newtonexcelbach.wordpress.com

Cubic Splines

Posted on

Cubic splines are used to fit a smooth curve to a series of points with a piecewise series of cubic polynomial curves. In addition to their use in interpolation, they are of particular interest to engineers because the spline is defined as the shape that a thin flexible beam (of constant flexural stiffness) would take up if it was constrained to pass through the defined points. This post will present an Excel User Defined Function (UDF) to generate a "natural" cubic spline for any series of 3 or more points. Later posts will look at alternative spline formulations, and applications of the cubic spline to structural analysis.

A cubic spline is defined as the curve that for any two adjacent internal points:

1. The curve passes exactly through both points

2. The slope of the curve at the end points is equal to the slope of the adjacent segments

3. The curvature of the curve at the end points is equal to the curvature of the adjacent segments

Alternative provisions for the end segments will generate different spline curves over the full extent of the curve. The most common provision for the ends is that the curvature is zero at both ends. This is known as a "natural cubic spline". In a structural analysis context this corresponds to a beam that is free to rotate at both ends, but is constrained in position at the ends and a number of internal points.

Further details of the theory of cubicl splines, and an algorithm for generating natural cubic splines are given in this Wikipedia article.

An excel spreadsheet with a UDF for generating cubic splines, based on the algorithm in the Wikipedia article, can be downloaded from: CSpline2.zip

The download is open source, and full VBA code for the UDF is freely accessible.

Example screen shots from this file are shown below:

	A	B	C	D	E	F	G	H
1	**Cubic Spline Function**							
2								
3	Fit a cubic spline to a a series of XY coordinates							
4	**=CSplineA(X Coords, Y Coords, Interpolation X Coords, Optional Output type)**							
5	**Input:**							
6	X Coords	} X and Y coordinates of the line, x must be in ascending order						
7	X Coords	}						
8	Interpolation X Coords	X coordinates where output is required						
9	Output Type	1 (default) for Y values, slope, curvature and radius						
10		2 for polynomial coefficients and related data						
11								
12	**Output:**							
13	**Output is a 6 or 8 column array, enter as an array function:**							
14	**To enter an array function:**		Enter the function in the top left cell of the output range					
15			Select the entire output range					
16			Press F2					
17			Press Ctrl-shift-enter					
18		**Out = 1**		**Out = 2 (see algorithm and fuction code)**				
19	Column 1	Spline Y values		h(i) = x(i+1) - x(i)				
20	Column 2	Slope		alpha(i) = (3 / h(i) * (Y(i+1) - Y(i))) - (3 / h(i-1) * (Y(i) - Y(i-1)))				
21	Column 3	Curvature		l(i) = 2 * (x(i+1) - x(i-1)) - h(i-1) * mu(i-1); l(1) = l(n) = 1				
22	Column 4	Radius		mu(i) = h(i) / l(i); mu(1) = 0				
23	Column 5	Arc length		z(i) = (alpha(i) - h(i-1) * z(i-1) / l(i); z(1) = z(n) = 0				
24	Column 6	Chord Length		b = (y(i+1) - y(i)) / h(i) - h(i) * c(i+1) + 2* c(i) / 3				
25	Columns 7 and 8 are only generated if "out" = 2							
26	Column 7			c = z(i) - mu(i) * c(i+1); c(n) = 0				
27	Column 8			d = (c(i+1) - c(i)) / (3 * h(i))				
28								
29				Y(i) = X^3 * d(j) + X^2 * c(j) + X * b(j) + y(j)				
30				Where X = Xint(i) - x(j)				

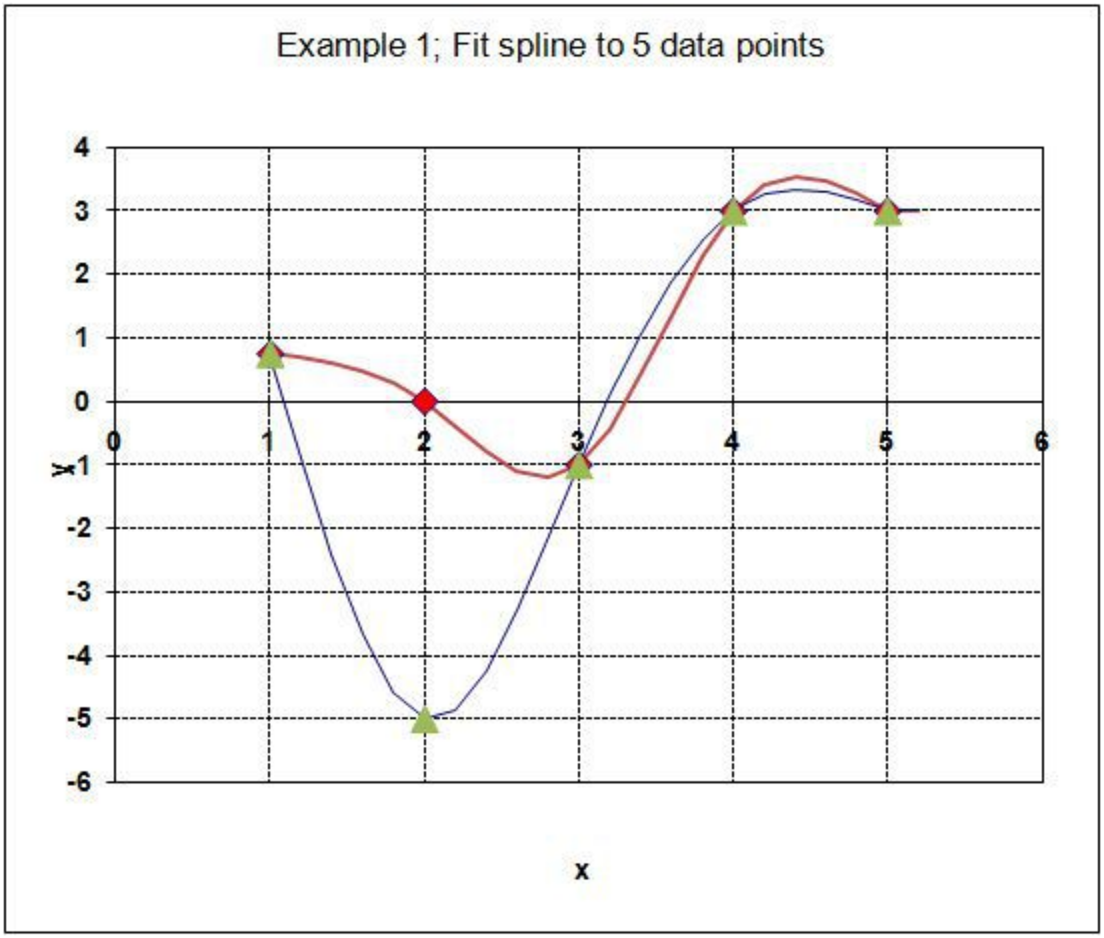
Csplinea Function

D35 {=csplinea(A35:A39,B35:B39,C35:C56)}

	A	B	C	D	E	F	G	H
32	Example 1; Fit spline to 5 data points							
33	Data1		Spline1		Data2		Spline2	
34	X	Y	X	Y	X	Y	X	Y
35	1	0.75	1	0.75	1	0.75	1	0.75
36	2	0	1.2	0.70	2	-5	1.2	-0.89
37	3	-1	1.4	0.62	3	-1	1.4	-2.40
38	4	3	1.6	0.49	4	3	1.6	-3.68
39	5	3	1.8	0.29	5	3	1.8	-4.58
40			2	0.00			2	-5.00
41			2.2	-0.39			2.2	-4.85
42			2.4	-0.79			2.4	-4.24
43			2.6	-1.10			2.6	-3.30
44			2.8	-1.20			2.8	-2.17
45			3	-1.00			3	-1.00
46			3.2	-0.42			3.2	0.09
47			3.4	0.42			3.4	1.06
48			3.6	1.37			3.6	1.88

49			3.8	2.28			3.8	2.54
50			4	3.00			4	3.00
51			4.2	3.41			4.2	3.26
52			4.4	3.55			4.4	3.34
53			4.6	3.48			4.6	3.30
54			4.8	3.28			4.8	3.17
55			5	3.00			5	3.00
56			5.2	3.00			5.2	3.00

Example 1; Fit spline to 5 data points



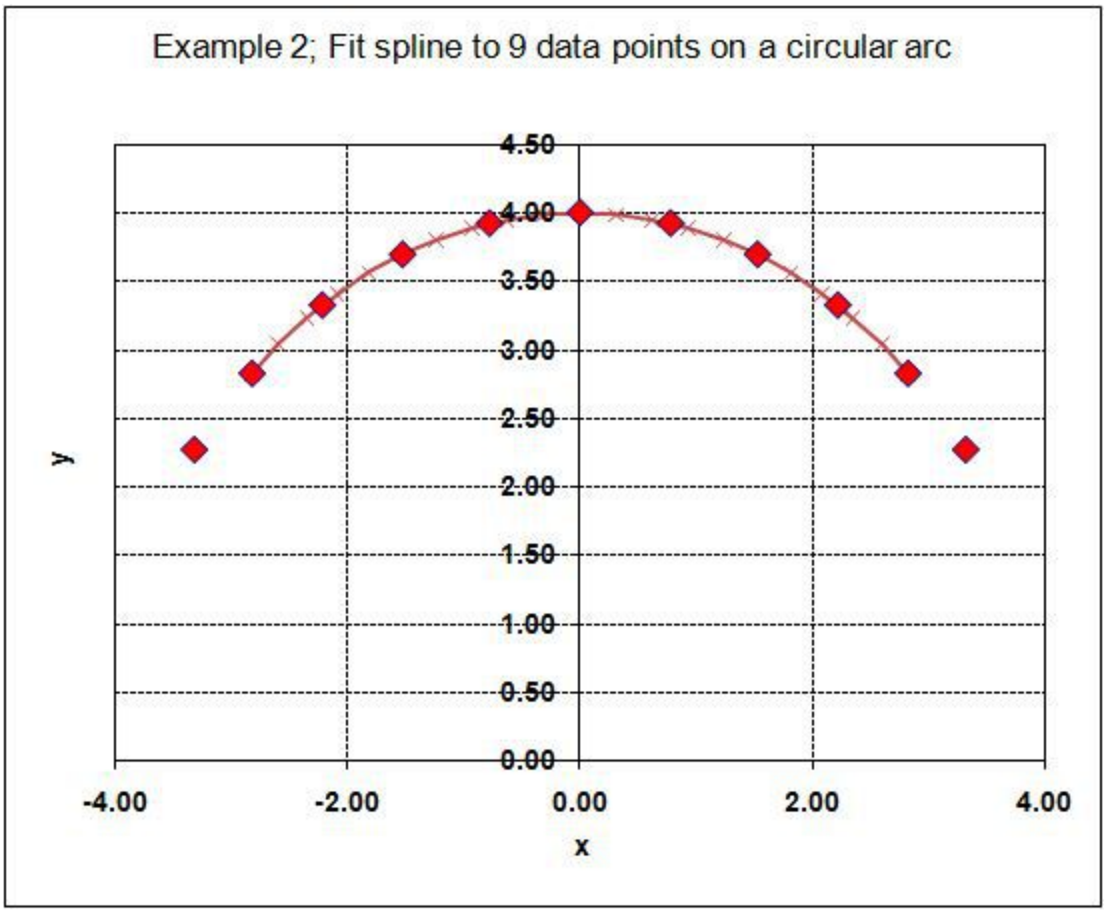
Example 1; Fit spline to 5 data points

D62 {=csplinea(A62:A72,B62:B72,C62:C82)}

	A	B	C	D	E	F	G	H	I
59	Example 2; Fit spline to 9 data points on a circular arc								
60	Data1		Spline1						
61	X	Y	X	Y	Slope	Curvature	Radius	Arc Length	Chord Length
62	-3.33	2.27	-2.828427	2.828	1.0027	-0.6927	-4.1000	-	-
63	-2.83	2.83	-2.597792	3.042	0.8551	-0.5875	-3.8767	0.3145	0.3145
64	-2.22	3.33	-2.351141	3.236	0.7240	-0.4751	-3.9606	0.3140	0.3139
65	-1.53	3.70	-2.089994	3.410	0.6128	-0.3966	-4.0676	0.3139	0.3139
66	-0.78	3.92	-1.815962	3.564	0.5097	-0.3557	-3.9758	0.3142	0.3141
67	0.00	4.00	-1.530734	3.696	0.4144	-0.3130	-4.0516	0.3142	0.3141

68	0.78	3.92	-1.236068	3.804	0.3251	-0.2928	-3.9704	0.3142	0.3141
69	1.53	3.70	-0.933781	3.890	0.2397	-0.2721	-3.9963	0.3141	0.3141
70	2.22	3.33	-0.625738	3.951	0.1585	-0.2589	-4.0096	0.3142	0.3141
71	2.83	2.83	-0.313836	3.988	0.0787	-0.2534	-3.9835	0.3142	0.3141
72	3.33	2.27	0	4.000	0.0000	-0.2478	-4.0348	0.3142	0.3141
73			0.3138364	3.988	-0.0787	-0.2534	-3.9835	0.3142	0.3141
74			0.6257379	3.951	-0.1585	-0.2589	-4.0096	0.3142	0.3141
75			0.9337815	3.890	-0.2397	-0.2721	-3.9963	0.3142	0.3141
76			1.236068	3.804	-0.3251	-0.2928	-3.9704	0.3141	0.3141
77			1.5307337	3.696	-0.4144	-0.3130	-4.0516	0.3142	0.3141
78			1.815962	3.564	-0.5097	-0.3557	-3.9758	0.3142	0.3141
79			2.0899943	3.410	-0.6128	-0.3966	-4.0676	0.3142	0.3141
80			2.351141	3.236	-0.7240	-0.4751	-3.9606	0.3139	0.3139
81			2.5977922	3.042	-0.8551	-0.5875	-3.8767	0.3140	0.3139
82			2.8284271	2.828	-1.0027	-0.6927	-4.1000	0.3145	0.3145
83									

Example 2; Fit spline to 9 data points on a circular arc



Example 2; Fit spline to 9 data points on a circular arc

"Dummy" data points at each end allow the curvature at the start and end points to be adjusted to the required value.

B88 {=csplinea(A62:A72,B62:B72,C62:C82,2)}

Polynomial coefficients from example 2

h	alpha	l	mu	z	b	c	d
0.50	0.000E+00	1.000	0.000	0.000E+00	1.175E+00	0.000E+00	-2.321E-01
0.61	-8.906E-01	2.207	0.275	-4.035E-01	1.003E+00	-3.463E-01	7.596E-02
0.69	-8.585E-01	2.429	0.285	-2.528E-01	6.666E-01	-2.082E-01	2.491E-02
0.75	-6.935E-01	2.687	0.279	-1.930E-01	4.144E-01	-1.565E-01	1.143E-02
0.78	-6.146E-01	2.852	0.274	-1.647E-01	1.988E-01	-1.308E-01	2.934E-03
0.78	-5.909E-01	2.908	0.268	-1.590E-01	-1.388E-17	-1.239E-01	-2.934E-03
0.75	-6.146E-01	2.852	0.263	-1.720E-01	-1.988E-01	-1.308E-01	-1.143E-02
0.69	-6.935E-01	2.686	0.257	-2.101E-01	-4.144E-01	-1.565E-01	-2.491E-02
0.61	-8.585E-01	2.417	0.251	-2.950E-01	-6.666E-01	-2.082E-01	-7.596E-02
0.50	-8.906E-01	2.055	0.242	-3.463E-01	-1.003E+00	-3.463E-01	2.321E-01
0.00	0.000E+00	1.000	0.000	0.000E+00	0.000E+00	0.000E+00	0.000E+00

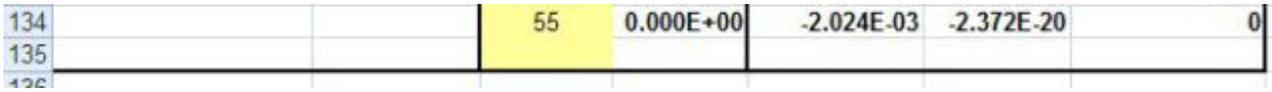
Example 2; Fit spline to 9 data points on a circular arc

B88 {=csplinea(A62:A72,B62:B72,C62:C82,2)}

Example 3; Fit spline to the deflected shape of a 3 span beam

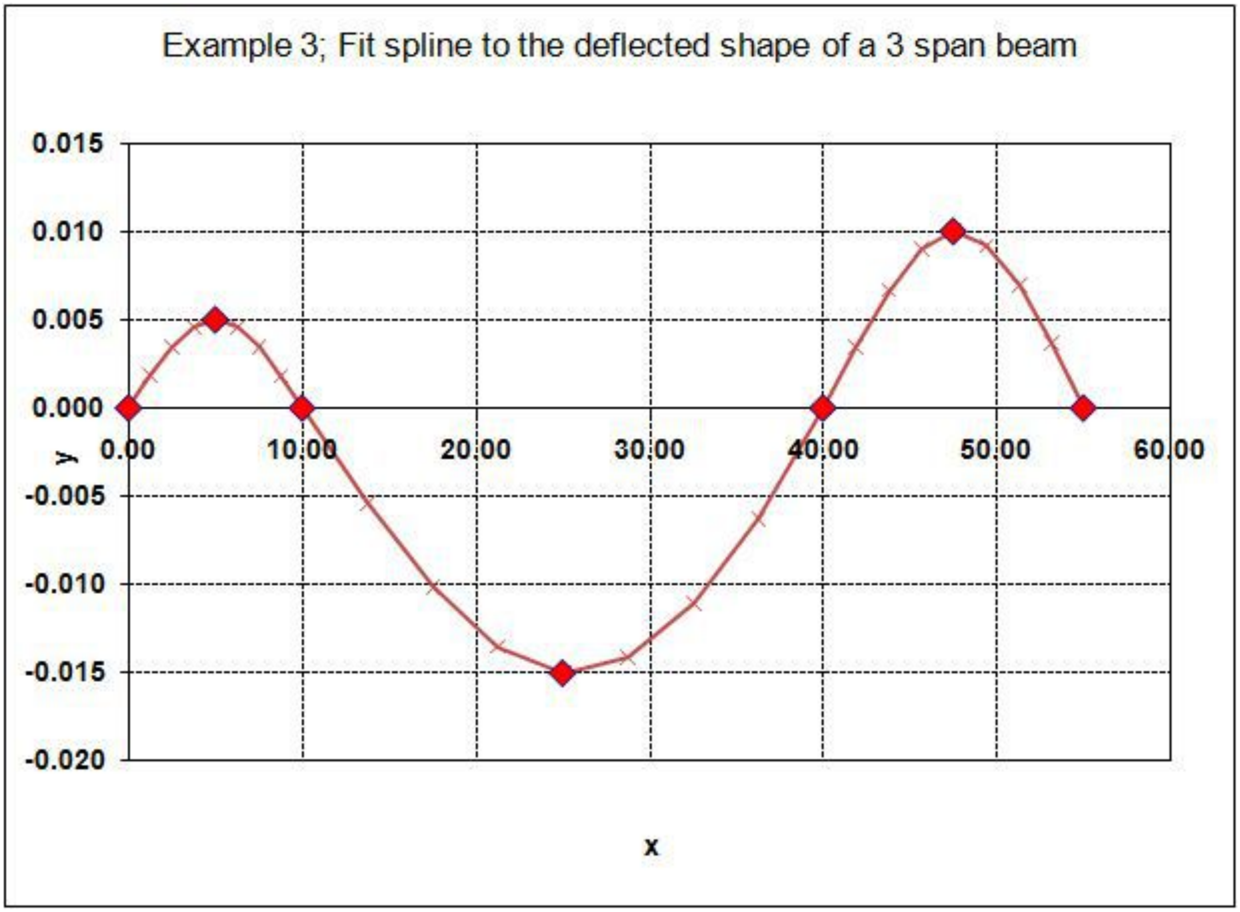
Span lengths	m	10	30	15	
Mid-span deflection	mm	5	-15	10	1.00E-03
Beam EI	20000000				

Data1 X	Data1 Y	Spline1 X	Spline1 Y	Slope	Curvature	Bending Moment
0.00	0.000	0	0.000E+00	1.502E-03	0.000E+00	0
5.00	0.005	1.25	1.838E-03	1.408E-03	-1.505E-04	-3,010
10.00	0.000	2.5	3.441E-03	1.125E-03	-3.010E-04	-6,021
25.00	-0.015	3.75	4.573E-03	6.551E-04	-4.515E-04	-9,031
40.00	0.000	5	5.000E-03	-3.436E-06	-6.021E-04	-12,041
47.50	0.010	6.25	4.565E-03	-6.607E-04	-4.495E-04	-8,990
55.00	0.000	7.5	3.428E-03	-1.127E-03	-2.969E-04	-5,938
		8.75	1.827E-03	-1.403E-03	-1.443E-04	-2,887
		10	0.000E+00	-1.488E-03	8.247E-06	165
		13.75	-5.422E-03	-1.377E-03	5.086E-05	1,017
		17.5	-1.013E-02	-1.107E-03	9.347E-05	1,869
		21.25	-1.352E-02	-6.761E-04	1.361E-04	2,722
		25	-1.500E-02	-8.591E-05	1.787E-04	3,574
		28.75	-1.413E-02	5.365E-04	1.533E-04	3,065
		32.5	-1.110E-02	1.064E-03	1.278E-04	2,557
		36.25	-6.268E-03	1.495E-03	1.024E-04	2,048
		40	0.000E+00	1.832E-03	7.698E-05	1,540
		41.875	3.477E-03	1.828E-03	-8.041E-05	-1,608
		43.75	6.672E-03	1.530E-03	-2.378E-04	-4,756
		45.625	9.031E-03	9.366E-04	-3.952E-04	-7,904
		47.5	1.000E-02	4.811E-05	-5.526E-04	-11,052
		49.375	9.200E-03	-8.585E-04	-4.144E-04	-8,289
		51.25	6.943E-03	-1.506E-03	-2.763E-04	-5,526
		53.125	3.714E-03	-1.895E-03	-1.381E-04	-2,763



134			55	0.000E+00	-2.024E-03	-2.372E-20	0
135							

Example 3; Fit spline to the deflected shape of a 3 span beam



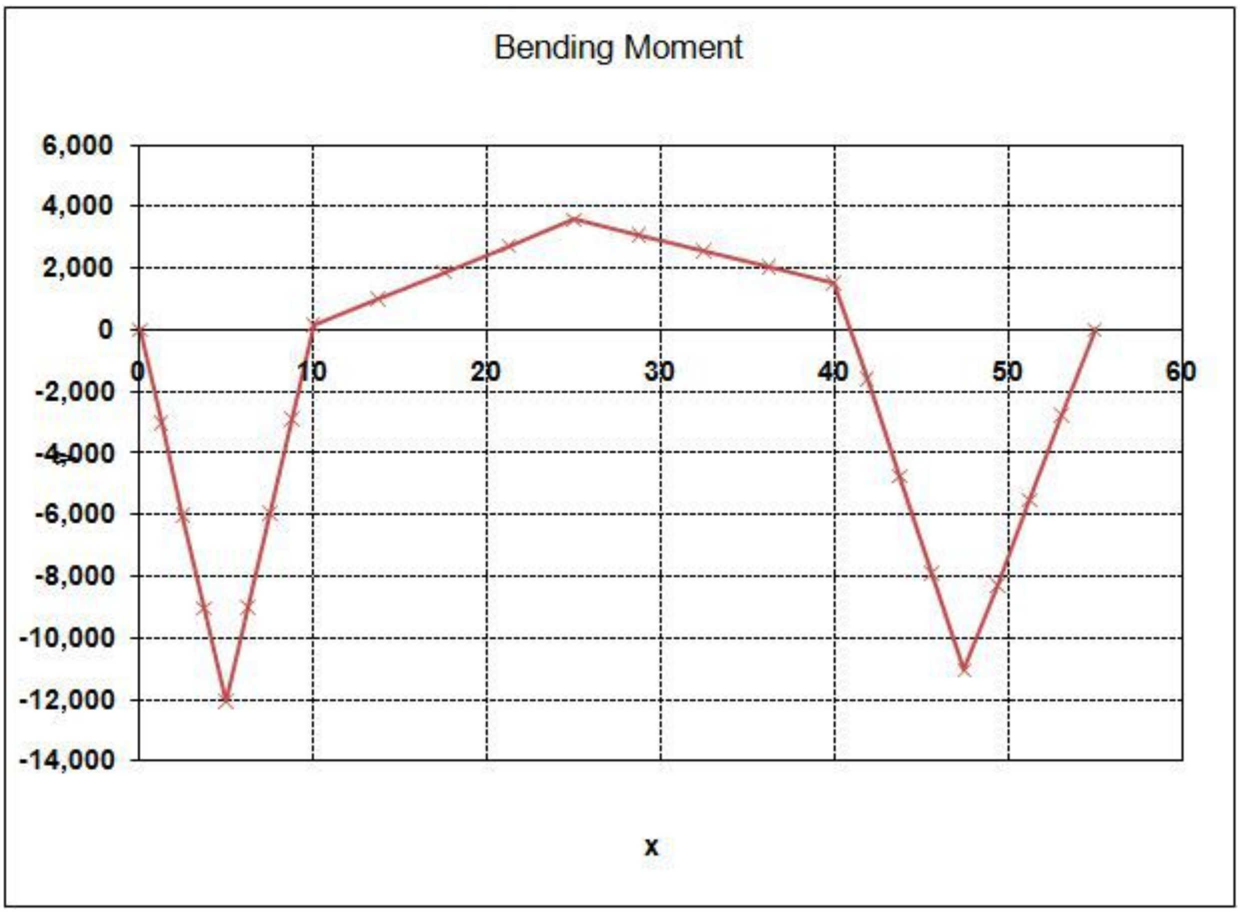
Example 3; Fit spline to the deflected shape of a 3 span beam

B140 {=csplinea(A110:A116,B110:B116,C110:C134,2)}

Polynomial coefficients from example 3

h	alpha	l	mu	z	b	c	d
5.00	0.000E+00	1.000	0.000	0.000E+00	1.502E-03	0.000E+00	-2.007E-05
5.00	-6.000E-03	20.000	0.250	-3.000E-04	-3.436E-06	-3.010E-04	2.034E-05
15.00	0.000E+00	38.750	0.387	3.871E-05	-1.488E-03	4.124E-06	1.894E-06
15.00	6.000E-03	54.194	0.277	1.000E-04	-8.591E-05	8.935E-05	-1.130E-06
7.50	1.000E-03	40.848	0.184	-1.224E-05	1.832E-03	3.849E-05	-1.399E-05
7.50	-8.000E-03	28.623	0.262	-2.763E-04	4.811E-05	-2.763E-04	1.228E-05
0.00	0.000E+00	1.000	0.000	0.000E+00	0.000E+00	0.000E+00	0.000E+00

Polynomial coefficients from example 3



Example 3; Bending Moments

Bending moments are calculated by multiplying the curvature at each point by the beam flexural stiffness, EI.