Mod.1 Lecture 4 - Exercises

Stochastic Differential Equations

X_t is a Brownian Motion (Wiener Process) and dX_t or $dX(t)$ is its increment. $X_0 = 0$.

1. The change in a share price $S(t)$ satisfies
$$dS = A(S,t)\,dX_t + B(S,t)\,dt,$$
for some functions A and B. If $f = f(S,t)$, then Itô's lemma gives the following SDE
$$df = \left(\frac{\partial f}{\partial t} + B\frac{\partial f}{\partial S} + \frac{1}{2}A^2\frac{\partial^2 f}{\partial S^2}\right)dt + A\frac{\partial f}{\partial S}dX_t.$$
Can A and B be chosen so that a function $g = g(S)$ has a change which has zero drift, but non-zero diffusion? State any appropriate conditions.

2. Show that $F(X_t) = \arcsin(2aX_t + \sin F_0)$ is a solution of the SDE
$$dF = 2a^2(\tan F)\left(\sec^2 F\right)dt + 2a(\sec F)\,dX_t,$$
where F_0 and a is a constant. The following standard result may be used
$$\frac{d}{dx}\sin^{-1}ax = \frac{a}{\sqrt{1-a^2x^2}}$$

3. Show that
$$\int_0^t X_\tau\left(1 - e^{-X_\tau^2}\right)dX_\tau = \overline{F}(X_t) + \int_0^t G(X_\tau)\,d\tau.$$
where the functions $\overline{F}$ and G should be determined.

4. Consider the process
$$d(\log y) = (\alpha - \beta\log y)\,dt + \delta dX_t.$$
The parameters $\alpha,\ \beta,\ \delta$ are constant. Show that y satisfies
$$\frac{dy}{y} = \left(\alpha - \beta\log y + \frac{1}{2}\delta^2\right)dt + \delta dX_t.$$

5. Show that
$$G = e^{t + ae^{X_t}}$$
is a solution of the stochastic differential equation
$$dG(t) = G\left(1 + \tfrac{1}{2}(\ln G - t) + \tfrac{1}{2}(\ln G - t)^2\right)dt + G(\ln G - t)\,dX,$$
where a is a constant.

6. The Ornstein-Uhlenbeck process satisfies the spot rate SDE given by
$$dr_t = \kappa(\theta - r_t)\,dt + \sigma dX_t,\ \ r_0 = u,$$
where κ, θ and σ are constants. Solve this SDE by setting $Y_t = e^{\kappa t}r_t$ and using Itô's lemma to show that
$$r_t = \theta + (u - \theta)e^{-\kappa t} + \sigma\int_0^t e^{-\kappa(t-s)}dX_s.$$