Semantic Web & Ontologies: Description Logics

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Contents

1 Reminder

An ontology serves as a formal representation of knowledge in a given domain, and its key structure components are as follow:

1. Language

   An ontology is defined using formal languages that provide the syntax and rules for expressing knowledge. Two commonly used ontology languages are RDF (Resource Description Framework) and OWL (Web Ontology Language). These languages allow us to create structured representations of knowledge with formal semantics.

2. TBox (Terminological Box)

   The TBox is responsible for defining the vocabulary of the ontology, i.e., the "schema" or "structure" of the domain knowledge. It contains:

   - **Concepts (Classes)**: Abstract groups or sets that define categories or types of things in the domain. For example, in a medical ontology, concepts could include "Person," "Doctor," "Patient." Also called, Atomic concepts and they are grouped into a set N_C.
   - **Roles (Properties)**: Relationships between concepts, such as "hasDiagnosis" or "treatsPatient." Roles describe how concepts relate to each other and they are grouped into a set N_R.
   - **Axioms**: These are logical statements or constraints that describe the structure of concepts and their relationships. Examples include class hierarchies (e.g., "Every doctor is a person") or restrictions (e.g., "A person can have at most one diagnosis at a time").

3. ABox (Assertion Box)

   The ABox contains facts about specific instances of the concepts and roles defined in the TBox. It is the "data" part of the ontology. For example:

   - **Individuals (Instances)**: Specific entities in the domain. In the medical example, "Alice" could be an individual instance of the concept "Person."
   - **Assertions**: Statements that describe how individuals relate to concepts and roles. For instance, "Alice is a Doctor" or "Alice treatsPatient Bob."

4. Reasoning

   This involves using automated reasoning techniques to infer new knowledge from the existing information in the ontology (TBox and ABox). Reasoners can perform tasks such as:

   - **Consistency Checking**: Ensuring that the ontology doesn't contain contradictory information.
   - **Concept Satisfiability**: Checking if a class can have any valid instances.
   - **Classification**: Determining the hierarchical relationships between classes.
   - **Instance Checking**: Verifying whether a particular individual belongs to a certain class.

2 Introduction to Description Logics (DL)

Description Logics (DL) are a family of formal knowledge representation languages, and they form the logical foundation of ontology languages like OWL. They are specifically designed for:

- Describing classes, roles/properties, and individuals/instances.
- Providing formal semantics, which means that the meaning of every statement is rigorously defined.

- Enabling $\boxed{\text{automated reasoning}}$, allowing computers to infer logical consequences from a given set of axioms.

For instance, a classe could be "Doctor", a role would be "treatsPatient" and an individualisai would be "Alice". Then the role or property could be expressed as treatsPatient(Doctor, Alice).

2.1 Attributive Language with Complements (ALC)

In Description Logics (DL), and specifically with ALC logic (Attributive Language with Complements), we can create complex concepts by combining atomic concepts using logical operators and quantifiers. Let's break down the elements and how we can construct ALC concepts.

2.2 What is ALC?

ALC is a basic form of description logic that supports the following:

- Atomic concepts: Basic categories (e.g., "Person"). If $A \in N_C$ then A is an ALC concept.
- Roles: Binary relations between individuals (e.g., "hasChild"). If C, D are ALC concepts and $r \in N_R$, then the following are ALC concepts:
  - Logical operators: conjunction ($C \sqcap D$ is an ALC concept), disjunction ($C \sqcup D$ is an ALC concept) and negation ($\neg C$ is an ALC concept).
  - Symbols $\top$ (top) and $\bot$ (bottom) represent fundamental concepts regarding truth values and set relationships.
    * $\top$ is defined as $A \sqcup \neg A$, meaning "A or not A." It represents a concept that is always true. In other words, any instance belongs to the concept represented by $\top$.
    * $\bot$ is defined as $A \sqcap \neg A$, meaning "A and not A." It represents a concept that is always false. There are no instances that can satisfy this concept.
  - Quantifiers: There 2 types, either existential or value restriction quantifiers which are both ALC concepts.
    * Existential restriction ($\exists\ r.C$): This means "there exists a role r pointing to a concept C" (e.g., $\exists\ hasChild.Student$ means *has a child who is a student*).
    * Value restriction ($\forall\ r.C$): This means "for all roles r, they point to a concept C" (e.g., $\forall\ hasChild.Student$ means *all children are students*).

You can build any combination of concepts, roles, and logical operators depending on the domain you're working with. In ALC, you define concepts using logical constructs, creating a clear structure for reasoning in complex systems.

2.3 ALC Semantics

In ALC semantics, an interpretation provides meaning to the concepts and roles within a description logic system. It formalizes the way in which concepts (classes) and roles (relationships) are mapped to elements of a domain, and how complex concepts are evaluated.

2.3.1 Interpretations

An interpretation $I = (\Delta^I, \cdot^I)$ consists of:

1. A non-empty domain Δ^I: This is the set of all possible individuals that the interpretation considers. These individuals are the entities over which concepts and roles are defined.

2. An extension mapping $\cdot^I$: This mapping defines how atomic concepts, roles, and individuals are interpreted within the domain. It tells us how to map concepts and roles to subsets and relations in Δ^I:

   - For every concept $A \in N_C$ (a concept name), $A^I \subseteq \Delta^I$ (i.e., concepts are interpreted as sets of individuals from the domain).
   - For every role $r \in N_R$ (a role name), $r^I \subseteq \Delta^I \times \Delta^I$ (i.e., roles are interpreted as binary relations between individuals).
   - For each individual a, $a^I \in \Delta^I$ (i.e., individuals are mapped to elements of the domain).

2.3.1.1 Extension Mapping

The extension mapping $\cdot^I$ is extended to **complex concepts** (formed by combining atomic concepts and roles) by the following rules:

1. $\boxed{\textbf{Conjunction } C \sqcap D}$

   - $(C \sqcap D)^I = C^I \cap D^I$
     This means that the interpretation of $C \sqcap D$ is the intersection of the interpretations of C and D. In other words, an individual belongs to $C \sqcap D$ if it belongs to both C and D.

2. $\boxed{\textbf{Disjunction } C \sqcup D}$

   - $(C \sqcup D)^I = C^I \cup D^I$
     The interpretation of $C \sqcup D$ is the union of the interpretations of C and D. An individual belongs to $C \sqcup D$ if it belongs to either C, D, or both.

3. $\boxed{\textbf{Negation } \neg C}$

   - $(\neg C)^I = \Delta^I \setminus C^I$
     The interpretation of $\neg C$ is the complement of the interpretation of C. An individual belongs to $\neg C$ if it does not belong to C.

4. $\boxed{\textbf{Existential restriction } \exists r.C}$

   - $(\exists r.C)^I = \{d \in \Delta^I \mid \text{there exists } e \in \Delta^I \text{ such that } (d,e) \in r^I \text{ and } e \in C^I\}$
     The interpretation of $\exists r.C$ is the set of individuals $d \in \Delta^I$ that are related via the role r to at least one individual e that belongs to the concept C.

5. $\boxed{\textbf{Value restriction } \forall r.C}$

   - $(\forall r.C)^I = \{d \in \Delta^I \mid \text{for all } e \in \Delta^I, \text{ if } (d,e) \in r^I, \text{ then } e \in C^I\}$
     The interpretation of $\forall r.C$ is the set of individuals $d \in \Delta^I$ such that all individuals e related to d via the role r must belong to the concept C.

2.3.1.2 Example of ALC with interpretation and extension mapping

Consider an example with the following concepts and roles:

$$
\begin{aligned}
Female^I &= \{x_1, x_2\} \\
Student^I &= \{x_3, x_4\} \\
hasChild^I &= \{(x_1, x_3), (x_2, x_4)\} \\
GoodScore^I &= \{x_3\}
\end{aligned}
$$

Now, let's evaluate the concept C := $Female \sqcap \forall hasChild.(Student \sqcap \exists hasScore.GoodScore)$.

$$\begin{aligned} C^I &= Female^I \cap (\forall\ \text{hasChild}.(\text{Student}\ \sqcap\ \exists\text{hasScore.GoodScore}))^I \\ &= Female^I \cap \{\ d \in \Delta^I \mid \forall e \in \Delta^I, (d,e) \in hasChild^I \implies e \in (\text{Student} \sqcap \exists\text{hasScore.GoodScore})^I\ \} \\ &= \underbrace{Female^I}_{\{x_1,x_2\}} \cap \{\ d \in \Delta^I \mid \forall e \in \Delta^I, (d,e) \in hasChild^I \implies e \in (\underbrace{\text{Student}^I}_{\{x_3,x_4\}} \cap \underbrace{(\exists\text{hasScore.GoodScore})^I}_{:=S}\ \} \end{aligned}$$

Where $S = \{d \in \Delta^I \mid$ there exists $e \in \Delta^I$ such that $(d,e) \in hasScore^I$ and $e \in GoodScore^I\}$. The interpretation S is only satisfied by the invididual x_3 which a child from the female individual x_1 from the $Female$ atomic concept. Therefore, we obtain:

$$\begin{aligned} C^I &= \underbrace{Female^I}_{\{x_1,x_2\}} \cap \{\ d \in \Delta^I \mid \forall e \in \Delta^I, (d,e) \in hasChild^I \implies e \in (\ \underbrace{\text{Student}^I}_{\{x_3,x_4\}} \cap \underbrace{(\exists\text{hasScore.GoodScore})^I}_{:=\ \{x_3\}}\ \} \\ &= \underbrace{Female^I}_{\{x_1,x_2\}} \cap \{\ d \in \Delta^I \mid \forall e \in \Delta^I, (d,e) \in hasChild^I \implies e \in (\ \underbrace{\text{Student}^I\ \cap\ (\exists\text{hasScore.GoodScore})^I}_{:=\ \{x_3\}}\ \} \\ &= \underbrace{Female^I}_{\{x_1,x_2\}} \cap \{(x_1,x_3)\} = \{x_1\} \end{aligned}$$

Only x_1 satisfies both being a female and having a child in $Student \sqcap \exists hasScore.GoodScore$. So, the concept evaluates to $\{x_1\}$.

2.4 Concept Definition

A concept definition is a formal way of describing a concept using logical expressions. It typically follows the structure $A = C$, where A is the concept name and C is the concept description that can include logical operators and other concepts. An interpretation I satisfies a concept definition $A = C$ if $A^I = C^I$.

2.4.0.1 Example

For instance, let's considere the following concept name $Heroine$ and the concept description $C = \text{Hero} \sqcap \text{Female}$. Given the interpretation I

$$\begin{aligned} \Delta^I &= \{a,b\} \\ \text{Heroine}^I &= \{a\} \\ \text{Hero}^I &= \{a,b\} \\ \text{Female}^I &= \{b\} \end{aligned}$$

To determine if the interpretation I satisfies the concept definition Heroine = Hero $\sqcap$ Female, we need to analyze the components of the definition and how they relate to the provided interpretation. To see if the interpretation I satisfies the concept definition, we check if:

$$\text{Heroine}^I = \text{Hero}^I \cap \text{Female}^I$$

- $\boxed{\text{Left Side}}$: $\text{Heroine}^I = \{a\}$
- $\boxed{\text{Right Side}}$: $\text{Hero}^I \cap \text{Female}^I = \{a,b\} \cap \{b\} = \{b\}$

Since $\{a\} \neq \{b\}$, interpretation I does not satisfy the concept definition Heroine = Hero $\sqcap$ Female.

To make I satisfy the concept definition, we need to ensure that there exists an individual that is both a "Hero" and "Female." We could just include an individual that is both a Hero and Female. Let's say we redefine

Heroine^I to include an individual c such that: c is a **Hero** and c is also **Female**. Now, we would need to consider a new interpretation I' defined as follows:

$$\begin{aligned} \Delta^I &= \{a, b, c\} \\ \text{Heroine}^I &= \{c\} \\ \text{Hero}^I &= \{a, c\}(\text{meaning both a and c are Heroes}) \\ \text{Female}^I &= \{b, c\}(\text{meaning both b and c are Female}) \end{aligned}$$

Now, $\text{Hero}^I \cap \text{Female}^I = \{a, c\} \cap \{b, c\} = \{c\} = \text{Heroine}^I$ and therefore we can conclude that the modified interpretation I' satisfies the concept definition Heroine $=$ Hero $\sqcap$ Female. We can also say the interpretation I' is a model of the concept definition.

2.5 General Concept Inclusion (GCI)

A General Concept Inclusion (GCI) is a formal expression of the form $C \sqsubseteq D$, where C and D are concepts. The expression asserts that every instance of concept C is also an instance of concept D (i.e., $C^I \subseteq D^I$ in any interpretation I).

2.5.1 Examples

Let's consider a few examples:

- $\boxed{\text{CGI 1}}$ Hero $\sqcap$ Villain $\sqsubseteq \bot$

  This states that no individual can be both a Hero and a Villain simultaneously (i.e., the concepts are disjoint). Any interpretation/model where the set of Heroes and Villains are distinct will satisfy this GCI. For instance, if:

$$\begin{aligned} \Delta^I &= \{a, b, c\} \\ \text{Hero}^I &= \{a\} \\ \text{Villain}^I &= \{b\} \end{aligned}$$

- $\boxed{\text{CGI 2}}$ Kitchen $\sqcup$ Bathroom $\sqsubseteq \exists$hasWashbasin.$\top$

  This means that every Kitchen and Bathroom must have at least one associated washbasin. An interpretation/model could be:

$$\begin{aligned} \Delta^I &= \{a, b\} \\ \text{Kitchen}^I &= \{a\} \\ \text{Bathroom}^I &= \{b\} \\ \text{hasWashbasin}^I &= \{(a, w), (b, w)\} \text{ for some washbasin w} \end{aligned}$$

- $\boxed{\text{Example 3}}$ Cold $\sqcap \exists$causedBy.Virus $\sqsubseteq$ Disease

  This states that if an individual has both a Cold and is caused by a Virus, then it is classified as a Disease. An interpretation/model could be:

$$\begin{aligned} \Delta^I &= \{a, b\} \\ \text{Cold}^I &= \{a\} \\ \text{causedBy}^I &= \{(a, v)\} \text{ for some virus v} \\ \text{Virus}^I &= \{v\} \end{aligned}$$

  Then Disease^I must include a.

2.6 Concept Assertion

A concept assertion is of the form $C(a)$, where C is a concept and a is an individual. A role assertion is of the form $r(a, b)$, where r is a role (relationship) between two individuals a and b.

An interpretation I is a model of the ABox A if it satisfies all the assertions. Specifically:

- For each concept assertion $C(a) \in A$, the individual a^I must belong to the concept C^I.
- For each role assertion $r(a, b) \in A$, the pair (a^I, b^I) must belong to the role r^I.

2.6.1 Example

Let's define the following assertions and Abox A:

- **Concept Assertions**: Hero(a) ("a is a Hero.") and Villain(b) ("b is a Villain.")
- **Role Assertion**: hasEnemy(a, b): "a has b as an enemy."

$$A = \{\text{Hero}(a), \text{Villain}(b), \text{hasEnemy}(a, b)\}$$

We consider the interpretation I:

$$\begin{aligned}\Delta I &= \{a^I, b^I\} \\ \text{Hero}^I &= \{a^I\} \\ \text{Villain}^I &= \{b^I\} \\ \text{hasEnemy}^I &= \{(a^I, b^I)\}\end{aligned}$$

This interpretation satisfies the ABox because:

$$\begin{aligned}a^I &\in \text{Hero}^I, \text{ satisfying Hero(a)} \\ b^I &\in \text{Villain}^I, \text{ satisfying Villain(b)} \\ (a^I, b^I) &\in \text{hasEnemy}^I, \text{ satisfying hasEnemy(a, b)}\end{aligned}$$

Comment 1 A TBox is a finite set of GCIs and/or concept definitions. It defines relationships between concepts in a knowledge representation system. An interpretation I is a model of the TBox if it satisfies all CGIs of the TBox.

Comment 2 An ABox is a finite set of assertions (both concept and role assertions) that describe facts about individuals in the domain of discourse. An interpretation I is a model of the ABox if it satisfies all assertions (concept & role) of the ABox.

3 Ontology

An **ontology** $O = (T, A)$ consists of:

1. A TBox T, which contains general concept inclusions (GCIs) and definitions.
2. An ABox A, which contains assertions about individuals.

3.1 Properties I

- An interpretation I is a model of the ontology O if it satisfies T and the ABox A.
- Inconsistency: The ontology O is inconsistent if O has no model.
- Incoherency: The ontology O is incoherent if O has an unsatisfiable concept.

  Given an ontology $O = (T, A)$ and a concept C, if $C^I = \emptyset$ for any interpretation/model I of O, we call C an unsatisfiable concept with respect to O. For instance, $C := A \sqcap \neg A)$ is an unsatisfiable concept with respect to any ontology.

  Example

  An incoherent ontology is one that contains contradictions, meaning it cannot have a model that satisfies all the statements in both the TBox and ABox. Let's define an ontology $O = (T, A)$ where:

  - TBox T: Human $\sqsubseteq \neg$Robot
  - ABox A: Human(a) and Robot(a)

  Why is this ontology incoherent? The TBox states that being a Human and a Robot are mutually exclusive (disjoint). However, the ABox asserts that individual a is both a Human and a Robot, creating a contradiction. Thus, there is no interpretation/model I that can satisfy both the TBox and the ABox, making the ontology incoherent.

3.2 Terminological Reasoning of an Ontology

Terminological reasoning in an ontology refers to determining relationships between concepts based on the TBox (the terminological component of the ontology). The main reasoning tasks are subsumption, equivalence, and entailment.

1. **Subsumption**

   Concept C is subsumed by concept D with respect to a TBox T if, in all models I of T, $C^I \subseteq D^I$. This means every instance of C is also an instance of D. For instance,

   - Let $T = \{\text{Dog} \sqsubseteq \text{Animal}\}$.
   - Subsumption: Dog $\sqsubseteq$ Animal because all dogs are animals in any model of T.

2. **Equivalence**

   Concepts C and D are equivalent with respect to a TBox T if, in all models I of T, $C^I = D^I$. This means C and D have the same instances in every model of T. For instance,

   - Let $T = \{\text{Bachelor} = \text{UnmarriedMan}\}$.
   - Equivalence: Bachelor $\equiv$ UnmarriedMan because they represent the same individuals in any model of T.

3. **Entailment**

   An ontology O entails a GCI $C \sqsubseteq D$, written $O \models C \sqsubseteq D$, if in all models I of O, $C^I \subseteq D^I$. Entailment can also apply to assertions like $C(a)$ or $r(a, b)$. For instance,

   - Let $O = \{\text{Dog} \sqsubseteq \text{Animal}, \text{Pet}(fido), \text{Dog}(fido)\}$.
   - Entailment: $O \models \text{Animal}(fido)$ because $fido$ is a dog, and all dogs are animals according to the ontology.

3.2.1 Mini Exercise: Equivalence

3.2.1.1 Exercise 1

Let's determine whether the concept A in the *TBox* T_1 defined as $T_1 = \{A = A_1 \sqcap A_2\}$ is equivalent to concept A in *TBox* T_2 defined as $T_2 = \{A = A_1, A = A_2\}$.

$\boxed{\textbf{Proof}}$ To prove or check equivalence, we need to check whether both TBoxes have the **same models**. In other words, for every interpretation I, the definitions in T_1 must imply those in T_2, and vice versa.

1. $\boxed{T_1 \implies T_2?}$

   In T_1, we have $A = A_1 \sqcap A_2$, meaning $A^I = (A_1 \sqcap A_2)^I = A_1^I \cap A_2^I$. This means every instance of A is both in A_1 and in A_2. However, it does not imply that $A_1^I = A_2^I$ for every interpretation I, since A_1 and A_2 could have additional elements outside A. Thus, $A^I = A_1^I \cap A_2^I$ ($\neq A = A_1^I = A_2^I$), therefore T_1 does not imply T_2, because A_1 and A_2 can be different sets, while A is just their intersection.

2. $\boxed{T_2 \implies T_1?}$

   In T_2, we have that $A = A_1$ and $A = A_2$, implying that $A_1^I = A_2^I = A^I$. This clearly implies that $A = A_1 \sqcap A_2$, because $A^I = A_1^I = A_2^I$, and hence $A^I = A_1^I \cap A_2^I$. So, T_2 does imply T_1.

T_2 is stronger than T_1 because it forces A_1 and A_2 to be the same set, whereas T_1 only forces A to be the intersection of A_1 and A_2, without requiring $A_1$and A_2 to be equal. Therefore, T_1 and T_2 are not equivalent.

3.2.1.2 Exercise 2

Let's determine whether the concept A in the *TBox* T_1 defined as $T_1 = \{A \sqsubseteq A_1 \sqcap A_2\}$ is equivalent to concept A in *TBox* T_2 defined as $T_2 = \{A \sqsubseteq A_1, A \sqsubseteq A_2\}$.

$\boxed{\textbf{Proof}}$ To prove or check equivalence, we need to check whether both TBoxes have the **same models**. In other words, for every interpretation I, the definitions in T_1 must imply those in T_2, and vice versa.

1. $\boxed{T_1 \implies T_2?}$

   In T_1, we have $A \sqsubseteq A_1 \sqcap A_2$, meaning $A^I \sqsubseteq (A_1 \sqcap A_2)^I \sqsubseteq A_1^I \cap A_2^I \implies A^I \sqsubseteq A_1^I$ and $A^I \sqsubseteq A_2^I$. Thus, T_1 does imply T_2.

2. $\boxed{T_2 \implies T_1?}$

   In T_2, we have that $A \sqsubseteq A_1$ and $A \sqsubseteq A_2$, implying that $A^I \sqsubseteq A_1^I$ and $A^I \sqsubseteq A_2^I$. This clearly implies that $A \sqsubseteq A_1 \sqcap A_2$, because $A^I \sqsubseteq A_1^I \sqcap A_2^I$. So, T_2 does imply T_1.

We can conclude that T_1 and T_2 are equivalent.

3.2.2 Mini Exercise: Entailment

Let's prove that the ontology $O = (T, A)$ with TBox T defined as $T = \{A \sqsubseteq B, B \sqsubseteq C\}$ entails $A \sqsubseteq C$ (i.e., $T \models A \sqsubseteq C$).

$\boxed{\textbf{Proof}}$ We need to show that in every interpretation I of T, the inclusion $A^I \subseteq C^I$ holds and we know that:

- $A \sqsubseteq B$ in T: This means that in every model I, $A^I \subseteq B^I$.
- $B \sqsubseteq C$ in T: This means that in every model I, $B^I \subseteq C^I$.

We can deduce:

1. From $A \sqsubseteq B$, we know that for any interpretation I, $A^I \subseteq B^I$. This means every instance of A is also an instance of B. Then, if $a \in A^I \implies a \in B^I$

2. From $B \sqsubseteq C$, we know that for any interpretation I, $B^I \subseteq C^I$. This means every instance of B is also an instance of C. Then if $a \in B^I \implies a \in C^I$

Combining these two inclusions: $a \in A^I \implies a \in B^I \implies a \in C^I$. Therefore, $A^I \subseteq C^I$ holds for all models I of T.

3.3 Properties II

Reminder: A concept C is **unsatisfiable** with respect to an ontology O if there is no interpretation I that satisfies C, i.e., there is no interpretation I such that $C^I \neq \emptyset$. A concept C is **satisfiable** with respect to an ontology O if there exists an interpretation I such that I satisfies both O and C, i.e., $C^I \neq \emptyset$.

3.3.1 Lemma 1: Unsatisfiability

Lemma 1 If C is unsatisfiable with respect to an ontology O, and $O \subseteq O'$, then C is unsatisfiable with respect to O'.

Proof We know that C is unsatisfiable with respect to O, which means that for every interpretation I, if I is an interpretation of O, then $C^I = \emptyset$. Besides, we also know that $O \subseteq O'$, meaning that O' contains all the axioms of O and possibly more. Let's prove it by contradiction.

- Suppose C is satisfiable with respect to O', i.e., there exists an interpretation I' that satisfies both O' and C, meaning $C^{I'} \neq \emptyset$. We will show that this leads to a contradiction.

- Since $O \subseteq O'$, every interpretation I' of O' is also an interpretation I' of O because O' contains all the axioms of O.

- By assumption, C is unsatisfiable with respect to O, meaning that for any model I of O, $C^I = \emptyset$. But since I' is a model of O', and therefore a model of O, we must have $C^{I'} = \emptyset$, which contradicts the assumption that $C^{I'} \neq \emptyset$.

In conclusion, the assumption that C is satisfiable with respect to O' leads to a contradiction. Hence, C must be unsatisfiable with respect to O'.

3.3.2 Lemma 2: Satisfiability

Lemma 2 If C is satisfiable with respect to an ontology O, and $O' \subseteq O$, then C is satisfiable with respect to O'.

Proof We know that C is satisfiable with respect to O, meaning there exists an interpretation I such that I satisfies C, i.e., $C^I \neq \emptyset$. Besides, we also know that $O' \subseteq O$, which means O' contains fewer or equal axioms compared to O. We must show that there is some interpretation I' that satisfies both O' and C, i.e., $C^{I'} \neq \emptyset$.

- Since $O' \subseteq O$, the ontology O' has fewer constraints (fewer axioms) than O. An interpretation that satisfies the more restrictive ontology O should also satisfy the less restrictive ontology O'.

- Since I is a model of O and satisfies C, i.e., $C^I \neq \emptyset$, and because $O' \subseteq O$, I must also be a model of O'. This is because O' contains only a subset of the axioms in O, so satisfying the axioms of O automatically satisfies the (fewer) axioms in O'.

Therefore, the interpretation I that satisfies C with respect to O also satisfies C with respect to O', meaning $C^I \neq \emptyset$ for some model of O'.

3.3.3 Mini Exercises: Unsatisfiability and Subsumption

3.3.3.1 Exercise 1

Let's prove that $(\forall r.A \sqcap \forall r.\neg A)$ is unsatisfiable, i.e., $(\forall r.A \sqcap \forall r.\neg A)^I = \emptyset$ for any model I of O. We need to show that this combination of concepts leads to an unsatisfiable concept, meaning no interpretation I can satisfy both conditions at the same time for any element x.

$\boxed{\textbf{Proof}}$

To do so, we will prove it by contradiction. Let's assume that $(\forall r.A \sqcap \forall r.\neg A)$ is satisfiable, i.e., it exists an interpretation I and some individual x in the domain Δ^I such that:

$$x \in (\forall r.A \sqcap \forall r.\neg A)^I = (\forall r.A)^I \cap (\forall r.\neg A)^I \implies x \in \underbrace{(\forall r.A)^I}_{(1)} \text{ and } x \in \underbrace{(\forall r.\neg A)^I}_{(2)}$$

Let's define each concept to understand the elements it contains:

1. By the definition of $(\forall r.A)^I = \{d \in \Delta^I \mid \text{for all } e \in \Delta^I, \text{ if } (d,e) \in r^I, \text{ then } e \in A^I\}$. This means that all individuals related to d via role r must belong to A^I.

2. Similarly, by the definition of $\forall r.\neg A = \{d \in \Delta^I \mid \text{for all } e \in \Delta^I, \text{ if } (d,e) \in r^I, \text{ then } e \in (\neg A)^I\}$, all individuals related to d via role r must belong to $(\neg A)^I$, which means they must not belong to A^I.

This poses a contradiction. The same individual e (related to d via role r) is required to both: belong to A^I (by $\forall r.A$), and not belong to A^I (by $\forall r.\neg A$). It is impossible for e to simultaneously satisfy both conditions, as no individual can both belong to A^I and not belong to A^I at the same time. We can conclude that our assumption is false, thus $(\forall r.A \sqcap \forall r.\neg A)$ is unsatisfiable.

3.3.3.2 Exercise 2

Let's prove that $(\forall r.A \sqcap \exists r.\neg A)$ is unsatisfiable, i.e., $(\forall r.A \sqcap \forall r.\neg A)^I = \emptyset$ for any model I of O.

$\boxed{\textbf{Proof}}$

Let's proceed by proof of contradiction, we assume that $(\forall r.A \sqcap \exists r.\neg A)$ is satisfiable, i.e., it exists an interpretation I and some individual x in the domain Δ^I such that:

$$x \in (\forall r.A \sqcap \exists r.\neg A)^I = (\forall r.A)^I \cap (\exists r.\neg A)^I \implies x \in \underbrace{(\forall r.A)^I}_{(1)} \cap \underbrace{(\exists r.\neg A)^I}_{(2)}$$

Once again, the interpretation of each concept is:

1. By the definition of $(\forall r.A)^I = \{d \in \Delta^I \mid \text{for all } e \in \Delta^I, \text{ if } (d,e) \in r^I, \text{ then } e \in A^I\}$. This means that all individuals related to d via role r must belong to A^I.

2. $(\exists r.\neg A)^I = \{d \in \Delta^I \mid \text{there exists } e \in \Delta^I \text{ such that } (d,e) \in r^I \text{ and } e \in (\neg A)^I\}$

This poses a contradiction, because from $x \in (\forall r.A)^I$, all objects e related to d through r must be in A^I, i.e., $e \in A^I$ and from $x \in (\exists r.\neg A)^I$, there must exist some e such that $e \notin A^I$. Hence, $(\forall r.A \sqcap \exists r.\neg A)^I = \emptyset$ for any interpretation I.

3.3.3.3 Exercise 3

Let's prove that $\exists r.(A \sqcap B)$ is subsumed by $\exists r.A$, which is equivalent to $(\exists r.(A \sqcap B) \sqsubseteq \exists r.A)$.

$\boxed{\textbf{Proof}}$

This means that for any interpretation I and any ontology, we have:

$$(\exists r.(A \sqcap B))^I \subseteq (\exists r.A)^I := \{d \in \Delta^I \mid \text{there exists } e \in \Delta^I \text{ such that } (d,e) \in r^I \text{ and } e \in A^I\}$$

Let's assume that we have an interpretation I and an object

$$x \in (\exists r.(A \sqcap B))^I = \{d \in \Delta^I \mid \text{there exists } e \in \Delta^I \text{ such that } (d,e) \in r^I \text{ and } e \in \underbrace{(A \sqcap B)^I}_{A^I \cap B^I}\}$$

This basically means that there exists some object e such that $(d,e) \in r^I$ and $e \in (A \sqcap B)^I = A^I \cap B^I$, which means that there exists some object e such that $(d,e) \in r^I$ and $e \in A^I$, thus $x \in (\exists r.A)^I$. As this holds for any interpretation I and any ontology, we can conclude that the subsumption holds.

3.3.3.4 Exercise 4

Let's prove that the following entailment holds for any model I and ontology O holds.

$$\{A \sqsubseteq \exists r.B,\ B \sqsubseteq C\} \models A \sqsubseteq \exists r.C$$

$\boxed{\textbf{Proof}}$

This means that we want to show that if the $TBox\ T = \{A \sqsubseteq \exists r.B, B \sqsubseteq C\}$ is true in all interpretations I, then $A \sqsubseteq \exists r.C$ must also be true in all interpretations. Let's assume an arbitrary interpretation I that satisfies the TBox T, then we have that:

- $A \sqsubseteq \exists r.B \longleftrightarrow A^I \subseteq (\exists r.B)^I = \{d \in \Delta^I \mid \text{there exists } e \in \Delta^I \text{ such that } (d,e) \in r^I \text{ and } e \in B^I\}$

  We deduce that from any $x \in A^I$, there is some $e \in B^I$ such that $(x,e) \in r^I$
- $B \sqsubseteq C \longleftrightarrow B^I \subseteq C^I$

  We infer that for any $e \in B^I$, we have $e \in C^I$.

Taking into account, the previous information,we need to show that for all $x \in A^I$, there exists some $e \in C^I$ such that $(d,e) \in r^I$ for $d \in \Delta^I$. Combining both inferences, we can deduce that, for any $x \in A^I$, there is some $e \in C^I$ such that $(x,e) \in r^I$, which is equivalent to stating that $x \in (\exists r.C)^I$. Hence, I satisfies $A \sqsubseteq \exists r.C$, and since I was an arbitrary interpretation, this holds for all interpretations.

3.3.3.5 Exercise 5

Let's evaluate whether the following subsumption holds or not.

$$\exists r.A \sqcap \exists r.B \sqsubseteq \exists r.(A \sqcap B)$$

$\boxed{\textbf{Proof}}$

We aim to see if $\exists r.A \sqcap \exists r.B \sqsubseteq \exists r.(A \sqcap B)$ holds, i.e., if an individual is related to some A through r and some B through r, does that guarantee that they are related to a single individual who is both in A and B

through r? Let's assume an arbitrary interpretation I and an object $x \in (\exists r.A \sqcap \exists r.B)^I$. Then:

$$(1)\ (\exists r.A \sqcap \exists r.B)^I = (\exists r.A)^I \cap (\exists r.B)^I$$
$$(2)\ (\exists r.A)^I = \{d \in \Delta^I \mid \text{there exists } e \in \Delta^I \text{ such that } (d,e) \in r^I \text{ and } e \in A^I\}$$
$$(3)\ (\exists r.B)^I = \{d \in \Delta^I \mid \text{there exists } e \in \Delta^I \text{ such that } (d,e) \in r^I \text{ and } e \in B^I\}$$
$$(4)\ (\exists r.(A \sqcap B))^I = \{d \in \Delta^I \mid \text{there exists } e \in \Delta^I \text{ such that } (d,e) \in r^I \text{ and } e \in \underbrace{(A \sqcap B)^I}_{A^I \cap B^I}\}$$

To show that the subsumption $\exists r.A \sqcap \exists r.B \sqsubseteq \exists r.(A \sqcap B)$ does not hold, we provide a counterexample where we consider:

- An individual x is related via r to two distinct individuals e_1 and e_2.
- $e_1 \in A^I$ and $e_2 \in B^I$.
- $e_1 \notin B^I$ and $e_2 \notin A^I$.

Therefore,

- If $x \in (\exists r.A)^I$, then there exists some individual e_1 such that $(x, e_1) \in r^I$ and $e_1 \in A^I$.
- If $x \in (\exists r.B)^I$, then there exists some individual e_2 such that $(x, e_2) \in r^I$ and $e_2 \in B^I$.

Since e_1 and e_2 are distinct and $e_1 \notin B^I$ and $e_2 \notin A^I$, there is no single individual e such that $(x, e) \in r^I$ and $e \in A^I \cap B^I$. Therefore:

$$x \in (\exists r.A \sqcap \exists r.B)^I \quad \text{but} \quad x \notin (\exists r.(A \sqcap B))^I$$

The existence of distinct individuals $e_1 \in A^I$ and $e_2 \in B^I$ such that $e_1 \neq e_2$ demonstrates that $x \in (\exists r.A)^I$ and $x \in (\exists r.B)^I$ does not guarantee $x \in (\exists r.(A \sqcap B))^I$.

Therefore, the subsumption $\exists r.A \sqcap \exists r.B \sqsubseteq \exists r.(A \sqcap B)$ does not hold.

3.3.3.6 Exercise 6

We are asked to prove the subsumption:

$$\forall r.A \sqcap \forall r.B \sqsubseteq \forall r.(A \sqcap B)$$

Proof

This means we want to show that for any interpretation I, the individuals that are related to only instances of A via the role r and are related to only instances of B via the role r, are also related to only instances that satisfy both A and B via r. Let's assume an arbitrary interpretation I and an individual $x \in (\forall r.A \sqcap \forall r.B)^I$. This means:

$$(1)\ (\forall r.A \sqcap \forall r.B)^I = (\forall r.A)^I \cap (\forall r.B)^I$$
$$(2)\ (\forall r.A)^I = \{d \in \Delta^I \mid \text{for all } e \in \Delta^I, \text{ if } (d,e) \in r^I, \text{ then } e \in A^I\}$$
$$(3)\ (\forall r.(A \sqcap B))^I = \{d \in \Delta^I \mid \text{for all } e \in \Delta^I, \text{ if } (d,e) \in r^I, \text{ then } e \in \underbrace{(A \sqcap B)^I}_{A^I \cap B^I}\}$$

We need to show that for all e, if $(x, e) \in r^I$, then $e \in A^I$ and $e \in B^I$. From $x \in (\forall r.A)^I$, we know that for all e such that $(x, e) \in r^I$, then $e \in A^I$ and from $x \in (\forall r.B)^I$, we know that for all e such that $(x, e) \in r^I$, then $e \in B^I$. Thus, for all e such that $(x, e) \in r^I$, we have both $e \in A^I$ and $e \in B^I$, which implies that $e \in (A \sqcap B)^I$. As this holds for any interpretation I, we conclude that the subsumption holds.

3.3.4 Assertional Reasoning

Assertional reasoning refers to drawing conclusions about specific individuals (or instances) based on the ontology. Given an ontology $O = (T, A)$, assertional reasoning involves determining whether certain assertions about individuals hold in all possible interpretations (models) of the ontology. This leads to two common problems:

1. **Instance Problem (Concept Membership)**: Deciding if an individual a is an instance of a concept C with respect to the ontology O. Formally, this means checking if $a^I \in C^I$ for all interpretations (models) I that satisfy the ontology O. If this holds, we say that $O \models C(a)$, meaning a is inferred to be an instance of C.

2. **Role Membership Problem**: Deciding if a relationship (or role) $r(a, b)$ holds between two individuals a and b with respect to the ontology O. This involves checking if $(a^I, b^I) \in r^I$ for all interpretations I that satisfy O. If this holds, we say that $O \models r(a, b)$, meaning the role assertion is inferred to hold between a and b.

3.3.4.1 Example

Let's consider the ontology $O = \{\mathcal{T}, \mathcal{A}\} = \{A \sqsubseteq B, A(a)\}$ in which we are asked to determine if $O \models B(a)$, i.e., whether a is an instance of concept B in all models of O. By the definition of the TBox $\mathcal{T}$ axiom $A \sqsubseteq B$, any individual that is an instance of A must also be an instance of B. Since the ABox contains $A(a)$ (i.e., a is an instance of A), we can infer that a is also an instance of B. Thus, $O \models B(a)$.