Dis^n of τ

$$\tau = \inf\{t \mid N_t > 0\}$$

CDF : $F(t)$

survial : $S(t) = 1 - F(t)$

pdf : $f(t) = F'(t)$

$$\Pr\{N_{t+h} = 1 \mid N_t = 0\} = \lambda h + o(h)$$

$$\equiv \Pr\{t < \tau < t+h \mid \tau > t\} = \lambda h + o(h)$$

$$\lambda = \lim_{h \to 0^+} \frac{\Pr\{ t < \tau \le t+h \mid \tau > t \}}{h} + \cancel{\frac{o(h)}{h}}$$

$$\downarrow$$

$$0$$

conditional instan- default rate

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

$$\lambda = \lim_{h \to 0^+} \frac{1}{h} \frac{S(t) - S(t+h)}{S(t)}$$

$$= -\frac{1}{S(t)} \lim_{h \to 0^+} \frac{S(t+h) - S(t)}{h}$$

$$\lambda = -\frac{S'(t)}{S(t)} = -\frac{d \log S(t)}{dt}$$

$$-\lambda t = \log S(t) - \log S(0)$$

survial f^n $\quad S(t) = e^{-\lambda t}$

CFD $\quad F(t) = 1 - S(t) = 1 - e^{-\lambda t}$

pdf $\quad f(t) = \lambda e^{-\lambda t}$

$$T \sim \exp(\lambda)$$

1/1

$$Pr(\tau < T) = F(T) = 1 - e$$

$$Pr(t < \tau < T) = F(T) - F(t) = e^{-\lambda t} - e^{-\lambda T}$$

$$= e^{-\lambda t}\left(1 - e^{-\lambda(T-t)}\right)$$

$$= Pr(\tau > t)\ Pr(t < \tau < T \mid \tau > t)$$

$$Pr(\tau = t) = 0$$

$$Pr(t < \tau < t+dt) = f(t)\,dt = \lambda e^{-\lambda t}\,dt$$

$$= e^{-\lambda t} \cdot \lambda dt$$

$$= Pr(\tau > t)\ Pr(t < \tau < t+h \mid \tau > t)$$

3 BPEs

	Intensity	Int rate	recovery	hedging	Correlation
1	Const	Stoch	0	only int rate	0
2	Stoch	Stoch	0	hedge both int & default	$\rho > 0$
3	Stoch	Stoch	RMV	hedg both int & default	$\rho > 0$

Case I

risky ZCB : $V(r, t; p)$

p: const intensity

riskfree : $Z(r, t)$

int rate : $dr = u(r,t)\,dt + \sigma(r,t)\,dX$

$$dZ = \left(\frac{\partial Z}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 Z}{\partial r^2}\right)dt + \frac{\partial Z}{\partial r}\,dr$$

$$= \mathcal{L}(Z)\,dt + \frac{\partial Z}{\partial r}\,dr$$

$$dV = \mathcal{L}(V)\,dt + \frac{\partial Z}{\partial r}\,dr$$

hedging portfolio

$$\Pi = V - \Delta Z$$

only hedg int. rate risk

(1) No default $t \to t + dt$ $1 - P\,dt$

$$d\pi = dV - \Delta\, dZ$$

$$= \mathcal{L}(V)dt + \frac{\partial V}{\partial r}dr - \Delta\left[\mathcal{L}(Z)dt + \frac{\partial Z}{\partial r}dr\right]$$

$$= \left(\mathcal{L}(V) - \Delta\,\mathcal{L}(Z)\right)dt + \left(\frac{\partial V}{\partial r} - \Delta\frac{\partial Z}{\partial r}\right)dr$$

$$\Delta = \frac{\partial V}{\partial r} \Big/ \frac{\partial Z}{\partial r}$$

$$= \left(\mathcal{L}(V) - \Delta\,\mathcal{L}(Z)\right)dt$$

2 / 2

② default $t \to t + dt$ $\quad p\,dt$

$$\pi = -V + O(\sqrt{dt})$$

$$d\pi = -V + \underbrace{O(\sqrt{dt})}_{\text{risk-free bond}}$$

$$E(d\pi) = r\,\pi\,dt$$

$$E(d\pi) = (1 - \cancel{p\,dt})\left(\mathcal{L}(V) - \delta f(z)\right)dt + p\,dt\left(-V + \cancel{O(\sqrt{dt})}\right)$$

$$= \left(\mathcal{L}(V) - pV - \delta f(z)\right)dt$$

$$\left(f(v) - pv - \delta f(z) \right) \cancel{dt}$$

$$= r\left(v - \delta z \right) \cancel{dt}$$

$$\Rightarrow \quad f(v) - (r+p)\, v = \delta \left[f(z) - r z \right]$$

$$RHS = \frac{\partial v}{\partial r} \Big/ \frac{\partial z}{\partial r} \left[-(u - \lambda w) \frac{\partial z}{\partial r} \right]$$

PE
th
ree bond

$$\rightarrow \quad f(z) - r z = \frac{\partial z}{\partial t} + \frac{1}{2} w^2 \frac{\partial^2 z}{\partial r^2} - r z = \underline{-(u - \lambda w) \frac{\partial z}{\partial r}}$$

$\rightarrow \mathcal{L}(z) - rz = \frac{\partial z}{\partial t} + \frac{1}{2}w^2 \frac{\partial^2 z}{\partial r^2} - rz = -(u - \lambda w)\frac{\partial z}{\partial r}$

free bond

$$RHS = -(u-\lambda w)\frac{\partial v}{\partial r}$$

$$\boxed{\mathcal{L}(v) + (u-\lambda w)\frac{\partial v}{\partial r} - (r+p)v = 0}$$

risky bond pricing equation

yield sp

$$V(t,T) = E^{Q}\left[e^{-\int_t^T r_s + p_s \, ds} \,\middle|\, \mathcal{F}_t \right]$$

$$= \underline{e^{-\int_t^T p_s \, ds}} \; E^{Q}\left[e^{-\int_t^T r_s \, ds} \,\middle|\, \mathcal{F}_t \right]$$

$$= S_t(T) \; Z(t,T)$$

$$YtM = -\frac{\log V(t,T)}{T-t} = -\frac{\log S_t(T)}{T-t} - \frac{\log Z(t,T)}{T-t}$$

$$= \frac{\int_t^T p_s \, ds}{T-t} + y_f.$$

$$sp = \frac{\int_t^T p_s \, ds}{T-t}$$

$$\overbrace{-\frac{\partial}{\partial T}\log V(t,T) = -\frac{\partial}{\partial T}\log S_t(T) - \frac{\partial}{\partial T}\log Z(t,T)}^{\text{forward rate sp}}$$

$$= p_T + f(t,T)$$

forward sp $= p_T$.

implied PD

$$V(t,T) = S_t(T)\, Z(t,T)$$

$$S_t(T) = \frac{V(t,T)}{Z(t,T)} = e^{-(y-y_f)(T-t)}$$

Case II

Hedging $\quad \pi = V - \Delta Z - \Delta_1 V_1$

$$dV(r, p, t) = \left(\frac{\partial V}{\partial t} + \frac{1}{2} w^2 \frac{\partial^2 V}{\partial r^2} + \frac{1}{2} \delta^2 \frac{\partial^2 V}{\partial p^2} + \rho \sigma \delta \frac{\partial^2 V}{\partial r \partial p} \right) dt + \frac{\partial V}{\partial p} dp + \frac{\partial V}{\partial r} dr$$

$$dp = \gamma(r, p, t)\, dt + \delta(r, p, t)\, dX_1$$

$$= \mathcal{L}'(V)\, dt + \frac{\partial V}{\partial p} dp + \frac{\partial V}{\partial r} dr$$

$$dV_1 = \mathcal{L}'(V_1)\, dt + \frac{\partial V_1}{\partial p} dp + \frac{\partial V_1}{\partial r} dr$$

① No default $\quad t \to t + dt$

$$d\pi = \left[f'(v) - \sigma f(z) - \sigma_1 f'(v_1) \right] dt +$$

$$\left[\frac{\partial v}{\partial r} - \sigma \frac{\partial z}{\partial r} - \sigma_1 \frac{\partial v_1}{\partial r} \right] dr +$$

$$\left[\frac{\partial v}{\partial p} - 0 - \sigma_1 \frac{\partial v_1}{\partial p} \right] dp$$

$$\sigma_1 = \frac{\partial v}{\partial p} \Big/ \frac{\partial v_1}{\partial p} \qquad \sigma = \left(\frac{\partial v}{\partial r} - \sigma_1 \frac{\partial v_1}{\partial r} \right) \Big/ \frac{\partial z}{\partial r}$$

$$\Rightarrow \quad d\pi = \left[f'(v) - \sigma f(z) - \sigma_1 f'(v_1) \right] dt$$

① default $t \to t+dt$

$$d\pi = -V + \Delta_1 V_1 + o(\sqrt{dt})$$

$$E(d\pi) = (1-\cancel{p}dt)\left(f'(v) - \Delta f(z) - \Delta_1 f'(v_1)\right) dt +$$

$$p\,dt\left(-v + \Delta_1 V_1 + \cancel{o(\sqrt{dt})}\right)$$

$$= \left[f'(v) - pv - \Delta f(z) - \Delta_1\left(f'(v_1) - p v_1\right)\right] dt$$

$$E(d\pi) = r\pi\, dt$$

$$= r\left[V - \Delta z - \Delta_1 V_1\right] dt.$$

$$= r\left[\, V - \partial z - \partial_1 V_1 \,\right] dt$$

$$\to \quad f'(v) - (r+p)v - \partial_1\left[f'(v_1) - (r+p)v_1 \right]$$

$$= \partial\left[f(z) - r z \right]$$

$$RHS = \left(\frac{\partial v}{\partial r} - \partial_1 \frac{\partial v_1}{\partial r} \right) \Big/ \cancel{\frac{\partial z}{\partial r}} \left[-(u - \lambda w) \cancel{\frac{\partial z}{\partial r}} \right]$$

$$= -(u - \lambda w)\left(\frac{\partial v}{\partial r} - \partial_1 \frac{\partial v_1}{\partial r} \right)$$

$$f'(v) - (r+p) + (u - \lambda w)\frac{\partial v}{\partial r} = \partial_1\left[f'(v_1) - (r+p)v_1 + (u-\lambda w)\frac{\partial v_1}{\partial r} \right.$$

$$= -(u$$

$$f'(v) - (r+p)v + (u-\lambda w)\frac{\partial v}{\partial r} = \delta_1 \left[f'(v_1) - (r+p)v_1 + (u-\lambda w)\frac{\partial v_1}{\partial r} \right]$$

$$\frac{f'(v) - (r+p)v + (u-\lambda w)\frac{\partial v}{\partial r}}{\frac{\partial v}{\partial p}} = \frac{f'(v_1) - (r+p)v_1 + (u-\lambda w)\frac{\partial v_1}{\partial r}}{\frac{\partial v_1}{\partial p}}$$

$$= a(r, p, t)$$

$$= -(\gamma - \lambda' \delta)$$

- $$f'(v) + (u-\lambda w)\frac{\partial v}{\partial r} + (\gamma - \lambda'\delta)\frac{\partial v}{\partial p} - (r+p)v = 0$$

$$\frac{f'(v) - (r+p) \text{[illegible]}}{\frac{\partial v}{\partial p}} = \frac{f(v_1) - (r+p)v_1 + (u - \lambda w)\frac{\partial v_1}{\partial r}}{\frac{\partial v_1}{\partial p}}$$

$$= a(r, p, t)$$

$$= -(\gamma - \lambda' \delta)$$

$$f'(v) + (u - \lambda w)\frac{\partial v}{\partial r} + (\gamma - \lambda' \delta)\frac{\partial v}{\partial p} - (r+p)v = 0$$

$\beta \rho \tau$

5 / 5

$u - \lambda w$: risk neutral drift of r

$\gamma - \lambda' \delta$: risk neutral drift of p

λ : mpor for int rate

λ' : mpor for default (intensity) risk

Case IV

θ: constant recovery rate

$l := 1-\theta$ loss given default

if default:

$$d\pi = -lV + l\,\Delta_1 V_1 + o(\sqrt{dt})$$

$$E(d\pi) = (1-\rho\,dt)\left(f'(v) - \Delta f(z) - \Delta_1 f'(v_1)\right) + \rho\,dt\left(-\underline{l}V + \underline{l}\,\Delta_1 V_1 + \cancel{o(\sqrt{dt})}\right)$$

$$= r\left(V - \Delta z - \Delta_1 V_1\right).$$

Recover on Treasury Value $\Rightarrow$ implied PD

$$V(0,t) = (1-F(t))\,Z(0,t) + F(t)\,\theta\,Z(0,t)$$

$$\frac{V(0,t)}{Z(0,t)} = 1 - F(t) + \theta F(t)$$

$s = y - y_f$

$$e^{-st} = 1 - (1-\theta)F(t)$$

$$F(t) = \frac{1-e^{-st}}{1-\theta}$$

$$s = -\frac{1}{t}\log\left[1-(1-\theta)F(t)\right]$$

Two-factor Vasicek Interest Model

$$V(t,T) = \exp\{A(t,T) - B(t,T)x - C(t,T)y\}$$

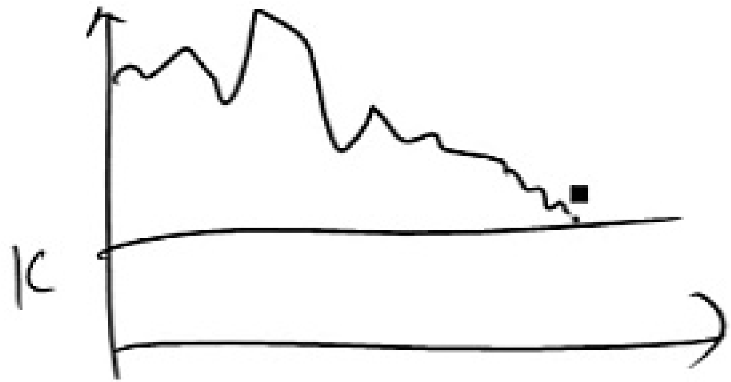
$$\frac{\partial V}{\partial t} = (\dot{A} - \dot{B}x - \dot{C}y)V$$

$$\frac{\partial V}{\partial x} = -BV \qquad \frac{\partial V}{\partial y} = -CV$$

$$\frac{\partial^2 V}{\partial x^2} = B^2 V \qquad \frac{\partial^2 V}{\partial y^2} = C^2 V \qquad \frac{\partial^2 V}{\partial x \partial y} = BCV$$

3 ODEs
$$\begin{cases} \dot{A} + \frac{1}{2}\sigma^2 B^2 + \frac{1}{2}\eta^2 C^2 + \rho\sigma\eta BC - \phi = 0 \\ \dot{B} - aB + 1 = 0 \quad \Rightarrow \\ \dot{C} - bC + 1 = 0 \quad \Rightarrow \end{cases}$$

$$\underbrace{A(T,T)}_{0} - A(t,T) = \int_t^T \frac{1}{2}\sigma^2 B^2 + \frac{1}{2}\eta^2 C^2 + \rho\sigma\eta BC - \phi(s)\, ds$$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