Credit Risk Lecture Notes

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Credit Risk

Act I: An Introduction to Credit Risk

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1 Introduction

Credit. The short, simple word represents a powerful financial concept. Credit stands for the idea that a person or company can use somebody else's money to support their own finances. Modern society has developed around the idea of credit, because it enables people to have or invest in the things they want today, but can't afford to pay for until tomorrow - possibly as a result of the initial investment itself. It can be said that to a large degree, credit is the oil that greases the machinery of the world economy. And credit risk? Well, before answering that question we must first delve into the credit markets, its instruments, its participants, and methodologies, in order to have the necessary information. In this introductory lecture we overview these topics and conclude with an attempt to define credit risk.

In the following we follow closely the exposition of Chacko G, et al.(2006) and Georgakopoulos (2004).

1.1 Types of Credit

Credit can come in several forms. Three of the ways that a person or company can use somebody else's money to support their own finances.

Currency. A loan is just one type of credit. There is another much more common type of credit that we all enjoy often without thinking about

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it: currency. We all seem to agree, for instance, that a 100 bill is worth exactly 100, although the bill itself, as the saying goes, is not worth the paper it is printed on. However, the value of the bill is backed by the country's credit. It enjoys the full faith and credit of the country's government. The government sustains the value of the currency and people use it based on the country's credit or sovereign debt.

Loans. In the most typical case, the lender (a bank, a company, or an individual) gives money or property to a borrower. The borrower agrees to return the fund or property at some future point in time, and for the use of the money, the borrower has to pay the lender a fee or interest.

Bonds. Debt obligations are a way for both companies and governments to raise money. They issue debt in the form of bonds, whidh are purchased be institutions and individuals. The bond issuer promise to return the initial sum or principal at a determined future date, otherwise known as the expiration or maturity date. The issuer also normally agrees to pay interest or the coupon at a fixed or floating rate on various, fixed dates.

In its simplest form, a bond is a contract between a lender and a bor rower under which the borrower promisa to repay a loan with interest. Bonds, however, come with many additional features, based on who issued the bond, for how long the bond is valid, what type of coupor rate is used, and if there are any redemption features attached to the bond.

Naturally the issuer of the bond is a major determinant of a bond's expected return and risk. It is easy to understand that companies are seen as riskier. Providers of debt than governments: companies are are likely to go bankrupt than countries. In the international debt markets, securities issued by the US. government are considered to have the lowest default risk of all. It is seen as so low that US. Treasury Bonds are generally referred to as risk-free bonds or default-free bonds.

Government bonds go by different names depending on their origin. In the US., they are called Treasury Bonds, because they are issued by the US. Treasury. Japanese government bonds are normally referred to by their abbreviation, JGBs, whereas British bonds are called Gilts.

In addition to issuing bonds in their local currency, governments can also issue them in foreign currencies. Examples are Eurobonds (denominated in a currency other than that of the country in which they are issued; note that the notation is related to neither the continent Europe nor the currency euro), foreign bonds (when the issuer is not domiciled in the country in which the bond is sold and traded), and global bonds (which are both foreign bonds and Eurobonds).

As mentioned previously, bonds are loans, and like most loans they have a final expiration date, or a term-to-maturity. At this date, the issuer redeems the bond buyer by paying back the principal investment (also known as retiring the bond). In the case of government debt, bonds with a maturity shorter than one year are generally considered short term. Bonds with a maturitybetween 1 and 10 years are viewed as intermediate term, and long-term bonds are those with a maturity of 10 years or more. Using the US. as an example, short-term bonds are known as Treasury Bills (or T-Bills), intermediate-term papers as Treasury Notes, and long-term debt as Treasury Bonds.

In the case of corporate debt, short-term bonds between 2 and 270 days are usually referred to as commercial papers, whereas intermediate and long-term bonds are simply known as corporate bonds.

In addition to returning the principal at maturity, the bond issuer normally compensates the buyer by periodic interest payments, known as coupon payments. Note that if the bond does not make coupon payments, and only pays back the principal value at maturity, it is known as a zero-coupon bond. Most coupons are based on a fixed rate, although floating rates can also be used, where the coupon rate varies according to the movements of an underlying benchmark such as the LIBOR.

Finally, bonds can come with specific redemption features. For example, they can be callable (the bond issuer has the right to redeem, or call back, the bond prior to its maturity date, normally for a higher price) or puttable (the opposite of callable, meaning the bondholders have the right to sell their bonds back to the issuer at a predetermined price). Convertible bonds also exist, which give bondholders the right but not the obligation to convert their bonds into a predetermined number of equity shares at or prior to the bond's maturity.

Credit derivatives. Credit derivatives are a derivative security that has a payoff which is conditioned on the occurrence of a *credit event*. The credit event is defined with respect to a *reference credit* (or several reference credits), and the *reference credit assets* issued by the reference credit. If the credit event has occurred, the *default payment* has to be made by one of the counterparties. Besides the default payment, a credit derivative can have further payoffs that are not default contingent.

The market for credit derivatives was created in the early 1990s in London and New York. The largest share in the market is taken up by the credit default swaps (CDSs) and their variations such as first-to-default swaps (FtDs). The second largest group are portfolio-related credit derivatives like collateralized loan obligations (CLOs), portfolio tranche protection and synthetic collateralized debt obligations (CDOs). Finally, there are more exotic credit derivatives like credit spread options and hybrid instruments.

Participants in the market for credit derivatives are mostly banks, motivated by regulatory capital arbitrage, funding arbitrage and trading motives. Insurances and reinsurances and investment funds also have a large market share and often are the ultimate suppliers of credit protection to the market:

they use credit derivatives as investments, or for the credit risk management of bond portfolios. Hedge funds are entering the credit derivatives business in increasing numbers because of the opportunities to gain on relative value trades between different markets and by the high leverage that many credit derivatives transactions allow.

1.2 Who Defaults?

As we have already stated, not paying back your loan is known as defaulting on the loan. As history shows us, it can happen to all types of borrowers: individuals, companies, and governments alike.

Individuals. Excessive credit card use, poor investment management, and lowered real estate value are common causes for bringing individuals to the brink of personal bankruptcy.

Companies. Just like individuals, companies file for bankrupcy when their costs exceed their revenues and available capital. One of the sure signs of an upcoming corporate default is when a firm stops paying coupons on the bonds they have issued. Unable to meet this financial obligation, the actual default is not far away.

Countries. For the most part, bonds issued by governments are immune from default. If the government needs more money to pay its debt, it can just print more. However, municipalities and sometimes countries do occasionally default on their debt. For instance, in 1978, the city of Cleveland, Ohio defaulted on 15 million in loans it owed to six different banks, after refusing to sell dhe city's electric plant to solve its acute cash shortage. On a national level, Mexico suspended all its debt payments in 1914, and then again in 1982. More recendy, Russia defaulted on loans in 1998, Turkey in 2001, and Argentina in 2002, in what was the biggest sovereign default in history at 141 billion of public debt owed to national and international banks and financial institutions. In situations like these, the International Monetary Fund (IMF) typically works with the country government to develop a partial repayment plan.

1.3 What Causes Defaults?

Not having enough money to pay your loans is an easy-to-understand reason for defaulting on a loan. Any obligor might default for this reason. There are several other so-called credit events that might lead to default. Typical credit events indude:

- Bankruptcy, when a company or organization is dissolved or becomes insolvent and is unable to pay its debts.

- Failure to pay within a reasonable amount of time after the due date and after reminders from the receiver.
- Significant downgrading of credit rating.
- Credit event after merger, which renders the new merged entity financially weaker than the original entity.
- Government action or market disruptions, typically confiscation of assets or effects of wars.

Credit events are often categorized as being driven by either market risk or company-specific risk. Market risk can be a change in overall interest rate levels or industry dynamics; company-specific risk relates to events concerning only the firm itself.

Among events that do not qualify as credit events are falling share prices, smaller than anticipated share dividends, non-significant reductions in the company's credit rating, failures by the company to pay for products or services, or accidental failures by the company to make a payment on time, provided the payment is eventually made within a suitable time period.

1.4 Default Process

Although a country can default on selected loans without declaring bankruptcy (as Argentina did in 2002) most companies that default on a bond almost automatically go into full bankruptcy. When a company declares bankruptq and defaults on all its due loans and credits, the liquidation process gathers whatever can be saved in the form of financial assets. How much that can be gathered relative to all outstanding debt is known as the recovery rate. All debt bank loans, bonds, credit lines, and so on is then ranked by seniority to decide which debtors to pay back first. The debt is traditionally broken up into two major parts: senior and junior debt, with senior debt ranked ahead of junior. For any new debt contract, such as a bond, a company is required to indicate if the new debt is junior or senior to already outstanding debt. Creditors with junior debt do not get paid until the senior debt holders have been paid in full. Senior corporate bonds thus carry less risk for investors than junior bonds, but also have a lower profit potential. An investor that plans to purchase a bond from an issuer that issues several hopes of bonds needs to consider the seniority of each bond, as this affects the recovery rate.

If bankruptcy actually takes place, debt holders have priority over stock and equity holders. The company's suppliers and providers should be paid first, and only after that should the company's owners be given whatever might be left. This rhymes with the general investment philosophy that investing in a company's equity (its stock) comes with higher risk and return than investing in its debt such as bonds.

The seniority of debt and the resulting cascade of cash flow are often referred to as the *debt waterfall* .

2 Modeling Credit Risk: A Guided Tour

In this section we briefly review the mathematical modeling approaches used to quantify credit risk. These can be divided into (a) Traditional Approaches, and (b) Modern Approaches.

2.1 Traditional Approaches

Traditional methods try to estimate the probability of default (denoted PD), rather than the potential losses in the event of default (denoted LGD). Furthermore, these models typically specify bankruptcy filing, default, or liquidation, thereby ignoring consideration of the downgrades and upgrades in credit quality that are measured in mark to market models. The three broad categories of traditional models used to estimate the probability of default are:

- Expert systems
- Rating systems
- Credit scoring models

Expert Systems. Historically, bankers have relied on expert systems to assess credit quality. These are based on, the Character (reputation), the Capital (leverage), the Capacity (earnings volatility), the Collateral, and the Cycle (macroeconomic) conditions. Evaluation of these variables is performed by human experts, who may be inconsistent and subjective in their assessments. Moreover, traditional expert systems specify no weighting scheme that would order these systems in terms of their relative importance in forecasting the probability of default.

Rating Systems. External credit ratings provided by firms specializing in credit analysis were first offered in the U.S. by Moody's in 1909. Agency ratings are opinions based on extensive human analysis of both the quantitative and qualitative performance of a firm. Companies with agency-rated debt tend to be large and publicly traded. Moody's primary business is providing credit opinions on financial obligations for investors. These ratings are well-accepted by the investment community, and extend not only to commercial firms but municipal, sovereign, and other obligors. The credit opinions are statements about loss given default and default probability, specifically expected loss, and thus act as combined default prediction and exposure models. The Office of the Comptroller of the Currency (OCC) in the U.S. has long required banks to use internal ratings systems to rank the credit quality of loans in their portfolios. However, the rating system has been rather crude, with most loans rated as Pass/Performing and only a minority of loans differentiated according to the four non-performing classifications (listed in order of declining credit quality): other assets especially mentioned (OAEM), substandard, doubtful, and loss. Similarly, the National Association of Insurance Commissioners (NAIC) requires insurance companies to rank their assets using a rating schedule with six classifications corresponding to the following credit ratings: A and above, BBB, BB, B, below B, and default. Many banks have instituted internal ratings systems in preparation for the BIS New Capital Accords.

Credit scoring models. The most commonly used traditional credit risk measurement methodology is the multiple discriminant credit scoring analysis pioneered by Altman (1968). This model is a multivariate approach built on the values of both ratio-level and categorical univariate measures. These values are combined and weighted to produce a credit risk score that best discriminates between firms that default and those that do not. The Z-Score model was constructed using multiple discriminant analysis, a multivariate technique that analyzes a set of variables to maximize the between group variance while minimizing the within group variance. This is a sequential process in which the analyst includes or excludes variables based on various statistical criteria.

In order to arrive at a final profile of variables, the following procedures were utilized: (1)observation of the statistical significance of various alternative functions, including determination of the relative contributions of each independent variable; (2) evaluation of intercorrelations among the relevant variables; (3) observation of the predictive accuracy of the various profiles; and (4) judgment of the analyst. From the original set of 22 variables the final Z-Score model chosen was the following discriminant function of five variables:

$$Z = 0.012X_1 + 0.014X_2 + 0.033X_3 + 0.006X_4 + 0.999X_5 \tag{1}$$

where: X_1 = working capital/total assets, X_2 = retained earnings/total assets,X_3 = earnings before interest and taxes/total assets, X_4 = market value equity/book value of total liabilities, X_5 = sales/total assets, and Z = overall index.

2.2 Modern Approaches

Modern methodologies of credit risk measurement can be divided in two alternative approaches with respect to their relationship with the asset pricing literature of academic finance: the structural approach pioneered by Merton (1974) and a reduced form approach utilizing intensity-based models to estimate stochastic hazard rates, pioneered by Jarrow and Turnbull (1995), Jarrow, Lando, and Turnbull (1997), and Duffie and Singleton (1998, 1999). These two schools of thought propose differing methodologies to accomplish the estimation of default probabilities. The structural approach models the economic process of default, whereas reduced form models decompose risky debt prices in order to estimate the random intensity process underlying default.

Structural Models. Merton (1974) models equity in a firm as a call option on the firm's assets (A) with a strike price equal to the liabilities of the firm (D). If at expiration (coinciding to the maturity of the firm's liabilities - the firms liabilities are assumed to be comprised of pure discount debt instruments) the market value of the firm's assets is greater than the value of its debt, then the firm's shareholders will exercise the option to repurchase the company's assets by repaying the debt. However, if the market value of the firm's assets is less than the value of its debt ($A < D$), then the option will not be exercised and the firm's shareholders will default. Thus, the probability of default until expiration (set equal to the maturity date of the firm's pure discount debt, typically assumed to be one year) is equal to the likelihood that the option will expire unexercised. To determine the probability of default we value the call option. We use an iterative method to estimate the unobserved variables that determine the value of the equity call option. Extensions to the basic Merton setup include: the allowance for default to occur anytime before maturity, a time-dependent barrier, and the addition of jumps to the firm process. See also Black &Cox (1976).

Intensity models. Default occurs after adequate early warning in Merton's structural model. That is, default occurs only after a gradual descent (diffusion) in asset values to the default point (equal to the debt level). This process implies that the probability of default steadily approaches zero as the time to maturity declines, something not observed in empirical term structures of credit spreads. More realistic credit spreads are obtained from reduced form or intensity-based models. That is, whereas structural models view default as the outcome of a gradual process of deterioration in asset values, intensity-based models view default as a sudden, unexpected event, thereby generating probability of default estimates that are more consistent with empirical observations.

In contrast to structural models, intensity-based models do not specify the economic process leading to default. Default is modelled as a point process. Defaults occur randomly with a probability determined by the intensity of a hazard function. Intensity-based models decompose observed credit spreads on defaultable debt to ascertain both the probability of default (conditional on there being no default prior to time t) and the LGD (which is

1 minus the recovery rate). Thus, intensity-based models are fundamentally empirical, using observable risky debt prices (and credit spreads) in order to ascertain the stochastic jump process governing default.

Jarrow and Turnbull (1995) assume that the recovery rate is a known fraction of the bond's face value at maturity date, whereas Duffie and Singleton (1998) assume that the recovery rate is a known fraction of the bond's value just prior to default. In Duffie and Singleton (1999), both PD and LGD are modeled as a function of economic state variables.

3 What is Credit Risk?

Now that we know what credit is, who it is that can default on his credit obligations, and how the default process is carried out, we have the foundations for actually defining credit risk. Using the terrninology we have developed in this lecture we can now say:

Credit risk is the risk of loss arisinng from some credit event with the counterparty.

We have also seen that there are many types of counterparts (individual companies, and sovereign governments) and many types of obligation (customer credit to financial derivatives transactions), which means the credit can take many forms. Common to all forms of credit, though, is the risk of default: that an obligor does not honour his or her payment obligations. As you've noticed, the word default is closely tied to the idea of credit risk, and credit risk is also often referred to as default risk. In the next lectures we will go into the details of how is credit risk modelled from a mathematical point of view.

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Credit Risk
Act II: Intensity Models

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1 Introduction

There are two main routes to modeling default risks the structural approach and the reduced-form or intensity, based approach. In the reduced-form approach, the dynamics of the default are exogenously given by a default rate (intensity). Intensity-based models focus directly on describing the conditional probability of default without the definition of the exact default event. The use of a Poisson process framework to describe default captures the idea that the timing of a default takes the investor by surprise. In this lecture we review the mathematical background necessary to construct this type of models and subsequently apply these ideas to the pricing of bonds and credit default swaps.

In the following we follow closely the exposition of Schönbucher (2003) and Bomfim (2004). The original sources of the models are Jarrow and Turnbull (1995), Jarrow, Lando, and Turnbull,(1997), and Duffie and Lando (2001).

2 Models

All processes and random variables we introduce are defined on a complete filtered probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, where Ω is the set of possible states of nature, the filtration $(\mathcal{F}_t)_{t\geq 0}$ represents the information structure of the

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setup and $\mathbb{P}$ is the probability measure that attaches probabilities to the events in Ω.

2.1 Stopping times

To model the *arrival risk* of a credit event we need to model an unknown random point in time $\tau \in \mathbb{R}_+$. If τ is the time of some event, we want that *at the time of the event it is known that this event has occurred.* This means that at every time t we know if τ has already occurred or not:

$$\{\tau \leq t\} \in \mathcal{F}_t, \, \forall t \geq 0. \tag{1}$$

This property defines the random variable τ as a *stopping time.* Equation (1) says that we can observe the event at the time it occurs. In order to represent a stopping time with a stochastic process, we define its *indicator process* that jumps from zero to one at the stopping time:

$$N_\tau(t) := \mathbf{1}_{\tau \leq t}. \tag{2}$$

For default risk modeling we use the *default indicator function* (the indicator function of the default event) and *survival indicator function* (one minus the default indicator function).

Survival indicator function

$$I(t) = 1_{\{\tau > T\}} = \begin{cases} 1 \; if \tau > T \\ 0 \; if \tau \leq T \end{cases}$$

2.2 The hazard rate

We give now a formal definition of the hazard rate and its connection with the probability of default. Let τ be a stopping time and $F(T) = \mathbb{P}(\tau \leq T)$ its distribution function. Assume that $F(T) < 1 \forall T$ and that $F(T)$ has a density $f(T)$. The *hazard rate function* h of τ is defined as:

$$h(T) := \frac{f(T)}{1 - F(T)}. \tag{3}$$

At later points in time $t > 0$ with $\tau > t$, the *conditional hazard rate* is defined as:

$$h(t,T) := \frac{f(t,T)}{1 - F(t,T)}, \tag{4}$$

where $F(t,T) := \mathbb{P}(\tau \leq T | \mathcal{F}_t)$ is the conditional distribution of τ given the information at time t, and $f(t,T)$ is the corresponding density. The hazard rate of default gives the finest possible resolution of the likelihood of default in an infinitesimally small time interval $[t, t + dt]$:

$$h(t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \mathbb{P}(\tau \leq t + \Delta t | \tau > t). \tag{5}$$

Knowledge of the hazard rate function allows a reconstruction of $F(t)$ and $F(t,T)$:

$$F(t) = 1 - e^{-\int_0^t h(s)ds} \text{ and } F(t,T) = 1 - e^{-\int_t^T h(t,s)ds}. \tag{6}$$

2.3 Point processes

A stopping time is the mathematical description of *one* event, a point process is a generalization to multiple events. A *point process* is a collection of points in time:

$$\{\tau_i,\, i \in \mathbb{N}\} = \{\tau_1, \tau_2, \ldots\}.$$

We assume that we have indexed these points in time in ascending order ($\tau_i < \tau_{i+1}$). We further assume that they are all stopping times, that they are all different and that there is only a finite number of such points over any finite time horizon. The point processes provide a good mathematical framework to analyse several events, i.e multiple defaults. We can turn this collection of time points into a stochastic process using the associated *counting process*:

$$N(t) := \sum_i \mathbf{1}_{\tau_i \le t}.$$

$N(t)$ counts the number of time points of the point process that lie before t. If all τ_i are greater than zero, a sample path of $N(t)$ would be a step function that starts at zero and increases by one at each τ_i. $N(t)$ contains all the information that is contained in the point process $\{\tau_i,\, i \in \mathbb{N}\}$ and viceversa. The advantage of using $N(t)$ is that we now have a *stochastic process.*

2.4 Poisson processes

In this section we introduce a mathematical framework for modeling default times using a Poisson process. Its main property is that the probability of a jump over a small time step is approximately proportional to the length of this time interval.

2.4.1 A model for default arrival risk

The following assumption describes the way in which the default arrival risk is modeled in all intensity-based default risk models. Let $N(t)$ be a counting process [1] with intensity $\lambda(t)$. The time of default τ is the time of the first jump of N, i.e.

$$\tau = \inf\{t \in \mathbb{R}_+ \mid N(t) > 0\}. \tag{7}$$

The survival probabilities in this case are given by:

$$P(t,T) = \mathbb{P}(N(T) - N(t) = 0 | \mathcal{F}_t). \tag{8}$$

[1] A counting process is a non-decreasing, integer-valued process $N(t)$ with $N(0) = 0$.

2.4.2 Construction of a Poisson process

A Poisson process $N(t)$ is an increasing process in the integers $0, 1, 2, 3, \ldots$, and to each of them we associate a the following *times of the jumps* $\tau_1, \tau_2, \tau_3, \ldots$ and the probability of a jump in the next instant. Let us assume that the probability of a jump in the next small time interval Δt is proportional to Δt:

$$\mathbb{P}(N(t+\Delta t) - N(t) = 1) = \lambda \Delta t. \tag{9}$$

Furthermore we suppose that jumps by more than 1 do not occur and that jumps in disjoint time intervals happen independently of each other. So the probability of the process remaining constant (i.e. not jumping) is

$$\mathbb{P}(N(t+\Delta t) - N(t) = 0) = 1 - \lambda \Delta t \tag{10}$$

and over the interval $[t, 2\Delta t]$ is

$$\begin{aligned}\mathbb{P}(N(t+2\Delta t) - N(t) = 0) &= \mathbb{P}(N(t+\Delta t) - N(t) = 0) \cdot \\ &\quad \mathbb{P}(N(t+2\Delta t) - N(t+\Delta t) = 0) = (1-\lambda\Delta t)^2.\end{aligned}$$

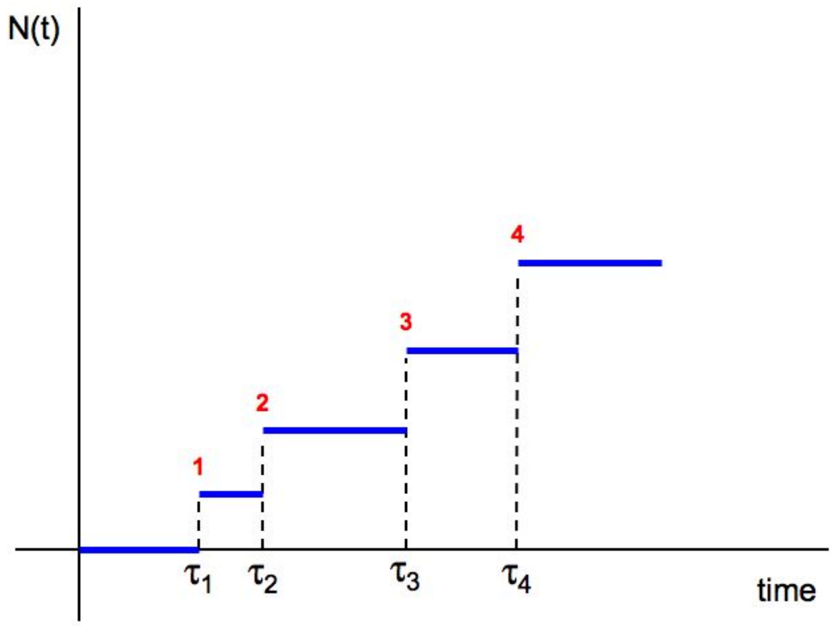

Figure 1: Poisson process.

Now we subdivide the interval $[t, T]$ into n subintervals of length $\Delta t = (T-t)/n$; in each of these subintervals the process N has a jump with probability $\lambda\Delta t$. If we conduct n independent binomial experiments, the probability of *no* jump at all in $[t, T]$ is given by:

$$\mathbb{P}(N(T) = N(t)) = (1 - \lambda\Delta t)^n = \left(1 - \frac{\lambda(T-t)}{n}\right)^n.$$

Because $(1 + x/n)^n \to e^x$ as $n \to \infty$, this converges to:

$$\mathbb{P}(N(T) = N(t)) \to e^{-\lambda(T-t)}.$$

As regards the probability of exactly *one* jump in $[t, T]$, there are n possibilities of having exactly one jump, giving a probability of

$$\begin{aligned}
\mathbb{P}(N(T) - N(t) = 1) &= n \cdot \lambda\Delta t \cdot (1 - \lambda\Delta t)^{n-1} \\
&= n \cdot \lambda \frac{T-t}{n} \cdot \frac{\left(1 - \frac{1}{n}\lambda(T-t)\right)^n}{\left(1 - \frac{1}{n}\lambda(T-t)\right)} \\
&= \frac{\lambda(T-t)}{1 - \frac{1}{n}\lambda(T-t)} \left(1 - \frac{1}{n}\lambda(T-t)\right)^n.
\end{aligned}$$

Again, using the limit result for the exponential function and the fact that the term in the denominator converges to 1 in the limit,

$$\mathbb{P}(N(T) - N(t) = 1) \to \lambda(T-t)e^{-\lambda(T-t)},\ n \to \infty.$$

In the same fashion the limit probabilities of *two* jumps are

$$\mathbb{P}(N(T) - N(t) = 2) = \frac{1}{2}\lambda^2(T-t)^2 e^{-\lambda(T-t)}$$

and for n jumps

$$\mathbb{P}(N(T) - N(t) = n) = \frac{1}{n!}\lambda^n(T-t)^n e^{-\lambda(T-t)}.$$

The above derivation is in fact the main ingredient of a Poisson process. Its formal definition is as follows.

A Poisson process with intensity $\lambda > 0$ is a non-decreasing, integer-valued process with initial value $N(0) = 0$ whose increments are independent and satisfy, for all $0 \leq t < T$,

$$\mathbb{P}(N(T) - N(t) = n) = \frac{1}{n!}\lambda^n(T-t)^n e^{-\lambda(T-t)}. \tag{11}$$

Poisson processes are usually used to model rare events and discretely countable events such as defaults. Usually one models the time of default of a firm as *the time of the first jump of a Poisson process.* The parameter λ in the construction of the Poisson process is called the *intensity* of the process. Other properties of the Poisson process are:

- The Poisson process has no memory. The probability of n jumps in $[t, t+s]$ is independent of $N(t)$ and the history of N before t.
- The inter-arrival times of a Poisson process $(\tau_{n+1} - \tau_n)$ are exponentially distributed with density of the time of the next jump of N given by
$$\mathbb{P}((\tau_{n+1} - \tau_n) \in tdt) = \lambda e^{-\lambda t} dt.$$
- Two or more jumps at exactly the same time have probability zero.

If we use the preceding development we can arrival of default as the first jump of $N(t)$) with a Poisson process with intensity λ, the survival probability can be obtained from equation (11):

$$P(t, T) = e^{-\lambda(T-t)}.$$

2.4.3 Inhomogeneous Poisson processes

If we let the intensity λ of the Poisson process be a function of time $\lambda(t)$, we reach an *inhomogeneous* Poisson process, whose properties are very similar to the homogeneous Poisson process, described above. Starting from the jump probability

$$\mathbb{P}(N(t + \Delta t) - N(t) = 1) = \lambda(t)\Delta t,$$

we calculate the probability of *no* jump in $[t, T]$:

$$\begin{aligned}
\mathbb{P}(N(T) - N(t) = 0) &= \prod_{i=1}^{n} (1 - \lambda(t + i\Delta t)\,\Delta t), \\
\ln \mathbb{P}(N(T) - N(t) = 0) &= \sum_{i=1}^{n} \ln(1 - \lambda(t + i\Delta t)\,\Delta t) \\
&\approx \sum_{i=1}^{n} -\lambda(t + i\Delta t)\,\Delta t \\
&\to -\int_t^T \lambda(s)ds \text{ as } \Delta t \to 0, \\
\mathbb{P}(N(T) - N(t) = 0) &\to \exp\left(-\int_t^T \lambda(s)ds\right) \text{ as } \Delta t \to 0.
\end{aligned}$$

Similarly we can derive the general formula for the probability of n jumps, thus obtaining the following:

An inhomogeneous Poisson process with intensity function $\lambda(t) > 0$ is a non-decreasing, integer-valued process with initial value $N(0) = 0$ whose increments are independent and satisfy

$$\mathbb{P}(N(T) - N(t) = n) = \frac{1}{n!}\left(\int_t^T \lambda(s)ds\right)^n exp\left(-\int_t^T \lambda(s)ds\right). \quad (12)$$

The intensity $\lambda(t)$ is a non-negative function of time only. Again, we can derive the survival probability for an inhomogeneous Poisson process using equation (12):

$$P(t,T) = e^{-\int_t^T \lambda(s)ds}$$

and the corresponding hazard rate of default:

$$h(t,T) = \lambda(T).$$

Now the default hazard rate depends on the time horizon T: the term structure of hazard rates is not flat, but given by $\lambda(T)$, so we can reach every term structure we desire.

3 Example: Bonds

We can now use the intensity framework outlined above to price bonds subject to credit risk. Consider the case of coupon-bearing bonds that have a nonzero recovery value upon default. Let R, $(0 \leq R \leq 1)$, denote the recovery value of the bond. Let us also assume that R is deterministic and that the bond holder receives R on the coupon payment date that immediately follows a default. Let $PD_{[T_{i-1},T_i]}$ denote the (risk-neutral) probability of a default occurring between coupon payment dates T_{i-1} and T_i, based on all information available at time t. Intuitively, it is not hard to see that this probability should be equal to the probability of surviving through time T_{i-1} minus the probability of surviving through T_i, i.e. :

$$PD_{[T_{i-1},T_i]} = P(t,T_{i-1}) - P(t,T_i)$$

In the case that the bond issuer defaults between T_{i-1} and T_i, then the bond holder will receive R at time T_i. What is the time-t value of the recovery payment received on that date? We can think of this payment as the present discounted value of a zero-coupon bond that pays R at time T_i with probability $P(t,T_{i-1}) - P(t,T_i)$ and zero otherwise. Using the risk-neutral valuation framework and $Z(t,T_i)$ as the value of a risk-less zero-coupon bond, the value of such a hypothetical risky bond would be

$$Z(t,T_i)\left[P(t,T_{i-1}) - P(t,T_i)\right]R$$

which is the discounted present value of the recovery payment associated with a default between times T_{i-1} and T_i. For a nonzero recovery bond with N payment dates, coupon C and face value F there are N possible dates for the recovery payment R to take place - each corresponding to a different default scenario. Today's value of the bond recovery payment is the weighted sum of all the recovery payments associated with all possible default scenarios, where the weights are given by the risk-neutral probabilities of each scenario actually taking place, i.e.

$$\sum_{i=1}^{N} Z(t,T_i)\left[P(t,T_{i-1}) - P(t,T_i)\right] R$$

The total value of the bond is just the sum of the present discounted values of the bond's coupon and principal payments, plus the bond's recovery payment:

$$\bar{B}(t,T_N) = \left[\sum_{i=1}^{N} Z(t,T_i)P(t,T_i)C + Z(t,T_N)P(t,T_N)\right] F \tag{13}$$

$$+\sum_{i=1}^{N} Z(t,T_i)\left[P(t,T_{i-1}) - P(t,T_i)\right] R \tag{14}$$

The previous equation can be interpreted as follows: the market value of a coupon-bearing defaultable bond is the probability-weighted sum of the present values of all possible cash flows associated with the bond.

4 Example: Credit Default Swaps (CDS)

In the same manner as with defaultable bonds, we can use the intensity framework to price derivative securities subject to credit risk. In this example we consider the most widespread type. In this section we discuss credit default swaps. An example Term Sheet can be found at the end of the Lecture.

4.1 Structure

A *credit default swap* (CDS) is a credit derivative which enables the investors to isolate the default risk of an obligor. The basic structure is as follows. **B** agrees to pay the default payment to **A** *if a default has happened*. The default payment is structured to replace the loss that a lender would incur upon a credit event of the reference entity **C**. If there is no default of the reference security until the maturity of the default swap, counterparty **B** pays nothing.

Counterparty **A** pays a fee for the default protection. Generally, the fee is a regular fee at intervals until default or maturity. If a default occurs between two fee payment dates, **A** still has to pay the fraction of the next fee payment that has accrued until the time of default. CDSs can differ in the specification of the default payment. Possible alternatives are:

- Physical delivery of one or several of the reference assets against repayment at par;

- Notional minus post-default market value of the reference asset;
- A pre-agreed fixed payoff, irrespective of the recovery rate.

The credit default swap contains a clean isolation of obligor **C**'s default risk. If the protection buyer (**A**) has an underlying exposure to **C**, he holds the market risk, but he is hedged against the default risk, while the protection seller (**B**) can assume the credit risk alone. By changing the set of reference securities in the CDS, the counterparties can agree to focus more on the default risk of an individual bond issued by **C**, or they can widen the coverage to any of **C**'s obligations, thus covering the obligor's default risk completely.

To identify a credit default swap, we need the following information:

1. The reference obligor and his reference assets;
2. The definition of the credit event that is to be insured;
3. The notional of the CDS;
4. The start of the CDS and the start of protection;
5. The maturity date;
6. The CDS spread;
7. The frequency and day count convention for the spread payments;
8. The payment at the credit event and its settlement.

The reference obligor in our case is **C**, his default risk is the object of the CDS contract. It is also necessary to specify a set of reference, usually a set of bond of a given seniority class issued by the reference obligor. Reference assets are necessary to:

- Determine some default events (missed payment on the reference assets);
- Specify the set of deliverable assets in default (for physical delivery);
- Determine a basis for the price and recovery determination mechanism in default (for cash settlement).

The event that is to be insured against is a default of the reference obligor, but, because of the large payments involved, the definition of what constitutes a default has to be made more precise, and a mechanism for the determination of the default event must be given. The standard definition of default includes:

- bankruptcy, filing for protection,

- failure to pay,
- obligation default, obligation acceleration,
- repudiation/moratorium,
- restructuring.

Of these, bankruptcy and filing for protection refer to the reference obligor himself, while the others are defined with respect to the reference obligations.

Notional values of CDSs vary from one million USD up to several hundred million, with smaller sizes for lower credit quality.

The starting date of most CDSs is three trading days after the trading date. We can specify a later starting date and in this case we speak of *forward credit default swaps*.

Most credit default swaps are quoted for a benchmark time-to-maturity of five years, but dealers also quote prices for other times to maturity, ranging from 1 to 10 years.

The credit default swap spread is the *price* of the default protection that has to be paid by the protection buyer to the protection seller. The cash payment amount is the CDS spread multiplied by the notional, adjusted for the day count convention. Typical payment terms are quarterly or semi-annually with an actual/360 day count convention. The first fee is usually payable at the end of the first period and if a default happens between two fee payment dates, the accrued fee up to the time of default must be paid to the protection seller.

4.2 Pricing

In this section we provide a simplified pricing model for credit default swaps.

Let us suppose that there are N periods, indexed by $n = 1, \ldots, N$. Without loss of generality, each period is of length Δt, expressed in units of years. Thus, time intervals are $\{(0, \Delta t), (\Delta t, 2\Delta t), \ldots, ((N-1)\Delta t, N\Delta t)\}$. The corresponding end of period maturities are $T_n = n\Delta t$.

Risk free forward interest rates are denoted $r((n-1)\Delta t, n\Delta t) \equiv r(T_{n-1}, T_n)$, i.e. the rate over the n^{th} period. We write these one-period forward rates in short form as r_n, as the forward rate applicable to the n^{th} time interval. The discount factors may be written as functions of forward rates, i.e.

$$D(0, T_n) = \exp\left(-\sum_{k=1}^{n} r_k \Delta t\right). \qquad (15)$$

For a given obligor, we suppose that default is likely with an hazard rate $\lambda_n \equiv \lambda(T_{n-1}, T_n)$, constant over forward period n. Given these default intensities, the survival function of the obligor is defined as

$$P(T_n) = \exp\left(-\sum_{k=1}^{n} \lambda_k \Delta t\right), \tag{16}$$

assuming that at time zero the obligor is solvent, i.e. $P(T_0) = P(0) = 1$.

In our canonical framework, the buyer of the CDS (**A**) purchases credit protection against the default of the reference security and, in return, pays a periodic payment to the seller (**B**) (*Premium leg*). These periodic payments continue until maturity or until the reference instrument default, in which event the seller pays to the buyer the loss on default of the reference security (*Default leg*).

4.2.1 Premium leg

We denote the $N-$period CDS spread as S_N, stated as an annualized percentage of the nominal value of the contract. Without loss of generality, we set the nominal value to 1 €. We assume that defaults occur only at the end of the period, so the premiums will be paid until the end of the period. Since the premium payments are made as long as the reference security survives, the expected present value of the premiums paid (PL_N) is as follows:

$$PL_N = S_N \Delta t \sum_{n=1}^{N} P(T_{n-1}) D(0, T_n). \tag{17}$$

This accounts for the expected present value of payments made from the buyer **A** to the seller **B**.

4.2.2 Default leg

The other possible payment of the CDS arises in the event of default, and goes from the seller to the buyer. The expected present value of this payment depends on the recovery rate in the event of default, which we denote as R. The loss payment on default is then equal to $(1-R)$ for every 1 € of notional principal. This implicitly assumes that the recovery of par convention is used.

The expected loss payment in period n is based on the probability of default in period n, conditional on no default in a prior period. This probability is given by the probability of surviving until period $n-1$ and then defaulting in period n:

$$P(T_{n-1})(1 - e^{-\lambda_n \Delta t}).$$

Therefore, the expected present value of loss payments (DL_N) equals the following:

$$DL_N = \sum_{n=1}^{N} P(T_{n-1})(1 - e^{-\lambda_n \Delta t}) D(0, T_n)(1 - R). \tag{18}$$

4.2.3 CDS fair spread

The fair pricing of the $N-$period CDS, i.e. the fair quote of the spread S_N, must be such that the expected present value of payments made by buyer and seller are equal, i.e. $PL_N = DL_N$. Thus we obtain

$$S_N = \frac{\sum_{n=1}^{N} P(T_{n-1})(1 - e^{-\lambda_n \Delta t}) D(0, T_n)(1 - R)}{\Delta t \sum_{n=1}^{N} P(T_{n-1}) D(0, T_n)}. \quad (19)$$

4.3 Identifying default hazard rates

In equations (17) and (18) the spread S_N and the discount factors $D(0, T_n)$ are observable in the default risk and government bond markets, respectively. However, the default hazard rates λ_n are not directly observed and need to be inferred from the observable variables.

Since there are N periods, we may use N CDSs of increasing maturity, each with spread S_n, and impose $PL_n = DL_n$, $n = 1, \ldots, N$. Thus we have N equations with as many unknowns, ~~which can be identified~~ in a recursive manner using bootstrapping. Let us ~~detail~~ some of this procedure.

1. Starting with the one-period $(N = 1)$ CDS with a spread S_1 per annum, we equate payments on the swap as follows:

$$\begin{aligned} PL_1 &= DL_1 \\ S_1 \Delta t P(T_0) D(0, T_1) &= \left(1 - e^{-\lambda_1 \Delta t}\right) D(0, T_1)(1 - R) \\ S_1 \Delta t &= \left(1 - e^{-\lambda_1 \Delta t}\right)(1 - R). \end{aligned}$$

   This results in an identification of λ_1, which is:

$$\lambda_1 = -\frac{1}{\Delta t} \ln\left(\frac{1 - R - S_1 \Delta t}{1 - R}\right), \quad (20)$$

   which also provides the survival function for the first period, i.e. $P(T_1) = \exp(-\lambda_1 \Delta t)$.

2. We now use the $2-$period CDS to extract the hazard rate for the second period, whose spread is denoted as S_2. We set $PL_2 = DL_2$ and obtain the following equation which can be solved for λ_2:

$$S_2 \Delta t \sum_{n=1}^{2} P(T_{n-1}) D(0, T_n) = \sum_{n=1}^{2} P(T_{n-1}) \left(1 - e^{-\lambda_n \Delta t}\right) D(0, T_n)(1-R).$$

   Expanding this equation, we have

$$\begin{aligned} S_2 \Delta t [P(T_0) D(0, T_1) + P(T_1) D(0, T_2)] &= P(T_0) \left(1 - e^{-\lambda_1 \Delta t}\right) D(0, T_1)(1 - R) \\ &\quad + P(T_1) \left(1 - e^{-\lambda_2 \Delta t}\right) D(0, T_2)(1 - R). \end{aligned}$$

Rearranging this equation delivers the value of λ_2, i.e.

$$\lambda_2 = -\frac{1}{\Delta t} \ln \left(\frac{L_1}{L_2} \right), \tag{21}$$

where

$$\begin{aligned} L_1 &= P(T_0)\left(1 - e^{-\lambda_1 \Delta t}\right) D(0,T_1)(1-R) \\ &\quad + P(T_1)D(0,T_2)(1-R) - S_2 \Delta t [D(0,T_1) + P(T_1)D(0,T_2)] \\ L_2 &= P(T_1)D(0,T_2)(1-R) \end{aligned}$$

and $P(T_0) = 1$.

3. In general, we can write down the expression for the k^{th} default hazard rate:

$$\lambda_k = -\frac{1}{\Delta t} \ln \left(\frac{P(T_{k-1})D(0,T_k)(1-R) + \sum_{n=1}^{k-1} G_n - S_k \Delta t \sum_{n=1}^{k} H_n}{P(T_{k-1})D(0,T_k)(1-R)} \right), \tag{22}$$

where

$$\begin{aligned} G_n &= P(T_{n-1})\left(1 - e^{-\lambda_n \Delta t}\right) D(0,T_n)(1-R), \\ H_n &= P(T_{n-1})D(0,T_n). \end{aligned}$$

Thus, we begin with λ_1 and, through a process of bootstrapping, we arrive at all λ_n, $n = 1, \ldots, N$.

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Par Credit Default Swap Spread Approximation from Default Probabilities

Purpose: There have been considerable client inquiries on how default probabilities are calculated from credit default swap spreads using our pricing analytic. This is a quantitative process that is not easy to explain intuitively. As such, rather than explain the calculation of default probabilities from credit default swap spreads, this paper focuses on the reverse – approximating par credit default swap spreads from default probabilities.

Definition: A credit default swap is an agreement in which one party buys protection for losses occurring due to a credit event of a reference entity up to the maturity date of the swap. The buyer of protection on a bond, a loan, or a class of bonds or loans, will typically buy protection on the notional of the asset and, upon the occurrence of a credit event, would deliver an obligation of the reference credit in exchange for the protection payout. The protection buyer pays a periodic fee for this protection up to the maturity date, unless a credit event triggers the contingent payment. If such trigger happens, the buyer of protection only needs to pay the accrued fee up to the day of the credit event (standard credit default swap).

Determining the Par Spread: A credit default swap has two valuation legs: fee and contingent. For a par spread, the net present value of both legs must equal to zero.

The valuation of the fee leg is approximated by:

$$PV\ of\ No\ Default\ Fee\ Pmts = S_N \cdot Annuity_N =$$

$$S_N \sum_{i=1}^{N} DF_i \cdot PND_i \cdot \Delta_i$$

where, S_N is the Par Spread for maturity N

DF_i is the Riskless Discount Factor from T_o to T_i

PND_i is the No Default Probability from T_o to T_i

Δ_i is the Accrual Period from T_{i-1} to T_i

If accrual fee is paid upon default, then the valuation of the fee leg is approximated by:

$$PV\ of\ No\ Default\ Fee\ Pmts + PV\ of\ Default\ Accruals =$$

$$S_N \cdot Annuity_N + S_N \cdot Default\ Accrual_N =$$

$$S_N \sum_{i=1}^{N} DF_i \cdot PND_i \cdot \Delta_i + S_N \sum_{i=1}^{N} DF_i \cdot (PND_{i-1} - PND_i) \cdot \frac{\Delta_i}{2}$$

where, $(PND_{i-1} – PND_i)$ is the Probability of a Credit Event occurring during period T_{i-1} to T_i

$\frac{\Delta_i}{2}$ is the Average Accrual from T_{i-1} to T_i

The valuation of the contingent leg is approximated by:

$$PV\ of\ Contingent = Contingent_N =$$

$$(1-R) \sum_{i=1}^{N} DF_i \cdot (PND_{i-1} - PND_i)$$

where, R is the Recovery Rate of the reference obligation

Therefore, for a par credit default swap,

$$Valuation\ of\ Fee\ Leg = Valuation\ of\ Contingent\ Leg$$

or

$$S_N \sum_{i=1}^{N} DF_i \cdot PND_i \cdot \Delta_i + S_N \sum_{i=1}^{N} DF_i \cdot (PND_{i-1} - PND_i) \cdot \frac{\Delta_i}{2} =$$

$$(1-R) \sum_{i=1}^{N} DF_i \cdot (PND_{i-1} - PND_i)$$

or

$$S_N = \frac{(1-R)\sum_{i=1}^{N} DF_i \cdot (PND_{i-1} - PND_i)}{\sum_{i=1}^{N} DF_i \cdot PND_i \cdot \Delta_i + DF_i \cdot (PND_{i-1} - PND_i) \cdot \frac{\Delta_i}{2}}$$

Example:

Recovery 30%

Period (i)	**Yld$_i$**	**DF$_i$**	**PND$_i$**	**Annuity$_N$**	**Default Accrual$_N$**	**Contingent$_N$**	**Approx S$_N$ (A/360)**
0.25	2.35%	99.41%	96.43%	0.240	0.004	0.025	1008
0.5	2.33%	98.84%	93.05%	0.470	0.008	0.048	988
0.75	2.39%	98.25%	89.76%	0.690	0.012	0.071	995
1	2.52%	97.63%	86.56%	0.901	0.016	0.093	998
1.25	2.70%	96.98%	83.91%	1.104	0.019	0.111	972
1.5	2.87%	96.28%	81.51%	1.300	0.022	0.127	945
1.75	3.05%	95.55%	79.41%	1.490	0.025	0.141	915
2	3.22%	94.78%	77.30%	1.673	0.028	0.155	896
2.25	3.37%	93.99%	75.79%	1.851	0.030	0.165	863
2.5	3.52%	93.17%	74.32%	2.024	0.032	0.175	837
2.75	3.67%	92.31%	72.87%	2.192	0.034	0.184	813
3	3.82%	91.44%	71.45%	2.355	0.036	0.193	794
3.25	3.92%	90.55%	70.66%	2.515	0.037	0.198	763
3.5	4.02%	89.64%	69.90%	2.672	0.038	0.203	737
3.75	4.12%	88.72%	69.14%	2.825	0.039	0.208	714
4	4.22%	87.79%	68.37%	2.975	0.040	0.213	695
4.25	4.30%	86.85%	68.22%	3.123	0.040	0.214	665
4.5	4.37%	85.91%	68.06%	3.269	0.040	0.215	639
4.75	4.45%	84.96%	67.91%	3.413	0.040	0.216	615
5	4.52%	84.00%	67.76%	3.555	0.040	0.217	594

Credit Risk
Act III: Structural Models

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1 Introduction

The structural approach to credit risk models says that a firm defaults when the market value of its assets is less than the obligations or debt it has to pay. Structural models are therefore sometimes also referred to as asset value models. These models look at a company's balance sheet and its capital structure to assess its creditworthiness. However, one of the inherent problems with this approach is that the value of a company assets is hard to observe directly. The annual report only provides an accounting version of the company's real assets. However, the market value of a company's stock equity is normally observable, as is its debt. It has been shown that default can be modeled as an option on a the company's debt that means we can use option pricing theory to infer the market value of the firm's assets.This approach of linking option pricing theory with the assessment of risky debt was pioneered by Black and Scholes (1973) and Merton (1974). It is often referred to as the Merton model, after Robert C Merton.

In the following we follow closely the exposition of Giesecke(2004) and Bielecki and Rutkowski (2004).

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2 Models

2.1 Merton Model

The basis of the structural approach, which goes back to Black & Scholes (1973) and Merton (1974), is that corporate liabilities are contingent claims on the assets of a firm. The market value of the firm is the fundamental source of uncertainty driving credit risk.

Consider a firm with market value V , which represents the expected discounted future cash flows of the firm. The firm is financed by equity and a zero coupon bond with face value K and maturity date T. The firm's contractual obligation is to repay the amount K to the bond investors at time T. Debt covenants grant bond investors absolute priority: if the firm cannot honour its payment obligation, then bond holders will immediately take over the firm. Hence the default time τ is a discrete random variable given by

$$\tau = \begin{cases} T\ if\ V_T < K \\ \infty\ otherwise \end{cases}$$

See Figure 1 for an illustration. To calculate the probability of default, we make assumptions about the distribution of assets at debt maturity under probability P. The standard model for the evolution of asset prices over time is geometric Brownian motion:

$$\frac{dV_t}{V_t} = \mu dt + \sigma dW_t \quad for\ V_0 > 0$$

where μ is a drift parameter, σ is a volatility parameter, and W is a standard Brownian motion. The default probabilities $p(T)$ are given by

$$p(T) = P\left[V_T < K\right] = N\left(\frac{\log L - mT}{\sigma\sqrt{T}}\right)$$

where $L = K/V_0$ is the initial leverage ratio and N is the standard normal distribution function and $m = \mu - \frac{1}{2}\sigma^2$.

$$N(x) = \tfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du \quad \forall x \in$$

Thus if the asset value V_T exceeds or equals the face value K of the bonds, the bond holders will receive their promised payment K and the shareholders will get the remaining $V_T - K$. However, if the value of assets V_T is less than K, the ownership of the firm will be transferred to the bondholders, who lose the amount $K - V_T$. Equity is worthless because of limited liability. Summarizing, the value of the bond issue B_T^T at time T is given by

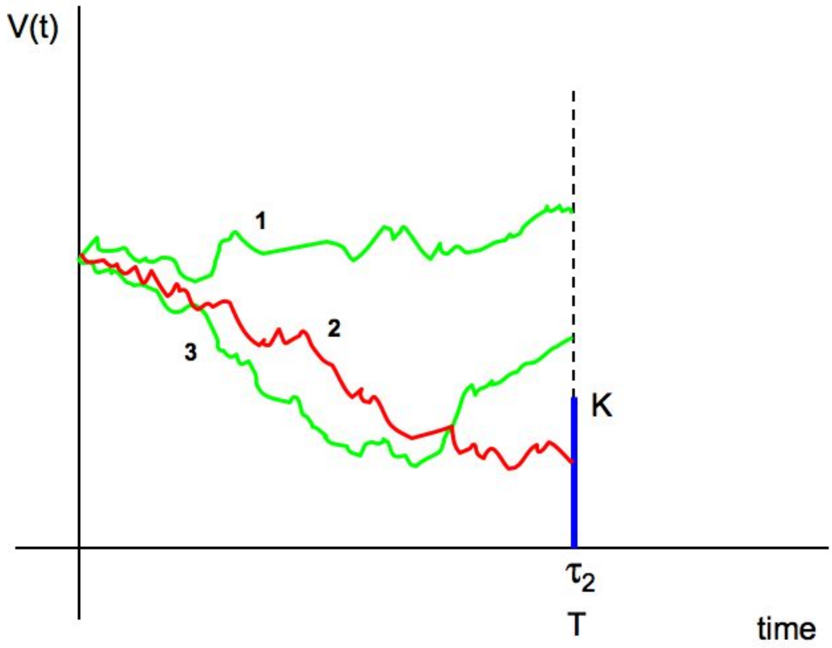

Figure 1: The Merton model.

$$B_T^T = \min(K, V_T) = K - \max(K - V_T, 0)$$

This payoff is equivalent to that of a portfolio composed of a default-free loan with face value K maturing at T and a short European put position on the assets of the firm with strike K and maturity T. The value of the equity E_T at time T is given by

$$E_T = \max(K - V_T, 0)$$

which is equivalent to the payoff of a European call option on the assets of the firm with strike K and maturity T.

Pricing equity and credit risky debt reduces to pricing European options. We consider the classical Black-Scholes setting and that the value of the firm is a traded asset. The equity value is given by the Black-Scholes call option formula C:

$$E_0 = C\left(\sigma, T, K, r, V_0\right) = V_0 N(d_+) - e^{-rT} K N(d_-)$$

where

$$d_\pm = \frac{(r \pm \frac{1}{2}\sigma^2)T - \log L}{\sigma\sqrt{T}}$$

While risk free zero coupon bond prices are just Ke^{-rT} with T being the bond maturity, the value of the corresponding credit-risky bonds is

$$B_0^T = Ke^{-rT} - P\left(\sigma, T, K, r, V_0\right)$$

where P is the Black-Scholes vanilla put option formula. We note that the value of the put is just equal to the present value of the default loss suffered by bond investors. This is the discount for default risk relative to the risk free bond, which is $Kexp(-rT)$. This yields

$$B_0^T = V_0 - V_0 N(d_+) - e^{-rT} K N(d_-)$$

which together with (3) cofirms the market value identity

$$V_0 = E_0 + B_0^T$$

The credit spread is the difference between the yield on a defaultable bond and the yield an otherwise equivalent default-free zero bond. It gives the excess return demanded by bond investors to bear the potential default losses. Since the yield y(t; T) on a bond with price $b(t;T)$ satisfies $b(t;T) = exp(y(t;T)(Tt))$, we have for the credit spread $S(t;T)$ at time t,

$$S(t,T) = \frac{1}{T-t} \log\left(\frac{B_0^T}{\bar{B}_0^T}\right), \quad T > t$$

where $Z_t^T = Ke^{-r(T-t)}$ is the price of a default-free bond maturing at T. The term structure of credit spreads is the schedule of $S(t;T)$ against T. In the Black-Scholes setting, we have $B_0^T = V_0 - V_0 N(d_+) - e^{-rT} K N(d_-)$ and obtain

$$S(0,T) = \frac{1}{T} \log\left(N(d_-) + \frac{1}{L} e^{-rT} N(-d_+)\right), \quad T > 0$$

which is a function of maturity T, asset volatility $sigma$ (the firms business risk), the initial leverage ratio L, and riskfree rates r.

2.2 Black-Cox Model

In the classical approach, the firm value can dwindle to almost nothing without triggering default. This is unfavorable to bondholders, as noted by Black & Cox (1976). Bond indenture provisions often include safety covenants that give bond investors the right to reorganize a firm if its value falls below a given barrier.

Suppose the constant default barrier D . Then the default time τ is a continuous random variable given by

$$\tau = \inf\left\{t > 0 : V_T < D\right\}$$

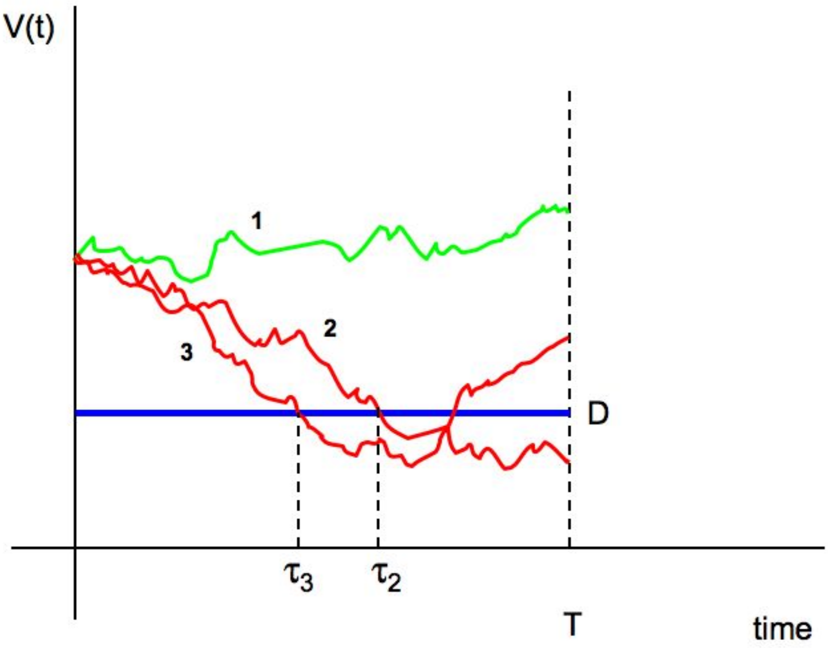

Figure 2: The Black&Cox model.

See Figure 3. In the Black-Scholes setting with asset dynamics (2), default probabilities are calculated as

Since the distribution of the historical low of an arithmetic Brownian motion, it can be shown that

$$p(T) = N\left(\frac{\log\left(\frac{D}{V_0}\right) - mT}{\sigma\sqrt{T}}\right) + \left(\frac{D}{V_0}\right)^{\frac{2m}{\sigma^2}} N\left(\frac{\log\left(\frac{D}{V_0}\right) + mT}{\sigma\sqrt{T}}\right)$$

We check whether this default definition is consistent with the payoff to investors. We need to consider two scenarios. The first is when $D \leq K$. If the firm value never falls below the barrier D over the term of the bond, then bond investors receive the face value $K < V_0$ and the equity holders receive the remaining $V_T - K$. However, if the firm value falls below the barrier at some point during the bond's term, then the firm defaults. In this case the firm stops operating, bond investors take over its assets D and equity investors receive nothing. Bond investors are fully protected: they receive at least the face value K upon default and the bond is not subject to default risk any more.

This anomaly does not occur if we assume $D < K$ so that bond holders are both exposed to some default risk and compensated for bearing that risk.

If the firm has not gone below the barrier and $VT \leq K$, then bond investors receive the face value K and the equity holders receive the remaining $V_T - K$. If the barrier remains untouched but $V_T < K$, then the firm defaults, since the remaining assets are not sufficient to pay for the debt in full. Bond investors collect the remaining assets V_T and equity becomes worthless. If the barrier is breached, then the firm defaults as well. Bond investors receive $D < K$ at default and equity becomes worthless.

However, if the barrier is below the bond's face value, then our earlier definition does not reflect economic reality anymore: it does not capture the situation when the firm is in default because $V_T < K$ although the barrier was never breached. We discuss a remedy to avoid this inconsistency by re-defining default.

2.3 Black-Cox: default redefined

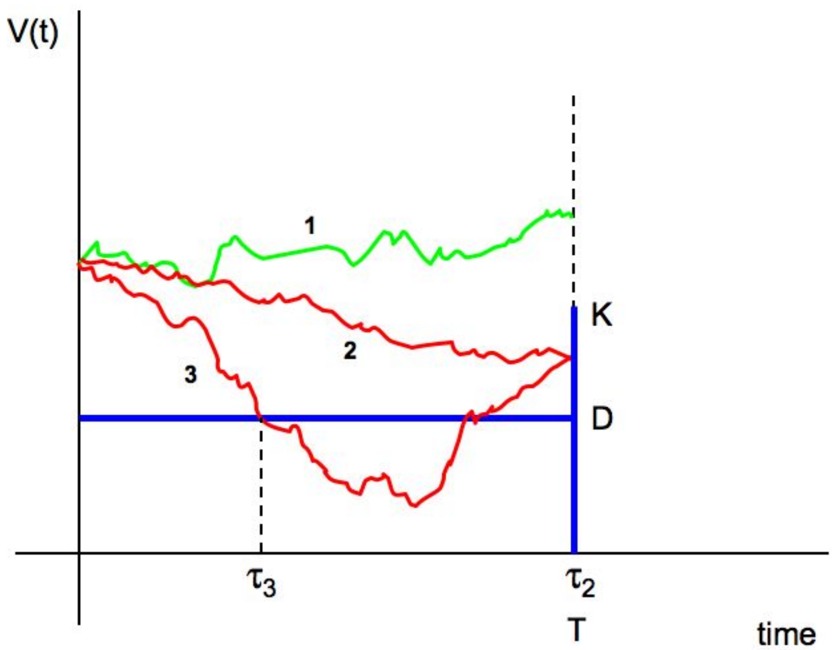

Figure 3: Modified Black-Cox model

We redefine default as firm value falling below the barrier $D < K$ at any time before maturity *or* firm value falling below face value K at maturity. Formally, the default time is now given by

$$\tau = \min\left\{\tau^1, \tau^2\right\}$$

In other words, the default time is defined as the minimum of the Black & Cox default time and Merton's default time. This definition of default is consistent with the payoff to equity and bonds. Even if the firm value does not fall below the barrier, if assets are below the bond's face value at maturity the firm defaults, see Figure 4. We obtain for the corresponding default probabilities

$$p(T) = 1 - P\left[\min\left\{\tau^1, \tau^2\right\} > T\right]$$
$$p(T) = 1 - P\left[\tau^1 > T, \tau^2 > T\right]$$

Using the joint distribution of an arithmetic Brownian and its running minimum, we obtain

$$p(T) = N\left(\frac{\log\left(\frac{D}{V_0}\right) - mT}{\sigma\sqrt{T}}\right) + \left(\frac{D}{V_0}\right)^{\frac{2m}{\sigma^2}} N\left(\frac{\log\left(\frac{D^2}{KV_0}\right) + mT}{\sigma\sqrt{T}}\right)$$

The corresponding payoff to equity investors at maturity is

$$E_T = \max(K - V_T, 0)\ 1_{\{M_T \geq D\}}$$

where 1_A is the indicator function of the event A. The equity position is equivalent to a European down-and-out call option position on firm assets V with strike K, barrier $D < K$, and maturity T. Pricing equity reduces to pricing European barrier options. In the Black-Scholes setting with constant interest rates and standard asset dynamics (2), we found

$$E_0 = C\left(\sigma, T, K, r, V_0\right) - V_0\left(\frac{D}{V_0}\right)^{\frac{2r}{\sigma^2}+1} N(h_+) + Ke^{-rT}\left(\frac{D}{V_0}\right)^{\frac{2r}{\sigma^2}+1} N(h_-)$$

where C is the vanilla call value and where

$$h_{\pm} = \frac{(r \pm \frac{1}{2}\sigma^2)T + \log\left(\frac{D^2}{KV_0}\right)}{\sigma\sqrt{T}}$$

The corresponding payoff to bond investors at maturity is

$$B_T^T = K - (K - V_T)^+ + (V_T - K)^+ 1_{\{M_T \geq D\}}$$

This position is equivalent to a portfolio composed of a risk free loan with face value K maturing at T, a short European put on the firm with strike K and maturity T and a long European down-and-in call on the firm with strike K and maturity T. In the Black&Cox approach bonds are worth at least as much as in the classical approach. In the rst-passage model bond investors have additionally a barrier option on the firm that knocks in if the firm defaults before the maturity T. Correspondingly,

$$B_0^T = Ke^{-rT} - P(\sigma, T, K, r, V_0) + DIC(\sigma, T, K, D, r, V_0)$$

where P is the vanilla put and DIC the down-and-in option value. The combined value of the option positions gives the present value of the default loss suffered by bond investors. We get

$$B_0^T = V_0 - C(\sigma, T, K, r, V_0) + V_0 \left(\frac{D}{V_0}\right)^{\frac{2r}{\sigma^2}+1} N(h_+) + Ke^{-rT} \left(\frac{D}{V_0}\right)^{\frac{2r}{\sigma^2}+1} N(h_-)$$

which again implies the value identity $V_0 = S_0 + B_0^T$.

2.4 Time-varying barrier

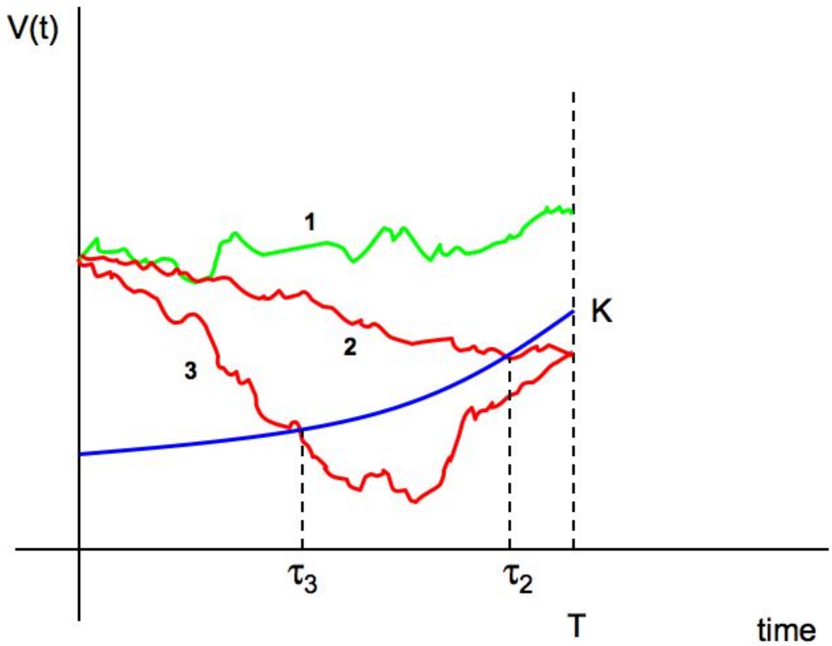

Figure 4: Time-varying barrier

The second way to avoid the inconsistency discussed above is to introduce a time-varying default barrier $D(t) \leq K$ for all $t \leq T$. For some constant $k > 0$, consider the deterministic function

$$D(t) = Ke^{-k(T-t)}$$

which can be thought of as the face value of the debt, discounted back to time t at a continuously compounding rate k. The firm defaults at

$$\tau = \inf\{t > 0 : V_T < D(t)\}$$

Observing that

$$\{V_T < D(t)\} = \{(m-k)t + \sigma W_t < \log L - kT\}$$

we have for the default probability

$$p(T) = P\left[\min_{t \le T}((m-k)t + \sigma W_t) < \log L - kT\right]$$

Now we have reduced the problem to calculating the distribution of the historical low of an arithmetic Brownian motion

$$p(T) = N\left(\frac{\log L - mT}{\sigma\sqrt{T}}\right) + \left(Le^{-kT}\right)^{-\frac{2}{\sigma^2}(m-k)} N\left(\frac{\log L - (m-2k)T}{\sigma\sqrt{T}}\right)$$

The corresponding equity position is a European down-and-out call option on firm assets with strike K, time-varying barrier $D(t)$, and maturity T:

$$E_T = (V_T - K, 0)^+ \ 1_{\{M_T^k \ge D\}}$$

where M_T^k is the running minimum of the firm value with adjusted arithmetic growth rate mk and $D = Kexp(-kT)$. Merton (1973) gives a closed-form expression for E_0. The bond position is given by

$$B_T^T = K - (K - V_T)^+ + (V_T - K)^+ 1_{\{M_T^k \ge D\}}$$

Bond values B_0^T can be calculated in a parallel manner as before.

3 Discussion

The calibration of structural models is problematic, as the value of the firm is not directly observable in the market. The face value and the maturity of the debt is not easy to estimate from the balance sheet given the complexity of the capital structure of company. Indeed, wet often have a mixture of short, medium, and long-term debts as well as different seniorities. The barrier level, in the case of a first-passage approach, is another parameter that is not easy to estimate, and its definition is generally ad hoc conditions the occurrence of default event.

Another drawback of the approaches described above is the fact that the default cannot happen immediately. This has been addressed by random barriers models such as *Credit Grades*. However, this adds even more complexity in terms of the calibration as we need to calibrate the volatility of the barrier level in addition to all other parameters.

Nevertheless, this approach could be useful to provide predictive tools related to upcoming default events. Indeed, a pre-default event could be defined as the first time the asset value is below a certain level higher than, but close to, they default barrier, and users of structural models observe the evolution of the so-sailed 'distance to default" or the marginal default probabilities. From a mathematical point of view, this means that the default time τ is predictable, a stopping-time with respect to the asset filtration. Unfortunately, market reality is different, as we do witness spread movements as well as jumps to default that happen in a surprising way.

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Credit Risk

Act IV: Case Study: The Subprime Crisis

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1 Introduction

In this last lecture we will put together what we have studied so far by applying it to the current Subprime Crisis. We will first briefly define the current situation, read an interpretation of it offered by Richard Bookstaber and discuss in detail the mathematical tools that may help us in analysing the event. We focus on two key tools: the modeling of the correlated defaults and the pricing structure of CDOs. Armed with market data and models we will make a simple, yet realistic analysis the crisis during the lesson.

2 The Problem

The subprime mortgage financial crisis was the sharp rise in foreclosures in the subprime mortgage market that began in the United States in 2006 and became a global financial crisis in July 2007. Rising interest rates increased newly-popular adjustable rate mortgages and property values suffered declines from the demise of the housing bubble, leaving home owners unable to meet financial commitments and lenders without a means to recoup their losses. Many observers believe this has resulted in a severe credit crunch,threatening the solvency of a number of marginal private banks and other financial institutions.The sharp rise in foreclosures after the housing bubble caused several major subprime mortgage lenders, such as New Century Financial Corporation, to shut down or file for bankruptcy, with some

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accused of actively encouraging fraudulent income inflation on loan applications, leading to the collapse of stock prices for many in the subprime mortgage industry, and drops in stock prices of some large lenders like Countrywide Financial[1].This has been associated with declines in stock markets worldwide, several hedge funds becoming worthless, coordinated national bank interventions, contractions of retail profits,and bankruptcy of several mortgage lenders.Observers of the meltdown have cast blame widely. Some, like Senate Banking, Housing, and Urban Affairs Committee chairman Chris Dodd of Connecticut, have highlighted the predatory lending practices of subprime lenders and the lack of effective government oversight [2]. Others have charged mortgage brokers with steering borrowers to unaffordable loans, appraisers with inflating housing values, and Wall Street investors with backing subprime mortgage securities without verifying the strength of the portfolios. Borrowers have also been criticized for over-stating their incomes on loan applications [3] and entering into loan agreements they could not meet [4]. The effects of the meltdown spread beyond housing and disrupted global financial markets as investors, largely deregulated foreign and domestic hedge funds, were forced to re-evaluate the risks they were taking and consumers lost the ability to finance further consumer spending, causing increased volatility in the fixed income, equity, and derivative markets.

3 An Interpretation

Blowing up the Lab on Wall Street [1]

Looks like Wall Street's mad scientists have blown up the lab again. The subprime mess that is cutting so wide a swath through financial markets can be traced to the alchemy of creating collateralized debt obligations (CDOs) compounded by the enormous amount of leverage applied by big hedge funds. CDOs are derivatives are synthetic financial instruments derived from another asset.

Here's the recipe for a CDO: you package a bunch of low-rated debt like subprime mortgages and then break the package into pieces, called tranches. Then, you pay to play. Some of the pieces are the first in line to get hit by any defaults, so they offer relatively high yields; others are last to get hit, with correspondingly lower yields. The alchemy begins when rating agencies such as Standard & Poor's and Fitch Ratings wave their magic wand over these top tranches and declare them to be a golden AAA rated. Top shelf. If you want to own AAA debt, CDOs have been about the only place to go; hardly any corporation can muster the credit worthiness to garner an AAA

[1] This section is taken verbatim from the article by Richard Bookstaber, *Time Magazine* , Thursday, Aug. 16, 2007, available on http://www.time.com/time/business/article/0,8599,1653556,00.html

rating anymore. Here's where the potion gets its poison potential.

Some individual parts of CDOs are about as base as bonds while others are not even investment grade. The assumption has been that even if the toxic waste bonds really stink, the quality tranches can keep the CDO above water. And life goes on.

The problem is that CDOs were untested; there was not much history to suggest CDOs would behave the same way as AAA corporate bonds. After all, the last few stress-free years have not exactly provided much of a testing ground for what can go wrong until, that is, subprime mortgages started their death march. Suddenly, investors realized things can actually head south in a big way, even stuff completely unrelated to CDOs. Like your stocks.

It's not the first time this has happened, yet Wall Street still isn't getting the message. One August day nine years ago, Russian bonds defaulted. A surprising result of this default was the spectacular failure of Long-Term Capital Management (LTCM), a hedge fund in Greenwich, Conn. Surprising because LTCM had nary a penny in Russian bonds. They nearly took the global financial structure with them.

Today we're seeing another improbable linkage. A number of hedge funds are failing; others are seeing returns plunge. Among these is Goldman Sachs's flagship Global Alpha Fund, which burned a quarter of its $10 billion value over the last few weeks. And just as LTCM was free of the Russian debt that precipitated its collapse, Global Alpha was not a player in subprime junk.

Indeed, Global Alpha's problems have not come from mortgages at all, but from a portfolio of stocks removed from its bread-and-butter exposure? The root of the problem is high leverage. For example, when this debacle hit, one of Goldman's funds was leveraged 6 to 1, so every dollar of investor capital claimed six dollars of positions. This is the dry kindling for a market firestorm. When things go bad for a highly leveraged hedge fund, it gets a margin call and has to sell assets to reduce its exposure. Naturally, as it sells, prices drop. The falling prices mean a further decline in the fund's collateral, forcing yet more selling. And so goes the downward cycle. Hedge funds that hold the toxic CDOs can easily undermine those that don't. It can be difficult to sell the stuff that's causing the problem; those markets are beyond redemption. So if you can't sell what you want to sell, you sell what you can sell. The fund looks at its other holdings, focusing on the more liquid positions and reduces its exposure there. This causes pressure on these markets, markets that have nothing to do with the original problem, other than the fact that they happened to be held by the fund that got in trouble. Now that these markets are feeling the heat, other highly leveraged funds with similar exposure will have to sell. This leads to another cycle of selling, but in what was up to that point a healthy market unrelated to the initial turmoil.

Figure 1: A poster in a US neighborhood affected by the subprime crisis.

As the subprime crisis propagates, it doesn't matter that some instruments are fundamentally strong and others are weak. What matters is who owns what, who is under pressure and what else they own. Hedge funds are constantly shifting their exposure, so it is difficult to predict the course a crisis will take. But if you are a highly leveraged fund precariously perched as these dominos fall as Goldman's are today and as LTCM was in 1998 you become part of the game. And if you are both highly leveraged and big, the problem that started in one insignificant little segment will now become your problem, and a much bigger one. Again, it's all about leverage. This is the case for crises in the past and will be the case for crises in the future. A world in which highly leveraged hedge funds share similar strategies makes it inevitable that what we are seeing now will occur again. And the more complex the strategies, the more surprising the linkages that will emerge. Yet, incredibly, despite the risk this poses, no one keeps watch over leverage. No regulator knows how much leverage the hedge funds have or how that leverage is changing.

The lesson this time around with Global Alpha is the same as it was with LTCM. But we seem to be slow on the uptake. These funds hired the best and the brightest, yet they became embroiled in crises largely of their own making. If it could happen to them, it will happen again. And we'll all share in the consequences. Again.

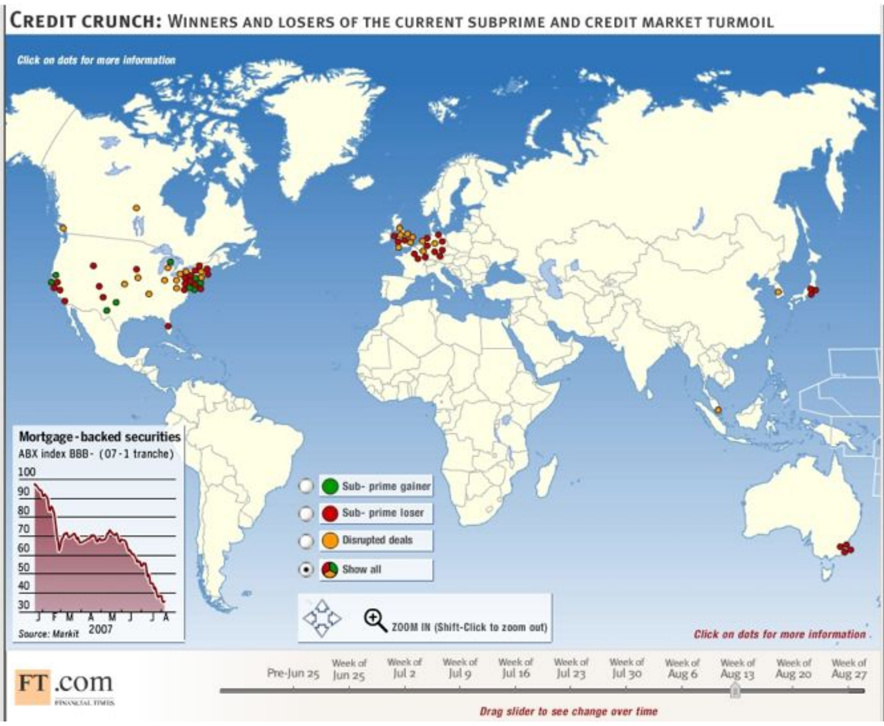

Figure 2: Winners and Losers in the Subprime Crises, source Financial Times, www.ft.com

4 Mathematical Tool 1: Correlated Defaults

In the following we follow closely the exposition of Schönbucher (2003), Galiani (2003) and Li (2000).

There are many models which attempt to describe default dependency. However these models present a series of problems. Some models catch default dependency in a very intuitive way, but cannot draw the timing of the default times in the portfolio. Others are able to describe default timing, but are difficult to calibrate and implement or have a very large number of parameters. In this section we illustrate how to combine the individual default risk and the default dependency structure by treating them separately, using *copula functions*.

4.1 Basic concepts on copula functions

Our aim is to model the default times of several obligors. We should be able to model the default dynamics of a single obligor and, in the meantime, the dependence structure of the defaults *between* the obligors. A regards the latter issue we present the fundamental concept of *copula functions*.

A function $C : [0, 1]^I \to [0, 1]$ is a copula if:

1. There exist random variables $U_1, \ldots, U_I$ taking values in $[0, 1]$ such that C is their distribution function;

2. C has uniform marginal distributions, i.e.

$$C(1, \ldots, 1, u_i, 1, \ldots, 1) = u_i, \; u_i \in [0, 1] \; \forall i \leq I.$$

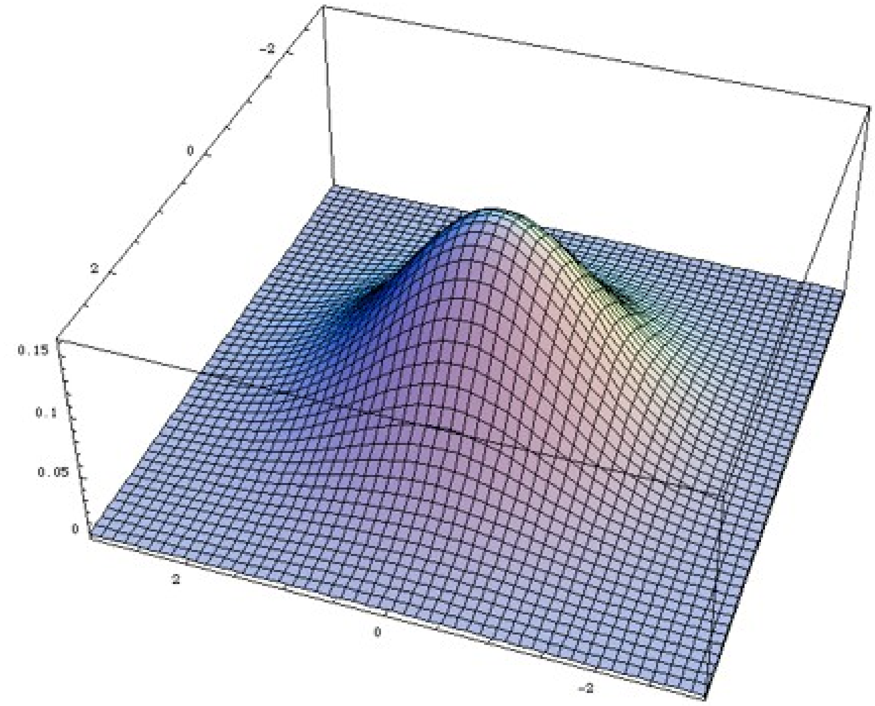

Figure 3: 2D Gaussian distribution with $\rho = 0$

The starting point for understanding the most important result about copulas, the Sklar's theorem, is the following property for continuous distribution functions F.

Let X denote a random variable with continuous distribution function F, then $Z = F(X)$ has a uniform distribution on $[0, 1]$. If U is a random variable with uniform distribution on $[0, 1]$, then $Y = F^{-1}(U)$ has the distribution function $F(\cdot)$.

If we consider a set of I real-valued random variables, all information we want is completely described by the joint distribution function of these random variables. The joint distribution function F of the random variables $X_1, X_2, \ldots, X_I$ is

$$F(\mathbf{x}) = \mathbb{P}(X_1 \leq x_1, X_2 \leq x_2, \ldots, X_I \leq x_I), \; \mathbf{x} = (x_1, \ldots x_I),$$

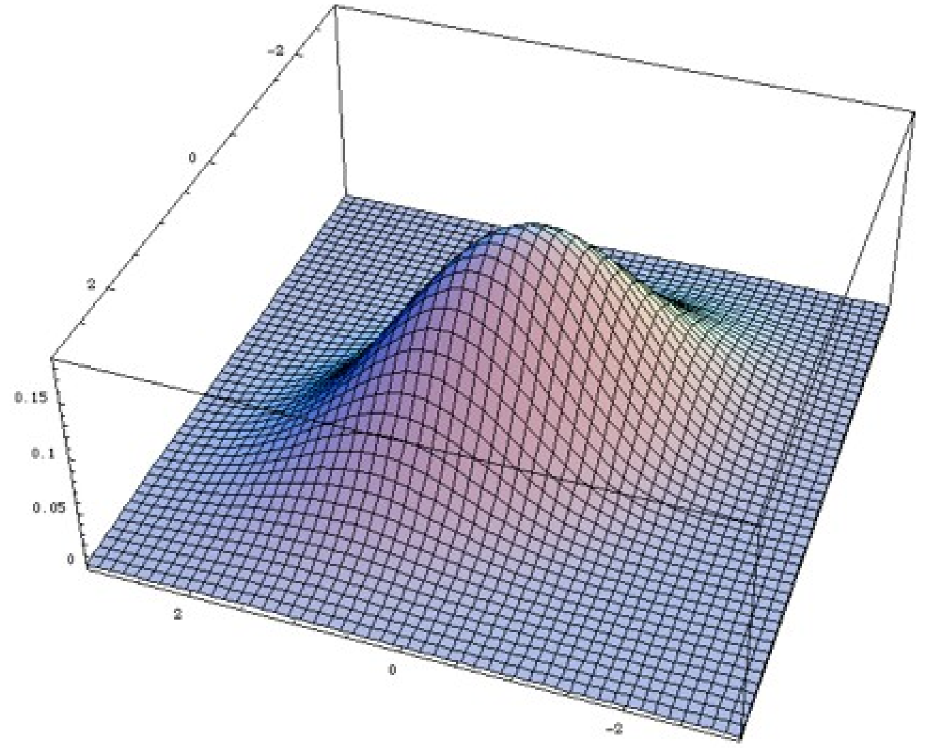

Figure 4: 2D Gaussian distribution with $\rho = 50\%$

while the marginal distribution functions $F_i(\cdot)$ of the X_i for $i = 1, \dots, I$ are

$$F_i(x) := \mathbb{P}(X_i \leq x).$$

The idea of the analysis of dependency with copula functions is that the joint distribution function F can be separated into two parts. The first part is represented by the marginal distribution functions of the random variables and the other part is the dependence structure between the random variables which is described by the copula function. Sklar's theorem shows how this decomposition can be obtained for any set of random variables.

Let $X_1, \dots, X_I$ be random variables with marginal distribution functions $F_1, F_2, \dots, F_I$ and joint distribution function F. Then there exists an $I-$dimensional copula C such that, for all $\mathbf{x} \in \mathbb{R}^I$:

$$F(\mathbf{x}) = C(F_1(x_1), F_2(x_2), \dots, F_I(x_I)) = C(\mathbf{F}(\mathbf{x})),$$

i.e. C is the distribution function of $\mathbf{F}(\mathbf{x}) := (F_1(x_1), F_2(x_2), \dots, F_I(x_I))$. If $F_1, F_2, \dots, F_I$ are continuous, then C is unique.

Sklar's theorem states that, for any multivariate distribution, the univariate marginal distributions and the dependence structure can be sepa-

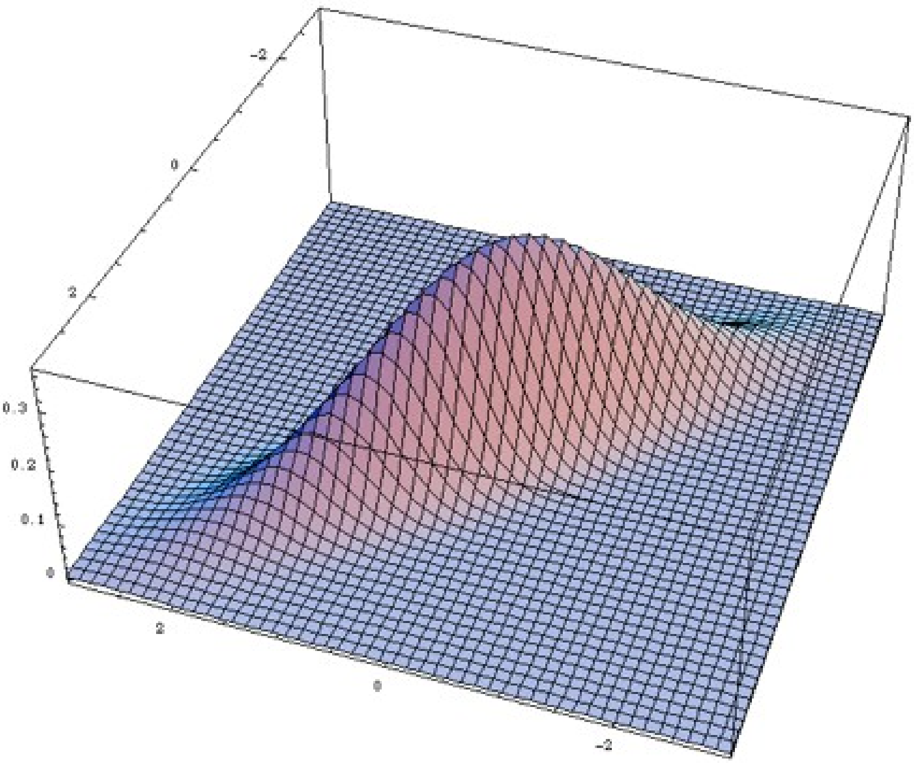

Figure 5: 2D Gaussian distribution with $\rho = 90\%$

rated. The dependence structure is completely characterized by the copula C.

4.2 Gaussian copula

In many applications, the Gaussian copula plays an important role.

Let $X_1, \ldots, X_I$ be normally distributed random variables with means $\mu_1, \ldots, \mu_I$, standard deviations $\sigma_1, \ldots, \sigma_I$ and variance-covariance matrix Σ. Then the distribution function $C_\Sigma(u_1, \ldots, u_I)$ of the random variables

$$U_i := \Phi\left(\frac{X_i - \mu_i}{\sigma_i}\right), \ i = 1, \ldots, I, \tag{1}$$

is said the *Gaussian copula to the variance-covariance matrix* Σ, where $\Phi(\cdot)$ denotes the cumulative univariate standard normal distribution function.

In order to obtain the explicit form of the Gaussian copula, we need the following

Let $X_1, \ldots, X_I$ be random variables with marginal distribution functions $F_1, F_2, \ldots, F_I$ and joint distribution function F. An $I-$dimensional copula

C can be built as:

$$C(u_1, \dots, u_I) = F(F_1^{-1}(u_1), \dots, F_I^{-1}(u_I)). \tag{2}$$

Applying equation (2) to the Gaussian copula, we can write:

$$C(u_1, \dots, u_I) = \Phi_\Sigma[\phi^{-1}(u_1), \dots, \phi^{-1}(u_I)], \tag{3}$$

thus obtaining

$$C_\Sigma(u_1, \dots, u_I) = \frac{1}{(2\pi)^{\frac{I}{2}}\sqrt{\det \Sigma}} \int_{-\infty}^{\phi^{-1}(u_1)} \dots \int_{-\infty}^{\phi^{-1}(u_I)} \exp\left(-\frac{1}{2}\mathbf{x}^T\Sigma^{-1}\mathbf{x}\right) d\mathbf{x}, \tag{4}$$

with $\mathbf{x} = (x_1, \dots, x_I)$.

The following algorithm generates random samples from the Gaussian copula with variance-covariance matrix Σ:

1. Generate $X_1, \dots, X_I$ which are jointly distributed with mean zero, standard deviation one and variance-covariance matrix Σ.

2. Let
$$U_i := \Phi(X_i),\ i = 1, \dots, I, \tag{5}$$
where $\Phi(\cdot)$ is the univariate cumulative normal distribution function. Then we say that $U_1, \dots, U_I$ are $C_\Sigma-$distributed.

The structure of copula functions allows us to draw random variables with a Gaussian copula that are not normally distributed in their marginal distribution. For example, we can generate random variables that are exponentially distributed in the margin, but have a Gaussian copula function, by generating the U_i as above and setting $Y_i := -\ln U_i$. Then $\mathbb{P}(Y_i \leq y) = 1 - e^{-y}$: the Y_i have an exponential distribution, but a Gaussian dependence structure.

There are $\frac{1}{2}(I-1)(I-2)$ free parameters in a Gaussian copula, i.e. the pairwise correlations of the X_i. This allows us to specify the dependence structure by specifying the pairwise dependence between the different obligors. If the number of obligors is large, this may be too much complex. So we can impose a factor structure to reduce the number of unknowns.

5 Default modeling with copula functions

The copula transformation is an extremely useful tool to connect the *times* of the defaults. Let us consider two obligors, with survival probability functions $P_1(\cdot)$ and $P_2(\cdot)$, which we assume to be continuous. Obligor 1 defaults before a certain time T_1 if $U_1 \geq P_1(T_1)$ and obligor 2 defaults if $U_2 \geq P_2(T_1)$. If we change the time horizon to a later time $T_2 > T_1$, the survival probabilities will be such that $P_1(T_1) > P_1(T_2)$ and $P_2(T_1) > P_2(T_2)$, i.e. we decrease the default thresholds. Let us denote by τ_1 and τ_2 the default times of obligors 1 and 2:

$$\tau_1 \in [T_0, T_1] \Leftrightarrow U_1 \in [P_1(T_0), P_1(T_1)],$$

$$\tau_1 \in [T_0, T_2] \Leftrightarrow U_1 \in [P_1(T_0), P_1(T_2)].$$

Thus

$$\tau_1 \in [T_1, T_2] \Leftrightarrow U_1 \in [P_1(T_1), P_1(T_2)].$$

The same results hold for obligor 2. Now, if we set $T_2 = T_1 + \Delta t$ and let $T_2 \to T_1$ (i.e. $\Delta t \to 0$), we obtain

$$\begin{aligned} \tau_1 \in [T_1, T_1 + \Delta t] \quad &\Leftrightarrow \quad U_1 \in [P_1(T_1), P_1(T_1 + \Delta t)] \\ \tau_1 = T_1 \quad &\Leftrightarrow \quad U_1 = P(T_1) \\ \tau_1 \quad &= \quad P^{-1}(U_1). \end{aligned} \tag{6}$$

Equation (6) tells us how we can determine the *exact* time of default of obligor 1 from a given value for U_1 and a given survival probability function $P_1(\cdot)$. In general, we can state the following algorithm, first proposed by Li in [12] for the special case of the Gaussian copula.

Assume we are given:

- A term structure of survival probabilities $P_i(0, T)$ for each obligor $i = 1, \ldots, I$.
- A copula function $C : [0, 1]^I \to [0, 1]$.
- An algorithm to sample random vectors $U_1, \ldots, U_I$ with the copula C as joint distribution function.

Then the simulation algorithm for the joint defaults events is as follows:

1. Draw $U_1, \ldots, U_I$ from the copula C.
2. Obtain the default times $\tau_1, \ldots, \tau_I$ from
$$\tau_i = P_i^{-1}(U_i). \tag{7}$$
3. Given this scenario of default times, simulate recoveries to each default, evaluate payoffs of the securities, etc.

In a nutshell, we take the marginal distributions of the default times (given by the survival probabilities) and link them with the copula $C(\cdot)$ to reach a joint, $I-$dimensional distribution of the default times.

5.1 Incorporating intensity models

An important specification of the copula model uses marginal survival probabilities coming from an intensity-based model.

- The marginal survival functions are given by a specification of a term structure of default intensities, which is calibrated to market data or described according to historical hazard rates. Thus, as discussed in section **??**,
$$P_i(T) = \exp\left(- \int_0^T \lambda_i(s) ds \right). \tag{8}$$
If the default intensities are chosen to be constant, we obtain
$$P_i(T) = \exp\left(-\lambda_i T\right). \tag{9}$$

- A Gaussian copula is chosen for the default dependency, usually in the form of a factor model. The simplest choice is a one-factor model with constant weights: even if it is very simple, it often already gives quite realistic results. If more realism is desired, a factor model can be chosen where, in addition to the economic-wide factor of the one-factor model, further factors are introduced, one for each component.

- The default times are given by finding the τ_i such that
$$\int_0^{\tau_i} \lambda_i(t) dt = -\ln U_i, \tag{10}$$
or, in case of constant hazard rate,
$$\tau_i = -\frac{\ln U_i}{\lambda_i}. \tag{11}$$

6 Mathematical Tool 2: Pricing CDOs

In the following we follow closely the exposition of Schönbucher (2003), Gibson (2003) and Peixoto (2004).

6.1 Structure

Collateralized debt obligations (CDOs) are a form of credit derivative offering exposure to a basket of credit-sensitive instruments: loans, bonds and credit default swaps. The assets are sold to a company that is set up exclusively for this purpose, and investors are offered the opportunity to invest in notes issued by this company. These *obligations* are *collateralized* by the underlying *debt* portfolio, hence the name CDO.

The financial innovation lies in the design of the payoff structure of the notes, which is created in order to offer risk/return profiles that are specifically targeted to the risk appetite and investment restrictions of different investor groups. In particular, even if the underlying portfolio is mostly unrated, it is possible to enhance the credit rating of most of the notes to high investment-grade ratings by concentrating the default risk in a small loss layer. The notes of a well-designed CDO can sometimes be sold for a cumulative price which is higher than the sum of the market value of the underlying assets.

CDO transactions can have several motivations:

- *Arbitrage* CDOs aim to arbitrage the price difference between the components of the underlying portfolio and the sale price of the CDO notes. Arbitrage CDOs have traded assets like bonds or CDSs exposures as underlying securities.

- *Balance sheet* CDOs allow banks to free up regulatory capital that is tied up in the underlying loan portfolio. Balance sheet CDOs usually reference a loan portfolio.

To demonstrate the basic structure, we first describe a very simple collateralized bond obligation (CBO). The components are as follows:

1. The underlying portfolio is composed of defaultable bonds issued by issuers $\mathbf{C}_i$ with notional amounts K_i, $i = 1, \ldots, I$. The total notional is $K = \sum_{i=1}^{I} K_i$.

2. The portfolio is transferred into a specially created company, the special purpose vehicle (SPV).

3. The SPV issues notes in "slices" of varying risk or subordination:

   (a) an EQUITY tranche with notional K_E,

   (b) several MEZZANINE tranches with notional K_{M1}, K_{M2}, K_{M3}, etc.,

   (c) a SENIOR tranche with notional K_S.

4. If during the existence of the contract one of the bonds defaults, the recovery payments are reinvested in default-free securities.

5. At maturity, the portfolio is liquidated and the proceeds are distributed to the tranches, according to their seniority.

The key point is the final redistribution of the portfolio value according to the seniority of the notes. First, the senior tranche is served. If the senior tranche can be fully repaid, then the most senior mezzanine tranche is repaid. If this tranche can also be fully repaid, then the next tranches are paid off in the order of their seniority, until finally the equity tranche is paid whatever is left of the portfolio's value.

Instead of viewing the payoffs of the different tranches as a function of the final value of the portfolio, it is more intuitive to view payoffs as a function of its *losses*.

1. The first losses hit the equity tranche alone. Until the cumulative loss amount has reached the equity's notional K_E, the other tranches are protected by the equity tranche.

2. Cumulative losses exceeding K_E affect the first mezzanine tranche, until its notional is used up.

3. After this, the subsequent mezzanine tranches are hit in the order of their seniority.

4. Only when all other tranches have absorbed their part of losses, the senior tranche begins suffering losses.

So the payment structure is tranched such that it is possible to buy exposure to a certain portion of the loss distribution of the pool. The tranches of lower seniority serve as a loss protection cushion for the tranches of higher seniority. The larger the degree of this subordination, the better the protection of the senior tranches and the higher their credit quality.

The basic idea of this derivative has been refined in several directions:

- Instead of traded bonds, the portfolio can also consist of loans. In this case we speak of a *collateralized loan obligation* (CLO).

- A *synthetic* CDO is constructed using credit default swaps instead of bonds and loans in the underlying portfolio. This gives the structure a larger degree of flexibility. Not all tranches must be sold off in the form of notes: protection can also be purchased in the form of pure derivatives transactions. The structure can be set up without the legally complicated transfer of the ownership of loans and it is also not affected by features of the specification of the underlying loans or bonds.

The maturity of the transaction is between 5 and 10 years. This means that the notes must pay regular coupons, which requires a modification of the distribution of the payoffs of the tranches. At every coupon payment date, the cash flow (i.e. coupon payments minus default losses) from the collateral portfolio is distributed to serve the coupon payments and to fill up a reserve for future coupon payments. The nature of the CDO payoff exposes mezzanine and senior tranches also to timing and clustering risk: if the times of defaults are clustered in the same coupon interval, a mezzanine or senior tranche could miss a coupon payment, even if the same tranche would have a full coupon payment if the defaults had been more evenly distributed over time.

In the majority of CDOs, tranches above equity level are rated by rating agencies such as Standard and Poor's or Moody's. The requirements of the rating agencies have a strong influence on the structuring of the tranches and the composition of the portfolio.

6.2 Pricing

A synthetic CDO tranche on a given reference portfolio is defined by an interval of losses, $[A, B]$, that the CDO tranche "investor" is responsible for. The endpoints of the interval are also referred to as *attachment* and *detachment* point, respectively.

A synthetic CDO tranche can be valued as a swap contract between two parties. The "investor" (counterparty **B**) receives periodic spread payments from the "issuer" (the *premium leg*) and makes contingent payments to the issuer (counterparty **A**) when defaults affect the tranche (the *default leg*).

The value of the tranche, from the perspective of the investor, is the expected present value of the premium leg less the expected present value of the default leg.

Let us assume the following:

- ρ is the correlation between credit i and $j \neq i$;
- All payments occur on periodic dates T_n, $n = 1, \ldots, N$, where N is the number of coupon payments;
- The investor makes a payment on the next periodic payment date after the default occurs, not on the date of the default itself;
- M is the number of CDSs in the portfolio;
- Each credit either takes no loss or a loss of $K_m(1 - R_m)$ (*loss given default*), where K_m is the notional amount and R_m is the recovery rate of the m^{th} credit, $m = 1, \ldots, M$;
- We express the attachment and detachment points as percentages of the total portfolio value $K = \sum_{m=1}^{M} K_m$, i.e. $A = a\%K$ and $B = b\%K$.
- K_m and R_m are equal across credits, i.e. $K_m = K$ and $R_m = R$, $m = 1, \ldots, M$;
- $\mathbb{P}(X(T_n) = m)$ is the probability of having m defaults ($m = 1, \ldots, M$) at coupon payment T_n, $n = 1, \ldots, N$.).

6.2.1 Expected loss

The expected default loss on the tranche (A, B) up to payment date T_n, denoted as $EL_{AB}(T_n)$, $n = 1, \ldots, N$, is an important quantity for computing the expected present values of the premium and default legs. It is computed as

$$EL_{AB}(T_n) = \sum_{m=1}^{M} \mathbb{P}(X(T_n) = m) \max(\min(m\,K(1-R), B) - A, 0). \quad (12)$$

6.2.2 Premium leg

The expected present value of the premium leg (PL) is the discounted sum of the spread payments the tranche investor expects to receive:

$$PL = S_{AB} \sum_{n=1}^{N} D(0, T_n)\Delta_n[(B - A) - EL_{AB}(T_n)], \quad (13)$$

where $D(0, T_n)$ is the risk-free discount factor for payment date T_n, Δ_n is the accrual factor for payment date T_n ($\Delta_n \approx T_n - T_{n-1}$) and S_{AB} is the spread paid to the tranche investor. The term $(B - A) - EL_{AB}(T_n)$ is the expected tranche principal outstanding on payment date T_n. It reflects the decline in principal as defaults affect the tranche.

6.2.3 Default leg

The expected present value of the default leg (DL) is the discounted sum of the expected payments the tranche investor must take when defaults affect the tranche:

$$DL = \sum_{n=1}^{N} D(0, T_n)(EL_{AB}(T_n) - EL_{AB}(T_{n-1})). \quad (14)$$

6.2.4 CDO tranche fair spread

The present value of the tranche (PV), from the perspective of the tranche investor, is

$$PV = PL - DL. \quad (15)$$

Like other swaps, a synthetic CDO tranche will have the spread set at inception so the swap's present value is zero. Setting $PV = 0$, we obtain that the fair spread is equal to

$$S_{AB} = \frac{\sum_{n=1}^{N} D(0, T_n)(EL_{AB}(T_n) - EL_{AB}(T_{n-1}))}{\sum_{n=1}^{N} D(0, T_n)\Delta_n[(B - A) - EL_{AB}(T_n)]}. \quad (16)$$

6.3 Monte Carlo simulation

We now present another way to price a synthetic CDO tranche, exploiting the Gaussian copula framework and performing N_{sim} Monte Carlo simulations. The idea is to generate random variables which represent the default times of the names in the portfolio, considering their correlation structure. We briefly resume the main steps:

1. Compute the Cholesky factorization [2] of the variance-covariance matrix Σ between the credits, which satisfies $\Sigma = \Theta\Theta^T$;
2. Simulate a set of M independent random variables $\mathbf{Z} = (Z_1, \ldots, Z_M)^T$ from a normal standard distribution;
3. Set $\mathbf{X} = \Theta\mathbf{Z}$;
4. Determine $U_m = \Phi(X_m)$, $m = 1, \ldots, M$;
5. Compute the default times as
$$\tau_m = -\frac{\ln(U_m)}{\lambda_m},\ m = 1, \ldots, M,$$
where λ_m denotes the default hazard rate of the m^{th} credit, supposed constant in time.

The loss given default of each name at a certain time t is given by

$$K_m(1 - R_m)\mathbf{1}_{\tau_m \leq t},$$

while the total loss is the sum of the losses over all M credits:

$$L(t) = \sum_{m=1}^{M} K_m(1 - R_m)\mathbf{1}_{\tau_m \leq t}. \tag{17}$$

Now we are interested in knowing how the total loss (17) affects a tranche with attachment point A and detachment point B. In fact, from the payment subordination, we know that the tranche $[A, B]$ suffers a loss at time t if and only if $A \leq L(t) \leq B$. So the tranche loss, for a given Monte Carlo scenario h, $h = 1, \ldots, N_{sim}$, is:

$$L_{AB}^h(t) = \max(\min(L(t), B) - A, 0). \tag{18}$$

Now we repeat this procedure N_{sim} times and name

$$EL_{AB}(T_n) = \frac{1}{N_{sim}} \sum_{h=1}^{N_{sim}} L_{AB}^h(T_n),\ n = 1, \ldots, N. \tag{19}$$

[2] Cholesky factorization is a matrix decomposition of a symmetric positive-definite matrix into a lower triangular matrix and the transpose of the lower triangular matrix. The lower triangular matrix is the Cholesky triangle of the original, positive-definite matrix.

Following the previous construction, we have that the Premium and Default legs can be obtained using again equations (13) and (14), respectively, while the fair spread is given by (16).

Note that the correlation ρ is equal for all the credits in the portfolio, the correlation matrix must take the following form:

$$\begin{pmatrix} 1 & \rho & \rho & \cdots & \rho \\ \rho & 1 & \rho & \cdots & \rho \\ \rho & \rho & 1 & \cdots & \rho \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho & \rho & \rho & \cdots & 1 \end{pmatrix}. \quad (20)$$

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