Angular Correlation Function

$$F = 1 + a\frac{\boldsymbol{p_e} \cdot \boldsymbol{p_\nu}}{E_e E_\nu} + b\frac{m_e}{E} + c\left(\frac{\boldsymbol{p_e} \cdot \boldsymbol{p_\nu}}{3E_e E_\nu} - \frac{(\boldsymbol{p_e} \cdot \boldsymbol{j})(\boldsymbol{p_\nu} \cdot \boldsymbol{j})}{E_e E_\nu}\right)$$
$$+ \frac{\boldsymbol{J}}{J} \cdot \left(A\frac{\boldsymbol{p_e}}{E_e} + B\frac{\boldsymbol{p_\nu}}{E_\nu} + D\frac{\boldsymbol{p_e} \times \boldsymbol{p_\nu}}{E_e E_\nu}\right)$$

Spherical Coordinates ($\boldsymbol{J}$ parallel to positive Z axis)

$$\boldsymbol{\beta_e} = (r = \beta_e; \theta = \theta_e; \phi = 0),\ \cos(\theta_e) \equiv z_e,\ \beta_e = \frac{|\boldsymbol{p_e}|}{E} = \sqrt{1 - \frac{m_e^2}{E^2}}$$

$$\boldsymbol{\beta_\nu} = (r = 1; \theta = \theta_\nu; \phi = \phi), \quad \cos(\theta_\nu) \equiv z_\nu$$

$$\boldsymbol{\beta_e} \cdot \boldsymbol{\beta_\nu} = \beta_e(\cos\theta_e \cos\theta_\nu + \sin\theta_e \sin\theta_\nu \cos\phi) =$$

$$\beta_e(z_e z_\nu + \sqrt{1 - z_e^2}\sqrt{1 - z_\nu^2}\cos\phi)$$

$$\boldsymbol{\beta_e} \cdot \boldsymbol{j} = \beta_e \cos\theta_e = \beta_e z_e$$

$$\boldsymbol{\beta_\nu} \cdot \boldsymbol{j} = \cos\theta_\nu = z_\nu$$

$$\boldsymbol{j} \cdot (\boldsymbol{\beta_e} \times \boldsymbol{\beta_\nu}) = \beta_e \sin\theta_e \sin\theta_\nu \sin\phi = \beta_e \sqrt{1 - z_e^2}\sqrt{1 - z_\nu^2}\sin\phi$$

Angular Correlation Factor

$$(\boldsymbol{\beta_e} \cdot \boldsymbol{j})(\boldsymbol{\beta_\nu} \cdot \boldsymbol{j}) = z_e z_\nu$$

Putting all together:

$$F = 1 + a\beta(z_e z_\nu + \sqrt{1 - z_e^2}\sqrt{1 - z_\nu^2}\cos\phi) + b\frac{m_e}{E} +$$

$$+ c\beta\left(-\frac{2}{3}z_e z_\nu + \frac{1}{3}\sqrt{1 - z_e^2}\sqrt{1 - z_\nu^2}\right) + A\beta z_e + B z_\nu +$$

$$+ D\beta\sqrt{1 - z_e^2}\sqrt{1 - z_\nu^2}\sin\phi =$$

$$= 1 + b\frac{m_e}{E} + \left(a - \frac{2}{3}c\right)\beta z_e z_\nu + A\beta z_e + B z_\nu +$$

$$+ \beta\sqrt{1 - z_e^2}\sqrt{1 - z_\nu^2}\left(\left(a + \frac{c}{3}\right)\cos\phi + D\sin\phi\right)$$

Single Variable: c

Angular Correlation Factor
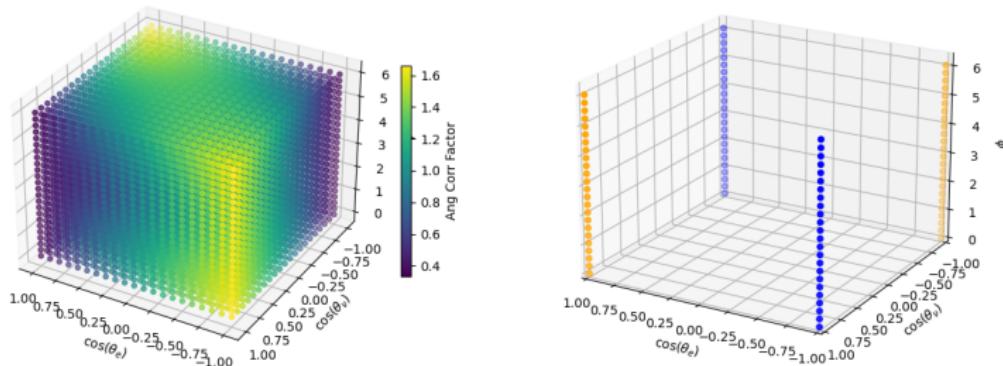
Figure: (Right) Values of the angular correlation Factor with c = 1, E = 5000 keV and rest of variables 0. (Left) Location of maximum (blue, value = 1.995) and minimum (orange, value = 0.005)

Single Variable: c

Angular Correlation Factor
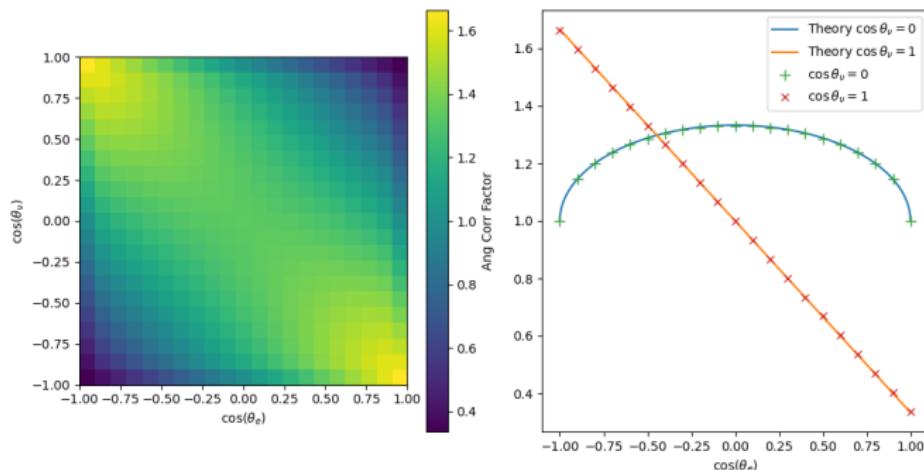
Figure: (Right) 2D projection of previous 3D image at $\phi = 0$ (Left) 1D projections at $\phi = 0$, and either $z_\nu = 0$ or $z_\nu = 1$

Single Variable c

Sampling
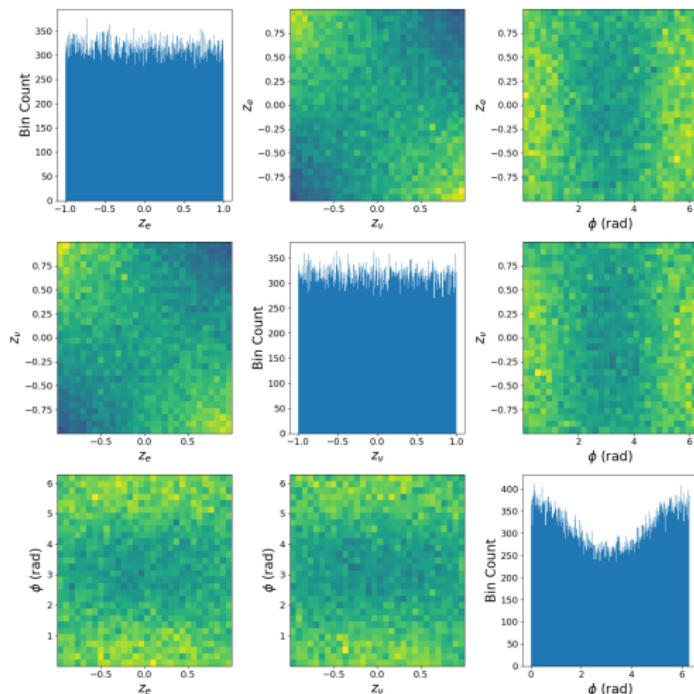
Figure: Pair plots for N = 100000 decays with c = 1, E = 1000 keV

Single Variable c

Marginal distributions

For z_e (and z_ν by symmetry of the expresions), we can observe reason why the marginal distribution becomes constant:

$$f(z_e) = N \int_{-1}^{1} dz_\nu \int_0^{2\pi} d\phi F =$$

$$= N \int_{-1}^{1} dz_\nu \int_0^{2\pi} d\phi \left(1 + c\beta \left(-\frac{2}{3} z_e z_\nu + \frac{1}{3}\sqrt{1 - z_e^2}\sqrt{1 - z_\nu^2}\cos\phi \right) \right)$$

$$= N \int_{-1}^{1} dz_\nu \int_0^{2\pi} d\phi = 4\pi N = N$$

Single Variable c

Marginal distributions

For ϕ, we can derive the expected shape:

$$f(\phi) = N \int_{-1}^{1} dz_\nu \int_{-1}^{1} dz_e F$$

$$= N \int_{-1}^{1} dz_\nu \int_{-1}^{1} dz_e \left(1 + c\beta \left(-\frac{2}{3} z_e z_\nu + \frac{1}{3} \sqrt{1 - z_e^2} \sqrt{1 - z_\nu^2} \cos \phi \right) \right)$$

$$= N \left(4 + c\beta \left(\frac{\pi}{2} \right)^2 \cos \phi / 3 \right) = N \left(1 + c\beta \frac{\pi^2}{48} \cos \phi \right)$$

Single Variable c

Marginal distributions
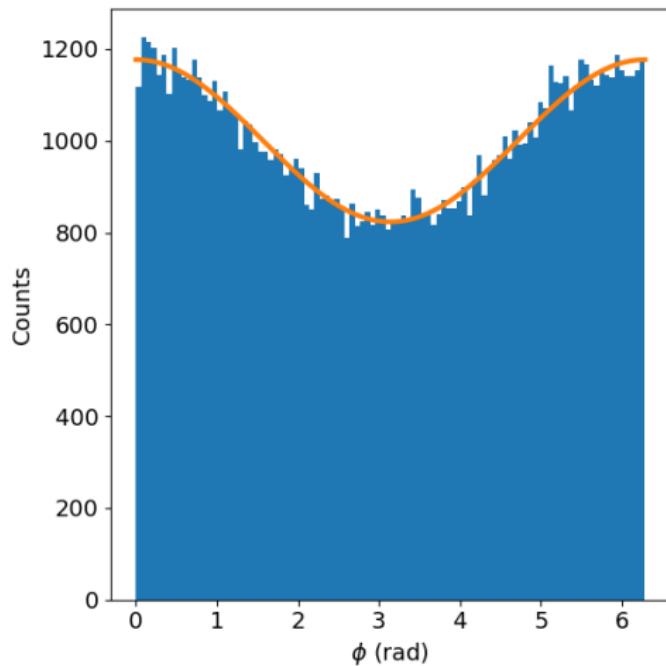
Figure: Histogram showing the values of ϕ with c = 1, E = 1000 keV for N = 100000 decays, and curve showing the theoretical distribution

Two variables: c and B

Since term proportional to c depends on E, we can consider different ratios by either:

▶ Fixing $c = B = 1$ and modifying the energy

▶ Same as before, but now $B = -1$

▶ Fixing $B = 1$ and $E \gg m_e \to \beta_e \approx 1$ and modifying $c > B$

We recall

$$F = 1 + c\beta_e \left(-\frac{2}{3} z_e z_\nu + \frac{1}{3} \sqrt{1 - z_e^2} \sqrt{1 - z_\nu^2} \cos\phi \right) + B z_\nu$$

Maxima and minima with $z_e = \pm 1, z_\nu \pm 1 \to \boldsymbol{p_e} \parallel \boldsymbol{p_\nu} \parallel \boldsymbol{J}$

Two variables: c and B
Angular Correlation Factor
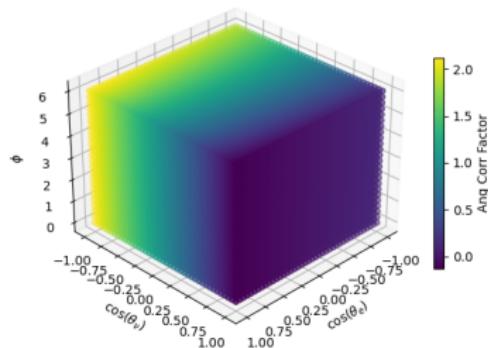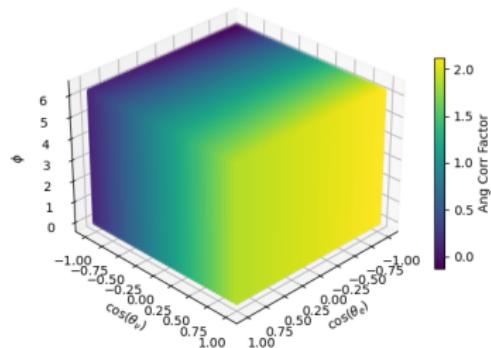
Figure: Values of the angular correlation Factor with (Right) $c = 1$, $B = 1$ and (Left) $c = 1$, $B = -1$; with $E = 520$ keV and rest of variables 0 for both.

Two variables: c and B

Angular Correlation Factor
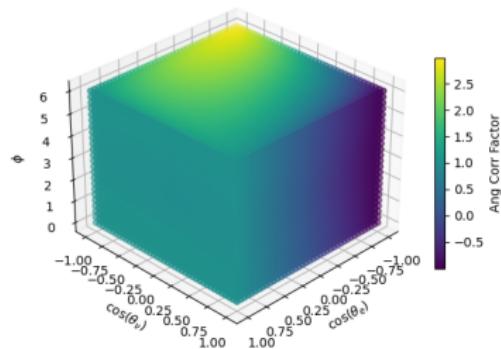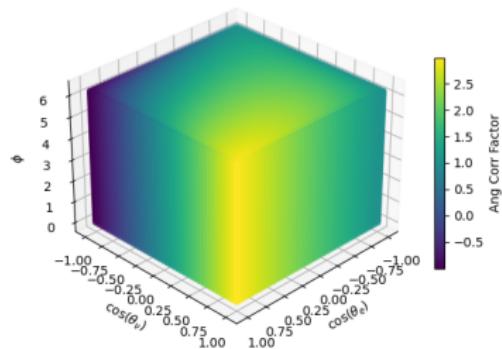
Figure: Values of the angular correlation Factor with (Right) c = 1, B = 1 and (Left) c = 1, B = -1; with E = 5000 keV and rest of variables 0 for both.

Two variables: c and B

Angular Correlation Factor
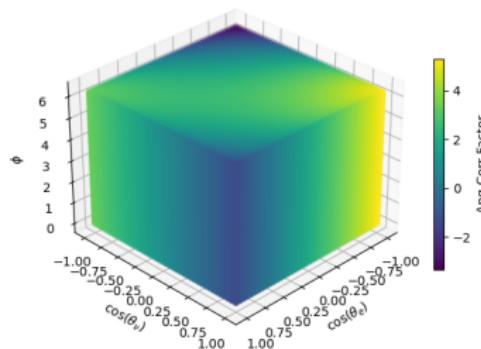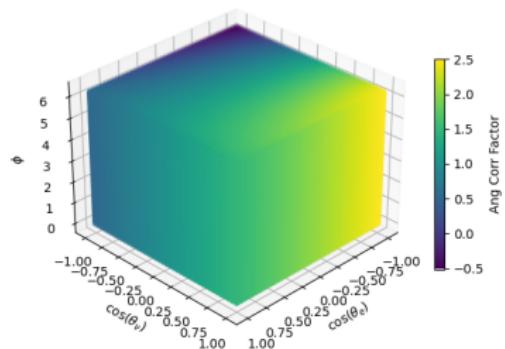
Figure: Values of the angular correlation Factor with (Right) a = c, B = 1, E = 800 keV and (Left) c = 5, B = 1, E = 5000 keV; with the rest of variables 0 for both.

Two variables: c and B

Maximum and Minimum
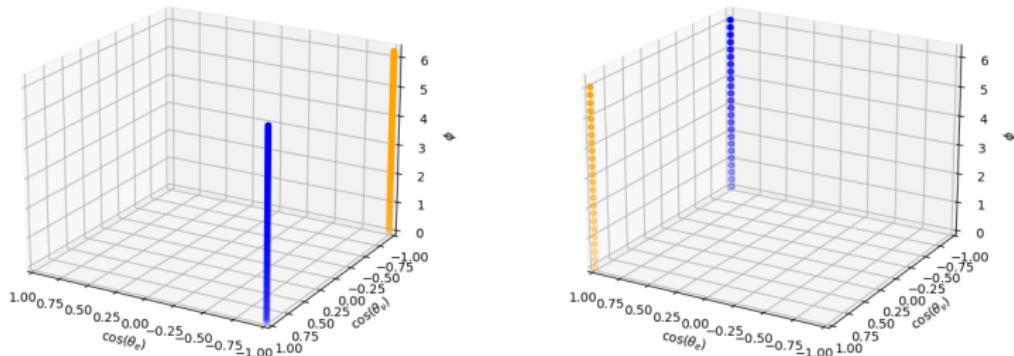
Figure: Location of maximum (blue, value = 2.66318) and minimum (orange, value = -0.66318) for (Right) c = B = 1, E = 5000 keV and (Left) c = 1, B = -1, E = 5000 keV

Two variable: c and B

Sampling
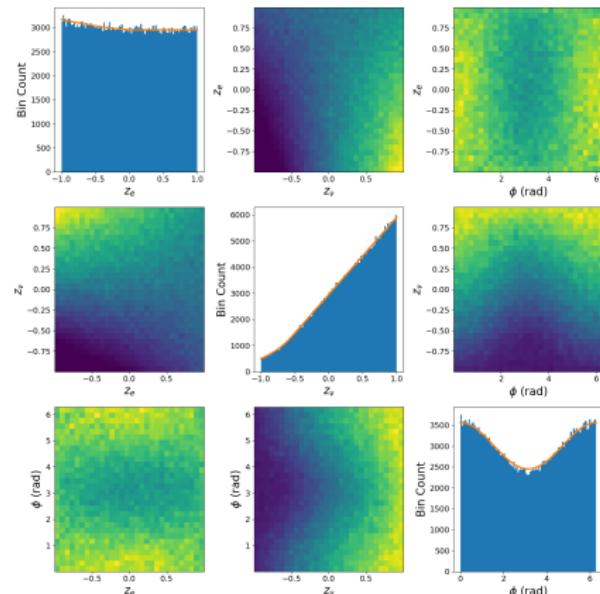
Figure: Pairplot with the marginal distributions for a simulation of N = 300000 decays with c = B = 1, E = 5000 keV. The 1 variable histograms show the theoretical distribution obtained from numerically integrating F with the constrain $F > 0$

Two variables: c and A

Since both terms proportional to a depends on E, we can consider
only consider different ratios by changing one (c), while leaving the
other (A) fixed. For convenience $E \gg m_e$.

$$F = 1 + \beta_e \left(c \left(-\frac{2}{3} z_e z_\nu + \frac{1}{3} \sqrt{1 - z_e^2} \sqrt{1 - z_\nu^2} \cos \phi \right) + A z_e \right)$$

Maximum and minimum with $z_e = \pm 1, z_\nu \pm 1 \rightarrow \boldsymbol{p_e} \parallel \boldsymbol{p_\nu} \parallel \boldsymbol{J}$

Two variables: c and A

Angular Correlation Factor

Figure: Values of the angular correlation Factor with (Right) A = 1, c = 0.25 and (Left) A = 1, c = -0.25, with E = 100000 keV and rest of variables 0 for both.

Two variables: c and A

Angular Correlation Factor

Figure: Values of the angular correlation Factor with (Right) A = 1, c = 1 and (Left) A = 1, c = -1, with E = 100000 keV and rest of variables 0 for both.

Two variables: c and A

Angular Correlation Factor

Figure: Values of the angular correlation Factor with (Right) A = 1, c = 4 and (Left) A = 1, c = -4, with E = 100000 keV and rest of variables 0 for both.

Two variables: c and A
Maximum and Minimum

Figure: Location of maximum (blue, value = 2.66664) and minimum
(orange, value = -0.66664) for (Right) A = c = 1, E = 100000 keV and
(Left) A = 1, c = -1, E = 100000 keV

Two variable: c and A
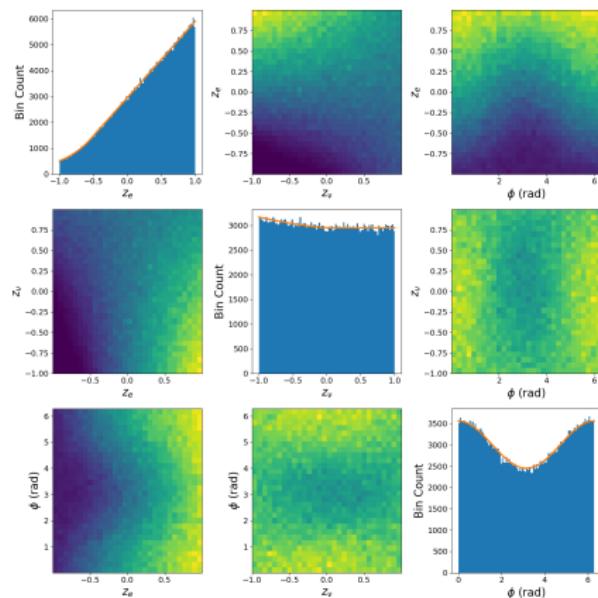
Sampling

Figure: Pairplot with the marginal distributions for a simulation of $N = 300000$ decays with $c = A = 1$, $E = 100000$ keV. The 1 variable histograms show the theoretical distribution obtained from numerically integrating F with the constrain $F > 0$

Two variables: c and a

Since both terms proportional to a depends on E, we can consider only consider different ratios by changing one (c), while leaving the other (A) fixed. For convenience $E \gg m_e$.

$$F = 1 + \left(a - \frac{2}{3}c\right)\beta z_e z_\nu + \beta\sqrt{1 - z_e^2}\sqrt{1 - z_\nu^2}\left(a + \frac{c}{3}\right)\cos\phi$$

Maximum and minimum depends on the relative signs of a and c

▶ c and a opposite sign: $z_e = \pm 1, z_\nu = \pm 1 \rightarrow \boldsymbol{p_e} \parallel \boldsymbol{p_\nu} \parallel \boldsymbol{J}$

▶ c and a same sign: $z_e = 0, z_\nu = 0 \rightarrow \boldsymbol{p_e} \parallel \boldsymbol{p_\nu} \perp \boldsymbol{J}$

Two variables: c and a
Angular Correlation Factor

Figure: Values of the angular correlation Factor with (Right) a = 1, c = 0.25 and (Left) a = 1, c = -0.25, with E = 100000 keV and rest of variables 0 for both.

Two variables: c and A

Angular Correlation Factor
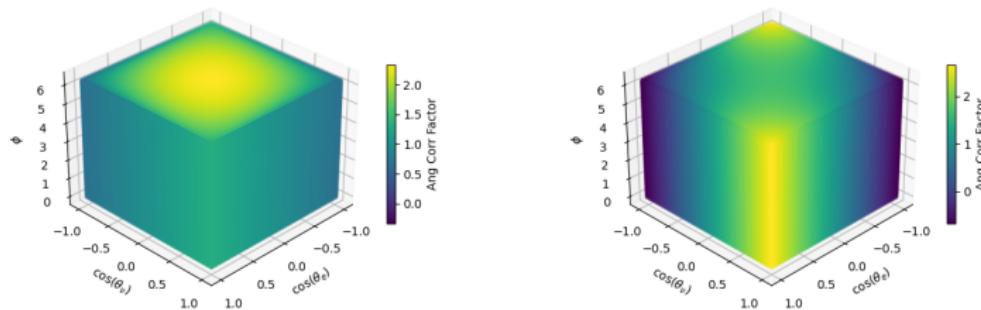
Figure: Values of the angular correlation Factor with (Right) a = 1, c = 1 and (Left) a = 1, c = -1, with E = 100000 keV and rest of variables 0 for both.

Two variables: c and A

Angular Correlation Factor

Figure: Values of the angular correlation Factor with (Right) a = 1, c = 4 and (Left) a = 1, c = -4, with E = 100000 keV and rest of variables 0 for both.

Two variables: c and a
Maximum and Minimum

Figure: Location of maximum (blue) and minimum (orange) for (Right) a = c = 1, E = 100000 keV (values 2.33332, -0.33332) and (Left) a = 1, c = -1, E = 100000 keV (values 2.66664, -0.66664)

Two variable: c and a

Sampling

Figure: Pairplot with the marginal distributions for a simulation of N = 300000 decays with c = a = 1, E = 100000 keV. The 1 variable histograms show the theoretical distribution obtained from numerically integrating F with the constrain $F > 0$

Two variables: c and D

Angular Correlation Factor

Since both terms proportional to a depends on E, we can consider only consider different ratios by changing one (c), while leaving the other (D) fixed. For convenience $E \gg m_e$.

$$F = 1 + \beta_e \left(-\frac{2}{3} c z_e z_\nu + \left(\frac{1}{3} c \cos \phi + D \sin \phi \right) \sqrt{1 - z_e^2} \sqrt{1 - z_\nu^2} \right)$$

Maxima and minima depend on the ratio between D and c:

▶ If $c^2 < 3D^2$: maximum at $z_e = z_\nu = 0$ and

$$\tan \phi = \frac{3D}{c}$$

$$F = 1 + \beta_e \sqrt{D^2 + \left(\frac{c}{3} \right)^2}$$

Two variables: c and D
Angular Correlation Factor

Since both terms proportional to a depends on E, we can consider only consider different ratios by changing one (c), while leaving the other (D) fixed. For convenience $E \gg m_e$.

$$F = 1 + \beta_e \left(-\frac{2}{3} c z_e z_\nu + \left(\frac{1}{3} c \cos\phi + D \sin\phi \right) \sqrt{1 - z_e^2} \sqrt{1 - z_\nu^2} \right)$$

Maxima and minima depend on the ratio between D and c:

- If $c^2 > 3D^2$: maximum at $z_e = z_\nu = \pm 1$ and

$$F = 1 + \frac{2}{3} \beta |c|$$

We look only at properties of the extrema

Two variables: c and D

Maximum and Minimum
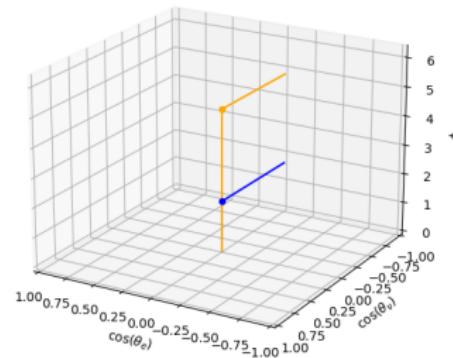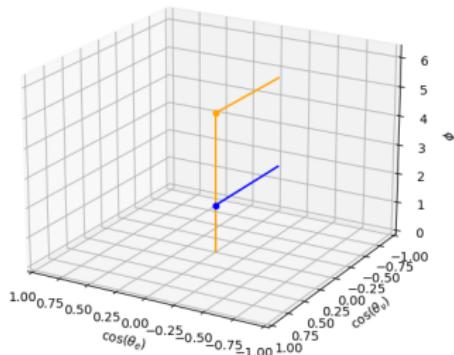
Figure: Positions of the maximum and minimum with Factor with (Right)
D = 1, c = 0.25 and (Left) D = 1, c = -0.25, with E = 100000 keV and
rest of variables 0 for both.

Two variables: c and D

Maximum and Minimum

Figure: Positions of the maximum and minimum with with (Right) D = 1, c = 1 and (Left) D = 1, c = -1, with E = 100000 keV and rest of variables 0 for both.

Two variables: c and D

Maximum and Minimum
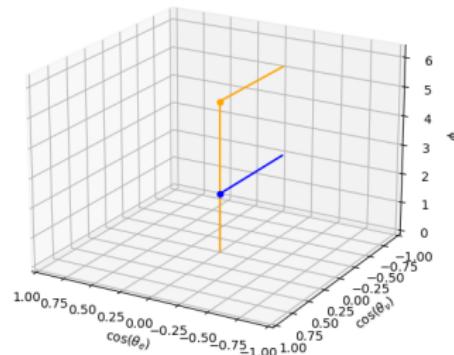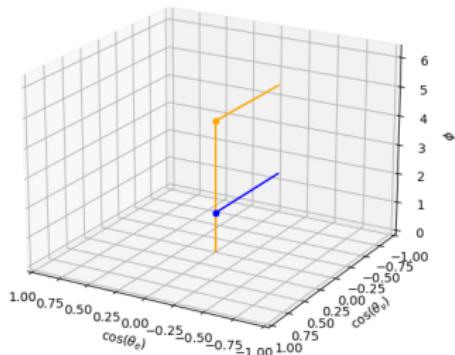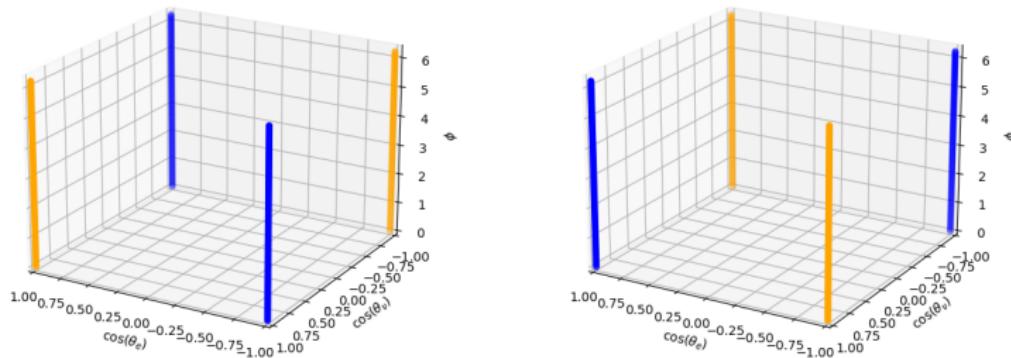
Figure: Positions of the maximum and minimum with (Right) $D = 1$, $c = 2$ and (Left) $D = 1$, $c = -2$, with $E = 100000$ keV and rest of variables 0 for both.

Two variables: c and D

Behaviour of maximum
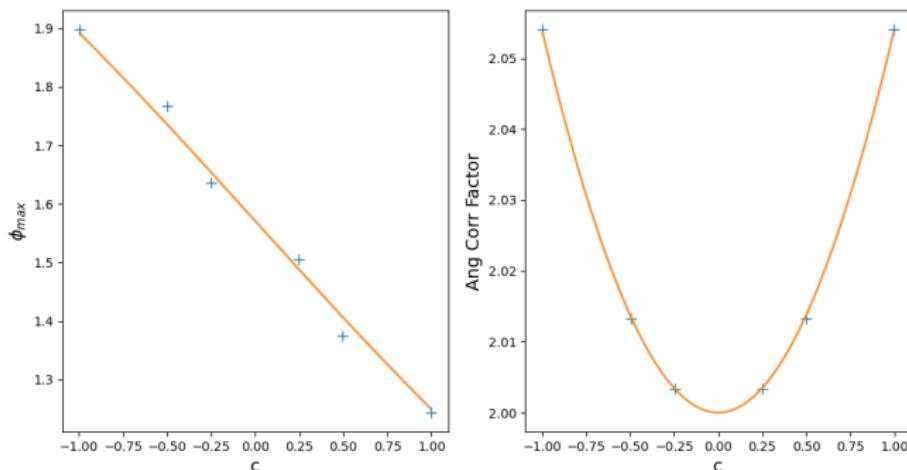
Figure: Behaviour of the ϕ coordinate for the maximum and the maximum value of the angular correlation factor for diferent values of c. Note discrepancies are a result of a sampling too coarse.

Two variable: c and D

Sampling
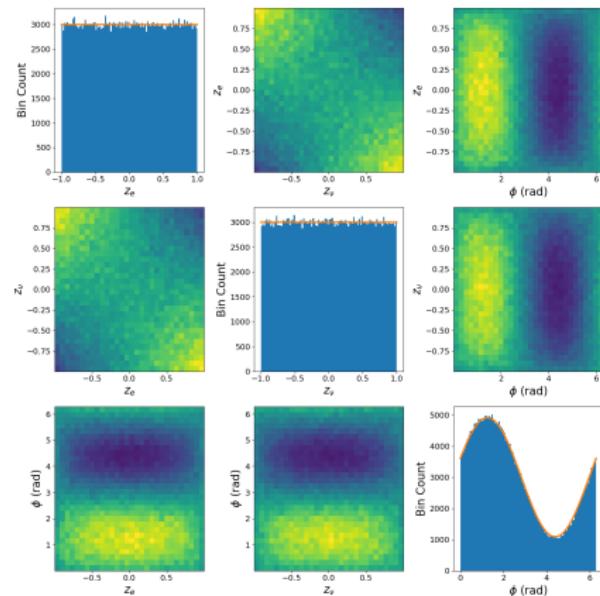
Figure: Pairplot with the marginal distributions for a simulation of $N = 300000$ decays with $c = D = 1$, $E = 100000$ keV. The 1 variable histograms show the theoretical distribution obtained from numerically integrating F with the constrain $F > 0$

Maximum of F

Single variable: a
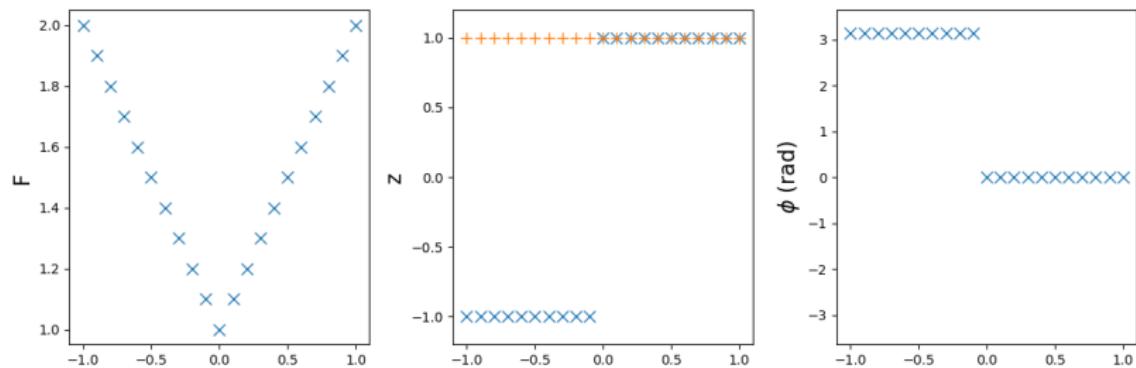
Figure: Characteristics of the maximum of F for variable a, E = 100000 keV, and rest of parameters 0

Maximum of F

Single variable: c
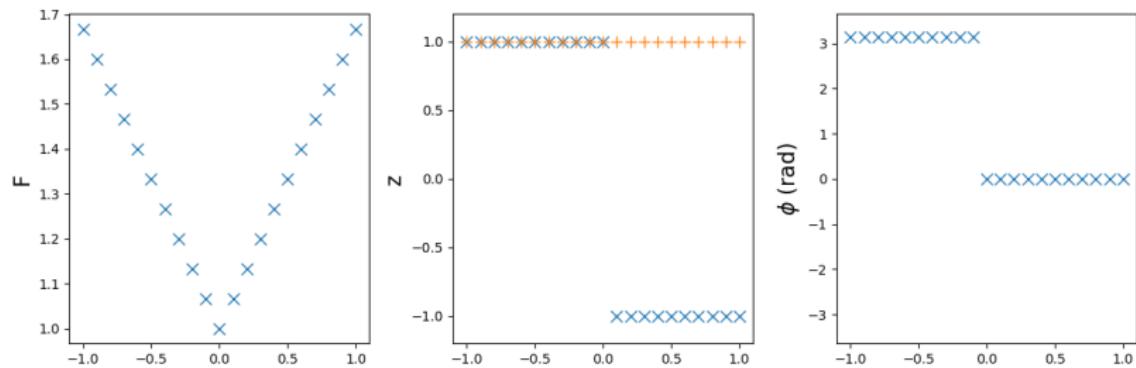
Figure: Characteristics of the maximum of F for variable c, E = 100000 keV, and rest of parameters 0
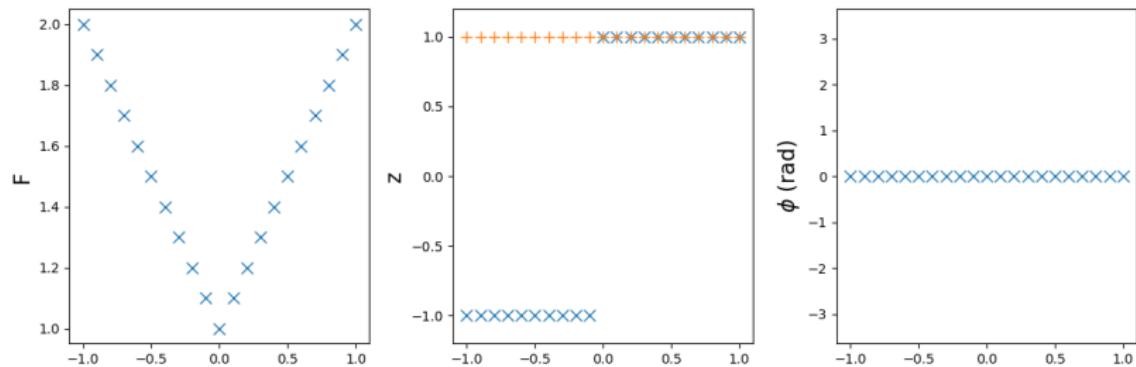
Maximum of F

Single variable: A

Figure: Characteristics of the maximum of F for variable A, E = 100000 keV, and rest of parameters 0

Maximum of F

Single variable: B
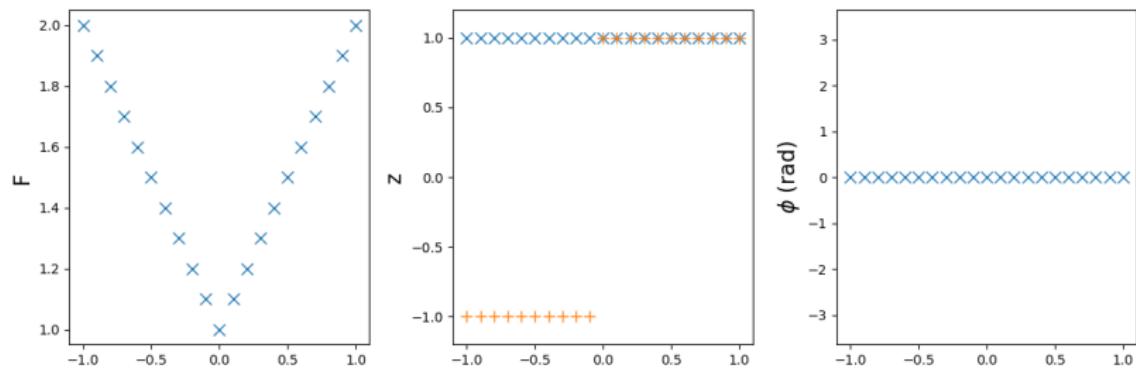
Figure: Characteristics of the maximum of F for variable B, E = 100000 keV, and rest of parameters 0

Maximum of F

Single variable: D
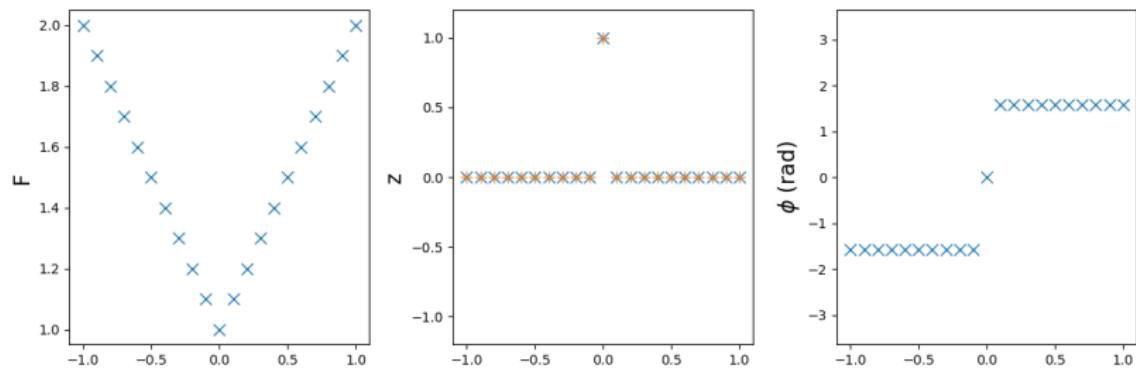
Figure: Characteristics of the maximum of F for variable D, E = 100000 keV, and rest of parameters 0

Maximum of F

Two variable: a and c
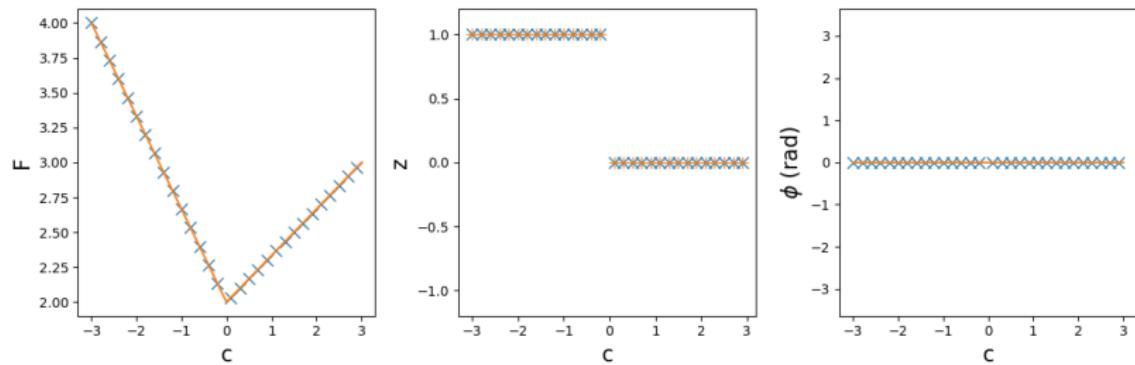
Figure: Characteristics of the maximum of F for a = 1, variable c, E = 100000 keV, and rest of parameters 0

Maximum of F

Two variable: a and A
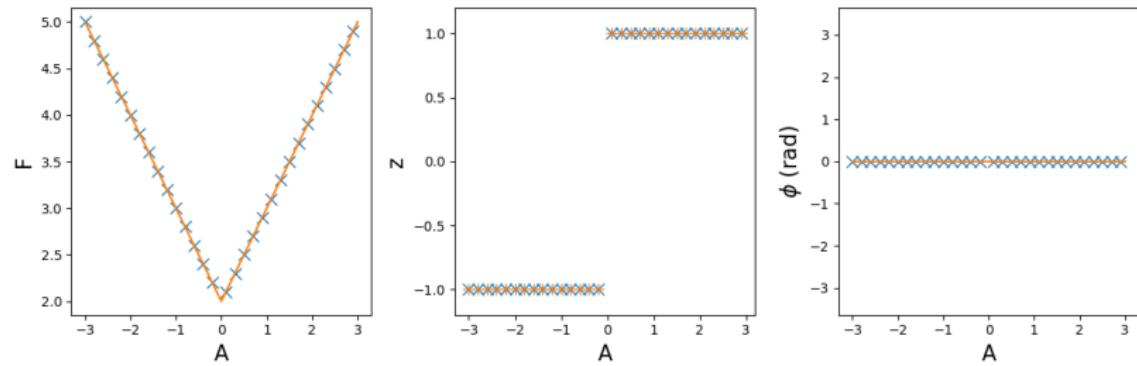
Figure: Characteristics of the maximum of F for a = 1, variable A, E = 100000 keV, and rest of parameters 0
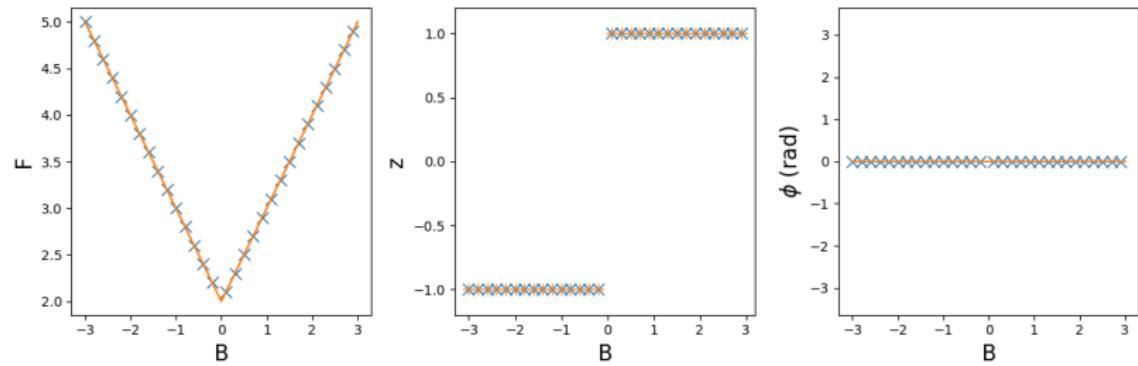
Maximum of F

Two variable: a and B

Figure: Characteristics of the maximum of F for a = 1, variable B, E = 100000 keV, and rest of parameters 0

Maximum of F
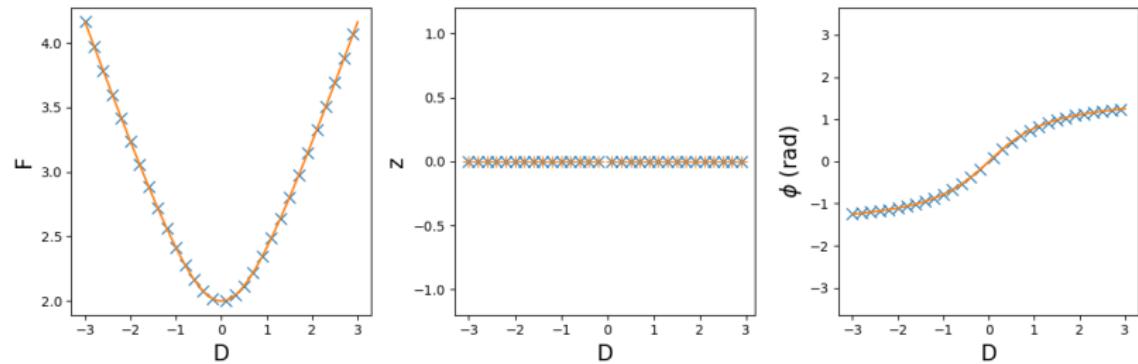
Two variable: a and D

Figure: Characteristics of the maximum of F for a = 1, variable D, E = 100000 keV, and rest of parameters 0

Maximum of F
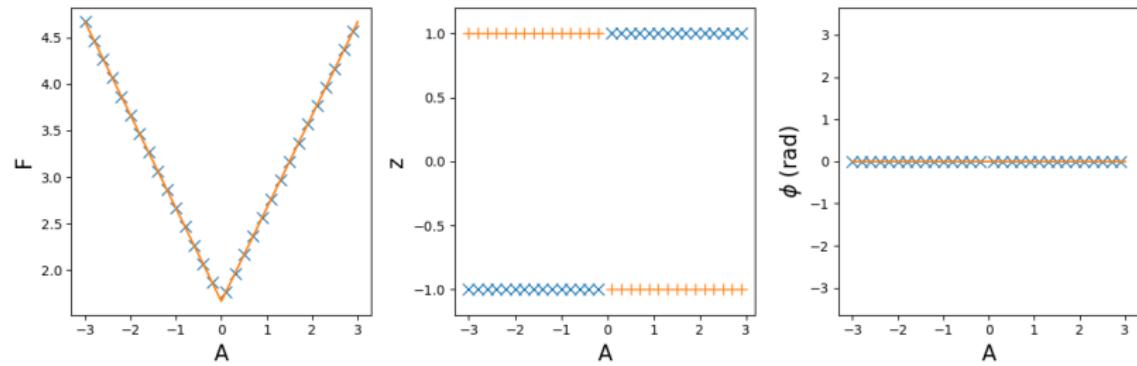
Two variable: c and A

Figure: Characteristics of the maximum of F for c = 1, variable A, E = 100000 keV, and rest of parameters 0

Maximum of F
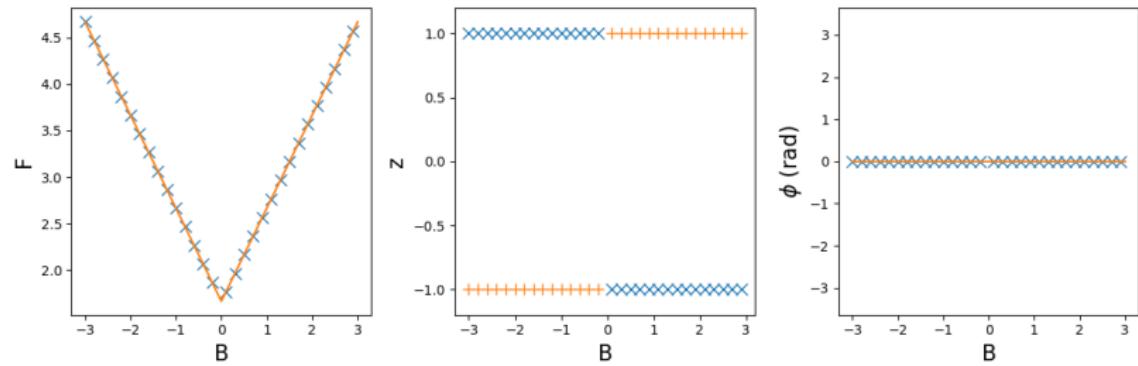
Two variable: c and B

Figure: Characteristics of the maximum of F for c = 1, variable B, E = 100000 keV, and rest of parameters 0

Maximum of F
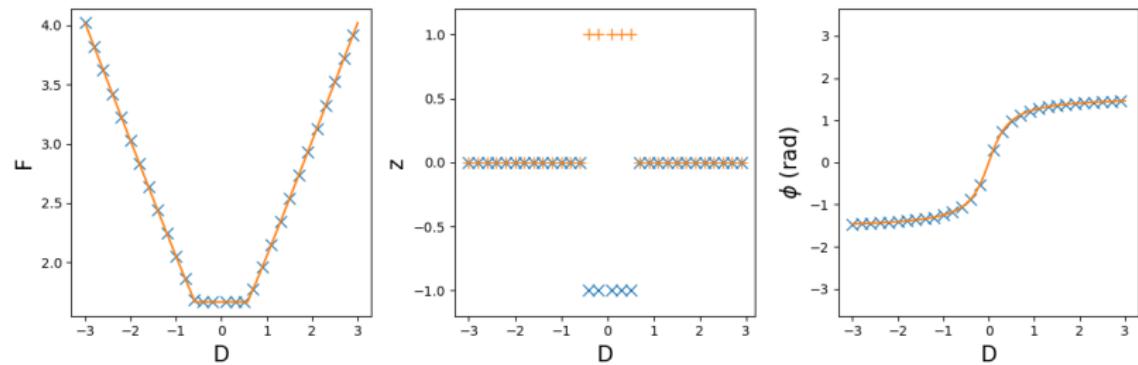
Two variable: c and D

Figure: Characteristics of the maximum of F for c = 1, variable D, E = 100000 keV, and rest of parameters 0
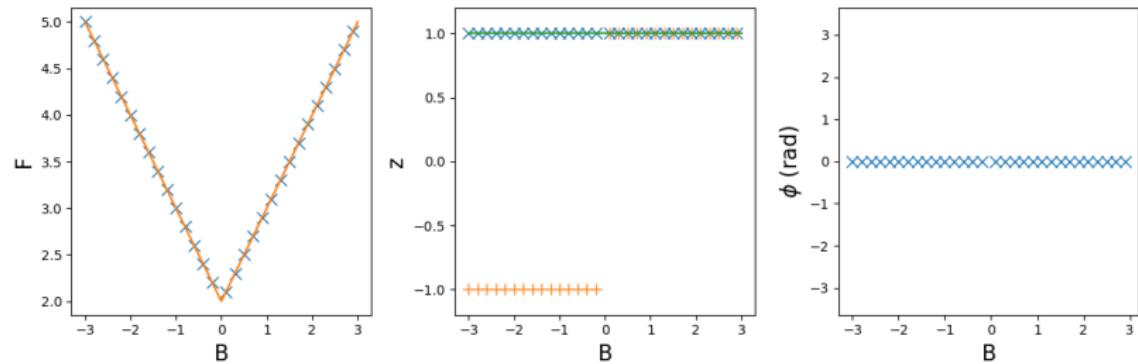
Maximum of F

Two variable: A and B

Figure: Characteristics of the maximum of F for A = 1, variable B, E = 100000 keV, and rest of parameters 0

Maximum of F
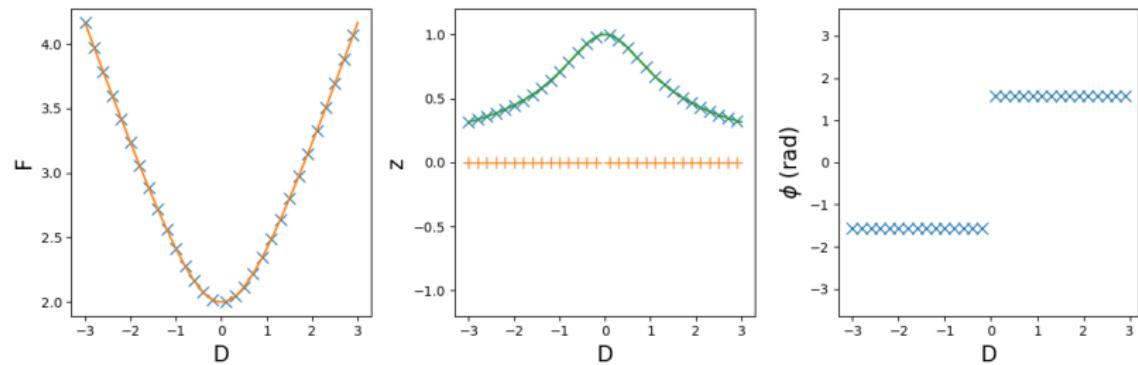
Two variable: A and D

Figure: Characteristics of the maximum of F for A = 1, variable D, E = 100000 keV, and rest of parameters 0

Maximum of F

Two variable: B and D
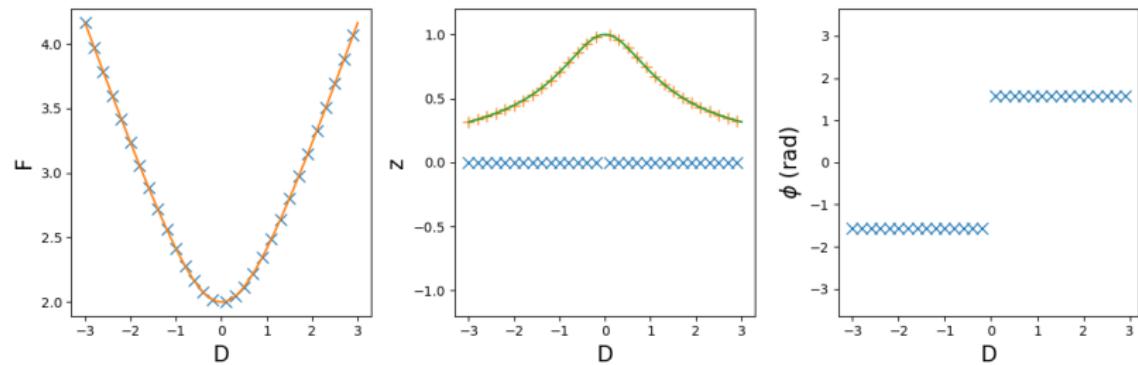
Figure: Characteristics of the maximum of F for B = 1, variable D, E = 100000 keV, and rest of parameters 0