

# SOLUTIONS Mod3.1 PG 1

## Black-Scholes Model - Solutions

Throughout this exercise you may use assume (where appropriate) the following results without proof

$$\begin{aligned} d_1 &= \frac{\log(S/E) + (r - D + \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}}, \\ d_2 &= \frac{\log(S/E) + (r - D - \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}} \quad \text{and} \\ N(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\phi^2/2} d\phi \end{aligned}$$

where  $S \geq 0$  is the spot price,  $t \leq T$  is the time,  $E > 0$  is the strike,  $T > 0$  the expiry date,  $r \geq 0$  the interest rate,  $D$  is the dividend yield and  $\sigma$  is the volatility of  $S$ .

1. The Black-Scholes formula for a European call option  $C(S, t)$  is given by

$$C(S, t) = S \exp(-D(T - t))N(d_1) - E \exp(-r(T - t))N(d_2).$$

- a) By differentiating with respect to  $S$  and  $\sigma$  show that the delta and vega are given by

$$\Delta = e^{(-D(T-t))}N(d_1), \quad \text{and} \quad v = \sqrt{\frac{T-t}{2\pi}} S e^{(-D(T-t))} e^{\left(-\frac{d_1^2}{2}\right)}.$$

Note:

$$\frac{\partial d_1}{\partial S} = \frac{\partial d_2}{\partial S} \quad \text{and} \quad \frac{\partial d_2}{\partial \sigma} = \frac{\partial d_1}{\partial \sigma} - \sqrt{T - t}$$

So

$$\begin{aligned} \Delta &= \frac{\partial C}{\partial S} \\ &= e^{(-D(T-t))}N(d_1) + S e^{(-D(T-t))} \frac{1}{\sqrt{2\pi}} e^{\left(-\frac{d_1^2}{2}\right)} \frac{\partial d_1}{\partial S} - E \exp(-r(T - t)) \frac{1}{\sqrt{2\pi}} e^{\left(-\frac{d_2^2}{2}\right)} \frac{\partial d_2}{\partial S} \\ &= e^{(-D(T-t))}N(d_1) + \frac{1}{\sqrt{2\pi}} \frac{\partial d_1}{\partial S} \underbrace{\left( S e^{(-D(T-t))} e^{\left(-\frac{d_1^2}{2}\right)} - E e^{(-r(T-t))} e^{\left(-\frac{d_2^2}{2}\right)} \right)}_{=0} \\ &= e^{(-D(T-t))}N(d_1) \quad \text{because the term in the bracket above is zero.} \end{aligned}$$

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$$\begin{aligned}
 v &= \frac{\partial C}{\partial \sigma} \\
 &= S e^{(-D(T-t))} \frac{1}{\sqrt{2\pi}} e^{\left(-\frac{d_1^2}{2}\right)} \frac{\partial d_1}{\partial \sigma} - E e^{(-r(T-t))} \frac{1}{\sqrt{2\pi}} e^{\left(-\frac{d_2^2}{2}\right)} \frac{\partial d_2}{\partial \sigma} \\
 &= S e^{(-D(T-t))} \frac{1}{\sqrt{2\pi}} e^{\left(-\frac{d_1^2}{2}\right)} \left( \frac{\partial d_2}{\partial \sigma} + \sqrt{T-t} \right) - \frac{1}{\sqrt{2\pi}} E e^{(-r(T-t))} e^{\left(-\frac{d_2^2}{2}\right)} \frac{\partial d_2}{\partial \sigma} \\
 &= \sqrt{\frac{T-t}{2\pi}} S e^{(-D(T-t))} e^{\left(-\frac{d_1^2}{2}\right)} + \frac{\partial d_2}{\partial \sigma} \frac{1}{\sqrt{2\pi}} \left[ \underbrace{S e^{(-D(T-t))} e^{\left(-\frac{d_1^2}{2}\right)} - E e^{(-r(T-t))} e^{\left(-\frac{d_2^2}{2}\right)}}_{=0} \right] \\
 &= \sqrt{\frac{T-t}{2\pi}} S e^{(-D(T-t))} e^{\left(-\frac{d_1^2}{2}\right)} \left( = \sqrt{\frac{T-t}{2\pi}} E e^{(-r(T-t))} e^{\left(-\frac{d_2^2}{2}\right)} \right)
 \end{aligned}$$

2. Given that  $S$  is defined by the SDE

$$dS = a(S, t) dt + b(S, t) dW \quad (2.1)$$

where  $a$  and  $b$  are given functions of  $S$  and  $t$ , show using Itô's lemma that any function  $V(S, t)$  satisfies the SDE

$$dV = \left( \frac{\partial V}{\partial t} + a \frac{\partial V}{\partial S} + \frac{1}{2} b^2 \frac{\partial^2 V}{\partial S^2} \right) dt + b \frac{\partial V}{\partial S} dW$$

where we have assumed that all partial derivatives exist.  
Hence derive the partial differential equation

$$\frac{\partial V}{\partial t} + \frac{1}{2} b^2 \frac{\partial^2 V}{\partial S^2} = r \left( V - S \frac{\partial V}{\partial S} \right) \quad (2.2)$$

for the fair price of an option based on a security  $S$  which satisfies (2.1) with  $r$  the risk-free interest rate.

Show (by substitution) that  $V(S, t) = e^{-\alpha t} S^2$  is a solution of (2.2) provided

$$b^2 = (\alpha - r) S^2$$

and  $\alpha$  is a constant.

The first part of this problem is trivial. Follow the derivation of the BSE as done in the notes. The only difference here is that  $a(S, t)$  and  $b(S, t)$  replace  $\mu S$  and  $\sigma S$ . For the second part simply substitute  $V(S, t) = e^{-\alpha t} S^2$  in (2.2); the following terms are needed

$$\begin{aligned}
 \frac{\partial V}{\partial t} &= -\alpha e^{-\alpha t} S^2; \quad \frac{\partial V}{\partial S} = 2e^{-\alpha t} S; \quad \frac{\partial^2 V}{\partial S^2} = 2e^{-\alpha t} \\
 -\alpha e^{-\alpha t} S^2 + \frac{1}{2} b^2 \times 2e^{-\alpha t} &= r(e^{-\alpha t} S^2 - 2e^{-\alpha t} S^2) \\
 -\alpha S^2 + b^2 &= -r S^2 \rightarrow b^2 = (\alpha - r) S^2.
 \end{aligned}$$

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3. The Black-Scholes formula for a European call option  $C(S, t)$  is

$$C(S, t) = S \exp(-D(T-t))N(d_1) - E \exp(-r(T-t))N(d_2)$$

From this expression, find the Black-Scholes value of the call option in the following limits:

- a. (time tends to expiry)  $t \rightarrow T^-$ ,  $\sigma > 0$  (*this depends on  $S/E$* );  
 $\exp(-r(T-t))$ ,  $\exp(-D(T-t)) \rightarrow 1$

$$d_{12} \rightarrow \frac{\log(S/E)}{\sigma\sqrt{T-t}} + O(\sqrt{T-t}) \rightarrow \begin{cases} \infty & S > E \\ 0 & S = E \\ -\infty & S < E \end{cases} \quad \text{so} \quad C \rightarrow \begin{cases} S - E & S > E \\ 0 & S = E \\ 0 & S < E \end{cases}$$

- b. (volatility tends to zero)  $\sigma \rightarrow 0^+$ ,  $t < T$ ; (*this depends on  $S \exp(-D(T-t)) / E \exp(-r(T-t))$* )

$$\begin{aligned} d_{12} &\rightarrow \frac{\log(S/E) + (r-D)(T-t)}{\sigma\sqrt{T-t}} + O(\sigma) \\ &= \frac{\log(S \exp(-D(T-t)) / E \exp(-r(T-t)))}{\sigma\sqrt{T-t}} + O(\sigma) \\ &\rightarrow \begin{cases} \infty & S e^{(-D(T-t))} > E e^{(-r(T-t))} \\ 0 & S e^{(-D(T-t))} = E e^{(-r(T-t))} \\ -\infty & S e^{(-D(T-t))} < E e^{(-r(T-t))} \end{cases} \\ \text{so } C &\rightarrow \max[S e^{(-D(T-t))} - E e^{(-r(T-t))}, 0] \end{aligned}$$

4. Suppose  $S$  evolves according to the SDE

$$dS = \mu S dt + S^\alpha dW$$

where  $\mu$  and  $\alpha$  are positive constants. Given that the interest rate is zero, derive the corresponding Black-Scholes partial differential equation (PDE) for the option based upon this asset  $S$  (you are not required to solve any equation). Write this PDE in terms of the greeks.

We know that if  $S$  evolves according to the stochastic differential equation (SDE)

$$dS = a(S, t) dt + b(S, t) dW$$

and  $V = V(S, t)$  then Itô gives

$$dV = \left( \frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2} S^{2\alpha} \frac{\partial^2 V}{\partial S^2} \right) dt + S^\alpha \frac{\partial V}{\partial S} dW$$

Then set up a portfolio  $\Pi = V - \Delta S \Rightarrow$  in one time-step (we hold  $\Delta$  fixed)  
 $d\Pi = dV - \Delta dS$ .

So

$$d\Pi = \left( \frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2} S^{2\alpha} \frac{\partial^2 V}{\partial S^2} \right) dt + S^\alpha \frac{\partial V}{\partial S} dW - \Delta (\mu S dt + S^\alpha dW)$$

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therefore we take  $\Delta = \frac{\partial V}{\partial S}$  to eliminate the risk associated with the portfolio (to cancel out terms with  $dX$ ), which gives

$$d\Pi = \left( \frac{\partial V}{\partial t} + \frac{1}{2} S^{2\alpha} \frac{\partial^2 V}{\partial S^2} \right) dt$$

portfolio now riskless. No arbitrage tells us that we are guaranteed return at risk free rate, so

$$\left( \frac{\partial V}{\partial t} + \frac{1}{2} S^{2\alpha} \frac{\partial^2 V}{\partial S^2} \right) dt = r\Pi dt$$

and  $r = 0$  so

$$\frac{\partial V}{\partial t} + \frac{1}{2} S^{2\alpha} \frac{\partial^2 V}{\partial S^2} = 0.$$

Using greeks  $\frac{\partial V}{\partial t} = \Theta$  and  $\frac{\partial^2 V}{\partial S^2} = \Gamma$  allows us to write this pde as

$$\Theta + \frac{1}{2} S^{2\alpha} \Gamma = 0.$$

5. The value of an option  $V(S, t)$  satisfies the Black-Scholes equation. Write the option value in the form

$$V(S, t) = \exp(-r(T - t))q(S, t). \quad (5.1)$$

Show that the function  $q(S, t)$  satisfies the equation

$$\frac{\partial q}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 q}{\partial S^2} + (r - D)S \frac{\partial q}{\partial S} = 0.$$

This is the backward Kolmogorov equation, used for calculating the expected value of stochastic quantities.

Substitute

$$\begin{aligned} \frac{\partial V}{\partial t} &= \exp(-r(T - t)) \frac{\partial}{\partial t} q(S, t) + rV(S, t), \\ \frac{\partial V}{\partial S} &= \exp(-r(T - t)) \frac{\partial q}{\partial S} \quad \& \\ \frac{\partial^2 V}{\partial S^2} &= \exp(-r(T - t)) \frac{\partial^2 q}{\partial S^2} \end{aligned}$$

from (5.1) into the BSE, all the exponentials cancel out and the above equation is left.

Thus the value of an option can be expressed in the form

$$V(S, t) = \exp(-r(T - t)) E[\text{Payoff}(S)]$$

where  $E[x]$  means the expected value of  $x$ . This is not a real expectation, but taken under the risk-neutral random walk (so  $r$  replaces  $\mu$ ) and forms the basis of Monte Carlo methods applied to finance. More on this later.