The Collector: Know Your Weapon—Part 2*

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BSD trader Soldier, last time I told you about delta and gamma Greeks. Today I'll enlighten you in on Vega, theta, and probability Greeks.
New trader Sir, I already know Vega.
BSD trader Soldier, if you want to speculate on an increase in implied volatility what type of options offer the most bang for the bucks?
New trader At-the-money options with long time to maturity.
BSD trader Soldier, you are possibly wrong on strikes and time! Now start with 20 push-ups while I start to tell you about Vega.
New trader Yes, Sir!

1 Refreshing notation on the BSM formula

Let me also this time refresh your memory of the Black–Scholes–Merton (BSM) formula

$$c = Se^{(b-r)T}N(d_1) - Xe^{-rT}N(d_2)$$
$$p = Xe^{-rT}N(-d_2) - Se^{(b-r)T}N(-d_1),$$

where

$$d_1 = \frac{\ln(S/X) + (b + \sigma^2/2)T}{\sigma\sqrt{T}},$$
$$d_2 = d_1 - \sigma\sqrt{T},$$

*Thanks to Jørgen Haug for useful comments.

and

$$\begin{aligned}
S &= \text{asset price}\\
X &= \text{strike price}\\
r &= \text{risk-free interest rate}\\
b &= \text{cost-of-carry rate of holding the underlying security}\\
T &= \text{time to expiration in years}\\
\sigma &= \text{volatility of the relative price change of the underlying asset price}\\
N(x) &= \text{the cumulative normal distribution function}
\end{aligned}$$

2 Vega Greeks

2.1 Vega

Vega,[1] also known as kappa, is the option's sensitivity to a small change in the implied volatility. Vega is equal for put and call options.

$$\text{Vega} = \frac{\partial c}{\partial \sigma} = \frac{\partial p}{\partial \sigma} = Se^{(b-r)T} n(d_1)\sqrt{T} > 0.$$

Implied volatility is often considered the market's best estimate of expected volatility for the duration of the option. It can also be interpreted as a basket of adjustments to the BSM formula, for factors that the formula doesn't take into account; demand and supply for that particular strike and maturity, stochastic volatility, jumps, and more. For instance a sudden increase in the Black–Scholes implied volatility for an out-of-the-money strike does not necessary imply that investors expect higher volatility. The increase can just as well be due to an option 'arbitrageur' expecting higher volatility of volatility.

Vega local maximum When trying to profit from moves in implied volatility it is useful to know where the option has the maximum Vega value for a given time to maturity. For a given strike price Vega attains its maximum when the asset price is

$$S = Xe^{(-b+\sigma^2/2)T}.$$

At this asset price we also have in-the-money risk neutral probability symmetry (which I come back to later). Moreover, at this asset price the generalized Black–Scholes–Merton (BSM) formula simplifies to

$$c = Se^{(b-r)T} N(\sigma\sqrt{T}) - \frac{Xe^{-rT}}{2},$$

$$p = \frac{Xe^{-rT}}{2} - Se^{(b-r)T} N(-\sigma\sqrt{T}).$$

Similarly, the strike that maximizes Vega given the asset price is

$$X = Se^{(b+\sigma^2/2)T}.$$

Vega global maximum Some years back a BSD trader called me late one evening, close to freaking out. He had shorted long-term options, which he hedged by going long short-term options. To his surprise the long-term options' Vega increased as time went by. After looking at my 3D Vega chart I confirmed that this was indeed the expected behavior. For options with long term to maturity the maximum Vega is not necessarily increasing with longer time to maturity, as many traders believe. Indeed, Vega has a global maximum at time

$$T_{\overline{V}} = \frac{1}{2r},$$

and asset price

$$S_{\overline{V}} = Xe^{(-b+\sigma^2/2)T_{\overline{V}}} = Xe^{\frac{-b+\sigma^2/2}{2r}}.$$

At this global maximum, Vega itself, described by Alexander (Sasha) Adamchuk,[2] is equal to the following simple expression

$$\mathrm{Vega}(S_{\overline{V}}, T_{\overline{V}}) = \frac{X}{2\sqrt{re\pi}}.$$

Figure 1 shows the graph of Vega with respect to the asset price and time. The intuition behind the Vega-top (Vega-mountain) is that the effect of discounting at some point in time dominates volatility (Vega): the lower the interest rate, the lower the effect of discounting, and the higher the relative effect of volatility on the option price. As the risk-free-rate goes to zero the time for the global maximum goes to infinity, that is we will have no global maximum when the risk-free

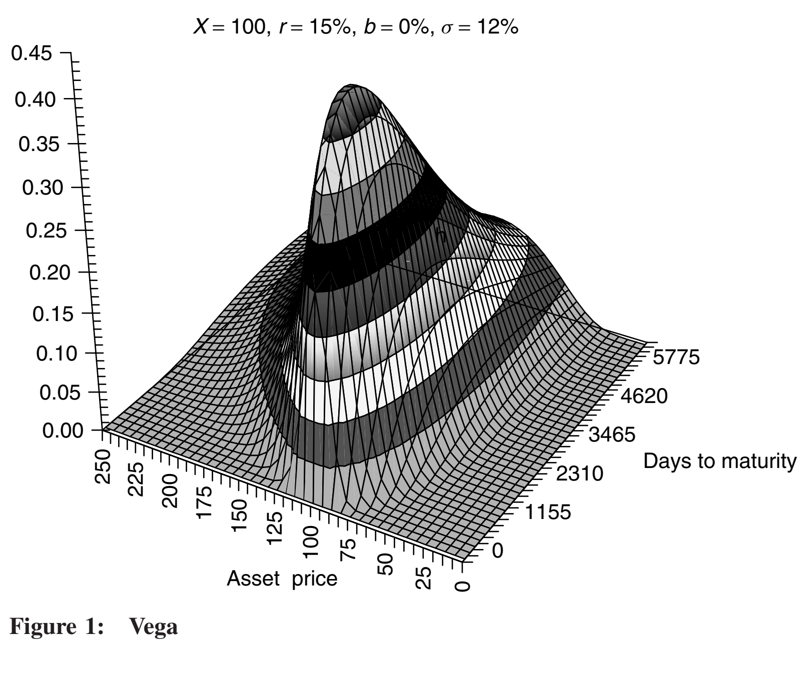

Figure 1: Vega

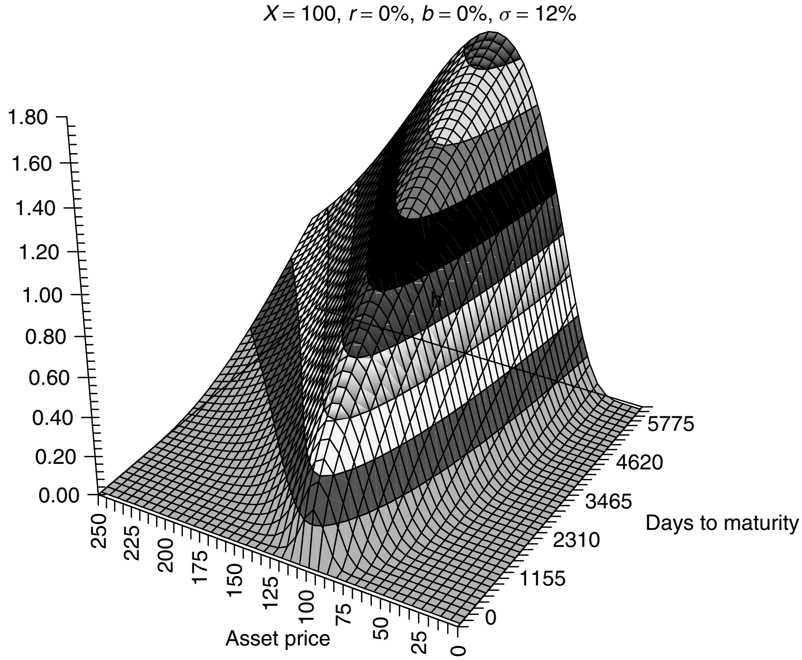

Figure 2: Vega

rate is zero. Figure 2 is the same as Figure 1 but with zero interest rate. The effect of Vega being a decreasing function of time to maturity typically kicks in only for options with very long times to maturity—unless the interest rate is very high. It is not, however, uncommon for caps and floors traders to use the Black-76 formula to compute Vegas for options with 10 to 15 years to expiration (caplets).

2.2 Vega symmetry

For options with different strikes we have the following Vega symmetry

$$\text{Vega}(S, X, T, r, b, \sigma) = \frac{X}{Se^{bT}} \text{Vega}\left(S, \frac{(Se^{bT})^2}{X}, T, r, b, \sigma\right).$$

As for the gamma symmetry, see Haug (2003), this symmetry is independent of the options being calls or puts—at least in theory.

2.3 Vega–gamma relationship

The following is a simple and useful relationship between Vega and gamma, described by Taleb (1997) amongst others:

$$\text{Vega} = \Gamma \sigma S^2 T.$$

2.4 Vega from delta

Given that we know the delta, what is the Vega? Vega and delta are related by a simple formula described by Wystrup (2002):

$$\text{Vega} = Se^{(b-r)T}\sqrt{T}n\left[N^{-1}(e^{(r-b)T}|\Delta|)\right],$$

where $N^{-1}(\cdot)$ is the inverted cumulative normal distribution, $n()$ is the normal density function, and Δ is the delta of a call or put option. Using the Vega–gamma relationship we can rewrite this relationship to express gamma as a function of the delta

$$\Gamma = \frac{e^{(b-r)T}n[N^{-1}(e^{(r-b)T}|\Delta|)]}{S\sigma\sqrt{T}}.$$

Relationships, such as the above ones, between delta and other option sensitivities are particularly useful in the FX options markets, where one often considers a particular delta rather than strike.

2.5 VegaP

The traditional textbook Vega gives the dollar change in option price for a percentage *point* change in volatility. When comparing the Vega risk of options on different assets it makes more sense to look at percentage changes in volatility. This metric can be constructed simply by multiplying the standard Vega with $\frac{\sigma}{10}$, which gives what is known as VegaP (percentage change in option price for a 10% change in volatility):

$$\text{VegaP} = \frac{\sigma}{10}Se^{(b-r)T}n(d_1)\sqrt{T} \geq 0.$$

VegaP attains its local and global maximum at the same asset price and time as for Vega. Some options systems use traditional textbook Vega, while others use VegaP.

When comparing Vegas for options with different maturities (calendar spreads) it makes more sense to look at some kind of weighted Vega, or alternatively Vega bucketing,[3] because short-term implied volatilities are typically more volatile than long-term implied volatilities. Several options systems implement some type of Vega weighting or Vega bucketing (see Haug 1993 and Taleb 1997 for more details).

2.6 Vega leverage, Vega elasticity

The percentage change in option value with respect to percentage point change in volatility is given by

$$\text{VegaLeverage}_{\text{call}} = \text{Vega}\frac{\sigma}{\text{call}} \geq 0,$$
$$\text{VegaLeverage}_{\text{put}} = \text{Vega}\frac{\sigma}{\text{put}} \geq 0.$$

The Vega elasticity is highest for out-of-the-money options. If you believe in an increase in implied volatility you will therefore get maximum bang for your bucks by buying out-of-the-money

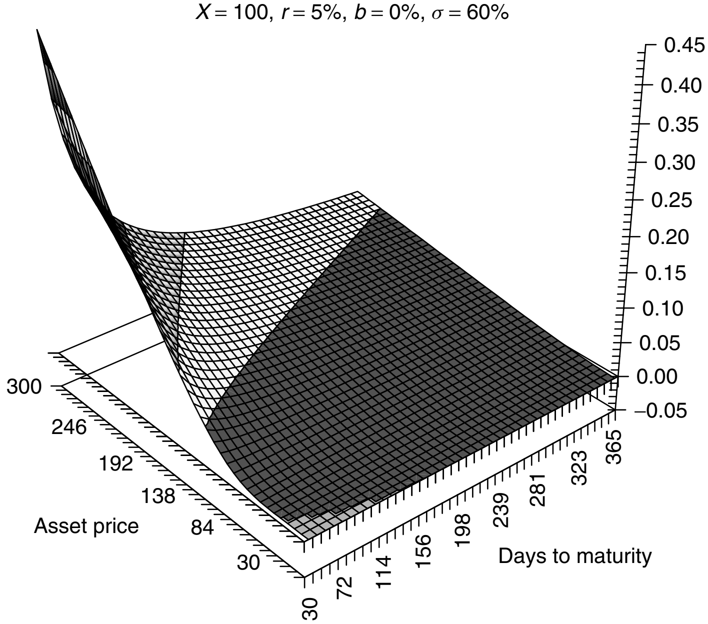

Figure 3: Vega leverage

options. Several traders I have met will typically tell you to buy at-the-money options when they want to speculate on higher implied volatility, to maximize Vega. There are several advantages to buying out-of-the-money options in such a scenario. One is the higher Vega-leverage. Another advantage is that you often also get a positive DvegaDvol (and also DgammaDvol), a measure we will have a closer look at below. The drawbacks of deep-out-of-the-money options are faster time decay (in percent of premium), and typically lower liquidity. Figure 3 illustrates the Vega leverage of a put option.

2.7 DvegaDvol, Vomma

DvegaDvol, also known as Vega convexity, Vomma (see Webb 1999), or Volga, is the sensitivity of Vega to changes in implied volatility. Together with DgammaDvol, see Haug (2003), Vomma is in my view one of the most important Greeks. DvegaDvol is given by

$$\text{DvegaDvol} = \frac{\partial^2 c}{\partial \sigma^2} = \frac{\partial^2 p}{\partial \sigma^2} = \text{Vega}\left(\frac{d_1 d_2}{\sigma}\right) \leq\geq 0.$$

For practical purposes, where one 'typically' wants to look at Vomma for the change of one percentage *point* in the volatility, one should divide Vomma by 10 000.

In case of DvegaPDvol we have

$$\text{DvegaPDvol} = \text{VegaP}\left(\frac{d_1 d_2}{\sigma}\right) \leq\geq 0.$$

Options far out-of-the money have the highest Vomma. More precisely given the strike price, Vomma is positive outside the interval

$$(S_L = Xe^{(-b-\sigma^2/2)T}, S_U = Xe^{(-b+\sigma^2/2)T}).$$

Given the asset price the Vomma is positive outside the interval (relevant only before conducting the trade)

$$(X_L = Se^{(b-\sigma^2/2)T}, X_U = Se^{(b+\sigma^2/2)T}).$$

If you are long options you typically want to have as high positive DvegaDvol as possible. If short options, you typically want negative DvegaDvol. Positive DvegaDvol tells you that you will earn more for every percentage point increase in volatility, and if implied volatility is falling you will lose less and less—that is, you have positive Vega convexity.

While DgammaDvol is most relevant for the volatility of the actual volatility of the underlying asset, DvegaDvol is more relevant for the volatility of the implied volatility. Although the volatility of implied volatility and the volatility of actual volatility will typically have high correlation, this is not always the case. DgammaDvol is relevant for traditional dynamic delta hedging under stochastic volatility. DvegaDvol trading has little to do with traditional dynamic delta hedging. DvegaDvol trading is a bet on changes on the price (changes in implied vol) for uncertainty in:

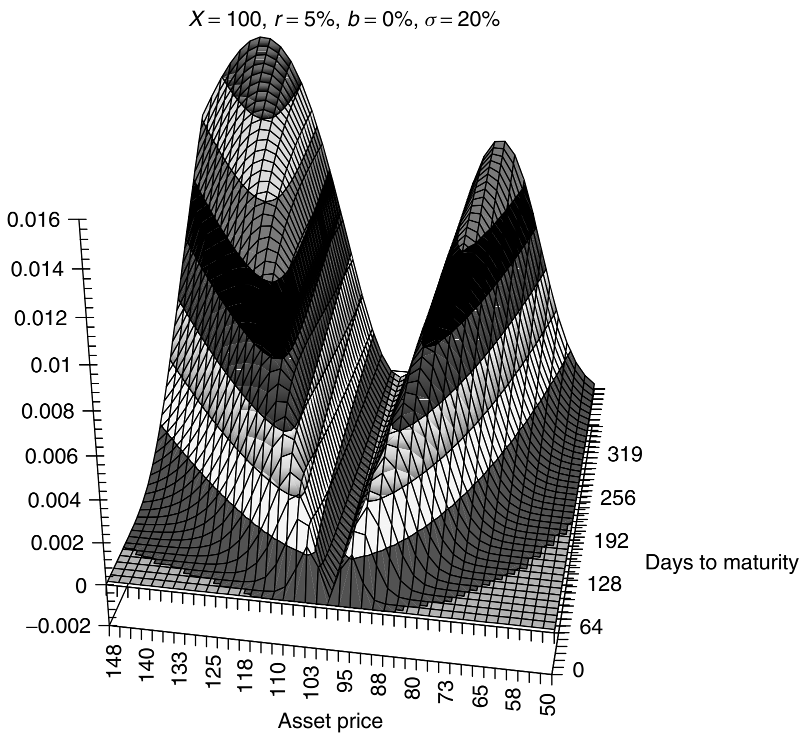

Figure 4: DvegaDvol

supply and demand, stochastic actual volatility (remember this is correlated to implied volatility), jumps and any other model risk: factors that affect the option price, but that are not taken into account in the Black–Scholes formula. A DvegaDvol trader does not necessarily need to identify the exact reason for the implied volatility to change. If you think the implied volatility will be volatile in the short term you should typically try to find options with high DvegaDvol. Figure 4 shows the graph of DvegaDvol for changes in asset price and time to maturity.

2.8 DvegaDtime

DvegaDtime is the change in Vega with respect to changes in time. Since we typically are looking at decreasing time to maturity we express this as minus the partial derivative

$$\text{DvegaDtime} = -\frac{\partial \text{Vega}}{\partial T} = \text{Vega}\left(r - b + \frac{bd_1}{\sigma\sqrt{T}} - \frac{1 + d_1 d_2}{2T}\right) \leq\geq 0$$

For practical purposes, where one 'typically' wants to express the sensitivity for a one percentage point change in volatility to a one day change in time, one should divide the DvegaDtime by 36 500, or 25 200 if you look at trading days only. Figure 5 illustrates DvegaDtime. Figure 6 shows DvegaDtime for a wider range of parameters and a lower implied volatility, as expected from Figure 1 we can see here that DvegaDtime actually can be positive.

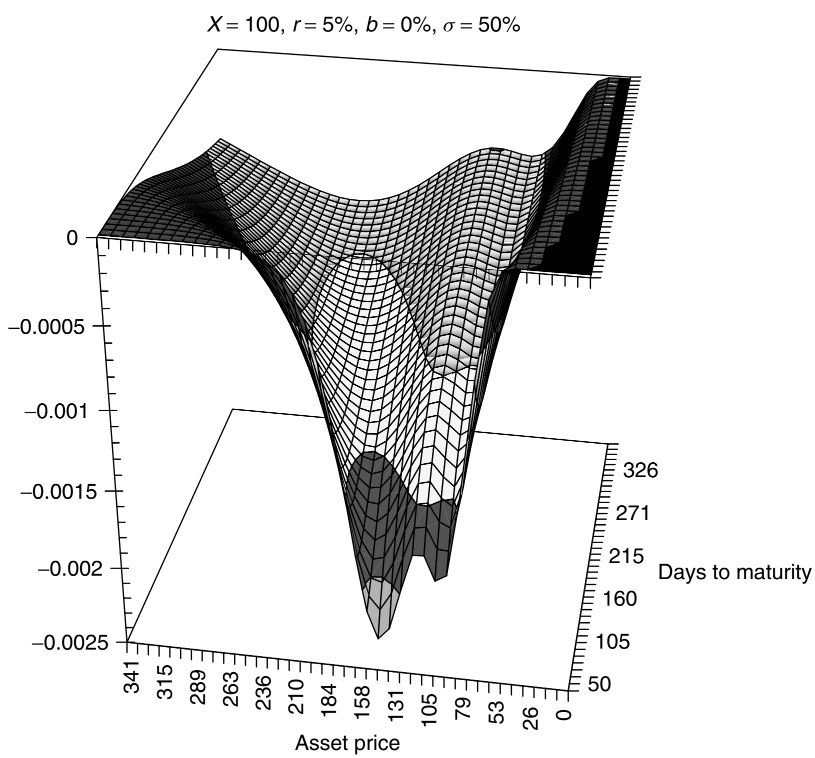

Figure 5: DvegaDtime

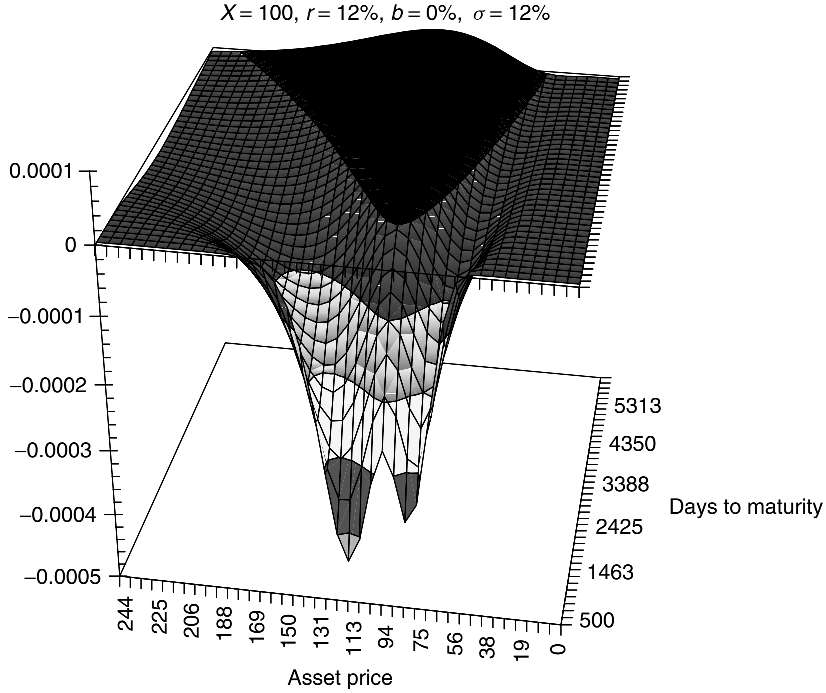

Figure 6: DvegaDtime (Vanna)

3 Theta Greeks

3.1 Theta

Theta is the option's sensitivity to a small change in time to maturity. As time to maturity decreases, it is normal to express theta as minus the partial derivative with respect to time.

Call

$$\Theta_{\text{call}} = -\frac{\partial c}{\partial T} = -\frac{Se^{(b-r)T}n(d_1)\sigma}{2\sqrt{T}} - (b-r)Se^{(b-r)T}N(d_1) - rXe^{-rT}N(d_2) \leq\geq 0.$$

Put

$$\Theta_{\text{put}} = -\frac{\partial p}{\partial T} = -\frac{Se^{(b-r)T}n(d_1)\sigma}{2\sqrt{T}} + (b-r)Se^{(b-r)T}N(-d_1) + rXe^{-rT}N(-d_2) \leq\geq 0.$$

Drift-less theta In practice it is often also of interest to know the drift-less theta, θ, which measures time decay without taking into account the drift of the underlying or discounting. In other words the drift-less theta isolates the effect time-decay has on uncertainty, assuming unchanged volatility. The uncertainty or volatilities effect on the option consists of time and volatility. In that case we have

$$\theta_{\text{call}} = \theta_{\text{put}} = \theta = -\frac{Sn(d_1)\sigma}{2\sqrt{T}} \leq 0.$$

3.2 Theta symmetry

In the case of drift-less theta for options with different strikes we have the following symmetry, for both puts and calls,

$$\theta(S, X, T, 0, 0, \sigma) = \frac{X}{S}\theta\left(S, \frac{S^2}{X}, T, 0, 0, \sigma\right)$$

Theta–Vega relationship There is a simple relationship between Vega and drift-less theta

$$\theta = -\frac{\text{Vega} \times \sigma}{2T}.$$

Bleed-offset volatility A more practical relationship between theta and Vega is what is known as bleed-offset vol. It measures how much the volatility must increase to offset the theta-bleed/time decay. Bleed-offset vol can be found simply by dividing the one-day theta by Vega, $\frac{\Theta}{\text{Vega}}$. In the case of positive theta you can actually have negative offset vol. Deep-in-the-money European options can have positive theta, in this case the offset-vol will be negative.

Theta–gamma relationship There is a simple relationship between drift-less gamma and drift-less theta

$$\Gamma = \frac{-2\theta}{S^2\sigma^2}.$$

4 Rho Greeks

4.1 Rho

Rho is the option's sensitivity to a small change in the risk-free interest rate.

Call

$$\rho_{\text{call}} = \frac{\partial c}{\partial r} = TXe^{-rT}N(d_2) > 0,$$

in the case the option is on a future or forward (that is b will always stay 0) the rho is given by

$$\rho_{\text{call}} = \frac{\partial c}{\partial r} = -Tc < 0.$$

Put

$$\rho_{\text{put}} = \frac{\partial p}{\partial r} = -TXe^{-rT}N(-d_2) < 0$$

in the case the option is on a future or forward (that is b will always stay 0) the rho is given by

$$\rho_{\text{put}} = \frac{\partial c}{\partial r} = -Tp < 0.$$

4.2 Cost-of-carry

This is the option's sensitivity to a marginal change in the cost-of-carry rate.

Cost-of-carry call

$$\frac{\partial c}{\partial b} = TSe^{(b-r)T}N(d_1) > 0.$$

Cost-of-carry put

$$\frac{\partial p}{\partial b} = -TSe^{(b-r)T}N(-d_1) < 0.$$

5 Probability Greeks

In this section we will look at risk neutral probabilities in relation to the BSM formula. Keep in mind that such risk adjusted probabilities could be very different from real world probabilities.[4]

5.1 In-the-money probability

In the (Black and Scholes 1973, Merton 1973) model, the risk neutral probability for a call option finishing in-the-money is

$$\zeta_c = N(d_2) > 0,$$

and for a put option

$$\zeta_p = N(-d_2) > 0.$$

This is the risk neutral probability of ending up in-the-money at maturity. It is not identical to the real world probability of ending up in-the-money. The real probability we simply cannot extract from options prices alone. A related sensitivity is the strike-delta, which is the partial derivatives of the option formula with respect to the strike price

$$\frac{\partial c}{\partial X} = -e^{-rT}N(d_2) > 0,$$
$$\frac{\partial p}{\partial X} = e^{-rT}N(-d_2) > 0.$$

This can be interpreted as the discounted risk neutral probability of ending up in-the-money (assuming you take the absolute value of the call strike-delta).

Probability mirror strikes For a put and a call to have the same risk neutral probability of finishing in-the-money, we can find the probability symmetric strikes

$$X_p = \frac{S^2}{X_c} e^{(2b-\sigma^2)T}, \qquad X_c = \frac{S^2}{X_p} e^{(2b-\sigma^2)T},$$

where X_p is the put strike, and X_c is the call strike. This naturally reduces to $N[d_2(X_c)] = N[d_2(X_p)]$. A special case is $X_c = X_p$, a probability mirror straddle (probability-neutral straddle). We have this at

$$X_c = X_p = Se^{(b-\sigma^2/2)T}.$$

At this point the risk neutral probability of ending up in-the-money is 0.5 for both the put and the call. Standard puts and calls will not have the same value at this point. The same value for a put and a call occurs when the options are at-the-money forward, $X = S^{bT}$. However, for a cash-or-nothing option (see Reiner and Rubinstein 1991b, Haug 1997) we will also have value-symmetry for puts and calls at the risk neutral probability strike. Moreover, at the probability-neutral straddle we will also have Vega symmetry as well as zero Vomma.

Strikes from probability Another interesting formula returns the strike of an option, given the risk neutral probability p_i of ending up in-the-money. The strike of a call is given by

$$X_c = S \exp[-N^{-1}(p_i)\sigma\sqrt{T} + (b - \sigma^2/2)T],$$

where $N^{-1}(x)$ is the inverse cumulative normal distribution. The strike for a put is given by

$$X_p = S \exp[N^{-1}(p_i)\sigma\sqrt{T} + (b - \sigma^2/2)T].$$

5.2 DzetaDvol

Zeta's sensitivity to change in the implied volatility is given by

$$\frac{\partial \zeta_c}{\partial \sigma} = \frac{\partial \zeta_p}{\partial \sigma} = -n(d_2)\left(\frac{d_1}{\sigma}\right) \leq\geq 0$$

and for a put

$$\frac{\partial \zeta_p}{\partial \sigma} = \frac{\partial \zeta_p}{\partial \sigma} = n(d_2)\left(\frac{d_1}{\sigma}\right) \leq\geq 0.$$

Divide by 100 to get the associated measure for percentage point volatility changes.

5.3 DzetaDtime

The in-the-money risk neutral probability's sensitivity to moving closer to maturity is given by

$$-\frac{\partial \zeta_c}{\partial T} = n(d_2)\left(\frac{b}{\sigma\sqrt{T}} - \frac{d_1}{2T}\right) \leq\geq 0,$$

and for a put

$$-\frac{\partial \zeta_p}{\partial T} = -n(d_2)\left(\frac{b}{\sigma\sqrt{T}} - \frac{d_1}{2T}\right) \leq\geq 0.$$

Divide by 365 to get the sensitivity for a one-day move.

5.4 Risk neutral probability density

BSM second partial derivatives with respect to the strike price yield the risk neutral probability density of the underlying asset, see Breeden and Litzenberger (1978) (this is also known as the strike gamma)

$$\mathrm{RND} = \frac{\partial^2 c}{\partial X^2} = \frac{\partial^2 p}{\partial X^2} = \frac{n(d_2)e^{-rT}}{X\sigma\sqrt{T}} \geq 0.$$

Figure 7 illustrates the risk neutral probability density with respect to variable time and asset price. With the same volatility for any asset price this is naturally the log-normal distribution of the asset price, as evident from the graph.

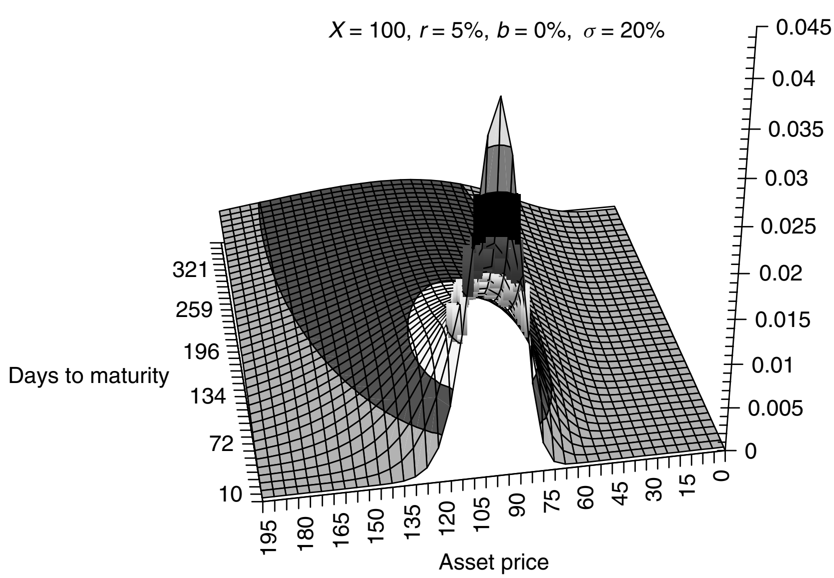

Figure 7: Risk neutral density

5.5 From in-the-money probability to density

Given the in-the-money risk-neutral probability, p_i, the risk neutral probability density is given by

$$\text{RND} = \frac{e^{-rT} n[N^{-1}(p_i)]}{X\sigma\sqrt{T}},$$

where $n()$ is the normal density function.

5.6 Probability of ever getting in-the-money

For in-the-money options the probability of ever getting in-the-money (hitting the strike) before maturity naturally equals unity, since we are already in-the-money. The risk neutral probability for an out-of-the-money call ever getting in-the-money is[5]

$$p_c = (X/S)^{\mu+\lambda} N(-z) + (X/S)^{\mu-\lambda} N(-z + 2\lambda\sigma\sqrt{T}).$$

Similarly, the risk neutral probability for an out-of-the-money put ever getting in-the-money (hitting the strike) before maturity is

$$p_p = (X/S)^{\mu+\lambda} N(z) + (X/S)^{\mu-\lambda} N(z - 2\lambda\sigma\sqrt{T}),$$

where

$$z = \frac{\ln(X/S)}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T}, \qquad \mu = \frac{b - \sigma^2/2}{\sigma^2}, \qquad \lambda = \sqrt{\mu^2 + \frac{2r}{\sigma^2}}.$$

This is equal to the barrier hit probability used for computing the value of a rebate, developed by Reiner and Rubinstein (1991a). Alternatively, the probability of ever getting in-the-money before maturity can be calculated in a very simple way in a binomial tree, using Brownian bridge probabilities.

End of Part 2

BSD trader Sergeant, that is all for now. You now know the basic operation of the Black–Scholes weapon.
New trader Did I hear you right? ‘Sergeant’?
BSD trader Yes. Now that you know the basics of the Black–Scholes weapon, I have decided to promote you.
New trader Thank you, Sir, for teaching me all your tricks.
BSD trader Here’s a three million loss limit. Time for you to start trading.
New trader Only three million?

FOOTNOTES & REFERENCES

1. While the other sensitivities have names that correspond to Greek letters Vega is the name of a star.

2. Described by Adamchuk on the Wilmott forum www.wilmott.com on February 6, 2002.
3. Vega bucketing simply refers to dividing the Vega risk into time buckets.
4. Risk neutral probabilities are simply real world probabilities that have been adjusted for risk. It is therefore not necessary to adjust for risk also in the discount factor for cash flows. This makes it valid to compute market prices as simple expectations of cash flows, with the *risk adjusted probabilities*, discounted at the *risk less interest rate*—hence the common name 'risk neutral' probabilities, which is somewhat of a misnomer.
5. This analytical probability was first published by Reiner and Rubinstein (1991a) in the context of barrier hit probability.

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