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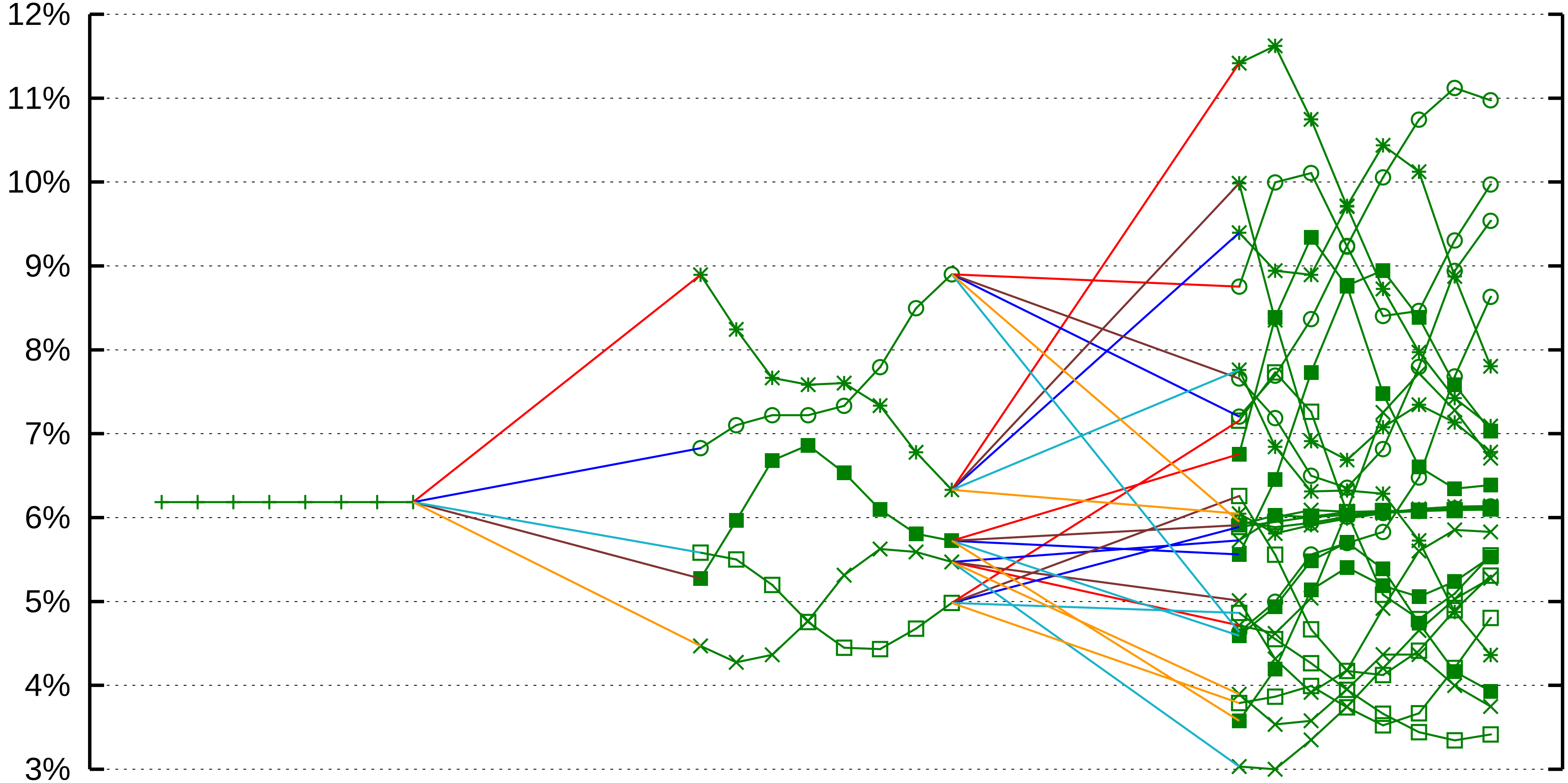
THE LIBOR MARKET MODEL



*OTC Analytics

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I. The market view of the yield curve

Financial markets are created by the activities of agents dealing in tradeable securities.

The *yield curve* is given by the distillation of many different market-observable prices into one condensed representation. The distillation procedure is called *yield curve stripping*. This usually involves market quotes from several considerably distinct market segments:

- Interbank cash deposit rates
- Government bond repo rates (this is how the bank of England sets interest rates)
- Interest rate futures. These are the exchange-traded equivalent to *Forward Rate Agreements* (FRAs).
- Swap rates (usually quoted by brokerage services)

Some of these market quotes are subject to credit risk, and for others it can practically be ignored. Some of these quotes are subject to convexity adjustments, and others are not.

In the financial market, there is no such thing as traded yield curve, but there are many interest rate related tradeable contracts.

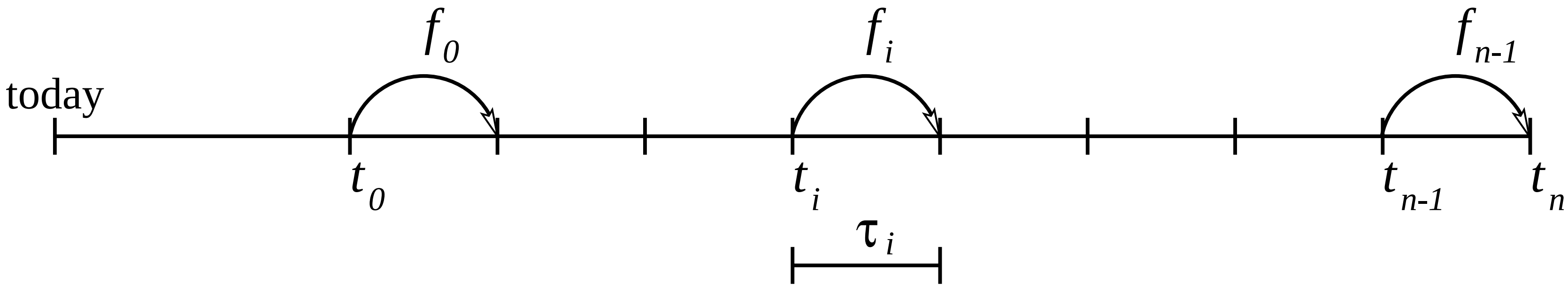
Since most of these contracts are sufficiently distinct, their prices are only very weakly linked by strict no-arbitrage requirements.

The yield curve is a theoretical construct. What matters financially is the trading of securities and contingent claims.

II. Yield curve discretisation

The concept of a *market model* is to describe directly the dynamics of observable market quotes of financially tradeable contracts, rather than to fall back to a hidden process driving the entirety of the fixed income market.

A *Libor market model* is based on the discretisation of the yield curve into *discrete spanning forward rates*.



Each forward rate immediately represents the (modelled) market quote for an associated Forward Rate Agreement (FRA).

A forward rate agreement quote f_i for period $t_i \longrightarrow t_{i+1}$ with accrual factor $\tau_i \simeq t_i - t_{i+1}$ means:

- Upon deposit of a notional at time t_i, at the later time t_{i+1} the notional plus interest amounting to $f_i \cdot \tau_i$ times the notional is returned to the depositor.

- For a borrower of money, the effetive funding discount factor over the (forward) interval $t_i \longrightarrow t_{i+1}$ is given by $1/\left(1 + f_i\tau_i\right)$.

The fair value of a Forward Rate Agreement on rate f_i struck at K is $P(t, t_{i+1}) \cdot (f_i - K)\,\tau_i$ where $P(t, t_{i+1})$ is the value at time t of a zero coupon bond paying one domestic currency unit at time t_{i+1}.

At time t_i, the value becomes $(f_i - K)\,\tau_i/\left(1 + f_i\tau_i\right)$.

III. Standard Libor market model dynamics

In the standard Libor market model for discretely compounded interest rates, we assume that each of n spanning forward rates f_i evolves according to the stochastic differential equation

$$\frac{\mathrm{d}f_i}{f_i} = \mu_i(\boldsymbol{f}, t)\mathrm{d}t + \sigma_i(t)\mathrm{d}\widetilde{W}_i \ . \qquad (1)$$

This ensures that all interest rates remain positive at all times. The drift terms are yet to be determined.

Correlation is incorporated by the fact that the n standard Wiener processes in equation (1) satisfy

$$\mathsf{E}\left[\mathrm{d}\widetilde{W}_i\,\mathrm{d}\widetilde{W}_j\right] = \varrho_{ij}\mathrm{d}t \ . \qquad (2)$$

The elements of the instantaneous covariance matrix $C(t)$ of the n forward rates are thus

$$c_{ij}(t) = \sigma_i(t)\sigma_j(t)\varrho_{ij} \ . \qquad (3)$$

Using a decomposition of $C(t)$ into a pseudo-square root $\tilde{A}$ such that

$$C = \tilde{A}\tilde{A}^\top \ , \tag{4}$$

we can transform equation (1) to

$$\frac{\mathrm{d}f_i}{f_i} = \mu_i \mathrm{d}t + \sum_j \tilde{a}_{ij} \mathrm{d}W_j \tag{5}$$

with $\mathrm{d}W_j$ being n independent standard Wiener processes where dependence on time has been omitted for clarity.

The matrix A may sometimes be referred to as the *driver* or as the *dispersion*[1] matrix.

[1]Karatzas and Shreve [KS91], page 284.

IV. Numéraire and measure

A fundamental principle of financial mathematics is that of *relative* or *numéraire denominated valuation*.

- Select a tradeable security whose value $N(t)$ can be readily determined at any point in time of interest for the valuation of the given contingent claim at hand. This security will be called the *numéraire*.

- Establish an equivalent martingale measure in which the values of *all* tradeable securities *relative to the numéraire* are martingales.

- Carry out all calculations as values *relative to the numéraire*.

- Multiply the numéraire-denominated result by today's value of the numéraire asset.

See also en.wikipedia.org/wiki/Numéraire, the 1995 article by Geman, Karoui, and Rochet [GKR95], or [BBW01] for a less formal exposition.

In other words, the value $v(t)$ of a contingent claim that pays a sequence of conditional cashflows $c_j(t_j)$ is to be computed as

$$v(t) \quad = \quad N(0) \cdot \mathsf{E}_{\mathbb{Q}_N}\left[\sum_j \frac{c_j(t_j)}{N(t_j)}\right] \quad . \tag{6}$$

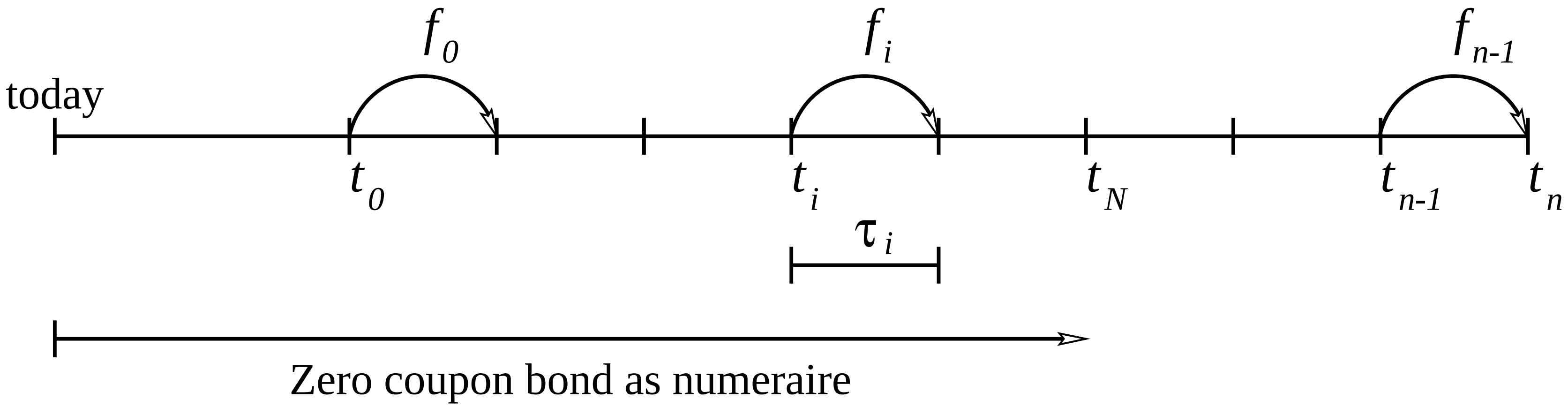
- *Discounting is but one very special case of relative valuation induced by choosing a cash account as the numéraire.*

- *The choice of numéraire is arbitrary and does not change absolute values.*

- A good choice of numéraire can make life a lot easier.
  - To value a caplet, a zero coupon bond makes a good numéraire.
  - For a swaption, the associated annuity is the natural choice.
  - For an option to exchange one asset for another [Mar78], select one of the two assets as numéraire.

V. The drift

The common choice of numéraire is some zero coupon bond that pays one currency unit at t_N.



The drifts μ_i in equations (1) and (5) can then be calculated with the following Ansatz:

$$\mathsf{E}_{t_N}\left[\mathrm{d}\left(\frac{f_i P_{i+1}}{P_N}\right)\right] = 0, \tag{7}$$

where P_i, $\forall\, i = 0, \ldots, n$ are the t_i discount bonds and $\mathsf{E}_{t_N}[\cdot]$ the expectation operator under the equivalent martingale measure induced by the choice of the discount bond P_N as numéraire.

Under the latter assumption, by the fundamental theorem of asset pricing, for the market to be free of arbitrage, all ratios of tradeable assets divided by the numéraire value have to form martingales [HP81], i.e. we also require

$$\mathsf{E}_{t_N}\left[\mathrm{d}\left(\frac{P_i}{P_N}\right)\right] = 0, \qquad \forall\, i = 0, \ldots, n, \tag{8}$$

since the discount bonds are assumed to be traded assets.

Now, introducing the bond ratio

$$X_i := P_{i+1}/P_N \tag{9}$$

and invoking Itô's formula on equation (7) yields

$$\mathsf{E}_{t_N}[X_i \mathrm{d}f_i + f_i \mathrm{d}X_i + \mathrm{d}f_i \mathrm{d}X_i] = 0\ . \tag{10}$$

Since $\mathrm{d}X$ is drift-free, this reduces to

$$\mathsf{E}_{t_N}[X_i\,\mathrm{d}f_i + \mathrm{d}X_i\,\mathrm{d}f_i] \;=\; \mathsf{E}_{t_N}[X_i \mu_i f_i \mathrm{d}t] + \mathsf{E}_{t_N}[\mathrm{d}X_i\,\mathrm{d}f_i] \;=\; 0\ . \tag{11}$$

In the following, we will use the instantaneous relative covariance brackets $[a, b]$ defined[2] by the instantaneous drift of the product of the infinitesimal *relative* increments of any two stochastic processes a and b, i.e.

$$[a, b] \;:=\; \mathsf{E}\left[\frac{\mathrm{d}a}{a}\frac{\mathrm{d}b}{b}\right]\Big/\,\mathrm{d}t\;. \tag{12}$$

The definition (12) immediately gives us

$$[a, bc] = [a, b] + [a, c] \qquad \text{and} \qquad [a, b/c] = [a, b] - [a, c]\;. \tag{13}$$

Using this notation, we return to the derivation of the drift of the discrete forward rates. From equation (11), we obtain

$$\mu_i = -\left[f_i, \frac{P_{i+1}}{P_N}\right] \tag{14}$$

[2] Rebonato [Reb98] refers to the bracket $[a, b]$ as the *Vaillant bracket* quoting a company-internal report of Barclays BZW by N. Vaillant in 1995.

whose computation is aided by the bracket notation:

$$\begin{aligned}
\left[f_i, \frac{P_{i+1}}{P_N}\right] &= [f_i, P_{i+1}] - [f_i, P_N] \\
&= \left[f_i, \Pi_{k=0}^{i}(1+f_k\tau_k)^{-1}\right] - \left[f_i, \Pi_{k=0}^{N-1}(1+f_k\tau_k)^{-1}\right] \\
&= -\sum_{k=0}^{i}[f_i, 1+f_k\tau_k] + \sum_{k=0}^{N-1}[f_i, 1+f_k\tau_k] \,.
\end{aligned} \tag{15}$$

By the definition of the bracket (12) and the dynamics of the individual forward rates (1), each of the terms in the sums on the right hand side of equation (15) is easily computed:

$$[f_i, 1+f_k\tau_k] = \frac{f_k\tau_k}{1+f_k\tau_k}\sigma_i\sigma_k\varrho_{ik} \tag{16}$$

Finally, cancellation of summation terms leads to the drift formulæ

$$\mu_i(\boldsymbol{f}(t),t) = \begin{cases} -\sigma_i \sum\limits_{k=i+1}^{N-1} \frac{f_k(t)\tau_k}{1+f_k(t)\tau_k}\sigma_k\varrho_{ik} & \text{for} \quad i < N-1 \\ 0 & \text{for} \quad i = N-1 \\ \sigma_i \sum\limits_{k=N}^{i} \frac{f_k(t)\tau_k}{1+f_k(t)\tau_k}\sigma_k\varrho_{ik} & \text{for} \quad i \geq N \end{cases} \tag{17}$$

Note that this means that the drift of all of the forward rates but one are indirectly stochastic, i.e. it is stochastic due to its explicit dependence on the stochastic forward rates themselves.

When $i = N-1$, i.e. for a drift-free forward rate f_i, we call the numéraire associated with the pricing measure the *natural numéraire* of the forward rate f_i.

VI. Factor reduction

It is possible to drive the evolution of the n forward rates with fewer underlying independent standard Wiener processes than there are forward rates, say only m of them.

In this case, the coefficient matrix $\tilde{A} \in \mathbb{R}^{n\times n}$ is to be replaced by $A \in \mathbb{R}^{n\times m}$ which must satisfy

$$\sum_{j=1}^{m} a_{ij}^2 = c_{ii} \tag{18}$$

in order to retain the calibration of the options on the FRAs, i.e. the caplets. In practice, this can be done very easily by calculating the decomposition as in equation (4) as before and rescaling according to

$$a_{ij} = \tilde{a}_{ij}\sqrt{\frac{c_{ii}}{\sum\limits_{k=1}^{m}\tilde{a}_{ik}^2}}\ . \tag{19}$$

The effect of this procedure is that the individual variances of each of the rates are still correct, even if we have reduced the number of driving factors to one, but the effective covariances will differ.

Using fewer factors than discrete forward rates means

- a destruction of
  - either the term structure of instantaneous volatility of FRAs
  - or the correlation structure of the FRAs
  - or both
- that simultaneous calibration to market instruments of different nature such as caplets and swaptions becomes practically impossible
- the model loses its *market* feature and becomes a *factor* model
- virtually no speed gain unless you have significantly fewer than $n/4$ factors

VII. Parametrisation of volatility and correlation

Stable calibration of any market model relies on the specification of a robust yet flexible a *reference volatility structure*. We call a specification of instantaneous volatility *time-homogeneous* or *stationary* if the volatility of any forward rate f_T that will fix at time T depends on calendar time t only in terms of $T-t$, i.e.

$$\sigma_T(t) = \sigma(T-t)\ . \tag{20}$$

One cannot fit many market prices with this strict assumption.

In fact, there are frequently good economic reasons why time-homogeneity may not be given for the term structure of instantaneous volatility of forward rates.

In practice, we may want to use an initial parametrised fit in order to find the values a, b, c, and d, such that (only) the caplet implied volatilities resulting from the instantaneous FRA volatility

$$\sigma_i(t) = k_i \left[(a + b\cdot(t_i - t))\cdot \mathrm{e}^{-c\cdot(t_i-t)} + d \right] \cdot \mathbf{1}_{\{t<t_i\}} \tag{21}$$

are perfectly matched to the caplet volatility entries in the swaption matrix with all of the adjustment coefficients k_i being as near to 1 as possible. Note that this is only to obtain a reasonable skeleton for the term strucure of FRA volatility. The so obtained parameters a, b, c, and d then determine the *reference* or *skeleton* term structure of instantaneous volatility

$$\sigma_T^{\text{reference}}(t) = \left[(a + b \cdot (T - t)) \cdot \mathrm{e}^{-c \cdot (T-t)} + d\right] \cdot \mathbf{1}_{\{t<T\}} \,. \tag{22}$$

As for the instantaneous correlation between forward rates, a parametrisation that is economically, econometrically, and analytically appealing is the functional form

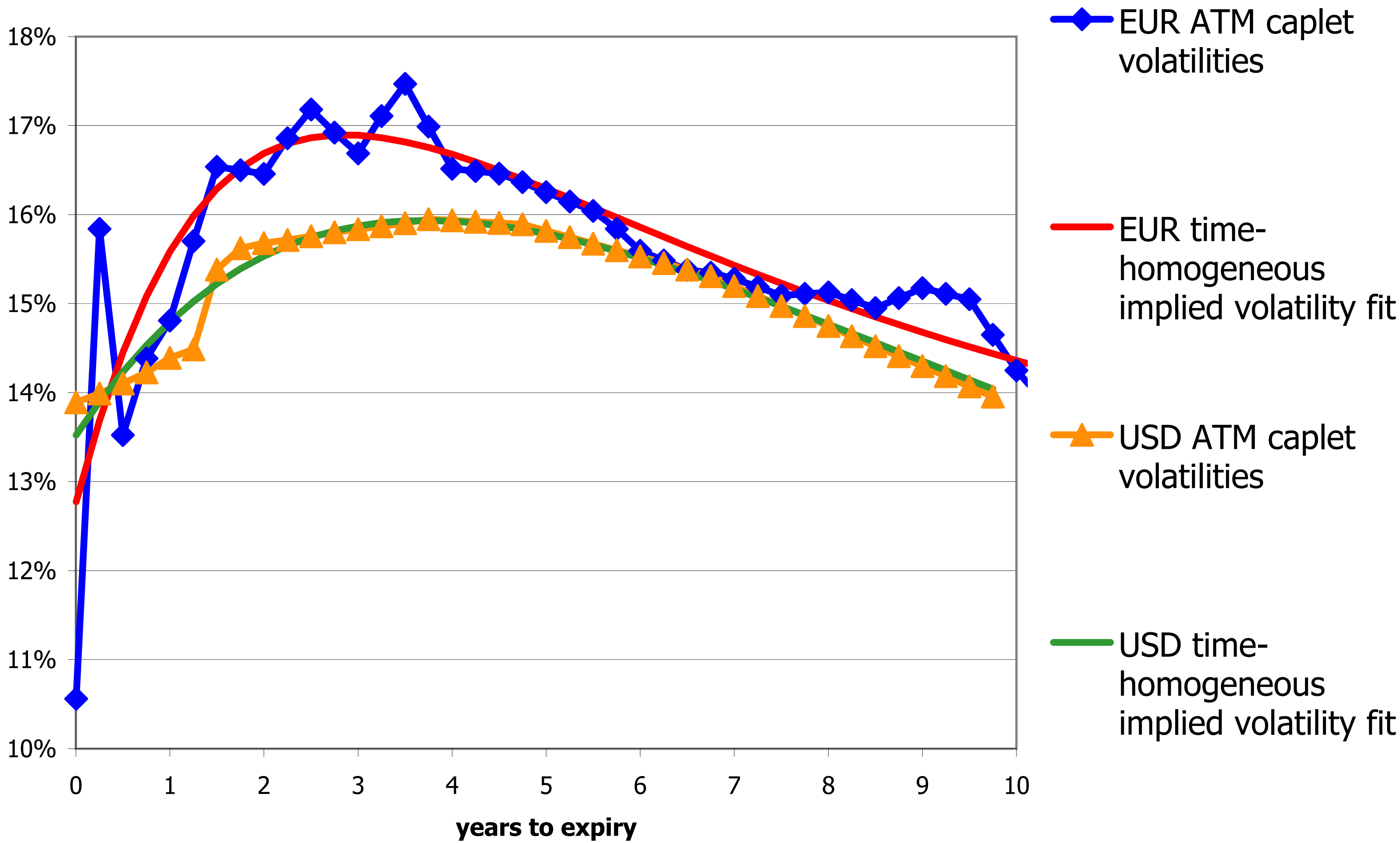
$$\varrho_{ij} \;=\; \mathrm{e}^{-\beta \cdot (t_i - t_j)} \tag{23}$$

with t_i and t_j, as before, being the expiry times of caplets #i and #j. A good value for the overall correlation coefficient is $\beta \approx 0.1$.

Since the instantaneous correlation function doesn't actually depend on calendar time t, integrated FRA/FRA covariances can be computed analytically:

$$
\begin{aligned}
\int \varrho_{ij}\sigma_i(t)\sigma_j(t)\,\mathrm{d}t \quad = \quad & \mathrm{e}^{-\beta\left|t_i-t_j\right|} \cdot k_i k_j \cdot \frac{1}{4c^3} \cdot \\
& \cdot\Bigg(4ac^2d\Big[\mathrm{e}^{c(t-t_j)}+\mathrm{e}^{c(t-t_i)}\Big]+4c^3d^2t \\
& \quad -4bcd\mathrm{e}^{c(t-t_i)}\Big[c(t-t_i)-1\Big]-4bcd\mathrm{e}^{c(t-t_j)}\Big[c(t-t_j)-1\Big] \\
& \quad +\mathrm{e}^{c(2t-t_i-t_j)}\Bigg(2a^2c^2+2abc\Big[1+c(t_i+t_j-2t)\Big] \\
& \qquad\qquad +b^2\Big[1+2c^2(t-t_i)(t-t_j)+c(t_i+t_j-2t)\Big]\Bigg)\Bigg)
\end{aligned}
\tag{24}
$$

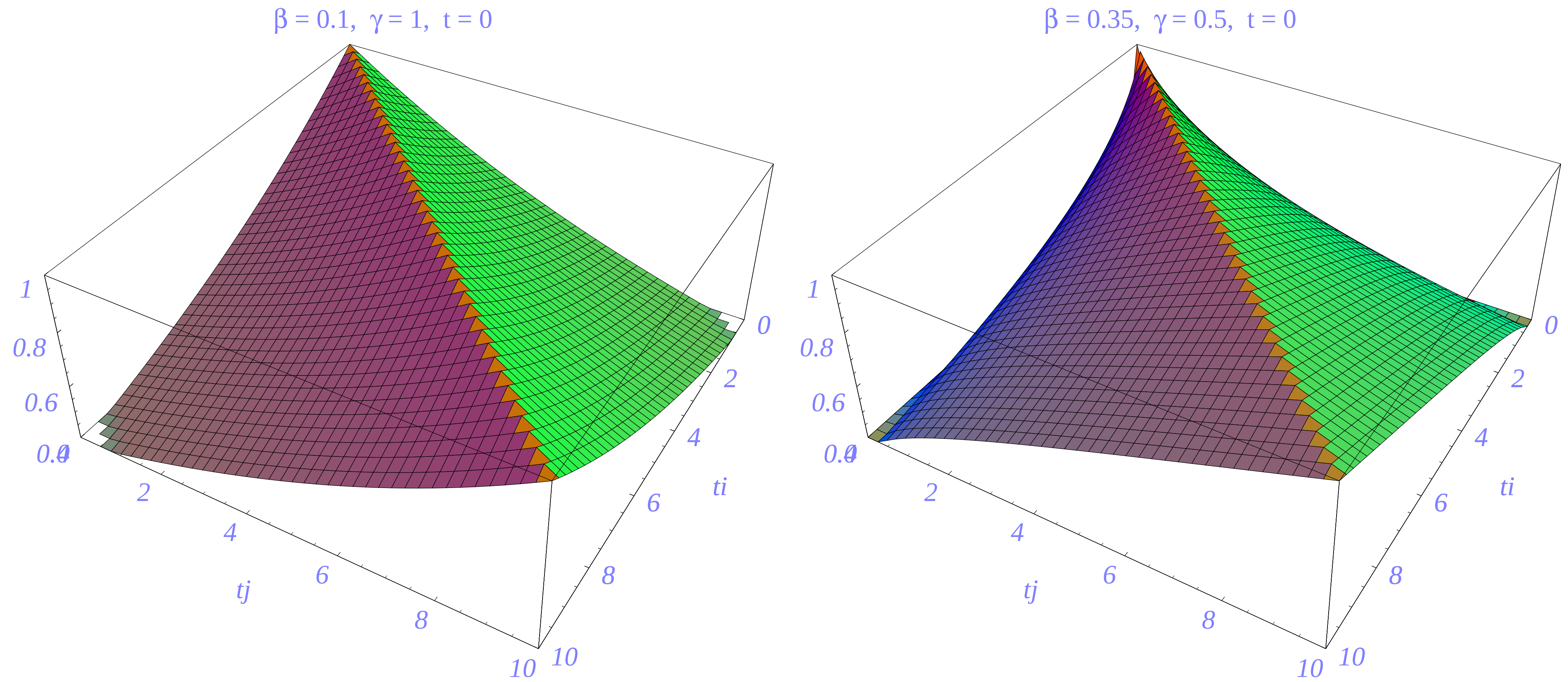
The quality of a reference fit of the implied volatilities consistent with equation (21) for a typical yield curve and caplet market both in EUR and USD is usually very good.



An alternative to the time independent correlation function (23) is

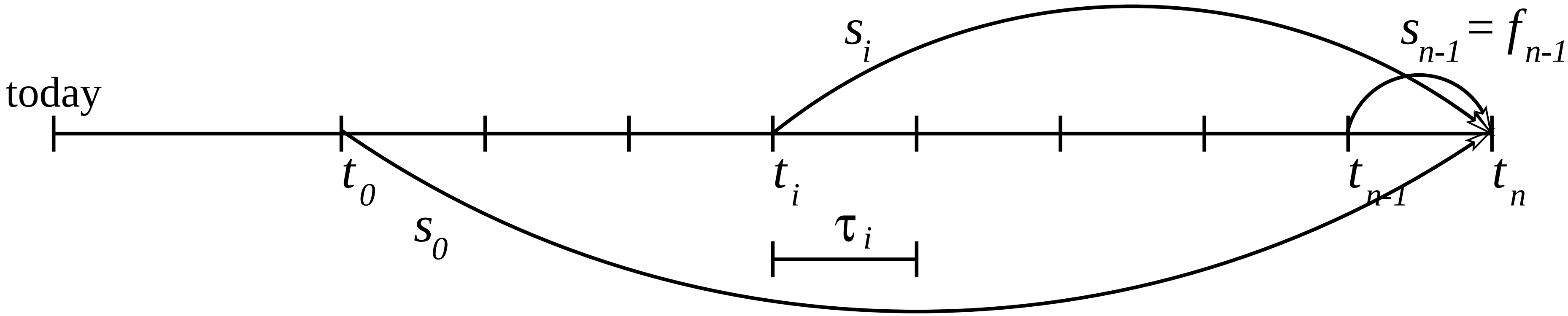
$$\varrho_{ij}(t) \;=\; (1-\eta)\cdot \mathrm{e}^{-\beta\left|(t_i-t)^{\gamma}-(t_j-t)^{\gamma}\right|} \;+\; \eta \tag{25}$$

with $\eta \in [-1, 1]$. Clearly, for $\gamma = 1$ and $\eta = 0$ this functional form is identical to (23). For the functional form (25), suitable parameters are $\gamma \approx 0.5$, $\beta \approx 0.35$, and $\eta \approx 0$.



VIII. Calibration to European swaptions

A forward swap rate s_i



(starting with the reset time of the forward rate f_i) can be written as the ratio

$$s_i = \frac{A_i}{B_i} \tag{26}$$

of the floating leg value

$$A_i = \sum_{j=i}^{n-1} P_{j+1} f_j \tau_j N_j \quad \text{for} \quad i = 0 \ldots n-1 \tag{27}$$

and the annuity

$$B_i = \sum_{j=i}^{n-1} P_{j+1} \tau_j N_j \quad \text{for} \quad i = 0 \ldots n-1 \ . \tag{28}$$

N_j is the notional associated with accrual period τ_j.

Since the market convention of price quotation for European swaptions uses the concept of implied Black volatilities for the forward swap rate, it seems appropriate to think of the swap rates' covariance matrix in relative terms just as much as for the forward rates themselves.

For a set of coterminal swaps all ending with a final payment at t_n, the elements of the swap rate covariance matrix C^s can therefore be written as

$$\begin{aligned} C^s_{ij} &= \left\langle \frac{\mathrm{d}s_i}{s_i} \cdot \frac{\mathrm{d}s_j}{s_j} \right\rangle \Big/ \mathrm{d}t \\ &= \sum_{k=0,l=0}^{n-1,n-1} \frac{\frac{\partial s_i}{\partial f_k} \cdot \frac{\partial s_j}{\partial f_l}}{s_i \cdot s_j} \cdot f_k f_l \cdot \left\langle \frac{\mathrm{d}f_k}{f_k} \frac{\mathrm{d}f_l}{f_l} \right\rangle \Big/ \mathrm{d}t \\ &= \sum_{k=0,l=0}^{n-1,n-1} \frac{\partial s_i}{\partial f_k} \frac{f_k}{s_i} \cdot C^f_{kl} \cdot \frac{f_l}{s_j} \frac{\partial s_j}{\partial f_l} \ . \end{aligned} \tag{29}$$

Defining the elements of the matrix $Z^{f\to s}$ by

$$Z^{f\to s}_{ik} = \frac{\partial s_i}{\partial f_k}\frac{f_k}{s_i}\;, \tag{30}$$

the mapping from the FRA covariance matrix C^{FRA} to the swap rate covariance matrix C^s can be seen as a matrix multiplication:

$$C^s = Z^{f\to s} \cdot C^f \cdot {Z^{f\to s}}^\top \quad . \tag{31}$$

Equations (30) and (31) are the basis of all fast constructive calibration algorithms.

When the floating and fixed payments of a swap occur simultaneously with the same frequency, it is possible to find a simple formula for the swap rate coefficients $Z^{f\to s}_{ik}$. Using

$$\frac{\partial P_{i+1}}{\partial f_k} = -P_{i+1}\frac{\tau_k}{1+f_k\tau_k}\cdot \mathbf{1}_{\{k\geq i\}}\;, \tag{32}$$

where $\mathbf{1}_{\{k \geq i\}}$ is one if $k \geq i$ and zero otherwise, and equations (27), (28), and (26), we have

$$\frac{\partial s_i}{\partial f_k} = \left\{ \frac{P_{k+1}\tau_k N_k}{B_i} - \frac{\tau_k}{1+f_k\tau_k} \cdot \frac{A_k}{B_i} + \frac{\tau_k}{1+f_k\tau_k} \cdot \frac{A_i B_k}{B_i^2} \right\} \cdot \mathbf{1}_{\{k \geq i\}} \, . \tag{33}$$

This enables us to calculate the elements of the forward rate to swap rate covariance transformation matrix $Z^{f \to s}$ to obtain the expression

$$Z_{ik}^{f \to s} = \left[\underbrace{\frac{P_{k+1} N_k f_k \tau_k}{A_i}}_{\text{constant weights approximation}} + \underbrace{\frac{(A_i B_k - A_k B_i) f_k \tau_k}{A_i B_i (1+f_k\tau_k)}}_{\text{shape correction}} \right] \cdot \mathbf{1}_{\{k \geq i\}} \, . \tag{34}$$

The second term inside the square brackets of equation (34) is a *shape correction*. Rewriting it as

$$\frac{(A_i B_k - A_k B_i) f_k \tau_k}{A_i B_i (1 + f_k \tau_k)} = \frac{f_k \tau_k}{A_i B_i (1 + f_k \tau_k)} \cdot \sum_{l=i}^{k-1} \sum_{m=k}^{n-1} P_{l+1} P_{m+1} N_l N_m \tau_l \tau_m (f_l - f_m) \quad (35)$$

highlights that it is a weighted average over inhomogeneities of the yield curve.

In fact, for a flat yield curve, all of the terms $(f_l - f_m)$ are identically zero and the mapping matrix $Z^{f \to s}$ is equivalent to the so-called constant-weights approximation.

In practice the yield curve is never entirely flat which makes it necessary to compute the swap rate coefficients via the full derivative calculation (30).

When floating and fixed schedules differ, we have to compute the partial dependencies of the swap rates' floating payments, floating payment discount factors, and fixed payment discount factors individually.

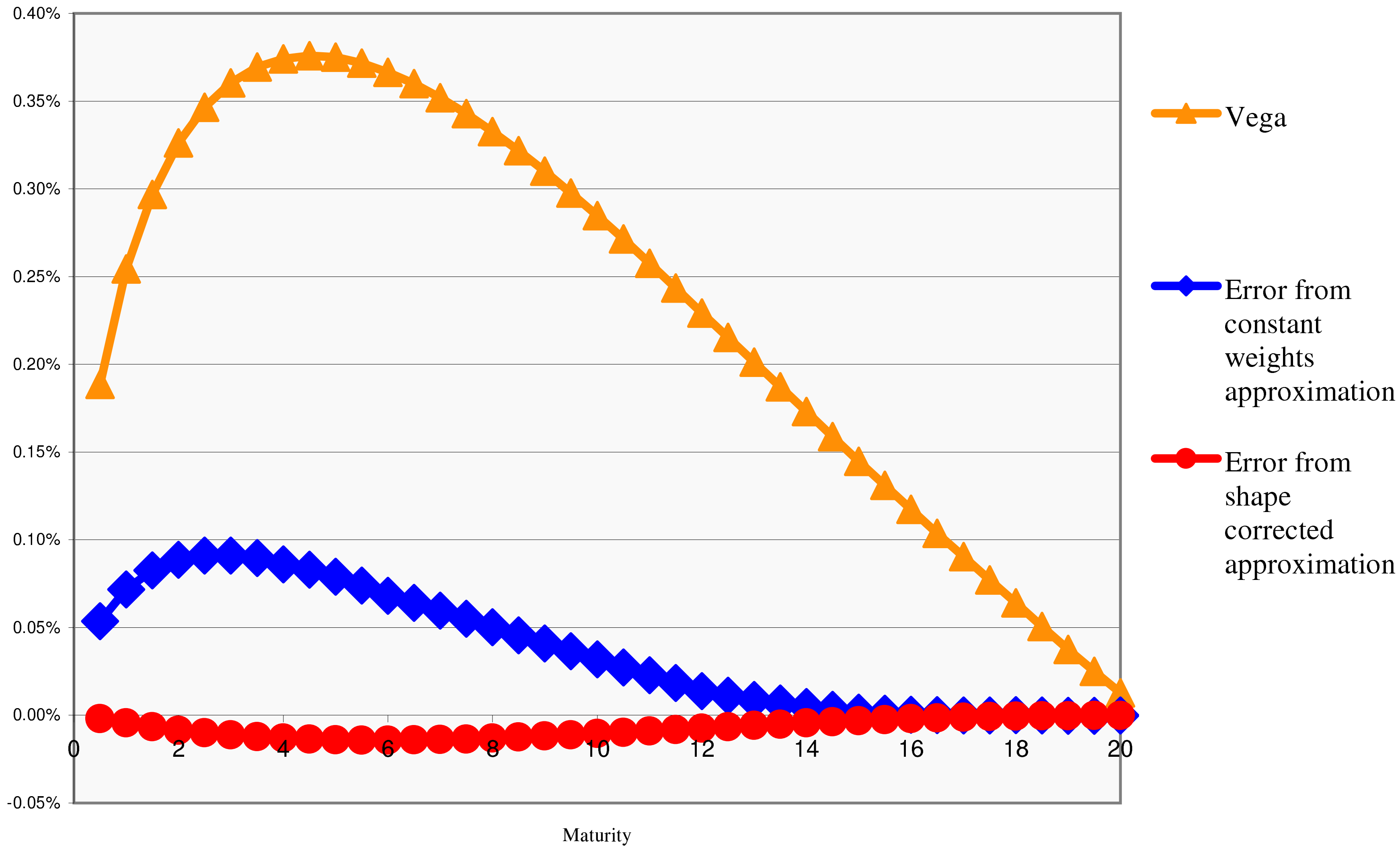
As things stand at this point, we have a map between the instantaneous FRA/FRA covariance matrix and the instantaneous swap/swap covariance matrix.

Unfortunately, though, the map involves the state of the yield curve at any one given point in time via the matrix Z.

The price of a European swaption, though, does not just depend on one single realised state or even path of instantaneous volatility. It is much more appropriate to think about some kind of *path integral average volatility*. Using arguments of factor decomposition and equal probability of up and down moves (in log space), it can be shown [JR00, Kaw02, Kaw03] that the specific structure of the map allows us to approximate the effective implied swaption volatilities by simply using today's state of the yield curve for the calculation of the mapping matrix Z:

$$\hat{\sigma}_{s_i}(t,T) = \sqrt{\sum_{k=i,l=i}^{n-1} Z_{ik}^{f\to s}(0) \cdot \frac{\int_t^T \sigma_k(t')\sigma_l(t')\varrho_{kl}\mathrm{d}t'}{T-t} \cdot Z_{il}^{f\to s}(0)} \qquad (36)$$

This approximate equivalent implied volatility can now be substituted into the Black swaption formula to produce a price *without the need for a single simulation*. In practice, the formula (36) works remarkably well.



We can now design a non-iterative calibration procedure that connects the stepwise covariance matrices of the logarithms of the realisations of the forward rates directly to the calibration volatilities of a set of European swaptions (including caplets). For any given time step from t to T, populate the time-unscaled FRA/FRA covariance matrix

$$C^f_{kl} = \frac{\int_{t'=t}^{T} \sigma_k(t')\sigma_l(t')\varrho_{kl}(t')\mathrm{d}t'}{T-t} \; . \tag{37}$$

Next, map this matrix into a time-unscaled swap/swap covariance matrix using the Z matrix calculated from the initial state of the yield curve

$$C^s = Z \cdot C^f \cdot Z^\top \; . \tag{38}$$

Note that this swap rate/swap rate covariance matrix is associated with forward swap rates that expire at times equal to or later than T. Its diagonal elements are the mean square volatilities of the n swap rates over the time step $t \to T$. For $t = 0$ and $T = t_1$, we notice that the diagonal element C^s_{11} represents the square of the FRA-covariance matrix implied Black volatility of the first swaption, which, if the model was already calibrated, should equate the market implied volatility of the swaption expiring at time t_1 denoted by $\sigma_{s_1}^{\text{market}}$. Since variances are additive, we have

$$C_{ij}^{s}(0,t_k)\cdot t_k = C_{ij}^{s}(0,t_{k-1})\cdot t_{k-1} + C_{ij}^{s}(t_{k-1},t_k)\cdot(t_k - t_{k-1}) \quad \text{for} \quad k \geq \max(i,j)\ . \tag{39}$$

In other words, we can compute the time-integrated (smaller) covariance matrix for a set of swaptions expiring at a later date by adding a subset of the (larger) time-integrated covariance matrix to an earlier date and the time-integrated covariance matrix from that earlier date to the later date.

This additive feature of covariances means that we can accomplish calibration of each swaption individually by rescaling the whole swap rate covariance matrix such that the diagonal elements, when averaged to the expiry date of any individual swaption, match the square of the respective market given implied volatility.

For this purpose, define the diagonal matrix Ξ by

$$\Xi_{gh} = \frac{\hat{\sigma}_{s_h}^{\text{market}}}{\hat{\sigma}_{s_h}(0, t_h)} \cdot \delta_{gh} \tag{40}$$

with δ_{gh} being the Kronecker symbol (which is zero unless $g = h$ when it is one) and $\hat{\sigma}_{s_h}(0, t_h)$ calculated from the FRA instantaneous volatility parametrisation through equation (36). The calibrated swap rate/swap rate covariance matrix for any time step $t \to T$ is thus given by

$$C^{s}_{\text{calibrated}} = \Xi \cdot Z \cdot C^{f} \cdot Z^{\top} \cdot \Xi \, . \tag{41}$$

When Z is invertible, we can therefore define the calibration matrix

$$M := Z^{-1} \cdot \Xi \cdot Z \tag{42}$$

and express the entire calibration procedure as the simple operation

$$C^{f}_{\text{calibrated}} = M \cdot C^{f}_{\text{parametric}} \cdot M^{\top} \, . \tag{43}$$

In order to use the matrix $C^f_{\text{calibrated}}$ for the evolution of the yield curve over the time step $t \longrightarrow T$ from a set of standard normal variates, we now simply need to compute a pseudo-square root $A^f_{\text{calibrated}}$ such that

$$C^f_{\text{calibrated}} = A^f_{\text{calibrated}} \, {A^f_{\text{calibrated}}}^\top \tag{44}$$

just as we would have done without calibration to swaptions.

In practice, a user may wish to specify not exactly as many swaptions as there are forward rates to calibrate to. Instead, it may be desirable to specify fewer, or even more than n swaption volatilities. In this case, the swap rate coefficient matrix Z may be over- or under-determined. Either way, it is still possible to find a matrix M that can be used in equation (43). To find it, let us first consider the singular value decomposition [PTVF92] of the transpose of Z :

$$Z^\top = U \cdot W \cdot V^\top \tag{45}$$

In the underdetermined case, the diagonal matrix W will have some zero entries on the diagonal. Let us define W' as the diagonal matrix whose diagonal elements are the inverse of the corresponding elements in W where they are nonzero, and zero otherwise. The matrix product $(W'W)$ then has unit elements wherever W has nonzero entries, and formally constitutes a *projection* matrix by virtue of the fact that its repeated application to any target vector has the same result as a single multiplication, i.e.

$$(W'W)^k \cdot X = (W'W) \cdot X \quad \forall\, k \geq 1 \text{ and } \forall\, X \ . \tag{46}$$

Now, the calibration procedure in the present framework amounts to the identification of $A^f_{\text{calibrated}}$ that satisfies

$$Z \cdot A^f_{\text{calibrated}} = \Xi \cdot Z \cdot A^f_{\text{parametric}} \tag{47}$$

but remains as close to the original $A^f_{\text{parametric}}$ as possible, i.e. to find the matrix $A^f_{\text{calibrated}}$ that meets equation (47) and simultaneously minimises

$$\left|\left|A^f_{\text{calibrated}} - A^f_{\text{parametric}}\right|\right| \tag{48}$$

for some suitable matrix norm. Now, denote the Moore-Penrose inverse [Alb72] of Z as $\widetilde{Z^{-1}}$, and write the product of $\widetilde{Z^{-1}}$ times Z itself as Q :

$$Q := {U^\top}^{-1} W'V^\top \cdot VWU^\top \tag{49}$$

By the aid of the orthogonality conditions satisfied by the constituents U and V of the singular value decomposition of Z, and by the fact that $(W'W)$ is a projection, both Q and the matrix

$$P := \mathbf{1} - Q \tag{50}$$

are also projection operators. In fact, P is the projection onto the kernel of Z. The desired matrix $A^f_{\text{calibrated}}$ can thus be found by adding the projection of $A^f_{\text{parametric}}$ onto the kernel of Z and the Moore-Penrose solution to equation (47), i.e.

$$A^f_{\text{calibrated}} = P \cdot A^f_{\text{parametric}} + \widetilde{Z^{-1}} \cdot \Xi \cdot Z \cdot A^f_{\text{parametric}} \ . \tag{51}$$

Since $A^f_{\text{parametric}}$ appears as the last multiplicand in both of the summands on right hand side, we can rewrite this as

$$\begin{aligned} A^f_{\text{calibrated}} &= (\mathbf{1} - UW'WU^\top + UW'V^\top \Xi Z) \cdot A^f_{\text{parametric}} \\ &= \left(\mathbf{1} - UW'\left(WU^\top - V\top\Xi Z\right)\right) \cdot A^f_{\text{parametric}} \\ &= \left(\mathbf{1} - UW'\left(1 - V^\top \Xi V\right) WU^\top\right) \cdot A^f_{\text{parametric}} \ . \end{aligned} \tag{52}$$

This means, the sought calibration matrix M is given by

$$M = \mathbf{1} - U \cdot W' \cdot \left(1 - V^\top \cdot \Xi \cdot V\right) \cdot W \cdot U^\top \ . \tag{53}$$

The key to this calibration procedure in the underdetermined case is that a *minimal* solution to the raw calibration problem (47) is combined with as much of the original covariance information as possible that has no effect on the calibration problem. In more formal terms, we combine the minimal solution to the calibration problem with the projection of the desired covariance structure onto the calibration kernel.

When Z is overdetermined, the correction matrix M cannot achieve calibration to all the desired market prices. Instead, the calibration procedure based on the linear algebraic operations above will result in a least squares fit in some suitable norm by virtue of the use of the singular value decomposition of Z.

Within the limits of the approximation (36), the operation given in equation (43) will provide calibration to European swaption prices whilst retaining as much calibration to the caplets as is possible without violating the overall FRA/FRA correlation structure too much.

IX. Long time steps using the predictor-corrector scheme

In order to price an exotic interest rate derivative in a Monte Carlo framework, we need to evolve the set of forward rates $\boldsymbol{f}$ from its present values into the future according to the stochastic differential equation

$$\mathrm{d}f_i(t) = f_i(t) \cdot \mu_i(\boldsymbol{f}(t), t)\mathrm{d}t + f_i(t) \cdot \sum_{j=1}^{m} a_{ij}\mathrm{d}W_j \tag{54}$$

driven by an m-dimensional standard Wiener process $\boldsymbol{W}$.

The drift terms given by equation (17) are clearly state-dependent and thus indirectly stochastic which forces us to use a numerical scheme to solve equation (54) along any one path.

A simple explicit Euler scheme would be

$$f_i^{\mathsf{Euler}}(\boldsymbol{f}(t), t+\Delta t) = f_i(t) + f_i(t) \cdot \mu_i(\boldsymbol{f}(t), t)\Delta t + f_i(t) \cdot \sum_{j=1}^{m} a_{ij}(t) z_j \sqrt{\Delta t} \tag{55}$$

with z_j being m independent normal variates. This would imply that we approximate the drift as constant over the time step $t \ \rightarrow \ t + \Delta t$.

Moreover, this scheme effectively means that we are using a normal distribution for the evolution of the forward rates over this time step.

Whilst we may agree to the approximation of a piecewise constant (in time) drift coefficient μ_i, the normal distribution may be undesirable, especially if we envisage to use large time steps Δt for reasons of computational efficiency.

However, when we assume piecewise constant drift, we might as well carry out the integration over the time step Δt analytically and use the scheme

$$f_i^{\text{Constant drift}}(\boldsymbol{f}(t), t + \Delta t) = f_i(t) \cdot \mathrm{e}^{\mu_i(\boldsymbol{f}(t),t)\Delta t - \frac{1}{2}c_{ii} + \sum\limits_{j=1}^{m} a_{ij} z_j} \tag{56}$$

whereby the time step scaling by $\sqrt{\Delta t}$ for A and by Δt for C has been absorbed into the respective matrices. In other words, we have set $A' := A \cdot \sqrt{\Delta t}$ and $C' := C \cdot \Delta t$ and dropped the primes. Equation (56) can also be viewed as the Euler scheme in logarithmic coordinates.

The above procedure works very well as long as the time steps Δt are not too long and is widely used and also referred to in publications [And00, GZ99].

Since the drift term appearing in the exponential function in equation (56) is in some sense a stochastic quantity itself, we will begin to notice that we are ignoring Jensen's inequality when the term $\mu_i \Delta t$ becomes large enough.

This happens when we choose a big step Δt, or the forward rates themselves or their volatility are large. Therefore, we should use a *hybrid predictor-corrector* method which models *only the drift* as indirectly stochastic.

A method that works very well in practice is as follows.

1. Given a current evolution of the yield curve denoted by $\boldsymbol{f}(t)$, we calculate the predicted solution $\boldsymbol{f}^{\text{Constant drift}}(\boldsymbol{f}(t), t + \Delta t)$ using one m-dimensional normal variate draw $\boldsymbol{z}$ following equation (56).

2. We recalculate the drift using this evolved yield curve. The predictor-corrector approximation $\tilde{\mu}_i$ for the drift is then given by the average of

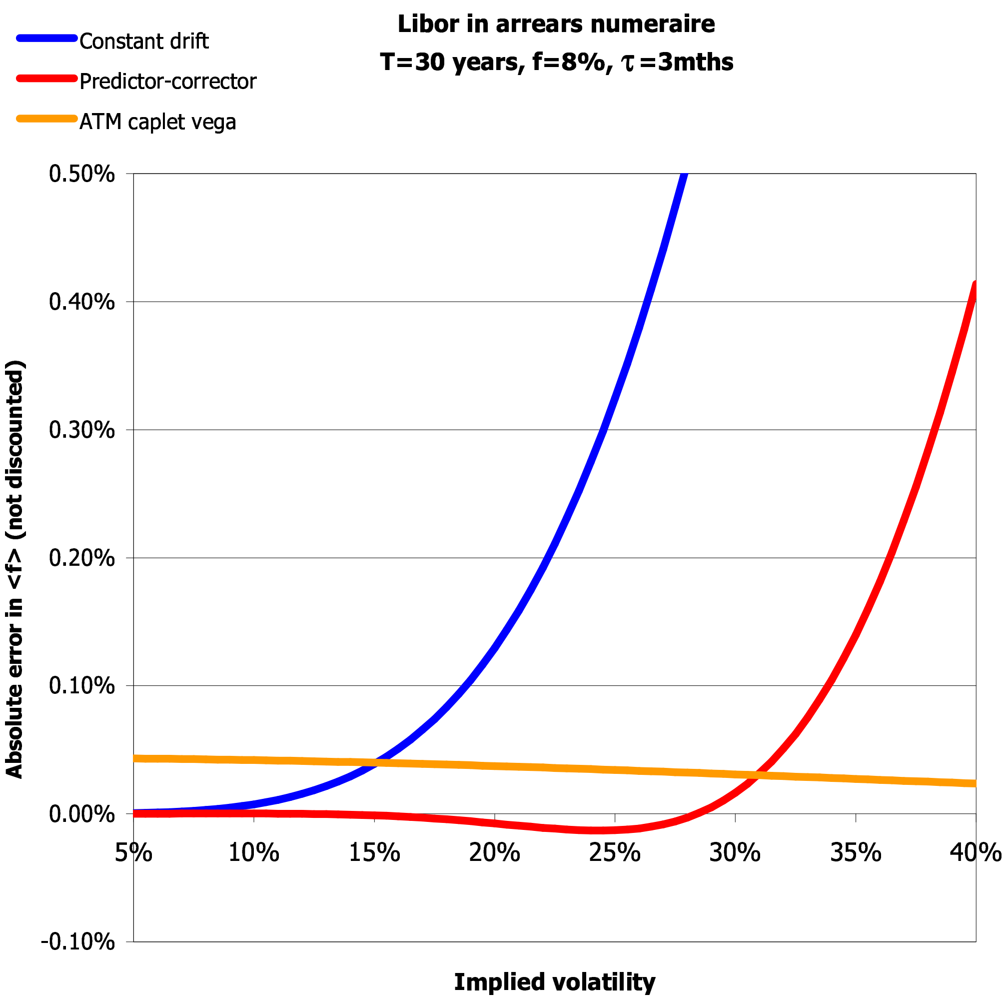
these two calculated drifts, i.e.

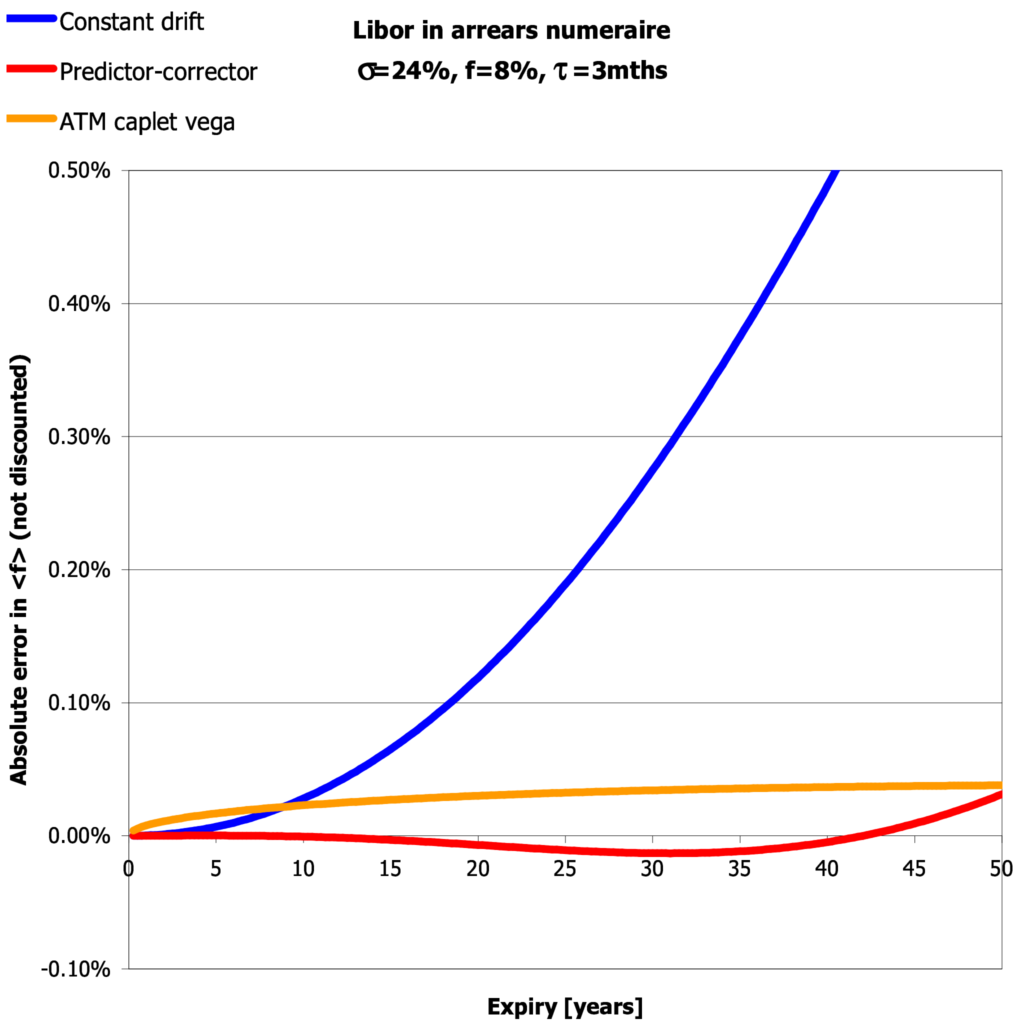
$$\tilde{\mu}_i(\boldsymbol{f}(t), t \;\rightarrow\; t + \Delta t) = {}^1\!/_2 \Big\{ \mu_i(\boldsymbol{f}(t), t) + \mu_i(\boldsymbol{f}^{\text{Constant drift}}(\boldsymbol{f}(t), t + \Delta t), t) \Big\} \;. \tag{57}$$

3. The predictor-corrector evolution is given by

$$f_i^{\text{Predictor-corrector}}(\boldsymbol{f}(t), t + \Delta t) = f_i(t) \cdot \mathrm{e}^{\tilde{\mu}_i(\boldsymbol{f}(t), t \,\rightarrow\, t + \Delta t)\Delta t - \frac{1}{2}c_{ii} + \sum\limits_{j=1}^{m} a_{ij} z_j} \tag{58}$$

wherein we re-use the same normal variate draw $\boldsymbol{z}$, i.e. we only correct the drift of the predicted solution.





The stability of the predictor-corrector drift method as a function of volatility level (left) and time to expiry (right) for the Libor-in-arrears convexity.

⇒ The predictor-corrector drift approximation is highly accurate!

X. Bermudan swaptions

A Bermudan swaption contract denoted by 'X-non-call-Y' gives the holder the right to enter into a swap at a prespecified strike rate K on a number of exercise opportunities.

The first exercise opportunity in this case would be Y years after inception. The swap that can be entered into has always the same terminal maturity date, namely X, independent on when exercise takes place.

A Bermudan swaption that entitles the holder to enter into a swap in which he pays the fixed rate is known as *payer's*, otherwise as *receiver's*.

For the owner of a payer's Bermudan swaption, the present value of exercising at time t_j is given by the intrinsic value $I(t_j)$ of the swap to be entered into at that time

$$I(t_j) = \sum_{k=j}^{n-1} \left[P_{k+1}(t_j) \cdot (f_k(t_j) - K)\, \tau_k \right] \; . \tag{59}$$

In order to decide optimally about early exercise at time t_j, the holder compares the present intrinsic value with the expected profit to be made by not exercising at that time. Thus, the t_j-value of the Bermudan swaption $V(t_j)$ is given by

$$V(t_j) = \begin{cases} \max\left\{I(t_j), \mathsf{E}_{t_j}[V(t_{j+1})]\right\} & \text{for} \quad j = 1 \ldots n-2 \\ \max\left\{I(t_j), 0\right\} & \text{for} \quad j = n-1 \end{cases} \quad . \tag{60}$$

In the marketplace, many variations are common such as differing payment frequencies between fixed and floating leg, margins on top of the floating payment, varying notionals (*roller coaster* or *amortizing* swaptions are not uncommon), time-varying strike of the swap to enter into, cross-currency payoff (quanto), and many more.

XI. The Bermudan swaption exercise domain

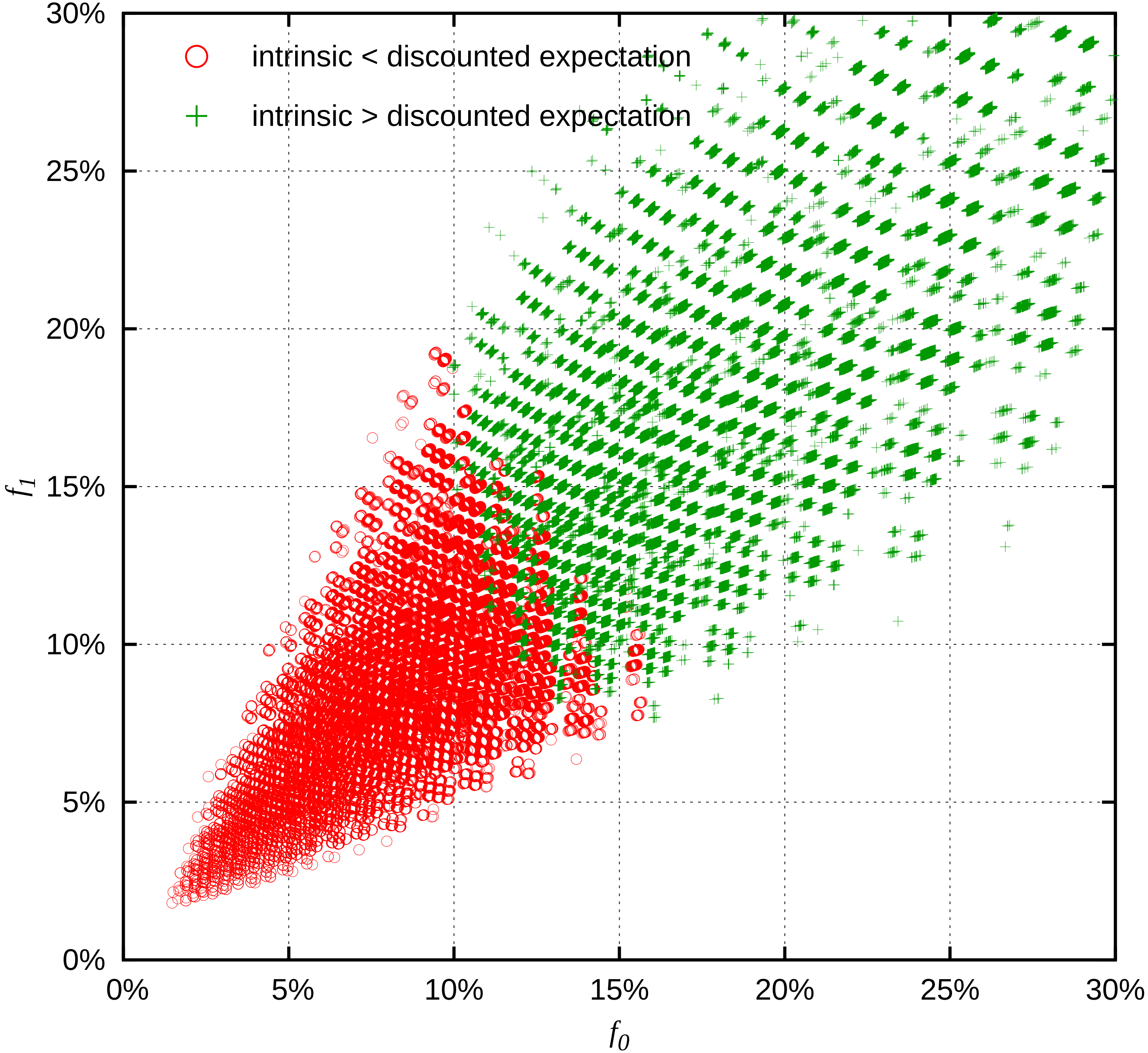
There are many ways to describe a given discretised yield curve unambiguously:

- a complete set of discrete spanning forward rates $\{f_i\}$;
- a complete set of discrete coterminal (or, indeed, coinitial) swap rates $\{s_i\}$;
- a complete set of discount factors $\{P_i\}$;

and more.

Given any complete specification of the discretised yield curve, we can project points in yield curve space onto any alternative coordinate choice we please.

It depends on our choice of projection to what extent the domain of optimal exercise and the domain of optimal continuation appear to overlap.


intrinsic < discounted expectation
intrinsic > discounted expectation
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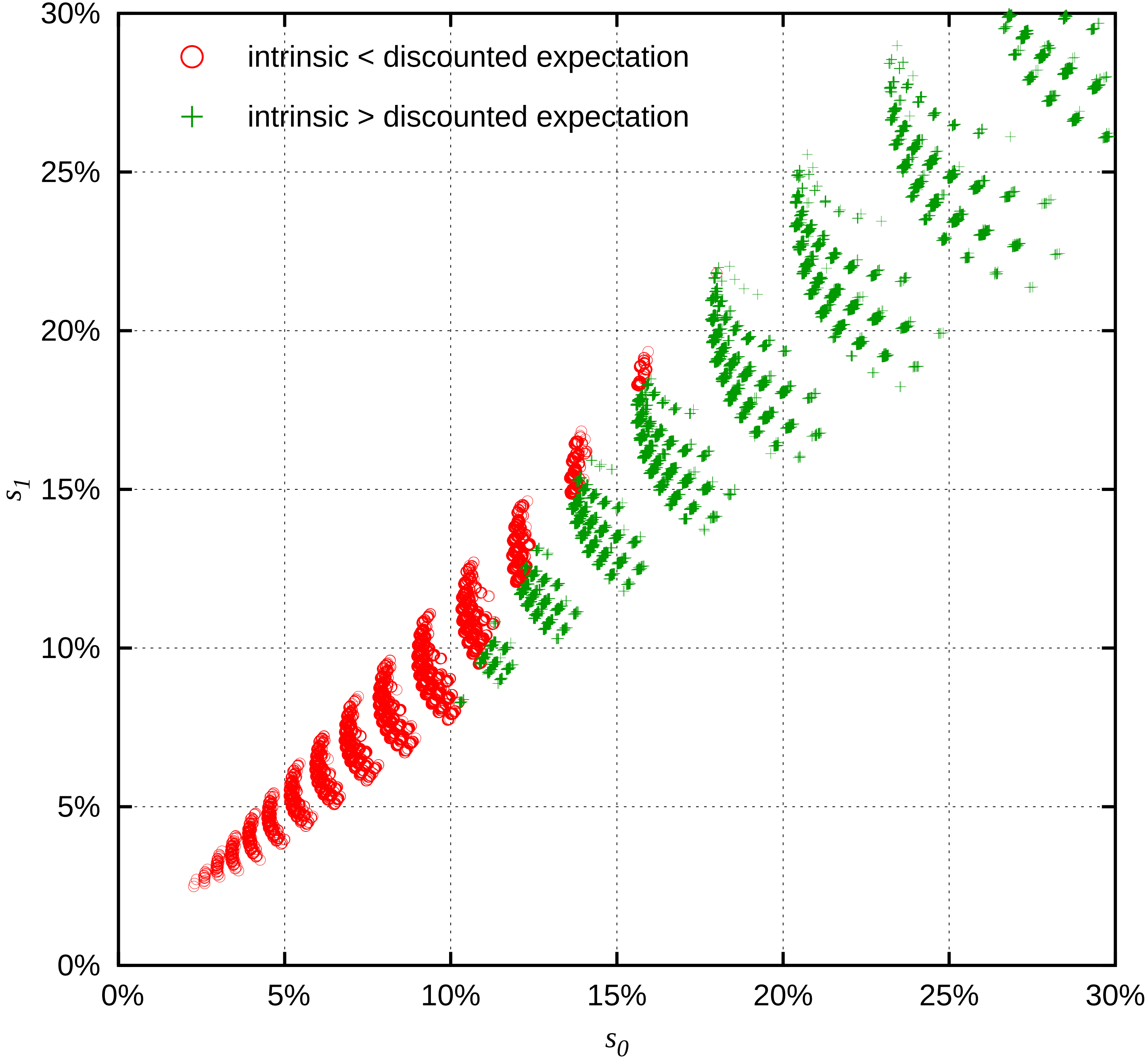
f_1
f_0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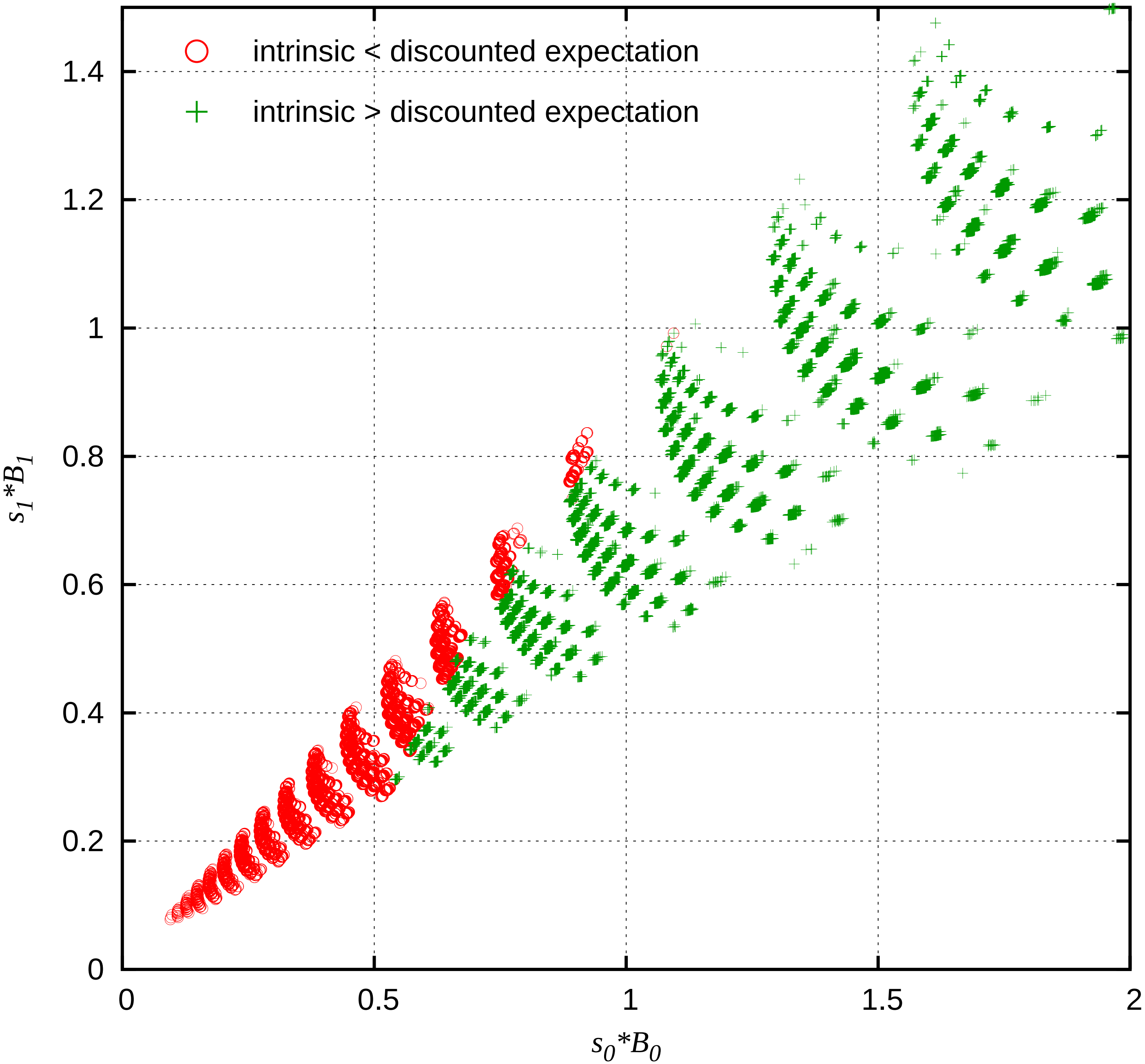
intrinsic < discounted expectation
intrinsic > discounted expectation
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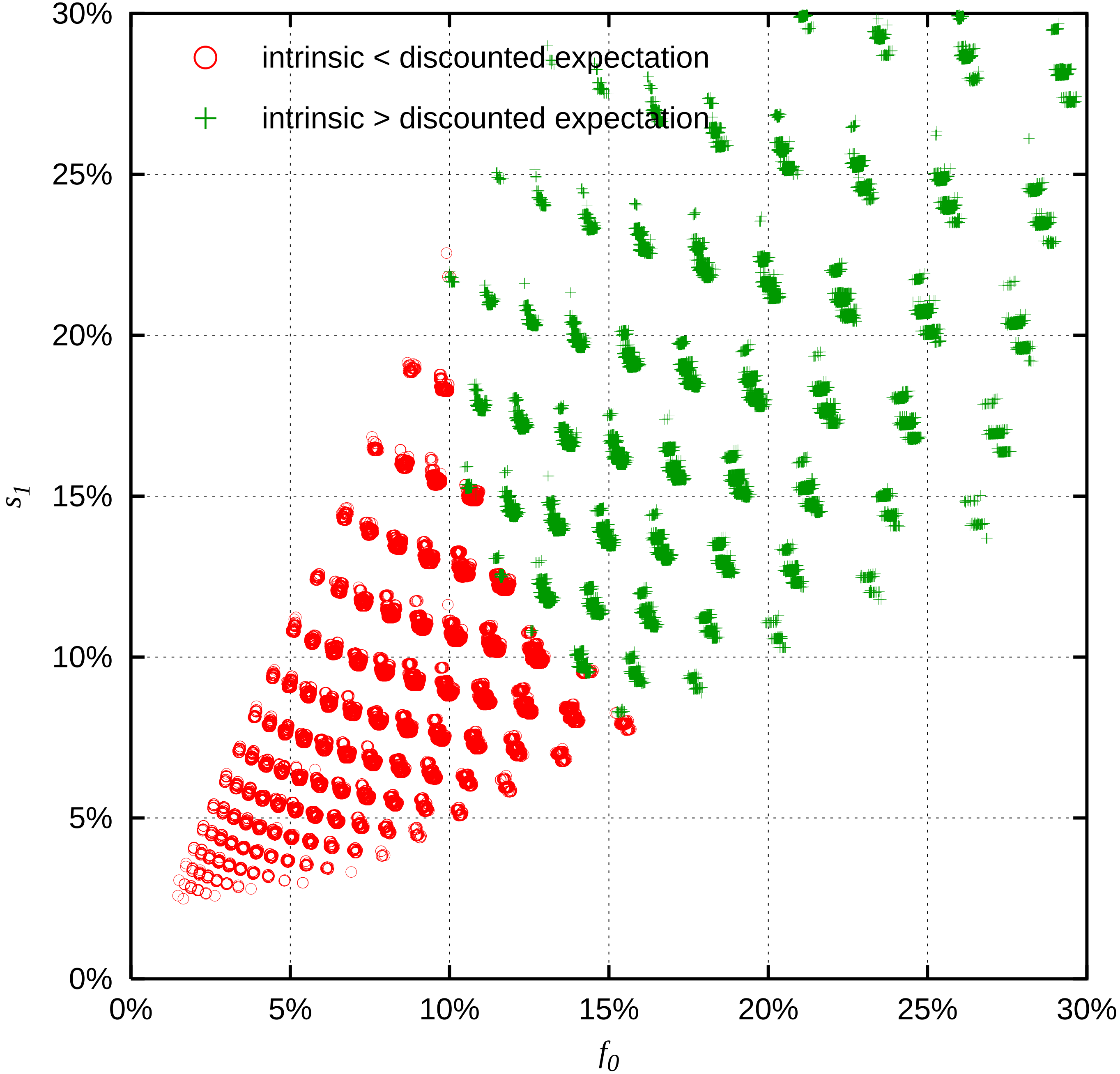
s_1*B_1
s_0*B_0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XII. Bermudan swaption exercise boundary parametrisation

Taking into account all of the heuristic observations about the shape of the exercise boundary in various projections for many different shapes of the yield curve and volatility structures, the following function can be chosen as the basis for the subsequent exercise decision strategy in the Monte Carlo simulation:

$$\mathcal{E}_i(\boldsymbol{f}(t_i)) = \phi_\pm \cdot \left(f_i(t_i) - \left[p_{i1} \cdot \frac{s_{i+1}(0)}{s_{i+1}(t_i) + p_{i2}} + p_{i3} \right] \right) \tag{61}$$

with

$$\phi_\pm = \left\{ \begin{array}{ccc} +1 & \text{for} & \text{payer's swaptions} \\ -1 & \text{for} & \text{receiver's swaptions} \end{array} \right\} \tag{62}$$

This function is hyperbolic in s_{i+1} and depends on three coefficients, the initial (i.e. at the calendar time of evaluation or inception of the derivative contract) value of $f_i(0)$ and $s_{i+1}(0)$, and their respective evolved values as given by the simulation procedure. For a more general discussion, see [JA10].

Since we have to make an exercise decision at each exercise opportunity time t_i, we allow for a new set of exercise function coefficients for each such time slice.

For non-standard Bermudan swaptions that have payments in between exercise dates, we use the shortest swap rate from t_i to the next exercise time instead of f_i. The parametric exercise decision given an evolved yield curve is then simply to exercise if $\mathcal{E}_i > 0$.

At the very last exercise opportunity at time t_{n-1} we have exact knowledge if exercise is optimal, namely when the residual swap is in the money. This easily integrates into the parametric description given by equation (61) by setting $p_{(n-1)\ 1}$ and $p_{(n-1)\ 2}$ to zero and $p_{(n-1)\ 3}$ to the strike:

$$\begin{aligned} p_{(n-1)\ 1} &= 0 \\ p_{(n-1)\ 2} &= 0 \\ p_{(n-1)\ 3} &= K \end{aligned} \tag{63}$$

XIII. The Bermudan Monte Carlo algorithm

- For a *training set* $\mathbb{P}^{\text{Training}}$ of N_{Training} evolutions of the yield curve into the future out to the last exercise time t_{n-1}, precalculate:

$$\mathbb{P}^{\text{Training}} = \{\boldsymbol{f}_{jk}\} \ , \ j = 1 \dots N_{\text{Training}} \, , k = 0 \dots n-1 \tag{64}$$

- For each evolution of the yield curve, precalculate and store the residual intrinsic value I_{jk} in the chosen numéraire as seen at time t_k.

- Carry out $n-1$ optimisations, one for each exercise opportunity t_i apart from the last one[3] in order to determine the best values to use for the coefficients p_{ij}.

The optimisations are to be done in reverse order, starting with the penultimate exercise time t_{n-2}.

[3]On the very last exercise opportunity, the optimal exercise parameters are given by equation (63) whence no optimisation is required for them.

Prior to each optimisation, we assign a path-value $v_j, j = 1 \ldots N_{\text{Training}}$ to each evolution path in the training set $\mathbb{P}^{\text{Training}}$ which represents the value of the Bermudan swaption on this path if no exercise occurs up until and including t_i.

The path-value vector $\boldsymbol{v}$ is initialised to be zero in all its elements before we enter the following loop which counts down in the time index variable i from $(n-2)$ to 0:

1. For each path $\boldsymbol{f}_{j\,(\cdot)}$ in $\mathbb{P}^{\text{Training}}$,

   if $(\mathcal{E}_{i+1}(\boldsymbol{f}_{ji}) > 0)$ **and** $(I_{j\ (i+1)} > 0)$,

   re-assign $v_j := I_{j\ (i+1)}$,

   else

   leave v_j unchanged.

2. Optimise the average of the exercise-decision dependent value

$$U_i(\boldsymbol{p}_i) = \frac{1}{N_{\text{Training}}} \sum_{j=1}^{N_{\text{Training}}} \left\{ \begin{array}{lll} I_{ji} & \textbf{if} & (\mathcal{E}_i(\boldsymbol{f}_{ji}\,;\,\boldsymbol{p}_i) > 0) \\ v_j & \textbf{else} & \end{array} \right\} \tag{65}$$

   over the three parameters p_{i1}, p_{i2}, and p_{i3}. Specifically, one can use the Broyden-Fletcher-Goldfarb-Shanno multi-dimensional variable metric method for this optimisation [PTVF92].

   It is worth noting that, since *absolutely all values are precalculated and stored*, the function to be optimised given by equation (65) requires merely N_{Training} evaluations of the exercise decision function $\mathcal{E}_i(\boldsymbol{f}_{ji}\,;\,\boldsymbol{p}_i)$ and the same number of additions and is thus *linear in the number of training paths and independent on the dimensionality or maturity of the problem*.

3. Decrement i by 1

4. **if** $(i \geq 0)$ continue with step 1.

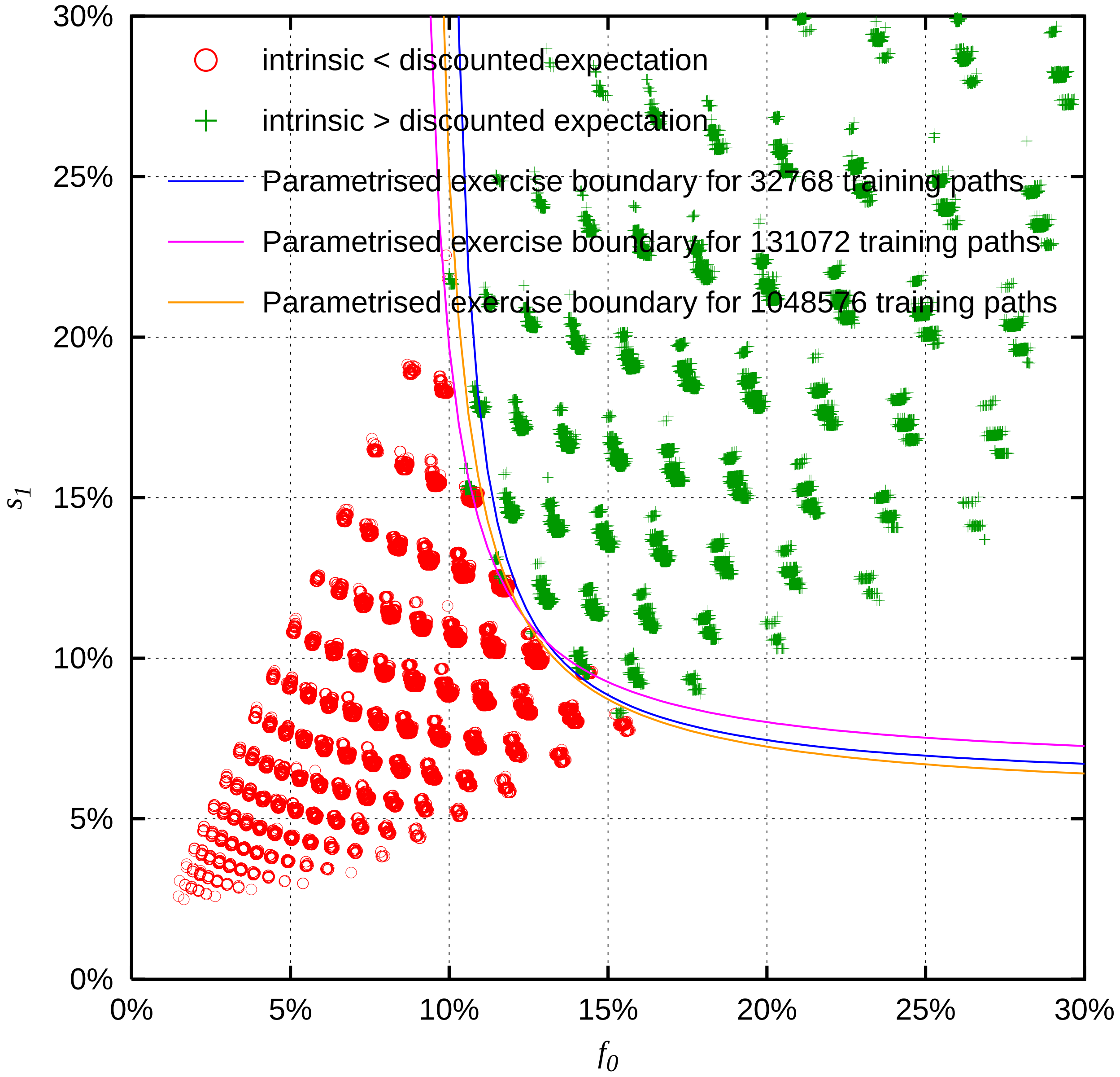
The final value $U_0(\boldsymbol{p}_0)$ gives then an estimate of the value of the Bermudan swaption with a slight upward bias.

Therefore, we finally re-run the simulation with a new set of N_{Sampling} yield curve evolutions using the established exercise strategy parametrisation given by the set of n exercise decision functions $\mathcal{E}_i$.

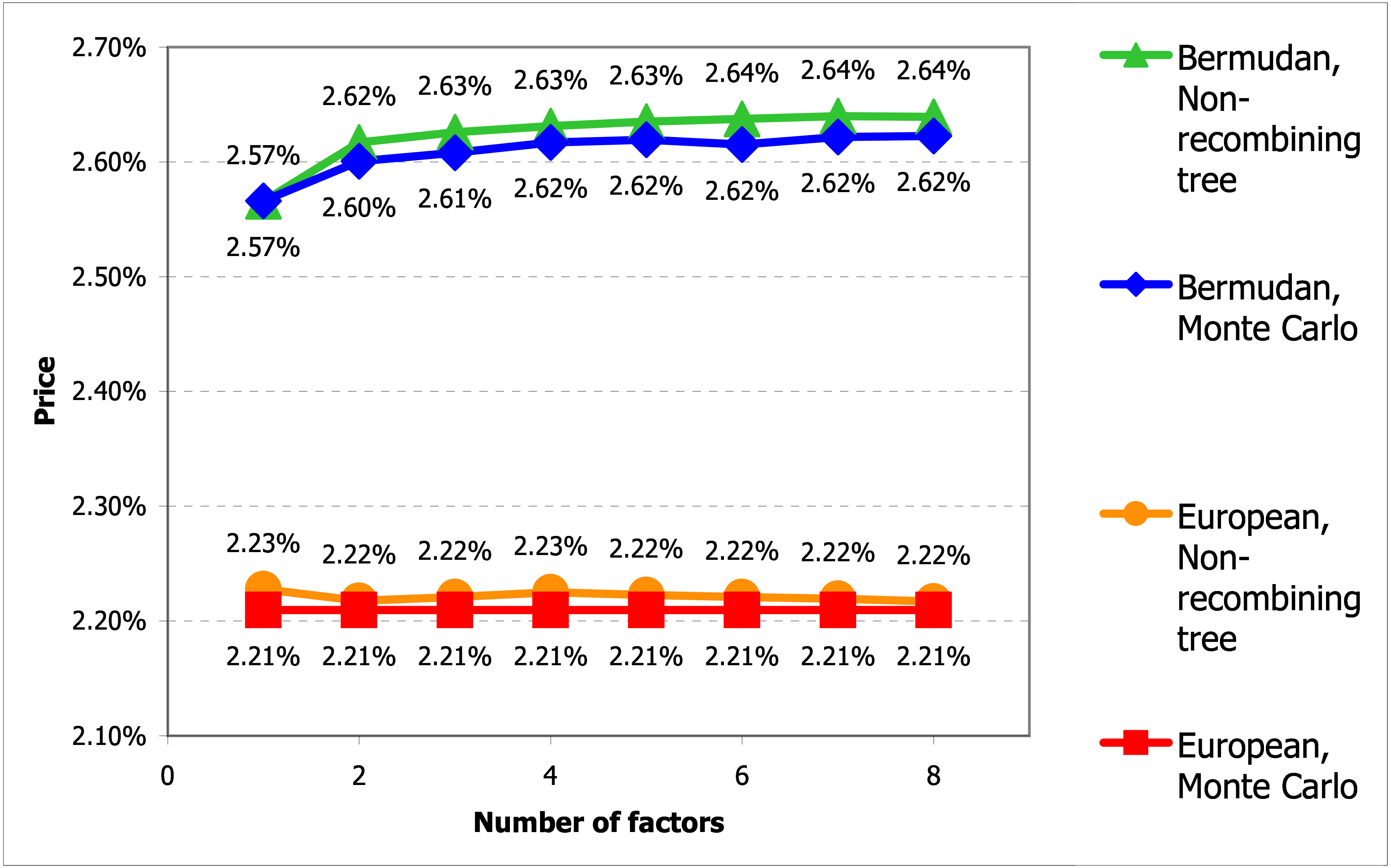
$N_{\text{Sampling}} \simeq 2N_{\text{Training}}$ is typically well sufficient, especially when the driving number generator method was a low-discrepancy sequence.

In practice, we may use a specific choice of coordinates for the specification of the exercise decision function, e.g. (f_i, s_{i+1}). Naturally, since we then have $\mathcal{E}_i = \mathcal{E}_i(f_i, s_{i+1})$, we only store (f_i, s_{i+1}) for each path and each exercise horizon.

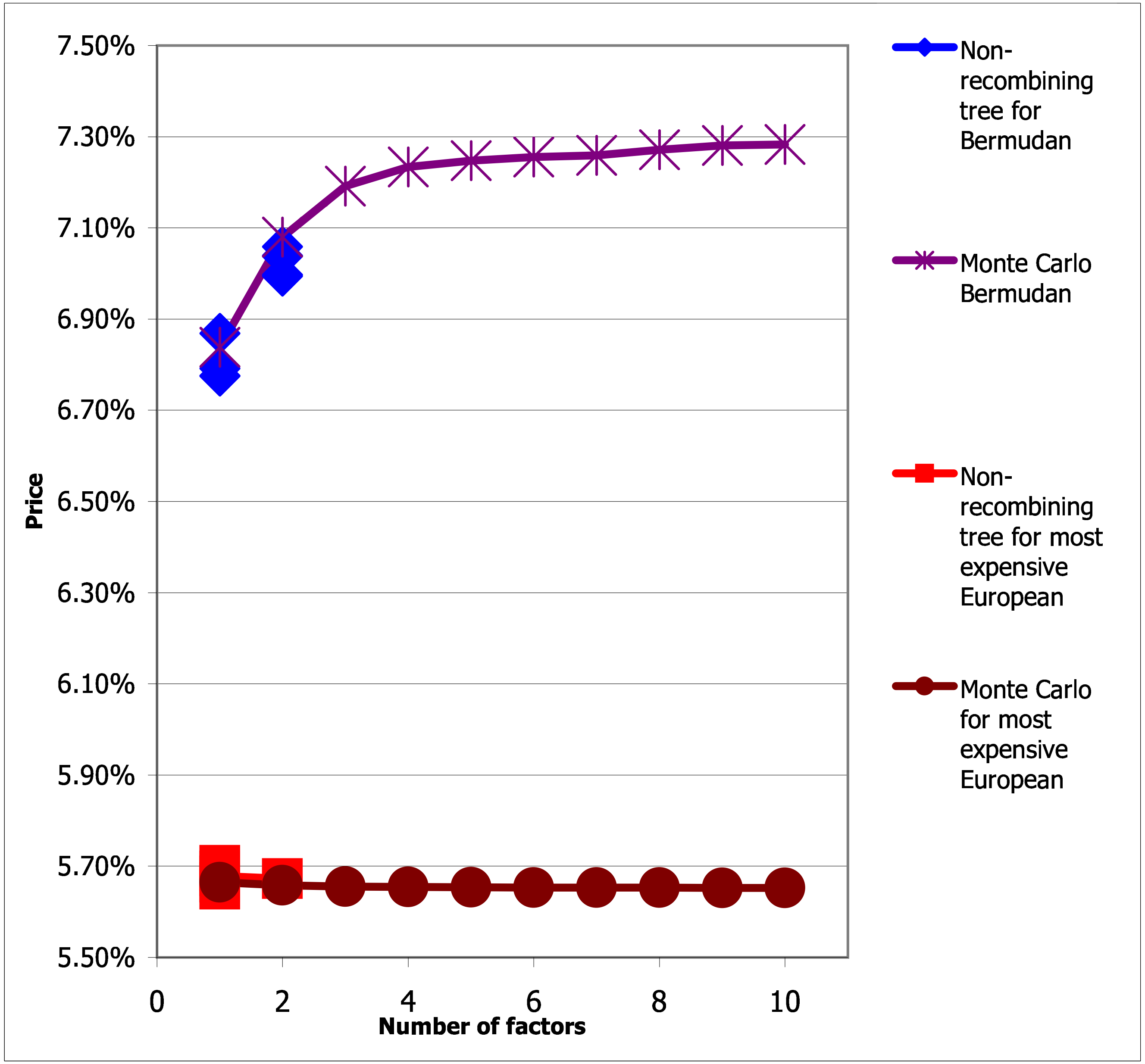
Only storing what is needed for the decision $(\mathcal{E}_i(\boldsymbol{f}_{ji}\,;\,\boldsymbol{p}_i) > 0)$ for each path at each exercise horizon greatly reduces the memory requirements and increases the speed of the algorithm!



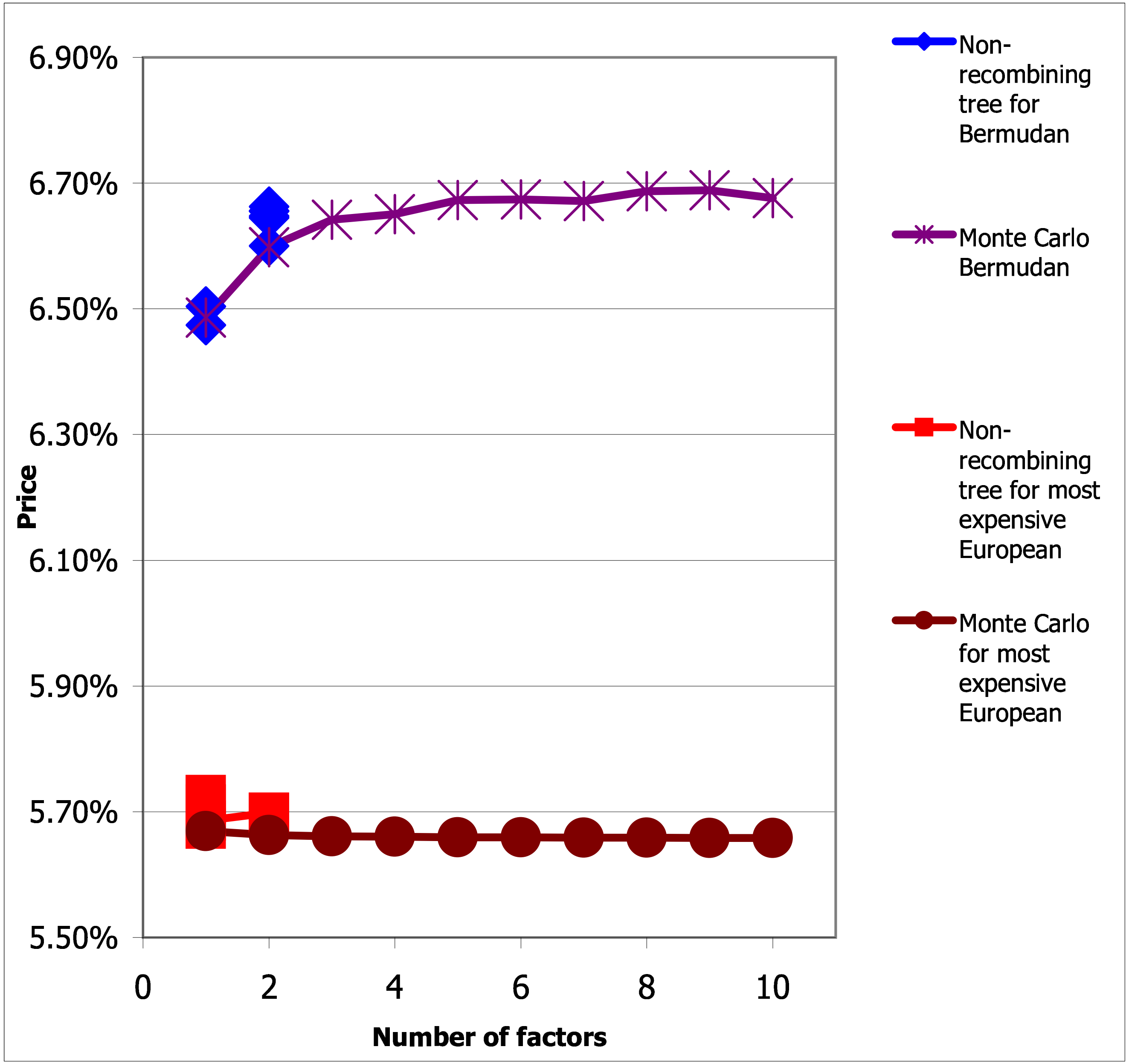
The exercise domain in the f_i-s_{i+1} projection of the evolved yield curve at $t_i = 2$, together with the parametrised exercise boundary resulting from training with different sizes of the training set.



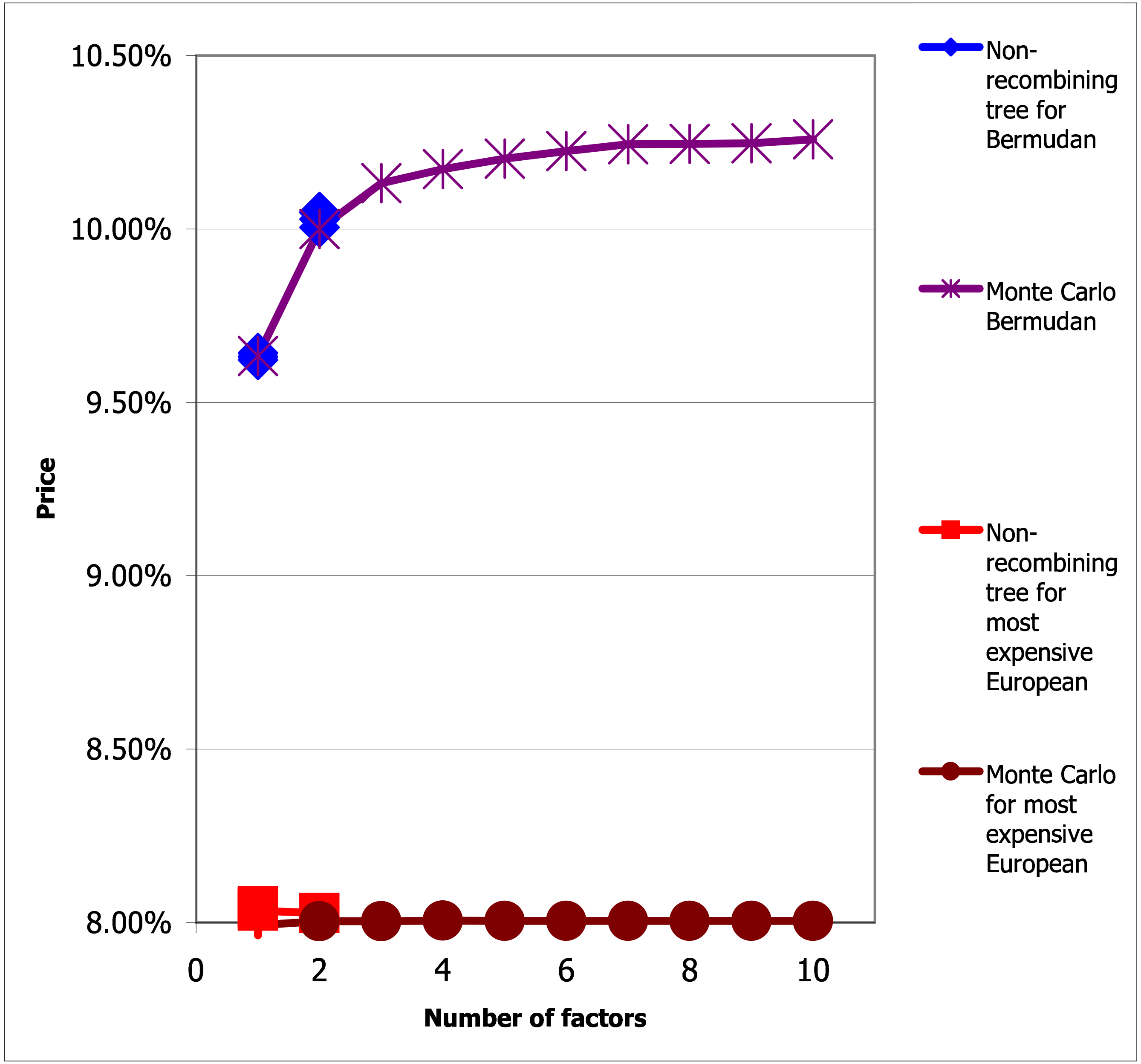
Bermudan swaption prices from the Monte Carlo model in comparison to those obtained from a non-recombining tree model for a 6-non-call-2 semi-annual payer's swaption.



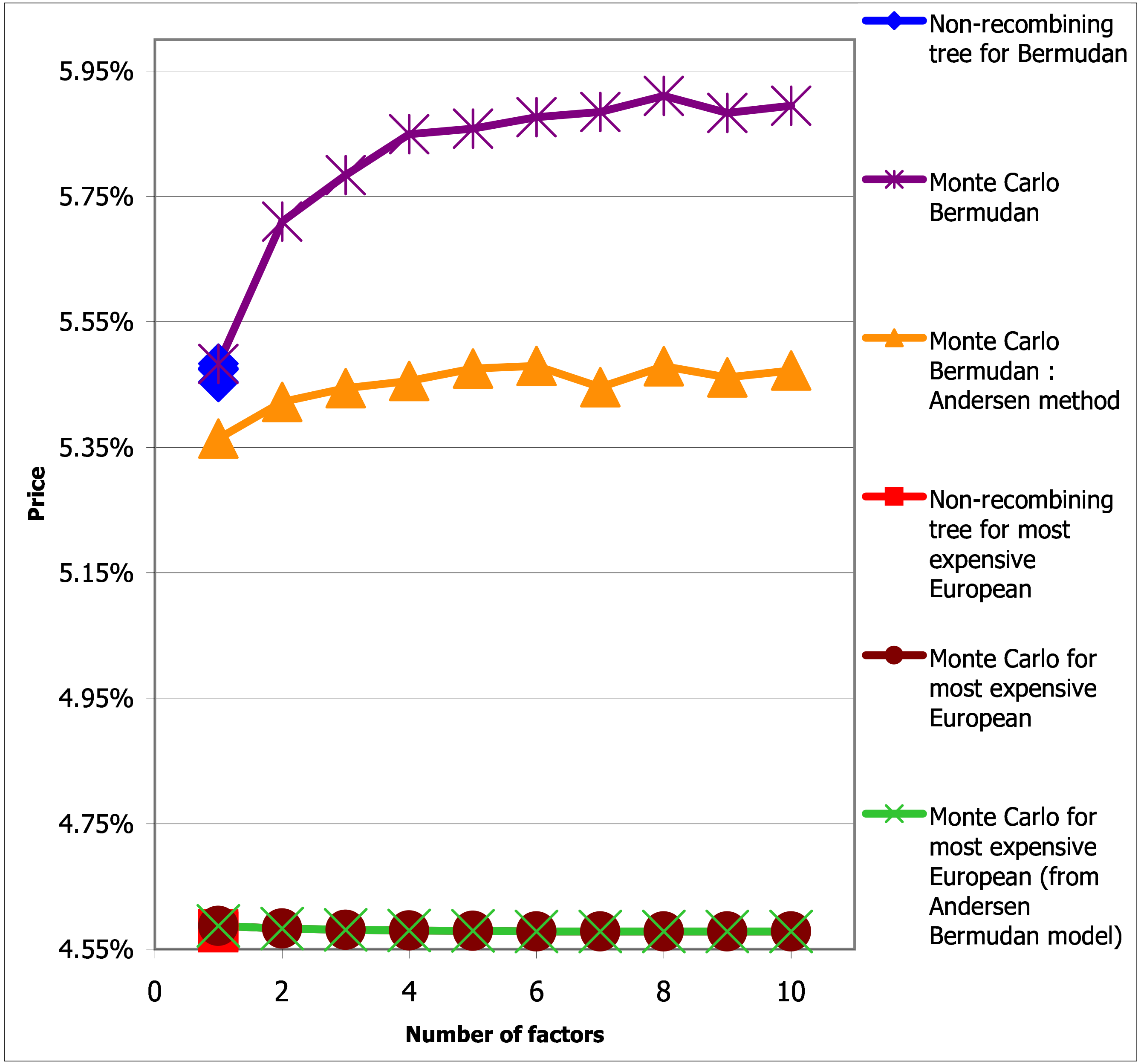
15-non-call-5 annual payer's



15-non-call-5 annual receiver's



15-non-call-5 annual payer's for steeply upwards sloping yield curve



20-non-call-10 semi-annual payer's in comparison to Andersen's method I [And00]

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