CQF Module 1 Answers

July 2016

1 Mathematical Methods

1. Consider the linear parabolic partial differential equation

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+a\frac{\partial u}{\partial x}+bu,$$

for the function $u(x,t)$; where a and b are constants. By using a substitution of the form

$$u(x,t)=e^{\alpha x+\beta t}v(x,t),$$

and suitable choice of α and β, show that the PDE can be reduced to the heat equation

$$\frac{\partial v}{\partial t}=\frac{\partial^2 v}{\partial x^2}.$$

This PDE has the same structure as the Black-Scholes Equation and the working here is used in part to reduce it to a one dimensional heat equation - hence very useful problem (much more on this later). Start by differentiating

$$\begin{aligned}\frac{\partial u}{\partial t}&=\left(\beta v+\frac{\partial v}{\partial t}\right)e^{\alpha x+\beta t}\\ \frac{\partial u}{\partial x}&=\left(\alpha v+\frac{\partial v}{\partial x}\right)e^{\alpha x+\beta t}\\ \frac{\partial^2 u}{\partial x^2}&=\left(\alpha^2 v+2\alpha\frac{\partial v}{\partial x}+\frac{\partial^2 v}{\partial x^2}\right)e^{\alpha x+\beta t}\end{aligned}$$

Substituting into the PDE we have

$$\left(\beta v+\frac{\partial v}{\partial t}\right)=\left(\alpha^2 v+2\alpha\frac{\partial v}{\partial x}+\frac{\partial^2 v}{\partial x^2}\right)+a\left(\alpha v+\frac{\partial v}{\partial x}\right)+bv,$$

rearrange to give

$$\frac{\partial v}{\partial t}=\frac{\partial^2 v}{\partial x^2}+(2\alpha+a)\frac{\partial v}{\partial x}+\left(\alpha^2+a\alpha+b-\beta\right)v.$$

To eliminate $\frac{\partial v}{\partial x}$ and v requires setting, in turn,

$$\begin{aligned}2\alpha+a&=0\\ \alpha^2+a\alpha+b-\beta&=0.\end{aligned}$$

Hence the choice is

$$\alpha=-\frac{1}{2}a \text{ and } \beta=b-\frac{1}{4}a^2$$

2. Consider the probability density function $f(x)$ given by

$$f(x)=\begin{cases}Ax^2\exp\left(-\lambda x^2\right) & x>0\\ 0 & x\leq 0\end{cases}.$$

Deduce that

$$A=4\sqrt{\frac{\lambda^3}{\pi}}.$$

Show that

$$\mathbb{E}[X]=\frac{2}{\sqrt{\pi\lambda}}.$$

By using integration by parts, or otherwise, deduce that for $n=0,1,2,...$ the even moments of this distribution are given by

$$\mathbb{E}\left[X^{2n}\right]=\frac{1.3...(2n+1)}{(2\lambda)^n}$$

and the odd moments are given by

$$\mathbb{E}\left[X^{2n+1}\right] = \frac{2}{\sqrt{\pi}} \frac{(n+1)!}{\lambda^{(2n+1)/2}}.$$

Start by using $\int_0^\infty f(x)\,dx = 1$. Hence $A\int_0^\infty x^2 \exp\left(-\lambda x^2\right) dx = 1$. Using the substitution $u = \sqrt{\lambda}x \rightarrow du = \sqrt{\lambda}dx$ which gives

$$\lambda^{-3/2}\int_0^\infty u^2 \exp\left(-u^2\right) du = 1 \text{ becomes } \lambda^{-3/2} I = 1.$$

A result we will use is

$$\int_0^\infty \exp\left(-x^2\right) dx = \sqrt{\pi}/2.$$

To evaluate I write

$$I = \int_0^\infty u\left(u \exp\left(-u^2\right)\right) du$$

and note from integration by substitution that

$$\int_0^\infty u \exp\left(-u^2\right) du = -\tfrac{1}{2}e^{-u^2} + c.$$

Integration by parts gives

$$\int_0^\infty u\left(u \exp\left(-u^2\right)\right) du = -\tfrac{u}{2}e^{-u^2}\Big|_0^\infty + \tfrac{1}{2}\int_0^\infty e^{-u^2} du = \tfrac{1}{4}\sqrt{\pi}$$

where we have used the earlier result.

$$\lambda^{-3/2} A \frac{\sqrt{\pi}}{4} = 1 \rightarrow A = \frac{4\lambda^{3/2}}{\sqrt{\pi}}.$$

To show $\mathbb{E}[X] = \frac{2}{\sqrt{\pi\lambda}}$, write $I_n = \mathbb{E}[X^n] =$

$$\frac{4\lambda^{3/2}}{\sqrt{\pi}}\int_0^\infty x^{n+2}e^{-\lambda x^2} dx; \text{ so } I_0 = 1.$$

$$\begin{aligned}
I_1 &= A\int_0^\infty x^3 e^{-\lambda x^2} dx = A\left(\frac{1}{\lambda^2}\int_0^\infty u^3 e^{-u^2} du\right) \\
&= \frac{A}{\lambda^2}\left(-\tfrac{1}{2}u^2 e^{-u^2}\Big|_0^\infty + \int_0^\infty u e^{-x^2} du\right) \\
&= \frac{A}{\lambda^2}\left(-\tfrac{1}{2}e^{-u^2}\Big|_0^\infty\right) = \frac{A}{2\lambda^2} = \frac{2}{\sqrt{\pi\lambda}}.
\end{aligned}$$

To consider odd and even moments, want to show

$$\mathbb{E}\left[X^{2n}\right] = \frac{1.3...(2n+1)}{(2\lambda)^n};\ \mathbb{E}\left[X^{2n+1}\right] = \frac{2}{\sqrt{\pi}}\frac{(n+1)!}{\lambda^{(2n+1)/2}}.$$

Put $I_n =$

$$\begin{aligned}
A\int_0^\infty x^{n+2}e^{-\lambda x^2} dx &= A\left(-\frac{1}{2\lambda}x^{1+n}e^{-\lambda x^2}\Big|_0^\infty + \frac{1+n}{2\lambda}\int_0^\infty x^n e^{-\lambda x^2} dx\right) \\
&= \frac{n+1}{2\lambda} I_{n-2} \\
I_{n+2} &= \frac{n+3}{2\lambda} I_n
\end{aligned}$$

$$I_2 = \frac{3}{2\lambda}I_0 = \frac{3}{2\lambda}; I_4 = \frac{3\times 5}{(2\lambda)^2};\ I_6 = \frac{3\times 5\times 7}{(2\lambda)^3}, ...$$

$$I_3 = \frac{4}{2\lambda}I_1 = \frac{4}{\sqrt{\pi}\lambda^{3/2}};\ I_5 = \frac{6}{2\lambda}I_3 = \frac{3}{\lambda}\times\frac{4}{\sqrt{\pi}\lambda^{3/2}} = \frac{2}{\sqrt{\pi}}\frac{1.2.3}{\lambda^{5/2}};\ I_7 = \frac{8}{2\lambda}I_5 = \frac{2}{\sqrt{\pi}}\frac{4!}{\lambda^{7/2}}, ...$$

Hence result.

2 Stochastic Calculus

$W,\ W(t), W_t$ all refer to standard Brownian motion

1. a. Itô's lemma can be used to deduce the following formula for stochastic differential equations and stochastic integrals

$$\int_0^t \frac{\partial F}{\partial W}\, dW(\tau) = F(W(t), t) - F(W(0), 0) - \int_0^t \left(\frac{\partial F}{\partial \tau} + \frac{1}{2}\frac{\partial^2 F}{\partial W^2}\right) d\tau$$

for a function $F(W(\tau), \tau)$ where $dW(\tau)$ is an increment of a Brownian motion.
If $W(0) = 0$ evaluate

$$\int_0^t \tau^2 \sin W dW(\tau).$$

$$\begin{array}{ll} \downarrow \dfrac{\partial F}{\partial W} = t^2 \sin W \longrightarrow & F = -t^2 \cos W \downarrow \\ \dfrac{\partial^2 F}{\partial X^2} = t^2 \cos W & \dfrac{\partial F}{\partial t} = -2t \cos W \end{array}$$

and substitute into the integral formula

$$\int_0^t \tau^2 \sin W\, dW(\tau) = -t^2 \cos W - \int_0^t \left(-2\tau \cos W + \frac{1}{2}\tau^2 \cos W\right) d\tau$$

b. Suppose the stochastic process $S(t)$ evolves according to Geometric Brownian Motion (GBM), where

$$dS = \mu S dt + \sigma S dW.$$

Obtain a SDE $df(S, t)$ for each of the following functions

i $f\ (S, t) = \alpha^t + \beta t S^n \qquad \alpha,\ \beta$ are constants

$$\begin{aligned} \frac{\partial f}{\partial t} &= \alpha^t \log a + \beta S^n;\ \frac{\partial f}{\partial S} = n\beta t S^{n-1};\ \frac{\partial^2 f}{\partial S^2} = n(n-1)\beta t S^{n-2} \\ df &= \left(\alpha^t \log a + n\mu\beta t S^n + \frac{1}{2} n(n-1)\beta t \sigma^2 S^n\right) dt + \sigma n \beta t S^n dW \end{aligned}$$

ii $f(S, t) = \log tS + \cos tS$

$$\begin{aligned} \frac{\partial f}{\partial t} &= \frac{1}{t} - S \sin tS;\ \frac{\partial f}{\partial S} = \frac{1}{S} - t \sin tS;\ \frac{\partial^2 f}{\partial S^2} = -\frac{1}{S^2} - t^2 \cot tS \\ df &= \left(\frac{1}{t} - S \sin tS + \mu S\left(\frac{1}{S} - t \sin tS\right) + \frac{1}{2}\sigma^2 S^2\left(-\frac{1}{S^2} - t^2 \cot tS\right)\right) dt + \sigma S\left(\frac{1}{S} - t \sin tS\right) dW \end{aligned}$$

2. Consider the following SDE

$$d\sigma = a dt + b dW,$$

where $a = a(\sigma, t),\ b = b(\sigma, t)$. The Forward Kolmogorov Equation (FKE), for the transition PDF $p = p(\sigma, t; \sigma', t')$ is

$$\frac{\partial p}{\partial t'} = \frac{1}{2}\frac{\partial^2}{\partial \sigma'^2}(b^2 p) - \frac{\partial}{\partial \sigma'}(ap),$$

where the primed variables refer to future states. The steady state solution is given by setting $\frac{\partial p}{\partial t'} = 0$. By considering suitable conditions, show that the steady state solution is given by

$$p(\sigma') = \frac{A}{b^2} e^{\int \frac{2a}{b^2} d\sigma'},$$

where A is an arbitrary constant. (During your working you may drop the primed notation).The Forward Kolmogorov Equation for the transition PDF $p(\sigma_t, t; \sigma'_t, t')$ is

$$\frac{\partial p}{\partial t'} = \frac{1}{2}\frac{\partial^2}{\partial \sigma_t'^2}(b^2(\sigma'_t, t')p) - \frac{\partial}{\partial \sigma'_t}(a(\sigma'_t, t')p),$$

where the primed variables refer to future states. The steady state solution is given by setting $\frac{\partial p}{\partial t'} = 0$. Considering the boundary conditions that as $\sigma'_t \to \pm\infty$, $p \to 0$ and $\dfrac{\partial p}{\partial \sigma'_t} \to 0$, show that the steady state solution is given by

$$p(\sigma'_t) = \frac{A}{b^2} e^{\int \frac{2a}{b^2} d\sigma'_t},$$

where the constant A is **not** to be calculated. (During your working you may drop the primed notation). At steady state $\frac{\partial p}{\partial t} = 0$. So the Forward Kolmogorov Equation is

$$\frac{1}{2}\frac{d^2}{d\sigma^2}(b^2 p) - \frac{d}{d\sigma}(ap) = 0$$

Integrating once we have

$$\frac{1}{2}\frac{d}{d\sigma}(b^2 p) = ap + C$$

Now using the boundary condition $\frac{\partial p}{\partial \sigma} \to 0$, we note that $C = 0$, so

$$\frac{1}{2}\frac{d}{d\sigma}(b^2 p) = ap$$

Now writing the above equation in the following way

$$\frac{d(b^2 p)}{b^2 p} = \frac{2a}{b^2} d\sigma$$

Integrating once again

$$\ln(b^2 p) + C = \int \frac{2a}{b^2} d\sigma$$

where C is a constant. Taking exponentials

$$B(b^2 p) = \exp\left(\int \frac{2a}{b^2} d\sigma\right),$$

where $B = e^C$

$$p(\sigma') = \frac{A}{b^2} e^{\displaystyle\int \frac{2a}{b^2} d\sigma}.$$

where $A = 1/B$.

3. Consider a function $V(t, S_t, r_t)$ where the two stochastic processes S_t and r_t evolve according to a two factor model given by

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dW_t^{(1)} \\ dr_t &= \gamma (m - r_t)\, dt + c\, dW_t^{(2)}, \end{aligned}$$

in turn. and where $\mathbb{E}\left[dW_t^{(1)} dW_t^{(2)}\right] = \rho dt$. The parameters μ, σ, γ, m and c are constant. Let $V(t, S_t, r_t)$ be a function on $[0, T]$ with $V(0, S_0, r_0) = v$. Using Itô, deduce the integral form for $V(T, S_T, r_T)$.
Begin by writing a 3D Taylor expansion for $V(t, S_t, r_t)$

$$\begin{aligned} & V(t+dt, S_t + dS, r_t + dv) - V(t, S_t, r_t) \\ = \; & \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial r}dr + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}dS^2 + \frac{1}{2}\frac{\partial^2 V}{\partial r^2}dr^2 + \frac{\partial^2 V}{\partial r \partial S}dr dS \end{aligned}$$

Since $dW_i^2 \to dt$ in the mean square limit for $i = 1, 2$, we see that

$$dS_t^2 \to \sigma^2 S_t^2 dt,$$
$$dr_t^2 \to c^2 dt,$$

Also, since $dW_t^{(1)} dW_t^{(2)} = \rho dt$, we see that

$$dS_t dr_t \to \rho c \sigma S_t dt$$

This gives us a *bivariate* version of Itô's Lemma, the SDE for V is given by

$$\begin{aligned} dV &= \left(\frac{\partial V}{\partial t} + \mu S_t \frac{\partial V}{\partial S} + \gamma (m - r_t) \frac{\partial V}{\partial r} \right. \\ &\quad \left. + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} + \frac{1}{2} c^2 \frac{\partial^2 V}{\partial r^2} + \rho c \sigma S_t \frac{\partial^2 V}{\partial r \partial S} \right) dt \\ &\quad + \sigma S_t \frac{\partial V}{\partial S} dW_t^{(1)} + c \frac{\partial V}{\partial r} dW_t^{(2)} \end{aligned}$$

Integrating over $[0, t]$, we get

$$\begin{aligned} V(t, S_t, r_t) &= \underbrace{V(0, S_0, r_0)}_{=v} + \int_0^t \left(\frac{\partial V}{\partial \tau} + \mu S_\tau \frac{\partial V}{\partial S} + \gamma (m - r_\tau) \frac{\partial V}{\partial r} \right. \\ &\quad \left. + \frac{1}{2} \sigma^2 S_\tau^2 \frac{\partial^2 V}{\partial S^2} + \frac{1}{2} c^2 \frac{\partial^2 V}{\partial r^2} + \rho c \sigma S_\tau \frac{\partial^2 V}{\partial r \partial S} \right) d\tau \\ &\quad + \int_0^t \sigma S_\tau \frac{\partial V}{\partial S} dW_\tau^{(1)} + \int_0^t c \frac{\partial V}{\partial r_\tau} dW_\tau^{(2)} \end{aligned}$$

4. A spot rate r_t, evolves according to the popular form

$$dr_t = u(r_t)\, dt + \nu r_t^\beta dW_t, \qquad (*)$$

where ν and β are constants. Suppose such a model has a **steady state transition probability density function** $p_\infty(r)$ that satisfies the forward Fokker Planck Equation.
Show that this implies the drift structure of $(*)$ is given by

$$u(r_t) = \nu^2 \beta r_t^{2\beta - 1} + \frac{1}{2} \nu^2 r_t^{2\beta} \frac{d}{dr} (\log p_\infty).$$

The forward F.P equation for $dr = u(r, t)\, dt + w(r, t)\, dW_t$ is

$$\frac{\partial p}{\partial t} = \frac{1}{2} \frac{\partial^2}{\partial r^2} \left(w^2(r, t)\, p(r, t) \right) - \frac{\partial}{\partial r} \left(u(r, t)\, p(r, t) \right)$$

for the probability density $p(r, t)$. The steady state equation for our model becomes

$$\frac{1}{2} \nu^2 \frac{d^2}{dr^2} \left(r^{2\beta} p_\infty(r) \right) - \frac{d}{dr} \left(u(r)\, p_\infty(r) \right) = 0$$

This can be simply integrated once to give

$$\begin{aligned} \frac{1}{2} \nu^2 \frac{d}{dr} \left(r^{2\beta} p_\infty(r) \right) - \left(u(r)\, p_\infty(r) \right) &= \text{const} \\ \frac{1}{2} \nu^2 \left(r^{2\beta} \frac{dp_\infty}{dr} \right) + \nu^2 \beta r^{2\beta - 1} p_\infty(r) - \left(u(r)\, p_\infty(r) \right) &= \text{const} \end{aligned}$$

The constant of integration is zero because as r becomes large

$$\left. \begin{matrix} p_\infty(r) \\ \frac{dp_\infty}{dr} \end{matrix} \right\} \longrightarrow 0$$

$$\begin{aligned} u(r)\, p_\infty(r) &= \frac{1}{2}\nu^2 r^{2\beta}\frac{dp_\infty}{dr} + \nu^2\beta r^{2\beta-1} p_\infty(r) \\ u(r) &= \frac{1}{2}\nu^2 r^{2\beta}\frac{1}{p_\infty(r)}\frac{dp_\infty}{dr} + \nu^2\beta r^{2\beta-1} \end{aligned}$$

We can write $\frac{1}{p_\infty}\frac{dp_\infty}{dr}$ as $\frac{d}{dr}(\log p_\infty)$

$$u(r) = \frac{1}{2}\nu^2 r^{2\beta}\frac{d}{dr}(\log p_\infty) + \nu^2\beta r^{2\beta-1}$$

5. The ordinary differential equation

$$\mu S\frac{du}{dS}+\tfrac{1}{2}\sigma^2S^2\frac{d^2u}{dS^2}=-1,$$

for the function $u(S)$ is to be **solved** with boundary conditions

$$\begin{aligned}u(S_0)&=0\\u(S_1)&=0.\end{aligned}$$

μ and σ are constants. Show that the solution is given by

$$u(S)=\frac{1}{\frac{1}{2}\sigma^2-\mu}\left(\log(S/S_0)-\frac{1-(S/S_0)^{1-2\mu/\sigma^2}}{1-(S_1/S_0)^{1-2\mu/\sigma^2}}\log(S_1/S_0)\right)$$

Hint: When solving for the particular integral, assume a solution of the form $C\log S$, where C is a constant.
Start by considering the homogeneous part of the equation i.e.

$$\mu S\frac{du}{dS}+\frac{1}{2}\sigma^2S^2\frac{d^2u}{dS^2}=0$$

and look for a solution of the form $u(S)=S^{\lambda}$. The resulting A.E. is

$$\lambda^2+\left(\tfrac{2\mu}{\sigma^2}-1\right)\lambda=0\longrightarrow\lambda=0,\left(1-\tfrac{2\mu}{\sigma^2}\right)$$

$$u(S)=A+BS^{1-2\mu/\sigma^2}$$

For convenience write $\nu=1-2\mu/\sigma^2$. For the homogeneous part, consider a solution $u(S)=C\log S$ where the coefficient C is to be determined, so

$$u'(S)=C/S;\quad u''(S)=-C/S^2;\quad \text{substitute this into the DE}$$

$$C\left(\mu-\frac{1}{2}\sigma^2\right)=-1\to C=\frac{1}{\frac{1}{2}\sigma^2-\mu}$$

so the general solution becomes

$$u(S)=A+BS^{\nu}+\frac{1}{\frac{1}{2}\sigma^2-\mu}\log S. \tag{A}$$

Now apply the boundary conditions:

$$u(S_0)=A+BS_0^{\nu}+\frac{1}{\frac{1}{2}\sigma^2-\mu}\log S_0=0 \tag{a}$$

$$u(S_1)=A+BS_1^{\nu}+\frac{1}{\frac{1}{2}\sigma^2-\mu}\log S_1=0 \tag{b}$$

$(b)-(a)$ gives

$$B=-\frac{1}{\frac{1}{2}\sigma^2-\mu}\frac{\log(S_1/S_0)}{S_1^{\nu}-S_0^{\nu}}.$$

Substituting back into (a)

$$A=-\frac{1}{\frac{1}{2}\sigma^2-\mu}\left(\log S_0-\frac{\log(S_1/S_0)}{S_1^{\nu}-S_0^{\nu}}S_0^{\nu}\right)$$

Now substitute back in (A) and after some algebraic manipulation the result is obtained.

3 Binomial Model

1. A share price is currently £15. At the end of three months, it will be either £13 or £17. By constructing a hedged portfolio, calculate the value of a three-month European option with payoff $\max\left(S^2 - 159, 0\right)$, where S is the share price at the end of three months. The risk-free rate is 5% per annum with continuous compounding.

$$\text{Stock} \quad \begin{array}{ccc} & & 17 \\ & \nearrow & \\ 15 & & \\ & \searrow & \\ & & 13 \end{array} \qquad \text{Option:} \quad \begin{array}{ccc} & & 130 \\ & & \left(= \max\left(17^2 - 159\right)\right) \\ & \nearrow & \\ V & & \\ & \searrow & \\ & & 10 \\ & & \left(= \max\left(13^2 - 159\right)\right) \end{array}$$

Now set up a Black-Scholes hedged portfolio, $V - \Delta S$, then binomial tree for its value is

$$\begin{array}{cc} & 130 - 17\Delta \\ V - 15\Delta & \\ & 10 - 13\Delta \end{array}$$

For risk-free portfolio choose Δ such that $130 - 17\Delta = 10 - 13\Delta \Rightarrow \Delta = 30$. So in absence of arbitrage and the risk free return of 5% p.a,

$$\begin{aligned} V - 15\Delta &= e^{-0.05\times 3/12}\left(130 - 17\Delta\right) \\ &= 30\left(15 - 17e^{-0.0125}\right) + 130e^{-0.0125}, \end{aligned}$$

and $V = 74.72$.

2. Consider the following model risk-free interest rate $r = 0$:

ω	$S(0)$	$S(1)$	$S(2)$
ω_1	S	aS	a^2S
ω_2	S	aS	S
ω_3	S	$a^{-1}S$	S
ω_4	S	$a^{-1}S$	$a^{-2}S$

S is the initial asset value at $t = 0$ and $a > 1$ is a constant. Asset:

$$\begin{array}{ccccc} & & & & a^2S \\ & & & \nearrow & \\ & & aS & & \\ & \nearrow & & \searrow & \\ S(0) = S & & & & S \\ & \searrow & & \nearrow & \\ & & \frac{1}{a}S & & \\ & & & \searrow & \\ & & & & \frac{1}{a^2}S \end{array}$$

Option:

$$\begin{array}{ccccc} & & & & a^2S - S \\ & & & \nearrow & \\ & & V_1 & & \\ & \nearrow & & \searrow & \\ V_3 & & & & 0 \\ & \searrow & & \nearrow & \\ & & V_2 & & \\ & & & \searrow & \\ & & & & 0 \end{array}$$

(a) In this model, replicate the **European call** option with strike equal to the initial asset value S over the two periods and so find the fair price of the option.
Replicate backwards over each one-period:

$$\text{Asset:}\quad \begin{array}{ccc} & & a^2S \\ & \nearrow & \\ aS & & \\ & \searrow & \\ & & S \end{array} \qquad \text{Option:}\quad \begin{array}{ccc} & & a^2S-S \\ & \nearrow & \\ V_1 & & \\ & \searrow & \\ & & 0 \end{array}$$

Replicate with ϕ units of stock and ψ bonds; so solve

$$\left.\begin{array}{c} a^2S\phi+\psi=a^2S-S \\ S\phi+\psi=0 \end{array}\right\} \quad \rightarrow \quad \phi=1, \psi=-S$$
$$V_1 \quad = \quad 1\times aS-S\times 1=(a-1)\,S.$$

$$\text{Asset:}\quad \begin{array}{ccc} & & S \\ & \nearrow & \\ \frac{1}{a}S & & \\ & \searrow & \\ & & \frac{1}{a^2}S \end{array} \qquad \text{Option:}\quad \begin{array}{ccc} & & 0 \\ & \nearrow & \\ V_2 & & \\ & \searrow & \\ & & 0 \end{array}$$

Clearly $V_2=0$ as both final nodes are worthless.

$$\text{Asset:}\quad \begin{array}{ccc} & & aS \\ & \nearrow & \\ S & & \\ & \searrow & \\ & & \frac{1}{a}S \end{array} \qquad \text{Option:}\quad \begin{array}{ccc} & & (a-1)\,S \\ & \nearrow & \\ V_3 & & \\ & \searrow & \\ & & 0 \end{array}$$

Usual replication with ϕ units of stock and ψ bonds; so solve

$$\left.\begin{array}{c} aS\phi+\psi=(a-1)\,S \\ \frac{1}{a}S\phi+\psi=0 \end{array}\right\} \quad \rightarrow \quad \phi=\frac{a\,(a-1)}{a^2-1}, \psi=-\frac{(a-1)\,S}{a^2-1}$$
$$\therefore \quad V_3=\frac{a\,(a-1)}{a^2-1}S-\frac{(a-1)\,S}{a^2-1}\times 1=\frac{a-1}{a+1}S.$$

Hence the price of the European call struck at S is $\frac{a-1}{a+1}S$.

b. Find all the one period risk-neutral probabilities and the corresponding probability measure $\mathbb{Q}$ on $\Omega=\{\omega_1,\omega_2,\omega_3,\omega_4\}$. Confirm that $\mathbb{E}^{\mathbb{Q}}[X]$ is the fair price.
These are when $r=0$,

$$\begin{aligned} q\,(\text{up}) &= \frac{s-s_d}{s_u-s_d} \\ q\,(\text{down}) &= \frac{s_u-s}{s_u-s_d} \end{aligned}$$

For each time-step we have the probabilities:

$$\begin{aligned} q\,(\text{up}) &= \frac{S-\frac{1}{a}S}{aS-\frac{1}{a}S}=\frac{1}{a+1}, \\ q\,(\text{up}) &= \frac{aS-S}{aS-\frac{1}{a}S}=\frac{a}{a+1}. \end{aligned}$$

$\therefore$

$$\begin{aligned}\omega_1 &= \frac{1}{(a+1)^2}\\ \omega_2 &= \frac{a}{(a+1)^2}\\ \omega_3 &= \frac{a}{(a+1)^2}\\ \omega_3 &= \frac{a^2}{(a+1)^2}.\end{aligned}$$

So the expected value is:

$$\mathbb{E}^{\mathbb{Q}}[X] = \sum_{\omega} p(\omega) X(\omega) = p(\omega_1)\left(a^2 S - S\right) + 0 + 0 + 0 = \frac{a-1}{a+1}S,$$

as before!

(b) Now consider a model where in each period the asset can either double or half. Show that the value of an option struck at the initial asset value S is $S/3$.
This is a special case of the above model when $a = 2$. Substituting in $a = 2$ into the option gives

$$\frac{2-1}{2+1}S = \frac{1}{3}S.$$

3. Repeat problem 1. by a replicating strategy. By calculating the risk-neutral probabilities obtain a price using an expectation.
Using earlier tree diagrams, the replicating strategy gives

$$\begin{aligned}17\phi + \psi e^{rt} &= 130\\ 13\phi + \psi e^{rt} &= 10\end{aligned}$$

hence $\phi = 30$. $\psi = -375.3$. Substituting into (where s is the initial stock price)

$$V_0 = \frac{x_u - x_d}{s_u - s_d}s + e^{-r}\frac{x_d s_u - x_u s_d}{s_u - s_d}$$

and

$$x_u = 130;\ x_d = 10;\ s_u = 17;\ s_d = 13$$

gives 74.56.
The risk-neutral probabilities

$$\begin{aligned}q(\text{up}) &= \frac{e^{rt}s - s_d}{s_u - s_d} = 0.54875\\ q(\text{down}) &= \frac{s_u - e^{rt}s}{s_u - s_d} = 0.45125\end{aligned}$$

The option value as an expectation is

$$e^{-rt}\left(q(\text{up}) \times x_u + q(\text{down}) \times x_d\right) = 0.98763\,(75.85) = 74.9$$

4. A share price is currently £80. At the end of three months, it will be either £84 or £76. Ignoring interest rates, calculate the value of a three-month **digital** call option with strike price £79. So strike $K = 80$

$$\text{Stock}\quad \begin{array}{ccc} & & 84\\ & \nearrow & \\ 80 & & \\ & \searrow & \\ & & 76\end{array} \qquad \text{Option:}\quad \begin{array}{ccc} & & 1\\ & & (84 > K)\\ & \nearrow & \\ V & & \\ & \searrow & \\ & & 0\\ & & (76 < K)\end{array}$$

Now set up a Black-Scholes hedged portfolio, $V - \Delta S$, then binomial tree for its value is

$$\begin{array}{ccc} & & 1 - 84\Delta \\ V - 80\Delta & & \\ & & -76\Delta \end{array}$$

For risk-free portfolio choose Δ such that $1 - 84\Delta = -76\Delta \Rightarrow \Delta = 1/8$. So in absence of arbitrage, $V - 80\Delta = 1 - 84\Delta$, and $V = 0.5$.