2 Linear Algebra Problem Sheet

1. Find the transpose A^{T} of the matrix:

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 2 & 3 & 4 & 5 \\ 4 & 4 & 4 & 4 \end{pmatrix}$$

$$A^T = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 3 & 4 \\ 1 & 4 & 4 \\ 0 & 5 & 4 \end{pmatrix}$$

2. Let $A = \begin{pmatrix} 1 & 2 \\ 4 & -3 \end{pmatrix}$; Find $2A$; A^2; A^3

$$2A = \begin{pmatrix} 2 & 4 \\ 8 & -6 \end{pmatrix}; \quad A^2 = \begin{pmatrix} 9 & -4 \\ -8 & 17 \end{pmatrix}; \quad A^3 = \begin{pmatrix} -7 & 30 \\ 60 & -67 \end{pmatrix}$$

3. Calculate $(2A - BC)^{\mathrm{T}}$ for

$$A = \begin{pmatrix} 2 & 0 \\ 1 & 1 \\ 3 & 1 \end{pmatrix}; \; B = \begin{pmatrix} 1 & 1 & -1 & 1 \\ 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix} \; C = \begin{pmatrix} 1 & 1 \\ -1 & 2 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$$

$$2A = \begin{pmatrix} 4 & 0 \\ 2 & 2 \\ 6 & 2 \end{pmatrix}; \; BC = \begin{pmatrix} 1 & 3 \\ 3 & 3 \\ 0 & 3 \end{pmatrix}; \; (2A - BC) = \begin{pmatrix} 3 & -3 \\ -1 & -1 \\ 6 & -1 \end{pmatrix};$$

$$(2A - BC)^{\mathrm{T}} = \begin{pmatrix} 3 & -1 & 6 \\ 3 & -1 & -1 \end{pmatrix}$$

4. Calculate all possible products between the following matrices

$$(1, -1, 2, 0)\,; \; \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}; \; \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ -1 & 1 \\ 0 & 3 \end{pmatrix}; \; \begin{pmatrix} 1 & -1 & 0 & 1 \\ 2 & 1 & -1 & 2 \end{pmatrix}$$

$$(1, -1, 2, 0) \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ -1 & 1 \\ 0 & 3 \end{pmatrix} = (-3, 3)\,; \; \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 & 1 \\ 2 & 1 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 5 & 1 & -2 & 5 \\ -1 & -2 & 1 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \\ -1 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 4 \\ 5 & 10 \end{pmatrix}; \; \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

5. Calculate all the minors and cofactors of $A = \begin{pmatrix} 1 & 2 & -1 \\ -1 & 0 & 1 \\ 3 & 2 & 1 \end{pmatrix}$

$$\begin{aligned}
M_{11} &= \begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} = -2;\ M_{12} = -\begin{vmatrix} -1 & 1 \\ 3 & 1 \end{vmatrix} = 4;\ M_{13} = \begin{vmatrix} -1 & 0 \\ 3 & 2 \end{vmatrix} = -2 \\
M_{21} &= \begin{vmatrix} 2 & -1 \\ 2 & 1 \end{vmatrix} = -4;\ M_{22} = \begin{vmatrix} 1 & -1 \\ 3 & 1 \end{vmatrix} = 4;\ M_{23} = -\begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} = 4 \\
M_{31} &= \begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} = 2;\ M_{32} = -\begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} = 0;\ M_{33} = \begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} = 2
\end{aligned}$$

6. Evaluate the determinant $|A|$ of

$$A = \begin{pmatrix} t-2 & 4 & 3 \\ 1 & t+1 & -2 \\ 0 & 0 & t-4 \end{pmatrix}.$$

Determine those values of t for which $|A| = 0$.

$$|A| = (t+2)(t-3)(t-4)$$

and $|A| = 0$ gives $t = -2,\ 3,\ 4.$

7. Reduce to echelon form where

$$A = \begin{pmatrix} 1 & 3 & -1 & 2 \\ 0 & 11 & -5 & 3 \\ 2 & -5 & 3 & 1 \\ 4 & 1 & 1 & 5 \end{pmatrix}; \qquad A = \begin{pmatrix} 0 & 1 & 3 & -2 \\ 0 & 4 & -1 & 3 \\ 0 & 0 & 2 & 1 \\ 0 & 5 & -3 & 4 \end{pmatrix}$$

First and second matrices in turn become

$$\begin{pmatrix} 1 & 3 & -1 & 2 \\ 0 & 11 & -5 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \qquad \begin{pmatrix} 0 & 1 & 3 & -2 \\ 0 & 0 & -13 & 11 \\ 0 & 0 & 0 & 35 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

8. Solve the linear system

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$$

use row reduction which gives the augmented matrix in echelon form:

$$\left(\begin{array}{ccc|c} 2 & 1 & 1 & 7 \\ 0 & 3 & 1 & 9 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

which gives $(x, y, z) = (1, 2, 3)$.

9. What is the condition on $a, \ b, \ c$ so that the following linear system has a solution

$$\begin{aligned} x+2y-3z &= a \\ 2x+6y-11z &= b \\ x-2y+7z &= c \end{aligned}$$

the system becomes

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & a \\ 2 & 6 & -11 & b \\ 1 & -2 & 7 & c \end{array}\right)$$

which after row reduction becomes

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & a \\ 0 & 2 & -5 & b-2a \\ 0 & 0 & 0 & c+2b-5a \end{array}\right)$$

For a solution to exist, the last row tells us that the left hand side must equal zero. So the condition is

$$c+2b-5a=0$$

10. A matrix A is orthogonal if $A^{-1}=A^T$. Show that A is orthogonal where

$$A=\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

i.e. $A^{\mathrm{T}}=A^{-1}$.

$$\begin{aligned} A^{\mathrm{T}} &= \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}; \\ |A| &= 1; \ \mathrm{adj}A=\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}^{\mathrm{T}}; \ A^{-1}=\frac{1}{|A|}\mathrm{adj}A=\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

11. Show that

$$\begin{vmatrix} y-z & z-x & x-y \\ z-x & x-y & y-z \\ x-y & y-z & z-x \end{vmatrix} = 0;$$

$$\begin{vmatrix} yz & x & x^2 \\ zx & y & y^2 \\ xy & z & z^2 \end{vmatrix} = \begin{vmatrix} 1 & x^2 & x^3 \\ 1 & y^2 & y^3 \\ 1 & z^2 & z^3 \end{vmatrix}$$

$$\begin{vmatrix} y-z & z-x & x-y \\ z-x & x-y & y-z \\ x-y & y-z & z-x \end{vmatrix} \overset{R_1 \longrightarrow R_1+R_2+R_3}{\sim} \begin{vmatrix} 0 & 0 & 0 \\ z-x & x-y & y-z \\ x-y & y-z & z-x \end{vmatrix} = 0$$

$$\begin{vmatrix} yz & x & x^2 \\ zx & y & y^2 \\ xy & z & z^2 \end{vmatrix} \underset{R_2 \longrightarrow yR_2;\ R_3 \longrightarrow zR_3}{\overset{R_1 \longrightarrow xR_1}{\sim}} \frac{1}{xyz} \begin{vmatrix} xyz & x^2 & x^3 \\ zxy & y^2 & y^3 \\ xyz & z^2 & z^3 \end{vmatrix}$$

$$= \frac{xyz}{xyz} \begin{vmatrix} 1 & x^2 & x^3 \\ 1 & y^2 & y^3 \\ 1 & z^2 & z^3 \end{vmatrix} = \begin{vmatrix} 1 & x^2 & x^3 \\ 1 & y^2 & y^3 \\ 1 & z^2 & z^3 \end{vmatrix}$$

12. Solve the following linear system for all values of λ

$$\begin{aligned} 4x_1 - 2x_2 - 7x_3 &= \lambda^2 - 1 \\ x_1 + x_2 - 4x_3 &= \lambda^2 + 2 \\ -5x_1 + 3x_2 + 8x_3 &= \lambda \end{aligned}$$

$$\left(\begin{array}{ccc|c} 4 & -2 & -7 & \lambda^2 - 1 \\ 1 & 1 & -4 & \lambda^2 + 2 \\ -5 & 3 & 8 & \lambda \end{array}\right)$$

which becomes in echelon form

$$\left(\begin{array}{ccc|c} 1 & 1 & -4 & \lambda^2 + 2 \\ 0 & 2 & -3 & \lambda^2 + 3 \\ 0 & 0 & 0 & \lambda^2 + \lambda - 2 \end{array}\right)$$

i.e.

$$\begin{aligned} x_1 + x_2 - 4x_3 &= \lambda^2 + 2 \\ 2x_2 - 3x_3 &= \lambda^2 + 3 \\ 0 &= \lambda^2 + \lambda - 2 \end{aligned}$$