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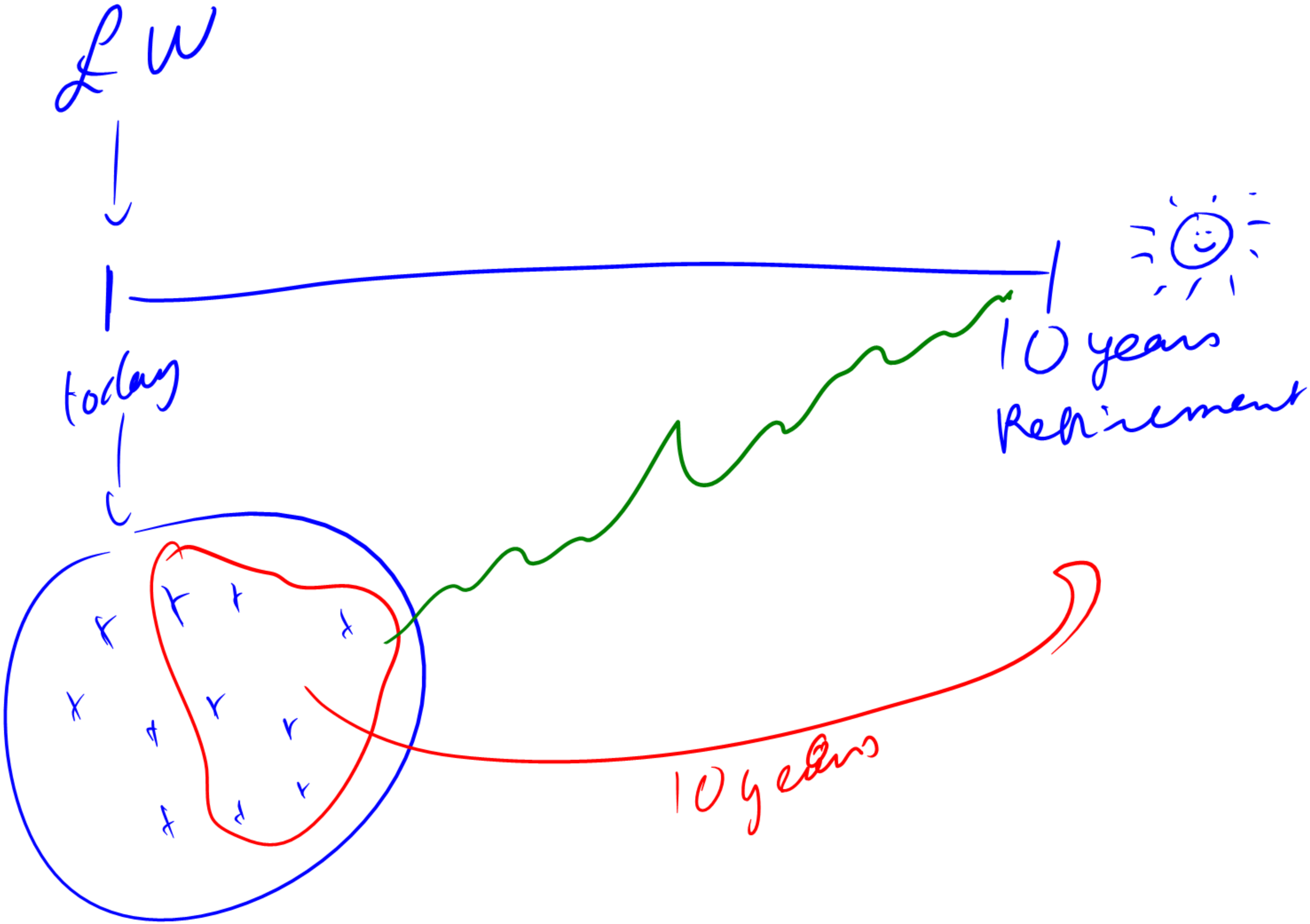
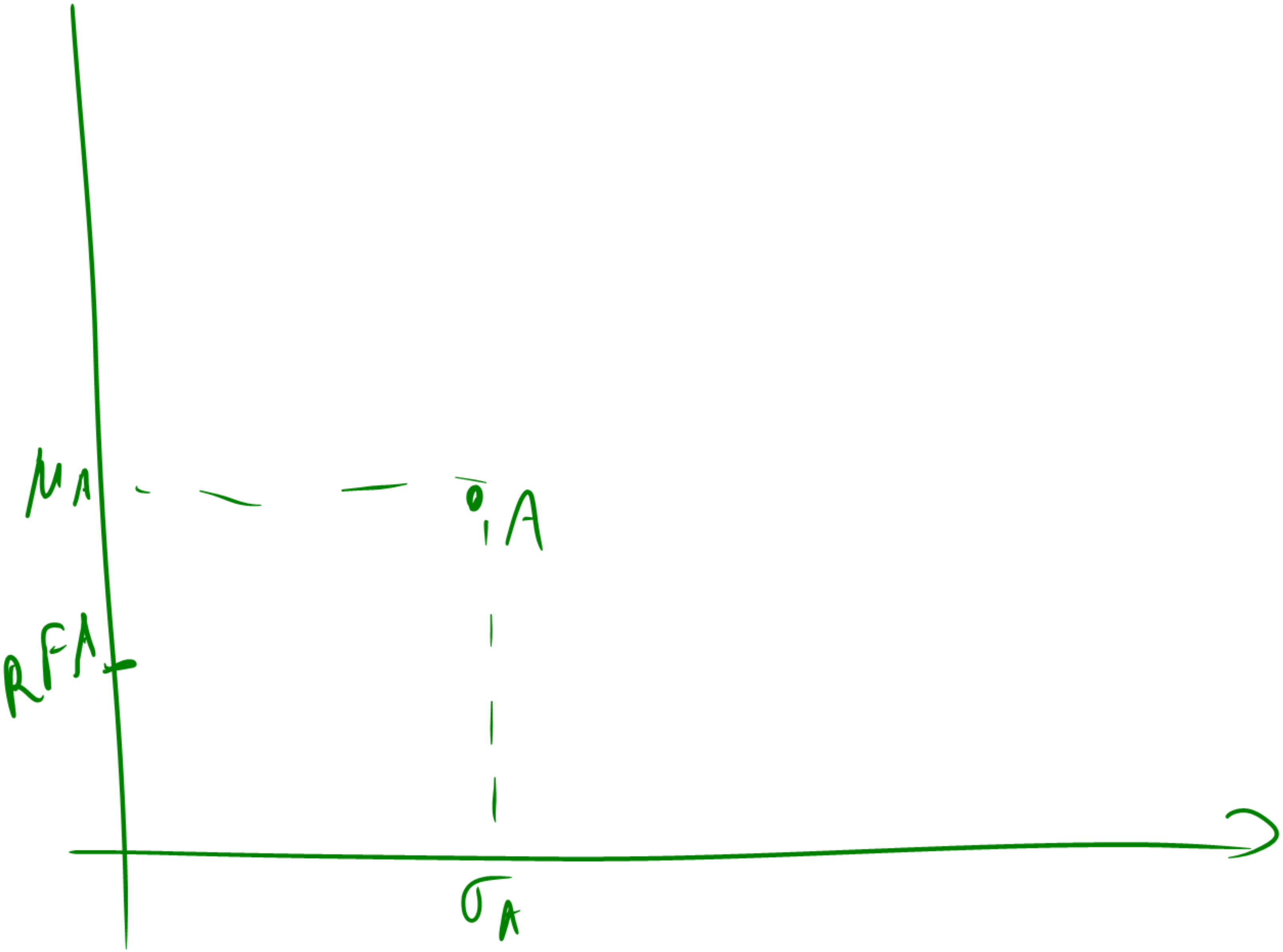
Module 2. Lecture 1
white board


£W
today
10 years
Retirement
10 years

1 risky asset



- invest a proportion w_A of my wealth in A.
- $1 - w_A$ in the risk-free asset.

$$\mu_\Pi = \mathbb{E}\left[w_A R_A + (1-w_A) R_F \right]$$

$$= w_A \mathbb{E}[R_A] + (1-w_A) R_F$$

$$= w_A \mu_A + (1-w_A) R_F$$

$$\boxed{\mu_\Pi = R_F + w_A \underbrace{(\mu_A - R_F)}_{\text{Risk Premium}}}$$

Return equation

Risk

$$\sigma_\pi^2 = \mathbb{V}\left[w_A R_A + \cancel{(1-w_A) R_F}\right]$$

$$= \mathbb{V}\left[w_A R_A\right]$$

$$= w_A^2\, \mathbb{V}\left[R_A\right]$$

$$\boxed{\sigma_\pi^2 = w_A^2\, \sigma_A^2}$$

$$\sigma_\pi = \sqrt{\sigma_\pi^2} = \boxed{w_A \sigma_A}$$

2 risky assets $\rightarrow w_A$ invested in A

$\rightarrow w_B = 1 - w_A$ invested in B

Return : $\mu_\pi = \mathbb{E}\left[w_A R_A + (1 - w_A) R_B \right]$

$= w_A \mathbb{E}[R_A] + (1 - w_A) \mathbb{E}[R_B]$

$= w_A \mu_A + (1 - w_A) \mu_B$

$$\boxed{\mu_\pi = \mu_B + w_A (\mu_A - \mu_B)}$$

Recall:

$$\sigma_\Pi^2 = \mathbb{V}[w_A R_A + w_B R_B]$$

$$= \mathbb{V}[w_A R_A] + \mathbb{V}[(1-w_A) R_B]$$

$$+ \mathrm{Cov}[w_A R_A, (1-w_A) R_B]$$

$$= w_A^2 \mathbb{V}[R_A] + (1-w_A)^2 \mathbb{V}[R_B]$$

$$+ 2 w_A (1-w_A) \mathrm{Cov}[R_A, R_B]$$

$$\mathbb{V}[X+Y]$$

$$= \mathbb{V}[X] + \mathbb{V}[Y]$$

$$+ \underbrace{2\,\mathrm{Cov}[X,Y]}_{2 \rho_{X,Y} \sigma_X \sigma_Y}$$

$$\rho_{X,Y} := \frac{\mathrm{Cov}[X,Y]}{\sigma_X \sigma_Y}$$

$$\boxed{\sigma_\Pi^2 = w_A^2 \sigma_A^2 + (1-w_A)^2 \sigma_B^2 + 2 w_A w_B \rho \sigma_A \sigma_B}$$

$$\hookrightarrow \sigma_\Pi = \sqrt{w_A^2 \sigma_A^2 + (1-w_A)^2 \sigma_B^2 + 2 w_A w_B \rho \sigma_A \sigma_B}$$

$$-1 \leq \rho \leq 1$$

replaced the ρ by -1

$$w_A^2\sigma_A^2 + (1-w_A)^2\sigma_B^2 - 2w_A(1-w_A)\sigma_A\sigma_B$$

$$\leq$$

$$\sigma_\pi^2$$

$$\leq$$

replaced ρ by $+1$

$$w_A^2\sigma_A^2 + (1-w_A)^2\sigma_B^2 + 2w_A(1-w_A)\sigma_A\sigma_B$$

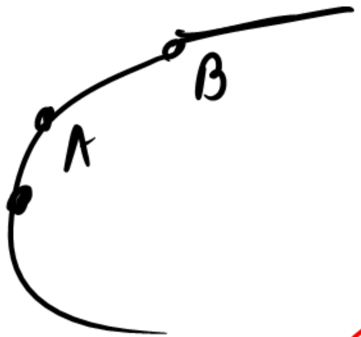
$$(w_A\sigma_A + (1-w_A)\sigma_B)^2$$

$$\sigma_\pi \leq w_A\sigma_A + (1-w_A)\sigma_B$$

2 Risky + the RFA

① - Risky portfolio P → w_A^P in asset A

→ $w_B^P = 1 - w_A^P$ in asset B



② - Overall portfolio : w invested in P

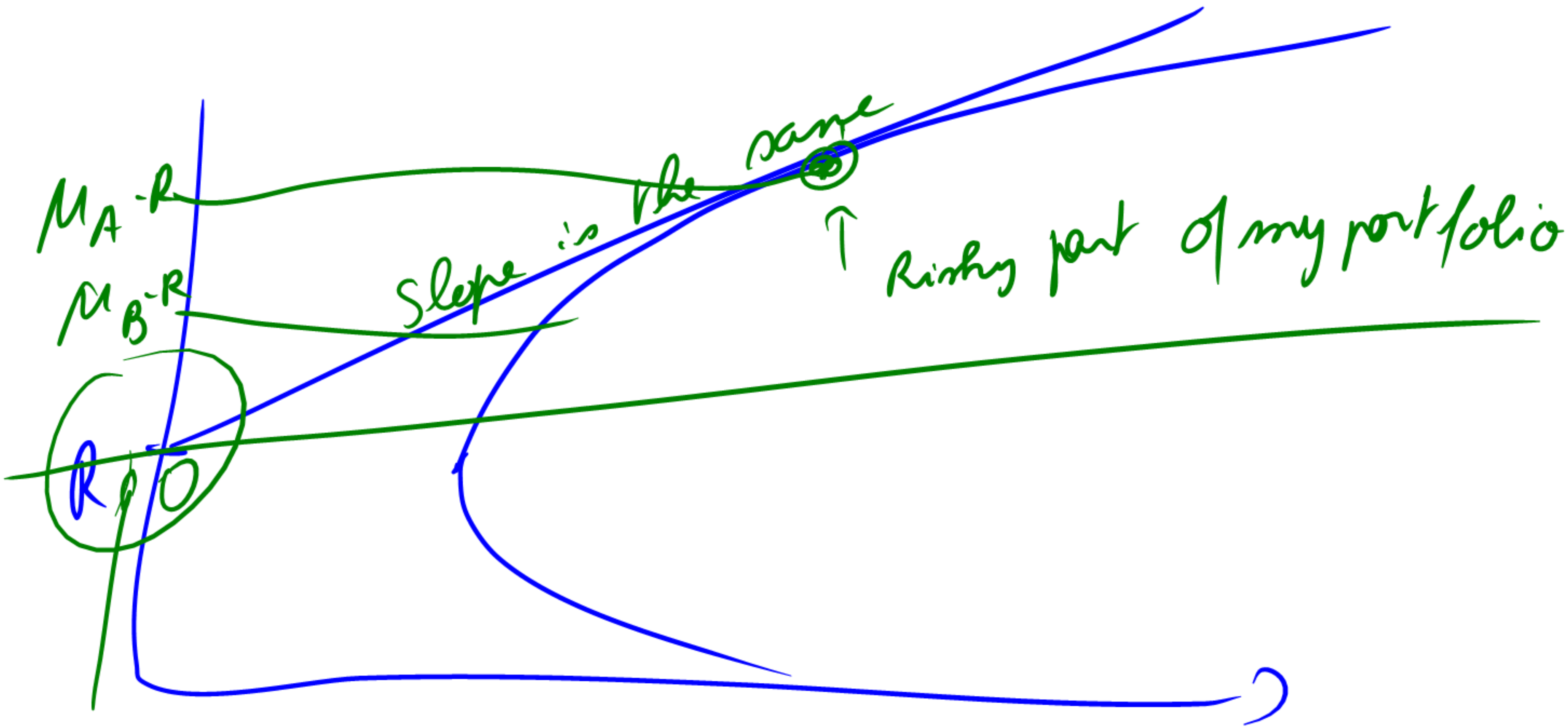
$(1-w)$ invested in the RFA

Check : $w_A = w \times w_A^P$

$w_B = w \times (1 - w_A^P)$

Risk-free weight = $(1-w)$

100%   100%



μA - R
μB - R
Slope is the same
Risky part of my portfolio
R
0

$$\cancel{w_A + w_B = 1}$$

$$\sum_{i=1}^{N} w_i = 1$$

$$w = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_i \\ \vdots \\ w_N \end{pmatrix}$$ N-element vector $$1_N = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

$$w^T \cdot 1_N = \begin{pmatrix} w_1 & w_2 & \dots & w_N \end{pmatrix} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = w_1 + w_2 + \dots + w_N = \sum_{i=1}^{N} w_i$$

$$\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_N \end{pmatrix}$$

$$(\mu_1\ \mu_2 \cdots \mu_N)\begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{pmatrix}$$

$$\mu_\pi = w^T\mu = \mu^T w = (w^T\mu)^T$$

$$= (w_1\ w_2 \cdots w_N)\begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_N \end{pmatrix}$$

$$= w_1\mu_1 + w_2\mu_2 + \cdots w_N\mu_N = \sum_{i=1}^{N} w_i\mu_i$$

$$\sigma_\Pi^2 = \boxed{w^T \Sigma' w}$$

$N = 2$

$$\Sigma' = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_2\sigma_1 & \sigma_2^2 \end{pmatrix}$$

$\rho\sigma_1\sigma_2 = \rho\sigma_2\sigma_1$

→ sym.

→ Positive definite

$1 \times 2 \qquad 2 \times 2$

$$\boxed{\begin{pmatrix} w_1 & w_2 \end{pmatrix} \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

$$\left(w_1\sigma_1^2 + w_2\rho\sigma_1\sigma_2 \,,\; w_1\rho\sigma_1\sigma_2 + w_2\sigma_2^2 \right) \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

$$= w_1^2\sigma_1^2 + \underbrace{w_1 w_2\rho\sigma_1\sigma_2 + w_1 w_2\rho\sigma_1\sigma_2}_{2w_1 w_2\rho\sigma_1\sigma_2} + w_2^2\sigma_2^2$$

$$\mu = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_N \end{pmatrix}$$

$$\underbrace{\mu - \mathbb{1}_N R}_{\text{vector of risk premia}} = \begin{pmatrix} \mu_1 - R \\ \mu_2 - R \\ \vdots \\ \mu_N - R \end{pmatrix}$$