TenSyGrid

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Abstract

1 Tensor Notation

A tensor is an N-way array where its values are accesed by N indeces. In this sense, a vector is an order 1 tensor and a matrix an order 2 tensor. A tensor of order n in the real numbers can de defined as $F \in \mathbb{R}^{I_1 \times \ldots \times I_n}$, where I_i is the dimension of the i_{th} order.

1.1 Tensor Operations

Let $F \in$

$$F \otimes G := \tag{1}$$

$$< F, G > \tag{2}$$

2 Multilinear models

A multi linear equation takes the form of:

$$p(x) = \sum_{i_n=0}^{i_n=1} \ldots \sum_{i_1=0}^{i_1=1} a_{i_1,\ldots,i_n} x_1^{i_1} \ldots x_n^{i_n} \tag{3}$$

$$\tag{4}$$

We can stablish a relationship between the coefficients of a multilinear polinomial in $x\mathbb{R}^n$ and the entries of a tensor $F \in \mathbb{R}^{\times_2 n}$. The following equation describes the relationship between the multi linear polynomial $p(x)$ and a tensor F.

$$F_{i_1,\ldots,i_n} = a_{i_1,\ldots,i_n} \tag{5}$$

2.1 Multilinear Time Independent Models

Multilinear Time Independent models or (MTI) describes a system and its evolution over time. The values used to define the system's operations are the following:

- $x \in \mathbb{R}^n$ the state.
- $y \in \mathbb{R}^p$ the output.
- $u \in \mathbb{R}^m$ the input.

Additionally, we need to consider the system's equations. Using the previous tensor notation we can describe the system equations explicitly as follows, where $F \in \mathbb{R}^{\times_2(n+m)\times n}$, $G \in \mathbb{R}^{\times_2(n+m)\times p}$:

$$\dot{x} =< F|M(x,u) > y \qquad\qquad =< G|M(x,u) > \tag{6}$$

Another way to define an MTI's state evolution is through the use of an implicit multilinear equation. In this case the tensor $H \in \mathbb{R}^{\times_2(2n+m+p)\times n+p}$ defines the state evolution. Additionnally, we would need to add the conditions $det(\partial_{\dot{x}} H) \neq 0$, $det(\partial_y H) \neq 0$ for the system to be properly defined (implicit function theorem). Otherwise, more equations would be need to define the system completely.

$$< H|M(\dot{x}, x, u, y) >= 0 \tag{7}$$

It is clear that the explicit formulation can be transformed into the implicit one such that the resulting system is: Where $\overline{H}$ is chosen appropriately to represent the $(n+p)$ multilinear equations $\dot{x}- < F|M(x,u) >= 0$ and $y- < G|M(x,u) >= 0$.

$$< \overline{H}|M(\dot{x}, x, u, y) >= 0 \tag{8}$$

3 Numerical Methods

3.1 Numerical Methods for ODE's

ODEs can be represented by the following implicit equation:

$$f(x, \dot{x}, t) = 0 \tag{9}$$

One specific method to solve a ystem of ODE are implicit methods. This is a type of family of methods that discretizes the time variable into a set of timesteps $\mathcal{T} := \{1, ..., T\}$. An initial value x_0 for x is given and then the following values are found by solving an equation involving the function in **??**. More specifically, we solve the equation $g(x_{t+1}, x_t) = 0$ where x_t is known, x_{t+1} is the unknown, anf g has the following expression:

$$g(x_{t+1}, x_t) = f(x_{t+1}, \frac{x_{t+1} - x_t}{\Delta t}, t+1) \tag{10}$$

We want to adapt this method to solve an explicit DAE as in **??**. To do so we define a new function $\overline{F}(x, u)$ that can be used as the implicit function.

$$\overline{F}(\dot{x}, x, z, u) = \begin{pmatrix} \dot{x} - < F|M(x, z, u) > \\ < G|M(x, z, u) > \end{pmatrix} \tag{11}$$

$$\overline{G}(x_{t+1}, x_t, z_{t+1}, u_{t+1}) = \overline{F}(\frac{x_{t+1} - x_t}{\Delta t}, x_t, z_{t+1}, u_{t+1}) \tag{12}$$

We can describe the method in algorithmic form as follows:

4 Case Study 1

For this case study we define a non linear EDO of the following form: Where $x \in \mathcal{C}(\mathbb{R}^n, \mathbb{R})$, $A, B, C \in \mathbb{R}^{n \times n}$.

$$\dot{x}(t) = Ax(t) + B(x(t) \odot x(t)) + x^T(t)Cx(t) \tag{13}$$

The equation represents an EDO with a linear component A and a quadratic component B. The objective is to transform this ODE into a multi linear DAE. For this purpose, we add n auxiliary variables in the form of the vector v as well as n algebraic equations.

$$B(x(t) \odot x(t)) \tag{14}$$

$$\Leftrightarrow \begin{cases} B(u(t) \odot x(t)) \\ u(t) = x(t) \end{cases} \tag{15}$$

This yields the following multi linear DAE system:

$$\dot{x}(t) = Ax(t) + B(u(t) \odot x(t)) \tag{16}$$

$$0 = u(t) - x(t) \tag{17}$$

$$A = \begin{pmatrix} -0.5 & 0.1 & 0 & 0 & 0 \\ 0.5 & -0.2 & 0.1 & 0 & 0 \\ 0 & 0 & -0.3 & 0 & 0 \\ 0 & 0 & 0 & 0.4 & -0.1 \\ 0 & 0 & 0 & 0 & -0.2 \end{pmatrix},$$

$$\text{B} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$
C = \begin{pmatrix} 0 & 0.05 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.1 \end{pmatrix} \quad (18)
$$

Finally we transform into the tenorsorized formulation of an iMTI. Where $F, G \in \mathbb{R}^{\times_2 2n \times n}$

$$
\dot{x}(t) =< F|M(x,u) > \quad (19)
$$

$$
0 =< G|M(x,u) > \quad (20)
$$

$$
F_{i_1,\ldots,i_n,j_1,\ldots j_n,k} = \begin{cases} A_{k,l} & \text{if } i_l = 1 \text{ and } \sum_{m=1}^{n} i_m + j_m = 1 \\ B_{k,l}+ & \text{if } i_l = 1, j_l = 1 \text{ and } \sum_{m=1}^{n} i_m + j_m = 2 \\ 0 & \text{otherwise} \end{cases} \quad (21)
$$

$$
G_{i_1,\ldots,i_n,j_1,\ldots j_n,k} = \begin{cases} 1 & \text{if } i_l = 1, l = k \text{ and } \sum_{m=1}^{2} i_m + j_m = 1 \\ -1 & \text{if } j_l = 1, l = k \text{ and } \sum_{m=1}^{2} i_m + j_m = 1 \\ 0 & \text{otherwise} \end{cases} \quad (22)
$$

$$
(23)
$$

5 Case Study 2

For this case study we define a non linear EDO of the following form: Where $x \in \mathcal{C}(\mathbb{R}^n, \mathbb{R})$, $A_i \in \mathbb{R}^{n \times n} \quad \forall i$.

$$
\dot{x}_i(t) = x^T(t) A_i x(t) \quad \forall i \in \{1, ..., n\} \quad (24)
$$

Using the same procedure done in the first case study we build the explicit multilinear tensor equations defining the model.

$$
\dot{x}(t) =< F|M(x,u) > \quad (25)
$$

$$
0 =< G|M(x,u) > \quad (26)
$$

$$
F_{i_1,\ldots,i_n,j_1,\ldots j_n,k} = \begin{cases} A_{k,l,m} + A_{k,m,l} & \text{if } l \neq m, \quad i_l = 1, i_m = 1 \text{ and } \sum_{m=1}^{n} i_m + j_m = 2 \\ A_{k,l,l} & \text{if } i_l = 1, j_l = 1 \text{ and } \sum_{m=1}^{n} i_m + j_m = 2 \\ 0 & \text{otherwise} \end{cases}
$$

$$
(27)
$$

$$
G_{i_1,\ldots,i_n,j_1,\ldots j_n,k} = \begin{cases} 1 & \text{if } i_l = 1, l = k \text{ and } \sum_{m=1}^{n} i_m + j_m = 1 \\ -1 & \text{if } j_l = 1, l = k \text{ and } \sum_{m=1}^{n} i_m + j_m = 1 \\ 0 & \text{otherwise} \end{cases} \quad (28)
$$

$$
(29)
$$

Let's us now analize the DAE's index. By virtue of the equations being in explicit form the first equation are already a ODE so they are of differential index 0. On the other hand the algbraic equations consist exclusively of equations of the form $u_i = x_i$, therefore differentiating the equation once already gives a EDO. This means that the DAE in this form is of index 1.