# Derivatives

# Valuing a Derivative Using a One-Period Binomial Model

## The Binomial Model: Example

Hightest Capital believes that a particular non-dividend-paying stock is currently trading at $50 and is considering the sale of a one-year European call option at an exercise price of $55. Answer the following questions:

1. If the stock price is expected to either go up or down by 20% over the next year, what price should Hightest expect to receive for the sold call option? Assume a risk-free rate of 5%.
2. How would the call option price change if the stock price were expected to either go up or down by 40% over the next year?





## The Binomial Model: Example

3. If Hightest had a more optimistic outlook on the future stock price (i.e., they estimated a higher probability of the option ending up in-the-money), how would the expected call option price change?
4. What would be the price of a one-year put option at an exercise price of $55 if the stock price were expected to change by 20%?

# The Binomial Model: Solution 1

One-period binomial tree for stock price

Call option X = \$55 $R_f$ = 5%

**Today** | **1 Year** | Payoff

$S_1^+ =$ $\quad$ $C_1^+ =$

$S_0$ = \$50

Up move

$S_1^- =$ $\quad$ $C_1^- =$

Down move







# Hedge Ratio: Solution 1

Create a risk-free portfolio by combining long stock and short calls.

Use the hedge ratio to compute the units of stock per short call:

$$\text{units of stock} = \frac{\left(C_1^+ - C_1^-\right)}{\left(S_1^+ - S_1^-\right)} =$$

Units of long stock per short call

## Hedge Ratio: Solution 1

Suppose we have shorted 1 call options:

Units of long stock = 0.25

Consider the payoffs at $T_1$:

| In upstate |
|---|
| Long stock |
| Short calls |
| |

-3





## Hedge Ratio: Solution 1

Consider the payoffs at $T_1$:

| In downstate |
|---|
| Long stock |
| Short calls |
| |

The value is the same in either state at $T_1$.

-3

## Option Value: Solution 1

Because the portfolio of long stock and short calls has the same value at time T=1, it is considered to be risk free and hence must only generate a risk-free rate of return:

Portfolio value at $T_1$ (= $10)

$$\frac{V_1}{V_0} - 1 = 0.05$$

Portfolio value at $T_0$

-1





## Option Value: Solution 1

Compute the value of $V_0$:

$$\frac{\$10}{V_0} - 1 = 0.05$$

Portfolio value at T=0:

Long stock =

Short call =

-4

## The Binomial Model: Solution 2

One-period binomial tree for stock price

Call option X = $55 $R_f$ = 5%

| Today | 1 Year | Payoff |
|---|---|---|
| | $S_1^+ =$ | $C_1^+ =$ |
| $S_0$ = $50 | | |
| | $S_1^- =$ | $C_1^- =$ |

Up move

Down move





## Hedge Ratio: Solution 2

Create a risk-free portfolio by combining long stock and short calls.
Use the hedge ratio to compute the units of stock per short call:

$$\text{units of stock} = \frac{\left(C_1^+ - C_1^-\right)}{\left(S_1^+ - S_1^-\right)} =$$

Units of long stock per short call → =

## Portfolio Payoffs: Solution 2

| In upstate | |
|---|---|
| Long stock | |
| Short calls | |
| | |

| In downstate | |
|---|---|
| Long stock | |
| Short calls | |
| | |







## Option Value: Solution 2

Compute the value of $V_0$:

$$\frac{\$11.25}{V_0} - 1 = 0.05$$

Portfolio value at T=0:

Long stock =

Short call =

## Option Value: Solution 3

Since the actual probabilities of an up or a down move in the underlying asset do not affect the no-arbitrage value of the option, the option price that Hightest may charge should not change. Hightest can offset the risk of selling the call by purchasing h* units of the underlying asset, so any directional views on the stock price do not affect the hedge position.




## Option Value: Solution 4 (20%)

Using the put-call parity:

$S_0 + p_0 = c_0 + PV(X)$

$p_0 =$

-2

## Example: Risk-Neutral Pricing

Using Hightest +/– 20% scenario: $S_0$ = \$50, Rf = 5%

$$U = \text{up-move factor} = 1.20$$
$$D = \text{down-move factor} = 0.80$$
$$\pi_U = \text{risk} - \text{neutral probability of up-move} = \frac{1+R_f-D}{U-D} = 0.625$$
$$\pi_D = \text{risk} - \text{neutral probability of down-move} = 1-\pi_U = 0.375$$





## Example: Risk-Neutral Pricing

$$\frac{(\text{prob}_{up} \times \text{payoff}_{up}) + ([1\text{-prob}_{up}] \times \text{payoff}_{down})}{1+Rf} = \text{option value (premium)}$$

$$\frac{(\text{prob}_{up} \times \$5) + ([1\text{-prob}_{up}] \times \$0)}{1.05} = \$2.98$$

$$\frac{(\text{prob}_{up} \times \$5)}{1.05} = \$2.98$$

$$\$3.129 = (\text{prob}_{up} \times \$5)$$

$$\frac{\$3.129}{\$5} = \text{prob}_{up} = 0.625$$

# Example: Risk-Neutral Pricing

Today

1 Year

$S^{+} = \$50 \times 1.20 = \$60$

$\pi_U = 0.625$

$c^{+} = \max(0, \$60.00 - \$55) = \$5$

$S_0 = \$50$

$S^{-} = \$50 \times 0.80 = \$40$

$\pi_D = 0.375$

$c^{-} = \max(0, \$40 - \$55) = \$0$

-2



# Example: Risk-Neutral Pricing

Call value = PV of cash flows (discounted at $R_f$):

$c_0 =$ ____________________ = ________ =

-2

# Solutions

## The Binomial Model: Solution 1

One-period binomial tree for stock price

Call option X = \$55 $R_f$ = 5%

**Today** | **1 Year** | Payoff

$S_1^+ =$ \$60.00 | $C_1^+ =$ 60 – 55 = 5

Up move

$S_0$ = \$50

$S_1^- =$ \$40.00 | $C_1^- =$ 40 – 55 = 0

Down move

## Hedge Ratio: Solution 1

Create a risk-free portfolio by combining long stock and short calls.

Use the hedge ratio to compute the units of stock per short call:

$$\text{units of stock} = \frac{\left(C_1^+ - C_1^-\right)}{\left(S_1^+ - S_1^-\right)} = \frac{(\$5.00 - \$0)}{(\$60.00 - \$40.00)}$$

Units of long stock per short call → $= 0.25$

-2





## Hedge Ratio: Solution 1

Suppose we have shorted 1 call option:

Units of long stock = 0.25

Consider the payoffs at $T_1$:

| In Upstate | | |
|---|---|---|
| Long stock | 0.25 × \$60.00 | \$15 |
| Short calls | 1 × –\$5 | –\$5 |
| | | \$10 |

-3

## Hedge Ratio: Solution 1

Consider the payoffs at $T_1$:

| In Downstate | | |
|---|---|---|
| Long stock | 0.25 × $40.00 | $10 |
| Short calls | 1 × $0 | 0 |
| | | $10 |

The value is the same in either state at $T_1$

-3



## Option Value: Solution 1

Compute the value of $V_0$:

$$\frac{\$10}{V_0} - 1 = 0.05$$

$$\frac{\$10}{1.05} = V_0 = \$9.52$$

Portfolio value at T=0:

Long stock = 0.25 × $50 = $12.50

Short call = 12.50 – 9.52 = 2.98

-4

## The Binomial Model: Solution 2

One-period binomial tree for stock price

Call option X = \$55 $R_f$ = 5%

| Today | 1 Year | Payoff |
|---|---|---|
| | $S_1^+ = \$70.00$ | $C_1^+ =$ 70 - 55 = 15 |
| $S_0 = \$50$ | Up move | |
| | $S_1^- = \$30.00$ | $C_1^- =$ 30 - 55 = 0 |
| | Down move | |

-3





## Hedge Ratio: Solution 2

Create a risk-free portfolio by combining long stock and short calls.
Use the hedge ratio to compute the units of stock per short call:

$$\text{units of stock} = \frac{(C_1^+ - C_1^-)}{(S_1^+ - S_1^-)} = \frac{(\$15.00 - \$0)}{(\$70.00 - \$30.00)}$$

Units of long stock per short call → = 0.375

-2

# Portfolio Payoffs: Solution 2

| **In Upstate** | | |
|---|---|---|
| Long stock | 0.375 × $70.00 | $26.25 |
| Short calls | 1 × –$15 | –$15 |
| | | $11.25 |

| **In Downstate** | | |
|---|---|---|
| Long stock | 0.375 × $30.00 | $11.25 |
| Short calls | 1 × $0 | 0 |
| | | $11.25 |

-6





# Option Value: Solution 2

Compute the value of $V_0$:

$$\frac{\$11.25}{V_0} - 1 = 0.05$$

$$\frac{\$11.25}{1.05} = V_0 = \$10.71$$

Portfolio value at T=0:

Long stock = 0.375 × $50 = $18.75

Short call = 18.75 – 10.71 = 8.04

-4

## Option Value: Solution 4 (20%)

Using the put-call parity:

$S_0 + p_0 = c_0 + PV(X)$

$\$50 + p_0 = \$2.98 + (\$55 / 1.05)$

$p_0 = \$2.98 + \$52.38 - \$50 = \$5.36$

-2





## Example: Risk-Neutral Pricing

Call value = PV of cash flows (discounted at $R_f$):

$$c_0 = \frac{(\$5 \times 0.625) + (\$0 \times 0.375)}{1.05} = \frac{\$3.13}{1.05} = \$2.98$$

-2