[Mod2.4] Market Risk Measurement Methods Solutions pg1

CQF Value at Risk

Solutions

1. Consider a position of £5 million in a single asset X with daily volatility of 1%. What are the annualised and 10-day standard deviations? Using the Normal factor calculate 99%/10day VaR in money terms.

   Solution:
   In order to annualise volatility we use the additivity of variance,

   $$\sigma_{1Y} = \sqrt{\sigma_{1D}^2 \times 252} = \sigma_{1D}\sqrt{252} = 0.01 \times \sqrt{252} \approx 0.16$$

   Notice that 1% daily volatility equates approximately to 16% volatility per annum.

   In order to calculate Value at Risk we need the value of Factor which corresponds to the $c = 99\%$ confidence. Using tables for the Normal Distribution we identify the factor value that cuts 1% on the left tail as $\Phi(-2.33) = 0.01$.

   $$\mathrm{VaR}_{99\%/10D} = \Phi^{-1}(1 - 0.99) \times \sigma_{10D} \times \Pi = 2.33 \times 0.01 \times \sqrt{10} \times £5\,\text{million} = £368,405$$

   where Π is portfolio value (for one asset).

2. Now, consider a portfolio of two assets X and Y, £100,000 investment each. The daily volatilities of both assets are 1% and correlation between their returns is $\rho_{XY} = 0.3$. Calculate 99%/5day Analytical VaR (in money terms) for this portfolio.

   Solution:
   The standard deviation in money terms is $\sigma_X = \sigma_Y = £1000$, which is 1% from £100,000. The variance of the portfolio's daily change is

   $$\begin{aligned} \sigma_\Pi^2 &= \sigma_X^2 + 2\rho_{XY}\,\sigma_X\sigma_Y + \sigma_Y^2 \\ \sigma_\Pi^2 &= 1000^2 + 2 \times 0.3 \times 1000 \times 1000 + 1000^2 = 2.6 \times 10^6 \end{aligned}$$

   which gives the standard deviation for the portfolio (its daily change) $\sigma_\Pi = £1,612.45$.

   Scaling to 5 days and using the factor value for $c = 99\%$ confidence, the result is

   $$\mathrm{VaR}_{99\%/5D} = 2.33 \times 1612.45 \times \sqrt{5} = £8,401.$$

   Question 1 and 2 calculations assume that portfolio value (its cumulative P&L) follows the Normal Distribution. 99% VaR risk measure represents any move beyond 2.33 standard deviations, however we do not know how worse the move (loss) can be.

The exercises have been edited by CQF Faculty, Richard Diamond, R.Diamond@fitchlearning.com

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3. Assume that P&L of an investment portfolio is a random variable that follows Normal distribution $X \sim N(\mu, \sigma^2)$. Use the definition of *VaR as a percentile* to derive analytical expression for VaR calculation.

   Hint: Start with probability argument for the P&L (loss) X exceeding VaR(X) threshold and convert X to a Standard Normal variable ϕ.

   Solution:
   The probability of loss $X < 0$ being worse than VaR < 0 is

   $$\Pr\left(X \leq \mathrm{VaR}(X)\right) \quad = \quad 1 - c$$

   Note that if P&L X is a random variable then VaR(X) is also a random variable. In order to use the well-known Normal Distribution functions, we have to work with the Standard Normal variable

   $$\begin{aligned} \Pr(\phi \leq \frac{\mathrm{VaR}(X) - \mu}{\sigma}) &= 1 - c \qquad \Longrightarrow \\ \mathrm{VaR}(X) &= \mu + \Phi^{-1}(1-c) \times \sigma \end{aligned}$$

   Inverse CDF for a probability distribution is known as 'percentile function'.

4. Assume 'elliptical markets': asset returns are Normally distributed or close. What percentage of returns are outside two standard deviations from the mean? Consider the left tail.

   Within that tail, what is the mean of standardised returns – that is, what is an average tail loss? Provide analytical solutions for abstract μ, σ using a simplifying assumption of Standard Normal Distribution.

   PDF for Normal Distribution $N(\mu, \sigma^2)$ is $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$.

   Solution:
   The percentage of returns outside n standard deviations (in general) on the left tail is the cumulative density function $\Phi(x)$, which is an integral over probability density

   $$\int_{-\infty}^{\mu - n\sigma} f(x)dx = \int_{-\infty}^{\mu - n\sigma} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \Phi(\mu - n\sigma)$$

   Note that while $\mathrm{VaR}_{c\%}(X)$ is a percentile and given by ICDF, the percentage 'cut on the tail' $1-c$ is given by the CDF. Example: for confidence level of 99%, the percentile that 'cuts' 1% of observations is $\Phi^{-1}(1 - 0.99) \approx 2.32635$.

   At times, $\Phi(\mu - n\sigma) = 1 - c$, is substituted in integration limits using any of the following:

   $$\mu - n\sigma = \Phi^{-1}(1-c) = \mathrm{VaR}_c$$

The mean of the values that fall within that tail (ie, cut by the percentile threshold) is

$$
\begin{aligned}
\frac{\int_{-\infty}^{\mu-n\sigma} x\, f(x)dx}{\int_{-\infty}^{\mu-n\sigma} f(x)dx} &= \frac{1}{\Phi(\mu-n\sigma)}\int_{-\infty}^{\mu-n\sigma} \frac{x}{\sigma\sqrt{2\pi}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \quad \dagger \\
& \quad \text{to simplify assume Standard Normal, } \Phi(\mu-n\sigma)=\Phi(-n) \\
&= \frac{1}{\Phi(-n)}\int_{-\infty}^{-n} \frac{x}{\sqrt{2\pi}}\, e^{-x^2/2} dx \quad \text{ready for calculation using } n=2\text{, see below.}
\end{aligned}
$$

To derive a general solution lets **swap variables** in original problem †

$$\mu - n\sigma = x \quad \Leftrightarrow \quad -n = \frac{x-\mu}{\sigma} = z \quad \Leftrightarrow \quad -dn = \frac{dx}{\sigma} = dz$$

$$
\begin{aligned}
&= \frac{1}{\Phi(x)}\int_{-\infty}^{x} \frac{\mu+z\sigma}{\sqrt{2\pi}}\, e^{-z^2/2}\, dz \qquad \Leftrightarrow \qquad \frac{1}{\Phi(x)}\int_{-\infty}^{x} \frac{\mu-n\sigma}{\sqrt{2\pi}}\, e^{-n^2/2}\,(-dn) \\
&= \frac{1}{\Phi(x)}\left(\mu\,\Phi(x) + \frac{\sigma}{\sqrt{2\pi}}\int_{-\infty}^{x} z e^{-z^2/2}\, dz\right) \\
&= \mu + \frac{\sigma}{\Phi(x)}\,\frac{-1}{\sqrt{2\pi}} e^{-x^2/2}
\end{aligned}
$$

which relates to $\quad \mathrm{ES}_c = \mu + \sigma\dfrac{\phi(\mathrm{VaR}_c)}{1-c} \quad$ cited in textbooks,

where standartised $\mathrm{VaR}_c = \Phi^{-1}(1-c) = x$ is in line with the above.

For the Standard Normal Distribution $N(\mu=0, \sigma^2=1)$ with density $\phi(z)$, we find that the percentage of returns outside two standard deviations on the left tail is

$$\int_{-\infty}^{-2}\phi(z) = \int_{-\infty}^{-2}\frac{1}{\sqrt{2\pi}}\, e^{-z^2/2}\, dz = \Phi(-2) = 0.02275$$

The mean of the values falling in this tail is

$$\frac{1}{\Phi(-2)}\int_{-\infty}^{-2}\frac{z}{\sqrt{2\pi}}\, e^{-z^2/2}\, dz = -\frac{1}{\Phi(-2)\sqrt{2\pi}} e^{-2} \approx -2.37$$

$$\text{using} \qquad \int z e^{-z^2/2} = -e^{-z^2/2}$$

The -2.37 figure is the mean of *standartised returns* and therefore, itself is a Standard Normal variable or Z-score. It also has a definition of Expected Shortfall (ES), *an average tail loss* given that the loss is *below* the VaR threshold. 'Given' reveals conditionality and ES is mathematically known as Conditional Value at Risk.

5. Recall the example of three bonds A,B and C from the Market Risk Measurement (Value at Risk) lecture: each bond has a face value of £1,000 payable at maturity and the independent probability of default 0.5%, when the loss is the face value in full.

   (a) For the portfolio equally invested in bonds A, B and C, why the 99% VaR is £1,000?

   (b) Calculate the Expected Shortfall (within the 1% tail) for the bond A (or B or C).

   (c) Calculate the Expected Shortfall (within the 1% tail) of a portfolio equally invested in bonds A, B and C.

   (d) Compare results from (b) and (c) to conclude whether ES is *sub-additive*.

Solution:

(a) Remember that VaR is value (loss) associated with the percentile. It was calculated that 98.5th percentile loss is £0 and adding the probability of the next outcome (1 default) to it gives $0.985 + 0.01485 = 0.99985$th percentile, so the minimum loss at $99th$ percentile is £1,000.

	Loss	Likelihood		
No defaults	£0	$\approx 98.51\%$	$0.995 \times 0.995 \times 0.995$	for (not ABC)
1 default	£1,000	$\approx 14.85\%$	$3 \times 0.005 \times 0.995^2$	for A (not BC) + B (not AC) + C (not AB)
2 defaults	£2,000	0.0074625%	$3 \times 0.005^2 \times 0.995$	for AB (not C) + BC (not A) + AC (not B)
3 defaults	£3,000	0.0000125%	$0.005 \times 0.005 \times 0.005$	for ABC

Table 1: Loss Distribution

For the Loss Distribution to work, we can have only none or 1 or 2 or 3 defaults.

(b) Within the 1% tail, there is a 0.5%/1% likelihood of losing £1,000. The conditional probability of loss is $\frac{\text{Pr}}{1-c}$ and so,

$$\text{ES} = \frac{\text{Pr}}{1-c} \times \text{Loss} = \frac{0.5\%}{1\%} \times 1000 = £500.$$

(c) We already established in (a) that 99% VaR for the portfolio is £1,000. Because ES (CVaR) is an average of VaRs on the tail, it can't be less than this value.

For ES for a continuously distributed variable we integrate,

$$\text{ES}_{1-c} = \frac{1}{1-c} \int_0^{1-c} \text{VaR}_\gamma(X) d\gamma$$

where VaR_γ is loss which in our case could be £3,000, £2,000 £1,000 and $d\gamma$ are 'chunks' of density.

ES ≡ Average loss on the tail ≡ Expected value over the loss distribution

For this discrete case, we substitute integration with summation. Instead of the tail, we have the Conditional Loss Distribution and chunks of its 100% density over which we calculate our expected value. Remember, the ES is an average loss.

$$\mathbf{ES} = 0.992525 \times 1000 + 0.0074625 \times 2000 + 0.0000125 \times 3000 = \pounds\mathbf{1,007.49}$$

How did we obtained those densities?

	Loss	Likelihood
3 defaults	£3,000	0.0000125%/1%
2 defaults	£2,000	0.0074625%/1%
1 default	£1,000	≈ 0.9925 †

Table 2: Conditional Loss Distribution

where Pr(1 default) = 1 - Pr(2 defaults) - Pr(3 defaults) so,

$$1 - 0.0074625 - 0.0000125 = 0.992525. \quad \dagger$$

We assume no other outcomes since we already chosen 99% VaR as the loss of £1,000 and ES is an integral over VaR values, an average loss on the tail.

If we take those conditional probabilities and multiply them by the respective marginal probabilities from (taken from (a) and rounded), the result 0.0149 is equal to the independent probability of at least one default.

$$\begin{aligned} 0.9925 \times 0.015 + 0.0074625 \times 0.0075 + 0.0000125 \times 0.00001 &\approx 0.0149 \\ 1 - \text{PrSurv} = 1 - \times 0.995^3 &\approx 0.0149. \end{aligned}$$

(d) ES for the portfolio of ABC is noticeably less than 3 × ES of each bond = £1,500 and so it is sub-additive. This holds for most cases, except a few that are only of academic interest.

6. VaR calculation for a portfolio of derivatives often done as a breakdown of the P&L into contributions from Delta, Gamma and Vega greeks. The contributions sum up linearly (algebraically) across greeks for each option. The total VaR is also calculated as a simple sum across positions.

However, it might be necessary to base VaR calculation on the asset price and take into account cross-asset movement (correlation). Consider a formula for Analytical VaR

$$\text{Factor} \times \sqrt{\delta t} \sqrt{\sum_{j=1}^{N} \sum_{i=1}^{N} \rho_{ij}\, \sigma_i \sigma_j\, \Delta_i S_i\, \Delta_j S_j}$$

Delta approximates the change in value over δt. What are the key assumptions of this calculation?

Solution:

(a) We assume the asset follows the log-random walk defined by $dS = \mu S dt + \sigma S dX$ SDE where $dX \to \phi\sqrt{\delta t}$ requires random Normal variable generation. If we study sample asset returns data, we are likely to see that an empirical returns distribution has fatter tails and a higher peak when compared to the Normal distribution.

(b) All relevant volatilities and correlations are known and robust. Robustness means the low variance of a statistical estimate, ie, if you throw out/add in some observations, the change in robust estimate would be insignificant. Empirical studies find strong heteroskedasticity of volatility (it goes up and down wildly over time), while correlation changes in sine-wave pattern between upper and lower bounds. These stylised facts are true for samples of various length, eg, 30D, 60D, 6M, etc.

(c) This analytical calculation is *linear*. It assumes that the *sensitivity* of options portfolio to the change in the underlying is approximated linearly, in this case by delta of the option.

7. What are the two main numerical methods used for the Empirical VaR estimation? What are their drawbacks?

Solution:

- **Monte Carlo** method requires generation of Normally distributed random numbers and relies on their low latency (evenness).
- **Bootstrapping** (or Historic Simulation) method uses actual asset price movements taken from historical data (eg, the last two years). More precisely, bootstrapping means sampling from *standardised historic residuals*

$$Z_t^* = \frac{u_{t,Hist}}{\sqrt{\sigma^2_{t,GARCH}}}$$

  where $u_{t,Hist}$ is econometric notation for return and standard deviation $\sigma_{t,GARCH}$ is smoothed by the application of a GARCH filter.

The criticisms of the Monte-Carlo method often return to the assumption of a log-normal random walk. To introduce correlation, a solution from multi-factor PDE or factorisation (eg, by Cholesky decomposition on covariance matrix) would be required. Coupled together with the need to run tens to hundreds thousands of simulations (to secure a reliable projection) it means that the MC method can be computationally slow.

The main criticism of the Bootstrapping method is that it too requires a large amount of data. Historical data include values that correspond to very different economic conditions. Often, precisely the data required for adequate estimation suffers from 'structural breaks', such as missing prices/assets not being traded.