Mod.1 Lecture 2 - Solutions to the Exercises

Taylor expansions and Transition Density Functions

This is a non-assessed **problem sheet.**

1. Expand $(2+x)^{-2}$ in ascending powers of x up to and including the term in x^3, and state the set of values of x for which the expansion is valid. Hence find the coefficient of x^3 in the expansion of $\dfrac{1+x^2}{(2+x)^2}$.
   Using a binomial expansion

$$\begin{aligned}(2+x)^{-2} &= 2^{-2}\left(1+\frac{x}{2}\right)^{-2} = \frac{1}{4}\left(1+(-2)\frac{x}{2}+\frac{(-2)(-3)}{2!}\left(\frac{x}{2}\right)^2+\frac{(-2)(-3)(-4)}{3!}\left(\frac{x}{2}\right)^3\right)\\ &= \frac{1}{4}\left(1-x+\frac{3}{4}x^2-\frac{1}{2}x^3\right) = \left(\frac{1}{4}-\frac{x}{4}+\frac{3}{16}x^2-\frac{1}{8}x^3\right)\end{aligned}$$

   The expansion is valid provided $\left|\frac{x}{2}\right| < 1$, i.e. $|x| < 2$ which is $-2 < x < 2$.
   The next part relies on the earlier expansion. So

$$\begin{aligned}\frac{1+x^2}{(2+x)^2} &= \left(1+x^2\right)(2+x)^{-2}\\ &= \left(1+x^2\right)\left(\frac{1}{4}-\frac{x}{4}+\frac{3}{16}x^2-\frac{1}{8}x^3\right)\\ &= \frac{1}{4}-\frac{x}{4}+\frac{3}{16}x^2-\frac{1}{8}x^3+x^2\left(\frac{1}{4}-\frac{x}{4}+\frac{3}{16}x^2\right)\\ &= \frac{1}{4}-\frac{x}{4}+\frac{3}{16}x^2-\frac{1}{8}x^3+\frac{x^2}{4}-\frac{x^3}{4}+\end{aligned}$$

2. Find the Maclaurin series for $\ln(1+x)$ and hence that for $\ln\left(\dfrac{1+x}{1-x}\right)$.
   We know $\ln\left(\dfrac{1+x}{1-x}\right) = \ln(1+x)-\ln(1-x)$. The Maclurin series for $\ln(1+x)$ is standard and given in my calculus books.

$$\ln(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-...$$

   so simply replace x by $-x$ to get

$$\ln(1-x) = -x-\frac{x^2}{2}-\frac{x^3}{3}-...$$

   Hence subtracting gives $\ln(1+x)-\ln(1-x) =$

$$2\left(x+\frac{x^3}{3}+\frac{x^5}{3}+...\right) = 2\sum_{n=1}^{\infty}\frac{x^{2n-1}}{2n-1}.$$

3. Find the Taylor series expansions of the following functions about $x = 0$ (*by first using a Binomial expansion in part* **a**) *and then considering how the function in part b) is related to that in part* **a**)).

(a) $f(x) = \dfrac{1}{1+x}$.

(b) $g(x) = \ln(1+x)$.

$$\frac{1}{1+x} = (1+x)^{-1} = 1 - x + x^2 - x^3 + \dots\dots\dots + (-1)^n x^n = \sum_{n=0}^{\infty} (-1)^n x^n$$

Now

$$\ln(1+x) = \int_0^x \frac{1}{1+t} dt,$$

so we integrate each term in the series for $(1+x)^{-1}$ giving

$$\begin{aligned} \frac{1}{1+x} &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \dots\dots\dots + (-1)^{n+1} \frac{x^n}{n} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}. \end{aligned}$$

4. Find the first 4 terms of the Taylor series for the following functions centered at $a = 1$. **Hint: The expansion will have powers of** $(x-1)$:

(a) $f(x) = \ln x$

$$f'(x) = \frac{1}{x},\ f''(x) = -\frac{1}{x^2},\ f^{(3)}(x) = \frac{2}{x^3},\ f^{(4)}(x) = -\frac{6}{x^4} \text{ and so}$$

$$\begin{aligned} \ln x &= \ln 1 + (x-1) \times 1 + \frac{(x-1)^2}{2!} \times (-1) + \frac{(x-1)^3}{3!} \times (2) + \frac{(x-1)^4}{4!} \times (-6) + \dots \\ &= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \dots. \\ f(x) &= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n} \end{aligned}$$

(b) $g(x) = \dfrac{1}{x}$

$$g'(x) = -\frac{1}{x^2},\ g''(x) = \frac{2}{x^3},\ g^{(3)}(x) = -\frac{6}{x^4},\ g^{(4)}(x) = \frac{24}{x^5} \text{ to give}$$

$$\begin{aligned} \frac{1}{x} &= 1 + (x-1) \times (-1) + \frac{(x-1)^2}{2!} \times (2) + \frac{(x-1)^3}{3!} \times (-6) + \frac{(x-1)^4}{4!} \times (24) + \dots \\ &= 1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4 + \dots. \\ g(x) &= \sum_{n=1}^{\infty} (x-1)^{n-1} \end{aligned}$$

5. Find all first order partial derivatives

(a) $f(x,y) = 2x^4y^3 - xy^2 + 3y + 1$.

$$\begin{aligned} f_x &= 8x^3y^3 - y^2 \\ f_y &= 6x^4y^2 - 2xy + 3 \end{aligned}$$

(b) $f(x,y,z) = xyze^{xyz}$.

$$\begin{aligned}
f_x &= xyz(yz)e^{xyz} + (yz)e^{xyz} = yze^{xyz}(xyz+1) \\
f_y &= xyz(xz)e^{xyz} + (xz)e^{xyz} = xze^{xyz}(xyz+1) \\
f_z &= xyz(xy)e^{xyz} + (xy)e^{xyz} = xye^{xyz}(xyz+1)
\end{aligned}$$

(c) $f(x,y,z) = \left(y^2+z^2\right)^x$. Hint: $\frac{d}{dx}a^x = a^x \ln a$; where $a > 0$.

$$\begin{aligned}
f_x &= \left(y^2+z^2\right)^x \ln\left(y^2+z^2\right) \\
f_y &= x\left(y^2+z^2\right)^{x-1} \times 2y = 2yx\left(y^2+z^2\right)^{x-1} \\
f_z &= x\left(y^2+z^2\right)^{x-1} \times 2z = 2xz\left(y^2+z^2\right)^{x-1}
\end{aligned}$$

6. Consider a **symmetric** random walk which starts with a marker placed at a point x at time s; written (x,s). Suppose at a later time $t > s$ the marker is at y; the future state denoted (y,t). The marker can move in step sizes of δy in a time step of δt. At the previous step the marker must have been at one of $(y-\delta y, t-\delta t)$ or $(y+\delta y, t-\delta t)$. The transition probability density function of the position y of the diffusion at a later time t, is written $p(x,s;y,t)$. Derive the Forward Equation

$$\frac{\partial p}{\partial t} = \tfrac{1}{2}\frac{\partial^2 p}{\partial y^2}. \tag{6.1}$$

You may omit the dependence on (x,s) in your working as they will not change. Assume a solution of (6.1) exists and takes the following form

$$p(y,t) = t^{-1/2} f(\eta)\,;\ \eta = \frac{y}{t^{1/2}}.$$

Solve (6.1) to show that a particular solution of this is

$$p(x,s;y,t) = \frac{1}{\sqrt{2\pi(t-s)}} \exp\left(-\frac{(y-x)^2}{2(t-s)}\right).$$

You may use the result $\int_{\mathbb{R}} e^{-x^2}dx = \sqrt{\pi}$, in your working.

$$p(y',t') = \tfrac{1}{2}p(y'+\delta y, t'-\delta t) + \tfrac{1}{2}p(y'-\delta y, t'-\delta t)$$

Taylor series expansion gives

$$\begin{aligned}
p(y'+\delta y, t'-\delta t) &= p(y',t') - \frac{\partial p}{\partial t'}\delta t + \frac{\partial p}{\partial y'}\delta y + \tfrac{1}{2}\frac{\partial^2 p}{\partial y'^2}\delta y^2 + \ldots \\
p(y'-\delta y, t'-\delta t) &= p(y',t') - \frac{\partial p}{\partial t'}\delta t - \frac{\partial p}{\partial y'}\delta y + \tfrac{1}{2}\frac{\partial^2 p}{\partial y'^2}\delta y^2 + \ldots
\end{aligned}$$

Substituting into the above

$$\begin{aligned}
p(y',t') &= \tfrac{1}{2}\left(p(y',t') - \frac{\partial p}{\partial t'}\delta t + \frac{\partial p}{\partial y'}\delta y + \tfrac{1}{2}\frac{\partial^2 p}{\partial y'^2}\delta y^2\right) \\
&\quad + \tfrac{1}{2}\left(p(y',t') - \frac{\partial p}{\partial t'}\delta t - \frac{\partial p}{\partial y'}\delta y + \tfrac{1}{2}\frac{\partial^2 p}{\partial y'^2}\delta y^2\right)
\end{aligned}$$

$$0 = -\frac{\partial p}{\partial t'}\delta t + \tfrac{1}{2}\frac{\partial^2 p}{\partial y'^2}\delta y^2$$

$$\frac{\partial p}{\partial t'} = \tfrac{1}{2}\frac{\delta y^2}{\delta t}\frac{\partial^2 p}{\partial y'^2}$$

Now take limits. This only makes sense if $\frac{\delta y^2}{\delta t}$ is $O(1)$, i.e. $\delta y^2 \sim O(\delta t)$ and letting $\delta y,\ \delta t \longrightarrow 0$ gives the equation

$$\frac{\partial p}{\partial t'} = \tfrac{1}{2}\frac{\partial^2 p}{\partial y'^2}$$

To solve, write

$$p(y,t) = t^{-1/2} f(\eta)$$

therefore

$$\frac{\partial p}{\partial y} = \frac{\partial p}{\partial \eta}\frac{\partial \eta}{\partial y} = t^{-1/2} f'(\eta) \times t^{-1/2} = t^{-1} f'(\eta)$$

$$\frac{\partial^2 p}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial p}{\partial y}\right) = \frac{\partial}{\partial y}\left(t^{-1} f'(\eta)\right) = t^{-3/2} f''(\eta)$$

$$\begin{aligned}\frac{\partial p}{\partial t} &= t^{-1/2}\frac{\partial}{\partial t} f(\eta) - \frac{1}{2} t^{-3/2} f(\eta) \\ &= t^{-1/2}\left(-\frac{1}{2} y t^{-3/2}\right) f'(\eta) - \frac{1}{2} t^{-3/2} f(\eta) \\ &= -\frac{1}{2}\eta t^{-3/2} f'(\eta) - \frac{1}{2} t^{-3/2} f(\eta)\end{aligned}$$

and then substituting

$$\begin{aligned}\frac{\partial p}{\partial t} &= -\frac{1}{2} t^{-3/2}\left(\eta f'(\eta) + f(\eta)\right) \\ \frac{\partial^2 p}{\partial y^2} &= t^{-3/2} f''(\eta)\end{aligned}$$

gives

$$-\frac{1}{2} t^{-3/2}\left(\eta f'(\eta) + f(\eta)\right) = \frac{1}{2} t^{-3/2} f''(\eta)$$

simplifying to the ODE

$$-\left(f + \eta f'\right) = f''.$$

We have an exact derivative on the lhs, i.e. $\dfrac{d}{d\eta}(\eta f) = f + \eta f'$, hence

$$-\frac{d}{d\eta}(\eta f) = f''$$

and we can integrate once to get

$$-\eta f = f' + K.$$

We set $K = 0$ (see class notes for justification) in order to get the correct solution, i.e.

$$-\eta f = f'$$

which can be solved as a simple first order variable separable equation:

$$f(\eta) = A \exp\left(-\tfrac{1}{2}\eta^2\right)$$

A is a normalizing constant, so write

$$\underbrace{A\int_{\mathbb{R}} \exp\left(-\tfrac{1}{2}\eta^2\right) d\eta}_{=\sqrt{\pi}} = 1 \rightarrow A = \frac{1}{\sqrt{2\pi}}$$

$$u(y,t) = t^{-1/2} f(\eta) \text{ becomes } u(y,t) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{y^2}{2t}\right).$$

If the random variable y has value x at time s then we can generalize to

$$p(x,s;y,t) = \frac{1}{\sqrt{2\pi(t-s)}} \exp\left(-\frac{(y-x)^2}{2(t-s)}\right)$$