Unit 6 Stochastic Process

Definition and classification; Markov Process: Markov chain, Matrix approach, Steady- State distribution; counting process: Binomial process, Poisson process; Simulation of stochastic process; Queuing system: Main component of queuing system, Little's law; Bernoulli single server queuing process: system with limited capacity; M/M/1 system: Evaluating the system performance.

Definitions and classifications:

DEFINITION: A stochastic process is a random variable that also depends on time. It is therefore a function of two arguments, $X(t, \omega)$,

Where; • $t \in T$ is time, with T being a set of possible times, usually $[0,\infty)$, $(-\infty,\infty)$, $\{0,1,2,...\}$, or $\{...,-2,-1,0,1,2,...\}$;

• $\omega \in \Omega$, as before, is an outcome of an experiment, with Ω being the whole sample space.

Values of $X(t,\omega)$ are called states.

DEFINITION: Stochastic process $X(t, \omega)$ is discrete-state if variable $X_t(\omega)$ is discrete for each time t, and it is a continuous-state if $X_t(\omega)$ is continuous.

DEFINITION: Stochastic process $X(t, \omega)$ is a discrete-time process if the set of times T is discrete, that is, it consists of separate, isolated points. It is a continuous-time process if T is a connected, possibly unbounded interval.

Markov processes:

Stochastic process X(t) is Markov if for any $t_1 < t_2 < ... < t_n < t$ and any sets A: $A_1, A_2, ..., A_n$. $P\{X(t) \in A \mid X(t_1) \in A_1,..., X(t_n) \in A_n\} = P\{X(t) \in A \mid X(t_n) \in A_n\}$.

In other words, knowing the present, we get no information from the past that can be used to predict the future,

$$P\{\text{future} \mid \text{past, present}\} = P\{\text{future} \mid \text{present}\}$$

Then, for the future development of a Markov process, only its present state is important, and it does not matter how the process arrived to this state.

Markov chain:

DEFINITION: A Markov chain is a discrete-time, discrete-state Markov stochastic process.

Introduce a few convenient simplifications. The time is discrete, so let us define the time set as $T = \{0, 1, 2,...\}$. We can then look at a Markov chain as a random sequence $\{X(0), X(1), X(2),...\}$.

The state set is also discrete, so let us enumerate the states as 1, 2,..., n. Sometimes we'll start enumeration from state 0, and sometimes we'll deal with a Markov chain with infinitely many (discrete) states, then we'll have $n = \infty$.

The Markov property means that only the value of X(t) matters for predicting X(t + 1), so the conditional probability $p_{ij}(t) = P\{X(t + 1) = j \mid X(t) = i\}$

$$= P\{X(t + 1) = j \mid X(t) = i, X(t-1) = h, X(t-2) = g,...\}$$

depends on i, j, and t only and equals the probability for the Markov chain X to make a transition from state i to state j at time t.

Transition probability:

DEFINITION: Probability $p_{ij}(t) = P\{X(t + 1) = j \mid X(t) = i\}$

$$= P\{X(t + 1) = j \mid X(t) = i, X(t-1) = h, X(t-2) = g,...\}$$

is called a transition probability. Probability $p_{ij}^{(h)}(t) = P\{X(t + h) = j \mid X(t) = i\}$ of moving from state i to state j by means of h transitions is an h-step transition probability.

In other word: A Markov chain is homogeneous if all its transition probabilities are independent of t. Being homogeneous means that transition from i to j has the same probability at any time. Then $p_{ij}(t) = p_{ij}$ and $p_{ij}^{(h)}(t) = p_{ij}^{(h)}$.

<u>Characteristics of a Markov chain:</u>

What do we need to know to describe a Markov chain?

By the Markov property, each next state should be predicted from the previous state only. Therefore, it is sufficient to know the distribution of its initial state X(0) and the mechanism of transitions from one state to another.

The distribution of a Markov chain is completely determined by the initial distribution P0 and one-step transition probabilities pij. Here P0 is the probability mass function of X_0,

$$P_0(x) = P\{X(0) = x\} \text{ for } x \in \{1,2,...,n\}$$

Based on this data, we would like to compute:

- h-step transition probabilities $p_{ij}^{(h)}$;
- P_h, the distribution of states at time h, which is our forecast for X (h);
- the limit of $p_{ij}^{(h)}$ and Ph as $h \to \infty$, which is our long-term forecast.

Indeed, when making forecasts for many transitions ahead, computations will become rather lengthy, and thus, it will be more efficient to take the limit.

$P_{ij} = P\{X(t + 1) = j \mid X(t) = i\}$, transition probability

$P_{ij}^{(h)} = P\{X(t + h) = j \mid X(t) = i\}$, h-step transition probability

$P_t(x) = P\{X(t) = x\}$, distribution of X (t), distribution of states at time t

$P_0(x) = P\{X(0) = x\}$, initial distribution

Thank you!!!