Stochastic Interest Rate Modeling

In this lecture. . .

- stochastic models for interest rates
- how to derive the pricing equation for many fixed-income products
- the structure of many popular one-factor interest rate models
- the theoretical framework for multi-factor interest rate modeling
- popular two-factor models

By the end of this lecture you will

- be able to derive the pricing equation for fixed-income instruments with one and two random factors

- appreciate the meaning of the market price of interest rate risk

- know the names of many popular interest rate models

Introduction

Whenever you price non linear derivatives products you require a model for the randomness in the underlying.

This is as true for fixed-income instruments as it is for equity derivatives.

In this lecture we see the ideas behind modeling interest rates using stochastic differential equations to model randomness.

We begin by having one source of randomness, the spot interest rate.

This is the subject of **one-factor interest rate modeling**.

- The model will allow the short-term interest rate, the spot rate, to follow a random walk.

This model leads to a parabolic partial differential equation for the prices of bonds and other interest rate derivative products.

Later we will consider modeling the fixed-income world using *two* sources of randomness.

The spot interest rate

The ‘spot rate’ that we will be modeling is a very loosely-defined quantity, meant to represent the yield on a bond of infinitesimal maturity.

In practice one should take this rate to be the yield on a liquid finite-maturity bond, say one of one month.

Bonds with one *day* to expiry do exist but their price is not necessarily a guide to other short-term rates.

Stochastic interest rates

Since we cannot realistically forecast the future course of an interest rate, it is natural to model it as a random variable.

- We are going to model the behavior of r, the interest rate received by the shortest possible deposit.

From this we will see the development of a model for all other rates.

The interest rate for the shortest possible deposit is commonly called the **spot interest rate**.

Let us suppose that the interest rate r is governed by a stochastic differential equation of the form

- $$dr = u(r,t)\,dt + w(r,t)\,dX.$$

The functional forms of $u(r,t)$ and $w(r,t)$ determine the behavior of the spot rate r.

For the present we will not specify any particular choices for these functions.

The pricing equation for the general model

When interest rates are stochastic a fixed-income instrument has a price of the form $V(r,t)$.

- We are not modeling a *traded* asset; the traded asset (the bond, say) is a derivative of our independent variable r.

Pricing a fixed-income instrument presents new technical problems, and is in a sense harder than pricing an option since *there is no underlying asset with which to hedge*.

$$V(S,t) \longrightarrow V(r,t;T)$$

- The only way to construct a hedged portfolio is by hedging one bond with a bond of a different maturity.

$$\Pi = V(r, t) -$$

Hedging one bond with another

We set up a portfolio containing two bonds with different maturities T_1 and T_2.

The bond with maturity T_1 has price $V_1(r, t; T_1)$ and the bond with maturity T_2 has price $V_2(r, t; T_2)$.

We hold one of the former and a number $-\Delta$ of the latter.

We have

$$\Pi = V_1 - \Delta V_2.$$

$V_1(r, t; T_1)$ $V_2(r, t; T_2)$

The change in this portfolio in a time dt is given by

$$d\Pi = \frac{\partial V_1}{\partial t}dt + \frac{\partial V_1}{\partial r}dr + \tfrac{1}{2}w^2\frac{\partial^2 V_1}{\partial r^2}dt - \Delta\left(\frac{\partial V_2}{\partial t}dt + \frac{\partial V_2}{\partial r}dr + \tfrac{1}{2}w^2\frac{\partial^2 V_2}{\partial r^2}dt\right),$$

where we have applied Itô's lemma to functions of r and t.

Which of these terms are random?

$$\left(\frac{\partial V_1}{\partial r} - \Delta\frac{\partial V_2}{\partial r}\right)dr \qquad \Delta = \frac{\partial V_1/\partial r}{\partial V_2/\partial r}$$

Once you've identified them you'll see that the choice

$$\Delta = \frac{\partial V_1}{\partial r} \Big/ \frac{\partial V_2}{\partial r}$$

eliminates all randomness in $d\Pi$. This is because it makes the coefficient of dr zero.

$\Downarrow$

No Arb

$d\Pi = r\Pi\, dt$

We then have

$$d\Pi = \left(\frac{\partial V_1}{\partial t} + \tfrac{1}{2} w^2 \frac{\partial^2 V_1}{\partial r^2} - \left(\frac{\partial V_1}{\partial r} \bigg/ \frac{\partial V_2}{\partial r} \right) \left(\frac{\partial V_2}{\partial t} + \tfrac{1}{2} w^2 \frac{\partial^2 V_2}{\partial r^2} \right) \right) dt$$

$$= r\Pi \, dt = r \left(V_1 - \left(\frac{\partial V_1}{\partial r} \bigg/ \frac{\partial V_2}{\partial r} \right) V_2 \right) dt,$$

where we have used arbitrage arguments to set the return on the portfolio equal to the risk-free rate. This risk-free rate is just the spot rate.

Collecting all V_1 terms on the left-hand side and all V_2 terms on the right-hand side we find that

$$(r,t)\quad \underbrace{\frac{\frac{\partial V_1}{\partial t}+\frac{1}{2}w^2\frac{\partial^2 V_1}{\partial r^2}-rV_1}{\frac{\partial V_1}{\partial r}}}_{\substack{\text{depends on } V_1(r,t;T_1)\\ \text{Indep of } T_2}} = \underbrace{\frac{\frac{\partial V_2}{\partial t}+\frac{1}{2}w^2\frac{\partial^2 V_2}{\partial r^2}-rV_2}{\frac{\partial V_2}{\partial r}}}_{\substack{\text{depends on } V_2(r,t;T_2)\\ \text{Indep. of } T_1}}. \quad (r,t)$$

At this point the distinction between the equity and interest-rate worlds starts to become apparent.

- This is *one* equation in *two* unknowns.

Fortunately, the left-hand side is a function of T_1 but not T_2 and the right-hand side is a function of T_2 but not T_1.

The only way for this to be possible is for both sides to be independent of the maturity date.

Furthermore, neither side can have any dependence of the specific contract at all.

Both sides must be equal to a universal constant.

Fn. indep. of T_i $(i = 1, 2)$

$F(r, t)$

The previous statement is not strictly true...

Both sides can be functions of the 'variables' in the model, r and t, since these are common to all instruments.

Dropping the subscript from V, we have

V_1, V_2

$$\frac{\frac{\partial V}{\partial t} + \frac{1}{2} w^2 \frac{\partial^2 V}{\partial r^2} - rV}{\frac{\partial V}{\partial r}} = a(r,t)$$

$V(r, t; T)$

for some function $a(r,t)$.

We shall find it convenient to write

$$a(r,t) = w(r,t)\lambda(r,t) - u(r,t);$$

for a given $u(r,t)$ and non-zero $w(r,t)$ this is always possible.

The fixed-income pricing equation is therefore

B.P.E.

$$\frac{\partial V}{\partial t}+\frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2}+(u-\lambda w)\frac{\partial V}{\partial r}-rV=0. \qquad (1)$$

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV=0$$

This is another parabolic partial differential equation.

Structurally, it is very similar to the Black–Scholes equation.

The only differences between this equation and the Black–Scholes equation are

- in the coefficients of the gamma and delta terms

- that the variable has another name, r instead of S

As with equity derivatives, we must tell the equation which contract we are solving by specifying a final condition, corresponding to the payoff at maturity, T.

For example, the final condition for a zero-coupon bond is

$$V(r,T) = 1.$$

$V(r,T;T) = 1$ Redemption value

$K(r,t)\,dt$ $V(r,t_c^-) = V(r,t_c^+) + K(r,t_c)$

The market price of risk?

$dr = u\,dt + w\,dX$

Imagine that you hold an unhedged position in one bond with maturity date T.

Itô on $V(r,t)$

In a time step dt this bond changes in value by

$$dV = w\frac{\partial V}{\partial r}dX + \underbrace{\left(\frac{\partial V}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} + u\frac{\partial V}{\partial r}\right)}\,dt.$$

This may be written as

$$dV = w\frac{\partial V}{\partial r}dX + \left(w\lambda\frac{\partial V}{\partial r} + rV\right)dt,$$

or

$$dV - rV\,dt = w\frac{\partial V}{\partial r}(dX + \lambda\,dt).$$

This expression contains a deterministic term in dt and a random term in dX.

The deterministic term may be interpreted as the excess return above the risk-free rate for accepting a certain level of risk.

- In return for taking the risk the portfolio profits by $\lambda\,dt$ per unit of risk, dX. The function λ is called the **market price of risk**.

Interpreting the market price of risk, and risk neutrality

The bond pricing equation contains references to the functions $u - \lambda w$ and w. The former is the coefficient of the first-order derivative with respect to the spot rate, and the latter appears in the coefficient of the diffusive, second-order derivative.

The four terms in the equation represent, in order as written,

$dr = u\,dt + w\,dX$

- time decay, $\partial V/\partial t$
- diffusion, w^2
- drift and $(u - \lambda w)$
- discounting. $-rV$

$(u - \lambda w)$

The equation is similar to the backward equation for a probability density function except for the final discounting term.

- As such we can interpret the solution of the bond pricing equation as the expected present value of all cashflows.

This idea should be familiar from equity derivatives.

$$V(S,t) = e^{-r(T-t)}\,\mathbb{E}\big[\underbrace{\text{Payoff}}\big]$$

As with equity options, this expectation is not with respect to the *real* random variable, but instead with respect to the *risk-neutral* variable.

$$dS/S = \mu\, dt + \sigma\, dX \longrightarrow dS/S = r\, dt + \sigma\, dX$$

- There is this difference because the drift term in the equation is not the drift of the real spot rate u, but the drift of another rate, called the **risk-neutral spot rate**. This rate has a drift of $u - \lambda w$.

$$dr = u\, dt + w\, dX \longleftarrow \text{Real world}$$

$$\Rightarrow dr = (u - \lambda w)\, dt + w\, dX \longleftarrow \text{Risk neutral}$$

Recall that for equities we have a real drift rate denoted by μ. However, we price as if the asset grows with a rate r, the risk-free rate.

For fixed-income products the real growth of the spot interest rate may be $u(r,t)$ but we price as if it were $u(r,t)-\lambda(r,t)w(r,t)$.

The latter is the **risk-adjusted drift rate**.

When pricing interest rate derivatives it is important to model, and price, using the risk-neutral rate.

The risk-neutral spot rate evolves according to

$$dr = (u - \lambda w)dt + w\, dX.$$

We can use Monte Carlo simulations for pricing fixed-income products, but we ensure that we simulate the risk-neutral spot rate process.

The relationship between prices and expectations

In the equity world we can write

e^{-rt}

$$V(S,t) = e^{-r(T-t)} E^*[\text{Payoff}],$$

when interest rates are constant.

We can write

$$V(S,t) = e^{-\int_t^T r(\tau)\, d\tau} E^*[\text{Payoff}],$$

when rates are deterministic, $r(t)$,. . .

When we have a fixed-income product and rates are stochastic the present value term must go inside the expectation...

$$V(S,t) = E^* \left[e^{-\int_t^T r(\tau)\, d\tau} \text{ Payoff} \right].$$

But the idea is the same...

The value of a derivative is the risk-neutral expectation of the present value of the payoff.

Rule of Thumb:

We need the new market-price-of-risk term because our modeled variable, r, is not traded.

- If a modeled quantity is traded then the risk-neutral growth rate is r

  rS

- If a modeled quantity is not traded then the risk-neutral growth rate is

$$\text{real growth} - \text{market price of risk} \times \text{volatility}$$

$$u - \lambda w$$

Tractable models and solutions of the pricing equation

We have built up the pricing equation for an arbitrary model.

That is, we have not specified the risk-neutral drift, $u - \lambda w$, or the volatility, w.

How can we choose these functions to give us a good model?

There are two ways to proceed:

- Choose a model that matches reality as closely as possible

- Choose a model which is easy to solve

Let us examine some choices for the risk-neutral drift and volatility that lead to tractable models, that is, models for which the solution of the pricing equation for zero-coupon bonds can be found analytically.

We will discuss these models and see what properties we like or dislike.

Named models

There are many interest rate models, associated with the names of their inventors.

- Vasicek $dr = (\eta - \gamma r)\,dt + \sqrt{\beta}\,dX$
- Cox, Ingersoll & Ross $dr = (\eta - \gamma r)\,dt + \sqrt{\beta r}\,dX$
- Ho & Lee $dr = \eta(t)\,dt + \sqrt{\beta}\,dX$
- Hull & White HW I Vasicek with all parameters time dep

and many more. HW II CIR with time dep parameters

Vasicek

$$\frac{\partial V}{\partial t} + \frac{1}{2} w^2 \frac{\partial^2 V}{\partial r^2} + (u - \lambda w)\frac{\partial V}{\partial r} - rV = 0$$

The Vasicek model (for the risk-neutral spot rate) takes the form

- $$dr = \underbrace{(\eta - \gamma r)}_{u - \lambda w} dt + \underbrace{\beta^{1/2}}_{w} dX.$$

Thus the pricing equation is

$$\frac{\partial V}{\partial t} + \frac{1}{2}\beta \frac{\partial^2 V}{\partial r^2} + (\eta - \gamma r)\frac{\partial V}{\partial r} - rV = 0.$$

And the final condition for a zero-coupon bond is

$$V(r, T) = 1. \qquad V(r, T; T) = 1$$

We are very lucky that this can be solved exactly.

And the form of the solution is very simple.

$$e^{A(t)-rB(t)}.$$

Affine sol^n,

Let's confirm this by substituting this expression into the partial differential equation

$\Rightarrow$ sol^n, of the P.D.E for particular form of $A(t)$ and $B(t)$

First of all we need some derivatives:

$$\frac{\partial}{\partial t}\left(e^{A(t)-rB(t)}\right) = \left(\dot{A}(t) - r\dot{B}(t)\right) e^{A(t)-rB(t)}, \quad = (\dot{A} - r\dot{B})V$$

where $\dot{}$ means d/dt.

$$\frac{\partial}{\partial r}\left(e^{A(t)-rB(t)}\right) = -B(t)e^{A(t)-rB(t)}, \quad = -BV$$

and

$$\frac{\partial^2}{\partial r^2}\left(e^{A(t)-rB(t)}\right) = B(t)^2 e^{A(t)-rB(t)}. \quad = B^2 V$$

Substituting these expressions into the pricing equation for the Vasicek model we get

$$\left(\dot{A}(t) - r\dot{B}(t)\right) e^{A(t)-rB(t)} + \tfrac{1}{2}\beta B(t)^2 e^{A(t)-rB(t)}$$

$$-(\eta - \gamma r)B(t)e^{A(t)-rB(t)} - re^{A(t)-rB(t)} = 0.$$

There is a common factor of $e^{A(t)-rB(t)}$. Divide by that and what is left is linear in r:

$$\underbrace{\left(\dot{A}(t) + \tfrac{1}{2}\,\beta B(t)^2 - \eta B(t)\right)}_{2} + r\,\underbrace{\left(-\dot{B}(t) + \gamma B(t) - 1\right)}_{1} = 0.$$

$\frac{dB}{dt} = \gamma B - 1$; $\frac{dA}{dt} = \eta B - \frac{1}{2}\beta B^2$

Certificate in Quantitative Finance

Both of the expressions in parentheses must be zero.

We have two ordinary differential equations for $A(t)$ and $B(t)$.

In order for the final condition to be satisfied we need

$$V(r,T;T) = e^{A(T)-rB(T)} = 1$$

$$e^{A(T;T)-rB(T;T)}$$

and so

$$A(T) = B(T) = 0.$$

$$\forall r$$

$$A(T) - rB(T) = 0$$

$B(T)=0$

$$\frac{dB}{dt} = \gamma B - 1$$

The solution is

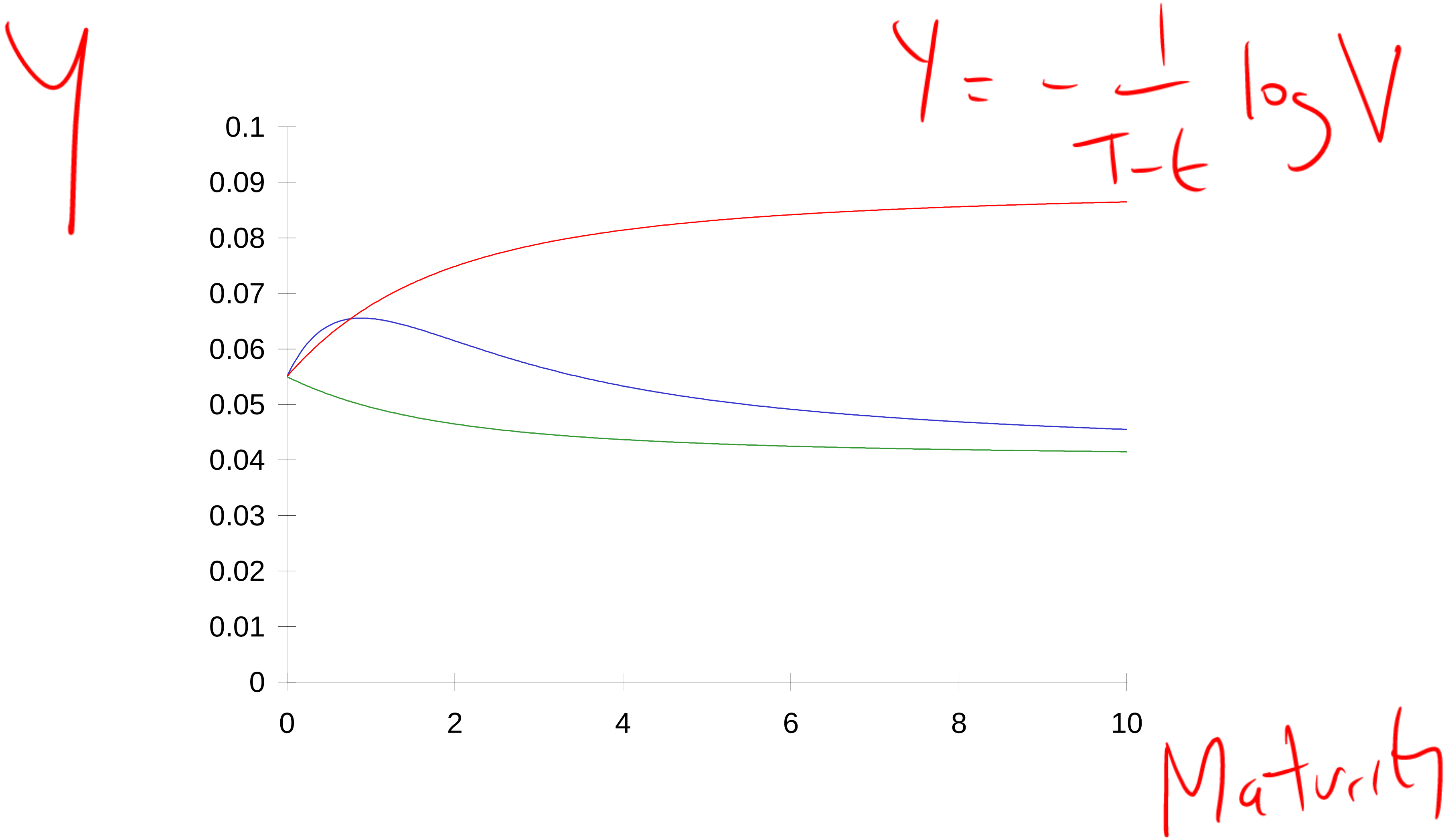
$$\int_t^T \frac{dB}{\gamma B - 1} = \int_t^T d\xi$$

$$B = \frac{1}{\gamma}(1 - e^{-\gamma(T-t)})$$

and

$$A = \frac{1}{\gamma^2}(B(t;T) - T + t)(\eta\gamma - \tfrac{1}{2}\beta) - \frac{\beta B(t;T)^2}{4\gamma}.$$

$e^{A - rB}$



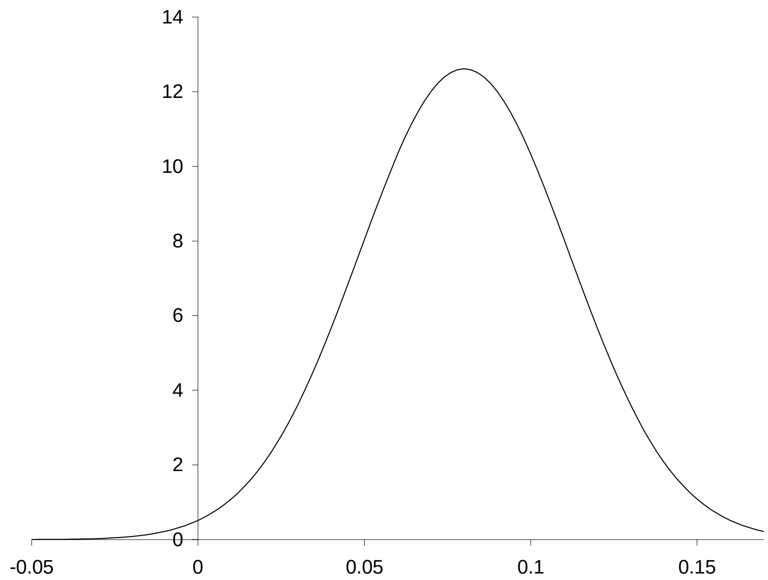
Three types of yield curve given by the Vasicek model.

The Fokker–Planck equation can be used to find the probability distribution for the risk-neutral spot rate.

The steady-state probability density function for the Vasicek model is

- $$P_\infty(r) = \sqrt{\frac{\gamma}{\beta\pi}} e^{-\frac{\gamma}{\beta}\left(r-\frac{\eta}{\gamma}\right)^2}.$$

Thus, in the long run, the spot rate is Normally distributed.



The spot rate can go negative!

Cox, Ingersoll & Ross

The CIR model takes the form

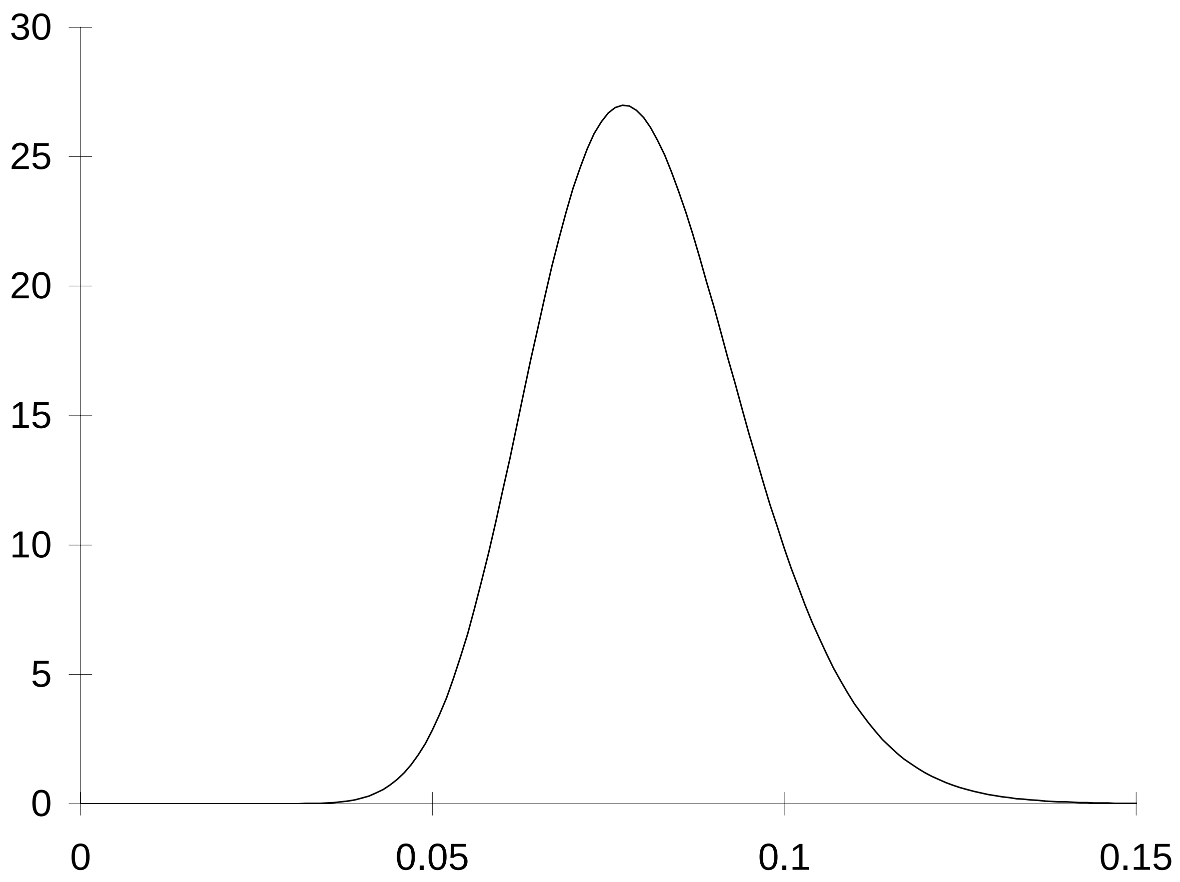
- $$dr = \underbrace{(\eta - \gamma r)}_{u-\lambda w}\, dt + \overbrace{\sqrt{\alpha r}}^{w}\, dX.$$

The spot rate is mean reverting and if $\eta > \alpha/2$ the spot rate stays positive.

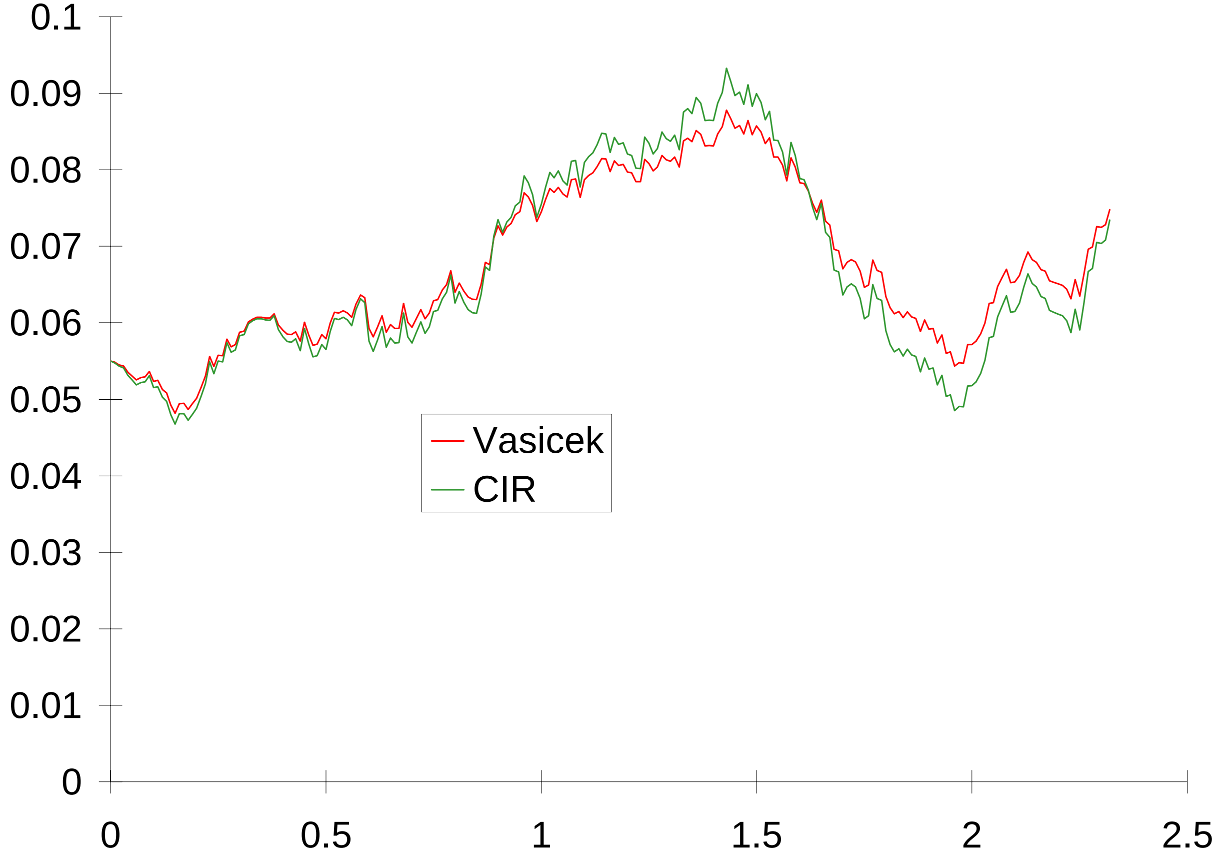
The value of a zero-coupon bond is again of the form

$$e^{A(t)-rB(t)},$$

for different (and more complicated) functions A and B.



The steady-state probability density function for the risk-neutral spot rate in the CIR model.



A simulation of the Vasicek and CIR models using the same random numbers.

Ho & Lee

$u - \lambda w$

Ho & Lee have

A $\quad$ w

- $$dr = \eta(t)dt + \beta^{1/2}dX.$$

The value of zero-coupon bonds is again given by

$$e^{A(t)-rB(t)}$$

$V = e^{A - rB} = Z_M$

where

$$B = T - t$$

and

$$A = -\int_t^T \eta(s)(T-s)ds + \frac{1}{6}\beta(T-t)^3.$$

This model was the first ‘no-arbitrage model’ of the term structure of interest rates.

By this is meant that the careful choice of the function $\eta(t)$ will result in theoretical zero-coupon bonds prices, output by the model, which are the same as market prices.

- This technique is also called **yield curve fitting**.

This careful choice is

$$\eta(t) = -\frac{\partial^2}{\partial t^2} \log Z_M(t^*; t) + \beta(t - t^*)$$

where today is time $t = t^*$.

In this $Z_M(t^*; T)$ is the market price today of zero-coupon bonds with maturity T.

$$dr = \eta(t)\, dt + \beta^{1/2}\, dX$$

Hull & White

Hull & White have extended both the Vasicek and the CIR models to incorporate time-dependent parameters:

$$1) \quad dr = (\eta(t) - \gamma(t)r)dt + \beta(t)^{1/2}dX$$

Vasicek

$$2) \quad dr = (\eta(t) - \gamma(t)r)dt + \sqrt{\alpha(t)r}\, dX$$

CIR

This time dependence again allows the yield curve (and even a volatility structure) to be fitted.

A more general model

Assume that $u - \lambda w$ and w take the form

$$u(r,t) - \lambda(r,t)w(r,t) = \eta(t) - \gamma(t)r,$$

$$w(r,t) = \sqrt{\alpha(t)r + \beta(t)}.$$

We have chosen u and w in the stochastic differential equation for r to take special functional forms for a very special reason.

With these choices the solution for the zero-coupon bond is of the simple form

- $$V(r,t) = e^{A(t)-rB(t)}. \qquad (2)$$

We will see how this works in a moment but first let's look at the properties of this random walk.

By suitably restricting these time-dependent functions, we can ensure that the random walk for r has the following nice properties:

- Positive interest rates

- Mean reversion

Positive interest rates:

Except for a few pathological cases interest rates are positive.

With the above model the spot rate can be bounded below by a positive number if $\alpha(t) > 0$ and $\beta \leq 0$. The lower bound is $-\beta/\alpha$.

Note that r can still go to infinity, but with probability zero.

Mean reversion:

Examining the drift term, we see that for large r the (risk-neutral) interest rate will tend to decrease towards the mean, which may be a function of time.

When the rate is small it will move up on average.

We also want the lower bound to be non-attainable, we don't want the spot interest rate to get forever stuck at the lower bound or have to impose further conditions to say how fast the spot rate moves away from this value.

This requirement means that

$$\eta(t) \geq -\beta(t)\gamma(t)/\alpha(t) + \alpha(t)/2.$$

- The model with all of α, β, γ and η non-zero is the most general stochastic differential equation for r which leads to a solution of the form (2).

Let's see how this works.

e^{A-rB} into B.P.E

Substitute (2) into the pricing equation (1). This gives

$$\frac{\partial A}{\partial t} - r\frac{\partial B}{\partial t} + \tfrac{1}{2}w^2B^2 - (u - \lambda w)B - r = 0. \qquad (3)$$

Some of these terms are functions of t and T (i.e. A and B) and others are functions of r and t (i.e. u and w).

Differentiating (3) with respect to r gives

$$-\frac{\partial B}{\partial t} + \tfrac{1}{2}B^2\frac{\partial}{\partial r}(w^2) - B\frac{\partial}{\partial r}(u - \lambda w) - 1 = 0.$$

Differentiate again with respect to r and divide through by B:

t, T

$$\tfrac{1}{2}B\frac{\partial^2}{\partial r^2}(w^2) - \frac{\partial^2}{\partial r^2}(u - \lambda w) = 0.$$

r, t

$w^2 = \alpha r + \beta$ $\qquad$ $u - \lambda w = \eta - \gamma r$

In this, only B can depend on the bond maturity T, therefore we must have

$$1 \quad \frac{\partial^2}{\partial r^2}(w^2) = 0, \quad 2 \quad -\frac{\partial^2}{\partial r^2}(u - \lambda w) = 0.$$

Therefore $u - \lambda w$ and w must be linear in r as proposed.

$1: \frac{\partial^2}{\partial r^2}(w^2) = 0$ $\qquad$ $\int: \frac{\partial}{\partial r}(w^2) = \alpha(t)$

$\int: w^2 = \alpha(t) r + \beta(t)$

$2 \; -\frac{\partial^2}{\partial r^2}(u - \lambda w) = 0$ $\qquad$ $\int: +\frac{\partial}{\partial r}(u - \lambda w) = -\gamma(t)$

$\int \; (u - \lambda w) = \eta(t) - \gamma(t) r$

The equations for A and B are

$$\frac{\partial A}{\partial t} = \eta(\cancel{t})B - \tfrac{1}{2}\beta(\cancel{t})B^2 \tag{4}$$

and

$$\frac{\partial B}{\partial t} = \tfrac{1}{2}\alpha(\cancel{t})B^2 + \gamma(\cancel{t})B - 1. \tag{5}$$

In order to satisfy the final data that $Z(r, T; T) = 1$ we must have

$$A(T; T) = 0 \quad \text{and} \quad B(T; T) = 0.$$

Solution for constant parameters

The solution for arbitrary α, β, γ and η is found by integrating the two ordinary differential equations (4) and (5).

(Generally speaking, though, when these parameters are time dependent this integration cannot be done explicitly.)

The simplest case is when α, β, γ and η are all constant. Then

$$\frac{\alpha}{2}A = a\psi_2 \log(a-B) + (\psi_2 + \tfrac{1}{2}\beta) b \log((B+b)/b) - \tfrac{1}{2}B\beta - a\psi_2 \log a,$$

and

$$B(t;T) = \frac{2(e^{\psi_1(T-t)} - 1)}{(\gamma + \psi_1)(e^{\psi_1(T-t)} - 1) + 2\psi_1},$$

where

$$b, a = \frac{\pm\gamma + \sqrt{\gamma^2 + 2\alpha}}{\alpha},$$

and

$$\psi_1 = \sqrt{\gamma^2 + 2\alpha} \quad \text{and} \quad \psi_2 = \frac{\eta - a\beta/2}{a+b}.$$

When all four of the parameters are constant it is obvious that both A and B are functions of only the one variable $\tau = T - t$, and not t and T individually; this would not necessarily be the case if any of the parameters were time dependent.

A wide variety of yield curves can be predicted by the model. As $\tau \to \infty$,

$\tau = T - t$

$$B \to \frac{2}{\gamma + \psi_1}$$

and the yield curve Y has long term behavior given by

$$Y \to \frac{2}{(\gamma + \psi_1)^2} \left(\eta(\gamma + \psi_1) - \beta \right).$$

Thus for constant and fixed parameters the model leads to a fixed long-term interest rate, independent of the spot rate.

The probability density function, $P(r,t)$, for the risk-neutral spot rate satisfies

$$\frac{\partial P}{\partial t} = \tfrac{1}{2}\frac{\partial^2}{\partial r^2}(w^2 P) - \frac{\partial}{\partial r}((u - \lambda w)P).$$

In the long term this settles down to a distribution, $P_\infty(r)$, that is independent of the *initial* value of the rate.

This distribution satisfies the ordinary differential equation

- $$\tfrac{1}{2}\frac{d^2}{dr^2}(w^2 P_\infty) = \frac{d}{dr}((u - \lambda w)P_\infty).$$

The solution of this for the general model with constant parameters is

$$P_\infty(r) = \frac{\left(\frac{2\gamma}{\alpha}\right)^k}{\Gamma(k)} \left(r + \frac{\beta}{\alpha}\right)^{k-1} e^{-\frac{2\gamma}{\alpha}(r+\frac{\beta}{\alpha})}$$

where

$$k = \frac{2\eta}{\alpha} + \frac{2\beta\gamma}{\alpha^2}$$

and $\Gamma(\cdot)$ is the gamma function. The boundary $r = -\beta/\alpha$ is non-attainable if $k > 1$.

The mean of the steady-state distribution is

$$\frac{\alpha k}{2\gamma} - \frac{\beta}{\alpha}.$$

Multi-factor interest rate modeling

The simple one-factor stochastic spot interest rate models cannot hope to capture the rich yield-curve structure found in practice: from a given spot rate at a given time they will predict the whole yield curve.

Generally speaking, the one source of randomness, the spot rate, may be good at modeling the overall level of the yield curve but it will not necessarily model shifts in the yield curve that are substantially different at different maturities.

For some instruments this may not be important. For example, for instruments that depend on the *level* of the yield curve it may be sufficient to have one source of randomness, i.e. one factor.

More sophisticated products depend on the difference between yields of different maturities and for these products it is important to model the tilting of the yield curve.

One way to do this is to invoke a second factor, a second source of randomness.

Theoretical framework

Assume that simple interest rate depend on two variables r, the spot interest rate, and another independent variable l where

$V(r, l, t; T)$

$$dr = u\,dt + w\,dX_1$$

$dX_1\,dX_2 = \rho\,dt$

and

$$dl = p\,dt + q\,dX_2.$$

Simple instruments will then have prices which are functions of r, l and t, $V(r, l, t)$.

All of u, w, p and q are allowed to be functions of r, l and t. The correlation coefficient ρ between dX_1 and dX_2 may also depend on r, l and t.

Note that we have not said what l is.

It could be another interest rate, a long rate, say, or the yield curve slope at the short end, or the volatility of the spot rate, for example.

We set up the framework in general and look at specific models later.

Since we have two sources of randomness now, in pricing one bond we must hedge with *two* others to eliminate the risk:

$$\circledast \qquad \Pi = V(r,l,t;T) - \Delta_1 V_1(r,l,t;T_1) - \Delta_2 V_2(r,l,t;T_2).$$

$dr^2 = w^2 dt$

$dl^2 = q^2 dt$

$dr\,dl = \rho w q\,dt$

$t \to t + dt$

$d\Pi = dV - \Delta_1 dV_1 - \Delta_2 dV_2$

Itô on $V(r,l,t)$ is

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial r}dr + \frac{\partial V}{\partial l}dl + \frac{1}{2}\frac{\partial^2 V}{\partial r^2}dr^2 + \frac{1}{2}\frac{\partial^2 V}{\partial l^2}dl^2 + \frac{\partial^2 V}{\partial r\,\partial l}dr\,dl$$

The change in the value of this portfolio is given by

$$d\Pi = (\mathcal{L}(V) - \Delta_1\mathcal{L}(V_1) - \Delta_2\mathcal{L}(V_2))\,dt$$
$$+ \underbrace{\left(\frac{\partial V}{\partial r} - \Delta_1\frac{\partial V_1}{\partial r} - \Delta_2\frac{\partial V_2}{\partial r}\right)}_{=0} dr + \underbrace{\left(\frac{\partial V}{\partial l} - \Delta_1\frac{\partial V_1}{\partial l} - \Delta_2\frac{\partial V_2}{\partial l}\right)}_{=0} dl, \quad (6)$$

with the obvious notation for V, V_1 and V_2.

Here

$$\mathcal{L}(V) = \frac{\partial V}{\partial t} + \tfrac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} + \rho w q\frac{\partial^2 V}{\partial r \partial l} + \tfrac{1}{2}q^2\frac{\partial^2 V}{\partial l^2}.$$

$$d\Pi = r\,\Pi\,dt = (*)$$

$$\left[\mathcal{L}(V) - \Delta_1\mathcal{L}(V_1) - \Delta_2\mathcal{L}(V_2)\right]dt = r\left[V - \Delta_1 V_1 - \Delta_2 V_2\right]dt$$

Now choose Δ_1 and Δ_2 to make the coefficients of dr and dl in (6) equal to zero. The corresponding portfolio is risk free and should earn the risk-free rate of interest, r.

We thus have the three equations

$$\frac{\partial V}{\partial r} - \Delta_1 \frac{\partial V_1}{\partial r} - \Delta_2 \frac{\partial V_2}{\partial r} = 0,$$

$$\frac{\partial V}{\partial l} - \Delta_1 \frac{\partial V_1}{\partial l} - \Delta_2 \frac{\partial V_2}{\partial l} = 0$$

and

$$\mathcal{L}'(V) - \Delta_1 \mathcal{L}'(V_1) - \Delta_2 \mathcal{L}'(V_2) = 0$$

where

$$\mathcal{L}'(V) = \mathcal{L}(V) - rV.$$

These are three simultaneous equations for Δ_1 and Δ_2. As such, this system is over-prescribed and for the equations to be consistent we require

$$\det(\mathbf{M}) = 0$$

where

$$\mathbf{M} = \begin{pmatrix} \mathcal{L}'(V) & \mathcal{L}'(V_1) & \mathcal{L}'(V_2) \\ \partial V/\partial r & \partial V_1/\partial r & \partial V_2/\partial r \\ \partial V/\partial l & \partial V_1/\partial l & \partial V_2/\partial l \end{pmatrix}.$$

$$\mathcal{L}'(V) = \alpha \frac{\partial V}{\partial r} + \beta \frac{\partial V}{\partial \lambda}$$

The first row of the matrix **M** is a linear combination of the second and third rows.

We can therefore write

$$\mathcal{L}'(V) = (\lambda_r w - u)\frac{\partial V}{\partial r} + (\lambda_l q - p)\frac{\partial V}{\partial l}$$

where the two functions $\lambda_r(r, l, t)$ and $\lambda_l(r, l, t)$ are the market prices of risk for r and l respectively, and are again independent of the maturity of any bond.

In full, we have

$$\frac{\partial V}{\partial t}+\frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2}+\rho w q\frac{\partial^2 V}{\partial r\partial l}+\frac{1}{2}q^2\frac{\partial^2 V}{\partial l^2}+(u-\lambda_r w)\frac{\partial V}{\partial r}+(p-\lambda_l q)\frac{\partial V}{\partial l}-rV=0. \tag{7}$$

The model for interest rate derivatives is defined by the choices of w, q, ρ, and the risk-adjusted drift rates $u-\lambda_r w$ and $p-\lambda_l q$.

This is yet another parabolic partial differential equation.

It has more variables than other partial differential equations we have seen.

Popular models

In this section we see some popular models.

Most of these models are popular because the pricing equations (7) for these models have explicit solutions.

In these models sometimes the second factor is the long rate and sometimes it is some other, usually unobservable, variable.

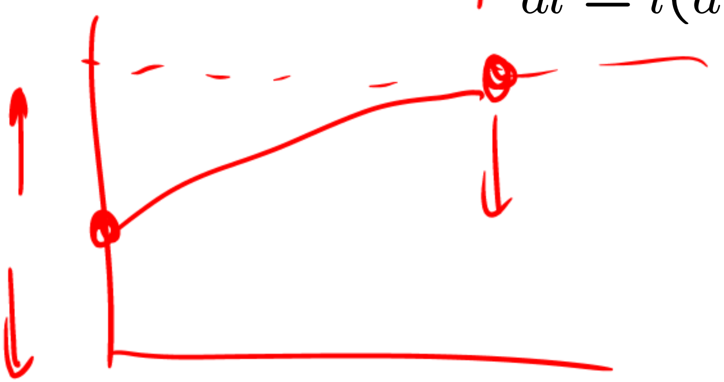
Brennan and Schwartz (1982)

In the Brennan and Schwartz model the risk-adjusted spot rate satisfies

$$dr = (a_1 + b_1(l - r))dt + \sigma_1 r\, dX_1$$

and the long rate satisfies

$$dl = l(a_2 - b_2 r + c_2 l)dt + \sigma_2 l\, dX_2.$$



Brennan and Schwartz choose the parameters statistically.

Because of the relatively complicated functional forms of the terms in these equations there are no simple solutions of the bond pricing equation.

The random terms in these two stochastic differential equations are of the lognormal form, but the drift terms are more complicated than that, having some mean reversion character.

The main problem with the Brennnan and Schwartz model is that it can blow up in a finite time, meaning that rates can go to infinity.

Fong and Vasicek (1991)

Fong & Vasicek consider the following model for risk-adjusted variables:

Vol

$$dr = a(\bar{r} - r)dt + \sqrt{\xi}\,dX_1$$

and

$$d\xi = b(\bar{\xi} - \xi)dt + c\sqrt{\xi}\,dX_2.$$

Thus they model r, the risk-adjusted spot rate, and ξ the square root of the volatility of the spot rate.

The latter cannot be observed, and this is an obvious weakness of the model. But it also makes it harder to show that the model is wrong.

The simple linear mean reversion and the square roots in these equations results in explicit equations for simple interest rate products.

Longstaff and Schwartz (1992)

Longstaff & Schwartz consider the following model for risk-adjusted variables:

$$dx = a(\bar{x} - x)dt + \sqrt{x}\, dX_1$$

and

$$dy = b(\bar{y} - y)dt + \sqrt{y}\, dX_2,$$

where the spot interest rate is given by

$$r = cx + dy.$$

Again, the simple nature of the terms in these equations results in explicit equations for simple interest rate products.

$u - \lambda w = \eta(t) - \gamma(t) r$

$w = \sqrt{\alpha(t) r + \beta(t)}$

General affine model

If r and l satisfy the following:

- the risk-adjusted drifts of both r and l are linear in r and l (but can have an arbitrary time dependence)
- the random terms for both r and l are both square roots of functions linear in r and l (but can have an arbitrary time dependence)
- the stochastic processes for r and l are uncorrelated

then. . .

... the two-factor pricing equation (7) has a solution for a zero-coupon bond of the form

$$V(r, l, t; T) = e^{A(t)-B(t)r-\underline{\underline{C(t)l}}}.$$

The ordinary differential equations for A, B and C must in general be solved numerically.

Summary

Please take away the following important ideas

- interest rates can be modeled as stochastic variables
- whenever a modeled quantity is not traded the pricing equation contains a market price of risk term
- the pricing equation is another partial differential equation, similar in form to the Black–Scholes equation