25.

(a) Fórmula de Rodigues: $\left[L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^2}(x^n \cdot e^{-x})\right]$

$$L_2(x) = \frac{e^x}{2!}\frac{d^2}{dx^2}(x^2 e^{-x}) = \frac{e^x}{2}\frac{d}{dx}(2xe^{-x} - x^2e^{-x})$$

$$= \frac{e^x}{2}\left[(2e^{-x} - 2xe^{-x}) - (2xe^{-x} - x^2e^{-x})\right]$$

$$= \frac{e^x}{2}\left[2e^{-x} - 2xe^{-x} - 2xe^{-x} + x^2e^{-x}\right]$$

$$= \frac{1}{2}(2 - 2x - 2x + x^2) = \frac{1}{2}(x^2 - 4x + 2)$$

$\hookrightarrow L_2(x) = \frac{1}{2}(x^2 - 4x + 2)$

(b) Encontrar raices $[x_0, x_1]$ del Polinomio $[L_2(x)]$ de orden 2.

Ceros: $L_2(x) = 0 = \frac{1}{2}(x^2 - 4x + 2) = \frac{1}{2}x^2 - 2x + 1$

$0 = \frac{1}{2}x^2 - 2x + 1$, Fórmula cuadrática: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1/2)(1)}}{2(1/2)} = \frac{2 \pm \sqrt{4-2}}{1} = 2 \pm \sqrt{2}$$

$x_0 = 2 - \sqrt{2}$, $x_1 = 2 + \sqrt{2}$

Raices: $x_0 = 2 - \sqrt{2}$, $x_1 = 2 + \sqrt{2}$

(c) Encontrar Pesos de la Cuadratura integrando las bases cardinales con la función de peso de Laguerre $\sigma(x) = e^{-x}$

$$w_0 = \int_0^\infty \sigma(x)\left(\frac{x - x_1}{x_0 - x_1}\right)dx, \quad w_1 = \int_0^\infty \sigma(x)\left(\frac{x - x_0}{x_1 - x_0}\right)dx$$

$$w_0 = \int_0^\infty e^{-x}\left(\frac{x - (2+\sqrt{2})}{2 - \sqrt{2} - (2+\sqrt{2})}\right)dx = \int_0^\infty e^{-x}\left(\frac{x - 2 - \sqrt{2}}{-2\sqrt{2}}\right)dx$$

$$= \frac{1}{2\sqrt{2}} \int_0^{\infty} e^{-x}(2+\sqrt{2}-x)\,dx$$

$$= \frac{\sqrt{2}}{4}\left\{(2+\sqrt{2})\underbrace{\int_0^{\infty} e^{-x}\,dx}_{I_1} - \underbrace{\int_0^{\infty} e^{-x}x\,dx}_{I_2}\right\}$$

$$I_1 = -\left[e^{-x}\right]_0^{\infty} = \left[e^{-x}\right]_{\infty}^{0} = \left[\frac{1}{1} - 0\right] = \underline{1}$$

$$I_2 = \int_0^{\infty} x e^{-x}\,dx \qquad \begin{matrix} u = x, & dv = e^{-x}dx \\ du = dx, & v = -e^{-x} \end{matrix}$$

$$= \left[-xe^{-x}\right]_0^{\infty} - \int_0^{\infty} -e^{-x}\,dx = \left[xe^{-x}\right]_{\infty}^{0} + \int_0^{\infty} e^{-x}\,dx$$

$$= \left[\frac{(0)}{e^{(0)}} - \left(\overset{\infty}{\lim_{a\to\infty}\frac{a}{e^a}}\right)\right] + \left[-e^{-x}\right]_0^{\infty} = \left[-\overset{(L'HOP)\ 0}{\lim_{a\to\infty}\frac{1}{e^a}}\right] + \left[e^{-x}\right]_{\infty}^{0}$$

$$= \left[\frac{1}{e^{(0)}} - \lim_{b\to\infty}\frac{1}{e^b}\right] = \underline{1}$$

$$\hookrightarrow w_0 = \frac{\sqrt{2}}{4}\left((2+\sqrt{2})(1) - (1)\right) = \frac{\sqrt{2}}{4}(1+\sqrt{2}) = \underline{\frac{\sqrt{2}+2}{4}}$$

$$w_0 = \frac{x_1}{4} = \frac{\sqrt{2}+2}{4}$$

$$w_1 = \int_0^{\infty} e^{-x}\left(\frac{x-(2-\sqrt{2})}{2+\sqrt{2}-(2-\sqrt{2})}\right)dx = \int_0^{\infty} e^{-x}\,\frac{x-2+\sqrt{2}}{2\sqrt{2}}$$

$$= \frac{1}{2\sqrt{2}}\left\{(\sqrt{2}-2)\underbrace{\int_0^{\infty} e^{-x}\,dx}_{I_1} + \underbrace{\int_0^{\infty} xe^{-x}\,dx}_{I_2}\right\} \quad \Bigg\| \begin{matrix} I_1 = 1 \\ I_2 = 1 \end{matrix}$$

$$= \frac{\sqrt{2}}{4}\left[(\sqrt{2}-2)(1) + (1)\right] = \frac{\sqrt{2}}{4}\left[\sqrt{2}-1\right] = \underline{\frac{2-\sqrt{2}}{4}}$$

$$w_1 = \frac{x_0}{4} = \frac{2-\sqrt{2}}{4}$$

Pesos:

$$w_0 = \frac{2+\sqrt{2}}{4} = \frac{x_1}{4}, \quad w_1 = \frac{2-\sqrt{2}}{4} = \frac{x_0}{4}$$

(d) Mostrar que la regla es exacta para polinomio de grado 3.

$$f(x) = x^3 \to \int_0^\infty e^{-x} x^3 \, dx = \sum_{i=0}^{1} w_i f(x_i)$$

1. $\sum_{i=0}^{1} w_i f(x_i)$.

dado: $x_0 = 2-\sqrt{2}$, $x_1 = 2+\sqrt{2}$

$w_0 = \frac{x_1}{4}$, $w_1 = \frac{x_0}{4}$

$$\sum_{i=0}^{1} w_i f(x_i) = w_0 f(x_0) + w_1 f(x_1) = \frac{x_1}{4}(x_0)^3 + \frac{x_0}{4}(x_1)^3$$

$$= \frac{1}{4}\left((2+\sqrt{2})(2-\sqrt{2})^3 + (2-\sqrt{2})(2+\sqrt{2})^3\right)$$

$$= \frac{1}{4}\left[(2+\sqrt{2})\left((2)^3+3(2)^2(-\sqrt{2})+3(2)(-\sqrt{2})^2+(-\sqrt{2})^3\right) + (2-\sqrt{2})\left((2)^3+3(2)^2(\sqrt{2})+3(2)(\sqrt{2})^2+(\sqrt{2})^3\right)\right]$$

$$= \frac{1}{4}\left[(2+\sqrt{2})(8-12\sqrt{2}+12-2\sqrt{2}) + (2-\sqrt{2})(8+12\sqrt{2}+12+2\sqrt{2})\right]$$

$$= \frac{1}{4}\left[(2+\sqrt{2})(20-14\sqrt{2}) + (2-\sqrt{2})(20+14\sqrt{2})\right]$$

$$= \frac{1}{4}\left[40-\cancel{28\sqrt{2}}+\cancel{20\sqrt{2}}-28+40+\cancel{28\sqrt{2}}-\cancel{20\sqrt{2}}-28\right] = \frac{1}{4}[80-56]$$

$$= \frac{1}{4}[24] = \underline{6}$$

2. $\int_0^\infty e^{-x} x^3 \, dx = \Gamma(x)$

La función gamma es una extención del concepto de factorial a los números reales, y por definición, es la siguiente:

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} \, dt \text{ , donde } \Gamma(x) = (x-1)!$$

Entonces, según la función gamma:

$$\int_0^\infty e^{-x} x^3 \, dx = (3)! = 3\cdot 2\cdot 1 = \underline{6}$$

Por lo tanto, como $\left[\sum_{i=0}^{1} w_i f(x_i) = 6\right]$ y $\left[\int_0^{\infty} e^{-x} x^3 dx = 6\right]$ fueron mostrados anteriormente, se muestra que la regla de cuadratura de Laguerre es exacta para un polinomio de grado 3 $[f(x) = x^3]$ y es exactamente $\boxed{6}$.