Covariance function from a Poisson PDE

The two-dimensional Poisson equation in a rectangular Cartesian coordinate sytem has the form

$$\frac{\partial^2 v(x,y)}{\partial x^2} + \frac{\partial^2 v(x,y)}{\partial y^2} = -\sum_{\forall i} s_i f_i(x,y) \tag{1}$$

1 Solving the Poisson equation

The exact solution for Eq. (1) is subject to particular boundary conditions. For a first boundary value problem with domain $0 \leq x \leq a$, $0 \leq y \leq b$, and conditions given by

$$\begin{aligned} v(x=0,y) &= f_1(y) \\ v(x=a,y) &= f_2(y) \\ v(x,y=0) &= f_3(x) \\ v(x,y=b) &= f_4(x) \end{aligned}$$

the solution to this equation is given by [1]

$$\begin{aligned} v(x,y) = \sum_{\forall i} s_i & \int_0^a \int_0^b f_i(\xi,\eta) G(x,y,\xi,\eta) \mathrm{d}\eta \mathrm{d}\xi \\ & + \int_0^b f_1(\eta) H_1(x,y,\xi,\eta)\mathrm{d}\eta - \int_0^b f_2(\eta) H_2(x,y,\xi,\eta)\mathrm{d}\eta \\ & + \int_0^a f_3(\xi) H_3(x,y,\xi,\eta)\mathrm{d}\xi - \int_0^a f_4(\eta) H_4(x,y,\xi,\eta)\mathrm{d}\xi, \end{aligned} \tag{2}$$

where the Green function $G(x,y,\xi,\eta)$ and the functions $H_1(x,y,\xi,\eta)$, $H_2(x,y,\xi,\eta)$, $H_3(x,y,\xi,\eta)$ and $H_4(x,y,\xi,\eta)$ are given by

$$G(x,y,\xi,\eta) = \frac{4}{ab}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{\sin(p_n x)\sin(q_m y)\sin(p_n\xi)\sin(q_m\eta)}{p_n^2+q_m^2}, \tag{3}$$

$$H_1(x,y,\xi,\eta) = \frac{\partial}{\partial\xi}G(x,y,\xi,\eta)\bigg|_{\xi=0}, \quad H_2(x,y,\xi,\eta) = \frac{\partial}{\partial\xi}G(x,y,\xi,\eta)\bigg|_{\xi=a},$$

$$H_3(x,y,\xi,\eta) = \frac{\partial}{\partial\eta}G(x,y,\xi,\eta)\bigg|_{\eta=0}, \quad H_4(x,y,\xi,\eta) = \frac{\partial}{\partial\eta}G(x,y,\xi,\eta)\bigg|_{\eta=b},$$

where $p_n = \frac{n\pi}{a}$, and $q_m = \frac{m\pi}{b}$.
Assuming $f_1(y) = f_2(y) = f_3(x) = f_4(x) = 0$, equation (2) is equal to

$$v(x,y) = \sum_{\forall i} s_i \int_0^a \int_0^b f_i(\xi,\eta) G(x,y,\xi,\eta)\mathrm{d}\eta\mathrm{d}\xi \tag{4}$$

where $G(x,y,\xi,\eta)$ is given by (3).

2 Covariance matrix bewteen the outputs

We assume that $f_i(x,y)$ are independent Gaussian processes with mean value equal to zero and covariance matrix equal to $K_i(x,x',y,y') = K_i(x,x')K_i(y,y')$. Since the PDE equation is linear, the $v_q(x,y)$ are Gaussian processes with covariance matrix given by

$$\operatorname{cov}\left[v_q(x,y), v_s(x',y')\right] = \mathrm{E}\left[\sum_{\forall i} s_{qi}\int_0^a\int_0^b f_i(\xi,\eta)G(x,y,\xi,\eta)\mathrm{d}\xi\mathrm{d}\eta\right. \tag{5}$$

$$\left.\sum_{\forall k} s_{sk}\int_0^a\int_0^b f_k(\xi',\eta')G(x',y',\xi',\eta')\mathrm{d}\xi'\mathrm{d}\eta'\right] \tag{6}$$

Then, the covariance $\operatorname{cov}\left[v_q(x,y), v_s(x',y')\right]$ is given as

$$\sum_{\forall i} s_{qi} \sum_{\forall k} s_{sk} \int_0^a \int_0^a \int_0^b \int_0^b G(x,y,\xi,\eta) G(x',y',\xi',\eta') \operatorname{E}\left[f_i(\xi,\eta) f_k(\xi',\eta')\right] \mathrm{d}\xi' \mathrm{d}\xi \mathrm{d}\eta' \mathrm{d}\eta$$

With $\operatorname{E}\left[f_i(\xi,\eta) f_k(\xi',\eta')\right] = K_{ik}(\xi,\xi',\eta,\eta')\delta_{ik} = K_{ik}(\xi,\xi')K_{ik}(\eta,\eta')\delta_{ik}$, the last expression can be written as

$$\sum_{\forall i} s_{qi} s_{si} \int_0^a \int_0^a \int_0^b \int_0^b G(x,y,\xi,\eta) G(x',y,\xi',\eta') K_i(\xi,\xi') K_i(\eta,\eta') \mathrm{d}\xi' \mathrm{d}\xi \mathrm{d}\eta' \mathrm{d}\eta$$

Using the expression (3) for $G(x,y,\xi,\eta)$ and SE kernels for the covariances of the latent processes, we have

$$\begin{aligned}
\sum_{\forall i} s_{qi} s_{si} \int_0^a \int_0^a \int_0^b \int_0^b \Bigg\{ \left[\frac{4}{ab} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\sin(p_n x) \sin(q_m y) \sin(p_n \xi) \sin(q_m \eta)}{p_n^2 + q_m^2} \right] \times \\
\left[\frac{4}{ab} \sum_{n'=1}^{\infty} \sum_{m'=1}^{\infty} \frac{\sin(p_{n'} x') \sin(q_{m'} y') \sin(p_{n'} \xi') \sin(q_{m'} \eta')}{p_{n'}^2 + q_{m'}^2} \right] \Bigg\} \times \\
\exp\left[-\frac{(\xi - \xi')^2}{\sigma_x^2} \right] \exp\left[-\frac{(\eta - \eta')^2}{\sigma_y^2} \right] \mathrm{d}\xi' \mathrm{d}\xi \mathrm{d}\eta' \mathrm{d}\eta
\end{aligned}$$

The above expression can be separated in two different sets of integrals

$$\begin{aligned}
\sum_{\forall i} s_{qi} s_{si} \Bigg\{ \frac{16}{(ab)^2} \sum_{\forall n} \sum_{\forall n'} \sum_{\forall m} \sum_{\forall m'} \frac{1}{(p_n^2 + q_m^2)(p_{n'}^2 + q_{m'}^2)} \sin(p_n x) \sin(p_{n'} x') \sin(q_m y) \sin(q_{m'} y') \times \\
\left\{ \int_0^a \int_0^a \sin(p_n \xi) \sin(p_{n'} \xi') \exp\left[-\frac{(\xi - \xi')^2}{\sigma_x^2} \right] \mathrm{d}\xi' \mathrm{d}\xi \right\} \left\{ \int_0^b \int_0^b \sin(q_m \eta) \sin(q_{m'} \eta') \exp\left[-\frac{(\eta - \eta')^2}{\sigma_y^2} \right] \mathrm{d}\eta' \mathrm{d}\eta \right\} \Bigg\}.
\end{aligned}$$

In this sense, we have

$$\operatorname{cov}\left[v_q(x,y), v_s(x',y')\right] = \sum_{\forall i} s_{qi} s_{si} \left\{ \frac{16}{(ab)^2} \sum_{\forall n} \sum_{\forall n'} \sum_{\forall m} \sum_{\forall m'} \frac{C^x_{v_q,v_s}(n,n') \sin(p_n x) \sin(p_{n'} x')\, C^y_{v_q,v_s}(m,m') \sin(q_m y) \sin(q_{m'} y')}{(p_n^2 + q_m^2)(p_{n'}^2 + q_{m'}^2)} \right\}$$

for which

$$C^x_{v_q,v_s}(n,n') = \int_0^a \int_0^a \sin(p_n \xi) \sin(p_{n'} \xi') \exp\left[-\frac{(\xi - \xi')^2}{\sigma_x^2} \right] \mathrm{d}\xi' \mathrm{d}\xi \tag{7}$$

$$C^y_{v_q,v_s}(m,m') = \int_0^b \int_0^b \sin(q_m \eta) \sin(q_{m'} \eta') \exp\left[-\frac{(\eta - \eta')^2}{\sigma_y^2} \right] \mathrm{d}\eta' \mathrm{d}\eta. \tag{8}$$

The expressions above follow an expression that has already been obtained for the solution of the Heat equation in one space variable. See [2] for details. In that case, the one space Heat equation involves the solution of the following double integral

$$C_{y_q,y_s}(n,m) = \int_0^l \int_0^l \sin(w_n \xi) \sin(w_m \xi') \exp\left[-\frac{(\xi - \xi')^2}{\sigma_x^2} \right] \mathrm{d}\xi' \mathrm{d}\xi,$$

where l is the spatial length, $w_n = \frac{n\pi}{l}$, $w_m = \frac{m\pi}{l}$, and σ_x is the lenght-scale for the SE kernel assumed for the latent function. It can be shown [2] that the solution for $C_{y_q,y_s}(n,m)$ follows as

$$C_{y_q,y_s}(n,m) = \begin{cases} \left(\dfrac{\sigma_x l}{\sqrt{\pi}(m^2 - n^2)} \right) \left\{ n e^{\left(\frac{\gamma_m \sigma_x}{2}\right)^2} \mathcal{I}\left[\mathcal{H}(\gamma_m, l)\right] - m e^{\left(\frac{\gamma_n \sigma_x}{2}\right)^2} \mathcal{I}\left[\mathcal{H}(\gamma_n, l)\right] \right\} & \text{if } n \text{ and } m \text{ are both even or both odd} \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

for $n \neq m$, where $\gamma_n = j w_n$, $\gamma_m = j w_m$, and

$$\mathcal{H}(\zeta, \upsilon) = \operatorname{erf}\left(\frac{\upsilon}{\sigma} + \frac{\sigma\zeta}{2} \right) - \operatorname{erf}\left(\frac{\sigma\zeta}{2} \right).$$

If $n = m$, the following expression must be used instead

$$C_{y_q,y_s}(n) = \frac{\sigma_x \sqrt{\pi}\, l}{2} e^{\left(\frac{\gamma_n \sigma_x}{2}\right)^2} \left\{ \mathcal{R}\left[\mathcal{H}(\gamma_n, l)\right] - \mathcal{I}\left[\mathcal{H}(\gamma_n, l)\right] \left[\frac{\sigma_x^2 n\pi}{2l^2} + \frac{1}{n\pi} \right] \right\} + \frac{\sigma_x^2}{2} \left[e^{-\left(\frac{l}{\sigma_x}\right)^2} \cos(n\pi) - 1 \right]$$

In the expression for $C_{y_q,y_s}(n,n')$, the term $e^{\left(\frac{\gamma_n\sigma_x}{2}\right)^2}\mathcal{I}\left[\mathcal{H}(\gamma_n,l)\right]$ can be replaced for a more numerical stable expression given in terms of the complex error function $w(z)$,

$$e^{\left(\frac{\gamma_n\sigma_x}{2}\right)^2}\mathcal{I}\left[\mathcal{H}(\gamma_n,l)\right]=\mathcal{I}\left[w(jz_1^{\gamma_n})-e^{-\left(\frac{l}{\sigma_x}\right)^2}e^{-\gamma_n l}w(jz_2^{\gamma_n})\right],$$

with $z_1^{\gamma_n}=\frac{\sigma_x\gamma_n}{2}$ and $z_2^{\gamma_n}=\frac{l}{\sigma_x}+\frac{\sigma_x\gamma_n}{2}$. A similar expression for $e^{\left(\frac{\gamma_n\sigma_x}{2}\right)^2}\mathcal{R}\left[\mathcal{H}(\gamma_n,l)\right]$ can be obtained when $n=m$.
Derivative for $C_{y_q,y_s}(n,m)$ in terms of σ_x can be found in [2].

3 Covariance matrix bewteen the outputs and the latent functions

The covariance matrix between the outputs $v_q(x,y)$ and the latent functions $f_i(x,y)$ is given by

$$\operatorname{cov}\left[v_q(x,y),f_i(x',y')\right]=\mathrm{E}\left[\sum_{\forall k}s_{qk}\int_0^a\int_0^b f_k(\xi,\tau)G(x,y,\xi,\eta)\mathrm{d}\xi\mathrm{d}\eta f_i(x',y')\right] \tag{10}$$

Then, the covariance $\operatorname{cov}\left[v_q(x,y),f_i(x',y')\right]$ is given as

$$\sum_{\forall k}s_{qk}\int_0^a\int_0^b G(x,y,\xi,\eta)\,\mathrm{E}\left[f_k(\xi,\eta)f_i(x',y')\right]\mathrm{d}\xi\mathrm{d}\eta$$

Using the factorized form for the covariance of the latent functions, the last expression can be written as

$$s_{qi}\int_0^a\int_0^b G(x,y,\xi,\eta)K_i(\xi,x')K_i(\eta,y')\mathrm{d}\xi\mathrm{d}\eta$$

With the expression (3) for $G(x,y,\xi,\eta)$ and SE kernels for the covariances of the latent processes, we have

$$s_{qi}\int_0^a\int_0^b\left\{\left[\frac{4}{ab}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\sin\left(p_nx\right)\sin\left(q_my\right)\sin\left(p_n\xi\right)\sin\left(q_m\eta\right)}{p_n^2+q_m^2}\right]\exp\left[-\frac{\left(\xi-x'\right)^2}{\sigma_x^2}\right]\exp\left[-\frac{\left(\eta-y'\right)^2}{\sigma_y^2}\right]\right\}\mathrm{d}\xi\mathrm{d}\eta$$

Again the above expression can be separated in two different sets of integrals

$$\frac{4s_{qi}}{ab}\sum_{\forall n}\sum_{\forall m}\frac{\sin\left(p_nx\right)\sin\left(q_my\right)}{p_n^2+q_m^2}\left\{\int_0^a\sin\left(p_n\xi\right)\exp\left[-\frac{\left(\xi-x'\right)^2}{\sigma_x^2}\right]\mathrm{d}\xi\right\}\left\{\int_0^b\sin\left(q_m\eta\right)\exp\left[-\frac{\left(\eta-y'\right)^2}{\sigma_y^2}\right]\mathrm{d}\eta\right\}$$

We have

$$\operatorname{cov}\left[v_q(x,y),f_i(x',y')\right]=\frac{4s_{qi}}{ab}\sum_{\forall n}\sum_{\forall m}\frac{\sin\left(p_nx\right)C^x_{v_q,f_i}(x',n)\sin\left(q_my\right)C^y_{v_q,f_i}(y',m)}{p_n^2+q_m^2}$$

in which

$$C^x_{v_q,f_i}(x',n)=\int_0^a\sin\left(p_n\xi\right)\exp\left[-\frac{\left(\xi-x'\right)^2}{\sigma_x^2}\right]\mathrm{d}\xi$$
$$C^y_{v_q,f_i}(y',n)=\int_0^b\sin\left(q_m\eta\right)\exp\left[-\frac{\left(\eta-y'\right)^2}{\sigma_y^2}\right]\mathrm{d}\eta.$$

A similar expression to the ones above was obtained when solving the covariance between the output of the Heat equation and the input function. See [2]. That expression is given as

$$C_{y_q,f_i}(x',n)=\int_0^l\sin\left(w_n\xi\right)\exp\left[-\frac{\left(\xi-x'\right)^2}{\sigma_x^2}\right]\mathrm{d}\xi.$$

Its solution follows as

$$C_{y_q,f_i}(x',n)=\frac{\sigma_x\sqrt{\pi}}{2}\exp\left(\frac{\gamma_n\sigma_x}{2}\right)^2\mathcal{I}\left[\exp\left(\gamma_nx'\right)\mathcal{H}(\gamma_n,x',l)\right],$$

where

$$\mathcal{H}(\zeta,\upsilon,\varphi)=\operatorname{erf}\left(\frac{\upsilon}{\sigma}+\frac{\sigma\zeta}{2}\right)-\operatorname{erf}\left(\frac{\upsilon-\varphi}{\sigma}+\frac{\sigma\zeta}{2}\right).$$

In the expression for $C_{y_q,f_i}(x',n)$, the term $\exp\left(\frac{\gamma_n\sigma_x}{2}\right)^2\mathcal{I}\left[\exp\left(\gamma_nx'\right)\mathcal{H}(\gamma_n,x',l)\right]$ can be replaced for a more numerical stable expression given in terms of the complex error function $w(z)$,

$$e^{\left(\frac{\gamma_n\sigma_x}{2}\right)^2}\mathcal{I}\left[e^{\gamma_nx'}\mathcal{H}(\gamma_n,x',l)\right]=\mathcal{I}\left[e^{-\left(\frac{x'-l}{\sigma_x}\right)^2}e^{\gamma_nl}w(jz_2^{\gamma_n,x'})-e^{-\left(\frac{x'}{\sigma_x}\right)^2}w(jz_1^{\gamma_n,x'})\right],$$

with $z_1^{\gamma_n,x'}=\frac{x'}{\sigma_x}+\frac{\sigma_x\gamma_n}{2}$ and $z_2^{\gamma_n,x'}=\frac{x'-l}{\sigma_x}+\frac{\sigma_x\gamma_n}{2}$.
The derivative for $C_{y_q,f_i}(x',n)$ in terms of σ_x can also be found in [2].

References

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