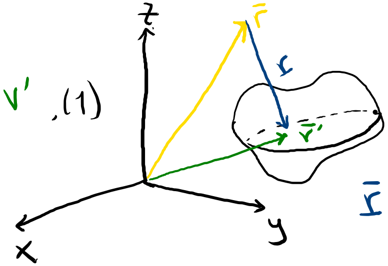
Problema 1.

$$\bar{E}(\bar{r}) = \frac{1}{4\pi\varepsilon_0}\int_{V'} \frac{\hat{\mathfrak{r}}}{\mathfrak{r}^2}\,\rho(\bar{r}')\,dv' \quad ,(1)$$



$\bar{\mathfrak{r}} = \bar{r} - \bar{r}'$

$\bar{r}$: la posición en donde se mide el campo eléctrico

$\bar{r}'$: punto donde se evalúa la densidad

$\rho(\bar{r}) \neq 0$ si $\bar{r} \in V$

$\rho(\bar{r}) = 0$ si $\bar{r} \notin V$

$$\bar{E}(\bar{r}) = \frac{1}{4\pi\varepsilon_0}\int_{\mathbb{R}^3} \frac{\hat{\mathfrak{r}}}{\mathfrak{r}^2}\,\rho(\bar{r}')\,dv'$$

$$\bar{\nabla}\cdot\bar{E}(\bar{r}) = \frac{1}{4\pi\varepsilon_0}\int_{\mathbb{R}^3} \nabla\cdot\left(\frac{\hat{\mathfrak{r}}}{\mathfrak{r}^2}\right)\rho(\bar{r}')\,dv' \qquad \bar{v} \equiv \frac{\hat{\mathfrak{r}}}{\mathfrak{r}^2} : \mathbb{R}^3 \to \mathbb{R}^3$$

$$\mathfrak{r} = \bar{r} - \bar{r}'$$

$$\nabla\cdot\left(\frac{\hat{\mathfrak{r}}}{\mathfrak{r}^2}\right)$$

$$\bar{\nabla}\cdot\bar{A} = \frac{\partial}{\partial x}A_x + \frac{\partial}{\partial y}A_y + \frac{\partial}{\partial z}A_z.$$

$$\bar{\nabla}\cdot\bar{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r) + \frac{1}{r\sin\theta}\frac{\partial(A_\theta\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial\varphi}$$

$$\bar{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{1}{r^2}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}(1) = 0.$$

Teorema de Gauss* establece que

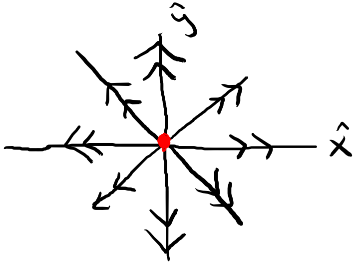
$$\int_V \bar{\nabla}\cdot\bar{A} = \oint_S \bar{A}\cdot d\bar{s}$$

. Consideremos una superficie esférica de radio R

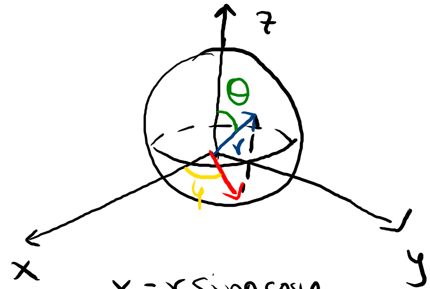
$$\oint_S \bar{A}\cdot d\bar{s} = \int_0^{2\pi}\int_0^{\pi}\left(\frac{\hat{r}}{\cancel{R^2}}\right)\cdot(\cancel{R^2}\sin\theta\,d\theta\,d\varphi\,\hat{r})$$

$$d\bar{S} = R^2\sin\theta\,d\theta\,d\varphi\,\hat{r}$$

$$= \int_0^{2\pi}\int_0^{\pi}\sin\theta\,d\theta\,d\varphi = 2(2\pi) = 4\pi.$$



$\frac{\hat{r}}{r^2}$



$x = r\sin\theta\cos\varphi$
$y = r\sin\theta\sin\varphi$
$z = r\cos\theta$

$\theta \in [0, \pi]$
$\varphi \in [0, 2\pi]$

El problema está en $r=0$

$\frac{\hat{r}}{r^2} \xrightarrow{r\to 0}$ "∞" $\oint_{R} A\cdot d\bar{s} = 4\pi$

$$\bar{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) = 0 \quad \forall\ \bar{r} \neq \bar{0}$$

Estas son las características de ua distribución conocida como la Delta de Dirac

$$\delta^3(\bar{r}) = \delta(x)\delta(y)\delta(z)$$

$$\delta(x) \equiv \begin{cases} 0 & \text{si } x \neq 0 \\ \infty & \text{si } x = 0 \end{cases}, \quad \int_{-\infty}^{\infty} \delta(x)\,dx = 1.$$

Por lo tanto $\nabla\cdot\left(\frac{\hat{r}}{r^2}\right) = 4\pi\delta^3(\bar{r})$, en términos de

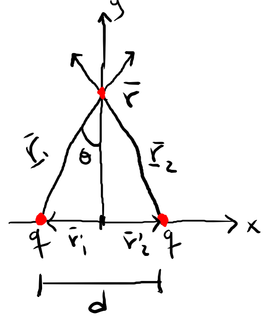
$\hat{\mathscr{r}}$: $\nabla\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = 4\pi\delta^3(\bar{\mathscr{r}})$

$$\nabla\cdot\bar{E}(\bar{r}) = \frac{1}{4\pi\varepsilon_0}\int_{\mathbb{R}^3} 4\pi\delta^3(\bar{\mathscr{r}})\rho(\bar{r}')\,dV' = \frac{1}{\cancel{4\pi}\varepsilon_0}\int_{\mathbb{R}^2} \cancel{4\pi}\,\delta(\bar{r}-\bar{r}')\rho(\bar{r}')\,dV' = \frac{1}{\varepsilon_0}\rho(\bar{r})$$

$$\boxed{\nabla\cdot\bar{E}(\bar{r}) = \frac{\rho(\bar{r})}{\varepsilon_0}}$$

Problema 2.

Vamos a identificar los vectores posición del problema



$$\cos\theta = \frac{z}{r}$$

$\bar{r} = (0,0,z)$

$\bar{r}_1' = (-\frac{d}{2},0,0)$ $\Rightarrow \bar{r}_1 = \bar{r} - \bar{r}_1' = (+\frac{d}{2},0,z)$

$\bar{r}_2' = (\frac{d}{2},0,0)$ $\quad \bar{r}_2 = \bar{r} - \bar{r}_2' = (-\frac{d}{2},0,z)$

El campo generado por una carga eléctrica puntual está dado por:

$$\bar{E}(\bar{r}) = \frac{1}{4\pi\varepsilon_0} q \frac{\hat{r}}{r^2} \qquad \hat{r} = \frac{\bar{r}}{r}$$

$$r_1 = \sqrt{\frac{d^2}{4} + z^2} = r_2$$

$$\bar{E}_1(\bar{r}) = \frac{1}{4\pi\varepsilon_0} q \frac{\hat{r}_1}{r_1^2} = \frac{1}{4\pi\varepsilon_0} q \frac{\bar{r}_1}{r_1^3}$$

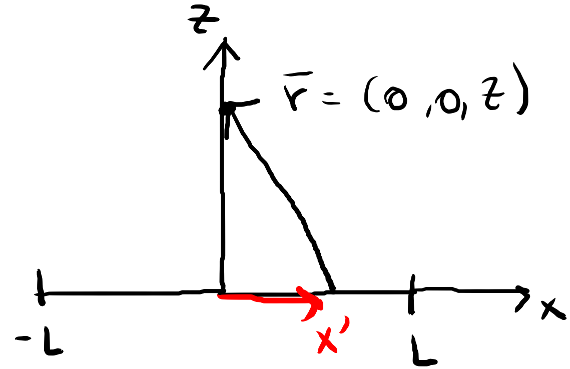
$$\bar{E}_2(\bar{r}) = \frac{1}{4\pi\varepsilon_0} q \frac{\bar{r}_2}{r_2^3}$$

$$\bar{E}_{Total}(\bar{r}) = \bar{E}_1(\bar{r}) + E_2(\bar{r}) = \frac{q}{4\pi\varepsilon_0} \frac{1}{r^3} [\bar{r}_1 + \bar{r}_2] = \frac{q}{4\pi\varepsilon_0} \frac{2z}{r^3} \hat{z}$$

$$\bar{E}_{total}(\bar{r}) = \frac{q}{4\pi\varepsilon_0} \frac{1}{r^2} \frac{2z}{r}$$

$$\boxed{\bar{E}_{total}(\bar{r}) = \frac{q\cos\theta}{2\pi\varepsilon_0 r^2} \hat{z}}$$

Problema 3.



$$\bar{E}(\bar{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\lambda(\bar{r}')}{\mathscr{r}^2}\hat{\mathscr{r}}\, dl'$$

$$dq = \lambda\, dl$$

$$\bar{r} = (0,0,z) \quad , \quad \bar{r}' = (x',0,0) \qquad \bar{r}-\bar{r}' = (-x', 0, z) = \bar{\mathscr{r}}$$

$$\mathscr{r} = \sqrt{x^2+z^2} \qquad dl' = dx'$$

$$\bar{E}(\bar{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\lambda\, dx'}{x^2+z^2}\frac{(-x',0,z)}{\sqrt{x'^2+z^2}}$$

$$① \quad E_x = \frac{\lambda}{4\pi\varepsilon_0}\int_{-L}^{L}\frac{-x'\,dx'}{(x^2+z^2)^{3/2}} \quad , \quad E_y = 0$$

$$E_z = \frac{\lambda z}{4\pi\varepsilon_0}\int_{-L}^{L}\frac{dx'}{(x^2+z^2)^{3/2}} \quad ②$$

① $E_x = \frac{\lambda}{4\pi\varepsilon_0}\int_{-L}^{L} \frac{-x'\,dx'}{\underbrace{(x'^2+z^2)^{3/2}}}$

$$f(x') = \frac{-x'}{(x'^2+z^2)^{3/2}}$$

$$-f(x') = f(-x')$$

$$f(-x') = \frac{x'}{(x'^2+z^2)^{3/2}}$$

$$-f(x') = \frac{x'}{(x'^2+z^2)^{3/2}} \Rightarrow E_x = 0$$

② $E_z = \frac{z\lambda}{4\pi\varepsilon_0}\int_{-L}^{L} \frac{dx'}{(x'^2+z^2)^{3/2}}$

La integral a resolver es de la forma

$$\int \frac{dx}{(x^2+a^2)^{3/2}} = \int \frac{dx}{a^3\left(\frac{x^2}{a^2}+1\right)^{3/2}} = \frac{1}{a^3}\int \frac{dx}{\left(1+\frac{x^2}{a^2}\right)^{3/2}}$$

$\underline{1+\tan^2\alpha = \sec^2\alpha}$, utilizamos el camb. de variable:

$$\frac{x^2}{a^2} = \tan^2\alpha \Rightarrow \boxed{\begin{aligned} x &= a\tan\alpha \\ dx &= a\sec^2\alpha\, d\alpha \end{aligned}}$$

$$\int \frac{dx}{(x^2+a^2)^{3/2}} \to \frac{1}{a^3}\int \frac{a\sec^2\alpha\, d\alpha}{\sec^3\alpha} \qquad \sec\alpha \equiv \frac{1}{\cos\alpha}$$

$$= \frac{1}{a^2}\int \cos\alpha\, d\alpha = \frac{1}{a^2}\sin\alpha$$

Para regresar a la variable original, notamos que

$$x = a\tan\alpha \Rightarrow x = a\frac{\sin\alpha}{\cos\alpha}, \quad \underline{\sin\alpha = \frac{x\cos\alpha}{a}}$$

$$\sin^2\alpha + \cos^2\alpha = 1 \Rightarrow \frac{\sin^2\alpha}{\cos^2\alpha} + 1 = \frac{1}{\cos^2\alpha}$$

$$\tan^2\alpha + 1 = \frac{1}{\cos^2\alpha} \Rightarrow \cos\alpha = \frac{1}{\sqrt{1+\tan^2\alpha}} = \frac{1}{\sqrt{1+\frac{x^2}{a^2}}}$$

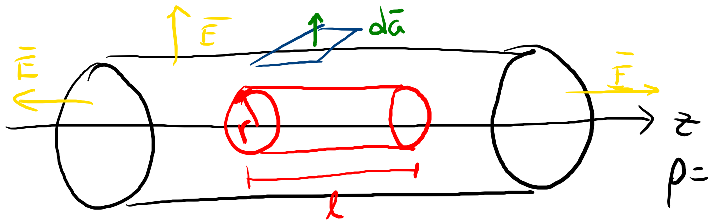
$$\int \frac{dx}{(x^2+a^2)^{3/2}} = \frac{1}{a^3}x\cos\alpha = \frac{x}{a^3}\frac{1}{\sqrt{1+x^2/a^2}} = \frac{x}{a^2}\frac{1}{\sqrt{a^2+x^2}}$$

$$E_z = \frac{\lambda \cancel{z}}{4\pi\varepsilon_0}\left[\frac{x}{z^{\cancel{2}}\sqrt{z^2+a^2}}\right]\Bigg|_{-L}^{L} = \frac{\lambda}{4\pi\varepsilon_0 z}\left[\frac{L}{\sqrt{L^2+z^2}} - \frac{(-L)}{\sqrt{L^2+z^2}}\right]$$

$$= \frac{L\lambda}{2\pi\varepsilon_0 z\sqrt{L^2+z^2}}$$

$$\underline{E(\vec{r}) = \frac{L\lambda}{2\pi\varepsilon_0 z\sqrt{L^2+z^2}}\hat{z}}$$

Problema 4.



Solución:

$\rho = kr$

Ley de Gauss:

$$\oint_S \bar{E}\cdot d\bar{a} = \frac{Q_{enc}}{\varepsilon_0} \qquad Q_{enc} = \int \rho\, dv \,,\quad dv = r'\,dr'\,d\phi'\,dz'$$

$r' \in [0, r]$

$\phi' \in [0, 2\pi]$

$z' \in [0, l]$

$$Q_{enc} = \int (kr')(r'\,dr'\,d\phi'\,dz')$$

$$= k\int_0^{\ell}\int_0^{2\pi}\int_0^{r} r'^2\,dr'\,d\phi'\,dz' = k\,\ell\,2\pi\int_0^{r} r'^2\,dr'$$

$$\boxed{Q_{enc} = \frac{k\,\ell\,2\pi}{3}\,r^3}$$

$$\oint_S \bar{E}\cdot d\bar{a} = \oint_S |E|\,da\,\cancel{\cos\theta}^{\,1} \qquad \bar{E} \parallel d\bar{a}\,,\ \theta = 0$$

$$= |E|\int da = |E|\,2\pi r\ell$$

$$|E|\,2\pi r \ell = \frac{k\,\ell\,2\pi\,r^3}{3\,\varepsilon_0}$$

$$|E| = \frac{k r^2}{3\varepsilon_0}\,,\qquad \boxed{\bar{E} = \frac{k r^2}{3\varepsilon_0}\,\hat{r}}$$