Lookback Options

In this lecture. . .

- the features that make up a lookback option

- how to put the lookback option into the Black–Scholes framework for both the continuous- and discrete-sampling cases

Introduction

The dream contract has to be one that pays the difference between the highest and the lowest asset prices realized by an asset over some period. Any speculator is trying to achieve such a trade.

- The contract that pays this is an example of a **lookback option**, an option that pays off some function of the realized maximum and/or minimum of the underlying asset over some prescribed period.

We can price these contracts in the Black–Scholes environment quite easily, theoretically. There are two cases to consider, whether the maximum/minimum is measured continuously or discretely.

M - S

M - E

Types of payoff

For the basic lookback contracts, the payoff comes in two varieties, like the Asian option.

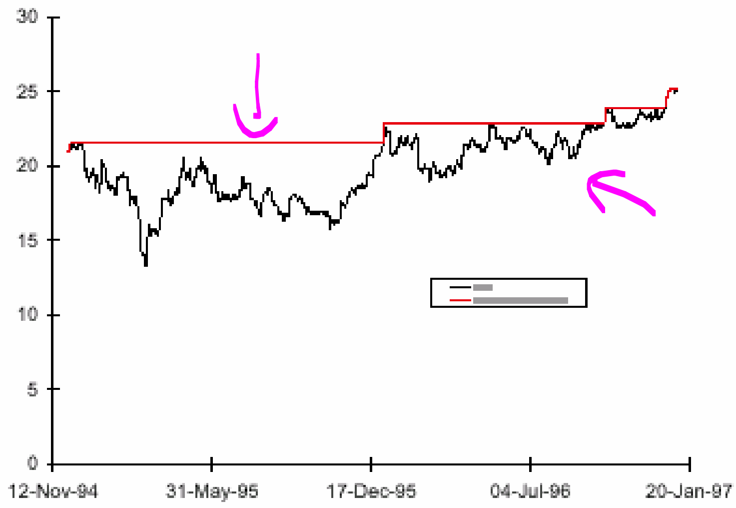
- The *rate* and the *strike* option, also called the **fixed strike** and the **floating strike** respectively.

These have payoffs that are the same as vanilla options except that in the strike option the vanilla exercise price is replaced by the maximum. In the rate option it is the asset value in the vanilla option that is replaced by the maximum.

Continuous measurement of the maximum

Introduce the new variable M as the realized maximum of the asset from the start of the sampling period $t = 0$, say, until the current time t:

$$M = \max_{0 \le \tau \le t} S(\tau).$$



An obvious point about this plot, but one that is worth mentioning, is that the asset price is always below the maximum. (This will not be the case when we come to examine the discretely-sampled case.)

- The value of our lookback option is a function of three variables, $V(S, M, t)$ but now we have the restriction

$$0 \leq S \leq M.$$

This observation will also lead us to the correct partial differential equation for the option's value, and the boundary conditions. We derive the equation in a heuristic fashion that can be made rigorous.

- When $S < M$ the maximum cannot change and so the variable M satisfies the stochastic differential equation

$$dM = 0.$$

- While $0 \leq S < M$ the governing equation must be Black–Scholes

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0,$$

with M as a 'parameter' and only for $S < M$.

- The behaviour of the option value when $S = M$ tells us the *boundary condition* to apply there. The boundary condition is

$$\frac{\partial V}{\partial M} = 0 \quad \text{on} \quad S = M.$$

The reason for this boundary condition is that the option value is insensitive to the level of the maximum when the asset price is *at* the maximum. This is because the probability of the present maximum still being the maximum at expiry is zero.

Finally, we must impose a condition at expiry to reflect the payoff. As an example, consider the lookback rate call option. This has a payoff given by

$$\max(M - E, 0).$$

The lookback strike put has a payoff given by

$$\max(M - S, 0).$$

σ + Math + numerics

Discrete measurement of the maximum

The discretely-sampled maximum is shown below. The asset price go above the maximum. Discrete sampling, as well as being more practical than continuous sampling, is used to decrease the value of a contract.

When the maximum is measured at discrete times we must first define the updating rule, from which follows the jump condition to apply across the sampling dates.

- If the maximum is measured at times t_i then the updating rule is simply

$$M_i = \max(S(t_i), M_{i-1}).$$

- The jump condition is then

$$V(S, M, t_i^-) = V(S, \max(S, M), t_i^+).$$

Note that the Black–Scholes equation is to be solved for all S, it is no longer constrained to be less than the maximum.

Numerics

US Put

Similarity reduction

The general lookback option with a payoff depending on one path-dependent quantity is a three-dimensional problem. The three dimensions are asset price, the maximum and time. The numerical solution of this problem is more time consuming than a two-dimensional problem. However, there are some special, and important, cases when the dimensionality of the problem can be reduced.

This reduction relies on some symmetry properties in the equation and is not something that can be applied to all, or, indeed, many, lookback contracts. It is certainly possible if the payoff takes the form

$$M^{\alpha} P(S/M). \tag{1}$$

For example, this is true for the lookback strike put:

$$\max(M - S, 0) = M \max\left(1 - \frac{S}{M}, 0\right).$$

If the payoff takes the form (1), then the substitution

$$\xi = \frac{S}{M}$$

leads to a problem for $W(\xi, t)$ where

$$V(S, M, t) = M^{\alpha} W(\xi, t)$$

where W satisfies the Black–Scholes equation with final condition

$$W(\xi, T) = P(\xi)$$

and the boundary condition

$$\frac{\partial W}{\partial \xi} - \alpha W = 0 \quad \text{on} \quad \xi = 1.$$