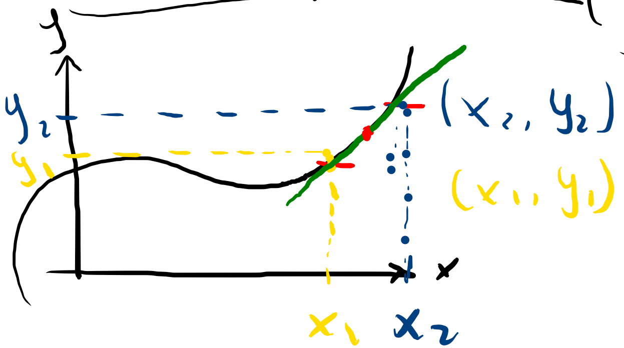
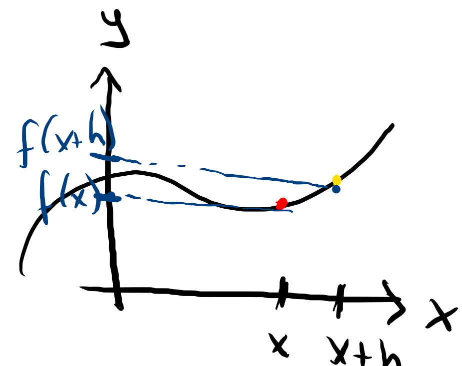
Cálculo diferencial ($\frac{d}{dx}$, funciones continuas, límites de funciones, teoremas importantes)



$$\underline{m} = \frac{y_2 - y_1}{x_2 - x_1}$$



Paso 1: $f(x+h)$

Paso 2: $f(x+h) - f(x)$

Paso 3: $\dfrac{f(x+h) - f(x)}{h}$

Paso 4: $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = f'(x) = \dfrac{df}{dx}, D_x f \ldots$

Derivadas famosas:

1) $f(x) = x^n$, $f'(x) = nx^{n-1}$

2) $f(x) = \sin x$, $f'(x) = \cos x$

3) $f(x) = \cos x$, $f'(x) = -\sin x$

4) $f(x) = e^x$, $f'(x) = e^x$

5) $f(x) = \ln x$, $f'(x) = \dfrac{1}{x}$

Ejercicios

(a) $f(x) = (x^2+2)^2$, $f'(x) = 2(x^2+2)[2x]$

$h(x) = x^2+2$ $\quad h'(x) = 2x$ $\quad f'(x) = 2(x^2+2)2x = 4x(x^2+2)$

$g(x) = x^2$, $\quad g'(x) = 2x$

$g(h(x)) = g(x^2+2) = (x^2+2)^2 = f(x)$.

$f(x) = g(h(x))$ Regla de la cadena

$f'(x) = g'(h(x)) \cdot h'(x)$

(b) $f(x) = (x^2+2)^{100}$,

$h(x) = x^2+2$, $h'(x) = 2x$

$g(x) = x^{100}$ $\quad g'(x) = 100x^{99}$ $\quad \underline{f'(x) = 200x(x^2+2)^{99}}$

(c) $f(x) = \underbrace{x^{10}}_{g(x)} \underbrace{(x^2+1)^{10}}_{h(x)}$

$f(x) = g(x)h(x)$ Regla de Leibnitz

$f'(x) = g(x)h'(x) + g'(x)h(x)$

$g'(x) = 10x^9 \qquad h'(x) = 10(x^2+1)^9(2x) = 20x(x^2+1)^9$

$$f'(x) = x^{10}\,20x(x^2+1)^9 + 10x^9(x^2+1)^{10}$$

$$\underbrace{f'(x) = 20x^{11}(x^2+1)^9 + (x^2+1)^{10}10x^9}$$

$$f'(x) = \cancel{10}\cdot 2\,\cancel{x^9}\,x^2\,(x^2\cancel{+1})^9 + (x^2+1)(x^2\cancel{+1})^9\,\cancel{10}\,\cancel{x^9}$$

$$= 10x^9(x^2+1)^9\left[2x^2 + x^2 + 1\right] =$$

$$f'(x) = 10x^9(x^2+1)^9\left[3x^2+1\right]$$

(d) $f(x) = \frac{x}{\sqrt{x+1}}$.

$\cdot)\ x^{-m} = \frac{1}{x^m}$

$\cdot)\ x^{\frac{m}{n}} = \sqrt[n]{x^m}$

$f(x) = x \cdot \frac{1}{(x+1)^{1/2}} = x(x+1)^{-1/2}$

$f'(x) = x\left[-\frac{1}{2}(x+1)^{-3/2}(1)\right] + (1)(x+1)^{-1/2}$

$= -\frac{x}{2(x+1)^{3/2}} + \frac{1}{(x+1)^{1/2}} = -\frac{x}{2\sqrt{(x+1)^3}} + \frac{1}{\sqrt{x+1}}$

$= \frac{1}{\sqrt{x+1}} - \frac{x}{2(x+1)\sqrt{x+1}} = \frac{2(x+1) - x}{2(x+1)\sqrt{x+1}}$

$f'(x) = \frac{x+2}{2(x+1)\sqrt{x+1}}$

(e) $f(x) = \dfrac{x^3}{\sqrt{x^2-1}}$

$$g'(x) = 3x^2$$

$$h'(x) = -\frac{1}{\cancel{2}}(x^2-1)^{-3/2} \cdot \cancel{2}x$$

$$f(x) = \frac{x^3}{(x^2-1)^{1/2}} = \underbrace{x^3}_{g(x)} \cdot \underbrace{(x^2-1)^{-1/2}}_{h(x)}$$

$ab = ba$ $\quad a, b \in \mathbb{R}$

$$\cdot \frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

$$f'(x) = -x(x^2-1)^{-3/2} x^3 + 3x^2 (x^2-1)^{-1/2}$$

$$= \frac{-x^4}{\underbrace{(x^2-1)^{3/2}}_{(x^2-1)(x^2-1)^{1/2}}} + \frac{3x^2}{(x^2-1)^{1/2}} = \frac{-x^4}{\underbrace{(x^2-1)(x^2-1)^{1/2}}} + \frac{3x^2}{(x^2-1)^{1/2}}$$

$$1 + \frac{1}{2} = \frac{3}{2}$$

$$= \frac{-x^4 + 3x^2(x^2-1)}{(x^2-1)(x^2-1)^{1/2}}$$

$$f'(x) = \frac{-x^4 + 3x^4 - 3x^2}{(x^2-1)(x^2-1)^{1/2}} = \frac{2x^4 - 3x^2}{(x^2-1)\sqrt{x^2-1}}$$

(f) $y = x^{1/n}$ Deriva implícitamente $\boxed{dy/dx}$, $y = y(x)$

1) $y - x^{1/n} = 0$

$\frac{1}{n} - 1 = \frac{1-n}{n}$

2) $\frac{dy}{dx} - \frac{1}{n} x^{\frac{1}{n}-1} = 0$

3) $y' - \frac{1}{n} x^{\frac{1-n}{n}} = 0$

4) $\underbrace{y' = \frac{1}{n} x^{\frac{1-n}{n}}}$

$f(x) = y$, $f'(x) = y'$

$g(x) = x^{1/3}$, $g'(x) = \frac{1}{3} x^{-2/3}$

(g) $x^{1/3} + y^{1/3} = 1$

$g(f(x)) = g(y)$

$= y^{1/3}$

1) $x^{1/3} + \underbrace{y^{1/3}} - 1 = 0$

2) $\frac{1}{3} x^{-2/3} + \underbrace{\frac{1}{3} y^{-2/3} y'} = 0$

3) $y' = \underline{\qquad}$

$y^{1/3} = g(f(x))$

$\frac{dy^{1/3}}{dx} = g'(f(x)) f'(x) = \frac{1}{3} y^{-2/3} y'$

(h) $\sin x + \sin y = 1/2$

$\sin x + \sin y - 1/2 = 0$

$\cos x + \cos(y)y' = 0$

$$\boxed{y' = -\frac{\cos x}{\cos y}}$$

$\sin y = g(f(x))$

$f(x) = y$, $f'(x) = y'$

$g(x) = \sin x$, $g'(x) = \cos x$

$g(f(x)) = g(y) = \sin y$

$\sin(f(x)) = \sin(y) = h(x) = g(f(x))$

$h'(x) = g'(f(x))f'(x) = \cos(y)y'$

(i) $m = \dfrac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$ $\qquad \dfrac{dm}{dv}$, $m = f(v)$ Derivar implícitamente

factor de Lorentz. Relatividad especial

$= m_0 \gamma \qquad , \gamma \equiv \dfrac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \left(1 - v^2/c^2\right)^{-1/2}, \quad \gamma^2 = \dfrac{1}{\left(1-\frac{v^2}{c^2}\right)}$

$m = m_0 \gamma \qquad \dfrac{d\gamma}{dv} = -\cancel{\tfrac{1}{2}}\left(1-\dfrac{v^2}{c^2}\right)^{-3/2}\left(\dfrac{-\cancel{2}v}{c^2}\right) = \dfrac{v}{c^2}\left(1-\dfrac{v^2}{c^2}\right)^{-3/2}$

$m - m_0\gamma = 0$

$$= \frac{v}{c^2}\frac{1}{\left(1-\frac{v^2}{c^2}\right)^{3/2}}$$

$\dfrac{dm}{dv} - m_0\dfrac{d\gamma}{dv} = 0$

$$= \frac{v}{c^2}\frac{1}{\left(1-\frac{v^2}{c^2}\right)\left(1-\frac{v^2}{c^2}\right)^{1/2}}$$

$m' - m_0\gamma' = 0$

$\underbrace{m' = m_0 \dfrac{v}{c^2}\gamma^3} \qquad \gamma' = \dfrac{v}{c^2}\gamma\dfrac{1}{\left(1-v^2/c^2\right)} = \dfrac{v}{c^2}\gamma\gamma^2 = \dfrac{v}{c^2}\gamma^3$

(j) $f(x) = e^{-x^2}$, $g(x) = [\ln(x)]^2$

$$\not\Updownarrow$$

$h(x) = \ln(x^2)$

$f(x) = h(g(x))$ $\quad g(x) = x^2$, $\quad h(x) = e^{-x}$

$h(g(x)) = h(x^2) = e^{-x^2}$

$g'(x) = 2x$, $h'(x) = e^{-x}(-1) = -e^{-x}$

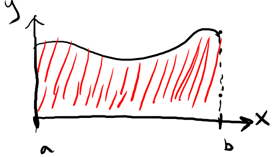
$f'(x) = h'(g(x))\,g'(x) = -e^{-x^2}\,2x = -2x\,e^{-x^2}$

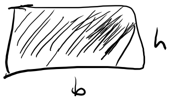
$g(x) = [\ln(x)]^2$, $\quad 2\ln(x)\,\frac{1}{x} = \frac{2}{x}\ln(x) = g'(x)$

$h(x) = \ln(x^2) = g(f(x))$, $\quad f(x) = x^2$, $g(x) = \ln(x)$

$g(f(x)) = g(x^2) = \ln(x^2)$, $\quad f'(x) = 2x$, $\quad g'(x) = \frac{1}{x}$

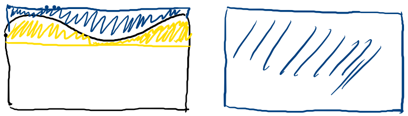
$h'(x) = g'(f(x))\,f'(x) = \frac{1}{x^2}\,2x = \frac{2}{x}$

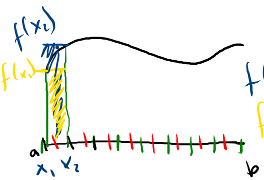




Riemann

$A(\square) = b \cdot h$





$f(x_2)(x_2 - x_1)$

$f(x_1)(x_2 - x_1)$

$\sum_{i=1}^{n} f(x_i)(x_i - x_{i-1}) > A(\ )$

$\sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1}) < A(\ )$

$$\lim_{x_i - x_{i-1} = 0} \sum_{i=1}^{n} f(x_i)\underline{(x_i - x_{i-1})} = \int_a^b f(x)\,dx$$

- $\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$
- $\int \cos x\,dx = \sin x + c$
- $\int \sin x\,dx = -\cos x + c$
- $\int e^x\,dx = e^x + c$
- $\int \frac{dx}{x} = \ln(x) + c$
- Cambio de variable

$u = x, \quad du = dx$

$dv = e^x, \quad v = e^x$

Int. por partes.

$$\int x e^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x + C = e^x(x-1) + c$$

Camb. de variable

$u = x + 1, \ du = dx$

$$\int \frac{1}{\sqrt{x+1}}\,dx \rightarrow \int \frac{1}{\sqrt{u}}\,du = \int u^{-1/2}\,du = \frac{u^{1/2}}{1/2} + c$$

$= 2u^{1/2} + c$

$= 2\sqrt{u} + C$

$\rightarrow 2\sqrt{x+1} + c$

$$\int \frac{dx}{\sqrt{x+1}} = \boxed{2\sqrt{x+1} + C}$$