Date: 2078 – 04 – 10 (Sunday)

Example 1: In some town, each day is either sunny or rainy. A sunny day is followed by another sunny day with probability 0.7, whereas a rainy day is followed by a sunny day with probability 0.4.

It rains on Monday. Make forecasts for Tuesday, Wednesday, and Thursday.

Solution: Weather conditions in this problem represent a homogeneous Markov chain with 2 states: state 1 = "sunny" and state 2 = "rainy." Transition probabilities are:

$$P_{11} = 0.7,\ P_{12} = 0.3,\ P_{21} = 0.4,\ P_{22} = 0.6,$$

where P_{12} and P_{22} were computed by the complement rule. If it rains on Monday, then Tuesday is sunny with probability $P_{21} = 0.4$ (making a transition from a rainy to a sunny day), and Tuesday is rainy with probability $P_{22} = 0.6$. We can predict a 60% chance of rain.

Wednesday forecast requires 2-step transition probabilities, making one transition from Monday to Tuesday, X(0) to X(1), and another one from Tuesday to Wednesday, X(1) to X(2). We'll have to condition on the weather situation on Tuesday and use the Law of Total Probability from p. 31,

$$\begin{aligned}
P_{21}^{(2)} &= P\{\text{Wednesday is sunny} \mid \text{Monday is rainy}\}\\
&= \sum_{i=1}^{2} P\{X(1) = i \mid X(0) = 2\} P\{X(2) = 1 \mid X(1) = i\}\\
&= P\{X(1) = 1 \mid X(0) = 2\} P\{X(2) = 1 \mid X(1) = 1\} + P\{X(1) = 2 \mid X(0) = 2\} P\{X(2) = 1 \mid X(1) = 2\}\\
&= P_{21} P_{11} + P_{22} P_{21} = (0.4)(0.7) + (0.6)(0.4) = 0.52.
\end{aligned}$$

By the Complement Rule, $P^{(2)}_{22} = 0.48$, and thus, we predict a 52% chance of sun and a 48% chance of rain on Wednesday.

For the Thursday forecast, we need to compute 3-step transition probabilities p(3) ij because it takes 3 transitions to move from Monday to Thursday. We have to use the Law of Total Probability conditioning on both Tuesday and Wednesday. For example, going from rainy Monday to sunny Thursday means going from rainy Monday to either rainy or sunny Tuesday, then to either rainy or sunny Wednesday, and finally, to sunny Thursday,

$$P_{21}^{(3)} = \sum_{i=1}^{2} \sum_{j=1}^{2} \boldsymbol{P_{2i} P_{ij} P_{j1}}$$

This corresponds to a sequence of states 2 → i → j → 1. However, we have already computed 2-step transition probabilities $P_{21}^{(2)}$ and $P_{22}^{(2)}$, describing transition from Monday to Wednesday. It remains to add one transition to Thursday, hence,

$$P_{21}^{(3)} = P^{(2)}_{21} P_{11} + P^{(2)}_{22} P_{21} = (0.52)(0.7) + (0.48)(0.4) = 0.556.$$

$$[P_{21}^{(3)} = \sum_{i=1}^{2} [\boldsymbol{P_{2i} P_{i1} P_{11} + P_{2i} P_{i2} P_{21}}]$$

$$= \boldsymbol{P_{21} P_{11} P_{11} + P_{21} P_{12} P_{21} + P_{22} P_{21} P_{11} + P_{22} P_{22} P_{21}}]$$

So, we predict a 55.6% chance of sun on Thursday and a 44.4% chance of rain.

The following transition diagram reflects the behavior of this Markov chain. Arrows represent all possible one-step transitions, along with the corresponding probabilities. Check this diagram against the transition probabilities stated in Example 6.7. To obtain, say, a 3-step transition probability $p_{2\,1}^{(2)}$; find all 2-arrow paths from state 2 "rainy" to state 1 "sunny." Multiply probabilities along each path and add over all 2-step paths

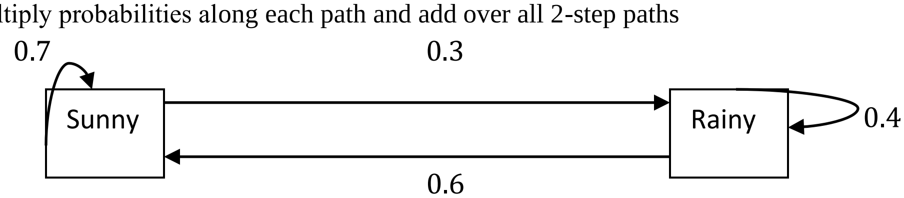


FIGURE 6.2: Transition diagram for the Markov chain in Example 6.7.

Let sunny = 1, Rainy = 2

Transition probability matrix P = $\begin{bmatrix} 0.7 & 0.3 \\ 0.6 & 0.4 \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$; $p_{11} + p_{12}$=1 & $p_{21} + p_{22}$=1

Example 6.8 (Weather, continued): Suppose now that it does not rain yet, but meteorologists predict an 80% chance of rain on Monday. How does this affect our forecasts?

In Example 6.7, we have computed forecasts under the condition of rain on Monday. Now, a sunny Monday (state 1) is also possible. Therefore, in addition to probabilities $p^{(h)}_{2j}$ we also need to compute $p^{(h)}_{1j}$ (say, using the transition diagram, see Figure 6.2),

$P^{(2)}_{11}$= (0.7)(0.7) + (0.3)(0.4) = 0.61,

$P^{(3)}_{11}$= (0.7)3 + (0.7)(0.3)(0.4) + (0.3)(0.4)(0.7) + (0.3)(0.6)(0.4) = 0.583.

The initial distribution $P_0(x)$ is given as,

$P_0(1)$ = P { sunny Monday } = 0.2, $P_0(2)$ = P { rainy Monday }= 0.8.

Then, for each forecast, we use the Law of Total Probability, conditioning on the weather on Monday,

$P_1(1) = P\{X(1) = 1\} = P_0(1)p_{11} + P_0(2)p_{21} = 0.46$ for Tuesday

$P_2(1) = P\{X(2) = 1\} = P_0(1)p^{(2)}_{11} + P_0(2)p^{(2)}_{21} = 0.538$ for Wednesday

$P_3(1) = P\{X(3) = 1\} = P_0(1)p^{(3)}_{11} + P_0(2)p^{(3)}_{21} = 0.5614$ for Thursday

These are probabilities of a sunny day (state 1), respectively, on Tuesday, Wednesday, and Thursday. Then, the chance of rain (state 2) on these days is $p_1(2) = 0.54$, $P_2(2) = 0.462$, and $P_3(2) = 0.4386$.

Matrix approach:

All one-step transition probabilities $p_{i\,j}$ can be conveniently written in an n×n transition probability matrix

$$P = \begin{bmatrix} p_{11} & p_{12} & --- & p_{1n} \\ p_{21} & p_{22} & --- & p_{2n} \\ - & - & --- & - \\ p_{n1} & p_{n2} & --- & p_{nn} \end{bmatrix}$$

The entry on the intersection of the i-th row and the j-th column is pij, the transition probability from state i to state j. From each state, a Markov chain makes a transition to one and only one state. States destinations are disjoint and exhaustive events, therefore, each row total equals 1,

$$P_{i\,1} + p_{i\,2} + \ldots + p_{i\,n} = 1.$$

We can also say that probabilities $p_{i\,1}, p_{i\,2}, \ldots, p_{i\,n}$ form the conditional distribution of X(1), given X(0), so they have to add to 1.

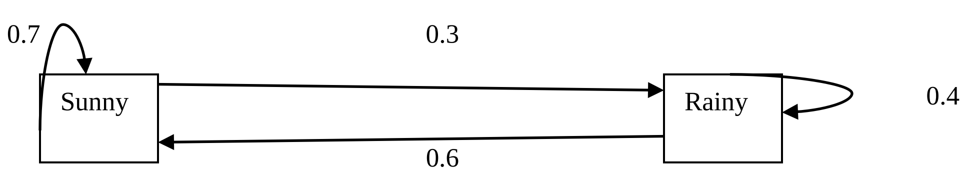
In general, this does not hold for column totals. Some states may be "more favorable" than others, then they are visited more often the others, thus their column total will be larger. Matrices with property (6.3) are called stochastic.

Similarly, h-step transition probabilities can be written in an h-step transition probability matrix

$$P^{(h)} = \begin{bmatrix} p_{11}{}^{(h)} & p_{12}{}^{(h)} & --- & p_{1n}{}^{(h)} \\ p_{21}{}^{(h)} & p_{22}{}^{(h)} & --- & p_{2n}{}^{(h)} \\ - & - & --- & - \\ p_{n1}{}^{(h)} & p_{n2}{}^{(h)} & --- & p_{nn}{}^{(h)} \end{bmatrix}$$

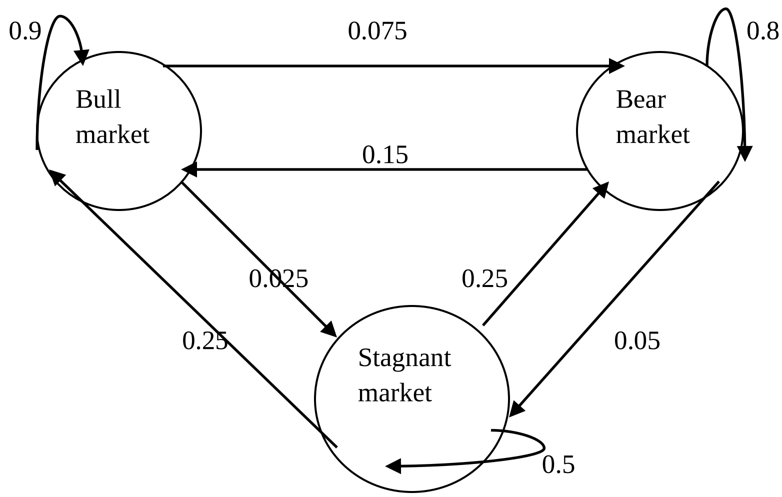
This matrix is also stochastic because each row represents the conditional distribution of X(h), given X(0) (which is a good way to check our results if we compute P(h) by hand).

The following transition diagram reflects the behavior of this Markov chain. Arrows represent all possible one-step transitions, along with the corresponding probabilities. Check this diagram against the transition probabilities stated in Example 6.7. To obtain, say, a 3-step transition probability $p_{2\,1}{}^{(2)}$; find all 2-arrow paths from state 2 "rainy" to state 1 "sunny." Multiply probabilities along each path and add over all 2-step paths.



Let sunny = 1, Rainy = 2

Transition probability matrix P = $\begin{bmatrix} 0.7 & 0.3 \\ 0.6 & 0.4 \end{bmatrix}$



Let Bull market = 1, Bear market = 2 and Stagnant market = 3.

Transition probability matrix

$$P = \begin{bmatrix} 0.9 & 0.075 & 0.025 \\ 0.15 & 0.8 & 0.05 \\ 0.25 & 0.25 & 0.5 \end{bmatrix}$$

n-Step transition probability:

Probability $p_{ij}(t) = P\{X(t+1) = j \mid X(t) = i\}$

$$= P\{X(t+1) = j \mid X(t) = i, X(t-1) = h, X(t-2) = g, ...\}$$

is called a transition probability. Probability $p_{ij}^{(n)}(t) = P\{X(t+n) = j \mid X(t) = i\}$ of moving from state i to state j by means of h transitions is an h-step transition probability.

$$P_{ij}^{(n)} = \sum_{k=1}^{m} p_{ik}^{(n-1)} p_{kj}^{(1)}$$

$$P_{ij}^{(2)} = \sum_{k=1}^{m} p_{ik} p_{kj}$$

$$P_{ij}^{(3)} = \sum_{k=1}^{m} \sum_{l=1}^{m} p_{ik} p_{kl} \; p_{lj}$$

Example 1:

Find 2 step and 3 step transition probability matrix from the transition probability matrix P = $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Solution: Here,

2 step transition probability matrix.

$$P^{(2)} = P.P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1+0 & 0+0 \\ 0+0 & 1+0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

3 step transition probability matrix.

$$P^{(3)} = P^{(2)}.P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1+0 & 0+0 \\ 0+0 & 1+0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Example 2:

Find 2 step and 3 step transition probability matrix from the transition probability matrix

$$P = \begin{bmatrix} 0 & 1 & 0 \\ q & 0 & p \\ 0 & 1 & 0 \end{bmatrix}$$

Solution: Here,

2 step transition probability matrix.

$$P^{(2)} = P.P = \begin{bmatrix} 0 & 1 & 0 \\ q & 0 & p \\ 0 & 1 & 0 \end{bmatrix}.\begin{bmatrix} 0 & 1 & 0 \\ q & 0 & p \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} q & 0 & p \\ q & q+p & 0 \\ 0 & 0 & p \end{bmatrix} = \begin{bmatrix} q & 0 & p \\ q & 1 & 0 \\ q & 0 & p \end{bmatrix}$$

3 step transition probability matrix.

$$P^{(3)} = P^{(2)}.P = \begin{bmatrix} q & 0 & p \\ q & 1 & 0 \\ q & 0 & p \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ q & 0 & p \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & q+p & 0 \\ q & 0 & p \\ 0 & q+p & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ q & 0 & p \\ 0 & 1 & 0 \end{bmatrix}$$

Example 3: If $P = \begin{bmatrix} 0.5 & 0.5 \\ 0.4 & 0.6 \end{bmatrix}$. Find $P_{12}^{(2)}$ and $P_{22}^{(2)}$.

Solution: Here;

Given,

$$P = \begin{bmatrix} 0.5 & 0.5 \\ 0.4 & 0.6 \end{bmatrix}.$$

$$P_{12}^{(2)} = \sum_{k=1}^{2} p_{1k}p_{k2} = P_{11}P_{12}+P_{12}P_{22} = 0.5x0.5+0.4x0.6 = 0.25+0.30 = 0.55$$

$$P_{22}^{(2)} = \sum_{k=1}^{2} p_{2k}p_{k2} = P_{21}P_{12}+P_{22}P_{22} = 0.4x0.5+0.6x0.6= 0.2+0.36 = 0.56$$

Example 3: If $P = \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix}$. Find $P_{11}^{(3)}$ and $P_{21}^{(3)}$.

Solution: Here;

Given,

$$P = \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix}.$$

$$P_{1\,1}^{(3)} = \sum_{k=1}^{2}\sum_{l=1}^{2} p_{1k}p_{kl}p_{l1} = P_{11}P_{11}P_{11} + P_{11}P_{12}P_{21} + P_{12}P_{21}P_{11} + P_{12}P_{22}P_{21}$$

$$= 0.7x0.7x0.7 + 0.7x0.3x0.4 + 0.3x0.4x0.7 + 0.3x0.6x0.4$$

$$= 0.583$$

$$P_{2\,1}^{(3)} = \sum_{k=1}^{2}\sum_{l=1}^{2} p_{2k}p_{kl}\ \ p_{l1}$$

$$= P_{21}P_{11}P_{11} + P_{21}P_{12}P_{21} + P_{22}P_{21}P_{11} + P_{22}P_{22}P_{21}$$

$$= 0.4x0.7x0.7 + 0.4x0.3x0.4 + 0.6x0.4x0.7 + 0.6x0.6x0.4$$

$$= 0.196 + 0.048 + 0.168 + 0.144$$

$= 0.556$

Example 5: In some town each day is either sunny or rainy. A sunny day is followed by another sunny day with probability 0.7, whereas a rainy day is followed by sunny day with probability 0.4. It trains on Monday Make forecast for Tuesday, Wednesday and Thursday.

Solution: Here;

Let state 1 = sunny and state 2 = rainy

Transition probabilities are $P_{11} = 0.7$, $P_{12} = 0.3$, $P_{21} = 0.4$, $P_{22} = 0.6$

1step transition probability matrix

$$P = \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix}$$

For Tuesday rainy chance (P_{22}) = 0.6 =60%

For Tuesday sunny chance (P_{21}) = 0.4 =40%

For Wednesday

$$P^{(2)} = PP = \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix} \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix} = \begin{bmatrix} 0.49 + 0.12 & 0.21 + 0.18 \\ 0.28 + 0.24 & 0.12 + 0.36 \end{bmatrix} = \begin{bmatrix} 0.61 & 0.40 \\ 0.52 & 0.48 \end{bmatrix}$$

For Wednesday rainy chance (P_{22}) = 0.48 =48%

For Wednesday sunny chance (P_{21}) = 0.52 =52%

For Thursday

$$P^{(3)} = P^{(2)}P = \begin{bmatrix} 0.61 & 0.4 \\ 0.52 & 0.48 \end{bmatrix} \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix} = \begin{bmatrix} 0.427 + 0.16 & 0.183 + 0.24 \\ 0.364 + 0.192 & 0.156 + 0.288 \end{bmatrix} = \begin{bmatrix} 0.632 & 0.423 \\ 0.556 & 0.444 \end{bmatrix}$$

For Thursday rainy chance (P_{22}) = 0.444 =44.4%

For Thursday sunny chance (P_{21}) = 0.556 =55.6%

Example 6: Find 2 step and 3 step transition probability matrix from the transition probability matrix

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution: Here, 2 step transition probability matrix.

$$P^{(2)} = P.P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3 step transition probability matrix.

$$P^{(3)} = P^{(2)}.P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thank you!!!