Stochastic Calculus and Itô's lemma

Throughout this problem sheet, you may assume that X_t is a Brownian Motion (Wiener Process) and dX_t is its increment; and $X_0 = 0$. SDE is Stochastic Differential Equation.

1. Let ϕ be a random variable which follows a standardised normal distribution, i.e. $\phi \sim N(0,1)$. Calculate the expected value and variance given by $\mathbb{E}[\psi]$ and $\mathbb{V}[\psi]$, in turn, where $\psi = \sqrt{dt}\phi$. dt is a small time-step. **Note: No integration is required.**

2. Consider the following examples of SDEs for a diffusion process G. Write these in standard form, i.e.
$$dG = A(G,t)dt + B(G,t)dX_t.$$
Give the drift and diffusion for each case.

   a. $df + dX_t - dt + 2\mu t f dt + 2\sqrt{f} dX_t = 0$
   b. $\dfrac{dy}{y} = (A + By)\, dt + (Cy)\, dX_t$
   c. $dS = (\nu - \mu S)dt + \sigma dX_t + 4dS$

3. Use Itô's lemma to obtain a SDE for each of the following functions:

   a. $f(X_t) = (X_t)^n$
   b. $y(X_t) = \exp(X_t)$
   c. $g(X_t) = \ln X_t$
   d. $h(X_t) = \sin X_t + \cos X_t$
   e. $f(X_t) = a^{X_t}$, where the constant $a > 1$

4. Using the formula below for stochastic integrals, for a function $F(X_t, t)$,
$$\int_0^t \frac{\partial F}{\partial X_t} dX_t = F(X_t, t) - F(X_0, 0) - \int_0^t \left(\frac{\partial F}{\partial \tau} + \frac{1}{2}\frac{\partial^2 F}{\partial X_t^2} \right) d\tau$$
show that we can write

   a. $\displaystyle\int_0^t X_\tau^3 dX_\tau = \tfrac{1}{4} X_t^4 - \tfrac{3}{2}\int_0^t X_\tau^2 d\tau$
   b. $\displaystyle\int_0^t \tau dX_t = tX_t - \int_0^t X_\tau d\tau$
   c. $\displaystyle\int_0^t (X_t + \tau)\, dX_t = \tfrac{1}{2} X_t^2 + tX_t - \int_0^t \left(X_\tau + \tfrac{1}{2}\right) d\tau$

5. Consider a diffusion process S_t which follows Geometric Brownian Motion
$$\frac{dS_t}{S_t} = \mu dt + \sigma dX_t.$$
Use Itô's Lemma to show that the SDE dV for $V = \log(tS)$ is given by
$$dV = \left(\frac{1}{t} + \mu - \frac{1}{2}\sigma^2 \right) dt + \sigma dX_t.$$

6. Consider a function $V(t, S_t, r_t)$ where the two stochastic processes S_t and r_t evolve according to a two factor model given by

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dX_t^{(1)} \\ dr_t &= \gamma (m - r_t) \, dt + c dX_t^{(2)}, \end{aligned}$$

in turn and where

$$dX_t^{(1)} dX_t^{(2)} = \rho dt.$$

The parameters μ, σ, γ, m and c are constant. Let $V(t, S_t, r_t)$ be a function on $[0, T]$ with $V(0, S_0, r_0) = v$. Using Itô, deduce the integral form for $V(T, S_T, r_T) =$

$$v + \int_0^T \left(\frac{\partial V}{\partial t} + \mu S_t \frac{\partial V}{\partial S} + \gamma (m - r_t) \frac{\partial V}{\partial r_t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} + \frac{1}{2} c^2 \frac{\partial^2 V}{\partial r_t^2} + \rho \sigma c S_t \frac{\partial^2 V}{\partial S \partial r_t} \right) dt$$
$$+ \int_0^T \sigma S_t \frac{\partial V}{\partial S} dX_t^{(1)} + \int_0^T c \frac{\partial V}{\partial r_t} dX_t^{(2)}.$$