CQF Final Project Topics:

Inputs for CVA

- Practice Lecture (by Alonso)
- Theory Lecture (by J. Gregory)

June 2016 Cohort – Workshop II

IRS $LIBOR_{6M}$, over 5Y

From: Project Brief.

Forward LIBOR, Discounting Factors (OIS) as inputs for CVA Calculation

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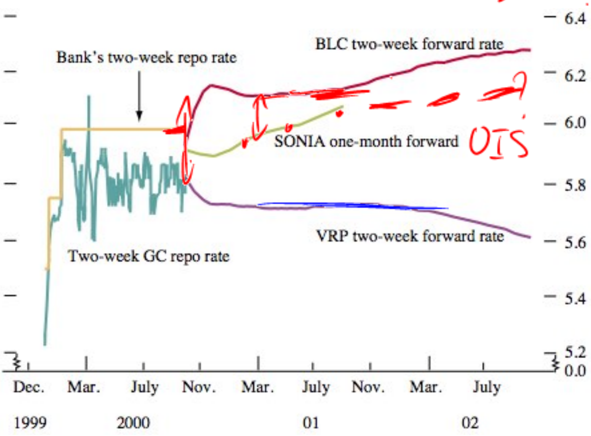
Yield Curves

To infer interest rate expectations, Bank of England builds a variety of forward curves:

- Bank Liability Curve (BLC) from LIBOR-linked instruments.
- Government Liability Curve (GLC) from Gilts, the industry equivalent is swap-based curve. ECB
- General Collateral two-week repo curve.
- SONIA and recently OIS, historic rates obtained by geometric averaging of O/N rates.

Each forward curve implies its own $Z(0; T_{i+1})$ and $L(t; T_i, T_{i+1})$.



LIBOR 2w

Fwd - (Spread)

OIS

(Fed Funds)

$r(t) \to f(t)$

After the BOE historic data (2000), BLC curve given by swaps on 6M LIBOR, carrying a credit risk $O(25bps)$ vs. SONIA – the spread.

OIS Spread over LIBOR

As we saw, evaluating the **spreads** is not the new idea. The words 'over LIBOR' are omitted which creates a lot of confusion.

3M LIBOR – 3M OIS

- LIBOR indicates money-market rates for **actual loans** between banks for up to 18M.
- OIS targets the Federal Funds Rate, an average over the period. The rate reflects uncollaterised borrowing overnight.

The spread for LIBOR fixings vs. OIS prices reveals the short-term **credit risk in the financial system**.

Dual Curve Pricing

We discount with $Z_f\,(0; T_{i+1})$ implied by **the same** forward curve as for $L(t; T_i, T_{i+1})$ to match the forward risk-neutral measure.

- Equivalent to discounting using AA-rated bonds.

However, risk-neutral valuation requires to discount at the rate earned by all economic agents, whereas their *funding cost varies*.

- OIS curve 'today' gives static $\mathrm{DF}_{\mathrm{OIS}}(0; T_{i+1})$.
- To implement the dual curve *pricing* of LIBOR-linked derivatives, use $Z_{\mathrm{f\,-\,OIS}}\,(0; T_{i+1})$ that subtracts the LIBOR-OIS spread.

For CVA we need future curves: if today the spread for $L(0;4,4.5) - OIS = 20$bps, then set $L(1;3,3.5) - OIS = 20$bps.

- Both, LIBOR and spot OIS are discrete rates and projected over time using $(1 + \tau L)$. To match credit risk between 3M LIBOR and O/N OIS, we add this spread typically constant for all tenors.

  This manual correction for mismatching expectations works better if OIS presented as Forward OIS.

  Matching of the curves is not the practical use for OIS-LIBOR spread. Implying an OIS curve is the use.

OIS Discount Factors

Observe the difference between DF taken under the forward measure and DF_{OIS} representing the risk-neutral measure $\mathbb{Q}$.

	0.00	0.5	1.0	1.5	2.0
Implied Z(0; T)	1.0000	0.9967	0.9927	0.9880	0.9824
Implied Spot		0.6663%	0.7301%	0.8043%	0.8883%
OIS Spot		0.4389%	0.4575%	0.4998%	0.5602%
OIS DF	1.0000	0.9978	0.9954	0.9925	0.9889

Forward LIBOR curve carries the credit risk, even in spot rate terms.

OIS-LIBOR Spreads

We choose constant **LIBOR - Fwd OIS** spread, an average across tenors, and calculate the Implied OIS curve for the next time.

		0.5	1.0	1.5	2.0
LIBOR-OIS (spot)	61.10	**0.2285%**	0.3381%	0.4552%	0.5833%
LIBOR - Fwd OIS	35.78	**0.2179%**	0.2798%	0.2935%	0.3259%
Fwd Inst -Fwd OIS	35.36	**0.2168%**	0.2783%	0.2912%	0.3227%
	Average, bps. Evaluate the range as well				
LIBOR-Fwd OIS	spread	35.78	bps		

$$L_{i,6\mathrm{M}} - \text{6M to OIS spread} \qquad \forall i$$

Fwd LIBOR		0.6617%	0.9422%	1.2346%	1.5090%
However, no new OIS curve available (or the new OIS data is stale)!					
Implied Fwd OIS		0.3039%	0.5843%	0.8768%	1.1512%

Forward LIBOR - a discrete rate

Market models work with **a. the discretisation** of the yield curve of spanning forward rates, which means they expire each 3M or 6M basis.

Each forward rate immediately represents **b. the market quote** for a Forward Rate Agreement (FRA), its today's strike.

LIBOR forward curve is always constructed from discrete rates and so, **c. the discrete rate maths** applies.

Forward LIBOR (cont.)

An instantaneous forward curve (actual or simulated), gives the **Forward LIBOR** discrete rates by

$$L_i = \frac{1}{T_{i+1} - T_i}\left[\frac{1}{Z(T_i, T_{i+1})} - 1\right]$$

$$\text{where} \quad Z(T_i, T_{i+1}) = \frac{Z(0, T_{i+1})}{Z(0, T_i)}$$

The same formula comes in variety of notations, for example, in Gatarek et al. (2006) on LMM in practice

$$L(T_0, T_{n-1}, T_n) = \left[\frac{B(T_0, T_{n-1})}{B(T_0, T_n)} - 1\right]\frac{1}{\delta_{n-1,n}}$$

From instantaneous forward rates

Forward-starting bond price $Z(T_i, T_{i+1})$ is a result of **the integration** over the instantaneous curve (BOE/HJM output):

$$Z(T_i, T_{i+1}) = \exp\left(-\int_{T_i}^{T_{i+1}} \bar{f}(t,\tau)d\tau\right)$$

To obtain ZCB price starting now, integrate over 0, T but that is typically done on spot curves.

A link between inst. forward rate $\bar{f}(t,\tau)$ and discrete Forward LIBOR

$$f_i = \frac{1}{T_{i+1} - T_i}\left[\exp\left(\int_{T_i}^{T_{i+1}} \bar{f}(t,\tau)d\tau\right) - 1\right]$$

or $L = m\left(e^{\bar{f}/m} - 1\right)$ where $m = \frac{1}{6M} = 2$ is comp. frequency.

Forward LIBOR (static curve)

BoE (one curve)

HJM SDE (many curves)

1. Forward Curve (6M increments)

0.5 increment Delta T, semi-annually

Tenor, T	**0.00**	**0.5**	**1.0**	**1.5**	**2.0**
Today, t=0		0.6663%	0.7940%	0.9527%	1.1402%

2. Zero-Coupon Bond (AA rated)

Calculated using integration 'under the curve' (discretised integration is summation times Delta T).

	0.00	**0.5**	**1.0**	**1.5**	**2.0**
Implied Z(0; T)	1.0000	0.9967	0.9927	0.9880	0.9824

3. Forward LIBOR **Resetting at time T**

Calculated using FRA formula.

	0.00	**0.5**	**1.0**	**1.5**	**2.0**
L(0; T, T+0.5)		0.6674%	0.7956%	0.9550%	1.1434%

L - k

Recap

Discrete Forward LIBOR is $L_i(t) = f_i$. LMM model evolves the discrete rate:

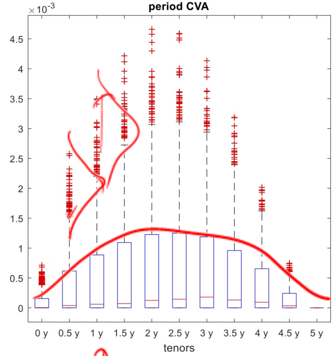
- Mercurio notation $L(t; T_i, T_{i+1}) = L_i(t)$, expiring T_{i+1}
- Gatarek notation $L(T_0; T_{n-1}, T_n)$
- Pena BGM notation $L_i(T_n)$, where i refers to discrete tenor and $T_n \in \{T_0, T_1, T_2, \dots, T_N\}$ simulated time in 3M increments

The instantaneous curve is usually taken at time $t = T_i$ and so

$$Z(T_i, T_{i+1}) = \exp\left(-\int_{T_i}^{T_{i+1}} \bar{f}(T_i, \tau)d\tau\right).$$

For CVA Calculation:

The deliverables of CVA IRS calculation are (a) MtM simulations of exposure or forward rates, (b) Expected Exposure **EE**, and (c) Potential Future Exposure **PFE** from the positive side of distribution.



$(L_{6M}(1.0) - K)$

EE D1

D2

97.5th

Here is an example of exposure analytics (CVA) that encompasses both EE and multiple PFE. From: Fernando R. Liorente, CQF Delegate

For CVA Calculation (Cont):

For a vanilla IRS you will need the input of 6M LIBOR $L(T, T + 6M)$ (these are Forward LIBOR rates).

[D1.] From **the forward curve TODAY**, it is possible to infer one statistic set of $\mapsto\ L(T, T + 6M)$ – explore “Yield Curve v3.xlsm”.

- The approach is not ideal. Long-term LIBOR rates incorporate the increasing credit risk of unsecured borrowing.
- You will not be able to vary $L(T, T + 6M)$.

The use of one and the same set of DF with multiple simulated curves is not consistent.

For CVA Calculation (Cont):

[D2.] HJM/LMM output (simulated inst./discrete forward rates) gives more flexibility in building the Exposure Profile:

Simulations deliver a sample of the same-period rates, from which the median and 97.5, 99th percentile rates can be picked for the PFE.

- MtM values are (discounted) swap cashflows form the initial curve $\mapsto [0, 5Y]$, then the curve of Forward LIBORs $\mapsto [0, 4.5]$ simulated at $t = 0.5$, then $\mapsto [0, 4]$ simulated at $t = 1$.

  HJM output is in rows: at each re-set point use the shorter curve: $t = 1$ the curve is over $[0, 4Y]$

  While Fwd LIBOR are taken off the curve, the range of curve movements generated by the HJM potentially allows using tenor column $\tau = 0.5$ only. Credit risk on the swap payment is **maximum** 6M.

Discounting Factor (to match Forward LIBOR)

- Given the continuous forward curve (HJM output), it is possible to obtain ZCB by integrating over the curve, also

$$Z_{f,inst}\,(T_i, T_{i+1}) = \frac{Z(0, T_{i+1})}{Z(0, T_i)}$$

- Given a discrete evolution of Forward LIBOR (LMM output) $L_k(T_n)$, the implied DF obtained by

$$\begin{aligned} Z_f\,(0, T_{i+1}) &= \prod_{k=n}^{i} \frac{1}{1+\tau_k L_k(T_n)} && (1) \\ &= \mathrm{DF}[0, T_1] \times \mathrm{DF}[T_1, T_2] \times \ldots \times \mathrm{DF}[T_i, T_{i+1}] && (2) \end{aligned}$$

Both approaches will result in <u>the mathematically same</u> DF Z_f (due to piecewise constant assumption over inst. fwd curve).

OIS Discounting (Implied OIS)

Using **one static set** of DF from an OIS curve today with re-simulated Forward LIBOR **curves** is inconsistent.

OIS Discounting (Implied OIS)

LIBOR $+$ **OIS**. Simulating Forward OIS in $\|$ to Forward LIBOR

- LMM for LIBOR vs. HJM for Forward OIS configuration (equivalence of the models - a question)
- HJM vs. HJM would be consistent but less useable.

LIBOR $\Rightarrow$ **Implied OIS**. OIS Discounting means **subtracting the spread** from the Forward LIBOR to obtain the parallel implied OIS curve and DFs from it.

- OIS-LIBOR spread makes the curves compatible in terms of the credit risk, but that is used in reverse to imply the OIS curve from each simulated Forward LIBOR curve.

Stochastic Spread(s) (via HJM)

Stochastic spread. OIS-LIBOR spread is customarily between discrete-time variables. Fwd-Fwd spread gives better estimation.

We use rate differences, analysed with the PCA to calibrate the HJM

$$\Delta f = f_{t+1} - f_t \qquad \text{separate for each tenor (column)}$$

$$f_{Fwd,t} - f_{FwdOIS,t} \quad \text{is also spread} \qquad \Delta f_s$$

then proceed with PCA on covariance of $\Delta f_s^i, \Delta f_s^j$ where i, j are tenors.

HJM SDE allow evolving df_s^j a stochastic spread for each tenor j, and will be sensitive to volatility input: high volatility during a credit crunch.

Recap of methods

If you have no OIS data: take a swap curve (discrete Forward LIBOR) and strip the implied OIS curve:

$$L_{i,6\mathrm{M}} - \text{6M to OIS spread} \qquad \forall i$$

Within the current market practice, implying the OIS DF by using the constant spread is another inconsistency.

Empirically the spreads tend to exhibit a 'humped' shape –

- spreads for each tenor together form **a basis curve**.

OIS today (no spread) 1 ✓	**Implied OIS** (constant spread) 2 ✓	**Stochastic Basis** 3 ✓
static DF	re-simulated curves	re-simulated curves and spreads

Multiple curve pricing

The consistent multiple curve pricing is desired that reflects tenor swaps and OIS-LIBOR basis (funding in 1M, 3M, 6M LIBOR).

Castagna et al., 2015 on links between basis and credit risk, for instance, **spot credit spread**

$$s(0,\tau) = \frac{1}{\tau}\frac{(\mathrm{LGD} + r(0,\tau)\tau)\ \mathrm{PD}(0,\tau)}{1 - \mathrm{PD}(0,\tau)}$$

where $PD(0,\tau) = 1 - \mathrm{PrSurv} = 1 - P_{CDS}(0,T)$, also notice $r(\tau)\tau = \int_0^T \bar{f}(\tau)d\tau$ by integration over an HJM-simulated curve $\bar{f}(t,\tau)$.

Forward credit spread $s(t, t+\tau)$ calculation relies on $PD(t, t+\tau)$ easily computable.

It is <u>not required</u> that you implement these results in CVA calculation as they are not yet part of the market practice.

Summary for CVA calculation

There is **no one correct way** to obtain inputs for your CVA calculation on a vanilla IRS or derivatives pricing in general.

- Whether to use OIS or not (means with collateral): depends on whether reliable spreads are available (3M, 6M to OIS spreads) or can be simulated stochastically.
- The practical model gives flexibility in Fwd LIBOR simulation, combined with Implied OIS or stochastic basis.

The alternative not detailed in these slides but worth noting is simulation of $r(t)$ by a one-factor model, while making volatility $\sigma \to \sigma(t)$ and fitting it to market data (caps, swaptions).

Yield Curve Construction

Some Essentials

OIS-LIBOR Spread - problem continues

The dual curve pricing involves adding OIS-LIBOR spread linearly and there are options such as:

- Spot OIS to discrete Forward LIBOR
- Fwd OIS to discrete Forward LIBOR
- Fwd OIS to Fwd Instantaneous rate

Fwd OIS also refers to instantaneous forward bootstrapped rate.

Seemingly, adding the spread to inst. forward curve can't be recommended as forward curve is smoothed. But Fwd Inst LIBOR - Fwd OIS spread appears to give the best curve estimation.

Assumptions in Curve Construction

Any inst. forward curve will show **zig-zag instability**, bootstrapped by continuous maths below

$$f(t,T) = -\frac{\partial}{\partial T}\ln Z(0;T) \overset{Disc}{=} -\frac{\ln Z_{i+1} - \ln Z_i}{T_{i+1} - T_i}$$

Smoothing/interpolation is required because a rough curve is unuseable. In fact, BOE applies Variable Roughness Penalty.

Hagan & West (2005) give the relationship below 'to recover' spot rates. But there are multiple $\bar{f}(t,\tau_i)$ and only one process $r(t)$

$$r(\tau)\tau = \int_0^T \bar{f}(\tau)d\tau$$

effectively the same as bond pricing by integration over the curve

$$Z(T_i, T_{i+1}) = \exp\left(-\int_{T_i}^{T_{i+1}} \bar{f}(T_i,\tau)d\tau\right)$$

Linear Interpolation

The linear interpolation is applied to either spot or forward rates: for rate at any tenor τ located as $\tau_i < \tau < \tau_{i+1}$ (not just a midpoint)

$$\bar{f}(\tau) = \frac{\tau - \tau_i}{\tau_{i+1} - \tau_i}\bar{f}(\tau_{i+1}) + \underbrace{\frac{\tau_{i+1} - \tau}{\tau_{i+1} - \tau_i}\bar{f}(\tau_i)}$$

as τ goes to τ_{i+1}, the impact of $\bar{f}_i$ reduces.

Think of the impact of adding a constant to the spot curve

- it seems as a big isssue but the fwd curve is smoothed or re-interpolated anyway,
- for Forward LIBOR, we make the piecewise linear assumption: $f_{i,Inst}$ covers the interval $[T_i, T_{i+1}]$;
- this local assumption means accumulating error – underestimating the convexity (concavity) of the curve.

Raw interpolation (over Log DF)

Raw interpolation means being linear on the log of discount factors $Z(0; T_i), Z(0; T_{i+1})$. The method has ready formulation in terms of $r(t)$

$$r(t)t = r_i\tau_i + (\tau - \tau_i)\underbrace{\frac{r_{i+1}\tau_{i+1} - r_i\tau_i}{\tau_{i+1} - \tau_i}}$$

where the underlined discrete forward rate is an average of the inst. forward rates – same done for $Z(T_i, T_{i+1})$ calculation!

$$\frac{r_{i+1}\tau_{i+1} - r_i\tau_i}{\tau_{i+1} - \tau_i} = \frac{1}{T_{i+1} - T_i}\int_{T_i}^{T_{i+1}} \bar{f}(\tau)d\tau \quad \Rightarrow \quad \frac{1}{n}\sum_{j=1}^{n}\bar{f}_j$$

Hagan & West (2005) note that raw interpolation is stable and serves to validate more advanced (monotone-preserving) interpolation methods (see Section 7, *Ibid.*).

Recap

- Since we cannot interpolate finitely enough – there is no continuity in forward curve and so differentiation is not feasible.
- Integration over curve is possible, in fact smoothing of the curve (BOE VRP) should improve integration accuracy.

The golden rules are: interpolation is done **from known inputs**. **Interpolated values can't be inputs to further interpolation**.

Please see Interpolation tab of the "Yield Curve v3" spreadsheet and explore methods suggested by Hagan & West (2005).

Submission date is Monday,

9 January 2017.

Don’t Extend Your Luck!