Equity

$$dS_t = \mu S_t\,dt + \sigma S_t\,dX(t)$$

$$dS_t = \mu S_t\,dt + \sqrt{\nu(t)}\,S_t\,dX_1(t)$$

$dX_1\,dX_2 \to \rho\,dt$

$$d\nu(t) = a\left(k - \nu(t)\right)dt + c\sqrt{\nu(t)}\,dX_2(t)$$

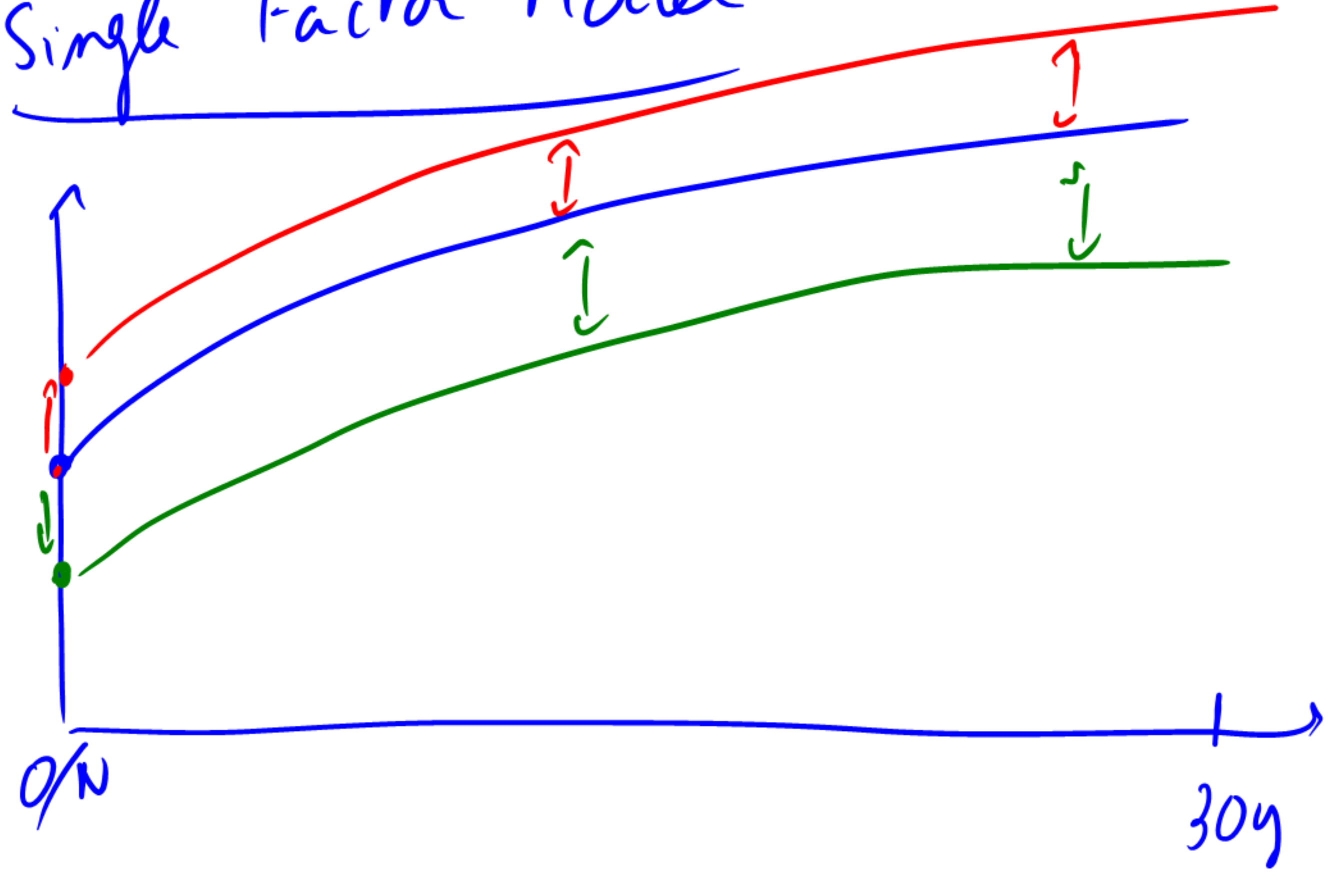
$$dS_t = \mu S_t\,dt + \sigma S_t\,dX_t + \xi S_t\,dJ(t)$$

A Rough Guide to Interest Rate Models:

① Single Factor Model.

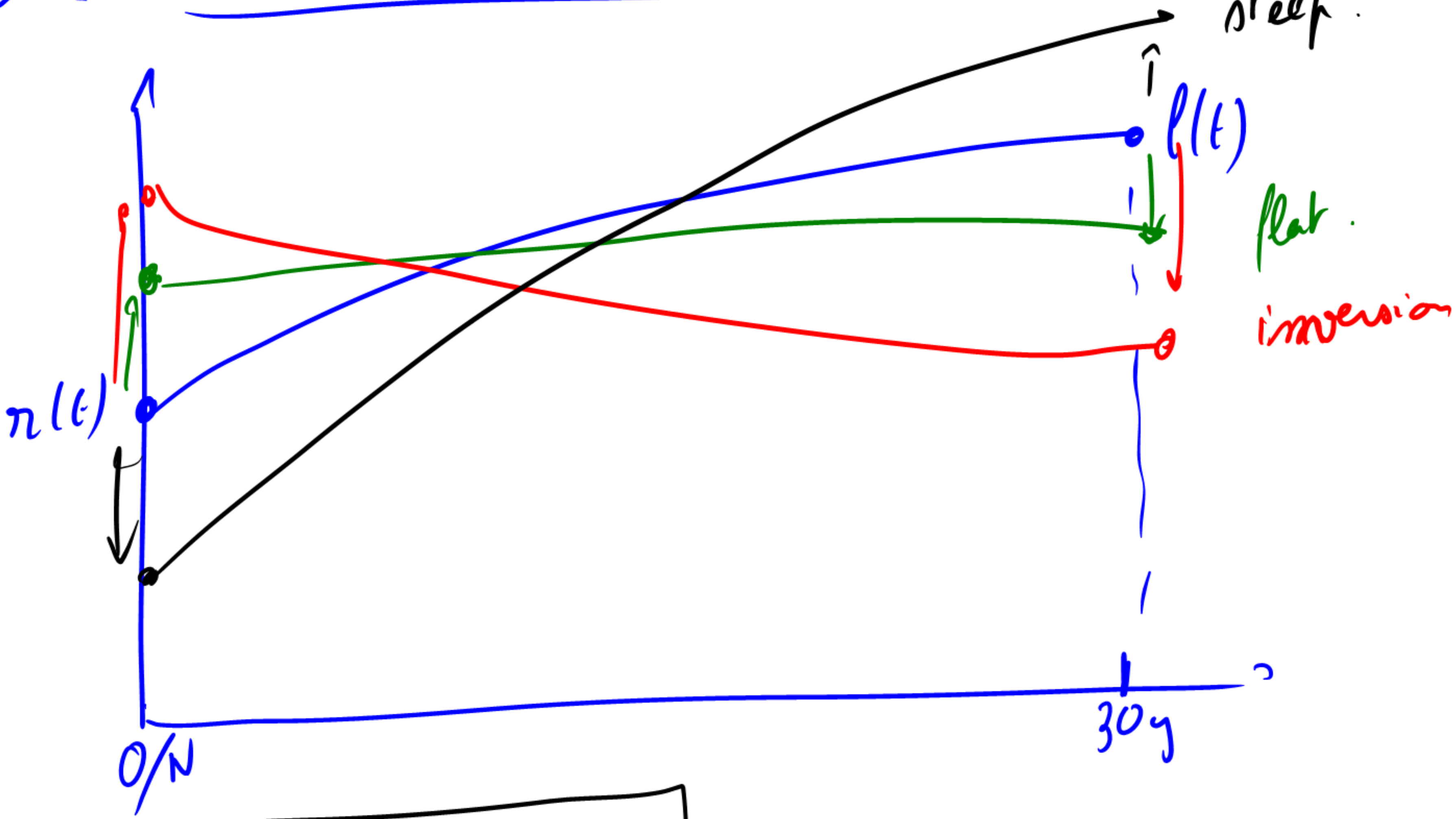
Parallel Shifts

1 factor → level
yield curve



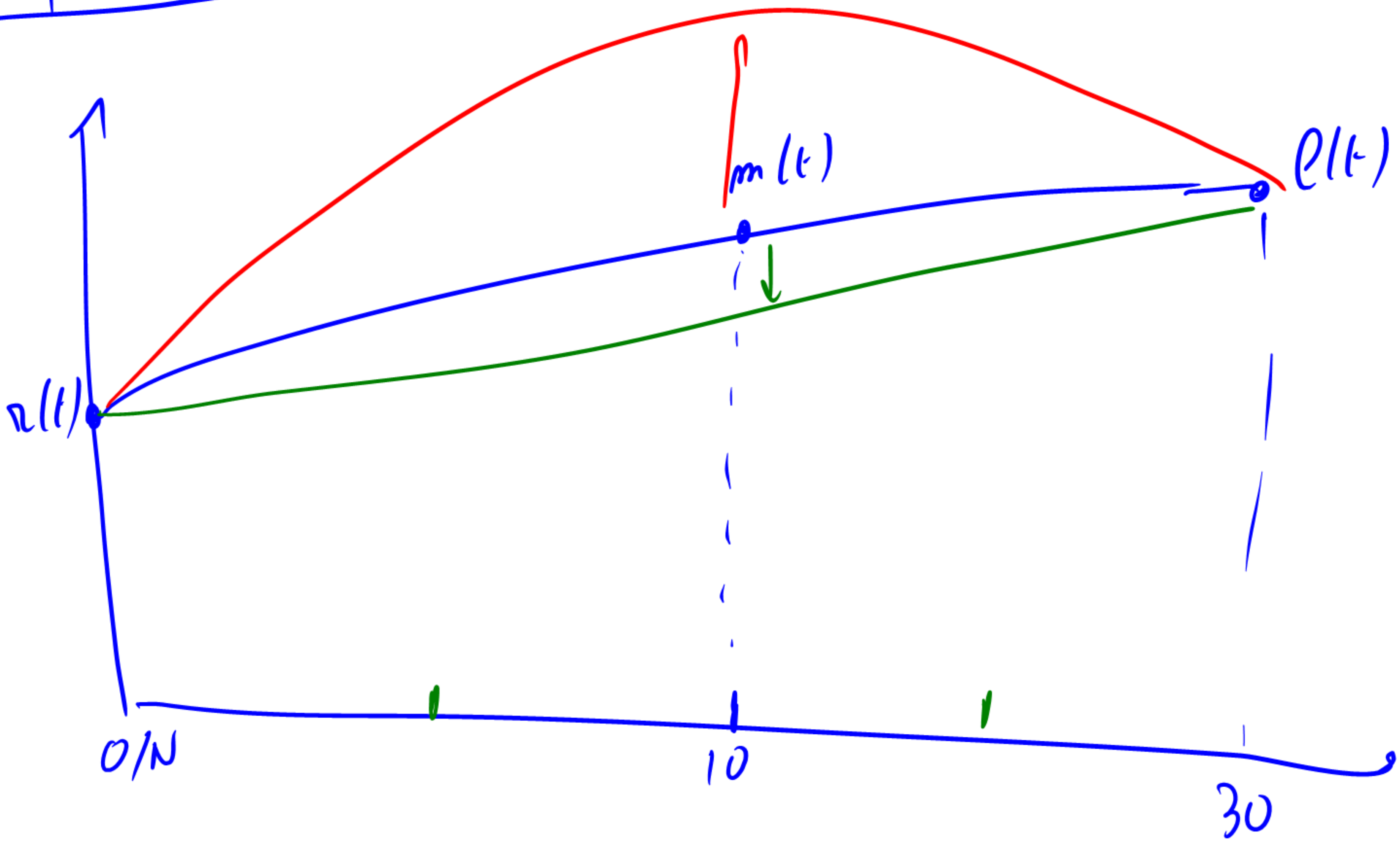
Varicek model works in negative interest rate environments, because it goes negative.

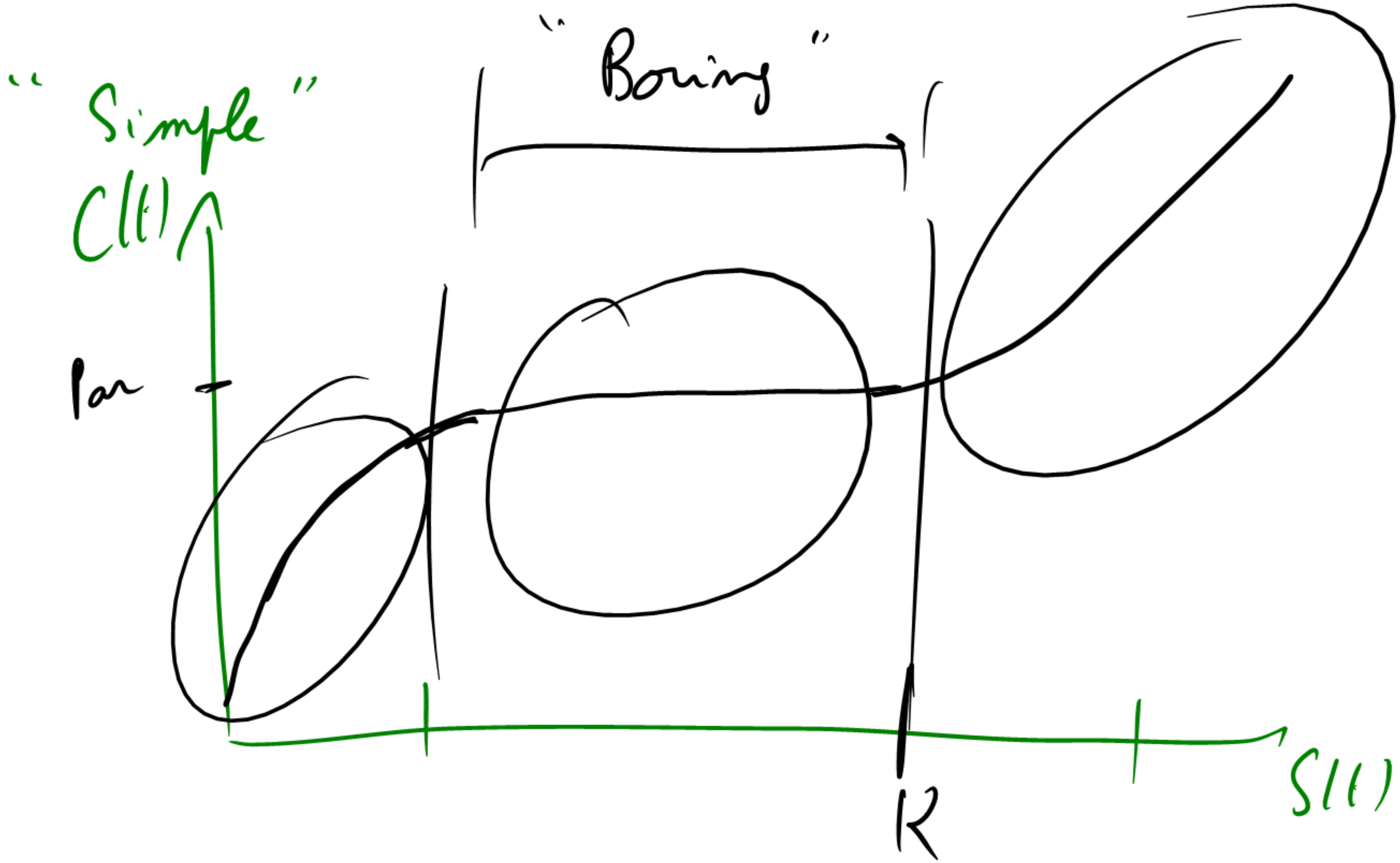
② - Two Factor Model



slope of the yield curve

3 factors




"Simple"
"Boring"
C(t)
Par
R
S(t)

Variance Decomposition / PCA:

1st factor (shift in yield curve) $\rightarrow \approx$ 85% of the variance

2nd factor (slope) $\approx$ 9% "

3rd factor (curvature) $\approx$ 4% "

98%

By Girsanov, $X^{\theta}(t) = X(t) + \int_0^t \theta(s)\,ds$

$$\Rightarrow dX^{\theta}(t) = dX(t) + \theta(t)\,dt \iff \boxed{dX(t) = dX^{\theta}(t) - \theta(t)\,dt}$$

Now, under $\mathbb{P}$: $dr(t) = \mu_t\,dt + \sigma_t \boxed{dX(t)}$

Under $\mathbb{Q}^{\theta}$: $dr(t) = \mu_t\,dt + \sigma_t\left(dX^{\theta}(t) - \theta(t)\,dt\right)$

$$dr(t) = \left(\mu_t - \sigma_t\,\theta(t)\right)dt + \sigma_t\,dX^{\theta}(t)$$

Pricing Here!

$$\mathbb{P} \longrightarrow \mathbb{Q}^{\theta}$$

I.R. deterministic

$$A(t) = e^{\int_0^t r(s)\,ds}$$

$$B(t,T) = e^{-\int_t^T r(s)\,ds}$$

r is constant

$$\to B(t,T) = e^{-r(T-t)}.$$

Stochastic

$$A(t) = e^{\int_0^t r(s)\,ds}$$

$$dr(t) = \dots$$

$$B(t,T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(s)\,ds} \middle| \mathcal{F}_t\right]$$

Feynman-Kac:

$$B(t,T) = \mathbb{E}^{\theta}\left[e^{-\int_t^T r(s)\,ds} \times 1 \,\middle|\, \mathcal{F}_t \right]$$

$$\left\{ dr(t) = \left(\mu_t - \sigma_t \theta(t)\right) dt + \sigma_t \, dX(t) \right.$$

Then

$$\begin{cases} \dfrac{\partial B}{\partial t} + \left(\mu_t - \sigma_t \theta(t)\right) \dfrac{\partial B}{\partial r} + \dfrac{1}{2} \sigma_t^2 \dfrac{\partial^2 B}{\partial r^2} - rB = 0 \\ B(T,T) = 1 \end{cases}$$

* $dn(t) = r(t)\,dX(t)$ under $\mathbb{P}$

$$dn(t) = r(t)\left(dX^{\theta}(t) - \theta(t)\,dt\right)$$

$$dn(t) = -r(t)\theta(t)\,dt + r(t)\,dX^{\theta}(t) \quad \text{under } \mathbb{Q}$$

* $dn^{-1}(t)$?

$$Z^*(t,T) = \Pi(t)\, \eta^{-1}(t)$$

$$d\Pi(t) = r(t)\left[-\theta(t)dt + dX^{\theta}(t)\right]$$

$$d\eta^{-1}(t) = \theta(t)\,\eta^{-1}(t)\, dX^{\theta}(t)$$

By the Itô product rule

$$dZ^*(t,T) = d\left(\Pi(t)\cdot \eta^{-1}(t)\right)$$

$$= \underbrace{d\Pi(t)\cdot \eta^{-1}(t) + \Pi(t)\cdot d\eta^{-1}(t)}_{\text{standard product rule}} + \underbrace{r(t)\,\theta(t)\,\eta^{-1}(t)\,dt}_{\text{cross variation}}$$

$$= r(t)\left[-\theta(t)dt + dX^{\theta}(t)\right]\eta^{-1}(t) + \Pi(t)\,\theta(t)\,\eta^{-1}(t)\,dX^{\theta}(t) + r(t)\,\theta(t)\,\eta^{-1}(t)\,dt$$

$$= \left[-r(t)\theta(t)\eta^{-1}(t) + \cancel{r(t)\theta(t)\eta^{-1}(t)}\right]dt + \left[r(t)\,\eta^{-1}(t) + \Pi(t)\,\theta(t)\,\eta^{-1}(t)\right]dX^{\theta}(t)$$

$$dZ^*(t,T) = \eta^{-1}(t)\left[r(t) + \Pi(t)\,\theta(t)\right]dX^{\theta}(t)$$

$$dZ^*(t,T) = \eta^{-1}(t)\left[r(t) + \Pi(t)\theta(t)\right]dX^{\theta}(t)$$

$$\Pi(t) = Z^*(t,T)\eta(t)$$

$$dZ^*(t,T) = \left[\eta^{-1}(t)r(t) + Z^*(t,T)\theta(t)\right]dX^{\theta}(t)$$

Next Step: $\boxed{B(t,T) = A(t)Z^*(t,T)}$

$$Z^*(t,T) = \frac{B(t,T)}{A(t)}$$

because

$$A(t) = e^{\int_0^t r(s)\,ds}$$

$$dA(t) = r(t)A(t)dt + 0$$

$$dB(t,T) = d\left(Z^*(t,T)\, A(t)\right)$$

$$= dZ^*(t,T) \cdot A(t) + dA(t) \cdot Z^*(t,T)$$

$$= \left[r(t)\eta^{-1}(t) + Z^*(t,T)\,\theta(t) \right] A(t)\, dX^{\theta}(t)$$

diffusion

$$+ \underbrace{r(t) A(t)\, Z^*(t,T)}_{B(t,T)}\, dt$$

drift

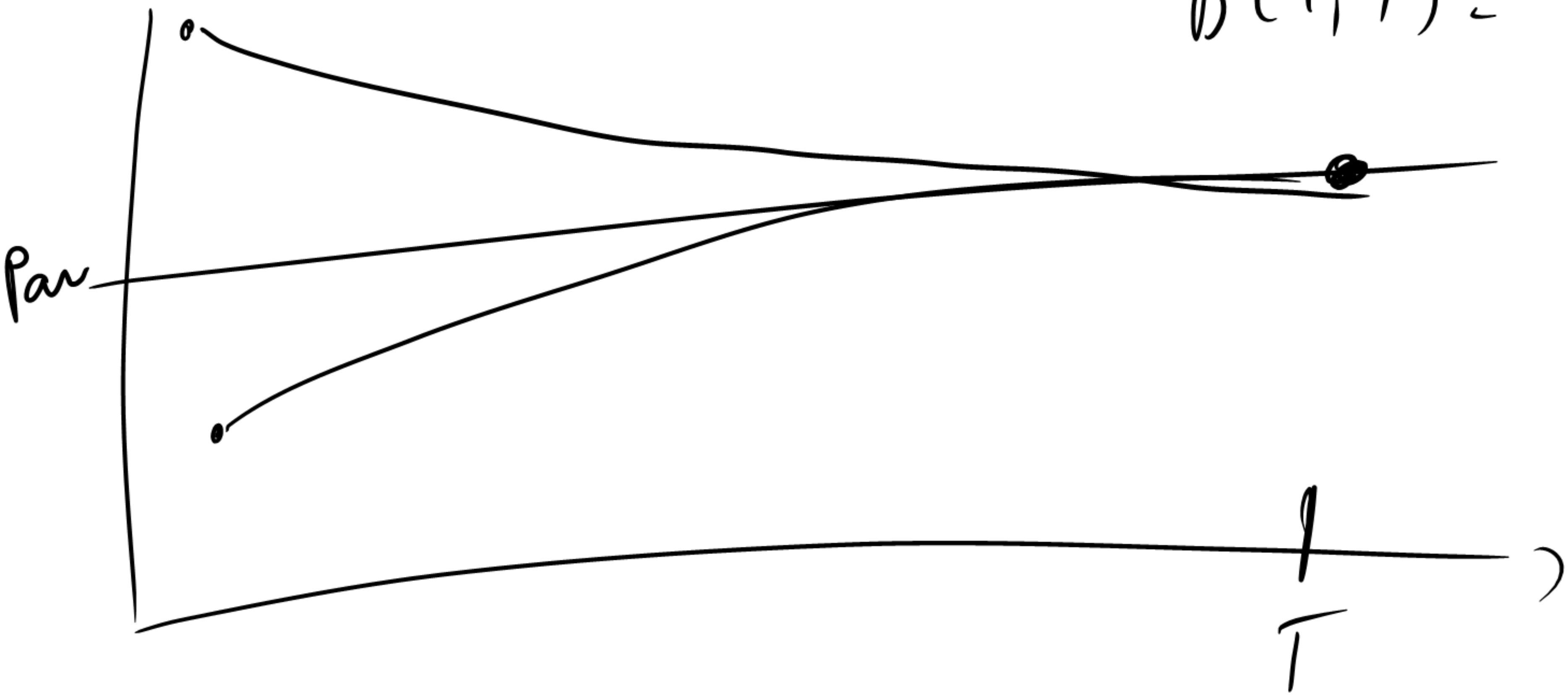
$$= r(t)\, B(t,T)\, dt + \left[r(t)\eta^{-1}(t) A(t) + \underbrace{Z^*(t,T) A(t)}_{B(t,T)}\, \theta(t) \right] dX^{\theta}(t)$$

$$= r(t)\, B(t,T)\, dt + \left[\frac{r(t)\eta^{-1}(t)}{Z^*(t,T)} + \theta(t) \right] B(t,T)\, dX^{\theta}(t)$$

$$b^{\theta}(t,T) = \left[\frac{r(t)\eta^{-1}(t)}{Z^*(t,T)} + \theta(t) \right]$$

$$dB(t,T) = r(t)\, B(t,T)\, dt + b^0(t,T)\, B(t,T)\, dX^0(t)$$

$$B(T,T) = 1$$



The market price of risk:

Under $\mathbb{Q}^\theta$: $\dfrac{dB(t,T)}{B(t,T)} = r(t)\,dt + b^\theta(t,T)\,dx^\theta(t)$

$$dx^\theta(t) = dx(t) + \theta(t)dt$$

Under $\mathbb{P}$: $\dfrac{dB(t,T)}{B(t,T)} = r(t)dt + b^\theta(t,T)\left(dx(t) + \theta(t)dt\right)$

$$\frac{dB(t,T)}{B(t,T)} = \left[r(t) + b^\theta(t,T)\,\theta(t)\right]dt + b^\theta(t,T)\,dx(t)$$

Money market Rate

Risk Premium

$b^\theta(t,T)$ = # of units of volatility (amount of risk taken)

$\theta(t)$ = mkt price of risk per unit (price of risk)

Equity World

$$\theta = \frac{\mu - r}{\sigma}$$

$$V(t) = A(t)\, \mathbb{E}^{Q}\left[\frac{Y_T}{A(T)} \,\middle|\, \mathcal{F}_t \right]$$

Call Option

$$Y_T = \left(B(T,U) - K \right)^+$$

$$= \left(B(T,U) - K \right) \mathbb{1}_{\{B(T,U) > K\}}$$

$$c(t) = A(t)\,\mathbb{E}^{Q}\left[\frac{(B(T,U)-K)}{A(T)}\mathbb{1}\{B(T,U)>K\}\,\middle|\,\mathcal{F}_t\right]$$

$$= A(t)\,\mathbb{E}^{Q}\left[\frac{B(T,U)}{A(T)}\mathbb{1}\{B(T,U)>K\}\,\middle|\,\mathcal{F}_t\right] - A(t)\,\mathbb{E}^{Q}\left[\frac{K}{A(T)}\mathbb{1}\{B(T,U)>K\}\,\middle|\,\mathcal{F}_t\right]$$

$$B(T,U) = A(T)\,\mathbb{E}^{Q}\left[\frac{1}{A(U)}\,\middle|\,\mathcal{F}_T\right]$$

$$c(t) = A(t)\,\mathbb{E}^{Q}\left[\frac{\cancel{A(T)}\,\mathbb{E}^{Q}\left[\frac{1}{A(U)}\,\middle|\,\cancel{\mathcal{F}_T}\right]\mathbb{1}\{B(T,U)>K\}}{\cancel{A(T)}}\,\middle|\,\mathcal{F}_t\right] + \dots$$

By the Tower Property,

$$c(t) = A(t)\,\mathbb{E}^{Q}\left[\frac{1}{A(U)}\mathbb{1}\{B(U,T)>K\}\,\middle|\,\mathcal{F}_t\right]$$

$$- K A(t)\,\mathbb{E}^{Q}\left[\frac{1}{A(T)}\mathbb{1}\{B(U,T)>K\}\,\middle|\,\mathcal{F}_t\right]$$

I.R. are deterministic

① Forward $\downarrow Y(T)$

$t \qquad\qquad T$

② $Y(t) \rightarrow Y(t)e^{\int_t^T r(s)ds}$

$$= \frac{Y(t)}{e^{-\int_t^T r(s)ds}}$$

$B(t,T)$

Forward Price

$$= \frac{Y(t)}{B(t,T)}$$

I.R. stochastic

$$B(t,T) = A(t)\, \mathbb{E}^{Q}\left[\frac{1}{A(T)} \Big| \mathcal{F}_t\right]$$

$$= \mathbb{E}^{Q}\left[e^{-\int_t^T r(s)ds} \Big| \mathcal{F}_t\right]$$

$$F_Y(t,T) = \frac{Y(t)}{B(t,T)}$$

$$F_Y(t,T) = \frac{A(t)\, \mathbb{E}^{Q}\left[\frac{Y}{A(T)} \Big| \mathcal{F}_t\right]}{B(t,T)}$$

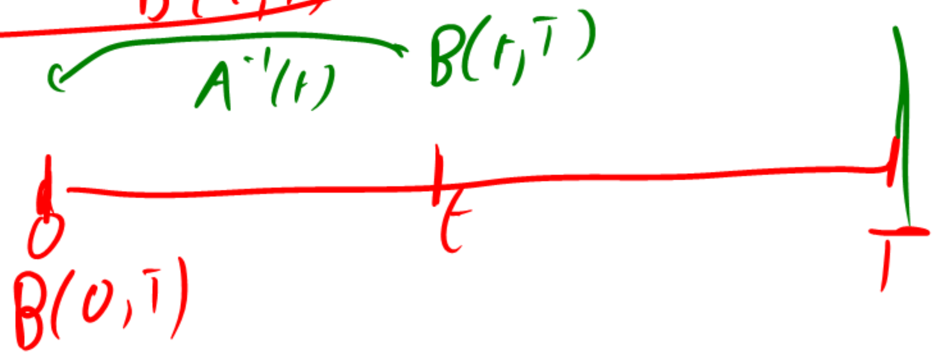
$$\lambda_t = \mathbb{E}^{\mathbb{Q}}\left[\frac{\overbrace{A(0)}^{1}\ \overbrace{B(T,T)}^{1}}{A(T)\ B(0,T)} \,\middle|\, \mathcal{F}_t \right]$$

known at t

$$\lambda_t = \frac{1}{B(0,T)} \mathbb{E}\left[\frac{1}{A(T)} \,\middle|\, \mathcal{F}_t \right] \times \frac{A(t)}{A(t)}$$

$$= \frac{1}{B(0,T)\,A(t)} \cdot \underbrace{A(t)\, \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{A(T)} \,\middle|\, \mathcal{F}_t \right]}_{B(t,T)}$$

$$= \frac{A^{-1}(t)\ B(t,T)}{B(0,T)}$$



Is λ_t an exponential martingale?

$$\lambda_t = \frac{B(t,T)}{\cancel{B(0,T)}\,\cancel{A(t)}}$$

$$B(t,T) = \cancel{B(0,T)}\,\cancel{A(t)}\exp\left\{-\frac{1}{2}\int_0^t b^2(s,T)\,ds + \int_0^t b(s,T)\,dX^{\mathbb{Q}}(s)\right\}$$

$$\lambda_t = \exp\left\{-\frac{1}{2}\int_0^t b^2(s,T)\,ds + \int_0^t b(s,T)\,dX^{\mathbb{Q}}(s)\right\}$$

as long as $b(s,T)$ remains finite, λ_t is an exponential martingale $\rightarrow$ R-N derivative

$$\mathbb{P} \xrightarrow{\eta_t} \mathbb{Q} \xrightarrow{\lambda_t} \mathbb{P}^T$$

$ Forward!!!

Portfolio Risk

Pricing Spot Products

Pricing Forward Products

$$X(t) \xrightarrow{\text{add drift}} X^{\mathbb{Q}}(t) \xrightarrow{\text{add drift}} X^T(t)$$

$$dX^{Q}(t) = dX(t) + \theta(t)\,dt$$

$$dX^{T}(t) = dX^{Q}(t) - b(t,T)\,dt$$

$$= dX(t) + \theta(t)\,dt - b(t,T)\,dt$$

$$dX^{T}(t) = dX(t) + \big(\theta(t) - b(t,T)\big)\,dt$$

To go from the classic FAPF to the N & I recipe:

$$V(t) = A(t)\,\mathbb{E}^{\mathbb{Q}}\left[\frac{Y}{A(T)} \,\middle|\, \mathcal{F}_t\right]$$

Recall that

$$\lambda_T = \frac{A^{-1}(T)\,B(T,T)}{B(0,T)} = \frac{1}{A(T)\,B(0,T)}$$

$$\lambda_t = \frac{B(t,T)}{A(t)\,B(0,T)}$$

So,

$$V(t) = A(t)\,B(0,T)\,\mathbb{E}^{\mathbb{Q}}\left[\frac{Y}{A(T)\,B(0,T)} \,\middle|\, \mathcal{F}_t\right]$$

$$= A(t)\,B(0,T)\,\mathbb{E}^{\mathbb{Q}}\left[Y\lambda_T \,\middle|\, \mathcal{F}_t\right]$$

$$= A(t)\,B(0,T) \times \mathbb{E}^{T}\left[Y \,\middle|\, \mathcal{F}_t\right] \underbrace{\mathbb{E}^{\mathbb{Q}}\left[\lambda_T \,\middle|\, \mathcal{F}_t\right]}_{\text{RN} \to \text{martingale}}$$

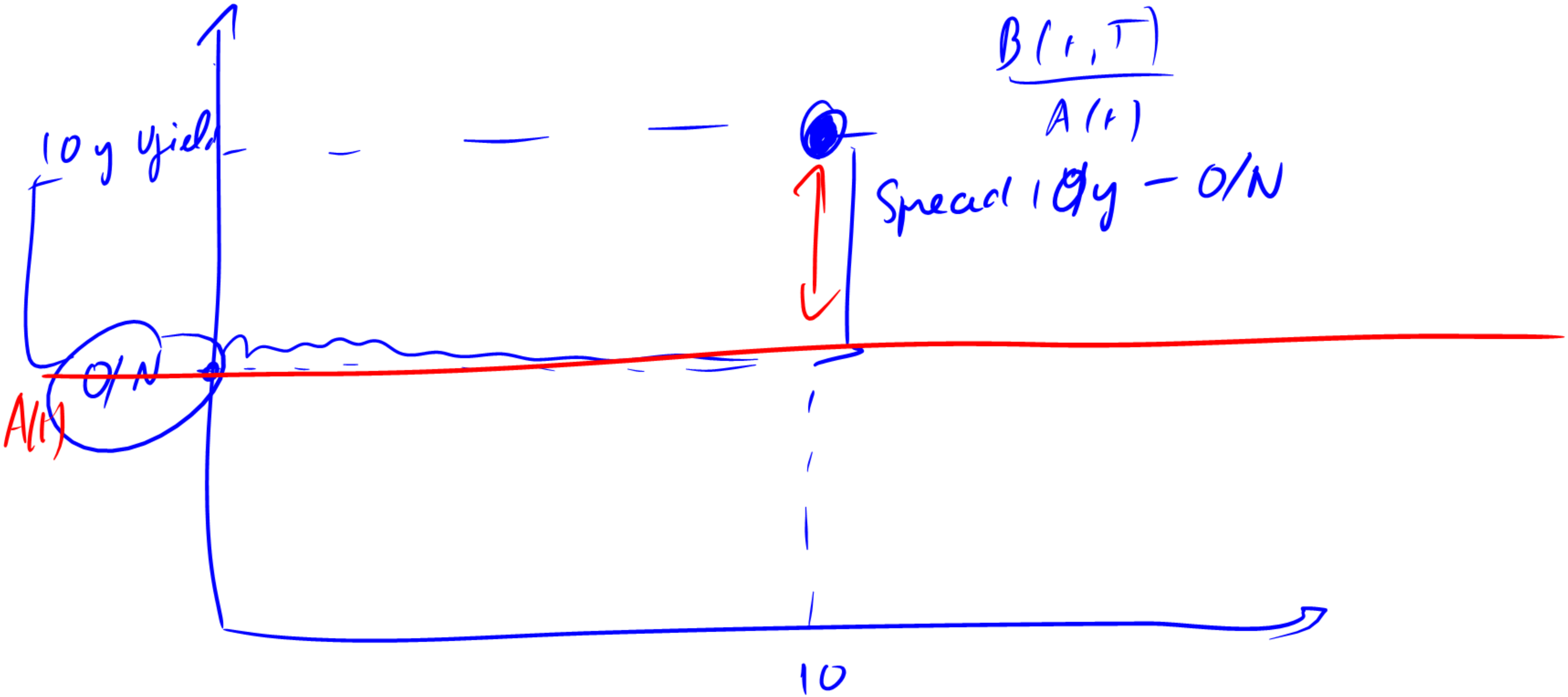
$$\mathbb{E}^{\mathbb{Q}}\left[\lambda_T \,\middle|\, \mathcal{F}_t\right] = \lambda_t$$

$$V(t) = A(t)\,B(0,T) \times \mathbb{E}^{T}\left[\,Y \mid \mathcal{F}_t\right] \times \mathbb{E}^{Q}\left[\lambda_T \mid \mathcal{F}_t\right]$$

$$= A(t)\,B(0,T) \times \mathbb{E}^{T}\left[Y \mid \mathcal{F}_t\right] \lambda_t$$

$$= \cancel{A(t)}\,B(\cancel{0,T}) \times \mathbb{E}^{T}\left[Y \mid \mathcal{F}_t\right] \frac{B(t,T)}{\cancel{A(t)}\,B(\cancel{0,T})}$$

$$\boxed{V(t) = B(t,T)\,\mathbb{E}^{T}\left[Y \mid \mathcal{F}_t\right]}$$



B(t,T) / A(t)
10 y yield
Spread 10y - O/N
O/N
A(t)
10