CQF Lecture 5.2 Structural Models

Solutions

1 Merton (1974): Model Calibration

The solution of this problem requires numerical root-finding for a system of two non-linear equations for two uknowns V_0 and σ_V.

$$E_0 = V_0 N(d_1) - De^{-rT} N(d_2) \tag{1}$$

$$\sigma_E E_0 = N(d_1)\sigma_V V_0 \tag{2}$$

The first equation is the expression of the firm's equity as a call option on the value of the assets. The strike for such option is equal to the repayment required by the debt.

Obtaining the second equation is less straightforward. It relies on the assumption that both, the firm's assets V_t and equity E_t are the processes with the same source of randomness dW_t – that is, the same risk factor.

$$dV_t = rV_t dt + \sigma_V V_t dW_t \tag{3}$$

$$dE_t = rE_t dt + \sigma_E E_t dW_t \tag{4}$$

Applying Itô's lemma to SDE (4) for a generalised payoff $G = f(E, t)$ gives the result below.

$$dG = \left(\frac{\partial G}{\partial E} rE + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial E^2}\sigma_E^2 E^2 \right) dt + \frac{\partial G}{\partial E}\sigma_E E dW$$

The similar result can be drawn for SDE (3) and payoff $E = f(V, t)$. Equating diffusion parts gives (under the same risk factor assumption)

$$\begin{aligned} \sigma_E E_t \frac{\partial G}{\partial E} &= \sigma_V V_t \frac{\partial G}{\partial V} \\ \sigma_E E_t &= \frac{\partial E_t}{\partial V_t} V_t \sigma_V \end{aligned}$$

where $\dfrac{\partial G}{\partial V} = \dfrac{\partial G}{\partial E}\dfrac{\partial E}{\partial V}$ by the chain rule and 'delta' $\frac{\partial E_t}{\partial V_t} = N(d_1)$.

The solution for the firm's asset volatility becomes $\sigma_V = \dfrac{1}{N(d_1)}\dfrac{E_t}{V_t}\sigma_E$.

1.1 Part a

Using the parameters from the statement of the problem and using a numerical method to solve the non-linear system, we obtain:

$V_0 = 12.4572$ and $\sigma_V = 17.83\%$. Using these values the probability of default is calculated as $1 - N(d_2) = 7.77\%$ where the meaning of $N(d_2)$ is probability of the call option being in the money (the firm's asset value will be above debt at maturity).

1.2 Part b

The solution for various values of equity volatility is:

- $\sigma_E = 10\%$: $V_0 = 12.5123, \sigma_V = 2.39\%$
- $\sigma_E = 20\%$: $V_0 = 12.5123, \sigma_V = 4.79\%$
- $\sigma_E = 30\%$: $V_0 = 12.5123, \sigma_V = 7.19\%$
- $\sigma_E = 40\%$: $V_0 = 12.5116, \sigma_V = 9.61\%$
- $\sigma_E = 50\%$: $V_0 = 12.5068, \sigma_V = 12.11\%$
- $\sigma_E = 60\%$: $V_0 = 12.4914, \sigma_V = 14.82\%$

2 Default redefined

In this simple exercise, the firm's value process dV can be simulated using the Euler scheme for discretisation of dW as usual. Monte Carlo algorithm allows to check both, the risk-neutral Merton PD as well as conditional Black-Cox PD. To approximate the probability of default, the number of paths going in default is divided by the total number of simulations. Below you can find a Matlab algorithm and some graphs demonstrating the results.

For the inputs given within the code, the estimated PD computed by the Monte-Carlo method is 59% (using 10,000 simulations and 100 timesteps). This can be attributed to the relatively high volatility of the firm's assets $\sigma_V = 40\%$.

2.1 MATLAB Code

```
%%% Monte Carlo Simulation Default Probabilities
%%% Default re-defined: Merton + Black & Cox Models
%%% Alonso Pena,   30.4.2009

clear all; close all;

%------------------------------------------------------------------
% STEP 0: Initialize input data
%------------------------------------------------------------------

V0 = 100;
ru = 0.05;
sigma=0.40;
K = 90; % debt at maturity
D = 80; % debt early value, constant assumption
T = 1;
M=101; % total num timesteps
nsim = 1000;  % total num mc simulations
default_count=0;
dt=T/M;

%%% MAIN Monte Carlo simulations
for i = 1:nsim

default_flag=false;
VOld=V0;

for j=1:M % time integration

%------------------------------------------------------------------
% STEP 1: Generate draw from N(0,1), std normal distribution
%------------------------------------------------------------------
phi = randn;

%------------------------------------------------------------------
% STEP 2: Integrate SDE for one timestep
%------------------------------------------------------------------
VNew = VOld + ru*VOld*dt + sigma*VOld*sqrt(dt)*phi;
V(i,j)=VNew;
```

```
%-----------------------------------------------------------------
% STEP 3: Check if barrier touched (i.e. default) before maturity
%-----------------------------------------------------------------
%%% BLACK-COX
if (VNew<D)
  default_flag=true;
end

VOld=VNew;

end % time integration

VT(i) =VNew;

%-----------------------------------------------------------------
% STEP 4: Check if barrier touched (i.e. default) at maturity
%-----------------------------------------------------------------
%%% MERTON
if VT(i)<K
    default_flag=true;
end

%-----------------------------------------------------------------
% STEP 5: Update default flag for each simulation
%-----------------------------------------------------------------
if  default_flag==true;
 default_count=default_count+1;
end

end % simulations

%-----------------------------------------------------------------
% STEP 6: Plot the mean of distribution ST and percent default
% probability
%-----------------------------------------------------------------
meanVT=mean(VT) % mean of distribution
default_count/nsim % percent defaults

%-----------------------------------------------------------------
%%% STEP 7: figures (optional)
%-----------------------------------------------------------------
figure(1)
plot(1:M,V(1:100,:),'b-','LineWidth',1,'Color','blue')
xlabel('TIME: t');
```

```
ylabel('FIRM VALUE: V(t)');
axis([0 M 0 300])
grid

figure(2)
hist(VT,30)
xlabel('V(T)'); ylabel('count');
% title(parameters)
grid

figure(3)
x = [default_count nsim-default_count];
% explode = [1 0];
% pie3(x,explode)
pie3(x)
title('percent defaults [red] and no default [green] ');
colormap(prism)
```

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2.2 Simulation Results

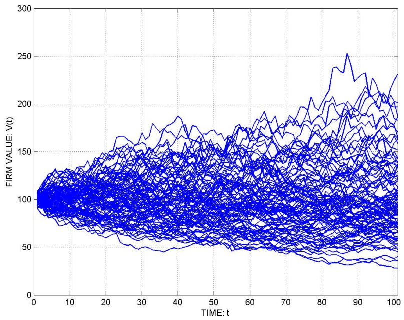

Figure 1: 100 Monte Carlo Paths

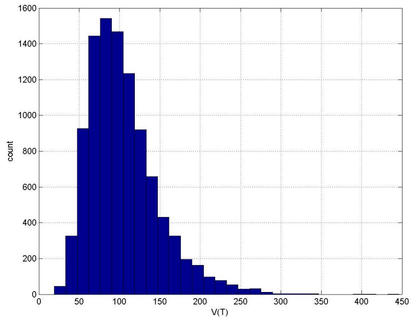

Figure 2: Histogram of $V(T)$

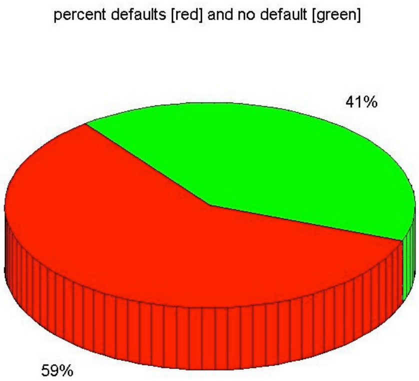

Figure 3: Defaults in MC Paths