CQF Module 1 Assignment

Instructions:

This is a non-assessed assignment. It is designed to help consolidate some of the key material covered in module 1.
Important topics of use through out the CQF covered during the maths primer are also included. The aim should be to work through the numerous problems in this sheet to help measure understanding and performance. Complete solutions will follow.

All queries should be directed to Riaz Ahmad by e-mail riaz.ahmad@fitchlearning.com

1 Basic Maths

1. Consider the linear parabolic partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + a\frac{\partial u}{\partial x} + bu,$$

for the function $u(x,t)$; where a and b are constants. By using a substitution of the form

$$u(x,t) = e^{\alpha x + \beta t} v(x,t),$$

and suitable choice of α and β, show that the PDE can be reduced to the heat equation

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}.$$

2. Consider the probability density function $f(x)$ given by

$$f(x) = \begin{cases} Ax^2 \exp\left(-\lambda x^2\right) & x > 0 \\ 0 & x \leq 0 \end{cases}.$$

Deduce that

$$A = 4\sqrt{\frac{\lambda^3}{\pi}}.$$

Show that

$$\mathbb{E}[X] = \frac{2}{\sqrt{\pi\lambda}}.$$

By using integration by parts, or otherwise, deduce that for $n = 0, 1, 2, \ldots$ the even moments of this distribution are given by

$$\mathbb{E}\left[X^{2n}\right] = \frac{1.3\ldots(2n+1)}{(2\lambda)^n}$$

and the odd moments are given by

$$\mathbb{E}\left[X^{2n+1}\right] = \frac{2}{\sqrt{\pi}}\frac{(n+1)!}{\lambda^{(2n+1)/2}}.$$

2 Stochastic Calculus

$W,\ W(t), W_t$ all refer to standard Brownian motion

1. a. Itô's lemma can be used to deduce the following formula for stochastic differential equations and stochastic integrals

$$\int_0^t \frac{\partial F}{\partial W}\, dW(\tau) = F(W(t), t) - F(W(0), 0) - \int_0^t \left(\frac{\partial F}{\partial \tau} + \frac{1}{2} \frac{\partial^2 F}{\partial W^2} \right) d\tau$$

for a function $F(W(\tau), \tau)$ where $dW(\tau)$ is an increment of a Brownian motion.
If $W(0) = 0$ evaluate

$$\int_0^t \tau^2 \sin W \, dW(\tau).$$

b. Suppose the stochastic process $S(t)$ evolves according to Geometric Brownian Motion (GBM), where

$$dS = \mu S dt + \sigma S dW.$$

Obtain a SDE $df(S, t)$ for each of the following functions

i $f\ (S, t) = \alpha^t + \beta t S^n \qquad \alpha, \beta$ are constants
ii $f(S, t) = \log tS + \cos tS$

2. Consider the following SDE

$$d\sigma = adt + bdW,$$

where $a = a(\sigma, t)$, $b = b(\sigma, t)$. The Forward Kolmogorov Equation (FKE), for the transition PDF $p = p(\sigma, t; \sigma', t')$ is

$$\frac{\partial p}{\partial t'} = \frac{1}{2} \frac{\partial^2}{\partial \sigma'^2}(b^2 p) - \frac{\partial}{\partial \sigma'}(ap),$$

where the primed variables refer to future states. The steady state solution is given by setting $\frac{\partial p}{\partial t'} = 0$. By considering suitable conditions, show that the steady state solution is given by

$$p(\sigma') = \frac{A}{b^2} e^{\int \frac{2a}{b^2} d\sigma'},$$

where A is an arbitrary constant. (During your working you may drop the primed notation).

3. Consider a function $V(t, S_t, r_t)$ where the two stochastic processes S_t and r_t evolve according to a two factor model given by

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dW_t^{(1)} \\ dr_t &= \gamma (m - r_t) dt + c dW_t^{(2)}, \end{aligned}$$

in turn. and where $\mathbb{E}\left[dW_t^{(1)} dW_t^{(2)}\right] = \rho dt$. The parameters μ, σ, γ, m and c are constant. Let $V(t, S_t, r_t)$ be a function on $[0, T]$ with $V(0, S_0, r_0) = v$. Using Itô, deduce the integral form for $V(T, S_T, r_T)$.

4. A spot rate r_t, evolves according to the popular form

$$dr_t = u(r_t)\,dt + \nu r_t^{\beta} dW_t, \qquad (*)$$

where ν and β are constants. Suppose such a model has a **steady state transition probability density function** $p_\infty(r)$ that satisfies the forward Fokker Planck Equation.
Show that this implies the drift structure of $(*)$ is given by

$$u(r_t) = \nu^2 \beta r_t^{2\beta-1} + \frac{1}{2}\nu^2 r_t^{2\beta} \frac{d}{dr}(\log p_\infty).$$

5. The ordinary differential equation

$$\mu S \frac{du}{dS} + \tfrac{1}{2}\sigma^2 S^2 \frac{d^2u}{dS^2} = -1,$$

for the function $u(S)$ is to be **solved** with boundary conditions

$$\begin{aligned} u(S_0) &= 0 \\ u(S_1) &= 0. \end{aligned}$$

μ and σ are constants. Show that the solution is given by

$$u(S) = \frac{1}{\frac{1}{2}\sigma^2 - \mu}\left(\log(S/S_0) - \frac{1-(S/S_0)^{1-2\mu/\sigma^2}}{1-(S_1/S_0)^{1-2\mu/\sigma^2}}\log(S_1/S_0)\right)$$

Hint: When solving for the particular integral, assume a solution of the form $C\log S$, where C is a constant.

3 Binomial Model

1. A share price is currently £15. At the end of three months, it will be either £13 or £17. By constructing a hedged portfolio, calculate the value of a three-month European option with payoff $\max\left(S^2 - 159, 0\right)$, where S is the share price at the end of three months. The risk-free rate is 5% per annum with continuous compounding.

2. Consider the following model risk-free interest rate $r = 0$:

ω	$S(0)$	$S(1)$	$S(2)$
ω_1	S	aS	$a^2 S$
ω_2	S	aS	S
ω_3	S	$a^{-1}S$	S
ω_4	S	$a^{-1}S$	$a^{-2}S$

   S is the initial asset value at $t = 0$ and $a > 1$ is a constant.

   (a) In this model, replicate the **European call** option with strike equal to the initial asset value S over the two periods and so find the fair price of the option.

   b. Find all the one period risk-neutral probabilities and the corresponding probability measure $\mathbb{Q}$ on $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$. Confirm that $\mathbb{E}^{\mathbb{Q}}[X]$ is the fair price.

   (b) Now consider a model where in each period the asset can either double or half. Show that the value of an option struck at the initial asset value S is $S/3$.

3. Repeat problem 1. by a replicating strategy. By calculating the risk-neutral probabilities obtain a price using an expectation.

4. A share price is currently £80. At the end of three months, it will be either £84 or £76. Ignoring interest rates, calculate the value of a three-month **digital** call option with strike price £79.