CQF Lecture 5.3 Intensity Models

Solutions

1. (a) **Default intensity** λ_T is a rate of probability of instantaneous default at time T, given that the firm survived until that time.

   (b) We derive the multiplication rule for independent increments from the first principles:

$$\begin{aligned} S(t,T) \times S(T,T+s) &= \Pr(\tau > T | \tau > t) \times \Pr(\tau > T+s | \tau > T) \\ &= \frac{\Pr(\tau > T) \times \Pr(\tau > T+s)}{\Pr(\tau > t) \times \Pr(\tau > T)} \\ &= S(t,T+s) \end{aligned}$$

   (c) If time period s is small then $S(T,T+s) = \underline{e^{-\lambda_T s} \approx 1 - \lambda_T s} + O(s^2)$.
   This is **a very useful** Taylor series expansion, often implied in derivations.

$$\begin{aligned} S(t,T+s) &= S(t,T)(1-\lambda_T\, s) \\ \frac{1}{s}\frac{S(t,T+s)-S(t,T)}{S(t,T)} &= -\lambda_T \\ \frac{\partial \log S(s)}{\partial s} &= -\lambda_s \end{aligned}$$

   Solve the equation by integrating both side from t to T

$$\log S(t,T) - \log S(t,t) = -\int_t^T \lambda_s ds$$

   With initial condition $S(t,t) = 1$,

$$S(t,T) = \exp\left\{-\int_t^T \lambda_s ds\right\}$$

   If intensity λ is constant, the distribution of survival (default waiting, inter-arrival times) is Exponential with the cdf $F(\tau) = 1 - e^{-\lambda\tau}$.

   Notes: The result matches derivation of distribution for τ on Slide 14 of Intensity Models lecture. Another illustration of the surface for survival probability $S(t,T)$ as a function of hazard rate λ_s and maturity $T = t + \tau$ is presented on Slide 36 in Credit Default Swaps lecture.

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(d) By definition *pdf* for default probability $\mathrm{PD}(t,T)$ can be written as

$$\begin{aligned}
\frac{\partial \mathrm{PD}(t,T)}{\partial T} &= \lim_{h\to 0^+} \frac{\mathrm{PD}(t,T+h)-\mathrm{PD}(t,T)}{h} \\
&= \lim_{h\to 0^+} \frac{S(t,T)-S(t,T+h)}{h} \qquad \text{(using PD} = 1-S) \\
&= \lim_{h\to 0^+} \frac{S(t,T)\,(1-S(T,T+h))}{h} \qquad \text{(using multiplication rule)} \\
&= \lim_{h\to 0^+} \frac{S(t,T)\mathrm{PD}(T,T+h)}{h} \qquad (S(t;T) \text{ is taken out of limit}) \\
&= \lambda_T S(t,T) \qquad (\text{since } h\to 0^+ \quad \lambda_T = \lambda_{T+h} \text{ or simply } \lambda_h)
\end{aligned}$$

Integrating from t to T and using dummy variable $s \equiv h$ for a small time period

$$\mathrm{PD}(t,T) = \int_t^T \lambda_s S(t,s)ds.$$

Using the TSE identified above, the probability of default over a small timestep can be expressed as

$$\mathrm{PD} = 1-S \approx 1-(1-\lambda_T s) = \lambda_T s \equiv pdt$$

On the small time scale the probability of default p is proportional to intensity λ_T.

2. Constructing a hedging portfolio of **a risky bond** $V(r,p,t)$ gives

$$\begin{aligned}
d\Pi &= dV - \Delta Z \\
&= (\frac{\partial V}{\partial t} + \tfrac{1}{2}w^2\frac{\partial^2 V}{\partial r^2})dt + \frac{\partial V}{\partial r}dr - \Delta\left((\frac{\partial Z}{\partial t} + \tfrac{1}{2}w^2\frac{\partial^2 Z}{\partial r^2})dt + \frac{\partial Z}{\partial r}dr\right) \\
&= \left(\frac{\partial V}{\partial t} + \tfrac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} - \Delta(\frac{\partial Z}{\partial t} + \tfrac{1}{2}w^2\frac{\partial^2 Z}{\partial r^2})\right)dt + (\frac{\partial V}{\partial r} - \Delta\frac{\partial Z}{\partial r})dr
\end{aligned}$$

First, we choose $\Delta = \frac{\partial V}{\partial r}/\frac{\partial Z}{\partial r}$ to eliminate interest rate risk (same as we did in bond pricing derviations). Second we have to account for default event: if it happens the value of the hedging portfolio will jump by $-V$ with probability pdt, so in terms of expected values

$$d\Pi = \left(\frac{\partial V}{\partial t} + \tfrac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} - \Delta(\frac{\partial Z}{\partial t} + \tfrac{1}{2}w^2\frac{\partial^2 Z}{\partial r^2})\right)dt - pVdt$$

Set $d\Pi = r\Pi dt$, leads to

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} - (r+p)V = \frac{\partial V}{\partial r}/\frac{\partial Z}{\partial r}(\frac{\partial Z}{\partial t} + \tfrac{1}{2}w^2\frac{\partial^2 Z}{\partial r^2} - rZ).$$

Which is

$$\frac{\frac{\partial V}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} - (r+p)V}{\frac{\partial V}{\partial r}} = \frac{\frac{\partial Z}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 Z}{\partial r^2} - rZ}{\frac{\partial Z}{\partial r}}.$$

The only way this equation holds is that both sides are independent of V and Z (remember the argument from interest rate modeling). In general affine model, we equate both sides to a function $a(r,t) = w(r,t)\lambda(r,t) - u(r,t)$. Therefore, omitting some familiar steps, we can arrive at the BPE of the form

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} + (u-\lambda w)\frac{\partial V}{\partial r} - (r+p)V = 0.$$

Note: here λ has a meaning of the market price of risk (see Stochastic Interest Rates Modeling lecture).

3. In this exercise, we manipulate a simple **transition matrix** with 'default' and 'no default' states in order to model migration (change) of credit ratings. Then we use the ratings transition information to represent a price of a risky bond.

   (a) Solving for intensity matrix

   $$\mathbf{Q} = \frac{\mathbf{I}-\mathbf{P}}{dt} = \frac{1}{dt}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1-pdt & pdt \\ 0 & 1 \end{pmatrix}\right] = \begin{pmatrix} p & -p \\ 0 & 0 \end{pmatrix}.$$

   (b) Substituting known matricies **V, Q** into equivalent transition density function (derived from a backward equation)

   $$\frac{d\mathbf{V}}{dt} = (r\mathbf{I} + \mathbf{Q})\,\mathbf{V}$$

   gives the following system of two linear ODEs to be solved

   $$\begin{cases} \frac{\partial V}{dt} = (r+p)V - pV_1 \\ \frac{\partial V_1}{dt} = rV_1 \end{cases}$$

   To solve ODEs we need final conditions $V(T) = 1$ and $V_1(T) = \theta$. First, we solve the equation for $V_1(t)$ and use its result to solve for $V(t)$.

   $$\begin{aligned} d\log V_1(t) &= rdt \qquad \text{(integrating from t to T)} \\ \log V_1(T) - \log V_1(t) &= r(T-t) \qquad \text{(using } V_1(T) = \theta) \end{aligned}$$

   we obtain the solution for

   $$V_1(t) = \theta e^{-r(T-t)}.$$

   Using the result to solve for $V(t)$ gives

   $$\frac{\partial V}{dt} - (r+p)V = -p\theta e^{-r(T-t)}$$

   Choosing Integrating Factor (IF) $e^{-(r+p)}$ and multiplying both sides by IF gives

   $$\frac{d}{dt}\left(V(t)e^{-(r+p)t}\right) = -p\theta e^{-rT-pt}$$

   Integrating both sides from t to T gives

   $$V(T)e^{-(r+p)T} - V(t)e^{-(r+p)t} = -p\theta e^{-rT}\int_t^T e^{-ps}ds$$

   $$e^{-(r+p)T} - V(t)e^{-(r+p)t} = \theta\left(e^{-T(r+p)} - e^{-(rT+pt)}\right)$$

   where $-p$ is conveniently cancelled by integration that gives $\frac{1}{-p}$. The solution is

   $$V(t) = (1-\theta)e^{-(r+p)(T-t)} + \theta e^{-r(T-t)}$$

   (c) Naming the risky bond Z_I, riskless bond Z_0 and Loss Given Default, LGD $= 1 - \theta$

   $$Z_I = (1 - \text{RR}) \times Z_0 \times (1\text{ - PD}) + \text{RR} \times Z_0.$$

   For each unit of the notional, a risky bond pays out Recovery Rate θ plus the amount inversely proportional to the probability of default.