Experiment

Early 1970's

60% → 101

100

40% → 99

C^1 0.5

C^2 Expected payoff = 0.6

Locked in 0.05 0.55

Expected return

$$\frac{0.6-0.55}{0.55} \approx 9\%$$

C^3 0.53

0.52

0.51 ... 0.5

no arbitrage

Replication

Cash

$$\begin{tikzcd}[row sep=small] & 1 \\ 1 \arrow[ur] \arrow[dr] & \\ & 1 \end{tikzcd}$$

Stock

$$\begin{tikzcd}[row sep=small] & 101 \\ 100 \arrow[ur] \arrow[dr] & \\ & 99 \end{tikzcd}$$

$$\begin{tikzcd}[row sep=small] & -\frac{99}{2} \\ -\frac{99}{2} \arrow[ur] \arrow[dr] & \\ & -\frac{99}{2} \end{tikzcd}$$

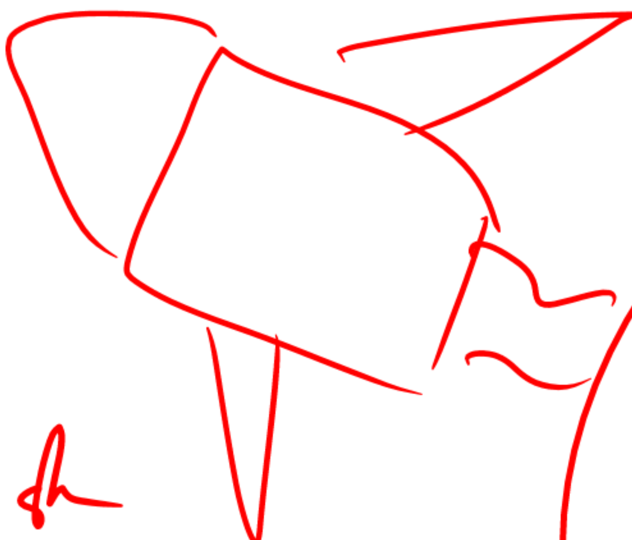
$$\begin{tikzcd}[row sep=small] & \frac{101}{2} \\ \frac{100}{2} \arrow[ur] \arrow[dr] & \\ & \frac{99}{2} \end{tikzcd}$$

$$\begin{tikzcd}[row sep=small] & 1 \\ \boxed{0.5} \arrow[ur] \arrow[dr] & \\ & 0 \end{tikzcd}$$

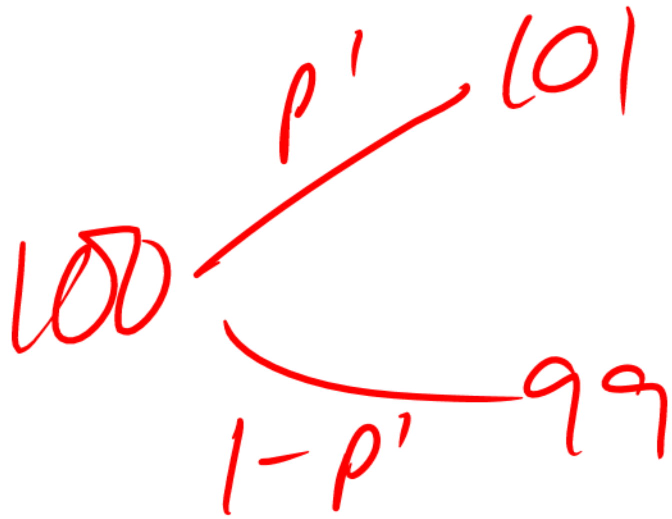
risk neutral world.

Real world

- risk averse
- Drift → return for risk



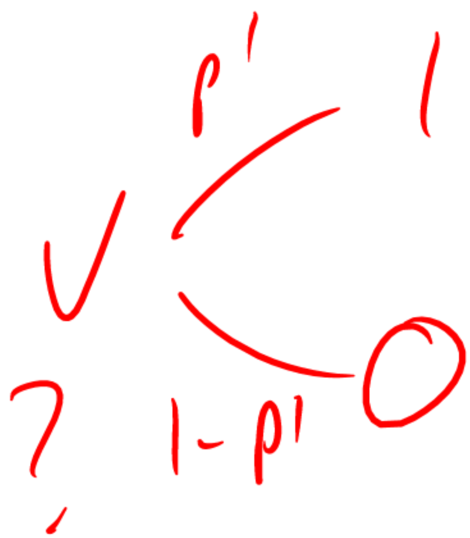
- No Drift
- No concept of risk
- Abstract prob theory



$$101 \times p' + (1-p')99$$

$$= 100$$

$$\boxed{p' = 0.5}$$



$$0.5 \times 1 + (0.5) \times 0$$

$$\boxed{V = 0.5}$$

$p' =$ risk-neutral probability

$$\begin{tikzcd}[row sep=small]
 & 103 \\
100 \arrow[ur, "p'"] \arrow[dr, "1-p'"'] & \\
 & 98
\end{tikzcd}$$

$$103p' + (1-p')98 = 100$$

$$5p' = 2$$

$$\underline{\underline{p' = 0.4}}$$

Risk neutral pricing

$$\begin{tikzcd}[row sep=small]
 & 3 \\
V \arrow[ur, "0.4"] \arrow[dr, "0.6"'] & \\
 & 0
\end{tikzcd}$$

$$3 \times 0.4 + 0 \times 0.6$$

$$\boxed{V = 1.2}$$

Delta Hedging

PDE

$\downarrow$

Black Scholes

$\downarrow$

F.D

Risk Neutral Pricing

Expectations in risk neutral world

$\downarrow$

Fundamental Asset Pricing formula.

$\downarrow$ MC

Lognormal

$$dS = \mu S\,dt + \sigma S\,dX$$

mean change in asset price

① $\mu S\,dt$

variance

② $\sigma^2 S^2\,dt$

Binomial

$$\begin{tikzcd} & uS \\ S \arrow[ur, "p"] \arrow[dr, "1-p"'] & \\ & vS \end{tikzcd}$$

$$= puS + (1-p)vS - S$$

$=$ variance

$$
\begin{tikzcd}[row sep=small]
 & uS \\
S \arrow[ur] \arrow[dr] & \\
 & vS
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}[row sep=small]
 & V^+ - \Delta uS \\
V - \Delta S \arrow[ur] \arrow[dr] & \\
 & V^- - \Delta vS
\end{tikzcd}
$$

① Hedge

$$V^+ - \Delta uS = V^- - \Delta vS$$

$$\Delta = \frac{V^+ - V^-}{(u-v)S}$$

② portfolio value at expiry

$$V^+ - \frac{(V^+ - V^-)}{(u-v)S} \cdot uS$$

$$= V^+ - \frac{u(V^+ - V^-)}{u-v} \quad *$$

③ P.V Equation

$$V - \Delta S = \frac{1}{1+r\delta t}\left[V^+ - \frac{u(V^+ - V^-)}{u-v}\right]$$

$$V = \Delta S + \frac{1}{1+r\delta t}\left[\quad\right]$$

$$\boxed{V = \frac{V^+ - V^-}{u-v} + \frac{1}{1+r\delta t}\left(\frac{uV^- - vV^+}{u-v}\right)}$$