Mean field game with congestion effect

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Overview

1. Theoritical Results
2. Results from simulation

Hamiltonian assumptions

Since $H_0(p, \mu) = \frac{1}{\beta}\frac{|p|^\beta}{(c_0+c_1\mu)^\alpha}$, a sensible choice for the discrete Hamiltonian is

$$\tilde{H}(p_1, p_2, \mu) = \frac{1}{\beta}\frac{\left((p_1)_-^2 + (p_2)_+^2\right)^{\frac{\beta}{2}}}{(c_0 + c_1\mu)^\alpha}, \tag{1}$$

where X_+, resp. X_- stand for the positive (resp. negative) part of X: $X = X_+ - X_-$ and $|X| = X_+ + X_-$, and where we set $X_+^2 = (X_+)^2$ and $X_-^2 = (X_-)^2$.

Finite differences

To take into account Neumann boundary conditions, we introduce ghost nodes $x_{-1} = -h$, $x_{N_h} = 1 + h$ and set:

$$U^n_{-1} = U^n_0, \quad U^n_{N_h} = U^n_{N_h-1}, \quad M^n_{-1} = M^n_0, \quad M^n_{N_h} = M^n_{N_h-1}. \tag{2}$$

We introduce the following finite difference operators:

$$\partial_t w(t_n, x) \leftrightarrow (D_t W)^n = \frac{W^{n+1} - W^n}{\Delta t}, \quad n \in \{0, \dots, N_T - 1\}, W \in \mathbb{R}^{N_T+1}, \tag{3}$$

$$\partial_x w(t, x) \leftrightarrow (DW)_i = \frac{W_{i+1} - W_i}{h}, \quad i \in \{0, \dots, N_h - 1\}, W \in \mathbb{R}^{N_h}, \tag{4}$$

$$\partial_{xx} w(t, x_i) \leftrightarrow (\Delta_h W)_i = \frac{W_{i+1} - 2W_i + W_{i-1}}{h^2}, \quad i \in \{0, \dots, N_h - 1\}, W \in \mathbb{R}^{N_h}. \tag{5}$$

Defining gradient operators:

$$[\nabla_h W]_i = ((DW)_i, (DW)_{i-1}), \quad i \in \{0, \dots, N_h - 1\}, W \in \mathbb{R}^{N_h}. \tag{6}$$

Finite differences (2)

Considering a matrix $W \in \mathbb{R}^{(N_T+1)\times N_h}$, we define:

$$\partial_x W \leftrightarrow \frac{1}{h}\begin{bmatrix} -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix} W.$$

This matrix is denoted as D.

Considering Neumann conditions:

$$\partial_{xx} W \leftrightarrow \frac{1}{h^2}\begin{bmatrix} -1 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \cdots & 0 \\ 0 & 1 & -2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -2 & 1 \\ 0 & 0 & \cdots & 1 & -1 \end{bmatrix} W.$$

Finite difference (3)

This matrix is denoted as D_2.

Gradient operator with Neumann conditions ($(DW)_{i-1}$):

$$\frac{1}{h}\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & -1 & 1 \end{bmatrix} W.$$

Closed Form for residuals and Jacobian

We consider the following discrete HJB :

$$\begin{cases} -(D_t U_i)^n - \nu(\Delta_h U^n)_i + \tilde{H}(M_i^{n+1}, [\nabla_h U^n]_i) = g(x_i) + \tilde{f}_0(M_i^{n+1}), & 0 \leq i < N_h, 0 \leq n < N_T \\ U_{-1}^n = U_0^n, & 0 \leq n < N_T, \\ U_{N_h}^n = U_{N_h-1}^n, & 0 \leq n < N_T, \\ U_i^{N_T} = \varphi(M_i^{N_T}), & 0 \leq i < N_h. \end{cases} \tag{7}$$

Closed Form for residuals and Jacobian

Thus we can compute the residuals and diagonal elements of the Jacobian associated : we have:

$$
\begin{aligned}
F_i = & -\frac{U_i^{n+1} - U_i^n}{\Delta t} - \nu \frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{h^2} \\
& + \frac{1}{h^\beta \beta} \frac{1}{(c_0 + c_1 M_i^{n+1})^\alpha} \left((U_{i+1}^n - U_i^n)_-^2 + (U_i^n - U_{i-1}^n)_+^2 \right)^{\beta/2} + C
\end{aligned}
\tag{8}
$$

Closed Form for residuals and Jacobian

The banded jacobian :

$$\frac{\partial F_i}{\partial U_{i-1}^n} = -\frac{\nu}{h^2} - \frac{1}{h^\beta}\frac{1}{(c_0 + c_1 M_i^{n+1})^\alpha}(U_i^n - U_{i-1}^n)_+ \tag{9}$$

$$\times \left((U_{i+1}^n - U_i^n)_-^2 + (U_i^n - U_{i-1}^n)_+^2\right)^{\beta/2-1}, \tag{10}$$

$$\frac{\partial F_i}{\partial U_i^n} = \frac{1}{\Delta t} + \frac{2\nu}{h^2} + \frac{1}{h^\beta}\frac{1}{(c_0 + c_1 M_i^{n+1})^\alpha} \tag{11}$$

$$\times \left((U_{i+1}^n - U_i^n)_- + (U_i^n - U_{i-1}^n)_+\right) \tag{12}$$

$$\times \left((U_{i+1}^n - U_i^n)_-^2 + (U_i^n - U_{i-1}^n)_+^2\right)^{\beta/2-1}, \tag{13}$$

$$\frac{\partial F_i}{\partial U_{i+1}^n} = -\frac{\nu}{h^2} - \frac{1}{h^\beta}\frac{1}{(c_0 + c_1 M_i^{n+1})^\alpha}(U_{i+1}^n - U_i^n)_- \tag{14}$$

$$\times \left((U_{i+1}^n - U_i^n)_-^2 + (U_i^n - U_{i-1}^n)_+^2\right)^{\beta/2-1}. \tag{15}$$

Closed Form for residuals and Jacobian

We can extract for the jacobian associated with KFP the following equations :

$$(J_{\tilde{H}})_{i,i-1} = -\frac{1}{h^\beta}\frac{1}{(c_0 + c_1 M_i^{n+1})^\alpha}(U_i^n - U_{i-1}^n)_+ \left[(U_{i+1}^n - U_i^n)_-^2 + (U_i^n - U_{i-1}^n)_+^2\right]^{\frac{\beta}{2}-1} \tag{16}$$

$$\begin{aligned}(J_{\tilde{H}})_{i,i} = \frac{1}{h^\beta}\frac{1}{(c_0 + c_1 M_i^{n+1})^\alpha}&\left[(U_{i+1}^n - U_i^n)_- + (U_i^n - U_{i-1}^n)_+\right] \\ &\times \left[(U_{i+1}^n - U_i^n)_-^2 + (U_i^n - U_{i-1}^n)_+^2\right]^{\frac{\beta}{2}-1}\end{aligned} \tag{17}$$

$$(J_{\tilde{H}})_{i,i+1} = -\frac{1}{h^\beta}\frac{1}{(c_0 + c_1 M_i^{n+1})^\alpha}(U_{i+1}^n - U_i^n)_- \left[(U_{i+1}^n - U_i^n)_-^2 + (U_i^n - U_{i-1}^n)_+^2\right]^{\frac{\beta}{2}-1} \tag{18}$$

The KFP Equation

We now consider the discrete version of the KFP equation (forward scheme):

$$(D_t M_i)^n - \nu(\Delta_h M^{n+1})_i - \mathcal{T}_i(U^n, M^{n+1}, \tilde{M}^{n+1}) = 0, \quad 0 \le i < N_h, \quad 0 \le n < N_T, \tag{19}$$

$$M_{-1}^n = M_0^n, \quad M_{N_h}^n = M_{N_h-1}^n, \quad 0 < n \le N_T, \tag{20}$$

$$M_i^0 = \bar{m}_0(x_i), \quad 0 \le i < N_h, \tag{21}$$

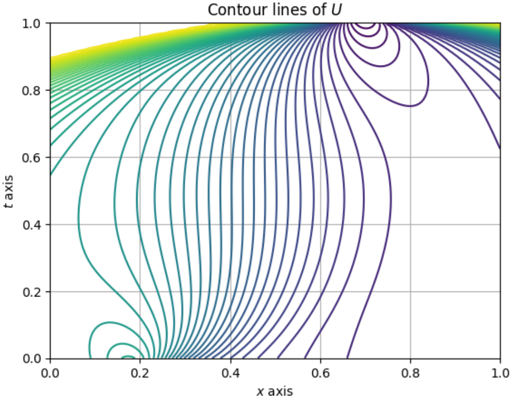
Transformation of the transport operator

Let A be the associated matrix. Then $A = (-J_{\tilde{H}})^T$. Formally derivation the $\tilde{H}$ we can obtain :

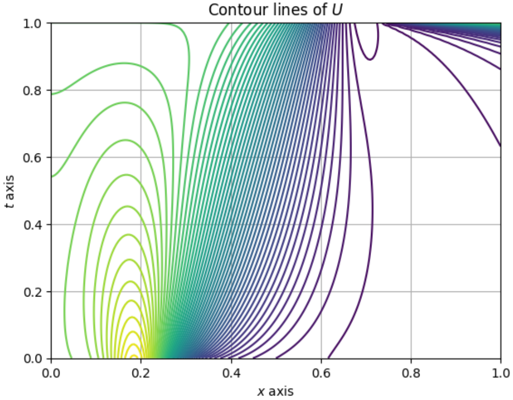
$$
\begin{aligned}
-A^T_{i,i} &= (J_{\tilde{H}})_{i,i}, \\
-A^T_{i,i-1} &= (J_{\tilde{H}})_{i,i-1}, \\
-A^T_{i,i+1} &= (J_{\tilde{H}})_{i,i+1}.
\end{aligned}
$$

Thus, solving for M^{n+1} reduces to solving:

$$
\left(I_{N_h} - \nu \Delta t D_2 + \Delta t (J_{\tilde{H}})^T \right) M^{n+1} = M^n. \tag{22}
$$

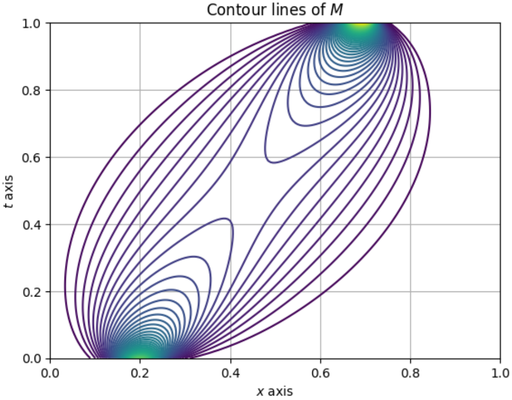


$\beta = 2,\ c_0 = 0.1,\ c_1 = 1,\ \alpha = 0.5,\ \sigma = 0.02$

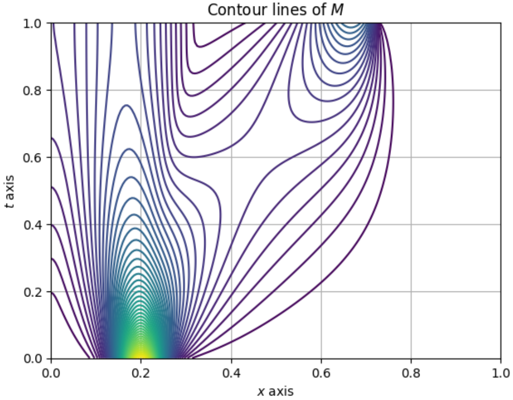


$\beta = 2,\ c_0 = 0.1,\ c_1 = 5,\ \alpha = 1,\ \sigma = 0.02$

Results Effect of c_1 (2)

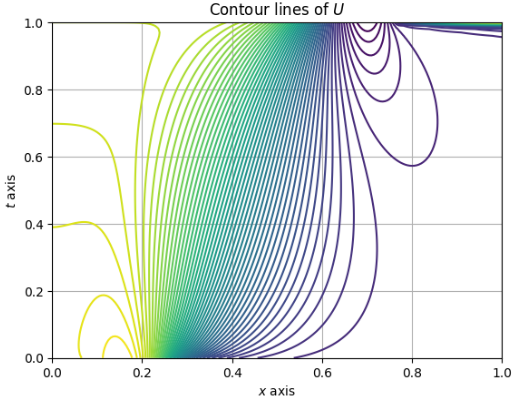


$\beta = 2,\ c_0 = 0.1,\ c_1 = 1,\ \alpha = 0.5,\ \sigma = 0.02$

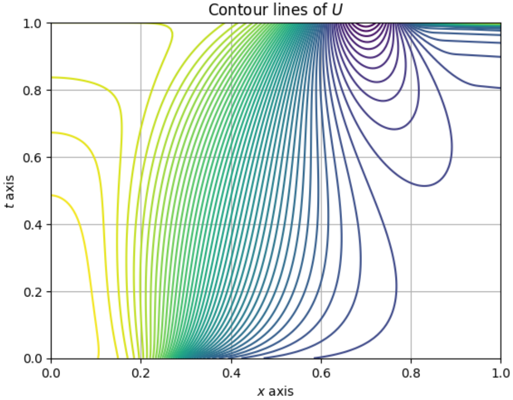


$\beta = 2,\ c_0 = 0.1,\ c_1 = 5,\ \alpha = 1,\ \sigma = 0.02$

Results Effect of c_1 (3)

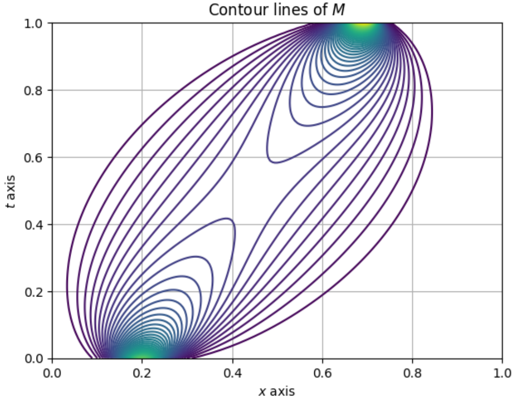


$\beta = 2,\ c_0 = 0.01,\ c_1 = 2,\ \alpha = 1.2,\ \sigma = 0.1$

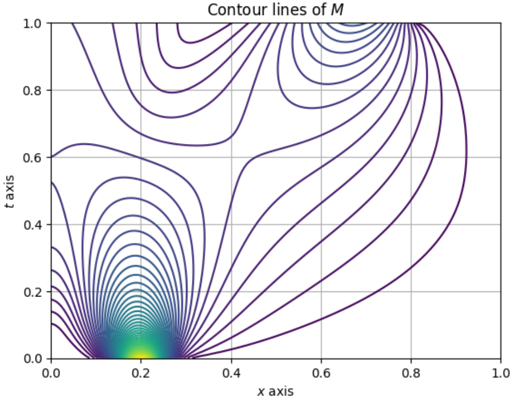


$\beta = 2,\ c_0 = 0.01,\ c_1 = 2,\ \alpha = 1.5,\ \sigma = 0.2$

Results Effect of c_1 (4)

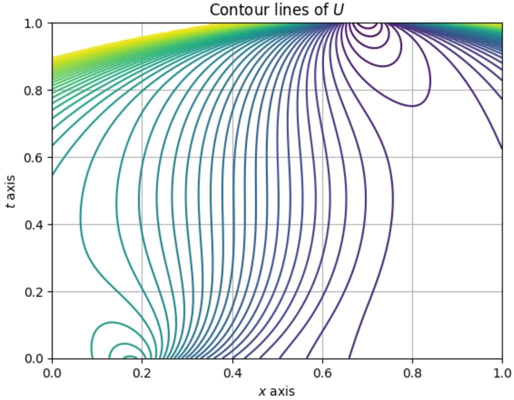


$\beta = 2,\ c_0 = 0.01,\ c_1 = 2,\ \alpha = 1.2,\ \sigma = 0.1$



$\beta = 2,\ c_0 = 0.01,\ c_1 = 2,\ \alpha = 1.5,\ \sigma = 0.2$

Results Effect of c_0 (1)



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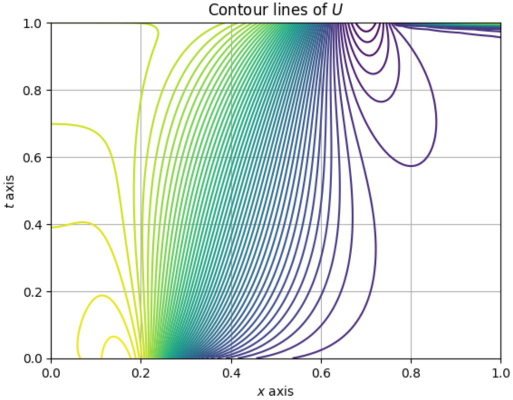
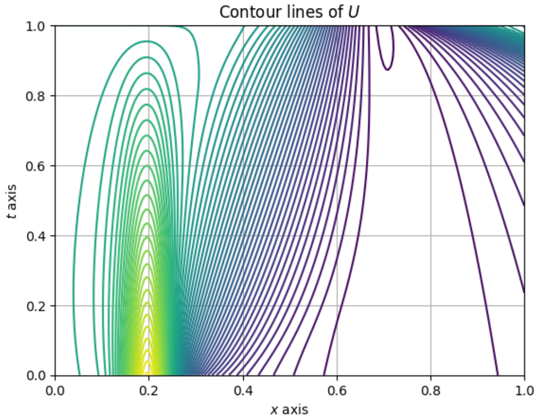


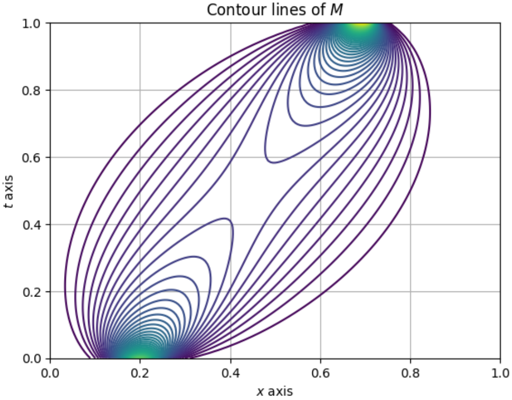
$\beta = 2,\ c_0 = 0.01,\ c_1 = 2,\ \alpha = 1.2,\ \sigma = 0.1$

Results Effect of c_0 (2)

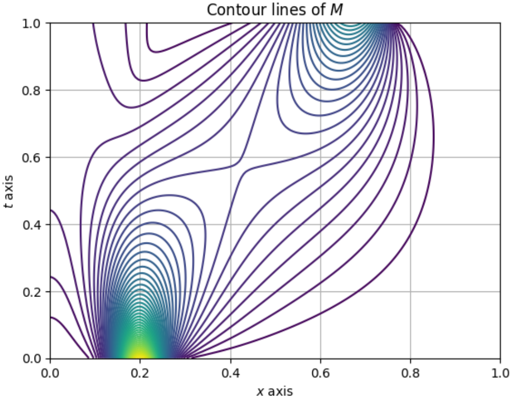


$$\beta = 2,\ c_0 = 1,\ c_1 = 3,\ \alpha = 2,\ \sigma = 0.002$$

Results Effect of c_0 (3)

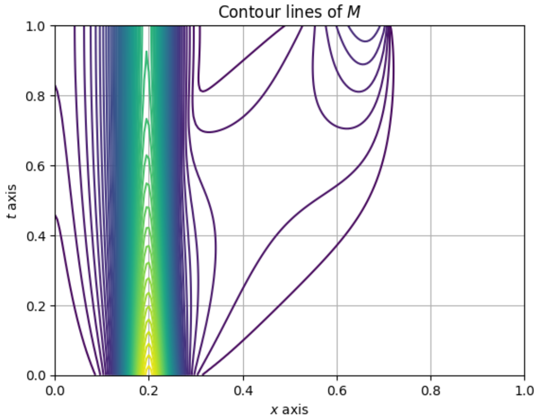


$\beta = 2,\ c_0 = 0.1,\ c_1 = 1,\ \alpha = 0.5,\ \sigma = 0.02$



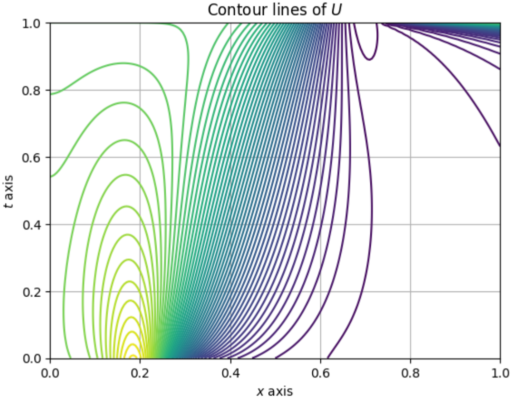
$\beta = 2,\ c_0 = 0.01,\ c_1 = 2,\ \alpha = 1.2,\ \sigma = 0.1$

Results Effect of c_0 (4)

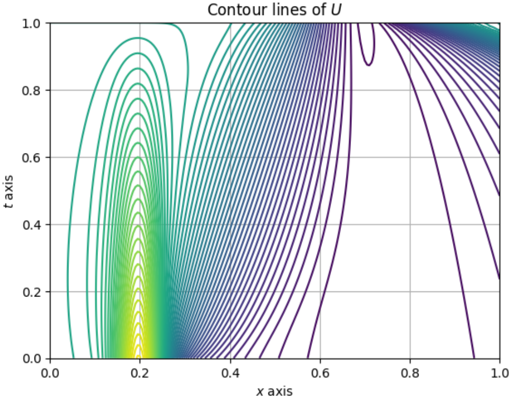


$$\beta = 2,\ c_0 = 1,\ c_1 = 3,\ \alpha = 2,\ \sigma = 0.002$$

Results Effect of α (1)



$\beta = 2,\ c_0 = 0.1,\ c_1 = 5,\ \alpha = 1,\ \sigma = 0.02$



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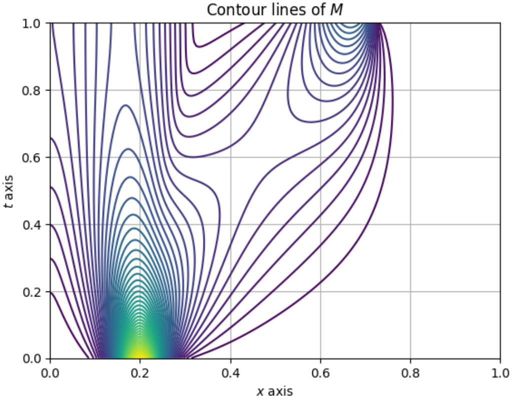
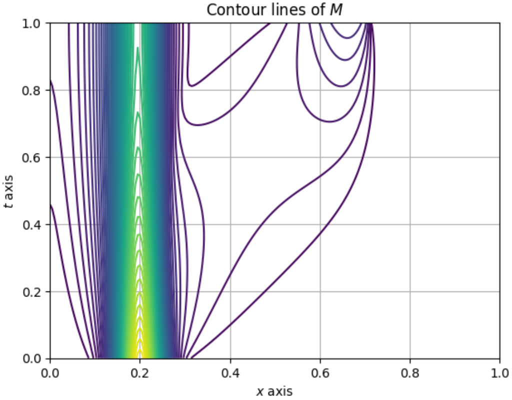
Results Effect of α (2)

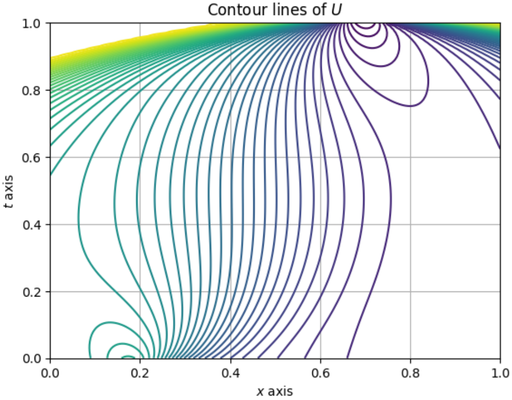


$\beta = 2,\ c_0 = 0.1,\ c_1 = 5,\ \alpha = 1,\ \sigma = 0.02$

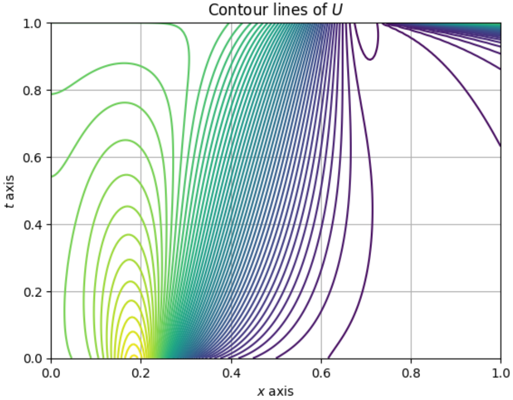


$\beta = 2,\ c_0 = 1,\ c_1 = 3,\ \alpha = 2,\ \sigma = 0.002$

Results Effect of σ (1)

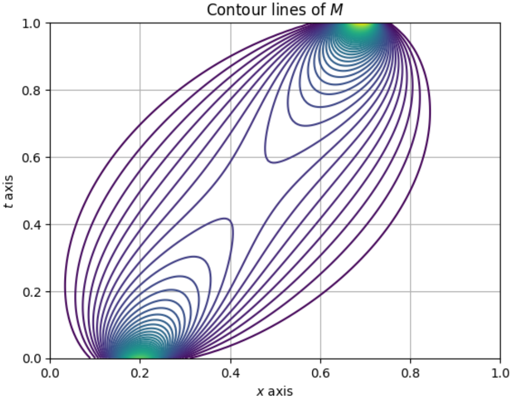


$\beta = 2,\ c_0 = 0.1,\ c_1 = 1,\ \alpha = 0.5,\ \sigma = 0.02$

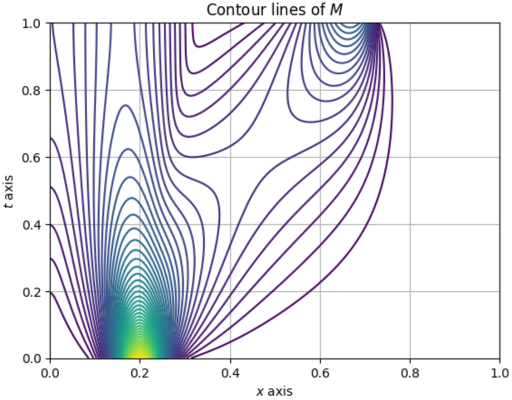


$\beta = 2,\ c_0 = 0.1,\ c_1 = 5,\ \alpha = 1,\ \sigma = 0.02$

Results Effect of σ (2)

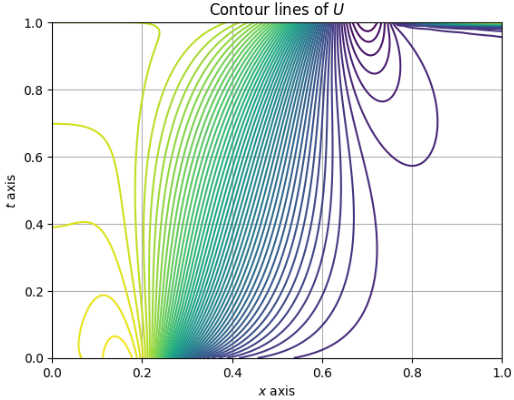


$\beta = 2,\ c_0 = 0.1,\ c_1 = 1,\ \alpha = 0.5,\ \sigma = 0.02$

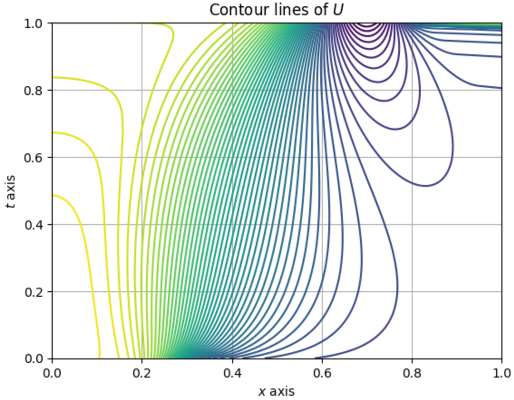


$\beta = 2,\ c_0 = 0.1,\ c_1 = 5,\ \alpha = 1,\ \sigma = 0.02$

Results Effect of σ (3)



$\beta = 2,\ c_0 = 0.01,\ c_1 = 2,\ \alpha = 1.2,\ \sigma = 0.1$



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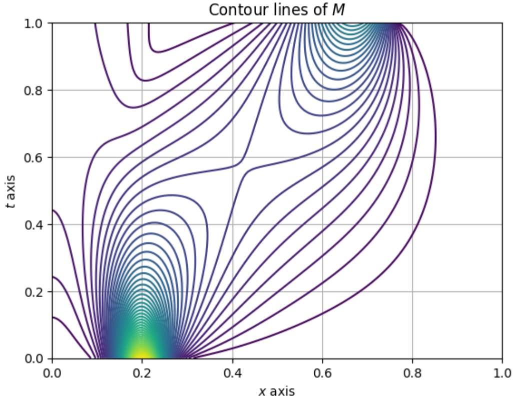
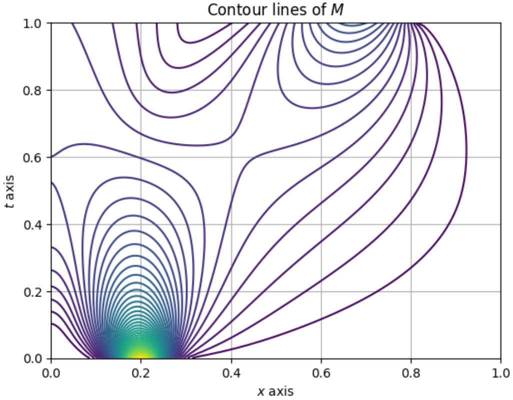
Results Effect of σ (4)

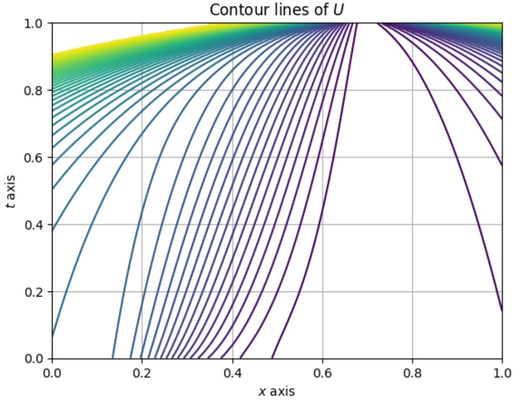


$\beta = 2,\ c_0 = 0.01,\ c_1 = 2,\ \alpha = 1.2,\ \sigma = 0.1$

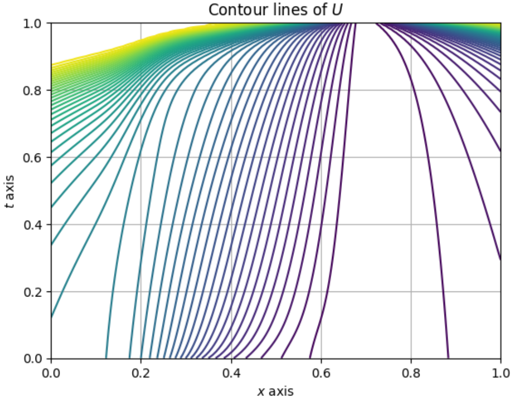


$\beta = 2,\ c_0 = 0.01,\ c_1 = 2,\ \alpha = 1.5,\ \sigma = 0.2$

MFG vs MFC : a comparison (1)

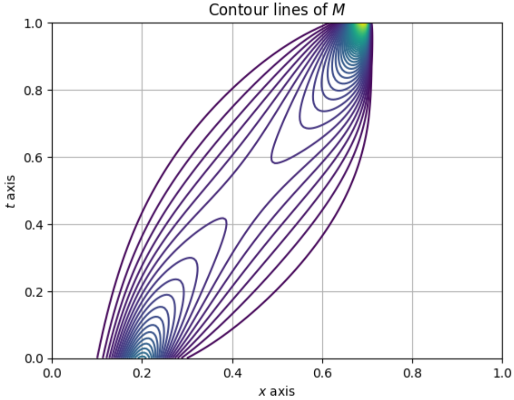


$\beta = 2,\ c_0 = 0.1,\ c_1 = 1,\ \alpha = 0.5,\ \sigma = 0.02$



$\beta = 2,\ c_0 = 0.1,\ c_1 = 1,\ \alpha = 0.5,\ \sigma = 0.02$

MFG vs MFC : a comparison (2)



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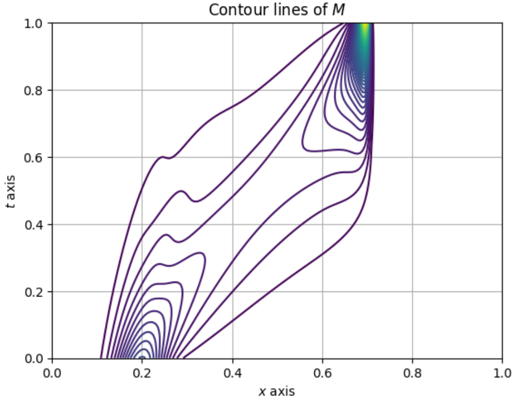

$\beta = 2,\ c_0 = 0.1,\ c_1 = 1,\ \alpha = 0.5,\ \sigma = 0.02$

The End