$$\mu = \mathbb{E}[X] = \sum x_i f_i$$

$$\sigma^2 = \mathbb{V}(X) = \mathbb{E}[(X-\mu)^2]$$

$$= \mathbb{E}[X^2] - \mu^2$$

$$\sigma^2 = \underbrace{\sum x_i^2 f_i}_{\text{2nd moment}} - \mu^2 = 0$$

$$\boxed{dX^2 \to dt}$$

$$(dX)^2 = dt$$ Quadratic Variation

$$(X_{t_i} - X_{t_{i-1}})^2 \simeq t_i - t_{i-1}$$

$$\frac{\left(X_{t_i}-X_{i-1}\right)^2}{t_i-t_{i-1}} \approx 1$$

Itô's Lemma

$$\boxed{dF = \frac{dF}{dx}\,dx + \frac{1}{2}\frac{d^2F}{dx^2}\,dt}$$

where $f = F(X)$

$F = X^2$, X^n, $\sin X$, $\log X$, e^X

$F = X^2$ $\frac{dF}{dX} = 2X$; $\frac{d^2F}{dX^2} = 2$ subst in Itô

$$\boxed{dF = 2X\,dX + dt}$$

$F = e^X$ subst in Itô

$$\boxed{df = e^X \, dX} + \frac{1}{2} e^X \, dt$$

$$\text{drift} = \frac{1}{2} e^X \; ; \; \text{diffusion} = e^X$$

Basic Itô :

$O(\sqrt{dt})$

$$dF = \underbrace{\boxed{\frac{dF}{dx} \, dX}}_{\text{diffusion}} + \underbrace{\boxed{\frac{1}{2} \frac{d^2F}{dX^2}}}_{\text{drift}} dt$$

$$F(t,X) = t^2 X \quad ; \quad t + X^{\wedge} \, ;$$

$$t e^{X} \quad ; \quad t^2 + \log X$$

$$t \to t + dt \quad ; \quad X \to X + dX$$

$$F(t+dt, X+dX) = F(t,X) + \frac{\partial F}{\partial t} dt + \frac{\partial F}{\partial X} dX + \frac{1}{2} \frac{\partial^2 F}{\partial X^2} dX^2 + \ldots$$

Know $dX^2 = dt$

$$dF = \left(\frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial X^2}\right)dt + \frac{\partial F}{\partial X}dX$$

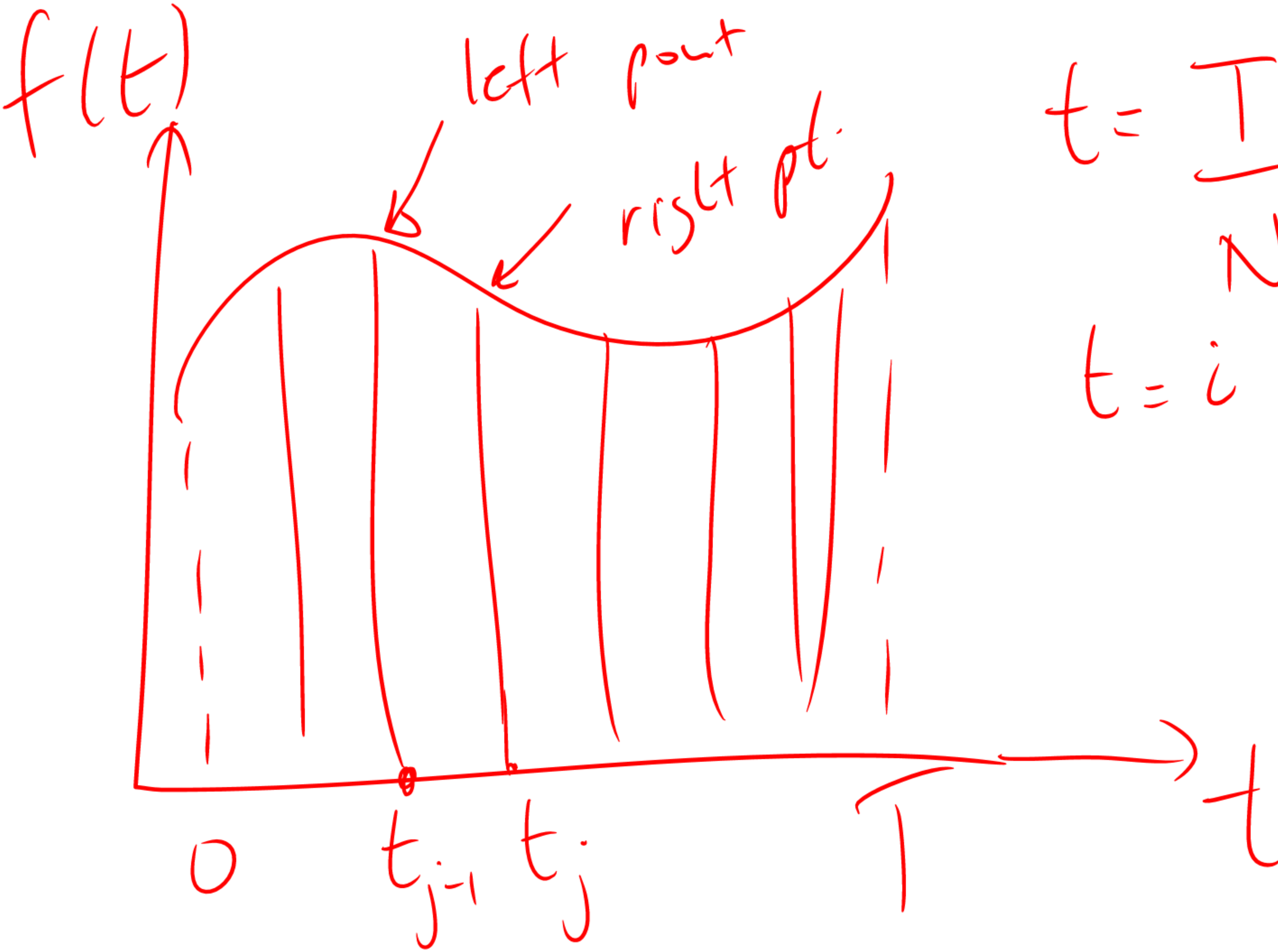
2D Itô's Lemma

$$dG = A(G,t)\,dt + B(G,t)\,dX$$

Integrate over $[0,T]$

$$\int_0^T dG = \int_0^T A(G,t)\,dt + \int_0^T B(G,t)\,dX$$

$$G_T = G_0 + \int_0^T A(G,t)\,dt + \int_0^T B(G,t)\,dX$$



$$t = \frac{T}{N}$$

$$t = i\,\delta t$$

$$\int_0^T f(t)\,dt = \lim_{N \to \infty} \sum_0^{N-1} f_{t_i}\left[t_i - t_{i-1}\right]$$

$$\int_0^T f(t)\,dt = \lim_{N \to \infty} \sum_0^{N-1} f_{t_{i-1}}\left[t_i - t_{i-1}\right]$$

Mean Sq. Limit or

Mean Sq Cgce

$$\mathbb{E}[dX^2] = dt$$

$$dS = \mu S\, dt + \sigma S\, dX$$

$$V = V(S) \qquad S \to S + dS$$

$$V(S+dS) = V(S) + \frac{dV}{dS}\, dS + \frac{1}{2}\frac{d^2V}{dS^2}\, dS^2$$

$$dV = \frac{dV}{dS}\, dS + \frac{1}{2}\frac{d^2V}{dS^2}\, \boxed{dS^2}$$

$$(\mu S\,dt + \sigma S\,dX)^2 = \underbrace{\mu^2 S^2 dt^2}_{\to 0} + \underbrace{\sigma^2 S^2 dt} + \underbrace{2\mu\sigma S^2 dt\,dX}_{\to O(dt^{3/2})}$$

$$\boxed{dS^2 = \sigma^2 S^2 dt}$$

$$dV = \underbrace{\left(\mu S \frac{dV}{dS} + \frac{1}{2}\sigma^2 S^2 \frac{d^2V}{dS^2}\right)dt}_{\text{DRIFT}} + \underbrace{\sigma S \frac{dV}{dS}\,dX}_{\text{Diffusion}}$$

$$dF = \left(\frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}\right)dt + \frac{\partial F}{\partial x}dX$$

$$\frac{\partial F}{\partial x}dX = dF - \left(\frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}\right)dt$$

Integrate over $[0, t]$

$$\int_0^t \frac{\partial F}{\partial x}dX = \underbrace{\int_0^t dF}_{\text{Recall } F = F(X,t)} - \int_0^t \left(\frac{\partial F}{\partial s} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}\right)ds$$

$$\boxed{X(0) = 0}$$

$$\underbrace{\int_0^t \frac{\partial F}{\partial X}\, dX}_{\text{stoch integral}} = F(X(t),t) - F(X(0),0) - \int_0^t \left(\frac{\partial F}{\partial s} + \frac{1}{2}\frac{\partial^2 F}{\partial X^2} \right) ds$$

A few slides earlier for $V = V(S)$

where S evolved according to GBM

$$dV = \left(\mu S \frac{dV}{dS} + \frac{1}{2} \sigma^2 S^2 \frac{d^2V}{dS^2} \right) dt + \sigma S \frac{dV}{dS} dX$$

But what if $V = V(S, t)$?

i.e. $S \to S + dS$ $\quad t \to t + dt$

2D TSE $\longrightarrow V(t+dt, S+dS) = V(S,t) +$

$$\frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} dS^2 \quad \sigma^2 S^2 dt$$

$$dV = \underbrace{\left(\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right)}_{\text{drift}} dt$$

$$\underbrace{+ \ \sigma S \frac{\partial V}{\partial S}}_{\text{diffusion}} dX$$