Part II
Calibration

Index of Notations for Part II

T_i	–	Time expressed as a dd/mm/yy
$B(0, T_i)$	–	Discount factor for the period $0 \div T_i$
$\sigma^{cap}\left(T_i, T_j\right) = \sigma_{i,j}^{cap}$	–	Volatility of cap for option maturing at T_i and length $T_i \div T_j$
N	–	Principal notional
$L\left(t, T_i, T_j\right)$	–	LIBOR forward rate at time t for period $T_i \div T_j$
$\delta_{i,j}$	–	Day count fraction for period $T_i \div T_j$
X	–	Strike price
W_i	–	Payoff at time T_i
$S_{i,j}(t) = S\left(t, T_i, T_j\right)$	–	Forward swap rate at time t for period $T_i \div T_j$
$c\left(t, T_{n-1}, T_n, \sigma_{n-1,n}^{cpl}\right)$	–	Caplet value for caplet covering period $T_{n-1} \div T_n$ with caplet volatility equal to $\sigma_{n-1,n}^{cpl}$
$\sigma^{cpl}(t, T_{n-1}, T_n) = \sigma_{n-1,n}^{cpl}$	–	Caplet volatility for a caplet covering period $T_{n-1} \div T_n$
$\sigma^{inst}(t, T_{i-1}, T_i)$	–	Instantaneous volatility of the forward rate $L(t, T_{i-1}, T_i)$
$\sigma^{inst}\left(t, T_{i-1,i}, T_{k,l}\right)$	–	Piecewise constant instantaneous volatility of forward rate $L(t, T_{i-1}, T_i)$ at time interval $T_k \div T_l$, where $k < l \leq i-1$
Σ_inst	–	Matrix of instantaneous volatilities
Σ_cpl	–	Matrix of caplet volatilities
$\sigma^{swpt}\left(t, T_i, T_j\right) = \sigma_{i,j}^{swpt}$	–	Swaption volatility for swaption with maturity at T_i and underlying swap length $T_i \div T_j$

7
Calibration Algorithms to Caps and Floors

7.1 INTRODUCTION

There is a wide range of various calibration algorithms for the LIBOR Market Model when used in practice. A lot of them are described in books and articles (see Rebonato (2002), Brigo and Mercurio (2001)), however there is still a lack of detailed algorithms presenting the step-by-step procedure of calibration clearly.

In this chapter we describe at the beginning some preliminary theory. We present the methodology of cap valuation. We demonstrate how to price caplets, how to derive ATM strikes from caps and finally we present the full algorithm for stripping caplet volatilities from cap quotes. All the theory is enriched by detailed examples taken from the real market.

All the market data are taken from a particular working day. The data contains interest rates for EUR taken from both the deposit and IRS markets and also ATM cap volatilities.

The next part of the chapter is dedicated to the application of non-parametric calibration algorithms to caps. For that purpose we use an algorithm that utilizes the derivation of caplet volatilities from cap volatilities, what is hard to find anywhere. It may seem to be easy but in our opinion the presentation of a detailed algorithm is necessary.

Taking into account the current situation of the market, we present some procedures of calibration without the time homogeneity assumption and then afterwards with the time homogeneity assumption. We compare both algorithms and results are produced. We present the calibration algorithms first with piecewise constant instantaneous volatilities depending on the time to maturity and then with a dependency on the maturity of the underlying forward rate. Both algorithms were presented in Brigo and Mercurio (2001) but in this book we present them in a more detailed way allowing them to be understood by readers of all levels. We present examples that the time homogeneity assumption does not work properly in certain market conditions.

7.2 MARKET DATA

One of the goals of the chapter is to present detailed algorithms and results of calibration. To do this we take into account one set of market data. All the market data is taken from 21 January 2005. We take into account following rates in EUR: the discount factors bootstrapped to form par interest rates (LIBOR's, FRA, IRS), at-the-money cap volatilities and swaption volatilities. Table 7.1 presents the discount factors and cap volatilities for a set of particular days.

Table 7.1 Market data from 21 January 2005: discount factors and cap volatilities

Tenor T_i	Date	Discount factor $B(0, T_i)$	Cap volatility $\sigma^{cap}(T_0, T_i)$
$t=0$	21-01-2005	1.0000000	N/A
T_0	25-01-2005	0.9997685	N/A
T_{SN}	26-01-2005	0.9997107	N/A
T_{SW}	01-02-2005	0.9993636	N/A
T_{2W}	08-02-2005	0.9989588	N/A
T_{1M}	25-02-2005	0.9979767	N/A
T_{2M}	25-03-2005	0.9963442	N/A
T_{3M}	25-04-2005	0.9945224	N/A
T_{6M}	25-07-2005	0.9890361	N/A
T_{9M}	25-10-2005	0.9832707	N/A
T_{1Y}	25-01-2006	0.9772395	0.1641
T_{2Y}	25-01-2007	0.9507588	0.2137
T_{3Y}	25-01-2008	0.9217704	0.2235
T_{4Y}	26-01-2009	0.8908955	0.2188
T_{5Y}	25-01-2010	0.8589736	0.2127
T_{6Y}	25-01-2011	0.8262486	0.2068
T_{7Y}	25-01-2012	0.7928704	0.2012
T_{8Y}	25-01-2013	0.7595743	0.1958
T_{9Y}	27-01-2014	0.7261153	0.1905
T_{10Y}	26-01-2015	0.6942849	0.1859
T_{12Y}	25-01-2017	0.6348348	0.1806
T_{15Y}	27-01-2020	0.5521957	0.1699
T_{20Y}	27-01-2025	0.4345583	0.1567

where

$B(0, T_i)$ – Discount factor for time period $0 \div T_i$

$\sigma^{cap}(T_0, T_i)$ – Market volatility of cap option starting at time T_0 and maturing at T_i.

The various discount factors are bootstrapped from interbank deposits and FRA quotations for short term below one year and IRS prices for long term above one year.

Some of the readers may not be familiar with nature of the cap mechanism. For that case it seems to be worth showing how to interpret the cap quotes.

Below we present the nature of caps and how to determine resets and payments. Let us start from cap covering one period. Usual periods are semiannual, however on the market there are also caps with quarterly periods. Payments from a cap are illustrated in Figure 7.1.

In Figure 7.1 $t=0$ means the pricing day (today). Date T_1 is the first reset date where LIBOR rate covering the period $T_1 \div T_2$ will be determined. The payment associated with this LIBOR rate occurs in date T_2 and equals W_2.

$$W_2 = N\left(L\left(T_1, T_1, T_2\right) - X\right)^+ \delta_{1,2}.$$

In general we have

$$W_n = N\left(L\left(T_{n-1}, T_{n-1}, T_n\right) - X\right)^+ \delta_{n-1,n} \tag{7.1}$$

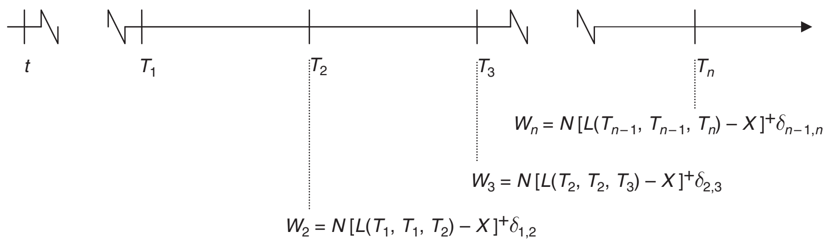

Figure 7.1 Payments from cap option covering n periods.

where: N is notional value of the cap, $L\,(T_{n-1}, T_{n-1}, T_n)$ is the LIBOR rate resetting at T_{n-1} and covering the period $T_{n-1} \div T_n$, X is the strike price of the cap, symbol $()^{+}$ denotes value from the brackets if greater than zero and zero otherwise and finally $\delta_{n-1,n}$ denotes year fraction of period $T_{n-1} \div T_n$ computed according to one of well defined day count basis, e.g. actual/360.

It is very important that we consider at this stage cap contracts functioning in the described manner. If the payment W_2 occurs at a different moment than T_2 and the payoff function remains unchanged then we will have not a plain-vanilla instrument, but an exotic one. Analogously, payments $W_3, W_4, \ldots, W_n$ are defined accordingly.

Payments constructed as described above constitute caplets. So, we can say, that a cap option is a set of caplets. Later in the text we describe how to obtain caplet volatilities from cap volatilities, which is the first and necessary step to any calibration of cap options.

Now we can move to swaptions to present the nature of these contracts. Table 7.2 presents market quotations of at-the-money swaption volatilities. By the 'at-the-money swaption volatility' we mean such volatility for which strike price is equal to forward swap rate. The definition of the forward swap rate will be presented later in the chapter.

Table 7.2 Market data from 21 January 2005: swaption volatilities

	Underlying IRS length									
Option maturity	$1Y$	$2Y$	$3Y$	$4Y$	$5Y$	$6Y$	$7Y$	$8Y$	$9Y$	$10Y$
T_{1Y}	0.2270	0.2300	0.2210	0.2090	0.1960	0.1860	0.1760	0.1690	0.1630	0.1590
T_{2Y}	0.2240	0.2150	0.2050	0.1940	0.1830	0.1740	0.1670	0.1620	0.1580	0.1540
T_{3Y}	0.2090	0.2010	0.1900	0.1800	0.1700	0.1630	0.1580	0.1550	0.1520	0.1500
T_{4Y}	0.1950	0.1870	0.1770	0.1680	0.1600	0.1550	0.1510	0.1480	0.1470	0.1450
T_{5Y}	0.1820	0.1740	0.1650	0.1580	0.1510	0.1480	0.1450	0.1430	0.1420	0.1400
T_{6Y}	0.1746	0.1674	0.1590	0.1524	0.1462	0.1436	0.1410	0.1394	0.1384	0.1368
T_{7Y}	0.1672	0.1608	0.1530	0.1468	0.1414	0.1392	0.1370	0.1358	0.1348	0.1336
T_{8Y}	0.1598	0.1542	0.1470	0.1412	0.1366	0.1348	0.1330	0.1322	0.1312	0.1304
T_{9Y}	0.1524	0.1476	0.1410	0.1356	0.1318	0.1304	0.1290	0.1286	0.1276	0.1272
T_{10Y}	0.1450	0.1410	0.1350	0.1300	0.1270	0.1260	0.1250	0.1250	0.1240	0.1240

Interpretation of ATM swaption quotes

The number 22.70 % in first column and first row means market swaption volatility with maturity equal to 1 year ($T_{1Y} = T_0$) (see column on the left) on the underlying swap starting in 1 year (T_{1Y}) and maturing in 2 years ($T_{2Y} = T_1$) from now.

We can define this volatility in the following way: $\sigma_{0,1}^{swpt} = \sigma^{swpt}(t, T_0, T_1) = 22.70\,\%$. Figure 7.2 presents a schema for payments from a swaption with maturity $T_{1Y} = T_0$ with an underlying swap period $T_0 \div T_1$.

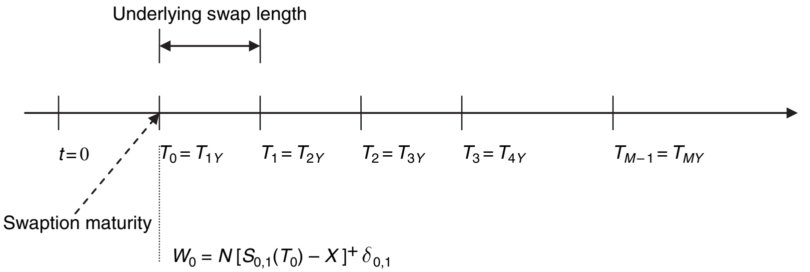

Figure 7.2 Payment from a swaption with maturity T_0 covering the period $T_0 - T_1$

Where X means strike price for the swaption and $S_{0,1}(T_0)$ is a forward swap rate that will be determined at $T_{1Y} = T_0$ and covering the period $T_0 \div T_1$.

Generally we can write $\sigma_{n,N}^{MKT}$ which means the market swaption volatility for a swaption maturing at T_n with underlying swap length $T_n \div T_N$.

We have presented in this section all necessary basic data which allows us to construct various calibration techniques. In the next section we start to present the calibration algorithms to cap options.

7.3 CALIBRATION TO CAPS

Calibration algorithms to caps are the simplest algorithms used in practice and do not require the use of optimization techniques. However, one should be careful and aware that such a calibration technique will definitely not be enough to solve the complicated pricing problems. Although the calibration is simple and almost straightforward it will be useful for later purposes to present it in a more detailed way. This is important because if one wants to obtain a good understanding of any calibration procedure for a LIBOR Market Model it is necessary to good understand how caps are quoted on the market and how to obtain caplet prices from cap quotes.

The calibration of the LIBOR Market Model requires knowing how to price caps, and more precisely, caplets in particular cap. To start with we should remember, that market prices of caplets are valued using the standard Black formula.

7.3.1 Caplet values

The LIBOR rate covering the period $T_{n-1} \div T_n$ resetting in T_{n-1} can be expressed from the perspective of today in terms of deterministic discount factors for periods $t \div T_{n-1}$ and $t \div T_n$ and year fraction $\delta_{n-1,n}$. So we have:

$$L(T_{n-1}, T_{n-1}, T_n) = F(T_0, T_{n-1}, T_n) = \left(\frac{B(T_0, T_{n-1})}{B(T_0, T_n)} - 1 \right) \frac{1}{\delta_{n-1,n}} \tag{7.2}$$

where $F(T_0, T_{n-1}, T_n)$ is an interest rate determined at T_0 covering the period $T_{n-1} \div T_n$.

Having this we can now determine the caplet price for the period $T_{n-1} \div T_n$ with payment at T_n and strike X in the following way:

$$c\left(T_0, T_{n-1}, T_n, \sigma^{cpl}_{n-1,n}\right) = B\left(T_0, T_n\right) \delta_{n-1,n}\left[F\left(T_0, T_{n-1}, T_n\right) N\left(d_1\right) - XN\left(d_2\right)\right] \tag{7.3}$$

where:

$$N\left(d_1\right) = \frac{\ln\left(\frac{F(T_0,T_{n-1},T_n)}{X}\right) + \frac{{\sigma^{cpl}_{n-1,n}}^2 \delta_{0,n-1}}{2}}{\sigma^{cpl}_{n-1,n}\sqrt{\delta_{0,n-1}}}$$

$$N\left(d_2\right) = \frac{\ln\left(\frac{F(T_0,T_{n-1},T_n)}{X}\right) - \frac{{\sigma^{cpl}_{n-1,n}}^2 \delta_{0,n-1}}{2}}{\sigma^{cpl}_{n-1,n}\sqrt{\delta_{0,n-1}}}$$

and $N()$ denotes standard normal distribution function and $\sigma^{cpl}_{n-1,n}$ the market volatility of caplet covering the period $T_{n-1} \div T_n$. This is a good moment to present an example of caplet pricing.

Example 7.1 Caplet value

We compute the caplet value taking real market data from 21 January 2005. The characteristics of the caplet is presented in Table 7.3:

Table 7.3 Caplet characteristic from example 7.1

Parameter	Value
$t = 0$	21-01-2005
T_0	25-01-2005
T_{n-1}	25-01-2006
T_n	25-04-2006
$B\left(t, T_{n-1}\right)$	0.9774658
$B\left(t, T_n\right)$	0.9712884
X	2.361 %
$\sigma^{cpl}_{n-1,n}$	20.15 %

Taking the data from Table 7.3 we have caplet value for unit value of currency EUR:

$$c\left(T_0, T_{n-1}, T_n, \sigma^{cpl}_{n-1,n}\right) = 0.000733039$$

End of example 7.1

The next step is to determine ATM strikes for cap options.

7.3.2 ATM strikes for caps

Let us define the forward swap rate, the rate of fixed leg of IRS which makes the contract fair in the context of present time. For computational reasons our IRS contract has length $T_s \div T_N$. The present value of the floating leg is given by:

$$PV\,(\text{Floating Leg}) = \sum_{i=s+1}^{N} B\left(T_0, T_i\right) L\left(T_{i-1}, T_{i-1}, T_i\right) \delta_{i-1,i}.$$

Analogously, the present value of the fixed leg is given by:

$$PV\,(\text{Fixed Leg}) = \sum_{i=s+1}^{N} B\,(T_0, T_i)\, S\,(T_0, T_s, T_N)\,\delta_{i-1,i}.$$

Assuming that the frequency of the floating payment is the same as the frequency of the fixed payments we can write:

$$PV\,(\text{Floating Leg}) = PV\,(\text{Fixed Leg}) \Leftrightarrow$$
$$\sum_{i=s+1}^{N} B\,(T_0, T_i)\, L\,(T_{i-1}, T_{i-1}, T_i)\,\delta_{i-1,i} = \sum_{i=s+1}^{N} B\,(T_0, T_i)\, S\,(T_0, T_s, T_N)\,\delta_{i-1,i}.$$

So the forward swap rate can be written as

$$S\,(T_0, T_s, T_N) = \frac{\sum\limits_{i=s+1}^{N} B\,(T_0, T_i)\, L\,(T_{i-1}, T_{i-1}, T_i)\,\delta_{i-1,i}}{\sum\limits_{i=s+1}^{N} B\,(T_0, T_i)\,\delta_{i-1,i}}.$$

The LIBOR rate $L\,(T_{i-1}, T_{i-1}, T_i)$ in above equation can be changed to the forward LIBOR rate.

$$S\,(T_0, T_s, T_N) = \frac{\sum\limits_{i=s+1}^{N} B\,(T_0, T_i)\left(\frac{B(T_0,T_{i-1})}{B(T_0,T_i)} - 1\right)\frac{1}{\delta_{i-1,i}}\delta_{i-1,i}}{\sum\limits_{i=s+1}^{N} B\,(T_0, T_i)\,\delta_{i-1,i}} = \frac{B\,(T_0, T_s) - B\,(T_0, T_N)}{\sum\limits_{i=s+1}^{N} B\,(T_0, T_i)\,\delta_{i-1,i}}. \quad (7.4)$$

For practical reasons it is important that $\delta_{i-1,i}$ denotes the year fraction of the fixed leg of given IRS. But in our particular case of calibration presented later we will use the same frequency for both the fixed and floating legs.

The forward swap rate derived above will be used to constitute the definition of an at-the-money (ATM) cap. We are saying that a particular cap is ATM if the strike price is equal to the forward swap rate. More precisely, let us consider a cap covering the period $T_s \div T_n$. Payments from that cap can be written:

$$\sum_{i=s+1}^{N} W_i = \sum_{i=s+1}^{N} N\,(L\,(T_{s-1}, T_{s-1}, T_s) - X)^{+}\,\delta_{s-1,s}$$

If strike price X in above equation is equal to forward swap rate $S\,(t, T_s, T_N)$ the cap is said to be ATM. Let us move now to present an example presenting computations of ATM strikes for cap options.

Example 7.2 ATM strikes for caps

Let us compute ATM strikes for a series of caps maturing from one year up to 20 years. Having computed the discount factors (given in Table 7.1) we can determine the

ATM cap strikes taking into consideration that the ATM cap options may be constructed in two ways:

1. Cap starts at date T_0, first reset rate is on T_{3M}, first payment is on T_{6M} based on a 3-month LIBOR resetting on T_{3M} and covering the period $T_{3M} \div T_{6M}$. All the other caplet periods are based on similar three-monthly spaced intervals.
2. Cap starts at date T_0, first reset rate is on T_{3M}, first payment is on T_{6M} based on a 3-month LIBOR resetting on T_{3M} and covering the period $T_{3M} \div T_{6M}$. The caplet periods up to and including 1 year are based on similar three-monthly spaced intervals. Above one year there is change of caplet interval from three-months to six-months. So the reset moment T_{1Y} determines a 6-month LIBOR rate covering the period $T_{1Y} \div T_{18M}$ which makes the caplet payment at T_{18M}. All the other caplet periods are based on similar six-monthly spaced intervals.

In this example we will use the first case.

Table 7.4 shows the computations for ATM strikes for caps from 1 year up to 20 years.

Table 7.4 ATM strikes for caps, preliminary computations

Time T_i	Date	Year fraction ACT/360 $\delta_{i-1,i}$	Discount factor (DF) $B(T_0, T_i)$	Year fraction* DF $\delta_{i-1,i}B(T_0, T_i)$	Cumulative sum $\sum_{j=6M}^{i} \delta_{j-1,j}B(T_0, T_j)$	Difference of DF $B(T_0, T_{3M}) - B(T_0, T_i)$	Forward swap rate (ATM cap strike) $S(T_0, T_{3M}, T_i)$
3M	25-01-2005	0.25000	0.9947527				
6M	25-07-2005	0.25278	0.9892651	0.2500642	0.2500642	0.0054876	2.19 %
9M	25-10-2005	0.25556	0.9834984	0.2513385	0.5014027	0.0112543	2.24 %
1Y	25-01-2006	0.25556	0.9774658	0.2497968	0.7511995	0.0172869	2.30 %
1Y 3M	25-04-2006	0.25000	0.9712884	0.2428221	0.9940216	0.0234643	2.36 %
1Y 6M	25-07-2006	0.25278	0.9648035	0.2438809	1.2379025	0.0299492	2.42 %
1Y 9M	25-10-2006	0.25556	0.9580084	0.2448244	1.4827269	0.0367443	2.48 %
2Y	25-01-2007	0.25556	0.9509789	0.2430279	1.7257548	0.0437737	2.54 %
2Y 3M	25-04-2007	0.25000	0.9440868	0.2360217	1.9617765	0.0506659	2.58 %
2Y 6M	25-07-2007	0.25278	0.9369436	0.2368385	2.1986150	0.0578091	2.63 %
2Y 9M	25-10-2007	0.25556	0.9295484	0.2375513	2.4361663	0.0652043	2.68 %
3Y	25-01-2008	0.25556	0.9219838	0.2356181	2.6717844	0.0727689	2.72 %
3Y 3M	25-04-2008	0.25278	0.9145031	0.2311661	2.9029504	0.0802496	2.76 %
3Y 6M	25-07-2008	0.25278	0.9068886	0.2292413	3.1321917	0.0878640	2.81 %
3Y 9M	27-10-2008	0.26111	0.8990590	0.2347543	3.3669460	0.0956937	2.84 %
4Y	26-01-2009	0.25278	0.8911017	0.2252507	3.5921967	0.1036510	2.89 %
4Y 3M	27-04-2009	0.25278	0.8833709	0.2232965	3.8154933	0.1113818	2.92 %
4Y 6M	27-07-2009	0.25278	0.8754579	0.2212963	4.0367896	0.1192947	2.96 %
4Y 9M	26-10-2009	0.25278	0.8673616	0.2192497	4.2560393	0.1273911	2.99 %
5Y	25-01-2010	0.25278	0.8591725	0.2171797	4.4732190	0.1355802	3.03 %
5Y 3M	26-04-2010	0.25278	0.8512070	0.2151662	4.6883852	0.1435457	3.06 %
5Y 6M	26-07-2010	0.25278	0.8430804	0.2131120	4.9014972	0.1516723	3.09 %
5Y 9M	25-10-2010	0.25278	0.8347939	0.2110173	5.1125146	0.1599588	3.13 %
6Y	25-01-2011	0.25556	0.8264399	0.2112013	5.3237159	0.1683127	3.16 %
7Y	25-01-2012	0.25556	0.7930540	0.2026694	6.1405422	0.2016987	3.28 %
8Y	25-01-2013	0.25556	0.7597502	0.1941584	6.9256882	0.2350025	3.39 %
9Y	27-01-2014	0.26111	0.7262834	0.1896407	7.6788386	0.2684693	3.50 %
10Y	26-01-2015	0.25278	0.6944457	0.1755404	8.3930044	0.3003070	3.58 %
11Y	25-01-2016	0.25278	0.6645450	0.1679822	9.0763113	0.3302076	3.64 %
12Y	25-01-2017	0.25556	0.6349818	0.1622731	9.7331116	0.3597709	3.70 %

Table 7.4 Continued

Time T_i	Date	Year fraction ACT/360 $\delta_{i-1,i}$	Discount factor (DF) $B(T_0, T_i)$	Year fraction* DF $\delta_{i-1,i}B(T_0, T_i)$	Cumulative sum $\sum_{j=6M}^{i} \delta_{j-1,j}B(T_0, T_j)$	Difference of DF $B(T_0, T_{3M}) - B(T_0, T_i)$	Forward swap rate (ATM cap strike) $S(T_0, T_{3M}, T_i)$
13Y	25-01-2018	0.25556	0.6068399	0.1550813	10.3590352	0.3879128	3.74 %
14Y	25-01-2019	0.25556	0.5792752	0.1480370	10.9567889	0.4154775	3.79 %
15Y	27-01-2020	0.26111	0.5523236	0.1442178	11.5300493	0.4424291	3.84 %
16Y	25-01-2021	0.25278	0.5273147	0.1332934	12.0726815	0.4674379	3.87 %
17Y	25-01-2022	0.25556	0.5030900	0.1285675	12.5919467	0.4916627	3.90 %
18Y	25-01-2023	0.25556	0.4795796	0.1225592	13.0870691	0.5151731	3.94 %
19Y	25-01-2024	0.25556	0.4567881	0.1167347	13.5588099	0.5379646	3.97 %
20Y	27-01-2025	0.26111	0.4346590	0.1134943	14.0115070	0.5600937	4.00 %

End of example 7.2

Having computed ATM strikes for cap options we can move to caplet bootstrapping.

7.3.3 Stripping caplet volatilities from cap quotes

The market volatility of caplets will be derived from cap volatilities quotations; to do that we need to introduce a stripping algorithm.

Let us start from a cap maturing in one year. Remembering that we have quarterly resets, so the effective date of the cap is $T_s = T_{3M}$, and payments are made at times: T_{6M}, T_{9M}, T_{1Y}. The volatility (precisely forward volatility) $\sigma^{cap}(t, T_{1Y})$ for a one year cap equals 16.41 %. The strike price $S(t, T_{3M}, T_{1Y})$ for this cap equals 2.301 %. However we need to make some assumptions if we want to compute caplet volatilities for the periods shorter than one year. To obtain this we generate two additional caps covering the periods: $T_{3M} \div T_{6M}$ and $T_{3M} \div T_{9M}$. The strike prices (ATM) for these caps equals the appropriate forward swap rates $S(t, T_{3M}, T_{6M})$ with value 2.194 % and $S(t, T_{3M}, T_{9M})$ with value 2.245 %. These strike rates can be obtained directly from yield curve. However, we have no volatilities for periods shorter than one year. To obtain these values, we use constant extrapolation, so we assume that: $\sigma^{cap}(t, T_{6M}) = \sigma^{cap}(t, T_{9M}) = \sigma^{cap}(t, T_{1Y})$. With this assumption we can compute 6-month caps using standard Black formula:

$$cap(t, T_{6M}) = B(t, T_{6M})\, \delta_{3M,6M} \lfloor F(t, T_{3M}, T_{6M})\, N\left(d_{1,6M}\right) - S(t, T_{3M}, T_{6M})\, N\left(d_{2,6M}\right) \rfloor$$

where:

$$N\left(d_{1,6M}\right) = \frac{\ln\left(\frac{F(t,T_{3M},T_{6M})}{X}\right) + \frac{\sigma^{cap}(t,T_{6M})^2 \delta_{t,3M}}{2}}{\sigma^{cap}(t, T_{6M})\sqrt{\delta_{t,3M}}}.$$

and

$$N\left(d_{2,6M}\right) = \frac{\ln\left(\frac{F(t,T_{3M},T_{6M})}{X}\right) - \frac{\sigma^{cap}(t,T_{6M})^2 \delta_{t,3M}}{2}}{\sigma^{cap}(t, T_{6M})\sqrt{\delta_{t,3M}}}.$$

Because the six month cap is built only from one caplet covering the period $T_{3M} \div T_{6M}$, the caplet volatility $\sigma^{caplet}(t, T_{3M}, T_{6M})$ for the period $T_{3M} \div T_{6M}$ is the same as the cap volatility $\sigma^{cap}(t, T_{6M})$ for the cap maturing at T_{6M} and equals 16.41 %.

Now we move to the next step, where we deal with the cap maturing at T_{9M}. We compute value of this cap again using the standard Black formula:

$$cap\,(t, T_{9M}) = B\,(t, T_{6M})\,\delta_{3M,6M}\left[F\,(t, T_{3M}, T_{6M})\,N\left(d_{1,6M}\right)\right.$$
$$\left. - S\,(t, T_{3M}, T_{9M})\,N\left(d_{2,6M}\right)\right] + B\,(t, T_{9M})\,\delta_{6M,9M}\left[F\,(t, T_{6M}, T_{9M})\,N\left(d_{1,9M}\right)\right.$$
$$\left. - S\,(t, T_{3M}, T_{9M})\,N\left(d_{2,9M}\right)\right]$$

where:

$$N\left(d_{1,9M}\right) = \frac{\ln\left(\frac{F(t,T_{6M},T_{9M})}{X}\right) + \frac{\sigma^{cap}(t,T_{6M})^2\delta_{t,6M}}{2}}{\sigma^{cap}\,(t, T_{9M})^2\sqrt{\delta_{t,6M}}}$$

and

$$N\left(d_{1,9M}\right) = \frac{\ln\left(\frac{F(t,T_{6M},T_{9M})}{X}\right) - \frac{\sigma^{cap}(t,T_{9M})^2\delta_{t,6M}}{2}}{\sigma^{cap}\,(t, T_{9M})^2\sqrt{\delta_{t,6M}}}.$$

Having the value of the cap maturing at T_{9M}, we can compute the sum of the caplet values for the periods $T_{3M} \div T_{6M}$ and $T_{6M} \div T_{9M}$ in the following way:

$$caplet\,(t, T_{3M}, T_{6M}) = B\,(t, T_{6M})\,\delta_{3M,6M}\left\lfloor F\,(t, T_{3M}, T_{6M})\,N\left(d_{1,6M}\right) - S\,(t, T_{3M}, T_{9M})\,N\left(d_{2,6M}\right)\right\rfloor$$

where

$$N\left(d_{1,6M}\right) = \frac{\ln\left(\frac{F(t,T_{3M},T_{6M})}{X}\right) + \frac{\sigma^{caplet}(t,T_{3M},T_{6M})^2\delta_{t,3M}}{2}}{\sigma^{caplet}\,(t, T_{3M}, T_{6M})\sqrt{\delta_{t,3M}}}$$

and

$$N\left(d_{2,6M}\right) = \frac{\ln\left(\frac{F(t,T_{3M},T_{6M})}{X}\right) - \frac{\sigma^{caplet}(t,T_{3M},T_{6M})^2\delta_{t,3M}}{2}}{\sigma^{caplet}\,(t, T_{3M}, T_{6M})\sqrt{\delta_{t,3M}}}.$$

In this case we input $\sigma^{caplet}\,(t, T_{3M}, T_{6M})$ as the value computed in the previous step of calibration (equaling 16.41 %). Next we compute the caplet value for the second period:

$$caplet\,(t, T_{6M}, T_{9M}) = B\,(t, T_{9M})\,\delta_{6M,9M}\left\lfloor F\,(t, T_{6M}, T_{9M})\,N\left(d_{1,9M}\right) - S\,(t, T_{3M}, T_{9M})\,N\left(d_{2,9M}\right)\right\rfloor$$

where:

$$N\left(d_{1,9M}\right) = \frac{\ln\left(\frac{F(t,T_{6M},T_{9M})}{S(t,T_{3M},T_{9M})}\right) + \frac{\sigma^{caplet}(t,T_{6M},T_{9M})^2\delta_{t,6M}}{2}}{\sigma^{caplet}\,(t, T_{6M}, T_{9M})\sqrt{\delta_{t,6M}}}$$

and

$$N\left(d_{1,9M}\right)=\frac{\ln\left(\frac{F(t,T_{6M},T_{9M})}{S(t,T_{3M},T_{9M})}\right)-\frac{\sigma^{caplet}(t,T_{6M},T_{9M})^2\delta_{t,6M}}{2}}{\sigma^{caplet}\left(t,T_{6M},T_{9M}\right)\sqrt{\delta_{t,6M}}}.$$

The final computation in this step of calculation is solving the equation with respect to $\sigma^{caplet}\left(t,T_{6M},T_{9M}\right)$

$$cap\left(t,T_{9M}\right)=caplet\left(t,T_{3M},T_{6M}\right)+caplet\left(t,T_{6M},T_{9M}\right).$$

Next we will obtain the caplet volatilities in the same way. For the broken periods greater then one year (e.g. one year and three months) we will be obliged to interpolate (usually using the linear method) the market quotes for cap volatilities.

Now we are able to write the complete algorithm for stripping caplet volatilities having market quotes for various cap volatilities which are ATM.

Algorithm 7.1 Caplet volatilities stripping

1. Determine all resets and maturity dates of all caplets. We deduce them from the market quotes of caps. Let us denote these moments (for 3-month intervals) as: $T_s = T_{3M}, T_{6M}, T_{9M}, \ldots, T_N$.
2. Generate the artificial caps according to the determined resets and maturities
   a. Compute the appropriate forward swap rates for ATM strikes of the caps for the periods: $T_s \div T_{6M}, T_s \div T_{9M}, \ldots, T_s \div T_N$
   b. Extrapolate using an interpolation method applied to observed market cap volatilities for all generated caps to obtain volatilities: $\sigma^{cap}\left(t,T_{6M}\right), \sigma^{cap}\left(t,T_{9M}\right), \ldots, \sigma^{cap}\left(t,T_N\right)$.
3. The first caplet volatility will be equal the first cap volatility, so $\sigma^{cap}\left(t,T_{6M}\right) = \sigma^{caplet}\left(t,T_{3M},T_{6M}\right)$.
4. Compute the market value for the cap whose maturity is longer by exactly one interval then previous cap, so $\sigma^{cap}\left(t,T_{6M+i}\right)=\sigma^{cap}\left(t,T_{9M}\right)$.
5. Having computed the previous caplet volatility (for last interval) we compute the implied caplet volatility for next interval solving the equation for the appropriate cap and sum of appropriate caplets, so $cap\left(t,T_{9M}\right)=\sum_{i=1}^{N} caplet\left(t,T_i,T_{i+1}\right)$.
6. Continue up to last cap reset. Increase the index in step (5).

End of algorithm 7.1

We now present an example of the stripping algorithm using our work just completed.

Example 7.3 Stripping caplet volatilities from cap quotes

Table 7.5 presents cap volatilities for periods from one year up to 20 years. Only cap volatilities for full years are taken directly from the market. Caps before 1 year are extrapolated using one year volatility as a constant. Caps for broken periods above 1 year are linearly interpolated. Strikes for ATM caps are taken from Table 7.4.

Table 7.5 Caplet volatilities stripped from cap volatilities

Tenor T_i	Market cap volatility $\sigma^{cap}(T_0, T_i)$	Caplet volatility $\sigma^{caplet}(T_0, T_{3M}, T_i)$	Time homogeneity test[1] $\sigma^{caplet}(T_0, T_{3M}, T_i)^2 \delta_{T_0,i-1}$
6M	0.1641	0.1641	0.0067
9M	0.1641	0.1641	0.0135
1Y	0.1641	0.1641	0.0204
1Y 3M	0.1765	0.2015	0.0412
1Y 6M	0.1889	0.2189	0.0606
1Y 9M	0.2013	0.2365	0.0848
2Y	0.2137	0.2550	0.1152
2Y 3M	0.2162	0.2212	0.0992
2Y 6M	0.2186	0.2255	0.1158
2Y 9M	0.2211	0.2298	0.1336
3Y	0.2235	0.2341	0.1527
3Y 3M	0.2223	0.2097	0.1338
3Y 6M	0.2212	0.2083	0.1429
3Y 9M	0.2200	0.2077	0.1530
4Y	0.2188	0.2051	0.1602
4Y 3M	0.2173	0.2007	0.1636
4Y 6M	0.2158	0.1982	0.1695
4Y 9M	0.2142	0.1959	0.1753
5Y	0.2127	0.1938	0.1810
6Y	0.2068	0.1859	0.2015
7Y	0.2012	0.1781	0.2171
8Y	0.1958	0.1700	0.2272
9Y	0.1905	0.1622	0.2335
10Y	0.1859	0.1570	0.2439
11Y	0.1833	0.1652	0.2976
12Y	0.1806	0.1602	0.3059
13Y	0.1770	0.1451	0.2723
14Y	0.1735	0.1380	0.2656
15Y	0.1699	0.1315	0.2587
16Y	0.1673	0.1353	0.2925
17Y	0.1646	0.1300	0.2872
18Y	0.1620	0.1243	0.2782
19Y	0.1593	0.1184	0.2666
20Y	0.1567	0.1131	0.2563

Note: [1] Results of these computations will be used later in the cap calibration algorithm.

End of example 7.3

We can present our computations graphically:

Figure 7.3 shows a typical pattern for cap volatilities and caplet volatilities as a function of maturity. In the case of cap volatility the maturity is the maturity of the cap, in the case of caplet volatility it is the maturity of the caplet. The cap volatilities are akin to cumulative averages of the caplet volatilities and therefore exhibit less variability. As indicated by Figure 7.3 we usually observe a hump in the volatilities. The peak of the hump is at about the 2 to 3 year point. There is no general agreement on the reason for the existence of the hump.

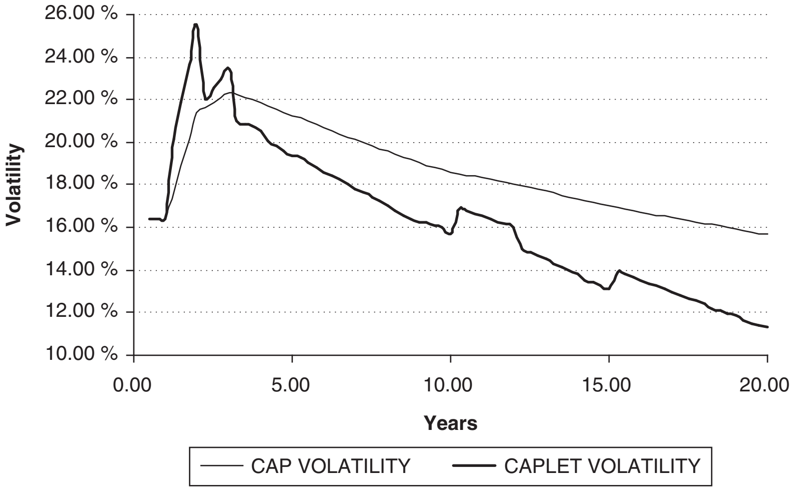

Figure 7.3 Cap and caplet volatility from 21 January 2005.

Having defined and computed caplet volatilities we can start to describe the key calibration algorithms.

7.4 NON-PARAMETRIC CALIBRATION ALGORITHMS

We have computed caplet volatility in the previous section and this is a good place to give an explanation of instantaneous volatility. The relationship between caplet volatility and instantaneous volatility of the forward rate $F(t, T_{i-1}, T_i)$ is defined as:

$$\sigma^{caplet}(t, T_{i-1}, T_i)^2 = \frac{1}{\delta_{t,T_{i-1}}} \int_t^{i-1} \sigma_i^2(t)\,dt$$

Having above equation in mind we can create many piecewise-constant instantaneous volatility structures.

Def. (Piecewise constant volatility). A volatility structure $\{\overline{\sigma}_i(.)\}_{i=1}^N$ is piecewise constant if

$$\overline{\sigma}_i(t) = (\text{const}),\ t \in (T_{i-1}, T_i)$$

Figure 7.4 below illustrates the nature of piecewise constant instantaneous volatility. We take, for example, volatility of forward rate covering the period $T_{6M} \div T_{9M}$.

The real value of forward rate $F(t, T_{6M}, T_{9M})$ will have uncertain value until time $t = T_{6M}$. Before time T_{6M} starting at T_0 we can derive the instantaneous volatility of the forward rate (e.g. from caplet volatilities). There is a practice in the market to assume, that instantaneous volatility will have constant value for a particular time period. In Figure 7.4 we assume a constant instantaneous volatility at periods $T_0 \div T_{3M}$ and $T_{3M} \div T_{6M}$.

As we will see later some of the structures can be impossible to create, because caplet volatilities may be not time-homogenous for a particular cap quotation taken from the real market.

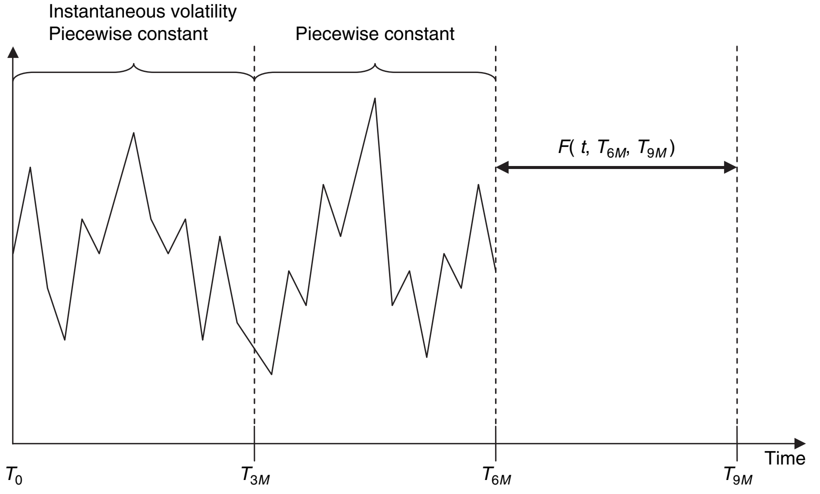

Figure 7.4 Piecewise constant instantaneous volatility.

Def. (Time homogeneity). Let us define a fixing to be one of the time points $T_1, \ldots, T_N$. Define $\vartheta: [0, T] \to \{1, \ldots, N\}$,

$$\vartheta(t) = \#\{\text{fixings in}\,(0, t)\}.$$

A volatility structure is said to be time homogeneous if it depends only on the index to maturity $i - \vartheta(t)$.

In our case, we can test time homogeneity in a simple way, by just multiplying the squared caplet volatilities by time. As we will see later the time homogeneity assumption does not hold for the market data used in our examples. We cannot assume that the instantaneous volatilities depend only on the time to maturity, because then some of piecewise-constant instantaneous volatilities might be negative.

We are ready to present now two approaches of LIBOR market model calibration to cap (precisely caplet) volatilities. Both are described in Brigo and Mercurio (2001), however we present more detailed algorithms and provide examples. The first of the algorithms is based upon the assumption that volatility depends only upon the time to maturity.

7.4.1 Piecewise constant instantaneous volatilities depending on the time to maturity

One possible way to determine instantaneous volatility is to assume that the piecewise constant instantaneous volatility depends only on the time to maturity. Figure 7.5 shows how piecewise constant instantaneous volatility depends on the time to maturity.

The interpretation of the Figure 7.5 is straightforward. The piecewise constant instantaneous volatility of the forward rate $F(t, T_{3M}, T_{6M})$ at the period $T_0 \div T_{3M}$ is the same as the piecewise constant instantaneous volatility of the forward rate $F(t, T_{6M}, T_{9M})$ at the period $T_{3M} \div T_{6M}$, and the same as the piecewise constant instantaneous volatility of the forward

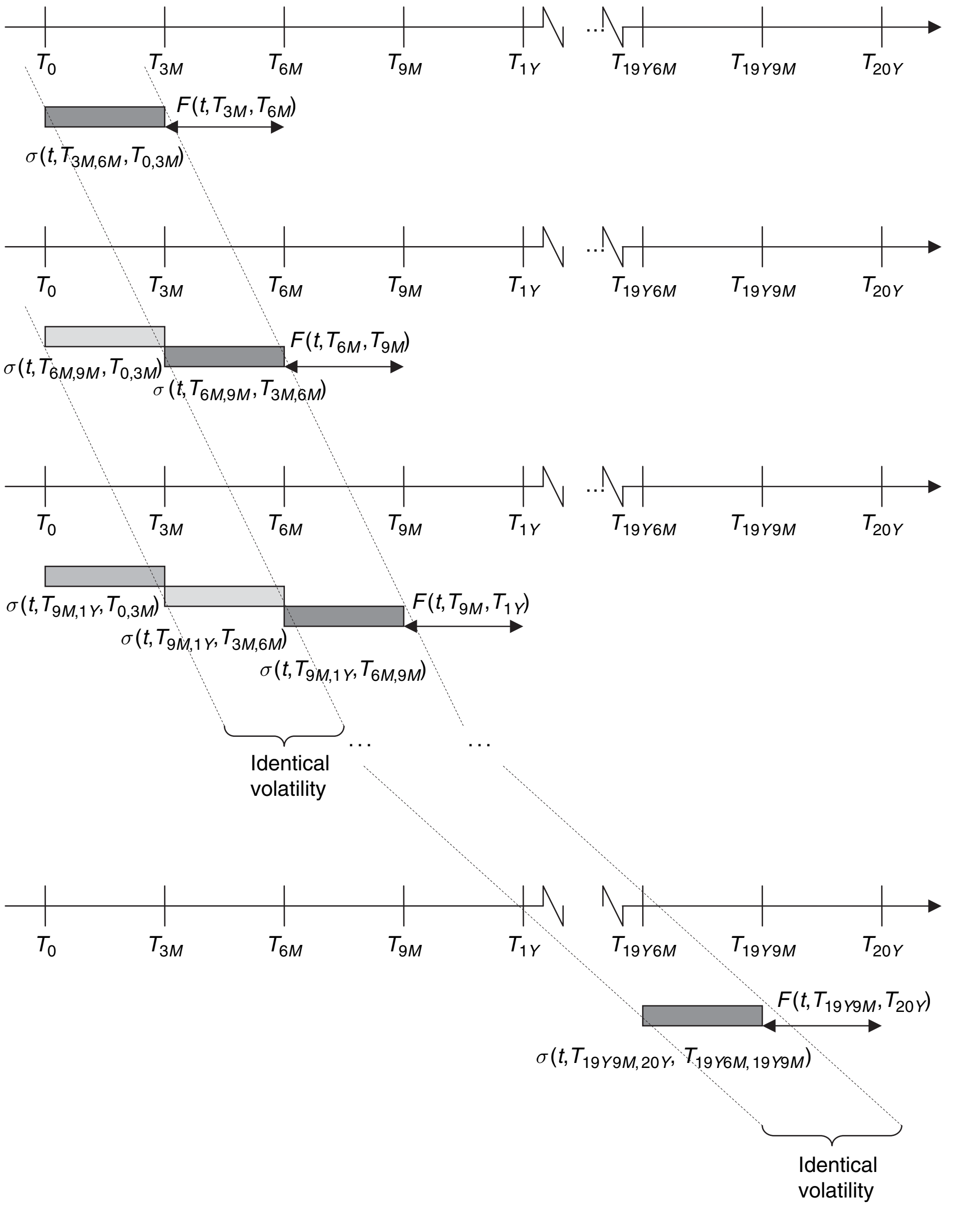

Figure 7.5 Piecewise constant instantaneous volatility dependent on time to maturity.

rate $F(t, T_{9M}, T_{1Y})$ at the period $T_{6M} \div T_{9M}$ and finally the same as the piecewise constant instantaneous volatility of the forward rate $F(t, T_{19Y9M}, T_{20Y})$ at the period $T_{19Y6M} \div T_{19Y9M}$. A similar situation exists for the other forward rates.

Our goal is to derive the instantaneous volatility matrix. So at the beginning we have to define an instantaneous volatility matrix to be computed. We present this as the lower triangular matrix below, based on a maximum caplet maturity at 20 years from now.

$$\mathbf{\Sigma\_inst} = \begin{bmatrix} \sigma^{inst}\left(t, T_{3M,6M}, T_{0,3M}\right) & --- & --- & \ldots & --- \\ \sigma^{inst}\left(t, T_{6M,9M}, T_{0,3M}\right) & \sigma^{inst}\left(t, T_{6M,9M}, T_{3M,6M}\right) & --- & \ldots & --- \\ \sigma^{inst}\left(t, T_{9M,1Y}, T_{0,3M}\right) & \sigma^{inst}\left(t, T_{9M,1Y}, T_{3M,6M}\right) & \sigma^{inst}\left(t, T_{9M,1Y}, T_{6M,9M}\right) & \ldots & --- \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \sigma^{inst}\left(t, T_{19.75Y,20Y}, T_{0,3M}\right) & \sigma^{inst}\left(t, T_{19.75Y,20Y}, T_{3M,6M}\right) & \sigma^{inst}\left(t, T_{19.75Y,20Y}, T_{6M,9M}\right) & \ldots & \sigma^{inst}\left(t, T_{19.75Y,20Y}, T_{19.75Y,20Y}\right) \end{bmatrix}$$

Elements of matrix **Σ_inst** have the following interpretations:

$\mathbf{\Sigma\_inst}(1,1) = \sigma^{inst}\left(t, T_{3M,6M}, T_{0,3M}\right)$ – instantaneous volatility related to the forward rate $F(t, T_{3M}, T_{6M})$ for time interval $T_0 \div T_{3M}$

$\mathbf{\Sigma\_inst}(2,1) = \sigma^{inst}\left(t, T_{6M,9M}, T_{0,3M}\right)$ – instantaneous volatility related to the forward rate $F(t, T_{6M}, T_{9M})$ for time interval $T_0 \div T_{3M}$

$\mathbf{\Sigma\_inst}(2,2) = \sigma^{inst}\left(t, T_{6M,9M}, T_{3M,6M}\right)$ – instantaneous volatility related to the forward rate $F(t, T_{6M}, T_{9M})$ for time interval $T_{3M} \div T_{6M}$

...

The elements follow exactly the same scheme.

We also need to define the matrix of caplet volatilities. This matrix is presented below

$$\mathbf{\Sigma\_cpl} = \begin{bmatrix} \sigma^{caplet}\left(t, T_{3M,6M}\right) \\ \sigma^{caplet}\left(t, T_{6M,9M}\right) \\ \sigma^{caplet}\left(t, T_{9M,1Y}\right) \\ \ldots \\ \sigma^{caplet}\left(t, T_{19.75Y,20Y}\right) \end{bmatrix}$$

Elements of the matrix **Σ_cpl** have the following interpretations

$\mathbf{\Sigma\_cpl}(1) = \sigma^{caplet}(t, T_{3M,6M})$ – Market caplet volatility for time interval $T_{3M} \div T_{6M}$.

Finally we define the matrix of time **T**

$$\mathbf{T} = \begin{bmatrix} 0 \\ T_{3M} \\ T_{6M} \\ \ldots \\ T_{20Y} \end{bmatrix}.$$

Having that we can then define $\delta(i, j) = [\mathbf{T}(j) - \mathbf{T}(i)]/basis$ for $j > i$.

We can move now to present the algorithm for calibration.

Algorithm 7.2 Calibration to caplets – piecewise constant instantaneous volatility depending on time to maturity

For i = 1 to N // Number of caplet volatilities

LeftSide= 0

For j = 1 to i

Sum $= \mathbf{\Sigma}\_inst\,(i, j)^2\, \delta\,(j-1, j)$

LeftSide=LeftSide+Sum

Next j

RightSide $= \mathbf{\Sigma}\_cpl\,(i)^2\, \delta\,(0, i)$

{Function: SolvingEquation → RightSide = LeftSide//for $\mathbf{\Sigma}\_inst\,(i, 1)$}

For k = 1 to i

$\mathbf{\Sigma}\_inst\,(i+1, k+1) = \mathbf{\Sigma}\_inst\,(i, k)$ // Assignment

Next k

Next i

End of algorithm 7.2

Below is example for first three iterations of algorithm 7.2:

Example 7.4 Calibration to caplets – piecewise constant instantaneous volatility depending on time to maturity

Equation for $\mathbf{\Sigma\_inst}\,(1, 1)$:

$$\mathbf{\Sigma\_cpl}\,(1)^2\, \delta\,(0, 1) = \mathbf{\Sigma\_inst}\,(1, 1)^2\, \delta\,(0, 1) \Rightarrow \mathbf{\Sigma\_inst}\,(1, 1) = sqrt\left(\frac{\delta\,(0, 1)}{\delta\,(0, 1)} \mathbf{\Sigma\_cpl}\,(1)^2\right)$$

$$= \mathbf{\Sigma\_cpl}\,(1) = 16.41\,\%$$

$$\text{Assignments}: \mathbf{\Sigma\_inst}\,(2, 2) = \mathbf{\Sigma\_inst}\,(1, 1) = 16.41\,\%$$

Equation for $\mathbf{\Sigma\_inst}\,(2, 1)$:

$$\mathbf{\Sigma\_cpl}\,(2)^2\, \delta\,(0, 2) = \mathbf{\Sigma\_inst}\,(2, 1)^2\, \delta\,(0, 1) + \mathbf{\Sigma\_inst}\,(2, 2)^2\, \delta\,(1, 2)$$

$$\Rightarrow \mathbf{\Sigma\_inst}\,(2, 1) = sqrt\left(\frac{\delta\,(0, 2)}{\delta\,(0, 1)} \mathbf{\Sigma\_cpl}\,(2)^2 - \frac{\delta\,(1, 2)}{\delta\,(0, 1)} \mathbf{\Sigma\_inst}\,(2, 2)^2\right)$$

$$= sqrt\left(\frac{0.50277778}{0.25} 16.41\,\%^2 - \frac{0.25277778}{0.25} 16.41\,\%^2\right) = 16.41\,\%$$

Assignments: $\mathbf{\Sigma\_inst}\,(3, 2) = \mathbf{\Sigma\_inst}\,(2, 1) = 16.41\,\%$, $\mathbf{\Sigma\_inst}\,(3, 3) = \mathbf{\Sigma\_inst}\,(2, 2) = 16.41\,\%$

Equation for **Σ_inst** (3, 1):

$$\mathbf{\Sigma\_cpl}\,(3)^2\,\delta\,(0,3) = \mathbf{\Sigma\_inst}\,(3,1)^2\,\delta\,(0,1) + \mathbf{\Sigma\_inst}\,(3,2)^2\,\delta\,(1,2) + \mathbf{\Sigma\_inst}\,(3,3)^2\,\delta\,(2,3)$$

$$\Rightarrow \mathbf{\Sigma\_inst}\,(3,1) = sqrt\left(\frac{\delta\,(0,3)}{\delta\,(0,1)}\mathbf{\Sigma\_cpl}\,(3)^2 - \frac{\delta\,(1,2)}{\delta\,(0,1)}\mathbf{\Sigma\_inst}\,(3,2)^2 - \frac{\delta\,(2,3)}{\delta\,(0,1)}\mathbf{\Sigma\_inst}\,(3,3)^2\right)$$

$$= sqrt\left(\frac{0.7583333}{0.25}16.41^2 - \frac{0.25277778}{0.25}16.41\,\%^2 - \frac{0.2599556}{0.25}16.41\,\%^2\right) = 16.41\,\%$$

Assignments: **Σ_inst** (4, 2) = **Σ_inst** (3, 1) = 16.41 %, **Σ_inst** (4, 3) = **Σ_inst** (3, 2) = 16.41 %, **Σ_inst** (4, 4) = **Σ_inst** (3, 3) = 16.41 %

End of example 7.4

Further computations should be done in similar way. Our results are presented in Table 7.6:

Table 7.6 Piecewise constant instantaneous volatilities depending on time to maturity

Tenor T_i	Date	Caplet volatility	Squared caplet volatility multiplied by time	Forward rate	Period 0; 3M	Period 3M; 6M	Period 6M; 9M	...	Sum of squared piecewise constant volatilities multiplied by time period
T_{6M}	25/07/2005	16.41%	0.0067322	$F_{3M,6M}(t)$	16.41 %			...	0.0067322
T_{9M}	25/10/2005	16.41 %	0.0135392	$F_{6M,9M}(t)$	16.41 %	16.41 %		...	0.0135392
T_{1Y}	25/01/2006	16.41 %	0.0204210	$F_{9M,1Y}(t)$	16.41 %	16.41 %	16.41 %	...	0.0204210
$T_{1.25Y}$	25/04/2006	20.15 %	0.0411662	$F_{1Y,1.25Y}(t)$	28.71 %	16.41 %	16.41 %	...	0.0411772
$T_{1.5Y}$	25/07/2006	21.89 %	0.0605620	$F_{1.25Y,1.5Y}(t)$	27.74 %	28.71 %	16.41 %	...	0.0605691
$T_{1.75Y}$	25/10/2006	23.65 %	0.0848306	$F_{1.5Y,1.75Y}(t)$	30.92 %	27.74 %	28.71 %	...	0.0848381
T_{2Y}	25/01/2007	25.50 %	0.1152388	$F_{1.75Y,2Y}(t)$	34.60 %	30.92 %	27.74 %	...	0.1152465
$T_{2.25Y}$	25/04/2007	22.12 %	0.0992180	$F_{2Y,2.25Y}(t)$	**0.00 %**	34.60 %	30.92 %	...	0.1155363

where:

Squared caplet volatility multiplied by time	$\sigma^{caplet}\left(T_0, T_{i,j}\right)^2 \delta_{T_0,i-1}$
Sum of squared piecewise constant volatilities multiplied by time period	$\Sigma\delta_{k,l}\sigma^{inst}\left(t, T_{i,j}T_{k,l}\right)^2$

The result of the time homogeneity assumption for instantaneous volatility is visible at tenor $T_{2.25Y}$. In such case a sum of squared piecewise constant volatilities multiplied by the time period is greater than squared caplet volatility multiplied by time, even if we put zero instantaneous volatility for forward rate $F_{2Y,2.25Y}(t)$ for time period 0,3M.

We can find an alternative way of calibrating BGM to cap options. A good choice will be piecewise constant instantaneous volatility depending on the maturity of the underlying forward rate, which we present below.

7.4.2 Piecewise constant instantaneous volatilities depending on the maturity of the underlying forward rate

Another way to determine the instantaneous volatility is assuming that the piecewise constant instantaneous volatility depends only on the maturity of underlying forward rate. Figure 7.6 presents the schema of the volatility dependent only on the maturity of the underlying forward rate.

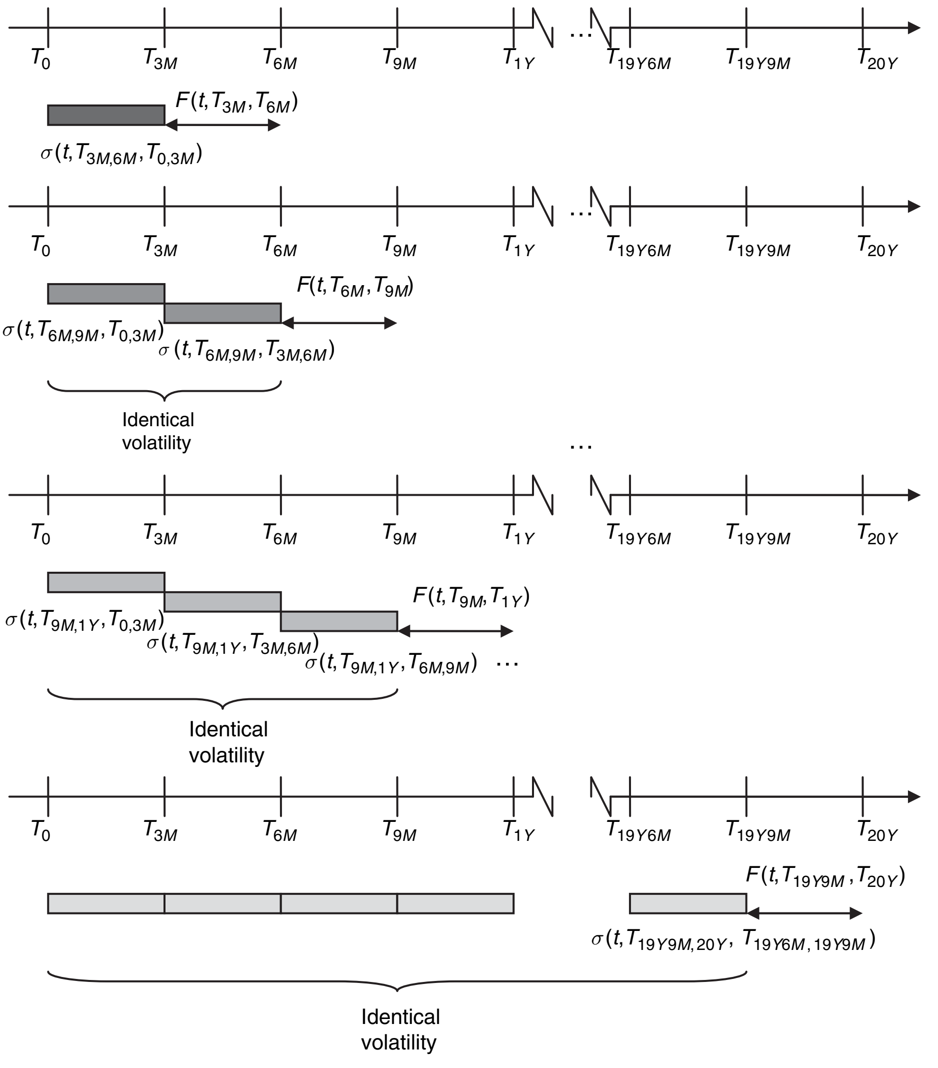

Figure 7.6 Piecewise constant instantaneous volatility dependent on the maturity of the underlying forward rate.

The piecewise constant instantaneous volatilities have identical values for all 3-month periods for a particular forward rate. For example for forward rate $F(t, T_{9Y}, T_{1Y})$ instantaneous volatility is the same in periods $T_0 \div T_{3M}, T_{3M} \div T_{6M}, T_{6M} \div T_{9M}$.

We now present our modified calibration algorithm.

Algorithm 7.3 Calibration to caplets – piecewise constant instantaneous volatility depending on the maturity of the underlying forward rate

For i = 1 to N // Number of caplet volatilities
 For j = 1 to i
  $\mathbf{\Sigma}\_inst\,(i, j) = \mathbf{\Sigma}\_cpl\,(i)$
 Next j
Next i

End of algorithm 7.3

Let us present an example for the first three iterations.

Example 7.5 Calibration to caplets – piecewise constant instantaneous volatility depending on the maturity of the underlying forward rate

Equation for $\mathbf{\Sigma\_inst}\,(1, 1)$:

$$\mathbf{\Sigma\_cpl}\,(1)^2\,\delta\,(0, 1) = \mathbf{\Sigma\_inst}\,(1, 1)^2\,\delta\,(0, 1) \Rightarrow \mathbf{\Sigma\_inst}\,(1, 1) = sqrt\left(\frac{\delta\,(0, 1)}{\delta\,(0, 1)}\mathbf{\Sigma\_cpl}\,(1)^2\right) = \mathbf{\Sigma\_cpl}\,(1) = 16.41\%$$

Equation for $\mathbf{\Sigma\_inst}\,(2, 1)$:

$$\mathbf{\Sigma\_cpl}\,(2)^2\,\delta\,(0, 2) = \mathbf{\Sigma\_inst}\,(2, 1)^2\,[\delta\,(0, 1) + \delta\,(1, 2)]$$
$$\Rightarrow \mathbf{\Sigma\_inst}\,(2, 1) = sqrt\left(\frac{\delta\,(0, 2)}{\delta\,(0, 1) + \delta\,(1, 2)}\mathbf{\Sigma\_cpl}\,(2)^2\right) = \mathbf{\Sigma\_cpl}\,(2) = 16.41\%$$

Assignments: $\mathbf{\Sigma\_inst}\,(2, 2) = \mathbf{\Sigma\_inst}\,(2, 1) = 16.41\,\%$

Equation for $\mathbf{\Sigma\_inst}\,(3, 1)$:

$$\mathbf{\Sigma\_cpl}\,(3)^2\,\delta\,(0, 3) = \mathbf{\Sigma\_inst}\,(3, 1)^2\,[\delta\,(0, 1) + \delta\,(1, 2) + \delta\,(2, 3)]$$
$$\Rightarrow \mathbf{\Sigma\_inst}\,(3, 1) = sqrt\left(\frac{\delta\,(0, 3)}{\delta\,(0, 1) + \delta\,(1, 2) + \delta\,(2, 3)}\mathbf{\Sigma\_cpl}\,(3)^2\right) = \mathbf{\Sigma\_cpl}\,(3) = 16.41\%$$

Assignments: $\mathbf{\Sigma\_inst}\,(3, 2) = \mathbf{\Sigma\_inst}\,(3, 1) = 16.41\,\%$, $\mathbf{\Sigma\_inst}\,(3, 3) = \mathbf{\Sigma\_inst}\,(3, 1) = 16.41\,\%$

End of example 7.5

Further computations should be done in similar way.

Table 7.7 presents the computed results for piecewise constant instantaneous volatilities depending on the maturity of the underlying forward rate.

Having analysed the two approaches for BGM calibration to caps it is important to notice that the time homogeneity assumption may lead to negative instantaneous volatilities but this is not the case for every business day. Banks and financial institutions may use a calibration

Table 7.7 Piecewise constant instantaneous volatilities depending on the maturity of the underlying forward rate

Tenor T_i	Date	Caplet volatility	Squared caplet volatility multiplied by time	Forward rate	Period 0; 3M	Period 3M; 6M	Period 6M; 9M	...	Sum of squared piecewise constant volatilities multiplied by time period
T_{6M}	25/07/2005	16.41%	0.0067322	$F_{3M,6M}(t)$	16.41%			...	0.0067322
T_{9M}	25/10/2005	16.41%	0.0135392	$F_{6M,9M}(t)$	16.41%	16.41%		...	0.0135392
T_{1Y}	25/01/2006	16.41%	0.0204210	$F_{9M,1Y}(t)$	16.41%	16.41%	16.41%	...	0.0204210
$T_{1.25Y}$	25/04/2006	20.15%	0.0411662	$F_{1Y,1.25Y}(t)$	20.15%	20.15%	20.15%	...	0.0411662
$T_{1.5Y}$	25/07/2006	21.89%	0.0605620	$F_{1.25Y,1.5Y}(t)$	21.89%	21.89%	21.89%	...	0.0605620
$T_{1.75Y}$	25/10/2006	23.65%	0.0848306	$F_{1.5Y,1.75Y}(t)$	23.65%	23.65%	23.65%	...	0.0848306
T_{2Y}	25/01/2007	25.50%	0.1152388	$F_{1.75Y,2Y}(t)$	25.50%	25.50%	25.50%	...	0.1152388
$T_{2.25Y}$	25/04/2007	22.12%	0.0992180	$F_{2Y,2.25Y}(t)$	22.12%	22.12%	22.12%	...	0.0992180
...	...	...	...	...	...	...	...	...	...

to caps based on the time homogeneity assumption but under the extra condition that all instantaneous volatilities will be positive for a particular day.

7.5 CONCLUSIONS

In this chapter we have presented in a detailed way all the necessary tools which help a beginner quantitative analyst to start learning calibration algorithms for the LIBOR Market Model.

At the beginning of the chapter we have gathered market data from a particular working day, using this data in all examples in the chapter. Such an approach seems to be very useful especially in the case where someone wants to compare results generated by different algorithms. This allows one to determine which calibration algorithm is better than another. However, one should be very careful in interpreting the conclusions because what is true for one particular day may not be true for another. Nevertheless the results may be a good starting point for further research.

After presenting the market data we have showed the nature of the cap mechanism. This is due to the fact that some readers may not be familiar with this and because such knowledge is fundamental for later cases. We demonstrated how to determine resets and payments in caps. Although the chapter was intended only for calibration to caps we have also presented the mechanism for swaptions. This will be used in Chapters 8 and 9 when we present calibration algorithms to swaptions and simultaneously to caps and swaptions.

Chapter 7 should be treated as introductory tool. We have presented the most popular algorithms allowing the user to calibrate LIBOR Market Models to cap options. In some market environments that approach seems to be sufficient. These algorithms may be used if we deal with those interest rate derivatives that depend mostly on behaviour of cap movements. Additionally one should assume that instantaneous correlations of forward rates will equal to one.

Both approaches presented in the chapter require caplet volatilities to be bootstrapped from cap options. Both can be used as a base for simultaneous calibration to caps and swaptions. However, as we have seen in some market circumstances the assumption of time homogeneity may lead to obtaining negative volatilities. Much safer is using approach based on assumption that volatilities depends on the maturity of the underlying forward rate presented in section 7.4.2 than on time to maturity presented in section 7.4.1.

In the next chapter we start to introduce non parametric calibration algorithms to caps and swaptions. First we introduce a separated approach. Next we move onto locally single factor approach. We also present calibration using historical correlations and extremely useful calibration to co-terminal swaptions.