Quantitative Methods

Hypothesis Testing

Intro and Exam Focus

- Hypothesis testing
  - *Single mean,* difference between means, mean difference, single variance, two variances
- *p*-value of a test
- Errors: Type I vs. Type II
- Parametric vs. nonparametric tests

Hypothesis Test: Example

- We wish to test the hypothesis that the true mean monthly return (**μ**) of a fund manager is 1% with a sample of size 101 and significance of 5%.

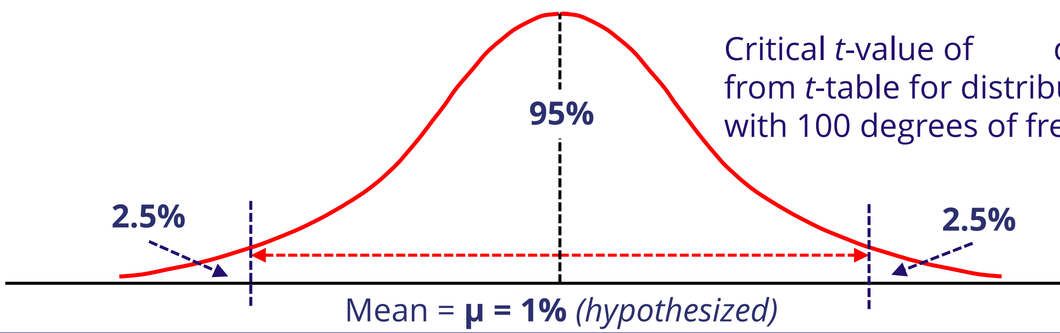
- Central limit theorem: if **μ**= 1%, then the distribution of sample means is a *t*-distribution with n – 1 degrees of freedom, mean = **μ**, and dispersion equal to the standard error, $SE_{\bar{x}} = \mathbf{s}/\sqrt{n}$.

Hypothesis Test: Solution

- We take a sample of size 101 and observe a sample mean, $\bar{X}$, of 1.5%
- Sample standard deviation, s, is 1.4%

df	One-Tailed Probabilities (*p*)		
	p = 0.10	**p = 0.05**	**p = 0.025**
90	1.291	1.662	1.987
100	1.290	1.660	1.984
110	1.289	1.659	1.982

Critical *t*-value of drawn from *t*-table for distribution with 100 degrees of freedom

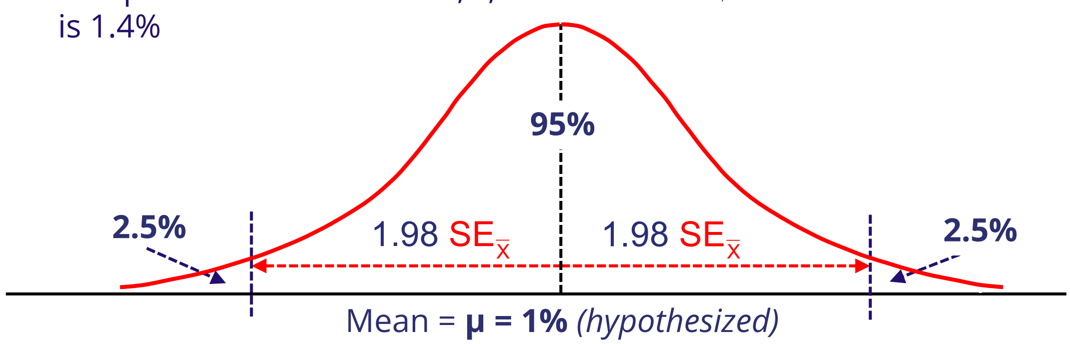


Hypothesis Test: Solution

- We take a sample of size 101 and observe a sample mean, $\bar{X}$, of 1.5%
- Sample standard deviation, s, is 1.4%

t-statistic of observed sample mean:

$$\frac{\bar{x}-\mu_0}{s/\sqrt{n}} = \qquad\qquad =$$



Steps in Hypothesis Testing

1. **State the hypothesis—the relation to be tested.**
   - Null hypothesis (H_0): $\mu = 1\%$
   - Alternative hypothesis (H_a): $\mu \neq 1\%$
2. **Select a test statistic and identify its distribution.**
   - The test statistic was the distance of the observed sample mean $\overline{X}$ from the hypothesized mean of 1% in standard errors. The distribution was a *t*-distribution (σ unknown).
3. **Specify the level of significance.**
   - The significance level for the test was 5%.

Steps in Hypothesis Testing (cont.)

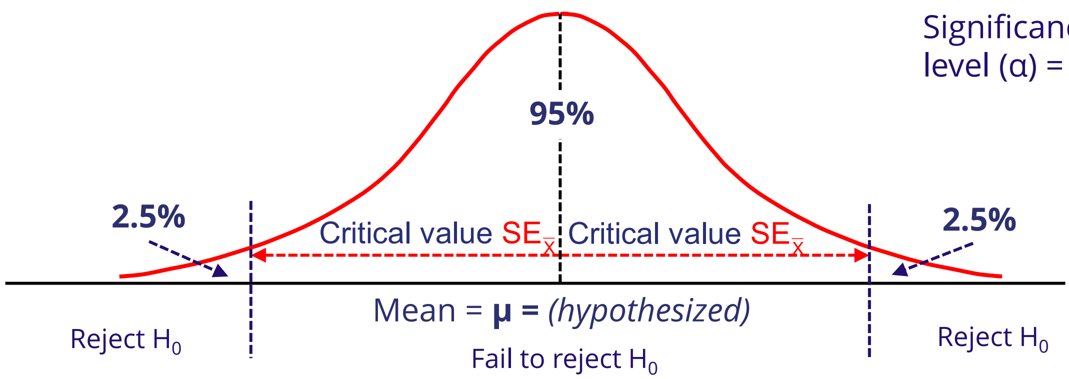
4. **State the decision rule for the hypothesis.**
   - If the test statistic is greater in magnitude than 1.98, then reject the null hypothesis.
5. **Collect the sample and calculate statistics.**
   - A test statistic of 3.6 was calculated based on a sample mean of 1.5% from a sample of size 101.
6. **Make a decision.**
   - Test statistic (3.6) > critical value$_{95\%}$ (1.98) → REJECT NULL

Two-Tailed Test

Use when testing if a population parameter is *different* from a specified value

H_0: $\mu = 0$ vs. H_a: $\mu \neq 0$

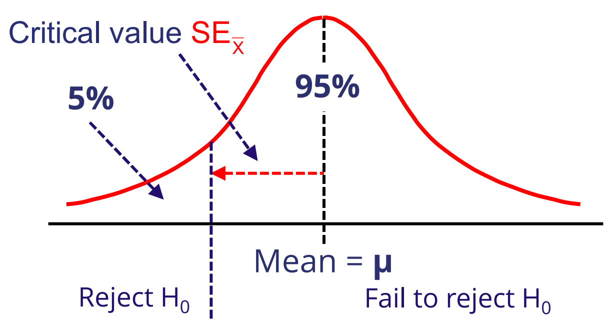
Significance level (α) = 5%



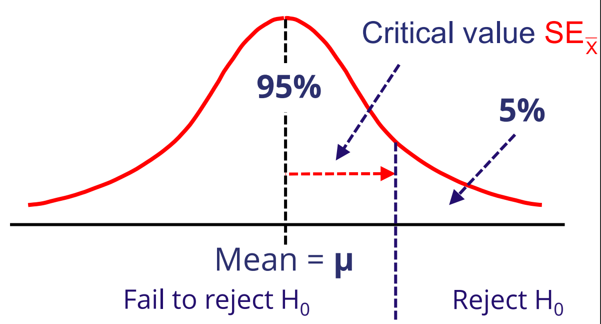
One-Tailed Test

Use when testing to see if a parameter is <u>above</u> or <u>below</u> a specified value

H_0: $\mu \geq 0$ vs. H_a: $\mu < 0$



H_0: $\mu \leq 0$ vs. H_a: $\mu > 0$



One-Tailed Test: Example

Data for a fund's monthly abnormal returns

Sample mean = 0.35% Sample size = 61

Sample std. dev. = 1.5%

Test the hypothesis that a fund's mean return is **less than or equal to** zero at the 5% significance level.

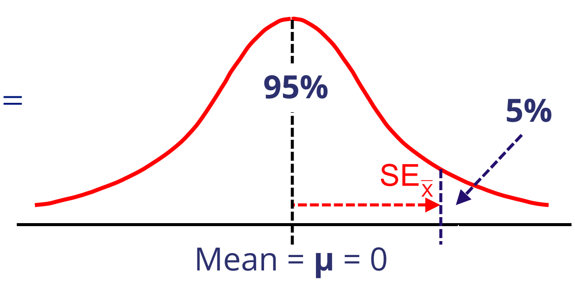
One-Tailed Test: Solution

One-tail test, H_0: $\mu \leq 0$ and H_a: $\mu > 0$

Reject H_0 if the test statistic is >   (is in 5% right tail)

df	One-Tailed Probabilities *(p)*		
	p = 0.10	p = 0.05	p = 0.025
60	1.296	1.671	2.000

$$t\text{-statistic} = \frac{\overline{x}_0 - 0}{s/\sqrt{n}} = \qquad =$$



H_0: evidence that mean monthly abnormal return

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Errors

Type I error: rejecting H_0 when it is actually true

[e.g., convicting an innocent person (null is innocent)]

Type II error: failing to reject H_0 when it is false

(e.g., failing to convict a guilty person)

Probability of Type I error = **significance level (α)**

Power of test is (1 – prob. of Type II Error (β))

p-Value

- The ***p*-value** of a test is the probability of getting the test statistic (or a result more extreme) if the null were true.

p-value < significance level → REJECT

- A *p*-value is the smallest level of significance at which the null can be rejected.
- Example—if the *p*-value of a test is 0.0213 or 2.13%:
  - We **can** reject null at 5% significance
  - We **can** reject null at 3% significance
  - We **cannot** reject null at 1% significance

Testing the Difference Between Means

For *independent* samples from different distributions:

- Same approach as for a single mean, except:
  - Hypothesis relates to μ1 – μ2
  - Test statistic is: $$\frac{(\bar{x}_1 - \bar{x}_2) - \text{hypothesized difference}}{\sqrt{\frac{s_p^2}{n_1} + \frac{s_p^2}{n_2}}}$$
  - Where s_p^2 is "pooled" standard error, calculated as: $$s_p^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}$$
  - *t*-test with $n_1 + n_2 - 2$ degrees of freedom

Testing the Difference Between Means: CFA Institute Example

Suppose we want to test whether the returns of the ACE High Yield Total Return Index, shown below, are different for two different time periods: Period 1 and Period 2.

	Period 1	Period 2
Mean	0.01775%	0.01134%
Standard deviation	0.31580%	0.38760%
Sample size	445 days	859 days

Is there a difference between the mean daily returns in Period 1 and in Period 2, using a 5% level of significance?

Testing the Difference Between Means: Solution

Is there a difference between the mean daily returns in Period 1 and in Period 2, using a 5% level of significance?

Solution:

State the hypotheses:

H_0: vs. H_a:

Identify appropriate test statistic:

t-distribution with = degrees of freedom

Specify critical value: with 5% significance and two tails:

t_{crit} = (large sample → *t*-values ≈ *z*-values)

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Testing the Difference Between Means: Solution

Calculate the test statistic:

$s_p^2 =$ =

Make decision: **test stat** **critical *t*-value (1.96) -** **null**

t-stat = =

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Testing the Mean Difference

For *dependent* samples from related distributions:

- Same approach as for a single mean, except:
  - Hypothesis relates to μ_d
  - Test statistic is:

$$\frac{(\bar{d}) - \text{hypothesized difference}}{\left(\frac{s_d}{\sqrt{n}}\right)}$$

  - *t*-test with n – 1 degrees of freedom

Testing a Single Variance

- Hypothesis relates to σ^2
- Test statistic given by:

$$\frac{(n - 1)s^2}{\text{hypothesized variance}}$$

- Chi-square (χ^2) test with n – 1 degrees of freedom

Testing a Single Variance: Example

A fund manager has a mandate specifying that monthly volatility should be a maximum of 2%. Since inception three years ago, the manager has achieved a mean monthly return of 1% and monthly standard deviation of 2.3%. Test whether this data implies that the true volatility of the manager breaches the mandate restriction with 5% significance.

Testing a Single Variance: Solution

One-tail test, H_0: $\sigma^2 \leq 4$ and H_a: $\sigma^2 > 4$

Reject H_0 if the test statistic is > (95th percentile of χ2 distribution with 36 – 1 = 35 degrees of freedom)

Degrees of Freedom	Probability in Right Tail 0.975	0.95	0.9	0.1	0.05	0.025
30	16.791	18.493	20.599	40.256	43.773	46.979
35	20.569	22.465	24.797	46.059	49.802	53.203
40	24.433	26.509	29.051	51.805	55.758	59.342

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Testing a Single Variance: Solution

$$\chi^2\text{-statistic} = \frac{(n-1)s^2}{\text{hypothesized variance}} = \qquad =$$

$H_{0:}$ that mean monthly volatility return > 2%.

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Testing Two Variances

- Hypothesis relates to σ_1^2 / σ_2^2
- Test statistic given by:

$$\frac{s_1^2}{s_2^2}$$

- *F*-test with $n_1 - 1$ degrees of freedom on numerator and $n_2 - 1$ on the denominator

Testing the Difference Between Variances: CFA Institute Example

You are investigating whether the population variance of returns on a stock market index changed after a change in market regulation. The first 418 weeks occurred before the regulation change, and the second 418 weeks occurred after the regulation change. You gather the data displayed below for 418 weeks of returns both before and after the change in regulation. You have specified a 5 percent level of significance.

	n	Mean Weekly Return (%)	Variance of Returns
Before regulation change	418	0.250	4.644
After regulation change	418	0.110	3.919

Are the variance of returns different before the regulation change versus after the regulation change?

Based on CFA Curriculum Volume 1, page 225, Example 3

Testing the Difference Between Variances: Solution

State the hypotheses:

$$H_0: \sigma^2_{before} = \sigma^2_{after} \text{ vs. } H_a: \sigma^2_{before} \neq \sigma^2_{after}$$

Identify appropriate test statistic:

- *F*-distribution with 417 df on numerator and 417 df on denominator
- *Specify critical value:* with 5% significance and two tails:

F_{crit} = 0.82512 and 1.21194 (from *F*-distribution tables)

Testing the Difference Between Variances: Solution

Calculate the test statistic:

$$F = \frac{\sigma^2_{before}}{\sigma^2_{after}} = \qquad =$$

Make decision: test stat (1.185) falls within 0.82512 and 1.21194

null

-2

Parametric vs. Nonparametric Tests

- **Parametric tests** are based on assumptions about population distributions and population parameters (e.g., *t*-test, *z*-test, *F*-test).
- **Nonparametric tests** make few, if any, assumptions about the population distribution and test things other than parameter values (e.g., runs tests, rank correlation tests).

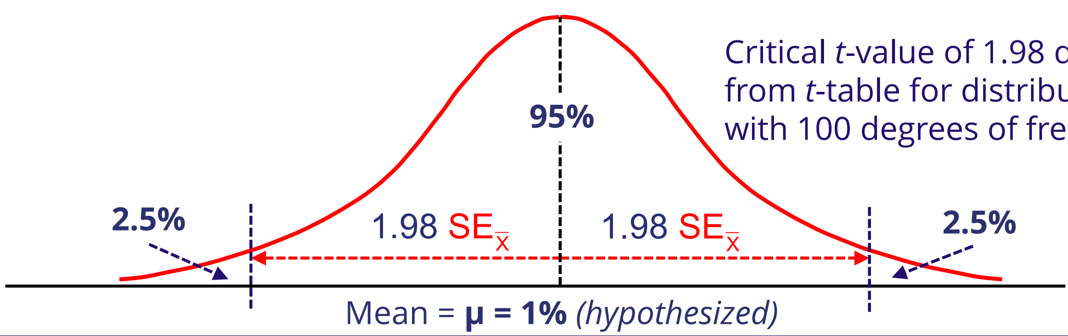
Solutions

Hypothesis Test: Solution

- We take a sample of size 101 and observe a sample mean, $\overline{X}$, of 1.5%
- Sample standard deviation, s, is 1.4%

df	One-Tailed Probabilities (p)		
	p = 0.10	p = 0.05	p = 0.025
90	1.291	1.662	1.987
100	1.290	1.660	1.984
110	1.289	1.659	1.982

Critical *t*-value of 1.98 drawn from *t*-table for distribution with 100 degrees of freedom



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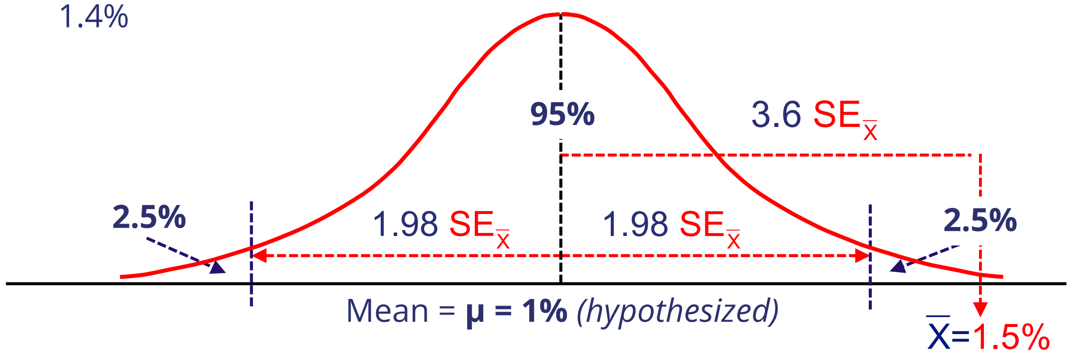
Hypothesis Test: Solution

- We take a sample of size 101 and observe a sample mean, $\overline{X}$, of 1.5%
- Sample standard deviation, s, is 1.4%

t-statistic of observed sample mean:

$$\frac{\overline{x}-\mu_0}{s/\sqrt{n}} = \frac{1.5-1.0}{1.4/\sqrt{101}} = 3.6$$



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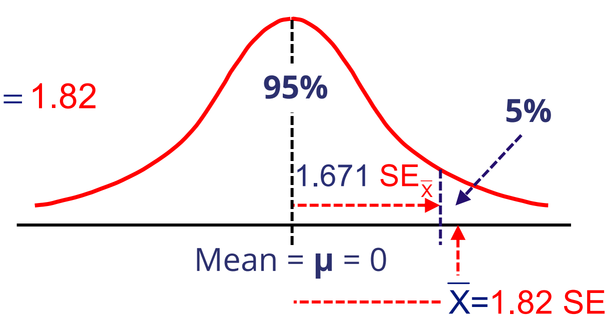
One-Tailed Test: Solution

One-tail test, H_0: $\mu \le 0$ and H_a: $\mu > 0$
Reject H_0 if the test statistic is > 1.671
(is in 5% right tail)

df	One-Tailed Probabilities (*p*) p = 0.10	p = 0.05	p = 0.025
60	1.296	1.671	2.000

$$\text{t-statistic} = \frac{\overline{x}_0-0}{s/\sqrt{n}} = \frac{0.35\%}{1.5\%/\sqrt{61}} = 1.82$$



Reject H_0: evidence that mean monthly abnormal return > 0%.

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Testing the Difference Between Means: Solution

Is there a difference between the mean daily returns in Period 1 and in Period 2, using a 5% level of significance?

Solution:

State the hypotheses:

$$H_0: \mu_1 = \mu_2 \text{ vs. } H_a: \mu_1 \neq \mu_2$$

Identify appropriate test statistic:

t-distribution with 445 + 859 – 2 = 1,302 degrees of freedom

Specify critical value: with 5% significance and two tails:

t_{crit} = 1.96 (large sample → t-values ≈ *z*-values)

-3

Testing the Difference Between Means: Solution

Calculate the test statistic:

$$s_p^2 = \frac{(445-1)(0.31580^2) + (859-1)(0.38760^2)}{445+859-2} = 0.1330$$

Make decision: **test stat (0.30) < critical t-value (1.96) → fail to reject null**

$$t\text{-stat} = \frac{(0.01775 - 0.01134) - 0}{\sqrt{\frac{0.1330}{445} + \frac{0.1330}{859}}} = 0.30$$

-3

Testing a Single Variance: Solution

One-tail test, H_0: $\sigma^2 \leq 4$ and H_a: $\sigma^2 > 4$

Reject H_0 if the test statistic is > 49.80 (95th percentile of χ2 distribution with 36 –1 = 35 degrees of freedom)

Degrees of Freedom	Probability in Right Tail					
	0.975	**0.95**	**0.9**	**0.1**	**0.05**	**0.025**
30	16.791	18.493	20.599	40.256	43.773	46.979
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-1

Testing a Single Variance: Solution

$$\chi^2\text{-statistic} = \frac{(n-1)s^2}{\text{hypothesized variance}} = \frac{35 \times 2.3^2}{2^2} = 46.29$$

Fail to reject H_0: no evidence that mean monthly volatility return > 2%.

-2

Testing the Difference Between Variances: Solution

Calculate the test statistic:

$$F = \frac{\sigma^2_{before}}{\sigma^2_{after}} = \frac{4.644}{3.919} = 1.185$$

Make decision: test stat (1.185) falls within 0.82512 and 1.21194

→ fail to reject null

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