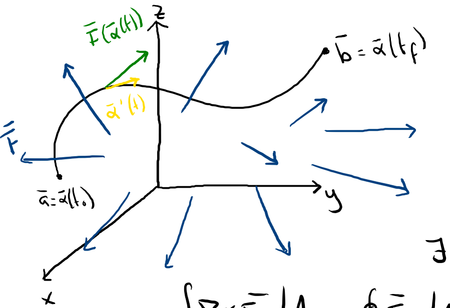
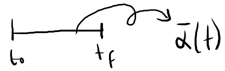
① Integral de trayectoria. $\bar{F}:\mathbb{R}^3 \to \mathbb{R}^3$



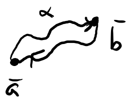


$$\int_{t_0}^{t_f} \bar{F}(\bar{\alpha}(t)) \cdot \bar{\alpha}'(t)\,dt = \int_{\bar{a}}^{\bar{b}} \bar{F} \cdot d\bar{\alpha}$$

$\nabla \times \bar{E} = 0$ Campo gradiente

$\exists\ f:\mathbb{R}^3 \to \mathbb{R}$ $\quad \boxed{\bar{E} = \nabla f}$

$$\int_S \nabla \times \bar{E}\, dA = \oint \bar{E} \cdot dl = 0$$



$V(\bar{r}) = -\int_{O}^{\bar{r}} \bar{E} \cdot d\bar{l}$: El potencial eléctrico

$$V(\bar{b}) - V(\bar{a}) = -\int_{O}^{\bar{b}} \bar{E} \cdot d\bar{l} + \int_{O}^{\bar{a}} \bar{E} \cdot d\bar{l} = -\int_{O}^{\bar{b}} \bar{E} \cdot d\bar{l} - \int_{\bar{a}}^{O} \bar{E} \cdot d\bar{l}$$

$$= -\int_{\bar{a}}^{\bar{b}} \bar{E} \cdot d\bar{l}$$

Por otro lado, por el teorema fundamental del cálculo

$$V(\bar{b}) - V(\bar{a}) = \int_{\bar{a}}^{\bar{b}} (\nabla V) \cdot d\bar{l}$$

Por lo tanto,

$$\int_{\bar{a}}^{\bar{b}} (\nabla V) \cdot d\bar{l} = -\int_{\bar{a}}^{\bar{b}} \bar{E} \cdot d\bar{l}$$

$$\nabla V = -\bar{E}$$

$$\boxed{\bar{E} = -\nabla V}$$

La diferencia de potencial, ΔV se mide en N.m. C o bien en J. C ≡ Volt.

$$W = \int_{\bar{a}}^{\bar{b}} \bar{F} \cdot d\bar{l} = -q \int_{\bar{a}}^{\bar{b}} \bar{E} \cdot d\bar{l} = q \int_{\bar{a}}^{\bar{b}} (\nabla V) \cdot d\bar{l} = q\left[V(\bar{b}) - V(\bar{a})\right]$$

$$W = q\, \Delta V$$

$$\boxed{\Delta V = \frac{W}{q}}$$

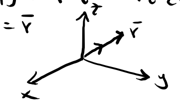
Problema 2

(a) $\bar{E} = k[xy\hat{x} + 2yz\hat{y} + 3xz\hat{z}]$

(b) $\bar{E} = k[y^2\hat{x} + (2xy+z^2)\hat{y} + 2yz\hat{z}]$

(a) $V(\bar{r}) = -\int_0^{\bar{r}} \bar{E}\cdot d\bar{l}$ $\quad \bar{r} = (x,y,z)$ $\quad \bar{\alpha}(t) = \bar{r}t$, $t\in[0,1]$

$\bar{\alpha}'(t) = \bar{r}$



$\bar{E}(\bar{\alpha}(t)) = k[xyt^2\hat{x} + 2yzt^2\hat{y} + 3xzt^2\hat{z}]$

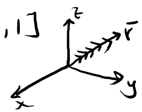
$\bar{E}(\bar{\alpha}(t)) = kt^2[xy\hat{x} + 2yz\hat{y} + 3xz\hat{z}]$

$\bar{E}(\bar{\alpha}(t))\cdot\bar{\alpha}'(t) = kt^2[x^2y + 2y^2z + 3xz^2]$

$V(\bar{r}) = -\int_0^{\bar{r}} \bar{E}\cdot d\bar{l} = -k[x^2y+2y^2z+3xz^2]\int_0^1 t^2\,dt$

$= -\frac{k}{3}[x^2y+2y^2z+3xz^2]$

Utilicemos otra trayectoria $\bar{\beta}(t) = \bar{r}t^2$, $\bar{\beta}'(t) = 2t\bar{r}$ $t\in[0,1]$



$\bar{E}(\bar{\beta}(t)) = k t^4[xy\hat{x} + 2yz\hat{y} + 3xz\hat{z}]$

$\bar{E}(\bar{\beta}(t))\cdot\bar{\beta}'(t) = 2t^5[x^2y + 2y^2z + 3xz^2]k$

$V(\bar{r}) = -\int_0^{\bar{r}} \bar{E}\cdot d\bar{l} = -2k[x^2y+2y^2z+3xz^2]\int_0^1 t^5\,dt$

$= -\frac{2k}{6}[x^2y+2y^2z+3xz^2] = -\frac{k}{3}[x^2y+2y^2z+3xz^2]$

$-\nabla V(\bar{r}) = -\frac{k}{3}(2xy+3z^2,\ x^2+4yz,\ 2y^2+6xz)$ X no es un campo eléctrico

(b) $\bar{E} = k[y^2\hat{x} + (2xy+z^2)\hat{y} + 2yz\hat{z}]$ $\quad \bar{\alpha}(t) = \bar{r}t$, $\bar{\alpha}'(t) = \bar{r}$ $t\in[0,1]$

$V(\bar{r}) = -\int_0^1 \bar{E}(\bar{\alpha}(t))\cdot\bar{\alpha}'(t)\,dt$

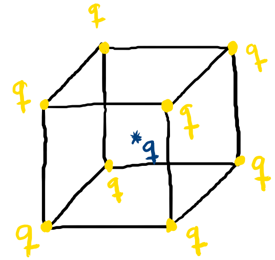
$\bar{E}(\bar{\alpha}(t)) = k[t^2y^2\hat{x} + t^2(2xy+z^2)y + 2t^2yz\hat{z}]$

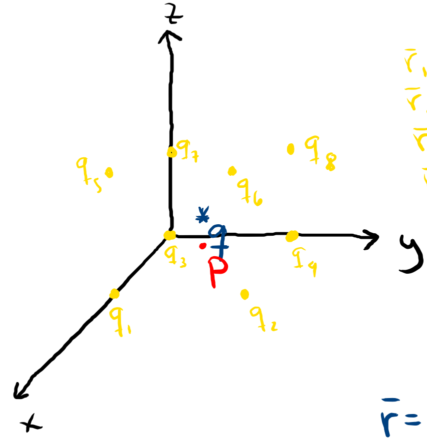
$\bar{E}(\bar{\alpha}(t))\cdot\bar{\alpha}'(t) = kt^2[y^2x + (2xy+z^2)y + 2yz^2]$

$V(\bar{r}) = -\frac{k}{3}[y^2x + (2xy+z^2)y + 2yz^2] = -k[xy^2 + yz^2] = V(\bar{r})$

$-\nabla V = k[y^2\hat{x} + (2yx+z^2)\hat{y} + 2zy\hat{z}] = \bar{E}(\bar{r})$

Problema 3





$\bar{r}_1' = (\ell,0,0)$
$\bar{r}_2' = (\ell,\ell,0)$
$\bar{r}_3' = (0,0,0)$
$\bar{r}_4' = (0,\ell,0)$
$\bar{r}_5' = (\ell,0,\ell)$
$\bar{r}_6' = (\ell,\ell,\ell)$
$\bar{r}_7' = (0,0,\ell)$
$\bar{r}_8' = (0,\ell,\ell)$

$\bar{r} = 1/2(\ell,\ell,\ell)$

$$r_i = \|\bar{r} - \bar{r}_i'\| = \frac{\ell\sqrt{3}}{2}$$

$$V_{total} = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{8}\frac{q}{r_i} = \frac{8q}{4\pi\varepsilon_0}\,\frac{2}{\ell\sqrt{3}} = \frac{4q}{\pi\varepsilon_0 \ell\sqrt{3}}.$$

$$\bar{P} = \left(\frac{\ell}{2}, \frac{\ell}{2}, z\right)$$

$$r_i = \|\bar{P} - \bar{r}_i'\| = \left\|\left(-\frac{\ell}{2}, \frac{\ell}{2}, z\right)\right\| = \sqrt{z^2 + \frac{\ell^2}{2}} \quad i \in \{1,2,3,4\}$$

$$r_i = \|\bar{P} - \bar{r}_i'\| = \left\|\left(-\frac{\ell}{2}, \frac{\ell}{2}, z - \frac{\ell}{2}\right)\right\| = \sqrt{(\ell - z)^2 + \frac{\ell^2}{2}} \quad i \in \{5,6,7,8\}$$

$$V_{total}(z) = \frac{q}{4\pi\varepsilon_0}\left[\frac{4}{\sqrt{z^2 + \frac{\ell^2}{2}}} + \frac{4}{\sqrt{(\ell - z)^2 + \frac{\ell^2}{2}}}\right]$$

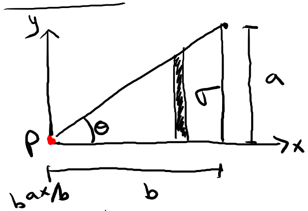
$$V_{total}\left(z = \frac{\ell}{2}\right) = \frac{q}{\pi\varepsilon_0}\left[\frac{1}{\sqrt{\frac{3\ell^2}{4}}} + \frac{1}{\sqrt{\frac{3\ell^2}{4}}}\right] = \frac{4q}{\pi\varepsilon_0\sqrt{3}\,\ell}.$$

$$V_{total}(z = 0) = \frac{q}{\pi\varepsilon_0}\left[\frac{\sqrt{2}}{\ell} + \frac{\sqrt{2}}{\ell\sqrt{3}}\right] = \frac{q\sqrt{2}}{\pi\varepsilon_0 \ell}\left[1 + \frac{1}{\sqrt{3}}\right].$$

Problema 4

$\sin\theta = \frac{a}{\sqrt{a^2+b^2}}$

$\cos\theta = \frac{b}{\sqrt{a^2+b^2}}$



$$V(\bar r) = \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma(\bar r')\, dA'}{r}$$

$r = \|\bar r - \bar r'\|$ $\quad \bar r = \bar 0 = (0,0,0)$

$\bar r' = (x, y, 0)$, $r = \|(-x,-y,0)\| =$

$y = \frac{ax}{b}$

$x \in [0, b]$

$x = 0$

$y = 0$

$y \in [0, \frac{ax}{b}]$

$x = b$

$y = a$

$$V(\bar r) = \frac{\sigma}{4\pi\varepsilon_0}\int_0\int_0 \frac{dy\,dx}{\sqrt{x^2+y^2}}$$

La integral respecto a la variable y es:

$$\int \frac{dy}{\sqrt{x^2+y^2}} = \frac{1}{x}\int \frac{dy}{\sqrt{1+\frac{y^2}{x^2}}}$$

$$\boxed{\begin{array}{l} \frac{y^2}{x^2} = \tan^2\alpha \\ y = x\tan\alpha \quad dy = x\sec^2\alpha\, d\alpha \end{array}}$$

$$\rightarrow \frac{1}{\cancel{x}}\int \frac{\cancel{x}\sec^2\alpha\, d\alpha}{\sec\alpha} = \int \sec\alpha\, d\alpha = \int \frac{\sec\alpha(\sec\alpha+\tan\alpha)}{\sec\alpha+\tan\alpha}\, d\alpha$$

$$= \int \frac{\sec^2\alpha + \sec\alpha\tan\alpha}{\sec\alpha+\tan\alpha}\, d\alpha$$

$$\boxed{\begin{array}{l} u = \sec\alpha + \tan\alpha \\ du = (\sec^2\alpha + \sec\alpha\tan\alpha)\, d\alpha \end{array}}$$

$$\rightarrow \int \frac{du}{u} = \ln(u)$$

$\tan\alpha = \frac{y}{x}$ $\quad \sec\alpha = \sqrt{1+\frac{y^2}{x^2}}$

$$\ln(u) \rightarrow \ln(\sec\alpha+\tan\alpha) \rightarrow \ln\left(\sqrt{\frac{x^2+y^2}{x^2}} + \frac{y}{x}\right) = \ln\left(\frac{1}{x}\left(\sqrt{x^2+y^2}+y\right)\right)$$

$$V(\bar r) = \frac{\sigma}{4\pi\varepsilon_0}\int_0^b \ln\left[\frac{1}{x}\left(\sqrt{x^2+y^2}+y\right)\right]\Bigg|_0^{\frac{ax}{b}} dx$$

$$= \frac{\sigma}{4\pi\varepsilon_0}\int_0^b \ln\left(\sqrt{1+\frac{a^2}{b^2}} + \frac{a}{b}\right) dx$$

$$= \frac{\sigma b}{4\pi\varepsilon_0}\ln\left(\frac{\sqrt{a^2+b^2}+a}{b}\right) = \frac{\sigma b}{4\pi\varepsilon_0}\ln\left(\frac{1+\frac{a}{\sqrt{a^2+b^2}}}{\frac{b}{\sqrt{a^2+2^2}}}\right)$$

$$\boxed{V(\theta) = \frac{\sigma b}{4\pi\varepsilon_0}\ln\left(\frac{1+\sin\theta}{\cos\theta}\right)}$$