CQF Module 4 - EXAM

Michael Adam Epstein

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1 Question 1

We wish to out the approximate value of a cashflow for a floorlet on the one month LIBOR, when using the Vasicek model. Show that this is given by

$$\max\left(r_f - r - \frac{1}{24}\,(\eta - \gamma r), 0\ \right),$$

where r_f is the floor rate and r the spot rate. *You MUST start by considering the yield curve power series expression given in the calibration and data analysis lecture.* **FULL working should be given for the series expansion.** [1]

1.1 Zero-Coupon Bond Pricing model

Let's beging by substituting the simplest solution (1.1) into the bond price equation (1.2)

$$Z(r,t;T) = e^{A(t;T)-rB(T-t)}, \quad \text{where} \quad t < T \tag{1.1}$$

The bond pricing equation is therefore

$$\frac{\partial V}{\partial t} + \frac{1}{2}W^2\frac{\partial^2 V}{\partial r^2} + (u - \lambda W)\frac{\partial V}{\partial r} - rV = 0 \tag{1.2}$$

Consider, $Z(r,t;T) =$ the value of a zero-coupon bond at time t, with maturity at time T and a principal of 1, dependent on r

$$Z(r,t;T) \to 0 \quad \text{as } r \to \infty$$

In order to satisfy a closed form solution, we'll require $Z(r,T;T) = 1$, so that

$$\begin{aligned} A(T;T) = 0 \quad \text{and} \quad B(T;T) = 0 \\ \text{if} \quad Z(r,t,T) = 1 \quad \text{then} \\ \frac{\partial Z}{\partial t} = \left(\frac{\partial A}{\partial t} - r\frac{\partial B}{\partial t}\right)Z, \\ \frac{\partial Z}{\partial r} = -BZ \\ \frac{\partial^2 Z}{\partial r^2} = B^2 Z \end{aligned} \tag{1.3}$$

Substituting into equation (1.2), We get

$$\left(\frac{\partial A}{\partial t}\right) = \left(\frac{\partial B}{\partial t}\right)Z\frac{1}{2}W^2B^2Z - (u - \lambda W)BZ - rZ = 0$$

[1]Special thanks to Dr. Wilmott: On Quantative Finance, Dr Riaz Ahmad, and Dr. Richard Diamond for the resources to complete.

Simplifying finally,

$$\frac{\partial A}{\partial t} - \frac{\partial B}{\partial t} + \frac{1}{2}W^2B^2 - (u - \lambda W)B - r = 0$$
$$A(T;T) = B(T;T) = 0$$

The floorlet cashflow approximate value is given by

$$\max(r_f - r_L, 0) \sim \max\Big(rf - r - \frac{1}{24}(\eta - \gamma r), 0\Big). \tag{1.4}$$

If we assume that the actual floating rate is the spot rate, i.e. $r_L \approx r$ (and this approximation may not be important), then a single caplet may be priced by solving

$$\frac{\partial V}{\partial t} + \frac{1}{2}\, W^2\, \frac{\partial^2 V}{\partial r^2} + (U - \lambda W)\frac{\partial V}{\partial r} - rV = 0, \tag{1.5}$$

with

$$V(r, T) = \max(r - r_C, 0).$$

A floor is similar to a cap except that the floor ensures that the interest rate is bounded below by r_f. A floor is made up of a sum of floorlets, each of which has a cashflow of

$$\max(r_f - r_L, 0)$$

A floorlet is a put on the spot rate. We can approximate r_L by r again, in which case the floorlet satisfies the bond pricing equation but with

$$V(r, T) = \max(rf - r, 0)$$

Errata 'Wilmott: On Quantitative Finance'

For some function of $a(r, t)$, a given $u(r, t)$ and $W(r, t) > 0$.

$$a(r, t) = W(r, t)\lambda(r, t) - u(r, t)$$

The function $\lambda(r, t)$ is yet to be specified.

THE MARKET PRICE OF RISK
To represent an unhedged position in a bond with maturity date T and an incremental time step dt using the function $\lambda(r, t)$, the bond changes in value by:

$$dV = W\frac{\partial V}{\partial r}dX + \Big(W\lambda\frac{\partial V}{\partial r} + rV\Big)dt.$$

where dX is a Brownian motion. dt is a deterministic term that can be interpreted as the excess return above the risk-free rate for accepting a certain level of risk. In return for taking the extra risk the portfolio profits by an extra λdt per unit of extra risk, dX. The function λ is therefore called the market price of risk

Where risk-neutral drift $u - \lambda W$ and W volatility terms are undefined, we'll use later analytically in the tractable Vasicek model (**??**) to ensure positive interest rates, mean reversion, and random walk we'll define r as possessing properties:

$$\text{Let} \quad u(r, t) - \lambda(r, t)W(r, t) = \eta(t) - \gamma(t)r \tag{1.6a}$$
$$\text{and} \quad W(r, t) = \sqrt{\alpha(t)r + \beta(t)}. \tag{1.6b}$$

We are describing a model for the risk-neutral spot rate. Where functions $\alpha, \beta, \gamma, \eta$, and λ in (1.6a) are a function of time. To ensure interest rates remain positive. let $\alpha(t) > 0$ and $\beta \le 0$ with a lower bound $-\beta/\alpha$. In case $\alpha(t) = 0$ we must take $\beta(t) \ge 0$. We impose further conditions for the rate of change with respect to the spot rate moving away from the long term mean. To ensure **Mean-reversion**, we restrict the lower bound requirement with $\eta(t) \ge -\beta(t)\gamma(t)/\alpha(t) + \alpha(t)/2$,

2 Question 2:

Consider the Black-Derman Toy (BDT) short-rate model given by

$$d\ln(r) = \left[\theta_t + \frac{\sigma'_t}{\sigma_t}\ln(r)\right]dt + \sigma_t dW_t$$

Using Itô, write down the BDT model as

$$dr = \alpha dt + \beta dW$$

2.1 Answer to Question 2

Let's consider The Ito for a second. We've used it to derive the Fokker plank, Kolomogorav and Black Scholes PDE. It's easily applicable to thel $(BDTmodel)$; although no explicit solution is given; let's apply ito to following and solve for d(r,t). given by the following expression:

$$dB = a(B,t)dt + b(B,t)dW_t \tag{2.1}$$

We're solving for a, b,and dB.; where dW_t is a standard BM process and is a function of time t. Consider V(B), the value of our bond, or for Black, BDT. If you prefer. Apply the typical Ito We saw many times! Consider a 1-factor model of the form. Where $r_t := \log(r_t)$
Where α_t is a deterministic function of time. Itô's Lemma may be applied to see that r_t is a geometric Brownian motion. This model was originally specified as a lattice model.

Where X_t is a $\mathbb{Q}$-Brownian motion. Then for all $t \in [0, T]$,

$$\begin{aligned}
dB =& \big(a(B,t)\frac{dV}{dB} + \frac{1}{2}b^2(B,t)\frac{\partial^2 V}{dB^2}\big)dt + b(B,t)\frac{dV}{dB}dX && (2.2)\\
&\text{Where} && (2.3)\\
quada =& \theta(t) + \frac{d(log\sigma(t))}{dt}\log r && (2.4)\\
b =& \sigma(t) && (2.5)\\
\Big[V =& e^B tendsto \frac{d}{V} d^2 V dB^2 = logt && (2.6)\\
dr =& \left(\theta(t) + \frac{d(\log\sigma(t))}{d} log\sigma(t)dtlogr + \frac{1}{2}\sigma^2(t)\right)dt + \sigma(t)rdW_t && (2.7)
\end{aligned}$$

Since V tends to e^B the short rate volatility; where $\beta = \sigma$ and time-independent, α becomes θ a constant, and the model is reduced to:

$$d\ln(r) = \theta(t)\ dt + \sigma dW_t$$

3 Answer to Question 3

We need to go back to our general formula for the BPE to solve this.

4 Solving the Bond Pricing Equation

Substitute the Bond Pricing equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}W^2\frac{\partial^2 V}{\partial r^2} + (u - \lambda W)\frac{\partial V}{\partial r} - rV = 0 \tag{4.1}$$

A bond has a payoff at maturity $t = T$ of one sum.
Where $V(r,T) \equiv Z(r,T)$ with the sum at maturity, $T = 1$

$$V = Z = e^{A(t;T) - rB(T-t)}$$

$$V = Z = \exp\big(A(t) - rB(t)\big) \quad \text{Where} \quad Z(r,t;T) = 1 \Rightarrow$$

$$= \exp(A(T;T) - rB(T;T)\big) = 1 \quad \text{This only happens when:}$$

$$\big(A(T;T) - rB(T;T)\big) = 0 \Rightarrow A(T;T) = B(T;T) = 0$$

$$Z_t = \Bigg(\dot{A} - r\dot{B}\Bigg) \quad Z_r = -BZ \quad Z_{rr} = B^2 Z \quad \text{Where } \cdot \equiv \frac{d}{dt}$$

Substituting in the BPE gives

$$\Rightarrow \dot{A} - r\dot{B} + \frac{1}{2}w^2B^2 - B - r = 0$$

$$\Rightarrow \Bigg(\dot{A} + \frac{1}{2}w^2B^2 - B\Bigg) - r\Bigg(\dot{B} + 1\Bigg) = 0$$

The result is two equations

$$\dot{B} + 1 = 0$$

$$\dot{A} + \frac{1}{2}w^2B^2 - B = 0$$

$$\frac{dB}{dt} = -1 \to \int_t^T dB = -\int_t^T d\tau$$

$$\to \underbrace{B(T;T)}_{=0} - B(t;T) = (T-t)$$

$$\because B(t;T) = (T-t)$$

Now the second equation becomes

$$\dot{A} = -\frac{1}{2}w^2B^2 + B$$

$$\frac{dA}{dt} = -\frac{1}{2}w^2(T-t)^2 + (T-t) \to$$

$$\int_t^T dA = -\frac{1}{2}w^2\int_t^T (T-\tau)^2 d\tau + (T-\tau)^2 d\tau$$

$$\underbrace{A(T;T)}_{=0} - A(T;T) = -\frac{1}{2}w^2\int_t^T (T-\tau)^2 d\tau + (T-\tau)^2 d\tau$$

$$\Rightarrow \quad A = \frac{w^2}{2}\int_t^T (T-\tau)^2 d\tau - \int_t^T (T-\tau) d\tau$$

$$= \quad \frac{w^2}{6}(T-\tau)^3 \frac{1}{2}(T-\tau)^2$$

Let Z_M denote the market In order to satisfy the final equation $Z(r,T;T) = 1$, we require:

$$A(T;T) = 0 \quad \text{and} \quad B(T;T) = 0$$

$$\text{if} \quad Z(r,t,T) = 1 \quad \text{then}$$

$$\frac{\partial Z}{\partial t} = \Big(\frac{\partial A}{\partial t} - r\frac{\partial B}{\partial t}\Big)Z,$$

$$\frac{\partial Z}{\partial r} = -BZ$$

$$\frac{\partial^2 Z}{\partial r^2} = B^2 Z \tag{4.2}$$

Substituting into equation (4.1), We get

$$\left(\frac{\partial A}{\partial t}\right) = \left(\frac{\partial B}{\partial t}\right)Z\frac{1}{2}W^2B^2Z - (u - \lambda W)BZ - rZ = 0$$

Simplifying,

$$\frac{\partial A}{\partial t} - \frac{\partial B}{\partial t} + \frac{1}{2}W^2B^2 - (u - \lambda W)B - r = 0$$
$$A(T;T) = B(T;T) = 0$$

Yield Curve fitting

The power series expansion for a zero coupon bond curve is defined by

$$Z(r,t;T) = 1 + a(r)(T-t) + \frac{1}{2}b(r)(T-t)^2 + \dots \tag{4.3}$$

by substituting 4.1

The real spot rate r satisfies the stochastic differential equation
$dr = u(r,t)dt + w(r,t)dX$

Model	$u(r,t) - \gamma(r,t)w(r,t)$	$w(r,t)$
Vasicek	$a - br$	c
CIR	$a - br$	$cr^{1/2}$
Ho & Lee	$a(t)$	c
Hull & White I	$a(t) - b(t)r$	$c(t)$
Hull & White II	$a(t) - b(t)r$	$c(t)r^{1/2}$
General affine	$a(t) - b(t)r$	$\big((c(t)r - d(t)\big)^{1/2}$

5 Question 3

Consider the spot rate r, which evolves according to the SDE

$$dr = u(r,t)dt + w(r,t)dW$$

The extended Hull and White model has drift and diffusion

$$u(r,t) = \eta(t) - \gamma r, \qquad w(r,t) = c,$$
$$\text{Giving us} \qquad dr = \big(\eta(t) - \gamma r\big)dt + cdX$$

in turn, where $\eta(t)$ is an arbitrary function of time t and γ and c are constants. Deduce that the value of a zero coupon bond, $Z(r,t;T)$ which has

$$Z(r,T;T) = 1$$

in the extended Hull and White model is given by

$$Z(r,t;T) = \exp\big(A(t;T) - rB(t;T)\big),$$

where

$$B(t;T) = \frac{1}{\gamma}\big(1 - e^{-\gamma(T-t)}\big)$$
$$A(t;T) = -\int_t^T \eta(\tau)B(\tau;T)dr + \frac{c^2}{2\gamma^2}\Big((T-t) + \frac{2}{\gamma}e^{-\gamma(T-t)} - \frac{1}{2\gamma}e^{-2\gamma(T-t)} - \frac{3}{2\gamma}\Big)$$

Where dX is an increment of Brownian motion

5.1 Answers to Question 3

Hull and White have extended the Vasicek model to incorporate time- dependent parameters.

$$dr = (\eta(t) - \gamma r)dt + cdW. \tag{5.1}$$

We're going to assume that γ and c are constants that were estimated statistically prior. We chose $\eta = \eta^*(t)$ at time t^* in order to make our theoretical prices coincide with the market prices of bonds denoted by Z_M .

Under this risk-neutral process the value of zero-coupon bonds, the solution of

$$Z(r, t; T) = e^{A(t;T) - rB(T-t)}, \quad \text{where} \quad t < T$$

Here the time-dependent parameter $\eta(t)$ can also be identified from the technique of yield curve fitting given as follows.

In order to fit the yield curve at time t^*, we must make $\eta^*(t)$ satisfy

$$\begin{aligned} A(t^*; T) &= -\int_{t^*}^{T} \eta^*(s) B(s; T) ds + \frac{c^2}{2\gamma^2}\Big(T - t^* + \frac{2}{\gamma} e^{-\gamma(T-t^*)} - \frac{1}{2\gamma} e^{-2\gamma(T-t^*)} - \frac{3}{2\gamma}\Big) \\ &= \log\big(Z_M(t^*; T)\big) + r^* B(t^*; T). \end{aligned} \tag{5.2}$$

This is an integral equation for $\eta^*(t)$ if we are given all of the other parameters and functions such as the market price of bonds $Z_M(t^*; T)$

If we differentiate (5.2) twice with respect to T we get

$$\eta^*(t) = -\frac{\partial^2}{\partial t^2} \log\big(Z_M(t^*; t)\big) - \gamma \frac{\partial}{\partial t} \log\big(Z_M(t^*; t)\big) + \frac{c^2}{2\gamma}\big(1 - e^{-2\gamma(t-t^*)}\big). \tag{5.3}$$

From this expression, we now solve for function $A(t; T)$,

$$\begin{aligned} A(t; T) = &\log\left(\frac{Z_M(t^*; T)}{Z_M(t^*; t)}\right) - B(t; T)\frac{\partial}{\partial t}\log\big(Z_M(t^*; T)\big) \\ &- \frac{c^2}{4\gamma^3}\big(e^{-\gamma(T-t^*)} - e^{-\gamma(t-t^*)}\big)^2\big(e^{2\gamma(t-t^*)} - 1\big). \end{aligned} \tag{5.4}$$

6 Question 4

Consider the process given by

$$dU_t = -\gamma\ U_t dt + \sigma dW_t; \quad \text{where:} \quad U_0 = u$$

where, γ, σ are constants. Hence, solve this equation for U_t and hence obtain the expectation $\mathbb{E}[U_t]$ and variance $\mathbb{V}[U_t]$.

6.1 Answer to Question 4

Consider, one dimensional Gaussian OU process $X = (X_t)$ where: $t \geq 0$ satisfies the following stochastic differential equation: 6.1

$$dU_t = -\gamma\ U_t dt + \sigma dW_t; \quad \text{where:} \quad U_0 = u \tag{6.1}$$

In equation 6.1, we observe an Ornstein-Uhlenbeck process [OU] proposed by Uhlenbeck and Ornstein (1930) as a modification to a standard continuous time random walk, driven by a Brownian motion with a drift term; a Wiener process denoted by W_t. The Ornstein-Uhlenbeck process is also a stationary Gaussian and Markovian process with a tendency to revert back to the mean over time t. In finance, the Vasicek model was introduced in 1977 attempting to capture the underlying behavior of interest rates changes over time t with a mean-reverting OU process driven by a Brownian motion.

$$dX_t = -\gamma(X_t - m)dt + \sigma dW_t, \quad \text{with} \quad X_0 = x \tag{6.2}$$

If X_t is the interest rate at time t and m is a reference value. To illustrate OU in Vasicek's model, the m represents the long term mean level of rates to which γ characterizes the velocity or speed of reversion around the the long term mean m. The standard deviation is denoted by σ as volatility followed by the Weiner process W_t.

$$\begin{aligned} \text{With: } \sigma > 0 \quad \text{and } \gamma > 0, \quad \text{Let: } U_t &= (X_t - m) \\ \text{We Get:} & \\ dU_t = dX_t &= -\gamma U_t dt + \sigma dW_t \end{aligned} \tag{6.3}$$

$$\begin{aligned} \text{So} \quad e^{\gamma t} dU_t + \gamma e^{\gamma t} &= \sigma e^{\gamma t} dW_t \\ \text{consequently,} \quad d(e^{\gamma t} U_t) &= \sigma e^{\gamma t} dW_t \\ \text{Let} \quad Z_t = e^{\gamma t} U_t \quad \text{where} \quad & (Z_0 = x_0 - m) \end{aligned}$$

$$\begin{aligned} \text{We obtain} \quad Z_t &= (x_0 - m) + \int_0^t \sigma e^{\gamma t} dW_s \\ \text{So,} \quad X_t &= Y_t + m = e^{-\gamma t} Z_t + m \\ &= e^{-\gamma t}\Big((x_0 - m) + \int_0^t \sigma e^{\gamma s} dW_s + m\Big) \\ &= m + e^{-\gamma t}(x_0 - m) + \sigma e^{-\gamma t} \int_0^t e^{\gamma s} dW_s \end{aligned} \tag{6.4}$$

X_t is a strong Markov solution to 6.2, So, we obtain that:

$$x_t \sim N\Big(m + e^{-\gamma t}(x_0 - m), \frac{\sigma^2}{2\gamma}(1 - e^{2\gamma t})\Big) \tag{6.5}$$

This distribution as $t \to \infty$ to the stationary distribution; $N\big(m, \frac{\sigma^2}{2\gamma}\big)$. The probability distribution X_t of approach an equilibrium probability distribution called the stationary distribution. This stationary distribution has a stationary density function. For a time changed Brownian motion another representation is here

$$X_t = m + e^{-\gamma t}(X_0 - m) + \sigma e^{-\gamma t} W_{e^{(2\gamma t - 1)/2\gamma}} \tag{6.6}$$

$$\mathbb{E}(X_t) = e^{-\gamma t} + m(1 - e^{-\gamma t}) \tag{6.7}$$

$$\mathbb{V}\mathrm{ar}(X_t) = \frac{\sigma^2}{2\gamma}(1 - e^{2\gamma t}) \tag{6.8}$$

7 Question 5

Please refer to the appended notes in the Excel file as well as all the documentation. It's very difficult to fit a yield curve. It's not hard in theory. It's a diffusion process. However, calibrating can be very tricky. You would need to use your volatility bootstraps and generate the three random columns of $\phi's$ and use as a gradient to curve fit the HJM model ϕ. With high speed computing & technology today, it's rather possible to get a decent approximation. I found it troublesome. It takes a while to model it really well. I didn't have the luxury of spending too much time on it. It could be a time consuming endeavor. Let's just say, Dr. Richard Diamond did all the hard work for us already. He made the Excel sheet. All that was left was to generate a random walker — with proper software you yield decent results. I couldn't get VBA to work much at all. In the very least. I was able to generate 1000 i;N paths — in VBA on mac. It took nearly 4 days; to get 1000 randnum generated in brand new excel.. it took nearly 45 minutes of complete freezing up to process — 1000 seeded results isn't enough for a decent approximation. Next time – I'll implement a different scripting language to do it. However — I was able to yield a decent diffusion curve. I've appended notes below.. Anti-thetic variate is very simple, but hard to integrate into Excel if you're on office 2016 or mac. I don't reccommend you upgrade your software folks! Good thing for BACKUP hard disks — or this test would be completely erased all along with my entire operating system this week. All said and done.. YES! It's possible to generate a Uniform (0,1) negatively correlated inverse variate —- not easily in excel.. 1000 — yes.. and that's a pretty bad calibration. It's awful — Sorry for the Delay.

Thanks!