The Greeks - Solutions

1. Use put-call parity to find the relationships bewteen the deltas(Δ), gammas(Γ), vegas($vega$), thetas(Θ), rhos(ρ) of European call and put options.
   Solution: Starting with put-call parity

$$C\left(S,t\right)-P\left(S,t\right)=S-Ee^{-r(T-t)} \qquad (*)$$

   Differentiating (*) wrt S gives

$$\frac{\partial C}{\partial S}-\frac{\partial P}{\partial S}=1$$

   i.e.

$$\Delta_C=1+\Delta_P \qquad (1)$$

   Now differentiate (*) for a 2nd time (i.e differentiate (1) once)

$$\frac{\partial^2 C}{\partial S^2}-\frac{\partial^2 P}{\partial S^2}=0,$$

   i.e.

$$\Gamma_C=\Gamma_P.$$

   We follow a similar approach for the other greeks.
   Diff. (*) wrt σ to obtain

$$\frac{\partial C}{\partial \sigma}=\frac{\partial P}{\partial \sigma},$$

   and wrt t gives

$$\frac{\partial C}{\partial t}-\frac{\partial P}{\partial t}=-rEe^{-r(T-t)}$$

$$\Rightarrow \Theta_C=\Theta_P-rEe^{-r(T-t)}.$$

   Finally, differentiate (*) wrt r

$$\frac{\partial C}{\partial r}-\frac{\partial P}{\partial r}=E\left(T-t\right)e^{-r(T-t)},$$

   i.e.

$$\rho_C=\rho_P+E\left(T-t\right)e^{-r(T-t)}.$$

2. Show that for a delta-neutral portfolio of options on a non-dividend paying stock, Π,

$$\Theta+\frac{1}{2}\sigma^2 S^2\Gamma=r\Pi.$$

   Solution: The portfolio satisfies the Black-Scholes equation

$$\frac{\partial \Pi}{\partial t}+\frac{1}{2}\sigma^2\frac{\partial^2 \Pi}{\partial S^2}+rS\frac{\partial \Pi}{\partial S}-r\Pi=0$$

and so

$$\Theta + rs\Delta + \frac{1}{2}\sigma^2 S^2 \Gamma = r\Pi$$

If the portfolio is is delta neutral, then $\Delta = 0$ and hence

$$\Theta + \frac{1}{2}\sigma^2 S^2 \Gamma = r\Pi$$

3. Show that

$$\frac{\partial \Delta}{\partial \sigma} = \frac{\partial\ vega}{\partial S}, \quad \frac{\partial \Gamma}{\partial \sigma} = \frac{\partial^2\ vega}{\partial S^2}, \quad \frac{\partial \Theta}{\partial \sigma} = \frac{\partial\ vega}{\partial t}, \quad \frac{\partial \Delta}{\partial r} = \frac{\partial \rho}{\partial S}.$$

Solution: We use the same method as in the previous question to change the derivatives, i.e.

$$\frac{\partial \Delta}{\partial \sigma} = \frac{\partial}{\partial \sigma}\left(\frac{\partial V}{\partial S}\right) = \frac{\partial}{\partial S}\left(\frac{\partial V}{\partial \sigma}\right) = \frac{\partial\ vega}{\partial S}$$

$$\frac{\partial \Gamma}{\partial \sigma} = \frac{\partial}{\partial \sigma}\left(\frac{\partial^2 V}{\partial S^2}\right) = \frac{\partial^2\ vega}{\partial S^2}$$

$$\frac{\partial \Theta}{\partial \sigma} = \frac{\partial}{\partial \sigma}(V_t) = \frac{\partial\ vega}{\partial t}$$

$$\frac{\partial \Delta}{\partial r} = \frac{\partial}{\partial r}\left(\frac{\partial V}{\partial S}\right) = \frac{\partial \rho}{\partial S}$$

4. The Black–Scholes formula for a European call option $C(S,t)$ is given by

$$C(S,t) = S\exp(-D(T-t))N(d_1) - E\exp(-r(T-t))N(d_2).$$

Show that the Speed of this option $\left(\frac{\partial \Gamma}{\partial S}\right)$ is given by

$$\text{Speed} = \frac{\partial^3 C}{\partial S^3} = -\frac{\Gamma}{S}\left(1 + \frac{d_1}{\sigma\sqrt{T-t}}\right)$$

You do not need to prove the result for Γ.
Solution: We know that gamma for this option (standard result) is

$$\Gamma = \frac{\exp(-D(T-t))N\,'(d_1)}{\sigma S\sqrt{T-t}} \text{ where } N\,'(d_1) = \frac{1}{\sqrt{2\pi}}\exp(-{d_1}^2/2)$$

$$\begin{aligned}
\frac{\partial \Gamma}{\partial S} &= \frac{\partial}{\partial S}\left\{\frac{\exp(-D(T-t))}{\sigma S\sqrt{T-t}}\exp(-{d_1}^2/2)\right\} \text{ there is also a factor } \frac{1}{\sqrt{2\pi}} \text{ through out} \\
&= \frac{\exp(-D(T-t))}{\sigma S\sqrt{T-t}}\frac{\partial}{\partial S}\left\{\exp(-{d_1}^2/2)\right\} + \exp(-{d_1}^2/2)\frac{\partial}{\partial S}\left\{\frac{\exp(-D(T-t))}{\sigma S\sqrt{T-t}}\right\} \\
&= \frac{\exp(-D(T-t))}{\sigma S\sqrt{T-t}}\exp(-{d_1}^2/2)\left\{\frac{-d_1}{\sigma S\sqrt{T-t}}\right\} - \frac{1}{S}\left\{\frac{\exp(-D(T-t))}{\sigma S\sqrt{T-t}}\exp(-{d_1}^2/2)\right\} \\
&= \Gamma\left[\frac{-d_1}{\sigma S\sqrt{T-t}} - \frac{1}{S}\right] = -\frac{1}{S}\Gamma\left[1 + \frac{d_1}{\sigma\sqrt{T-t}}\right].
\end{aligned}$$

5. Consider a delta-neutral portfolio of derivatives, Π. For a small change in the price of the underlying asset, δS, over a short time interval, δt, show that the change in the portfolio value, $\delta\Pi$, satisfies

$$\delta\Pi = \Theta\delta t + \frac{1}{2}\Gamma\delta S^2$$

where $\Theta = \dfrac{\partial\Pi}{\partial t}$ and $\Gamma = \dfrac{\partial^2\Pi}{\partial S^2}$.

Solution: Applying Itô's lemma to the value of the portfolio, Π :

$$\delta\Pi = \frac{\partial\Pi}{\partial S}\delta S + \frac{\partial\Pi}{\partial t}\delta t + \frac{1}{2}\frac{\partial^2\Pi}{\partial S^2}\delta S^2$$

to order δt. If the portfolio is delta-neutral, then $\Delta = \dfrac{\partial\Pi}{\partial S} = 0$, and so

$$\delta\Pi = \frac{\partial\Pi}{\partial t}\delta t + \frac{1}{2}\frac{\partial^2\Pi}{\partial S^2}\delta S^2 = \Theta\delta t + \frac{1}{2}\Gamma\delta S^2.$$

6. (a) By differentiating the Black–Scholes equation with respect to σ, show that the vega of an option, $vega$, satisfies the differential equation

$$\frac{\partial vega}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 vega}{\partial S^2} + rS\frac{\partial vega}{\partial S} - rvega + \sigma S^2\Gamma = 0$$

where $\Gamma = \partial^2 V/\partial S^2$. What is the final condition?

$$\frac{\partial}{\partial\sigma}\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0\right) \rightarrow$$

$$\frac{\partial}{\partial t}\left(\frac{\partial V}{\partial\sigma}\right) + \left\{\frac{1}{2}\sigma^2 S^2\frac{\partial^2}{\partial S^2}\left(\frac{\partial V}{\partial\sigma}\right) + \sigma S^2\frac{\partial^2 V}{\partial S^2}\right\} + rS\frac{\partial}{\partial S}\frac{\partial V}{\partial\sigma} - r\frac{\partial V}{\partial\sigma} = 0$$

and we know $vega = \frac{\partial V}{\partial\sigma}$ & $\Gamma = \frac{\partial^2 V}{\partial S^2}$, hence

$$\frac{\partial vega}{\partial t} + \left\{\frac{1}{2}\sigma^2 S^2\frac{\partial^2 vega}{\partial S^2} + \sigma S^2\Gamma\right\} + rS\frac{\partial vega}{\partial S} - rvega = 0$$

at $t = T$ $V(S,T) = \max(S-E)$, we want a condition for υ so

$$vega(S,T) = \frac{\partial}{\partial\sigma}V(S,T) = \frac{\partial}{\partial\sigma}\max(S-E) = 0$$

So final condition is $vega(S,T) = 0$.

(b) Similarly, find the PDE satisfied by ρ, the sensitivity of the option value to the interest rate.

We differentiate the BSE wrt r using the same method as above, in terms of switching around the derivative terms to obtain

$$\frac{\partial\rho}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2\rho}{\partial S^2} + rS\frac{\partial\rho}{\partial S} - r\rho + S\frac{\partial V}{\partial S} - V = 0$$

7. Writing $C(S,t) = f(t)\,S$ gives

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D)\,S\frac{\partial V}{\partial S} - rV = -f(t)\,S$$

and we now use the transformation $V = \phi(t)\,S$ to convert to an ode which is a function of t alone.

$$\frac{\partial V}{\partial t} = \phi'(t)\,S; \quad \frac{\partial V}{\partial S} = \phi; \quad \frac{\partial^2 V}{\partial S^2} = 0$$

For the final condition we know

$$\begin{aligned} V(S,T) &= S \equiv \phi(T)\,S \\ &\Longrightarrow \phi(T) = 1 \end{aligned}$$

So the original problem reduces to

$$\begin{aligned} \frac{d\phi}{dt} + (r-D)\,\phi - r\phi &= -f(t) \\ &\longrightarrow \frac{d\phi}{dt} - D\phi = -f \end{aligned}$$

which is a first order linear equation i.e. integrating factor method. I.F is e^{-Dt} so the ODE becomes

$$\begin{aligned} e^{-Dt}\frac{d\phi}{dt} - D\phi e^{-Dt} &= -fe^{-Dt} \\ \frac{d}{dt}\left(e^{-Dt}\phi\right) &= -fe^{-Dt} \\ \int_t^T d\left(e^{-D\tau}\phi(\tau)\right) &= -\int_t^T f(\tau)\,e^{-D\tau}d\tau \\ \left(e^{-D\tau}\phi(\tau)\right)\Big|_t^T &= -\int_t^T f(\tau)\,e^{-D\tau}d\tau \\ e^{-DT}\phi(T) - e^{-Dt}\phi(t) &= -\int_t^T f(\tau)\,e^{-D\tau}d\tau \end{aligned}$$

and we know $\phi(T) = 1$, hence

$$\begin{aligned} e^{-DT} - e^{-Dt}\phi(t) &= -\int_t^T f(\tau)\,e^{-D\tau}d\tau \\ e^{-Dt}\phi(t) &= e^{-DT} + \int_t^T f(\tau)\,e^{-D\tau}d\tau \\ \phi(t) &= e^{-D(T-t)} + \int_t^T f(\tau)\,e^{-D(\tau-t)}d\tau \end{aligned}$$

So the option price $V(S,t) = \phi(t)\,S$ and $\Delta(S,t) = \dfrac{\partial V}{\partial S} = \phi(t) =$

$$e^{-D(T-t)} + \int_t^T f(\tau)\,e^{-D(\tau-t)}d\tau.$$