An Introduction to Exotic and Path-dependent Options

In this lecture...

- how to classify options according to important features
- how to think about derivatives in a way that makes it easy to compare and contrast different contracts
- the names and contract details for many basic types of exotic options

Discrete cashflows

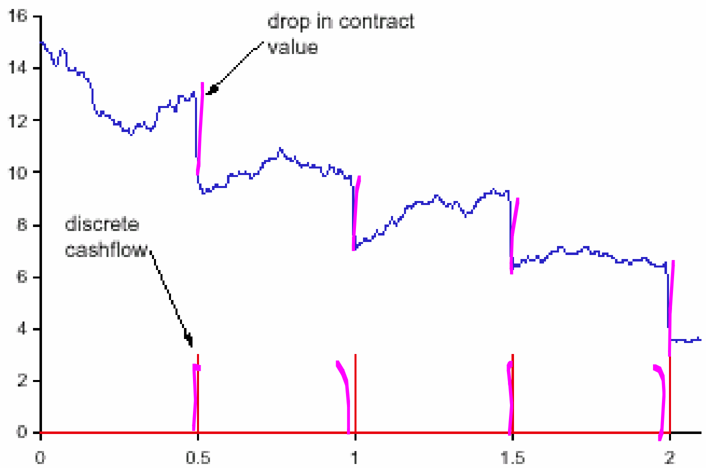
Imagine a contract that pays the holder an amount q at time t_q. The contract could be a bond and the payment a coupon.

If we use $V(t)$ to denote the contract value and t_q^- and t_q^+ to denote just before and just after the cashflow date then simple arbitrage considerations lead to

- $$V(t_q^-) = V(t_q^+) + q.$$

This is a **jump condition**.

The value of the contract jumps by the amount of the cashflow. The behaviour of the contract value across the payment date is shown in the figure.



A discrete cashflow and its effect on a contract value.

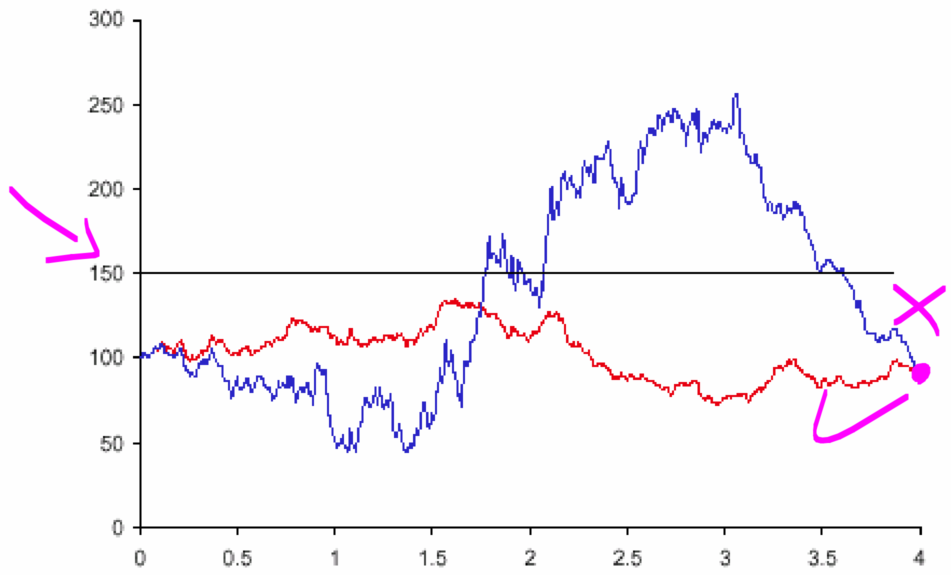
If the contract is contingent on an underlying variable so that we have $V(S,t)$ then we can accommodate cashflows that depend on the level of the asset S i.e. we could have $q(S)$.

Weak path dependence

- Options whose value depends on the asset history, but can still be written as $V(S,t)$ are said to be **weakly path dependent**.

One of the most common reasons for weak path dependence in a contract is a **barrier**. Barrier (or knock-in, or knock-out) options are triggered by the action of the underlying hitting a prescribed value at some time before expiry.

For example, as long as the asset remains below 150, the contract will have a call payoff at expiry. However, should the asset reach this level before expiry then the option becomes worthless; the option has 'knocked out.'



Two paths having the same value at expiry but with completely different payoffs.

Strong path dependence

Of particular interest, mathematical and practical, are the **strongly path-dependent contracts**. These have payoffs that depend on some property of the asset price path in addition to the value of the underlying at the present moment in time; in the equity option language, we cannot write the value as $V(S,t)$.

- The contract value is a function of at least one more independent variable.

Example:

The Asian option has a payoff that depends on the average value of the underlying asset from inception to expiry. We must keep track of more information about the asset price path than simply its present position.

The extra information that we need is contained in the ‘running average.’ This is the average of the asset price from inception until the present, when we are valuing the option.

Early exercise

Early exercise is a common feature. For example, the conversion of convertible bonds is mathematically identical to the early exercise of an American option.

- The key point about early exercise is that the holder of this valuable right should ideally act *optimally*, i.e. they must decide *when* to exercise or convert.

Time dependence

Here we are concerned with time dependence in the option contract. We can add such time dependence to any of the features described above.

For example, early exercise might only be permitted on certain dates or during certain periods. This intermittent early exercise is a characteristic of **Bermudan options**.

Similarly, the position of the barrier in a knock-out option may change with time. Every month it may be reset at a higher level than the month before.

- These contracts are referred to as **time inhomogeneous**.

Dimensionality

Dimensionality refers to the number of underlying independent variables.

- The vanilla option has two independent variables, S and t, and is thus two dimensional.

- The weakly path-dependent contracts have the same number of dimensions as their non-path-dependent cousins, i.e. a barrier call option has the same two dimensions as a vanilla call.

We can have two types of three-dimensional problem.

- The first occurs when we have a **second source of randomness**, such as a second underlying asset.

  We might, for example, have an option on the maximum of two equities.

- The other type of problem that is also three dimensional is the **strongly path-dependent** contract.

  The new independent variable is a measure of the path-dependent quantity on which the option is contingent.

The order of an option

The basic, vanilla options are of first order. Their payoffs depend only on the underlying asset, the quantity that we are *directly* modeling. Other, path-dependent, contracts can still be of first order if the payoff only depends only on properties of the asset price path.

- **Higher order** refers to options whose payoff, and hence value, is contingent on the value of *another* option.

The obvious first-order options are compound options, for example, a call option giving the holder the right to buy a put option. The compound option expires at some date T_1 and the option on which it is contingent, expires at a later time T_2. Technically speaking, such an option is weakly path dependent.

From a practical point of view, the compound option raises some important modeling issues.

- The payoff for the compound option depends on the *market* value of the underlying option, and not on the theoretical price.

If you hold a compound option, and want to exercise the first option then you must take possession of the underlying option. High order option values are very sensitive to the basic pricing model and should be handled with care.

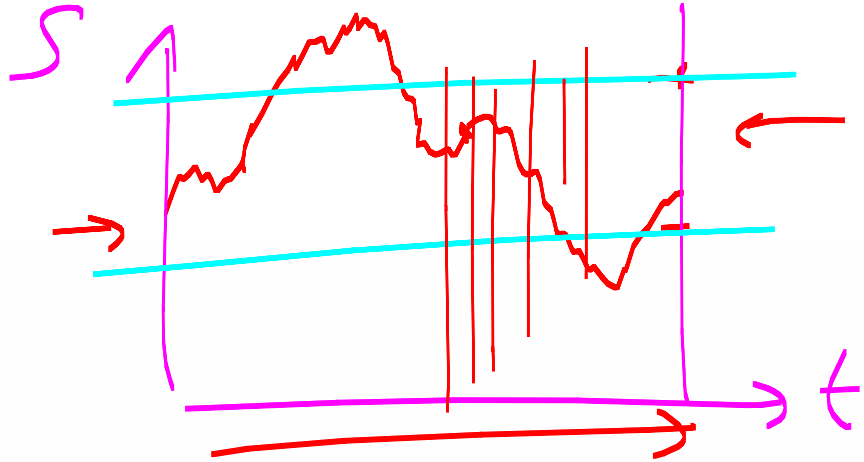
Decisions

Holding an American option you are faced with the decision whether and when to exercise your rights. The American option is the most common contract that contains within it a decision feature. Other contracts require more subtle and interesting decisions to be made.

Example: The passport option is an option on a trading account. You buy and sell some asset, if you are in profit on the expiry of the option you keep the money, if you have made a loss it is written off. The decisions to be made here are when to buy, sell or hold, and how much to buy, sell or hold.

Examples:

- Range notes
- Barrier options
- Asian options
- Lookback options



Range notes

Range notes are very popular contracts, existing on the 'lognormal' assets such as equities and currencies, and as fixed-income products.

- In its basic, equity derivative, form the range note pays at a rate of L all the time that the underlying lies within a given range, $S_l \le S \le S_u$.

That is, for every dt that the asset is in the range you receive $L\,dt$. Introducing $\mathcal{I}(S)$ as the function taking the value 1 when $S_l \le S \le S_u$ and zero otherwise, the range note satisfies

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV + L\mathcal{I}(S) = 0.$$

Barrier options

- **Barrier options** have a payoff that is contingent on the underlying asset reaching some specified level before expiry.

The critical level is called the barrier, there may be more than one. Barrier options are weakly path dependent.

Barrier options come in two main varieties, the ‘in’ barrier option (or **knock-in**) and the ‘out’ barrier option (or **knock-out**). The former only have a payoff if the barrier level is reached before expiry and the latter only have a payoff if the barrier is *not* reached before expiry.

These contracts are weakly path dependent.

Asian options

- **Asian options** have a payoff that depends on the average value of the underlying asset over some period before expiry.

They are strongly path dependent. Their value prior to expiry depends on the path taken.

The average used in the calculation of the option's payoff can be defined in many different ways.

It can be an **arithmetic average** or a **geometric average**, for example.

The data could be **continuously sampled**, so that every realized asset price over the given period is used. More commonly, for practical and legal reasons, the data is usually **sampled discretely**

Lookback options

- **Lookback options** have a payoff that depends on the realized maximum or minimum of the underlying asset over some period prior to expiry.

An extreme example, that captures the flavour of these contracts is the option that pays off the difference between that maximum realised value of the asset and the minimum value over the next year. Thus it enables the holder to buy at the lowest price and sell at the highest, every trader's dream.

Again the maximum or minimum can be calculated **continuously** or **discretely**, using every realized asset price or just a subset.