Covariance function from a Poisson PDE

The two-dimensional Poisson equation in a rectangular Cartesian coordinate sytem has the form

$$\frac{\partial^2 v(x,y)}{\partial x^2} + \frac{\partial^2 v(x,y)}{\partial y^2} = -f(x,y) \tag{1}$$

1 Solving the Poisson equation

The exact solution for Eq. (1) is subject to particular boundary conditions. For a first boundary value problem with domain $0 \leq x \leq a$, $0 \leq y \leq b$, and conditions given by

$$\begin{aligned} v(x=0,y) &= g_1(y) \\ v(x=a,y) &= g_2(y) \\ v(x,y=0) &= g_3(x) \\ v(x,y=b) &= g_4(x) \end{aligned}$$

the solution to this equation is given by [1]

$$\begin{aligned} v(x,y) = &\int_0^a \int_0^b f(\xi,\eta) G(x,y,\xi,\eta) \mathrm{d}\eta \mathrm{d}\xi \\ &+ \int_0^b g_1(\eta) H_1(x,y,\eta) \mathrm{d}\eta - \int_0^b g_2(\eta) H_2(x,y,\eta) \mathrm{d}\eta \\ &+ \int_0^a g_3(\xi) H_3(x,y,\xi) \mathrm{d}\xi - \int_0^a g_4(\eta) H_4(x,y,\xi) \mathrm{d}\xi, \end{aligned} \tag{2}$$

where the Green function $G(x,y,\xi,\eta)$ and the functions $H_1(x,y,\xi,\eta)$, $H_2(x,y,\xi,\eta)$, $H_3(x,y,\xi,\eta)$ and $H_4(x,y,\xi,\eta)$ are given by

$$\begin{aligned} G(x,y,\xi,\eta) &= \frac{4}{ab} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\sin(p_n x) \sin(q_m y) \sin(p_n \xi) \sin(q_m \eta)}{p_n^2 + q_m^2}, \\ H_1(x,y,\eta) &= \frac{\partial}{\partial \xi} G(x,y,\xi,\eta)\bigg|_{\xi=0}, \quad H_2(x,y,\eta) = \frac{\partial}{\partial \xi} G(x,y,\xi,\eta)\bigg|_{\xi=a}, \\ H_3(x,y,\xi) &= \frac{\partial}{\partial \eta} G(x,y,\xi,\eta)\bigg|_{\eta=0}, \quad H_4(x,y,\xi) = \frac{\partial}{\partial \eta} G(x,y,\xi,\eta)\bigg|_{\eta=b}, \end{aligned} \tag{3}$$

where $p_n = \frac{n\pi}{a}$, and $q_m = \frac{m\pi}{b}$.
Assuming $g_1(y) = g_2(y) = g_3(x) = g_4(x) = 0$, equation (2) is equal to

$$v(x,y) = \int_0^a \int_0^b f(\xi,\eta) G(x,y,\xi,\eta) \mathrm{d}\eta \mathrm{d}\xi \tag{4}$$

where $G(x,y,\xi,\eta)$ is given by (3).

Boundary conditions

We first compute the functions $H_1(x,y,\xi,\eta)$, $H_2(x,y,\xi,\eta)$, $H_3(x,y,\xi,\eta)$, and $H_4(x,y,\xi,\eta)$,

$$H_1(x,y,\eta) = \frac{\partial}{\partial \xi} G(x,y,\xi,\eta)\bigg|_{\xi=0} = \frac{4}{ab} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{p_n}{p_n^2 + q_m^2} \sin(p_n x) \sin(q_m y) \sin(q_m \eta)$$

$$H_2(x,y,\eta) = \frac{\partial}{\partial \xi} G(x,y,\xi,\eta)\bigg|_{\xi=a} = \frac{4}{ab} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{p_n \cos(n\pi)}{p_n^2 + q_m^2} \sin(p_n x) \sin(q_m y) \sin(q_m \eta)$$

$$H_3(x,y,\xi) = \frac{\partial}{\partial \eta}G(x,y,\xi,\eta)\bigg|_{\eta=0} = \frac{4}{ab}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{q_m}{p_n^2+q_m^2}\sin(p_n x)\sin(q_m y)\sin(p_n\xi)$$

$$H_4(x,y,\xi) = \frac{\partial}{\partial \eta}G(x,y,\xi,\eta)\bigg|_{\eta=b} = \frac{4}{ab}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{q_m\cos(m\pi)}{p_n^2+q_m^2}\sin(p_n x)\sin(q_m y)\sin(p_n\xi)$$

2 Covariance matrix bewteen the outputs

We assume that $f(x,y)$ is a Gaussian process with mean value equal to zero and covariance matrix equal to $K(x,x',y,y') = K(x,x')K(y,y')$. Since the PDE equation is linear, the $v(x,y)$ is a Gaussian process with covariance function given by

$$\mathrm{cov}\left[v(x,y), v(x',y')\right] = \mathrm{E}\left[\int_0^a\int_0^b f(\xi,\eta)G(x,y,\xi,\eta)\mathrm{d}\xi\mathrm{d}\eta, \int_0^a\int_0^b f(\xi',\eta')G(x',y',\xi',\eta')\mathrm{d}\xi'\mathrm{d}\eta'\right]. \tag{5}$$

Then, the covariance $\mathrm{cov}\left[v(x,y), v(x',y')\right]$ is given as

$$\int_0^a\int_0^a\int_0^b\int_0^b G(x,y,\xi,\eta)G(x',y',\xi',\eta')\,\mathrm{E}\left[f(\xi,\eta)f(\xi',\eta')\right]\mathrm{d}\xi'\mathrm{d}\xi\mathrm{d}\eta'\mathrm{d}\eta$$

With $\mathrm{E}\left[f(\xi,\eta)f(\xi',\eta')\right] = K(\xi,\xi',\eta,\eta') = K(\xi,\xi')K(\eta,\eta')$, the last expression can be written as

$$\int_0^a\int_0^a\int_0^b\int_0^b G(x,y,\xi,\eta)G(x',y,\xi',\eta')K(\xi,\xi')K(\eta,\eta')\mathrm{d}\xi'\mathrm{d}\xi\mathrm{d}\eta'\mathrm{d}\eta$$

Using the expression (3) for $G(x,y,\xi,\eta)$ and SE kernels for the covariances of the latent processes, we have

$$\begin{aligned}\int_0^a\int_0^a\int_0^b\int_0^b &\left\{\left[\frac{4}{ab}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\sin(p_n x)\sin(q_m y)\sin(p_n\xi)\sin(q_m\eta)}{p_n^2+q_m^2}\right]\times\right.\\ &\left.\left[\frac{4}{ab}\sum_{n'=1}^{\infty}\sum_{m'=1}^{\infty}\frac{\sin(p_{n'}x')\sin(q_{m'}y')\sin(p_{n'}\xi')\sin(q_{m'}\eta')}{p_{n'}^2+q_{m'}^2}\right]\right\}\times\\ &\exp\left[-\frac{(\xi-\xi')^2}{\sigma_x^2}\right]\exp\left[-\frac{(\eta-\eta')^2}{\sigma_y^2}\right]\mathrm{d}\xi'\mathrm{d}\xi\mathrm{d}\eta'\mathrm{d}\eta\end{aligned}$$

The above expression can be separated in two different sets of integrals

$$\begin{aligned}&\frac{16}{(ab)^2}\sum_{\forall n}\sum_{\forall n'}\sum_{\forall m}\sum_{\forall m'}\frac{1}{(p_n^2+q_m^2)(p_{n'}^2+q_{m'}^2)}\sin(p_n x)\sin(p_{n'}x')\sin(q_m y)\sin(q_{m'}y')\times\\ &\left\{\int_0^a\int_0^a\sin(p_n\xi)\sin(p_{n'}\xi')\exp\left[-\frac{(\xi-\xi')^2}{\sigma_x^2}\right]\mathrm{d}\xi'\mathrm{d}\xi\right\}\left\{\int_0^b\int_0^b\sin(q_m\eta)\sin(q_{m'}\eta')\exp\left[-\frac{(\eta-\eta')^2}{\sigma_y^2}\right]\mathrm{d}\eta'\mathrm{d}\eta\right\}.\end{aligned}$$

In this sense, we have

$$\mathrm{cov}\left[v(x,y), v(x',y')\right] = \frac{16}{(ab)^2}\sum_{\forall n}\sum_{\forall n'}\sum_{\forall m}\sum_{\forall m'}\frac{C_{v,v}^x(n,n')\sin(p_n x)\sin(p_{n'}x')\,C_{v,v}^y(m,m')\sin(q_m y)\sin(q_{m'}y')}{(p_n^2+q_m^2)(p_{n'}^2+q_{m'}^2)}$$

for which

$$C_{v,v}^x(n,n') = \int_0^a\int_0^a\sin(p_n\xi)\sin(p_{n'}\xi')\exp\left[-\frac{(\xi-\xi')^2}{\sigma_x^2}\right]\mathrm{d}\xi'\mathrm{d}\xi \tag{6}$$

$$C_{v,v}^y(m,m') = \int_0^b\int_0^b\sin(q_m\eta)\sin(q_{m'}\eta')\exp\left[-\frac{(\eta-\eta')^2}{\sigma_y^2}\right]\mathrm{d}\eta'\mathrm{d}\eta. \tag{7}$$

The expressions above follow an expression that has already been obtained for the solution of the Heat equation in one space variable. See [2] for details. In that case, the one space Heat equation involves the solution of the following double integral

$$C_{v,v}^x(n,m) = \int_0^l\int_0^l\sin(w_n\xi)\sin(w_m\xi')\exp\left[-\frac{(\xi-\xi')^2}{\sigma_x^2}\right]\mathrm{d}\xi'\mathrm{d}\xi,$$

where l is the spatial length, $w_n = \frac{n\pi}{l}$, $w_m = \frac{m\pi}{l}$, and σ_x is the lenght-scale for the SE kernel assumed for the latent function. It can be shown [2] that the solution for $C^x_{v,v}(n,m)$ follows as

$$C^x_{v,v}(n,m) = \begin{cases} \left(\frac{\sigma_x l}{\sqrt{\pi}(m^2-n^2)}\right)\left\{ne^{\left(\frac{\gamma_m\sigma_x}{2}\right)^2}\mathcal{I}\left[\mathcal{H}(\gamma_m,l)\right] - \ me^{\left(\frac{\gamma_n\sigma_x}{2}\right)^2}\mathcal{I}\left[\mathcal{H}(\gamma_n,l)\right]\right\} & \text{if } n \text{ and } m \text{ are both even or both odd} \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

for $n \neq m$, where $\gamma_n = jw_n$, $\gamma_m = jw_m$, and

$$\mathcal{H}(\zeta, \upsilon) = \operatorname{erf}\left(\frac{\upsilon}{\sigma} + \frac{\sigma\zeta}{2}\right) - \operatorname{erf}\left(\frac{\sigma\zeta}{2}\right).$$

The term $e^{\left(\frac{\gamma_m\sigma_x}{2}\right)^2}\mathcal{I}\left[\mathcal{H}(\gamma_m,l)\right]$ follows as

$$\begin{aligned} e^{\left(\frac{\gamma_m\sigma_x}{2}\right)^2}\mathcal{I}\left[\mathcal{H}(\gamma_m,l)\right] &= \mathcal{I}\left[e^{\left(\frac{\gamma_m\sigma_x}{2}\right)^2}\mathcal{H}(\gamma_m,l)\right] = \mathcal{I}\left\{e^{\left(\frac{\gamma_m\sigma_x}{2}\right)^2}\left[\operatorname{erf}\left(\frac{l}{\sigma_x}+\frac{\sigma_x\gamma_m}{2}\right) - \operatorname{erf}\left(\frac{\sigma_x\gamma_m}{2}\right)\right]\right\} \\ &= \mathcal{I}\left\{e^{\left(\frac{\gamma_m\sigma_x}{2}\right)^2}\left[\operatorname{erfc}\left(\frac{\sigma_x\gamma_m}{2}\right) - \operatorname{erfc}\left(\frac{l}{\sigma_x}+\frac{\sigma_x\gamma_m}{2}\right)\right]\right\} \\ &= \mathcal{I}\left\{\mathcal{W}(\sigma_x, m)\right\}, \end{aligned}$$

with

$$\mathcal{W}(\sigma_x,m) = e^{\left(\frac{\gamma_m\sigma_x}{2}\right)^2}\left[\operatorname{erfc}\left(\frac{\sigma_x\gamma_m}{2}\right) - \operatorname{erfc}\left(\frac{l}{\sigma_x}+\frac{\sigma_x\gamma_m}{2}\right)\right] = e^{\left(\frac{\gamma_m\sigma_x}{2}\right)^2}\left[\operatorname{erfc}\left(z_1^{\gamma_m}\right) - \operatorname{erfc}\left(z_2^{\gamma_m}\right)\right],$$

where

$$z_1^{\gamma_m} = \frac{\sigma_x\gamma_m}{2}, \quad z_2^{\gamma_m} = \frac{l}{\sigma_x} + \frac{\sigma_x\gamma_m}{2}.$$

This means that

$$C^x_{v,v}(n,m) = \begin{cases} \left(\frac{\sigma_x l}{\sqrt{\pi}(m^2-n^2)}\right)\left\{n\mathcal{I}\left[\mathcal{W}(\sigma_x,m)\right] - \ m\mathcal{I}\left[\mathcal{W}(\sigma_x,n)\right]\right\} & \text{if } n \text{ and } m \text{ are both even or both odd} \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

The derivative of $C^x_{v,v}(n,m)$ with respect to σ_x for $n \neq m$ follows as

$$\begin{aligned} \frac{\partial C^x_{v,v}(n,m)}{\partial\sigma_x} &= \left(\frac{l}{\sqrt{\pi}(m^2-n^2)}\right)\left\{n\mathcal{I}\left[\mathcal{W}(\sigma_x,m)\right] - m\mathcal{I}\left[\mathcal{W}(\sigma_x,n)\right]\right\} \\ &+ \left(\frac{l\sigma_x}{\sqrt{\pi}(m^2-n^2)}\right)\left\{n\mathcal{I}\left[\frac{\partial\mathcal{W}(\sigma_x,m)}{\partial\sigma_x}\right] - m\mathcal{I}\left[\frac{\partial\mathcal{W}(\sigma_x,n)}{\partial\sigma_x}\right]\right\}. \end{aligned} \tag{10}$$

where the derivative $\frac{\partial\mathcal{W}(\sigma_x,m)}{\partial\sigma_x}$ is

$$\frac{\gamma_m^2\sigma_x}{2}e^{\left(\frac{\gamma_m\sigma_x}{2}\right)^2}\left[\operatorname{erfc}\left(z_1^{\gamma_m}\right) - \operatorname{erfc}\left(z_2^{\gamma_m}\right)\right] - \frac{2}{\sqrt{\pi}}e^{\left(\frac{\gamma_m\sigma_x}{2}\right)^2}\left[\exp\left[-\left(z_1^{\gamma_m}\right)^2\right]\frac{\partial z_1^{\gamma_m}}{\partial\sigma_x} - \exp\left[-\left(z_2^{\gamma_m}\right)^2\right]\frac{\partial z_2^{\gamma_m}}{\partial\sigma_x}\right]$$

with

$$\frac{\partial}{\partial\sigma_x}\left(z_1^{\gamma_m}\right) = \frac{\gamma_m}{2}, \qquad \frac{\partial}{\partial\sigma_x}\left(z_2^{\gamma_m}\right) = -\frac{l}{\sigma_x^2} + \frac{\gamma_m}{2}.$$

If $n = m$, the following expression must be used instead

$$\begin{aligned} C^x_{v,v}(n) &= \frac{\sigma_x\sqrt{\pi}\,l}{2}e^{\left(\frac{\gamma_n\sigma_x}{2}\right)^2}\left\{\mathcal{R}\left[\mathcal{H}(\gamma_n,l)\right] - \mathcal{I}\left[\mathcal{H}(\gamma_n,l)\right]\left[\frac{\sigma_x^2 n\pi}{2l^2} + \frac{1}{n\pi}\right]\right\} + \frac{\sigma_x^2}{2}\left[e^{-\left(\frac{l}{\sigma_x}\right)^2}\cos(n\pi) - 1\right], \\ &= \frac{\sigma_x\sqrt{\pi}\,l}{2}\left\{e^{\left(\frac{\gamma_n\sigma_x}{2}\right)^2}\mathcal{R}\left[\mathcal{H}(\gamma_n,l)\right] - e^{\left(\frac{\gamma_n\sigma_x}{2}\right)^2}\mathcal{I}\left[\mathcal{H}(\gamma_n,l)\right]\left[\frac{\sigma_x^2 n\pi}{2l^2} + \frac{1}{n\pi}\right]\right\} + \frac{\sigma_x^2}{2}\left[e^{-\left(\frac{l}{\sigma_x}\right)^2}\cos(n\pi) - 1\right], \\ &= \frac{\sigma_x\sqrt{\pi}\,l}{2}\left\{\mathcal{R}\left[\mathcal{W}(\sigma_x,n)\right] - \mathcal{I}\left[\mathcal{W}(\sigma_x,n)\right]\left[\frac{\sigma_x^2 n\pi}{2l^2} + \frac{1}{n\pi}\right]\right\} + \frac{\sigma_x^2}{2}\left[e^{-\left(\frac{l}{\sigma_x}\right)^2}\cos(n\pi) - 1\right]. \end{aligned}$$

The derivative follows as

$$\begin{aligned} \frac{\partial C^x_{v,v}(n)}{\partial\sigma_x} &= \frac{\sqrt{\pi}\,l}{2}\left\{\mathcal{R}\left[\mathcal{W}(\sigma_x,n)\right] - \mathcal{I}\left[\mathcal{W}(\sigma_x,n)\right]\left[\frac{\sigma_x^2 n\pi}{2l^2} + \frac{1}{n\pi}\right]\right\} \\ &+ \frac{\sigma_x\sqrt{\pi}\,l}{2}\left\{\mathcal{R}\left[\frac{\partial\mathcal{W}(\sigma_x,n)}{\partial\sigma_x}\right] - \mathcal{I}\left[\frac{\partial\mathcal{W}(\sigma_x,n)}{\partial\sigma_x}\right]\left[\frac{\sigma_x^2 n\pi}{2l^2} + \frac{1}{n\pi}\right] - \mathcal{I}\left[\mathcal{W}(\sigma_x,n)\right]\left(\frac{\sigma_x n\pi}{l^2}\right)\right\} \\ &+ \sigma_x\left[e^{-\left(\frac{l}{\sigma_x}\right)^2}\cos(n\pi) - 1\right] + \frac{l^2}{\sigma_x}e^{-\left(\frac{l}{\sigma_x}\right)^2}\cos(n\pi). \end{aligned}$$

Gaussian processes over the boundary conditions

We assume that the boundary functions $f_1(y)$, $f_2(y)$, $f_3(x)$, and $f_4(x)$ are independent Gaussian processes with SE covariance function. This means that the covariance for $\text{cov}\left[v_q(x,y), v_s(x',y')\right]$ has additional terms that follow as

$$\begin{aligned}\text{cov}\left[v_q^{\text{BC}}(x,y), v_s^{\text{BC}}(x',y')\right] = &\int_0^b\int_0^b H_1(x,y,\eta)H_1(x',y',\eta')K_{f_1}(\eta,\eta')\text{d}\eta'\text{d}\eta\\ &+\int_0^b\int_0^b H_2(x,y,\eta)H_2(x',y',\eta')K_{f_2}(\eta,\eta')\text{d}\eta'\text{d}\eta\\ &+\int_0^a\int_0^a H_3(x,y,\xi)H_3(x',y',\xi')K_{f_3}(\xi,\xi')\text{d}\xi'\text{d}\xi\\ &+\int_0^a\int_0^a H_4(x,y,\xi)H_4(x',y',\xi')K_{f_4}(\xi,\xi')\text{d}\xi'\text{d}\xi.\end{aligned}$$

For the first covariance, we have

$$\int_0^b\int_0^b H_1(x,y,\eta)H_1(x',y',\eta')K_{f_1}(\eta,\eta')\text{d}\eta'\text{d}\eta = \frac{16}{(ab)^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n'=1}^{\infty}\sum_{m'=1}^{\infty}\frac{p_n p_{n'}}{(p_n^2+q_m^2)(p_{n'}^2+q_{m'}^2)}\times$$
$$\sin(p_n x)\sin(q_m y)\sin(p_{n'}x')\sin(q_{m'}y')\int_0^b\int_0^b \sin(q_m\eta)\sin(q_{m'}\eta')\exp\left[-\frac{(\eta-\eta')^2}{\sigma_{f_1}^2}\right]\text{d}\eta'\text{d}\eta.$$

Leading to

$$\frac{16}{(ab)^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n'=1}^{\infty}\sum_{m'=1}^{\infty}\frac{p_n p_{n'}\sin(p_n x)\sin(p_{n'}x')\,C^{f_1}_{v_q^{\text{BC}},v_s^{\text{BC}}}(m,m')\sin(q_m y)\sin(q_{m'}y')}{(p_n^2+q_m^2)(p_{n'}^2+q_{m'}^2)}.$$

Following a similar procedure for the second covariance, we obtain

$$\frac{16}{(ab)^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n'=1}^{\infty}\sum_{m'=1}^{\infty}\frac{p_n\cos(n\pi)p_{n'}\cos(n'\pi)\sin(p_n x)\sin(p_{n'}x')\,C^{f_2}_{v_q^{\text{BC}},v_s^{\text{BC}}}(m,m')\sin(q_m y)\sin(q_{m'}y')}{(p_n^2+q_m^2)(p_{n'}^2+q_{m'}^2)}.$$

For the third covariance, it follows

$$\frac{16}{(ab)^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n'=1}^{\infty}\sum_{m'=1}^{\infty}\frac{C^{f_3}_{v_q^{\text{BC}},v_s^{\text{BC}}}(n,n')\sin(p_n x)\sin(p_{n'}x')\left[q_m q_{m'}\right]\sin(q_m y)\sin(q_{m'}y')}{(p_n^2+q_m^2)(p_{n'}^2+q_{m'}^2)}.$$

Finally, for the fourth covariance, we get

$$\frac{16}{(ab)^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n'=1}^{\infty}\sum_{m'=1}^{\infty}\frac{C^{f_4}_{v_q^{\text{BC}},v_s^{\text{BC}}}(n,n')\sin(p_n x)\sin(p_{n'}x')\left[q_m\cos(m\pi)q_{m'}\cos(m'\pi)\right]\sin(q_m y)\sin(q_{m'}y')}{(p_n^2+q_m^2)(p_{n'}^2+q_{m'}^2)}.$$

3 Covariance matrix bewteen the outputs and the latent functions

The covariance matrix between the outputs $v(x,y)$ and the latent functions $f(x,y)$ is given by

$$\text{cov}\left[v(x,y), f(x',y')\right] = \text{E}\left[\int_0^a\int_0^b f(\xi,\tau)G(x,y,\xi,\eta)\text{d}\xi\text{d}\eta f(x',y')\right] \tag{11}$$

Then, the covariance $\text{cov}\left[v(x,y), f(x',y')\right]$ is given as

$$\int_0^a\int_0^b G(x,y,\xi,\eta)\,\text{E}\left[f(\xi,\eta)f(x',y')\right]\text{d}\xi\text{d}\eta$$

Using the factorized form for the covariance of the latent functions, the last expression can be written as

$$\int_0^a\int_0^b G(x,y,\xi,\eta)K(\xi,x')K(\eta,y')\text{d}\xi\text{d}\eta$$

With the expression (3) for $G(x,y,\xi,\eta)$ and SE kernels for the covariances of the latent processes, we have

$$\int_0^a\int_0^b\left\{\left[\frac{4}{ab}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\sin(p_n x)\sin(q_m y)\sin(p_n\xi)\sin(q_m\eta)}{p_n^2+q_m^2}\right]\exp\left[-\frac{(\xi-x')^2}{\sigma_x^2}\right]\exp\left[-\frac{(\eta-y')^2}{\sigma_y^2}\right]\right\}\text{d}\xi\text{d}\eta$$

Again the above expression can be separated in two different sets of integrals

$$\frac{4}{ab}\sum_{\forall n}\sum_{\forall m}\frac{\sin\left(p_n x\right)\sin\left(q_m y\right)}{p_n^2+q_m^2}\left\{\int_0^a \sin\left(p_n\xi\right)\exp\left[-\frac{\left(\xi-x'\right)^2}{\sigma_x^2}\right]\mathrm{d}\xi\right\}\left\{\int_0^b \sin\left(q_m\eta\right)\exp\left[-\frac{\left(\eta-y'\right)^2}{\sigma_y^2}\right]\mathrm{d}\eta\right\}$$

We have

$$\operatorname{cov}\left[v(x,y),f(x',y')\right]=\frac{4}{ab}\sum_{\forall n}\sum_{\forall m}\frac{\sin\left(p_n x\right)C_{v,f}^{x}(x',n)\sin\left(q_m y\right)C_{v,f}^{y}(y',m)}{p_n^2+q_m^2}$$

in which

$$C_{v,f}^{x}(x',n)=\int_0^a \sin\left(p_n\xi\right)\exp\left[-\frac{\left(\xi-x'\right)^2}{\sigma_x^2}\right]\mathrm{d}\xi$$

$$C_{v,f}^{y}(y',m)=\int_0^b \sin\left(q_m\eta\right)\exp\left[-\frac{\left(\eta-y'\right)^2}{\sigma_y^2}\right]\mathrm{d}\eta.$$

A similar expression to the ones above was obtained when solving the covariance between the output of the Heat equation and the input function. See [2]. That expression is given as

$$C_{v,f}^{x}(x',n)=\int_0^l \sin\left(w_n\xi\right)\exp\left[-\frac{\left(\xi-x'\right)^2}{\sigma_x^2}\right]\mathrm{d}\xi.$$

Its solution follows as

$$C_{v,f}^{x}(x',n)=\frac{\sigma_x\sqrt{\pi}}{2}\exp\left(\frac{\gamma_n\sigma_x}{2}\right)^2\mathcal{I}\left[\exp\left(\gamma_n x'\right)\mathcal{H}(\gamma_n,x',l)\right],$$

where

$$\mathcal{H}(\zeta,\upsilon,\varphi)=\operatorname{erf}\left(\frac{\upsilon}{\sigma}+\frac{\sigma\zeta}{2}\right)-\operatorname{erf}\left(\frac{\upsilon-\varphi}{\sigma}+\frac{\sigma\zeta}{2}\right).$$

In the expression for $C_{v,f}^{x}(x',n)$, the term $\exp\left(\frac{\gamma_n\sigma_x}{2}\right)^2\mathcal{I}\left[\exp\left(\gamma_n x'\right)\mathcal{H}(\gamma_n,x',l)\right]$ can be written as

$$\begin{aligned}
e^{\left(\frac{\gamma_n\sigma_x}{2}\right)^2}\mathcal{I}\left[e^{\gamma_n x'}\mathcal{H}(\gamma_n,x',l)\right]&=\mathcal{I}\left[e^{\left(\frac{\gamma_n\sigma_x}{2}\right)^2}e^{\gamma_n x'}\mathcal{H}(\gamma_n,x',l)\right]\\
&=\mathcal{I}\left\{e^{\left(\frac{\gamma_n\sigma_x}{2}\right)^2}e^{\gamma_n x'}\left[\operatorname{erf}\left(\frac{x'}{\sigma_x}+\frac{\sigma_x\gamma_n}{2}\right)-\operatorname{erf}\left(\frac{x'-l}{\sigma_x}+\frac{\sigma_x\gamma_n}{2}\right)\right]\right\}\\
&=\mathcal{I}\left\{e^{\left(\frac{\gamma_n\sigma_x}{2}\right)^2}e^{\gamma_n x'}\left[\operatorname{erfc}\left(\frac{x'-l}{\sigma_x}+\frac{\sigma_x\gamma_n}{2}\right)-\operatorname{erfc}\left(\frac{x'}{\sigma_x}+\frac{\sigma_x\gamma_n}{2}\right)\right]\right\}\\
&=\mathcal{I}\left\{e^{\left(\frac{\gamma_n\sigma_x}{2}\right)^2}e^{\gamma_n x'}\left[\operatorname{erfc}\left(z_2^{\gamma_n,x'}\right)-\operatorname{erfc}\left(z_1^{\gamma_n,x'}\right)\right]\right\}
\end{aligned}\tag{12}$$

with $z_1^{\gamma_n,x'}=\frac{x'}{\sigma_x}+\frac{\sigma_x\gamma_n}{2}$ and $z_2^{\gamma_n,x'}=\frac{x'-l}{\sigma_x}+\frac{\sigma_x\gamma_n}{2}$.

We then get

$$C_{v,f}^{x}(x',n)=\frac{\sigma_x\sqrt{\pi}}{2}\mathcal{I}\left\{e^{\left(\frac{\gamma_n\sigma_x}{2}\right)^2}e^{\gamma_n x'}\left[\operatorname{erfc}\left(z_2^{\gamma_n,x'}\right)-\operatorname{erfc}\left(z_1^{\gamma_n,x'}\right)\right]\right\},$$

To find the derivative for $C_{v,f}^{x}(x',n)$ in terms of σ_x, we first have

$$\frac{\partial z_1^{\gamma_n,x'}}{\partial\sigma_x}=-\frac{x'}{\sigma_x^2}+\frac{\gamma_n}{2},\qquad\qquad \frac{\partial z_2^{\gamma_n,x'}}{\partial\sigma_x}=-\frac{x'-l}{\sigma_x^2}+\frac{\gamma_n}{2}.$$

The derivative follows

$$\frac{\partial C_{v,f}^{x}(x',n)}{\partial\sigma_x}=\frac{\sqrt{\pi}}{2}\mathcal{I}\left[\widetilde{\mathcal{W}}(\sigma_x)\right]+\frac{\sigma_x\sqrt{\pi}}{2}\mathcal{I}\left[\frac{\partial\widetilde{\mathcal{W}}(\sigma_x)}{\partial\sigma_x}\right],\tag{13}$$

where

$$\widetilde{\mathcal{W}}(\sigma_x)=e^{\left(\frac{\gamma_n\sigma_x}{2}\right)^2}e^{\gamma_n x'}\left[\operatorname{erfc}\left(z_2^{\gamma_n,x'}\right)-\operatorname{erfc}\left(z_1^{\gamma_n,x'}\right)\right].$$

The derivative $\frac{\partial\widetilde{\mathcal{W}}(\sigma_x)}{\partial\sigma_x}$ is

$$\frac{\gamma_n^2\sigma_x}{2}e^{\left(\frac{\gamma_n\sigma_x}{2}\right)^2}e^{\gamma_n x'}\left[\operatorname{erfc}\left(z_2^{\gamma_n,x'}\right)-\operatorname{erfc}\left(z_1^{\gamma_n,x'}\right)\right]-\frac{2}{\sqrt{\pi}}e^{\left(\frac{\gamma_n\sigma_x}{2}\right)^2}e^{\gamma_n x'}\left[\exp\left[-\left(z_2^{\gamma_n,x'}\right)^2\right]\frac{\partial z_2^{\gamma_n,x'}}{\partial\sigma_x}-\exp\left[-\left(z_1^{\gamma_n,x'}\right)^2\right]\frac{\partial z_1^{\gamma_n,x'}}{\partial\sigma_x}\right]$$

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