Quantitative Methods

Portfolio Mathematics

Intro and Exam Focus

- Calculating portfolio return and variance (for two or three assets)
- Using conditional probabilities in probability trees
- Bayes' formula

Portfolio Expected Return and Variance

Expected return of a portfolio of two assets, A and B:

$$E(R_p) = w_A E(R_A) + w_B E(R_B)$$

Portfolio variance: $Var(R_p) = w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + 2w_A w_B Cov_{AB}$

Note link to correlation (ρ): $Cov_{AB} = \rho_{AB}\sigma_A\sigma_B$

$$Var(R_p) = w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + 2w_A w_B \rho_{AB}\sigma_A\sigma_B$$

Portfolio Expected Return & Variance: CFA Institute Example

Key information about a three-asset portfolio is presented below:

Asset Class	Weight	Expected Return (%)
S&P 500	0.5	13
U.S. long-term corporate bonds	0.25	6
MSCI EAFE	0.25	15

Compute the portfolio's expected return:

$$E(R_P) = \qquad + \qquad + \qquad =$$

Portfolio Expected Return & Variance: CFA Institute Example

Key information about a three-asset portfolio is presented below:

Covariance Matrix	S&P 500	U.S. Long-Term Corporate Bonds	MSCI EAFE
S&P 500	400	45	189
U.S. long-term corporate bonds	45	81	38
MSCI EAFE	189	38	441

Compute the portfolio's variance:

$$\sigma_P^2 = \left(0.5^2 \times 400\right) + \left(0.25^2 \times 81\right) + \left(0.25^2 \times 441\right) + \left(2 \times 0.5 \times 0.25 \times 45\right)$$

$$+\left(2 \times 0.5 \times 0.25 \times 189\right) + \left(2 \times 0.25 \times 0.25 \times 38\right) =$$

$$\sigma_P = \qquad = \qquad \text{or}$$

Covariance Based on Joint Probability: CFA Institute Example

Joint probability function of BankCorp and NewBank returns:

Returns	R_B = 20%	R_B = 16%	R_B = 10%
R_A = 25%	0.20		
R_A = 12%		0.50	
R_A = 10%			0.30

Joint probabilities

$E(R_A)$ = 0.20(25%) + 0.50(12%) + 0.30(10%) = 14%

$E(R_B)$ = 0.20(20%) + 0.50(16%) + 0.30(10%) = 15%

$$\mathbf{Cov_{AB} = \Sigma_A \Sigma_B\, P(R_A,R_B)[R_A - E(R_A][(R_B - E(R_B)]}$$

Covariance Based on Joint Probability: Solution

Returns	R_B = 20%	R_B = 16%	R_B = 10%
R_A = 25%	0.20		
R_A = 12%		0.50	
R_A = 10%			0.30

$E(R_B)$ = 15%

$E(R_A)$ = 14%

Cov$_{AB}$ = 0.2 (25 – 14) (20 – 15)
+ 0.5 (12 – 14) (16 – 15)
+ 0.3 (10 – 14) (10 – 15)=

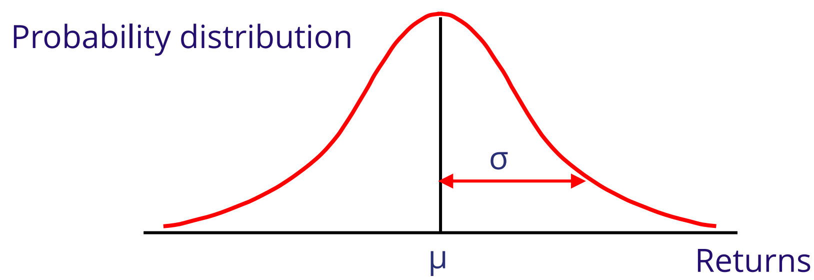
More precisely (if decimal returns had been used)

=

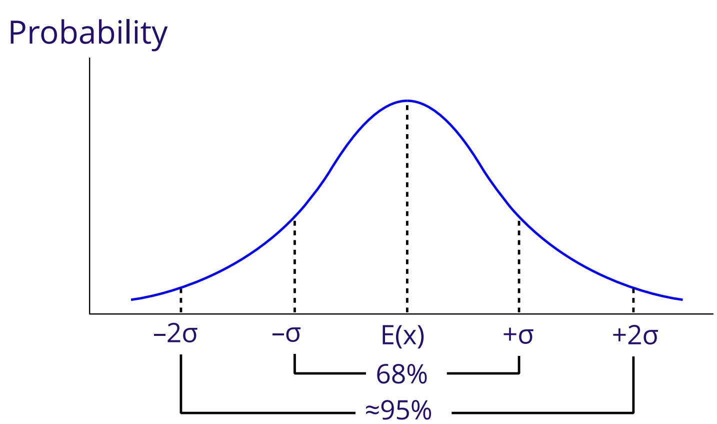
-4

The Normal Distribution

- Completely described by **mean and variance**
- **Symmetric** about the mean (skewness = 0)
- **Kurtosis** (a measure of peakedness/fat tails) = 3



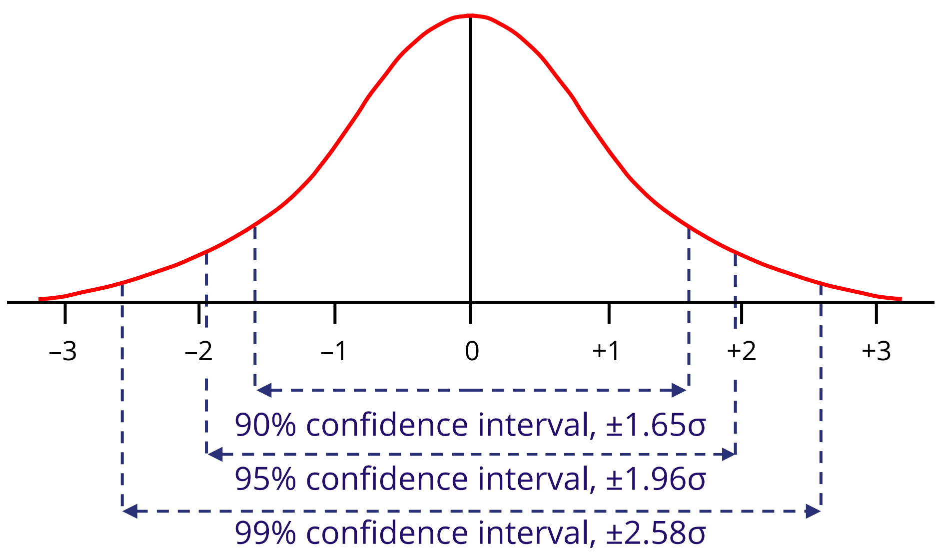
Normal Distribution: Key Distances



Normal Distribution: Key Distances (cont.)

Number of Standard Deviations	Area in One Tail	Two-Tailed Confidence Level
1.65	5%	90%
1.96	2.5%	95%
2.33	1%	98%
2.58	0.5%	99%

Confidence Intervals



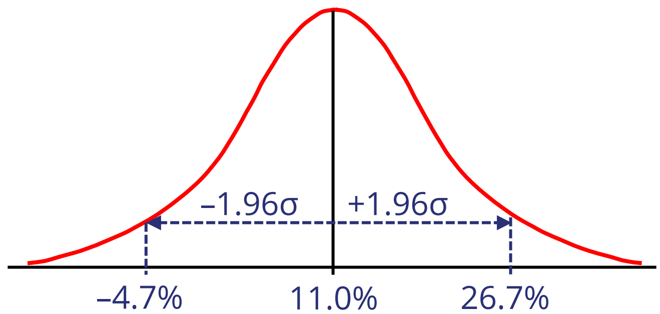
-3

Normal Distribution: Example

The mean annual return (normally distributed) on a portfolio is estimated to be 11%, with a standard deviation of 8%.

Calculate a 95% confidence interval for the next year's return.

11% ±   =   to



-2

Standard Normal Distribution

A normal distribution that has been standardized so that **mean = 0 and standard deviation = 1**

To standardize a random variable, calculate the *z*-value

Subtract the mean (so mean = 0) and divide by standard deviation (so σ = 1):

$$z = \frac{X - \mu}{\sigma}$$

Z is the number of standard deviations of a value, *X*, from the mean

Using *z*-Tables: Example

- A *z*-table gives cumulative probabilities for the standard normal distribution.

Example: The EPS for a large group of firms is normally distributed and has **μ = $4.00** and **σ = $1.50**. Find the probability that a randomly selected firm's earnings are less than $3.70.

z = =

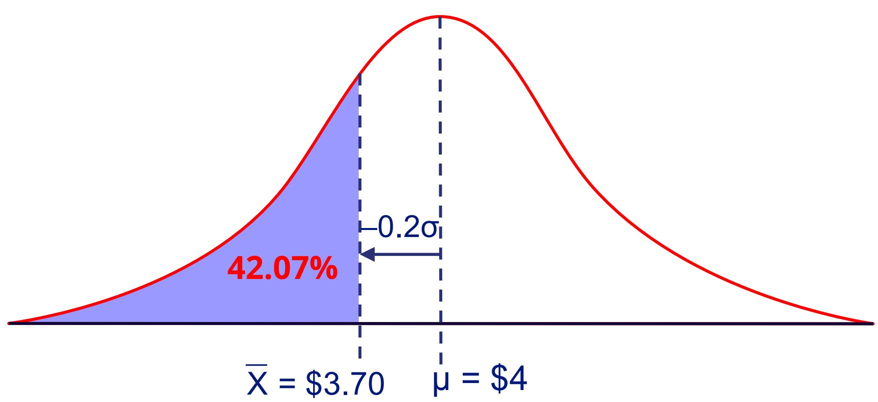
z	0.00	0.01
0.0	0.5000	0.4960
–0.1	0.4602	0.4562
–0.2	0.4207	0.4168

The table shows CDF for std. normal distribution, =

-3

Using *z*-Tables (cont.)

There is a 42.07% probability that the EPS of a randomly selected firm will be more than 0.20 standard deviations **below** the mean (i.e., < $3.70):



-1

Shortfall Risk and Safety-First Ratio

Shortfall risk: probability that a portfolio return or value will be below a target return or value

Roy's safety-first ratio: number of std. dev. target is below the expected return/value

$$\text{SF ratio} = \frac{\left[E(R_P) - R_L\right]}{\sigma_P}, \text{ where } R_L = \text{threshold/target return}$$

If R_L = risk-free rate, SF ratio is the same as Sharpe ratio

Shortfall Risk and Safety-First Ratio: CFA Institute Example

Which of the following portfolios has the lowest probability of earning less than 2%?

Portfolio	1	2
Expected annual return (%)	12	14
Standard deviation of return	15	16

$SFRatio_1 = \quad = \qquad SFRatio_2 = \quad =$

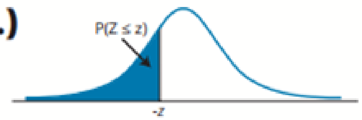
-3

Safety-First Ratio: Solution

CUMULATIVE *Z*-TABLE (CONT.)

Standard Normal Distribution

$P(Z \leq z) = N(z)$ for $z \geq 0$



z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641
–0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
–0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
–0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
–0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
–0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
–0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
–0.7	0.2420	0.2389	0.2358	0.2327	0.2297	0.2266	0.2236	0.2207	0.2177	0.2148
–0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
–0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611

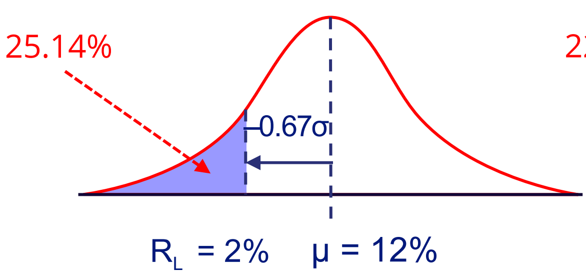
Portfolio 1 ≈

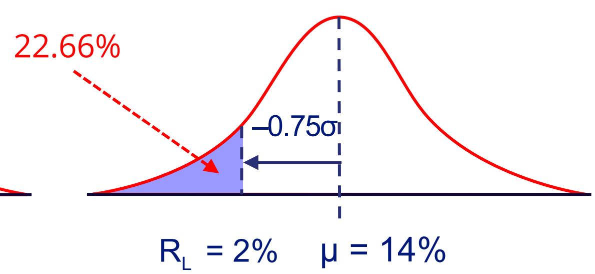
Portfolio 2 =

Safety-First Ratio: Solution

Portfolio 1 ≈ 0.2514

Portfolio 2 ≈ 0.2266





Solutions

Portfolio Expected Return & Variance: CFA Institute Example

Key information about a three-asset portfolio is presented below:

Asset Class	Weight	Expected Return (%)
S&P 500	0.5	13
U.S. long-term corporate bonds	0.25	6
MSCI EAFE	0.25	15

Compute the portfolio's expected return:

$$E(R_P) = (0.5 \times 13\%) + (0.25 \times 6\%) + (0.25 \times 15\%) = 11.75\%$$

Portfolio Expected Return & Variance: CFA Institute Example

Key information about a three-asset portfolio is presented below:

Covariance Matrix	S&P 500	U.S. Long-Term Corporate Bonds	MSCI EAFE
S&P 500	400	45	189
U.S. long-term corporate bonds	45	81	38
MSCI EAFE	189	38	441

Compute the portfolio's variance:

$$\sigma_P^2 = \left(0.5^2 \times 400\right) + \left(0.25^2 \times 81\right) + \left(0.25^2 \times 441\right) + \left(2 \times 0.5 \times 0.25 \times 45\right)$$

$$+\left(2 \times 0.5 \times 0.25 \times 189\right) + \left(2 \times 0.25 \times 0.25 \times 38\right) = 195.875$$

$$\sigma_P = \sqrt{195.875} = 0.14 \text{ or } 14\%$$

-2

Covariance Based on Joint Probability: Solution

Returns	$R_B = 20\%$	$R_B = 16\%$	$R_B = 10\%$
$R_A = 25\%$	0.20		
$R_A = 12\%$		0.50	
$R_A = 10\%$			0.30

$E(R_B) = 15\%$

$E(R_A) = 14\%$

$$\mathbf{Cov_{AB}} = 0.2\,(25 - 14)\,(20 - 15) + 0.5\,(12 - 14)\,(16 - 15) + 0.3\,(10 - 14)\,(10 - 15) = 16$$

More precisely (if decimal returns had been used)

16 / 10,000 = 0.0016

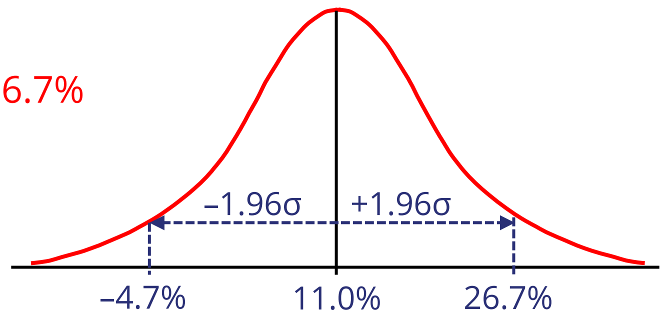
-4

Normal Distribution: Example

The mean annual return (normally distributed) on a portfolio is estimated to be 11%, with a standard deviation of 8%.

Calculate a 95% confidence interval for the next year's return.

11% ± (1.96)(8%) = –4.7% to 26.7%



-2

Using *z*-Tables: Example

- A *z*-table gives cumulative probabilities for the standard normal distribution.

Example: The EPS for a large group of firms are normally distributed and have **μ = $4.00** and **σ = $1.50**. Find the probability that a randomly selected firm's earnings are less than $3.70.

$$z = \frac{3.70 - 4.00}{1.50} = -0.20\sigma$$

z	0.00	0.01
0.0	0.5000	0.4960
–0.1	0.4602	0.4562
–0.2	0.4207	0.4168

The table shows CDF for std. normal distribution, N(–0.2) = 42.07%.

-3

Shortfall Risk and Safety-First Ratio: CFA Institute Example

Which of the following portfolios has the lowest probability of earning less than 2%?

Portfolio	1	2
Expected annual return (%)	12	14
Standard deviation of return	15	16

$$SFRatio_1 = \frac{12 - 2}{15} = 0.667\sigma$$

$$SFRatio_2 = \frac{14 - 2}{16} = 0.75\sigma$$

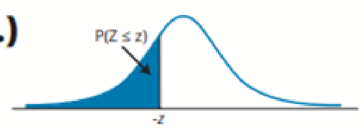
-3

Safety-First Ratio: Solution

CUMULATIVE Z-TABLE (CONT.)

Standard Normal Distribution

$P(Z \leq z) = N(z)$ for $z \geq 0$



z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641
–0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
–0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
–0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
–0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
–0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
–0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
–0.7	0.2420	0.2389	0.2358	0.2327	0.2297	0.2266	0.2236	0.2207	0.2177	0.2148
–0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
–0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611

Portfolio 1 ≈ 0.2514

Portfolio 2 = 0.2266

-2