1. The global minimum variance portfolio is obtained by optimising *s.t.* the budget constraint

$$\underset{\boldsymbol{w}}{\operatorname{argmin}} \frac{1}{2} \boldsymbol{w}' \boldsymbol{\Sigma} \boldsymbol{w} \qquad \text{s.t. } \boldsymbol{w}'\mathbf{1} = 1$$

1) Obtain analytical solution for optimal allocations w^*.
Provide full workings, analytical solution for the Lagrangian multiplier.
No computation necessary.

We will proceed using the Lagrangian to obtain a parameterization of the boundary of opportunity sets based on the return objective, ***m***.

Define the vector of returns, $\boldsymbol{\mu}$ as $\mu =$

0.04
0.08
0.12
0.15

We will solve one general problem for an arbitrary correlation matrix **R** and then, once it is solved, we will substitute for the three correlations matrices given in the exercise.

To solve the general problem, we are going to need the little covariance matrix decomposition outlined in class. Define the standard deviation matrix, S as

S =

0.07	0	0	0
0	0.12	0	0
0	0	0.18	0
0	0	0	0.26

then the covariance matrix Σ is given by: Σ = SRS

Finally define the weight vector w as

$$w = \begin{pmatrix} w_A \\ w_B \\ w_C \\ w_D \end{pmatrix}$$

Our optimization problem can be formulated as

$$\min_w \frac{1}{2} w^T SRSw$$

Subject to:

$$\begin{aligned} \mu^T w &= m \\ \mathbf{1}^T w &= 1 \end{aligned}$$

where the two constraints are respectively the return constraint and the budget equation.

Next, we form the Lagrange function: with two lagrange multipliers λ and γ:

$$L(w, \lambda, \gamma) = \frac{1}{2} w^T SRSw + \lambda(m - \mu^T w) + \gamma(1 - \mathbf{1}^T w)$$

and solve for the first order condition:

$$\begin{aligned} \frac{\partial L}{\partial w}(w, \lambda, \gamma) &= w^T \Sigma - \lambda \mu^T - \gamma \mathbf{1}^T = 0 \\ \frac{\partial L}{\partial \lambda}(w, \lambda, \gamma) &= m - \mu^T w = 0 \\ \frac{\partial L}{\partial \gamma}(w, \lambda, \gamma) &= (1 - \mathbf{1}^T w) = 0 \end{aligned}$$

We then get the optimal weight vector v^*

$$w^* = (SRS)^{-1}(\lambda\mu + \gamma\mathbf{1}) \tag{1}$$

where

$$\begin{cases} \lambda = \frac{Am-B}{AC-B^2} \\ \gamma = \frac{C-Bm}{AC-B^2} \end{cases}$$

and

$$\begin{cases} A = \mathbf{1}^T (SRS)^{-1} \mathbf{1} \\ B = \mu^T (SRS)^{-1} \mathbf{1} \\ C = \mu^T (SRS)^{-1} \mu \end{cases}$$

that's essentially as far as we can go without putting some values into the correlation matrix R.

With the Correlation structure:

$$\boldsymbol{R} = \begin{pmatrix} 1 & 0.2 & 0.5 & 0.3 \\ 0.2 & 1 & 0.7 & 0.4 \\ 0.5 & 0.7 & 1 & 0.9 \\ 0.3 & 0.4 & 0.9 & 1 \end{pmatrix}$$

Then:

A	1505.2608751
B	50.58862248
C	1.961294709

$$\begin{cases} A = \mathbf{1}^T (SRS)^{-1} \mathbf{1} \\ B = \mu^T (SRS)^{-1} \mathbf{1} \\ C = \mu^T (SRS)^{-1} \mu \end{cases}$$

so, for example, for a 10% return constraint, we would have

$\lambda =$	lambda	λ	0.2542605
$\gamma =$	gamma	γ	-0.0078808

$$\begin{cases} \lambda = \frac{Am-B}{AC-B^2} \\ \gamma = \frac{C-Bm}{AC-B^2} \end{cases}$$

and substituting into w^*, we would get:

w* =

5.87%
75.90%
-31.95%
50.18%

2. Consider the following optimization task for a targeted return $m = 10\%$, for which the net of allocations invested (borrowed) in a risk-free asset:

$$\underset{\boldsymbol{w}}{\operatorname{argmin}} \frac{1}{2}\boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w}, \qquad \text{s.t. } r + (\boldsymbol{\mu} - r\mathbf{1})'\mathbf{w} = 0.1$$

- Obtain analytical solution for the the Lagrangian multiplier and optimal allocations $\mathbf{w}^*$.
- For the target return of 3% and risk-free rate of 0.5% calculate optimal allocations $\mathbf{w}^*$ and portfolio standard deviation $\sigma_\Pi = \sqrt{\mathbf{w}^{*\prime}\boldsymbol{\Sigma}\mathbf{w}^*}$ using asset data and correlation above.

Subject to:

$$r^* + (\mu - r^*\mathbf{1})^T w = m$$

where $r^* = 0.05$ is the floor rate.
We form the Lagrange function:

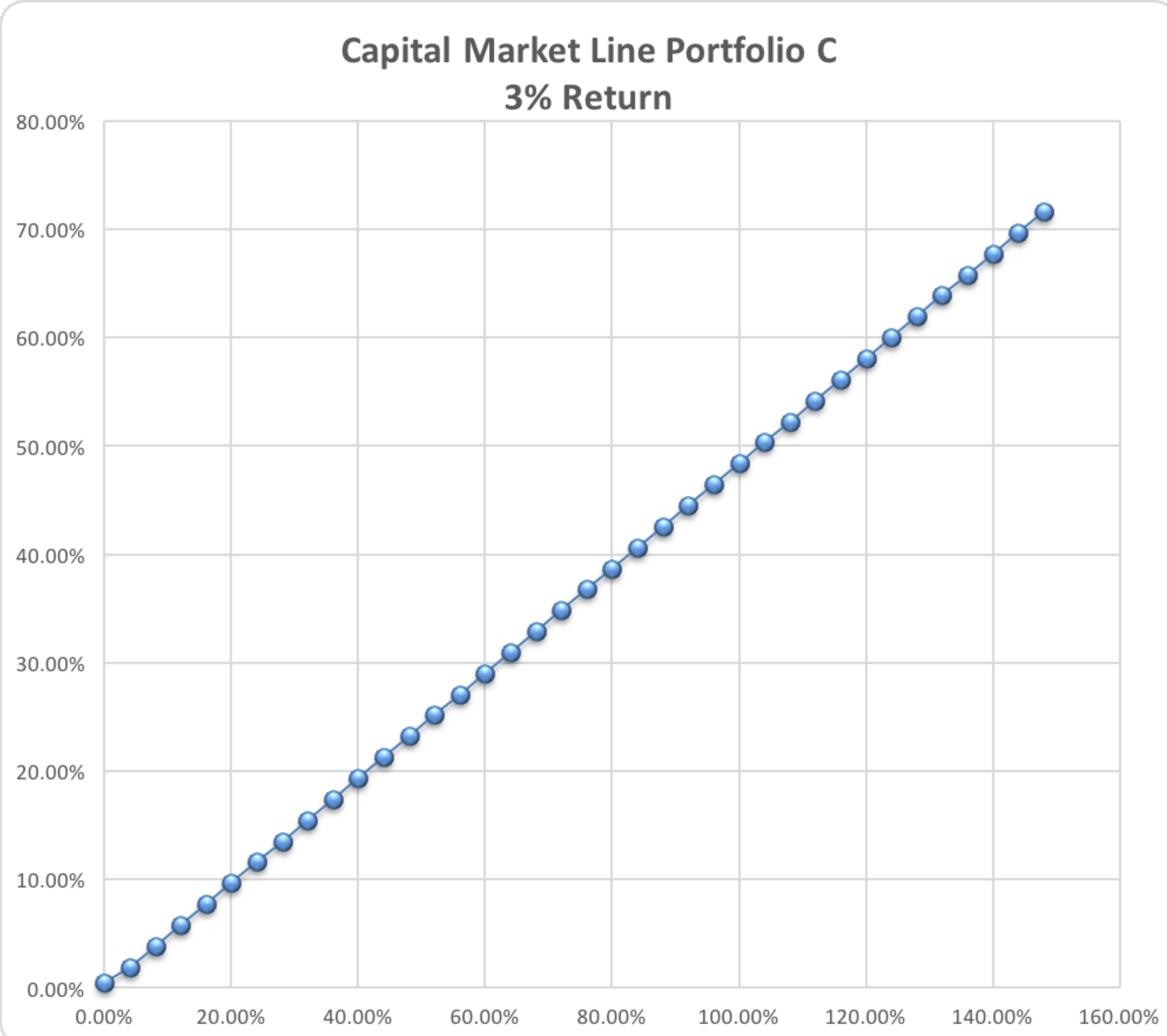
$$L(x, \lambda) = \frac{1}{2} w^T \Sigma w + \lambda(m - r^* - (\mu - r^*\mathbf{1})^T w)$$

3. Provide the definition of *a tangency portfolio* and calculate the slope of Capital Market Line. Briefly explain its role in evaluating investments. To calculate μ_T, σ_T for the risk-free rate 0.5% use ready formulae for $\mathbf{w_T}$ from the lecture (no derivation necessary here, only computation).

$$s = \frac{\mu_\Pi - r}{\sigma_\Pi}.$$

SLOPE OF CML

0.478968875 sharpe ratio



m = return constraint
"=MMULT(TRANSPOSE(B4:B7),C4:C7-P4:P7*O4:O7)"

portfolio return	$\mu' w$	2.50%	"=MMULT(TRANSPOSE(B4:B7),C4:C7-P4:P7*O4:O7)"
		3%	"=(RF+b13)"

There is one relatively easy way to do this without having to maximize this fairly complex looking criterion. From the lecture, recall that we already know how to obtain the portfolio maximizing the slope of a line: the **tangency portfolio** maximizes the slope of the CML. All we need to do is adapt this result to our question.

Our question can be reinterpreted as finding the tangency portfolio between a line going through the point $(0, 0.05)$ and the efficient frontier. The equation of the non-investable line can be obtained by varying m and solving the following optimization:

$$\min_w \frac{1}{2} w^T \Sigma w$$

Subject to:

$$r^* + (\mu - r^* \mathbf{1})^T w = m$$

where $r^* = 0.05$ is the floor rate.

We form the Lagrange function:

$$L(x, \lambda) = \frac{1}{2} w^T \Sigma w + \lambda(m - r^* - (\mu - r^* \mathbf{1})^T w)$$

We now solve for the first order condition by taking the derivative with respect to the vector w:

$$\frac{\partial L}{\partial w} = w^T \Sigma - \lambda(\mu - r^* \mathbf{1})^T = 0$$

Checking the second order condition, the Hessian of the objective function is still the covariance matrix, which is positive definite. Therefore, we have reached the optimal weight vector w^*:

$$w^* = \lambda \Sigma^{-1} (\mu - r^* \mathbf{1})$$

Substituting in the constraint equation, we get:

$$\lambda = \frac{m - r^*}{(\mu - r^* \mathbf{1})^T \Sigma^{-1} (\mu - r^* \mathbf{1})}$$

Finally, we get w^*:

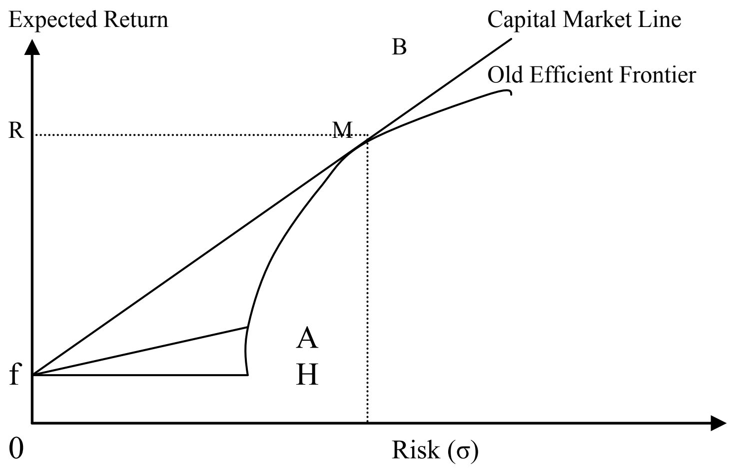
$$w^* = \frac{(m - r^*) \Sigma^{-1} (\mu - r^* \mathbf{1})}{(\mu - r^* \mathbf{1})^T \Sigma^{-1} (\mu - r^* \mathbf{1})}$$

Now, to find the tangency portfolio, simply recall that the tangency portfolio is 100% invested in the risky assets, i.e. $\mathbf{1}^T w = 1$ to finally obtain [1]:

$$w_t = \frac{\Sigma^{-1} (\mu - r^* \mathbf{1})}{B - A r^*}$$

Plugging in the values,

[1] To do so, substitute the definition of w^* into the budget equation in order to get m_t, the tangency portfolio's return. Next, substitute the equation for m_t back into the equation for w^* in order to deduce the tangency portfolio weights, w_t.



The capital market line (CML) is a tool that facilitates the search for the highest expected returns in relation to various levels of portfolio risk, measured by the standard deviation. The market portfolio point (M) lies where the CML and the efficient frontier line are tangent. To the left of that point is a combination of the risk- free assets and the market portfolio (some assets are invested in the market portfolio, with the remainder being lent at the risk- free rate). To the right of the tangency point is the area in which all of an investor's assets are invested in the market portfolio in addition to using leverage. Borrowed funds at the risk-free rate are used for this purpose. When depicted graphically, the vertical axis represents the portfolio's rate of return, while the horizontal axis represents the standard deviation of return. The risk-free rate, **r**f, is located at the vertical intercept.

The equation for the CML is:

EXHIBIT 28 – 2
Required Rate of Return – Capital Market Line

$$r_p = r_f + \sigma_p \left[\frac{r_m - r_f}{\sigma_m} \right]$$

Where:

r_p = Expected or required rate of return
r_f = Risk-free rate of return
r_m = Market rate of return
σ_p = Standard deviation of the portfolio
σ_m = Standard deviation of the market

Example Question

What is the investor's required rate of return on a portfolio using the CML model if the market return is 10% with a 4% standard deviation, the risk-free return is 6%, and the portfolio has a standard deviation of 3%

r_p = 6% + [3% x (10% – 6%)/4%)] = 9%

The slope of the capital market line in Figure 28-2 is determined by the part of the formula:

$$\frac{r_m - r_f}{\sigma_m}$$

The slope of the line will change when any of the three variables changes (the return of the market, the risk-free rate, or the standard deviation of the market).

The Capital Asset Pricing Model (CAPM) was formulated mostly by William Sharpe (1964) and was built upon the concepts laid down by Markowitz. It relates the risk as measured by beta to the required rate of return or expected level of return on a security. All unsystematic risk is assumed to be diversified away. Certain assumptions are built into the model and include the following:

- Rational investors use like information to formulate the efficient frontier.
- Investors can borrow and lend at the risk-free rate of return.
- Taxes, inflation, and transaction costs are equivalent to zero. There is no preference made for investment decisions on capital gains versus dividend distributions.

Mean Variance Optimization.

the primary goal of modern portfolio theory is the optimal asset allocation. The mean-variance optimization is a process that can assist investors in the asset allocation by balancing risks and returns.

4. For the tangency portfolio $\mathbf{w_T}$ and confidence level $c = 99\%$, calculate the risk measures:

- Analytical VaR (1D or 10D depends on the time scale of original returns/standard deviation data and is not defined here)

$$\mathrm{VaR}_c(X) = \boldsymbol{w}'\boldsymbol{\mu} + \text{Factor} \times \sqrt{\boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w}}$$

= ua + factor * Sa

$\mathrm{VaR}_c(X)$	**-2.324781874**	$\mathrm{VaR}_c(X) = \boldsymbol{w}'\boldsymbol{\mu} + \text{Factor} \times \sqrt{\boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w}}$

Portfolio Assumptions

μ	$\boldsymbol{w}'\boldsymbol{\mu}$	3.00%	"ua"	
σ	$\sigma_\Pi = \sqrt{\mathbf{w}^{*\prime}\boldsymbol{\Sigma}\mathbf{w}^*}$	5.22%	"Sa"	
Factor inv CDF	$\Phi^{-1}(1-c)$	**-2.326347874**	"factor"	normsinv(1-99)
X% Conf. Int.'c'		99%	"X"	
Z (NormsInv(X)		2.33	"Z"	

=Z+ua*Sa

$\mathrm{VaR}_c(X)$	**2.331566**	$\mathrm{VaR}_c(X)$

- Marginal Contributions to Risk measured by VaR $\mathrm{diag}(\boldsymbol{w}) \times \frac{dVaR}{d\boldsymbol{w}}$, where the derivative is

$$\frac{dVaR}{d\boldsymbol{w}} = \boldsymbol{\mu} + \frac{\boldsymbol{\Sigma}\boldsymbol{w}}{\sqrt{\boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w}}} \times \text{Factor}$$

Marginal Contributions to VaR	**-4.426605123**	ua+(d31/sa)*factor

$\boldsymbol{\Sigma}\boldsymbol{w}$ **10%**

$\sigma_\Pi = \sqrt{\mathbf{w}^{*\prime}\boldsymbol{\Sigma}\mathbf{w}^*}$ (10% / 5.22%) + 3.00% x (factor Z) = **4.493601533**

- Expected Shortfall according to

$$\text{ES}_c(X) = \mu_\Pi + \sigma_\Pi \frac{\phi(\text{VaR}_c)}{1-c}$$

where portfolio return is $\mu_\Pi = \boldsymbol{w'\mu}$ and portfolio risk $\sigma_\Pi = \sqrt{\boldsymbol{w'\Sigma w}}$ and $\phi()$ is Normal *pdf.*

$\Phi^{-1}(1-c)$

Normal PDF $\phi()$ **0.026749382**

=PHI(H22)

$\text{ES}_c(X) = \mu_\Pi + \sigma_\Pi \frac{\phi(\text{VaR}_c)}{1-c}$ **Expected Shortfall (ES)** **$0.17**

ua+Sa*(H28)/(1-X)

ua+ Sa* (phi Var_C/1-99%)

3% + 5.22% *(**0.026749382/1-99%) = ES**

B. Analytical Risk [21%, 7% each]

X refers to a random variable, while $X(t)$ represents the Brownian Motion.

1. Assume that P&L of an investment portfolio over time τ follows Normal distribution $N(\mu, \sigma^2\tau)$. Begin with the definition of VaR as a percentile $Pr(X \geq x) \leq c$ to obtain the known formula for Analytical VaR

$$\text{VaR}(X) = \mu + \sigma\sqrt{\tau} \times \Phi^{-1}(1-c).$$

For two correlated random variables K and M, the bivariate Normal distribution is defined as $K \sim N(\mu_k, \sigma_k)$, $M \sim N(\mu_m, \sigma_m)$ with ρ_{km}.

- Find the mean and variance for this joint Normal distribution (analytically).
- Prove the sub-additivity of Value at Risk, $\text{VaR}(K+M) \leq \text{VaR}(K) + \text{VaR}(M)$.

	Portfolio K	Portfolio M	Portfolio(K,M)
mu_Er	1.912	3.126	2.519
St_dev	3.71%	8.20%	5%
variance	0.00137715 2	0.006728921	0.002026518
weight	50%	50%	
cov. (k,m)			0.001522067
corr (k,m)	1	-0.5	50%

-4.534307905	Var 10day 99% portfolio K
-5.964795186	Var 10day 99% portfolio M
-7.46299377	Var 10day 99% portfolio KM
-10.49910309	Var 10day 99% VaR(K) + VaR(M).

Prove the sub-additivity of Value at Risk

VaR(K + M) ≤ VaR(K) + VaR(M).

K ~ N(μk,σk), M ~ N(μm,σm) with ρkm

Var(K+M)= -7.46% > -10.49% = VaR(K)+VaR(M) = -10.49%

Note: Subbadditive VaR was not proven in the experiment. The Non-subaddītivity of the experiment in several instances disproved.

Value at Risk is not additive

The fact that **correlations between individual risk factors enter the VAR calculation** is also the reason why **Value At Risk is not simply additive**. The VAR of a portfolio containing assets M and K does not equal the sum of VAR of asset M and VAR of asset K.

NonSubbadditive VaR example

For satisfying equation below refer

VaR(K+ M) = 9.8 < VaR(K) + VaR(M) = 3.1 + 3.1 = 6.2

Var(K+M) = -7.46% > VaR(K)+VaR(M) = -10.49%

2. Covariance matrix can be decomposed as $\boldsymbol{\Sigma} = \boldsymbol{A}\boldsymbol{A}'$ by Cholesky method. The resulting lower triangular matrix $\boldsymbol{A}$ is used for imposing correlation on a vector of independent Normal $\boldsymbol{X}$.

$$\boldsymbol{A} = \begin{pmatrix} \sigma_1 & 0 \\ \rho\sigma_2 & \sqrt{1-\rho^2}\sigma_2 \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

- For the two-variate case, show analytically what $\boldsymbol{\Sigma} = \boldsymbol{A}\boldsymbol{A}'$ is equal to.
- Write down the results for correlated $Y_1(t)$ and $Y_2(t)$ from $\boldsymbol{Y} = \boldsymbol{A}\boldsymbol{X}$.
- Does $Y_2(t)$ keep the properties of the Brownian Motion if $X_1(t), X_2(t)$ are Standard Normal? **Note:** consider the distribution of the $Y_2(t)$ increment, its variance.

$$\boldsymbol{A} = \begin{pmatrix} \sigma_1 & 0 \\ \rho\sigma_2 & \sqrt{1-\rho^2}\sigma_2 \end{pmatrix} \quad \text{and} \quad \boldsymbol{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

vector of independent Normal $\boldsymbol{X}$

$$\Sigma^{1/2} = \begin{pmatrix} \sigma_1 & 0 \\ \rho\sigma_2 & \sqrt{\sigma_2^2(1-\rho^2)} \end{pmatrix}$$

For the two-variate case, show analytically what Σ = AA′ is equal to.

Forward substitution

A is lower triangular with

(0.5*	$\boldsymbol{A}$
σ_1	0
$\rho\sigma_2$	$\sqrt{1-\rho^2}\sigma_2$
1	ρ_{12}
ρ_{21}	1

Backward substitution: A Transpose

A' is upper triangular

$\boldsymbol{A'}$		$\boldsymbol{\Sigma = AA'}$	
σ_1	$\rho\sigma_2$	σ_1^2	$\rho_{12}\sigma_1\sigma_2$
0	$\sqrt{1-\rho^2}\sigma_2$	$\rho_{21}\sigma_2\sigma_1$	σ_2^2
Identity Matrix			
1	0		
0	1		

so $Y_1 = \mu_1 + \sigma_1 * rannor1$ and $Y_2 = \mu_2 + \rho * \sigma_2 * rannor1 + \sqrt{\sigma_2^2(1-\rho^2)} * rannor2$, where $rannor1$ and $rannor2$ are two independent random variables. Thus, we can use the following code to generate the bivariate normal data:

mutually independent normal random variables

$\boldsymbol{Y = AX}$	$\boldsymbol{X}$	means	Variance X	Std Deviation
$Y_1(t)$	X_1	μ_1	σ_1^2	σ_1
$Y_2(t)$	X_2	μ_2	σ_2^2	σ_2

$\boldsymbol{Y = AX}$	$\boldsymbol{X}$	means	Variance X	Std Deviation	RHO
40	57	0	4	2	0.5
237.3683298	30	10	25	5	

BIVARIATE DATA BELOW CORRESPONDS WITH ABOVE TABLES	
$Y_1(t)$	mean_i+Sig_i*(RANDBETWEEN(1,100))
$Y_2(t)$	rho_x*sig_2*rand1+SQRT(sig_2*2-SIG2*rho_x*2)*rand2

Such that Y1 and Y2 are bivariate normally distributed.

3. Let's redefine the Cholesky result as a one-factor risk model, where $X_1(t) \equiv Z_t$ represents a common factor such as market, while $X_2(t) \equiv \epsilon_t$ for an idiosyncratic residual, $\epsilon_t \sim N(0,1)$ and $\sigma_1 = \sigma_2 = 1$.

$$Y_2(t) = \rho Z_t + \sqrt{1-\rho^2}\epsilon_t$$

Let's specify the probability of some event as $\Pr(Y_2 \leq -2.33)$. Provide all derivation steps to show that the probability conditional on market factor is calculated using the Normal *cdf* $\Phi()$ as

$$\Pr(\text{Event}|Z_t) = \Phi\left(\frac{-2.33 - \rho Z_t}{\sqrt{1-\rho^2}}\right)$$

Hint: simply rearrange around the idiosyncratic factor $\epsilon_t \sim N(0,1)$.

Mean	Std. Deviation
0	1
0	1

$$^{1|0}\Phi^{-1}_{K}(q) = {}^{0}{}_{M} - {}^{1|0}\Phi^{-1}_{1\,M}(1-q)$$

value-at-risk metrics represent a q-quantile of loss of sub(1) K

Let and denote cumulative distribution functions (CDFs) of K and M, conditional on information available at time 0.

The preceding superscripts 1|0 are a convention to alert you that the distributions are "for random variables at time 1 but conditional on information available at time 0."

q = quantile Value at risk

Inverse CDF "-2.33 represents the Z-Score. It's 99% confidence"

CDF M

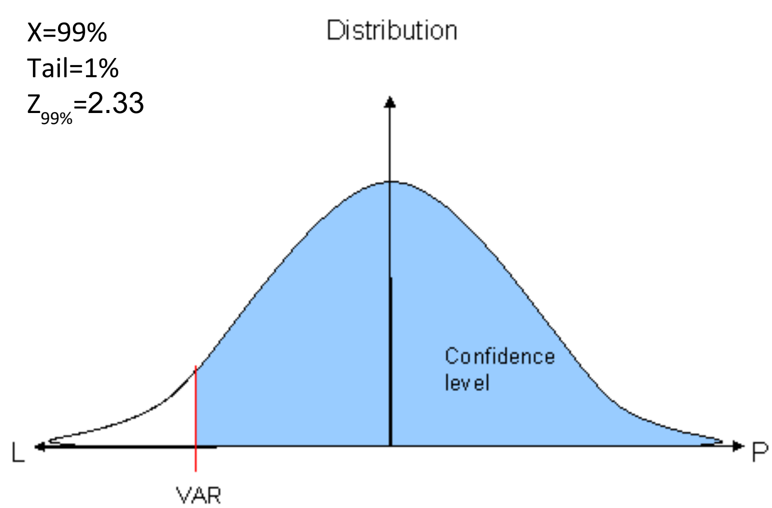
$^{1|0}\Phi_{1M}$

CDF K

$^{1|0}\Phi_{1K}$

Using the naming convention indicated in the text, name the following value-at-risk metric: Conditional 0.99 quantile of a portfolio's loss, measured in GBP over the next trading day.

Equals a One-day 99% GBP VaR.



The probability of loss $X < 0$ being worse than VaR < 0 is

$$\Pr\left(X \leq \mathrm{VaR}(X)\right) = 1 - c$$

Note that if P&L X is a random variable then VaR(X) is also a random variable. In order to use the well-known Normal Distribution functions, we have to work with the Standard Normal variable

$$\begin{aligned} \Pr(\phi \leq \frac{\mathrm{VaR}(X) - \mu}{\sigma}) &= 1 - c \qquad \Longrightarrow \\ \mathrm{VaR}(X) &= \mu + \Phi^{-1}(1-c) \times \sigma \end{aligned}$$

Within that tail, what is the mean of standardised returns – that is, what is an average tail loss? Provide analytical solutions for abstract μ, σ using a simplifying assumption of Standard Normal Distribution.

PDF for Normal Distribution $N(\mu, \sigma^2)$ is $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$.

Solution:

The percentage of returns outside n standard deviations (in general) on the left tail is the cumulative density function $\Phi(x)$, which is an integral over probability density

$$\int_{-\infty}^{\mu - n\sigma} f(x)dx = \int_{-\infty}^{\mu - n\sigma} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \Phi(\mu - n\sigma)$$

Note that while $\mathrm{VaR}_{c\%}(X)$ is a percentile and given by ICDF, the percentage 'cut on the tail' $1 - c$ is given by the CDF. Example: for confidence level of 99%, the percentile that 'cuts' 1% of observations is $\Phi^{-1}(1 - 0.99) \approx 2.32635$.

At times, $\Phi(\mu - n\sigma) = 1 - c$, is substituted in integration limits using any of the following:

$$\mu - n\sigma = \Phi^{-1}(1 - c) = \mathrm{VaR}_c$$

The mean of the values that fall within that tail (ie, cut by the percentile threshold) is

$$\frac{\int_{-\infty}^{\mu-n\sigma} x\,f(x)dx}{\int_{-\infty}^{\mu-n\sigma} f(x)dx} = \frac{1}{\Phi(\mu-n\sigma)}\int_{-\infty}^{\mu-n\sigma}\frac{x}{\sigma\sqrt{2\pi}}\,e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx \quad \dagger$$

to simplify assume Standard Normal, $\Phi(\mu - n\sigma) = \Phi(-n)$

$$= \frac{1}{\Phi(-n)}\int_{-\infty}^{-n}\frac{x}{\sqrt{2\pi}}\,e^{-x^2/2}dx \quad \text{ready for calculation using } n=2\text{, see below.}$$

To derive a general solution lets **swap variables** in original problem $\dagger$

$$\mu - n\sigma = x \quad \Leftrightarrow \quad -n = \frac{x-\mu}{\sigma} = z \quad \Leftrightarrow \quad -dn = \frac{dx}{\sigma} = dz$$

$$= \frac{1}{\Phi(x)}\int_{-\infty}^{x}\frac{\mu+z\sigma}{\sqrt{2\pi}}\,e^{-z^2/2}\,dz \qquad \Leftrightarrow \qquad \frac{1}{\Phi(x)}\int_{-\infty}^{x}\frac{\mu-n\sigma}{\sqrt{2\pi}}\,e^{-n^2/2}\,(-dn)$$

$$= \frac{1}{\Phi(x)}\left(\mu\,\Phi(x) + \frac{\sigma}{\sqrt{2\pi}}\int_{-\infty}^{x} ze^{-z^2/2}\,dz\right)$$

$$= \mu + \frac{\sigma}{\Phi(x)}\,\frac{-1}{\sqrt{2\pi}}e^{-x^2/2}$$

which relates to $\quad \mathrm{ES}_c = \mu + \sigma\dfrac{\phi(\mathrm{VaR}_c)}{1-c} \quad$ cited in textbooks,

where standartised $\mathrm{VaR}_c = \Phi^{-1}(1-c) = x$ is in line with the above.

For the Standard Normal Distribution $N(\mu = 0, \sigma^2 = 1)$ with density $\phi(z)$, we find that the percentage of returns outside two standard deviations on the left tail is

$$\int_{-\infty}^{-2}\phi(z) = \int_{-\infty}^{-2}\frac{1}{\sqrt{2\pi}}\,e^{-z^2/2}\,dz = \Phi(-2) = 0.02275$$

The mean of the values falling in this tail is

$$\frac{1}{\Phi(-2)}\int_{-\infty}^{-2}\frac{z}{\sqrt{2\pi}}\,e^{-z^2/2}\,dz = -\frac{1}{\Phi(-2)\sqrt{2\pi}}e^{-2} \approx -2.37$$

$$\text{using} \qquad \int ze^{-z^2/2} = -e^{-z^2/2}$$

The -2.37 figure is the mean of *standartised returns* and therefore, itself is a Standard Normal variable or Z-score. It also has a definition of Expected Shortfall (ES), *an average tail loss* given that the loss is *below* the VaR threshold. ‘Given’ reveals conditionality and ES is mathematically known as Conditional Value at Risk.

C. Value at Risk on FTSE 100 [23%]

Imagine that each morning you calculate 99%/10day VaR from available prior data only. Once ten days pass you compare that VaR number to the realised return and check if your prediction about the worst loss was breached. You are given a dataset of FTSE 100 index levels, continue in Excel.

B.1 Calculate the rolling 99%/10day Value at Risk for an investment in the market index using a sample standard deviation of log-returns, as follows:

- The rolling standard deviation for a sample of 21 is computed for days 1-21, 2-22, ..., there must be 21 observations in the sample. So, you have a time series of σ_t.
- Scale standard deviation to reflect a ten days move $\sigma_{10D} = \sqrt{10 \times \sigma^2}$ (we can add variances) and scale an average daily return as $\mu_{10D} = \mu \times 10$ where μ is a mean return of all data given.
- Calculate Value at Risk for each day t (starting on Day 21) as follows:

$$\text{VaR} = \mu_{10D} + \text{Factor} \times \sigma_{10D} \quad \dagger$$

  where Factor is a percentile of the Standard Normal Distribution that 'cuts' 1% on the tail.

In Excel, you will have a final column with VaR_t as a percentage since calculation is done on returns.

B.2 Calculate two numbers: (a) the percentage of VaR breaches and (b) conditional probability of breach in VaR, given that a breach was observed for the previous period.

- VaR is fixed at time t and compared to the realised return at time $t+10$. A breach occurs when a realised 10-day return $r_{10D,t} = \ln(S_{t+10}/S_t)$ is below the VaR_t quantity (negative scale).
- 20/08/2009 is the first day on which VaR_t computation is available. Number of breaches divided by number of observations will give the percentage of breaches.
- Plot time series of VaR_t and indicate breaches. Briefly discuss, are the breaches independent?

In Excel, you will add two columns for $r_{10D,t}$ and $0, 1$ indicator, where 1 means a breach.

B.3 Now that you know all the data in the end, construct the histogram and Q-Q plot for each, 1D and 10D log-returns. Briefly discuss if the Normal distribution of returns is a reasonable assumption.