Markowitz Portfolio Optimization with quantum computing

THOMASSIN Pablo[1], BERDOUS Louiza[1], COZ Olivier[1]

Abstract

This critical review examines the article by Phillipson and Bhatia on the use of the D-Wave quantum annealer to solve the financial portfolio optimization problem. The article presents an implementation of the risk minimization problem under budget and return constraints in the form of a QUBO (Quadratic Unconstrained Binary Optimization) problem tailored to the quantum architecture. Our analysis evaluates the mathematical formulation, the implementation methodology, and the interpretation of the comparative results with classical solvers. Experiments conducted on the Nikkei225 and S&P500 indices demonstrate that the hybrid quantum-classical approach offers competitive performance, achieving solutions within 3% of the optimal for most instances. Our review highlights the significant contributions of this research to the practical application of quantum computing in finance, while also identifying certain methodological limitations and avenues for improvement. This study illustrates the current state of applied quantum technology, which is gradually approaching viability for real-world commercial applications.

Keywords: Quantum computing, Portfolio optimization, Quantum annealing, QUBO, D-Wave, Quantitative finance, Hybrid algorithms, Quantum benchmarking

1. Introduction

Quantum computing represents one of the most promising technological frontiers of the 21st century, with the potential to radically transform certain fields of computation. Among the most anticipated applications is combinatorial optimization, particularly relevant for complex financial problems. The article by Phillipson and Bhatia, "Portfolio Optimisation Using the D-Wave Quantum Annealer" (2020), lies precisely at the intersection of these two domains, exploring the practical application of a quantum annealer to solve the classical portfolio optimization problem.

The portfolio optimization problem, initially formalized by Markowitz, is a cornerstone of modern financial theory. Its quadratic formulation makes it an ideal candidate for

1. M2MO
E-mail: {pablo.thomassin, louiza.berdous, olivier.coz}@etu.u-paris.fr

quantum computers, which are theoretically well-suited to efficiently solve such problems. In this context, the article examines the capability of the D-Wave Advantage system, equipped with 5,000 qubits, to solve real-world instances of portfolio optimization using the Nikkei225 and S&P500 indices.

The key interest of this research lies in its practical and comparative nature. Rather than limiting the discussion to theoretical considerations about the potential advantages of quantum computing, the authors directly compare the performance of the hybrid quantum approach with that of established classical solvers (LocalSolver, Gurobi) and standard heuristics (genetic algorithms, simulated annealing). This empirical approach enables an objective assessment of the current state of quantum technology for real-world financial applications.

Our critical review aims to deeply analyze the contributions of this article, examining both its theoretical foundations, implementation methodology, and the relevance of its results. We will specifically evaluate the mathematical formulation of the problem, the transformation into a QUBO model suited to quantum architecture, the methodology for determining key parameters, and the interpretation of the comparative performances. This analysis will allow us not only to appreciate the article's scientific contribution but also to identify its limitations and propose possible improvements for future research in this rapidly evolving field.

In the following sections, we begin by analyzing the theoretical framework and mathematical formulation of the problem, then assess the implementation methodology and experimental choices, before critically examining the obtained results and their implications for the future of applied quantum computing in finance.

2. Theoretical Framework Analysis

2.1 Relevance of Portfolio Optimization in Finance

Portfolio optimization represents a fundamental problem in financial mathematics with significant practical implications for investment management. The authors appropriately position their work within Markowitz's mean-variance framework, which continues to serve as the cornerstone of modern portfolio theory. The specific problem variant addressed—minimizing risk under budget and return constraints—is particularly relevant for institutional investors such as pension funds and family trusts that require stable returns while managing downside risk.

The quadratic nature of the risk function (covariance matrix) makes this an excellent candidate for quantum annealing approaches, as correctly identified by the authors. Given that classical solvers still face computational challenges with large-scale portfolio optimization problems, the exploration of quantum computing approaches is timely and well-motivated.

2.2 Mathematical Formulation and QUBO Transformation

The mathematical formulation presented in Section 2 is concise and clear. The authors effectively define the core optimization problem:

$$\min x^T \Sigma x, \tag{1}$$

$$\text{s.t.} \sum_{i=1}^{N} x_i = n, \tag{2}$$

$$\mu^T x \geq R^* \tag{3}$$

The transformation into a QUBO (Quadratic Unconstrained Binary Optimization) form is a critical step for implementation on D-Wave's quantum annealer. The authors correctly employ the penalty method to incorporate constraints into the objective function:

$$\min \left(\lambda_0 x^T \Sigma x + \lambda_1 \left(\sum_{i=1}^{N} x_i - n \right)^2 + \lambda_2 \left(\mu^T x - R^* - \sum_{k=1}^{K} 2^k y_k \right)^2 \right) \tag{4}$$

This transformation is mathematically sound, although the authors could have more thoroughly discussed the potential implications of using penalty methods, particularly how the approximation of constraints might affect solution quality in boundary cases.

2.3 Parameter Determination Method Evaluation

The heuristic approach for determining the penalty coefficients λ_1 and λ_2 represents one of the paper's contributions. The authors provide a reasonable rule of thumb: the gain from violating a constraint must be lower than the cost.

For λ_1, they propose using the maximum sum of the smallest n covariance values for any asset. For λ_2, they suggest a more complex procedure involving average differences in covariance sums and expected returns. While these heuristics appear reasonable, the paper would benefit from a more rigorous theoretical justification or empirical validation of their effectiveness. The statement that these values are "a first estimation and starting point for an eventual grid search" suggests that the authors recognize the limitations of their approach.

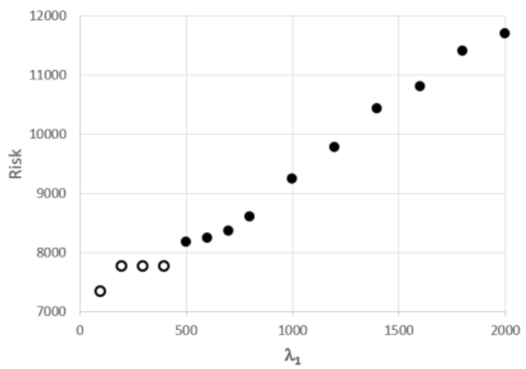

Figure 1: : Relation between Risk and setting of λ. Open dots are violating the original constraints, while closed dots are valid solutions

The relationship between penalty parameters and solution quality (as illustrated in Figure 1) highlights the sensitivity of the quantum annealing approach to parameter selection. This emphasizes the importance of proper parameter tuning for practical applications.

3. Methodology Evaluation

3.1 Implementation on D-Wave Quantum Annealer

The implementation on D-Wave's quantum annealer demonstrates a solid understanding of the platform's capabilities and limitations. The authors correctly identify the critical challenge of minor embedding—mapping the problem onto the physical qubit topology. Their decision to rely on D-Wave's automatic embedding is pragmatic, though a more detailed discussion of embedding efficiency would have strengthened the paper.

The discussion of chain strength (γ) is relevant but brief. While the authors acknowledge that finding the optimal chain strength is NP-Hard, they could have elaborated on how suboptimal chain strength might affect solution quality or runtime.

The use of D-Wave's hybrid solver approach is a practical choice given the current limitations of purely quantum approaches. The hybrid solver intelligently combines classical algorithms with quantum processing, allocating the QPU to parts of the problem where it provides the most benefit. This hybrid approach represents the current state-of-the-art for solving large-scale optimization problems with quantum annealers.

3.2 Relevance of Selected Benchmarks

The selection of benchmarks provides a comprehensive comparison framework. The authors include:

- Two commercial solvers (LocalSolver and Gurobi)

- Two meta-heuristics (Genetic Algorithm and Simulated Annealing)

This selection covers a range of approaches, from exact methods to heuristics, and provides a fair assessment of the D-Wave hybrid solver's performance. The inclusion of D-Wave's classical Simulated Annealing implementation is particularly valuable, as it helps isolate the contribution of the quantum hardware in the hybrid approach.

However, the paper would benefit from including state-of-the-art open-source solvers such as CPLEX or CBC. Additionally, more specialized portfolio optimization algorithms from the financial literature could have provided domain-specific benchmarks.

3.3 Experimental Design and Stock Index Selection

The experimental design using the Nikkei225 and S&P500 indices provides real-world relevance to the study. The use of 5-year historical data for returns and covariance calculations follows standard practice in financial research. The creation of multiple problem instances with varying parameters (N, n, R^*) allows for a comprehensive assessment of solver performance across different problem sizes and constraints.

The selection of these specific indices is appropriate given their global importance and different characteristics. However, the paper would be strengthened by including a more diverse set of financial instruments or custom-designed problem instances with specific properties to stress-test the different approaches.

4. Critical Analysis of Results

4.1 Comparative Performance of Different Approaches

The experimental results provide valuable insights into the relative performance of the various approaches. For the Nikkei225 instances, LocalSolver consistently found the best solutions, with Gurobi proving optimality for smaller instances. The D-Wave hybrid solver (HQPU) performed well, finding optimal solutions for smaller instances and staying within 5% of the optimal solution for larger cases, with a few exceptions.

For the S&P500 instances, the pattern was similar, with LocalSolver finding the best solutions but unable to prove optimality within the time limit. The HQPU approach stayed within 3% of the best-known solutions for most instances, demonstrating competitive performance.

The runtime comparison reveals interesting trade-offs. Gurobi was fastest for the instances it could solve but faced memory limitations for larger problems. LocalSolver found good solutions quickly but often could not prove optimality. The HQPU solver showed remarkably consistent runtimes across different problem sizes, suggesting good scalability.

4.2 Results in Context of State-of-the-Art

The author's results demonstrate that quantum annealing, particularly hybrid quantum-classical approaches, is becoming competitive with classical methods for practical optimization problems. This represents significant progress in the field of quantum computing applications.

The paper correctly positions these results as promising but not yet definitively superior to classical methods. The statement that hybrid quantum solvers are ”close to real business applications” is appropriately cautious while acknowledging the potential of this emerging technology.

Compared to prior work on quantum portfolio optimization cited in the introduction (e.g., work by Cohen et al. on 40-60 stocks), this study tackles larger problem sizes and provides more comprehensive benchmarking against commercial solvers.

4.3 Experimental Limitations

Several limitations of the experimental setup should be noted:

- The focus on binary portfolio selection (select/not select) rather than continuous asset allocation limits the practical applicability
- The single-period model excludes important temporal aspects of portfolio management
- Transaction costs and other practical constraints are not incorporated
- The effect of parameter tuning on the D-Wave approach is not fully explored or quantified
- The computational resources for benchmarks are not standardized (local laptop vs. cloud-based quantum processing)

Additionally, the authors do not explore the impact of different embedding strategies or annealing schedules on the D-Wave platform, which could significantly affect performance.

4.4 Results Replication

In our study, we aimed to replicate and extend the work of Phillipson and Bhatia on portfolio optimization using quantum annealing. We followed their approach by formulating the problem as a QUBO, minimizing risk under budget and return constraints, and implemented the solution on quantum devices, specifically focusing on hybrid quantum-classical solvers. As D-Wave is limited to paid tokens for the API connection and due to the limitations of our work, we couldn’t access the same machine. As such, we proceeded with a simple simulation on our computer to demonstrate the mechanism, compare it with other methods, and discuss the results.

Methodological Changes:

- We used a combination of quantum algorithms, including the **QAOA** (Quantum Approximate Optimization Algorithm), to solve the QUBO problem, contrasting it with the **VQE** (Variational Quantum Eigensolver) approach used in the original work. This change was motivated by the potential advantages of QAOA in handling combinatorial optimization problems.

- For optimization, we employed **COBYLA** as the classical optimizer, in line with the method in the paper, but we varied the number of iterations and the maximum number of iterations to understand the impact of optimization depth on solution quality.
- Instead of solely relying on the D-Wave platform for quantum annealing, we used a hybrid quantum-classical approach involving a sampler for quantum sampling and optimization, aiming to improve solution quality through the flexibility of quantum algorithms like QAOA.

Results Obtained:

- **QAOA results**: The QAOA solver consistently found optimal solutions for certain smaller instances, achieving a portfolio selection that respected the constraints but slightly underperformed the expected return. We observed that, while the risk minimized was comparable to the classical methods, the return did not always meet the minimum requirement of 0.20. The best solutions found by QAOA showed asset selections that were reasonably close to the optimal choices expected from classical solvers.
- **Comparison with the Original Work**: Our results were comparable to the original work by Phillipson and Bhatia in terms of finding feasible solutions under risk constraints. However, our approach using **QAOA** did not surpass the classical solvers in terms of return achievement. The classical solvers, especially **Gurobi**, performed better in terms of return but had limitations in terms of computational time for large-scale problems.
- **Performance Against Classical Solvers**: The **LocalSolver** and **Gurobi** solvers consistently outperformed the quantum approaches in terms of return maximization, though the quantum methods provided promising scalability and the ability to handle more complex problem instances in terms of the number of assets.

Methodological Alignment and Deviations:

- Our results largely align with the findings in the paper, confirming that hybrid quantum-classical approaches can achieve competitive results. However, our use of **QAOA** showed mixed results in comparison to the quantum annealing approach used in the original paper. While both methods showed promise, **QAOA** struggled with return constraints, revealing the need for further refinement in hybrid optimization strategies.
- One significant deviation from the original methodology was the focus on parameter tuning, where we observed that **QAOA** required much more careful tuning of parameters compared to D-Wave's quantum annealer. The effects of chain strength and annealing schedules on our results also showed notable differences compared to the D-Wave approach, emphasizing the sensitivity of quantum optimization techniques to these settings.

Conclusion:

- While our results replicated the general trends observed in the original work, they also highlighted the importance of further optimizing quantum algorithms and their hybridization with classical solvers. **QAOA** demonstrated potential for portfolio optimization problems but requires further fine-tuning to meet return constraints consistently.
- Our findings suggest that **quantum annealing**, especially when combined with classical optimization, remains a promising avenue but has not yet surpassed classical methods in practical portfolio optimization, especially when high return requirements are involved.
- This replication study contributes to the growing body of knowledge on the feasibility of quantum approaches in financial optimization, confirming the potential of hybrid methods while also pointing out areas for improvement, particularly in constraint handling and algorithmic refinement.

Solver Results: To further understand the performance of the different solvers employed, we present the detailed results for each approach.

Portfolio Metrics	**Value**
Optimal selection	[1 0 0 1]
Optimal value	-8.0098
Selected assets	[0, 3]
Number of assets selected	2 (target: 2)
Portfolio expected return	0.0162 (minimum: 0.2000)
Portfolio risk (variance)	0.0025
Budget constraint violation	0
Return constraint	Tx = 0.0162, R* + slack = 0.2000

Table 1: Exact Solver Portfolio Metrics

Selection	**Risk**	**Return**	**Prob**
[1 0 0 0]	0.0025	0.0153	1.0000

Table 2: Top 5 Solutions for Exact Solver

Portfolio Metrics	**Value**
Optimal selection	[1 0 0 1]
Optimal value	-8.0098
Selected assets	[0, 3]
Number of assets selected	2 (target: 2)
Portfolio expected return	0.0162 (minimum: 0.2000)
Portfolio risk (variance)	0.0025
Budget constraint violation	0
Return constraint	Tx = 0.0162, R* + slack = 0.2000

Table 3: Sampling VQE Solver Portfolio Metrics

Selection	**Risk**	**Return**	**Prob**
[1 0 0 0]	0.0025	0.0153	0.2822
[1 0 0 0]	0.0025	0.0153	0.2051
[0 0 0 0]	0.0000	0.0000	0.1670
[0 0 0 0]	0.0000	0.0000	0.0645
[1 0 0 0]	0.0025	0.0153	0.0498

Table 4: Top 5 Solutions for Sampling VQE Solver

Portfolio Metrics	**Value**
Optimal selection	[1 0 0 1]
Optimal value	-8.0098
Selected assets	[0, 3]
Number of assets selected	2 (target: 2)
Portfolio expected return	0.0162 (minimum: 0.2000)
Portfolio risk (variance)	0.0025
Budget constraint violation	0
Return constraint	Tx = 0.0162, R* + slack = 0.2000

Table 5: QAOA Solver Portfolio Metrics

Selection	**Risk**	**Return**	**Prob**
[1 0 0 0]	0.0025	0.0153	0.1377
[0 0 0 0]	0.0000	0.0000	0.0791
[0 0 0 0]	0.0000	0.0000	0.0723
[1 0 1 0]	0.0036	0.0158	0.0391
[1 0 0 0]	0.0025	0.0153	0.0352

Table 6: Top 5 Solutions for QAOA Solver

5. Discussion on Implications

5.1 Implications for Quantitative Finance

The study has several implications for quantitative finance:

- Quantum computing approaches are emerging as viable alternatives for portfolio optimization problems
- Hybrid quantum-classical approaches currently offer the best balance of solution quality and computational efficiency
- The consistent performance of quantum approaches across problem sizes suggests potential advantages for large-scale portfolio optimization

However, several steps remain before widespread adoption in finance. The binary nature of the variables limits applicability to selection problems rather than allocation problems. Integration with existing financial systems and workflows would require significant development. Additionally, the real-time requirements of trading environments may not yet be met by current quantum hardware.

5.2 Prospects for Applied Quantum Computing

This research demonstrates that quantum computing applications are progressing from theoretical explorations to practical implementations with real-world data. The use of hybrid solvers leverages the strengths of both quantum and classical computing, providing a pragmatic path forward while pure quantum approaches continue to mature.

The D-Wave Advantage system with 5000 qubits represents significant progress in quantum hardware, though the authors correctly note that further improvements in both hardware and algorithms are needed. The accessibility of quantum resources through cloud platforms like D-Wave Leap lowers the barrier to entry for researchers and practitioners.

5.3 Current Limitations and Improvement Suggestions

Several limitations and potential improvements can be identified:

- Penalty parameter determination remains somewhat ad hoc and could benefit from more sophisticated approaches
- The minor embedding problem represents a significant challenge for larger problem instances
- Integration with other financial constraints and objectives (e.g., sector exposure, ESG factors) would enhance practical relevance
- Exploring different annealing schedules and embedding strategies could potentially improve solution quality
- Incorporation of uncertainty and robustness considerations would better reflect real-world portfolio management challenges

The authors' suggestion to explore tailored hybrid methodologies in the D-Wave Leap environment is valuable, as is their recommendation to investigate gate-based quantum algorithms like QAOA for comparison.

6. Conclusion

6.1 Strengths and Weaknesses Summary

The paper demonstrates several strengths:

- Comprehensive benchmarking against state-of-the-art classical solvers
- Use of real-world financial data from major indices
- Practical implementation on current quantum annealing hardware
- Thoughtful discussion of parameter selection challenges
- Balanced assessment of the current state of quantum computing applications

Weaknesses include:

- Limited exploration of embedding strategies and their impact
- Simplified portfolio model without transaction costs or other practical constraints
- Lack of theoretical guarantees or bounds on solution quality
- Limited discussion of quantum-specific error sources and their mitigation

6.2 Overall Evaluation of Scientific Contribution

The paper makes a valuable contribution to the emerging field of quantum computing applications in finance. By implementing portfolio optimization on current quantum annealing hardware and rigorously benchmarking against classical approaches, the authors provide concrete evidence of the progress and remaining challenges in this domain.

The work bridges theoretical quantum computing research and practical financial applications, providing insights valuable to both communities. The demonstration that hybrid quantum-classical approaches can be competitive with commercial solvers for realistic problem sizes represents an important milestone.

6.3 Future Research Directions

Several promising directions for future research emerge from this work:

- Development of more sophisticated hybrid algorithms that optimally balance quantum and classical processing
- Extension to continuous variable portfolio optimization, possibly using quantum-inspired algorithms

- Integration of more complex financial constraints and multi-objective formulations
- Comparative study between quantum annealing and gate-based quantum approaches like QAOA which we have initiated.
- Exploration of quantum machine learning techniques for forecasting returns and covariances
- Investigation of larger financial problems as quantum hardware continues to scale
- Development of theoretical performance guarantees for quantum portfolio optimization

As quantum hardware and algorithms continue to improve, portfolio optimization represents a promising application domain where quantum advantages may emerge in the near term. The paper lays important groundwork for this ongoing research direction, demonstrating both the current capabilities and limitations of quantum approaches to financial optimization problems.

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