Quantitative Methods

The Time Value of Money in Finance

Intro and Exam Focus

- Valuation
  - Bonds: using the TVM keys of the calculator
  - Equity: Gordon's growth model
  - Cash flow additivity applications
    - Forward interest rates
    - Forward currency exchange rates
    - Option pricing

Fundamentals of Valuation

Value of investment today = present value of future cash flows discounted at the required return of investors

For a single cash flow of size FV at time *t* with periodic rate *r:*

$$PV_0 = \frac{FV_t}{(1+r)^t} \text{ or } FV_t \times \frac{1}{(1+r)^t}$$

Discount factor

Using continuously compounded rates:

$$PV_0 = FV_t e^{-rt}$$

Zero-Coupon Bond: Example

- A zero-coupon bond offers a single payment (FV) at maturity (t):

$$PV_0 = \frac{FV_t}{(1+r)^t}$$

A STRIP from a U.K. government bond offers a payment of £100 in 10 years' time. Assuming annual compounding, what should an investor demanding 5% return pay for this security?

Solution:

Price = =

Zero-Coupon Bond Implied Return: Example

- Given a price, the implied return of a zero-coupon bond can be calculated.

Three years later, the U.K. government bond STRIP in the previous example is trading at a price is £78.17. Calculate the implied return of the bond.

Solution:

Implied return = =

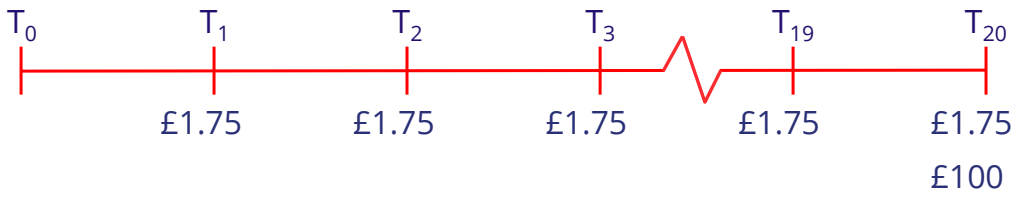
Calculator keystrokes: [÷] [=] [y^x] [1/x] [=] – 1 [=]

Coupon-Paying Bonds: Example

- A fixed-coupon bond offers a regular constant coupon payment with a payment of par at maturity.

A 10-year U.K. government bond has an annualized coupon rate of 3.5%, a yield of 4%, and the coupon is paid semiannually. What is the price of the bond?

Coupon-Paying Bonds: Solution



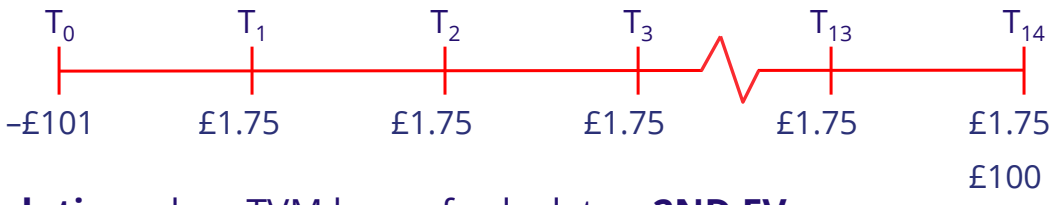
Solution: clear TVM keys of calculator: **2ND FV**

- **N**, **I/Y**, ; **PMT**) **FV**, CPT

Coupon-Paying Bonds Implied Return: Example

Three years later, the U.K. government bond in the previous example is trading at a price of £101 per £100 of par. Calculate the implied annualized return (yield) of the bond.

Coupon-Paying Bonds Implied Return: Solution



Solution: clear TVM keys of calculator: **2ND FV**

N, 3.5 / 2 = 1.75 **PMT**, 100 **FV** **PV**,

CPT **I/Y:**

Note this is the semiannual return of the bond.

Annualized quoted yield = =

Loan Cash Flows: Example

A homeowner takes out a $500,000 25-year mortgage with a fixed rate of 6%. Payments are made monthly.

1. Calculate the fixed monthly payment.
2. Calculate the split of interest and principal for the first and last month's payment.

Loan Cash Flows: Solution

Clear TVM keys of calculator: **2ND FV**

Fixed monthly payment:

N, **I/Y**,) **PV**, CPT **PMT**:

Interest and principal:

2. Use the amortization worksheet of the calculator: **2ND PV**

P1 = 1 ENTER ↓, P2 = 1 ENTER ↓ ↓, PRN = ↓ INT =

P1 = 300 ENTER ↓, P2 = 300 ENTER ↓ ↓, PRN = ↓ INT =

-3

Constant Perpetuities: Example

A perpetual constant periodic payment of size CF has a present value of:

$$PV_0 = \frac{CF}{r} \text{ or } CF \times \frac{1}{r}$$

Perpetuity factor

A preferred stock pays a constant dividend of €0.50 per share in perpetuity. Calculate the price of a share if investors demand a return of 8%.

Solution: price = = per share.

-1

Growing Perpetuities: CFAI Example

When perpetual cash flows grow at constant rate *g:*

$$PV_0 = \frac{CF_1}{r - g}$$

A Shipline currently pays a constant dividend of £1.50 per share which is expected to grow at a rate of 6% in perpetuity. Calculate the price of a share if investors require a return of 15%.

Solution: price= =

-1

2-Stage Growth: CFAI Example

Calculate the value of Shipline stock if:

- Last year's dividend = £1.50.
- Dividend will grow at 6% for 3 years.
- Dividend will grow at 2% in perpetuity after 3 years.
- Required return is 15%.

2-Stage Growth: Solution Approach 1

Step 1: Forecast the dividend for each year of rapid growth and for the first year of constant growth.

$D_1 = £1.50(1.06)^1 =$

$D_2 = £1.50(1.06)^2 =$

$D_3 = \$1.50(1.06)^3 =$

Identify the first dividend that will grow at a constant growth rate.

-1

2-Stage Growth: Solution Approach 1

Step 2: Use the constant growth model to find the value of the stock one period before the dividend that will grow at a constant rate (D_3).

Using: $P_2 = D_3 / (r - g)$

We have: $P_2 =$ =

-1

2-Stage Growth: Solution Approach 1

Step 3: Find the PV of expected dividends and of the expected future stock price.

$$P_0 = \frac{D_1}{1+r} + \frac{D_2}{(1+r)^2} + \frac{P_2}{(1+r)^2}$$

$P_0 =$ + + =

-4

Alternative Approach

Methodology

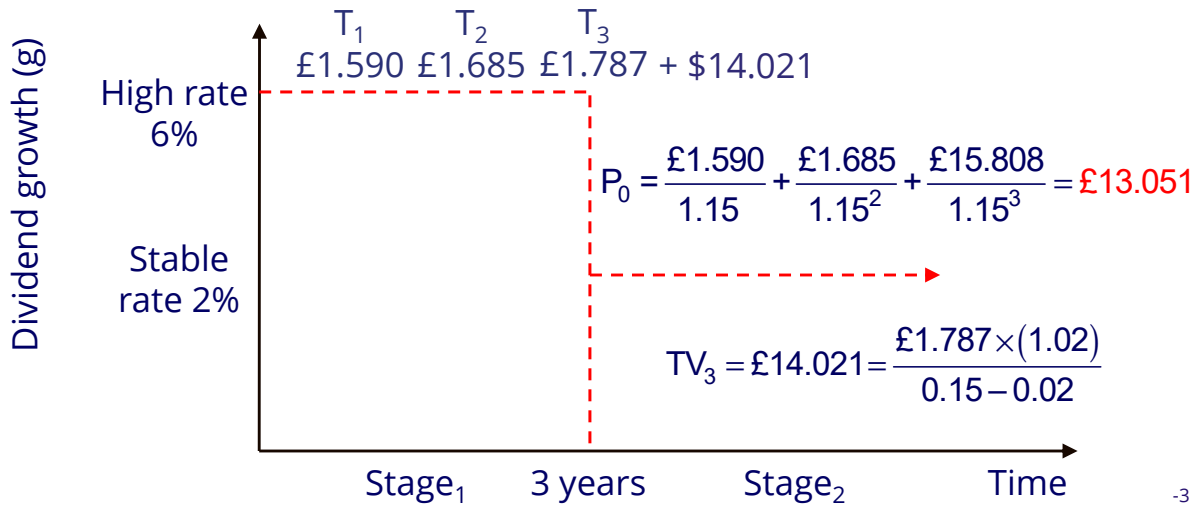
1. Individual estimation of supernormal dividends, *followed by . . .*
2. Calculation of a terminal value

Note: very important concept

V_0 = PV(dividends over first *n* years)
+ PV(terminal value)

From Gordon growth model approach

2-Stage Growth: Solution Approach 2



-3

Equity Implied Return and Growth Rates

$$P_0 = \frac{D_1}{(r - g)}$$

Rearranging for *r:* $$r = \frac{D_1}{P_0} + g$$

Rearranging for *g:* $$g = r - \frac{D_1}{P_0}$$

Equity Implied Return: CFAI Example

Coca-Cola Company stock trades at a share price of USD63.00 and its annualized expected dividend per share during the next year is USD1.76.

1. If an analyst expects Coca-Cola's dividend per share to increase at a constant 4 percent per year indefinitely, calculate the required return expected by investors.

Solution: r = + = or

The required return expected for Coca-Cola stock is percent given its current price, expected dividend, and expected dividend growth rate. Investor expectations of future stock returns are inferred by the combination of the current price, expected future cash flows, and the cash flow growth rate.

-2

Equity Implied Growth: CFAI Example

Coca-Cola Company stock trades at a share price of USD63.00 and its annualized expected dividend per share during the next year is USD1.76.

2. If the analyst believes that Coca-Cola stock investors should expect a return of 7 percent, calculate the implied dividend growth rate for Coca-Cola.

Solution: g= – = or

The implied dividend growth rate for Coca-Cola stock is percent given its expected return, price, and expected dividend. Given that a higher expected return is assumed in this question compared to the case in Question 1, the result is a higher implied dividend growth rate to justify Coca-Cola's stock price of USD63.00.

-2

Price-to-Earnings Ratio

$$P_0 = \frac{D_1}{(r-g)}$$

Dividing both sides by E_1:

$$\frac{P_0}{E_1} = \frac{D_1/E_1}{(r-g)} = \frac{\text{payout ratio}}{(r-g)}$$

Price-to-Earnings Ratio: CFAI Example

Suppose Coca-Cola stock trades at a forward price to earnings ratio of 28 and its expected dividend payout ratio is 70 percent. Analysts believe that Coca-Cola stock should earn a 9 percent return and that its dividends will grow by 4.50 percent per year indefinitely. Recommend a course of action for an investor interested in taking a position in Coca-Cola stock.

Solution: $\frac{P_0}{E_1} = \frac{D_1/E_1}{(r-g)} =$ =

Coca-Cola's forward price-to-earnings ratio of 28 is much greater than , which is computed from the equation. Investor expectations of cash flow growth and return are inconsistent with Coca-Cola's forward price to earnings ratio. **Specifically, an investor should consider a short position in Coca-Cola stock in the belief that its price should decline because its current price to earnings ratio is well above what its fundamentals imply**.

-2

Cash Flow Additivity

Principle of ***no arbitrage***

- "Two equivalent sets of cash flows must have the same present value"

Example for a coupon-paying bond

- Sum of PVs of individual cash flows = PV of bond with same cash flows

Applications

1. Forward interest rates
2. Forward currency exchange rates
3. Option pricing

Forward Interest Rates: CFAI Example

One and Two-Year U.S. Treasury STRIP Prices		
Date	**PV(1y)**	**PV(2y)**
31 May 2022	98.028	95.109
15 June 2022	97.402	93.937

What is the implied forward rate for Year 2 ($F_{1,1}$) at the end of May and mid-June?

Forward Interest Rates: Solution

Step 1: Calculate spot rates.

May 1y STRIP = r =

May 2y STRIP = = r

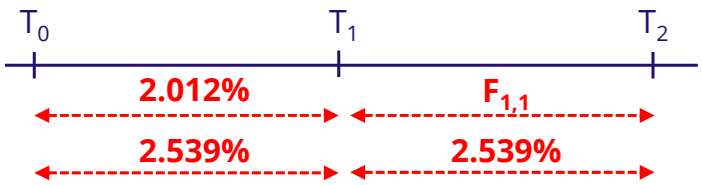
June 1y STRIP = r =

June 2y STRIP = = r =

-4

Forward Interest Rates: Solution

Step 2: Calculate forward rates (end of May).



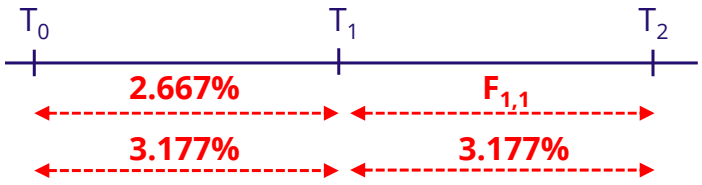
→ $F_{1,1}$ =

→ $F_{1,1}$ = or

-3

Forward Interest Rates: Solution

Step 3: Calculate forward rates (mid-June).



→ $F_{1,1}$ =

→ $F_{1,1}$ = or

-3

Forward Currency Exchange Rates: Example

Current USD/GBP exchange rate = 1.25 USD per GBP. Continuously compounded interest rates are 2% in USD and 2.5% in GBP.

Calculate the no-arbitrage 1-year forward exchange rate.

Solution:

Ratio of USD to GBP today: is USD125 to GBP100

After one year, USD125 will grow to USD125$e^{0.02}$ =

After one year, GBP100 will grow to GBP100$e^{0.025}$ =

1-year forward USD/GBP exchange rate =

=

-3

Option Pricing: CFAI Example

Today	1 Year	Option to Buy ("Call") at 50
S_0= CYN40	CYN 56.00 up-move	CYN= 6
	CYN 32.00 down-move	0

hedge ratio = = =

-2

Option Pricing: Solution

- Consider buying 0.25 shares and selling one call option.

Value of portfolio after up-move = – =

Value of portfolio after down-move = – =

Portfolio has a risk-free payoff of CYN at the end of one year.

If R_f = 5%, **value of portfolio today**

Hence, = $\therefore C_0$ =

-5

Solutions

Zero-Coupon Bond: Example

- A zero-coupon bond offers a single payment (FV) at maturity (t):

$$PV_0 = \frac{FV_t}{(1+r)^t}$$

A STRIP from a U.K. government bond offers a payment of £100 in 10 years' time. Assuming annual compounding, what should an investor demanding 5% return pay for this security?

Solution:

$$\text{Price} = \frac{£100}{(1.05)^{10}} = £61.39$$

-1

Zero-Coupon Bond Implied Return: Example

- Given a price, the implied return of a zero-coupon bond can be calculated.

Three years later, the U.K. government bond STRIP in the previous example is trading at a price is £78.17. Calculate the implied return of the bond.

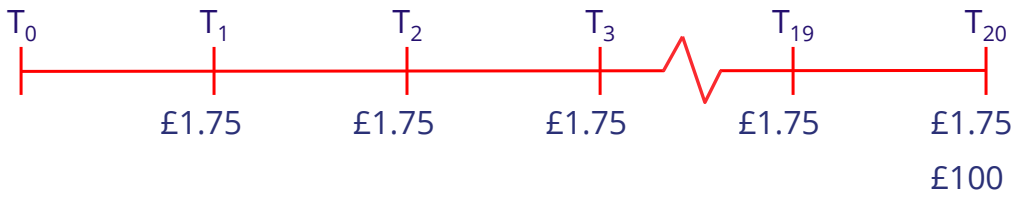
Solution:

Implied return = $(100 / 78.17)^{1/7} - 1 = 3.58\%$

Calculator keystrokes: 100 [÷] 78.17 [=] [y^x] 7 [1/x] [=] – 1 [=] 0.0358

-2

Coupon-Paying Bonds: Solution

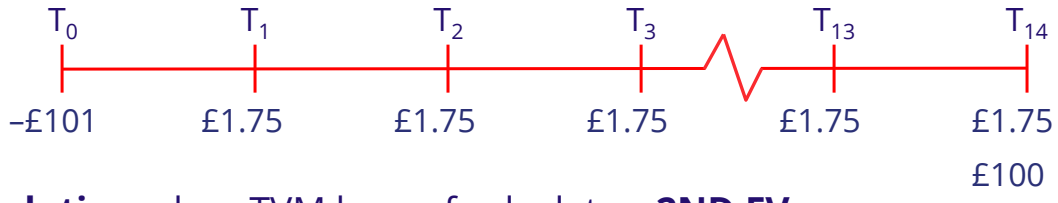


Solution: clear TVM keys of calculator: **2ND FV**

- 10 × 2 = 20 **N**, 4/2 = 2 **I/Y**, 3.5 / 2 = 1.75 **PMT**, 100 **FV**, CPT **PV**: –95.912

-2

Coupon-Paying Bonds Implied Return: Solution



Solution: clear TVM keys of calculator: **2ND FV**

7 × 2 = 14 **N**, 3.5/2 = 1.75 **PMT**, 100 **FV**, -101 **PV**,

CPT **I/Y:** 1.669%

Note this is the semiannual return of the bond.

Annualized quoted yield = 1.669% × 2 = 3.34%

-3

Loan Cash Flows: Example

A homeowner takes out a $500,000 25-year mortgage with a fixed rate of 6%. Payments are made monthly.

1. Calculate the fixed monthly payment.
2. Calculate the split of interest and principal for the first and last month's payment.

Loan Cash Flows: Solution

Clear TVM keys of calculator: **2ND FV**

Fixed monthly payment:

1. 25 × 12 = 300 **N**, 6/12 = 0.5 **I/Y**, 500000 **PV**, CPT **PMT** : –3,221.51

Interest and principal:

2. Use the amortization worksheet of the calculator: **2ND PV**

P1 = 1 ENTER ↓, P2 = 1 ENTER ↓ ↓, PRN = –721.51 ↓ INT = –2500

P1 = 300 ENTER ↓, P2 = 300 ENTER ↓ ↓, PRN = –3205.48 ↓ INT = –16.03

Constant Perpetuities: Example

A perpetual constant periodic payment of size CF has a present value of:

$$PV_0 = \frac{CF}{r} \text{ or } CF \times \frac{1}{r}$$

Perpetuity factor

A preferred stock pays a constant dividend of €0.50 per share in perpetuity. Calculate the price of a share if investors demand a return of 8%.

Solution: price = €0.50 / 0.08 = €6.25 per share

-1

Growing Perpetuities: CFAI Example

When perpetual cash flows grow at constant rate *g:*

$$PV_0 = \frac{CF_1}{r - g}$$

A Shipline currently pays a constant dividend of £1.50 per share which is expected to grow at a rate of 6% in perpetuity. Calculate the price of a share if investors require a return of 15%.

Solution:

$$\text{Price} = \frac{£1.50 \times 1.06}{0.15 - 0.06} = £17.67$$

-1

2-Stage Growth: Solution Approach 1

Step 1: Forecast the dividend for each year of rapid growth and for the first year of constant growth.

$D_1 = £1.50(1.06)^1 = £1.590$

$D_2 = £1.50(1.06)^2 = £1.685$

$D_3 = \$1.50(1.06)^3 = £1.787$

Identify the first dividend that will grow at a constant growth rate.

-1

2-Stage Growth: Solution Approach 1

Step 2: Use the constant growth model to find the value of the stock one period before the dividend that will grow at a constant rate (D_3).

Using: $P_2 = D_3 / (r - g)$

We have: $P_2 = \frac{£1.787}{0.15 - 0.02} = £13.746$

-1

2-Stage Growth: Solution Approach 1

Step 3: Find the PV of expected dividends and of the expected future stock price.

$$P_0 = \frac{D_1}{1+r} + \frac{D_2}{(1+r)^2} + \frac{P_2}{(1+r)^2}$$

$$P_0 = \frac{£1.590}{1.15} + \frac{£1.685}{(1.15)^2} + \frac{£13.746}{(1.15)^2} = \textbf{\$13.051}$$

-4

Equity Implied Return: CFAI Example

Coca-Cola Company stock trades at a share price of USD63.00 and its annualized expected dividend per share during the next year is USD1.76.

1. If an analyst expects Coca-Cola's dividend per share to increase at a constant 4 percent per year indefinitely, calculate the required return expected by investors.

Solution: $r = \frac{\$1.76}{\$63} + 0.04 = 0.0679$ or 6.79%

The required return expected for Coca-Cola stock is 6.79 percent given its current price, expected dividend, and expected dividend growth rate. Investor expectations of future stock returns are inferred by the combination of the current price, expected future cash flows, and the cash flow growth rate.

-2

Equity Implied Growth: CFAI Example

Coca-Cola Company stock trades at a share price of USD63.00 and its annualized expected dividend per share during the next year is USD1.76.

2. If the analyst believes that Coca-Cola stock investors should expect a return of 7 percent, calculate the implied dividend growth rate for Coca-Cola.

Solution: $g = 0.07 - \frac{\$1.76}{\$63} = 0.0421$ or 4.21%

The implied dividend growth rate for Coca-Cola stock is 4.21 percent given its expected return, price, and expected dividend. Given that a higher expected return is assumed in this question compared to the case in question 1, the result is a higher implied dividend growth rate to justify Coca-Cola's stock price of USD63.00.

-2

Based on CFA Curriculum Volume 1, Page 66, Example 10

Price-to-Earnings Ratio

$$P_0 = \frac{D_1}{(r - g)}$$

Dividing both sides by E_1:

$$\frac{P_0}{E_1} = \frac{D_1/E_1}{(r - g)} = \frac{\text{payout ratio}}{(r - g)}$$

Price-to-Earnings Ratio: CFAI Example

Suppose Coca-Cola stock trades at a forward price to earnings ratio of 28 and its expected dividend payout ratio is 70 percent. Analysts believe that Coca-Cola stock should earn a 9 percent return and that its dividends will grow by 4.50 percent per year indefinitely. Recommend a course of action for an investor interested in taking a position in Coca-Cola stock.

Solution: $\frac{P_0}{E_1} = \frac{D_1/E_1}{(r-g)} = \frac{0.7}{(0.09-0.045)} = 15.56$ $\qquad$ 28 > 15.56.

Coca-Cola's forward price-to-earnings ratio of 28 is much greater than 15.56, which is computed from the equation. Investor expectations of cash flow growth and return are inconsistent with Coca-Cola's forward price to earnings ratio. **Specifically, an investor should consider a short position in Coca-Cola stock in the belief that its price should decline because its current price to earnings ratio is well above what its fundamentals imply**.

Forward Interest Rates: Solution

Step 1: Calculate spot rates.

$$\text{May 1y STRIP} = \frac{\$100}{\$98.028} - 1 = r = 2.012\%$$

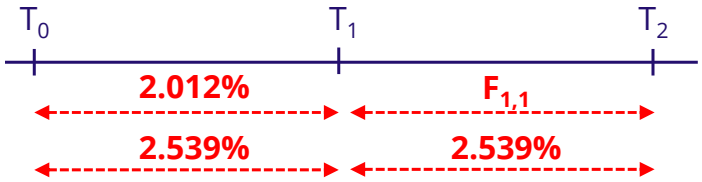
$$\text{May 2y STRIP} = \left(\frac{\$100}{\$95.109}\right)^{1/2} - 1 = r = 2.539\%$$

$$\text{June 1y STRIP} = \frac{\$100}{\$97.402} - 1 = r = 2.667\%$$

$$\text{June 2y STRIP} = \left(\frac{\$100}{\$93.937}\right)^{1/2} - 1 = r = 3.177\%$$

Forward Interest Rates: Solution

Step 2: Calculate forward rates (end of May).



$$(1.02012)(1+F_{1,1}) = (1.02539)^2$$

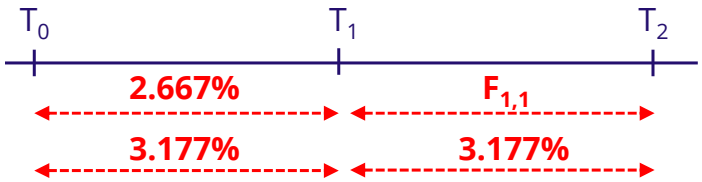
$$\rightarrow F_{1,1} = (1.02539)^2 / 1.02012 - 1$$

$$\rightarrow F_{1,1} = 0.03069 \text{ or } 3.069\%$$

-3

Forward Interest Rates: Solution

Step 3: Calculate forward rates (mid-June).



$$(1.02667)(1+F_{1,1}) = (1.03177)^2$$

$$\rightarrow F_{1,1} = (1.03177)^2 / 1.02667 - 1$$

$$\rightarrow F_{1,1} = 0.03689 \text{ or } 3.689\%$$

-3

Forward Currency Exchange Rates: Example

Current USD/GBP exchange rate = 1.25 USD per GBP. Continuously compounded interest rates are 2% in USD and 2.5% in GBP.

Calculate the no-arbitrage 1-year forward exchange rate.

Solution:

Ratio of USD to GBP today: USD125 to GBP100

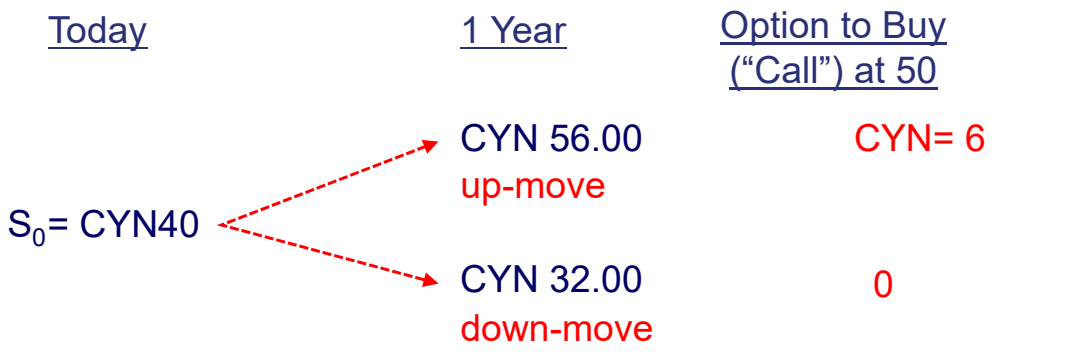
After one year, USD125 will grow to USD125$e^{0.02}$ = USD127.525

After one year, GBP100 will grow to GBP100$e^{0.025}$ = GBP102.532

1-year forward USD/GBP exchange rate = USD127.525 / GBP102.532 = 1.24

-3

Option Pricing: CFAI Example



$$\text{hedge ratio} = \frac{\text{payoff}_{up} - \text{payoff}_{down}}{\text{price}_{up} - \text{price}_{down}} = \frac{6 - 0}{56 - 32} = 0.25$$

-2

Option Pricing: Solution

- Consider buying 0.25 shares and selling one call option.

Value of portfolio after up-move = 0.25(CYN 56) – CYN 6 = CYN 8

Value of portfolio after down-move = 0.25 (CYN 32) – CYN 0 = CYN 8

Portfolio has a risk-free payoff of CYN 8 at the end of one year.

If R_f = 5%, **value of portfolio today = CYN 8 / 1.05 = 7.619**

Hence, 0.25(CYN 40) – C_0 = 7.619 $\therefore C_0$ = CYN 2.381