

CQF Exercises 4.3 Calibration

1. Substitute the fitted function for $A(t; T)$, using the Ho & Lee model, back into the solution of the bond pricing equation for a zero-coupon bond,

$$Z(r, t; T) = \exp(A(t; T) - r(T - t)).$$

What do you notice?

Solution:

With a Ho & Lee model, the form of the fitted function for $A(t; T)$ is

$$A(t; T) = \log\left(\frac{Z_M(t^*; T)}{Z_M(t^*; t)}\right) - (T - t) \frac{\partial}{\partial t} \log(Z_M(t^*; t)) - \frac{1}{2} c^2 (t - t^*) (T - t)^2.$$

Then

$$\begin{aligned} Z(t; T) &= e^{\log\left(\frac{Z_M(t^*; T)}{Z_M(t^*; t)}\right) - (T - t) \frac{\partial}{\partial t} \log(Z_M(t^*; t)) - \frac{1}{2} c^2 (t - t^*) (T - t)^2 - r(T - t)} \\ &= \frac{Z_M(t^*; T)}{Z_M(t^*; t)} e^{-(T - t) \left(\frac{\partial}{\partial t} \log(Z_M(t^*; t)) + \frac{1}{2} c^2 (t - t^*) (T - t) + r \right)}. \end{aligned}$$

We note that that when $t = t^*$

$$Z(t^*; T) = \frac{Z_M(t^*; T)}{Z_M(t^*; t^*)} e^{-(T - t^*) \left(\frac{\partial}{\partial t} \log(Z_M(t^*; t)) + \frac{1}{2} c^2 (t^* - t^*) (T - t^*) + r \right)} = Z_M(t^*; T).$$

2. Differentiate Equation (2) on page 19 of the lecture notes, twice to solve for the value of $\eta^*(t)$. What is the value of a zero-coupon bond with a fitted Vasicek model for the interest rate?

Solution:

We have

$$\begin{aligned} & - \int_{t^*}^T \eta^*(s) B(s; T) ds + \frac{c^2}{2\gamma^2} \left((T - t^*) + \frac{2}{\gamma} e^{-\gamma(T - t^*)} - \frac{1}{2\gamma} e^{-2\gamma(T - t^*)} - \frac{3}{2\gamma} \right) \\ &= \log(Z_M(t^*; T)) + r^* B(t^*; T). \end{aligned}$$

Differentiating with respect to T ,

$$\begin{aligned} & - \int_{t^*}^T \eta^*(s) \frac{\partial}{\partial T} B(s; T) ds - \eta^*(T) B(T; T) + \frac{c^2}{2\gamma^2} \left(1 - 2e^{-\gamma(T - t^*)} + e^{-2\gamma(T - t^*)} \right) \\ &= \frac{\partial}{\partial T} \log(Z_M(t^*; T)) + r^* \frac{\partial}{\partial T} B(t^*; T). \end{aligned}$$

Now

$$B(t; T) = \frac{1}{\gamma} \left(1 - e^{-\gamma(T - t)} \right) \quad \text{so} \quad B(T; T) = 0,$$

and

$$\frac{\partial}{\partial T} B(t; T) = e^{-\gamma(T - t)}.$$

Substituting back into the PDE

$$\begin{aligned} & - \int_{t^*}^T \eta^*(s) e^{-\gamma(T-s)} ds + \frac{c^2}{2\gamma^2} \left(1 - 2e^{-\gamma(T-t^*)} + e^{-2\gamma(T-t^*)} \right) \\ & = \frac{\partial}{\partial T} \log(Z_M(t^*; T)) + r^* e^{-\gamma(T-t^*)}. \end{aligned}$$

Differentiating again with respect to T ,

$$\begin{aligned} & -\eta^*(T) + \gamma \int_{t^*}^T \eta^*(s) e^{-\gamma(T-s)} ds + \frac{c^2}{2\gamma^2} \left(2\gamma e^{-\gamma(T-t^*)} - 2\gamma e^{-2\gamma(T-t^*)} \right) \\ & = \frac{\partial^2}{\partial T^2} \log(Z_M(t^*; T)) - \gamma r^* e^{-\gamma(T-t^*)}. \end{aligned}$$

Substituting for the integral from the previous equation, we find

$$\begin{aligned} & -\eta^*(T) + \gamma \left(\frac{c^2}{2\gamma^2} \left(1 - 2e^{-\gamma(T-t^*)} + e^{-2\gamma(T-t^*)} \right) \right) - \frac{\partial}{\partial T} \log(Z_M(t^*; T)) - r^* e^{-\gamma(T-t^*)} \\ & + \frac{c^2}{2\gamma^2} \left(2\gamma e^{-\gamma(T-t^*)} - 2\gamma e^{-2\gamma(T-t^*)} \right) \\ & = \frac{\partial^2}{\partial T^2} \log(Z_M(t^*; T)) - \gamma r^* e^{-\gamma(T-t^*)}. \end{aligned}$$

This simplifies to

$$\eta^*(T) = -\frac{\partial^2}{\partial T^2} \log(Z_M(t^*; T)) + \frac{c^2}{2\gamma} - \gamma \frac{\partial}{\partial T} \log(Z_M(t^*; T)) - \frac{c^2}{2\gamma} e^{-2\gamma(T-t^*)},$$

and

$$\eta^*(t) = -\frac{\partial^2}{\partial t^2} \log(Z_M(t^*; t)) - \gamma \frac{\partial}{\partial t} \log(Z_M(t^*; t)) + \frac{c^2}{2\gamma} \left(1 - e^{-2\gamma(t-t^*)} \right).$$

We then have

$$A(t; T) = - \int_t^T \eta^*(s) B(s; T) ds + \frac{c^2}{2\gamma^2} \left((T-t) + \frac{2}{\gamma} e^{-\gamma(T-t)} - \frac{1}{2\gamma} e^{-2\gamma(T-t)} - \frac{3}{2\gamma} \right)$$

and substituting for η^* and integrating, we find

$$= \log \left(\frac{Z_M(t^*; T)}{Z_M(t^*; t)} \right) - B(t; T) \frac{\partial}{\partial t} \log(Z_M(t^*; t)) - \frac{c^2}{4\gamma^3} \left(e^{-\gamma(T-t^*)} - e^{-\gamma(t-t^*)} \right) \left(e^{2\gamma(t-t^*)} - 1 \right).$$

We know the value of a zero-coupon bond is

$$Z(r, t; T) = \exp(A(t; T) - rB(t; T)),$$

with $A(t; T)$ given by the above, and

$$B(t; T) = \frac{1}{\gamma} \left(1 - e^{-\gamma(T-t)} \right).$$