Covariance function from a Poisson PDE

The two-dimensional Poisson equation in a rectangular Cartesian coordinate sytem has the form

$$\frac{\partial^2 v(x,y)}{\partial x^2} + \frac{\partial^2 v(x,y)}{\partial y^2} = -\sum_{\forall i} s_i f_i(x,y) \tag{1}$$

1 Solving the Poisson equation

The exact solution for Eq. (1) is subject to particular boundary conditions. For a first boundary value problem with domain $0 \leq x \leq a$, $0 \leq y \leq b$, and conditions given by

$$\begin{aligned} v(x=0,y) &= f_1(y) \\ v(x=a,y) &= f_2(y) \\ v(x,y=0) &= f_3(x) \\ v(x,y=b) &= f_4(x) \end{aligned}$$

the solution to this equation is given by [1]

$$\begin{aligned} v(x,y) = &\sum_{\forall i} s_i \int_0^a \int_0^b f_i(\xi,\eta) G(x,y,\xi,\eta) \mathrm{d}\eta \mathrm{d}\xi \\ &+ \int_0^b f_1(\eta) H_1(x,y,\eta) \mathrm{d}\eta - \int_0^b f_2(\eta) H_2(x,y,\eta) \mathrm{d}\eta \\ &+ \int_0^a f_3(\xi) H_3(x,y,\xi) \mathrm{d}\xi - \int_0^a f_4(\eta) H_4(x,y,\xi) \mathrm{d}\xi, \end{aligned} \tag{2}$$

where the Green function $G(x,y,\xi,\eta)$ and the functions $H_1(x,y,\xi,\eta)$, $H_2(x,y,\xi,\eta)$, $H_3(x,y,\xi,\eta)$ and $H_4(x,y,\xi,\eta)$ are given by

$$\begin{aligned} G(x,y,\xi,\eta) &= \frac{4}{ab} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\sin(p_n x) \sin(q_m y) \sin(p_n \xi) \sin(q_m \eta)}{p_n^2 + q_m^2}, \\ H_1(x,y,\eta) &= \frac{\partial}{\partial \xi} G(x,y,\xi,\eta) \bigg|_{\xi=0}, \quad H_2(x,y,\eta) = \frac{\partial}{\partial \xi} G(x,y,\xi,\eta) \bigg|_{\xi=a}, \\ H_3(x,y,\xi) &= \frac{\partial}{\partial \eta} G(x,y,\xi,\eta) \bigg|_{\eta=0}, \quad H_4(x,y,\xi) = \frac{\partial}{\partial \eta} G(x,y,\xi,\eta) \bigg|_{\eta=b}, \end{aligned} \tag{3}$$

where $p_n = \frac{n\pi}{a}$, and $q_m = \frac{m\pi}{b}$.
Assuming $f_1(y) = f_2(y) = f_3(x) = f_4(x) = 0$, equation (2) is equal to

$$v(x,y) = \sum_{\forall i} s_i \int_0^a \int_0^b f_i(\xi,\eta) G(x,y,\xi,\eta) \mathrm{d}\eta \mathrm{d}\xi \tag{4}$$

where $G(x,y,\xi,\eta)$ is given by (3).

Boundary conditions

We first compute the functions $H_1(x,y,\xi,\eta)$, $H_2(x,y,\xi,\eta)$, $H_3(x,y,\xi,\eta)$, and $H_4(x,y,\xi,\eta)$,

$$H_1(x,y,\eta) = \frac{\partial}{\partial \xi} G(x,y,\xi,\eta) \bigg|_{\xi=0} = \frac{4}{ab} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{p_n}{p_n^2 + q_m^2} \sin(p_n x) \sin(q_m y) \sin(q_m \eta)$$

$$H_2(x,y,\eta) = \frac{\partial}{\partial \xi} G(x,y,\xi,\eta) \bigg|_{\xi=a} = \frac{4}{ab} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{p_n \cos(n\pi)}{p_n^2 + q_m^2} \sin(p_n x) \sin(q_m y) \sin(q_m \eta)$$

$$H_3(x,y,\xi) = \frac{\partial}{\partial \eta} G(x,y,\xi,\eta)\bigg|_{\eta=0} = \frac{4}{ab} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{q_m}{p_n^2 + q_m^2} \sin(p_n x) \sin(q_m y) \sin(p_n \xi)$$

$$H_4(x,y,\xi) = \frac{\partial}{\partial \eta} G(x,y,\xi,\eta)\bigg|_{\eta=b} = \frac{4}{ab} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{q_m \cos(m\pi)}{p_n^2 + q_m^2} \sin(p_n x) \sin(q_m y) \sin(p_n \xi)$$

2 Covariance matrix bewteen the outputs

We assume that $f_i(x,y)$ are independent Gaussian processes with mean value equal to zero and covariance matrix equal to $K_i(x,x',y,y') = K_i(x,x')K_i(y,y')$. Since the PDE equation is linear, the $v_q(x,y)$ are Gaussian processes with covariance matrix given by

$$\operatorname{cov}[v_q(x,y), v_s(x',y')] = \mathrm{E}\left[\sum_{\forall i} s_{qi} \int_0^a \int_0^b f_i(\xi,\eta) G(x,y,\xi,\eta) \mathrm{d}\xi \mathrm{d}\eta \right. \tag{5}$$

$$\left. \sum_{\forall k} s_{sk} \int_0^a \int_0^b f_k(\xi',\eta') G(x',y',\xi',\eta') \mathrm{d}\xi' \mathrm{d}\eta' \right] \tag{6}$$

Then, the covariance $\operatorname{cov}[v_q(x,y), v_s(x',y')]$ is given as

$$\sum_{\forall i} s_{qi} \sum_{\forall k} s_{sk} \int_0^a \int_0^a \int_0^b \int_0^b G(x,y,\xi,\eta) G(x',y',\xi',\eta')\, \mathrm{E}[f_i(\xi,\eta) f_k(\xi',\eta')]\, \mathrm{d}\xi' \mathrm{d}\xi \mathrm{d}\eta' \mathrm{d}\eta$$

With $\mathrm{E}[f_i(\xi,\eta) f_k(\xi',\eta')] = K_{ik}(\xi,\xi',\eta,\eta')\delta_{ik} = K_{ik}(\xi,\xi')K_{ik}(\eta,\eta')\delta_{ik}$, the last expression can be written as

$$\sum_{\forall i} s_{qi} s_{si} \int_0^a \int_0^a \int_0^b \int_0^b G(x,y,\xi,\eta) G(x',y,\xi',\eta') K_i(\xi,\xi') K_i(\eta,\eta') \mathrm{d}\xi' \mathrm{d}\xi \mathrm{d}\eta' \mathrm{d}\eta$$

Using the expression (3) for $G(x,y,\xi,\eta)$ and SE kernels for the covariances of the latent processes, we have

$$\sum_{\forall i} s_{qi} s_{si} \int_0^a \int_0^a \int_0^b \int_0^b \left\{ \left[\frac{4}{ab} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\sin(p_n x) \sin(q_m y) \sin(p_n \xi) \sin(q_m \eta)}{p_n^2 + q_m^2} \right] \times \right.$$
$$\left. \left[\frac{4}{ab} \sum_{n'=1}^{\infty} \sum_{m'=1}^{\infty} \frac{\sin(p_{n'} x') \sin(q_{m'} y') \sin(p_{n'} \xi') \sin(q_{m'} \eta')}{p_{n'}^2 + q_{m'}^2} \right] \right\} \times$$
$$\exp\left[-\frac{(\xi - \xi')^2}{\sigma_x^2} \right] \exp\left[-\frac{(\eta - \eta')^2}{\sigma_y^2} \right] \mathrm{d}\xi' \mathrm{d}\xi \mathrm{d}\eta' \mathrm{d}\eta$$

The above expression can be separated in two different sets of integrals

$$\sum_{\forall i} s_{qi} s_{si} \left\{ \frac{16}{(ab)^2} \sum_{\forall n} \sum_{\forall n'} \sum_{\forall m} \sum_{\forall m'} \frac{1}{(p_n^2 + q_m^2)(p_{n'}^2 + q_{m'}^2)} \sin(p_n x) \sin(p_{n'} x') \sin(q_m y) \sin(q_{m'} y') \times \right.$$
$$\left. \left\{ \int_0^a \int_0^a \sin(p_n \xi) \sin(p_{n'} \xi') \exp\left[-\frac{(\xi - \xi')^2}{\sigma_x^2} \right] \mathrm{d}\xi' \mathrm{d}\xi \right\} \left\{ \int_0^b \int_0^b \sin(q_m \eta) \sin(q_{m'} \eta') \exp\left[-\frac{(\eta - \eta')^2}{\sigma_y^2} \right] \mathrm{d}\eta' \mathrm{d}\eta \right\} \right\}.$$

In this sense, we have

$$\operatorname{cov}[v_q(x,y), v_s(x',y')] = \sum_{\forall i} s_{qi} s_{si} \left\{ \frac{16}{(ab)^2} \sum_{\forall n} \sum_{\forall n'} \sum_{\forall m} \sum_{\forall m'} \frac{C^x_{v_q,v_s}(n,n') \sin(p_n x) \sin(p_{n'} x')\, C^y_{v_q,v_s}(m,m') \sin(q_m y) \sin(q_{m'} y')}{(p_n^2 + q_m^2)(p_{n'}^2 + q_{m'}^2)} \right\}$$

for which

$$C^x_{v_q,v_s}(n,n') = \int_0^a \int_0^a \sin(p_n \xi) \sin(p_{n'} \xi') \exp\left[-\frac{(\xi - \xi')^2}{\sigma_x^2} \right] \mathrm{d}\xi' \mathrm{d}\xi \tag{7}$$

$$C^y_{v_q,v_s}(m,m') = \int_0^b \int_0^b \sin(q_m \eta) \sin(q_{m'} \eta') \exp\left[-\frac{(\eta - \eta')^2}{\sigma_y^2} \right] \mathrm{d}\eta' \mathrm{d}\eta. \tag{8}$$

The expressions above follow an expression that has already been obtained for the solution of the Heat equation in one space variable. See [2] for details. In that case, the one space Heat equation involves the solution of the following double integral

$$C_{y_q,y_s}(n,m) = \int_0^l \int_0^l \sin(w_n\xi)\sin(w_m\xi')\exp\left[-\frac{(\xi-\xi')^2}{\sigma_x^2}\right]\mathrm{d}\xi'\mathrm{d}\xi,$$

where l is the spatial length, $w_n = \frac{n\pi}{l}$, $w_m = \frac{m\pi}{l}$, and σ_x is the lenght-scale for the SE kernel assumed for the latent function. It can be shown [2] that the solution for $C_{y_q,y_s}(n,m)$ follows as

$$C_{y_q,y_s}(n,m) = \begin{cases} \left(\frac{\sigma_x l}{\sqrt{\pi}(m^2-n^2)}\right)\left\{ne^{\left(\frac{\gamma_m\sigma_x}{2}\right)^2}\mathcal{I}\left[\mathcal{H}(\gamma_m,l)\right] - \ me^{\left(\frac{\gamma_n\sigma_x}{2}\right)^2}\mathcal{I}\left[\mathcal{H}(\gamma_n,l)\right]\right\} & \text{if } n \text{ and } m \text{ are both even or both odd} \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

for $n \neq m$, where $\gamma_n = jw_n$, $\gamma_m = jw_m$, and

$$\mathcal{H}(\zeta,\upsilon) = \mathrm{erf}\left(\frac{\upsilon}{\sigma}+\frac{\sigma\zeta}{2}\right) - \mathrm{erf}\left(\frac{\sigma\zeta}{2}\right).$$

If $n = m$, the following expression must be used instead

$$C_{y_q,y_s}(n) = \frac{\sigma_x\sqrt{\pi}\,l}{2}e^{\left(\frac{\gamma_n\sigma_x}{2}\right)^2}\left\{\mathcal{R}\left[\mathcal{H}(\gamma_n,l)\right] - \mathcal{I}\left[\mathcal{H}(\gamma_n,l)\right]\left[\frac{\sigma_x^2 n\pi}{2l^2}+\frac{1}{n\pi}\right]\right\} + \frac{\sigma_x^2}{2}\left[e^{-\left(\frac{l}{\sigma_x}\right)^2}\cos(n\pi)-1\right]$$

In the expression for $C_{y_q,y_s}(n,n')$, the term $e^{\left(\frac{\gamma_n\sigma_x}{2}\right)^2}\mathcal{I}\left[\mathcal{H}(\gamma_n,l)\right]$ can be replaced for a more numerical stable expression given in terms of the complex error function $w(z)$,

$$e^{\left(\frac{\gamma_n\sigma_x}{2}\right)^2}\mathcal{I}\left[\mathcal{H}(\gamma_n,l)\right] = \mathcal{I}\left[w(jz_1^{\gamma_n}) - e^{-\left(\frac{l}{\sigma_x}\right)^2}e^{-\gamma_n l}w(jz_2^{\gamma_n})\right],$$

with $z_1^{\gamma_n} = \frac{\sigma_x\gamma_n}{2}$ and $z_2^{\gamma_n} = \frac{l}{\sigma_x} + \frac{\sigma_x\gamma_n}{2}$. A similar expression for $e^{\left(\frac{\gamma_n\sigma_x}{2}\right)^2}\mathcal{R}\left[\mathcal{H}(\gamma_n,l)\right]$ can be obtained when $n = m$. Derivative for $C_{y_q,y_s}(n,m)$ in terms of σ_x can be found in [2].

Gaussian processes over the boundary conditions

We assume that the boundary functions $f_1(y)$, $f_2(y)$, $f_3(x)$, and $f_4(x)$ are independent Gaussian processes with SE covariance function. This means that the covariance for $\mathrm{cov}\left[v_q(x,y), v_s(x',y')\right]$ has additional terms that follow as

$$\begin{aligned}\mathrm{cov}\left[v_q^{\mathrm{BC}}(x,y), v_s^{\mathrm{BC}}(x',y')\right] &= \int_0^b\int_0^b H_1(x,y,\eta)H_1(x',y',\eta')K_{f_1}(\eta,\eta')\mathrm{d}\eta'\mathrm{d}\eta \\ &+ \int_0^b\int_0^b H_2(x,y,\eta)H_2(x',y',\eta')K_{f_2}(\eta,\eta')\mathrm{d}\eta'\mathrm{d}\eta \\ &+ \int_0^a\int_0^a H_3(x,y,\xi)H_3(x',y',\xi')K_{f_3}(\xi,\xi')\mathrm{d}\xi'\mathrm{d}\xi \\ &+ \int_0^a\int_0^a H_4(x,y,\xi)H_4(x',y',\xi')K_{f_4}(\xi,\xi')\mathrm{d}\xi'\mathrm{d}\xi.\end{aligned}$$

For the first covariance, we have

$$\int_0^b\int_0^b H_1(x,y,\eta)H_1(x',y',\eta')K_{f_1}(\eta,\eta')\mathrm{d}\eta'\mathrm{d}\eta = \frac{16}{(ab)^2}\sum_{n=1}^\infty\sum_{m=1}^\infty\sum_{n'=1}^\infty\sum_{m'=1}^\infty\frac{p_n p_{n'}}{(p_n^2+q_m^2)(p_{n'}^2+q_{m'}^2)}\times$$
$$\sin(p_n x)\sin(q_m y)\sin(p_{n'}x')\sin(q_{m'}y')\int_0^b\int_0^b\sin(q_m\eta)\sin(q_{m'}\eta')\exp\left[-\frac{(\eta-\eta')^2}{\sigma_{f_1}^2}\right]\mathrm{d}\eta'\mathrm{d}\eta.$$

Leading to

$$\frac{16}{(ab)^2}\sum_{n=1}^\infty\sum_{m=1}^\infty\sum_{n'=1}^\infty\sum_{m'=1}^\infty\frac{p_n p_{n'}\sin(p_n x)\sin(p_{n'}x')\,C^{f_1}_{v_q^{\mathrm{BC}},v_s^{\mathrm{BC}}}(m,m')\sin(q_m y)\sin(q_{m'}y')}{(p_n^2+q_m^2)(p_{n'}^2+q_{m'}^2)}.$$

Following a similar procedure for the second covariance, we obtain

$$\frac{16}{(ab)^2}\sum_{n=1}^\infty\sum_{m=1}^\infty\sum_{n'=1}^\infty\sum_{m'=1}^\infty\frac{p_n\cos(n\pi)p_{n'}\cos(n'\pi)\sin(p_n x)\sin(p_{n'}x')\,C^{f_2}_{v_q^{\mathrm{BC}},v_s^{\mathrm{BC}}}(m,m')\sin(q_m y)\sin(q_{m'}y')}{(p_n^2+q_m^2)(p_{n'}^2+q_{m'}^2)}.$$

For the third covariance, it follows

$$\frac{16}{(ab)^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n'=1}^{\infty}\sum_{m'=1}^{\infty}\frac{C^{f_3}_{v_q^{\mathrm{BC}},v_s^{\mathrm{BC}}}(n,n')\sin(p_n x)\sin(p_{n'}x')\left[q_m q_{m'}\right]\sin(q_m y)\sin(q_{m'}y')}{(p_n^2+q_m^2)(p_{n'}^2+q_{m'}^2)}.$$

Finally, for the fourth covariance, we get

$$\frac{16}{(ab)^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n'=1}^{\infty}\sum_{m'=1}^{\infty}\frac{C^{f_4}_{v_q^{\mathrm{BC}},v_s^{\mathrm{BC}}}(n,n')\sin(p_n x)\sin(p_{n'}x')\left[q_m\cos(m\pi)q_{m'}\cos(m'\pi)\right]\sin(q_m y)\sin(q_{m'}y')}{(p_n^2+q_m^2)(p_{n'}^2+q_{m'}^2)}.$$

3 Covariance matrix bewteen the outputs and the latent functions

The covariance matrix between the outputs $v_q(x,y)$ and the latent functions $f_i(x,y)$ is given by

$$\mathrm{cov}\left[v_q(x,y),f_i(x',y')\right]=\mathrm{E}\left[\sum_{\forall k}s_{qk}\int_0^a\int_0^b f_k(\xi,\tau)G(x,y,\xi,\eta)\mathrm{d}\xi\mathrm{d}\eta f_i(x',y')\right] \tag{10}$$

Then, the covariance $\mathrm{cov}\left[v_q(x,y),f_i(x',y')\right]$ is given as

$$\sum_{\forall k}s_{qk}\int_0^a\int_0^b G(x,y,\xi,\eta)\,\mathrm{E}\left[f_k(\xi,\eta)f_i(x',y')\right]\mathrm{d}\xi\mathrm{d}\eta$$

Using the factorized form for the covariance of the latent functions, the last expression can be written as

$$s_{qi}\int_0^a\int_0^b G(x,y,\xi,\eta)K_i(\xi,x')K_i(\eta,y')\mathrm{d}\xi\mathrm{d}\eta$$

With the expression (3) for $G(x,y,\xi,\eta)$ and SE kernels for the covariances of the latent processes, we have

$$s_{qi}\int_0^a\int_0^b\left\{\left[\frac{4}{ab}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\sin(p_n x)\sin(q_m y)\sin(p_n\xi)\sin(q_m\eta)}{p_n^2+q_m^2}\right]\exp\left[-\frac{(\xi-x')^2}{\sigma_x^2}\right]\exp\left[-\frac{(\eta-y')^2}{\sigma_y^2}\right]\right\}\mathrm{d}\xi\mathrm{d}\eta$$

Again the above expression can be separated in two different sets of integrals

$$\frac{4s_{qi}}{ab}\sum_{\forall n}\sum_{\forall m}\frac{\sin(p_n x)\sin(q_m y)}{p_n^2+q_m^2}\left\{\int_0^a\sin(p_n\xi)\exp\left[-\frac{(\xi-x')^2}{\sigma_x^2}\right]\mathrm{d}\xi\right\}\left\{\int_0^b\sin(q_m\eta)\exp\left[-\frac{(\eta-y')^2}{\sigma_y^2}\right]\mathrm{d}\eta\right\}$$

We have

$$\mathrm{cov}\left[v_q(x,y),f_i(x',y')\right]=\frac{4s_{qi}}{ab}\sum_{\forall n}\sum_{\forall m}\frac{\sin(p_n x)\,C^x_{v_q,f_i}(x',n)\sin(q_m y)\,C^y_{v_q,f_i}(y',m)}{p_n^2+q_m^2}$$

in which

$$C^x_{v_q,f_i}(x',n)=\int_0^a\sin(p_n\xi)\exp\left[-\frac{(\xi-x')^2}{\sigma_x^2}\right]\mathrm{d}\xi$$
$$C^y_{v_q,f_i}(y',n)=\int_0^b\sin(q_m\eta)\exp\left[-\frac{(\eta-y')^2}{\sigma_y^2}\right]\mathrm{d}\eta.$$

A similar expression to the ones above was obtained when solving the covariance between the output of the Heat equation and the input function. See [2]. That expression is given as

$$C_{y_q,f_i}(x',n)=\int_0^l\sin(w_n\xi)\exp\left[-\frac{(\xi-x')^2}{\sigma_x^2}\right]\mathrm{d}\xi.$$

Its solution follows as

$$C_{y_q,f_i}(x',n)=\frac{\sigma_x\sqrt{\pi}}{2}\exp\left(\frac{\gamma_n\sigma_x}{2}\right)^2\mathcal{I}\left[\exp(\gamma_n x')\,\mathcal{H}(\gamma_n,x',l)\right],$$

where

$$\mathcal{H}(\zeta,\upsilon,\varphi)=\mathrm{erf}\left(\frac{\upsilon}{\sigma}+\frac{\sigma\zeta}{2}\right)-\mathrm{erf}\left(\frac{\upsilon-\varphi}{\sigma}+\frac{\sigma\zeta}{2}\right).$$

In the expression for $C_{y_q,f_i}(x',n)$, the term $\exp\left(\frac{\gamma_n\sigma_x}{2}\right)^2\mathcal{I}\left[\exp\left(\gamma_n x'\right)\mathcal{H}(\gamma_n,x',l)\right]$ can be replaced for a more numerical stable expression given in terms of the complex error function $w(z)$,

$$e^{\left(\frac{\gamma_n\sigma_x}{2}\right)^2}\mathcal{I}\left[e^{\gamma_n x'}\mathcal{H}(\gamma_n,x',l)\right]=\mathcal{I}\left[e^{-\left(\frac{x'-l}{\sigma_x}\right)^2}e^{\gamma_n l}w(jz_2^{\gamma_n,x'})-e^{-\left(\frac{x'}{\sigma_x}\right)^2}w(jz_1^{\gamma_n,x'})\right],$$

with $z_1^{\gamma_n,x'}=\frac{x'}{\sigma_x}+\frac{\sigma_x\gamma_n}{2}$ and $z_2^{\gamma_n,x'}=\frac{x'-l}{\sigma_x}+\frac{\sigma_x\gamma_n}{2}$.
The derivative for $C_{y_q,f_i}(x',n)$ in terms of σ_x can also be found in [2].

References

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