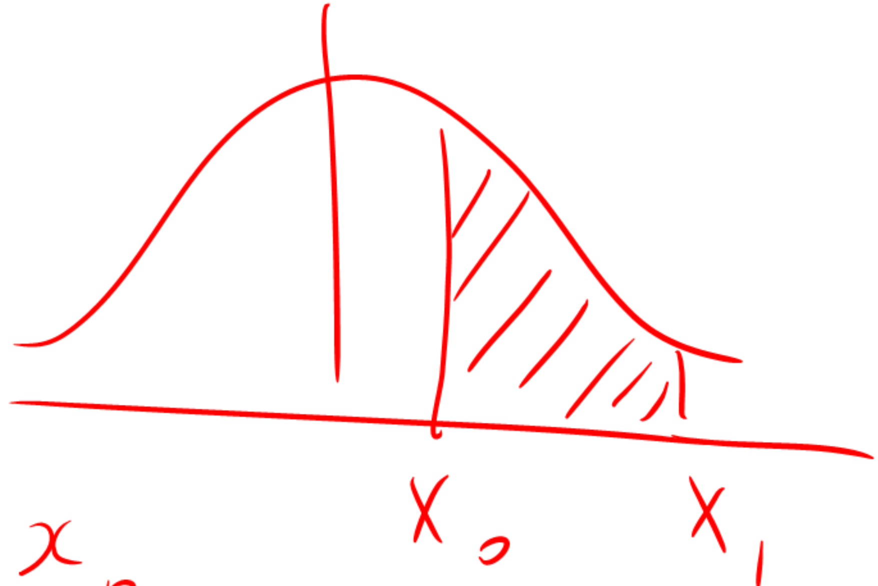
$$\int_{x_0}^{x_1} e^{-s^2}\, ds$$



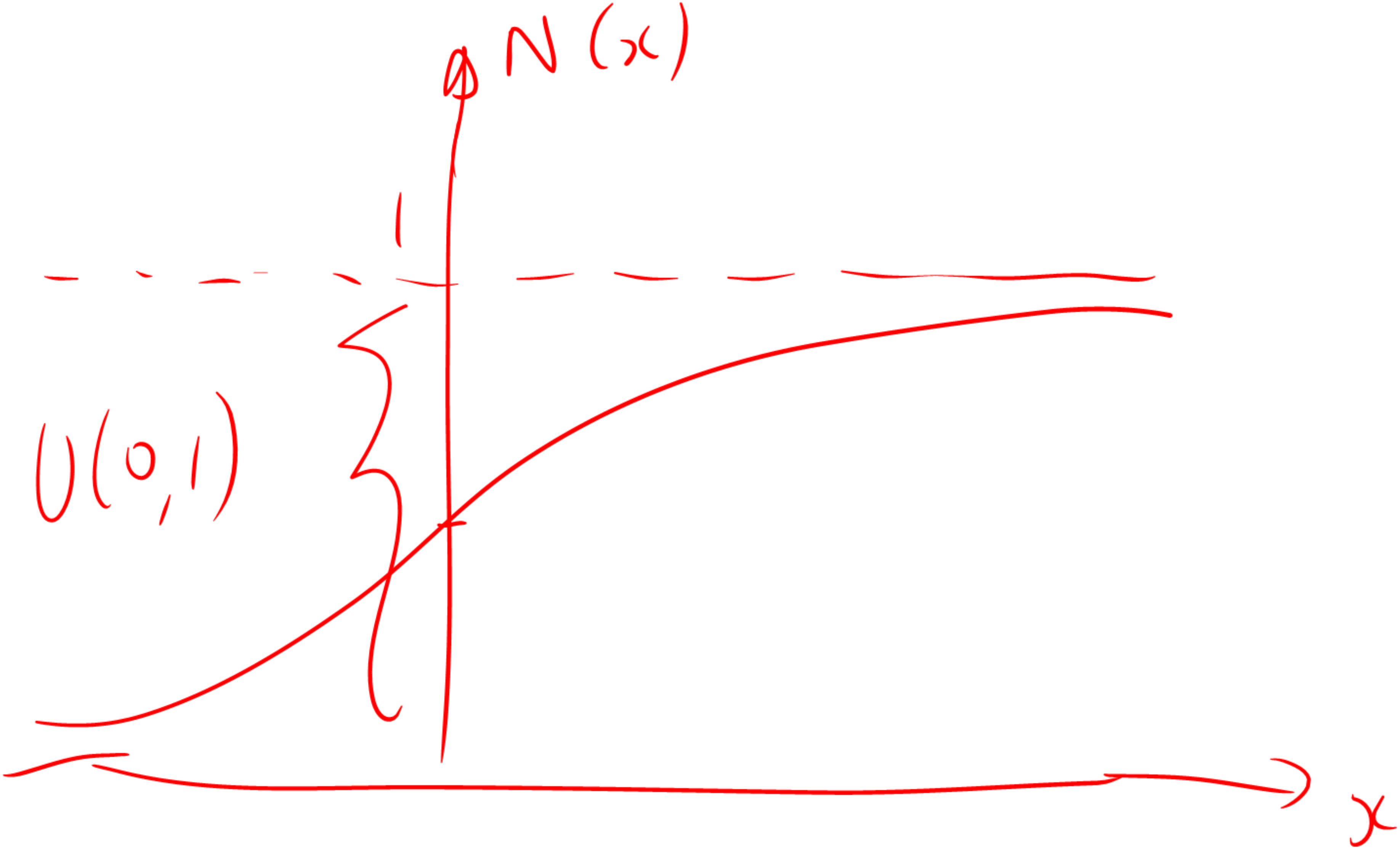
$$= \int_{-\infty}^{x_1} e^{-s^2} - \int_{-\infty}^{x_0} e^{-s^2}$$

$$= \left[\int_{-\infty}^{0} + \int_{0}^{x_1} \right] - \left[\int_{-\infty}^{0} + \int_{0}^{x_0} \right]$$

$$\frac{\sqrt{\pi}}{2}\left[\frac{2}{\sqrt{\pi}}\int_0^{x_1} e^{-s^2}\,ds - \frac{2}{\sqrt{\pi}}\int_0^{x_0} e^{-s^2}\,ds\right]$$

$$\frac{\sqrt{\pi}}{2}\left[\operatorname{erf}(x_1) - \operatorname{erf}(x_0)\right]$$

$$= \int_{x_0}^{x_1} e^{-s^2}\,ds$$


N(x)
1
U(0,1)
x

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}s^2}\, ds = \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{-\frac{1}{2}s^2}\, ds}_{0.5}$$

$$+ \underbrace{\frac{1}{\sqrt{2\pi}} \int_{0}^{x} e^{-\frac{1}{2}s^2}\, ds}$$

Put $u = s/\sqrt{2} \rightarrow \sqrt{2}\, du = ds$

$$\frac{1}{\sqrt{2\pi}} \int_{0}^{x/\sqrt{2}} e^{-u^2} \cdot \sqrt{2}\, du = \left(\frac{2}{\sqrt{\pi}} \int_{0}^{x/\sqrt{2}} e^{-u^2}\, du \right) \frac{1}{2}$$

$$= \frac{1}{2}\, \mathrm{erf}\left(x/\sqrt{2} \right)$$

$$\frac{1}{2}\left[1+\operatorname{erf}\left(x/\sqrt{2}\right)\right] = N(x)$$

We know $\underbrace{N(x)}_{y} \sim U(0,1)$

$$y = \frac{1}{2}\left(1+\operatorname{erf}\left(x/\sqrt{2}\right)\right)$$

$$\sqrt{2}\,\operatorname{erf}^{-1}(2y-1) = x \quad \therefore \; x \sim N(0,1)$$

Python erfinv(.)

is the inverse error function

$$x = (2**0.5)\,\text{erfinv}(2*U(0,1)-1)$$

'Inverse Transform Sampling'

$M = 2;\quad N = 3;\quad L = 4$

(M, N, L) M blocks of $(N \times L)$
matrices

$M \times N \times L$ elements

$A = \text{zeros}((M, N, L))$

Consistent $|A| \neq 0$

No. of eq^ns = No. of unknowns

Inconsistent $|A| = 0$

No. of eq^ns > No. of unknowns

$\nexists$ sol^n

No. of eq^ns < No. of unknowns $\Rightarrow$ ∞ sol^ns

$$V_n^{m+1} = F(\underbrace{V_{n-1}^{m}, V_n^{m}, V_{n+1}^{m}}_{\text{Known}})$$

$$G(V_{n-1}^{m+1}, V_n^{m+1}, V_{n+1}^{m+1}) = V_n^{m}$$

$$a_n V_{n-1}^{m+1} + b_n V_n^{m+1} + c_n V_{n+1}^{m+1} = V_n^{m}$$

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

R1 $|a| > |b| + |c|$ ✓

$|e| > |d| + |f|$ ✓

$|i| > |g| + |h|$ ✓

S.D.D $\Rightarrow$ Inv. exists

Numerical Integration

To approximate $\int_a^b f(x)\,dx = I$

Trapezoidal

$$h = \frac{b-a}{n}$$

$$I \approx \frac{h}{2}\left[\underbrace{f_0}_{a} + 2\sum_{i=1}^{n-1} f_i + \underbrace{f_n}_{b}\right]$$

Simpson

$$I \approx \frac{h}{3}\left[\underbrace{f_0}_{a} + 2\sum_{i=1}^{N-1} f_{2i} + 4\sum_{i=1}^{N} f_{2i-1} \quad \underbrace{f_N}_{b}\right]$$

Black-Scholes

$$C_B(S,t) = S N(d_1) - E e^{-r(T-t)} N(d_2)$$

$$d_{1,2} = \frac{\log\left(\frac{S}{E}\right) + \left(r \pm \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

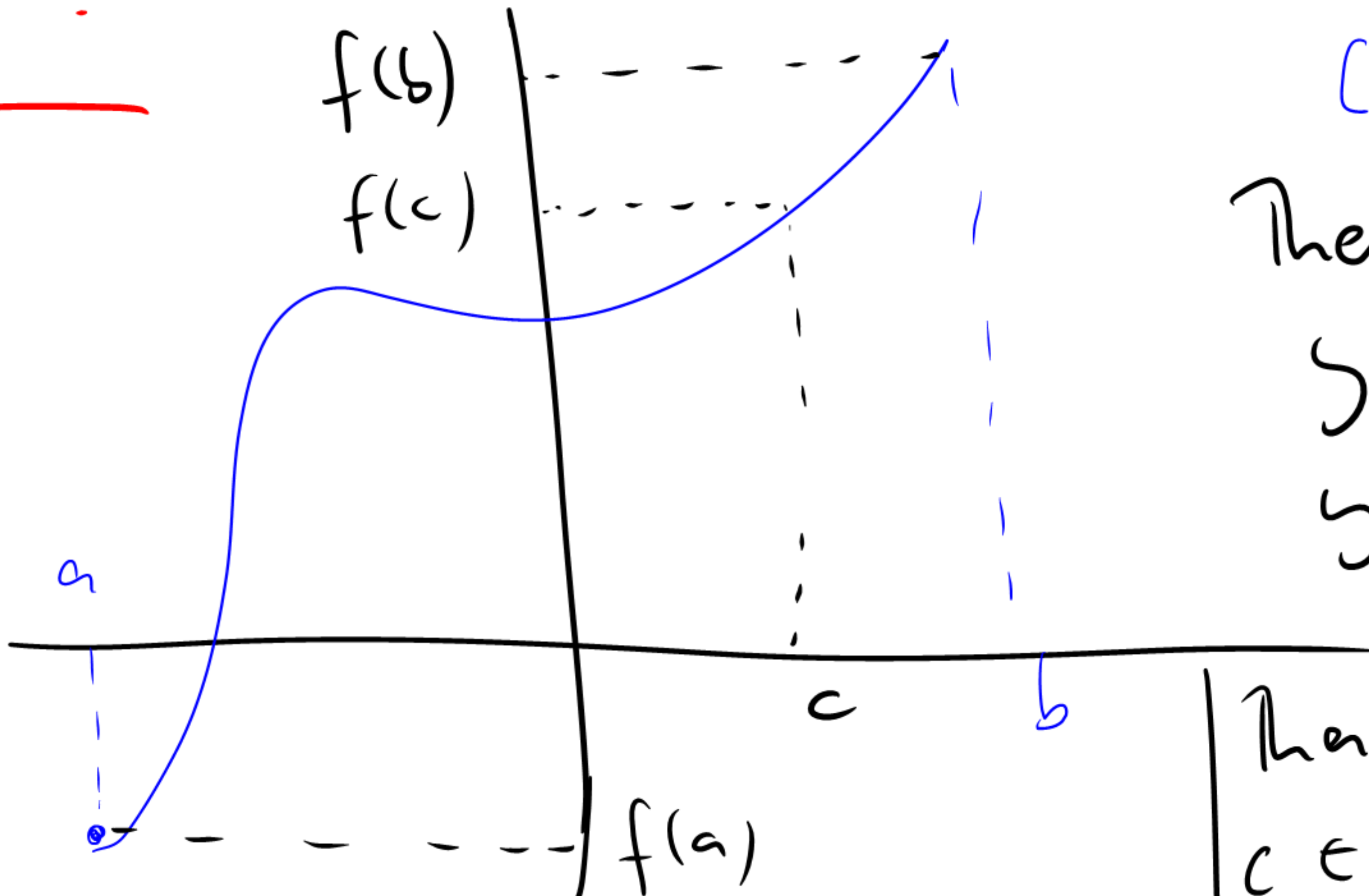
Market price C_M

$C_B(\sigma) = C_M \longrightarrow C_B(\sigma) - C_M = 0$

$\longrightarrow \sigma = \sigma_{imp}$.

Method of Bisection

I.V.T:



f(x) cts on [a,b]

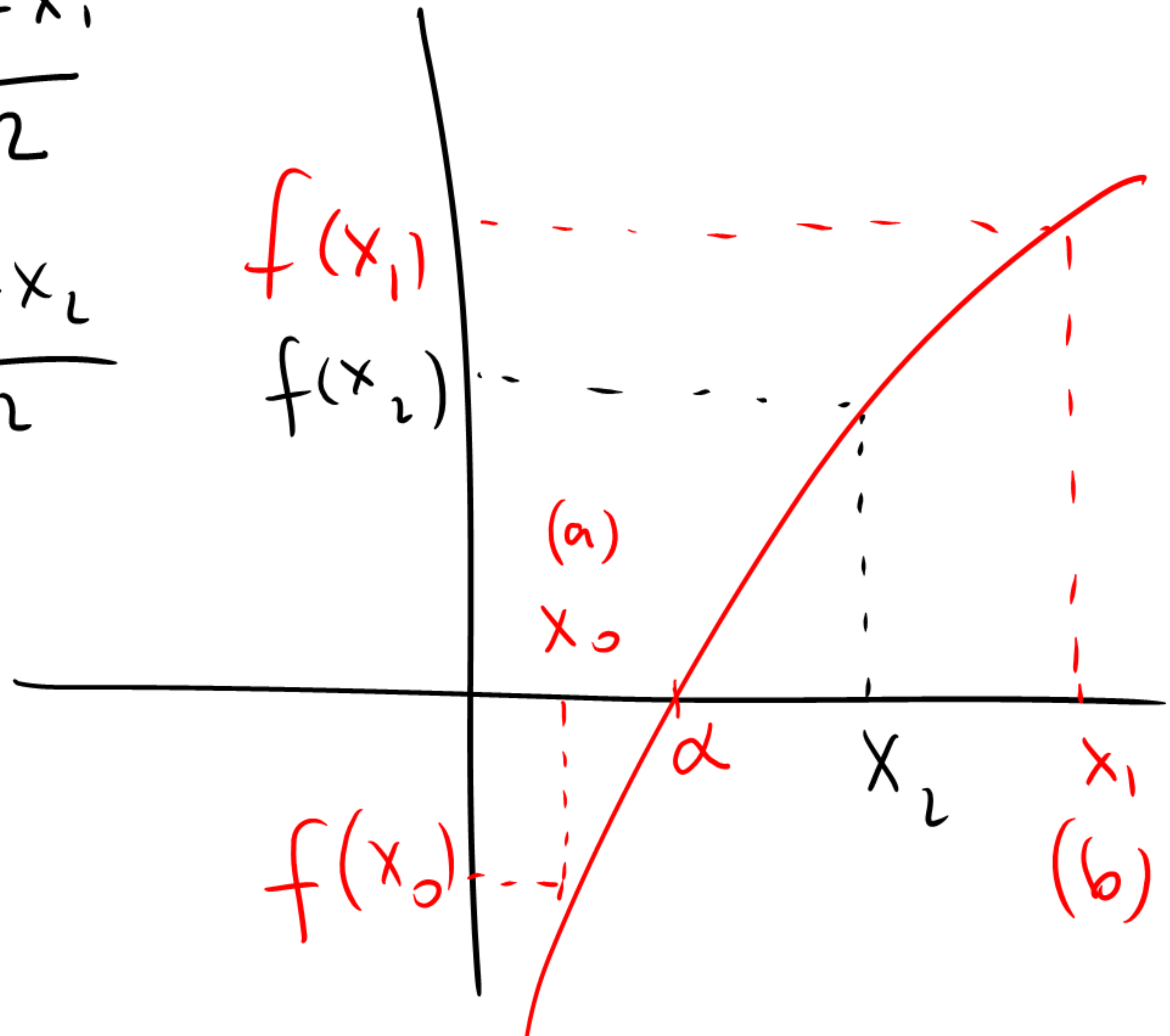
Then for any y s.t $y \in [f(a), f(b)]$

Then $\exists$ $c \in [a,b]$ s.t $f(c) = y$

More importantly if $f(a)f(b) < 0$ then root lies in interval.

$$x_2 = \frac{x_0 + x_1}{2}$$

$$x_3 = \frac{x_0 + x_2}{2}$$

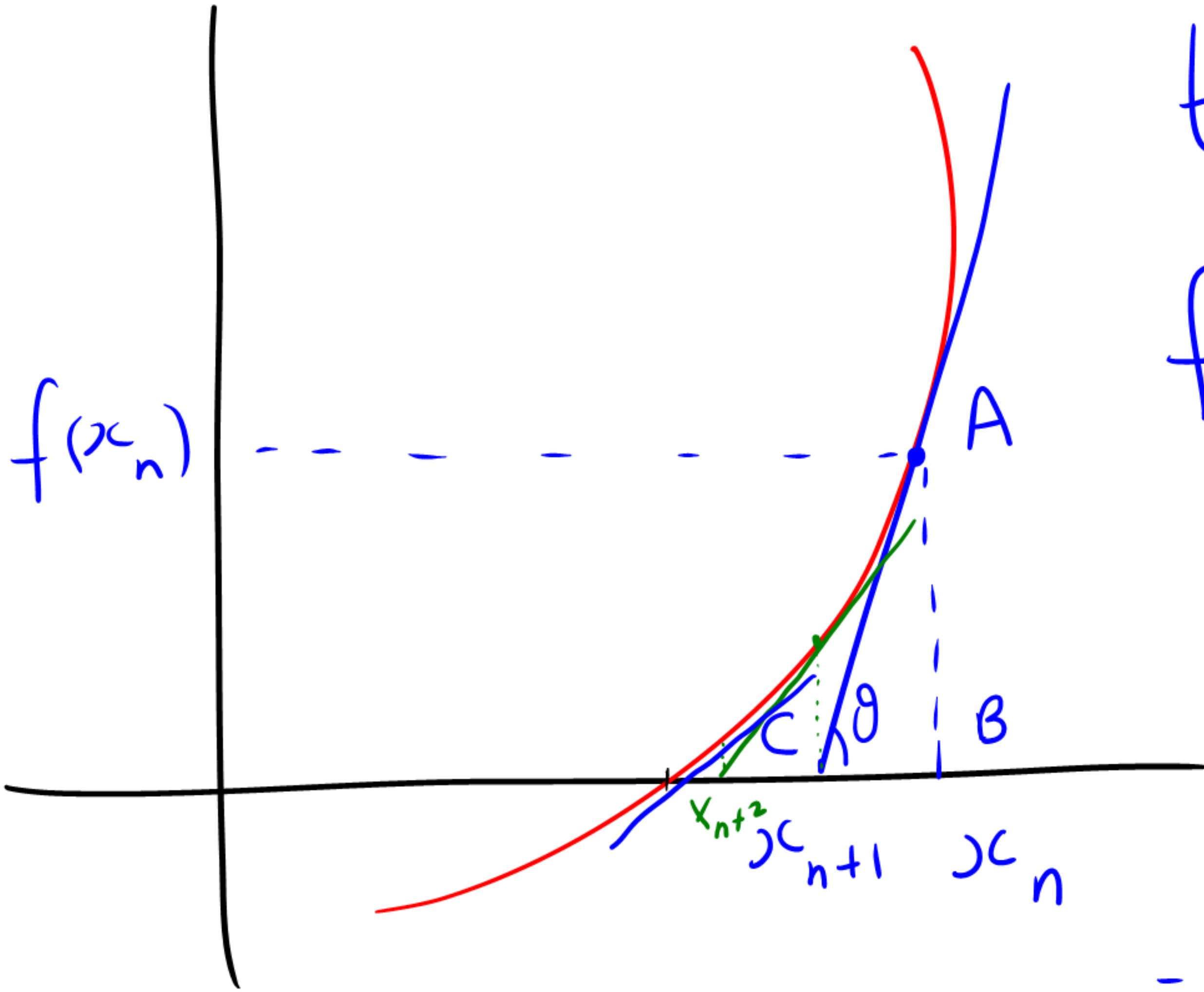


$f(x)$ cts on $[a,b]$ st $f(a)f(b) < 0$

After n-steps method returns a value c s.t

$$|c - \alpha| \leq \frac{|b-a|}{2^n}$$

Newton-Raphson



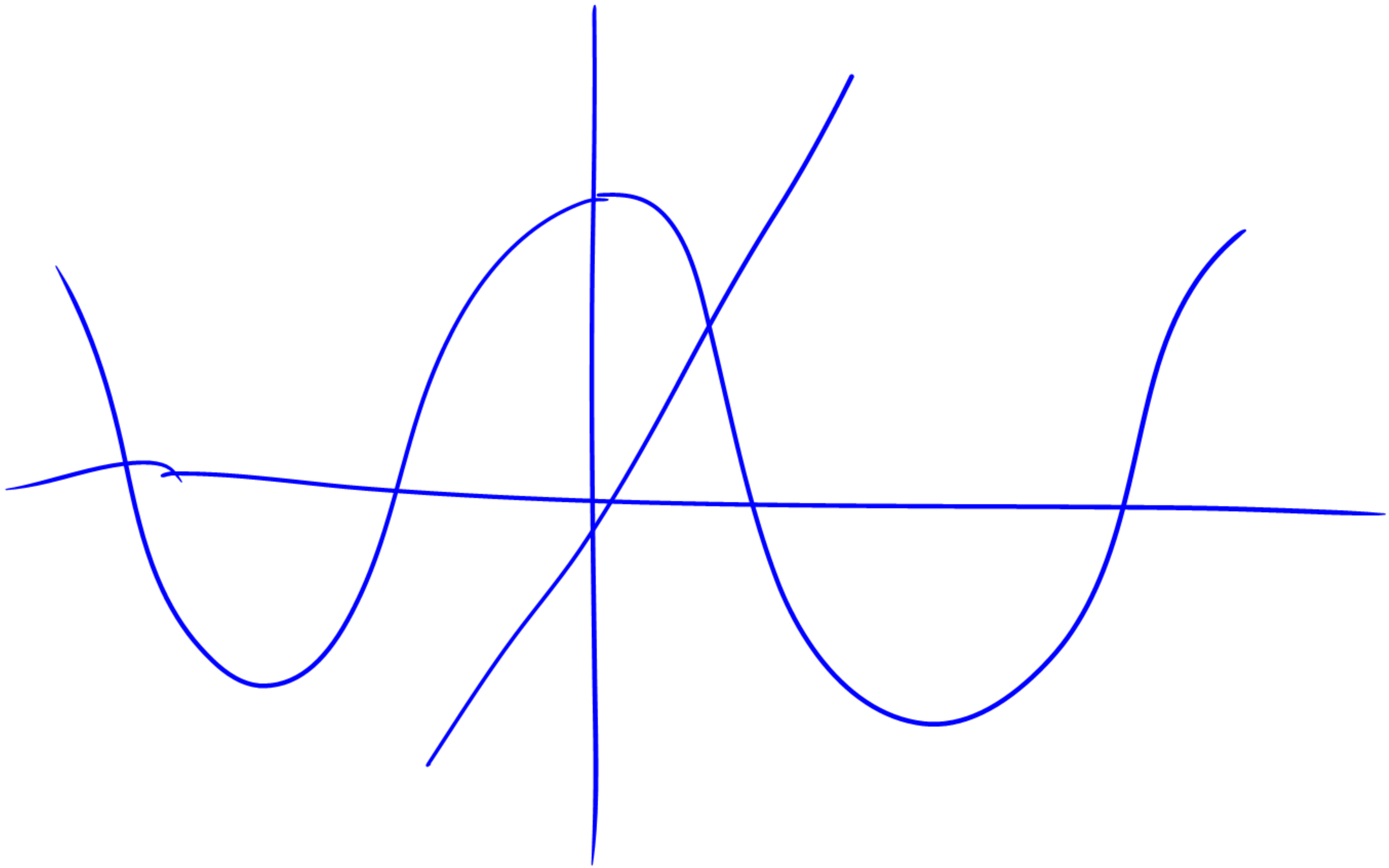
$$\tan\theta = \frac{AB}{BC}$$

$$f'(x_n) = \frac{f(x_n)}{x_n - x_{n+1}}$$

rearranging

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$n = 0, 1, 2, \ldots$



$$C - P = S - E e^{-r(T-t)}$$

$10.45 \qquad 5.57 \qquad 100 \qquad r = 5\%$

$(T - t) = 1$

$N(d_1)$ $\quad\quad$ $N(d_2)$

$\|$

$CDF(d_1)$

$CDF(d_2)$

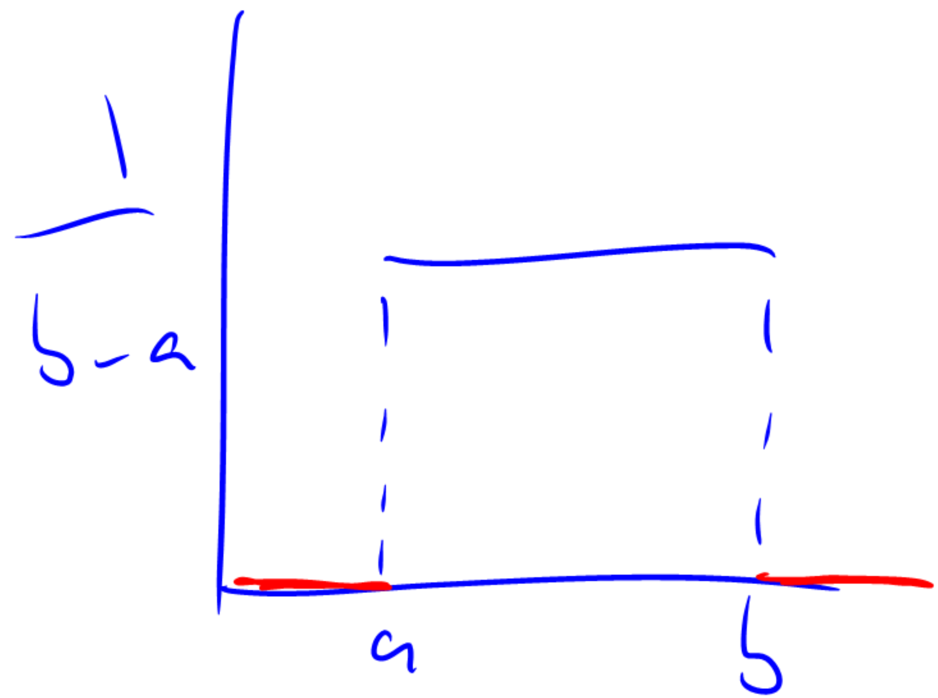
$$\int_{-\infty}^{d_1} \cdots \quad\quad \int_{-\infty}^{d_2}$$

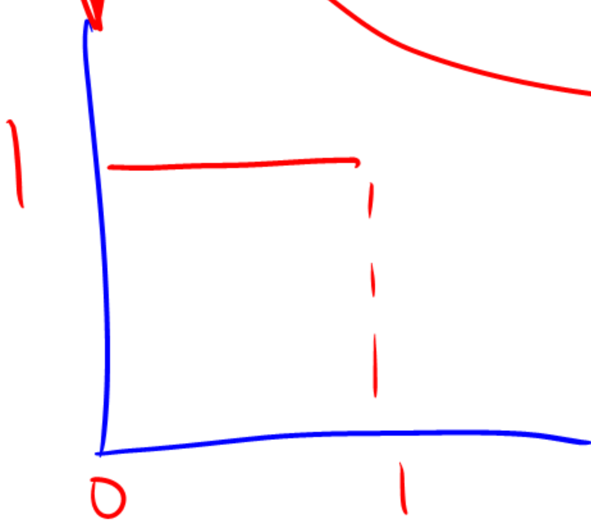
$$\left(\log \frac{S}{t}\right) + \left(\cdots \right) \Big/ \sigma\sqrt{T-t}$$

$U(0,1)$

$U(a,b)$

$U(a,b)$





Numerical Diff

$h = \delta x$

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$ fwd

$$f'(x) \approx \frac{f(x) - f(x-h)}{h}$$ backward

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}$$ centred

2 ways to simulate stock

1. $S_{i+1} = S_i \left(1 + r\,\delta t + \sigma \phi \sqrt{\delta t}\right)$

$\left(r - \frac{1}{2}\sigma^2\right)\delta t + \sigma \phi \sqrt{\delta t}$

2. $S_{t+\delta t} = S_t\, e$ ✓

$S_T = S_0 \exp\left[\left(r - \frac{1}{2}\sigma^2\right)T + \sigma \phi \sqrt{T}\right]$

$$\frac{\partial u}{\partial t} = \frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial x^2} \qquad u(x, t)$$

$$x : 0 \longrightarrow \bar{x} \qquad N \text{ steps} \qquad \delta x = \frac{\bar{x}}{N}$$

$$t : 0 \longrightarrow T \qquad M \text{ steps} \qquad \delta t = T/M$$

$$x = n\,\delta x \qquad 0 \leq n \leq N$$

$$t = m\,\delta t \qquad 0 \leq m \leq M$$

$$u(0, t) = 0 = u(\bar{x}, t)$$

$$u(x, 0) = f(x) = \sin x \ \left(\text{or } 1 - x^2\right)$$

2D T.S.E $u(x,t) = u(n\delta x, m\delta t)$

$= u_n^m$

For time:

$$\frac{\partial u}{\partial t} \simeq \frac{u_n^{m+1} - u_n^m}{\delta t}$$

For x:

$$\frac{\partial^2 u}{\partial x^2} \simeq \frac{u_{n-1}^m - 2u_n^m + u_{n+1}^m}{\delta x^2}$$

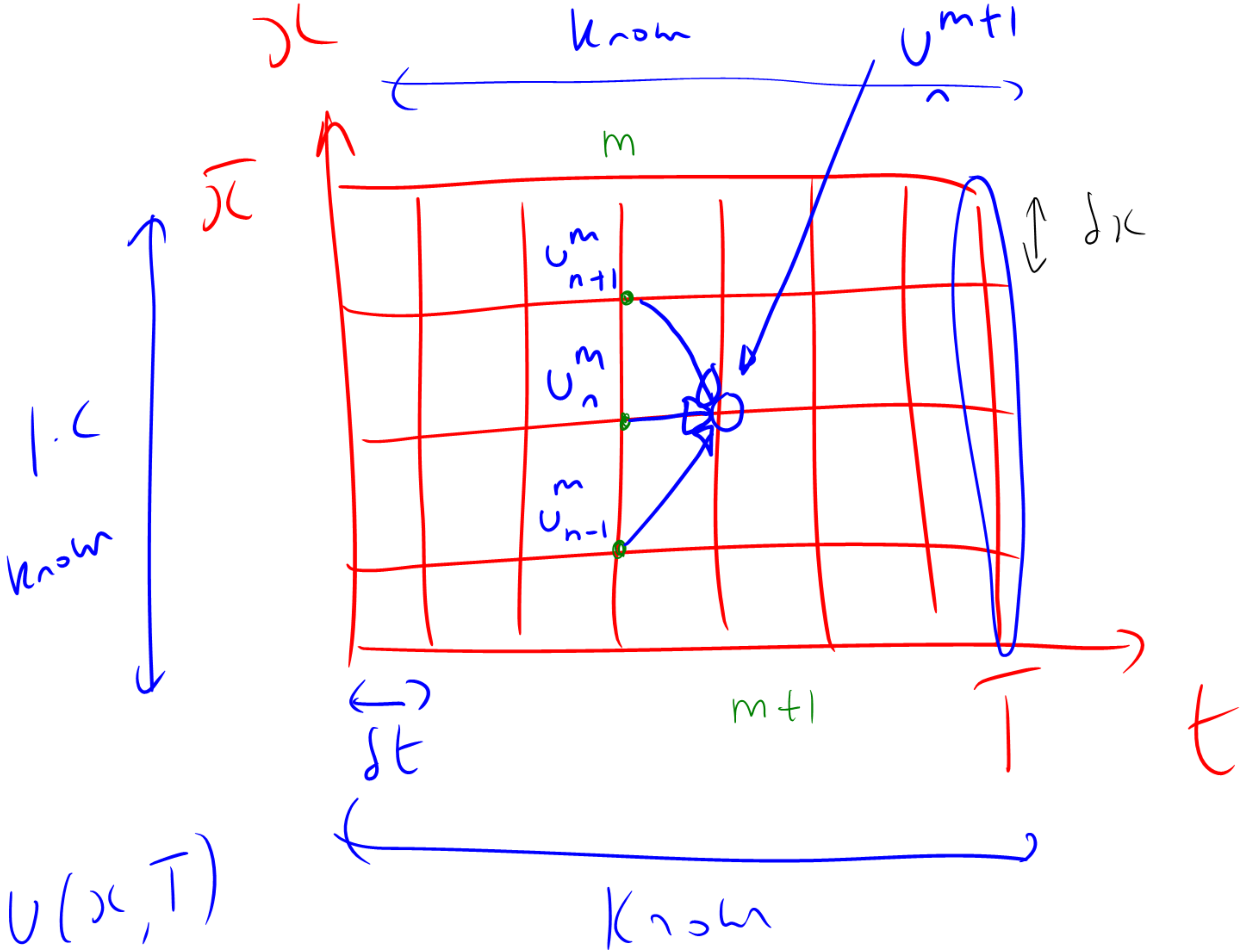
$$\frac{\partial u}{\partial t} = \frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial x^2}$$

$$u_n^{m+1} - u_n^m = \frac{1}{2}\sigma^2 \frac{\delta t}{\delta x^2}\left(u_{n-1}^m - 2u_n^m + u_{n+1}^m\right)$$

$$r = \frac{1}{2}\sigma^2 \frac{\delta t}{\delta x^2}$$

$$u_n^{m+1} = u_n^m + r u_{n-1}^m - 2r u_n^m + r u_{n+1}^m$$

$$= r u_{n-1}^m + (1-2r) u_n^m + r u_{n+1}^m$$


x
know
u^{m+1}_n
m
$\bar{x}$
u^m_{n+1}
δx
u^m_n
i.c
know
u^m_{n-1}
δt
m+1
T
t
Know
$u(x,T)$