Social and Graph Data Management
Node and Link Analysis

Benoît Groz (slides and content are mostly from Silviu Maniu)

December 8th, 2023

M2 Data Science

Estimating Node Worth

Nodes on the Web: pages (sites, Wikipedia, . . .), users (Twitter, Facebook), etc.

To use/find the nodes that are more “interesting” than others, we have to estimate the worth of each node.

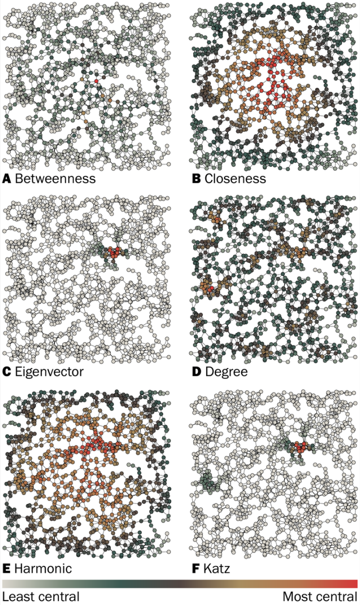
Can be used combined with textual (or profile) information to retrieve content in information retrieval – but also the *links* between information are important

Estimating Node Worth: centrality measures

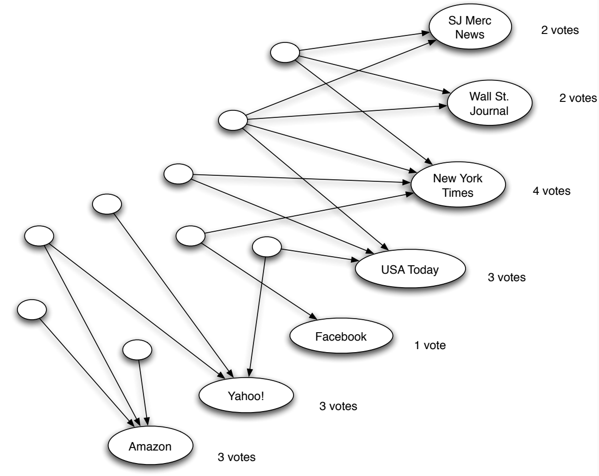
Rank of node v can be evaluated using Node centrality measures :

- Betweenness centrality : $\sum_{v_1, v_2 \neq v}$(# shortest paths from s to t that contain v)/(# shortest paths from s to t)
- Closeness : $1/(\sum_{v'} d(v, v'))$, Harmonic : $\sum_{v' \neq v} 1/d(v, v')$
- Spectral centrality: Katz : $\sum_k \sum_{v'} \alpha^k A^k_{v\,v'}$, Hits, Eigenvector, PageRank
- Degree centrality (weighted or not)

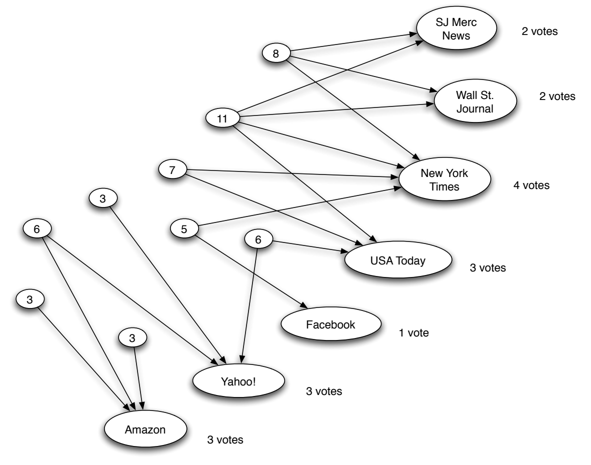
Source: `https://en.wikipedia.org/wiki/Centrality`



Degree Centrality



Weighted Degree Centrality



Repeated Improvement

Principle of Repeated Improvement: the pages which link to high-worth pages are high-worth themselves

Suggests a two-way relationship:

- nodes which are relevant are called authorities, and are linked to by good nodes
- nodes which link to many other high-authority nodes are called hubs

Hubs and Authorities

Each node $i \in V$ has attached an authority score a_i and a hub score h_i

Each score is update in relation to the other.

HITS Algorithm

1. Initialize: for each $i \in V$: $a_i = 1$, $h_i = 1$
2. Authority Update Rule: the authority score of a page is the sum of hub scores of the *incoming* neighbours

$$a_i = \sum_{j \in I(i)} h_j$$

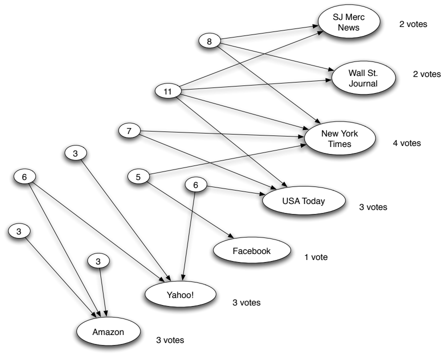
3. Hub Update Rule: the hub score of a page is the sum of the authority scores of the *outgoing* neighbours

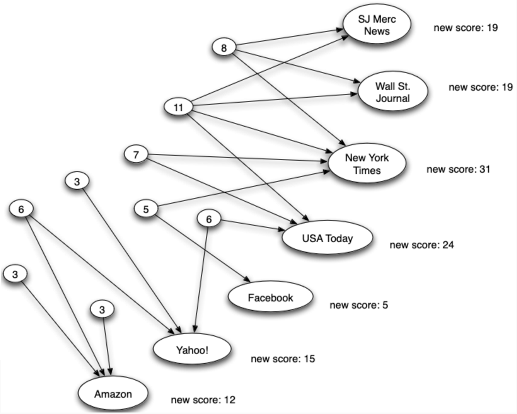
$$h_i = \sum_{j \in O(i)} a_j$$

4. Normalization the sum of all hub (or authority) scores must be the same between steps

In practice, the algorithm is repeated for a fixed number of steps k

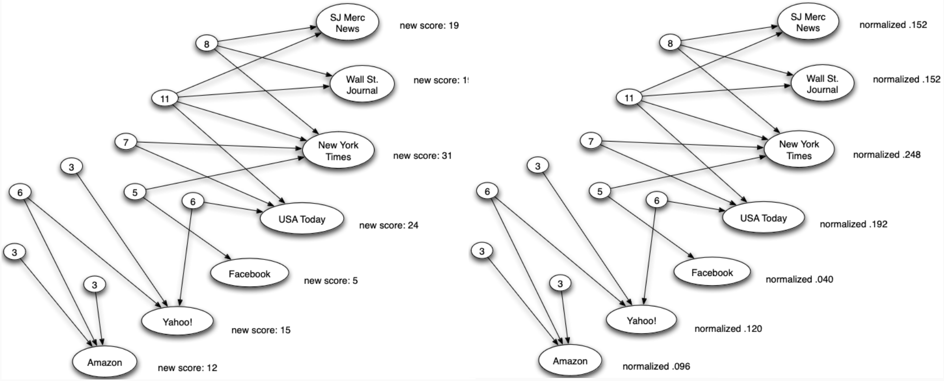
Applying HITS – Example 1a





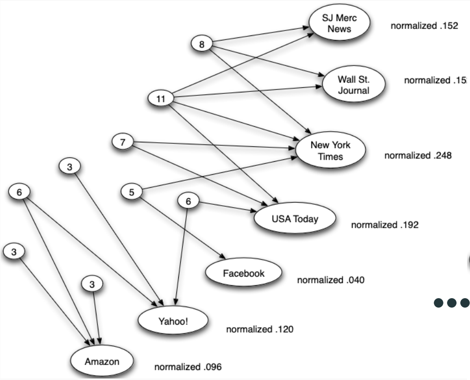
Updating authority scores

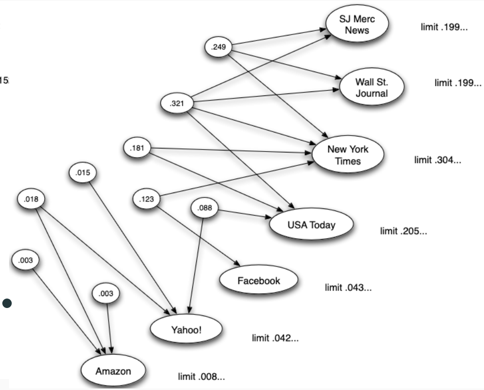
Applying HITS – Example 1b



Normalizing authority scores

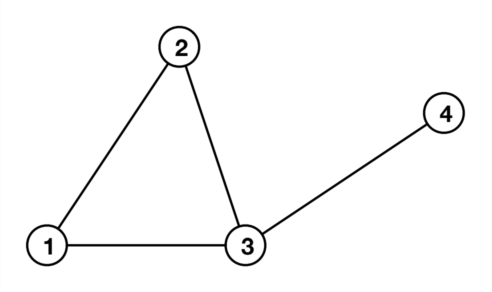
Applying HITS – Example 1c





Limit of hub and authority scores

Applying HITS – Example 2



HITS: Matrix View

In matrix form, at time $\boldsymbol{k}$, the vectors $\boldsymbol{a}$ and $\boldsymbol{h}$ have the following forms:

$$\boldsymbol{a}^{\langle k \rangle} = A^\top \boldsymbol{h}^{\langle k-1 \rangle} \quad \boldsymbol{h}^{\langle k \rangle} = A\boldsymbol{a}^{\langle k \rangle}$$

We can see that, e.g.:

$$\boldsymbol{h}^{\langle k \rangle} = AA^\top \boldsymbol{h}^{\langle k-1 \rangle},$$

which leads to the general formulas:

$$\boldsymbol{h}^{\langle k \rangle} = (AA^\top)^k \boldsymbol{h}^{\langle 0 \rangle}$$
$$\boldsymbol{a}^{\langle k \rangle} = (A^\top A)^{k-1} A^\top \boldsymbol{h}^{\langle 0 \rangle}$$

Applying spectral analysis, it can be shown that the normalized values converge when $\boldsymbol{k} \to \infty$

HITS in Practice

- computes two scores for each node, which might be hard to interpret
- it is meant for particular queries, and only a subset of pages, so query-time computation
- it is not usually used in current search engines

Table of contents

PageRank: Ranking Nodes in A Graph

Definition 1: The important nodes are the nodes that are linked to by other important nodes (recursive).

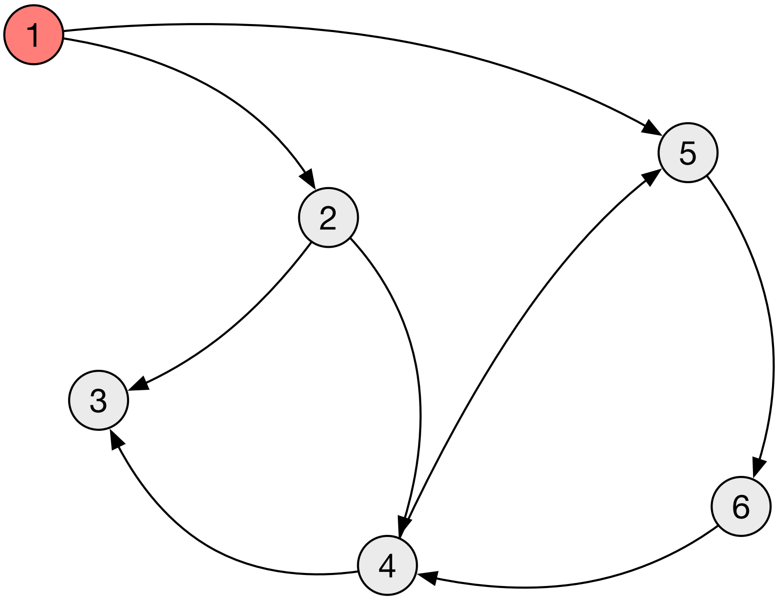
Definition 2 – the random surfer model, where the surfer walks on the graph:

1. the surfer starts at a node (*e.g.*, Google) and chooses randomly an outgoing node (*e.g.*, a page in the search results),
2. the surfer behaves in the same manner for other nodes,
3. at each step the surfer has a probability $1 - \alpha$ (damping factor) of jumping elsewhere randomly.

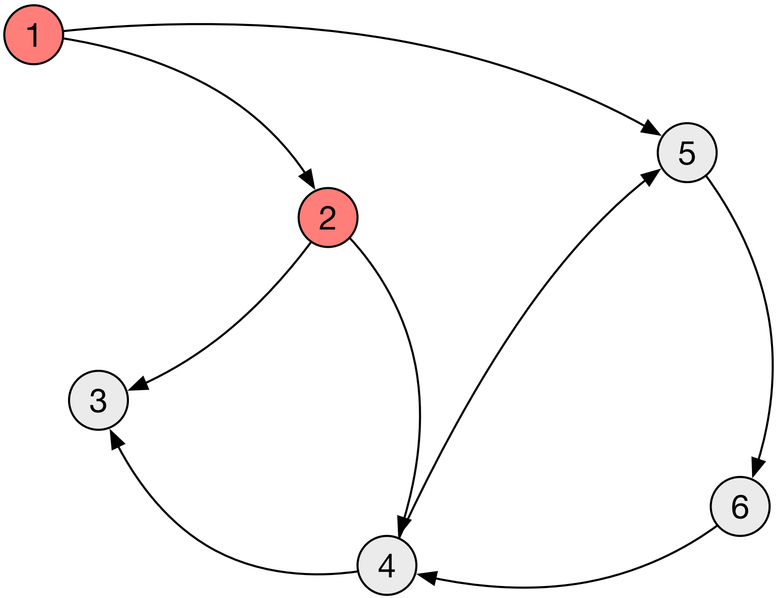
The importance of a page = the stationary probability that the surfer is on a page at time ∞.

The two definitions are equivalent.

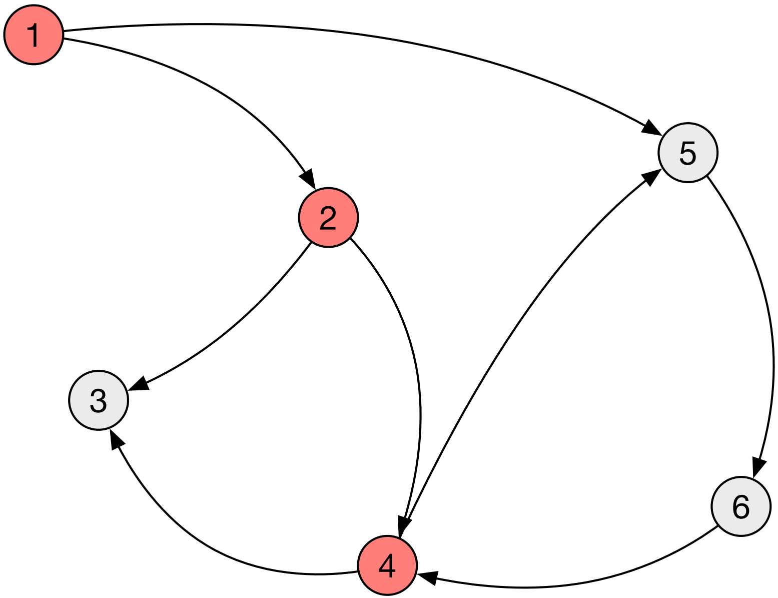
PageRank: Random Surfer



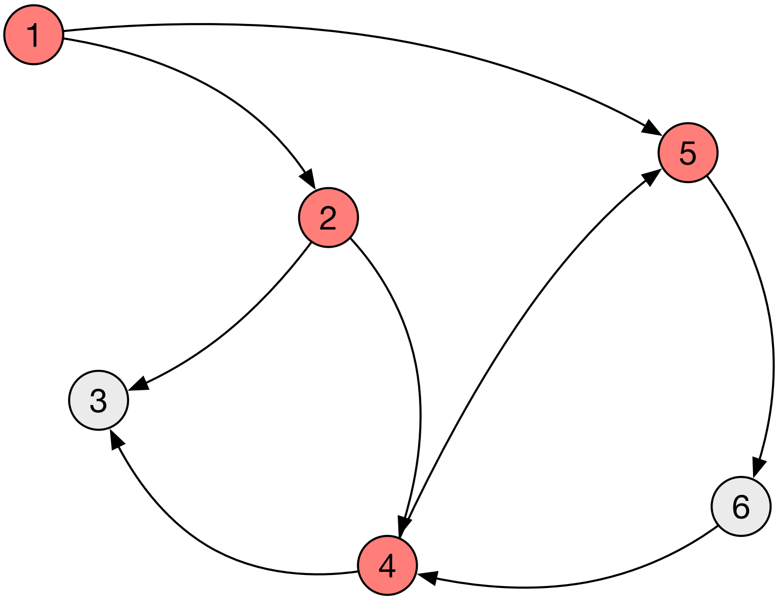
PageRank: Random Surfer



PageRank: Random Surfer



PageRank: Random Surfer



PageRank Equation and Algorithm

For a graph G having n nodes, where each node i has the incoming neighbours I_i and outgoing neighbours O_i:

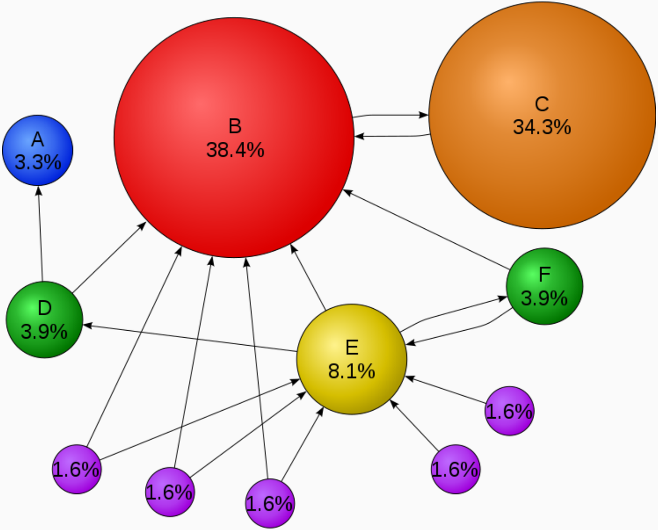
$$p(i) = \alpha \sum_{j \in I_i} \frac{p(j)}{|O_j|} + \frac{1-\alpha}{n}.$$

Algorithm for computing $p(i)$:

1. start with initial values of $p(i) = \frac{1}{n}$,
2. iteratively apply the equation for each node i,
3. stop when the probabilities converge (stationary).

Monte-Carlo approximation: simulate N walks and take $p(i) = \frac{v_i}{N}$, where v_i number of visits of page i among N walks.

PageRank Example



PageRanks with a damping factor 0.85

source: `https://commons.wikimedia.org/wiki/File:PageRanks-Example.svg`

PageRank: Matrix View

We need to define a transition matrix $\boldsymbol{M}$ and a teleportation vector $\boldsymbol{t}$.

The transition matrix is a stochastic matrix where:

$$m_{ij} = \frac{a_{ij}}{|O_i|},$$

where $\boldsymbol{a}_{ij}$ is the corresponding entry in the adjacency matrix; it is stochastic because $\sum_j m_{ij} = 1$.

The teleportation vector $\boldsymbol{t}$ is also stochastic:

$$\boldsymbol{t} = \begin{pmatrix} 1/n & \dots & 1/n \end{pmatrix}^\top$$

PageRank: Matrix View and Markov Chains

The matrix formulation views the process as a Markov chain:

- used to model stochastic sequences of events (states)
- current state/event depends *only* on the predecessor state
- transitions are distributions of the successor states

Ergodic Markov chain – always converges to a stationary distribution of probability of states:

- irreducible – there is a sequence of transitions from any state to another: no “partition” of the states
- aperiodic – no state in the class has period > 1, where period is $gcd(\{n \mid p_n(i, i) > 0\})$

PageRank: Matrix View

The PageRank vector $\boldsymbol{p}$ at step $\boldsymbol{k}$ is defined as:

$$\boldsymbol{p}^{\langle k \rangle} = \alpha M \boldsymbol{p}^{\langle k-1 \rangle} + (1 - \alpha)\boldsymbol{t}$$

It can be proved this converges to a known value $\boldsymbol{p}^{\langle \infty \rangle}$, the stationary distribution:

$$\boldsymbol{p}^{\langle \infty \rangle} = (I - \alpha M)^{-1}(1 - \alpha)\boldsymbol{t}$$

- intuitively, the proof shows that the teleportation vector allows the corresponding Markov chain to be ergodic

Variants of PageRank

Depending where the surfer teleports with probability $1 - \alpha$, we have different variants of PageRank:

- classic PageRank: the surfer can jump to any node.
- personalized PageRank: the surfer can only jump to their start page
$$\boldsymbol{t}_j = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \end{pmatrix}^\top$$
- topic-sensitive PageRank: the surfer can only jump to a set of same-topic pages

$$\boldsymbol{t} = \begin{pmatrix} 0 & 1/l & 1/l & \dots & 0 \end{pmatrix}^\top$$

Table of contents

The Link Prediction Problem

Social networks are evolving, and new relationships (links) appear all the time

Link Prediction Problem: predict which links are more likely to appear in a social network

Assumes that links can be predicted via analysis based only on the social network itself

Applications:

- new link recommendation (e.g., new friends)
- missing link inference
- analyzing network evolution

Link Scoring Function

We want to “guess” the score of potential links for a graph $G = (V, E)$, i.e., a function defined on the missing links $E' = (V \times V) \setminus E$:

$$\text{score} : E' \to \mathbb{R}^+$$

For a given i, score(i, j) established a ranking of all (unlinked) nodes j relative to i – best scores are the most likely new links

How can we define the score function using only the properties intrinsic to the network?

Node Neighbourhood Scores

- Common Neighbours, most straightforward just counts the number of common neighbours:

$$\text{score}(i, j) = |N(i) \cap N(j)|$$

- Jaccard coefficient, computes the “similarity” between the neighborhood sets

$$\text{score}(i, j) = \frac{|N(i) \cap N(j)|}{|N(i) \cup N(j)|}$$

- Preferential attachment, the score is proportional to the degrees of each node:

$$\text{score}(i, j) = k_i k_j$$

Path-Based Scores

- Inverse Distance, the score is inversely proportional to the distance between two nodes

$$\text{score}(i,j) = 1/d_{ij}$$

- Katz, where the score is a weighted sum of all the path between i and j

$$\text{score}(i,j) = \sum_{l=1}^{\infty} \beta^{l} |\text{paths}_{i,j}^{\langle l \rangle}|,$$

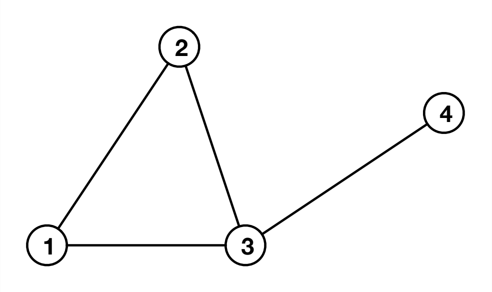
where $\beta \in (0,1)$

Random Walk-Based Scores

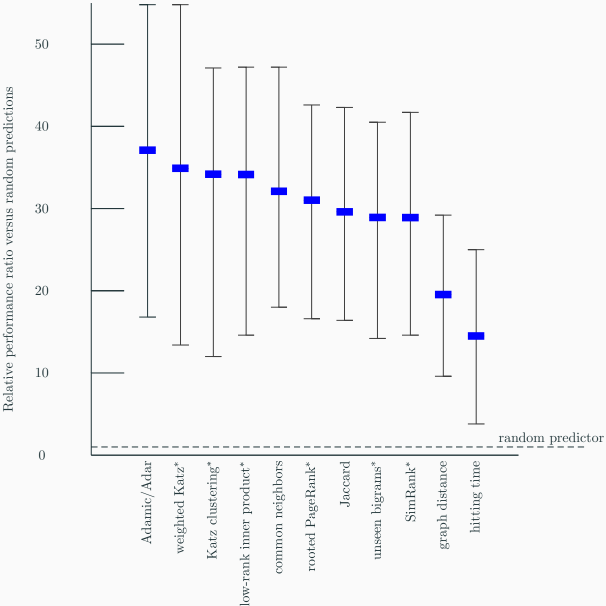
- Hitting time, $\text{score}(i, j) = H_{i,j}$ the time it takes a random walk from i to reach j
- Personalized PageRank, generally any PageRank-related measure in which the teleportation vector is rooted at i
- SimRank, a recursive definition based on the score of neighbours

$$\text{score}(i, j) = \gamma \frac{\sum_{a \in N(i)} \sum_{b \in N(j)} \text{score}(a, b)}{k_i k_j}$$

Applying the Scores – Example



Performance of Link Prediction



Acknowledgments

Figures in slides 4, 5, 6 and 10 are taken from the book "Networks, Crowds, and Markets", D. Easley and J. Kleinberg, Cambridge University Press, 2010.

The figure in slide 34 is taken
from [Liben-Nowell and Kleinberg, 2003].

Kleinberg, J. M. (1999).
Authoritative sources in a hyperlinked environment.
J. ACM, 46(5).

Liben-Nowell, D. and Kleinberg, J. (2003).
The link prediction problem for social networks.
In *Proceedings of the Twelfth International Conference on Information and Knowledge Management.*

Page, L., Brin, S., Motwani, R., and Winograd, T. (1998).
The PageRank citation ranking: Bringing order to the Web.
In *Proceedings of the 7th International World Wide Web Conference.*