Feynman-Kač :

Assume that (1) $V(t,S)$ solves a boundary value problem:

$$\frac{\partial V}{\partial t}(t,S) + \mu(t,S)\frac{\partial V}{\partial S}(t,S) + \frac{1}{2}\sigma^2(t,S)\frac{\partial^2 V}{\partial S^2} - rV(t,S) = 0$$

$$V(T,S) = G(S)$$

(2) and that the process S follows the dynamics:

$$dS = \mu(t,S)\,dt + \sigma(t,S)\,dX(t), \quad S(0) = S_0$$

Then, the function V has a representation as an expectation

$$V(t,S) = \mathbb{E}\left[e^{-r(T-t)}G(S_T) \,|\, \mathcal{F}_t\right]$$

$\mu(t,S) = rS$

$\sigma(t,S) = \sigma S$

In the Black-Scholes model we saw yesterday

$$V(t,S) = e^{-r(T-t)}\, \mathbb{E}^{Q}\left[G(S_T) \mid F_t \right]$$

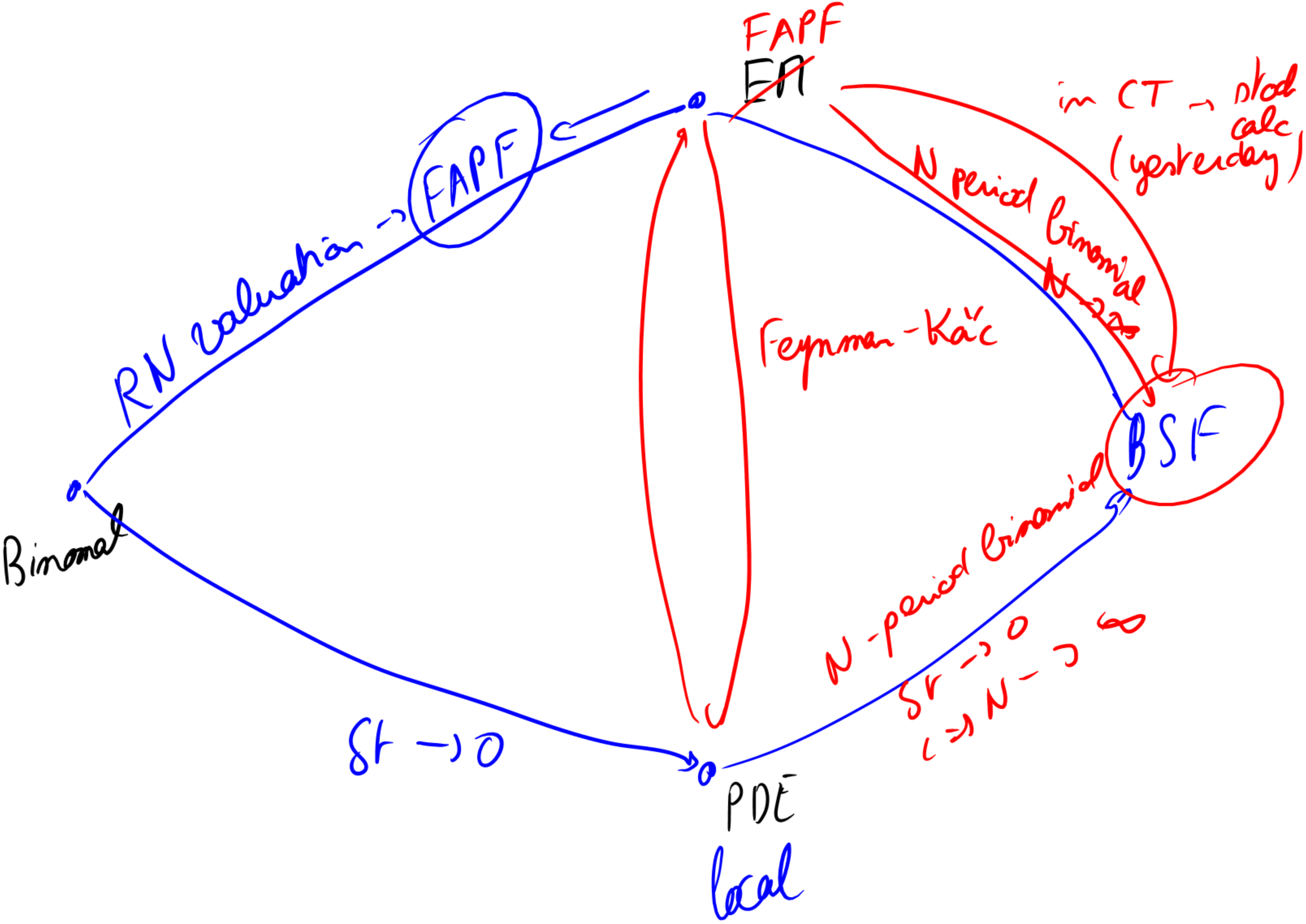
where S_t follows the dynamics

$$dS_t = r S_t\, dt + \sigma S_t\, dX^{Q}(t)$$

→ By Feynman Kač, $V(t,S)$ solves the boundary problem:

$$\begin{cases} \dfrac{\partial V}{\partial t} + r S \dfrac{\partial V}{\partial S} + \dfrac{1}{2} \sigma^2 S^2 \dfrac{\partial^2 V}{\partial S^2} - rV = 0 \\ V(T,S) = G(S) \end{cases}$$

$$X_t^{Q} = X_t^{P} + \int_0^t \theta(s)\, ds$$



FAPF
~~EM~~
RN valuation → FAPF
in CT → stoch calc (yesterday)
N period binomial
~~N → ∞~~
Feynman-Kač
BSF
Binomial
δt → 0
PDE
local
N-period binomial
δt → 0
⟺ N → ∞

Δ-hedging / No Arb

1) You create a portfolio

$$\Pi = V - \Delta S$$

Our objective is to hedge the risk so Π becomes a hedged portfolio

2) Π will be hedged if it is riskless, that is, its value in the up state and down state is the same

$$\Pi_u = \Pi_d \iff V_u - \Delta S_u = V_d - \Delta S_d$$

$$\Rightarrow \boxed{\Delta = \frac{V_u - V_d}{S_u - S_d}}$$

3) Our portfolio is now risk-free. So, to prevent arbitrage, it must earn the risk-free rate

$$\Pi_u = \Pi_d = (1 + r\delta t)\,\Pi_0$$

$$\Leftrightarrow \Pi_u = \frac{1}{D}\Pi_0$$

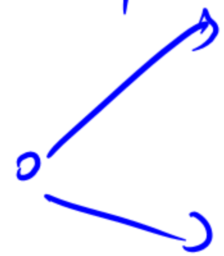
$$\Leftrightarrow V_u - \Delta S_u = \frac{1}{D}\left(V_0 - \Delta S_0\right)$$

$$\Leftrightarrow V_u - \frac{V_u - V_d}{(u-d)S_0} = \frac{1}{D}\left(V_0 - \frac{V_u - V_d}{(u-d)S_0} \cdot S_0\right)$$

$\Leftrightarrow$ . . $\Leftrightarrow$ formula (1)

Risk Neutral Valuation :

- On average, the stock must return the risk-free rate



$$\mathbb{E}^{RN}[S_T] = (1+r\delta t) S_0$$

$$\Leftrightarrow p^* S_u + (1-p^*) S_d = (1+r\delta t) S_0$$

$$\Leftrightarrow p^* S_0 u + (1-p^*) S_0 d = (1+r\delta t) S_0$$

$$\Leftrightarrow \boxed{p^* = \frac{\frac{1}{D} - d}{u-d}}$$

$\boxed{\text{RN probability}}$

For the option :

$$\mathbb{E}^{RN}[V_T] = (1+r\delta t)\, V_0$$

$$\Leftrightarrow p^* V_u + (1-p^*) V_d = (1+r\delta t) V_0$$

$$\Leftrightarrow V_0 = D \left[\underbrace{p^* V_u + (1-p^*) V_d}_{\text{Payoffs}} \right]$$

$$0 < p_1 < 1 \longrightarrow \mathbb{P}_1$$

$$0 < p_2 < 1 \longrightarrow \mathbb{P}_2$$

$\Big\}$ equivalent

Let's try to implement what we saw last night in the Rumble binomial model.

- $\mathbb{P}$ - measure

$$\begin{tikzcd}[column sep=small] & S_0 u \\ S_0 \arrow[ur, "p"] \arrow[dr] & \\ & S_0 d \end{tikzcd}$$

NOT a Martingale

- look for a measure $\mathbb{Q}$ such that the discounted stock price is a martingale $\Rightarrow$ find $\mathbb{Q}$ such that

$$\mathbb{E}^{\mathbb{Q}}\left[D S_T\right] = S_0$$

Martingale condition

$0 < q < 1$

$$\cancel{\mathbb{E}^{\mathbb{Q}}\left[D_T S_T \mid \mathcal{F}_t\right] = D_t S_t}$$

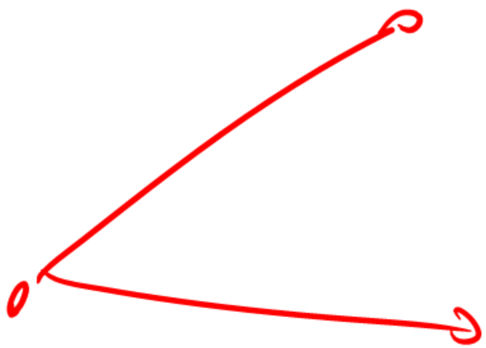
$\mathbb{Q} \to q$

$$= q D \cancel{S_0} u + (1-q) D \cancel{S_0} d = \cancel{S_0}$$

$$\Leftrightarrow \boxed{q = \frac{\frac{1}{D} - d}{u - d} = p^*}$$

① $p^* = q$ RN $\equiv$ EMM

② There is a UNIQUE EMM!!!



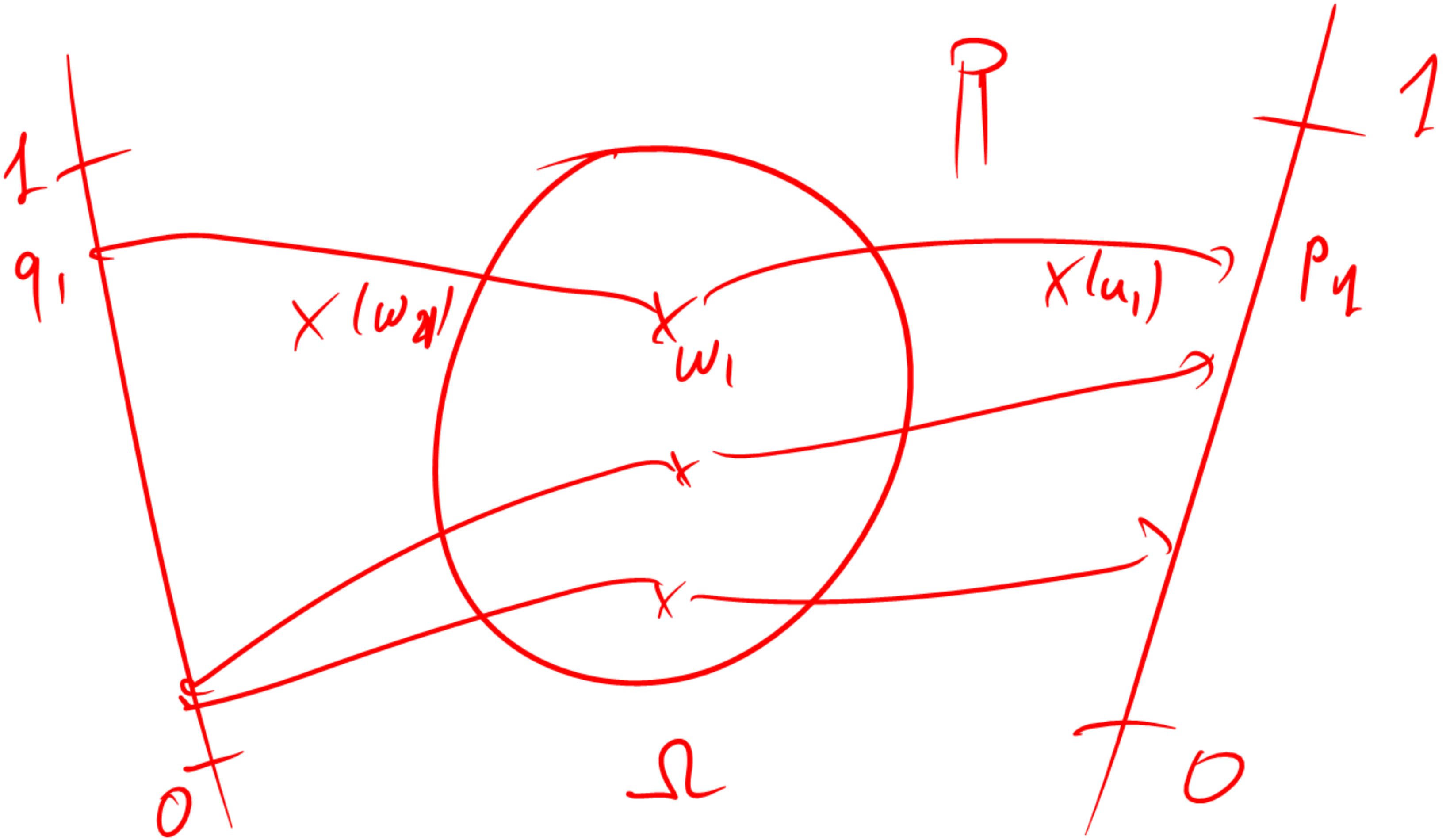
$p \in (0,1)$

$p_1 \in (0,1)$

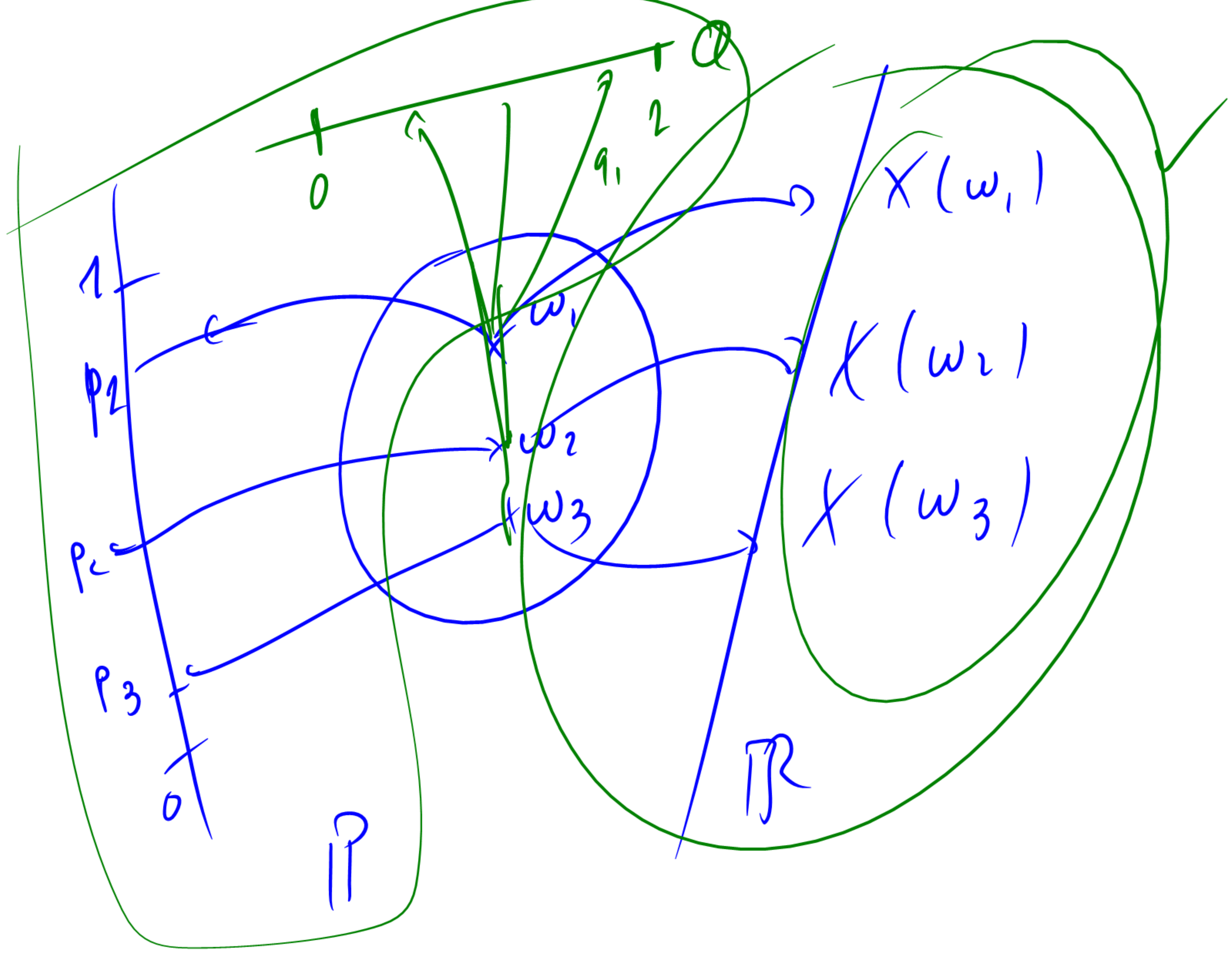
$p_2 \in (0,1)$

$\vdots$

$\frac{\mu - r}{\sigma}$



$\mathbb{P}$
1
q_1
$X(\omega_2)$
ω_1
$X(\omega_1)$
p_1
1
0
Ω
0



Ω
0
q₁
2
1
p_2
p_1
p_3
0
ω_1
ω_2
ω_3
$X(\omega_1)$
$X(\omega_2)$
$X(\omega_3)$
$\mathbb{P}$
$\mathbb{R}$

RN pricing formula

$$V_0 = D\left(\cancel{p^*}\,V_u + (1-\cancel{p^*})\,V_d\right)$$

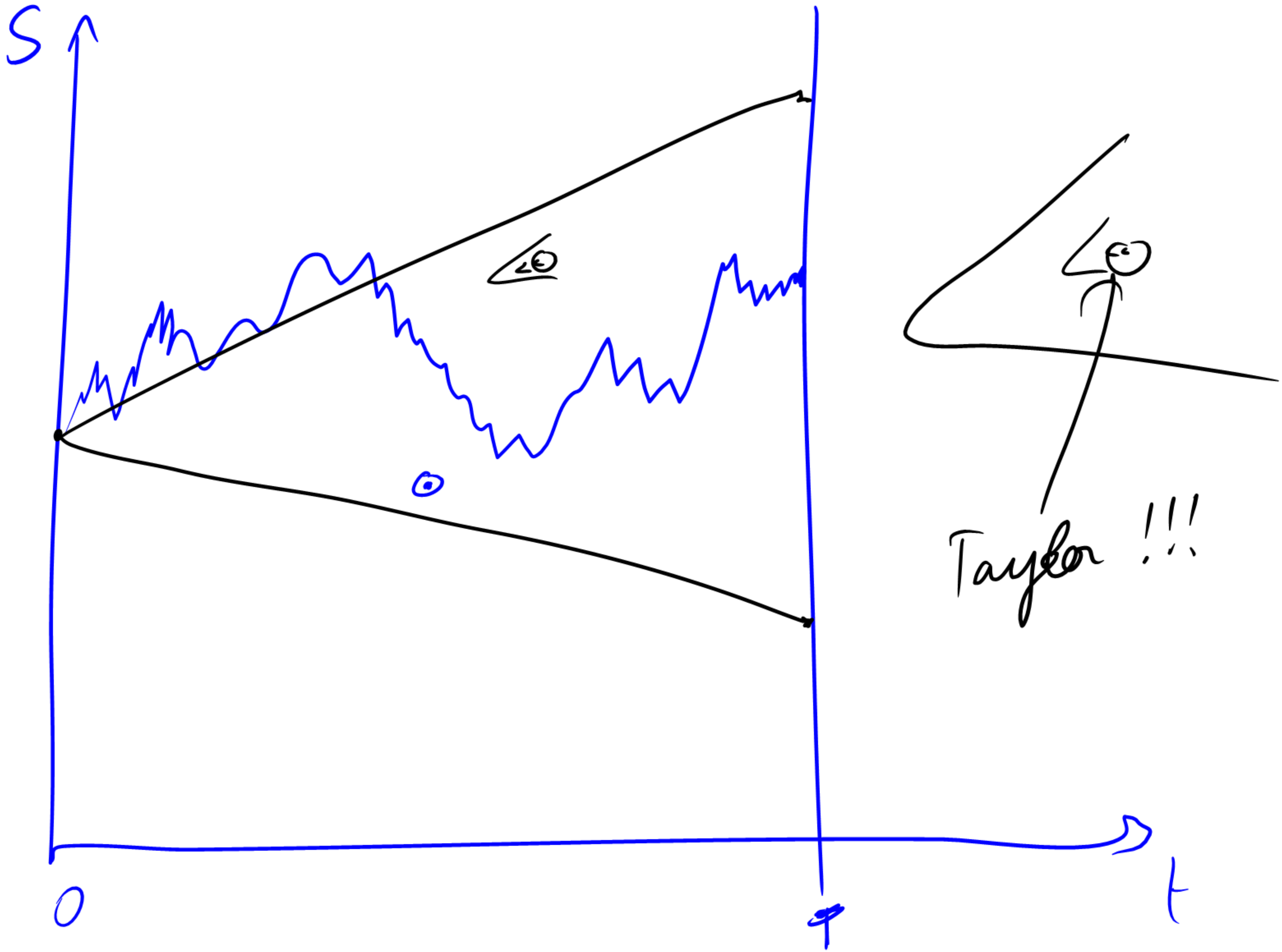
$q \qquad\qquad q$

$$= q\,(D V_u) + (1-q)\,(D V_d)$$

$$\boxed{V_0 = \mathbb{E}^{\mathbb{Q}}\left[D V_T\right]}$$

FAPF

1-period binomial model


S
0
T
t
Taylor !!!

$$u = e^{\sigma\sqrt{\delta t}} \approx 1 + \sigma\sqrt{\delta t} + \frac{1}{2}\sigma^2 \delta t$$

$$d = e^{-\sigma\sqrt{\delta t}} \approx 1 - \sigma\sqrt{\delta t} + \frac{1}{2}\sigma^2 \delta t$$

$$D = e^{-r\delta t} \approx 1 - r\delta t$$

$$u - d = e^{\sigma\sqrt{\delta t}} - e^{-\sigma\sqrt{\delta t}} \approx 2\sigma\sqrt{\delta t}$$

$$\frac{1}{D} - d = e^{r\delta t} - e^{-\sigma\sqrt{\delta t}} \approx \sigma\sqrt{\delta t} + \left(r - \frac{1}{2}\sigma^2\right)\delta t$$

$$u - \frac{1}{D} \approx \cancel{1} + \sigma\sqrt{\delta t} + \frac{1}{2}\sigma^2 \delta t - \cancel{1} - r\delta t$$

$$\approx \sigma\sqrt{\delta t} + \left(\frac{1}{2}\sigma^2 - r\right)\delta t$$

By Taylor,

$$V_u = V + \frac{\partial V}{\partial t}\delta t + \frac{\partial V}{\partial S}\delta S_u + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(\delta S)^2 + \dots$$

$$\begin{aligned}
\delta S_u &= S_u - S \\
&= S\left(e^{\sigma\sqrt{\delta t}} - 1\right) \\
&\simeq S\left(\cancel{1} + \sigma\sqrt{\delta t} + \frac{1}{2}\sigma^2\delta t - \cancel{1}\right) \\
&\simeq S\left(\sigma\sqrt{\delta t} + \frac{1}{2}\sigma^2\delta t\right)
\end{aligned}$$

$$\boxed{(\delta S_u)^2 \simeq S^2\sigma^2\delta t}$$

$$V_d = V + \frac{\partial V}{\partial t}\delta t + \frac{\partial V}{\partial S}\delta S_d + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(\delta S_d)^2$$

$$\begin{aligned}\delta S_d &= S_d - S \\ &= S e^{-\sigma\sqrt{\delta t}} - S \\ &= S\left(-\sigma\sqrt{\delta t} + \frac{1}{2}\sigma^2 \delta t\right)\end{aligned}$$

$$(\delta S_d)^2 = S^2 \sigma^2 \delta t$$

$k = 2$: applying the RN valuation formula

$$V(N-2, j) = D\left[p^* V(N-1, j+1) + (1-p^*) V(N-1, j)\right]$$

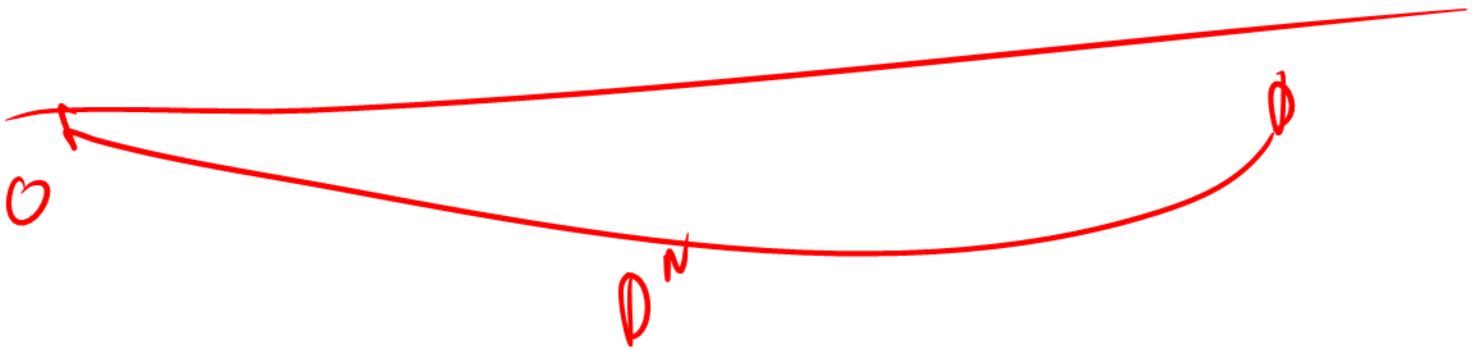
$$= D\Big[p^*\big(D(p^* V(N, j+2) + (1-p^*) V(N, j+1))\big) + (1-p^*)\big(D(p^* V(N, j+1) + (1-p^*) V(N, j))\big)\Big]$$

$$= D^2\left[1\, p^{*2} V(N, j+2) + 2 p^*(1-p^*) V(N, j+1) + 1\,(1-p^*)^2 V(N, j)\right]$$

$$D^N \sum_{i=0}^{N} C_N^i (p^*)^i (1-p^*) f(X_i)$$

$$= D^N \mathbb{E}^{B(N,p^*)}\left[f(X) \right]$$

$$\underbrace{\max\left[S(N,i) - K, 0 \right]}_{\text{Payoff}}$$



$$V_0 = D^N \mathbb{E}^{B(N,p^*)}\left[\underbrace{\max\left[S(N,i)-E,0\right]}_{(S(N,i)-E)\times \mathbb{1}\{S(N,i)-E>0}\right]$$

$$= \underbrace{D^N \mathbb{E}^{B(N,p^*)}\left[S(N,i)\,\mathbb{1}\{S(N,i)-E>0\}\right]}_{①\ \text{Tough!}}$$

$$\underbrace{+\ D^N \mathbb{E}^{B(N,p^*)}\left[E\,\mathbb{1}\{S(N,i)>E\}\right]}_{②\ \text{Easy!}}$$

② . $D^N \, \mathbb{E}^{B(N,p^*)}\left[E \, \mathbb{1}\{S(N) > E\} \right]$

$E D^N \, \mathbb{E}^{B(N,p^*)}\left[\mathbb{1}\{S(N) > E\} \right]$

$= E D^N \, P^*\left[S(N) > E \right]$ $\quad N \to \infty$

$\longrightarrow E e^{-rT} N(d_2)$

① $D^N \mathbb{E}^{B(N,p^*)}\left[S(N)\,\mathbb{1}_{\{S(N)>E\}}\right]$

$$= S_0 D^N \sum_{i=0}^{N} C_N^i (p^*)^i (1-p^*)^{N-i}\, u^i d^{N-i}\, \mathbb{1}_{\{S(N)>E\}} \quad (*)$$

$i + N - i = N$

Set $\bar{p} = p^* D u$

$(1-\bar{p}) = (1-p^*) d D$

$$(*)\quad S_0 \sum_{i=0}^{N} \underbrace{C_N^i (\bar{p})^i (1-\bar{p})^{N-i}}_{B(N,\bar{p})} \mathbb{1}_{\{S(N)>E\}}$$

$$= S_0 \boxed{\mathbb{E}^{B(N,\bar{p})}\left[\mathbb{1}_{\{S(N)>E\}}\right]} \xrightarrow{N\to\infty} SN(d_1)$$