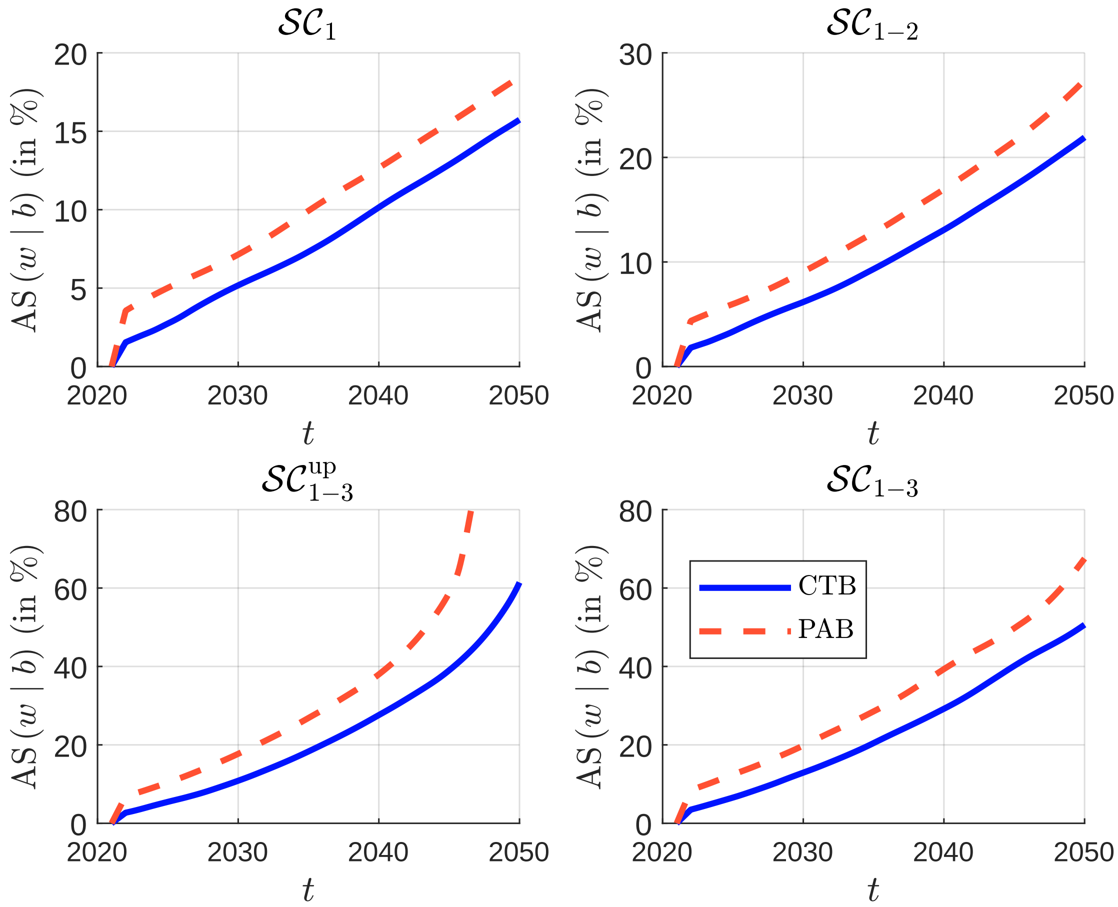
Figure 111: Duration-times-spread cost of dynamically decarbonized portfolios (Global Corporate, June 2022)



Source: MSCI (2022), Trucost (2022) & Barahhou *et al.* (2022)

Empirical results

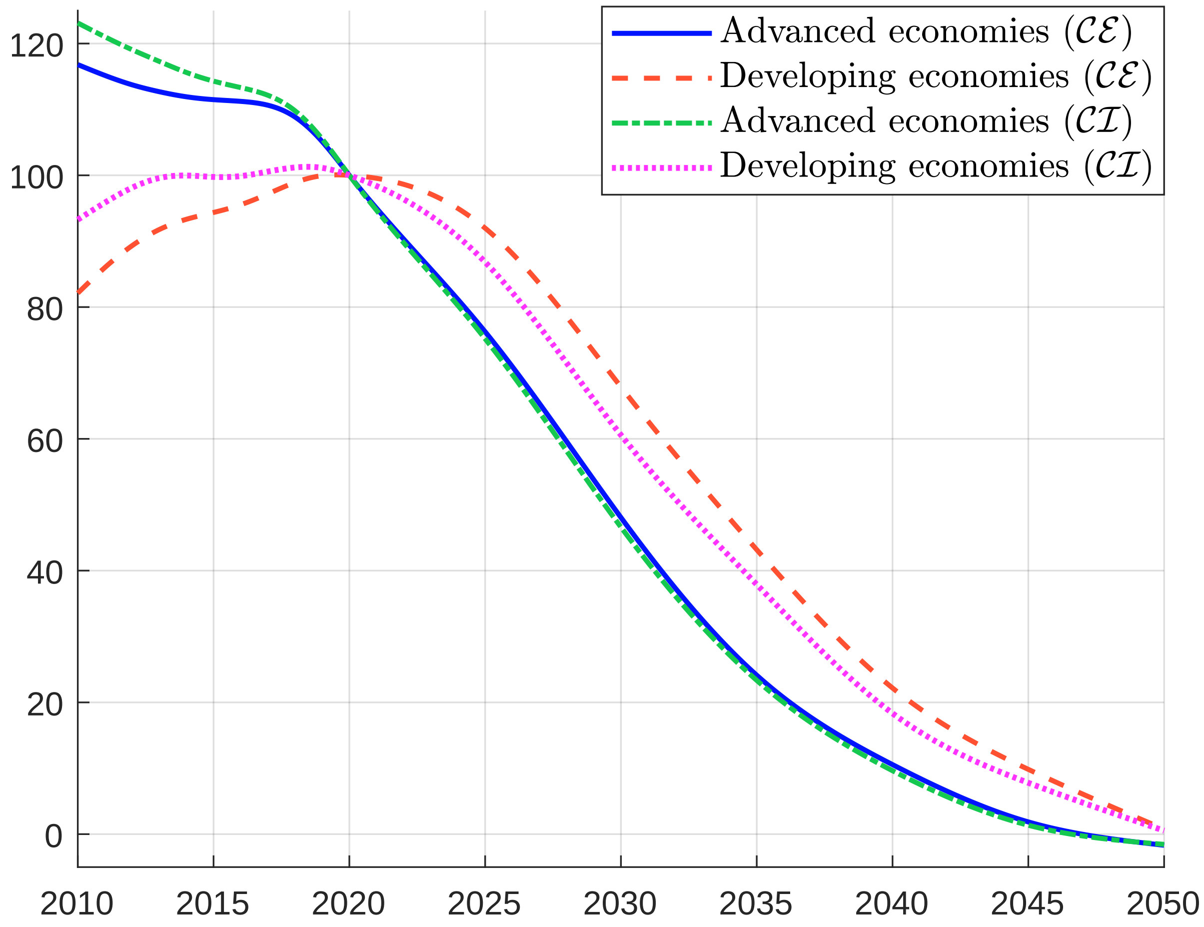
Figure 112: Active share of dynamically decarbonized portfolios (Global Corporate, June 2022)



Source: MSCI (2022), Trucost (2022) & Barahhou *et al.* (2022)

Empirical results

Figure 113: IEA decarbonization pathways



Empirical results

Table 92: First year of country exit from the NZE investment portfolio (GHG/GDP intensity metric)

Australia	2025	Finland	2029	Lithuania	2025	Romania	2029
Austria	2029	France	2029	Luxembourg	2029	Singapore	2029
Belgium	2028	Germany	2029	Mexico	2029	Slovakia	2025
Canada	2024	Hong Kong	2029	Malaysia	2028	Slovenia	2028
Chile	2029	Hungary	2029	Malta	2029	South Korea	2024
China	2028	Indonesia	2024	Netherlands	2029	Spain	2028
Colombia	2029	Ireland	2029	Norway	2029	Switzerland	2029
Cyprus	2029	Israel	2029	New Zealand	2024	Sweden	2029
Czechia	2024	Italy	2029	Peru	2029	Thailand	2025
Denmark	2029	Japan	2029	Poland	2029	United Kingdom	2029
Estonia	2025	Latvia	2028	Portugal	2028	United States	2028

Source: Barahhou *et al.* (2023, Table 9, page 26.

Empirical results

Table 93: Country exclusion year by intensity metric

Metric	$\frac{\text{GHG}}{\text{GDP}}$	$\frac{\text{GHG}}{\text{Population}}$	$\frac{CO_2 \text{ (production)}}{\text{GDP}}$	$\frac{CO_2 \text{ (consumption)}}{\text{Population}}$
China	2028	2031	2027	2031
France	2029	2032	2027	2031
Indonesia	2024	2032	2024	2031
Ireland	2029	2030	2027	2030
Japan	2029	2032	2027	2031
United States	2028	2030	2026	2029
United Kingdom	2029	2032	2027	2031
Sweden	2029	2032	2027	2031

Source: Barahhou *et al.* (2023, Table 14, page 31.

The core-satellite approach

The two building block approach

Decarbonizing the portfolio

- Net-zero decarbonization portfolio
- Net-zero transition portfolio
- Dynamic low-carbon portfolio

Financing the transition

- Net-zero contribution portfolio
- Net-zero funding portfolio
- Net-zero transformation portfolio

The core-satellite approach

The core-satellite approach

Decarbonized portfolio

- Carbon intensity
- Decarbonization pathway(**s**)
- Top-down approach
- Portfolio construction
- Net-zero **carbon** metrics

$+$

Transition portfolio

- Green intensity
- Financing the transition
- Bottom-up approach
- Security selection
- Net-zero **transition** metrics

$$1 - \alpha(t)$$

$$\alpha(t)$$

Core portfolio

A typical program for the equity bucket looks like this:

$$
\begin{aligned}
w^{\star}(t) &= \arg\min \frac{1}{2}\left(w - b(t)\right)^{\top} \Sigma(t) \left(w - b(t)\right) \\
& \text{s.t.} \quad \left\{
\begin{array}{l}
\mathcal{CI}(t, w) \leq \left(1 - \mathcal{R}(t_0, t)\right) \mathcal{CI}(t_0, b(t_0)) \\
\mathcal{CM}(t, w) \leq \mathcal{CM}^{\star}(t) \\
0 \leq w_i \leq \mathbb{1}\left\{\mathcal{CM}_i(t) \leq \mathcal{CM}^{+}\right\} \\
w \in \Omega_0 \cap \Omega
\end{array}
\right.
\end{aligned}
$$

Core portfolio

For the bond bucket, we get a similar optimization problem:

$$\begin{aligned} w^{\star}(t) &= \arg\min \mathcal{D}(w \mid b(t)) \\ &\text{s.t.} \left\{ \begin{array}{l} \mathcal{CI}(t, w) \leq (1 - \mathcal{R}(t_0, t)) \mathcal{CI}(t_0, b(t_0)) \\ \mathcal{CM}(t, w) \leq \mathcal{CM}^{\star}(t) \\ 0 \leq w_i \leq \mathbb{1}\left\{\mathcal{CM}_i(t) \leq \mathcal{CM}^{+}\right\} \\ w \in \Omega_0 \cap \Omega \end{array} \right. \end{aligned}$$

The electricity sector scenario in the core portfolio

The constraint to meet a reduction rate for a given sector $\mathcal{S}ector_j$ is:

$$\frac{\sum_{i=1}^{n} \mathbb{1}\left\{i \in \mathcal{S}ector_j\right\} w_i \mathcal{CI}_i}{\sum_{i=1}^{n} \mathbb{1}\left\{i \in \mathcal{S}ector_j\right\} w_i} = \mathcal{CI}\left(\mathcal{S}ector_j, \mathcal{R}_j\right)$$

where $\mathcal{CI}\left(\mathcal{S}ector_j, \mathcal{R}_j\right)$ is the carbon intensity target for the given sector:

$$\mathcal{CI}\left(\mathcal{S}ector_j, \mathcal{R}_j\right) = \left(1 - \mathcal{R}_j\right) \frac{\sum_{i=1}^{n} \mathbb{1}\left\{i \in \mathcal{S}ector_j\right\} b_i \mathcal{CI}_i}{\sum_{i=1}^{n} \mathbb{1}\left\{i \in \mathcal{S}ector_j\right\} b_i}$$

We deduce that:

$$\sum_{i=1}^{n} \mathbb{1}\left\{i \in \mathcal{S}ector_j\right\} w_i \mathcal{CI}_i = \mathcal{CI}\left(\mathcal{S}ector_j, \mathcal{R}_j\right) \sum_{i=1}^{n} \mathbb{1}\left\{i \in \mathcal{S}ector_j\right\} w_i$$

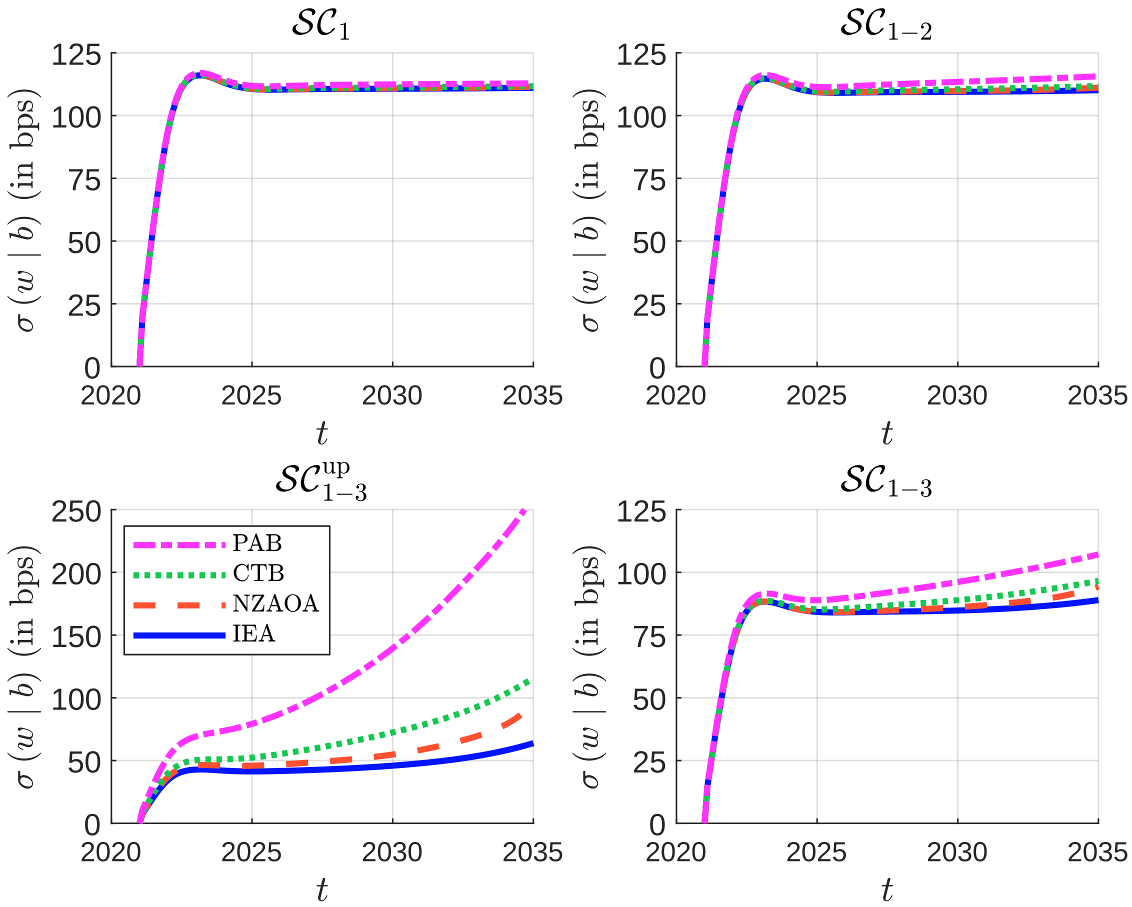
which is equivalent to the following constraint:

$$\sum_{i=1}^{n} \mathbb{1}\left\{i \in \mathcal{S}ector_j\right\} w_i \left(\mathcal{CI}_i - \mathcal{CI}\left(\mathcal{S}ector_j, \mathcal{R}_j\right)\right) = 0 \Leftrightarrow \left(\boldsymbol{s}_j \circ \left(\mathcal{CI}_i - \mathcal{CI}_j^{\star}\right)\right)^{\top} w = 0$$

where $\mathcal{CI}_j^{\star} = \mathcal{CI}\left(\mathcal{S}ector_j, \mathcal{R}_j\right)$

Core portfolio

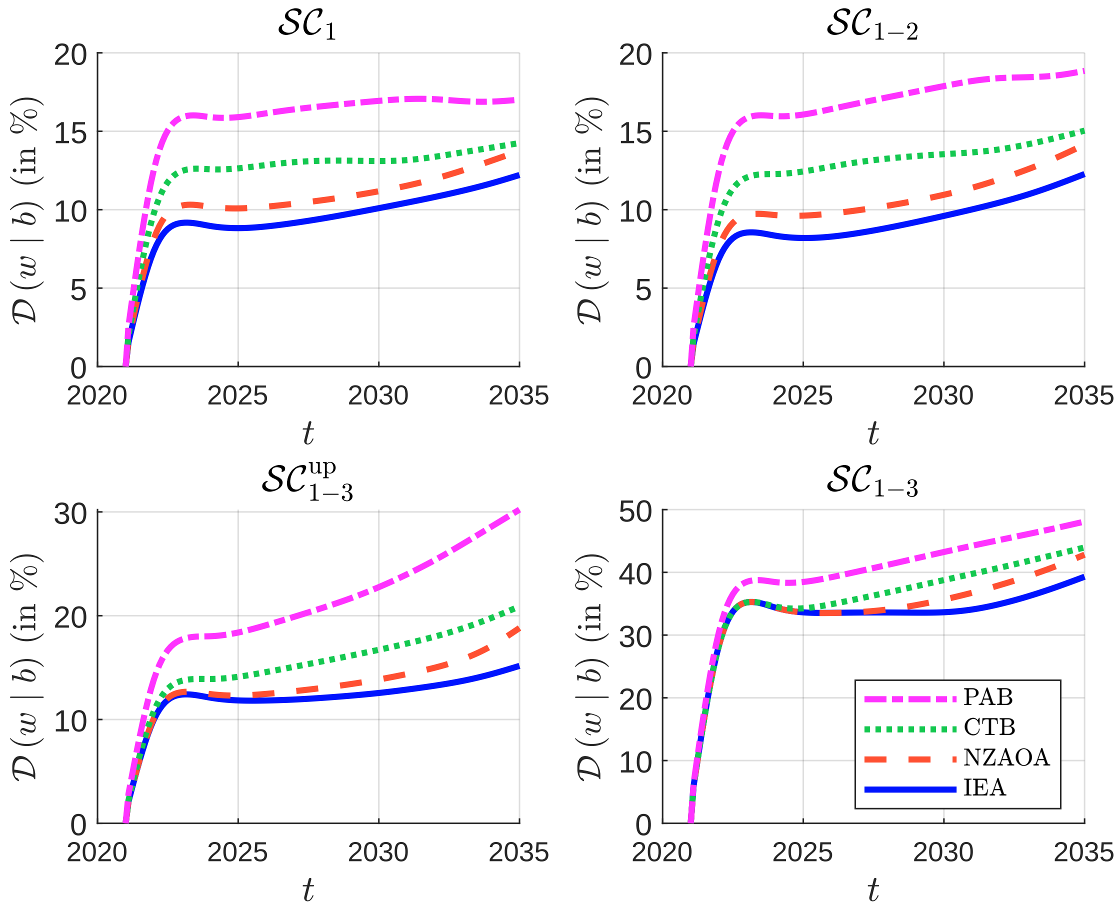
Figure 114: TE volatility of decarbonized portfolios (MSCI World, December 2021, $\mathcal{CM}^{\star} = -3.5\%$, $\mathcal{CM}^{+} = 10\%$, IEA NZE electricity sector scenario)



Source: Ben Slimane *et al.* (2023)

Core portfolio

Figure 115: Active risk of decarbonized portfolios (Global Corporate, December 2021, $\mathcal{CM}^{\star} = -3.5\%$, $\mathcal{CM}^{+} = 10\%$, IEA NZE electricity sector scenario)



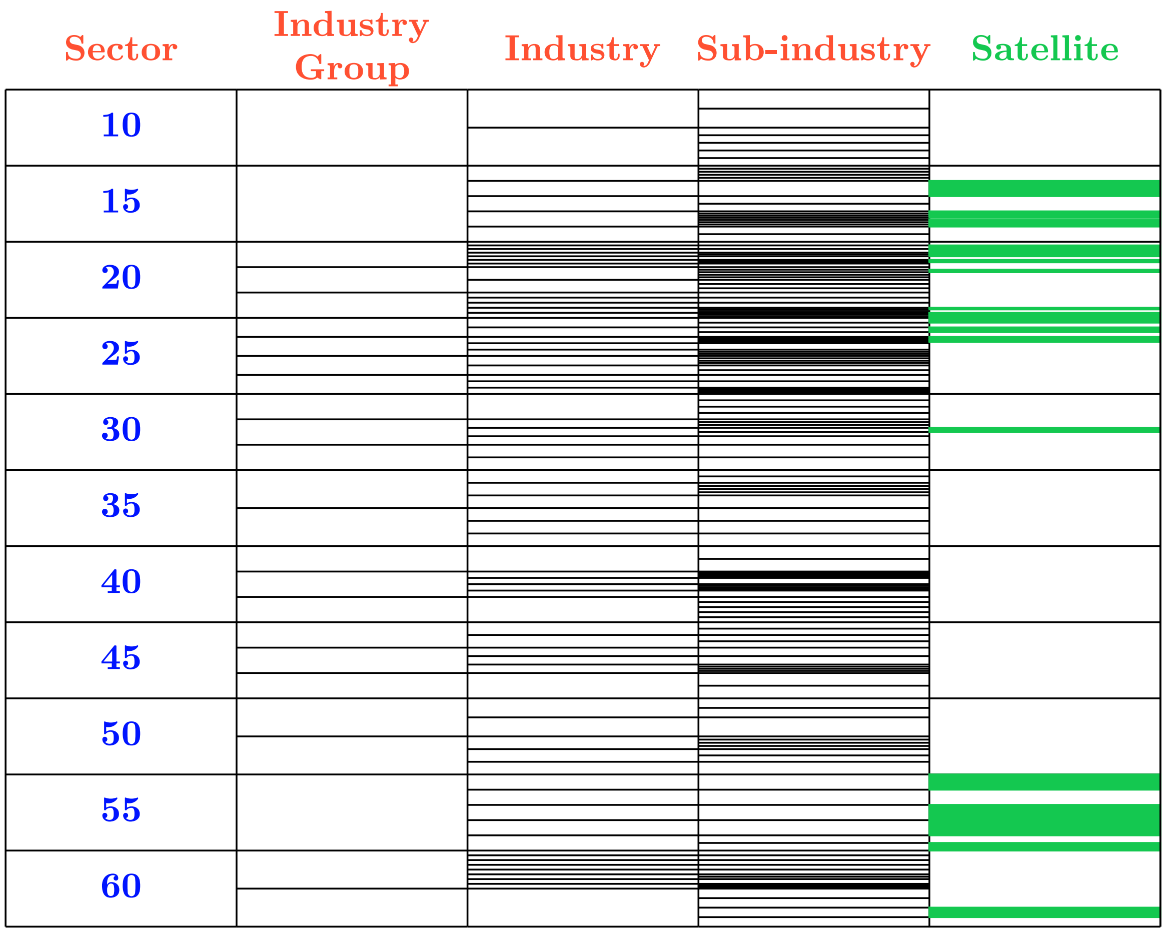
Source: Ben Slimane *et al.* (2023)

Satellite portfolio

- Green, sustainability and sustainability-linked bonds
- Green stocks
- Green infrastructure
- Sustainable real estate

Satellite portfolio

Figure 116: Narrow specification of the satellite investment universegeneral]GICS



Source: Ben Slimane *et al.* (2023).

Green bonds

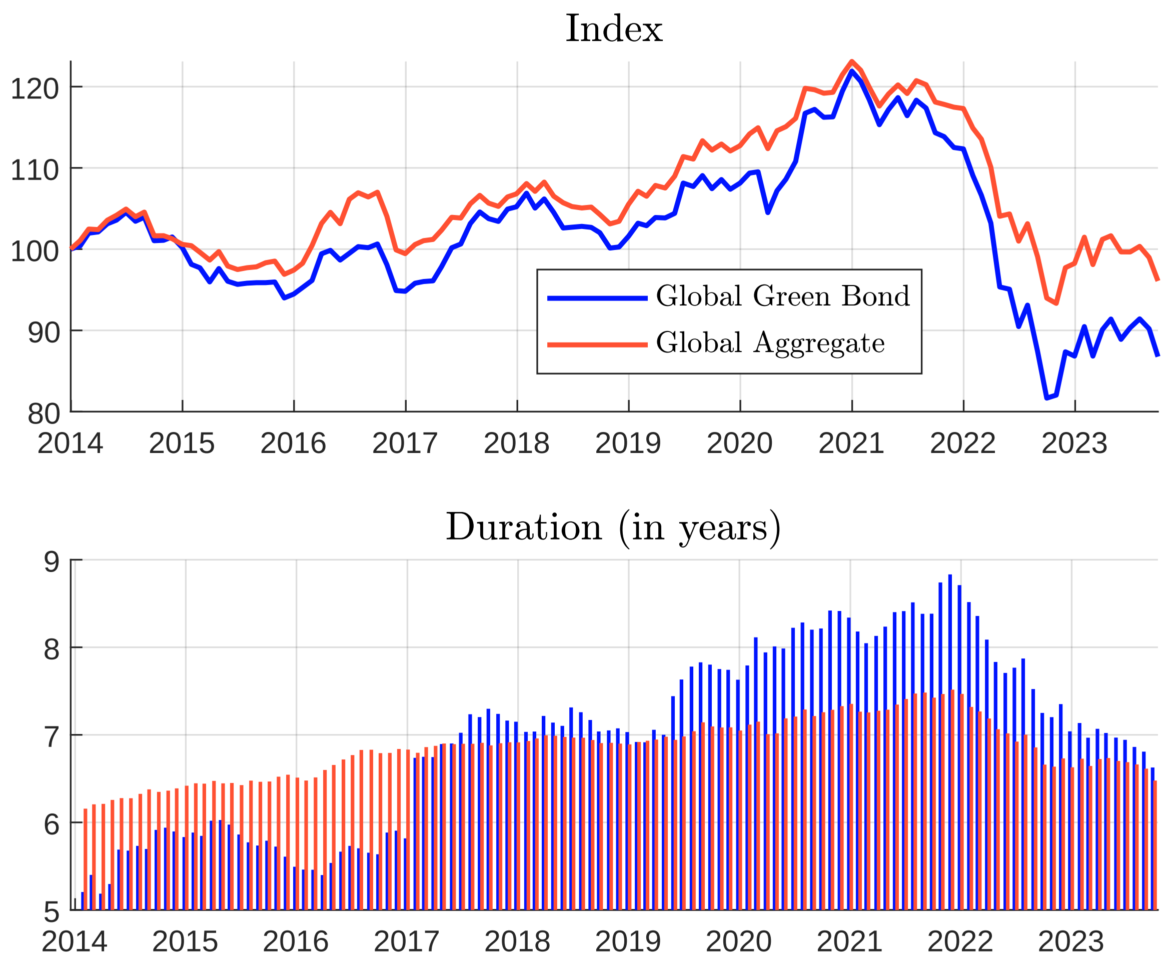
Table 94: GSS+ bond issuance

Year	Green		Social		Sustainability		SLB	
	#	$ bn	#	$ bn	#	$ bn	#	$ bn
2022	1 784	531.6	542	152.8	614	174.8	382	144.3
2021	1 971	686.1	554	242.1	646	233.2	343	161.5
2020	1 076	291.2	273	172.0	308	154.8	47	16.5
2019	877	268.0	99	22.2	333	85.2	18	8.9
2018	582	165.3	48	16.5	52	22.1	1	2.2
2017	472	160.9	46	11.8	17	9.2	1	0.2
2016	285	99.7	14	2.2	16	6.6	0	0.0

Source: Bloomberg (2023), GSS+ Instrument Indicator & Author's calculations.

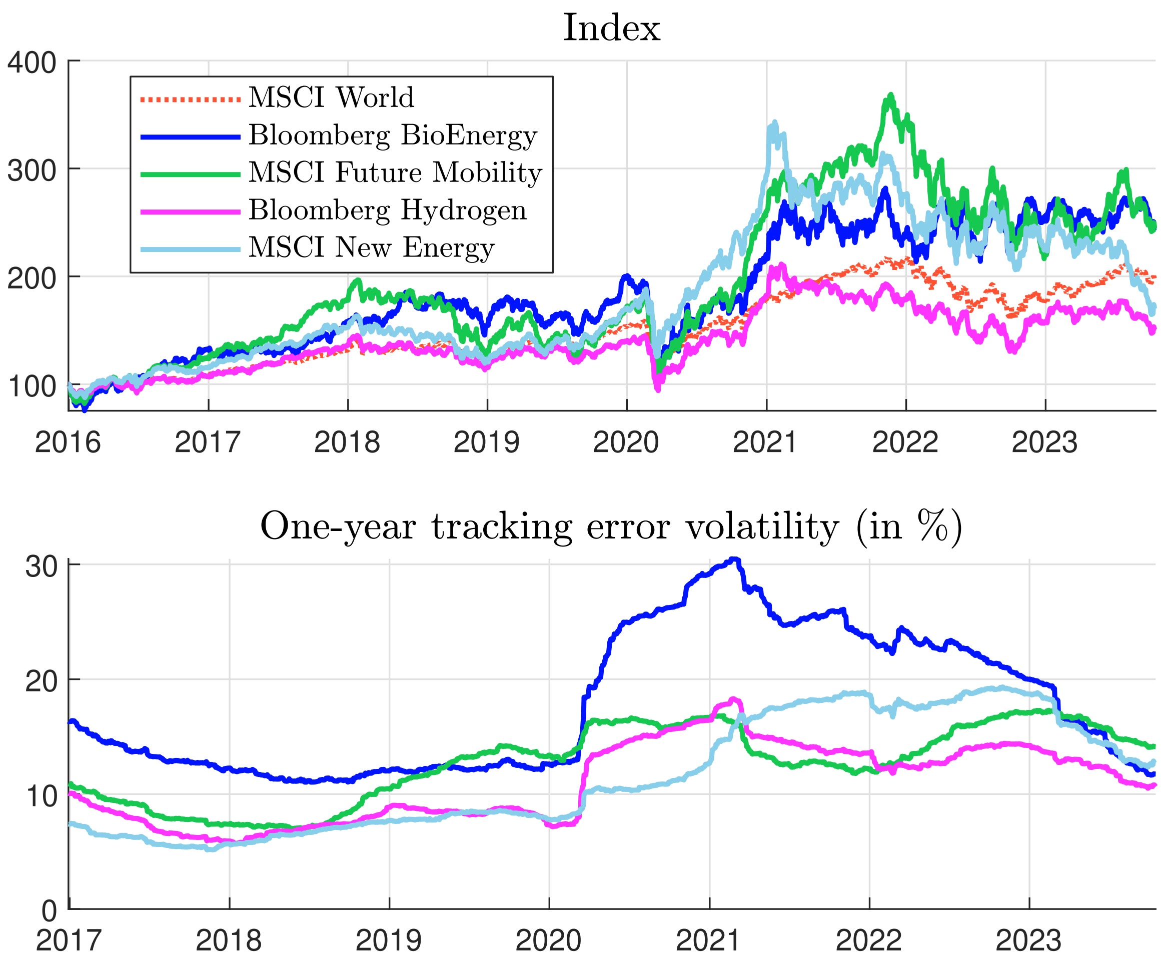
Green bonds

Figure 117: Performance and duration of the Bloomberg Global Green Bond and Aggregate indices



Green stocks

Figure 118: Performance and tracking error volatility of thematic equity indices



Green infrastructure

The European Commission defines green infrastructure as "*a strategically planned network of natural and semi-natural areas with other environmental features, designed and managed to deliver a wide range of ecosystem services, while also enhancing biodiversity*". Green infrastructure is implemented in a variety of sectors, from energy through energy transmission infrastructure, water through natural water retention measures or sustainable urban drainage systems, to the urban landscape with street trees to help sequester carbon or green roofs to help regulate the temperature of buildings. The cost of implementing green infrastructure is in the identification, mapping, planning and creation of the infrastructure, but the environmental, economic and social benefits make it worthwhile. Funds that assess infrastructure needs are emerging in the market and typically invest in owners of sustainable infrastructure assets as well as companies that are leaders in infrastructure investment. In addition to infrastructure funds, investors are also considering direct investments such as green car parks, water infrastructure and flood defences.

Sustainable real estate

- CRREM (Carbon Risk Real Estate Monitor) $\Rightarrow$ whole-building approach for in-use emissions
- GRESB $\Rightarrow$ GHG Protocol principles to the real estate industry (corporate approach)
- SBTi Building Guidelines
- PCAF/CRREM/GRESB joint technical guidance $\Rightarrow$ Accounting and reporting of financed GHG emissions from real estate operations (GHG Protocol)

Allocation process

The stock/bond mix allocation

- Let α_{equity} and α_{bond} be the proportions of stocks and bonds in the multi-asset portfolio
- Let $\alpha^{\mathrm{satellite}}$ be the weight of the satellite portfolio
- The core allocation is given by the vector $\left(\alpha_{\mathrm{equity}}^{\mathrm{core}}, \alpha_{\mathrm{bond}}^{\mathrm{core}}\right)$, while the satellite allocation is defined by $\left(\alpha_{\mathrm{equity}}^{\mathrm{satellite}}, \alpha_{\mathrm{bond}}^{\mathrm{satellite}}\right)$
- We have the following identities:

$$\left\{ \begin{array}{l} \alpha_{\mathrm{equity}} = \left(1 - \alpha^{\mathrm{satellite}}\right) \alpha_{\mathrm{equity}}^{\mathrm{core}} + \alpha^{\mathrm{satellite}} \alpha_{\mathrm{equity}}^{\mathrm{satellite}} \\ \alpha_{\mathrm{bond}} = \left(1 - \alpha^{\mathrm{satellite}}\right) \alpha_{\mathrm{bond}}^{\mathrm{core}} + \left(1 - \alpha^{\mathrm{satellite}}\right) \alpha_{\mathrm{bond}}^{\mathrm{satellite}} \end{array} \right.$$

Allocation process

The stock/bond mix allocation

- In general, the fund manager targets a strategic asset allocation at the portfolio level, *i.e.* the proportions α_{equity} and α_{bond} are given
- For example, a defensive portfolio corresponds to a 20/80 constant mix strategy, while the 50/50 allocation is known as a balanced portfolio. Another famous allocation rule is the 60/40 portfolio, which is 60% in stocks and 40% in bonds.
- The solution is to calculate the proportion of bonds in the core portfolio relative to the proportion of bonds in the satellite portfolio:

$$\alpha_{\text{bond}}^{\text{core}} = \frac{\alpha_{\text{bond}} - \alpha^{\text{satellite}} \alpha_{\text{bond}}^{\text{satellite}}}{1 - \alpha^{\text{satellite}}}$$

Allocation process

The stock/bond mix allocation

Example #11

We consider a 60/40 constant mix strategy. The satellite portfolio represents 10% of the net zero investments. We assume that the satellite portfolio has 70% exposure to green bonds.

Allocation process

The stock/bond mix allocation

We have $\alpha_{\text{equity}} = 60\%$, $\alpha_{\text{bond}} = 40\%$, $\alpha^{\text{core}} = 90\%$, $\alpha^{\text{satellite}} = 10\%$ and $\alpha_{\text{bond}}^{\text{satellite}} = 70\%$. We deduce that:

$$\alpha_{\text{bond}}^{\text{core}} = \frac{0.40 - 0.10 \times 0.70}{1 - 0.10} = \frac{33}{90} = 36.67\%$$

The core allocation is then $(63.33\%, 36.67\%)$, while the satellite allocation is $(30\%, 70\%)$. We check that:

$$\left\{ \begin{array}{l} \alpha_{\text{equity}} = 0.90 \times \left(1 - \dfrac{33}{90}\right) + 0.10 \times 0.30 = 60\% \\ \alpha_{\text{bond}} = 0.90 \times \dfrac{33}{90} + 0.10 \times 0.70 = 40\% \end{array} \right.$$

Allocation process

The stock/bond mix allocation

Table 95: Calculating the bond allocation in the core portfolio ($\alpha_{\text{bond}}^{\text{core}}$ in %)

Strategy		60/40			50/50			20/80		
$\alpha_{\text{bond}}^{\text{satellite}}$		70.0	80.0	90.0	70.0	80.0	90.0	70.0	80.0	90.0
$\alpha^{\text{satellite}}$	0%	40.0	40.0	40.0	50.0	50.0	50.0	80.0	80.0	80.0
	1%	39.7	39.6	39.5	49.8	49.7	49.6	80.1	80.0	79.9
	5%	38.4	37.9	37.4	48.9	48.4	47.9	80.5	80.0	79.5
	10%	36.7	35.6	34.4	47.8	46.7	45.6	81.1	80.0	78.9
	15%	34.7	32.9	31.2	46.5	44.7	42.9	81.8	80.0	78.2
	20%	32.5	30.0	27.5	45.0	42.5	40.0	82.5	80.0	77.5
	25%	30.0	26.7	23.3	43.3	40.0	36.7	83.3	80.0	76.7

Allocation process

Tracking error risk of the core-satellite portfolio

The tracking error volatility of the core-satellite portfolio has the following expression:

$$\sigma(w \mid b) = \sqrt{\tilde{\boldsymbol{\alpha}}^\top \tilde{\boldsymbol{\Sigma}}(w \mid b)\,\tilde{\boldsymbol{\alpha}}} = \sqrt{\left(\tilde{\boldsymbol{\alpha}} \circ \tilde{\boldsymbol{\sigma}}(w \mid b)\right)^\top \tilde{\boldsymbol{\rho}}(w \mid b)\left(\tilde{\boldsymbol{\alpha}} \circ \tilde{\boldsymbol{\sigma}}(w \mid b)\right)}$$

where:

- $\tilde{\boldsymbol{\alpha}}$ is the vector of allocation:

$$\tilde{\boldsymbol{\alpha}} = \begin{pmatrix} \left(1 - \alpha^{\mathrm{satellite}}\right)\alpha^{\mathrm{core}}_{\mathrm{equity}} \\ \left(1 - \alpha^{\mathrm{satellite}}\right)\alpha^{\mathrm{core}}_{\mathrm{bond}} \\ \alpha^{\mathrm{satellite}}\alpha^{\mathrm{satellite}}_{\mathrm{equity}} \\ \alpha^{\mathrm{satellite}}\alpha^{\mathrm{satellite}}_{\mathrm{bond}} \end{pmatrix}$$

- $\tilde{\boldsymbol{\rho}}(w \mid b)$ is the correlation matrix of $R(w) - R(b)$
- $\tilde{\boldsymbol{\sigma}}(w \mid b)$ is the vector of tracking error volatilities:

$$\tilde{\boldsymbol{\sigma}}(w \mid b) = \begin{pmatrix} \sigma\left(w^{\mathrm{core}}_{\mathrm{equity}} \mid b_{\mathrm{equity}}\right) \\ \sigma\left(w^{\mathrm{core}}_{\mathrm{bond}} \mid b_{\mathrm{bond}}\right) \\ \sigma\left(w^{\mathrm{satellite}}_{\mathrm{equity}} \mid b_{\mathrm{equity}}\right) \\ \sigma\left(w^{\mathrm{satellite}}_{\mathrm{bond}} \mid b_{\mathrm{bond}}\right) \end{pmatrix}$$

Allocation process

Tracking error risk of the core-satellite portfolio

Example #12

The tracking error volatilities are 2% for the core equity portfolio, 25 bps for the core bond portfolio, 20% for the satellite equity portfolio, and 3% for the satellite bond portfolio. To define the correlation matrix $\tilde{\rho}(w \mid b)$, we assume an 80% correlation between the two equity baskets, a 50% correlation between the two bond baskets, and a 0% correlation between the equity and bond baskets. We consider a 60/40 constant mix strategy. The satellite portfolio represents 10% of the net zero portfolio and has 70% exposure to green bonds

Allocation process

Tracking error risk of the core-satellite portfolio

We compute the tracking error covariance matrix $\tilde{\mathbf{\Sigma}}(w \mid b)$ as follows:

- The tracking error variance for the core equity portfolio is $\tilde{\mathbf{\Sigma}}_{1,1}(w \mid b) = 0.02^2$
- The tracking error variance for the satellite equity portfolio is $\tilde{\mathbf{\Sigma}}_{3,3}(w \mid b) = 0.20^2$
- The tracking error covariance for the two core portfolios is $\tilde{\mathbf{\Sigma}}_{1,2}(w \mid b) = 0 \times 0.02 \times 0.0025$
- The tracking error covariance for the core equity portfolio and the satellite equity portfolio is $\tilde{\mathbf{\Sigma}}_{1,3}(w \mid b) = 0.80 \times 0.02 \times 0.20$
- Etc.

Finally, we get:

$$\tilde{\mathbf{\Sigma}}(w \mid b) = \begin{pmatrix} 4 & 0 & 32 & 0 \\ 0 & 0.0625 & 0 & 0.375 \\ 32 & 0 & 400 & 0 \\ 0 & 0.375 & 0 & 9 \end{pmatrix} \times 10^{-4}$$

and $\sigma(w \mid b) = 1.68\%$ because $\tilde{\boldsymbol{\alpha}} = (57\%, 33\%, 3\%, 7\%)$

Allocation process

Tracking error risk of the core-satellite portfolio

Table 96: Estimation of the tracking error volatility of the core-satellite portfolio (in %)

	$\alpha^{\text{satellite}}$	Bond	Defensive	Balanced	60/40	Dynamic	Equity
Lower bound	10%	0.38	0.62	1.36	1.62	2.15	2.69
	20%	0.63	1.00	2.18	2.60	3.45	4.31
	30%	0.92	1.43	3.11	3.71	4.93	6.16
Upper bound	10%	0.53	1.18	2.16	2.49	3.15	3.80
	20%	0.80	1.76	3.20	3.68	4.64	5.60
	30%	1.07	2.34	4.24	4.87	6.13	7.40

Allocation process

Tracking error risk of the core-satellite portfolio

$$\frac{\partial \alpha^{\mathrm{satellite}}(t)}{\partial t} \geq 0$$

Course 2023-2024 in Portfolio Allocation and Asset Management
Lecture 6. Equity and Bond Portfolio Optimization with Green Preferences (Exercise)

Thierry Roncalli*

*Amundi Asset Management[23]

*University of Paris-Saclay

January 2024

[23] The opinions expressed in this presentation are those of the authors and are not meant to represent the opinions or official positions of Amundi Asset Management.

Agenda

- Lecture 1: Portfolio Optimization
- Lecture 2: Risk Budgeting
- Lecture 3: Smart Beta, Factor Investing and Alternative Risk Premia
- Lecture 4: Equity Portfolio Optimization with ESG Scores
- Lecture 5: Climate Portfolio Construction
- **Lecture 6: Equity and Bond Portfolio Optimization with Green Preferences**
- Lecture 7: Machine Learning in Asset Management

We consider an investment universe of 8 issuers. In the table below, we report the carbon emissions $\mathcal{CE}_{i,j}$ (in $\mathrm{ktCO_2e}$) of these companies and their revenues Y_i (in \$ bn), and we indicate in the last row whether the company belongs to sector $\mathcal{S}ector_1$ or $\mathcal{S}ector_2$:

Issuer	#1	#2	#3	#4	#5	#6	#7	#8
$\mathcal{CE}_{i,1}$	75	5 000	720	50	2 500	25	30 000	5
$\mathcal{CE}_{i,2}$	75	5 000	1 030	350	4 500	5	2000	64
$\mathcal{CE}_{i,3}$	24 000	15 000	1 210	550	500	187	30 000	199
Y_i	300	328	125	100	200	102	107	25
$\mathcal{S}ector$	1	2	1	1	2	1	2	2

The benchmark b of this investment universe is defined as:

$$b = (22\%, 19\%, 17\%, 13\%, 11\%, 8\%, 6\%, 4\%)$$

In what follows, we consider long-only portfolios.

Question 1

We want to compute the carbon intensity of the benchmark.

Question (a)

Compute the carbon intensities $\mathcal{CI}_{i,j}$ of each company i for the scopes 1, 2 and 3.

We have:

$$\mathcal{CI}_{i,j} = \frac{\mathcal{CE}_{i,j}}{Y_i}$$

For instance, if we consider the 8^{th} issuer, we have[24]:

$$\begin{aligned}
\mathcal{CI}_{8,1} &= \frac{\mathcal{CE}_{8,1}}{Y_8} = \frac{5}{25} = 0.20 \text{ tCO}_2\text{e/\$ mn} \\
\mathcal{CI}_{8,2} &= \frac{\mathcal{CE}_{8,2}}{Y_8} = \frac{64}{25} = 2.56 \text{ tCO}_2\text{e/\$ mn} \\
\mathcal{CI}_{8,3} &= \frac{\mathcal{CE}_{8,3}}{Y_8} = \frac{199}{25} = 7.96 \text{ tCO}_2\text{e/\$ mn}
\end{aligned}$$

[24]Because 1 $\text{ktCO}_2\text{e/\$}$ bn $= 1$ $\text{tCO}_2\text{e/\$ mn}$.

Since we have:

Issuer	#1	#2	#3	#4	#5	#6	#7	#8
$\mathcal{CE}_{i,1}$	75	5 000	720	50	2 500	25	30 000	5
$\mathcal{CE}_{i,2}$	75	5 000	1 030	350	4 500	5	2000	64
$\mathcal{CE}_{i,3}$	24 000	15 000	1 210	550	500	187	30 000	199
Y_i	300	328	125	100	200	102	107	25

we obtain:

Issuer	#1	#2	#3	#4	#5	#6	#7	#8
$\mathcal{CI}_{i,1}$	0.25	15.24	5.76	0.50	12.50	0.25	280.37	0.20
$\mathcal{CI}_{i,2}$	0.25	15.24	8.24	3.50	22.50	0.05	18.69	2.56
$\mathcal{CI}_{i,3}$	80.00	45.73	9.68	5.50	2.50	1.83	280.37	7.96

Question (b)

Deduce the carbon intensities $\mathcal{CI}_{i,j}$ of each company i for the scopes $1+2$ and $1+2+3$.

We have:

$$\mathcal{CI}_{i,1-2} = \frac{\mathcal{CE}_{i,1} + \mathcal{CE}_{i,2}}{Y_i} = \mathcal{CI}_{i,1} + \mathcal{CI}_{i,2}$$

and:

$$\mathcal{CI}_{i,1-3} = \mathcal{CI}_{i,1} + \mathcal{CI}_{i,2} + \mathcal{CI}_{i,3}$$

We deduce that:

Issuer	#1	#2	#3	#4	#5	#6	#7	#8
$\mathcal{CI}_{i,1}$	0.25	15.24	5.76	0.50	12.50	0.25	280.37	0.20
$\mathcal{CI}_{i,1-2}$	0.50	30.49	14.00	4.00	35.00	0.29	299.07	2.76
$\mathcal{CI}_{i,1-3}$	80.50	76.22	23.68	9.50	37.50	2.12	579.44	10.72

Question (c)

Deduce the weighted average carbon intensity (WACI) of the benchmark if we consider the scope $1 + 2 + 3$.

We have:

$$
\begin{aligned}
\mathcal{CI}(b) &= \sum_{i=1}^{8} b_i \mathcal{CI}_i \\
&= 0.22 \times 80.50 + 0.19 \times 76.2195 + 0.17 \times 23.68 + 0.13 \times 9.50 + \\
&\quad 0.11 \times 37.50 + 0.08 \times 2.1275 + 0.06 \times 579.4393 + 0.04 \times 10.72 \\
&= 76.9427 \text{ tCO}_2\text{e/\$ mn}
\end{aligned}
$$

Question (d)

We assume that the market capitalization of the benchmark portfolio is equal to \$10 tn and we invest \$1 bn.

Question (d).i

Deduce the market capitalization of each company (expressed in \$ bn).

We have:

$$b_i = \frac{MC_i}{\sum_{k=1}^{8} \mathrm{MC}_k}$$

and $\sum_{k=1}^{8} \mathrm{MC}_k = \$10$ tn. We deduce that:

$$\mathrm{MC}_i = 10 \times b_i$$

We obtain the following values of market capitalization expressed in \$ bn:

Issuer	#1	#2	#3	#4	#5	#6	#7	#8
MC_i	2 200	1 900	1 700	1 300	1 100	800	600	400

Question (d).ii

Compute the ownership ratio for each asset (expressed in bps).

Let W be the wealth invested in the benchmark portfolio b. The wealth invested in asset i is equal to $b_i W$. We deduce that the ownership ratio is equal to:

$$\varpi_i = \frac{b_i W}{\mathrm{MC}_i} = \frac{b_i W}{b_i \sum_{k=1}^{n} \mathrm{MC}_k} = \frac{W}{\sum_{k=1}^{n} \mathrm{MC}_k}$$

When we invest in a capitalization-weighted portfolio, the ownership ratio is the same for all the assets. In our case, we have:

$$\varpi_i = \frac{1}{10 \times 1000} = 0.01\%$$

The ownership ratio is equal to 1 basis point.

Question (d).iii

Compute the carbon emissions of the benchmark portfolio[a] if we invest $1 bn and we consider the scope $1 + 2 + 3$.

[a] We assume that the float percentage is equal to 100% for all the 8 companies.

Using the financed emissions approach, the carbon emissions of our investment is equal to:

$$\begin{array}{rcl}\mathcal{CE}\,(\$1\text{ bn}) & = & 0.01\% \times (75 + 75 + 24\,000) + \\ & & 0.01\% \times (5\,000 + 5\,000 + 15\,000) + \\ & & \ldots + \\ & & 0.01\% \times (5 + 64 + 199) \\ & = & 12.3045 \text{ ktCO}_2\text{e}\end{array}$$

Question (d).iv

Compare the (exact) carbon intensity of the benchmark portfolio with the WACI value obtained in Question 1.(c).

We compute the revenues of our investment:

$$\mathsf{Y}\left(\$1\text{ bn}\right) = 0.01\% \sum_{i=1}^{8} Y_i = \$0.1287\text{ bn}$$

We deduce that the exact carbon intensity is equal to:

$$\mathcal{CI}\left(\$1\text{ bn}\right) = \frac{\mathcal{CE}\left(\$1\text{ bn}\right)}{Y\left(\$1\text{ bn}\right)} = \frac{12.3045}{0.1287} = 95.6061 \text{ tCO}_2\text{e}/\$\text{ mn}$$

We notice that the WACI of the benchmark underestimates the exact carbon intensity of our investment by 19.5%:

$$76.9427 < 95.6061$$

Question 2

We want to manage an equity portfolio with respect to the previous investment universe and reduce the weighted average carbon intensity of the benchmark by the rate $\mathcal{R}$. We assume that the volatility of the stocks is respectively equal to 22%, 20%, 25%, 18%, 40%, 23%, 13% and 29%. The correlation matrix between these stocks is given by:

$$\rho = \begin{pmatrix} 100\% & & & & & & & \\ 80\% & 100\% & & & & & & \\ 70\% & 75\% & 100\% & & & & & \\ 60\% & 65\% & 80\% & 100\% & & & & \\ 70\% & 50\% & 70\% & 85\% & 100\% & & & \\ 50\% & 60\% & 70\% & 80\% & 60\% & 100\% & & \\ 70\% & 50\% & 70\% & 75\% & 80\% & 50\% & 100\% & \\ 60\% & 65\% & 70\% & 75\% & 65\% & 70\% & 60\% & 100\% \end{pmatrix}$$

Question (a)

Compute the covariance matrix Σ.

The covariance matrix $\Sigma = (\Sigma_{i,j})$ is defined by:

$$\Sigma_{i,j} = \rho_{i,j}\sigma_i\sigma_j$$

We obtain the following numerical values (expressed in bps):

$$\Sigma = \begin{pmatrix}
484.0 & 352.0 & 385.0 & 237.6 & 616.0 & 253.0 & 200.2 & 382.8 \\
352.0 & 400.0 & 375.0 & 234.0 & 400.0 & 276.0 & 130.0 & 377.0 \\
385.0 & 375.0 & 625.0 & 360.0 & 700.0 & 402.5 & 227.5 & 507.5 \\
237.6 & 234.0 & 360.0 & 324.0 & 612.0 & 331.2 & 175.5 & 391.5 \\
616.0 & 400.0 & 700.0 & 612.0 & 1600.0 & 552.0 & 416.0 & 754.0 \\
253.0 & 276.0 & 402.5 & 331.2 & 552.0 & 529.0 & 149.5 & 466.9 \\
200.2 & 130.0 & 227.5 & 175.5 & 416.0 & 149.5 & 169.0 & 226.2 \\
382.8 & 377.0 & 507.5 & 391.5 & 754.0 & 466.9 & 226.2 & 841.0
\end{pmatrix}$$

Question (b)

Write the optimization problem if the objective function is to minimize the tracking error risk under the constraint of carbon intensity reduction.

The tracking error variance of portfolio w with respect to benchmark b is equal to:

$$\sigma^2(w \mid b) = (w - b)^\top \Sigma (w - b)$$

The carbon intensity constraint has the following expression:

$$\sum_{i=1}^{8} w_i \mathcal{CI}_i \leq (1 - \mathcal{R}) \mathcal{CI}(b)$$

where $\mathcal{R}$ is the reduction rate and $\mathcal{CI}(b)$ is the carbon intensity of the benchmark. Let $\mathcal{CI}^\star = (1 - \mathcal{R}) \mathcal{CI}(b)$ be the target value of the carbon footprint. The optimization problem is then:

$$\begin{aligned} w^\star &= \arg\min \frac{1}{2}\sigma^2(w \mid b) \\ &\text{s.t.} \left\{ \begin{array}{l} \sum_{i=1}^{8} w_i \mathcal{CI}_i \leq \mathcal{CI}^\star \\ \sum_{i=1}^{8} w_i = 1 \\ 0 \leq w_i \leq 1 \end{array} \right. \end{aligned}$$

We add the second and third constraints in order to obtain a long-only portfolio.

Question (c)

Give the QP formulation of the optimization problem.

The objective function is equal to:

$$f(w) = \frac{1}{2}\sigma^2(w \mid b) = \frac{1}{2}(w-b)^\top \Sigma (w-b) = \frac{1}{2}w^\top \Sigma w - w^\top \Sigma b + \frac{1}{2} b^\top \Sigma b$$

while the matrix form of the carbon intensity constraint is:

$$\boldsymbol{\mathcal{CI}}^\top w \leq \boldsymbol{\mathcal{CI}}^\star$$

where $\boldsymbol{\mathcal{CI}} = (\boldsymbol{\mathcal{CI}}_1, \ldots, \boldsymbol{\mathcal{CI}}_8)$ is the column vector of carbon intensities. Since $b^\top \Sigma b$ is a constant and does not depend on w, we can cast the previous optimization problem into a QP problem:

$$\begin{aligned} w^\star &= \arg\min \frac{1}{2} w^\top Q w - w^\top R \\ & \text{s.t.} \quad \left\{ \begin{array}{l} Aw = B \\ Cw \leq D \\ w^- \leq w \leq w^+ \end{array} \right. \end{aligned}$$

We have $Q = \Sigma$, $R = \Sigma b$, $A = \mathbf{1}_8^\top$, $B = 1$, $C = \boldsymbol{\mathcal{CI}}^\top$, $D = \boldsymbol{\mathcal{CI}}^\star$, $w^- = \mathbf{0}_8$ and $w^+ = \mathbf{1}_8$.

Question (d)

$\mathcal{R}$ is equal to 20%. Find the optimal portfolio if we target scope $1 + 2$. What is the value of the tracking error volatility?

We have:

$$\begin{aligned}\mathcal{CI}(b) &= 0.22 \times 0.50 + 0.19 \times 30.4878 + \ldots + 0.04 \times 2.76 \\ &= 30.7305 \text{ tCO}_2\text{e/\$ mn}\end{aligned}$$

We deduce that:

$$\mathcal{CI}^{\star} = (1 - \mathcal{R})\,\mathcal{CI}(b) = 0.80 \times 30.7305 = 24.5844 \text{ tCO}_2\text{e/\$ mn}$$

Therefore, the inequality constraint of the QP problem is:

$$\begin{pmatrix} 0.50 & 30.49 & 14.00 & 4.00 & 35.00 & 0.29 & 299.07 & 2.76 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_7 \\ w_8 \end{pmatrix} \leq 24.5844$$

We obtain the following optimal solution:

$$w^{\star} = \begin{pmatrix} 23.4961\% \\ 17.8129\% \\ 17.1278\% \\ 15.4643\% \\ 10.4037\% \\ 7.5903\% \\ 4.0946\% \\ 4.0104\% \end{pmatrix}$$

The minimum tracking error volatility $\sigma(w^{\star} \mid b)$ is equal to 15.37 bps.

Question (e)

Same question if $\mathcal{R}$ is equal to 30%, 50%, and 70%.

Table 97: Solution of the equity optimization problem (scope $\mathcal{SC}_{1-2}$)

$\mathcal{R}$	0%	20%	30%	50%	70%
w_1	22.0000	23.4961	24.2441	25.7402	30.4117
w_2	19.0000	17.8129	17.2194	16.0323	9.8310
w_3	17.0000	17.1278	17.1917	17.3194	17.8348
w_4	13.0000	15.4643	16.6964	19.1606	23.3934
w_5	11.0000	10.4037	10.1055	9.5091	7.1088
w_6	8.0000	7.5903	7.3854	6.9757	6.7329
w_7	6.0000	4.0946	3.1418	1.2364	0.0000
w_8	4.0000	4.0104	4.0157	4.0261	4.6874
$\mathcal{CI}(w)$	30.7305	24.5844	21.5114	15.3653	9.2192
$\sigma(w \mid b)$	0.00	15.37	23.05	38.42	72.45

In Table 97, we report the optimal solution $w^{\star}$ (expressed in %) of the optimization problem for different values of $\mathcal{R}$. We also indicate the carbon intensity of the portfolio (in $\mathrm{tCO_2e}/\$\ \mathrm{mn}$) and the tracking error volatility (in bps). For instance, if $\mathcal{R}$ is set to 50%, the weights of assets #1, #3, #4 and #8 increase whereas the weights of assets #2, #5, #6 and #7 decrease. The carbon intensity of this portfolio is equal to 15.3653 $\mathrm{tCO_2e}/\$\ \mathrm{mn}$. The tracking error volatility is below 40 bps, which is relatively low.

Question (f)

We target scope $1 + 2 + 3$. Find the optimal portfolio if $\mathcal{R}$ is equal to $20\%, 30\%$, 50% and 70%. Give the value of the tracking error volatility for each optimized portfolio.

In this case, the inequality constraint $Cw \leq D$ is defined by:

$$C = \mathcal{CI}_{1-3}^{\top} = \begin{pmatrix} 80.5000 \\ 76.2195 \\ 23.6800 \\ 9.5000 \\ 37.5000 \\ 2.1275 \\ 579.4393 \\ 10.7200 \end{pmatrix}^{\top}$$

and:

$$D = (1 - \mathcal{R}) \times 76.9427$$

We obtain the results given in Table 98.

Table 98: Solution of the equity optimization problem (scope $\mathcal{SC}_{1-3}$)

$\mathcal{R}$	0%	20%	30%	50%	70%
w_1	22.0000	23.9666	24.9499	26.4870	13.6749
w_2	19.0000	17.4410	16.6615	8.8001	0.0000
w_3	17.0000	17.1988	17.2981	19.4253	24.1464
w_4	13.0000	16.5034	18.2552	25.8926	41.0535
w_5	11.0000	10.2049	9.8073	7.1330	3.5676
w_6	8.0000	7.4169	7.1254	7.0659	8.8851
w_7	6.0000	3.2641	1.8961	0.0000	0.0000
w_8	4.0000	4.0043	4.0065	5.1961	8.6725
$\mathcal{CI}(w)$	76.9427	61.5541	53.8599	38.4713	23.0828
$\sigma(w \mid b)$	0.00	21.99	32.99	104.81	414.48

Question (g)

Compare the optimal solutions obtained in Questions 2.(e) and 2.(f).

Figure 119: Impact of the scope on the tracking error volatility

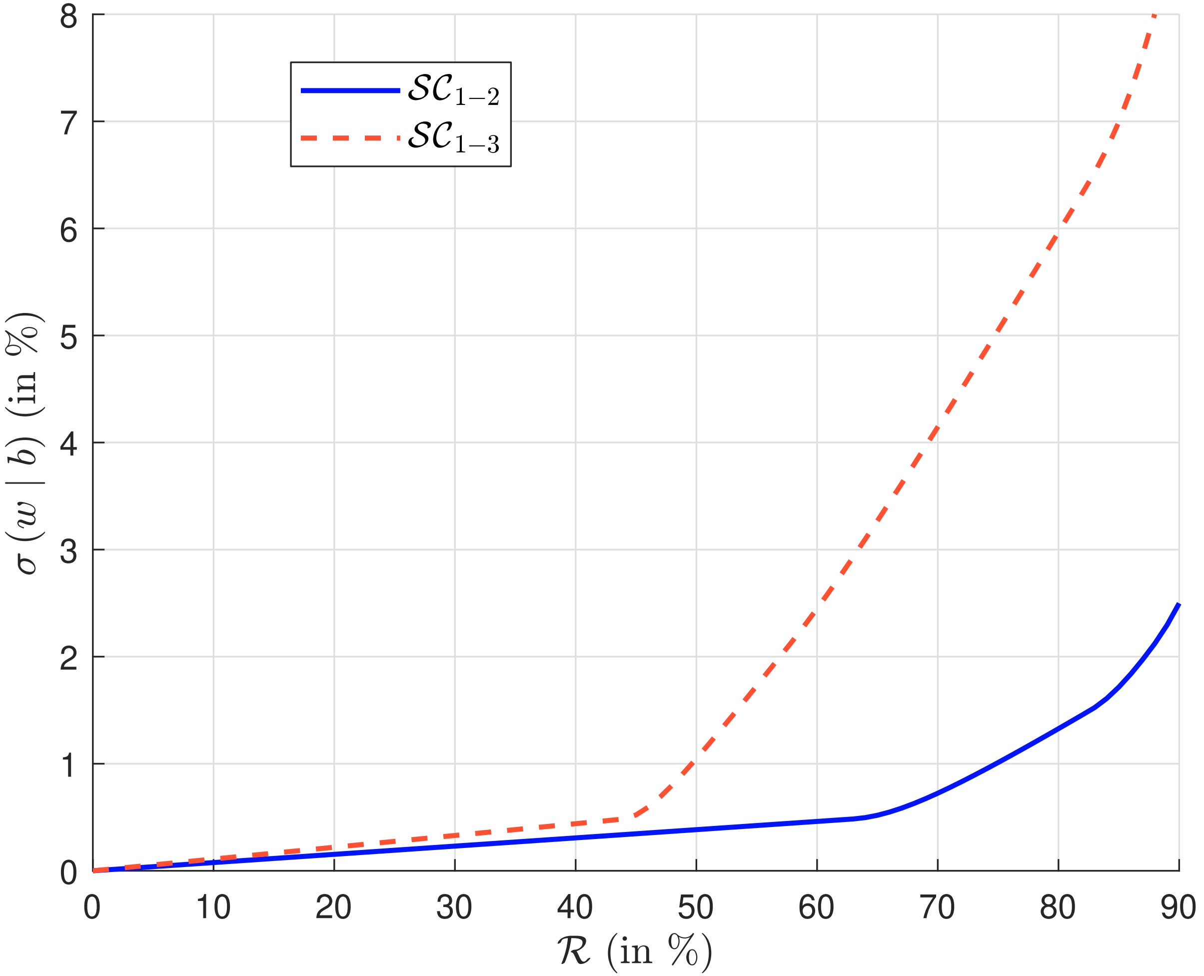
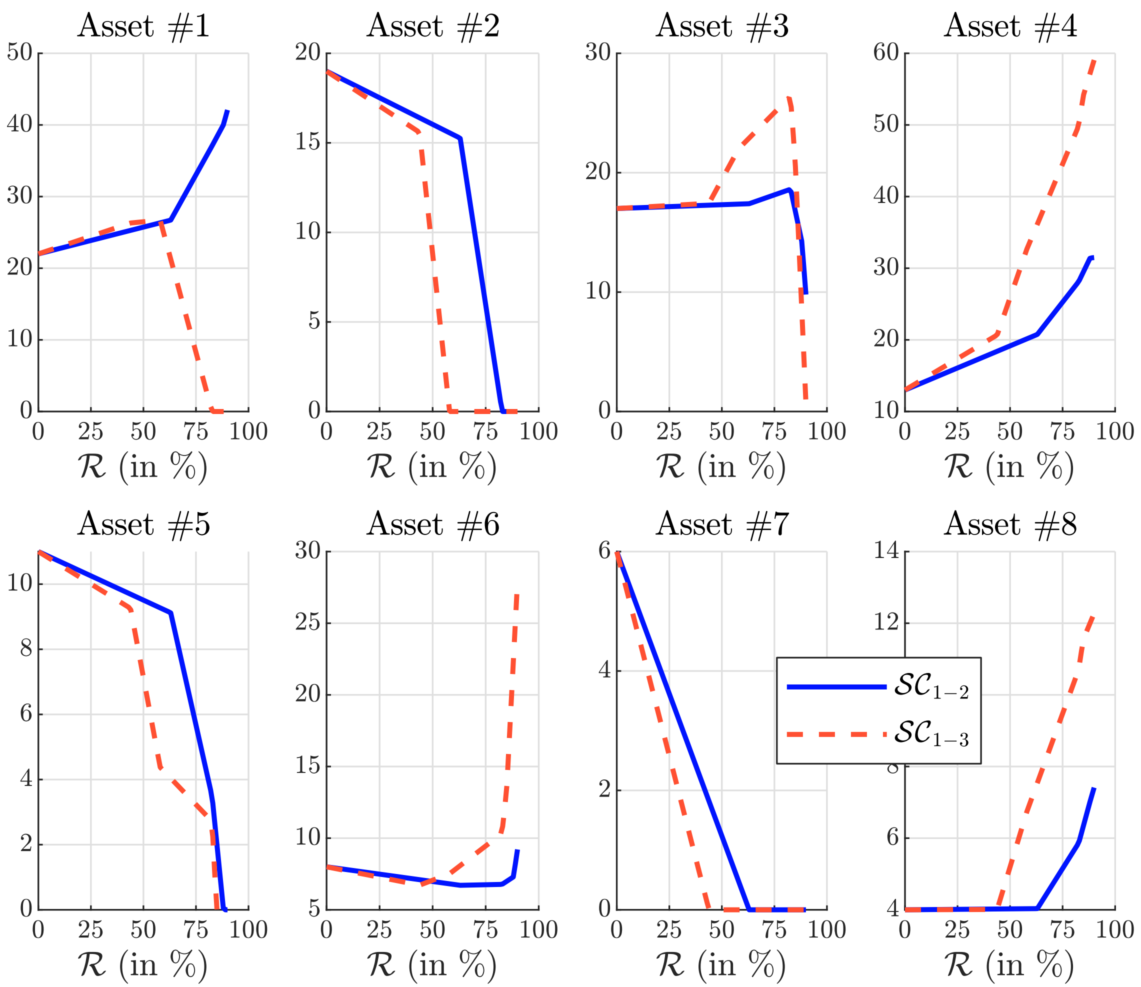


Figure 120: Impact of the scope on the portfolio allocation (in %)



In Figure 119, we report the relationship between the reduction rate $\mathcal{R}$ and the tracking error volatility $\sigma(w \mid b)$. The choice of the scope has little impact when $\mathcal{R} \leq 45\%$. Then, we notice a high increase when we consider the scope $1+2+3$. The portfolio's weights are given in Figure 120. For assets #1 and #3, the behavior is divergent when we compare scopes $1+2$ and $1+2+3$.

Question 3

We want to manage a bond portfolio with respect to the previous investment universe and reduce the weighted average carbon intensity of the benchmark by the rate $\mathcal{R}$. We use the scope $1 + 2 + 3$. In the table below, we report the modified duration MD_i and the duration-times-spread factor DTS_i of each corporate bond i:

Asset	#1	#2	#3	#4	#5	#6	#7	#8
MD_i (in years)	3.56	7.48	6.54	10.23	2.40	2.30	9.12	7.96
DTS_i (in bps)	103	155	75	796	89	45	320	245
$\mathcal{S}$*ector*	1	2	1	1	2	1	2	2

Question 3 (Cont'd)

We remind that the active risk can be calculated using three functions. For the active share, we have:

$$\mathcal{R}_{\mathrm{AS}}(w \mid b) = \sigma^2_{\mathrm{AS}}(w \mid b) = \sum_{i=1}^{n} (w_i - b_i)^2$$

We also consider the MD-based tracking error risk:

$$\mathcal{R}_{\mathrm{MD}}(w \mid b) = \sigma^2_{\mathrm{MD}}(w \mid b) = \sum_{j=1}^{n_{\mathcal{S}ector}} \left(\sum_{i \in \mathcal{S}ector_j} (w_i - b_i)\, \mathrm{MD}_i \right)^2$$

and the DTS-based tracking error risk:

$$\mathcal{R}_{\mathrm{DTS}}(w \mid b) = \sigma^2_{\mathrm{DTS}}(w \mid b) = \sum_{j=1}^{n_{\mathcal{S}ector}} \left(\sum_{i \in \mathcal{S}ector_j} (w_i - b_i)\, \mathrm{DTS}_i \right)^2$$

Question 3 (Cont'd)

Finally, we define the synthetic risk measure as a combination of AS, MD and DTS active risks:

$$\mathcal{R}(w \mid b) = \varphi_{\mathrm{AS}} \mathcal{R}_{\mathrm{AS}}(w \mid b) + \varphi_{\mathrm{MD}} \mathcal{R}_{\mathrm{MD}}(w \mid b) + \varphi_{\mathrm{DTS}} \mathcal{R}_{\mathrm{DTS}}(w \mid b)$$

where $\varphi_{\mathrm{AS}} \geq 0$, $\varphi_{\mathrm{MD}} \geq 0$ and $\varphi_{\mathrm{DTS}} \geq 0$ indicate the weight of each risk. In what follows, we use the following numerical values: $\varphi_{\mathrm{AS}} = 100$, $\varphi_{\mathrm{MD}} = 25$ and $\varphi_{\mathrm{DTS}} = 1$. The reduction rate $\boldsymbol{\mathcal{R}}$ of the weighted average carbon intensity is set to 50% for the scope $1 + 2 + 3$.

Question (a)

Compute the modified duration $\mathrm{MD}\,(b)$ and the duration-times-spread factor $\mathrm{DTS}\,(b)$ of the benchmark.

We have:

$$\begin{aligned}\mathrm{MD}(b) &= \sum_{i=1}^{n} b_i \, \mathrm{MD}_i \\ &= 0.22 \times 3.56 + 0.19 \times 7.48 + \ldots + 0.04 \times 7.96 \\ &= 5.96 \text{ years}\end{aligned}$$

and:

$$\begin{aligned}\mathrm{DTS}(b) &= \sum_{i=1}^{n} b_i \, \mathrm{DTS}_i \\ &= 0.22 \times 103 + 0.19 \times 155 + \ldots + 0.04 \times 155 \\ &= 210.73 \text{ bps}\end{aligned}$$

Question (b)

Let w_{ew} be the equally-weighted portfolio. Compute[a] $\mathrm{MD}\left(w_{\mathrm{ew}}\right)$, $\mathrm{DTS}\left(w_{\mathrm{ew}}\right)$, $\sigma_{\mathrm{AS}}\left(w_{\mathrm{ew}} \mid b\right)$, $\sigma_{\mathrm{MD}}\left(w_{\mathrm{ew}} \mid b\right)$ and $\sigma_{\mathrm{DTS}}\left(w_{\mathrm{ew}} \mid b\right)$.

[a] Precise the corresponding unit (years, bps or %) for each metric.

We have:

$$\left\{\begin{array}{l}\mathrm{MD}\left(w_{\mathrm{ew}}\right)=6.20 \text{ years} \\ \mathrm{DTS}\left(w_{\mathrm{ew}}\right)=228.50 \text{ bps} \\ \sigma_{\mathrm{AS}}\left(w_{\mathrm{ew}} \mid b\right)=17.03\% \\ \sigma_{\mathrm{MD}}\left(w_{\mathrm{ew}} \mid b\right)=1.00 \text{ years} \\ \sigma_{\mathrm{DTS}}\left(w_{\mathrm{ew}} \mid b\right)=36.19 \text{ bps}\end{array}\right.$$

Question (c)

We consider the following optimization problem:

$$
\begin{aligned}
w^{\star} &= \arg\min \frac{1}{2}\mathcal{R}_{\mathrm{AS}}\left(w \mid b\right) \\
&\text{s.t.} \left\{
\begin{array}{l}
\sum_{i=1}^{n} w_i = 1 \\
\mathrm{MD}\left(w\right) = \mathrm{MD}\left(b\right) \\
\mathrm{DTS}\left(w\right) = \mathrm{DTS}\left(b\right) \\
\boldsymbol{\mathcal{CI}}\left(w\right) \leq \left(1 - \boldsymbol{\mathcal{R}}\right) \boldsymbol{\mathcal{CI}}\left(b\right) \\
0 \leq w_i \leq 1
\end{array}
\right.
\end{aligned}
$$

Give the analytical value of the objective function. Find the optimal portfolio $w^{\star}$. Compute $\mathrm{MD}\left(w^{\star}\right)$, $\mathrm{DTS}\left(w^{\star}\right)$, $\sigma_{\mathrm{AS}}\left(w^{\star} \mid b\right)$, $\sigma_{\mathrm{MD}}\left(w^{\star} \mid b\right)$ and $\sigma_{\mathrm{DTS}}\left(w^{\star} \mid b\right)$.

We have:

$$\begin{array}{rcl}\mathcal{R}_{\mathrm{AS}}(w \mid b) & = & (w_1 - 0.22)^2 + (w_2 - 0.19)^2 + (w_3 - 0.17)^2 + (w_4 - 0.13)^2 + \\ & & (w_5 - 0.11)^2 + (w_6 - 0.08)^2 + (w_7 - 0.06)^2 + (w_8 - 0.04)^2\end{array}$$

The objective function is then:

$$f(w) = \frac{1}{2}\mathcal{R}_{\mathrm{AS}}(w \mid b)$$

The optimal solution is equal to:

$$w^\star = (17.30\%, 17.41\%, 20.95\%, 14.41\%, 10.02\%, 11.09\%, 0\%, 8.81\%)$$

The risk metrics are:

$$\left\{\begin{array}{l}\mathrm{MD}(w^\star) = 5.96 \text{ years} \\ \mathrm{DTS}(w^\star) = 210.73 \text{ bps} \\ \sigma_{\mathrm{AS}}(w^\star \mid b) = 10.57\% \\ \sigma_{\mathrm{MD}}(w^\star \mid b) = 0.43 \text{ years} \\ \sigma_{\mathrm{DTS}}(w^\star \mid b) = 15.21 \text{ bps}\end{array}\right.$$

Question (d)

We consider the following optimization problem:

$$\begin{aligned} w^{\star} &= \arg\min \frac{\varphi_{\mathrm{AS}}}{2} \mathcal{R}_{\mathrm{AS}}(w \mid b) + \frac{\varphi_{\mathrm{MD}}}{2} \mathcal{R}_{\mathrm{MD}}(w \mid b) \\ &\text{s.t.} \quad \left\{ \begin{array}{l} \sum_{i=1}^{n} w_i = 1 \\ \mathrm{DTS}(w) = \mathrm{DTS}(b) \\ \boldsymbol{\mathcal{CI}}(w) \leq (1 - \boldsymbol{\mathcal{R}}) \boldsymbol{\mathcal{CI}}(b) \\ 0 \leq w_i \leq 1 \end{array} \right. \end{aligned}$$

Give the analytical value of the objective function. Find the optimal portfolio $w^{\star}$. Compute $\mathrm{MD}(w^{\star})$, $\mathrm{DTS}(w^{\star})$, $\sigma_{\mathrm{AS}}(w^{\star} \mid b)$, $\sigma_{\mathrm{MD}}(w^{\star} \mid b)$ and $\sigma_{\mathrm{DTS}}(w^{\star} \mid b)$.

We have[25]:

$$\begin{aligned}
\mathcal{R}_{\mathrm{MD}}(w \mid b) &= \left(\sum_{i=1,3,4,6}(w_i - b_i)\,\mathrm{MD}_i\right)^2 + \left(\sum_{i=2,5,7,8}(w_i - b_i)\,\mathrm{MD}_i\right)^2 \\
&= \left(\sum_{i=1,3,4,6} w_i\,\mathrm{MD}_i - \sum_{i=1,3,4,6} b_i\,\mathrm{MD}_i\right)^2 + \\
&\quad \left(\sum_{i=2,5,7,8} w_i\,\mathrm{MD}_i - \sum_{i=2,5,7,8} b_i\,\mathrm{MD}_i\right)^2 \\
&= (3.56w_1 + 6.54w_3 + 10.23w_4 + 2.30w_6 - 3.4089)^2 + \\
&\quad (7.48w_2 + 2.40w_5 + 9.12w_7 + 7.96w_8 - 2.5508)^2
\end{aligned}$$

The objective function is then:

$$f(w) = \frac{\varphi_{\mathrm{AS}}}{2}\mathcal{R}_{\mathrm{AS}}(w \mid b) + \frac{\varphi_{\mathrm{MD}}}{2}\mathcal{R}_{\mathrm{MD}}(w \mid b)$$

[25] We verify that $3.4089 + 2.5508 = 5.9597$ years.

The optimal solution is equal to:

$$w^{\star} = (16.31\%, 18.44\%, 17.70\%, 13.82\%, 11.67\%, 11.18\%, 0\%, 10.88\%)$$

The risk metrics are:

$$\left\{\begin{array}{l} \mathrm{MD}\left(w^{\star}\right) = 5.93 \text{ years} \\ \mathrm{DTS}\left(w^{\star}\right) = 210.73 \text{ bps} \\ \sigma_{\mathrm{AS}}\left(w^{\star} \mid b\right) = 11.30\% \\ \sigma_{\mathrm{MD}}\left(w^{\star} \mid b\right) = 0.03 \text{ years} \\ \sigma_{\mathrm{DTS}}\left(w^{\star} \mid b\right) = 3.70 \text{ bps} \end{array}\right.$$

Question (e)

We consider the following optimization problem:

$$
\begin{aligned}
w^{\star} &= \arg\min \frac{1}{2}\mathcal{R}\left(w \mid b\right) \\
&\text{s.t.} \quad \left\{
\begin{array}{l}
\sum_{i=1}^{n} w_i = 1 \\
\mathcal{CI}\left(w\right) \leq \left(1 - \boldsymbol{\mathcal{R}}\right) \mathcal{CI}\left(b\right) \\
0 \leq w_i \leq 1
\end{array}
\right.
\end{aligned}
$$

Give the analytical value of the objective function. Find the optimal portfolio $w^{\star}$. Compute $\mathrm{MD}\left(w^{\star}\right)$, $\mathrm{DTS}\left(w^{\star}\right)$, $\sigma_{\mathrm{AS}}\left(w^{\star} \mid b\right)$, $\sigma_{\mathrm{MD}}\left(w^{\star} \mid b\right)$ and $\sigma_{\mathrm{DTS}}\left(w^{\star} \mid b\right)$.

We have[26]:

$$\begin{aligned}
\mathcal{R}_{\mathrm{DTS}}(w \mid b) &= \left(\sum_{i=1,3,4,6}(w_i - b_i)\,\mathrm{DTS}_i\right)^2 + \left(\sum_{i=2,5,7,8}(w_i - b_i)\,\mathrm{DTS}_i\right)^2 \\
&= (103w_1 + 75w_3 + 796w_4 + 45w_6 - 142.49)^2 + \\
&\quad (155w_2 + 89w_5 + 320w_7 + 245w_8 - 68.24)^2
\end{aligned}$$

The objective function is then:

$$f(w) = \frac{\varphi_{\mathrm{AS}}}{2}\mathcal{R}_{\mathrm{AS}}(w \mid b) + \frac{\varphi_{\mathrm{MD}}}{2}\mathcal{R}_{\mathrm{MD}}(w \mid b) + \frac{\varphi_{\mathrm{DTS}}}{2}\mathcal{R}_{\mathrm{DTS}}(w \mid b)$$

[26] We verify that $142.49 + 68.24 = 210.73$ bps.

The optimal solution is equal to:

$$w^\star = (16.98\%, 17.21\%, 18.26\%, 13.45\%, 12.10\%, 9.46\%, 0\%, 12.55\%)$$

The risk metrics are:

$$\left\{\begin{array}{l} \mathrm{MD}(w^\star) = 5.97 \text{ years} \\ \mathrm{DTS}(w^\star) = 210.68 \text{ bps} \\ \sigma_{\mathrm{AS}}(w^\star \mid b) = 11.94\% \\ \sigma_{\mathrm{MD}}(w^\star \mid b) = 0.03 \text{ years} \\ \sigma_{\mathrm{DTS}}(w^\star \mid b) = 0.06 \text{ bps} \end{array}\right.$$

Question (f)

Comment on the results obtained in Questions 3.(c), 3.(d) and 3.(e).

Table 99: Solution of the bond optimization problem (scope $\mathcal{SC}_{1-3}$)

Problem	Benchmark	3.(c)	3.(d)	3.(e)
w_1	22.0000	17.3049	16.3102	16.9797
w_2	19.0000	17.4119	18.4420	17.2101
w_3	17.0000	20.9523	17.6993	18.2582
w_4	13.0000	14.4113	13.8195	13.4494
w_5	11.0000	10.0239	11.6729	12.1008
w_6	8.0000	11.0881	11.1792	9.4553
w_7	6.0000	0.0000	0.0000	0.0000
w_8	4.0000	8.8075	10.8769	12.5464
$\mathrm{MD}(w)$	5.9597	5.9597	5.9344	5.9683
$\mathrm{DTS}(w)$	210.7300	210.7300	210.7300	210.6791
$\sigma_{\mathrm{AS}}(w \mid b)$	0.0000	10.5726	11.3004	11.9400
$\sigma_{\mathrm{MD}}(w \mid b)$	0.0000	0.4338	0.0254	0.0308
$\sigma_{\mathrm{DTS}}(w \mid b)$	0.0000	15.2056	3.7018	0.0561
$\mathcal{CI}(w)$	76.9427	38.4713	38.4713	38.4713

Question (g)

How to find the previous solution of Question 3.(e) using a QP solver?

The goal is to write the objective function into a quadratic function:

$$\begin{aligned} f(w) &= \frac{\varphi_{\mathrm{AS}}}{2}\mathcal{R}_{\mathrm{AS}}(w \mid b) + \frac{\varphi_{\mathrm{MD}}}{2}\mathcal{R}_{\mathrm{MD}}(w \mid b) + \frac{\varphi_{\mathrm{DTS}}}{2}\mathcal{R}_{\mathrm{DTS}}(w \mid b) \\ &= \frac{1}{2}w^\top Q(b)\, w - w^\top R(b) + c(b) \end{aligned}$$

where:

$$\begin{aligned} \mathcal{R}_{\mathrm{AS}}(w \mid b) &= (w_1 - 0.22)^2 + (w_2 - 0.19)^2 + (w_3 - 0.17)^2 + (w_4 - 0.13)^2 + \\ &\quad (w_5 - 0.11)^2 + (w_6 - 0.08)^2 + (w_7 - 0.06)^2 + (w_8 - 0.04)^2 \\ \mathcal{R}_{\mathrm{MD}}(w \mid b) &= (3.56w_1 + 6.54w_3 + 10.23w_4 + 2.30w_6 - 3.4089)^2 + \\ &\quad (7.48w_2 + 2.40w_5 + 9.12w_7 + 7.96w_8 - 2.5508)^2 \\ \mathcal{R}_{\mathrm{DTS}}(w \mid b) &= (103w_1 + 75w_3 + 796w_4 + 45w_6 - 142.49)^2 + \\ &\quad (155w_2 + 89w_5 + 320w_7 + 245w_8 - 68.24)^2 \end{aligned}$$

We use the analytical approach which is described in Section 11.1.2 on pages 332-339. Moreover, we rearrange the universe such that the first fourth assets belong to the first sector and the last fourth assets belong to the second sector. In this case, we have:

$$w = \left(\underbrace{w_1, w_3, w_4, w_6}_{\mathcal{S}ector_1}, \underbrace{w_2, w_5, w_7, w_8}_{\mathcal{S}ector_2} \right)$$

The matrix $Q(b)$ is block-diagonal:

$$Q(b) = \begin{pmatrix} Q_1 & \mathbf{0}_{4,4} \\ \mathbf{0}_{4,4} & Q_2 \end{pmatrix}$$

where the matrices Q_1 and Q_2 are equal to:

$$Q_1 = \begin{pmatrix} 11\,025.8400 & 8\,307.0600 & 82\,898.4700 & 4\,839.7000 \\ 8\,307.0600 & 6\,794.2900 & 61\,372.6050 & 3\,751.0500 \\ 82\,898.4700 & 61\,372.6050 & 636\,332.3225 & 36\,408.2250 \\ 4\,839.7000 & 3\,751.0500 & 36\,408.2250 & 2\,257.2500 \end{pmatrix}$$

and:

$$Q_2 = \begin{pmatrix} 25\,523.7600 & 14\,243.8000 & 51\,305.4400 & 39\,463.5200 \\ 14\,243.8000 & 8\,165.0000 & 29\,027.2000 & 22\,282.6000 \\ 51\,305.4400 & 29\,027.2000 & 104\,579.3600 & 80\,214.8800 \\ 39\,463.5200 & 22\,282.6000 & 80\,214.8800 & 61\,709.0400 \end{pmatrix}$$

The vector $R(b)$ is defined as follows:

$$R(b) = \begin{pmatrix} 15\,001.8621 \\ 11\,261.1051 \\ 114\,306.8662 \\ 6\,616.0617 \\ 11\,073.1996 \\ 6\,237.4080 \\ 22\,424.3824 \\ 17\,230.4092 \end{pmatrix}$$

Finally, the value of $c(b)$ is equal to:

$$c(b) = 12\,714.3386$$

Using a QP solver, we obtain the following numerical solution:

$$\begin{pmatrix} w_1 \\ w_3 \\ w_4 \\ w_6 \\ w_2 \\ w_5 \\ w_7 \\ w_8 \end{pmatrix} = \begin{pmatrix} 16.9796 \\ 18.2582 \\ 13.4494 \\ 9.4553 \\ 17.2102 \\ 12.1009 \\ 0.0000 \\ 12.5464 \end{pmatrix} \times 10^{-2}$$

We observe some small differences (after the fifth digit) because the QP solver is more efficient than a traditional nonlinear solver.

Question 4

We consider a variant of Question 3 and assume that the synthetic risk measure is:

$$\mathcal{D}(w \mid b) = \varphi_{\mathrm{AS}} \mathcal{D}_{\mathrm{AS}}(w \mid b) + \varphi_{\mathrm{MD}} \mathcal{D}_{\mathrm{MD}}(w \mid b) + \varphi_{\mathrm{DTS}} \mathcal{D}_{\mathrm{DTS}}(w \mid b)$$

where:

$$\begin{aligned}
\mathcal{D}_{\mathrm{AS}}(w \mid b) &= \frac{1}{2} \sum_{i=1}^{n} \left| w_i - b_i \right| \\
\mathcal{D}_{\mathrm{MD}}(w \mid b) &= \sum_{j=1}^{n_{\mathcal{S}ector}} \left| \sum\nolimits_{i \in \mathcal{S}ector_j} (w_i - b_i) \, \mathrm{MD}_i \right| \\
\mathcal{D}_{\mathrm{DTS}}(w \mid b) &= \sum_{j=1}^{n_{\mathcal{S}ector}} \left| \sum\nolimits_{i \in \mathcal{S}ector_j} (w_i - b_i) \, \mathrm{DTS}_i \right|
\end{aligned}$$

Question (a)

Define the corresponding optimization problem when the objective is to minimize the active risk and reduce the carbon intensity of the benchmark by $\mathcal{R}$.

The optimization problem is:

$$\begin{array}{rcl} w^{\star} & = & \arg\min \mathcal{D}\left(w \mid b\right) \\ & \text{s.t.} & \left\{ \begin{array}{l} \mathbf{1}_8^\top w = 1 \\ \boldsymbol{\mathcal{CI}}^\top w \leq \left(1 - \boldsymbol{\mathcal{R}}\right) \boldsymbol{\mathcal{CI}}\left(b\right) \\ \mathbf{0}_8 \leq w \leq \mathbf{1}_8 \end{array} \right. \end{array}$$

Question (b)

Give the LP formulation of the optimization problem.

We use the absolute value trick and obtain the following optimization problem:

$$
\begin{aligned}
w^{\star} &= \arg\min \frac{1}{2}\varphi_{\mathrm{AS}}\sum_{i=1}^{8}\tau_{i,w} + \varphi_{\mathrm{MD}}\sum_{j=1}^{2}\tau_{j,\mathrm{MD}} + \varphi_{\mathrm{DTS}}\sum_{j=1}^{2}\tau_{j,\mathrm{DTS}} \\
&\text{s.t.} \quad \left\{
\begin{array}{l}
\mathbf{1}_8^\top w = 1 \\
\mathbf{0}_8 \leq w \leq \mathbf{1}_8 \\
\boldsymbol{\mathcal{CI}}^\top w \leq (1-\boldsymbol{\mathcal{R}})\,\boldsymbol{\mathcal{CI}}(b) \\
\left|w_i - b_i\right| \leq \tau_{i,w} \\
\left|\sum_{i\in\boldsymbol{\mathcal{S}}ector_j}(w_i - b_i)\,\mathrm{MD}_i\right| \leq \tau_{j,\mathrm{MD}} \\
\left|\sum_{i\in\boldsymbol{\mathcal{S}}ector_j}(w_i - b_i)\,\mathrm{DTS}_i\right| \leq \tau_{j,\mathrm{DTS}} \\
\tau_{i,w} \geq 0, \tau_{j,\mathrm{MD}} \geq 0, \tau_{j,\mathrm{DTS}} \geq 0
\end{array}
\right.
\end{aligned}
$$

We can now formulate this problem as a standard LP problem:

$$\begin{aligned} x^{\star} &= \arg\min c^{\top} x \\ & \text{s.t.} \quad \left\{ \begin{array}{l} Ax = B \\ Cx \leq D \\ x^{-} \leq x \leq x^{+} \end{array} \right. \end{aligned}$$

where x is the 20×1 vector defined as follows:

$$x = \begin{pmatrix} w \\ \tau_w \\ \tau_{\mathrm{MD}} \\ \tau_{\mathrm{DTS}} \end{pmatrix}$$

The 20×1 vector c is equal to:

$$c = \begin{pmatrix} \mathbf{0}_8 \\ \frac{1}{2}\varphi_{\mathrm{AS}}\mathbf{1}_8 \\ \varphi_{\mathrm{MD}}\mathbf{1}_2 \\ \varphi_{\mathrm{DTS}}\mathbf{1}_2 \end{pmatrix}$$

The equality constraint is defined by $A = \begin{pmatrix} \mathbf{1}_8^\top & \mathbf{0}_8^\top & \mathbf{0}_2^\top & \mathbf{0}_2^\top \end{pmatrix}$ and $B = 1$. The bounds are $x^- = \mathbf{0}_{20}$ and $x^+ = \infty \cdot \mathbf{1}_{20}$.

For the inequality constraint, we have[27]:

$$Cx \leq D \Leftrightarrow \left(\begin{array}{cccc} I_8 & -I_8 & \mathbf{0}_{8,2} & \mathbf{0}_{8,2} \\ -I_8 & -I_8 & \mathbf{0}_{8,2} & \mathbf{0}_{8,2} \\ C_{\mathrm{MD}} & \mathbf{0}_{2,8} & -I_2 & \mathbf{0}_{2,2} \\ -C_{\mathrm{MD}} & \mathbf{0}_{2,8} & -I_2 & \mathbf{0}_{2,2} \\ C_{\mathrm{DTS}} & \mathbf{0}_{2,8} & \mathbf{0}_{2,2} & -I_2 \\ -C_{\mathrm{DTS}} & \mathbf{0}_{2,8} & \mathbf{0}_{2,2} & -I_2 \\ \boldsymbol{\mathcal{CI}}^\top & \mathbf{0}_{1,8} & 0 & 0 \end{array}\right) x \leq \left(\begin{array}{c} b \\ -b \\ \mathrm{MD}^\star \\ -\mathrm{MD}^\star \\ \mathrm{DTS}^\star \\ -\mathrm{DTS}^\star \\ (1-\boldsymbol{\mathcal{R}})\boldsymbol{\mathcal{CI}}(b) \end{array}\right)$$

where:

$$C_{\mathrm{MD}} = \left(\begin{array}{cccccccc} 3.56 & 0.00 & 6.54 & 10.23 & 0.00 & 2.30 & 0.00 & 0.00 \\ 0.00 & 7.48 & 0.00 & 0.00 & 2.40 & 0.00 & 9.12 & 7.96 \end{array}\right)$$

and:

$$C_{\mathrm{DTS}} = \left(\begin{array}{cccccccc} 103 & 0 & 75 & 796 & 0 & 45 & 0 & 0 \\ 0 & 155 & 0 & 0 & 89 & 0 & 320 & 245 \end{array}\right)$$

The 2×1 vectors $\mathrm{MD}^\star$ and $\mathrm{DTS}^\star$ are respectively equal to $(3.4089, 2.5508)$ and $(142.49, 68.24)$.

[27] C is a 25×8 matrix and D is a 25×1 vector.

Question (c)

Find the optimal portfolio when $\mathcal{R}$ is set to 50%. Compare the solution with this obtained in Question 3.(e).

We obtain the following solution:

$$\begin{aligned}
w^{\star} &= (18.7360, 15.8657, 17.8575, 13.2589, 11, 9.4622, 0, 13.8196) \times 10^{-2} \\
\tau_w^{\star} &= (3.2640, 3.1343, 0.8575, 0.2589, 0, 1.4622, 6, 9.8196) \times 10^{-2} \\
\tau_{\mathrm{MD}} &= (0, 0) \\
\tau_{\mathrm{DTS}} &= (0, 0)
\end{aligned}$$

Table 100: Solution of the bond optimization problem (scope $\mathcal{SC}_{1-3}$)

Problem	Benchmark	3.(e)	4.(c)
w_1	22.0000	16.9796	18.7360
w_2	19.0000	17.2102	15.8657
w_3	17.0000	18.2582	17.8575
w_4	13.0000	13.4494	13.2589
w_5	11.0000	12.1009	11.0000
w_6	8.0000	9.4553	9.4622
w_7	6.0000	0.0000	0.0000
w_8	4.0000	12.5464	13.8196
$\mathrm{MD}(w)$	5.9597	5.9683	5.9597
$\mathrm{DTS}(w)$	210.7300	210.6791	210.7300
$\sigma_{\mathrm{AS}}(w \mid b)$	0.0000	11.9400	12.4837
$\sigma_{\mathrm{MD}}(w \mid b)$	0.0000	0.0308	0.0000
$\sigma_{\mathrm{DTS}}(w \mid b)$	0.0000	0.0561	0.0000
$\mathcal{D}_{\mathrm{AS}}(w \mid b)$	0.0000	25.6203	24.7964
$\mathcal{D}_{\mathrm{MD}}(w \mid b)$	0.0000	0.0426	0.0000
$\mathcal{D}_{\mathrm{DTS}}(w \mid b)$	0.0000	0.0608	0.0000
$\mathcal{CI}(w)$	76.9427	38.4713	38.4713

In Table 100, we compare the two solutions[28]. They are very close. In fact, we notice that the LP solution matches perfectly the MD and DTS constraints, but has a higher AS risk $\sigma_{\mathrm{AS}}(w \mid b)$. If we note the two solutions $w^\star(\mathcal{L}_1)$ and $w^\star(\mathcal{L}_2)$, we have:

$$\left\{ \begin{array}{c} \mathcal{R}(w^\star(\mathcal{L}_2) \mid b) = 1.4524 < \mathcal{R}(w^\star(\mathcal{L}_1) \mid b) = 1.5584 \\ \mathcal{D}(w^\star(\mathcal{L}_2) \mid b) = 13.9366 > \mathcal{D}(w^\star(\mathcal{L}_1) \mid b) = 12.3982 \end{array} \right.$$

There is a trade-off between the $\mathcal{L}_1$- and $\mathcal{L}_2$-norm risk measures. This is why we cannot say that one solution dominates the other.

[28] The units are the following: % for the weights w_i, and the active share metrics $\sigma_{\mathrm{AS}}(w \mid b)$ and $\mathcal{D}_{\mathrm{AS}}(w \mid b)$; years for the modified duration metrics $\mathrm{MD}(w)$, $\sigma_{\mathrm{MD}}(w \mid b)$ and $\mathcal{D}_{\mathrm{MD}}(w \mid b)$; bps for the duration-times-spread metrics $\mathrm{DTS}(w)$, $\sigma_{\mathrm{DTS}}(w \mid b)$ and $\mathcal{D}_{\mathrm{DTS}}(w \mid b)$; $\mathrm{tCO_2e}/\$\ \mathrm{mn}$ for the carbon intensity $\mathrm{DTS}(w)$.

Course 2023-2024 in Portfolio Allocation and Asset Management

Lecture 7. Machine Learning in Asset Management

Thierry Roncalli*

*Amundi Asset Management[29]

*University of Paris-Saclay

January 2024

[29] The opinions expressed in this presentation are those of the authors and are not meant to represent the opinions or official positions of Amundi Asset Management.

Agenda

- Lecture 1: Portfolio Optimization
- Lecture 2: Risk Budgeting
- Lecture 3: Smart Beta, Factor Investing and Alternative Risk Premia
- Lecture 4: Equity Portfolio Optimization with ESG Scores
- Lecture 5: Climate Portfolio Construction
- Lecture 6: Equity and Bond Portfolio Optimization with Green Preferences
- **Lecture 7: Machine Learning in Asset Management**

Prologue

- Machine learning is a hot topic in asset management (and more generally in finance)
- Machine learning and data mining are two sides of the same coin

backtesting performance $\neq$ **live performance**

- Reaching for the stars: a complex/complicated process does not mean a good solution

Don't forget the 3 rules in asset management

1. It is difficult to make money
2. It is difficult to make money
3. It is difficult to make money

Prologue

- In this lecture, we focus on ML optimization algorithms, because they have proved their worth
- We have no time to study classical ML methods that can be used by quants to build investment strategies[30]

[30] Don't believe that they are always significantly better than standard statistical approaches!!!

Standard optimization algorithms

- Gradient descent methods
- Conjugate gradient (CG) methods (Fletcher–Reeves, Polak–Ribiere, etc.)
- Quasi-Newton (QN) methods (NR, BFGS, DFP, etc.)
- Quadratic programming (QP) methods
- Sequential QP methods
- Interior-point methods

Standard optimization algorithms

- We consider the following unconstrained minimization problem:

$$x^{\star} = \arg\min_{x} f(x) \tag{7}$$

where $x \in \mathbb{R}^n$ and $f(x)$ is a continuous, smooth and convex function

- In order to find the solution $x^{\star}$, optimization algorithms use iterative algorithms:

$$\begin{aligned} x^{(k+1)} &= x^{(k)} + \Delta x^{(k)} \\ &= x^{(k)} - \eta^{(k)} D^{(k)} \end{aligned}$$

where:

  - $x^{(0)}$ is the vector of starting values
  - $x^{(k)}$ is the approximated solution of Problem (7) at the k^{th} iteration
  - $\eta^{(k)} > 0$ is a scalar that determines the step size
  - $D^{(k)}$ is the direction

Standard optimization algorithms

- Gradient descent:

$$D^{(k)} = \nabla f\left(x^{(k)}\right) = \frac{\partial f\left(x^{(k)}\right)}{\partial x}$$

- Newton-Raphson method:

$$D^{(k)} = \left(\nabla^2 f\left(x^{(k)}\right)\right)^{-1} \nabla f\left(x^{(k)}\right) = \left(\frac{\partial^2 f\left(x^{(k)}\right)}{\partial x\, \partial x^\top}\right)^{-1} \frac{\partial f\left(x^{(k)}\right)}{\partial x}$$

- Quasi-Newton method:

$$D^{(k)} = H^{(k)} \nabla f\left(x^{(k)}\right)$$

where $H^{(k)}$ is an approximation of the inverse of the Hessian matrix

Standard optimization algorithms

What are the issues?

1. How to solve large-scale optimization problems?
2. How to solve optimization problems where there are multiple solutions?
3. How to just find an "*acceptable*" solution?

The case of neural networks and deep learning

$\Rightarrow$ Standard approaches are not well adapted

Machine learning optimization algorithms

Machine learning problems

- Non-smooth objective function
- Non-unique solution
- Large-scale dimension

Optimization in machine learning requires to reinvent numerical optimization

Machine learning optimization algorithms

We consider 4 methods:

- Cyclical coordinate descent (CCD)
- Alternative direction method of multipliers (ADMM)
- Proximal operators (PO)
- Dykstra's algorithm (DA)

Coordinate descent methods

The fall and the rise of the steepest descent method

In the 1980s:

- Conjugate gradient methods (Fletcher–Reeves, Polak–Ribiere, etc.)
- Quasi-Newton methods (NR, BFGS, DFP, etc.)

In the 1990s:

- Neural networks
- Learning rules: Descent, Momentum/Nesterov and Adaptive learning methods

In the 2000s:

- Gradient descent (by **observations**): Batch gradient descent (BGD), Stochatic gradient descent (SGD), Mini-batch gradient descent (MGD)
- Gradient descent (by **parameters**): Coordinate descent (CD), cyclical coordinate descent (CCD), Random coordinate descent (RCD)

Coordinate descent methods

Descent method

The descent algorithm is defined by the following rule:

$$x^{(k+1)} = x^{(k)} + \Delta x^{(k)} = x^{(k)} - \eta^{(k)} D^{(k)}$$

At the k^{th} Iteration, the current solution $x^{(k)}$ is updated by going in the opposite direction to $D^{(k)}$ (generally, we set $D^{(k)} = \partial_x f\left(x^{(k)}\right)$)

Coordinate descent method

Coordinate descent is a modification of the descent algorithm by minimizing the function along one coordinate at each step:

$$x_i^{(k+1)} = x_i^{(k)} + \Delta x_i^{(k)} = x_i^{(k)} - \eta^{(k)} D_i^{(k)}$$

$\Rightarrow$ The coordinate descent algorithm becomes a scalar problem

Coordinate descent methods

Choice of the variable i

1. Random coordinate descent (RCD)
   We assign a random number between 1 and n to the index i (Nesterov, 2012)
2. Cyclical coordinate descent (CCD)
   We cyclically iterate through the coordinates (Tseng, 2001):

$$x_i^{(k+1)} = \arg\min_x f\left(x_1^{(k+1)}, \ldots, x_{i-1}^{(k+1)}, x, x_{i+1}^{(k)}, \ldots, x_n^{(k)}\right)$$

Cyclical coordinate descent (CCD)

Example 1

We consider the following function:

$$f(x_1, x_2, x_3) = (x_1 - 1)^2 + x_2^2 - x_2 + (x_3 - 2)^4 e^{x_1 - x_2 + 3}$$

We have:

$$\begin{aligned}
D_1 = \frac{\partial f(x_1, x_2, x_3)}{\partial x_1} &= 2(x_1 - 1) + (x_3 - 2)^4 e^{x_1 - x_2 + 3} \\
D_2 = \frac{\partial f(x_1, x_2, x_3)}{\partial x_2} &= 2x_2 - 1 - (x_3 - 2)^4 e^{x_1 - x_2 + 3} \\
D_3 = \frac{\partial f(x_1, x_2, x_3)}{\partial x_3} &= 4(x_3 - 2)^3 e^{x_1 - x_2 + 3}
\end{aligned}$$

Cyclical coordinate descent (CCD)

The CCD algorithm is defined by the following iterations:

$$\left\{\begin{array}{l}
x_1^{(k+1)} = x_1^{(k)} - \eta^{(k)} \left(2\left(x_1^{(k)} - 1\right) + \left(x_3^{(k)} - 2\right)^4 e^{x_1^{(k)} - x_2^{(k)} + 3}\right) \\
x_2^{(k+1)} = x_2^{(k)} - \eta^{(k)} \left(2x_2^{(k)} - 1 - \left(x_3^{(k)} - 2\right)^4 e^{x_1^{(k+1)} - x_2^{(k)} + 3}\right) \\
x_3^{(k+1)} = x_3^{(k)} - \eta^{(k)} \left(4\left(x_3^{(k)} - 2\right)^3 e^{x_1^{(k+1)} - x_2^{(k+1)} + 3}\right)
\end{array}\right.$$

We have the following scheme:

$$\begin{array}{lcl}
\left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right) & \rightarrow & x_1^{(1)} \rightarrow \left(x_1^{(1)}, x_2^{(0)}, x_3^{(0)}\right) \rightarrow x_2^{(1)} \rightarrow \left(x_1^{(1)}, x_2^{(1)}, x_3^{(0)}\right) \rightarrow x_3^{(1)} \rightarrow \\
\left(x_1^{(1)}, x_2^{(1)}, x_3^{(1)}\right) & \rightarrow & x_1^{(2)} \rightarrow \left(x_1^{(2)}, x_2^{(1)}, x_3^{(1)}\right) \rightarrow x_2^{(2)} \rightarrow \left(x_1^{(2)}, x_2^{(2)}, x_3^{(1)}\right) \rightarrow x_3^{(2)} \rightarrow \\
\left(x_1^{(2)}, x_2^{(2)}, x_3^{(2)}\right) & \rightarrow & x_1^{(3)} \rightarrow \ldots
\end{array}$$

Cyclical coordinate descent (CCD)

Table 101: Solution obtained with the CCD algorithm ($\eta^{(k)} = 0.25$)

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$	$D_1^{(k)}$	$D_2^{(k)}$	$D_3^{(k)}$
0	1.0000	1.0000	1.0000			
1	-4.0214	0.7831	1.1646	20.0855	0.8675	-0.6582
2	-1.5307	0.8834	2.2121	-9.9626	-0.4013	-4.1902
3	-0.2663	0.6949	2.1388	-5.0578	0.7540	0.2932
4	0.3661	0.5988	2.0962	-2.5297	0.3845	0.1703
5	0.6827	0.5499	2.0758	-1.2663	0.1957	0.0818
6	0.8412	0.5252	2.0638	-0.6338	0.0989	0.0480
7	0.9205	0.5127	2.0560	-0.3172	0.0498	0.0314
8	0.9602	0.5064	2.0504	-0.1588	0.0251	0.0222
9	0.9800	0.5033	2.0463	-0.0795	0.0126	0.0166
∞	1.0000	0.5000	2.0000	0.0000	0.0000	0.0000

The lasso revolution

Least absolute shrinkage and selection operator (lasso)

The lasso method consists in adding a ℓ_1 penalty function to the least square problem:

$$\begin{aligned}\hat{\beta}^{\text{lasso}}(\tau) &= \arg\min \frac{1}{2}(Y - X\beta)^\top (Y - X\beta) \\ &\text{s.t.} \quad \|\beta\|_1 = \sum_{j=1}^{m} |\beta_j| \leq \tau\end{aligned}$$

This problem is equivalent to:

$$\hat{\beta}^{\text{lasso}}(\lambda) = \arg\min \frac{1}{2}(Y - X\beta)^\top (Y - X\beta) + \lambda \|\beta\|_1$$

We have:

$$\tau = \left\| \hat{\beta}^{\text{lasso}}(\lambda) \right\|_1$$

Solving the lasso regression problem

We introduce the parametrization:

$$\beta = \begin{pmatrix} I_m & -I_m \end{pmatrix} \begin{pmatrix} \beta^+ \\ \beta^- \end{pmatrix} = \beta^+ - \beta^-$$

under the constraints $\beta^+ \geq \mathbf{0}_m$ and $\beta^- \geq \mathbf{0}_m$. We deduce that:

$$\|\beta\|_1 = \sum_{j=1}^m \left| \beta_j^+ - \beta_j^- \right| = \sum_{j=1}^m \left| \beta_j^+ \right| + \sum_{j=1}^m \left| \beta_j^- \right| = \mathbf{1}_m^\top \beta^+ + \mathbf{1}_m^\top \beta^-$$

Solving the lasso regression problem

Augmented QP program of the lasso regression (λ-problem)

The augmented QP program is specified as follows:

$$\begin{aligned}
\hat{\theta} &= \arg\min \frac{1}{2}\theta^\top Q\theta - \theta^\top R \\
&\text{s.t.} \quad \theta \geq \mathbf{0}_{2m}
\end{aligned}$$

where $\theta = (\beta^+, \beta^-)$, $\tilde{X} = \begin{pmatrix} X & -X \end{pmatrix}$, $Q = \tilde{X}^\top \tilde{X}$ and $R = \tilde{X}^\top Y + \lambda \mathbf{1}_{2m}$. If we denote $T = \begin{pmatrix} I_m & -I_m \end{pmatrix}$, we obtain:

$$\hat{\beta}^{\text{lasso}}(\lambda) = T\hat{\theta}$$

Solving the lasso regression problem

Augmented QP program of the lasso regression (τ-problem)

If we consider the τ-problem, we obtain another augmented QP program:

$$\begin{aligned} \hat{\theta} &= \arg\min \frac{1}{2}\theta^\top Q\theta - \theta^\top R \\ &\text{s.t.} \quad \left\{ \begin{array}{l} C\theta \leq D \\ \theta \geq \mathbf{0}_{2m} \end{array} \right. \end{aligned}$$

where $Q = \tilde{X}^\top \tilde{X}$, $R = \tilde{X}^\top Y$, $C = \mathbf{1}_{2m}^\top$ and $D = \tau$. Again, we have:

$$\hat{\beta}(\tau) = T\hat{\theta}$$

Solving the lasso regression problem

We consider the linear regression:

$$Y = X\beta + \varepsilon$$

where Y is a $n \times 1$ vector, X is a $n \times m$ matrix and β is a $m \times 1$ vector. The optimization problem is:

$$\hat{\beta} = \arg\min f(\beta) = \frac{1}{2}(Y - X\beta)^\top (Y - X\beta)$$

Since we have $\partial_\beta f(\beta) = -X^\top (Y - X\beta)$, we deduce that:

$$\begin{aligned}
\frac{\partial f(\beta)}{\partial \beta_j} &= x_j^\top (X\beta - Y) \\
&= x_j^\top \left(x_j \beta_j + X_{(-j)}\beta_{(-j)} - Y\right) \\
&= x_j^\top x_j \beta_j + x_j^\top X_{(-j)}\beta_{(-j)} - x_j^\top Y
\end{aligned}$$

where x_j is the $n \times 1$ vector corresponding to the j^{th} variable and $X_{(-j)}$ is the $n \times (m-1)$ matrix (without the j^{th} variable)

Solving the lasso regression problem

At the optimum, we have $\partial_{\beta_j} f(\beta) = 0$ or:

$$\beta_j = \frac{x_j^\top Y - x_j^\top X_{(-j)}\beta_{(-j)}}{x_j^\top x_j} = \frac{x_j^\top \left(Y - X_{(-j)}\beta_{(-j)}\right)}{x_j^\top x_j}$$

CCD algorithm for the linear regression

We have:

$$\beta_j^{(k+1)} = \frac{x_j^\top \left(Y - \sum_{j'=1}^{j-1} x_{j'}\beta_{j'}^{(k+1)} - \sum_{j'=j+1}^{m} x_{j'}\beta_{j'}^{(k)}\right)}{x_j^\top x_j}$$

$\Rightarrow$ Introducing pointwise constraints is straightforward

Solving the lasso regression problem

The objective function becomes:

$$\begin{aligned} f(\beta) &= \frac{1}{2}(Y - X\beta)^\top (Y - X\beta) + \lambda \|\beta\|_1 \\ &= f_{\mathrm{OLS}}(\beta) + \lambda \|\beta\|_1 \end{aligned}$$

Since the norm is separable — $\|\beta\|_1 = \sum_{j=1}^m |\beta_j|$, the first-order condition is:

$$\frac{\partial\, f_{\mathrm{OLS}}(\beta)}{\partial\, \beta_j} + \lambda \partial |\beta_j| = 0$$

or:

$$\underbrace{\left(x_j^\top x_j\right)}_{c} \beta_j - \underbrace{x_j^\top \left(Y - X_{(-j)}\beta_{(-j)}\right)}_{v} + \lambda \partial\, |\beta_j| = 0$$

Derivation of the soft-thresholding operator

We consider the following equation:

$$c\beta_j - v + \lambda\partial\left|\beta_j\right| \in \{0\}$$

where $c > 0$ and $\lambda > 0$. Since we have $\partial\left|\beta_j\right| = \operatorname{sign}\left(\beta_j\right)$, we deduce that:

$$\beta_j^\star = \begin{cases} c^{-1}\left(v+\lambda\right) & \text{if } \beta_j^\star < 0 \\ 0 & \text{if } \beta_j^\star = 0 \\ c^{-1}\left(v-\lambda\right) & \text{if } \beta_j^\star > 0 \end{cases}$$

If $\beta_j^\star < 0$ or $\beta_j^\star > 0$, then we have $v+\lambda < 0$ or $v-\lambda > 0$. This is equivalent to set $|v| > \lambda > 0$. The case $\beta_j^\star = 0$ implies that $|v| \leq \lambda$. We deduce that:

$$\beta_j^\star = c^{-1} \cdot \mathcal{S}\left(v;\lambda\right)$$

where $\mathcal{S}\left(v;\lambda\right)$ is the soft-thresholding operator:

$$\begin{aligned} \mathcal{S}\left(v;\lambda\right) &= \begin{cases} 0 & \text{if } |v| \leq \lambda \\ v - \lambda\operatorname{sign}\left(v\right) & \text{otherwise} \end{cases} \\ &= \operatorname{sign}\left(v\right)\cdot\left(|v|-\lambda\right)_+ \end{aligned}$$

Solving the lasso regression problem

CCD algorithm for the lasso regression

We have:

$$\beta_j^{(k+1)} = \frac{1}{x_j^\top x_j} \mathcal{S}\left(x_j^\top \left(Y - \sum_{j'=1}^{j-1} x_{j'} \beta_{j'}^{(k+1)} - \sum_{j'=j+1}^{m} x_{j'} \beta_{j'}^{(k)} \right) ; \lambda \right)$$

where $\mathcal{S}(v; \lambda)$ is the **soft-thresholding operator**:

$$\mathcal{S}(v; \lambda) = \operatorname{sign}(v) \cdot (|v| - \lambda)_+$$

Solving the lasso regression problem

Table 102: Matlab code

```
for k = 1:nIters
    for j = 1:m
        x_j = X(:,j);
        X_j = X;
        X_j(:,j) = zeros(n,1);
        if lambda > 0
            v = x_j'*(Y - X_j*beta);
            beta(j) = max(abs(v) - lambda,0) * sign(v) / (x_j'*x_j);
        else
            beta(j) = x_j'*(Y - X_j*beta) / (x_j'*x_j);
        end
    end
end
```

Solving the lasso regression problem

Example 2

We consider the following data:

i	y	x_1	x_2	x_3	x_4	x_5
1	3.1	2.8	4.3	0.3	2.2	3.5
2	24.9	5.9	3.6	3.2	0.7	6.4
3	27.3	6.0	9.6	7.6	9.5	0.9
4	25.4	8.4	5.4	1.8	1.0	7.1
5	46.1	5.2	7.6	8.3	0.6	4.5
6	45.7	6.0	7.0	9.6	0.6	0.6
7	47.4	6.1	1.0	8.5	9.6	8.6
8	−1.8	1.2	9.6	2.7	4.8	5.8
9	20.8	3.2	5.0	4.2	2.7	3.6
10	6.8	0.5	9.2	6.9	9.3	0.7
11	12.9	7.9	9.1	1.0	5.9	5.4
12	37.0	1.8	1.3	9.2	6.1	8.3
13	14.7	7.4	5.6	0.9	5.6	3.9
14	−3.2	2.3	6.6	0.0	3.6	6.4
15	44.3	7.7	2.2	6.5	1.3	0.7

Solving the lasso regression problem

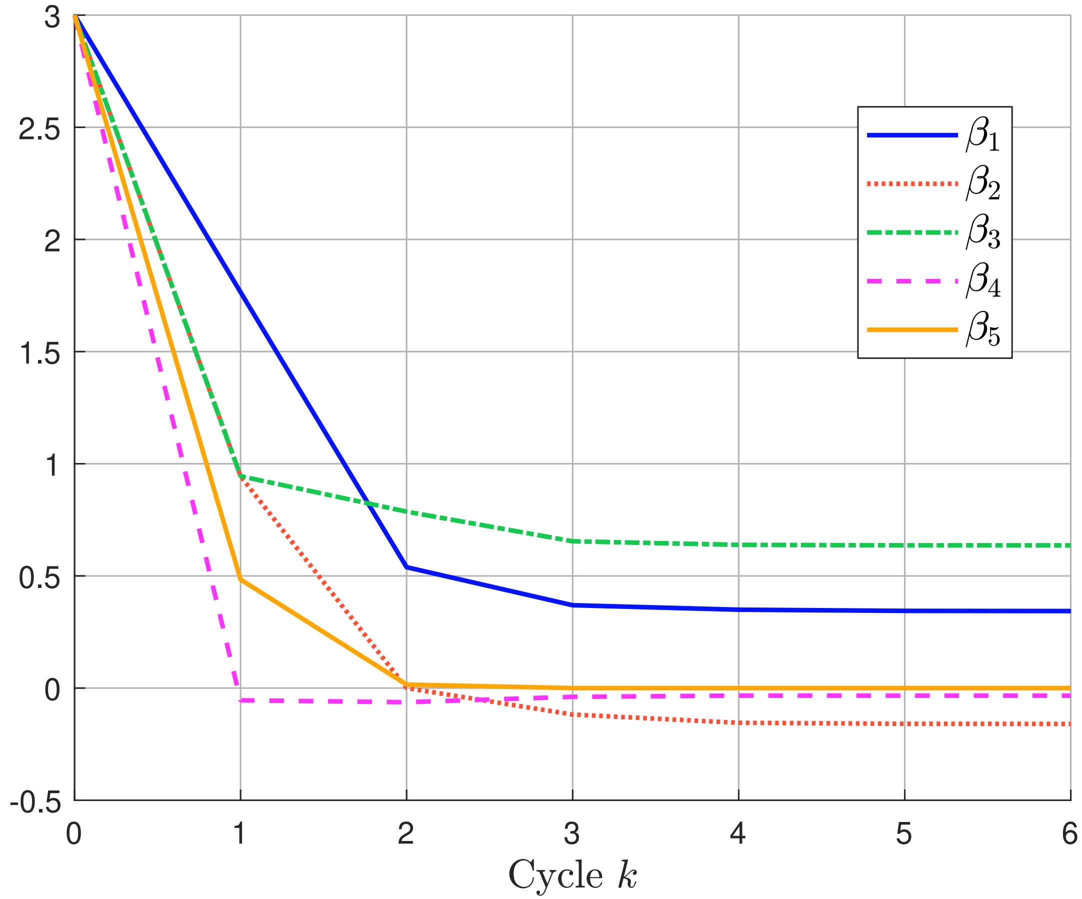


Figure 121: Convergence of the CCD algorithm (lasso regression, $\lambda = 2$)

Note: we start the CCD algorithm with $\beta_j^{(0)} = 0$ (don't forget to standardize the data!)

Solving the lasso regression problem

1. The dimension problem is $(2m, 2m)$ for QP and $(1, 0)$ for CCD!
2. CCD is faster for lasso regression than for linear regression (because of the soft-thresholding operator)!

Suppose $n = 50\,000$ and $m = 1\,000\,000$ (DNA sequence problem!)

Solving the lasso regression problem

Example 3

- We consider an experiment with $n = 100\,000$ observations and $m = 50$ variables.
- The design matrix X is built using the uniform distribution while the residuals are simulated using a Gaussian distribution and a standard deviation of 20%.
- The beta coefficients are distributed uniformly between -3 and $+3$ except four coefficients that take a larger value.
- We then standardize the data of X and Y.
- For initializing the coordinates, we use uniform random numbers between -1 and $+1$.

Solving the lasso regression problem

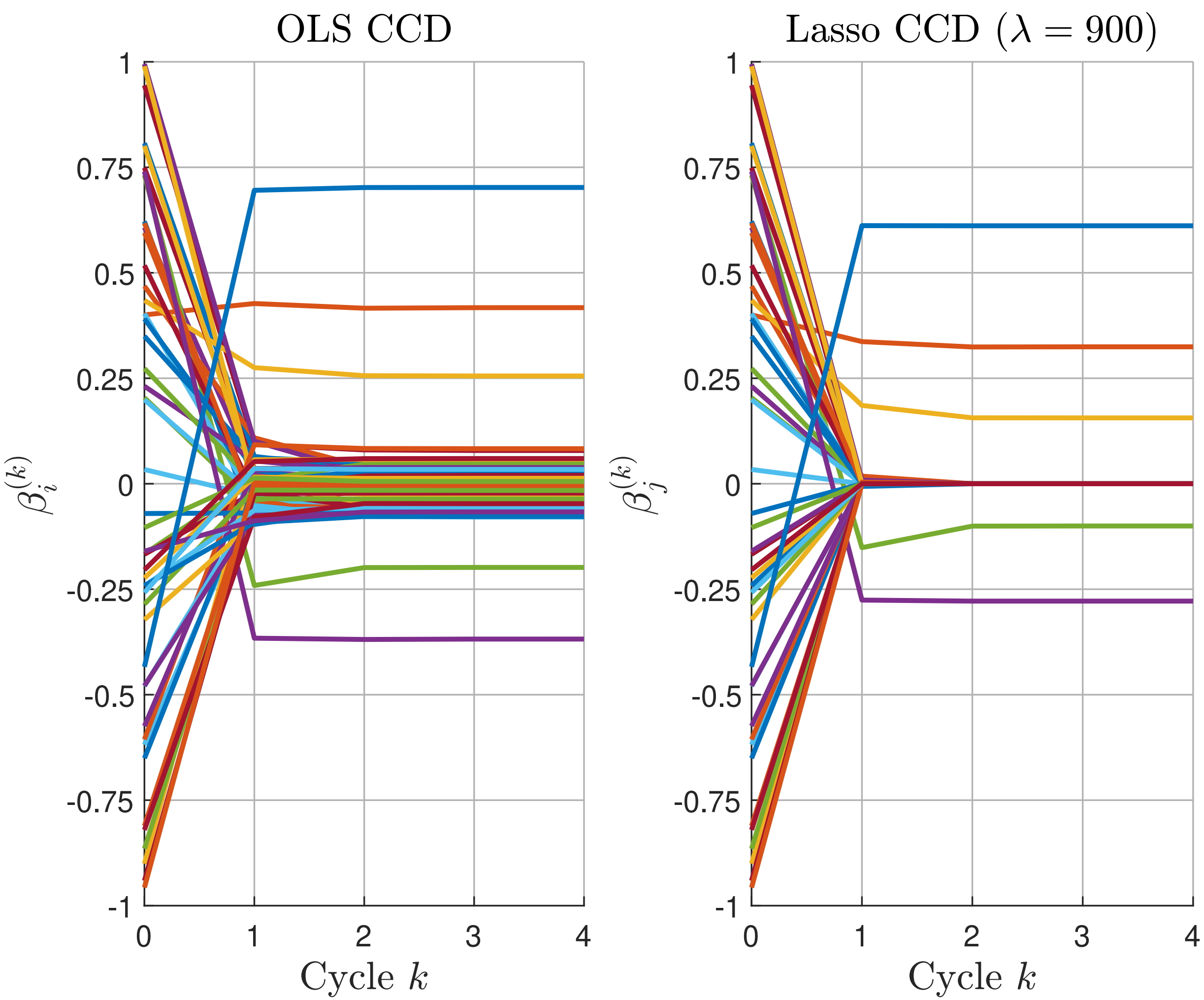


Figure 122: Convergence of the CCD algorithm (lasso vs linear regression)

Alternative direction method of multipliers

Definition

The alternating direction method of multipliers (ADMM) is an algorithm introduced by Gabay and Mercier (1976) to solve optimization problems which can be expressed as:

$$\begin{aligned} \{x^\star, y^\star\} &= \arg\min_{(x,y)} f_x(x) + f_y(y) \\ \text{s.t.} &\quad Ax + By = c \end{aligned}$$

The algorithm is:

$$\begin{aligned} x^{(k+1)} &= \arg\min_{x} \left\{ f_x(x) + \frac{\varphi}{2} \left\| Ax + By^{(k)} - c + u^{(k)} \right\|_2^2 \right\} \\ y^{(k+1)} &= \arg\min_{y} \left\{ f_y(y) + \frac{\varphi}{2} \left\| Ax^{(k+1)} + By - c + u^{(k)} \right\|_2^2 \right\} \\ u^{(k+1)} &= u^{(k)} + \left(Ax^{(k+1)} + By^{(k+1)} - c \right) \end{aligned}$$

Alternative direction method of multipliers

What is the underlying idea?

- Minimizing $f_x(x) + f_y(y)$ with respect to (x, y) is a difficult task
- Minimizing
$$g_x(x) = f_x(x) + \frac{\varphi}{2} \|Ax + By - c\|_2^2$$
with respect to x and minimizing
$$g_y(y) = f_y(y) + \frac{\varphi}{2} \|Ax + By - c\|_2^2$$
with respect to y is easier

Alternative direction method of multipliers

We use the following notations:

- $f_x^{(k+1)}(x)$ is the objective function of the x-update step:

$$f_x^{(k+1)}(x) = f_x(x) + \frac{\varphi}{2}\left\| Ax + By^{(k)} - c + u^{(k)} \right\|_2^2$$

- $f_y^{(k+1)}(y)$ is the objective function of the y-update step:

$$f_y^{(k+1)}(y) = f_y(y) + \frac{\varphi}{2}\left\| Ax^{(k+1)} + By - c + u^{(k)} \right\|_2^2$$

Alternative direction method of multipliers

When $A = I_n$ and $B = -I_n$, we have:

1.
$$Ax + By^{(k)} - c + u^{(k)} = x - y^{(k)} - c + u^{(k)} = x - v_x^{(k+1)}$$
where:
$$v_x^{(k+1)} = y^{(k)} + c - u^{(k)}$$

2.
$$Ax^{(k+1)} + By - c + u^{(k)} = x^{(k+1)} - y - c + u^{(k)} = v_y^{(k+1)} - y$$
where:
$$v_y^{(k+1)} = x^{(k+1)} - c + u^{(k)}$$

3.
$$\begin{aligned} f_x^{(k+1)}(x) &= f_x(x) + \frac{\varphi}{2}\left\| x - v_x^{(k+1)} \right\|_2^2 \\ f_y^{(k+1)}(y) &= f_y(y) + \frac{\varphi}{2}\left\| y - v_y^{(k+1)} \right\|_2^2 \end{aligned}$$

Alternative direction method of multipliers

- We consider a problem of the form:

$$x^{\star} = \arg\min_{x} g(x)$$

The idea is then to write $g(x)$ as a separable function:

$$g(x) = g_1(x) + g_2(x)$$

and to consider the following equivalent ADMM problem:

$$\begin{aligned} \{x^{\star}, y^{\star}\} &= \arg\min_{(x,y)} f_x(x) + f_y(y) \\ \text{s.t.} &\quad x = y \end{aligned}$$

where $f_x(x) = g_1(x)$ and $f_y(y) = g_2(y)$

Alternative direction method of multipliers

- We consider a problem of the form:

$$\begin{aligned} x^{\star} &= \arg\min_{x} g(x) \\ \text{s.t.} &\quad x \in \Omega \end{aligned}$$

We have:

$$\begin{aligned} \{x^{\star}, y^{\star}\} &= \arg\min_{(x,y)} f_x(x) + f_y(y) \\ \text{s.t.} &\quad x = y \end{aligned}$$

where $f_x(x) = g(x)$, $f_y(y) = \mathbb{1}_{\Omega}(y)$ and:

$$\mathbb{1}_{\Omega}(y) = \begin{cases} 0 & \text{if } \quad y \in \Omega \\ +\infty & \text{if } \quad y \notin \Omega \end{cases}$$

Alternative direction method of multipliers

Special case

$$\Omega = \left\{ x : x^- \leq x \leq x^+ \right\}$$

By setting $\varphi = 1$, the y-step becomes:

$$\begin{aligned} y^{(k+1)} &= \arg\min \left\{ \mathbb{1}_\Omega \left(y \right) + \frac{1}{2} \left\| x^{(k+1)} - y + u^{(k)} \right\|_2^2 \right\} \\ &= \mathbf{prox}_{f_y} \left(x^{(k+1)} + u^{(k)} \right) \end{aligned}$$

where the proximal operator is the box projection or the truncation operator:

$$\begin{aligned} \mathbf{prox}_{f_y} \left(v \right) &= x^- \odot \mathbb{1} \left\{ v < x^- \right\} + \\ & \quad v \odot \mathbb{1} \left\{ x^- \leq v \leq x^+ \right\} + \\ & \quad x^+ \odot \mathbb{1} \left\{ v > x^+ \right\} \\ &= \mathcal{T} \left(v; x^-, x^+ \right) \end{aligned}$$

Alternative direction method of multipliers

Special case

$$\Omega = \left\{ x : x^{-} \leq x \leq x^{+} \right\}$$

The ADMM algorithm is then:

$$\begin{aligned}
x^{(k+1)} &= \arg\min \left\{ g(x) + \frac{1}{2} \left\| x - y^{(k)} + u^{(k)} \right\|_2^2 \right\} \\
y^{(k+1)} &= \mathbf{prox}_{f_y} \left(x^{(k+1)} + u^{(k)} \right) \\
u^{(k+1)} &= u^{(k)} + \left(x^{(k+1)} - y^{(k+1)} \right)
\end{aligned}$$

$\Rightarrow$ Solving the constrained optimization problem consists in solving the unconstrained optimization problem, applying the box projection and iterating these steps until convergence

Alternative direction method of multipliers

Lasso regression

The λ-problem of the lasso regression has the following ADMM formulation:

$$\begin{aligned} \left\{\beta^\star, \bar{\beta}^\star\right\} &= \arg\min \frac{1}{2}(Y - X\beta)^\top (Y - X\beta) + \lambda \|\bar{\beta}\|_1 \\ \text{s.t.} \quad & \beta - \bar{\beta} = \mathbf{0}_m \end{aligned}$$

We have:

$$\begin{aligned} f_x(\beta) &= \frac{1}{2}(Y - X\beta)^\top (Y - X\beta) \\ &= \frac{1}{2}\beta^\top \left(X^\top X\right)\beta - \beta^\top \left(X^\top Y\right) + \frac{1}{2}Y^\top Y \end{aligned}$$

and:

$$f_y\left(\bar{\beta}\right) = \lambda \|\bar{\beta}\|_1$$

Alternative direction method of multipliers

The x-step is:

$$\beta^{(k+1)} = \arg\min_{\beta} \left\{ \frac{1}{2}\beta^\top \left(X^\top X\right)\beta - \beta^\top \left(X^\top Y\right) + \frac{\varphi}{2}\left\|\beta - \bar{\beta}^{(k)} + u^{(k)}\right\|_2^2 \right\}$$

Since we have:

$$\begin{aligned}\frac{\varphi}{2}\left\|\beta - \bar{\beta}^{(k)} + u^{(k)}\right\|_2^2 &= \frac{\varphi}{2}\beta^\top\beta - \varphi\beta^\top\left(\bar{\beta}^{(k)} - u^{(k)}\right) + \\ &\quad \frac{\varphi}{2}\left(\bar{\beta}^{(k)} - u^{(k)}\right)^\top\left(\bar{\beta}^{(k)} - u^{(k)}\right)\end{aligned}$$

we deduce that the x-update is a standard QP problem where:

$$f_x^{(k+1)}(\beta) = \frac{1}{2}\beta^\top\left(X^\top X + \varphi I_m\right)\beta - \beta^\top\left(X^\top Y + \varphi\left(\bar{\beta}^{(k)} - u^{(k)}\right)\right)$$

It follows that the solution is:

$$\begin{aligned}\beta^{(k+1)} &= \arg\min f_x^{(k+1)}(\beta) \\ &= \left(X^\top X + \varphi I_m\right)^{-1}\left(X^\top Y + \varphi\left(\bar{\beta}^{(k)} - u^{(k)}\right)\right)\end{aligned}$$

Alternative direction method of multipliers

The y-step is:

$$\begin{aligned}\bar{\beta}^{(k+1)} &= \arg\min_{\bar{\beta}} \left\{ \lambda \|\bar{\beta}\|_1 + \frac{\varphi}{2} \left\| \beta^{(k+1)} - \bar{\beta} + u^{(k)} \right\|_2^2 \right\} \\ &= \arg\min \left\{ \frac{1}{2} \left\| \bar{\beta} - \left(\beta^{(k+1)} + u^{(k)} \right) \right\|_2^2 + \frac{\lambda}{\varphi} \|\bar{\beta}\|_1 \right\}\end{aligned}$$

We recognize the soft-thresholding problem with $v = \beta^{(k+1)} + u^{(k)}$. We have:

$$\bar{\beta}^{(k+1)} = \mathcal{S}\left(\beta^{(k+1)} + u^{(k)}; \varphi^{-1}\lambda \right)$$

where:

$$\mathcal{S}(v; \lambda) = \operatorname{sign}(v) \cdot (|v| - \lambda)_+$$

Alternative direction method of multipliers

ADMM-Lasso algorithm (Boyd *et al.*, 2011)

Finally, the ADMM algorithm is made up of the following steps:

$$\left\{\begin{array}{l} \beta^{(k+1)} = \left(X^\top X + \varphi I_m\right)^{-1} \left(X^\top Y + \varphi\left(\bar{\beta}^{(k)} - u^{(k)}\right)\right) \\ \bar{\beta}^{(k+1)} = \mathcal{S}\left(\beta^{(k+1)} + u^{(k)}; \varphi^{-1}\lambda\right) \\ u^{(k+1)} = u^{(k)} + \left(\beta^{(k+1)} - \bar{\beta}^{(k+1)}\right) \end{array}\right.$$

Alternative direction method of multipliers

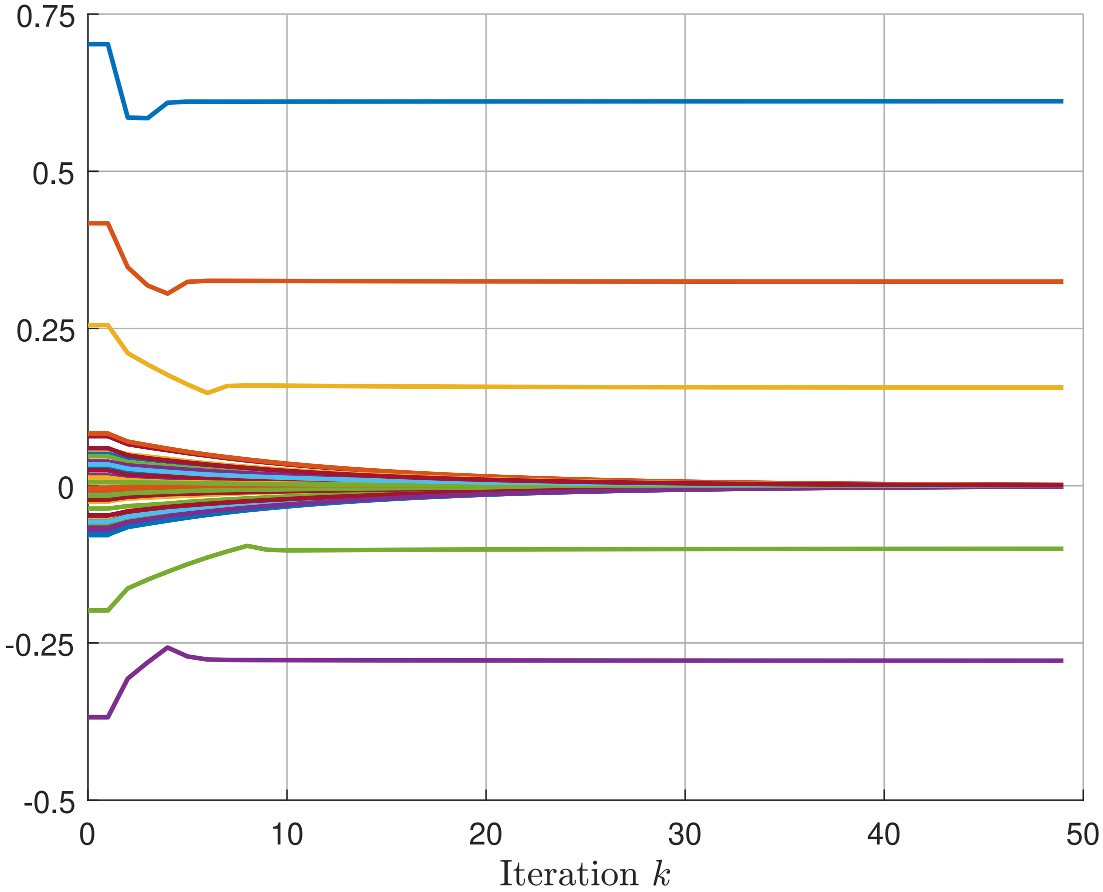


Figure 123: Convergence of the ADMM algorithm (Example 3, $\lambda = 900$)

Note: the initial values are the OLS estimates and we set $\varphi = \lambda$

Alternative direction method of multipliers

In practice, we use a time-varying parameter $\varphi^{(k)}$ (see Perrin and Roncalli, 2020).

Proximal operator

Definition

The proximal operator $\mathbf{prox}_f(v)$ of the function $f(x)$ is defined by:

$$\mathbf{prox}_f(v) = x^\star = \arg\min_x \left\{ f_v(x) = f(x) + \frac{1}{2}\|x - v\|_2^2 \right\}$$

Proximal operator

Example 4

We consider the scalar-valued logarithmic barrier function $f(x) = -\lambda \ln x$

Proximal operator

We have:

$$\begin{aligned} f_v(x) &= -\lambda \ln x + \frac{1}{2}(x-v)^2 \\ &= -\lambda \ln x + \frac{1}{2}x^2 - xv + \frac{1}{2}v^2 \end{aligned}$$

The first-order condition is $-\lambda x^{-1} + x - v = 0$. We obtain two roots with opposite signs:

$$x' = \frac{v - \sqrt{v^2 + 4\lambda}}{2} \text{ and } x'' = \frac{v + \sqrt{v^2 + 4\lambda}}{2}$$

Since the logarithmic function is defined for $x > 0$, we deduce that:

$$\mathbf{prox}_f(v) = \frac{v + \sqrt{v^2 + 4\lambda}}{2}$$

Proximal operator

In the case where $f(x) = \mathbb{1}_{\Omega}(x)$, we have:

$$
\begin{aligned}
\mathbf{prox}_f(v) &= \arg\min_x \left\{ \mathbb{1}_{\Omega}(x) + \frac{1}{2}\|x - v\|_2^2 \right\} \\
&= \arg\min_{x \in \Omega} \left\{ \|x - v\|_2^2 \right\} \\
&= \mathcal{P}_{\Omega}(v)
\end{aligned}
$$

where $\mathcal{P}_{\Omega}(v)$ is the standard projection of v onto Ω

Proximal operator

Table 103: Projection for some simple polyhedra

Notation	Ω	$\mathcal{P}_{\Omega}(v)$
$\mathcal{A}_{ffineset}[A, B]$	$Ax = B$	$v - A^{\dagger}(Av - B)$
$\mathcal{H}_{yperplane}[a, b]$	$a^{\top}x = b$	$v - \dfrac{(a^{\top}v - b)}{\Vert a \Vert_2^2} a$
$\mathcal{H}_{alfspace}[c, d]$	$c^{\top}x \leq d$	$v - \dfrac{(c^{\top}v - d)_+}{\Vert c \Vert_2^2} c$
$\mathcal{B}_{ox}[x^-, x^+]$	$x^- \leq x \leq x^+$	$\mathcal{T}(v; x^-, x^+)$

Source: Parikh and Boyd (2014)

Note: $A^{\dagger}$ is the Moore-Penrose pseudo-inverse of A, and $\mathcal{T}(v; x^-, x^+)$ is the truncation operator

Remark: No analytical formula for the (multi-dimensional) inequality constraint $Cx \leq D \Rightarrow$ it may be solved using the Dykstra's algorithm

Proximal operator

Separable sum

If $f(x) = \sum_{i=1}^{n} f_i(x_i)$ is fully separable, then the proximal of $f(v)$ is the vector of the proximal operators applied to each scalar-valued function $f_i(x_i)$:

$$\mathbf{prox}_f(v) = \begin{pmatrix} \mathbf{prox}_{f_1}(v_1) \\ \vdots \\ \mathbf{prox}_{f_n}(v_n) \end{pmatrix}$$

Proximal operator

If $f(x) = -\lambda \ln x$, we have:

$$\mathbf{prox}_f(v) = \frac{v + \sqrt{v^2 + 4\lambda}}{2}$$

In the case of the vector-valued logarithmic barrier $f(x) = -\lambda \sum_{i=1}^n \ln x_i$, we deduce that:

$$\mathbf{prox}_f(v) = \frac{v + \sqrt{v \odot v + 4\lambda}}{2}$$

Proximal operator

Moreau decomposition theorem

We have:

$$\mathbf{prox}_f(v) + \mathbf{prox}_{f^*}(v) = v$$

where f^* is the convex conjugate of f.

Application

If $f(x)$ is a ℓ_q-norm function, then $f^*(x) = \mathbb{1}_{\mathcal{B}_p}(x)$ where $\mathcal{B}_p$ is the ℓ_p unit ball and $p^{-1} + q^{-1} = 1$. Since we have $\mathbf{prox}_{f^*}(v) = \mathcal{P}_{\mathcal{B}_p}(v)$, we deduce that:

$$\mathbf{prox}_f(v) + \mathcal{P}_{\mathcal{B}_p}(v) = v$$

The proximal of the ℓ_p-ball can be deduced from the proximal operator of the ℓ_q-norm function.

Proximal operator

Table 104: Proximal of the ℓ_p-norm function $f(x) = \|x\|_p$

p	$\mathbf{prox}_{\lambda f}(v)$
$p = 1$	$\mathcal{S}(v; \lambda) = \operatorname{sign}(v) \odot (\lvert v \rvert - \lambda \mathbf{1}_n)_+$
$p = 2$	$\left(1 - \dfrac{\lambda}{\max(\lambda, \|v\|_2)}\right) v$
$p = \infty$	$\operatorname{sign}(v) \odot \mathbf{prox}_{\lambda \max x}(\lvert v \rvert)$

We have:

$$\mathbf{prox}_{\lambda \max x}(v) = \min(v, s^\star)$$

where $s^\star$ is the solution of the following equation:

$$s^\star = \left\{ s \in \mathbb{R} : \sum_{i=1}^{n} (v_i - s)_+ = \lambda \right\}$$

Proximal operator

Table 105: Proximal of the ℓ_p-ball $\mathcal{B}_p(c,\lambda) = \left\{x \in \mathbb{R}^n : \|x - c\|_p \leq \lambda\right\}$ when c is equal to $\mathbf{0}_n$

p	$\mathcal{P}_{\mathcal{B}_p(\mathbf{0}_n,\lambda)}(v)$	q
$p = 1$	$v - \operatorname{sign}(v) \odot \mathbf{prox}_{\lambda \max x}(\lvert v \rvert)$	$q = \infty$
$p = 2$	$v - \mathbf{prox}_{\lambda \|x\|_2}(v)$	$q = 2$
$p = \infty$	$\mathcal{T}(v; -\lambda, \lambda)$	$q = 1$

Proximal operator

Scaling and translation

Let us define $g(x) = f(ax + b)$ where $a \neq 0$. We have:

$$\mathbf{prox}_g(v) = \frac{\mathbf{prox}_{a^2 f}(av + b) - b}{a}$$

Application

We can use this property when the center c of the ℓ_p ball is not equal to $\mathbf{0}_n$. Since we have $\mathbf{prox}_g(v) = \mathbf{prox}_f(v - c) + c$ where $g(x) = f(x - c)$ and the equivalence $\mathcal{B}_p(\mathbf{0}_n, \lambda) = \{x \in \mathbb{R}^n : f(x) \leq \lambda\}$ where $f(x) = \|x\|_p$, we deduce that:

$$\mathcal{P}_{\mathcal{B}_p(c,\lambda)}(v) = \mathcal{P}_{\mathcal{B}_p(\mathbf{0}_n,\lambda)}(v - c) + c$$

Application to the τ-problem of the lasso regression

We have:

$$\begin{aligned} \hat{\beta}(\tau) &= \arg\min_{\beta} \frac{1}{2}(Y - X\beta)^\top (Y - X\beta) \\ \text{s.t.} &\quad \|\beta\|_1 \leq \tau \end{aligned}$$

The ADMM formulation is:

$$\begin{aligned} \left\{\beta^\star, \bar{\beta}^\star\right\} &= \arg\min_{(\beta,\bar{\beta})} \frac{1}{2}(Y - X\beta)^\top (Y - X\beta) + \mathbb{1}_\Omega\left(\bar{\beta}\right) \\ \text{s.t.} &\quad \beta = \bar{\beta} \end{aligned}$$

where $\Omega = \mathcal{B}_1(\mathbf{0}_m, \tau)$ is the centered ℓ_1 ball with radius τ

Application to the τ-problem of the lasso regression

1. The x-update is:

$$\begin{aligned}\beta^{(k+1)} &= \arg\min_{\beta} \left\{ \frac{1}{2} (Y - X\beta)^\top (Y - X\beta) + \frac{\varphi}{2} \left\| \beta - \bar{\beta}^{(k)} + u^{(k)} \right\|_2^2 \right\} \\ &= \left(X^\top X + \varphi I_m \right)^{-1} \left(X^\top Y + \varphi \left(\bar{\beta}^{(k)} - u^{(k)} \right) \right)\end{aligned}$$

where $v_x^{(k+1)} = \bar{\beta}^{(k)} - u^{(k)}$

Application to the τ-problem of the lasso regression

2. The y-update is:

$$
\begin{aligned}
\bar{\beta}^{(k+1)} &= \arg\min_{\bar{\beta}} \left\{ \mathbb{1}_{\Omega}\left(\bar{\beta}\right) + \frac{\varphi}{2} \left\| \beta^{(k+1)} - \bar{\beta} + u^{(k)} \right\|_2^2 \right\} \\
&= \mathbf{prox}_{f_y}\left(\beta^{(k+1)} + u^{(k)}\right) \\
&= \mathcal{P}_{\Omega}\left(v_y^{(k+1)}\right) \\
&= v_y^{(k+1)} - \operatorname{sign}\left(v_y^{(k+1)}\right) \odot \mathbf{prox}_{\tau \max x}\left(\left|v_y^{(k+1)}\right|\right)
\end{aligned}
$$

where $v_y^{(k+1)} = \beta^{(k+1)} + u^{(k)}$

Application to the τ-problem of the lasso regression

3. The u-update is:

$$u^{(k+1)} = u^{(k)} + \beta^{(k+1)} - \bar{\beta}^{(k+1)}$$

Application to the τ-problem of the lasso regression

ADMM-Lasso algorithm

The ADMM algorithm is :

$$\left\{ \begin{array}{l} \beta^{(k+1)} = \left(X^\top X + \varphi I_m\right)^{-1} \left(X^\top Y + \varphi \left(\bar{\beta}^{(k)} - u^{(k)}\right)\right) \\ \bar{\beta}^{(k+1)} = \left\{ \begin{array}{ll} \mathcal{S}\left(\beta^{(k+1)} + u^{(k)}; \varphi^{-1}\lambda\right) & (\lambda\text{-problem}) \\ \mathcal{P}_{\mathcal{B}_1(\mathbf{0}_m,\tau)}\left(\beta^{(k+1)} + u^{(k)}\right) & (\tau\text{-problem}) \end{array} \right. \\ u^{(k+1)} = u^{(k)} + \left(\beta^{(k+1)} - \bar{\beta}^{(k+1)}\right) \end{array} \right.$$

Remark

The ADMM algorithm is similar for λ- and τ-problems since the only difference concerns the y-step. However, the τ-problem is easier to solve with the ADMM algorithm from a practical point of view, because the y-update of the τ-problem is independent of the penalization parameter φ.

Derivation of the soft-thresholding operator

We consider the following equation:

$$cx - v + \lambda \partial \left| x \right| \in 0$$

where $c > 0$ and $\lambda > 0$. Since we have $\partial \left| x \right| = \operatorname{sign}(x)$, we deduce that:

$$x^{\star} = \left\{ \begin{array}{ll} c^{-1}(v + \lambda) & \text{if } x^{\star} < 0 \\ 0 & \text{if } x^{\star} = 0 \\ c^{-1}(v - \lambda) & \text{if } x^{\star} > 0 \end{array} \right.$$

If $x^{\star} < 0$ or $x^{\star} > 0$, then we have $v + \lambda < 0$ or $v - \lambda > 0$. This is equivalent to set $\left| v \right| > \lambda > 0$. The case $x^{\star} = 0$ implies that $\left| v \right| \leq \lambda$. We deduce that:

$$x^{\star} = c^{-1} \cdot \mathcal{S}(v; \lambda)$$

where $\mathcal{S}(v; \lambda)$ is the soft-thresholding operator:

$$\begin{array}{rcl} \mathcal{S}(v; \lambda) & = & \left\{ \begin{array}{ll} 0 & \text{if } \left| v \right| \leq \lambda \\ v - \lambda \operatorname{sign}(v) & \text{otherwise} \end{array} \right. \\ & = & \operatorname{sign}(v) \cdot \left(\left| v \right| - \lambda \right)_{+} \end{array}$$

Derivation of the soft-thresholding operator

We use the result on the separable sum

Remark

If $f(x) = \lambda \|x\|_1$, we have $f(x) = \lambda \sum_{i=1}^{n} |x_i|$ and $f_i(x_i) = \lambda |x_i|$. We deduce that the proximal operator of $f(x)$ is the vector formulation of the soft-thresholding operator:

$$\mathbf{prox}_{\lambda \|x\|_1}(v) = \begin{pmatrix} \operatorname{sign}(v_1) \cdot (|v_1| - \lambda)_+ \\ \vdots \\ \operatorname{sign}(v_n) \cdot (|v_n| - \lambda)_+ \end{pmatrix} = \operatorname{sign}(v) \odot (|v| - \lambda \mathbf{1}_n)_+$$

The soft-thresholding operator is the proximal operator of the ℓ_1-norm $f(x) = \|x\|_1$. Indeed, we have $\mathbf{prox}_f(v) = \mathcal{S}(v; 1)$ and $\mathbf{prox}_{\lambda f}(v) = \mathcal{S}(v; \lambda)$.

Dykstra's algorithm

We consider the following optimization problem:

$$\begin{aligned} x^\star &= \arg\min f_x(x) \\ \text{s.t.} &\quad x \in \Omega \end{aligned}$$

where Ω is a complex set of constraints:

$$\Omega = \Omega_1 \cap \Omega_2 \cap \cdots \Omega_m$$

We set $y = x$ and $f_y(y) = \mathbb{1}_\Omega(y)$. The ADMM algorithm becomes

$$\begin{aligned} x^{(k+1)} &= \arg\min \left\{ f_x(x) + \frac{\varphi}{2} \left\| x - y^{(k)} + u^{(k)} \right\|_2^2 \right\} \\ v^{(k)} &= x^{(k+1)} + u^{(k)} \\ y^{(k+1)} &= \mathcal{P}_\Omega\left(v^{(k)}\right) \\ u^{(k+1)} &= u^{(k)} + \left(x^{(k+1)} - y^{(k+1)}\right) \end{aligned}$$

How to compute $\mathcal{P}_\Omega(v)$?

Dykstra's algorithm

More generally, we consider the proximal optimization problem where the function $f(x)$ is the convex sum of basic functions $f_j(x)$:

$$x^\star = \arg\min_x \left\{ \sum_{j=1}^{m} f_j(x) + \frac{1}{2}\left\| x - v \right\|_2^2 \right\}$$

and the proximal of each basic function is known.

How to find the solution $x^\star$?

Dykstra's algorithm

The case $m = 2$

- We know the proximal solution of the ℓ_1-norm function $f_1(x) = \lambda_1 \|x\|_1$
- We know the proximal solution of the logarithmic barrier function $f_2(x) = \lambda_2 \sum_{i=1}^n \ln x_i$
- We don't know how to compute the proximal operator of $f(x) = f_1(x) + f_2(x)$:

$$\begin{aligned} x^\star &= \arg\min_x f_1(x) + f_2(x) + \frac{1}{2}\|x - v\|_2^2 \\ &= \mathbf{prox}_f(v) \end{aligned}$$

Dykstra's algorithm

The case $m = 2$

The Dykstra's algorithm consists in the following iterations:

$$\left\{ \begin{array}{l} x^{(k+1)} = \mathbf{prox}_{f_1} \left(y^{(k)} + p^{(k)} \right) \\ p^{(k+1)} = y^{(k)} + p^{(k)} - x^{(k+1)} \\ y^{(k+1)} = \mathbf{prox}_{f_2} \left(x^{(k+1)} + q^{(k)} \right) \\ q^{(k+1)} = x^{(k+1)} + q^{(k)} - y^{(k+1)} \end{array} \right.$$

where $x^{(0)} = y^{(0)} = v$ and $p^{(0)} = q^{(0)} = \mathbf{0}_n$

Dykstra's algorithm

The case $m = 2$

This algorithm is related to the Douglas-Rachford splitting framework:

$$\left\{\begin{array}{l} x^{\left(k+\frac{1}{2}\right)} = \mathbf{prox}_{f_1}\left(x^{(k)} + p^{(k)}\right) \\ p^{(k+1)} = p^{(k)} - \Delta_{1/2} x^{\left(k+\frac{1}{2}\right)} \\ x^{(k+1)} = \mathbf{prox}_{f_2}\left(x^{\left(k+\frac{1}{2}\right)} + q^{(k)}\right) \\ q^{(k+1)} = q^{(k)} - \Delta_{1/2} x^{(k+1)} \end{array}\right.$$

where $\Delta_h x^{(k)} = x^{(k)} - x^{(k-h)}$

Dykstra's algorithm

The case $m = 2$

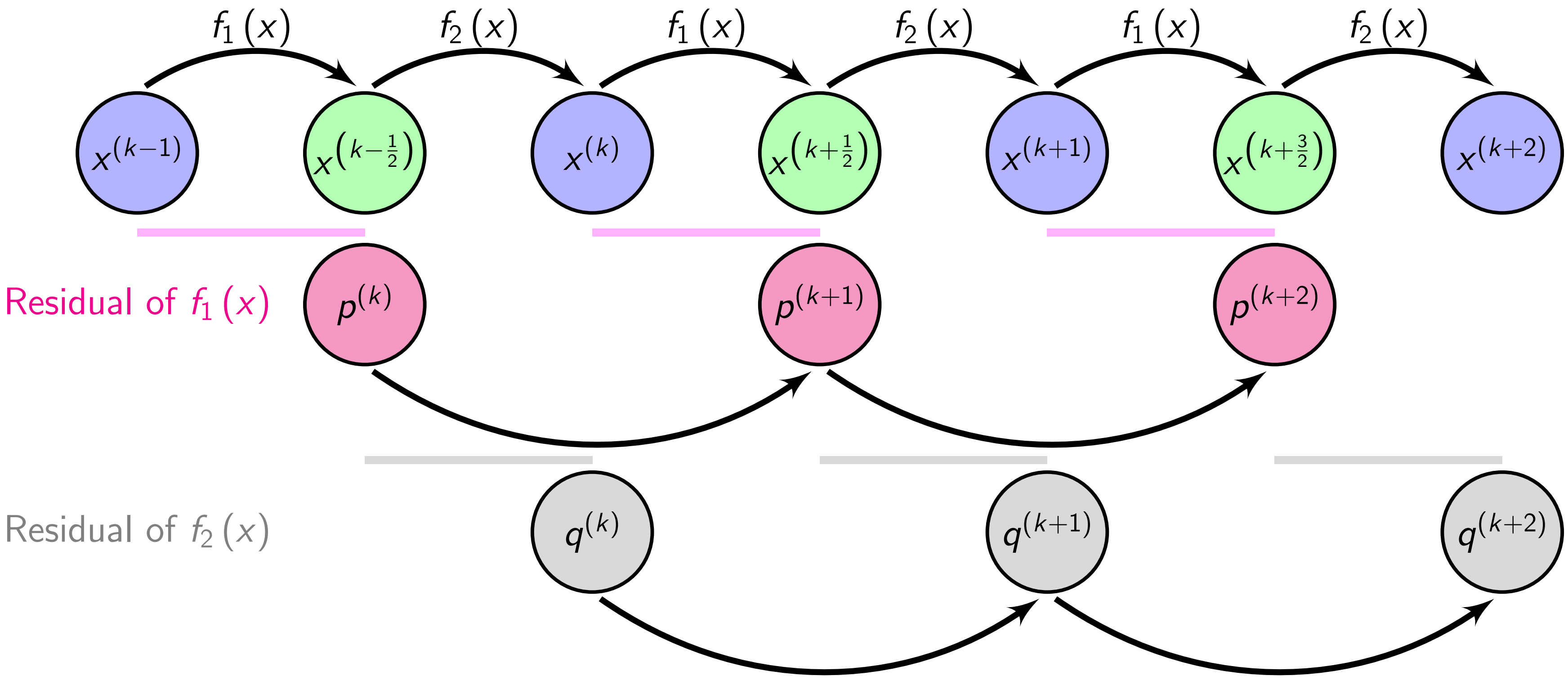


Figure 124: Splitting method of the Dykstra's algorithm

Dykstra's algorithm

The case $m > 2$

The case $m > 2$ is a generalization of the previous algorithm by considering m residuals:

1. The x-update is:

$$x^{(k+1)} = \mathbf{prox}_{f_{j(k)}} \left(x^{(k)} + z^{(k+1-m)} \right)$$

2. The z-update is:

$$z^{(k+1)} = x^{(k)} + z^{(k+1-m)} - x^{(k+1)}$$

where $x^{(0)} = v$, $z^{(k)} = \mathbf{0}_n$ for $k < 0$ and $j(k) = \mathrm{mod}(k+1, m)$ denotes the modulo operator taking values in $\{1, \ldots, m\}$

Remark

The variable $x^{(k)}$ is updated at each iteration while the residual $z^{(k)}$ is updated every m iterations. This implies that the basic function $f_j(x)$ is related to the residuals $z^{(j)}$, $z^{(j+m)}$, $z^{(j+2m)}$, etc.

Dykstra's algorithm

The case $m > 2$

Tibshirani (2017) proposes to write the Dykstra's algorithm by using two iteration indices k and j. The main index k refers to the cycle, whereas the sub-index j refers to the constraint number

The Dykstra's algorithm becomes:

1. The x-update is:

$$x^{(k+1,j)} = \mathbf{prox}_{f_j}\left(x^{(k+1,j-1)} + z^{(k,j)}\right)$$

2. The z-update is:

$$z^{(k+1,j)} = x^{(k+1,j-1)} + z^{(k,j)} - x^{(k+1,j)}$$

where $x^{(1,0)} = v$, $z^{(k,j)} = \mathbf{0}_n$ for $k = 0$ and $x^{(k+1,0)} = x^{(k,m)}$

Dykstra's algorithm

The case $m > 2$

The Dykstra's algorithm is particularly efficient when we consider the projection problem:

$$x^{\star} = \mathcal{P}_{\Omega}(v)$$

where:

$$\Omega = \Omega_1 \cap \Omega_2 \cap \cdots \cap \Omega_m$$

Indeed, the Dykstra's algorithm becomes:

1. The x-update is:

$$x^{(k+1,j)} = \mathbf{prox}_{f_j}\left(x^{(k+1,j-1)} + z^{(k,j)}\right) = \mathcal{P}_{\Omega_j}\left(x^{(k+1,j-1)} + z^{(k,j)}\right)$$

2. The z-update is:

$$z^{(k+1,j)} = x^{(k+1,j-1)} + z^{(k,j)} - x^{(k+1,j)}$$

where $x^{(1,0)} = v$, $z^{(k,j)} = \mathbf{0}_n$ for $k = 0$ and $x^{(k+1,0)} = x^{(k,m)}$

Dykstra's algorithm

Successive projections of $\mathcal{P}_{\Omega_j}\left(x^{(k+1,j-1)}\right)$ do not work!

Successive projections of $\mathcal{P}_{\Omega_j}\left(x^{(k+1,j-1)} + z^{(k,j)}\right)$ do work!

Dykstra's algorithm

Table 106: Solving the proximal problem with linear inequality constraints

The goal is to compute the solution $x^{\star} = \mathbf{prox}_f(v)$ where $f(x) = \mathbb{1}_{\Omega}(x)$ and $\Omega = \{x \in \mathbb{R}^n : Cx \leq D\}$
We initialize $x^{(0,m)} \leftarrow v$
We set $z^{(0,1)} \leftarrow \mathbf{0}_n, \ldots, z^{(0,m)} \leftarrow \mathbf{0}_n$
$k \leftarrow 0$
repeat
  $x^{(k+1,0)} \leftarrow x^{(k,m)}$
  for $j = 1 : m$ **do**
    The x-update is:

$$x^{(k+1,j)} = x^{(k+1,j-1)} + z^{(k,j)} - \frac{\left(c_{(j)}^{\top} x^{(k+1;j-1)} + c_{(j)}^{\top} z^{(k,j)} - d_{(j)}\right)_{+}}{\left\|c_{(j)}\right\|_2^2} c_{(j)}$$

    The z-update is:

$$z^{(k+1,j)} = x^{(k+1,j-1)} + z^{(k,j)} - x^{(k+1,j)}$$

  end for
  $k \leftarrow k + 1$
until Convergence
return $x^{\star} \leftarrow x^{(k,m)}$

Dykstra's algorithm

Table 107: Solving the proximal problem with general linear constraints

The goal is to compute the solution $x^{\star} = \mathbf{prox}_f(v)$ where $f(x) = \mathbb{1}_{\Omega}(x)$, $\Omega = \Omega_1 \cap \Omega_2 \cap \Omega_3$, $\Omega_1 = \{x \in \mathbb{R}^n : Ax = B\}$, $\Omega_2 = \{x \in \mathbb{R}^n : Cx \leq D\}$ and $\Omega_3 = \{x \in \mathbb{R}^n : x^- \leq x \leq x^+\}$

We initialize $x_m^{(0)} \leftarrow v$

We set $z_1^{(0)} \leftarrow \mathbf{0}_n$, $z_2^{(0)} \leftarrow \mathbf{0}_n$ and $z_3^{(0)} \leftarrow \mathbf{0}_n$

$k \leftarrow 0$

repeat

$x_0^{(k+1)} \leftarrow x_m^{(k)}$

$x_1^{(k+1)} \leftarrow x_0^{(k+1)} + z_1^{(k)} - A^{\dagger}\left(Ax_0^{(k+1)} + Az_1^{(k)} - B\right)$

$z_1^{(k+1)} \leftarrow x_0^{(k+1)} + z_1^{(k)} - x_1^{(k+1)}$

$x_2^{(k+1)} \leftarrow \mathcal{P}_{\Omega_2}\left(x_1^{(k+1)} + z_2^{(k)}\right)$

$z_2^{(k+1)} \leftarrow x_1^{(k+1)} + z_2^{(k)} - x_2^{(k+1)}$

$x_3^{(k+1)} \leftarrow \mathcal{T}\left(x_2^{(k+1)} + z_3^{(k)}; x^-, x^+\right)$

$z_3^{(k+1)} \leftarrow x_2^{(k+1)} + z_3^{(k)} - x_3^{(k+1)}$

$k \leftarrow k + 1$

until Convergence

return $x^{\star} \leftarrow x_3^{(k)}$

▶ Previous algorithm

Dykstra's algorithm

Remark

Since we have:

$$\frac{1}{2}\left\|x-v\right\|_2^2=\frac{1}{2}x^\top x-x^\top v+\frac{1}{2}v^\top v$$

the two previous problems can be cast into a QP problem:

$$\begin{aligned} x^\star &= \arg\min_x \frac{1}{2}x^\top I_n x-x^\top v \\ \text{s.t.} &\quad x\in\Omega \end{aligned}$$

Dykstra's algorithm

Dykstra's algorithm versus QP algorithm

- The vector v is defined by the elements $v_i = \ln\left(1 + i^2\right)$
- The set of constraints is:

$$\Omega = \left\{ x \in \mathbb{R}^n : \sum_{i=1}^{n} x_i \leq \frac{1}{2}, \sum_{i=1}^{n} e^{-i} x_i \geq 0 \right\}$$

- Using a Matlab implementation, we find that the computational time of the Dykstra's algorithm when n is equal to 10 million is equal to the QP algorithm when n is equal to 12 500!
- The QP algorithm requires to store the matrix I_n — impossible when $n > 10^5$. For instance, the size of I_n is equal to 7450.6 GB when $n = 10^6$

Application to portfolio allocation

Table 108: Some objective functions used in portfolio optimization

Item	Portfolio	$f(x)$	Reference
(1)	MVO	$\frac{1}{2}x^\top \Sigma x - \gamma x^\top \mu$	Markowitz (1952)
(2)	GMV	$\frac{1}{2}x^\top \Sigma x$	Jagganathan and Ma (2003)
(3)	MDP	$\ln\left(\sqrt{x^\top \Sigma x}\right) - \ln\left(x^\top \sigma\right)$	Choueifaty and Coignard (2008)
(4)	KL	$\sum_{i=1}^n x_i \ln\left(x_i / \tilde{x}_i\right)$	Bera and Park (2008)
(5)	ERC	$\frac{1}{2}x^\top \Sigma x - \lambda \sum_{i=1}^n \ln x_i$	Maillard *et al.* (2010)
(6)	RB	$\mathcal{R}(x) - \lambda \sum_{i=1}^n \mathcal{RB}_i \cdot \ln x_i$	Roncalli (2015)
(7)	RQE	$\frac{1}{2}x^\top D x$	Carmichael *et al.* (2018)

Application to portfolio allocation

Table 109: Some regularization penalties used in portfolio optimization

Item	Regularization	$\mathfrak{R}(x)$	Reference
(8)	Ridge	$\lambda \left\Vert x - \tilde{x} \right\Vert_2^2$	DeMiguel *et al.* (2009)
(9)	Lasso	$\lambda \left\Vert x - \tilde{x} \right\Vert_1$	Brodie *at al.* (2009)
(10)	Log-barrier	$-\sum_{i=1}^n \lambda_i \ln x_i$	Roncalli (2013)
(11)	Shannon's entropy	$\lambda \sum_{i=1}^n x_i \ln x_i$	Yu *et al.* (2014)

Application to portfolio allocation

Table 110: Some constraints used in portfolio optimization

Item	Constraint	Ω
(12)	No cash and leverage	$\sum_{i=1}^{n} x_i = 1$
(13)	No short selling	$x_i \geq 0$
(14)	Weight bounds	$x_i^- \leq x_i \leq x_i^+$
(15)	Asset class limits	$c_j^- \leq \sum_{i \in \mathcal{C}_j} x_i \leq c_j^+$
(16)	Turnover	$\sum_{i=1}^{n} \lvert x_i - \tilde{x}_i \rvert \leq \boldsymbol{\tau}^+$
(17)	Transaction costs	$\sum_{i=1}^{n} \left(c_i^- \left(\tilde{x}_i - x_i \right)_+ + c_i^+ \left(x_i - \tilde{x}_i \right)_+ \right) \leq \boldsymbol{c}^+$
(18)	Leverage limit	$\sum_{i=1}^{n} \lvert x_i \rvert \leq \mathcal{L}^+$
(19)	Long/short exposure	$-\mathcal{LS}^- \leq \sum_{i=1}^{n} x_i \leq \mathcal{LS}^+$
(20)	Benchmarking	$\sqrt{\left(x - \tilde{x}\right)^\top \Sigma \left(x - \tilde{x}\right)} \leq \sigma^+$
(21)	Tracking error floor	$\sqrt{\left(x - \tilde{x}\right)^\top \Sigma \left(x - \tilde{x}\right)} \geq \sigma^-$
(22)	Active share floor	$\frac{1}{2} \sum_{i=1}^{n} \lvert x_i - \tilde{x}_i \rvert \geq \mathcal{AS}^-$
(23)	Number of active bets	$\left(x^\top x\right)^{-1} \geq \mathcal{N}^-$

Application to portfolio allocation

Most of portfolio optimization problems are a combination of:

1. an objective function (Table 108)
2. one or two regularization penalty functions (Table 109)
3. some constraints (Table 110)

Perrin and Roncalli (2020) solve **all these problems** using CCD, ADMM, Dykstra and the appropriate proximal functions. For that, they derive:

- the semi-analytical solution of the x-step for all objective functions
- the proximal solution of the y-step for all regularization penalty functions and constraints

Herfindahl-MV optimization

Formulation of the mathematical problem

- The second generation of minimum variance strategies uses a global diversification constraint
- The most popular solution is based on the Herfindahl index:

$$\mathcal{H}(x) = \sum_{i=1}^{n} x_i^2$$

- The effective number of bets is the inverse of the Herfindahl index:

$$\mathcal{N}(x) = \mathcal{H}(x)^{-1}$$

- The optimization program is:

$$\begin{aligned} x^\star &= \arg\min_x \frac{1}{2} x^\top \Sigma x \\ \text{s.t.} & \left\{ \begin{array}{l} \mathbf{1}_n^\top x = 1 \\ \mathbf{0}_n \leq x \leq x^+ \\ \mathcal{N}(x) \geq \mathcal{N}^- \end{array} \right. \end{aligned}$$

where $\mathcal{N}^-$ is the minimum number of effective bets.

Herfindahl-MV optimization

The QP solution

- The Herfindhal constraint is equivalent to:

$$\begin{aligned} \mathcal{N}(x) \geq \mathcal{N}^{-} & \Leftrightarrow & \left(x^\top x\right)^{-1} \geq \mathcal{N}^{-} \\ & \Leftrightarrow & x^\top x \leq \frac{1}{\mathcal{N}^{-}} \end{aligned}$$

- The QP problem is:

$$\begin{aligned} x^\star(\lambda) & = & \arg\min_x \frac{1}{2} x^\top \Sigma x + \lambda x^\top x = \frac{1}{2} x^\top \left(\Sigma + 2\lambda I_n\right) x \\ & \text{s.t.} & \left\{ \begin{array}{l} \mathbf{1}_n^\top x = 1 \\ \mathbf{0}_n \leq x \leq x^{+} \end{array} \right. \end{aligned}$$

  where $\lambda \geq 0$ is a scalar
- We have $\mathcal{N}(x) \in \left[\mathcal{N}\left(x^\star(0)\right), n\right]$
- The optimal value $\lambda^\star$ is found using the bi-section algorithm such that $\mathcal{N}\left(x^\star(\lambda)\right) = \mathcal{N}^{-}$

Herfindahl-MV optimization

The ADMM solution (first version)

- The ADMM form is:

$$\begin{aligned}\{x^\star, y^\star\} &= \arg\min_{(x,y)} \frac{1}{2} x^\top \Sigma x + \mathbb{1}_{\Omega_1}(x) + \mathbb{1}_{\Omega_2}(y) \\ \text{s.t.} &\quad x = y\end{aligned}$$

  where $\Omega_1 = \left\{x \in \mathbb{R}^n : \mathbf{1}_n^\top x = 1, \mathbf{0}_n \leq x \leq x^+\right\}$ and $\Omega_2 = \mathcal{B}_2\left(\mathbf{0}_n, \sqrt{\frac{1}{\mathcal{N}^-}}\right)$

- The x-update is a QP problem:

$$x^{(k+1)} = \arg\min_x \left\{\frac{1}{2} x^\top \left(\Sigma + \varphi I_n\right) x - \varphi x^\top \left(y^{(k)} - u^{(k)}\right) + \mathbb{1}_{\Omega_1}(x)\right\}$$

- The y-update is:

$$y^{(k+1)} = \frac{x^{(k+1)} + u^{(k)}}{\max\left(1, \sqrt{\mathcal{N}^-}\left\|x^{(k+1)} + u^{(k)}\right\|_2\right)}$$

Herfindahl-MV optimization

The ADMM solution (second version)

- A better approach is to write the problem as follows:

$$\begin{aligned} \{x^\star, y^\star\} &= \arg\min_{(x,y)} \frac{1}{2} x^\top \Sigma x + \mathbb{1}_{\Omega_3}(x) + \mathbb{1}_{\Omega_4}(y) \\ \text{s.t.} &\quad x = y \end{aligned}$$

  where $\Omega_3 = \mathcal{H}_{yperplane}\left[\mathbf{1}_n, 1\right]$ and $\Omega_4 = \mathcal{B}_{ox}\left[\mathbf{0}_n, x^+\right] \cap \mathcal{B}_2\left(\mathbf{0}_n, \sqrt{\frac{1}{\mathcal{N}^-}}\right)$

- The x-update is:

$$x^{(k+1)} = (\Sigma + \varphi I_n)^{-1} \left(\varphi \left(y^{(k)} - u^{(k)}\right) + \frac{1 - \mathbf{1}_n^\top (\Sigma + \varphi I_n)^{-1} \varphi \left(y^{(k)} - u^{(k)}\right)}{\mathbf{1}_n^\top (\Sigma + \varphi I_n)^{-1} \mathbf{1}_n} \mathbf{1}_n\right)$$

- The y-update is:

$$y^{(k+1)} = \mathcal{P}_{\mathcal{B}\mathrm{ox}-\mathcal{B}\mathrm{all}}\left(x^{(k+1)} + u^{(k)}; \mathbf{0}_n, x^+, \mathbf{0}_n, \sqrt{\frac{1}{\mathcal{N}^-}}\right)$$

  where $\mathcal{P}_{\mathcal{B}\mathrm{ox}-\mathcal{B}\mathrm{all}}$ corresponds to the Dykstra's algorithm given by Perrin and Roncalli (2020)

Herfindahl-MV optimization

Remark

If we compare the computational time of the three approaches, we observe that the best method is the second version of the ADMM algorithm:

$$\begin{aligned}\mathcal{CT}\left(\mathrm{QP}; n = 1000\right) &= 50 \times \mathcal{CT}\left(\mathrm{ADMM}_2; n = 1000\right) \\ \mathcal{CT}\left(\mathrm{ADMM}_1; n = 1000\right) &= 400 \times \mathcal{CT}\left(\mathrm{ADMM}_2; n = 1000\right)\end{aligned}$$

Herfindahl-MV optimization

The QP solution

Example 5

We consider an investment universe of eight stocks. We assume that their volatilities are 21%, 20%, 40%, 18%, 35%, 23%, **7%** and 29%. The correlation matrix is defined as follows:

$$\rho = \begin{pmatrix} 100\% & & & & & & & \\ 80\% & 100\% & & & & & & \\ 70\% & 75\% & 100\% & & & & & \\ 60\% & 65\% & 90\% & 100\% & & & & \\ 70\% & 50\% & 70\% & 85\% & 100\% & & & \\ 50\% & 60\% & 70\% & 80\% & 60\% & 100\% & & \\ 70\% & 50\% & 70\% & 75\% & 80\% & 50\% & 100\% & \\ 60\% & 65\% & 70\% & 75\% & 65\% & 70\% & 80\% & 100\% \end{pmatrix}$$

Herfindahl-MV optimization

Table 111: Minimum variance portfolios (in %)

$\mathcal{N}^{-}$	1.00	2.00	3.00	4.00	5.00	6.00	6.50	7.00	7.50	8.00
$x_1^\star$	0.00	3.22	9.60	13.83	15.18	15.05	14.69	14.27	13.75	12.50
$x_2^\star$	0.00	12.75	14.14	15.85	16.19	15.89	15.39	14.82	14.13	12.50
$x_3^\star$	0.00	0.00	0.00	0.00	0.00	0.07	2.05	4.21	6.79	12.50
$x_4^\star$	0.00	10.13	15.01	17.38	17.21	16.09	15.40	14.72	13.97	12.50
$x_5^\star$	0.00	0.00	0.00	0.00	0.71	5.10	6.33	7.64	9.17	12.50
$x_6^\star$	0.00	5.36	8.95	12.42	13.68	14.01	13.80	13.56	13.25	12.50
$x_7^\star$	100.00	68.53	52.31	40.01	31.52	25.13	22.92	20.63	18.00	12.50
$x_8^\star$	0.00	0.00	0.00	0.50	5.51	8.66	9.41	10.14	10.95	12.50
$\lambda^\star$ (in %)	0.00	1.59	3.10	5.90	10.38	18.31	23.45	31.73	49.79	∞

Note: the upper bound x^{+} is set to $\mathbf{1}_n$. The solutions are those found by the ADMM algorithm. We also report the value of $\lambda^\star$ found by the bi-section algorithm when we use the QP algorithm.

ERC portfolio optimization

We recall that:

$$x^{\star} = \arg \min_{x} \frac{1}{2} x^{\top} \Sigma x - \lambda \sum_{i=1}^{n} \ln x_i$$

and:

$$x_{\mathrm{erc}} = \frac{x^{\star}}{\mathbf{1}_n^{\top} x^{\star}}$$

ERC portfolio optimization

The CCD solution

- The first-order condition $(\Sigma x)_i - \lambda x_i^{-1} = 0$ implies that:

$$x_i^2 \sigma_i^2 + x_i \sigma_i \sum_{j \neq i} x_j \rho_{i,j} \sigma_j - \lambda = 0$$

- The CCD algorithm is:

$$x_i^{(k+1)} = \frac{-v_i^{(k+1)} + \sqrt{\left(v_i^{(k+1)}\right)^2 + 4\lambda\sigma_i^2}}{2\sigma_i^2}$$

where:

$$v_i^{(k+1)} = \sigma_i \sum_{j<i} x_j^{(k+1)} \rho_{i,j} \sigma_j + \sigma_i \sum_{j>i} x_j^{(k)} \rho_{i,j} \sigma_j$$

ERC portfolio optimization

The ADMM solution

- In the case of the ADMM algorithm, we set:

$$\begin{aligned} f_x(x) &= \frac{1}{2} x^\top \Sigma x \\ f_y(y) &= -\lambda \sum_{i=1}^{n} \ln y_i \\ x &= y \end{aligned}$$

- The x-update step is:

$$x^{(k+1)} = (\Sigma + \varphi I_n)^{-1} \varphi \left(y^{(k)} - u^{(k)} \right)$$

- The y-update step is:

$$y_i^{(k+1)} = \frac{1}{2} \left(\left(x_i^{(k+1)} + u_i^{(k)} \right) + \sqrt{\left(x_i^{(k+1)} + u_i^{(k)} \right)^2 + 4\lambda\varphi^{-1}} \right)$$

RB portfolio optimization

The RB portfolio is equal to:

$$x_{\mathrm{rb}} = \frac{x^{\star}}{\mathbf{1}_n^\top x^{\star}}$$

where $x^{\star}$ is the solution of the logarithmic barrier problem:

$$x^{\star} = \arg\min_{x} \mathcal{R}(x) - \lambda \sum_{i=1}^{n} \mathcal{RB}_i \cdot \ln x_i$$

λ is any positive scalar and $\mathcal{RB}_i$ is the risk budget allocated to Asset i

RB portfolio optimization

The CCD solution (SD risk measure)

- In the case of the standard deviation-based risk measure:

$$\mathcal{R}(x) = -x^\top (\mu - r) + \xi \sqrt{x^\top \Sigma x}$$

the first-order condition for defining the CCD algorithm is:

$$-(\mu_i - r) + \xi \frac{(\Sigma x)_i}{\sqrt{x^\top \Sigma x}} - \lambda \frac{\mathcal{RB}_i}{x_i} = 0$$

- It follows that $\xi x_i (\Sigma x)_i - (\mu_i - r) x_i \sigma(x) - \lambda \sigma(x) \cdot \mathcal{RB}_i = 0$ or equivalently:

$$\alpha_i x_i^2 + \beta_i x_i + \gamma_i = 0$$

where $\alpha_i = \xi \sigma_i^2$, $\beta_i = \xi \sigma_i \sum_{j \neq i} x_j \rho_{i,j} \sigma_j - (\mu_i - r) \sigma(x)$ and $\gamma_i = -\lambda \sigma(x) \cdot \mathcal{RB}_i$

RB portfolio optimization

The CCD solution (SD risk measure)

- The CCD algorithm is:

$$x_i^{(k+1)} = \frac{-\beta_i^{(k+1)} + \sqrt{\left(\beta_i^{(k+1)}\right)^2 - 4\alpha_i^{(k+1)}\gamma_i^{(k+1)}}}{2\alpha_i^{(k+1)}}$$

where:

$$\begin{cases} \alpha_i^{(k+1)} = \xi\sigma_i^2 \\ \beta_i^{(k+1)} = \xi\sigma_i\left(\sum_{j<i} x_j^{(k+1)}\rho_{i,j}\sigma_j + \sum_{j>i} x_j^{(k)}\rho_{i,j}\sigma_j\right) - (\mu_i - r)\,\sigma_i^{(k+1)}(x) \\ \gamma_i^{(k+1)} = -\lambda\sigma_i^{(k+1)}(x) \cdot \mathcal{RB}_i \\ \sigma_i^{(k+1)}(x) = \sqrt{\chi^\top \Sigma \chi} \\ \chi = \left(x_1^{(k+1)}, \ldots, x_{i-1}^{(k+1)}, x_i^{(k)}, x_{i+1}^{(k)} \ldots, x_n^{(k)}\right) \end{cases}$$

RB portfolio optimization

The ADMM solution (convex risk measure)

- We have:

$$\begin{aligned} \{x^{\star}, y^{\star}\} &= \arg\min_{x,y} \mathcal{R}(x) - \lambda \sum_{i=1}^{n} \mathcal{RB}_i \cdot \ln y_i \\ \text{s.t.} &\quad x = y \end{aligned}$$

- The ADMM algorithm is:

$$\left\{ \begin{array}{l} x^{(k+1)} = \mathbf{prox}_{\varphi^{-1}\mathcal{R}(x)} \left(y^{(k)} - u^{(k)}\right) \\ v_y^{(k+1)} = x^{(k+1)} + u^{(k)} \\ y^{(k+1)} = \frac{1}{2} \left(v_y^{(k+1)} + \sqrt{v_y^{(k+1)} \odot v_y^{(k+1)} + 4\lambda\varphi^{-1} \cdot \mathcal{RB}} \right) \\ u^{(k+1)} = u^{(k)} + x^{(k+1)} - y^{(k+1)} \end{array} \right.$$

Tips and tricks of portfolio optimization

- Full allocation — $\sum_{i=1}^{n} x_i = 1$:

$$\Omega = \mathcal{H}_{yperplane}\left[\mathbf{1}_n, 1\right]$$

We have:

$$\mathcal{P}_{\Omega}\left(v\right) = v - \left(\frac{\mathbf{1}_n^\top v - 1}{n}\right)\mathbf{1}_n$$

- Cash neutral — $\sum_{i=1}^{n} x_i = 0$:

$$\Omega = \mathcal{H}_{yperplane}\left[\mathbf{1}_n, 0\right]$$

We have:

$$\mathcal{P}_{\Omega}\left(v\right) = v - \left(\frac{\mathbf{1}_n^\top v}{n}\right)\mathbf{1}_n$$

Tips and tricks of portfolio optimization

- No short selling — $x \geq \mathbf{0}_n$:

$$\Omega = \mathcal{B}_{ox}\left[\mathbf{0}_n, \infty\right]$$

  We have:

$$\mathcal{P}_{\Omega}(v) = \mathcal{T}\left(v; \mathbf{0}_n, \infty\right)$$

- Weight bounds — $x^{-} \leq x \leq x^{+}$:

$$\Omega = \mathcal{B}_{ox}\left[x^{-}, x^{+}\right]$$

  We have:

$$\mathcal{P}_{\Omega}(v) = \mathcal{T}\left(v; x^{-}, x^{+}\right)$$

Tips and tricks of portfolio optimization

- μ-problem — $\mu(x) \geq \mu^{\star}$:

$$\Omega = \mathcal{H}_{alfspace}\left[-\mu, -\mu^{\star}\right]$$

We have:

$$\mathcal{P}_{\Omega}(v) = v + \frac{\left(\mu^{\star} - \mu^{\top} v\right)_{+}}{\left\|\mu\right\|_{2}^{2}} \mu$$

Tips and tricks of portfolio optimization

- σ-problem — $\sigma(x) \leq \sigma^{\star}$:

$$\Omega = \left\{ x : \sqrt{x^{\top}\Sigma x} \leq \sigma^{\star} \right\}$$

We have:

$$\begin{aligned} \sqrt{x^{\top}\Sigma x} \leq \sigma^{\star} & \Leftrightarrow & \sqrt{x^{\top}\left(LL^{\top}\right)x} \leq \sigma^{\star} \\ & \Leftrightarrow & \left\| y^{\top} y \right\|_2 \leq \sigma^{\star} \\ & \Leftrightarrow & y \in \mathcal{B}_2\left(\mathbf{0}_n, \sigma^{\star}\right) \end{aligned}$$

where $y = L^{\top}x$ and L is the Cholesky decomposition of Σ. It follows that the proximal of the y-update is the projection onto the ℓ_2 ball $\mathcal{B}_2\left(\mathbf{0}_n, \sigma^{\star}\right)$:

$$\begin{aligned} \mathcal{P}_{\Omega}(v) &=& v - \mathbf{prox}_{\sigma^{\star}\|x\|_2}(v) \\ &=& v - \left(1 - \frac{\sigma^{\star}}{\max\left(\sigma^{\star}, \|v\|_2\right)}\right) v \end{aligned}$$

Tips and tricks of portfolio optimization

- Leverage management — $\sum_{i=1}^{n} |x_i| \leq \mathcal{L}^+$:

$$\begin{aligned}\Omega &= \left\{x : \|x\|_1 \leq \mathcal{L}^+\right\} \\ &= \mathcal{B}_1\left(\mathbf{0}_n, \mathcal{L}^+\right)\end{aligned}$$

The proximal of the y-update is the projection onto the ℓ_1 ball $\mathcal{B}_1\left(\mathbf{0}_n, \mathcal{L}^+\right)$:

$$\mathcal{P}_{\Omega}(v) = v - \operatorname{sign}(v) \odot \mathbf{prox}_{\mathcal{L}^+ \max x}(|v|)$$

Tips and tricks of portfolio optimization

- Leverage management — $\mathcal{LS}^{-} \leq \sum_{i=1}^{n} x_i \leq \mathcal{LS}^{+}$:

$$\Omega = \mathcal{H}_{alfspace}\left[\mathbf{1}_n, \mathcal{LS}^{+}\right] \cap \mathcal{H}_{alfspace}\left[-\mathbf{1}_n, -\mathcal{LS}^{-}\right]$$

The proximal of the y-update is obtained with the Dykstra's algorithm by combining the two half-space projections.

- Leverage management — $\left|\sum_{i=1}^{n} x_i\right| \leq \mathcal{L}^{+}$:

$$\Omega = \left\{x : \left|\mathbf{1}_n^\top x\right| \leq \mathcal{L}^{+}\right\}$$

This is a special case of the previous result where $\mathcal{LS}^{+} = \mathcal{L}^{+}$ and $\mathcal{LS}^{-} = -\mathcal{L}^{+}$:

$$\Omega = \mathcal{H}_{alfspace}\left[\mathbf{1}_n, \mathcal{L}^{+}\right] \cap \mathcal{H}_{alfspace}\left[-\mathbf{1}_n, \mathcal{L}^{+}\right]$$

Tips and tricks of portfolio optimization

- Concentration management[31]
  Portfolio managers can also use another constraint concerning the sum of the k largest values:

$$f(x) = \sum_{i=n-k+1}^{n} x_{(i:n)} = x_{(n:n)} + \ldots + x_{(n-k+1:n)}$$

where $x_{(i:n)}$ is the order statistics of x: $x_{(1:n)} \leq x_{(2:n)} \leq \cdots \leq x_{(n:n)}$. Beck (2017) shows that:

$$\mathbf{prox}_{\lambda f(x)}(v) = v - \lambda \mathcal{P}_{\Omega}\left(\frac{v}{\lambda}\right)$$

where:

$$\Omega = \left\{x \in [0,1]^n : \mathbf{1}_n^\top x = k\right\} = \mathcal{B}_{ox}[\mathbf{0}_n, \mathbf{1}_n] \cap \mathcal{H}_{yperlane}[\mathbf{1}_n, k]$$

[31] An example is the 5/10/40 UCITS rule: A UCITS fund may invest no more than 10% of its net assets in transferable securities or money market instruments issued by the same body, with a further aggregate limitation of 40% of net assets on exposures of greater than 5% to single issuers.

Tips and tricks of portfolio optimization

- Entropy portfolio management
  Bera and Park (2008) propose using a cross-entropy measure as the objective function:

$$
\begin{aligned}
x^{\star} &= \arg\min_{x} \mathrm{KL}\left(x \mid \tilde{x}\right) \\
& \text{s.t.} \quad \left\{
\begin{array}{l}
\mathbf{1}_n^{\top} x = 1 \\
\mathbf{0}_n \leq x \leq \mathbf{1}_n \\
\mu\left(x\right) \geq \mu^{\star}, \sigma\left(x\right) \leq \sigma^{\star}
\end{array}
\right.
\end{aligned}
$$

  where $\mathrm{KL}\left(x \mid \tilde{x}\right)$ is the Kullback-Leibler measure:

$$
\mathrm{KL}\left(x \mid \tilde{x}\right) = \sum_{i=1}^{n} x_i \ln\left(x_i / \tilde{x}_i\right)
$$

  and $\tilde{x}$ is a reference portfolio

Tips and tricks of portfolio optimization

- Entropy portfolio management
  We have:

$$\mathbf{prox}_{\lambda\,\mathrm{KL}(v|\tilde{x})}(v) = \lambda \begin{pmatrix} W\left(\lambda^{-1}\tilde{x}_1 e^{\lambda^{-1}v_1 - \tilde{x}_1^{-1}}\right) \\ \vdots \\ W\left(\lambda^{-1}\tilde{x}_n e^{\lambda^{-1}v_n - \tilde{x}_n^{-1}}\right) \end{pmatrix}$$

  where $W(x)$ is the Lambert W function

Tips and tricks of portfolio optimization

Remark

Since the Shannon's entropy is equal to $\mathrm{SE}(x) = -\mathrm{KL}(x \mid \mathbf{1}_n)$, we deduce that:

$$\mathbf{prox}_{\lambda\,\mathrm{SE}(x)}(v) = \lambda \begin{pmatrix} W\left(\lambda^{-1} e^{\lambda^{-1} v_1 - 1}\right) \\ \vdots \\ W\left(\lambda^{-1} e^{\lambda^{-1} v_n - 1}\right) \end{pmatrix}$$

Tips and tricks of portfolio optimization

- Active share constraint — $\mathcal{AS}(x \mid \tilde{x}) \geq \mathcal{AS}^{-}$:

$$\mathcal{AS}(x \mid \tilde{x}) = \frac{1}{2}\sum_{i=1}^{n} |x_i - \tilde{x}_i| \geq \mathcal{AS}^{-}$$

We use the projection onto the complement $\bar{\mathcal{B}}_1(c, r)$ of the ℓ_1 ball and we obtain:

$$\mathcal{P}_{\Omega}(v) = v + \operatorname{sign}(v - \tilde{x}) \odot \frac{\max\left(2\mathcal{AS}^{-} - \left\| v - \tilde{x} \right\|_1, 0\right)}{n}$$

Tips and tricks of portfolio optimization

- Tracking error volatility — $\sigma(x \mid \tilde{x}) \leq \sigma^\star$:

$$\begin{aligned}\sigma(x \mid \tilde{x}) \leq \sigma^\star & \Leftrightarrow & \sqrt{(x-\tilde{x})^\top \Sigma (x-\tilde{x})} \leq \sigma^\star \\ & \Leftrightarrow & \|y\|_2 \leq \sigma^\star \\ & \Leftrightarrow & y \in \mathcal{B}_2(\mathbf{0}_n, \sigma^\star)\end{aligned}$$

where $y = L^\top x - L^\top \tilde{x}$. It follows that $Ax + By = c$ where $A = L^\top$, $B = -I_n$ and $c = L^\top \tilde{x}$. It follows that the proximal of the y-update is the projection onto the ℓ_2 ball $\mathcal{B}_2(\mathbf{0}_n, \sigma^\star)$:

$$\begin{aligned}\mathcal{P}_\Omega(v) &= v - \mathbf{prox}_{\sigma^\star \|x\|_2}(v) \\ &= v - \left(1 - \frac{\sigma^\star}{\max(\sigma^\star, \|v\|_2)}\right) v\end{aligned}$$

Tips and tricks of portfolio optimization

- Bid-ask transaction cost management:

$$\boldsymbol{c}\left(x \mid x_0\right) = \lambda \sum_{i=1}^{n} \left(c_i^{-}\left(x_{0,i} - x_i\right)_{+} + c_i^{+}\left(x_i - x_{0,i}\right)_{+}\right)$$

where c_i^{-} and c_i^{+} are the bid and ask transaction costs. We have:

$$\mathbf{prox}_{\boldsymbol{c}(x|x_0)}\left(v\right) = x_0 + \mathcal{S}\left(v - x_0; \lambda c^{-}, \lambda c^{+}\right)$$

where $\mathcal{S}\left(v; \lambda_{-}, \lambda_{+}\right) = \left(v - \lambda_{+}\right)_{+} - \left(v + \lambda_{-}\right)_{-}$ is the two-sided soft-thresholding operator.

Tips and tricks of portfolio optimization

- Turnover management:

$$\Omega = \left\{ x \in \mathbb{R}^n : \left\| x - x_0 \right\|_1 \leq \boldsymbol{\tau}^+ \right\}$$

The proximal operator is:

$$\mathcal{P}_\Omega (v) = v - \operatorname{sign}(v - x_0) \odot \min\left(\left| v - x_0 \right|, s^\star \right)$$

where $s^\star = \left\{ s \in \mathbb{R} : \sum_{i=1}^n \left(\left| v_i - x_{0,i} \right| - s \right)_+ = \boldsymbol{\tau}^+ \right\}$.

Pattern learning and self-automated strategies

Table 112: What works / What doesn't

	Bond Scoring	Stock Picking	Trend Filtering	Mean Reverting	Index Tracking	HF Tracking	Stock Classification	Technical Analysis
Lasso		☺	☺	☺	☹	☺		
NMF							☺	☹
Boosting		☺				☺		
Bagging		☺				☺		
Random forests	☺			☹				☹
Neural nets	☺					☹		
SVM	☺	☹	☹				☹	
Sparse Kalman					☹	☺		
K-NN	☹							
K-means	☺						☺	
Testing protocols[32]	☺	☺	☺	☺		☺		

Source: Roncalli (2014), Big Data in Asset Management, ESMA/CEMA/GEA meeting, Madrid.

[32] Cross-validation, training/test/probe sets, K-fold, etc.

Pattern learning and self-automated strategies

$$2021 \neq 2014$$

The evolution of machine learning in finance is fast, very fast!

Pattern learning and self-automated strategies

Some examples

- Natural Language Processing (NLP)
- Deep learning (DL)
- Reinforcement learning (RL)
- Gaussian process (GP) and Bayesian optimization (BO)
- Learning to rank (MLR)
- Etc.

Some applications

- Robo-advisory
- Stock classification
- $Q_1 - Q_5$ long/short strategy
- Trend-following strategies
- Mean-reverting strategies
- Scoring models
- Sentiment and news analysis
- Etc.

Market generators

- The underlying idea is to simulate artificial multi-dimensional financial time series, whose statistical properties are the same as those observed in the financial markets

$\approx$ **Monte Carlo simulation of the financial market**

- 3 main approaches:
  1. Restricted Boltzmann machines (RBM)
  2. Generative adversarial networks (GAN)
  3. Convolutional Wasserstein models (W-GAN)
- The goal is to:
  - improve the the risk management of quantitative investment strategies
  - avoid the over-fitting bias of backtesting

The current research shows that results are disappointed until now

Portfolio optimization with CCD and ADMM algorithms

Question 1

We consider the following optimization program:

$$x^{\star} = \arg\min \frac{1}{2} x^{\top} \Sigma x - \lambda \sum_{i=1}^{n} b_i \ln x_i$$

where Σ is the covariance matrix, b is a vector of positive budgets and x is the vector of portfolio weights.

Portfolio optimization with CCD and ADMM algorithms

Question 1.a

Write the first-order condition with respect to the coordinate x_i and show that the solution $x^{\star}$ corresponds to a risk-budgeting portfolio.

Portfolio optimization with CCD and ADMM algorithms

We have:

$$\mathcal{L}(x;\lambda) = \arg\min \frac{1}{2} x^\top \Sigma x - \lambda \sum_{i=1}^{n} b_i \ln x_i$$

The first-order condition is:

$$\frac{\partial\,\mathcal{L}(x;\lambda)}{\partial\, x_i} = (\Sigma x)_i - \lambda \frac{b_i}{x_i} = 0$$

or:

$$x_i \cdot (\Sigma x)_i = \lambda b_i$$

Portfolio optimization with CCD and ADMM algorithms

If we assume that the risk measure is the portfolio volatility:

$$\mathcal{R}(x) = \sqrt{x^\top \Sigma x}$$

the risk contribution of Asset i is equal to:

$$\mathcal{RC}_i(x) = \frac{x_i \cdot (\Sigma x)_i}{\sqrt{x^\top \Sigma x}}$$

We deduce that the optimization problem defines a risk budgeting portfolio:

$$\frac{x_i \cdot (\Sigma x)_i}{b_i} = \frac{x_j \cdot (\Sigma x)_j}{b_j} = \lambda \Leftrightarrow \frac{\mathcal{RC}_i(x)}{b_i} = \frac{\mathcal{RC}_j(x)}{b_j}$$

where the risk measure is the portfolio volatility and the risk budgets are $(b_1, \ldots, b_n)$.

Portfolio optimization with CCD and ADMM algorithms

Question 1.b

Find the optimal value $x_i^\star$ when we consider the other coordinates $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ as fixed.

Portfolio optimization with CCD and ADMM algorithms

The first-order condition is equivalent to:

$$x_i \cdot (\Sigma x)_i - \lambda b_i = 0$$

We have:

$$(\Sigma x)_i = x_i \sigma_i^2 + \sigma_i \sum_{j \neq i} x_j \rho_{i,j} \sigma_j$$

It follows that:

$$x_i^2 \sigma_i^2 + x_i \sigma_i \sum_{j \neq i} x_j \rho_{i,j} \sigma_j - \lambda b_i = 0$$

Portfolio optimization with CCD and ADMM algorithms

We obtain a second-degree equation:

$$\alpha_i x_i^2 + \beta_i x_i + \gamma_i = 0$$

where:

$$\begin{cases} \alpha_i = \sigma_i^2 \\ \beta_i = \sigma_i \sum_{j \neq i} x_j \rho_{i,j} \sigma_j \\ \gamma_i = -\lambda b_i \end{cases}$$

1. The polynomial function is convex because we have $\alpha_i = \sigma_i^2 > 0$
2. The product of the roots is negative:

$$x_i' x_i'' = \frac{\gamma_i}{\alpha_i} = -\frac{\lambda b_i}{\sigma_i^2} < 0$$

3. The discriminant is positive:

$$\Delta = \beta_i^2 - 4\alpha_i\gamma_i = \left(\sigma_i \sum_{j \neq i} \rho_{i,j}\sigma_j y_j\right)^2 + 4\lambda b_i \sigma_i^2 > 0$$

Portfolio optimization with CCD and ADMM algorithms

We always have two solutions with opposite signs. We deduce that the solution is the positive root of the second-degree equation:

$$\begin{aligned} x_i^\star &= x_i'' = \frac{-\beta_i + \sqrt{\beta_i^2 - 4\alpha_i\gamma_i}}{2\alpha_i} \\ &= \frac{-\sigma_i \sum_{j\neq i} x_j\rho_{i,j}\sigma_j + \sqrt{\sigma_i^2 \left(\sum_{j\neq i} x_j\rho_{i,j}\sigma_j\right)^2 + 4\lambda b_i\sigma_i^2}}{2\sigma_i^2} \end{aligned}$$

Portfolio optimization with CCD and ADMM algorithms

Question 1.c

We note $x_i^{(k)}$ the value of the i^{th} coordinate at the k^{th} iteration. Deduce the corresponding CCD algorithm. How to find the RB portfolio x_{rb}?

Portfolio optimization with CCD and ADMM algorithms

The CCD algorithm consists in iterating the following formula:

$$x_i^{(k)} = \frac{-\beta_i^{(k)} + \sqrt{\left(\beta_i^{(k)}\right)^2 - 4\alpha_i^{(k)}\gamma_i^{(k)}}}{2\alpha_i^{(k)}}$$

where:

$$\left\{\begin{array}{l} \alpha_i^{(k)} = \sigma_i^2 \\ \beta_i^{(k)} = \sigma_i \left(\sum_{j<i} \rho_{i,j}\sigma_j x_j^{(k)} + \sum_{j>i} \rho_{i,j}\sigma_j x_j^{(k-1)}\right) \\ \gamma_i^{(k)} = -\lambda b_i \end{array}\right.$$

The RB portfolio is the scaled solution:

$$x_{\mathrm{rb}} = \frac{x^\star}{\sum_{i=1}^n x_i^\star}$$

Portfolio optimization with CCD and ADMM algorithms

Question 1.d

We consider a universe of three assets, whose volatilities are equal to 20%, 25% and 30%. The correlation matrix is equal to:

$$\rho = \left(\begin{array}{rrr} 100\% & & \\ 50\% & 100\% & \\ 60\% & 70\% & 100\% \end{array} \right)$$

We would like to compute the ERC portfolio[a] using the CCD algorithm. We initialize the CCD algorithm with the following starting values $x^{(0)} = (33.3\%, 33.3\%, 33.3\%)$. We assume that $\lambda = 1$.

[a] This means that:

$$b_i = \frac{1}{3}$$

Portfolio optimization with CCD and ADMM algorithms

Question 1.d.i

Starting from $x^{(0)}$, find the optimal coordinate $x_1^{(1)}$ for the first asset.

Portfolio optimization with CCD and ADMM algorithms

We have:

$$\left\{\begin{array}{l}\alpha_1^{(1)} = 0.2^2 = 4\% \\ \beta_1^{(1)} = 0.02033 \\ \gamma_i^{(1)} = -0.333\%\end{array}\right.$$

We obtain:

$$x_1^{(1)} = 2.64375$$

Portfolio optimization with CCD and ADMM algorithms

Question 1.d.ii

Compute then the optimal coordinate $x_2^{(1)}$ for the second asset.

Portfolio optimization with CCD and ADMM algorithms

We have:

$$\begin{cases} \alpha_2^{(1)} = 0.25^2 = 6.25\% \\ \beta_2^{(1)} = 0.08359 \\ \gamma_2^{(1)} = -0.333\% \end{cases}$$

We obtain:

$$x_2^{(1)} = 1.73553$$

Portfolio optimization with CCD and ADMM algorithms

Question 1.d.iii

Compute then the optimal coordinate $x_3^{(1)}$ for the third asset.

Portfolio optimization with CCD and ADMM algorithms

We have:

$$\left\{\begin{array}{l} \alpha_3^{(1)} = 0.3^2 = 9\% \\ \beta_3^{(1)} = 0.18629 \\ \gamma_3^{(1)} = -0.333\% \end{array}\right.$$

We obtain:

$$x_3^{(1)} = 1.15019$$

Portfolio optimization with CCD and ADMM algorithms

Question 1.d.iv

Give the CCD coordinates $x_i^{(k)}$ for $k = 1, \ldots, 10$.

Portfolio optimization with CCD and ADMM algorithms

Table 113: CCD coordinates ($k = 1, \ldots, 5$)

k	i	$\alpha_i^{(k)}$	$\beta_i^{(k)}$	$\gamma_i^{(k)}$	$x_i^{(k)}$	CCD coordinates x_1	x_2	x_3
0						0.33333	0.33333	0.33333
1	1	0.04000	0.02033	−0.33333	2.64375	2.64375	0.33333	0.33333
1	2	0.06250	0.08359	−0.33333	1.73553	2.64375	1.73553	0.33333
1	3	0.09000	0.18629	−0.33333	1.15019	2.64375	1.73553	1.15019
2	1	0.04000	0.08480	−0.33333	2.01525	2.01525	1.73553	1.15019
2	2	0.06250	0.11077	−0.33333	1.58744	2.01525	1.58744	1.15019
2	3	0.09000	0.15589	−0.33333	1.24434	2.01525	1.58744	1.24434
3	1	0.04000	0.08448	−0.33333	2.01782	2.01782	1.58744	1.24434
3	2	0.06250	0.11577	−0.33333	1.56202	2.01782	1.56202	1.24434
3	3	0.09000	0.15465	−0.33333	1.24842	2.01782	1.56202	1.24842
4	1	0.04000	0.08399	−0.33333	2.02183	2.02183	1.56202	1.24842
4	2	0.06250	0.11609	−0.33333	1.56044	2.02183	1.56044	1.24842
4	3	0.09000	0.15471	−0.33333	1.24821	2.02183	1.56044	1.24821
5	1	0.04000	0.08395	−0.33333	2.02222	2.02222	1.56044	1.24821
5	2	0.06250	0.11609	−0.33333	1.56044	2.02222	1.56044	1.24821
5	3	0.09000	0.15472	−0.33333	1.24817	2.02222	1.56044	1.24817

Portfolio optimization with CCD and ADMM algorithms

Table 114: CCD coordinates ($k = 6, \ldots, 10$)

k	i	$\alpha_i^{(k)}$	$\beta_i^{(k)}$	$\gamma_i^{(k)}$	$x_i^{(k)}$	CCD coordinates x_1	x_2	x_3
0						0.33333	0.33333	0.33333
6	1	0.04000	0.08395	−0.33333	2.02223	2.02223	1.56044	1.24817
6	2	0.06250	0.11608	−0.33333	1.56045	2.02223	1.56045	1.24817
6	3	0.09000	0.15472	−0.33333	1.24816	2.02223	1.56045	1.24816
7	1	0.04000	0.08395	−0.33333	2.02223	2.02223	1.56045	1.24816
7	2	0.06250	0.11608	−0.33333	1.56046	2.02223	1.56046	1.24816
7	3	0.09000	0.15472	−0.33333	1.24816	2.02223	1.56046	1.24816
8	1	0.04000	0.08395	−0.33333	2.02223	2.02223	1.56046	1.24816
8	2	0.06250	0.11608	−0.33333	1.56046	2.02223	1.56046	1.24816
8	3	0.09000	0.15472	−0.33333	1.24816	2.02223	1.56046	1.24816
9	1	0.04000	0.08395	−0.33333	2.02223	2.02223	1.56046	1.24816
9	2	0.06250	0.11608	−0.33333	1.56046	2.02223	1.56046	1.24816
9	3	0.09000	0.15472	−0.33333	1.24816	2.02223	1.56046	1.24816
10	1	0.04000	0.08395	−0.33333	2.02223	2.02223	1.56046	1.24816
10	2	0.06250	0.11608	−0.33333	1.56046	2.02223	1.56046	1.24816
10	3	0.09000	0.15472	−0.33333	1.24816	2.02223	1.56046	1.24816

Portfolio optimization with CCD and ADMM algorithms

Question 1.d.v

Deduce the ERC portfolio.

Portfolio optimization with CCD and ADMM algorithms

The CCD algorithm has converged to the following solution:

$$x^{\star} = \begin{pmatrix} 2.02223 \\ 1.56046 \\ 1.24816 \end{pmatrix}$$

Since $\sum_{i=1}^{3} x_i^{\star} = 4.83085$, we deduce that:

$$x_{\mathrm{erc}} = \frac{1}{4.83085} \begin{pmatrix} 2.02223 \\ 1.56046 \\ 1.24816 \end{pmatrix} = \begin{pmatrix} 41.86076\% \\ 32.30189\% \\ 25.83736\% \end{pmatrix}$$

Portfolio optimization with CCD and ADMM algorithms

Question 1.d.vi

Compute the variance of the previous CCD solution. What do you notice? Explain this result.

Portfolio optimization with CCD and ADMM algorithms

We remind that the CCD solution is:

$$x^{\star} = \begin{pmatrix} 2.02223 \\ 1.56046 \\ 1.24816 \end{pmatrix}$$

We have:

$$\sigma^{2}\left(x^{\star}\right) = x^{\star\top}\Sigma x^{\star} = 1$$

We notice that:

$$\sigma^{2}\left(x^{\star}\right) = \lambda$$

Portfolio optimization with CCD and ADMM algorithms

At the optimum, we remind that:

$$\lambda = \frac{x_i^\star \cdot (\Sigma x^\star)_i}{b_i} = \frac{x_i^\star \cdot (\Sigma x^\star)_i}{n^{-1}}$$

We deduce that:

$$\begin{aligned}
\lambda &= \frac{1}{n} \sum_{i=1}^{n} \frac{x_i^\star \cdot (\Sigma x^\star)_i}{n^{-1}} \\
&= \sum_{i=1}^{n} x_i^\star \cdot (\Sigma x^\star)_i \\
&= x^{\star\top} \Sigma x^\star \\
&= \sigma^2 (x^\star)
\end{aligned}$$

It follows that the portfolio variance of the CCD solution is exactly equal to λ.

Portfolio optimization with CCD and ADMM algorithms

Question 1.d.vii

Verify that the CCD solution converges faster to the ERC portfolio when we assume that $\lambda = x_{\mathrm{erc}}^\top \Sigma x_{\mathrm{erc}}$.

Portfolio optimization with CCD and ADMM algorithms

We have:

$$\sigma(x_{\mathrm{erc}}) = \sqrt{x_{\mathrm{erc}}^\top \Sigma x_{\mathrm{erc}}} = 20.70029\%$$

and:

$$\sigma^2(x_{\mathrm{erc}}) = 4.28502\%$$

We obtain the results given in Table 115 when $\lambda = 4.28502\%$. If we compare with those given in Tables 113 and 114, it is obvious that the convergence is faster in the present case.

Portfolio optimization with CCD and ADMM algorithms

Table 115: CCD coordinates ($k = 1, \ldots, 5$)

k	i	$\alpha_i^{(k)}$	$\beta_i^{(k)}$	$\gamma_i^{(k)}$	$x_i^{(k)}$	CCD coordinates		
						x_1	x_2	x_3
0						0.33333	0.33333	0.33333
1	1	0.04000	0.02033	−0.01428	0.39521	0.39521	0.33333	0.33333
1	2	0.06250	0.02738	−0.01428	0.30680	0.39521	0.30680	0.33333
1	3	0.09000	0.03033	−0.01428	0.26403	0.39521	0.30680	0.26403
2	1	0.04000	0.01718	−0.01428	0.42027	0.42027	0.30680	0.26403
2	2	0.06250	0.02437	−0.01428	0.32133	0.42027	0.32133	0.26403
2	3	0.09000	0.03200	−0.01428	0.25847	0.42027	0.32133	0.25847
3	1	0.04000	0.01734	−0.01428	0.41893	0.41893	0.32133	0.25847
3	2	0.06250	0.02404	−0.01428	0.32295	0.41893	0.32295	0.25847
3	3	0.09000	0.03204	−0.01428	0.25835	0.41893	0.32295	0.25835
4	1	0.04000	0.01737	−0.01428	0.41863	0.41863	0.32295	0.25835
4	2	0.06250	0.02403	−0.01428	0.32302	0.41863	0.32302	0.25835
4	3	0.09000	0.03203	−0.01428	0.25837	0.41863	0.32302	0.25837
5	1	0.04000	0.01738	−0.01428	0.41861	0.41861	0.32302	0.25837
5	2	0.06250	0.02403	−0.01428	0.32302	0.41861	0.32302	0.25837
5	3	0.09000	0.03203	−0.01428	0.25837	0.41861	0.32302	0.25837

Portfolio optimization with CCD and ADMM algorithms

Question 2

We recall that the ADMM algorithm is based on the following optimization problem:

$$\begin{aligned}
\{x^\star, y^\star\} &= \arg\min f_x(x) + f_y(y) \\
\text{s.t.} &\quad Ax + By = c
\end{aligned}$$

Portfolio optimization with CCD and ADMM algorithms

Question 2.a

Describe the ADMM algorithm.

Portfolio optimization with CCD and ADMM algorithms

The ADMM algorithm consists in the following iterations:

$$\left\{\begin{array}{l} x^{(k+1)} = \arg\min_x \left\{ f_x(x) + \dfrac{\varphi}{2}\left\| Ax + By^{(k)} - c + u^{(k)} \right\|_2^2 \right\} \\ y^{(k+1)} = \arg\min_y \left\{ f_y(y) + \dfrac{\varphi}{2}\left\| Ax^{(k+1)} + By - c + u^{(k)} \right\|_2^2 \right\} \\ u^{(k+1)} = u^{(k)} + \left(Ax^{(k+1)} + By^{(k+1)} - c \right) \end{array}\right.$$

Portfolio optimization with CCD and ADMM algorithms

Question 2.b

We consider the following optimization problem:

$$\begin{aligned} w^{\star}(\gamma) &= \arg\min \frac{1}{2}(w-b)^{\top}\Sigma(w-b)-\gamma(w-b)^{\top}\mu \\ & \text{s.t.} \quad \left\{ \begin{array}{l} \mathbf{1}_n^{\top} w = 1 \\ \sum_{i=1}^{n} |w_i - b_i| \leq \boldsymbol{\tau}^{+} \\ \mathbf{0}_n \leq w \leq \mathbf{1}_n \end{array} \right. \end{aligned}$$

Portfolio optimization with CCD and ADMM algorithms

Question 2.b.i

Give the meaning of the symbols w, b, Σ, and μ. What is the goal of this optimization program? What is the meaning of the constraint $\sum_{i=1}^{n} |w_i - b_i| \leq \boldsymbol{\tau}^{+}$?

Portfolio optimization with CCD and ADMM algorithms

- w is the vector of portfolio weights:

$$w = (w_1, \ldots, w_n)$$

- b is the vector of benchmark weights:

$$b = (b_1, \ldots, b_n)$$

- Σ is the covariance matrix of asset returns
- μ is the vector of expected returns

Portfolio optimization with CCD and ADMM algorithms

The goal of the optimization problem is to tilt a benchmark portfolio by controlling the volatility of the tracking error:

$$\sigma\left(w \mid b\right) = \sqrt{\left(w - b\right)^{\top} \Sigma \left(w - b\right)}$$

and improving the expected excess return:

$$\mu\left(w \mid b\right) = \left(w - b\right)^{\top} \mu$$

This is a typical γ-problem when there is a benchmark

Portfolio optimization with CCD and ADMM algorithms

We remind that the turnover between the benchmark b and the portfolio w is equal to:

$$\boldsymbol{\tau}\left(w \mid b\right) = \sum_{i=1}^{n} \left|w_i - b_i\right|$$

Therefore, we impose that the turnover is less than an upper limit:

$$\boldsymbol{\tau}\left(w \mid b\right) \leq \boldsymbol{\tau}^{+}$$

Portfolio optimization with CCD and ADMM algorithms

Question 2.b.ii

What is the best way to specify $f_x(x)$ and $f_y(y)$ in order to find numerically the solution. Justify your choice.

Portfolio optimization with CCD and ADMM algorithms

The best way to specify $f_x(x)$ and $f_y(y)$ is to split the QP problem and the turnover constraint:

$$\begin{aligned}
\{x^\star, y^\star\} &= \arg\min_{x,y} f_x(x) + f_y(y) \\
\text{s.t.} \quad & x - y = \mathbf{0}_n
\end{aligned}$$

where:

$$\begin{aligned}
f_x(x) &= \frac{1}{2}(x-b)^\top \Sigma (x-b) - \gamma (x-b)^\top \mu + \mathbb{1}_{\Omega_1}(x) + \mathbb{1}_{\Omega_3}(x) \\
f_y(y) &= \mathbb{1}_{\Omega_2}(y) \\
\Omega_1(x) &= \left\{x : \mathbf{1}_n^\top x = 1\right\} \\
\Omega_2(y) &= \left\{y : \sum_{i=1}^n |y_i - b_i| \leq \boldsymbol{\tau}^+\right\} \\
\Omega_3(x) &= \{x : \mathbf{0}_n \leq x \leq \mathbf{1}_n\}
\end{aligned}$$

Indeed, the x-update step is a standard QP problem whereas the y-update step is the projection onto the ℓ_1-ball $\mathcal{B}_1(b, \boldsymbol{\tau}^+)$.

Portfolio optimization with CCD and ADMM algorithms

Question 2.b.iii

Give the corresponding ADMM algorithm.

Portfolio optimization with CCD and ADMM algorithms

We have:

$$\begin{aligned}
(*) &= \frac{1}{2}(x-b)^\top \Sigma (x-b) - \gamma (x-b)^\top \mu \\
&= \frac{1}{2}x^\top \Sigma x - x^\top \Sigma b + \frac{1}{2}b^\top \Sigma b - \gamma x^\top \mu + \gamma b^\top \mu \\
&= \frac{1}{2}x^\top \Sigma x - x^\top (\Sigma b + \gamma \mu) + \underbrace{\left(\gamma b^\top \mu + \frac{1}{2}b^\top \Sigma b\right)}_{\text{constant}}
\end{aligned}$$

Portfolio optimization with CCD and ADMM algorithms

If we note $v_x^{(k+1)} = y^{(k)} - u^{(k)}$, we have:

$$\begin{aligned}
\left\| x - y^{(k)} + u^{(k)} \right\|_2^2 &= \left\| x - v_x^{(k+1)} \right\|_2^2 \\
&= \left(x - v_x^{(k+1)} \right)^\top \left(x - v_x^{(k+1)} \right) \\
&= x^\top I_n x - 2x^\top v_x^{(k+1)} + \underbrace{\left(v_x^{(k+1)} \right)^\top v_x^{(k+1)}}_{\text{constant}}
\end{aligned}$$

Portfolio optimization with CCD and ADMM algorithms

It follows that:

$$\begin{aligned}
f_x^{(k+1)}(x) &= f_x(x) + \frac{\varphi}{2}\left\| x - y^{(k)} + u^{(k)} \right\|_2^2 \\
&= \frac{1}{2}(x-b)^\top \Sigma (x-b) - \gamma (x-b)^\top \mu + \\
&\quad \mathbb{1}_{\Omega_1}(x) + \mathbb{1}_{\Omega_3}(x) + \frac{\varphi}{2}\left\| x - y^{(k)} + u^{(k)} \right\|_2^2 \\
&= \frac{1}{2} x^\top (\Sigma + \varphi I_n) x - x^\top \left(\Sigma b + \gamma \mu + \varphi v_x^{(k+1)} \right) + \\
&\quad \mathbb{1}_{\Omega_1}(x) + \mathbb{1}_{\Omega_3}(x) + \text{constant}
\end{aligned}$$

Portfolio optimization with CCD and ADMM algorithms

We have:

$$\begin{aligned} f_y^{(k+1)}(y) &= \mathbb{1}_{\Omega_2}(y) + \frac{\varphi}{2}\left\| x^{(k+1)} - y + u^{(k)} \right\|_2^2 \\ &= \mathbb{1}_{\Omega_2}(y) + \frac{\varphi}{2}\left\| y - v_y^{(k+1)} \right\|_2^2 \end{aligned}$$

where $v_y^{(k+1)} = x^{(k+1)} + u^{(k)}$. We deduce that:

$$\begin{aligned} y^{(k+1)} &= \arg\min_y f_y^{(k+1)}(y) \\ &= \mathcal{P}_{\Omega_2}\left(v_y^{(k+1)}\right) \end{aligned}$$

where:

$$\Omega_2 = \mathcal{B}_1\left(b, \tau^+\right)$$

Portfolio optimization with CCD and ADMM algorithms

We remind that:

$$\begin{aligned}
\mathcal{P}_{\mathcal{B}_1(c,\lambda)}(v) &= \mathcal{P}_{\mathcal{B}_1(\mathbf{0}_n,\lambda)}(v-c)+c \\
\mathcal{P}_{\mathcal{B}_1(\mathbf{0}_n,\lambda)}(v) &= v-\operatorname{sign}(v)\odot\mathbf{prox}_{\lambda\max x}(|v|) \\
\mathbf{prox}_{\lambda\max x}(v) &= \min(v,s^\star)
\end{aligned}$$

where $s^\star$ is the solution of the following equation:

$$s^\star=\left\{s\in\mathbb{R}:\sum_{i=1}^{n}(v_i-s)_+=\lambda\right\}$$

Portfolio optimization with CCD and ADMM algorithms

We deduce that:

$$\begin{aligned}
\mathcal{P}_{\Omega_2}\left(v_y^{(k+1)}\right) &= \mathcal{P}_{\mathcal{B}_1(b,\boldsymbol{\tau}^+)}\left(v_y^{(k+1)}\right) \\
&= \mathcal{P}_{\mathcal{B}_1(\mathbf{0}_n,\boldsymbol{\tau}^+)}\left(v_y^{(k+1)} - b\right) + b \\
&= v_y^{(k+1)} - \operatorname{sign}\left(v_y^{(k+1)} - b\right) \odot \mathbf{prox}_{\boldsymbol{\tau}^+ \max x}\left(\left|v_y^{(k+1)} - b\right|\right) \\
&= v_y^{(k+1)} - \operatorname{sign}\left(v_y^{(k+1)} - b\right) \odot \min\left(\left|v_y^{(k+1)} - b\right|, s^\star\right)
\end{aligned}$$

where $s^\star$ is the solution of the following equation:

$$s^\star = \left\{ s \in \mathbb{R} : \sum_{i=1}^{n} \left(\left|v_{y,i}^{(k+1)} - b_i\right| - s\right)_+ = \boldsymbol{\tau}^+ \right\}$$

Portfolio optimization with CCD and ADMM algorithms

The ADMM algorithm becomes:

$$\left\{\begin{array}{l}
v_x^{(k+1)} = y^{(k)} - u^{(k)} \\
Q^{(k+1)} = \Sigma + \varphi I_n \\
R^{(k+1)} = \Sigma b + \gamma \mu + \varphi v_x^{(k+1)} \\
x^{(k+1)} = \arg\min_x \left\{\frac{1}{2} x^\top Q^{(k+1)} x - x^\top R^{(k+1)} + \mathbb{1}_{\Omega_1}(x) + \mathbb{1}_{\Omega_3}(x)\right\} \\
v_y^{(k+1)} = x^{(k+1)} + u^{(k)} \\
s^\star = \left\{ s \in \mathbb{R} : \sum_{i=1}^n \left(\left| v_{y,i}^{(k+1)} - b_i \right| - s\right)_+ = \tau^+ \right\} \\
y^{(k+1)} = v_y^{(k+1)} - \operatorname{sign}\left(v_y^{(k+1)} - b\right) \odot \min\left(\left| v_y^{(k+1)} - b \right|, s^\star\right) \\
u^{(k+1)} = u^{(k)} + x^{(k+1)} - y^{(k+1)}
\end{array}\right.$$

Portfolio optimization with CCD and ADMM algorithms

Question 2.c

We consider the following optimization problem:

$$
\begin{aligned}
w^{\star} &= \arg\min \left\| w - \tilde{w} \right\|_1 \\
&\text{s.t.} \quad \left\{
\begin{array}{l}
\mathbf{1}_n^\top w = 1 \\
\sqrt{(w-b)^\top \Sigma (w-b)} \leq \sigma^{+} \\
\mathbf{0}_n \leq w \leq \mathbf{1}_n
\end{array}
\right.
\end{aligned}
$$

Portfolio optimization with CCD and ADMM algorithms

Question 2.c.i

What is the meaning of the objective function $\left\|w-\tilde{w}\right\|_1$? What is the meaning of the constraint $\sqrt{(w-b)^\top \Sigma (w-b)} \leq \sigma^+$?

Portfolio optimization with CCD and ADMM algorithms

The objective function $\left\| w - \tilde{w} \right\|_1$ is the turnover between a given portfolio $\tilde{w}$ and the optimized portfolio w

The constraint $\sqrt{(w - b)^\top \Sigma (w - b)} \leq \sigma^+$ is a tracking error limit with respect to a benchmark b

Portfolio optimization with CCD and ADMM algorithms

Question 2.c.ii

Propose an equivalent optimization problem such that $f_x(x)$ is a QP problem. How to solve the y-update?

Portfolio optimization with CCD and ADMM algorithms

The optimization problem is equivalent to solve the following program:

$$\begin{aligned} w^{\star} &= \arg\min \frac{1}{2} (w - b)^{\top} \Sigma (w - b) + \lambda \| w - \tilde{w} \|_1 \\ \text{s.t.} & \left\{ \begin{array}{l} \mathbf{1}_n^{\top} w = 1 \\ \mathbf{0}_n \leq w \leq \mathbf{1}_n \end{array} \right. \end{aligned}$$

Portfolio optimization with CCD and ADMM algorithms

We deduce that:

$$f_x(x) = \frac{1}{2}(x-b)^\top \Sigma (x-b) + \mathbb{1}_{\Omega_1}(x) + \mathbb{1}_{\Omega_2}(x)$$

where:

$$\Omega_1(x) = \left\{x : \mathbf{1}_n^\top x = 1\right\}$$

and:

$$\Omega_2(x) = \left\{x : \mathbf{0}_n \leq x \leq \mathbf{1}_n\right\}$$

Portfolio optimization with CCD and ADMM algorithms

We have:

$$f_y(y) = \lambda \left\| w - \tilde{w} \right\|_1$$

We remind that:

$$\mathbf{prox}_{\lambda\|x\|_1}(v) = \mathcal{S}(v;\lambda) = \operatorname{sign}(v) \odot (|v| - \lambda \mathbf{1}_n)_+$$

and:

$$\mathbf{prox}_{f(x+b)}(v) = \mathbf{prox}_f(v+b) - b$$

The y-update step is then equal to:

$$\begin{aligned} y^{(k+1)} &= \mathbf{prox}_{\lambda\|w-\tilde{w}\|_1}\left(x^{(k+1)} + u^{(k)}\right) \\ &= \tilde{w} + \operatorname{sign}\left(x^{(k+1)} + u^{(k)} - \tilde{w}\right) \odot \left(\left|x^{(k+1)} + u^{(k)} - \tilde{w}\right| - \lambda \mathbf{1}_n\right)_+ \end{aligned}$$

because $f_y(y)$ is fully separable[33]

[33] Otherwise the scaling property does not work!

Regularized portfolio optimization

Exercise

We consider an investment universe with 6 assets. We assume that their expected returns are 4%, 6%, 7%, 8%, 10% and 10%,, and their volatilities are 6%, 10%, 11%, 15%, 15% and 20%. The correlation matrix is given by:

$$\rho = \left(\begin{array}{rrrrrr} 100\% & & & & & \\ 50\% & 100\% & & & & \\ 20\% & 20\% & 100\% & & & \\ 50\% & 50\% & 80\% & 100\% & & \\ 0\% & -20\% & -50\% & -30\% & 100\% & \\ 0\% & 20\% & 30\% & 0\% & 0\% & 100\% \end{array} \right)$$

Regularized portfolio optimization

Question 1

We restrict the analysis to long-only portfolios meaning that $\sum_{i=1}^{n} x_i = 1$ and $x_i \geq 0$.

Regularized portfolio optimization

Question 1.a

We consider the Herfindahl index $\mathcal{H}(x) = \sum_{i=1}^{n} x_i^2$. What are the two limit cases of $\mathcal{H}(x)$? What is the interpretation of the statistic $\mathcal{N}(x) = \mathcal{H}^{-1}(x)$?

Regularized portfolio optimization

We consider the following optimization problem:

$$\begin{aligned} x^{\star} &= \arg\min \mathcal{H}(x) \\ & \text{s.t.} \quad \sum_{i=1}^{n} x_i = 1 \end{aligned}$$

We deduce that the Lagrange function is:

$$\begin{aligned} \mathcal{L}(x;\lambda) &= \mathcal{H}(x) - \lambda \left(\sum_{i=1}^{n} x_i = 1 \right) \\ &= x^{\top} x - \lambda \left(\mathbf{1}_n^{\top} x - 1 \right) \end{aligned}$$

Regularized portfolio optimization

The first-order condition is:

$$\frac{\partial \mathcal{L}(x;\lambda)}{\partial x} = x - \lambda \mathbf{1}_n = \mathbf{0}_n$$

Since we have $\mathbf{1}_n^\top x - 1 = 0$, we deduce that:

$$\lambda = \frac{1}{\mathbf{1}_n^\top \mathbf{1}_n} = \frac{1}{n}$$

We conclude that the lower bound is reached for the equally-weighted portfolio:

$$x_{\mathrm{ew}} = \frac{1}{n} \cdot \mathbf{1}_n$$

and we have:

$$\mathcal{H}(x_{\mathrm{ew}}) = \frac{1}{n^2} \cdot \mathbf{1}_n^\top \mathbf{1}_n = \frac{1}{n}$$

Regularized portfolio optimization

Since the weights are positive, we have:

$$\begin{aligned} \mathcal{H}(x) &= \sum_{i=1}^{n} x_i^2 \\ &\leq \left(\sum_{i=1}^{n} x_i\right)^2 \\ &\leq 1 \end{aligned}$$

The upper bound is reached when the portfolio is concentrated on one asset:

$$\exists i : x_i = 1$$

Regularized portfolio optimization

We conclude that:

$$\frac{1}{n} \leq \mathcal{H}(x) \leq 1$$

The statistic $\mathcal{N}(x) = \mathcal{H}^{-1}(x)$ is the effective number of assets

Regularized portfolio optimization

Question 1.b

We consider the following optimization problem ($\mathcal{P}_1$):

$$\begin{aligned} x^\star(\lambda) &= \arg\min \frac{1}{2} x^\top \Sigma x + \lambda x^\top x \\ &\text{s.t.} \left\{ \begin{array}{l} \sum_{i=1}^n x_i = 1 \\ x_i \geq 0 \end{array} \right. \end{aligned}$$

What is the link between this constrained optimization program and the weight diversification based on the Herfindahl index?

Regularized portfolio optimization

The optimization problem $(\mathcal{P}_1)$ is equivalent to:

$$\begin{aligned} x^\star\left(\mathcal{H}^+\right) & = & \arg\min \frac{1}{2} x^\top \Sigma x \\ & & \text{s.t.} \quad \left\{ \begin{array}{l} \sum_{i=1}^n x_i = 1 \\ x_i \geq 0 \\ x^\top x \leq \mathcal{H}^+ \end{array} \right. \end{aligned}$$

We obtain a long-only minimum variance portfolio with a diversification constraint based on the Herfindahl index:

$$\mathcal{H}(x) \leq \mathcal{H}^+$$

We have the following correspondance:

$$\mathcal{H}^+ = \mathcal{H}\left(x^\star(\lambda)\right) = x^\star(\lambda)^\top x^\star(\lambda)$$

Given a value of λ, we can then compute the implicit constraint $\mathcal{H}(x) \leq \mathcal{H}^+$.

Regularized portfolio optimization

Question 1.c

Solve Program $(\mathcal{P}_1)$ when λ is equal to respectively 0, 0.001, 0.01, 0.05, 0.10 and 10. Compute the statistic $\mathcal{N}(x)$. Comment on these results.

Regularized portfolio optimization

Table 116: Solution of the optimization problem $(\mathcal{P}_1)$

λ	0.000	0.001	0.010	0.050	0.100	10.000
$x_1^\star(\lambda)$ (in %)	44.60	35.66	23.97	18.71	17.76	16.68
$x_2^\star(\lambda)$ (in %)	9.12	14.60	18.10	17.08	16.89	16.67
$x_3^\star(\lambda)$ (in %)	25.46	26.57	19.96	16.89	16.71	16.67
$x_4^\star(\lambda)$ (in %)	0.00	0.00	7.64	14.46	15.52	16.65
$x_5^\star(\lambda)$ (in %)	20.40	22.11	22.38	19.31	18.21	16.69
$x_6^\star(\lambda)$ (in %)	0.43	1.07	7.94	13.55	14.92	16.65
$\mathcal{H}(x^\star(\lambda))$	0.3137	0.2680	0.1923	0.1693	0.1675	0.1667
$\mathcal{N}(x^\star(\lambda))$	3.19	3.73	5.20	5.91	5.97	6.00

Regularized portfolio optimization

Question 1.d

Using the bisection algorithm, find the optimal value of $\lambda^{\star}$ that satisfies:

$$\mathcal{N}\left(x^{\star}\left(\lambda^{\star}\right)\right)=4$$

Give the composition of $x^{\star}\left(\lambda^{\star}\right)$. What is the interpretation of $x^{\star}\left(\lambda^{\star}\right)$?

Regularized portfolio optimization

The optimal solution is:

$$\lambda^\star = 0.002301$$

The optimal weights (in %) are equal to:

$$x^\star = \begin{pmatrix} 31.62\% \\ 17.24\% \\ 26.18\% \\ 0.00\% \\ 22.63\% \\ 2.33\% \end{pmatrix}$$

The effective number of bets $\mathcal{N}(x^\star)$ is equal to 4

Regularized portfolio optimization

Question 2

We consider long/short portfolios and the following optimization problem $(\mathcal{P}_2)$:

$$\begin{aligned} x^{\star}(\lambda) &= \arg\min \frac{1}{2} x^{\top} \Sigma x + \lambda \sum_{i=1}^{n} |x_i| \\ &\text{s.t. } \sum_{i=1}^{n} x_i = 1 \end{aligned}$$

Regularized portfolio optimization

Question 2.a

Solve Program ($\mathcal{P}_2$) when λ is equal to respectively 0, 0.0001, 0.001, 0.01, 0.05, 0.10 and 10. Comment on these results.

Regularized portfolio optimization

Table 117: Solution of the optimization problem $(\mathcal{P}_2)$

λ	0.000	0.0001	0.001	0.010	0.050	0.100	10.000
$x_1^\star(\lambda)$ (in %)	35.82	37.17	44.50	44.60	44.60	44.60	44.60
$x_2^\star(\lambda)$ (in %)	33.08	30.26	11.48	9.12	9.12	9.12	9.12
$x_3^\star(\lambda)$ (in %)	77.62	71.77	31.28	25.46	25.46	25.46	25.46
$x_4^\star(\lambda)$ (in %)	-53.48	-47.97	-7.16	0.00	0.00	0.00	0.00
$x_5^\star(\lambda)$ (in %)	20.83	20.56	19.90	20.40	20.40	20.40	20.40
$x_6^\star(\lambda)$ (in %)	-13.87	-11.78	0.00	0.43	0.43	0.43	0.43
$\mathcal{L}(x)$ (in %)	234.69	219.50	114.33	100.00	100.00	100.00	100.00

Regularized portfolio optimization

Question 2.b

For each optimized portfolio, calculate the following statistic:

$$\mathcal{L}(x) = \sum_{i=1}^{n} |x_i|$$

What is the interpretation of $\mathcal{L}(x)$? What is the impact of Lasso regularization?

Regularized portfolio optimization

$\mathcal{L}(x) = \sum_{i=1}^{n} |x_i|$ is the leverage ratio. Their values are reported in Table 117.

Regularized portfolio optimization

Question 3

We assume that the investor holds an initial portfolio $x^{(0)}$ defined as follows:

$$x^{(0)} = \begin{pmatrix} 10\% \\ 15\% \\ 20\% \\ 25\% \\ 30\% \\ 0\% \end{pmatrix}$$

We consider the optimization problem $(\mathcal{P}_3)$:

$$\begin{aligned} x^{\star}(\lambda) &= \arg\min \frac{1}{2} x^{\top} \Sigma x + \lambda \sum_{i=1}^{n} \left| x_i - x_i^{(0)} \right| \\ \text{s.t.} & \quad \sum_{i=1}^{n} x_i = 1 \end{aligned}$$

Regularized portfolio optimization

Question 3.a

Solve Program ($\mathcal{P}_3$) when λ is equal respectively to 0, 0.0001, 0.001, 0.0015 and 0.01. Compute the turnover of each optimized portfolio. Comment on these results.

Regularized portfolio optimization

Table 118: Solution of the optimization problem $(\mathcal{P}_3)$

λ	0.000	0.000	0.001	0.002	0.010
$x_1^\star(\lambda)$ (in %)	35.82	35.55	27.90	24.28	10.00
$x_2^\star(\lambda)$ (in %)	33.08	30.61	15.00	15.00	15.00
$x_3^\star(\lambda)$ (in %)	77.62	72.35	33.36	22.86	20.00
$x_4^\star(\lambda)$ (in %)	-53.48	-48.00	-5.20	7.87	25.00
$x_5^\star(\lambda)$ (in %)	20.83	21.51	28.94	30.00	30.00
$x_6^\star(\lambda)$ (in %)	-13.87	-12.02	0.00	0.00	0.00
$\tau\left(x^\star(\lambda) \mid x^{(0)}\right)$ (in %)	203.04	187.02	62.51	34.27	0.00

Regularized portfolio optimization

Question 3.b

Using the bisection algorithm, find the optimal value of $\lambda^\star$ such that the two-way turnover is equal to 60%. Give the composition of $x^\star(\lambda^\star)$.

Regularized portfolio optimization

The optimal solution is:

$$\lambda^\star = 0.00103$$

The optimal weights (in %) are equal to:

$$x^\star = \begin{pmatrix} 27.23\% \\ 15.00\% \\ 32.77\% \\ -4.30\% \\ 29.30\% \\ 0.00\% \end{pmatrix}$$

The turnover $\tau\left(x^\star \mid x^{(0)}\right)$ is equal to 60%

Regularized portfolio optimization

Question 3.c

Same question when the two-way turnover is equal to 50%.

Regularized portfolio optimization

The optimal solution is:

$$\lambda^{\star} = 0.00119$$

The optimal weights (in %) are equal to:

$$x^{\star} = \begin{pmatrix} 25.53\% \\ 15.00\% \\ 29.47\% \\ 0.00\% \\ 30.00\% \\ 0.00\% \end{pmatrix}$$

The turnover $\tau \left(x^{\star} \mid x^{(0)} \right)$ is equal to 50%

Regularized portfolio optimization

Question 3.d

What becomes the portfolio $x^{\star}(\lambda)$ when $\lambda \rightarrow \infty$? How do you explain this result?

Regularized portfolio optimization

We notice that:

$$\lim_{\lambda \to \infty} x^{\star}(\lambda) = x^{(0)}$$

This is normal since we have:

$$\begin{aligned} x^{\star}(\lambda) &= \arg\min \frac{1}{2} x^{\top} \Sigma x + \lambda \sum_{i=1}^{n} \left| x_i - x_i^{(0)} \right| \\ &\text{s.t.} \quad \sum_{i=1}^{n} x_i = 1 \end{aligned}$$

We deduce that:

$$\begin{aligned} x^{\star}(\infty) &= \arg\min \sum_{i=1}^{n} \left| x_i - x_i^{(0)} \right| \\ &\text{s.t.} \quad \sum_{i=1}^{n} x_i = 1 \end{aligned}$$

The solution is $x^{\star}(\infty) = x^{(0)}$

Main references

- Beck, A. (2017)
  First-Order Methods in Optimization, MOS-SIAM Series on Optimization, 25, SIAM.

- Coqueret, G., and Guida, T. (2020)
  Machine Learning for Factor Investing, Chapman and Hall/CRC Financial Mathematics Series.

- Perrin, S., and Roncalli, T. (2020)
  Machine Learning Algorithms and Portfolio Optimization, in Jurczenko, E. (Ed.), *Machine Learning in Asset Management: New Developments and Financial Applications*, Wiley, pp. 261-328, `arxiv.org/abs/1909.10233`.

References I

BOURGERON, T., LEZMI, E., and RONCALLI, T. (2018)
Robust Asset Allocation for Robo-Advisors, *arXiv*,
`arxiv.org/abs/1902.07449`.

BOYD, S., PARIKH, N., CHU, E., PELEATO, B., and ECKSTEIN, J. (2010)
Distributed Optimization and Statistical Learning via the Alternating Direction Method of Multipliers, *Foundations and Trends® in Machine learning*, 3(1), pp. 1-122.

GABAY, D., and MERCIER, B. (1976)
A Dual Algorithm for the Solution of Nonlinear Variational Problems via Finite Element Approximation, *Computers & Mathematics with Applications*, 2(1), pp. 17-40.

References II

GONZALVEZ, J., LEZMI, E., RONCALLI, T., and XU, J. (2019)
Financial Applications of Gaussian Processes and Bayesian Optimization, *arXiv*, `arxiv.org/abs/1903.04841`.

GRIVEAU-BILLION, T., RICHARD, J-C., and RONCALLI, T. (2013)
A Fast Algorithm for Computing High-dimensional Risk Parity Portfolios, *SSRN*, `www.ssrn.com/abstract=2325255`.

JURCZENKO, E. (2020)
Machine Learning in Asset Management: New Developments and Financial Applications, Wiley.

KONDRATYEV, A., and SCHWARZ, C. (2020)
The Market Generator, *SSRN*, `www.ssrn.com/abstract=3384948`.

References III

Lezmi, E., Roche, J., Roncalli, T., and Xu, J. (2020)
Improving the Robustness of Trading Strategy Backtesting with Boltzmann Machines and Generative Adversarial Networks, *arXiv*, `https://arxiv.org/abs/2007.04838`.

Parikh, N., and Boyd, S. (2014)
Proximal Algorithms, *Foundations and Trends® in Optimization*, 1(3), pp. 127-239.

Tibshirani, R. (1996)
Regression Shrinkage and Selection via the Lasso, *Journal of the Royal Statistical Society B*, 58(1), pp. 267-288.

References IV

TIBSHIRANI, R.J. (2017)
Dykstra's Algorithm, ADMM, and Coordinate Descent: Connections, Insights, and Extensions, in Guyon, I., Luxburg, U.V., Bengio, S., Wallach, H., Fergus, R., Vishwanathan, S., and Garnett, R. (Eds), *Advances in Neural Information Processing Systems*, 30, pp. 517-528.

TSENG, P. (2001)
Convergence of a Block Coordinate Descent Method for Nondifferentiable Minimization, *Journal of Optimization Theory and Applications*, 109(3), pp. 475-494.