Mod.1 Lecture 2 - Exercises

Taylor expansions and Transition Density Functions

This is a non-assessed **problem sheet.**

1. Expand $(2+x)^{-2}$ in ascending powers of x up to and including the term in x^3, and state the set of values of x for which the expansion is valid. Hence find the coefficient of x^3 in the expansion of $\dfrac{1+x^2}{(2+x)^2}$.

2. Find the Maclaurin series for $\ln(1+x)$ and hence that for $\ln\left(\dfrac{1+x}{1-x}\right)$.

3. Find the Taylor series expansions of the following functions about $x=0$ (*by first using a Binomial expansion in part* **a**) *and then considering how the function in part b) is related to that in part* **a**)).

   (a) $f(x) = \dfrac{1}{1+x}$.

   (b) $g(x) = \ln(1+x)$.

4. Find the first 4 terms of the Taylor series for the following functions centred at $a=1$. **Hint: The expansion will have powers of** $(x-1)$:

   (a) $f(x) = \ln x$

   (b) $g(x) = \dfrac{1}{x}$

5. Find all first order partial derivatives

   (a) $f(x,y) = 2x^4y^3 - xy^2 + 3y + 1$.

   (b) $f(x,y,z) = xyze^{xyz}$.

   (c) $f(x,y,z) = \left(y^2+z^2\right)^x$. Hint: $\frac{d}{dx}a^x = a^x \ln a$; where $a>0$.

6. Consider a **symmetric** random walk which starts with a marker placed at a point x at time s; written (x,s). Suppose at a later time $t>s$ the marker is at y; the future state denoted (y,t). The marker can move in step sizes of δy in a time step of δt. At the previous step the marker must have been at one of $(y-\delta y, t-\delta t)$ or $(y+\delta y, t-\delta t)$. The transition probability density function of the position y of the diffusion at a later time t, is written $p(x,s;y,t)$. Derive the Forward Equation

$$\frac{\partial p}{\partial t} = \frac{1}{2}\frac{\partial^2 p}{\partial y^2}. \tag{6.1}$$

   You may omit the dependence on (x,s) in your working as they will not change. Assume a solution of (6.1) exists and takes the following form

$$p(y,t) = t^{-1/2} f(\eta)\,;\ \ \eta = \frac{y}{t^{1/2}}.$$

   Solve (6.1) to show that a particular solution of this is

$$p(x,s;y,t) = \frac{1}{\sqrt{2\pi(t-s)}} \exp\left(-\frac{(y-x)^2}{2(t-s)}\right).$$

   You may use the result $\int_{\mathbb{R}} e^{-x^2}dx = \sqrt{\pi}$, in your working.