<u>Steady-state distribution:</u>

DEFINITION:

A collection of limiting probabilities $\pi_x = \lim_{h \to 0} P_h(x)$ is called a steady-state distribution of a Markov chain X (t).

When limit exist, it can be used as a forecast of the distribution of X after many transitions. When steady state distribution exists $\pi P = \pi$ and $\sum_x \pi_x = 1$.

<u>Example 6:</u>

Obtain steady state distribution of a Markov Chain having transition probability matrix P = $\begin{bmatrix} 0.2 & 0.8 \\ 0.5 & 0.5 \end{bmatrix}$.

<u>Solution:</u> Here;

$$P = \begin{bmatrix} 0.2 & 0.8 \\ 0.5 & 0.5 \end{bmatrix}.$$

Let $\pi = (\pi_1 \quad \pi_2)$

Now, $\pi P = \pi$

Or, $[\pi_1 \quad \pi_2]\begin{bmatrix} 0.2 & 0.8 \\ 0.5 & 0.5 \end{bmatrix} = [\pi_1 \quad \pi_2]$

Or, $[0.2\pi_1 + 0.5\pi_2 \quad 0.8\pi_1 + 0.5\pi_2] = [\pi_1 \quad \pi_2]$

Hence, $0.2\pi_1 + 0.5\pi_2 = \pi_1$ ----------(i)

$0.8\pi_1 + 0.5\pi_2 = \pi_2$ ----------(ii)

From (i), we get

$0.5\pi_2 = \pi_1 - 0.2\pi_1$

Or; $0.5\pi_2 = 0.8\pi_1$

Or; $\pi_2 = 8/5\ \pi_1$

$= 1.6\pi_1$

Since, $\pi_1 + \pi_2 = 1$

Or; $\pi_1 + 1.6\pi_1 = 1$

Or; $2.6\pi_1 = 1$

Or; $\pi_1 = 1/2.6 = 5/13$

$\pi_2 = \frac{1.6}{2.6} = \frac{8}{13}$.

Hence, in long-run probability of state 1 is $\frac{5}{13}$ and state 2 is $\frac{8}{13}$.

Example 7: (Shared device, continued). In Example 6.9 on p. 139, Obtain steady state distribution of a Markov Chain having the transition probability matrix for the number of concurrent users is

$$P = \begin{bmatrix} \mathbf{0.64} & \mathbf{0.32} & \mathbf{0.04} \\ \mathbf{0.40} & \mathbf{0.50} & \mathbf{0.10} \\ \mathbf{0.25} & \mathbf{0.50} & \mathbf{0.25} \end{bmatrix}$$

Solution: Here;

$$P = \begin{bmatrix} 0.64 & 0.32 & 0.04 \\ 0.40 & 0.50 & 0.10 \\ 0.25 & 0.50 & 0.25 \end{bmatrix}$$

The steady state equation for the Markov Chain

Let $\pi = (\pi_0 \quad \pi_1 \quad \pi_2)$

Now, $\pi P = \pi$

Or, $[\pi_0 \quad \pi_1 \quad \pi_2] \begin{bmatrix} 0.64 & 0.32 & 0.04 \\ 0.40 & 0.50 & 0.10 \\ 0.25 & 0.50 & 0.25 \end{bmatrix} = [\pi_0 \quad \pi_1 \quad \pi_2]$

Or, $[0.64\pi_0+0.40\pi_1+0.25\pi_2 \quad 0.32\pi_0+0.50\pi_1+0.50\pi_2 \quad 0.04\pi_0+0.10\pi_1+0.25\pi_2]=[\pi_0 \quad \pi_1 \quad \pi_2]$

Hence, $0.64\pi_0 + 0.40\pi_1 + 0.25\pi_2 = \pi_0$ ----------(i)

$0.32\pi_0 + 0.50\pi_1 + 0.50\pi_2 = \pi_1$ ----------(ii)

$0.04\pi_0 + 0.10\pi_1 + 0.25\pi_2 = \pi_2$ -----------(iii)

From equation (i), we have

$0.40\pi_1 + 0.25\pi_2 = 0.36\pi_0$

Or, $0.40\pi_1 = 0.36\pi_0 – 0.25\pi_2$

Or, $\pi_1 = (0.36\pi_0 – 0.25\pi_2)/0.40$------------------(iv)

From equation (ii), we have

$0.32\pi_0 + 0.50\pi_1 + 0.50\pi_2 = \pi_1$

Or, $0.50\pi_1 = 0.32\pi_0 + 0.50\pi_2$

Or, $\pi_1 = (0.32\pi_0 + 0.50\pi_2)/0.50$-----------(v)

From equation (iii), we have

$0.04\pi_0 + 0.10\pi_1 + 0.25\pi_2 = \pi_2$

Or, $0.10\pi_1 = 0.75\pi_2 - 0.04\pi_0$

Or, $\pi_1 = 7.5\pi_2 – 0.4\pi_0$ ----------------(vi)

Solving equation (iv) and (v), we get

$(0.36\pi_0 – 0.25\pi_2)/0.40 = (0.32\pi_0 + 0.50\pi_2)/0.50$

Or, $0.9\pi_0 – 0.625\pi_2 = 0.64\pi_0 + \pi_2$

Or, $0.26\pi_0 = 1.625\pi_2$

Or, $\pi_0 = 6.25\pi_2$ ------------------(vii)

Solving equation (vi) and (vii), we get

$\pi_1 = 7.5\pi_2 – 0.4\pi_0$

Or, $\pi_1 = 7.5\pi_2 – 0.4 \times 6.25\pi_2$

Or, $\pi_1 = 7.5\pi_2 – 0.4 \times 6.25\pi_2$

Or, , $\pi_1 = 5\pi_2$ ----------(viii)

Since, $\pi_0 + \pi_1 + \pi_2 = 1$

Or, $6.25\pi_2 + 5\pi_2 + \pi_2 = 1$

Or, $12.25\pi_2 = 1$

Or, $\pi_2 = 0.0816$

From equation (viii), we get

$\pi_1 = 5\pi_2 = 5 \times 0.0816 = 0.4082$

From equation (vii), we get

$\pi_0 = 6.25\pi_2 = 6.25 \times 0.0816 = 0.5102$

Hence, in long-run probability of state 0 is $\mathbf{0.5102}$, state 1 is 0.4082 and state 2 is 0.0816.

Counting process:

A large number of situations can be described by counting processes. As becomes clear from their name, they count. What they count differs greatly from one process to another. These may be counts of arrived jobs, completed tasks, transmitted messages, detected errors, scored goals, and so on.

DEFINITION: A stochastic process X is counting if X (t) is the number of items counted by the time t.

As time passes, one can count additional items; therefore, sample paths of a counting process are always non-decreasing. Also, counts are nonnegative integers, $X(t) \in \{0, 1, 2, 3, ...\}$. Hence, all counting processes are discrete-state.

A stochastic process X is called counting if x (t) is the number of items counted by the time t. On passes of time additional items cannot be counted. Counts are non negative integers. $X(t) \in \{0, 1, 2, ----\}$. Hence counting are discrete state. Hence $\{X(t), t \geq 0\}$ is called a counting process if

(i) $X(0) = 0$.

(ii) For $0<t_1<t_2< ------- < t_n <t$

$$X(0) < X(t_1) < X(t_2) < ------- < X(t_n) < X(t)$$

(iii) $X(t_i) - X(t_{(i-1)})$ denotes the number of events in (t_{i-1}, t_i)

(iv) $X(t_i) - X(t_{(i-1)})$ are independently distributed.

Example: (E-mails and attachments). Figure 6.6 shows sample paths of two counting process, X(t) being the number of transmitted e-mails by the time t and A(t) being the number of transmitted attachments. According to the graphs, e-mails were transmitted at t = 8, 22, 30, 32, 35, 40, 41, 50, 52, and 57 min. The e-mail counting process X(t) increments by 1 at each of these times. Only 3 of these e-mails contained attachments. One attachment was sent at t = 8, five more at t = 35, making the total of A (35) = 6, and two more attachments at t = 50, making the total of A(50) = 8.

Two classes of counting processes will be discussed in detail, a discrete-time **Binomial process** and a continuous-time **Poisson process**.

Binomial process:

Binomial process X (n) is the number of successes in the first n independent Bernoulli trials, where n = 0,1,2,....

It is a discrete-time discrete-space counting stochastic process. Moreover, it is Markov, and therefore, a Markov chain.

λ = arrival rate

Δ = frame size

p = probability of arrival (success) during one frame (trial)

$X(t/\Delta)$ = number of arrivals by the time t

T = inter arrival time

The inter arrival period consists of a Geometric number of frames Y, each frame taking Δ seconds. Hence the inter arrival time can be computed as

$$T = Y\Delta$$

It is a rescaled Geometric random variable taking possible values Δ, 2Δ, 3Δ, etc. Its expectation and variance are

$\lambda = p/\Delta$

$n = t/\Delta$

$X(n) = \text{Binomial}(n, p)$

$Y = \text{Geometric}(p)$

$T = Y\Delta$

$E(T) = E(Y\Delta) = \Delta E(Y) = \Delta/p = 1/\lambda$

$\text{Var}(T) = V(Y\Delta) = \Delta^2 V(Y) = (1-p)\left(\frac{\Delta}{P}\right)^2 = \frac{1-P}{\lambda^2}$.

Example 7:

Suppose that a number of defects coming from an assembly line can be modeled as a Binomial counting process with frames of one-half-minute length and probability $p = 0.02$ of a defect during each frame.

i. Find the probability of going more than 20 minutes without a defect.
ii. Determine the arrival rate in units of defects per hour.
iii. If the process is stopped for inspection each time a defect is found, on average how long will the process run until it is stopped?

Solution: Here;

Let X_n = Number of defects in n frames.

$P = 0.02$

$\Delta = 0.5$ minutes.

T = time between two successive defects.

Now;

(i) For $t = 20$ minutes

$n = \frac{t}{\Delta} = \frac{20}{0.5} = 40$

$P\{X(40) = 0\} = C(40, 0)\,(0.02)^0\,(1-0.02)^{40} = 0.446$

(ii) Since 1 hour has 120 frames, λ = no. of defects per hour = n p = 120x0.02 = 2.4 defects per hour.

(iii) $E(T) = \frac{\Delta}{\lambda} = \frac{0.5}{0.02} = 250.5$

Example 8:

Customers come to a self-service gas station at the rate of 20 per hour. Their arrivals are modeled by a Binomial counting process.

i. How many frames per hour should we choose, and what should be the length of each frame if the probability of an arrival during each frame is to be 0.05?
ii. With this frames, find the expected value and standard deviation of the time between arrivals at the gas station.

Solution: Here;

Arrival rate $\lambda = 20\ \text{hr}^{-1}$

(i) $p = 0.05$

$\Delta = p/\lambda = 0.05/20\ hr^{-1} = (1/400)$ hrs = 9 seconds.

Also, n = number of frames in 1 hr = $1hr/\Delta = 400$ frames.

(ii) Let T = inter-arrival time. E (T) = $\Delta/P = 9/0.05 = 180$ SEC = 180/60 = 3 min. SD (T) = $(\Delta/P)\sqrt{1-P} = 180\sqrt{1-0.05} = 175.44$ SEC. = 175.44/60 = 2.92 min.

Example 9:

Jobs are sent to a mainframe computer at a rate of 4 jobs per minute. Arrivals are modeled by a Binomial counting process.

i. Choose a frame size that makes the probability of a new jobs received during each frame equal to 0.1
ii. Using the chosen frame compute the probability of more than 4 jobs received during one minute
iii. What is probability of more than 20 jobs during 5 minutes
iv. What is average inter arrival time and variance?
v. What is probability that next job does not arrive during next 30 seconds?

Solution: Here;

$\lambda = 4$ jobs per minute, $p = 0.1$

(i) $\Delta = p/\lambda = 0.1/4 = 0.025$ min

For $t = 1$, $n = t/\Delta = 1/0.025 = 40$ frames.

$n = 40$, $p = 0.1$

(ii) $P\{X(n) > 4\} = 1 - P\{X(n) \leq 4\}$

$= 1 - [\sum_{x=0}^{4} 40_{C_x}(0.1)^x 0.9^{40-x}]$

$= 1 - 4\ (0.9)^{40} + 40(0.1)(0.9)^{39} + 780(0.1)^2(0.9)^{38}$

$+ 9880(0.1)^3\ (0.9)^{37} + 91350(0.1)^4(0.9)^{36}$

$= 0.37$

(iii) $P\{X(n) > 20\} = P\{X(n) \geq 20.5\}$; Using continuity correction

$= 1 - P\{X(n) \leq 20.5\}$

$= 1 - P\{\frac{X(n)-\mu}{\sigma} \leq \frac{20.5-\mu}{\sigma}\}$

$= P\left\{\frac{X(n)-np}{\sqrt{npq}} > \frac{20.5-200x0.1}{\sqrt{200x0.1x0.9}}\right\}$

$= P(Z > 0.12)$

$= 0.5 - P(0 < Z < 0.12)$

$= 0.5 - 0.0478$

$= 0.4522$

(iv) E (T) = 1/ λ = ¼ = 0.25 min = 15 sec.

$V(T) = \frac{1-p}{\lambda^2} = \frac{0.9}{4^2} = 0.056$

(v) T = Y Δ = 0.025 Y

P (T>30 sec)= P (T>0.5 min)

= P [0.025Y>0.5]

= P (Y>20)

$= \sum_{K=21}^{\infty}(1-P)^{K-1}P$

Example 10:

Jobs are sent to a mainframe computer at a rate of 2 jobs per minute. Arrivals are modeled by a Binomial counting process.

(a) Choose such a frame size that makes the probability of a new job during each frame equal 0.1.

(b) Using the chosen frames, compute the probability of more than 3 jobs received during one minute.
(c) Compute the probability of more than 30 jobs during 10 minutes. (d) What is the average interarrival time, and what is the variance?

(e) Compute the probability that the next job does not arrive during the next 30 seconds

Solution:

(a) We have,

λ = 2 min−1 and p = 0.1. Then

Δ = p/ λ

= 0.05 min or 3 sec.

(b) During t = 1 min, we have n = t/Δ = 20 frames. The number of jobs during this time is Binomial(n = 20,p = 0.1). From Table A2 (in the Appendix),

P {X(n) > 3}= 1−P {X(n) ≤3}= 1−0.8670 = 0.1330.

(c) Here, n = 10/0.05 = 200 frames, and we use Normal approximation to Binomial(n,p) distribution (recall p. 94). With a proper continuity correction, and using Table A4,

P {X(n) > 30} = P {X(n) > 30.5}

$= P\left\{\frac{\boldsymbol{X(n)-np}}{\sqrt{\boldsymbol{np(1-P)}}} > \frac{\boldsymbol{30.5-(200)(0.1)}}{\sqrt{\boldsymbol{(200)(0.1)(1-0.1)}}}\right\}$

= P {Z > 2.48}= 1−0.9934 = 0.0066.

Comparing questions (b) and (c), notice that 3 jobs during 1 minute is not the same as 30 jobs during 10 minutes!

(d) E(T) = 1/λ = 1/2 min or 30 sec. Intuitively, this is rather clear because the jobs arrive at a rate of two per minute. The interarrival time has variance

$\mathrm{Var}(T) = \frac{\boldsymbol{1-P}}{\boldsymbol{\lambda^2}} = \frac{\boldsymbol{0.9}}{\boldsymbol{2^2}} = 0.225$

(e) For the interarrival time T = Y Δ = Y (0.05) and a Geometric variable Y ,

P {T > 0.5 min} = P {Y (0.05) > 0.5}

= P {Y > 10}

$= \sum_{\boldsymbol{K=11}}^{\infty}\boldsymbol{(1-P)^{K-1}P}$

$= (1-P)^{10} = (0.9)^{10}$

= 0.3138

Alternatively, this is also the probability of 0 arrivals during n = t/Δ = 0.5/0.05 = 10 frames, which is also $(1-p)^{10}$.

Thank you!!!

Poisson Process:

DEFINITION: Poisson process is a continuous-time counting stochastic process obtained from a Binomial counting process when its frame size Δ decreases to 0 while the arrival rate λ remains constant.

Let $X(t)$ = No. of arrivals occurring until time t.

T= inter arrival time

T_k= the of k^{th} arrival

$X(t) = \text{Poisson}(\lambda t)$

$T = \text{Exponential}(\lambda)$

$T_k = \text{Gamma}(k, \lambda)$

$E\,X(t) = n\,p = \frac{tp}{\Delta} = \lambda t$

$V\,X(t) = \lambda t \qquad F_T(t) = 1 - e^{-\lambda t}$

Probability of k^{th} arrival before time t

$P\{T_k \leq t\} = P\{X(t) \geq k\}$

$P\{T_k > t\} = P\{X(t) < k\}$

Example 10:

Customers arrive at a shop at the rate of 2 per minute. Find (i) expected number of customers in a 5 min period (iii) the variance of the number of customers in the same period (iii) the probability that there will be at least one customer.

Solution: Here;

Number of hits $k = 5000$, $\lambda = 5\ \text{min}^{-1}$

Expected time $= \frac{k}{\lambda} = \frac{5000}{5} = 1000$ minutes

Standard deviation $(\sigma) = \frac{\sqrt{k}}{\lambda} = 14.14$

$$P\{T_k < 12\text{ hr.}\} = P\{T_k < 720\}$$

$$= P\left\{\frac{T_k - \mu}{\sigma} < \frac{720 - \mu}{\sigma}\right\}$$

$$= P\left\{Z < \frac{720 - 1000}{14.14}\right\}$$

$$= P(Z < -19.44)$$

$$= 0$$

Example 11:

Customers arrive at a shop at the rate of 2 per minute. Find (i) expected number of customers in a 5 minute period (ii) the variance of the number of customers in the same period (iii) the probability that there will be at least one customer.

Solution: Here;

$\lambda = 2$

$t = 5$

(i) $E(X) = \lambda t = 5 \times 2 = 10$

(ii) $V(X) = \lambda t = 5 \times 2 = 10$

(iii) $P\{X(5) \geq 1\} = 1 - P\{X(5) < 1\}$

$= 1 - P\{X(5) = 0\}$

$= 1 - e^{-10}$

$= 0.999$

Example 12:

Shipments of paper arrive at a printing shop according to a Poisson process at a rate of 0.5 shipments per day.

(i) Find the probability that the printing shop receives more than two shipments in a day.

(ii) If there are more than four days between shipments, all the paper will be used up and the presses will be idle. What is the probability that this will happen?

Solution: Here;

Arrival time; $\lambda = 0.5$ per day

X (t) = No. of arrival (shipments) in t days, it is Poisson (0.5 t)

T = Inter-arrival time measured in days, it is Exponential (0.5).

(i) $P[X(1) > 2] = 1 - P[X(1) \leq 2]$

$= 1 - [P[X(1) = 0] + P[X(1) = 1] + P[X(1) = 2]]$

$= 1 - \left[\frac{e^{-0.5}(0.5)^0}{0!} + \frac{e^{-0.5}(0.5)^1}{1!} + \frac{e^{-0.5}(0.5)^2}{2!}\right]$

$= 1 - e^{-0.5}[1 + 0.5 + 0.125]$

$= 1 - 0.6065 \times 1.625$

$= 0.014.$

(ii) $P[T > 4] = \int_4^\infty 0.5e^{-0.5\,t}\, dt$

$= 0.5\left[\frac{e^{-0.5\,t}}{-0.5}\right]_4^\infty$

$= e^{-0.5 \times 4}$

$$= 0.135$$