Math238 Formula Sheet

Part 1. Linear System, Augmented Matrix, Gauss-Jordan Elimination

1 A system of linear equations is said to be homogeneous if the constant terms are all zero.

2. Elementary Row Operations:
   - Interchange two rows
   - Multiply a row by some nonzero constant • Add a multiple of a row to another row.

3. This matrix which have following properties is in reduced row-echelon form
   1. All zero rows are at the bottom.
   2. The first non-zero entry of every non-zero row is a 1 (leading one).
   3. Leading ones go from left to right.
   4. All entries above and below any leading one are zero.

4. Gauss-Jordan Elimination is a procedure of a series of elementary row operations which will reduce the augmented matrix to a reduced row-echelon form.

Part 2. Matrices

1. A matrix is a rectangular array of numbers. The numbers in the array are called the entries in the matrix.
2. Two matrices are defined to be equal if they have the same size and their corresponding entries are equal.
3. If A and B are matrices of the same size, then the sum A+(or -)B is the matrix obtained by adding (or subtracting) the entries of B to the corresponding entries of A.
4. If A is any matrix and c is any scalar, then the product cA is the matrix obtained by multiplying each entry of the matrix A by c.
5. If A is an m×r matrix and B is an r×n matrix, then the product AB is the m×n matrix whose entry in row i and column j of AB, single out row i from the matrix A and column j from the matrix B . Multiply the corresponding entries from the row and column together and then add up the resulting products.
6. **Transpose**: The transpose operator $A^\top$ swaps rows and columns. If $A \in \mathbb{R}^{m\times n}$ then $A^\top \in \mathbb{R}^{n\times m}$ and $(A^\top)_{ij} = A_{ji}$.
   - $(A^\top)^\top = A$.
   - $(AB)^\top = B^\top A^\top$.
7. If A is a square matrix, and if a matrix B of the same size can be found such that AB=BA=I , then A is said to be ***invertible*** and B is called an ***inverse*** of A . If no such matrix B can be found, then A is said to be ***singular*** .

$$(AB)^{-1} = B^{-1}A^{-1}$$

8. *If* A *is an invertible* n×n *matrix ,then for each* n×1 *matrix* **b** *,the system of equations* A**X** =**b** *has exactly one solution ,namely,* $X = A^{-1}b$.
9.

The matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible if $ad - bc \neq 0$, *in which case the inverse is given by the formula*

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} \dfrac{d}{ad - bc} & -\dfrac{b}{ad - bc} \\ -\dfrac{c}{ad - bc} & \dfrac{a}{ad - bc} \end{bmatrix}$$

10. Theorem:
    (*a*) *The transpose of a lower triangular matrix is upper triangular, and the transpose of an upper triangular matrix is lower triangular.*
    (*b*) *The product of lower triangular matrices is lower triangular, and the product of upper triangular matrices is upper triangular.*
    (*c*) *A triangular matrix is invertible if and only if its diagonal entries are all nonzero.*
    (*d*) *The inverse of an invertible lower triangular matrix is lower triangular, and the inverse of an invertible upper triangular matrix is upper triangular.*

Part 3. Determinant

1. Cofactor expansion: $\det(A) = a_{j1}C_{j1} + a_{j2}C_{j2} + \cdots . + a_{jn}C_{jn}$
2. If A is an nxn triangular matrix (upper triangular, lower triangular, or diagonal), then det(A) is the product of the entries on the main diagonal of the matrix ; that is, det(A) = $a_{11}a_{22}\ldots a_{nn}$.

Properties of the determinant function

Given an $n \times n$ matrix A

- If A has a zero row or zero column then $\det(A) = 0$.
- If we get matrix B by interchanging two rows of A then $\det(B) = -\det(A)$.
- If we get matrix B by multipying one row of A by $k \neq 0$ then $\det(B) = k\det(A)$.
- If we get matrix B by adding a multiple of a row to another of matrix A then $\det(B) = \det(A)$.
- $\det(kA) = k^n \det(A)$.
- $\det(A^T) = \det(A)$.
- $\det(AB) = \det(A)\det(B)$
- $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

Part 4. Vectors in 2-Space, 3-Space, and n-Space

Definition(Dot Product)

If $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ then the **dot product** of $\mathbf{u}$ and $\mathbf{v}$ is defined by

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n$$

If θ is the angle between $\mathbf{u}$ and $\mathbf{v}$ then $\mathbf{u} \cdot \mathbf{v} = ||\mathbf{u}||\ ||\mathbf{v}|| \cos\theta$

Two vectors are orthogonal to each other if and only if their dot product is zero.

Point-normal equations of the line and plane:

$$a(x - x_0) + b(y - y_0) = 0 \quad \textbf{[line]}$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \quad \textbf{[plane]}$$

THEOREM 3.3.1

(*a*) *If a and b are constants that are not both zero, then an equation of the form*

$$ax + by + c = 0$$

represents a line in R^2 *with normal* $\mathbf{n} = (a, b)$.

(*b*) *If a, b, and c are constants that are not all zero, then an equation of the form*

$$ax + by + cz + d = 0$$

represents a plane in R^3 *with normal* $\mathbf{n} = (a, b, c)$.

***Vector forms* of a line and plane:**

Lines: If $\mathbf{x}_0$ and $\mathbf{v}$ are vectors in R^n, and if $\mathbf{v}$ is nonzero, then the equation

$$\mathbf{x} = \mathbf{x}_0 + t\mathbf{v}$$

defines the ***line through*** $\mathbf{x}_0$ ***that is parallel to*** $\mathbf{v}$. In the special case where $\mathbf{x}_0 = \mathbf{0}$, the line is said to ***pass through the origin***.

Planes: If $\mathbf{x}_0$, $\mathbf{v}_1$, and $\mathbf{v}_2$ are vectors in R^n, and if $\mathbf{v}_1$ and $\mathbf{v}_2$ are not collinear, then the equation

$$\mathbf{x} = \mathbf{x}_0 + t_1\mathbf{v}_1 + t_2\mathbf{v}_2$$

defines the ***plane through*** $\mathbf{x}_0$ ***that is parallel to*** $\mathbf{v}_1$ ***and*** $\mathbf{v}_2$. In the special case where $\mathbf{x}_0 = \mathbf{0}$, the plane is said to ***pass through the origin***.

Projection:

$$\text{proj}_{\mathbf{a}}\mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a} \quad (\textit{vector component of } \mathbf{u} \textit{ along } \mathbf{a})$$

$$\mathbf{u} - \text{proj}_{\mathbf{a}}\mathbf{u} = \mathbf{u} - \frac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a} \quad (\textit{vector component of } \mathbf{u} \textit{ orthogonal to } \mathbf{a})$$

Distance:

(*a*) *In* R^2 *the distance* D *between the point* $P_0(x_0, y_0)$ *and the line* $ax + by + c = 0$ *is*

$$D = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$$

(*b*) *In* R^3 *the distance* D *between the point* $P_0(x_0, y_0, z_0)$ *and the plane* $ax + by + cz + d = 0$ *is*

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Cross Product If $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ are vectors in 3-space, then the ***cross product*** $\mathbf{u} \times \mathbf{v}$ is the vector defined by

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1)$$

or, in determinant notation,

$$\mathbf{u} \times \mathbf{v} = \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right)$$

THEOREM 3.5.1 Relationships Involving Cross Product and Dot Product

If $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are vectors in 3-space, then*

(*a*) $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0$ ($\mathbf{u} \times \mathbf{v}$ *is orthogonal to* $\mathbf{u}$)

(*b*) $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0$ ($\mathbf{u} \times \mathbf{v}$ *is orthogonal to* $\mathbf{v}$)

(*c*) $\|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2$ (*Lagrange's identity*)

(*d*) $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$ (*relationship between cross and dot products*)

(*e*) $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u}$ (*relationship between cross and dot products*)

THEOREM 3.5.2 Properties of Cross Product

If $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are any vectors in 3-space and* k *is any scalar, then:*

(*a*) $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$

(*b*) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$

(*c*) $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w})$

(*d*) $k(\mathbf{u} \times \mathbf{v}) = (k\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (k\mathbf{v})$

(*e*) $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$

(*f*) $\mathbf{u} \times \mathbf{u} = \mathbf{0}$

Area of a Parallelogram and Volumn of the parallelepiped

(*a*) *The absolute value of the determinant*

$$\det \begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}$$

is equal to the area of the parallelogram in 2-space determined by the vectors $\mathbf{u} = (u_1, u_2)$ *and* $\mathbf{v} = (v_1, v_2)$. *(See Figure 3.4.7a.)*

(*b*) *The absolute value of the determinant*

$$\det \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}$$

is equal to the volume of the parallelepiped in 3-space determined by the vectors $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$, *and* $\mathbf{w} = (w_1, w_2, w_3)$. *(See Figure 3.4.7b.)*

Part 5. Real Vector Spaces

1.Vector space:

DEFINITION Let V be an arbitrary nonempty set of objects on which two operatior are defined: addition, and multiplication by numbers called ***scalars***. By ***addition*** w mean a rule for associating with each pair of objects $\mathbf{u}$ and $\mathbf{v}$ in V an object $\mathbf{u}+$ called the ***sum*** of $\mathbf{u}$ and $\mathbf{v}$; by ***scalar multiplication*** we mean a rule for associating wit each scalar k and each object $\mathbf{u}$ in V an object $k\mathbf{u}$, called the ***scalar multiple*** of $\mathbf{u}$ by If the following axioms are satisfied by all objects $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ in V and all scalars k an m, then we call V a ***vector space*** and we call the objects in V ***vectors***.

1. If $\mathbf{u}$ and $\mathbf{v}$ are objects in V, then $\mathbf{u}+\mathbf{v}$ is in V.
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$
3. $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$
4. There is an object $\mathbf{0}$ in V, called a ***zero vector*** for V, such that $\mathbf{0}+\mathbf{u}=\mathbf{u}+\mathbf{0}=$ for all $\mathbf{u}$ in V.
5. For each $\mathbf{u}$ in V, there is an object $-\mathbf{u}$ in V, called a ***negative*** of $\mathbf{u}$, such th: $\mathbf{u}+(-\mathbf{u})=(-\mathbf{u})+\mathbf{u}=\mathbf{0}$.
6. If k is any scalar and $\mathbf{u}$ is any object in V, then $k\mathbf{u}$ is in V.
7. $k(\mathbf{u}+\mathbf{v})=k\mathbf{u}+k\mathbf{v}$
8. $(k+m)\mathbf{u}=k\mathbf{u}+m\mathbf{u}$
9. $k(m\mathbf{u})=(km)(\mathbf{u})$
10. $1\mathbf{u}=\mathbf{u}$

Subspace: A subset W of a vector space V is called a ***subspace*** of V if W is itself a vector space under the addition and scalar multiplication defined on V.

Linear Independence:

DEFINITION If $S=\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_r\}$ is a set of two or more vectors in a vector space V, then S is said to be a ***linearly independent set*** if no vector in S can be expressed as a linear combination of the others. A set that is not linearly independent is said to be ***linearly dependent***.

THEOREM *A nonempty set $S=\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_r\}$ in a vector space V is linearly independent if and only if the only coefficients satisfying the vector equation*

$$k_1\mathbf{v}_1+k_2\mathbf{v}_2+\cdots+k_r\mathbf{v}_r=\mathbf{0}$$

are $k_1=0, k_2=0,\ldots,k_r=0$.

Definition(Basis)

A basis of a subspace S of $\mathbb{R}^n$ is a set of vectors that span S and are linearly independant.

Theorem. Let V be a finite-dimensional vector space and $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ any basis.

a) If a set has more than n vector, then it is linearly dependent.
b) If a set has fewer than n vector, then it does not span V.

THEOREM 4.5.3 Plus/Minus Theorem

Let S be a nonempty set of vectors in a vector space V.

(a) *If S is a linearly independent set, and if $\mathbf{v}$ is a vector in V that is outside of span(S), then the set $S\cup\{\mathbf{v}\}$ that results by inserting $\mathbf{v}$ into S is still linearly independent.*

(b) *If $\mathbf{v}$ is a vector in S that is expressible as a linear combination of other vectors in S, and if $S-\{\mathbf{v}\}$ denotes the set obtained by removing $\mathbf{v}$ from S, then S and $S-\{\mathbf{v}\}$ span the same space; that is,*

$$\operatorname{span}(S)=\operatorname{span}(S-\{\mathbf{v}\})$$

Theorem: Let S be finite set of vectors in a finite-dimensional vector space V.

a) If S spans V but is not a basis for V, then S can be reduced to a basis for V by removing appropriate vectors from S.

b) If S is a linearly independent set that is not already a basis for V, then S can be enlarged to a basis for V by inserting appropriate vectors into S.

Change of Basis:

$$[\text{new basis} \mid \text{old basis}] \xrightarrow{\text{row operations}} [I \mid \text{transition from old to new}]$$

If we change the basis for a vector space V from an old basis B to a new basis B', suppose P is the transition matrix, then for each vector **v** in V, the old coordinate vector $[\mathbf{v}]_B$ is related to the new coordinate vector $[\mathbf{v}]_{B'}$ by the equation
$[\mathbf{v}]_{B'} = P[\mathbf{v}]_B$

<u>**Definition**</u>(Row Space, Column Space, Null Space)
Let A be an $m \times n$ matrix,

- The row space of $A = \text{span}(\text{Rows of } A)$.
- The Column space of $A = \text{span}(\text{Columns of } A)$.
- The Null space is the subspace of $\mathbb{R}^n$ spanned by the solutions of the homogeneous system $A\mathbf{x} = \mathbf{0}$.

<u>**Definition**</u>(Rank)
The rank of a matrix A (denoted by rank(A))is the dimension of its row space (or column space since they're equal)

<u>**Definition**</u>(Nullity)
The nullity of a matrix A (denoted by nullity(A)), is the dimension of its Null space.

<u>**Theorem**</u> (*The Rank Theorem*)
For any $A_{m\times n}$,

$$\text{rank}(A) + \text{nullity}(A) = n.$$

Part. 6. Linear Transformation, Eigenvalue and Eigenvectors

Definition: A matrix transformation or linear transformation T_A: $R^n \to R^m$ is defined by equations of the form $Y = AX$, $Y \in R^m$, $X \in R^n$, A is mxn matrix. The matrix A=$[a_{ij}]$ is called the standard matrix for the linear transformation.

DEFINITION 1 If A is an $n \times n$ matrix, then a nonzero vector $\mathbf{x}$ in R^n is called an ***eigenvector*** of A (or of the matrix operator T_A) if $A\mathbf{x}$ is a scalar multiple of $\mathbf{x}$; that is,

$$A\mathbf{x} = \lambda\mathbf{x}$$

for some scalar λ. The scalar λ is called an ***eigenvalue*** of A (or of T_A), and $\mathbf{x}$ is said to be an ***eigenvector corresponding to*** $\boldsymbol{\lambda}$.

THEOREM 5.1.1 *If A is an $n \times n$ matrix, then λ is an eigenvalue of A if and only if it satisfies the equation*

$$\det(\lambda I - A) = 0 \tag{1}$$

*This is called the **characteristic equation** of A.*

Theorem: If A is an n×n triangular matrix (upper triangular, low triangular, or diagonal), then the eigenvalues of A are entries on the main diagonal of A.

THEOREM 5.1.3 *If A is an $n \times n$ matrix, the following statements are equivalent.*

(*a*) *λ is an eigenvalue of A.*

(*b*) *λ is a solution of the characteristic equation $\det(\lambda I - A) = 0$.*

(*c*) *The system of equations $(\lambda I - A)\mathbf{x} = \mathbf{0}$ has nontrivial solutions.*

(*d*) *There is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$.*

Definition(Similar Matrices)
Given A and B two $n \times n$ matrices. A is said to be similar to B (written $A \sim B$) if there is an invertible matrix P such that $P^{-1}AP = B$.

Definition(Diagonalizable matrix)
An $n \times n$ matrix A is diagonalizable if there is a diagonal matrix D that is similar to A.
i.e. If there is a diagonal matrix D and an invertible matrix P such that $D = P^{-1}AP$.

Theorem: If an n×n matrix A has n distinct eigenvalues, then A is diagonalizable.

THEOREM 5.1.5 Equivalent Statements

If A is an $n \times n$ matrix, then the following statements are equivalent.

(*a*) *A is invertible.*

(*b*) *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution.*

(*c*) *The reduced row echelon form of A is I_n.*

(*d*) *A is expressible as a product of elementary matrices.*

(*e*) *$A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.*

(*f*) *$A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$.*

(*g*) $\det(A) \neq 0$.

(*h*) *The column vectors of A are linearly independent.*

(*i*) *The row vectors of A are linearly independent.*

(*j*) *The column vectors of A span R^n.*

(*k*) *The row vectors of A span R^n.*

(*l*) *The column vectors of A form a basis for R^n.*

(*m*) *The row vectors of A form a basis for R^n.*

(*n*) *A has rank n.*

(*o*) *A has nullity 0.*

(*p*) *The orthogonal complement of the null space of A is R^n.*

(*q*) *The orthogonal complement of the row space of A is $\{\mathbf{0}\}$.*

(*r*) *The kernel of T_A is $\{\mathbf{0}\}$.*

(*s*) *The range of T_A is R^n.*

(*t*) *T_A is one-to-one.*

(*u*) *$\lambda = 0$ is not an eigenvalue of A.*