Fixed Income Derivatives: Managing Smile Risk

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Martingales

- Martingales

  $M(t) = M(t,*) \;=\;$ random process

  M can depend on anything known by time t

- Given a probability measure Q, $M(t,*)$ is a Martingale iff

  $$M(t,*) \;= E\{M(T,*) \mid t\} \qquad \text{for all } T > t$$

  given all info known by time t

  "for all times $T > t$, the average of $M(T)$ is $M(t)$"

- Martingale representation theorem: If $M(t)$ is a Martingale then

$$dM = A(t,*)dW(t)$$

for some $A(t,*)$, where dW is Brownian motion under Q

Arbitrage free pricing

• Choose any positive, tradable security that has no cash flow as the *numeraire*. Then there is a probability measure Q such that $V(t,*)/N(t,*)$ is a Martingale for all tradable securities. Here $V(t,*)$ is the security price and $N(t,*)$ is the value of the numeraire:

$$\frac{V(t)}{N(t)} = E\left\{ \frac{V(T,*)}{N(T,*)} \,\middle|\, t \right\} \qquad \underline{\text{all}}\ T > t$$

no cash flow

or

$$\frac{V(t)}{N(t)} = E\left\{ \left(\frac{V(T,*)}{N(T,*)} + \int_t^T \frac{C(t',*)}{N(t',*)} \right) \,\middle|\, t \right\} \qquad \underline{\text{all}}\ T > t$$

cash flow $C(t,*)$

– complete markets: Harrison & Pliska *et al* – *hedging arguments* (BS)
– incomplete markets: *law of one price*

• $N(t) = e^{\int_0^t r(t')dt'}$ *money market numeraire*

$$V(t) = N(t) E\left\{ \frac{V(T)}{N(T)} \,\middle|\, t \right\} = E\left\{ e^{-\int_t^T r(t')dt'} V(T) \,\middle|\, t \right\} \qquad \underline{\text{all}}\ T > t$$

risk neutral measure

• $N(t) = Z(t, t_{set})$ *zero coupon bond, maturity* T_{set}

$$V(t) = Z(t, T_{set}) E\left\{ \frac{V(T)}{Z(T, T_{set})} \,\middle|\, t \right\} \qquad \underline{\text{all}}\ T > t$$

forward measure for T_{set}

Swaptions

- $V_{sw}(t) = R_f \sum_{j=1}^{n} \alpha_j Z(t,t_j) + Z(t,t_n) - Z(t,t_0) = [R_f - R_s(t)]L(t)$ receiver / payer

$$R_s(t) = \frac{Z(t,t_0) - Z(t,t_n)}{L(t)}; \qquad L(t) = \sum_{j=1}^{n} \alpha_j Z(t,t_j)$$

- $V_{swptn}(t_{ex}) = [R_f - R_s(t_{ex})]^{+} L(t_{ex})$ at t_{ex} payoff

- Choose $N(t) = L(t) = \sum_{j=1}^{n} \alpha_j Z(t,t_j)$

$$\frac{V_{swptn}(0)}{L(0)} = E\left\{ \frac{V_{swptn}(T)}{L(T)} \,\middle|\, 0 \right\} \quad \text{for any } T > 0$$ use $T = t_{ex}$

$$V_{swptn}(0) = L(0) E\{[R_f - \underline{R_s(t_{ex})}]^{+} \mid 0\}$$

- $R_s(t)$ is a Martingale: $\xleftarrow{\text{theory}}$ $\xrightarrow{\text{model}}$

$$dR_s = A(t,*)dW = \underline{\sigma R_s dW}$$

$$L(0) = \sum_j \alpha_j D(t_j); \qquad R_s^0 = \frac{D(t_0) - D(t_n)}{\sum_j \alpha_j D(t_j)}$$

Backwards Kolmogorov equation

• If $U(t,\mathbf{x}) = E\{F(\mathbf{X}(T) \mid \mathbf{X}(t) = \mathbf{x}\}$

where

$$dX_j = \mu_j(t,\mathbf{X})dt + a_j(t,\mathbf{X})dW_j \qquad j = 1,2,\ldots$$

with

$$dW_j dW_k = \rho_{jk}dt$$

Then $U(t,\mathbf{x})$ is the solution of

$$\frac{\partial U}{\partial t} + \sum_j \mu_j(t,\mathbf{x})\frac{\partial U}{\partial x_j} + \tfrac{1}{2}\sum_{jk} \rho_{jk}a_j(t,\mathbf{x})a_k(t,\mathbf{x})\frac{\partial^2 U}{\partial x_j \partial x_k} = 0$$

for $t < T$, with

$$U(t,\mathbf{x}) = F(\mathbf{x}) \qquad \text{at } t = T$$

• $\tilde{V}(t,R) = E\{[R_f - R_s(t_{ex})]^+ \mid R_s(t) = R\}$

$$\underline{dR_s = \sigma R_s dW}$$

So

$$\frac{\partial \tilde{V}}{\partial t} + \tfrac{1}{2}\sigma^2 R^2 \frac{\partial^2 V}{\partial R^2} = 0 \qquad \text{for } t < t_{ex}$$

$$\tilde{V} = [R_f - R]^+ \qquad \text{at } t = t_{ex}$$

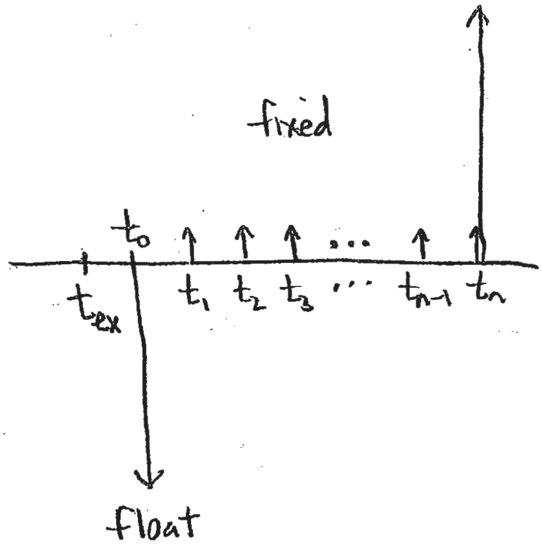
or

$$\tilde{V}(0,R_s^0) = R_f \mathrm{N}(d_1) - R_s^0 \mathrm{N}(d_2)$$

$$d_{1,2} = \frac{\log R_f/R_s^0 \pm \frac{1}{2}\sigma^2 t_{ex}}{\sigma t_{ex}^{1/2}}$$

<u>Black's formula</u>

Swaption prices

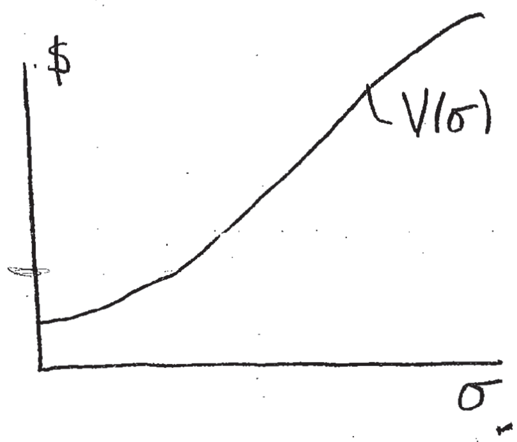


$$V_{swptn}(0) = L_0[R_f \mathrm{N}(d_1) - R_s^0 \mathrm{N}(d_2)]$$

$$d_{1,2} = \frac{\log R_f/R_s^0 \pm \frac{1}{2}\sigma^2 t_{ex}}{\sigma t_{ex}^{1/2}}$$

$$L_0 = \sum_j \alpha_j D(t_j); \quad R_s^0 = \frac{D(t_0) - D(t_n)}{\sum_j \alpha_j D(t_j)}$$

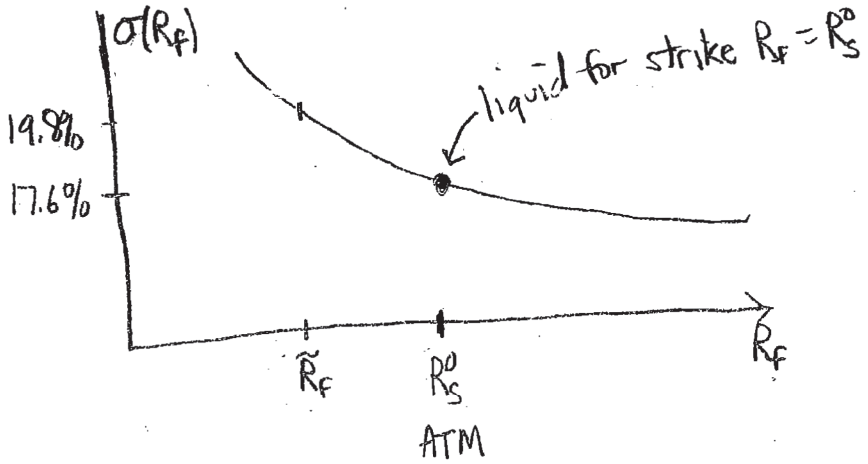
– Payer: $R_f \leftrightarrows R_s^0$



- Market price quoted as σ
  "3 into 7 receiver @ 8%"
  "17.6% vol"

implied vol

- Smile



Managing vol books

Preb Yamae
ICAP
TTKL ←
Eurbrks
Eurb

- Swaption/caplet vol matrix

t_{ex}\tenor	3m	1y	2y	3y	$\cdots$	10y
1m	0.182	0.180	0.17	0.176	$\cdots$	0.160
3m	0.177	0.175	0.174	0.170	$\cdots$	0.157
6m	0.173	0.172	0.174	0.165	$\cdots$	0.155
1y	0.171	0.170	0.170	0.167	$\cdots$	0.150
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$
10y	0.164	0.161	0.160	0.158	$\cdots$	0.142

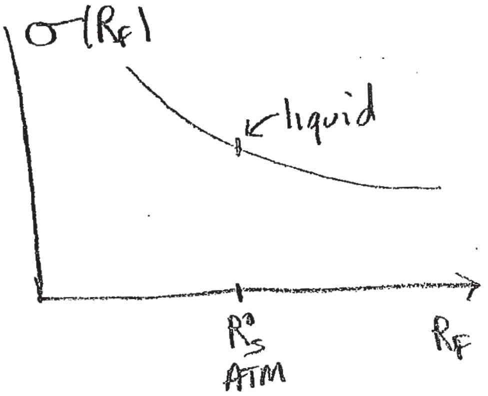
– vols are for ATM options ($R_f = R_s^0$) liquid
– broker quotes, options on EDF's
– interpolate for off-grid swaptions

- Find V_{book}

– bump vol, find ΔV_{book} → buy/sell swaption → vega neutral
– bump YC instrument (e.g, EDF), find ΔV_{book} → buy/sell instrument

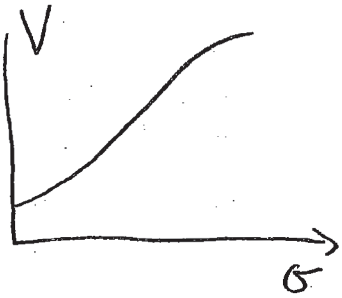
→ delta neutral

– off market ($R_f \neq R_s^0$)?



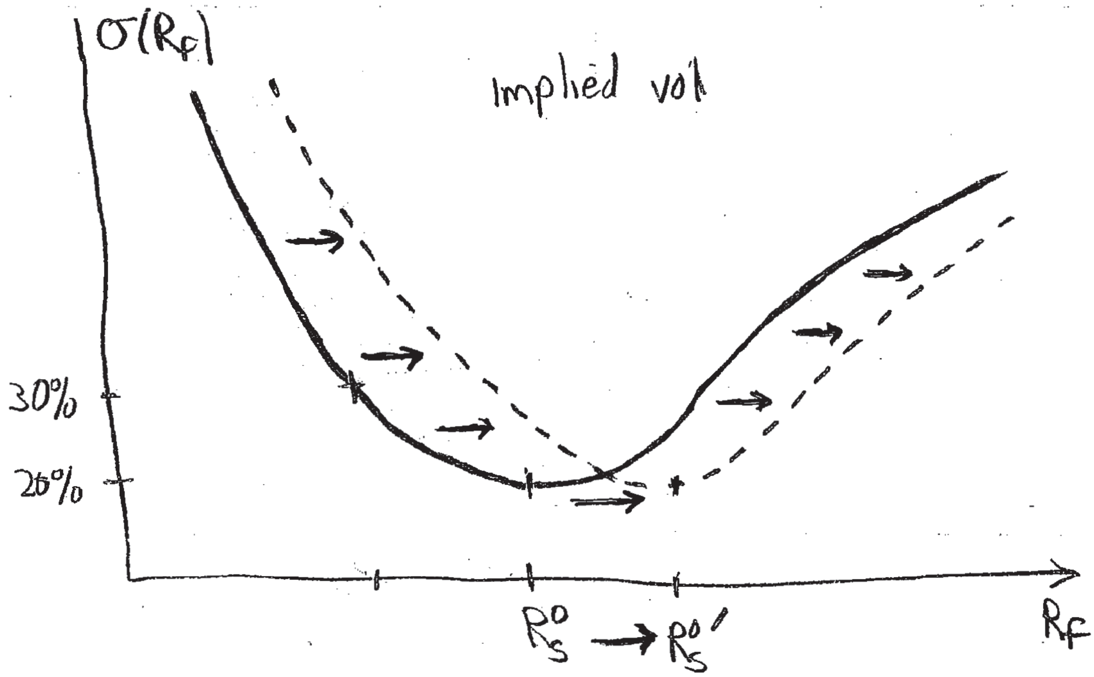
VolMatID	USDVM.001	10/25/2001	USD	10/25/2001					
tex\tenor	3M	1Y	2Y	3Y	4Y	5Y	7Y	10Y	15Y
1M	5.25	12.25	13.50	14.13	14.50	14.75	14.50	14.25	13.25
3M	7.55	13.00	14.13	14.38	14.63	14.75	14.63	14.50	13.50
6M	11.44	14.25	14.88	15.00	15.13	15.25	15.00	14.75	13.50
9M	15.76	15.50	15.63	15.56	15.50	15.50	15.31	15.13	13.75
1Y	16.20	16.75	16.38	16.13	15.88	15.75	15.63	15.50	14.00
18M	16.78	17.25	16.75	16.56	16.31	16.13	15.88	15.63	14.13
2Y	19.25	17.75	17.13	17.00	16.75	16.50	16.13	15.75	14.25
3Y	19.25	17.75	17.13	17.00	16.75	16.50	16.13	15.63	14.00
4Y	19.00	17.50	16.88	16.63	16.38	16.13	15.63	15.13	13.63
5Y	18.50	17.38	16.50	16.13	15.88	15.50	15.13	14.63	13.13
7Y	16.25	15.38	14.88	14.38	14.13	13.88	13.50	13.25	11.88
10Y	14.00	13.50	13.00	12.50	12.25	12.00	11.38	11.00	9.88
15Y	11.00	11.00	10.75	10.63	10.50	10.25	9.88	9.38	8.75

Implied vols



- $V(R_s^0) = L_0[R_f \mathrm{N}(d_1) - R_0\, \mathrm{N}(d_2)]$

$$d_{1,2} = \frac{\log R_f/R_s^0 + \frac{1}{2}\sigma^2 t_{ex}}{\sigma t_{ex}^{1/2}}$$



- *Different* model at each strike R_f

vega $= \dfrac{\partial}{\partial\sigma} V(0, R_s^0, R_f)$ *hedge all vega with* 1 *ATM swaption* expensive

$\Delta = \dfrac{\partial}{\partial R_s^0} V(0, R_s^0, R_f)$ *hedge all* Δ *with* 1 *ATM swaption* cheap

$R_s^0 \to R_s^0 + \delta, \quad \sigma \to \sigma + \ ?$ systematic change in σ with R_s^0

– *Need single model for all strikes*

Local vol models (Dupire/Derman-Kani)

- $V(R_s^0, R_f) = L_0 E\{[R_f - R_s(t_{ex})]^+ \mid R_s(0) = R_s^0\}$

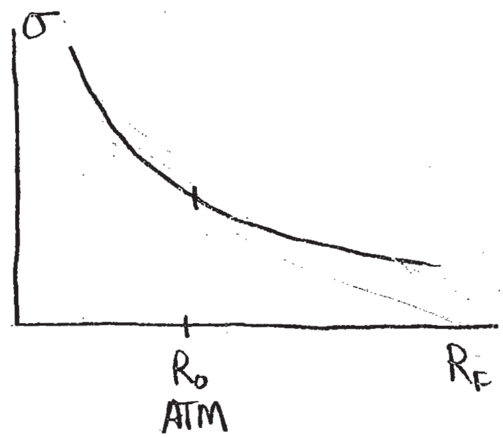
$$dR_s = C(t,*)dW = A(t,R_s)R_s dW = A(R_s)R_s dW$$

new model

theory ← → model

- given $A(R_s)$, solve for $V(R_s^0, R_f)$
- find implied vol $\sigma_{imp}(R_f)$ from $V(R_s^0, R_f)$
- vary $A(R_s)$ until $\sigma_{imp}(R_f)$ matches market for all R_f

single self consistent model for all strikes R_f



- Singular perturbation analysis:

$$V(R_s^0, R_f) = L_0[R_f \mathrm{N}(d_1) - R_s^0 \mathrm{N}(d_2)]$$

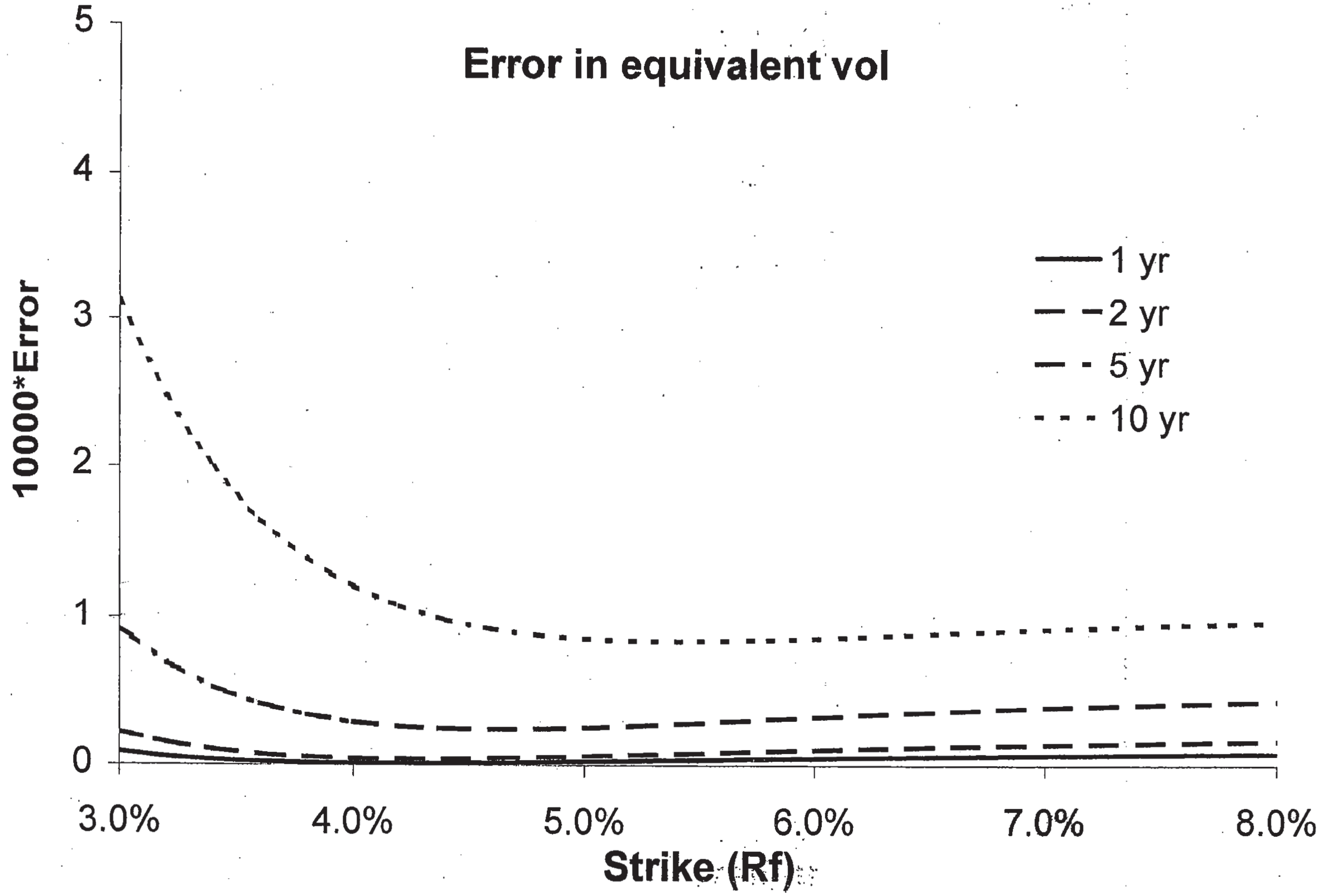
$$d_{1,2} = \frac{\log R_f/R_s^0 + \frac{1}{2}\sigma^2 t_{ex}}{\sigma t_{ex}^{1/2}}$$

with

small: 1% or less

$$\sigma_{imp}(R_s^0, R_f) = A(R_{av})\Big\{1 + \tfrac{1}{24}(\gamma_2 - 2\gamma_1^2 - 2\gamma_1)\frac{(R_s^0 - R_f)^2}{R_{av}^2} + \tfrac{1}{24}(2\gamma_2 - \gamma_1^2 + 2\gamma_1)A^2(R_{av})t_{ex} + \cdots\Big\}$$

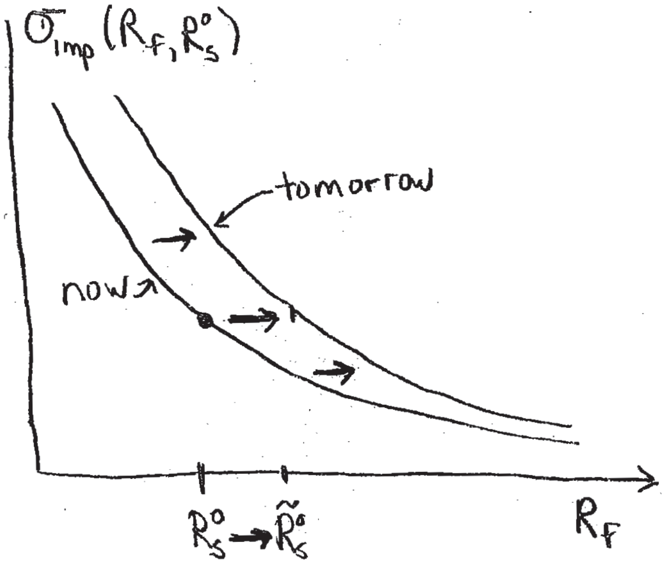
$$R_{av} = \tfrac{1}{2}(R_f + R_s^0); \quad \gamma_1 = \frac{R_{av}A'(R_{av})}{A(R_{av})}; \quad \gamma_2 = \frac{R_{av}^2 A''(R_{av})}{A(R_{av})}$$



units: 1 on vertical axis ⇒ 20.00% → 20.01%

Reprise

$$dR_s = aA(R_s)R_s dW$$



$$V(R_s^0, R_f) = BS(t_{ex}, R_s^0, R_f, \sigma_{imp})$$

$$\sigma_{imp}(R_s^0, R_f) = aA(R_{av})\{1 + \cdots$$

– calibration

$$\sigma_{market}(R_f) = aA(\tfrac{1}{2}(R_s^0 + R_f)\{1 + \cdots$$

so

$$aA(R_s) = \sigma_{market}(2R_{\cancel{f}} - R_s^0)\{1 - \cdots$$

iterate

new term

– hedging

$$\frac{\partial V}{\partial R_s^0} = \frac{\partial BS}{\partial R_s^0} + \frac{\partial BS}{\partial \sigma_{imp}} \frac{\partial \sigma_{imp}(R_s^0, R_f)}{\partial R_s^0}$$

for each R_f

new Δ   old Δ   old vega · systematic shift

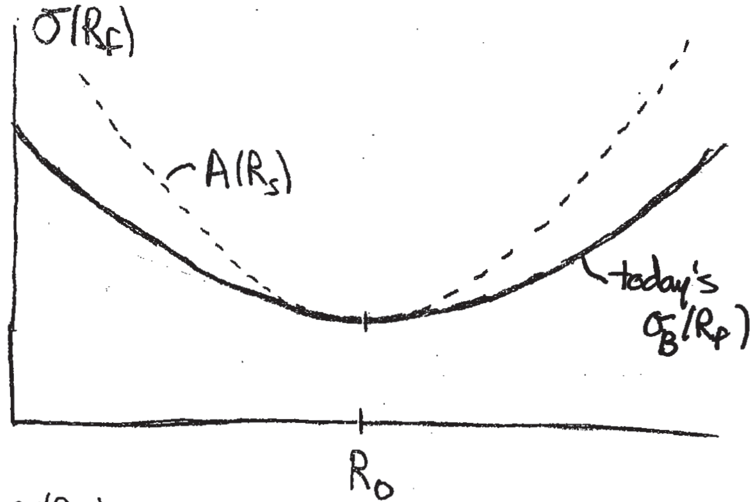
$$\frac{\partial V}{\partial a} = \frac{\partial BS}{\partial \sigma_{imp}} \frac{\partial \sigma_{imp}(R_s^0, R_f)}{\partial a}$$

for each R_f

new vega   old vega · scaling factor

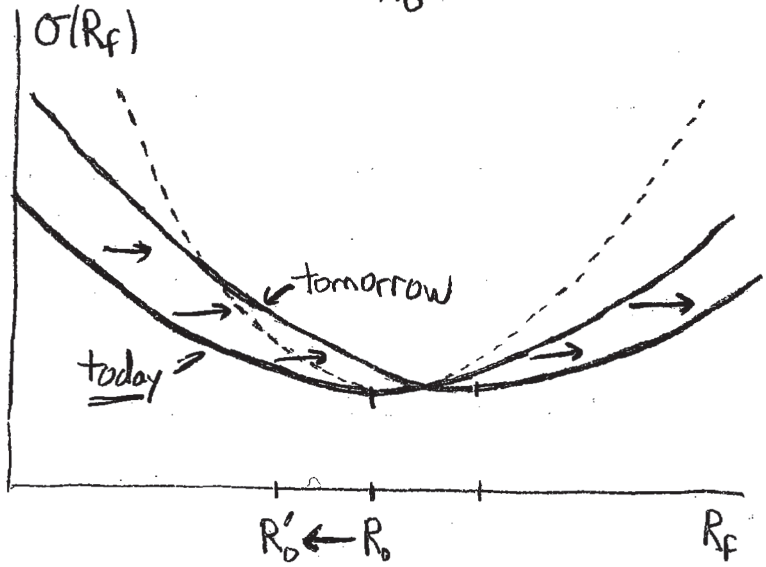
Does it work

$dR_s = A(R_s)R_s dW$
pick $A(R_s)$ to match
today's smile

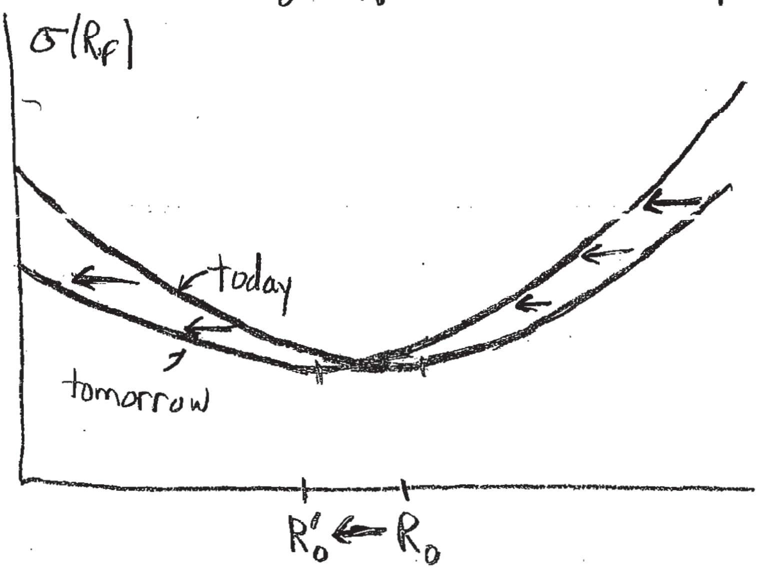


tomorrow predict:

if $R_s^0 \to R_s^0 + \delta$ then
the smile shifts the
other way *by δ*



experience says



Only reasonable models are
$A(R_s) = a(t)R_s^{\beta}$

scaling

Stochastic $\alpha\beta\rho$ model

- Fundamental theory

$$V = L\mathrm{E}\{[R_s(t_{ex}) - R_f]^+ | R_s(0) = R_s^0\}$$

$$dR_s = A(*)dW$$

- 1 factor Markovian models failed

– non-Markovian models: too messy

– use 2 factor Markovian model

– second factor: make vol stochastic

account for vega risk

- stochastic $\alpha\beta\rho$ model

$$dR_s = aR_s^\beta dW_1$$

$$da = \nu a dW_2$$

$$dW_1 dW_2 = \rho dt$$

- first factor: CEV model for rates

$\beta = 0?$ $\quad\beta = \frac{1}{2}?$ $\quad\beta = 1?$

– stochastic vol produces smile *(adverse selection)*

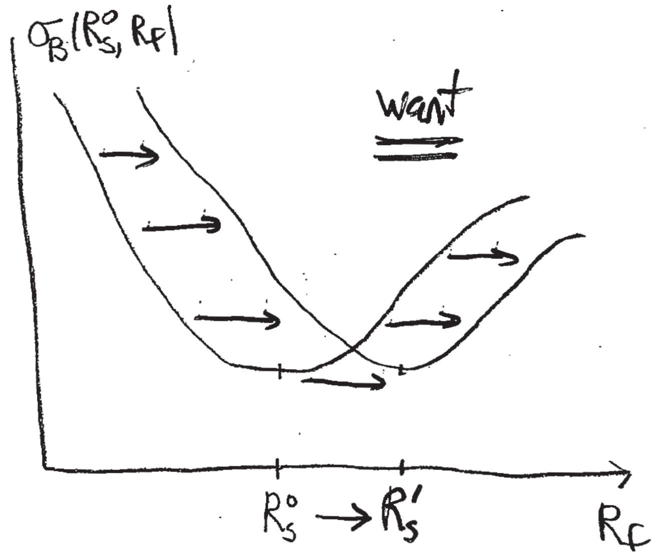
– correlate stochastic vol and rates

use correlation to set skew

SABR model

- SABR

$$
\begin{aligned}
dR_s &= aR_s^\beta dW_1 \\
d\alpha &= \nu\alpha dW_2 \\
dW_1 dW_2 &= \rho dt
\end{aligned}
$$



- option prices: Black's formula with

$$
\sigma_B \approx \underbrace{\frac{\alpha}{(R_s^0)^{1-\beta}}}_{\substack{\text{ATM vol} \\ \text{backbone}}} \left\{ 1 - (\underbrace{1-\beta}_{\substack{\text{beta} \\ \text{skew}}} - \underbrace{\rho\lambda}_{\substack{\text{vanna} \\ \text{skew}}}) \frac{R_f - R_s^0}{2R_s^0} + \underset{\text{volga smile}}{(2-3\rho^2)\lambda^2 \frac{(R_f - R_s^0)^2}{12(R_s^0)^2}} + \cdots \right.
$$

$$
\lambda = \frac{\nu (R_s^0)^{1-\beta}}{\alpha} = \frac{\text{vol of vol}}{\text{ATM vol}}
$$

- implied vol is very rough ...

Equivalent vol for the SABR model

• Accurate formula for all strikes R_f :

$$V(R_s^0, R_f) = BS(R_s^0, R_f, \sigma_B)$$

with

$$\sigma_B(R_s^0, R_f) = \frac{\alpha}{(R_s^0 R_f)^{(1-\beta)/2}\{1 + \frac{(1-\beta)^2}{24}\log^2 R_s^0/R_f\}} \cdot \frac{z}{x(z)} \cdot$$

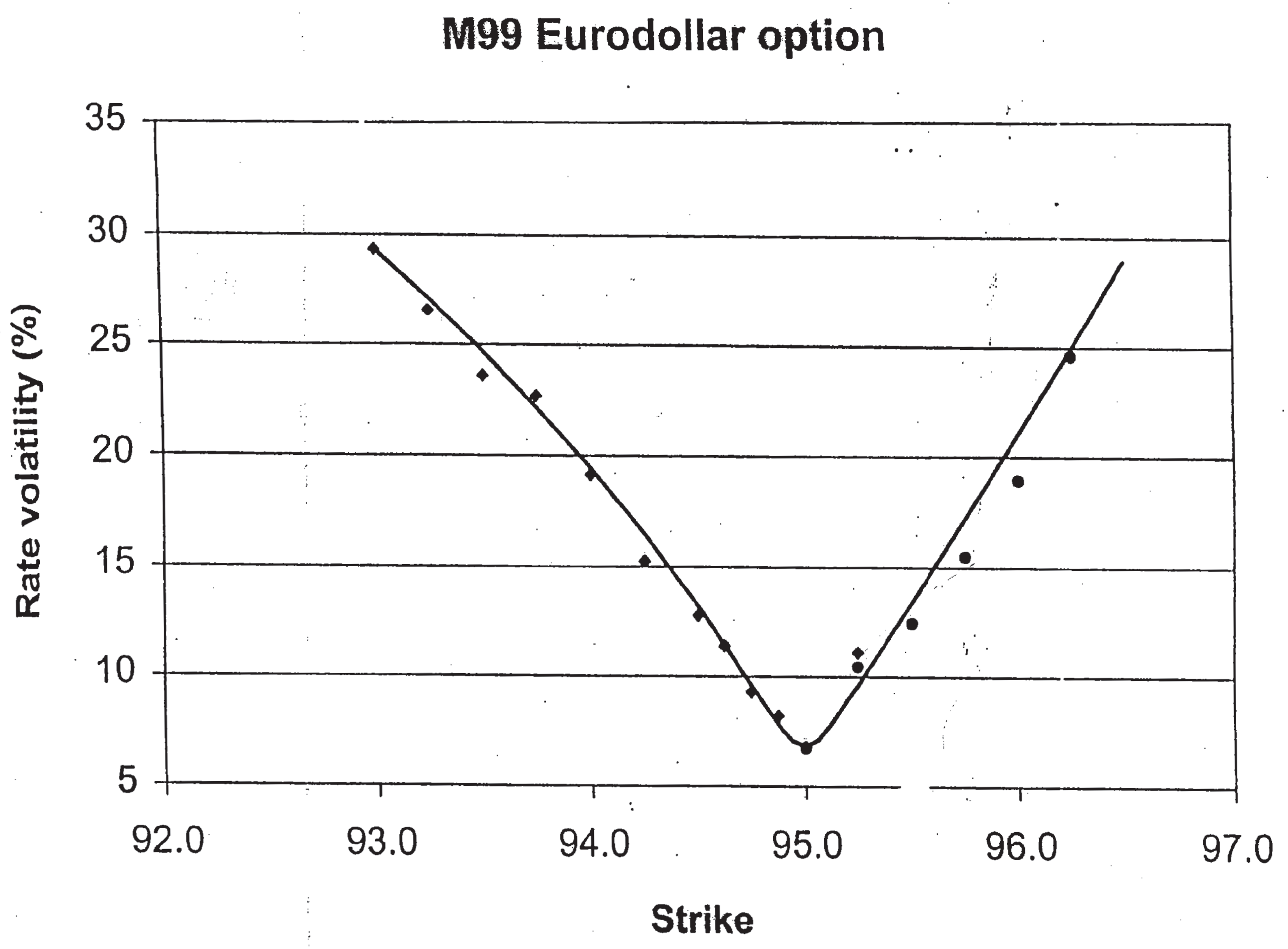
$$\left\{1 + [(1-\beta)^2 + 6\rho\beta\lambda + (2-3\rho^2)\lambda^2]\frac{\alpha^2 t_{ex}}{24(R_s^0 R_f)^{1-\beta}} + \cdots\right.$$

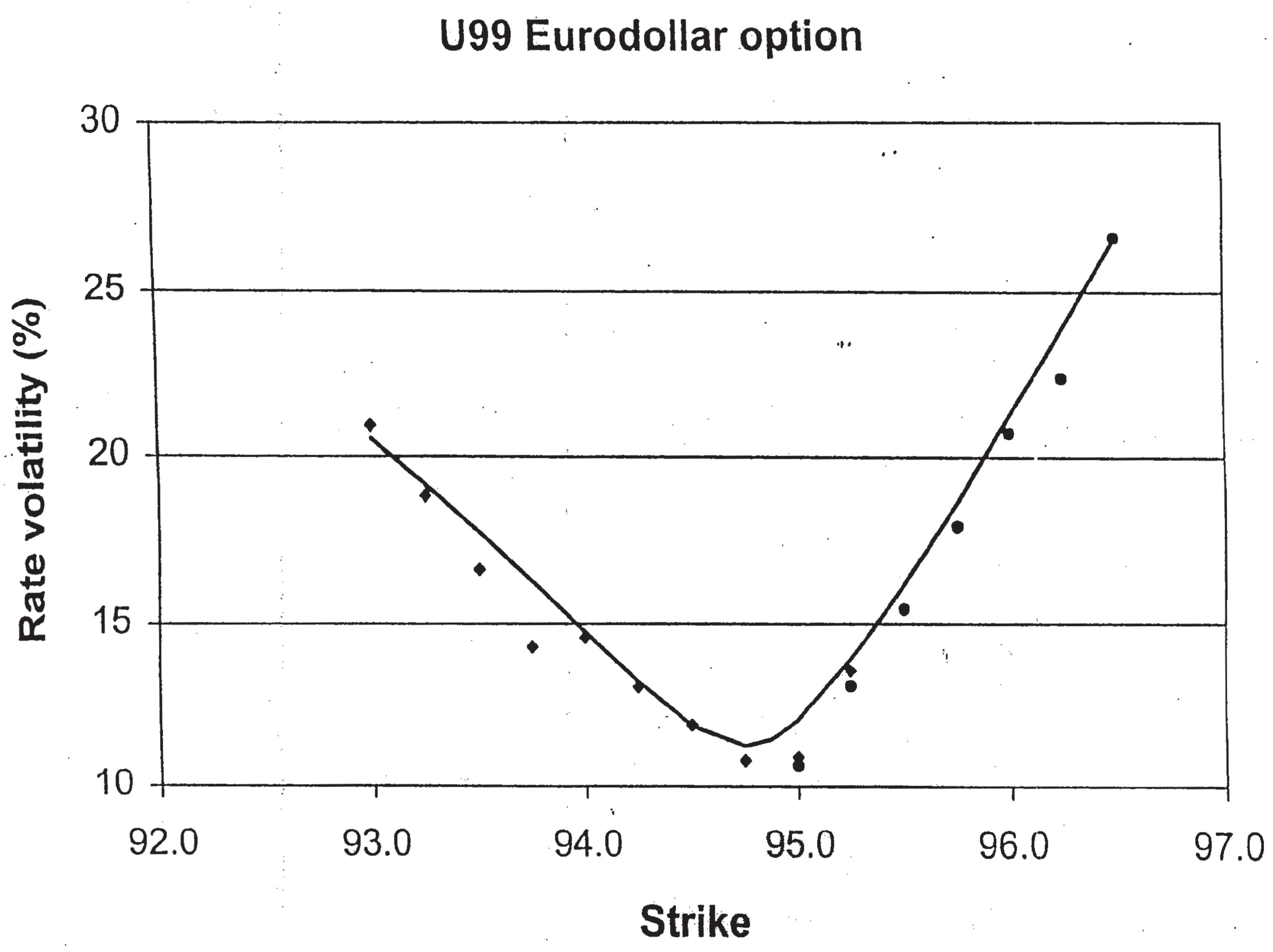
$$\lambda = \frac{\nu}{\alpha}(R_s^0 R_f)^{(1-\beta)/2} = \frac{\text{“vol of vol”}}{\text{“rate vol”}}$$

$$z = \frac{\nu}{\alpha}(R_s^0 R_f)^{(1-\beta)/2}\log R_s^0/R_f$$

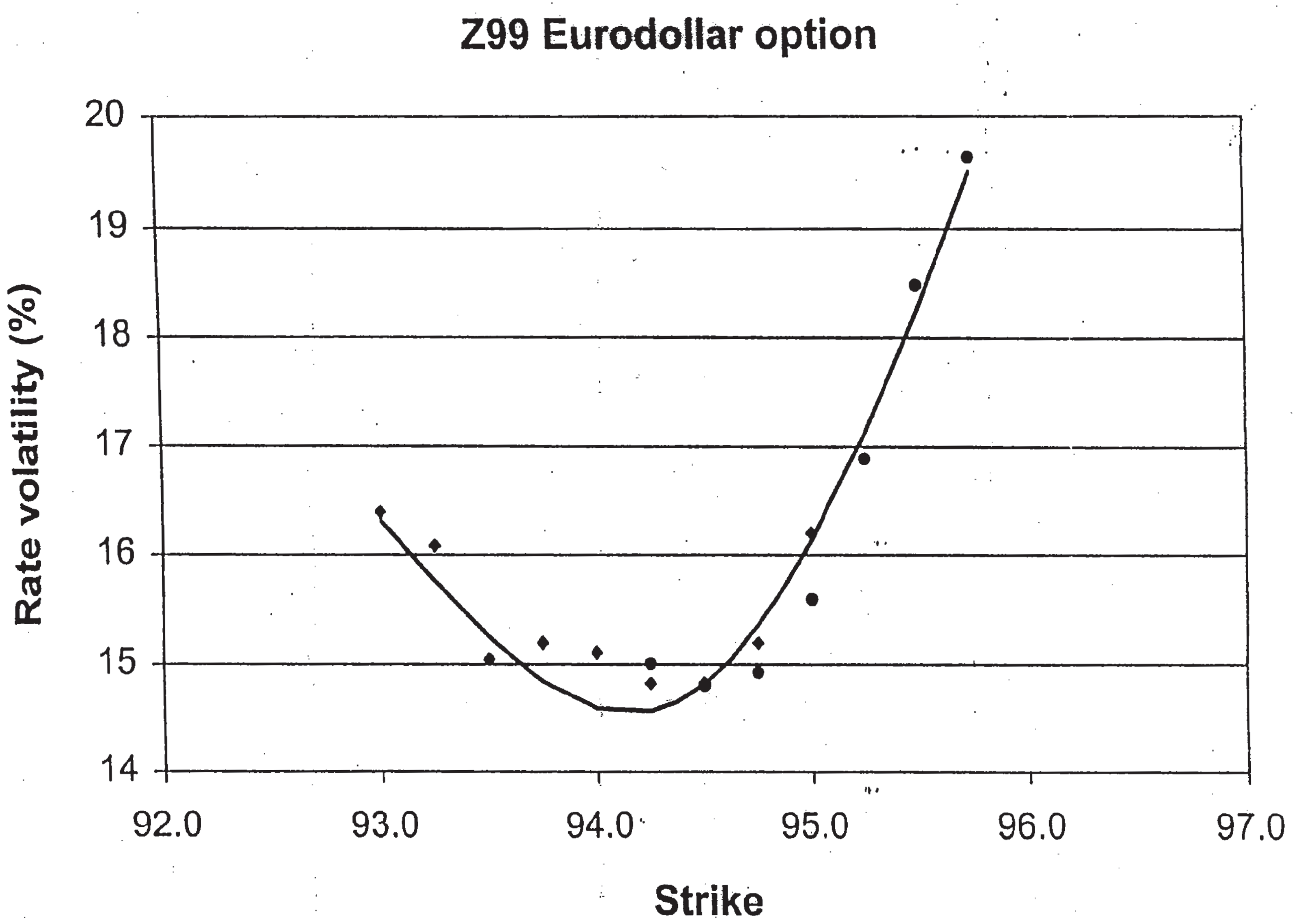
$$x(z) = \log\frac{z - \rho + \sqrt{1 - 2\rho z + z^2}}{1-\rho} \Leftrightarrow z = \sinh x - \rho(\cosh x - 1)$$

– closed form; extremely fast

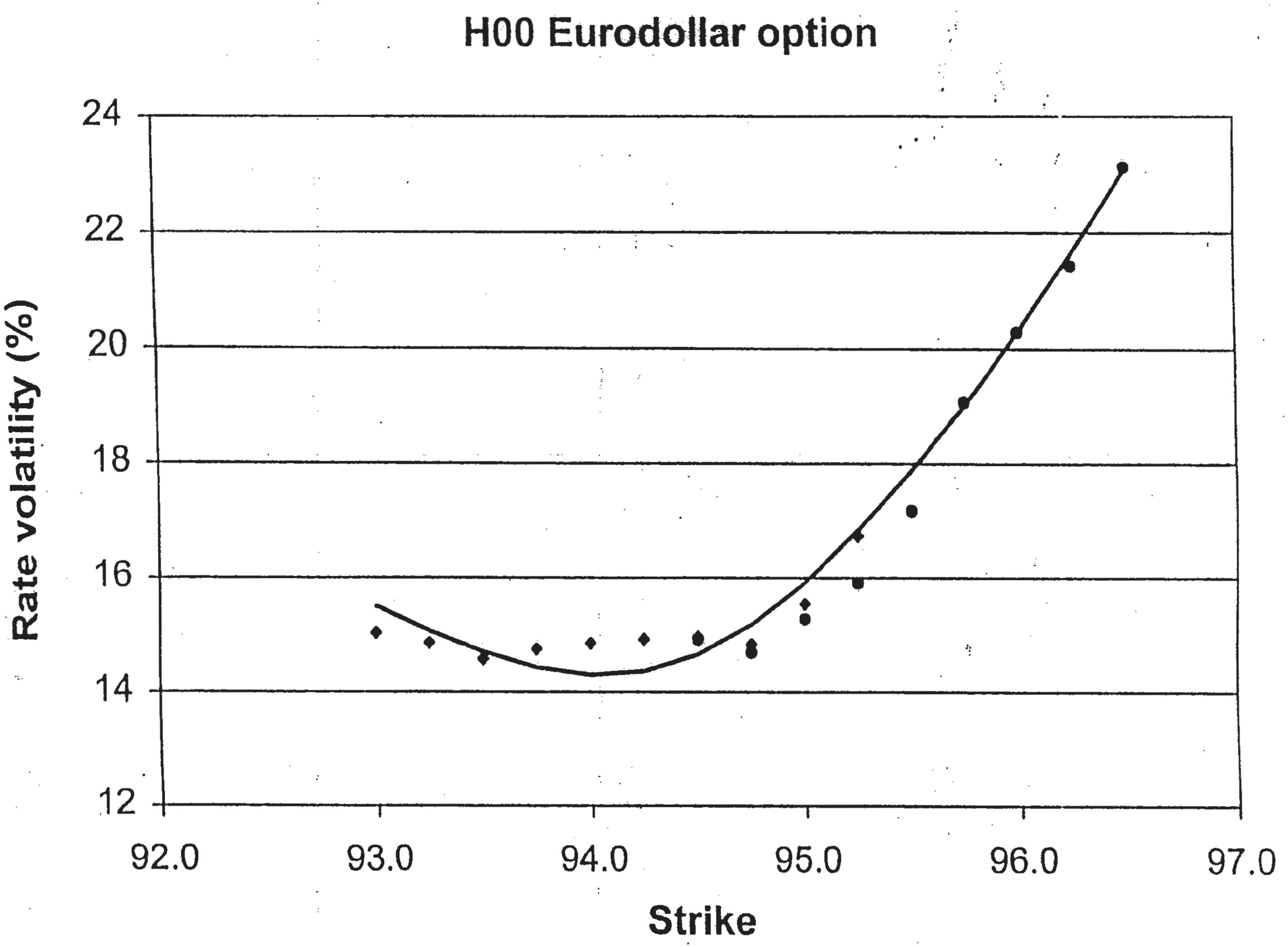

M99 Eurodollar option
Rate volatility (%)
35
30
25
20
15
10
5
92.0
93.0
94.0
95.0
96.0
97.0
Strike

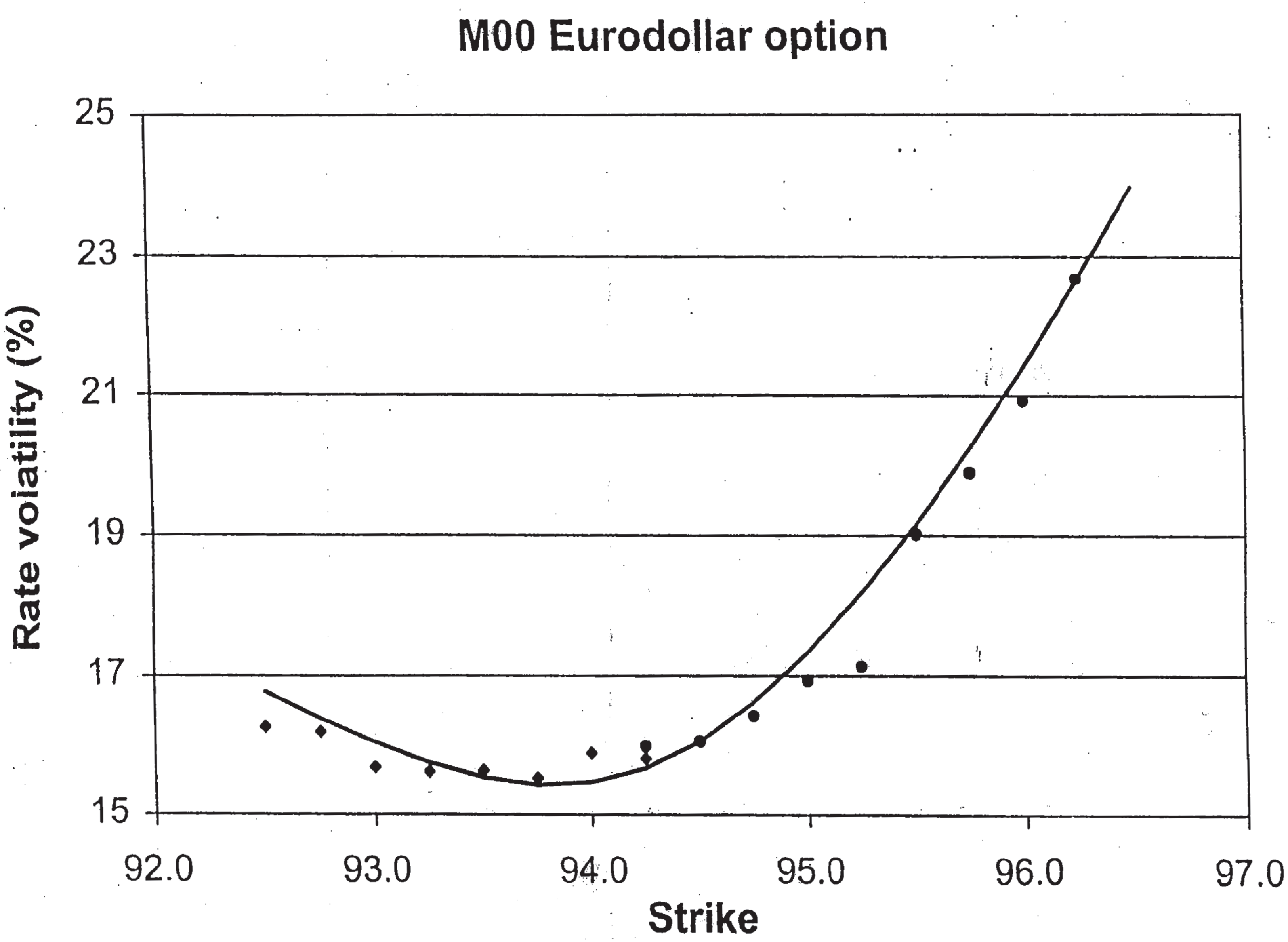


U99 Eurodollar option
Rate volatility (%)
30
25
20
15
10
92.0
93.0
94.0
95.0
96.0
97.0
Strike


Z99 Eurodollar option
Rate volatility (%)
20
19
18
17
16
15
14
92.0
93.0
94.0
95.0
96.0
97.0
Strike

Numerix


H00 Eurodollar option
Rate volatility (%)
24
22
20
18
16
14
12
92.0
93.0
94.0
95.0
96.0
97.0
Strike


M00 Eurodollar option
Rate volatility (%)
25
23
21
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17
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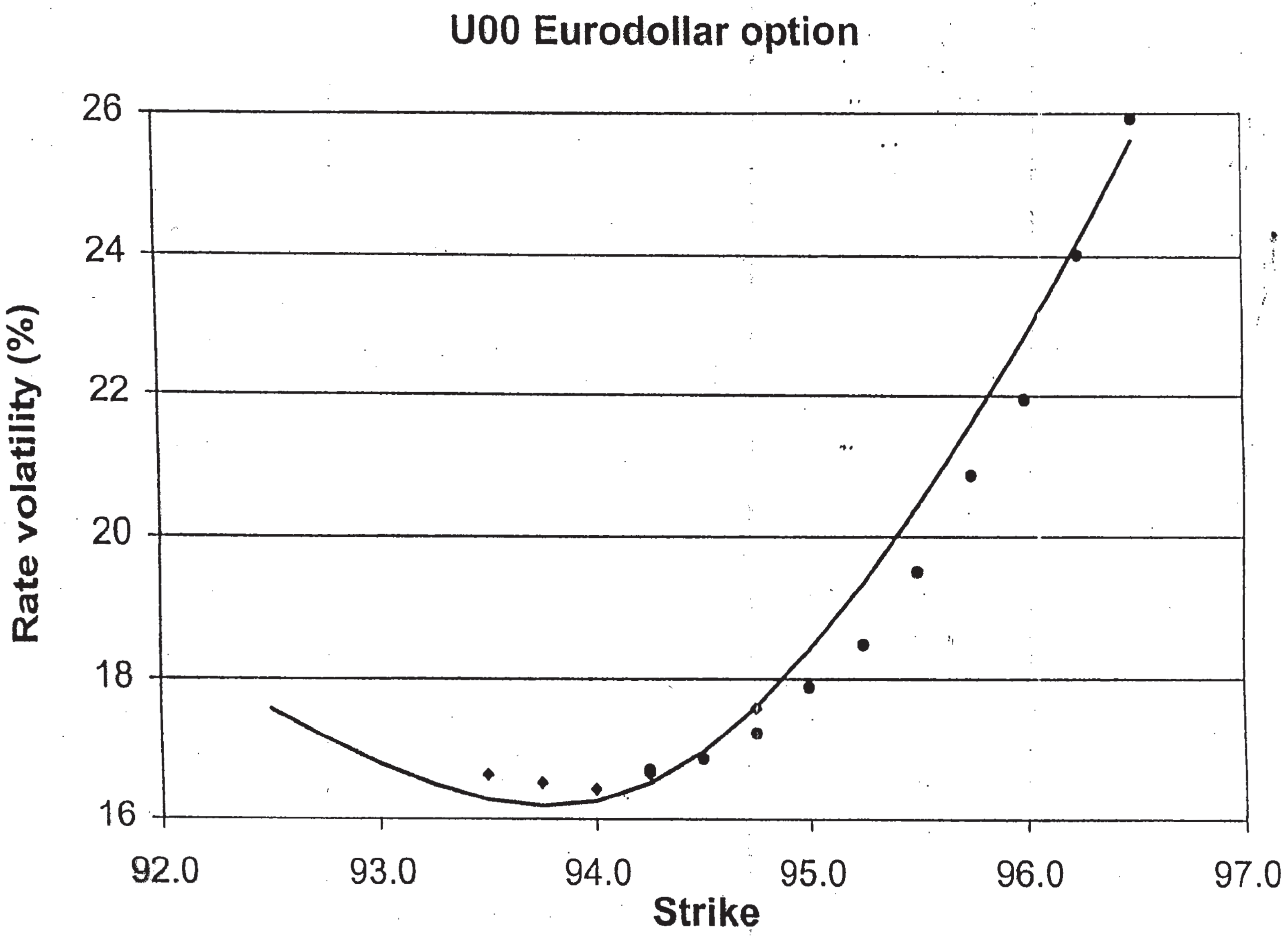
92.0
93.0
94.0
95.0
96.0
97.0
Strike



U00 Eurodollar option
Rate volatility (%)
16
18
20
22
24
26
92.0
93.0
94.0
95.0
96.0
97.0
Strike

NumeriX

Does it work?

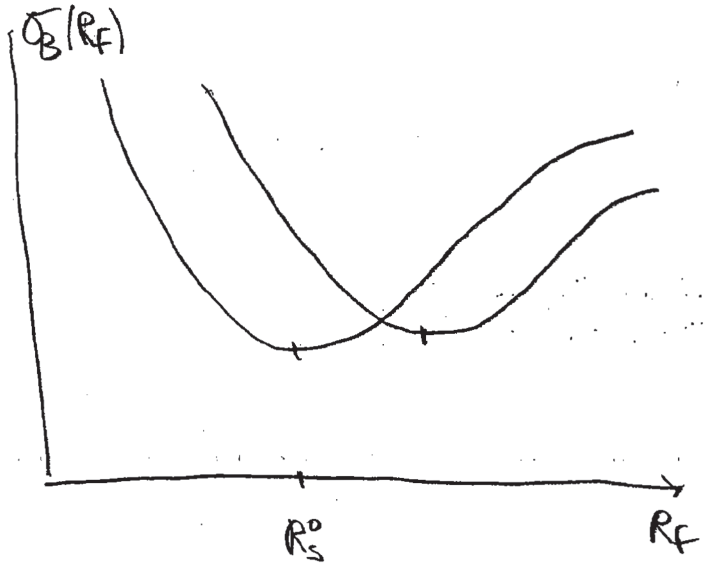
• SABR model

– matches fixed income market

Libor options, swaptions, caplets/floorlets, treasury options

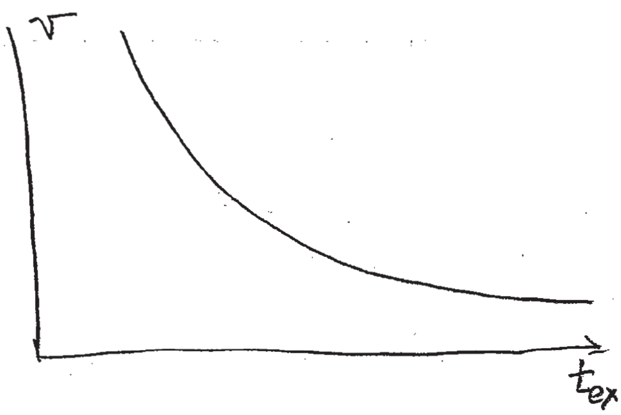
– gives stable hedges

– backbone & skew can be set independently



• Vol of vol vs. t_{ex}

suggests jump model (or truncated Levy model!)



Fitting parameters

- Smile/backbone:

$$\sigma \approx \sigma_{ATM}\left\{1 - (1 - \beta - \frac{\rho v}{\sigma_{ATM}})\frac{R_f - R_0}{2R_0} + (2 - 3\rho^2)\frac{v^2(R_f - R_0)^2}{\sigma_{ATM}^2 R_0^2} + \cdots\right.$$

with

$$\sigma_{ATM} = \frac{\alpha}{R_0^{(1-\beta)}}$$

– a or σ_{ATM} controls the ATM vol;
– ρ and β both control the skew, but β controls the "backbone"
– v controls the smile

– typically β is set from $\log \sigma_{ATM}$ vs $\log R_0$ studies ($\frac{1}{3}$, $\frac{1}{2}$, or 1)
– then ρ and v are fit to the smile

– typically a or σ_{ATM} are updated daily or every few hours,
– ρ and v are re-fitted every month or as needed
– β doesn't change

Managing smile risk

- Swaption prices are now from 4 matrices

σ_{ATM}	$3m$	$1y$	$2y$	$3y$	$\cdots$	$10y$
$1m$	5.25%	12.25%	13.50%	14.13%	$\cdots$	14.25%
$3m$	7.55%	13.00%	14.13%	14.38%	$\cdots$	14.50%
$6m$	11.44%	14.25%	14.18%	15.00%	$\cdots$	14.75%
$1y$	16.20%	16.75%	16.38%	16.13%	$\cdots$	15.50%
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$
$10y$	14.00%	13.50%	13.00%	12.50%	$\cdots$	11.00%

– similar matrices for β, ρ, ν
– parameter values are found by interpolating on matrix entries

- Risks

– books have swaptions at all different strikes
– exotic options give rise to risk at various strikes
Bermudan struck at 7.132% is calibrated to swaptions struck at 7.132%. Leads to vega risk at 7.132%

– bumping each parameter in turn, and re-valuing the book yields
a matrix of σ_{ATM} risks (vega),
a matrix of ρ risks (vanna)
a matrix of ν risks (volga)

Levy flights

$$dF = A(t,F)dt + B(t,F)dZ$$

- What is dZ?

– stationary, Markovian, independent increment

- Transition density:

$$g(T-t, y-x)dy = \text{prob}\{y < Z(T) \le y + dy \mid Z(t) = x\}$$

$$g(t,x) = \int_{-\infty}^{\infty} g(\tau,y)g(t-\tau, x-y)dy$$

- Fourier transform:

$$G(t,k) \equiv \int_{-\infty}^{\infty} e^{-ikx} g(t,x)dx$$

$$G(t,k) = G(\tau,k)G(t-\tau,k) \qquad \text{for each } k \text{ and } 0 \le \tau \le t$$

- For each k, solving the functional equation yields

$$G(t,k) = e^{-t\phi(k)} \qquad \text{for some } \phi(k)$$

- *No intrinsic time scale for dZ* $\Rightarrow$ similarity solution!

$$g(t,x) = \frac{1}{t^{1/\alpha}} h(x/t^{1/\alpha})$$

$$G(t,k) = \begin{cases} e^{-c_+ k^\alpha t} \\ e^{-c_- |k|^\alpha t} \end{cases} \qquad \text{for some } c_- = c_+^*$$

Truncated Levy distributions

$$g(t,x) = \frac{1}{t^{1/\alpha}} h(x/t^{1/\alpha}) \qquad \text{for } 1 \le \alpha < 2$$

$$G(t,k) = \begin{cases} e^{-c_+ k^\alpha t} & \text{for } k > 0 \\ e^{-c_- |k|^\alpha t} & \text{for } k < 0 \end{cases} \qquad \text{for some } c_- = c_+^*$$

$$g(t,x) \sim C_\pm \frac{t}{|x|^{1+\alpha}} \qquad \text{no variance!}$$

– experimentally, $\alpha \sim 1.4$

• $g(t,x)$ and option values $V(t,x)$ follow same equation

$$\frac{\partial V(t,x)}{\partial t} = \int K(y) \left\{ V(t,x-y) - V(x) + y \frac{\partial V(t,x)}{\partial y} \right\} dy$$

$$K(y) = \frac{C_\pm}{|y|^{1+\alpha}} \qquad \text{for } y \gtrless 0$$

– $K(y)$ = rate of jumps of size y
– no convergence at $t = t_{\exp}$

• truncate: $\quad K(y) = \dfrac{C_\pm e^{-\varepsilon|y|}}{|y|^{1+\alpha}} \qquad$ for $y \gtrless 0$

– core spreads out at at^γ, wings spread out slower, at $At^{1/2}$
– for $at^\gamma \ll (t/\varepsilon)^{1/2}$, strong smile/skew (Levy-like)
– for $at^\gamma \gg (t/\varepsilon)^{1/2}$, diffusion like (Brownian motion)
cross-over ~ 3 months
– three parameters: $c_+ = c_-^* = ce^{-i\eta}$, ε
intensity c governs atm vol, asymmetry η governs skew
ε governs both smile and crossover

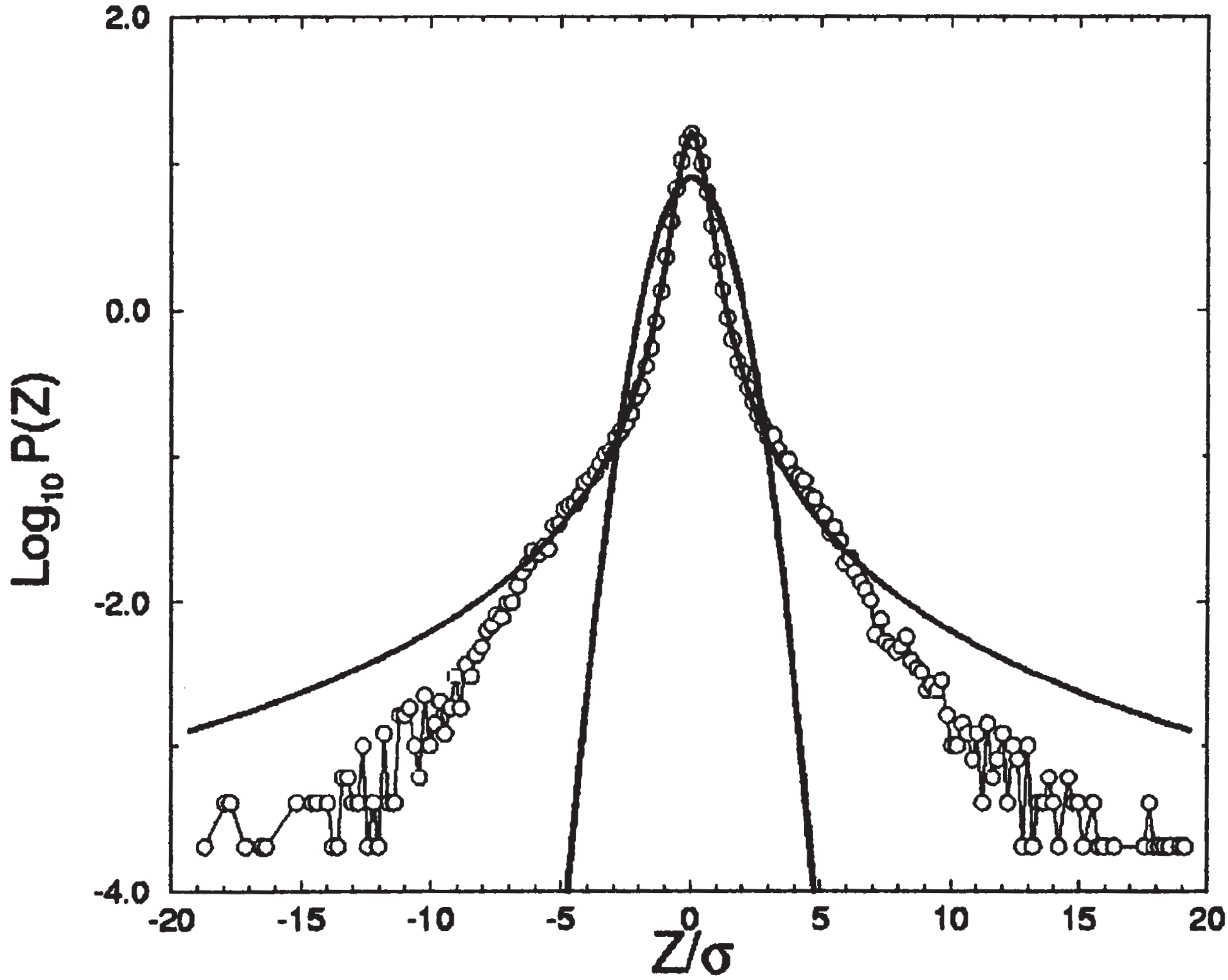


Fig. 5.6. Probability distribution $p_{\Delta t}(\ell)$ of the price variations ℓ [= for the S&P500, determined from all records between 1984 and million records). $p_{\Delta t}(\ell)$ [denoted by $P(Z)$] is plotted versus ℓ/σ where σ (= 0.0508) is the standard deviation calculated from the Gaussian distribution (*thick solid line*) corresponding to this value to the best fit with a Lévy distribution (*thin solid line*). The Lévy $\alpha = 1.4$ and $c = 3.75 \times 10^{-3}$, see (5.83)] gives a much better rep data for $\ell/\sigma \leq 6$. If $\ell/\sigma \geq 6$, the distribution of the index deca exponentially. Reproduced with permission from [105]

which decays from 1 (for $t = 0$) to 0 (for $t \to \infty$). This be zero for all $t > 0$ if the hypothesis were true. In practi