Exotic Solutions pg 1

1. Consider an option which pays a continuous cash-flow to the holder at a rate proportional to the square of the underlying asset's price, so that during a time interval dt the holder receives $S^2 dt$. Suppose that at expiry the value of the option is
$$V(S,T) = S^2.$$
The underlying evolution follows geometric Brownian motion
$$dS = \mu S dt + \sigma S dX.$$
Derive the Black-Scholes partial differential equation for this "power" option and show that it is
$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = -S^2.$$
By assuming a solution of the form
$$V(S,t) = \phi(t) S^2$$
show that
$$\phi(t) = \frac{1}{\sigma^2 + r}\left(\left(\sigma^2 + r + 1\right) e^{\left(\sigma^2+r\right)(T-t)} - 1\right).$$
Solution: Start with delta hedged portfolio
$$\Pi = V(S,t) - \Delta S.$$
with
$$dS = \mu S dt + \sigma S dX$$
over one time-step dt, where Δ is fixed from t to $t + dt$
$$d\Pi = \underbrace{dV - \Delta dS}_{\text{changes in } V \text{ and } S} + \underbrace{S^2 dt}_{\text{cash flow from holding } V}$$

$$\begin{aligned} d\Pi &= \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt + \\ &\quad \frac{\partial V}{\partial S} dS - \Delta dS + S^2 dt \end{aligned}$$
Only source of risk is in dS, so choose $\Delta = \frac{\partial V}{\partial S}$ to eliminate it.
$$\begin{aligned} d\Pi &= \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + S^2\right) dt \\ &= r\Pi\, dt \\ &= r(V - \Delta S)\, dt = r\left(V - S\frac{\partial V}{\partial S}\right) dt \end{aligned}$$
$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + S^2 = rV - rS\frac{\partial V}{\partial S}$$
$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = -S^2$$
At expiry we have $V(S,T) = S^2$. Look for a solution of the form $V(S,t) = \phi(t) S^2$, then

$$\begin{aligned}\frac{\partial V}{\partial t} &= \dot{\phi}(t)\,S^2,\ \frac{\partial V}{\partial S}=2S\phi(t)\,,\ \frac{\partial^2 V}{\partial S^2}=2\phi(t)\\ V(S,T) &= \phi(T)\,S^2=S^2\Longrightarrow\phi(T)=1\end{aligned}$$

$$\dot{\phi}(t)\,S^2+\sigma^2S^2\phi(t)+2rS^2\phi(t)-rS^2\phi(t)=-S^2\longrightarrow$$

$$\dot{\phi}(t)+\left(\sigma^2+r\right)\phi(t)=-1,\quad\phi(T)=1$$

use integrating factor

$$e^{\left(\sigma^2+r\right)t}$$

$$\begin{aligned}\frac{d}{dt}\left(\phi(t)\,e^{\left(\sigma^2+r\right)t}\right) &= -e^{\left(\sigma^2+r\right)t}\\ \int d\left(\phi(t)\,e^{\left(\sigma^2+r\right)t}\right) &= -\int e^{\left(\sigma^2+r\right)t}dt\\ \phi(t)\,e^{\left(\sigma^2+r\right)t} &= -\frac{e^{\left(\sigma^2+r\right)t}}{\left(\sigma^2+r\right)}+A\\ \phi(t) &= -\frac{1}{\left(\sigma^2+r\right)}+Ae^{-\left(\sigma^2+r\right)t}\end{aligned}$$

we know $\phi(T)=1$ so $A=\left(1+\frac{1}{\left(\sigma^2+r\right)}\right)e^{\left(\sigma^2+r\right)t}$.
Hence

$$\phi(t)=\frac{1}{\sigma^2+r}\left(\left(\sigma^2+r+1\right)e^{\left(\sigma^2+r\right)(T-t)}-1\right)$$

2. Consider separable solutions of the Black–Scholes equation

$$\frac{\partial V}{\partial t}+\tfrac{1}{2}\sigma^2S^2\frac{\partial^2V}{\partial S^2}+(r-D)\,S\frac{\partial V}{\partial S}-rV=0, \tag{2.1}$$

of the form

$$V(S,t)=f(S)g(t),$$

Show that (2.1) can be expressed as the following first order differential equation (2.2a) and Cauchy-Euler equation (2.2b)

$$\begin{aligned}\frac{dg}{dt}-\lambda g &= 0 &\text{(2.2a)}\\ \tfrac{1}{2}\sigma^2S^2f''+(r-D)\,Sf'+(\lambda-r)\,f &= 0, &\text{(2.2b)}\end{aligned}$$

for some (universal) constant λ, where the following notation is used

$$f'=\frac{df}{dS},\ f''=\frac{d^2f}{dS^2}.$$

You may assume that (2.2b) has a solution of the form $f(S)=S^\alpha$. Solve these to obtain the following solutions for (2.1) :

i for distinct roots of the A.E (2.2b) (A, B - constants)

$$V(S,t)=e^{\lambda t}S^{\frac{1}{2}-\frac{r-D}{\sigma^2}}\left(AS^{\alpha_+}+BS^{\alpha_-}\right)$$

ii for a repeated root of the A.E (2.2b) (ε, ζ - constants)

$$V(S,t)=e^{\left(\left(r+\frac{\sigma^2}{2}\left(\frac{r-D}{\sigma^2}+\frac{1}{2}\right)^2\right)t\right)}S^{\left(\frac{1}{2}-\frac{r-D}{\sigma^2}\right)}\left(\varepsilon+\zeta\log S\right)$$

where

$$\bar{d}_\pm=\pm\sqrt{\left(\frac{r-D}{\sigma^2}-\frac{1}{2}\right)^2-\frac{2\left(\lambda-r\right)}{\sigma^2}}.$$

Start by differentiating

$$\begin{aligned}\frac{\partial V}{\partial t} &= f\dot{g}\\ \frac{\partial V}{\partial S} &= gf'\\ \frac{\partial^2 V}{\partial S^2} &= gf''\end{aligned}$$

and substitute in the BSE

$$f\dot{g}+\frac{1}{2}\sigma^2S^2gf''+(r-D)\,Sgf'-rgf=0.$$

Dividing through by gf

$$\frac{f\dot{g}+\frac{1}{2}\sigma^2S^2gf''+(r-D)\,Sgf'-rgf}{gf}=0$$

and arrange so that t dependency on lhs and S dependency on rhs,

$$\frac{\dot{g}}{g}=\frac{-\frac{1}{2}\sigma^2S^2f''-(r-D)\,Sf'+rf}{f}=\lambda$$

$$\dot{g}\;=\lambda g\to g\,(t)=ke^{\lambda t}.$$

Secondly a 2nd order Cauchy-Euler equation:

$$\frac{1}{2}\sigma^2S^2f''+(r-D)\,Sf'+(\lambda-r)\,f=0$$

Putting

$$f(S)=S^{\alpha}$$

gives a quadratic in α

$$\alpha^2+\left(\frac{2\,(r-D)}{\sigma^2}-1\right)\alpha+\frac{2}{\sigma^2}\,(\lambda-r)=0$$

hence

$$\begin{aligned}\alpha_{\pm} &= \frac{1}{2}\left(1-\frac{2\,(r-D)}{\sigma^2}\right)\pm\frac{1}{2}\sqrt{\left(\frac{2\,(r-D)}{\sigma^2}-1\right)^2-\frac{8}{\sigma^2}\,(\lambda-r)}\\ \alpha_{\pm} &= \frac{1}{2}\left(1-\frac{2\,(r-D)}{\sigma^2}\right)\pm\frac{1}{2}\sqrt{\frac{4\,(r-D)^2}{\sigma^4}+1-\frac{4\,(r-D)}{\sigma^2}-\frac{8\,(\lambda-r)}{\sigma^2}}\\ &= \frac{1}{2}\left(1-\frac{2\,(r-D)}{\sigma^2}\right)\pm\sqrt{\frac{(r-D)^2}{\sigma^4}+\frac{1}{4}-\frac{(r-D)}{\sigma^2}-\frac{2\,(\lambda-r)}{\sigma^2}}\\ &= \frac{1}{2}\left(1-\frac{2\,(r-D)}{\sigma^2}\right)\pm\sqrt{\left(\frac{r-D}{\sigma^2}-\frac{1}{2}\right)^2-\frac{2\,(\lambda-r)}{\sigma^2}}\end{aligned}$$

<u>2 cases to consider</u>:
(1) Solution for distinct roots - $f\,(S)=aS^{\alpha_+}+bS^{\alpha_-}$

$$V\,(S,t)=e^{\lambda t}S^{\frac{1}{2}-\frac{r-D}{\sigma^2}}\left[AS^{\overline{d}_+}+BS^{\overline{d}_-}\right]\qquad A,\,B-\;\text{constants}$$

where

$$\overline{d}_+=\sqrt{\left(\frac{r-D}{\sigma^2}-\frac{1}{2}\right)^2-\frac{2\,(\lambda-r)}{\sigma^2}}\;;\quad \overline{d}_-=-\sqrt{\left(\frac{r-D}{\sigma^2}-\frac{1}{2}\right)^2-\frac{2\,(\lambda-r)}{\sigma^2}}$$

(2) Repeated Root - $f(S) = S^{\frac{1}{2}-\frac{r}{\sigma^2}}[a + b\log S]$. Now $\left(\frac{r-D}{\sigma^2} - \frac{1}{2}\right)^2 = \frac{2(\lambda - r)}{\sigma^2} \to \lambda = r + \frac{\sigma^2}{2}\left(\frac{r-D}{\sigma^2} - \frac{1}{2}\right)^2$ therefore

$$V(S,t) = \exp\left(\left(r + \frac{\sigma^2}{2}\left(\frac{r-D}{\sigma^2} - \frac{1}{2}\right)^2\right)t\right) S^{\left(\frac{1}{2}-\frac{r-D}{\sigma^2}\right)}[\varepsilon + \zeta \log S] \qquad \varepsilon, \zeta \text{ - constants}$$

3. Assume that an asset price S evolves according to the SDE

$$\frac{dS}{S} = (\mu - D)\,dt + \sigma dX,$$

where μ and σ are constants. In particular S pays out a continuous dividend stream equal to $DS\ dt$ during the infinitesimal time interval dt, where D the dividend yield is constant.
Now suppose a European style derivative security is written on this asset with the properties that at expiry the holder receives the asset and prior to expiry the derivative pays a continuous cash flow $C(S,t)\,dt$ during each time interval of length dt.
Show that the option price $V(S,t)$ satisfies the following partial differential equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D)\,S\frac{\partial V}{\partial S} - rV = -C(S,t)\,.$$

Solution:

$$\begin{aligned} \Pi &= V - \Delta S \\ d\Pi &= dV - \Delta dS - \Delta DSdt + C(S,t)\,dt, \end{aligned}$$

because we are short the stock. Using the usual hedging argument gives

$$\begin{aligned} d\Pi &= \left(\frac{\partial V}{\partial t} + \mu S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right)dt + \sigma S \frac{\partial V}{\partial S}dX \\ &\quad -\Delta(\mu S dt + \sigma S dX) - \Delta DSdt + C(S,t)\,dt \end{aligned}$$

Put $\Delta = \frac{\partial V}{\partial S}$ to eliminate risk and use no-arbitrage

$$\begin{aligned} d\Pi &= \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - DS\frac{\partial V}{\partial S} + C(S,t)\right)dt \\ &= r\left(V - S\frac{\partial V}{\partial S}\right)dt \end{aligned}$$

from which the BSE is obtained

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D)\,S\frac{\partial V}{\partial S} - rV = -C(S,t)$$

4. An asset S follows a Geometric Brownian Motion $dS = \mu S dt + \sigma S dW$, where μ and σ are constants. We wish to value an option that pays off at expiry T an amount which is a function of the path taken by the asset between time zero and expiry. Assuming that an option value V depends on S, t and a quantity

$$I(t) = \int_0^t f(S,\tau)\, d\tau,$$

where f is a specified function and r the risk free interest rate, the option pricing equation is

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + f(S,t)\frac{\partial V}{\partial I} + rS\frac{\partial V}{\partial S} - rV = 0,$$

for the function $V(S,I,t)$.
For an arithmetic strike Asian call option the payoff at time T is

$$\max\left(S - \frac{1}{T}\int_0^T S(t)\, dt, 0\right).$$

By writing the value of this option as

$$V(S,I,t) = SW(R,t),$$

where $R = I/S$, show that the partial differential equation for $W(R,t)$ is given by

$$\frac{\partial W}{\partial t} + \frac{1}{2}\sigma^2 R^2 \frac{\partial^2 W}{\partial R^2} + (1 - rR)\frac{\partial W}{\partial R} = 0.$$

Solution: If $V(S,I,t) = SW(R,t)$; $R = I/S$ then

$$\frac{\partial R}{\partial S} = -\frac{I}{S^2};\ \frac{\partial R}{\partial I} = \frac{1}{S};\ \frac{\partial}{\partial S} \equiv -\frac{I}{S^2}\frac{\partial}{\partial R}$$

$$\begin{aligned}
\frac{\partial V}{\partial t} &= S\frac{\partial W}{\partial t};\ \frac{\partial V}{\partial I} = S\frac{\partial W}{\partial R}\frac{\partial R}{\partial I} = \frac{\partial W}{\partial R} \\
\frac{\partial V}{\partial S} &= W + S\frac{\partial W}{\partial R}\frac{\partial R}{\partial S} = W - R\frac{\partial W}{\partial R} \\
\frac{\partial^2 V}{\partial S^2} &= \frac{\partial}{\partial S}\left(W - R\frac{\partial W}{\partial R}\right) = \frac{\partial R}{\partial S}\frac{\partial}{\partial R}\left(W - R\frac{\partial W}{\partial R}\right) \\
&= -\frac{I}{S^2}\frac{\partial}{\partial R}\left(W - R\frac{\partial W}{\partial R}\right) = -\frac{R}{S}\left(\frac{\partial W}{\partial R} - \frac{\partial W}{\partial R} - R\frac{\partial^2 W}{\partial R^2}\right) \\
&= \frac{R^2}{S}\frac{\partial^2 W}{\partial R^2}
\end{aligned}$$

Substituting in the pricing PDE

$$\begin{aligned}
S\frac{\partial W}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{R^2}{S}\frac{\partial^2 W}{\partial R^2} + S\frac{\partial W}{\partial R} + rS\left(W - R\frac{\partial W}{\partial R}\right) - rSW &= 0 \\
\frac{\partial W}{\partial t} + \frac{1}{2}\sigma^2 R^2 \frac{\partial^2 W}{\partial R^2} + \frac{\partial W}{\partial R} + r\left(W - R\frac{\partial W}{\partial R}\right) - rW &= 0 \\
\frac{\partial W}{\partial t} + \frac{1}{2}\sigma^2 R^2 \frac{\partial^2 W}{\partial R^2} + (1 - rR)\frac{\partial W}{\partial R} &= 0
\end{aligned}$$