Optimization for Machine Learning

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Contents

1 Exercise 1

Solve the following optimization problem:

$$\textbf{Mininimize: } f(x,y) = x^2 + y^2$$
$$\textbf{Subject to: } x + y \leq 1,\ x \geq 0,\ y \geq 0$$

1.1 Answers

Let's proceed step-by-step. First, we need to formulate the problem. We want to minimize the function:

$$f(x,y) = x^2 + y^2$$

Subject to:

- $g_1(x,y) = x + y - 1 \leq 0$
- $g_2(x) = -x \leq 0$
- $g_3(y) = -y \leq$

Then, we define the Lagrangian function $L(x,y,\lambda_1,\lambda_2,\lambda_3)$:

$$L(x,y,\lambda_1,\lambda_2,\lambda_3) = x^2 + y^2 + \lambda_1(x+y-1) + \lambda_2(-x) + \lambda_3(-y)$$

Here, $\lambda_1, \lambda_2, \lambda_3$ are the Lagrange multipliers for the constraints. The KKT conditions are:

- **Stationarity**:

$$\frac{\partial L}{\partial x} = 2x + \lambda_1 - \lambda_2 = 0, \quad \frac{\partial L}{\partial y} = 2y + \lambda_1 - \lambda_3 = 0$$

- **Primal feasibility**:

$$g_1(x,y) = x + y - 1 \leq 0, \quad g_2(x) = -x \leq 0, \quad g_3(y) = -y \leq 0$$

- **Dual feasibility**:

$$\lambda_1 \geq 0, \quad \lambda_2 \geq 0, \quad \lambda_3 \geq 0$$

- **Complementary slackness**:

$$\lambda_1(x+y-1) = 0, \quad \lambda_2(-x) = 0, \quad \lambda_3(-y) = 0$$

Thus, we can solve the KKT conditions:

$$\begin{cases} 2x + \lambda_1 - \lambda_2 = 0 \\ 2y + \lambda_1 - \lambda_3 = 0 \\ \lambda_1(x+y-1) = 0 \\ \lambda_2(-x) = 0 \\ \lambda_3(-y) = 0 \end{cases} \longleftrightarrow \begin{cases} 2x + \lambda_1 - \lambda_2 = 0 \\ 2y + \lambda_1 - \lambda_3 = 0 \\ \lambda_1 \cdot (x+y-1) = 0 \\ \lambda_2 \cdot x = 0 \\ \lambda_3 \cdot y = 0 \end{cases} \quad \text{where } \lambda_i \geq 0\ \forall i \in \{1,2,3\}$$

Now, we proceed case by case:

1. $\boxed{\lambda_i = 0\ \forall i \in \{1,2,3\}}$

$$\lambda_i = 0\ \forall i \in \{1,2,3\} \implies \begin{cases} 2x + \lambda_1 - \lambda_2 = 0 \\ 2y + \lambda_1 - \lambda_3 = 0 \\ \lambda_1(x+y-1) = 0 \\ \lambda_2(-x) = 0 \\ \lambda_3(-y) = 0 \end{cases} \longleftrightarrow \quad x = y = 0 \implies A = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \ \wedge\ \lambda_i = 0\ \forall i \in \{1,2,3\}$$

2. $\boxed{\lambda_1 > 0 \ \wedge \ \lambda_i = 0 \ \forall i \in \{2,3\}}$

$$\lambda_1 > 0 \wedge \lambda_i = 0 \ \forall i \in \{2,3\} \implies \begin{cases} 2x + \lambda_1 - \lambda_2 = 0 \\ 2y + \lambda_1 - \lambda_3 = 0 \\ \lambda_1(x+y-1) = 0 \\ \lambda_2(-x) = 0 \\ \lambda_3(-y) = 0 \end{cases} \longleftrightarrow \begin{cases} 2x + \lambda_1 = 0 \\ 2y + \lambda_1 = 0 \\ x = 1 - y \end{cases} \longleftrightarrow \begin{cases} 2 - 2y + \lambda_1 = 0 \longleftrightarrow 2y = 2 + \lambda_1 \\ 2y + \lambda_1 = 0 \\ x = 1 - y \end{cases}$$

Then, solving we obtain:

$$B = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \ \wedge \ \lambda_1 = -1 \ \wedge \lambda_i = 0 \ \forall i \in \{1,2,3\}$$

3. $\boxed{\lambda_1 > 0 \ \wedge \ \lambda_2 > 0 \ \wedge \ \lambda_3 = 0}$

$$\lambda_1 > 0 \wedge \lambda_2 > 0 \wedge \lambda_3 = 0 \implies \begin{cases} 2x + \lambda_1 - \lambda_2 = 0 \\ 2y + \lambda_1 - \lambda_3 = 0 \\ \lambda_1(x+y-1) = 0 \\ \lambda_2(-x) = 0 \\ \lambda_3(-y) = 0 \end{cases} \longleftrightarrow \begin{cases} 2x + \lambda_1 - \lambda_2 = 0 \\ 2y + \lambda_1 = 0 \\ x = 1 - y \\ x = 0 \end{cases} \longleftrightarrow \begin{cases} 2x + \lambda_1 - \lambda_2 = 0 \longleftrightarrow \lambda_2 = -2 \\ \lambda_1 = -2 \\ y = 1 \\ x = 0 \end{cases}$$

Then, solving we obtain the following candidate solution:

$$C = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \ \wedge \ \lambda_1 = -2 \ \wedge \lambda_2 = -2 \ \wedge \lambda_3 = 0$$

4. $\boxed{\lambda_1 > 0 \ \wedge \ \lambda_2 > 0 \ \wedge \ \lambda_3 > 0}$

$$\lambda_1 > 0 \ \wedge \ \lambda_2 > 0 \ \wedge \ \lambda_3 > 0 \implies \begin{cases} 2x + \lambda_1 - \lambda_2 = 0 \\ 2y + \lambda_1 - \lambda_3 = 0 \\ \lambda_1(x+y-1) = 0 \\ \lambda_2(-x) = 0 \\ \lambda_3(-y) = 0 \end{cases} \longleftrightarrow \begin{cases} 2x + \lambda_1 - \lambda_2 = 0 \\ 2y + \lambda_1 - \lambda_3 = 0 \\ x = 1 - y \\ x = 0 \\ y = 0 \end{cases} \longleftrightarrow \begin{cases} 2x + \lambda_1 - \lambda_2 = 0 \\ 2y + \lambda_1 - \lambda_3 = 0 \\ 0 = 1 \quad \text{Contradiction} \\ y = 0 \\ x = 0 \end{cases}$$

Then, there is no solution for this case.

We conclude that, after considering all cases, the solution candidates are:

$$A = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \ \wedge \ \lambda_i = 0 \ \forall i \in \{1,2,3\} \implies \text{The Lagrangian multipliers are non-negative.}$$

$$B = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \ \wedge \ \lambda_1 = -1 \ \wedge \lambda_i = 0 \ \forall i \in \{1,2,3\} \implies \text{The Lagrangian multipliers need to be non-negative.}$$

$$C = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \ \wedge \ \lambda_1 = -2 \ \wedge \lambda_2 = -2 \ \wedge \lambda_3 = 0 \implies \text{The Lagrangian multipliers need to be non-negative.}$$

Therefore, the optimal solution is candidate A:

$$x^* = 0, \ y^* = 0$$

The minimum value of the objective function is:

$$f(x^*, y^*) = 0.$$

2 Exercise 2

Solve the following optimization problem:

$$\textbf{Mininimize: } f(x_1, \ldots, x_n) = \textstyle\sum_{i=1}^n x_i^2$$
$$\textbf{Subject to: } \textstyle\sum_{i=1}^n x_i = c, \; x_i \geq 0, \text{ where } c > 0 \text{ and } \forall i = 1, \ldots, n$$

2.1 Answers

Let's proceed step-by-step. First, we need to formulate the problem. We want to minimize the function:

$$f(x_1, \ldots, x_n) = \sum_{i=1}^n x_i^2$$

Subject to:

- $g_0(x_1, \ldots, x_n) = \sum_{i=1}^n x_i - c \leq 0$
- $g_1(x_1, \ldots, x_n) = -x_1 \leq 0, \; \ldots, \; g_i(x_1, \ldots, x_n) = -x_i \leq 0, \; \ldots, \; g_n(x_1, \ldots, x_n) = -x_n \leq 0$

Then, we define the Lagrangian function $L(x_1, \ldots, x_n, \lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_n)$:

$$L(x_1, \ldots, x_n, \lambda_0, \lambda_1, \ldots, \lambda_n) = \sum_{i=1}^n x_i^2 + \lambda_0 \cdot (\sum_{i=1}^n x_i - c) + \lambda_1(-x_1) + \cdots + \lambda_n(-x_n)$$

Here, $\lambda_i \; \forall i \in \{0, \ldots, n\}$ are the Lagrange multipliers for the constraints. The KKT conditions are:

- **Stationarity**:
$$\frac{\partial L}{\partial x_1} = 2x_1 + \lambda_0 - \lambda_1 = 0, \quad \frac{\partial L}{\partial x_2} = 2x_2 + \lambda_0 - \lambda_2 = 0, \quad \ldots, \quad \frac{\partial L}{\partial x_n} = 2x_n + \lambda_0 - \lambda_n = 0$$

- **Primal feasibility**:
$$g_0(x_1, \ldots, x_n) = \sum_{i=1}^n x_i - c = 0, \; g_1(x_1, \ldots, x_n) = -x_1 \leq 0, \; \ldots, \; g_n(x_1, \ldots, x_n) = -x_n \leq 0$$

- **Dual feasibility**:
$$\lambda_i \geq 0 \; \forall i \in \{1, \ldots, n\}, \quad \lambda_0 \in \mathbb{R}$$

- **Complementary slackness**:
$$\lambda_0 \cdot (\sum_{i=1}^n x_i - c) = 0, \quad \lambda_1(-x_1) = 0, \quad \ldots, \quad \lambda_n(-x_n) = 0$$

Thus, we can solve the KKT conditions:

$$\begin{cases} 2x_1 + \lambda_0 - \lambda_1 = 0 \\ 2x_2 + \lambda_0 - \lambda_2 = 0 \\ \quad \ldots \\ 2x_n + \lambda_0 - \lambda_n = 0 \\ \lambda_0 \cdot (\sum_{i=1}^n x_i - c) = 0 \\ \lambda_1(-x_1) = 0 \\ \quad \ldots \\ \lambda_n(-x_n) = 0 \end{cases} \longleftrightarrow \begin{cases} 2x_1 + \lambda_0 - \lambda_1 = 0 \\ 2x_2 + \lambda_0 - \lambda_2 = 0 \\ \quad \ldots \\ 2x_n + \lambda_0 - \lambda_n = 0 \\ \lambda_1 \cdot x_1 = 0 \\ \quad \ldots \\ \lambda_n \cdot x_n = 0 \\ \lambda_0 \cdot (\sum_{i=1}^n x_i - c) = 0 \end{cases} \quad \text{where } \lambda_i \geq 0 \; \forall i \in \{1, \ldots, n\} \text{ and } \lambda_0 \in \mathbb{R}$$

Now, we proceed case by case:

1. $\boxed{\lambda_i = 0 \ \forall i \in \{0, \dots, n\}}$

$$\lambda_i = 0 \ \forall i \in \{0, \dots, n\} \implies \begin{cases} 2x_1 + \lambda_0 - \lambda_1 = 0 \\ 2x_2 + \lambda_0 - \lambda_2 = 0 \\ \quad \dots \\ 2x_n + \lambda_0 - \lambda_n = 0 \\ \lambda_1 \cdot x_1 = 0 \\ \quad \dots \\ \lambda_n \cdot x_n = 0 \\ \lambda_0 (\sum_{i=1}^n x_i - c) = 0 \end{cases} \longleftrightarrow x_1 = x_2 = \dots = x_n = 0$$

Then, the candidate solution is:

$$A = \begin{pmatrix} 0 \\ \dots \\ 0 \end{pmatrix} \ \wedge \ \lambda_i = 0 \ \forall i \in \{0, \dots, n\}$$

2. $\boxed{\lambda_0 \neq 0 \ \wedge \ \lambda_i = 0 \ \forall i \in \{1, \dots, n\}}$

$$\lambda_0 \neq 0 \ \wedge \ \lambda_i = 0 \ \forall i \in \{1, \dots, n\} \implies \begin{cases} 2x_1 + \lambda_0 = 0 \\ 2x_2 + \lambda_0 = 0 \\ \quad \dots \\ 2x_n + \lambda_0 = 0 \\ \sum_{i=1}^n x_i = c \end{cases} \longleftrightarrow \begin{cases} 2x_1 + \lambda_0 = 0 \\ x_1 = x_2 = \dots = x_n \\ \sum_{i=1}^n x_i = c \end{cases} \longleftrightarrow \begin{cases} \lambda_0 = \frac{-2c}{n} \\ x_1 = x_2 = \dots = x_n \\ x_1 = \frac{c}{n} \end{cases}$$

Then, the candidate solution is:

$$B = \begin{pmatrix} \frac{c}{n} \\ \dots \\ \frac{c}{n} \end{pmatrix} \ \wedge \ \lambda_0 = \frac{-2c}{n} \ \wedge \lambda_i = 0 \ \forall i \in \{1, \dots, n\}$$

3. $\boxed{\lambda_0 \neq 0 \ \wedge \ \lambda_1 > 0 \ \wedge \ \lambda_i = 0 \ \forall i \in \{2, \dots, n\}}$ ★

$$\bigstar \implies \begin{cases} 2x_1 + \lambda_0 - \lambda_1 = 0 \\ 2x_2 + \lambda_0 - \lambda_2 = 0 \\ \quad \dots \\ 2x_n + \lambda_0 - \lambda_n = 0 \\ \lambda_1 \cdot x_1 = 0 \\ \quad \dots \\ \lambda_n \cdot x_n = 0 \\ \lambda_0 (\sum_{i=1}^n x_i - c) = 0 \end{cases} \longleftrightarrow \begin{cases} 2x_1 + \lambda_0 - \lambda_1 = 0 \\ 2x_2 + \lambda_0 = 0 \\ \quad \dots \\ 2x_n + \lambda_0 = 0 \\ x_1 = 0 \\ \sum_{i=1}^n x_i = c \end{cases} \longleftrightarrow \begin{cases} \lambda_0 = \lambda_1 = -2x_2 = \frac{-2c}{n-1} \\ x_2 = \dots = x_n \\ x_1 = 0 \\ x_2 = \frac{c}{n-1} \end{cases}$$

Then, solving we obtain the following candidate solution:

$$C = \begin{pmatrix} 0 \\ \frac{c}{n-1} \\ \dots \\ \frac{c}{n-1} \end{pmatrix} \ \wedge \ \lambda_0 = \lambda_1 = \frac{-2c}{n-1} \ \wedge \lambda_i = 0 \ \forall i \in \{2, \dots, n\}$$

4. $\boxed{\lambda_0 \neq 0 \;\wedge\; \lambda_1, \lambda_2 > 0 \;\wedge \lambda_i = 0 \;\forall i \in \{3, \ldots, n\}}$ ★★

$$★★ \implies \begin{cases} 2x_1 + \lambda_0 - \lambda_1 = 0 \\ 2x_2 + \lambda_0 - \lambda_2 = 0 \\ \quad \ldots \\ 2x_n + \lambda_0 - \lambda_n = 0 \\ \lambda_1 \cdot x_1 = 0 \\ \quad \ldots \\ \lambda_n \cdot x_n = 0 \\ \lambda_0 (\sum_{i=1}^n x_i - c) = 0 \end{cases} \longleftrightarrow \begin{cases} 2x_1 + \lambda_0 - \lambda_1 = 0 \\ 2x_2 + \lambda_0 - \lambda_2 = 0 \\ 2x_3 + \lambda_0 = 0 \\ \quad \ldots \\ 2x_n + \lambda_0 = 0 \\ x_1 = 0 \\ x_2 = 0 \\ \sum_{i=1}^n x_i = c \end{cases} \longleftrightarrow \begin{cases} \lambda_0 = \lambda_1 = \lambda_2 = \frac{-2c}{n-2} \\ x_3 = \cdots = x_n \\ x_1 = 0 \\ x_2 = 0 \\ x_3 = \frac{c}{n-2} \end{cases}$$

Then, solving we obtain the following candidate solution:

$$D = \begin{pmatrix} 0 \\ 0 \\ \frac{c}{n-2} \\ \ldots \\ \frac{c}{n-2} \end{pmatrix} \;\wedge\; \lambda_0 = \lambda_1 = \lambda_2 = \frac{-2c}{n-2} \;\wedge \lambda_i = 0 \;\forall i \in \{3, \ldots, n\}$$

5. In the upcoming cases, we can observe the pattern and therefore discard all the candidates as the Lagrangian multipliers are negative, even though if we find a feasible solution. Notice as well, that candidates B and C are also discarded due to previous reasons. Only candidate A remains a possible true candidate.

6. $\boxed{\lambda_0 \neq 0 \;\wedge\; \lambda_1, \lambda_2, \ldots, \lambda_{n-1} > 0 \;\wedge \lambda_n = 0}$ ★★★

$$★★★ \implies \begin{cases} 2x_1 + \lambda_0 - \lambda_1 = 0 \\ 2x_2 + \lambda_0 - \lambda_2 = 0 \\ \quad \ldots \\ 2x_n + \lambda_0 - \lambda_n = 0 \\ \lambda_1 \cdot x_1 = 0 \\ \quad \ldots \\ \lambda_n \cdot x_n = 0 \\ \lambda_0 (\sum_{i=1}^n x_i - c) = 0 \end{cases} \longleftrightarrow \begin{cases} \lambda_0 - \lambda_1 = 0 \\ \lambda_0 - \lambda_2 = 0 \\ \quad \ldots \\ \lambda_0 - \lambda_{n-1} = 0 \\ 2x_n + \lambda_0 = 0 \\ x_1, \ldots, x_{n-1} = 0 \\ \sum_{i=1}^n x_i = c \end{cases} \longleftrightarrow \begin{cases} \lambda_0 = \lambda_1 = \cdots = \lambda_{n-1} \\ \lambda_0 = -2c \\ x_1, \ldots, x_{n-1} = 0 \\ x_n = c \end{cases}$$

Then, solving we obtain the following candidate solution:

$$\Omega = \begin{pmatrix} 0 \\ \ldots \\ 0 \\ c \end{pmatrix} \;\wedge\; \lambda_0 = \lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} = -2c \;\wedge \lambda_n = 0$$

As we can observe, this solution doesn't satisfy the dual feasibility as well, therefore it's discarded too.

We conclude that, after considering all cases, the only possible solution candidates are:

$$A = \begin{pmatrix} 0 \\ \ldots \\ 0 \end{pmatrix} \in \mathbb{R}^n \;\wedge\; \lambda_i = 0 \;\forall i \in \{1, 2, 3\} \implies \text{The Lagrangian multipliers are non-negative.}$$

$$B = \begin{pmatrix} \frac{c}{n} \\ \ldots \\ \frac{c}{n} \end{pmatrix} \;\wedge\; \lambda_0 = \frac{-2c}{n} \;\wedge \lambda_i = 0 \;\forall i \in \{1, \ldots, n\} \implies \text{The Lagrangian multipliers are non-negative.}$$

The values of the objective function for both candidates are:

$$f(A) = \sum_{i=1}^{n} 0^2 = 0.$$

$$f(B) = \sum_{i=1}^{n} \left(\frac{c}{n}\right)^2 = \frac{c^2}{n}.$$

Clearly, the solution is candidate A.

3 Exercise 3

Solve the following optimization problem for $Q = 2\mathcal{I},\ b = (-4,-6)^T,\ A = (1,1)$ and $b = 4$.

$$\textbf{Mininimize: } f(x,y) = x^T Q x + b^T x$$
$$\textbf{Subject to: } Ax = b,\ x \geq 0$$

Where:

- $x \in \mathbb{R}^n$ is the decision variable, $Q \in \mathbb{R}^{n \times n}$ is a symmetric positive definite matrix.
- $b \in \mathbb{R}^n$ is a coefficient vector, $A \in \mathbb{R}^{m \times n}$.

3.1 Answers

Let's proceed step-by-step. First, we need to formulate the problem. We want to minimize the function:

$$f(x,y) = x^T Q x + b^T x = 2x^2 + 2y^2 - 4x - 6y$$

Subject to:

- $g_0(x,y) = Ax - b = x + y - 4 = 0$
- $g_1(x,y) = -x \leq 0$

Then, we define the Lagrangian function $L(x, y, \lambda_0, \lambda_1)$:

$$L(x, y, \lambda_0, \lambda_1) = 2x^2 + 2y^2 - 4x - 6y + \lambda_0(x + y - 4) + \lambda_1(-x)$$

Here, $\lambda_i\ \forall i \in \{1, 2\}$ are the Lagrange multipliers for the constraints. The KKT conditions are:

- **Stationarity**:
$$\frac{\partial L}{\partial x} = 4x - 4 + \lambda_0 - \lambda_1 = 0, \quad \frac{\partial L}{\partial y} = 4y - 6 + \lambda_0 = 0$$
- **Primal feasibility**:
$$g_0(x,y) = x + y - 4 = 0,\ g_1(x,y) = -x \leq 0$$
- **Dual feasibility**:
$$\lambda_1 \geq 0, \quad \lambda_0 \in \mathbb{R}$$
- **Complementary slackness**:
$$\lambda_0(x + y - 4) = 0, \quad \lambda_1(-x) = 0$$

Thus, we can solve the KKT conditions:

$$\begin{cases} 4x - 4 + \lambda_0 - \lambda_1 = 0 \\ 4y - 6 + \lambda_0 = 0 \\ \lambda_0(x + y - 4) = 0 \\ \lambda_1(-x) = 0 \end{cases} \longleftrightarrow \begin{cases} 4x - 4 + \lambda_0 - \lambda_1 = 0 \\ 4y - 6 + \lambda_0 = 0 \\ \lambda_0(x + y - 4) = 0 \\ \lambda_1 \cdot x = 0 \end{cases} \quad \text{where } \lambda_1 \geq 0 \text{ and } \lambda_0 \in \mathbb{R}$$

Now, we proceed case by case:

1. $\boxed{\lambda_i = 0 \ \forall i \in \{0,1\}}$

$$\lambda_i = 0 \ \forall i \in \{0,1\} \implies \begin{cases} 4x - 4 + \lambda_0 - \lambda_1 = 0 \\ 4y - 6 + \lambda_0 = 0 \\ \lambda_0(x+y-4) = 0 \\ \lambda_1 \cdot x = 0 \end{cases} \longleftrightarrow \begin{cases} x = 1 \\ y = \frac{3}{2} \end{cases}$$

Then, the candidate solution is:

$$A = \begin{pmatrix} 1 \\ \frac{3}{2} \end{pmatrix} \ \wedge \ \lambda_i = 0 \ \forall i \in \{0,1\}$$

2. $\boxed{\lambda_0 \neq 0 \ \wedge \ \lambda_1 = 0}$

$$\lambda_0 \neq 0 \ \wedge \ \lambda_1 = 0 \implies \begin{cases} 4x - 4 + \lambda_0 - \lambda_1 = 0 \\ 4y - 6 + \lambda_0 = 0 \\ \lambda_0(x+y-4) = 0 \\ \lambda_1 \cdot x = 0 \end{cases} \longleftrightarrow \begin{cases} 4x - 4 + \lambda_0 = 0 \\ 4y - 6 + \lambda_0 = 0 \\ x = 4 - y \end{cases} \longleftrightarrow \begin{cases} \lambda_0 = -3 \\ x = \frac{7}{4} \\ y = \frac{9}{4} \end{cases}$$

Then, the candidate solution is:

$$B = \begin{pmatrix} \frac{7}{4} \\ \frac{9}{4} \end{pmatrix} \ \wedge \ \lambda_0 = -3 \ \wedge \lambda_1 = 0$$

3. $\boxed{\lambda_0 \neq 0 \ \wedge \ \lambda_1 > 0}$

$$\lambda_0 \neq 0 \ \wedge \ \lambda_1 > 0 \implies \begin{cases} 4x - 4 + \lambda_0 - \lambda_1 = 0 \\ 4y - 6 + \lambda_0 = 0 \\ \lambda_0(x+y-4) = 0 \\ \lambda_1 \cdot x = 0 \end{cases} \longleftrightarrow \begin{cases} \lambda_1 = -14 \\ \lambda_0 = -10 \\ y = 4 \\ x = 0 \end{cases}$$

Then, solving we obtain the following candidate solution:

$$C = \begin{pmatrix} 0 \\ 4 \end{pmatrix} \ \wedge \ \lambda_0 = -10 \ \wedge \ \lambda_1 = -14$$

We conclude that, after considering all cases, the only possible solution candidates are:

$$A = \begin{pmatrix} 1 \\ \frac{3}{2} \end{pmatrix} \ \wedge \ \lambda_i = 0 \ \forall i \in \{0,1\} \implies \text{Multipliers associated to inequality constraints are non-negative.}$$

$$B = \begin{pmatrix} \frac{7}{4} \\ \frac{9}{4} \end{pmatrix} \ \wedge \ \lambda_0 = -3 \ \wedge \lambda_1 = 0 \implies \text{Multipliers associated to inequality constraints are non-negative.}$$

$$C = \begin{pmatrix} 0 \\ 4 \end{pmatrix} \ \wedge \ \lambda_0 = -10 \ \wedge \ \lambda_1 = -14 \implies \text{The Lagrangian multipliers from inequalities need to be non-negative.}$$

The values of the objective function for both candidates are:

$$f(A) = 2(1)^2 + 2\left(\frac{3}{2}\right)^2 - 4(1) - 6\left(\frac{3}{2}\right) = 2 + \frac{9}{2} - 4 - 9 = -\frac{13}{2} = -6.5$$
$$f(B) = 2\left(\frac{7}{4}\right)^2 + 2\left(\frac{9}{4}\right)^2 - 4\left(\frac{7}{4}\right) - 6\left(\frac{9}{4}\right) = -\frac{17}{4} = -4.25$$

Clearly, the solution is candidate A.