

CQF Module 1 Answers

July 2016

1 Mathematical Methods

1. Consider the linear parabolic partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + a \frac{\partial u}{\partial x} + bu,$$

for the function $u(x, t)$; where a and b are constants. By using a substitution of the form

$$u(x, t) = e^{\alpha x + \beta t} v(x, t),$$

and suitable choice of α and β , show that the PDE can be reduced to the heat equation

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}.$$

This PDE has the same structure as the Black-Scholes Equation and the working here is used in part to reduce it to a one dimensional heat equation - hence very useful problem (much more on this later). Start by differentiating

$$\begin{aligned}\frac{\partial u}{\partial t} &= \left(\beta v + \frac{\partial v}{\partial t} \right) e^{\alpha x + \beta t} \\ \frac{\partial u}{\partial x} &= \left(\alpha v + \frac{\partial v}{\partial x} \right) e^{\alpha x + \beta t} \\ \frac{\partial^2 u}{\partial x^2} &= \left(\alpha^2 v + 2\alpha \frac{\partial v}{\partial x} + \frac{\partial^2 v}{\partial x^2} \right) e^{\alpha x + \beta t}\end{aligned}$$

Substituting into the PDE we have

$$\left(\beta v + \frac{\partial v}{\partial t} \right) = \left(\alpha^2 v + 2\alpha \frac{\partial v}{\partial x} + \frac{\partial^2 v}{\partial x^2} \right) + a \left(\alpha v + \frac{\partial v}{\partial x} \right) + bv,$$

rearrange to give

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + (2\alpha + a) \frac{\partial v}{\partial x} + (\alpha^2 + a\alpha + b - \beta) v.$$

To eliminate $\frac{\partial v}{\partial x}$ and v requires setting, in turn,

$$\begin{aligned}2\alpha + a &= 0 \\ \alpha^2 + a\alpha + b - \beta &= 0.\end{aligned}$$

Hence the choice is

$$\alpha = -\frac{1}{2}a \text{ and } \beta = b - \frac{1}{4}a^2$$

2. Consider the probability density function $f(x)$ given by

$$f(x) = \begin{cases} Ax^2 \exp(-\lambda x^2) & x > 0 \\ 0 & x \leq 0 \end{cases}.$$

Deduce that

$$A = 4\sqrt{\frac{\lambda^3}{\pi}}.$$

Show that

$$\mathbb{E}[X] = \frac{2}{\sqrt{\pi\lambda}}.$$

By using integration by parts, or otherwise, deduce that for $n = 0, 1, 2, \dots$ the even moments of this distribution are given by

$$\mathbb{E}[X^{2n}] = \frac{1.3 \dots (2n+1)}{(2\lambda)^n}$$

and the odd moments are given by

$$\mathbb{E}[X^{2n+1}] = \frac{2}{\sqrt{\pi}} \frac{(n+1)!}{\lambda^{(2n+1)/2}}.$$

Start by using $\int_0^\infty f(x) dx = 1$. Hence $A \int_0^\infty x^2 \exp(-\lambda x^2) dx = 1$. Using the substitution $u = \sqrt{\lambda}x \rightarrow du = \sqrt{\lambda}dx$ which gives

$$\lambda^{-3/2} \int_0^\infty u^2 \exp(-u^2) du = 1 \text{ becomes } \lambda^{-3/2} I = 1.$$

A result we will use is

$$\int_0^\infty \exp(-x^2) dx = \sqrt{\pi}/2.$$

To evaluate I write

$$I = \int_0^\infty u (u \exp(-u^2)) du$$

and note from integration by substitution that

$$\int_0^\infty u \exp(-u^2) du = -\frac{1}{2} e^{-u^2} + c.$$

Integration by parts gives

$$\int_0^\infty u (u \exp(-u^2)) du = -\frac{u}{2} e^{-u^2} \Big|_0^\infty + \frac{1}{2} \int_0^\infty e^{-u^2} du = \frac{1}{4} \sqrt{\pi}$$

where we have used the earlier result.

$$\lambda^{-3/2} A \frac{\sqrt{\pi}}{4} = 1 \rightarrow A = \frac{4\lambda^{3/2}}{\sqrt{\pi}}.$$

To show $\mathbb{E}[X] = \frac{2}{\sqrt{\pi\lambda}}$, write $I_n = \mathbb{E}[X^n] =$

$$\frac{4\lambda^{3/2}}{\sqrt{\pi}} \int_0^\infty x^{n+2} e^{-\lambda x^2} dx; \text{ so } I_0 = 1.$$

$$\begin{aligned} I_1 &= A \int_0^\infty x^3 e^{-\lambda x^2} dx = A \left(\frac{1}{\lambda^2} \int_0^\infty u^3 e^{-u^2} du \right) \\ &= \frac{A}{\lambda^2} \left(-\frac{1}{2} u^2 e^{-u^2} \Big|_0^\infty + \int_0^\infty u e^{-u^2} du \right) \\ &= \frac{A}{\lambda^2} \left(-\frac{1}{2} e^{-u^2} \Big|_0^\infty \right) = \frac{A}{2\lambda^2} = \frac{2}{\sqrt{\pi\lambda}}. \end{aligned}$$

To consider odd and even moments, want to show

$$\mathbb{E}[X^{2n}] = \frac{1.3 \dots (2n+1)}{(2\lambda)^n}; \quad \mathbb{E}[X^{2n+1}] = \frac{2}{\sqrt{\pi}} \frac{(n+1)!}{\lambda^{(2n+1)/2}}.$$

Put $I_n =$

$$\begin{aligned} A \int_0^\infty x^{n+2} e^{-\lambda x^2} dx &= A \left(-\frac{1}{2\lambda} x^{1+n} e^{-\lambda x^2} \Big|_0^\infty + \frac{1+n}{2\lambda} \int_0^\infty x^n e^{-\lambda x^2} dx \right) \\ &= \frac{n+1}{2\lambda} I_{n-2} \end{aligned}$$

$$I_{n+2} = \frac{n+3}{2\lambda} I_n$$

$$I_2 = \frac{3}{2\lambda} I_0 = \frac{3}{2\lambda}; I_4 = \frac{3 \times 5}{(2\lambda)^2}; I_6 = \frac{3 \times 5 \times 7}{(2\lambda)^3}, \dots$$

$$I_3 = \frac{4}{2\lambda} I_1 = \frac{4}{\sqrt{\pi\lambda}^{3/2}}; I_5 = \frac{6}{2\lambda} I_3 = \frac{3}{\lambda} \times \frac{4}{\sqrt{\pi\lambda}^{3/2}} = \frac{2}{\sqrt{\pi}} \frac{1.2.3}{\lambda^{5/2}}; I_7 = \frac{8}{2\lambda} I_5 = \frac{2}{\sqrt{\pi}} \frac{4!}{\lambda^{7/2}}, \dots$$

Hence result.

2 Stochastic Calculus

$W, W(t), W_t$ all refer to standard Brownian motion

1. a. Itô's lemma can be used to deduce the following formula for stochastic differential equations and stochastic integrals

$$\int_0^t \frac{\partial F}{\partial W} dW(\tau) = F(W(t), t) - F(W(0), 0) - \int_0^t \left(\frac{\partial F}{\partial \tau} + \frac{1}{2} \frac{\partial^2 F}{\partial W^2} \right) d\tau$$

for a function $F(W(\tau), \tau)$ where $dW(\tau)$ is an increment of a Brownian motion.

If $W(0) = 0$ evaluate

$$\begin{aligned} \int_0^t \tau^2 \sin W dW(\tau). \\ \downarrow \frac{\partial F}{\partial W} = t^2 \sin W \longrightarrow F = -t^2 \cos W \downarrow \\ \frac{\partial^2 F}{\partial W^2} = t^2 \cos W \quad \frac{\partial F}{\partial t} = -2t \cos W \end{aligned}$$

and substitute into the integral formula

$$\int_0^t \tau^2 \sin W dW(\tau) = -t^2 \cos W - \int_0^t \left(-2\tau \cos W + \frac{1}{2} \tau^2 \cos W \right) d\tau$$

- b. Suppose the stochastic process $S(t)$ evolves according to Geometric Brownian Motion (GBM), where

$$dS = \mu S dt + \sigma S dW.$$

Obtain a SDE $df(S, t)$ for each of the following functions

- i $f(S, t) = \alpha^t + \beta t S^n$ α, β are constants

$$\begin{aligned} \frac{\partial f}{\partial t} &= \alpha^t \log a + \beta S^n; \quad \frac{\partial f}{\partial S} = n\beta t S^{n-1}; \quad \frac{\partial^2 f}{\partial S^2} = n(n-1)\beta t S^{n-2} \\ df &= \left(\alpha^t \log a + n\mu\beta t S^n + \frac{1}{2} n(n-1)\beta t \sigma^2 S^n \right) dt + \sigma n\beta t S^n dW \end{aligned}$$

- ii $f(S, t) = \log tS + \cos tS$

$$\begin{aligned} \frac{\partial f}{\partial t} &= \frac{1}{t} - S \sin tS; \quad \frac{\partial f}{\partial S} = \frac{1}{S} - t \sin tS; \quad \frac{\partial^2 f}{\partial S^2} = -\frac{1}{S^2} - t^2 \cot tS \\ df &= \left(\frac{1}{t} - S \sin tS + \mu S \left(\frac{1}{S} - t \sin tS \right) + \frac{1}{2} \sigma^2 S^2 \left(-\frac{1}{S^2} - t^2 \cot tS \right) \right) dt + \sigma S \left(\frac{1}{S} - t \sin tS \right) dW \end{aligned}$$

2. Consider the following SDE

$$d\sigma = a dt + b dW,$$

where $a = a(\sigma, t)$, $b = b(\sigma, t)$. The Forward Kolmogorov Equation (FKE), for the transition PDF $p = p(\sigma, t; \sigma', t')$ is

$$\frac{\partial p}{\partial t'} = \frac{1}{2} \frac{\partial^2}{\partial \sigma'^2} (b^2 p) - \frac{\partial}{\partial \sigma'} (a p),$$

where the primed variables refer to future states. The steady state solution is given by setting $\frac{\partial p}{\partial t'} = 0$. By considering suitable conditions, show that the steady state solution is given by

$$p(\sigma') = \frac{A}{b^2} e^{\int \frac{2a}{b^2} d\sigma'},$$

where A is an arbitrary constant. (During your working you may drop the primed notation). The Forward Kolmogorov Equation for the transition PDF $p(\sigma_t, t; \sigma'_t, t')$ is

$$\frac{\partial p}{\partial t'} = \frac{1}{2} \frac{\partial^2}{\partial \sigma_t'^2} (b^2(\sigma'_t, t') p) - \frac{\partial}{\partial \sigma'_t} (a(\sigma'_t, t') p),$$

where the primed variables refer to future states. The steady state solution is given by setting $\frac{\partial p}{\partial t'} = 0$. Considering the boundary conditions that as $\sigma'_t \rightarrow \pm\infty$, $p \rightarrow 0$ and $\frac{\partial p}{\partial \sigma'_t} \rightarrow 0$, show that the steady state solution is given by

$$p(\sigma'_t) = \frac{A}{b^2} e^{\int \frac{2a}{b^2} d\sigma'_t},$$

where the constant A is **not** to be calculated. (During your working you may drop the primed notation). At steady state $\frac{\partial p}{\partial t} = 0$. So the Forward Kolmogorov Equation is

$$\frac{1}{2} \frac{d^2}{d\sigma^2} (b^2 p) - \frac{d}{d\sigma} (ap) = 0$$

Integrating once we have

$$\frac{1}{2} \frac{d}{d\sigma} (b^2 p) = ap + C$$

Now using the boundary condition $\frac{\partial p}{\partial \sigma} \rightarrow 0$, we note that $C = 0$, so

$$\frac{1}{2} \frac{d}{d\sigma} (b^2 p) = ap$$

Now writing the above equation in the following way

$$\frac{d(b^2 p)}{b^2 p} = \frac{2a}{b^2} d\sigma$$

Integrating once again

$$\ln(b^2 p) + C = \int \frac{2a}{b^2} d\sigma$$

where C is a constant. Taking exponentials

$$B(b^2 p) = \exp\left(\int \frac{2a}{b^2} d\sigma\right),$$

where $B = e^C$

$$p(\sigma') = \frac{A}{b^2} e^{\int \frac{2a}{b^2} d\sigma}.$$

where $A = 1/B$.

3. Consider a function $V(t, S_t, r_t)$ where the two stochastic processes S_t and r_t evolve according to a two factor model given by

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dW_t^{(1)} \\ dr_t &= \gamma(m - r_t) dt + c dW_t^{(2)}, \end{aligned}$$

in turn. and where $\mathbb{E}[dW_t^{(1)} dW_t^{(2)}] = \rho dt$. The parameters μ, σ, γ, m and c are constant. Let $V(t, S_t, r_t)$ be a function on $[0, T]$ with $V(0, S_0, r_0) = v$. Using Itô, deduce the integral form for $V(T, S_T, r_T)$.

Begin by writing a 3D Taylor expansion for $V(t, S_t, r_t)$

$$\begin{aligned} & V(t + dt, S_t + dS, r_t + dr) - V(t, S_t, r_t) \\ &= \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{\partial V}{\partial r} dr + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} dS^2 + \frac{1}{2} \frac{\partial^2 V}{\partial r^2} dr^2 + \frac{\partial^2 V}{\partial r \partial S} dr dS \end{aligned}$$

Since $dW_t^2 \rightarrow dt$ in the mean square limit for $i = 1, 2$, we see that

$$dS_t^2 \rightarrow \sigma^2 S_t^2 dt,$$

$$dr_t^2 \rightarrow c^2 dt,$$

Also, since $dW_t^{(1)} dW_t^{(2)} = \rho dt$, we see that

$$dS_t dr_t \rightarrow \rho c \sigma S_t dt$$

This gives us a *bivariate* version of Itô's Lemma, the SDE for V is given by

$$\begin{aligned} dV = & \left(\frac{\partial V}{\partial t} + \mu S_t \frac{\partial V}{\partial S} + \gamma (m - r_t) \frac{\partial V}{\partial r} \right. \\ & + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} + \frac{1}{2} c^2 \frac{\partial^2 V}{\partial r^2} + \rho c \sigma S_t \frac{\partial^2 V}{\partial r \partial S} \Big) dt \\ & + \sigma S_t \frac{\partial V}{\partial S} dW_t^{(1)} + c \frac{\partial V}{\partial r} dW_t^{(2)} \end{aligned}$$

Integrating over $[0, t]$, we get

$$\begin{aligned} V(t, S_t, r_t) = & \underbrace{V(0, S_0, r_0)}_{=v} + \int_0^t \left(\frac{\partial V}{\partial \tau} + \mu S_\tau \frac{\partial V}{\partial S} + \gamma (m - r_\tau) \frac{\partial V}{\partial r} \right. \\ & + \frac{1}{2} \sigma^2 S_\tau^2 \frac{\partial^2 V}{\partial S^2} + \frac{1}{2} c^2 \frac{\partial^2 V}{\partial r^2} + \rho c \sigma S_\tau \frac{\partial^2 V}{\partial r \partial S} \Big) d\tau \\ & + \int_0^t \sigma S_\tau \frac{\partial V}{\partial S} dW_\tau^{(1)} + \int_0^t c \frac{\partial V}{\partial r} dW_\tau^{(2)} \end{aligned}$$

4. A spot rate r_t , evolves according to the popular form

$$dr_t = u(r_t) dt + \nu r_t^\beta dW_t, \quad (*)$$

where ν and β are constants. Suppose such a model has a **steady state transition probability density function** $p_\infty(r)$ that satisfies the forward Fokker Planck Equation. Show that this implies the drift structure of $(*)$ is given by

$$u(r_t) = \nu^2 \beta r_t^{2\beta-1} + \frac{1}{2} \nu^2 r_t^{2\beta} \frac{d}{dr} (\log p_\infty).$$

The forward F.P equation for $dr = u(r, t) dt + w(r, t) dW_t$ is

$$\frac{\partial p}{\partial t} = \frac{1}{2} \frac{\partial^2}{\partial r^2} (w^2(r, t) p(r, t)) - \frac{\partial}{\partial r} (u(r, t) p(r, t))$$

for the probability density $p(r, t)$. The steady state equation for our model becomes

$$\frac{1}{2} \nu^2 \frac{d^2}{dr^2} (r^{2\beta} p_\infty(r)) - \frac{d}{dr} (u(r) p_\infty(r)) = 0$$

This can be simply integrated once to give

$$\begin{aligned} \frac{1}{2} \nu^2 \frac{d}{dr} (r^{2\beta} p_\infty(r)) - (u(r) p_\infty(r)) &= \text{const} \\ \frac{1}{2} \nu^2 \left(r^{2\beta} \frac{dp_\infty}{dr} \right) + \nu^2 \beta r^{2\beta-1} p_\infty(r) - (u(r) p_\infty(r)) &= \text{const} \end{aligned}$$

The constant of integration is zero because as r becomes large

$$\left. \begin{matrix} p_\infty(r) \\ \frac{dp_\infty}{dr} \end{matrix} \right\} \longrightarrow 0$$

$$\begin{aligned}
u(r)p_{\infty}(r) &= \frac{1}{2}\nu^2 r^{2\beta} \frac{dp_{\infty}}{dr} + \nu^2 \beta r^{2\beta-1} p_{\infty}(r) \\
u(r) &= \frac{1}{2}\nu^2 r^{2\beta} \frac{1}{p_{\infty}(r)} \frac{dp_{\infty}}{dr} + \nu^2 \beta r^{2\beta-1}
\end{aligned}$$

We can write $\frac{1}{p_{\infty}} \frac{dp_{\infty}}{dr}$ as $\frac{d}{dr} (\log p_{\infty})$

$$u(r) = \frac{1}{2}\nu^2 r^{2\beta} \frac{d}{dr} (\log p_{\infty}) + \nu^2 \beta r^{2\beta-1}$$

5. The ordinary differential equation

$$\mu S \frac{du}{dS} + \frac{1}{2} \sigma^2 S^2 \frac{d^2 u}{dS^2} = -1,$$

for the function $u(S)$ is to be **solved** with boundary conditions

$$\begin{aligned} u(S_0) &= 0 \\ u(S_1) &= 0. \end{aligned}$$

μ and σ are constants. Show that the solution is given by

$$u(S) = \frac{1}{\frac{1}{2}\sigma^2 - \mu} \left(\log(S/S_0) - \frac{1 - (S/S_0)^{1-2\mu/\sigma^2}}{1 - (S_1/S_0)^{1-2\mu/\sigma^2}} \log(S_1/S_0) \right)$$

Hint: When solving for the particular integral, assume a solution of the form $C \log S$, where C is a constant.

Start by considering the homogeneous part of the equation i.e.

$$\mu S \frac{du}{dS} + \frac{1}{2} \sigma^2 S^2 \frac{d^2 u}{dS^2} = 0$$

and look for a solution of the form $u(S) = S^\lambda$. The resulting A.E. is

$$\lambda^2 + \left(\frac{2\mu}{\sigma^2} - 1 \right) \lambda = 0 \longrightarrow \lambda = 0, \left(1 - \frac{2\mu}{\sigma^2} \right)$$

$$u(S) = A + BS^{1-2\mu/\sigma^2}$$

For convenience write $\nu = 1 - 2\mu/\sigma^2$. For the homogeneous part, consider a solution $u(S) = C \log S$ where the coefficient C is to be determined, so

$$u'(S) = C/S; \quad u''(S) = -C/S^2; \quad \text{substitute this into the DE}$$

$$C \left(\mu - \frac{1}{2} \sigma^2 \right) = -1 \rightarrow C = \frac{1}{\frac{1}{2} \sigma^2 - \mu}$$

so the general solution becomes

$$u(S) = A + BS^\nu + \frac{1}{\frac{1}{2} \sigma^2 - \mu} \log S. \tag{A}$$

Now apply the boundary conditions:

$$u(S_0) = A + BS_0^\nu + \frac{1}{\frac{1}{2} \sigma^2 - \mu} \log S_0 = 0 \tag{a}$$

$$u(S_1) = A + BS_1^\nu + \frac{1}{\frac{1}{2} \sigma^2 - \mu} \log S_1 = 0 \tag{b}$$

(b) - (a) gives

$$B = -\frac{1}{\frac{1}{2} \sigma^2 - \mu} \frac{\log(S_1/S_0)}{S_1^\nu - S_0^\nu}.$$

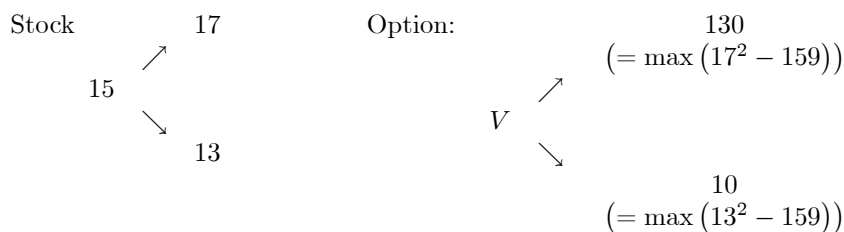
Substituting back into (a)

$$A = -\frac{1}{\frac{1}{2} \sigma^2 - \mu} \left(\log S_0 - \frac{\log(S_1/S_0)}{S_1^\nu - S_0^\nu} S_0^\nu \right)$$

Now substitute back in (A) and after some algebraic manipulation the result is obtained.

3 Binomial Model

1. A share price is currently £15. At the end of three months, it will be either £13 or £17. By constructing a hedged portfolio, calculate the value of a three-month European option with payoff $\max(S^2 - 159, 0)$, where S is the share price at the end of three months. The risk-free rate is 5% per annum with continuous compounding.



Now set up a Black-Scholes hedged portfolio, $V - \Delta S$, then binomial tree for its value is

$$\begin{array}{c} 130 - 17\Delta \\ V - 15\Delta \\ 10 - 13\Delta \end{array}$$

For risk-free portfolio choose Δ such that $130 - 17\Delta = 10 - 13\Delta \Rightarrow \Delta = 30$. So in absence of arbitrage and the risk free return of 5% p.a,

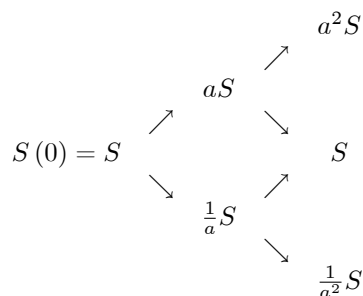
$$\begin{aligned} V - 15\Delta &= e^{-0.05 \times 3/12} (130 - 17\Delta) \\ &= 30 (15 - 17e^{-0.0125}) + 130e^{-0.0125}, \end{aligned}$$

and $V = 74.72$.

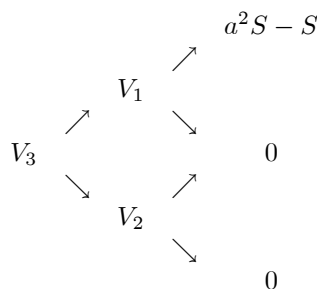
2. Consider the following model risk-free interest rate $r = 0$:

ω	$S(0)$	$S(1)$	$S(2)$
ω_1	S	aS	a^2S
ω_2	S	aS	S
ω_3	S	$a^{-1}S$	S
ω_4	S	$a^{-1}S$	$a^{-2}S$

S is the initial asset value at $t = 0$ and $a > 1$ is a constant. Asset:

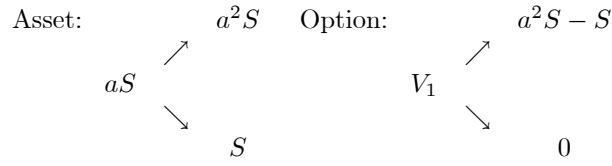


Option:



- (a) In this model, replicate the **European call** option with strike equal to the initial asset value S over the two periods and so find the fair price of the option.

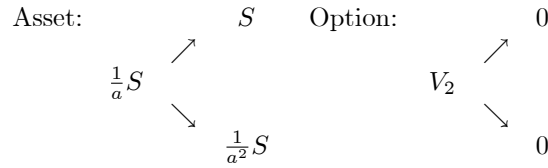
Replicate backwards over each one-period:



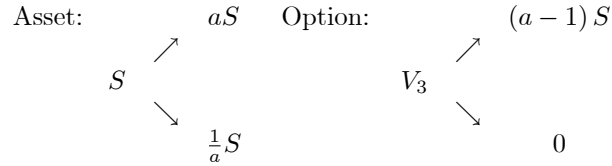
Replicate with ϕ units of stock and ψ bonds; so solve

$$\left. \begin{aligned} a^2 S \phi + \psi &= a^2 S - S \\ S \phi + \psi &= 0 \end{aligned} \right\} \rightarrow \phi = 1, \psi = -S$$

$$V_1 = 1 \times aS - S \times 1 = (a - 1)S.$$



Clearly $V_2 = 0$ as both final nodes are worthless.



Usual replication with ϕ units of stock and ψ bonds; so solve

$$\left. \begin{aligned} aS \phi + \psi &= (a - 1)S \\ \frac{1}{a}S \phi + \psi &= 0 \end{aligned} \right\} \rightarrow \phi = \frac{a(a - 1)}{a^2 - 1}, \psi = -\frac{(a - 1)S}{a^2 - 1}$$

$$\therefore V_3 = \frac{a(a - 1)}{a^2 - 1}S - \frac{(a - 1)S}{a^2 - 1} \times 1 = \frac{a - 1}{a + 1}S.$$

Hence the price of the European call struck at S is $\frac{a-1}{a+1}S$.

- b. Find all the one period risk-neutral probabilities and the corresponding probability measure \mathbb{Q} on $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$. Confirm that $\mathbb{E}^{\mathbb{Q}}[X]$ is the fair price.

These are when $r = 0$,

$$q(\text{up}) = \frac{s - s_d}{s_u - s_d}$$

$$q(\text{down}) = \frac{s_u - s}{s_u - s_d}$$

For each time-step we have the probabilities:

$$q(\text{up}) = \frac{S - \frac{1}{a}S}{aS - \frac{1}{a}S} = \frac{1}{a + 1},$$

$$q(\text{down}) = \frac{aS - S}{aS - \frac{1}{a}S} = \frac{a}{a + 1}.$$

∴

$$\begin{aligned}\omega_1 &= \frac{1}{(a+1)^2} \\ \omega_2 &= \frac{a}{(a+1)^2} \\ \omega_3 &= \frac{a}{(a+1)^2} \\ \omega_3 &= \frac{a^2}{(a+1)^2}.\end{aligned}$$

So the expected value is:

$$\mathbb{E}^{\mathbb{Q}}[X] = \sum_{\omega} p(\omega) X(\omega) = p(\omega_1)(a^2S - S) + 0 + 0 + 0 = \frac{a-1}{a+1}S,$$

as before!

- (b) Now consider a model where in each period the asset can either double or half. Show that the value of an option struck at the initial asset value S is $S/3$.

This is a special case of the above model when $a = 2$. Substituting in $a = 2$ into the option gives

$$\frac{2-1}{2+1}S = \frac{1}{3}S.$$

3. Repeat problem 1. by a replicating strategy. By calculating the risk-neutral probabilities obtain a price using an expectation.

Using earlier tree diagrams, the replicating strategy gives

$$\begin{aligned}17\phi + \psi e^{rt} &= 130 \\ 13\phi + \psi e^{rt} &= 10\end{aligned}$$

hence $\phi = 30$. $\psi = -375.3$. Substituting into (where s is the initial stock price)

$$V_0 = \frac{x_u - x_d}{s_u - s_d}s + e^{-r} \frac{x_d s_u - x_u s_d}{s_u - s_d}$$

and

$$x_u = 130; x_d = 10; s_u = 17; s_d = 13$$

gives 74.56.

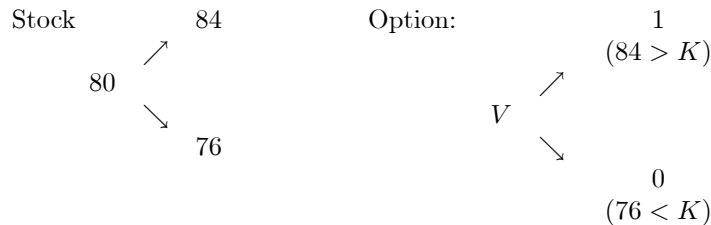
The risk-neutral probabilities

$$\begin{aligned}q(\text{up}) &= \frac{e^{rt}s - s_d}{s_u - s_d} = 0.54875 \\ q(\text{down}) &= \frac{s_u - e^{rt}s}{s_u - s_d} = 0.45125\end{aligned}$$

The option value as an expectation is

$$e^{-rt}(q(\text{up}) \times x_u + q(\text{down}) \times x_d) = 0.98763(75.85) = 74.9$$

4. A share price is currently £80. At the end of three months, it will be either £84 or £76. Ignoring interest rates, calculate the value of a three-month **digital** call option with strike price £79. So strike $K = 80$



Now set up a Black-Scholes hedged portfolio, $V - \Delta S$, then binomial tree for its value is

$$V - 80\Delta \quad \begin{array}{l} 1 - 84\Delta \\ -76\Delta \end{array}$$

For risk-free portfolio choose Δ such that $1 - 84\Delta = -76\Delta \Rightarrow \Delta = 1/8$. So in absence of arbitrage, $V - 80\Delta = 1 - 84\Delta$, and $V = 0.5$.