Poisson Process:

DEFINITION: Poisson process is a continuous-time counting stochastic process obtained from a Binomial counting process when its frame size Δ decreases to 0 while the arrival rate λ remains constant.

Let $X(t)$ = No. of arrivals occurring until time t.

T= inter arrival time

T_k= the of k^{th} arrival

$X(t) = \text{Poisson}(\lambda t)$

$T = \text{Exponential}(\lambda)$

$T_k = \text{Gamma}(k, \lambda)$

$E\,X(t) = n\,p = \frac{tp}{\Delta} = \lambda t$

$V\,X(t) = \lambda t \qquad F_T(t) = 1 - e^{-\lambda t}$

Probability of k^{th} arrival before time t

$P\{T_k \le t\} = P\{X(t) \ge k\}$

$P\{T_k > t\} = P\{X(t) < k\}$

Example 10:

Customers arrive at a shop at the rate of 2 per minute. Find (i) expected number of customers in a 5 min period (iii) the variance of the number of customers in the same period (iii) the probability that there will be at least one customer.

Solution: Here;

Number of hits $k = 5000, \lambda = 5\ \text{min}^{-1}$

Expected time $= \frac{k}{\lambda} = \frac{5000}{5} = 1000$ minutes

Standard deviation $(\sigma) = \frac{\sqrt{k}}{\lambda} = 14.14$

$$P\{T_k < 12\text{ hr.}\} = P\{T_k < 720\}$$

$$= P\left\{\frac{T_k-\mu}{\sigma} < \frac{720-\mu}{\sigma}\right\}$$

$$= P\left\{Z < \frac{720-1000}{14.14}\right\}$$

$$= P(Z<-19.44)$$

$$= 0$$

Example 11:

Customers arrive at a shop at the rate of 2 per minute. Find (i) expected number of customers in a 5 minute period (ii) the variance of the number of customers in the same period (iii) the probability that there will be at least one customer.

Solution: Here;

$\lambda = 2$

$t = 5$

(i) $E(X) = \lambda t = 5x2 = 10$

(ii) $V(X) = \lambda t = 5x2 = 10$

(iii) $P\{X(5) \geq 1\} = 1 - P\{X(5) < 1\}$

$= 1 - P\{X(5) = 0\}$

$= 1 - e^{-10}$

$= 0.999$

Example 12:

Shipments of paper arrive at a printing shop according to a Poisson process at a rate of 0.5 shipments per day.

(i) Find the probability that the printing shop receives more than two shipments in a day.

(ii) If there are more than four days between shipments, all the paper will be used up and the presses will be idle. What is the probability that this will happen?

Solution: Here;

Arrival time; $\lambda = 0.5$ per day

$X(t)$ = No. of arrival (shipments) in t days, it is Poisson (0.5 t)

T = Inter-arrival time measured in days, it is Exponential (0.5).

(i) $P[X(1) > 2] = 1 - P[X(1) \leq 2]$

$= 1 - [P[X(1) = 0] + P[X(1) = 1] + P[X(1) = 2]]$

$= 1 - \left[\frac{e^{-0.5}(0.5)^0}{0!} + \frac{e^{-0.5}(0.5)^1}{1!} + \frac{e^{-0.5}(0.5)^2}{2!}\right]$

$= 1 - e^{-0.5}[1 + 0.5 + 0.125]$

$= 1 - 0.6065 \times 1.625$

$= 0.014.$

(ii) $P[T > 4] = \int_4^{\infty} 0.5e^{-0.5\,t}\, dt$

$= 0.5\left[\frac{e^{-0.5\,t}}{-0.5}\right]_4^{\infty}$

$= e^{-0.5 \times 4}$

$= 0.135$