Infinite impulse response filters

Bilinear z-transform

Dr. Ing. Rodrigo Gonzalez

rodralez@frm.utn.edu.ar
rodrigo.gonzalez@ingenieria.uncuyo.edu.ar

Summary

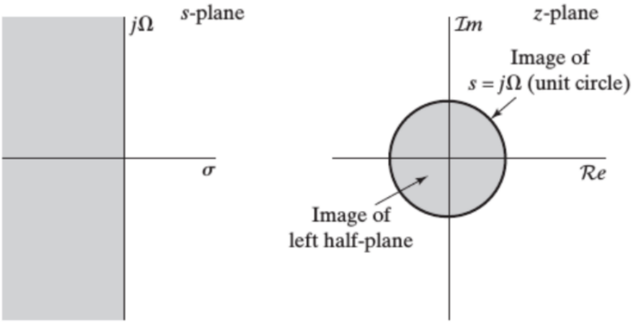
1. Classification of discrete filters
2. IIR filtering in frequency domain
3. Bilinear transform (Tustin's Method)
   - Map from s to z
   - Frequency pre-warping
   - Example of IIR design using bilinear transform
4. IIR structures
   - Direct form I IIR implementation
   - Direct form II IIR implementation
   - IIR cascade implementation
5. FIR vs IIR

Table: Classification of discrete filters

	Finite impulse response (FIR)	Infinite impulse response (IIR)
Filtering in time domain	Moving average	Leaky Integrator
Filtering in frequency domain	Windowed Filters Equiripple Minimax	Bilinear z-transform ZOH method

IIR filtering in frequency domain

- The main idea is to transform an analog filter to the discrete domain.
- From s domain to z domain.
- This way, all the theory behind analog filter can be reused to implement a filter in a computer (Butterworth filter, Chebyshev filters, Elliptic filter).



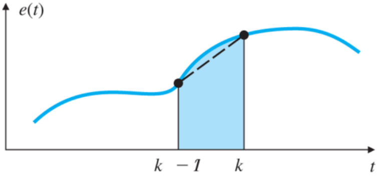
Bilinear transform (Tustin's Method)

Suppose the following integrator,

$$\frac{U(s)}{E(s)} = D_c(s) = \frac{1}{s}\,. \tag{1}$$

The area under $e(t)$ over $k \times T$ periods is,

$$u(k) = \int_0^{k-1} e(t)dt + \int_{k-1}^{k} e(t)dt\,. \tag{2}$$



Tustin's method uses the trapezoidal integration, to approximate $e(t)$ by a straight line between two samples. The technique is an algebraic transformation between variables s and z.

$$u(k) = u(k-1) + \frac{T}{2}\left[e(k-1) + e(k)\right], \tag{3}$$

$$U(z) = z^{-1}U(z) + \frac{T}{2}\left[z^{-1}E(z) + E(z)\right], \tag{4}$$

$$U(z)(1 - z^{-1}) = \frac{T}{2}\left[E(z)(1 + z^{-1})\right], \tag{5}$$

$$\Longrightarrow \frac{U(z)}{E(z)} = \frac{T}{2}\left(\frac{1 + z^{-1}}{1 - z^{-1}}\right) = \frac{1}{\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}}\,. \tag{6}$$

Comparing Eq. 1 and 6,

$$\boxed{s \approx \frac{2}{T}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)}\,. \tag{7}$$

Relationship between analog and digital frequencies

- Ω is the analog frequency, $-\infty, < \Omega < \infty$.
- ω, the "digital" frequency, $-\pi, < \omega < \pi$, i.e., $-2\pi f_s/2, < \omega < 2\pi f_s/2$.
- What is the relationship between Ω and ω.

Doing $s = j\Omega$, z should be evaluated in the unity circle, so, $z = r \cdot e^{j\omega} = \cdot e^{j\omega}$, with $r = 1$.

$$s = \frac{2}{T}\left(\frac{1 - e^{-j\omega}}{1 + e^{-j\omega}}\right) = \frac{2}{T}\left[\frac{2e^{-j\omega/2}(j\sin\omega/2)}{2e^{-j\omega/2}(\cos\omega/2)}\right] = j\frac{2}{T}\tan(\omega/2)\,. \tag{8}$$

Real and imaginary parts on both sides of Eq. 8 are,

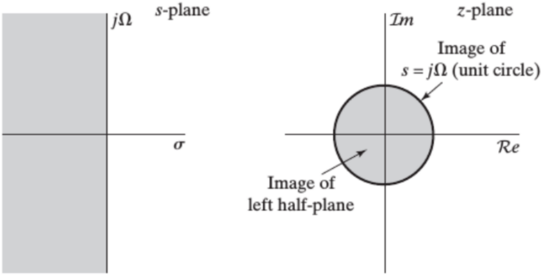
Since $s = \sigma + j\Omega$,

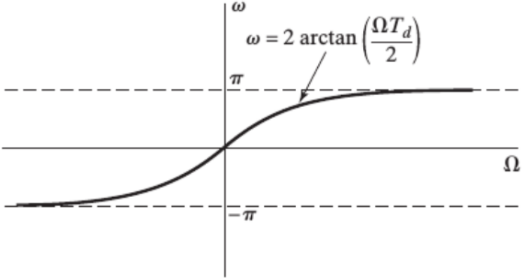
$$\sigma = 0\,, \tag{9}$$

$$\Omega = \frac{2}{T}\tan(\omega/2)\,, \tag{10}$$

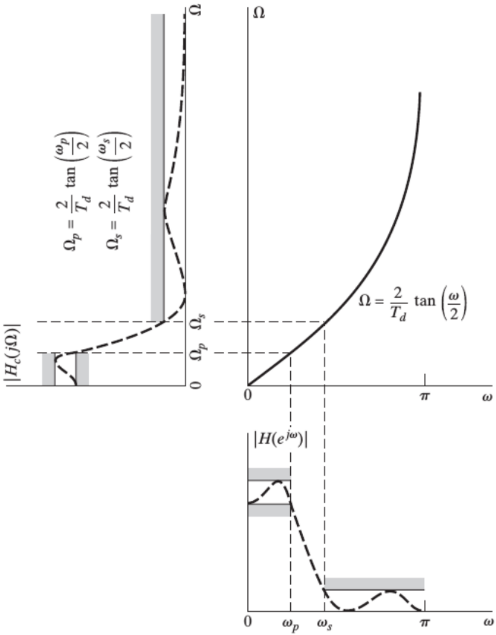
$$\Longrightarrow \omega = \arctan(\Omega T/2)\,. \tag{11}$$

Map from s to z





- Non-linear relationship between Ω and ω.
- Analog frequencies has to be adjust **before** analog filter design.



Example of IIR design using bilinear transform

1. Choose the analog filter that complains with the desired performance.

   For example, second-order Butterworth low-pass filter.

$$G(s) = \frac{\Omega_c^2}{s^2 + s\sqrt{2}\Omega_c + \Omega_c^2}$$

2. Cut-off digital frequency is normalized.

$$f_{dc} = 100 \text{ Hz}, \; f_s = 1000 \text{ Hz}, \; T = 0.001 \text{ s}.$$

$$\omega_c = 2\,\pi\,100/1000 = 0.628 \text{ rad/s}.$$

3. Pre-warp the analog frequencies.

$$\Omega_c = \frac{2}{T}\tan(\omega_c/2) = \frac{2}{0.001}\tan\left(\frac{0.628}{2}\right) = 649.839 \text{ rad/s}.$$

$$f_{ac} = 103.42 \text{ Hz}.$$

4 Replace s by the bilinear transform, $s \approx \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$.

$$H(s) = \frac{\Omega_c^2}{s^2 + s\sqrt{2}\Omega_c + \Omega_c^2}$$

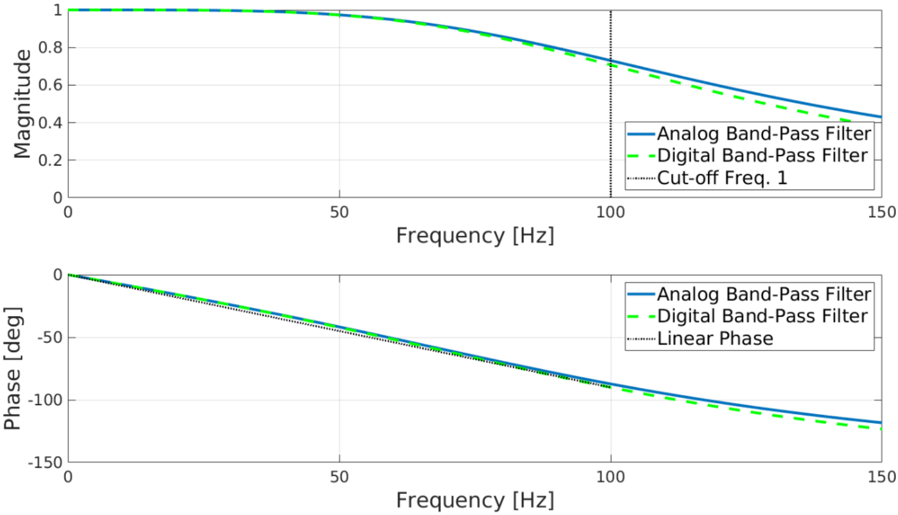
$$H(z) = \frac{(649.84)^2}{\left(\frac{2}{T}\right)^2 \left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\sqrt{2}\,(649.84) + (649.84)^2}$$

$$H(z) = \frac{0.067 + 0.135z^{-1} + 0.067z^{-2}}{1 - 1.143z^{-1} + 0.413z^{-2}}$$

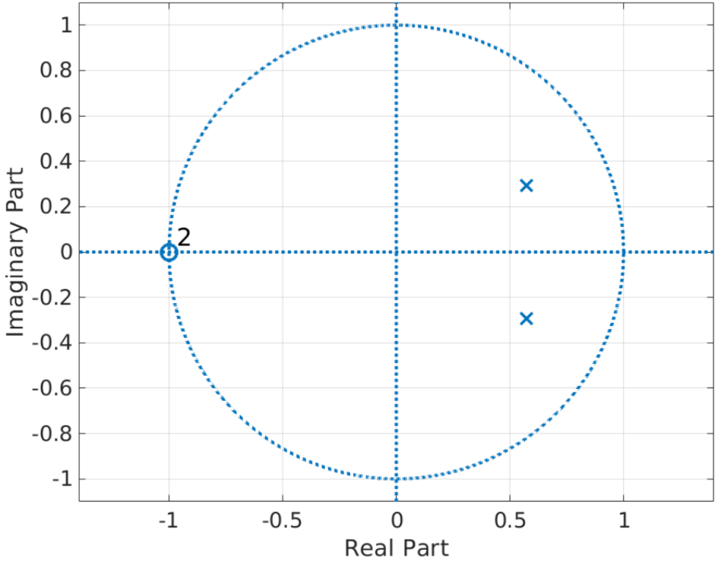
5 Invert the Z-transform to find the difference equation.

$$\begin{aligned} y[n] = {} & 0.067\,x[n] + 0.135\,x[n-1] + 0.067\,x[n-2]1+ \\ & + 1.143\,y[n-1] - 0.413\,y[n-2] \end{aligned} \tag{12}$$

Frequency and phase responses

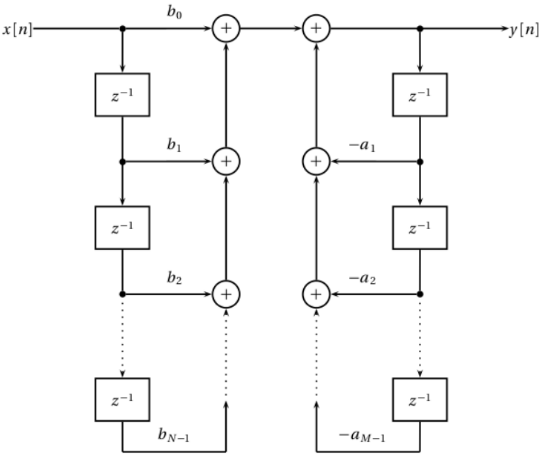


Z plane



Direct form I IIR implementation

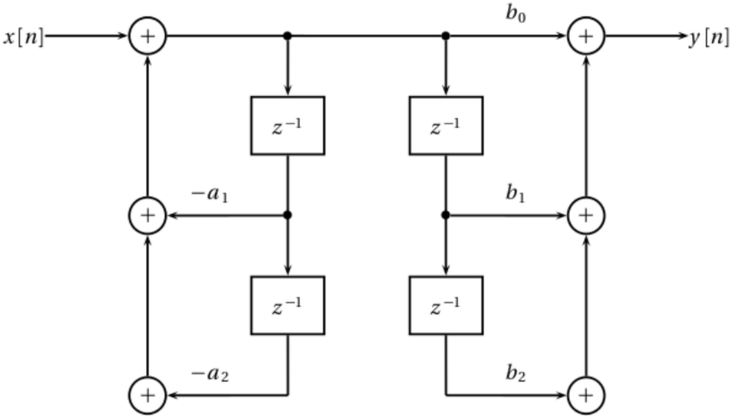
$$H(z) = \frac{b_0 + b_1 z^{-1} + \ldots + b_{N-1} z^{N-1}}{1 + a_1 z^{-1} + \ldots + a_{M-1} z^{M-1}}$$



Direct form I IIR implementation inverted

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

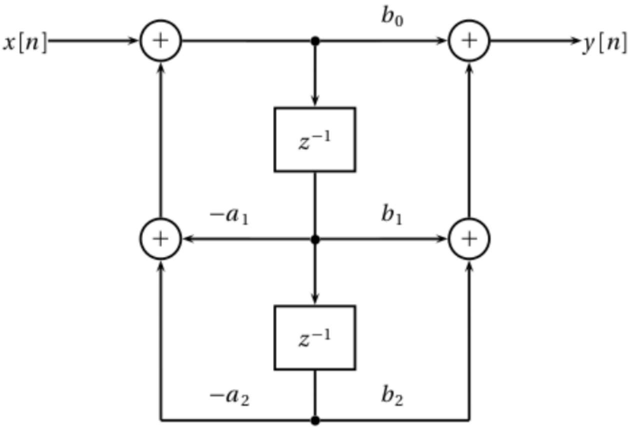
By the commutative properties of the z-transform, we can invert the order of computation to turn the Direct Form I structure into a new structure.



Direct form II IIR implementation

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

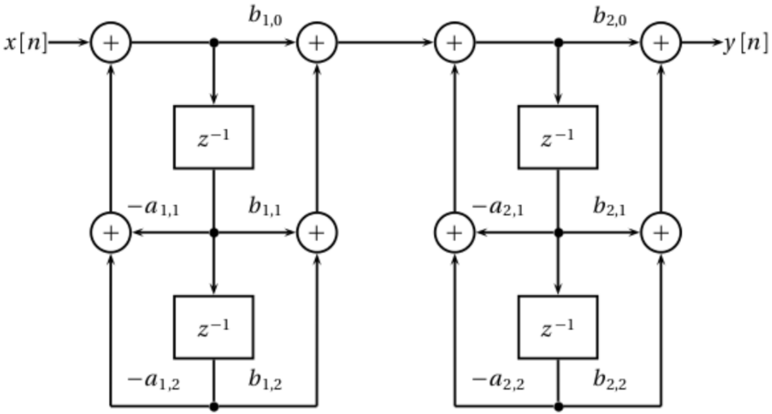
We can then combine the parallel delays together. This implementation is called Direct Form II; its obvious advantage is the reduced number of the required delay elements (hence of memory storage).



IIR cascade implementation

The cascade structure of N second-order sections is much less sensitive to quantization errors than the previous Direct form II of order $2 \cdot N$.

$$H(z) = \prod_{k=1}^{N} G_k \frac{b_{0k} + b_{1k} z^{-1} + b_{2k} z^{-2}}{1 + a_{1k} z^{-1} + a_{2k} z^{-2}}$$



FIR vs IIR

FIR, pros:

- Unconditional stability (no poles).
- Precise control of the phase response and, in particular, exact linear phase.
- Robustness with respect to finite numerical precision hardware.

FIR, cons:

- Longer input-output delay.
- Higher computational cost with respect to IIR solutions.

IIR, pros:

- Lower computational cost with respect to an FIR with similar behavior.
- Shorter input-output delay.
- Compact representation.

IIR, cons:

- Stability is not guaranteed.
- Phase response is difficult to control.
- Design is complex in the general case.
- Sensitive to numerical precision.

Bibliography

1 Paolo Prandoni and Martin Vetterli. Signal processing for communications. Taylor and Francis Group, LLC. 2008. Sections 7.3, and 7.4.2.

2 Oliver Hinton. Digital Signal Processing Resources for EEE305 Course. Chapter 5. www.staff.ncl.ac.uk/oliver.hinton/eee305/

3 Gene F. Franklin, J. David Powell and Abbas Emami-Naeini. *Feedback Control of Dynamic Systems*. 7th Edition. 2014. Section 8.3.