

# Derivatives

## Valuing a Derivative Using a One-Period Binomial Model

## The Binomial Model: Example

Hightest Capital believes that a particular non-dividend-paying stock is currently trading at \$50 and is considering the sale of a one-year European call option at an exercise price of \$55. Answer the following questions:

1. If the stock price is expected to either go up or down by 20% over the next year, what price should Hightest expect to receive for the sold call option? Assume a risk-free rate of 5%.
2. How would the call option price change if the stock price were expected to either go up or down by 40% over the next year?

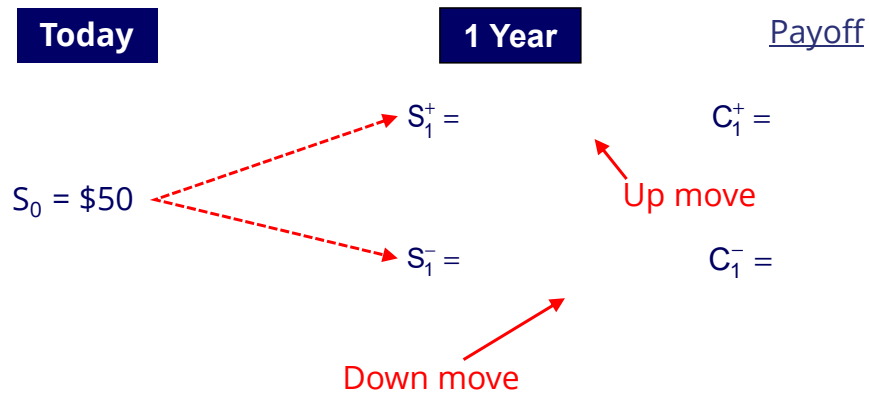
## The Binomial Model: Example

3. If Hightest had a more optimistic outlook on the future stock price (i.e., they estimated a higher probability of the option ending up in-the-money), how would the expected call option price change?
4. What would be the price of a one-year put option at an exercise price of \$55 if the stock price were expected to change by 20%?

## The Binomial Model: **Solution 1**

One-period binomial tree for stock price

Call option X = \$55  $R_f = 5\%$



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## Hedge Ratio: **Solution 1**

Create a risk-free portfolio by combining long stock and short calls.

Use the hedge ratio to compute the units of stock per short call:

$$\text{units of stock} = \frac{(C_1^+ - C_1^-)}{(S_1^+ - S_1^-)} =$$

Units of long stock per short call →

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## Hedge Ratio: **Solution 1**

Suppose we have shorted 1 call options:

Units of long stock = 0.25

Consider the payoffs at  $T_1$ :

In upstate	
Long stock	
Short calls	

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## Hedge Ratio: **Solution 1**

Consider the payoffs at  $T_1$ :

In downstate	
Long stock	
Short calls	

The value is the same in either state at  $T_1$ .

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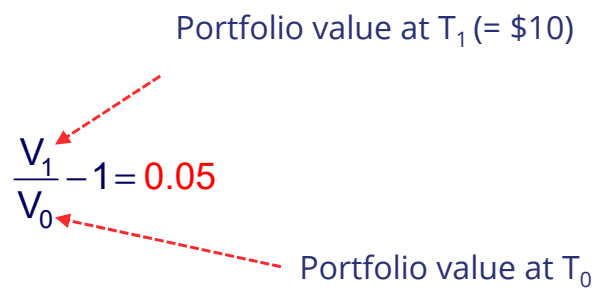
## Option Value: Solution 1

Because the portfolio of long stock and short calls has the same value at time  $T=1$ , it is considered to be **risk free** and hence must only generate a **risk-free rate of return**:

Portfolio value at  $T_1 (= \$10)$

$$\frac{V_1}{V_0} - 1 = 0.05$$

Portfolio value at  $T_0$



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## Option Value: Solution 1

Compute the value of  $V_0$ :

$$\frac{\$10}{V_0} - 1 = 0.05$$

Portfolio value at  $T=0$ :

Long stock =

Short call =

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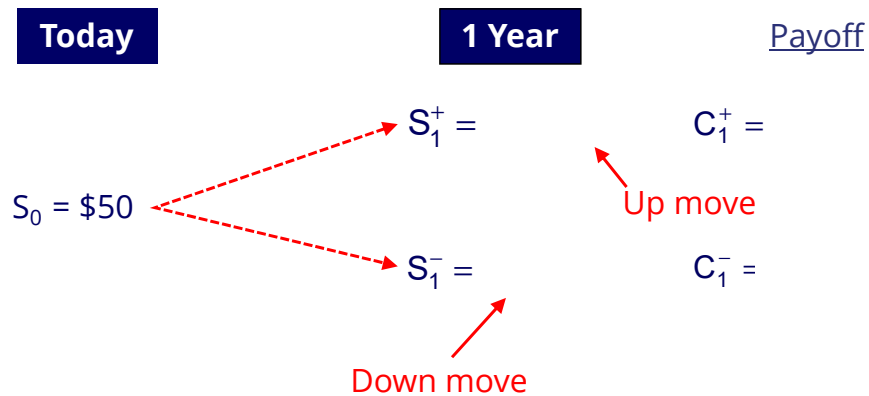
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## The Binomial Model: **Solution 2**

One-period binomial tree for stock price

Call option X = \$55  $R_f = 5\%$



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## Hedge Ratio: **Solution 2**

Create a risk-free portfolio by combining long stock and short calls.  
Use the hedge ratio to compute the units of stock per short call:

$$\text{units of stock} = \frac{(C_1^+ - C_1^-)}{(S_1^+ - S_1^-)} =$$

Units of long stock per short call  $\rightarrow$  =

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## Portfolio Payoffs: **Solution 2**

In upstate	
Long stock	
Short calls	
In downstate	
Long stock	
Short calls	

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## Option Value: **Solution 2**

Compute the value of  $V_0$ :

$$\frac{\$11.25}{V_0} - 1 = 0.05$$

Portfolio value at T=0:

Long stock =

Short call =

-4

## Option Value: **Solution 3**

Since the actual probabilities of an up or a down move in the underlying asset do not affect the no-arbitrage value of the option, the option price that Hightest may charge should not change. Hightest can offset the risk of selling the call by purchasing  $h^*$  units of the underlying asset, so any directional views on the stock price do not affect the hedge position.

## Option Value: **Solution 4 (20%)**

Using the put-call parity:

$$S_0 + p_0 = c_0 + PV(X)$$

$$p_0 =$$



## Example: Risk-Neutral Pricing

Using Hightest +/- 20% scenario:  $S_0 = \$50$ ,  $R_f = 5\%$

$U$  = up-move factor = 1.20

$D$  = down-move factor = 0.80

$$\pi_U = \text{risk-neutral probability of up-move} = \frac{1 + R_f - D}{U - D} = 0.625$$

$$\pi_D = \text{risk-neutral probability of down-move} = 1 - \pi_U = 0.375$$

## Example: Risk-Neutral Pricing

$$\frac{(\text{prob}_{up} \times \text{payoff}_{up}) + ([1 - \text{prob}_{up}] \times \text{payoff}_{down})}{1 + R_f} = \text{option value (premium)}$$

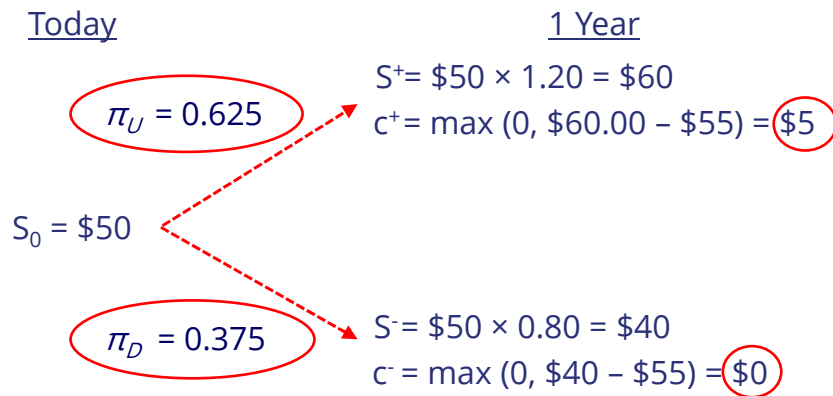
$$\frac{(\text{prob}_{up} \times \$5) + ([1 - \text{prob}_{up}] \times \$0)}{1.05} = \$2.98$$

$$\frac{(\text{prob}_{up} \times \$5)}{1.05} = \$2.98$$

$$\$3.129 = (\text{prob}_{up} \times \$5)$$

$$\frac{\$3.129}{\$5} = \text{prob}_{up} = 0.625$$

## Example: Risk-Neutral Pricing



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## Example: Risk-Neutral Pricing

Call value = PV of cash flows (discounted at  $R_f$ ):

$$C_0 = \frac{C^+ \pi_U + C^- \pi_D}{1 + R_f} = \frac{\$5 \times 0.625 + \$0 \times 0.375}{1 + 0.05} = \$4.76$$

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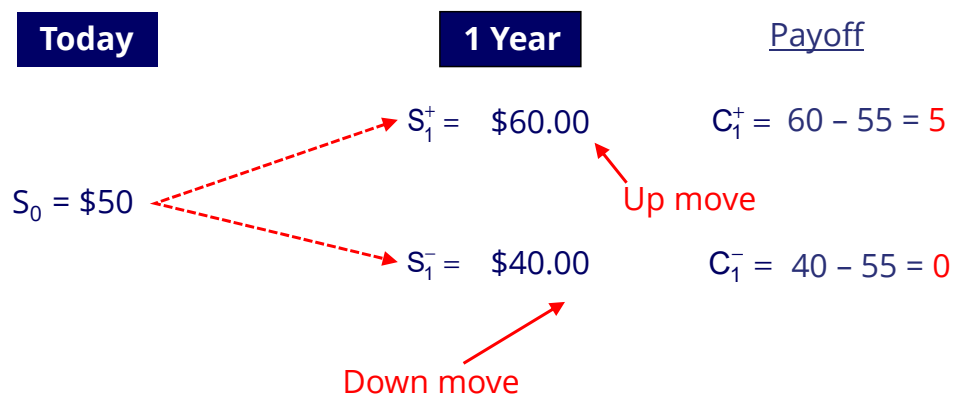
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## Solutions

### The Binomial Model: **Solution 1**

One-period binomial tree for stock price

Call option  $X = \$55$   $R_f = 5\%$



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## Hedge Ratio: **Solution 1**

Create a risk-free portfolio by combining long stock and short calls.

Use the hedge ratio to compute the units of stock per short call:

$$\text{units of stock} = \frac{(C_1^+ - C_1^-)}{(S_1^+ - S_1^-)} = \frac{(\$5.00 - \$0)}{(\$60.00 - \$40.00)}$$

Units of long stock per short call  $\rightarrow$  = **0.25**

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## Hedge Ratio: **Solution 1**

Suppose we have shorted 1 call option:

Units of long stock = 0.25

Consider the payoffs at  $T_1$ :

In Upstate		
Long stock	$0.25 \times \$60.00$	\$15
Short calls	$1 \times -\$5$	-\$5
		<hr/> \$10

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## Hedge Ratio: **Solution 1**

Consider the payoffs at  $T_1$ :

In Downstate		
Long stock	$0.25 \times \$40.00$	\$10
Short calls	$1 \times \$0$	0
		<hr/> \$10

The value is the same in either state at  $T_1$

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## Option Value: **Solution 1**

Compute the value of  $V_0$ :

$$\frac{\$10}{V_0} - 1 = 0.05$$

$$\frac{\$10}{1.05} = V_0 = \$9.52$$

Portfolio value at  $T=0$ :

Long stock =  $0.25 \times \$50 = \$12.50$

Short call =  $12.50 - 9.52 = 2.98$

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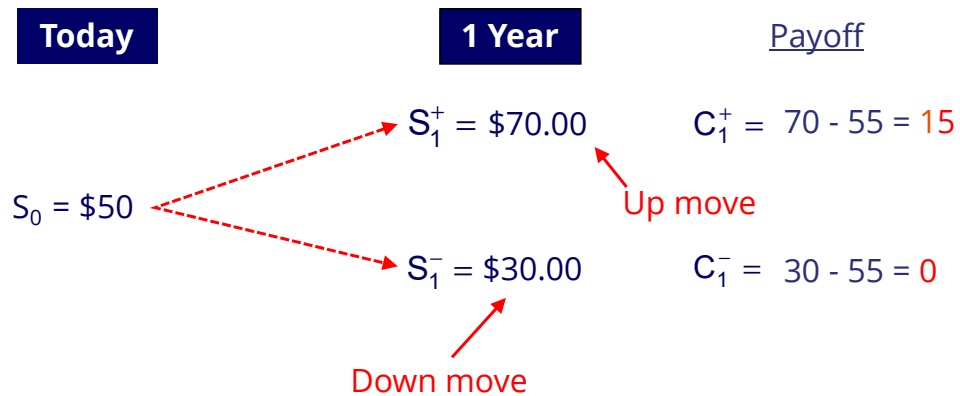
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## The Binomial Model: **Solution 2**

One-period binomial tree for stock price

Call option X = \$55  $R_f = 5\%$



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## Hedge Ratio: **Solution 2**

Create a risk-free portfolio by combining long stock and short calls.  
Use the hedge ratio to compute the units of stock per short call:

$$\text{units of stock} = \frac{(C_1^+ - C_1^-)}{(S_1^+ - S_1^-)} = \frac{(\$15.00 - \$0)}{(\$70.00 - \$30.00)}$$

Units of long stock per short call  $\rightarrow = 0.375$

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## Portfolio Payoffs: **Solution 2**

In Upstate		
Long stock	$0.375 \times \$70.00$	\$26.25
Short calls	$1 \times -\$15$	-\$15
		<hr/> \$11.25
In Downstate		
Long stock	$0.375 \times \$30.00$	\$11.25
Short calls	$1 \times \$0$	0
		<hr/> \$11.25

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## Option Value: **Solution 2**

Compute the value of  $V_0$ :

$$\frac{\$11.25}{V_0} - 1 = 0.05$$

$$\frac{\$11.25}{1.05} = V_0 = \$10.71$$

Portfolio value at T=0:

Long stock =  $0.375 \times \$50 = \$18.75$

Short call =  $18.75 - 10.71 = 8.04$

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## Option Value: **Solution 4 (20%)**

Using the put-call parity:

$$S_0 + p_0 = c_0 + PV(X)$$

$$\$50 + p_0 = \$2.98 + (\$55 / 1.05)$$

$$p_0 = \$2.98 + \$52.38 - \$50 = \text{\textcolor{red}{\$5.36}}$$

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## **Example: Risk-Neutral Pricing**

Call value = PV of cash flows (discounted at  $R_f$ ):

$$c_0 = \frac{(\$5 \times 0.625) + (\$0 \times 0.375)}{1.05} = \frac{\$3.13}{1.05} = \text{\textcolor{red}{\$2.98}}$$

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