1 Calculus Problem Sheet

1. Consider two functions $f(x) = 9x + 2$ and $g(x) = \frac{x}{9} - \frac{2}{9}$. Show that they are inverse functions of one another.
   This simply requires showing $f(g(x)) = g(f(x)) = x$

$$\begin{aligned} f(g(x)) &= 9\left(\frac{x}{9} - \frac{2}{9}\right) + 2 = x - 2 + 2 = x \\ g(f(x)) &= \frac{9x+2}{9} - \frac{2}{9} = x + \frac{2}{9} - \frac{2}{9} = x \end{aligned}$$

2. Obtain the inverse of the function $f(x) = x^{1/3} + 2$.

$$\begin{aligned} y &= x^{1/3} + 2 \longrightarrow x^{1/3} = y - 2 \\ x &= (y-2)^3 = g(y) \\ \therefore\ & f^{-1}(x) = g(x) = (x-2)^3 \end{aligned}$$

3. Calculate the following limits:

$$\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{(x+2)(x-2)}{x-2} = \lim_{x\to 2}(x+2) \longrightarrow 4$$

$$\lim_{x\to 1} \frac{x^2-x}{2x^2+5x-7} = \lim_{x\to 1} \frac{x(x-1)}{(2x+7)(x-1)} = \lim_{x\to 1} \frac{x}{(2x+7)} \longrightarrow \frac{1}{9}$$

$$\lim_{x\to -25} \frac{\sqrt{x}+5}{x-25} = \lim_{x\to -25} \frac{\sqrt{x}+5}{(\sqrt{x}-5)(\sqrt{x}+5)} = \lim_{x\to -25} \frac{\sqrt{x}+5}{(\sqrt{x}-5)(\sqrt{x}+5)} \longrightarrow \infty$$

$$\lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} \ :\ \text{see primer lecture notes}$$

$$\lim_{h\to -2} \frac{h^3+8}{h+2} = \lim_{h\to -2} \frac{(h+2)^3 - 6h(h+2)}{h+2} = \lim_{h\to -2} (h+2)^2 - 6h \longrightarrow 12$$

$$\lim_{t\to 1} \frac{(1/t)-1}{t-1} = \lim_{t\to 1} \frac{(1-t)/t}{t-1} = \lim_{t\to 1} \frac{-(t-1)/t}{t-1} \lim_{t\to 1} \frac{-1}{t} \longrightarrow -1$$

$$\lim_{x\to\sqrt{2}} \left(x^2+3\right)(x-4) = \lim_{x\to\sqrt{2}} \left(x^2+3\right) \lim_{x\to\sqrt{2}} (x-4) \longrightarrow 5\left(\sqrt{2}-4\right)$$

4. Using the definition of the derivative, show that for

$$\begin{aligned}
y &= 2x+1, \quad y' = 2 \\
y' &= \lim_{h\to 0}\frac{[2(x+h)+1]-(2x+1)}{h} = \lim_{h\to 0}\frac{2x+2h+1-2x-1}{h} = \lim_{h\to 0}\frac{2h}{h} = \lim_{h\to 0} 2 = 2 \\
f(x) &= \frac{1}{x-2}, \quad f'(x) = -\frac{1}{(x-2)^2} \\
f'(x) &= \lim_{h\to 0}\frac{\frac{1}{x-2+h}-\frac{1}{x-2}}{h} = \lim_{h\to 0}\frac{1}{h}\frac{(x-2)-(x-2+h)}{(x-2+h)(x-2)} = \lim_{h\to 0}\frac{1}{h}\frac{-h}{(x-2+h)(x-2)} \\
&= \lim_{h\to 0}\frac{-1}{(x-2+h)(x-2)} = -\frac{1}{(x-2)^2}
\end{aligned}$$

$$\begin{aligned}
g(x) &= |x-5|, \quad \text{no derivative exists at } x=5 \\
g'_+(x) &= \lim_{h\to 0^+}\frac{g(x+h)-g(x)}{h} \longrightarrow g'_+(5) = \lim_{h\to 0^+}\frac{g(5+h)-g(5)}{h} = \lim_{h\to 0^+}\frac{|5+h-5|-|0|}{h} \\
&= \lim_{h\to 0^+}\frac{|h|}{h} = \lim_{h\to 0^+}\frac{h}{h} = 1 \\
g'_-(x) &= \lim_{h\to 0^-}\frac{g(x+h)-g(x)}{h} \longrightarrow g'_-(5) = \lim_{h\to 0^-}\frac{g(5+h)-g(5)}{h} = \lim_{h\to 0^-}\frac{|5+h-5|-|0|}{h} \\
&= \lim_{h\to 0^-}\frac{|h|}{h} = \lim_{h\to 0^+}\frac{-h}{h} = -1 \\
g'_+(x) &= g'_-(x) \therefore \text{ the derivative does not exist at } x=5.
\end{aligned}$$

5. Differentiate the following functions y, to obtain $\frac{dy}{dx}$:
We know the Chain Rule:

$$y = f(u) \quad \text{where} \quad u = F(x) \quad \text{then} \quad \frac{dy}{dx} = \frac{df}{du}\frac{du}{dx}$$

$$\begin{aligned}
y &= \left(x^2-4x+2\right)^5 \\
\frac{dy}{dx} &= 5u^4.(2x-4) = 5\left(x^2-4x+2\right)^4(2x-4)
\end{aligned}$$

$$\begin{aligned}
y &= \frac{1}{(4x^2+6x-7)^3} = \left(4x^2+6x-7\right)^{-3} \\
\frac{dy}{dx} &= -6\left(4x^2+6x-7\right)^{-4}(4x+3)
\end{aligned}$$

$$y^4+3y-4x^3 = 5x+1$$

implicit differentiation:

$$\frac{dy}{dx} = \frac{5+12x^2}{4y^3+3}$$

$$\begin{aligned}
y &= \ln\sqrt[3]{(2x+5)^2} = \ln(2x+5)^{2/3} = \frac{2}{3}\ln(2x+5)) \\
\frac{dy}{dx} &= \frac{4}{3(2x+5)}
\end{aligned}$$

$$y = \cos(4-3x) : \frac{dy}{dx} = 3\sin(4-3x)$$

Now use the product rule:

$$y = x^2 \exp(x) : \frac{dy}{dx} = xe^x(2+x)$$

Next problem requires the quotient role

$$\begin{aligned}
y &= \frac{3x^2 - x + 2}{4x^2+5} \\
y' &= \frac{\left(4x^2+5\right)(6x-1) - 8x\left(3x^2-x+2\right)}{\left(4x^2+5\right)^2}
\end{aligned}$$

6. Calculate the following

$$\begin{aligned}
\int \sqrt{x}\left(x^2-4x+2\right)\, dx &= \int \left(x^{5/2} - 4x^{3/2} + 2x^{1/2}\right) dx \\
&= \frac{2}{7}x^{7/2} - \frac{8}{5}x^{5/2} + \frac{4}{3}x^{3/2} + C
\end{aligned}$$

$$\begin{aligned}
\int_4^1 \left(3\sqrt{x}+1\right)\left(\sqrt{x}-2\right) dx &= \\
\int_4^1 \left(3x - 5x^{1/2} - 2\right) dx &= \left[\frac{3}{2}x^2 - \frac{10}{3}x^{3/2} - 2x\right]_4^1 = \frac{41}{6}
\end{aligned}$$

$$\begin{aligned}
&\int_{-1}^{-2} \frac{(2s-7)}{s^3} ds \\
= &\int_{-1}^{-2} \frac{2}{s^2} + \frac{7}{2s^3} ds = \left[-\frac{2}{s}\right]_{-1}^{-2} + \left[\frac{7}{2s^2}\right]_{-1}^{-2} = -\frac{29}{8}
\end{aligned}$$

$$\begin{aligned}
\int_3^2 \frac{\left(x^2-1\right)}{(x-1)} dx &= \\
\int_3^2 (x+1)\, dx &= \left[\frac{x^2}{2} + x\right]_3^2 = -\frac{7}{2}
\end{aligned}$$

$$\begin{aligned}
\int_{-1}^5 |2x-3|\, dx &= \int_{-1}^{3/2} (3-2x)\, dx + \int_{3/2}^5 (2x-3)\, dx \\
&= \left[3x - x^2\right]_{-1}^{\frac{3}{2}} + \left[x^2 - 3x\right]_{\frac{3}{2}}^5 = \frac{37}{2}
\end{aligned}$$

$$\int \frac{5x-12}{x(x-4)}dx \quad : \quad \frac{5x-12}{x(x-4)} \equiv \frac{3}{x} + \frac{2}{(x-4)}$$
$$= \int \frac{3}{x} + \int \frac{2}{(x-4)} = 3\ln x + 2\ln(x-4) + C$$
$$= \ln x^3 (x-4)^2 + C$$

7. By using suitable substitutions (change of variable), evaluate the following

$$\int \left(3-x^4\right)^3 x^3 dx$$

let $z = 3 - x^4$ so that $dz = -4x^3 dx$ and the integral becomes

$$-\frac{1}{4}\int z^3 dz = -\frac{1}{16}\left(3-x^4\right)^4 + c$$

$$\int \frac{\left(x^2+x\right)}{\left(4-3x^2-2x^3\right)^4}dx$$

put $z = 4 - 3x^2 - 2x^3$ so that $dz = -6\left(x^2+x\right)dx$ to give

$$-\frac{1}{6}\int \frac{1}{z^4}dz = \frac{1}{18\left(4-3x^2-2x^3\right)^3} + c$$

$$\int \frac{\left(\sqrt{u}+3\right)^4}{\sqrt{u}}du$$

let $z = \sqrt{u} + 3 =$ so that $dz = \frac{1}{2\sqrt{u}}du$ and we have

$$2\int z^4 dz = 2\frac{z^5}{5} + c = \frac{2\left(\sqrt{u}+3\right)^5}{5} + c$$

$$\int \left(1+\frac{1}{u}\right)^{-3}\left(\frac{1}{u^2}\right)du$$

let $z = 1 + u^{-1}$ so that $dz = -\frac{1}{u^2}du$ and

$$-\int z^{-3}dz = \frac{1}{2}\left(1+\frac{1}{u}\right)^{-2}$$

$$\int xe^{x^2}dx$$

let $z = x^2$ so that $dz = 2xdx$, so

$$\frac{1}{2}\int e^z dz = \frac{1}{2}e^z + c = \frac{1}{2}e^{x^2} + c$$

$$\int (\sin x)\, e^{\cos x} dx$$

let $z = \cos x$ so that $dz = -(\sin x)\, dx$

$$-\int e^z dz = -e^z + c = -e^{\cos x} + c$$

8. If $f(x,y) = (x-y)\sin(3x+2y)$, determine $f_x,\ f_y,\ f_{xx},\ f_{yy},\ f_{xy},\ f_{yx}$. Now evaluate these expressions at $(0, \pi/3)$.

$$f_y = 2(x-y)\cos(3x+2y) - \sin(3x+2y)$$

$$f_x = \sin(3x+2y) + 3(x-y)\cos(3x+2y)$$
$$f_{xx} = 6\cos(3x+2y) - 9(x-y)\sin(3x+2y)$$

$$f_{yy} = -4\cos(3x+2y) - 4(x-y)\sin(3x+2y)$$
$$f_{xy} = f_{yx} = -\cos(3x+2y) - 6(x-y)\sin(3x+2y)$$

Now evaluate these expressions at $x = 0; y = \pi/3$.
We use $\sin(2\pi/3) = \sqrt{3}/2;\ \cos(2\pi/3) = -1/2$

$$\begin{aligned} f_x\left(0, \frac{\pi}{3}\right) &= \frac{\sqrt{3}}{2} + \frac{\pi}{2};\ f_y\left(0, \frac{\pi}{3}\right) = \frac{\pi}{3} - \frac{\sqrt{3}}{2} \\ f_{xx}\left(0, \frac{\pi}{3}\right) &= -3 + 3\pi\frac{\sqrt{3}}{2};\ f_{yy}\left(0, \frac{\pi}{3}\right) = 2 + \frac{2\pi}{\sqrt{3}} \\ f_{xy}\left(0, \frac{\pi}{3}\right) &= \frac{1}{2} + \pi\sqrt{3} \end{aligned}$$

9. Show that $z = \ln\left(z = \ln\left((x-a)^2 + (y-b)^2\right)\right)$ satisfies

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

except at (a,b). Write $z = \ln u;\ u = (x-a)^2 + (y-b)^2$

$$\begin{aligned} \frac{\partial z}{\partial x} &= \frac{2(x-a)}{u};\ \frac{\partial z}{\partial y} = \frac{2(y-a)}{u} \\ \frac{\partial^2 z}{\partial x^2} &= \frac{2u - 4(x-a)^2}{u^2};\ \frac{\partial^2 z}{\partial y^2} = \frac{2u - 4(y-b)^2}{u^2} \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} &= \frac{2u - 4(x-a)^2 + 2u - 4(y-b)^2}{u^2} \\ &= \frac{4\left((x-a)^2 + (y-b)^2\right) - 4(x-a)^2 - 4(y-b)^2}{u^2} = 0 \end{aligned}$$

10. Obtain Taylor series expansions for the following functions about the given point x_0. If no point is given, then expand about the point 0 (in which case you can use standard Taylor series expansions)

$$\begin{aligned} f(x) &= x^2 \sin x = x^2 \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - .... + (-1)^n \frac{x^{2n+1}}{(2n+1)!} \right) \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+3}}{(2n+1)!} \end{aligned}$$

$$\begin{aligned} f(x) &= \cos x;\ x_0 = \pi/3 \\ f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n \end{aligned}$$

$f(\pi/3) = 1/2;\ f'(\pi/3) = -\sqrt{3}/2;\ f''(\pi/3) = -1/2;\ f'''(\pi/3) = \sqrt{3}/2;\ f^{(4)}(\pi/3) = 1/2$

The Taylor series expansion about $x = \pi/3$ is thus $f(x) = \cos x =$

$\frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) - \frac{1}{4}\left(x - \frac{\pi}{3}\right)^2 + \frac{\sqrt{3}}{12}\left(x - \frac{\pi}{3}\right)^3 + \frac{1}{48}\left(x - \frac{\pi}{3}\right)^4 + ... + \frac{f^{(n)}(\pi/3)}{n!}\left(x - \frac{\pi}{3}\right)^n + ..$

$$\begin{aligned} f(x) &= \exp x;\ x_0 = -3 :\ \text{put } u = x + 3 \text{ and expand } e^u \text{ about } u = 0 \\ e^u &= \sum_{n=0}^{\infty} \frac{u^n}{n!} \longrightarrow e^{x+3} = \sum_{n=0}^{\infty} \frac{(x+3)^n}{n!} \\ e^x &= e^{-3} \sum_{n=0}^{\infty} \frac{(x+3)^n}{n!} = e^{-3} \left(1 + (x+3) + \tfrac{1}{2!}(x+3)^2 + \tfrac{1}{3!}(x+3)^3 + ... \right) \end{aligned}$$

$$\begin{aligned} f(x) &= \frac{1}{1-4x} = (1-4x)^{-1} \\ &= 1 + 4x + 16x^2 + 64x^2 + ... + 4^n x^n = \sum_{n=0}^{\infty} 4^n x^n \end{aligned}$$

this is a convergent series with $|x| < 1/4$.

$$\begin{aligned} f(x) &= \frac{3}{2x+5} = 3(2x+5)^{-1} = 3 \times 5^{-1} \left(1 + \frac{2}{5}x \right) \\ \left(1 + \frac{2}{5}x \right) &= 1 - \tfrac{2}{5}x + \tfrac{4}{25}x^2 - \tfrac{8}{125}x^3 + ... + (-1)^n \left(\tfrac{2}{5}\right)^n x^n \\ f(x) &= \tfrac{3}{5} \sum_{n=0}^{\infty} (-1)^n \left(\tfrac{2}{5}\right)^n x^n : \left|\tfrac{2}{5}x\right| < 1 \iff |x| < 5/2 \end{aligned}$$

$$\begin{aligned} f(x) &= \frac{x^2+1}{x-1} = -\left(x^2+1\right)(1-x)^{-1} = -\left(x^2+1\right) \sum_{n=0}^{\infty} x^n = -\sum_{n=0}^{\infty} x^{n+2} - \sum_{n=0}^{\infty} x^n \\ &= -1 - x - 2\sum_{n=2}^{\infty} x^n \end{aligned}$$

11. If $U(x,y,z) = 2x^2 - yz + xz^2$, where $x = 2\sin t$, $y = t^2 - t + 1$, $z = 3\exp(-t)$,

find $\dfrac{dU}{dt}$ at $t = 0$.

$$\begin{aligned}\frac{dx}{dt} &= 2\cos t;\ \frac{dy}{dt} = 2t - 1;\ \frac{dz}{dt} = -3e^{-t}\\ \frac{\partial U}{\partial x} &= 4x + z^2;\ \frac{\partial U}{\partial y} = -z;\ \frac{\partial U}{\partial z} = -y + 2xz\end{aligned}$$

$$\begin{aligned}\frac{dU}{dt} &= \frac{dx}{dt}\frac{\partial U}{\partial x} + \frac{dy}{dt}\frac{\partial U}{\partial y} + \frac{dz}{dt}\frac{\partial U}{\partial z} = (2\cos t)\left(4x + z^2\right) + (2t-1)(-z) + \left(-3e^{-t}\right)(-y + 2xz)\\ &= (2\cos t)\left(4\sin t + 9e^{-2t}\right) + (2t-1)\left(-3e^{-t}\right) + \left(-3e^{-t}\right)\left(-t^2 + t - 1 + 12e^{-t}\sin t\right)\\ \left.\frac{dU}{dt}\right|_{t=0} &= (2)(9) + (-1)(-3) + (-3)(-1) = 24\end{aligned}$$

12. Given $w = f(x,y)$; $x = r\cos\theta$, $y = r\sin\theta$; show that

$$\left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 = \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2$$

$$\begin{aligned}x_r &= \cos\theta;\ x_\theta = -r\cos\theta\\ y_r &= \sin\theta;\ y_\theta = r\cos\theta\end{aligned}$$

Now use chain rule II

$$\begin{aligned}\frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} = \cos\theta\frac{\partial w}{\partial x} + \sin\theta\frac{\partial w}{\partial y}\\ \left(\frac{\partial w}{\partial r}\right)^2 &= \cos^2\theta\left(\frac{\partial w}{\partial x}\right)^2 + \sin^2\theta\left(\frac{\partial w}{\partial y}\right)^2 + \sin 2\theta\frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \qquad (1)\end{aligned}$$

Similarly

$$\begin{aligned}\frac{\partial w}{\partial \theta} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \theta} = -r\sin\theta\frac{\partial w}{\partial x} + r\cos\theta\frac{\partial w}{\partial y}\\ \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 &= \sin^2\theta\left(\frac{\partial w}{\partial x}\right)^2 + \cos^2\theta\left(\frac{\partial w}{\partial y}\right)^2 - \sin 2\theta\frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \qquad (2)\end{aligned}$$

$(1) + (2)$ gives $\left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 =$

$$\left(\frac{\partial w}{\partial x}\right)^2\left(\sin^2\theta + \cos^2\theta\right) + \left(\frac{\partial w}{\partial y}\right)^2\left(\sin^2\theta + \cos^2\theta\right)$$

$$\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2$$