Coverage

- Interest Rates
- Time Value of Money
- Money Market Securities
- Interest Rate Swaps
- Currency Markets
- Bonds
- Commodities
- Equities
- Futures and Futures Pricing
- Options and Option Pricing

Discounting and Discount Factors

To discount future cash flows for analysis or valuation purposes we simply need to work backwards through the time value of money formula:

- $$PV = \frac{TV}{(1+r)^n}$$

This would give us the value today of a cash flow due in the future:

DF = 1/1.05^2 = 0.907029

PV = 50/1.05^2 = 45.35

PV = 50 x 0.907029 = 45.35

Point out that DFs can be strung together easily through multiplication to work out new ones. Eg.

1 year DF = 0.952380 (rates are 5%), 1fwd2 = 0.943396 (rates are 6%). 2 year DF = 0.95238 x 0.943396 = 0.89847 (rates are 5.50%)

Similarly you get easily get a FORWARD rate by dividing one spot discount factor over another Eg.

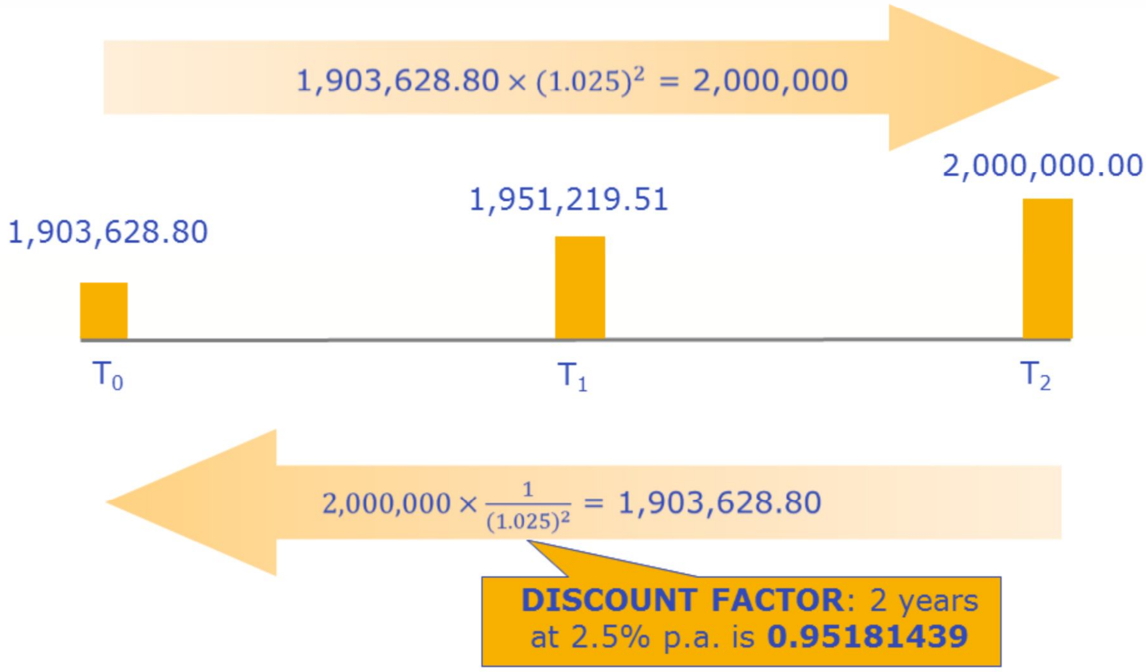
1fwd2 = 0.89847 / 0.952380 = 0.943396 (rates are 6%)

If we know what interest rates are, we can of course use this formula but it may be cumbersome since we have to work out the interest rate and time period for each flow. In practice its common to use discount factors (DFs) which are a single number we can multiply by to obtain PV. This is achieved by doing most of the “work” first in order to get the DFs. Quants and traders will have a spot curve, forward curve and a run of discount factors in front of them as all times, probably in Excel

It is often useful to quickly be able to multiply a cash flow by a number to generate PV. For this we need a discount factor (DF) which is related to the above equation as follows:

$$DF = \frac{1}{(1+r)^n}$$

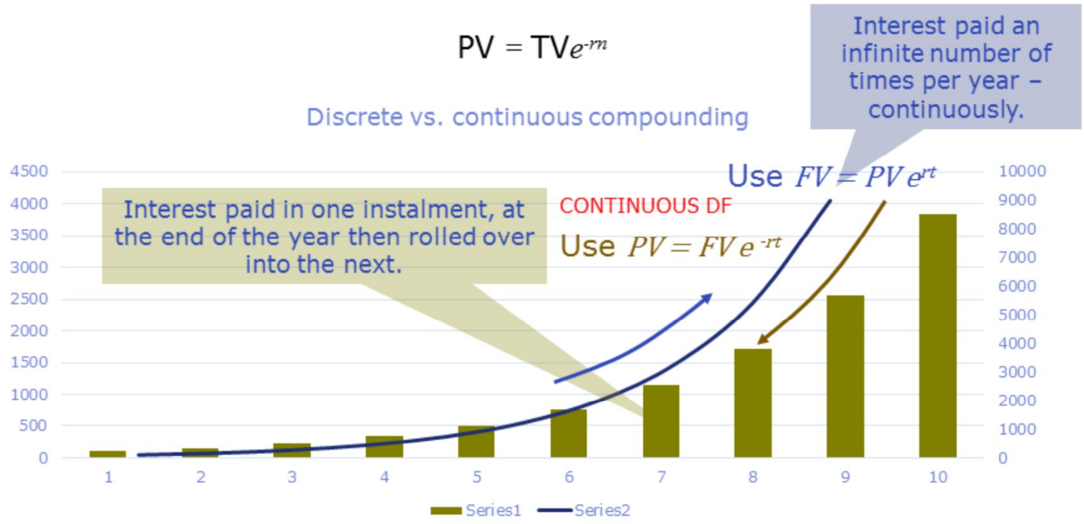
Discount factors are also powerful for quickly calculating forward rates


$1{,}903{,}628.80 \times (1.025)^2 = 2{,}000{,}000$
1,903,628.80
1,951,219.51
2,000,000.00
T_0
T_1
T_2
$2{,}000{,}000 \times \frac{1}{(1.025)^2} = 1{,}903{,}628.80$
DISCOUNT FACTOR: 2 years at 2.5% p.a. is **0.95181439**

Continuous Discounting Formula

If we want to discount using a continuous interest rate we use:

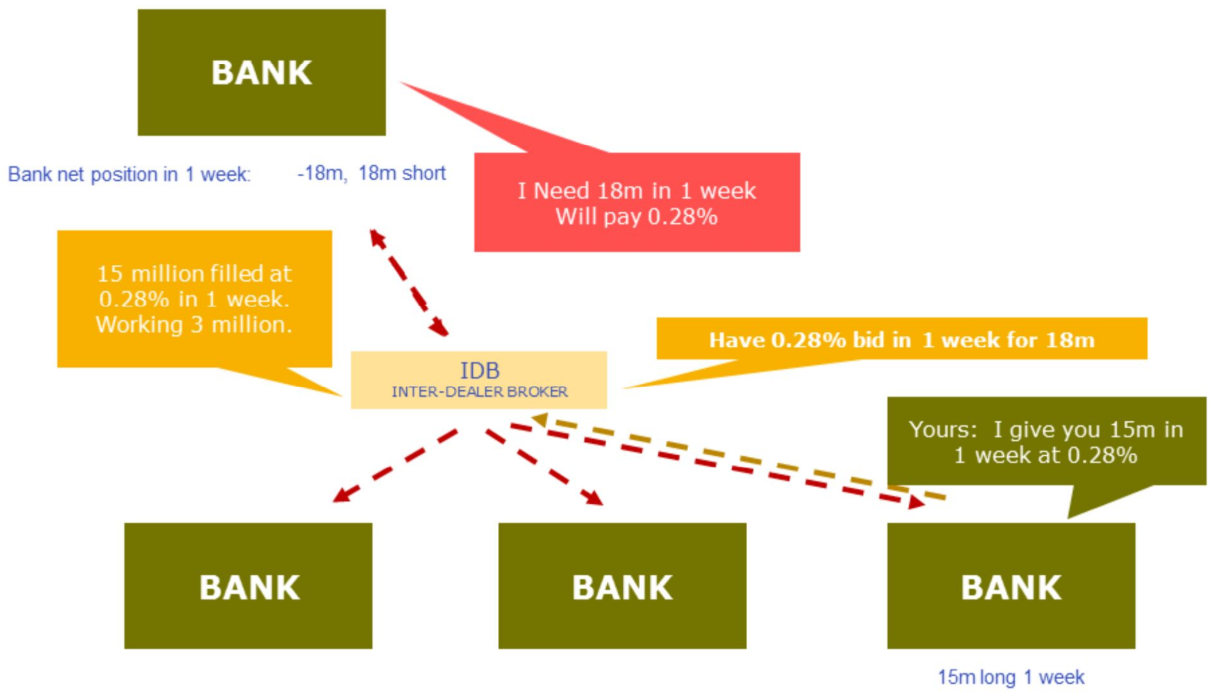
$$PV = TVe^{-rn}$$



Money markets

Borrowing and lending by corporations, funds and banks for periods of up to **12 MONTHS**. So, for the 1 week period:

The bank manages this legacy exposure in the wholesale or **interbank** money markets



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Reference rates

London interbank offered rate, Libor

Libor is set each working day in London by the Intercontinental Exchange (The ICE). It is calculated for USD, GBP, EUR, JPY and CHF for o/n, 1 week, 1, 2, 3, 6 and 12 months.

It is used as **a reference rate**, similar in some ways to closing prices in stocks.

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Calculating 1 week USD Libor

"At what rate could you borrow funds, were you to do so by asking for and then accepting interbank offers in a reasonable market size just prior to 11 am London time?"

USD Libor Panel (18 banks)	Rates (% p.a.)
Bank of America	0.1500
Barclays	~~0.1700~~
BNP Paribas	0.1620
Bank of Tokyo Mitsubishi UFJ	~~0.1900~~
Credit Agricole	0.1675
Citi	0.1700
Credit Suisse	0.1500
Deutschebank	0.1600
HSBC	0.1500
J.P.Morgan	~~0.1400~~
Lloyds	0.1500
Norinchukin Bank	~~0.2000~~
Rabobank	~~0.1200~~
Royal Bank of Canada	~~0.1400~~
RBS	0.1600
Soc Gen	~~0.1250~~
Sumitomo Mistui	~~0.2000~~
UBS	0.1450

Average of remaining 10 rates:
0.15645% p.a. SW USD Libor

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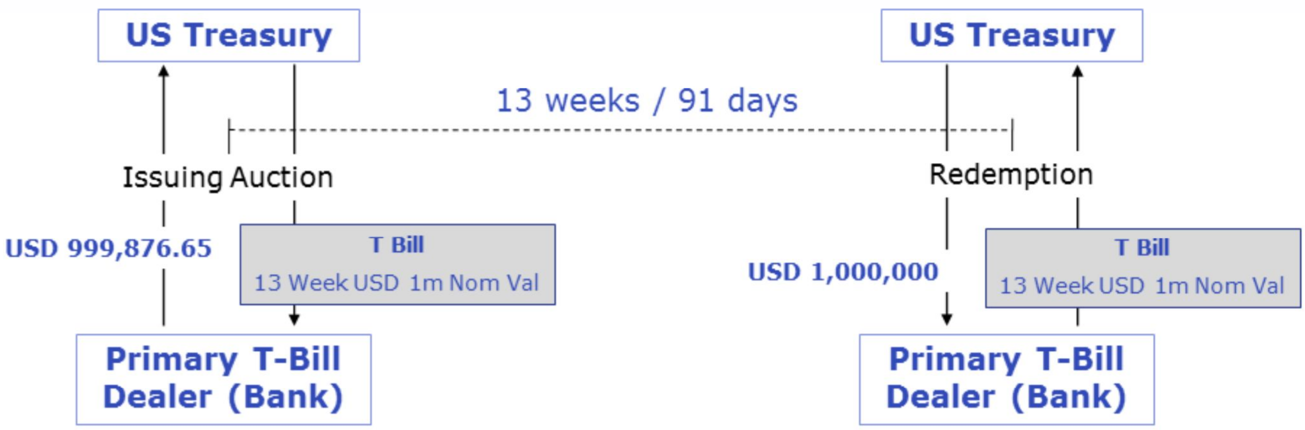
As at 9 October 2015.

Money Market Securities

Treasury Bills

Treasury Bills are issued buy governments to fund their short term cash flows.

- They are issued at a discount and redeemed at par;
- They are a proxy for a risk-free return for their currency



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Commercial Paper

US Commercial Paper (USCP) is an unsecured form of borrowing issued by major financial and non-financial corporations. Maturities average about 30 days; max 270.

Euro Commercial Paper (ECP) may have maturities up to 365 days ("Euro" refers to non-domestic currency issue)

Certificates of Deposit

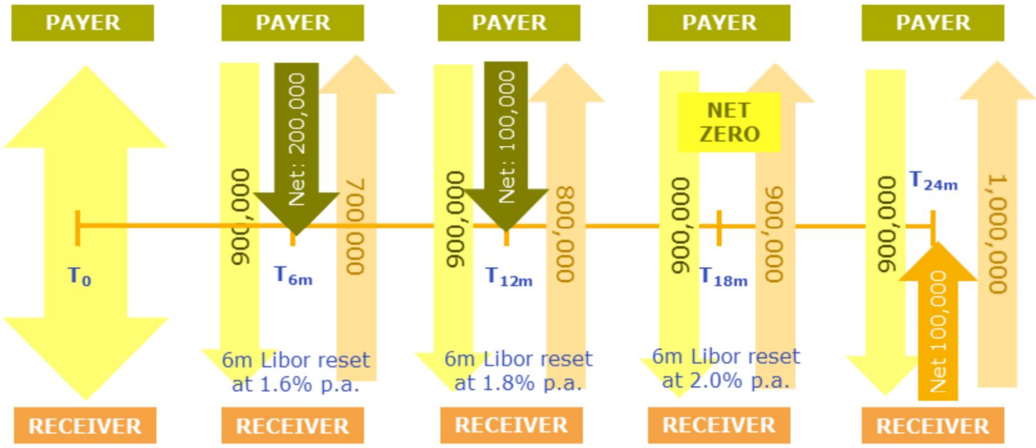
A CD is a certificate representing funds deposited at a bank for a named period at a FIXED INTEREST RATE. Note that CDs pay interest.

Euro Commercial Paper is an unsecured general obligation in the form of a promissory bearer note, issued on a discount or interest-bearing basis by large commercial and industrial organizations. Maturities of ECP range from a few days up to one year, with most being 182 days; the minimum amount is usually $500,000.

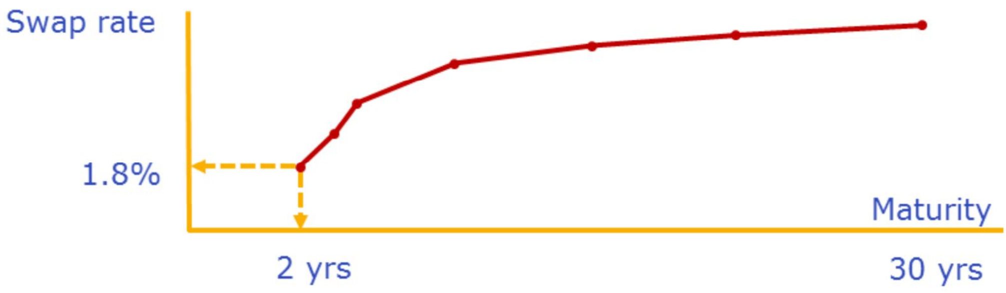
Derivatives: An interest rate swap (IRS)

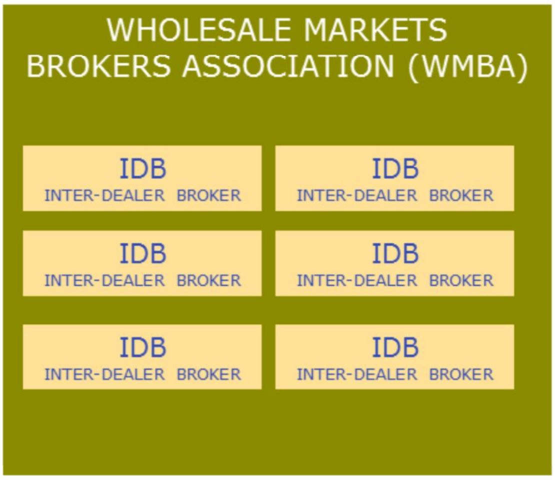
Agree swap:
- **2-year** semi-annual/6s;
- Notional 100m
- Swap rate is 1.8% p.a.

6m Libor set at 1.4% p.a.



A swap curve





Rate	Volume	
0.35	118,850,650	
0.40	140,000,000	
0.42	323,798,000	
0.43	850,000,000	**Weighted Average**
0.44	374,509,000	0.4655
0.45	2,936,852,609	
0.47	707,500,000	
0.48	1,773,200,000	
0.50	1,707,500,000	
0.53	550,000,000	
	9,482,210,259	

- **Euronia** (Euro Overnight Index Average);
- **Eonia** (Euro Overnight Index Average);
- **Fed Funds Effective Rate**
  – is a volume-weighted average of rates on trades arranged by major brokers as calculated by the Federal Reserve Bank of New York; and
- **Tonar** (Tokyo Overnight Average Rate)

The WMBA calculates an overnight indexed rate for the Euro – Euronia

There is also EONIA calculated by the European Central Bank;

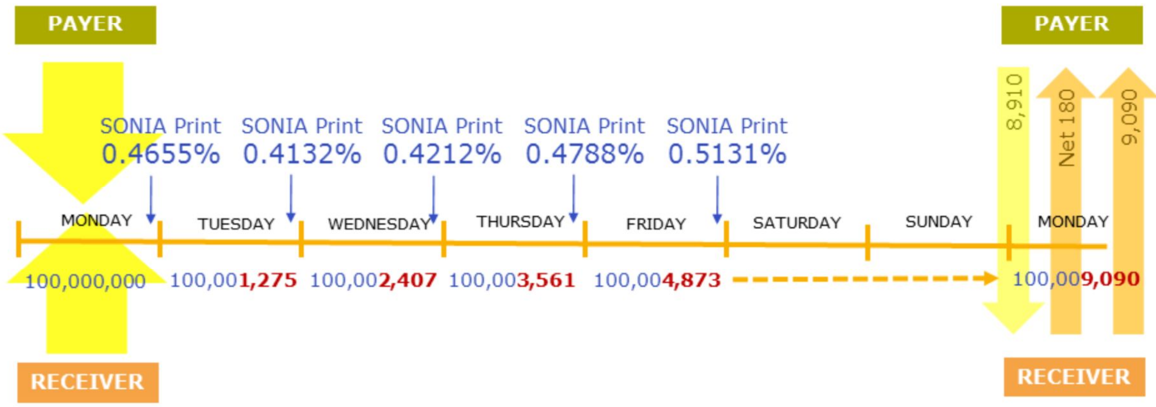
the Fed Funds Effective rate, for the US dollar is calculated by the New York fed. And

The Japanese Yen overnight index rate is TONAR – which is calculated by the Bank of Japan

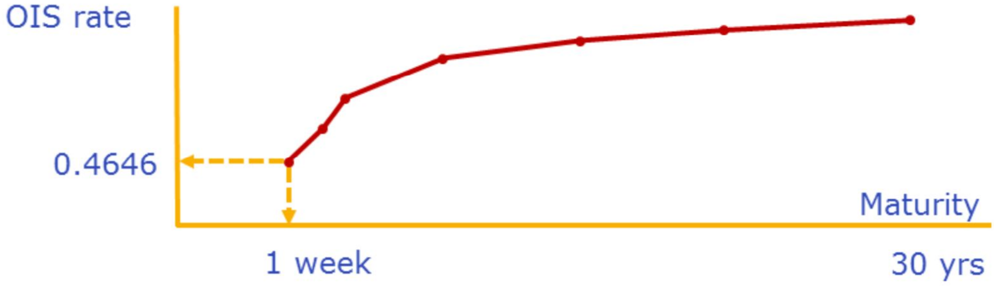
An overnight index swap (OIS)

Agree swap:

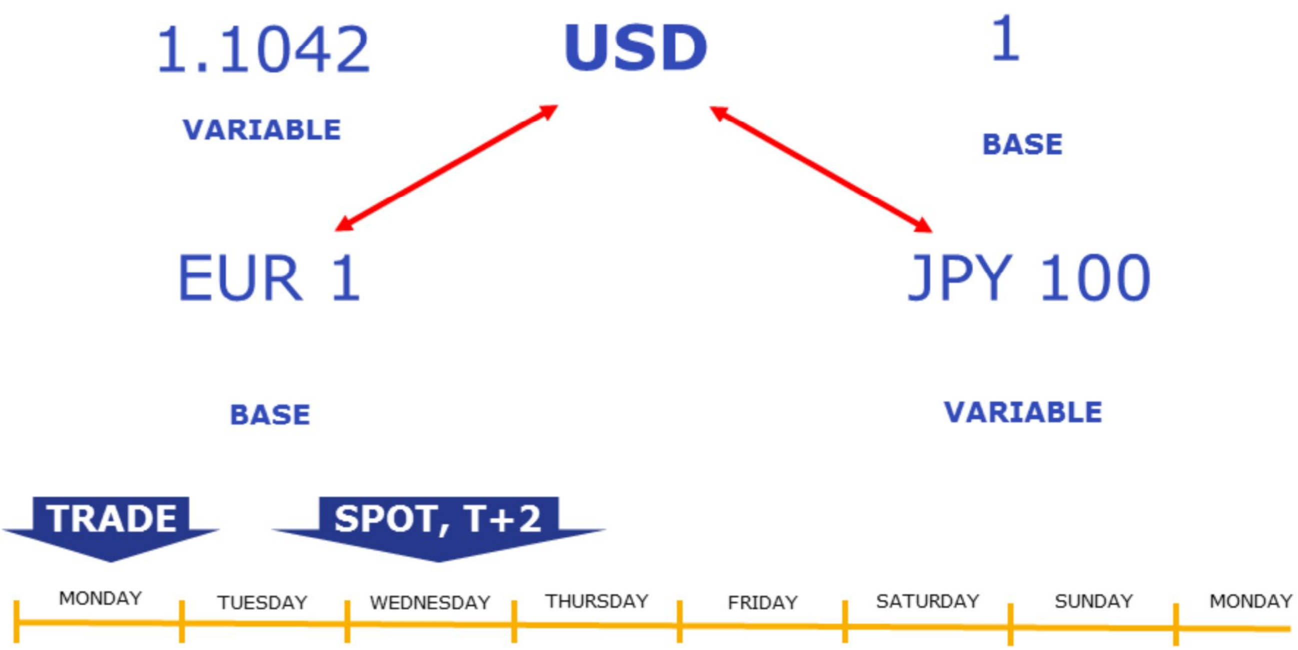
- 1 week SONIA swap;
- Notional 100m
- 1 week SONIA swap rate is 0.4646% p.a.



OIS Curve



Foreign Exchange (Currency) Markets



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Forward FX

T_0 — SPOT — SPOT +1 WEEK

1 week EUR is 0.5% p.a
1 week USD is 2.0% p.a.

M	T	W	Th	F	S	S	M	T	W

$$\text{USD } 1.0943 \xrightarrow{1.0943 \times \left(1 + 2\%\left(\frac{7}{360}\right)\right)} \text{USD } 1.0947$$

$$\text{EUR } 1.0000 \xrightarrow[1.0000 \times \left(1 + 0.5\%\left(\frac{7}{360}\right)\right)]{} \text{EUR } 1.0001$$

EUR USD 1.0946

1 week EUR USD forward points is +3

Government bonds

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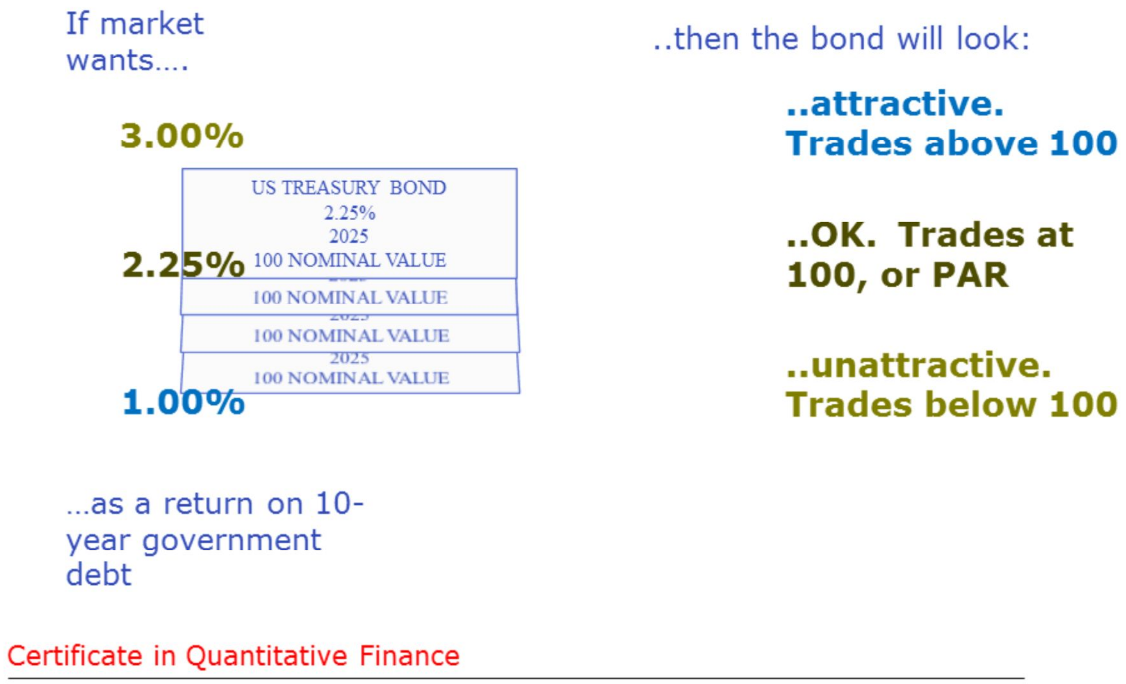
Where do they come from?



Government bonds are interest paying instruments

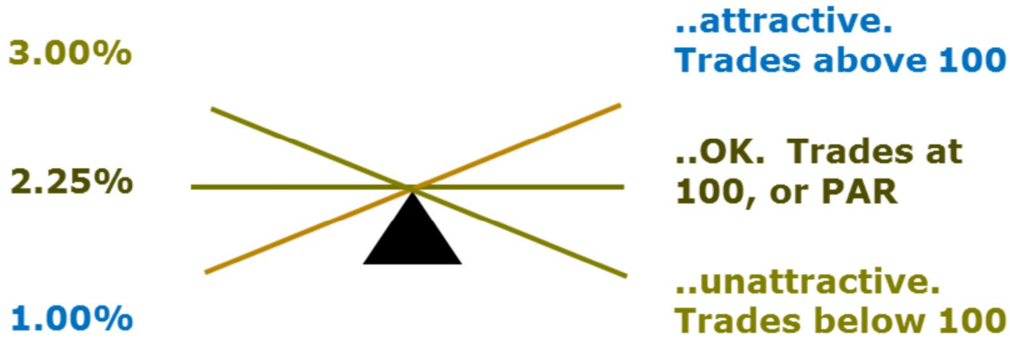
They are therefore vulnerable to changes in the interest rate climate.

This is a 10 year US T-bond. It is paying 2.25% of its face value – fine if the market's requirement for 10 years is 2.25%, but if the market wants more (it expects a period of economic growth and the threat of higher inflation)…

It will not look attractive – at least not at 100.

Its price will fall – but how far?

- So for a fixed coupon bond:



- There is an **inverse** relationship between the bond's price and its return, or **yield**

Return if the bond is trading at par

$$\frac{22.50}{100} = 22.50\% \text{ for the entire 10 years}$$

As an annual average: 2.25%

Return if the bond is below par at, say, 98:

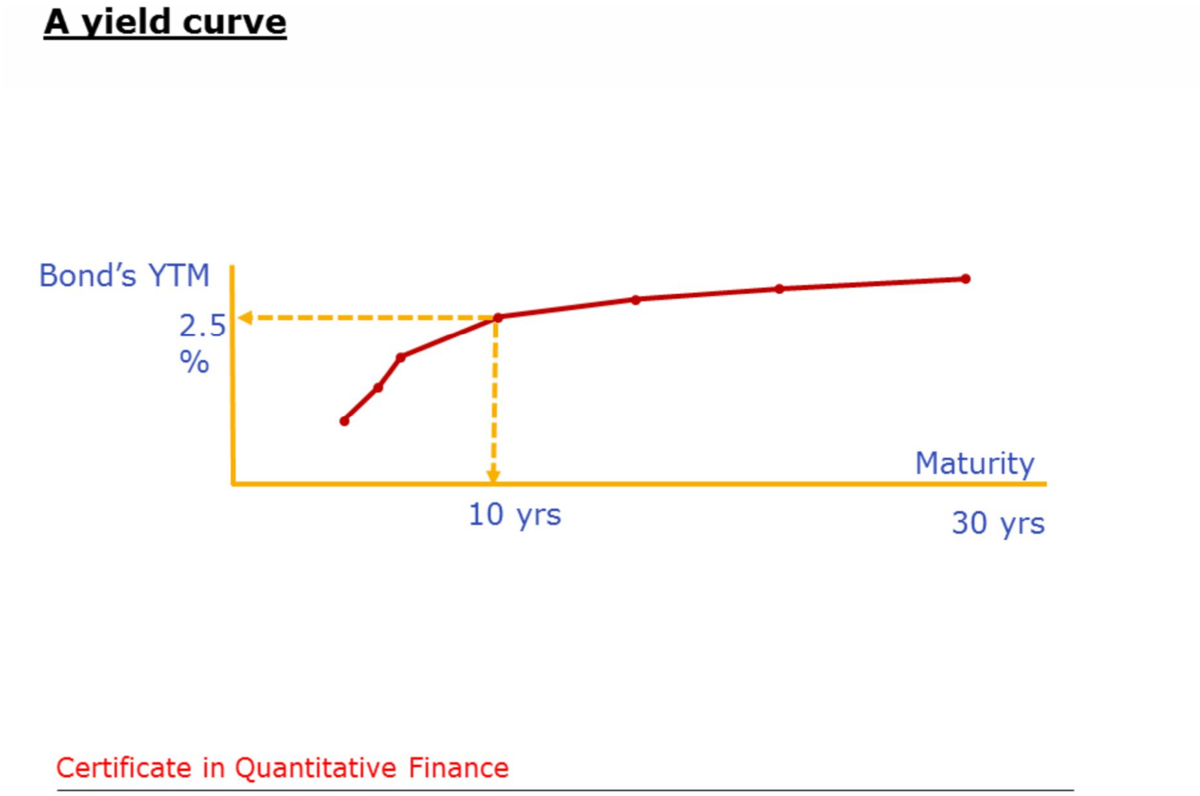
capital gain at redemption

$$\frac{22.50 + 2}{98} = 25.00\% \text{ for the entire 10 years}$$

As an annual average: 2.50%

This is the bond’s yield to maturity (an approximation called the simple, or Japanese yield)

- The bond is trading at 98;
- Its **coupon is 2.25%**;
- Its **yield to maturity (YTM) is 2.5%**



…if we plot a bond's YTM on a maturity axis….

Corporate Bonds - Credit

Bonds issued by companies are not free from the risk of default; their ability to pay may be doubtful. One way to assess an issuer's ability to repay their debt is their **credit rating**. Credit ratings are published by three credit rating agencies:

- Standard and Poors
  - AAA, AA+, AA, AA-, A+, A, A-, BBB+, BBB, BBB-, | BB+ ......
- Moody's
  - Aaa, Aa1, Aa2, Aa3, A1, A2, A3, Baa1, Baa2, Baa3, | Ba1 ......
- Fitch
  - AAA, AA+, AA, AA-, A+, A, A-, BBB+, BBB, BBB-, | BB+ ......

INVESTMENT GRADE

HIGH YIELD

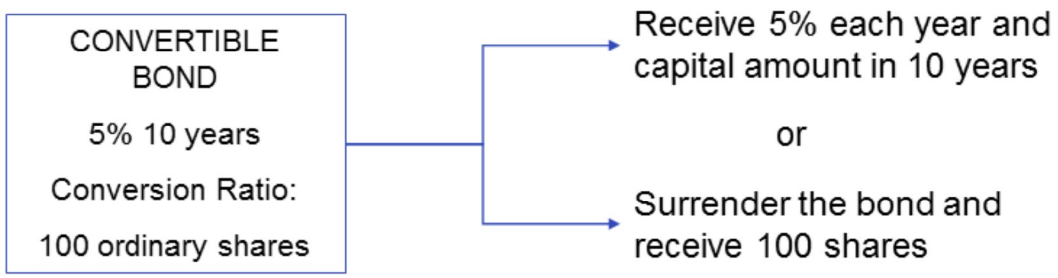
Each notch represents the same marginal difference in creditworthiness

Moody's	S&P	Fitch	
Investment Grade			
Aaa	AAA	AAA	
Aa1	AA+	AA+	
Aa2	AA	AA	
Aa3	AA-	AA-	Maximum safety
A1	A+	A+	
A2	A	A	Very high grade
A3	A-	A-	
Baa1	BBB+	BBB+	
Baa2	BBB	BBB	
Baa3	BBB-	BBB-	
Speculative			
Ba1	BB+	BB+	
Ba2	BB	BB	Low grade, speculative
Ba3	BB-	BB-	
B1	B+	B+	
B2	B	B	Highly speculative
B3	B-	B-	
Predominantly speculative			
	CCC+		
	CCC	CCC	
	CCC-		Substantial risk
Caa	CC	CC	
	C	C	Possibly in default
Ca	CI		No income being paid
C		DDD	DEFAULT
		DD	
	D	D	

Convertible Bonds

Another feature is convertibility. Convertible bonds offer:

- Bond returns of a fixed coupon and capital redemption; and
- A chance to surrender the bond and receive shares in the company.
- Example: A company issues a 10-year 5% convertible. The holder of the bond has two choices:



Repo Agreements

FEATURES OF REPOS

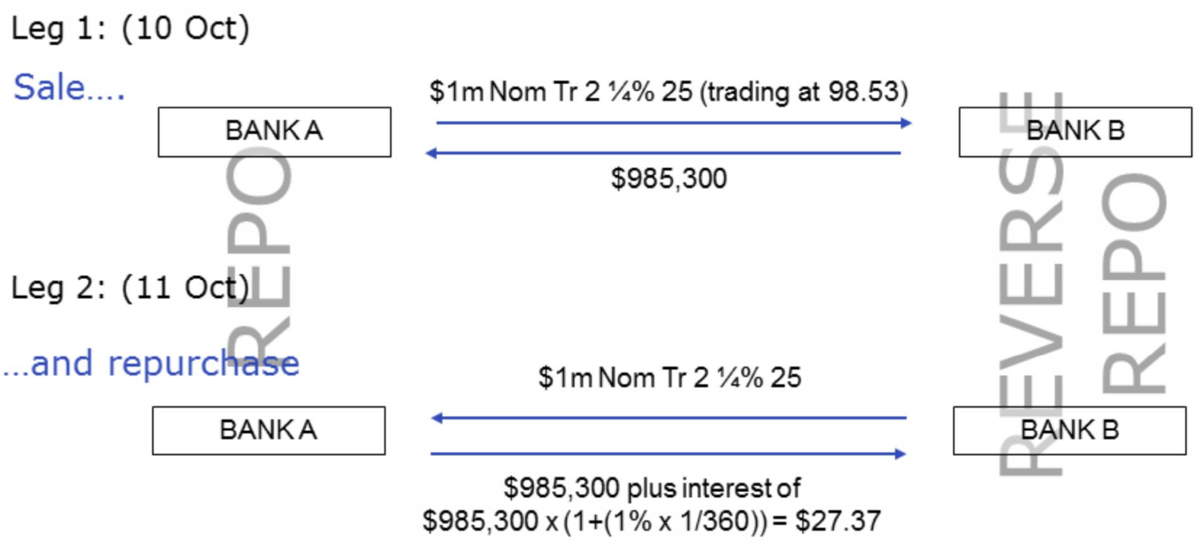
Definition

A transaction in which one party sells another a security. At the same time, as part of the same agreement the party agrees to repurchase identical securities on a specified date at a specified price.

- There are two legs to a repo
- The seller delivers securities and receives cash from the buyer
- The difference between the cash received by the seller in leg 1 and returned in leg 2 is known as the repo rate

mention motivation for repos. Shorting, Leverage etc and similarity to stock lending

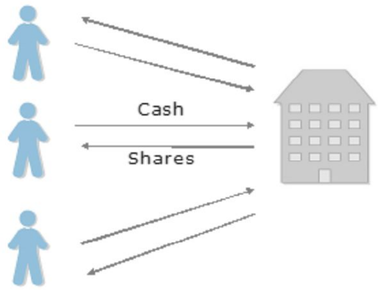
Example: An overnight repo agreed at 1% p.a.



Equities

Companies and Their Shareholders

A company issues shares in order to raise capital. The investors buy a share in the company's success.



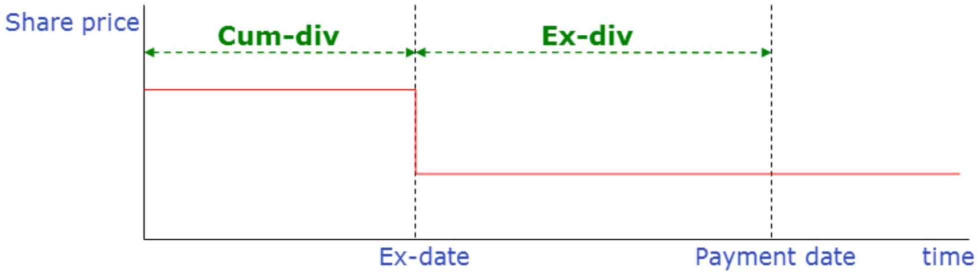
Points to note:

- The board of directors manage the company
- Authorised share capital versus issued share capital

Dividends

Companies sometimes pay dividends to their shareholders:

- The **ex-date** is a 'qualifying date'. You have to be the owner of the share on this day to receive the dividend on the up-coming payment date.
- During the period between the ex-date and the payment date the share is described as ex-div, or XD.
- Purchasing the share during the ex-div period means you will not recive the dividend on the payment date, as a result the share price will drop by the amount of the dividend (all else being equal) after the ex date:



Derivatives

Futures / Forwards

A futures contract is an agreement between two people to do a transaction on a date in the future, but at terms agreed now. A futures contract is an **obligation**.

Contract (no.1)

A **will** buy from B 100 Boeing at USD 135 per share on the third Wednesday in September

Terms

- A is the **buyer** / **long** – takes delivery
- B is the **seller** / **short** – makes delivery
- Boeing shares is the **underlying**
- 'third Wednesday in September' is the **delivery date**
- 135 per share is the September **futures price** (cf. cash or spot)
- **Contract size**: the contract is for 100 shares
- There is credit risk – this is mitigated by **margin**

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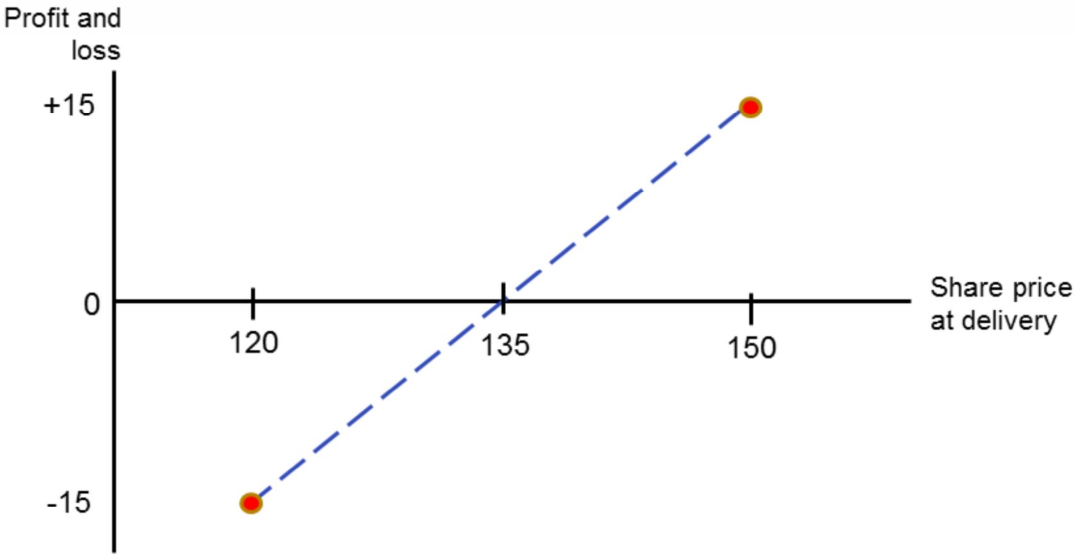
Note the buyer is the buyer *in* the contract, not *of* the contract…

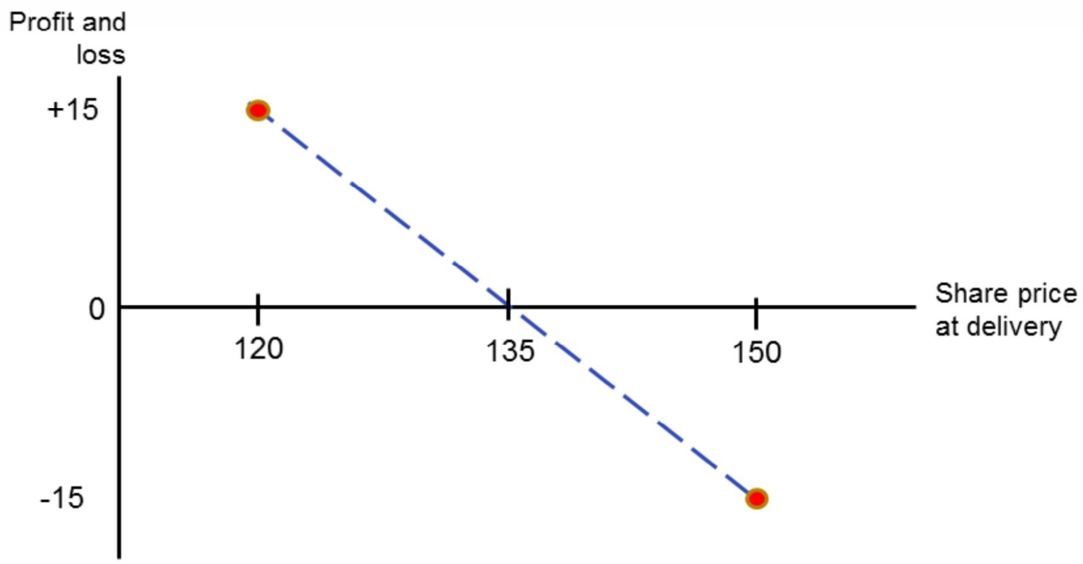
The Long Futures Position

- A's profit and loss *at expiry*, A is long September Boeing future at 135:



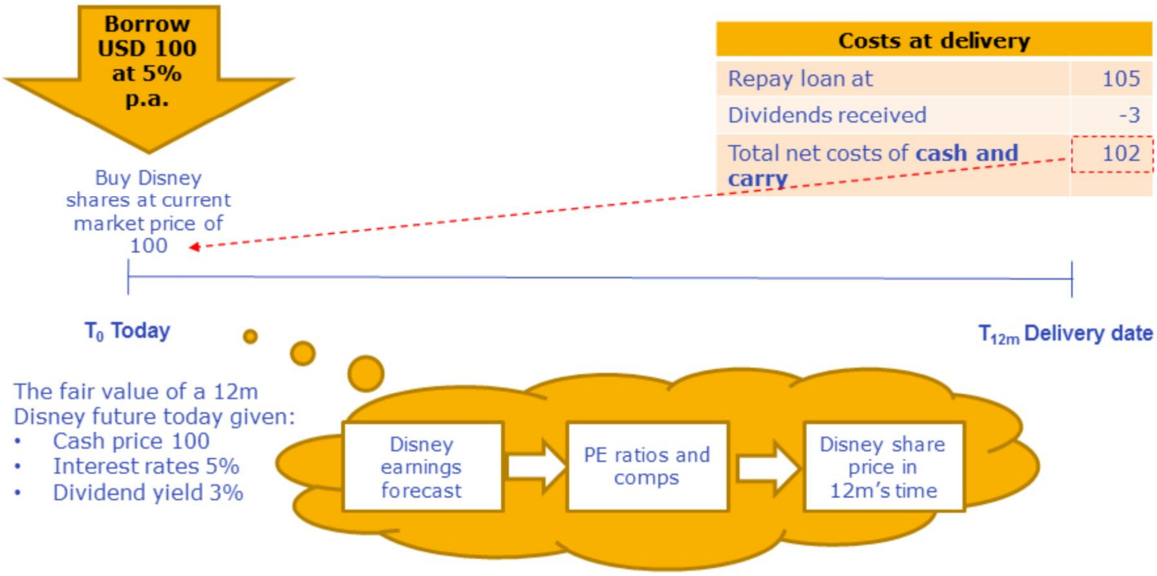
The Short Futures Position

- B's profit and loss *at expiry*, B is short September Boeing future at 135:



Pricing Futures & Forwards

Disney shares are trading at 100 with a dividend yield of 3%, 12m interest rates are 5%:



Costs at delivery	
Repay loan at	105
Dividends received	-3
Total net costs of **cash and carry**	102

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Pricing Futures & Forwards

Imagine you are asked to quote someone a price for delivering Disney shares in a year's time. How would you determine the 12-month delivery price? You could:

Make an educated guess; or

Buy the asset now and hold it until the delivery date (the less risky choice)

In the second instance you would need to know:

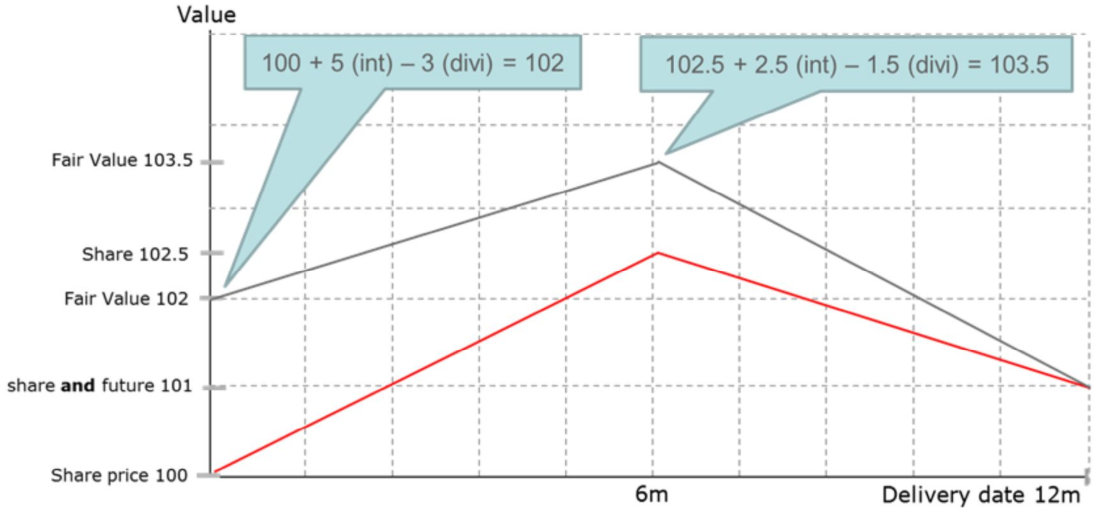
The price of the asset now (cash price)

Your costs of holding the asset until delivery

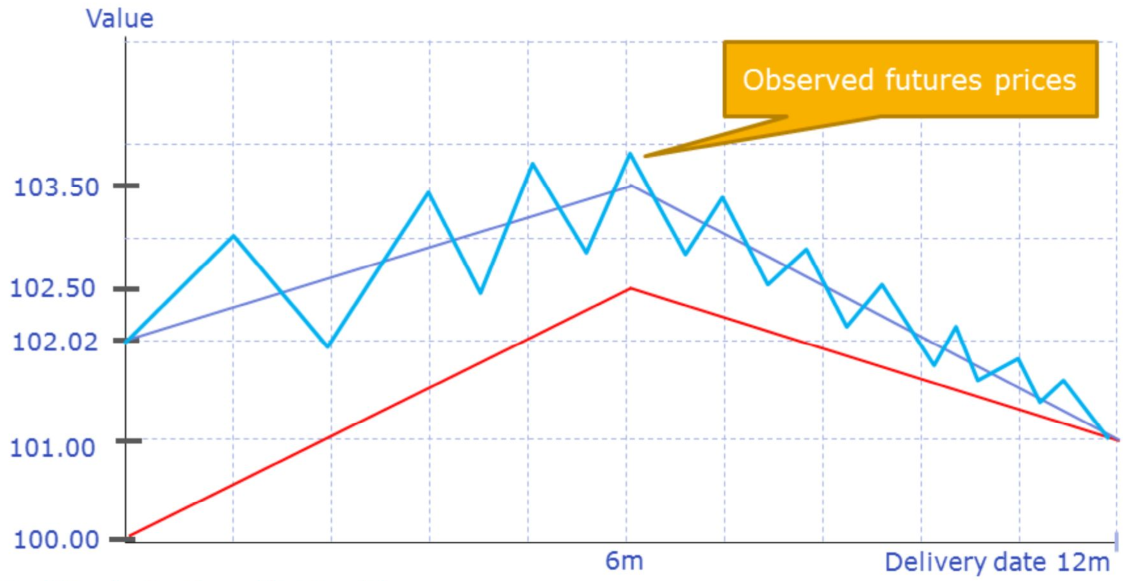
Any benefits obtained from holding the asset until delivery

Convergence

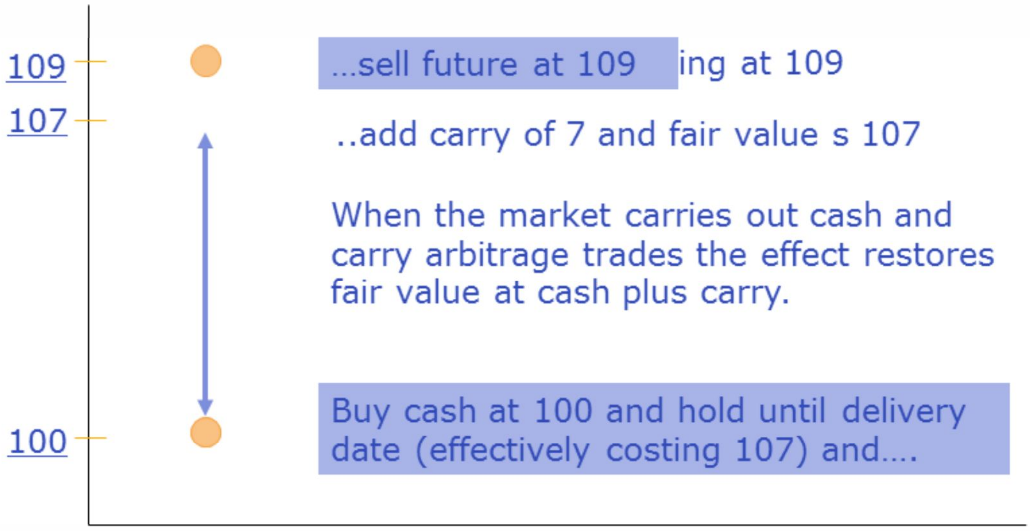
12m forward on an equity with a dividend of 3%, when interest rates are at 5% p.a.



- In reality markets will not trade at their fair value – the cash and futures markets have different volumes at different times as well futures markets being preferred by speculators for their low capital outlay..



Cash and carry arbitrage



Put more formerly the fair value of a future, F with a delivery date at time T, on an underlying asset trading today, t, at $S(t)$ when interest rates are r:

$$F = S(t)e^{r(T-t)}.$$

A First Example of No Arbitrage

It may be possible to generate a profit if the future is not trading at its fair value. This is known as cash and carry arbitrage.

- If the future is above fair value:
  - Buy cash and sell future today

- If the future is below fair value:
  - Sell cash and buy future today

This also illustrates the term 'pricing using the no arbitrage principle'.

- If future were trading at 770 rather than 759 what could we have done ?
  - Bought Spot, Paid insurance and storage, borrowed cash, sold future and locked in (770 - 759) risk free profit over the year
  - Wouldn't matter if Price of Coca fell because the future would hedge us

- REVERSE cash and carry relies on being able to SHORT the underlying which is not always possible (particularly if there is a convenience yield – holder of commodity will be holding because he has a use for the physical and wont be lending it out)
- If futures traded at 740, we could short the cocoa, MAY borrow it, put proceeds on deposit

Options

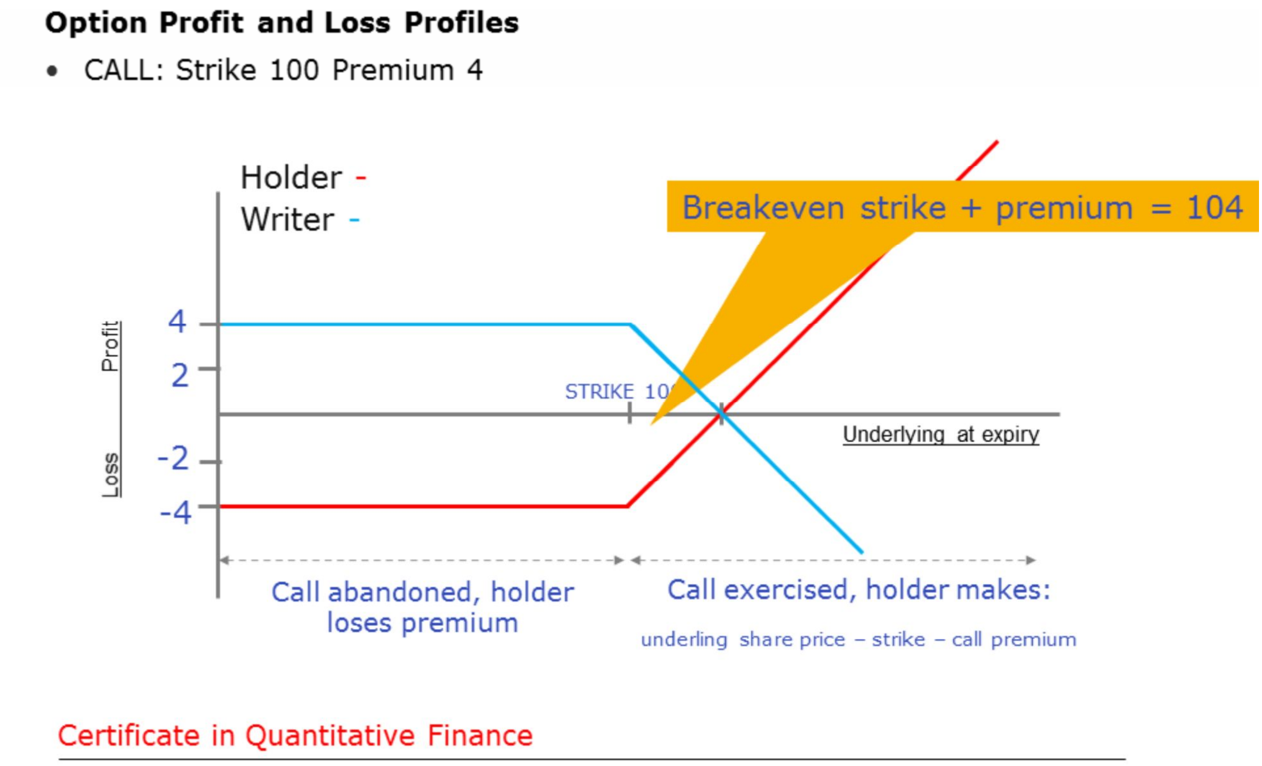
Call Options – The Right to Buy

Premium

Option

A *may* **buy from B** 1,000 Boeing shares at USD 100 on or before 3rd Wednesday in April

- **A** is the holder / buyer / long: has choices
- **B** is the writer / seller / short; may have obligations
- Boeing shares are the underlying
- The third Wednesday in April is the expiry
- 100 is the exercise price, or strike price
- Exercise styles
  - **A**merican: exercise **a**nytime up to expiry
  - **E**uropean: exercise at **e**xpiry only
- Premium – this is in the same units as the strike.
- A call option is insurance against the share price rising above the strike



This diagram is a profit and loss diagram for our call option above.

It shows what it ‘feels like’ for the holder and the writer in terms of consequences and their exposure at expiry.

1. First lets look at the holder – we start here because they have choices and we can rationalise their decisions.

what would the holder do if the shares, at expiry, were below the strike?

2. ..they would abandon the option (they are not gong to use an option to buy shares at 100 when they are in fact trading at, say, 85. if the holder abandons the option they lose their premium.
3. Would the holder exercise at 102?
4. …yes, they would buy the shares through the option at 100 and sell on the market at 102 making a profit of 2, thereby reducing their overall loses to -2.

Were would the holder break-even?

5. …at strike plus premium
6. So we can see that if, at expiry, the underlying is above the strike the call will be exercised. The profit made by the holder will be the underling share price – strike – call premium
7. Now let’s look at the writer’s profit and loss – we know that what they do will correspond with the holder’s decision..
8. If the holder abandons the option, the writer keeps the premium consequence-free

9. We said that at 102 the holder would exercise, this would mean that the writer would be handing over shares worth 102 at the strike of 100 and in so doing losing 2, bringing the profit down to 2.

10. Finally we know that for any value above the strike the holder will exercise and the writers P/L will be: Call premium – (underlying share price – strike price)

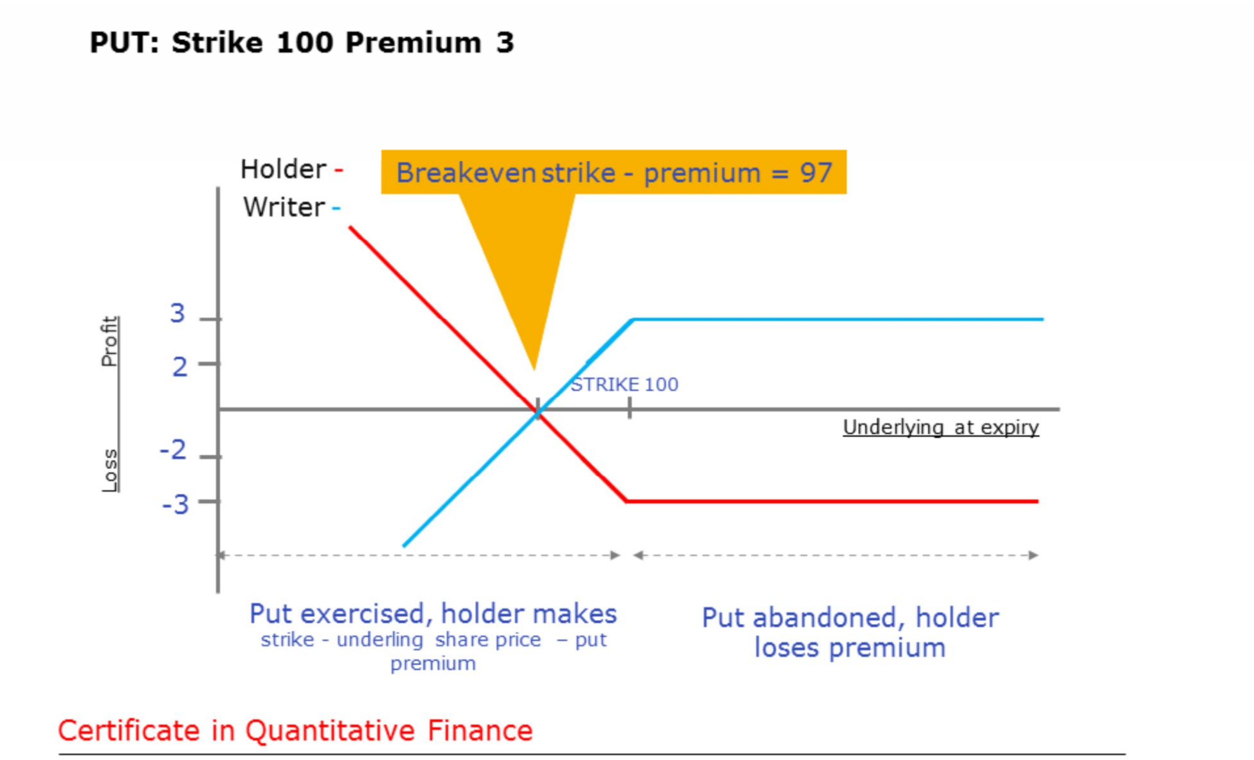
Put Options – The Right to Sell

Premium

Option

A *may* **sell to** B 1,000 Boeing shares at USD 100 on or before 3rd Wednesday in April

- A is the holder / buyer / long
- B is the writer / seller / short
- Boeing shares are the underlying
- The third Wednesday in April is the expiry
- 104 is the exercise price, or strike price
- Exercise styles
  - American: exercise anytime up to expiry
  - European: exercise at expiry only
  - Bermudan: exercise on certain days during the option's life
- Premium



This diagram is a profit and loss diagram for our put option above.

It shows what it 'feels like' for the holder and the writer in terms of consequences and their exposure at expiry.

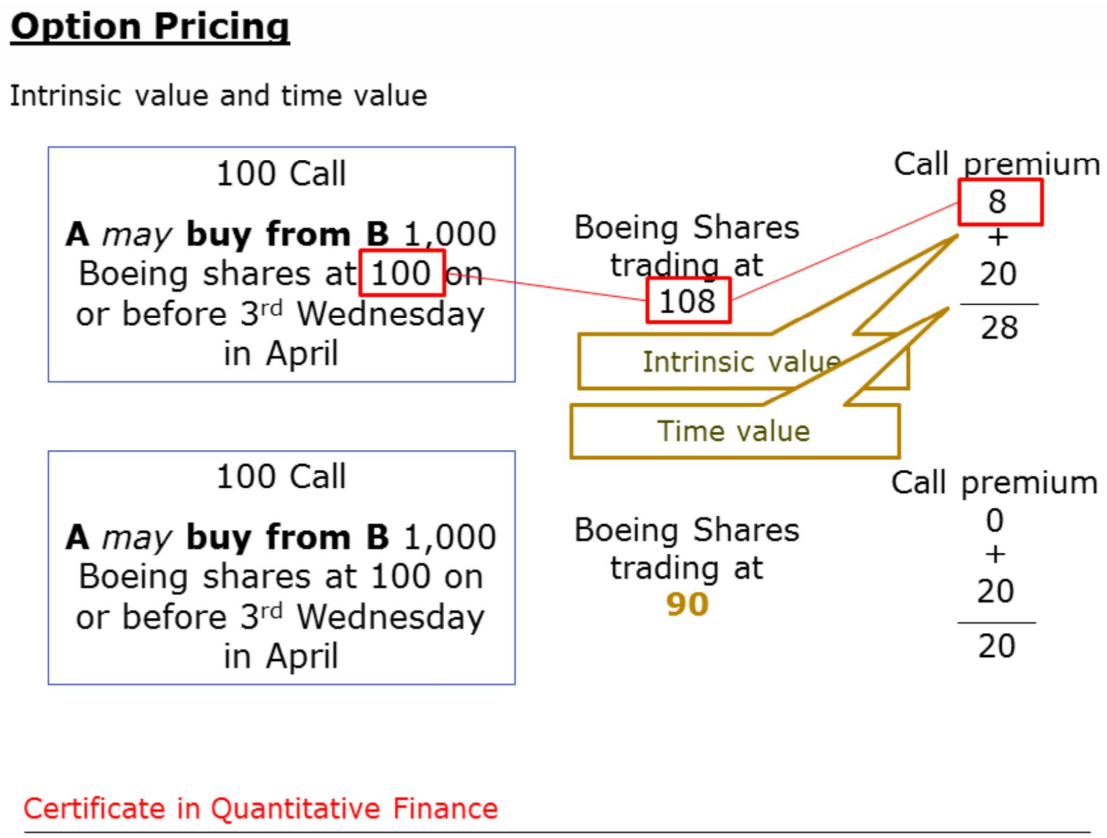
1. First lets look at the holder – we start here because they have choices and we can rationalise their decisions.

what would the holder do if the shares, at expiry, were above the strike?

2. ..they would abandon the option (they are not gong to use an option to sell shares at 100 when they are in fact trading at, say, 115. if the holder abandons the option they lose their premium.
3. Would the holder exercise at 98?
4. …yes, they would buy the shares in the market at 98 and deliver them through the put at 100 making a profit of 2, thereby reducing their overall loses to -2.

Were would the holder break-even?

5. …at strike minus premium
6. So we can see that if, at expiry, the underlying is below the strike the put will be exercised. The profit made by the holder will be the strike - underling share price – put premium
7. Now let's look at the writer's profit and loss – we know that what they do will correspond with the holder's decision..
8. If the holder abandons the option, the writer keeps the premium consequence-free
9. We said that at 98 the holder would exercise, this would mean that the writer would be buying shares worth 98 at the strike of 100 and in so doing losing 2, bringing the profit down to 2.
10. Finally we know that for any value above the strike the holder will exercise and the writers P/L will be: put premium – (strike price – underlying share price)

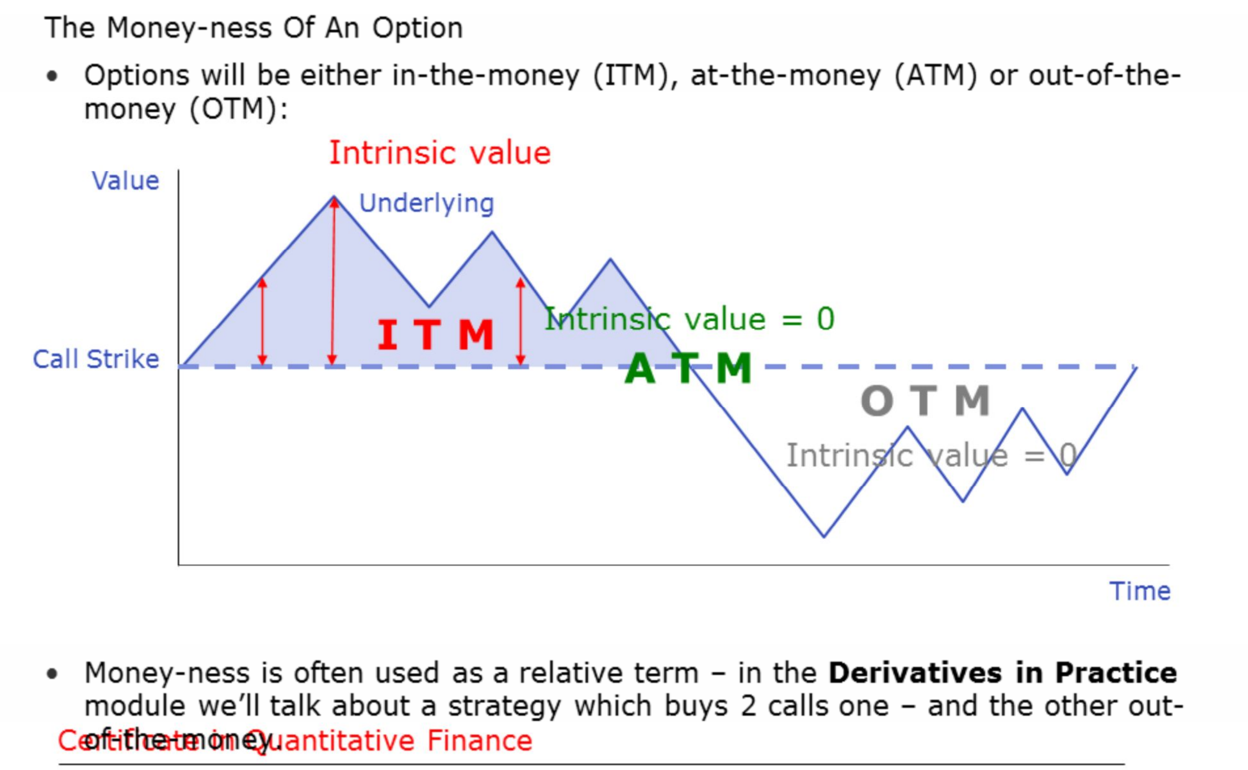


Now lets take a look at the premium. What drives its value? How is it calculated? The first step is to apply some common sense:

1. Here is our 100 April Boeing call from before and lets say Boeing shares are trading at 108. What is the minimum amount the writer should charge for this option?...
2. …..8: this is because if the holder exercised this instant they would make a profit of USD 8 per share.
3. This element of the premium we call intrinsic value. Because there is a chance that Boeing shares could rise some more over the remaining life of the option (and thereby add more intrinsic value) the writer would add
4. Time value to the premium. This is an involved calculation: we'd need to work out the probability of the shares trading above the strike at expiry as well as the most probable value of the shares at expiry *given* they are trading above the strike so we'll just make it up here and lets say its 20. this makes the total premium 28.

5. If Boeing shares are trading at 90 there is no intrinsic value (the writer will not give the holder a rebate at this level, because the only way a loss would be realised is if the holder exercised the option and inmposed the loss on

themselves!). The premium would be entirely made up of time value.

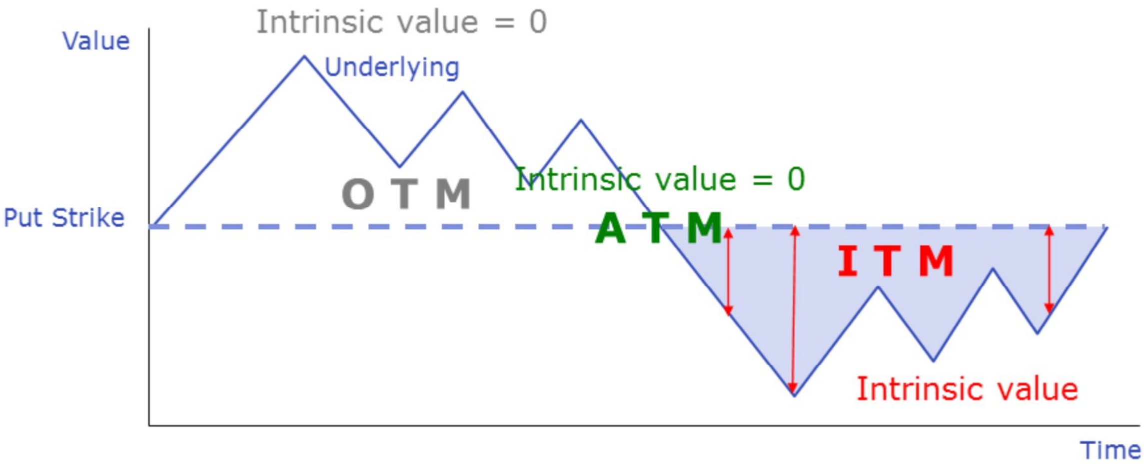


Here is a chart showing

1. The strike of a call option and
2. The underlying's price – notice that when the underlying is above the strike the call has intrinsic value. When an option has intrinsic value we refer to it as in-the-money.
3. When the underlying is below the strike the call is out-of-the-money and when the underlying and strike are at the same level we refer to it as at-the-money. Notice that the option has no intrinsic value when its ATM.
4. Money-ness is often used as a relative term – in the **Derivatives in Practice** module we'll talk about a strategy which buys 2 calls one – and the other out-of-the-money.

The Money-ness Of An Option

- Options will be either in-the-money (ITM), at-the-money (ATM) or out-of-the-money (OTM):



Factors that influence the Option Premium

There are five factors that drive option premiums:

- The price of the underlying — FACT
- The strike price of the option — FACT

IV

- The option's remaining life — FACT
- Interest rates (and dividends / foreign currency interest rates) — FACT
- The expected volatility of the underlying asset's returns — ???

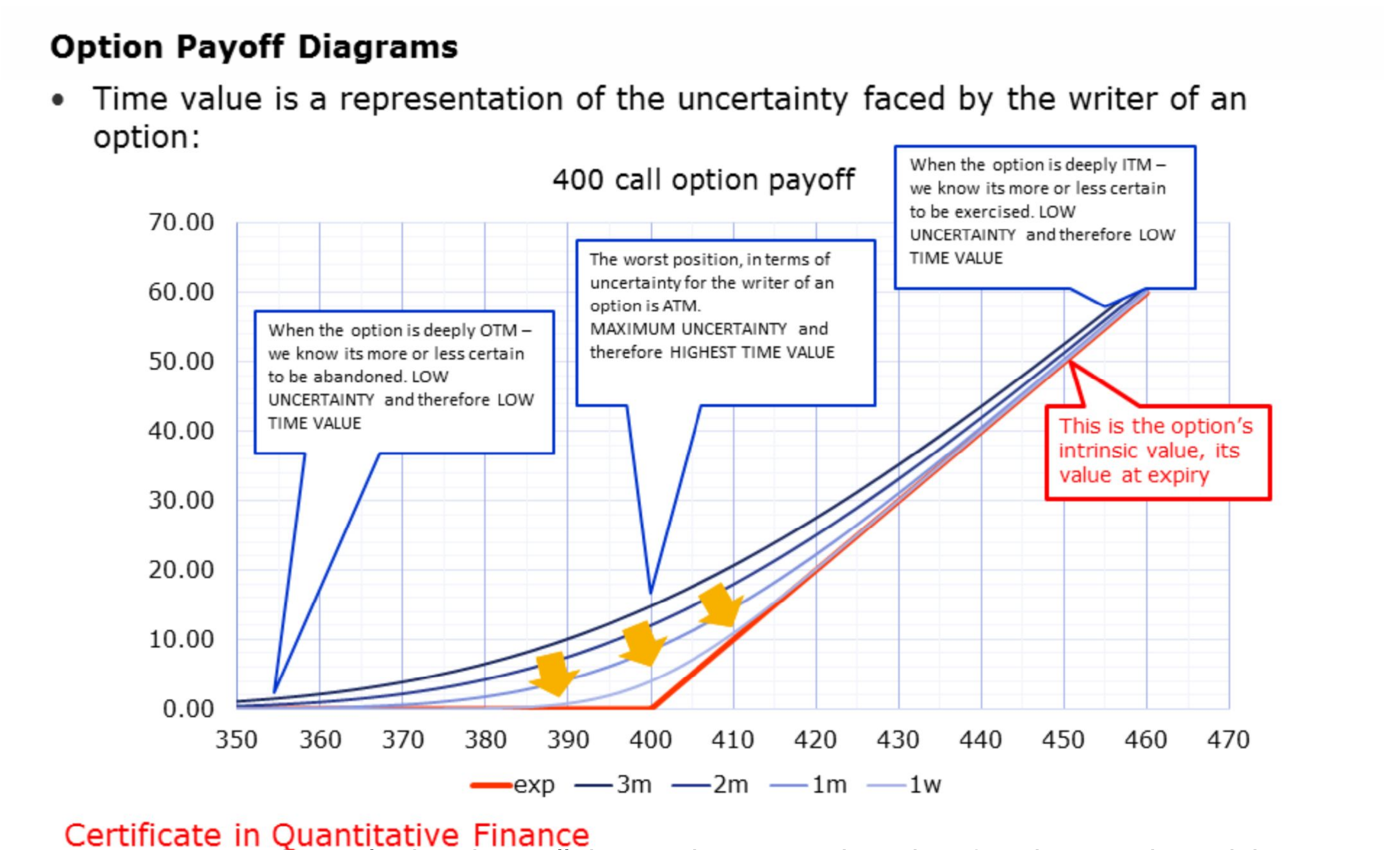
TV

These are usually input into a pricing model to calculate a return:

- The price of the underlying
- The strike price of the option
- The option's remaining life
- Interest rates
- Volatility

→ **BLACK-SCHOLES** $C = S_0N(d_1) - Ke^{-rt}N(d_2)$ → **Premium** value

Pricing Time Value



We are now going to look at how all that we have mentioned so far about option pricing can be seen in a payoff diagram. We can then use the pay-off diagram later to look at more ideas in relation to options.

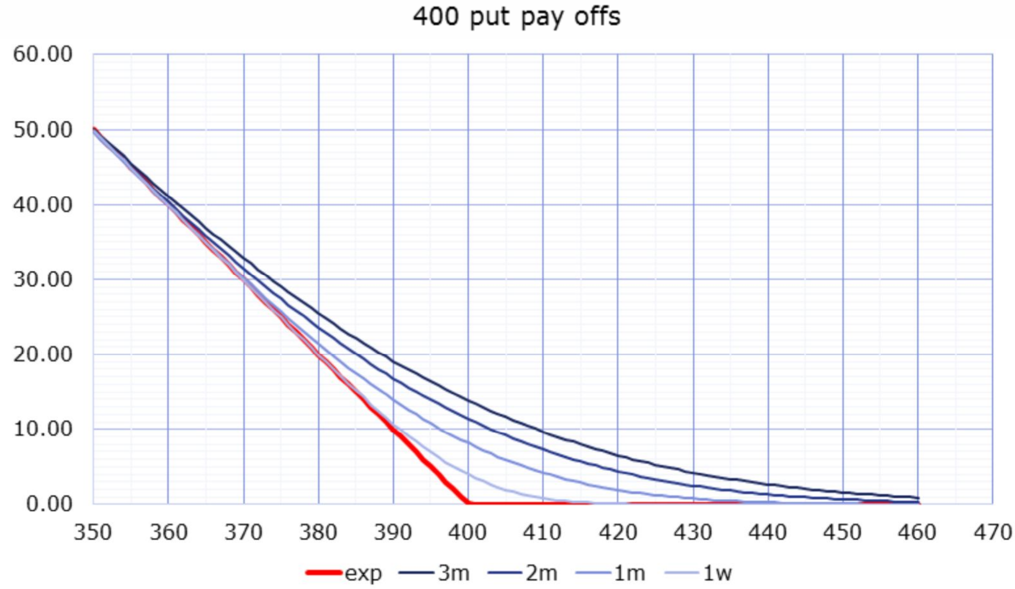
This option has a 400 strike so in the area to the left of 400 on the x axis the option is out of the money.

To the right of the strike it is in the money and its intrinsic value will increase at a 1:1 ratio with the underlying's price.

1. This is what the intrinsic value of the premium looks like – but we need to add time value. One of the key points here is that time value is a representation of the uncertainty faced by the writer of an option.
2. When the option is deeply OTM – we know its more or less certain to be abandoned. LOW UNCERTAINTY and therefore LOW TIME VALUE;
3. When the option is deeply ITM – we know its more or less certain to be exercised. LOW UNCERTAINTY and therefore, again LOW TIME VALUE;
4. The worst position, in terms of uncertainty for the writer of an option is ATM. MAXIMUM UNCERTAINTY and therefore HIGHEST TIME VALUE
5. So here's the overall premium line
6. As time passes, though the time value gap will diminish and the premium line will move towards the intrinsic value line
7. Here it is one month later

8. A month after that; and
9. With one week to expiry.

Put pay-off

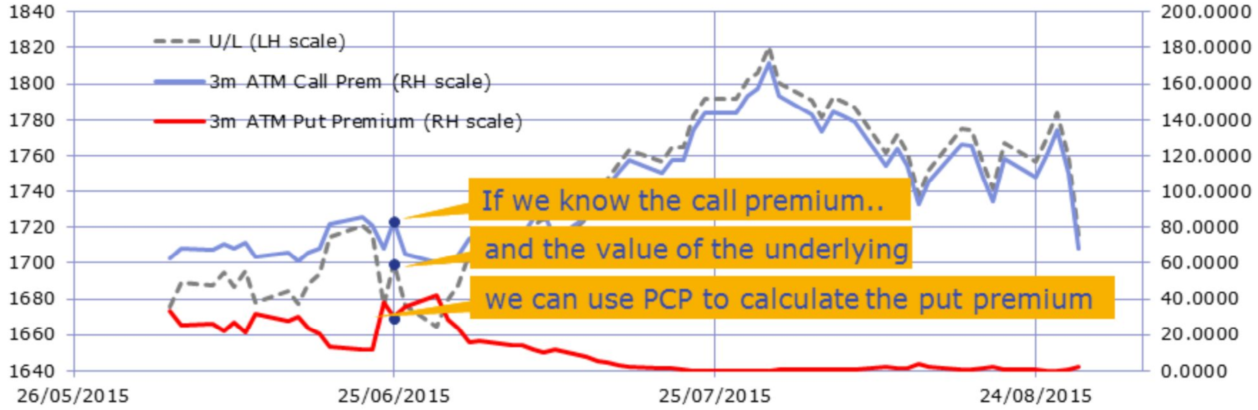

400 put pay offs
60.00
50.00
40.00
30.00
20.00
10.00
0.00
350
360
370
380
390
400
410
420
430
440
450
460
470
exp
3m
2m
1m
1w

Put-Call Parity (PCP)

What is PCP?

Put-call parity is the relationship between the premiums of a call and a put option with the same

- underlying,
- strike, and
- expiry



PCP Illustration

A non-dividend paying stock is currently trading at 42 (S_0). A 1-year European 40 call on the stock is trading at 10. So

- S_0 = 42
- T = 1
- C = 10
- X = 40

All we need is the interest rate for the period to the expiry of the option and put-call parity allows us to calculate the premium for a 1-year 40 European Put. Lets say

- $r_{1\text{year}}$ = 5%

Rationale (Discrete Compounding)

Consider two ways that guarantee owning the stock above in one year's time

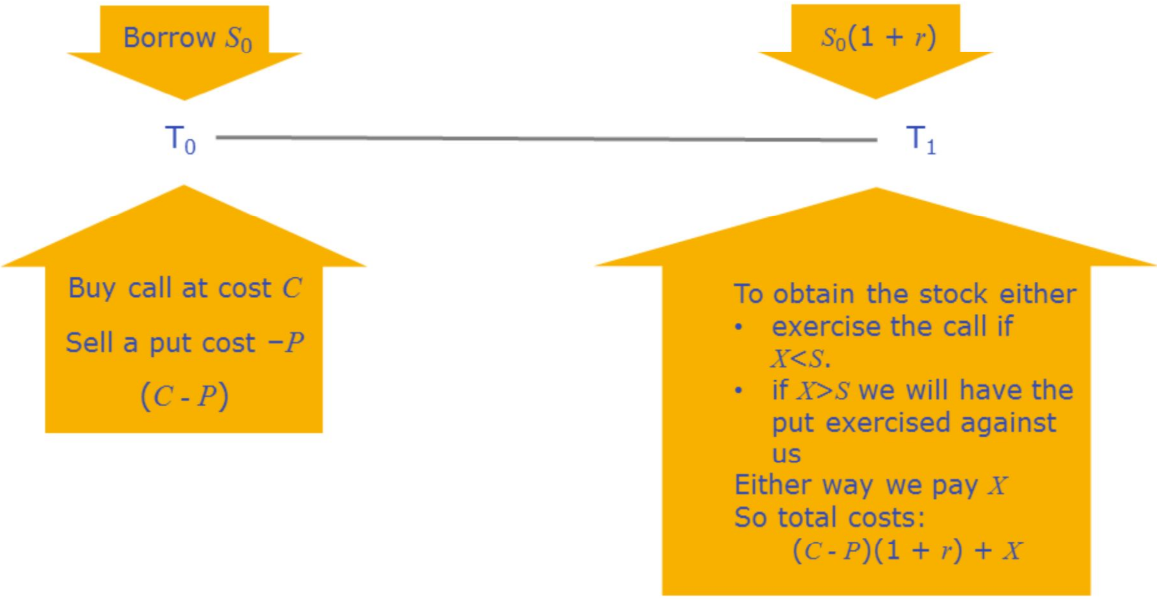
- Buy the stock now and hold it for a year. We do this by borrowing S_0 at r now, buying the stock and holding it. Our costs would be:

$$S_0(1 + r)$$

- Alternatively, we could buy a call and sell a put, both with the same strike, X, and with an expiry of 1-year. We borrow the net cost of the options, so our costs here would be:

$$(C - P)(1 + r) + X$$

Put call parity assumes these two journeys to the same point cost the same amount (we'll see if this is reasonable)


Borrow S_0
$S_0(1 + r)$
T_0
T_1
Buy call at cost C
Sell a put cost $-P$
$(C - P)$
To obtain the stock either
• exercise the call if $X<S$.
• if $X>S$ we will have the put exercised against us
Either way we pay X
So total costs:
$(C - P)(1 + r) + X$

Which would mean:

$$(C - P)(1 + r) + X = S_0(1 + r)$$

If we divide by $(1 + r)$ and re-arrange we get:

$$C - P = S_0 - \frac{X}{(1 + r)}$$

This equation fits our 1-year example, but if we say that t is the fraction of a year until expiry we could re-write the formula for any European option on a non-dividend stock using any expiry:

$$C - P = S_0 - \frac{X}{(1 + rt)}$$

If we put the data from our example above we can see what a put should be trading at:

$$P = C - S_0 + \frac{X}{(1+r)}$$

$$= 10 - 42 + \frac{40}{1.05}$$

$$= 6.09$$

Suppose the put was trading in the market at 5. What would you do?

…if the put was trading at 5 it is under-priced:

Action	Cash flow (in:+, out -)	Result if S>X	Result if S<X
Sell call	+10	Call exercised against us – hand over stock and receive 40	Call abandoned
Borrow X/(1+rt)	+38.09	Maturity proceeds of 40 to repay loan	Proceeds of exercised put
Buy put	-5	Abandon put	Exercise put – deliver stock at 40
Buy the underlying	-42	Delivered through Short call	Delivered through long put
Total	+1.09		

Appendix: Time value of money

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Compounding

Compounding single sums

Compounding is a process which determines the value of an amount invested now, for a given number of periods at a fixed periodic interest rate. Compounding requires:

- an amount to be invested now (PV, or present value);
- a maturity date – a number of periods away (n); and
- a periodic interest rate (r).

These together give us the value of the investment (FV) on the maturity date.

Illustration

Lets look at 100,000 invested for 1 year at a rate of 2% p.a. This would result in:

- 2,000 interest (PV x r); or
- a total maturity amount of 102,000: PV x (1+ r)

If we invested for a *further* year at 2% p.a. we would see

- Interest of 2,040 (102,000 x 2%); and
- A total maturity amount of 104,040: 102,000 x (1+ 2%).

For the 2 year investment we simply multiplied the PV amount by (1 + 2%) for the first year then (1+2%) for the second year. In other words:

- 100,000 x $(1+2\%)^2$

NB: The periodic rate is uniform for both periods.

We can formerly express this process as:

$$FV = PV\,(1+r)^n$$

Where FV is the future value of an amount PV invested today for n periods at an interest rate of r per period.

Example:

6,000 placed on deposit for five years at an annual interest rate of 6 ½% p.a. What is the final sum assuming compound interest?

$$6{,}000\,(1 + 0.065)^5 = 8{,}220.52$$

Compounding where interest is paid more than once a year (nominal vs. effective rates)

More often than not interest is *quoted* as an annual rate but *paid* in 2 semi-annual instalments, or 4 quarterly instalments, or 12 monthly instalments…etc. Here we **divide the (nominal) rate** by the frequency and **multiply the periods** by the frequency. Using the 100,000 at 2% for 1 year above:

- Semi annual interest payment (SA):

$$100{,}000 \times \left(1 + \frac{0.02}{2}\right)^{1 \times 2} = 102{,}010.00$$

  Nominal rate is 2% p.a., **effective rate** is 2.01% p.a

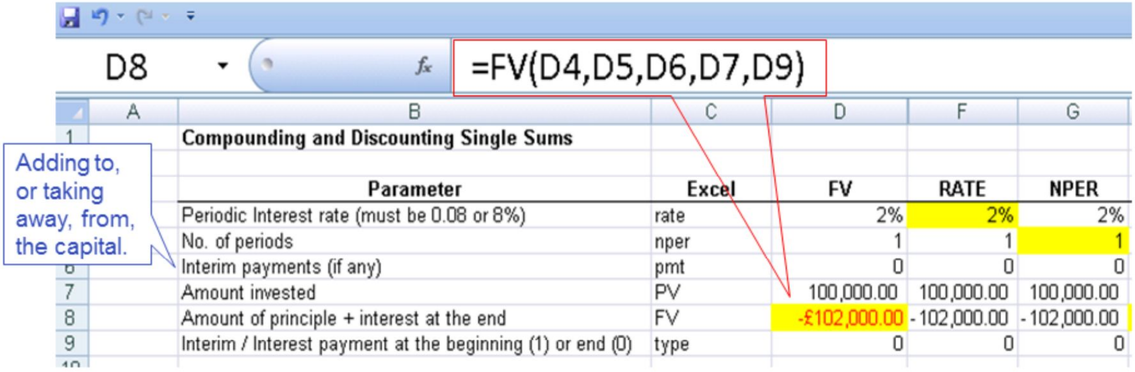
- Quarterly interest payment (Q):

$$100{,}000 \times \left(1 + \frac{0.02}{4}\right)^{1 \times 4} = 102{,}015.05$$

  Nominal rate is 2% p.a., effective rate is 2.01505% p.a

Compounding using Excel's FV function

Excel's FV function will calculate future values for you. Notice how the function assigns the future value a negative sign, this denotes that the *direction* of the FV cash flow is in the opposite direction to the PV cash flow.



	A	B	C	D	F	G
1		**Compounding and Discounting Single Sums**				
2						
3		**Parameter**	**Excel**	**FV**	**RATE**	**NPER**
4		Periodic Interest rate (must be 0.08 or 8%)	rate	2%	2%	2%
5		No. of periods	nper	1	1	1
6		Interim payments (if any)	pmt	0	0	0
7		Amount invested	PV	100,000.00	100,000.00	100,000.00
8		Amount of principle + interest at the end	FV	-£102,000.00	-102,000.00	-102,000.00
9		Interim / Interest payment at the beginning (1) or end (0)	type	0	0	0

Excel will also calculate any of the other parameters as an unknown. These are the RATE and NPER functions.

Compounding where payment frequency is continuous

There are certain circumstances where life is easier if we compound using the assumption that interest is paid continuously.

This would mean that the calculation for 100,000 invested for a year at 2% p.a. paid continuously would look like:

$$100{,}000 \times \left(1 + \frac{0.02}{\text{INFINITY}}\right)^{1 \times \text{INFINITY}}$$

To get round this we use a constant, e, which has the value 2.71828. so if r_c is the continuously compounded periodic rate:

- $FV = PVe^{rn}$

So:

$$100{,}000 \times 2.71828^{0.02 \times 1} = 102{,}020.13$$

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The function for e^x in excel is EXP(x):

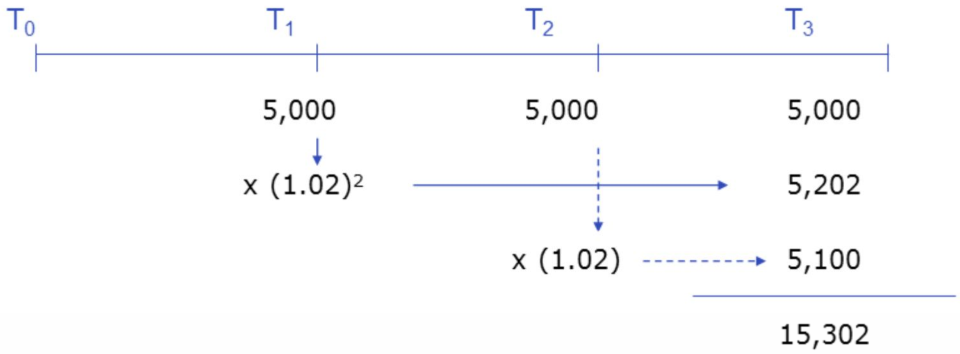
H8 | f_x | =H7*EXP(H4*H5)

B	C	H
Compounding and Discounting Single Sums		
		FV
Parameter	**Excel**	**Exp()**
Periodic Interest rate (must be 0.08 or 8%)	rate	2%
No. of periods	nper	1
Interim payments (if any)	pmt	0
Amount invested	PV	£100,000.00
Amount of principle + interest at the end	FV	£102,020.13
Interim / Interest payment at the beginning (1) or e	type	0

Compounding multiple uniform sums at a uniform rate

A series of uniform sums at equal intervals is called an annuity. This might be the regular contributions made to a pension fund for example. We can compound these using a uniform interest rate as follows:

- 3-year 5,000 annuity
- rate is 2% p.a.



To calculate the forward value of an annuity in Excel we use the FV function.
But here there is no PV amount – all we have are uniform interim payments.
If we do add a PV amount Excel will treat it as a cash flow:

=FV(D16,D17,D18,D19,D21)

Parameter	Excel	FV
Periodic Interest rate (must be 0.08 or 8%)	rate	2%
No. of periods	nper	3
Interim payments (if any)	pmt	5,000.00
Amount invested	PV	0.00
Amount of principle + interest at the end	FV	-£15,302.00
Interim / Interest payment att he beginning (1) or end (0)	type	0

Compounding for money market periods

Money markets periods are those between overnight and twelve months in length. Money market rates are quoted as annual nominal rates:

If the 1 week rate is quoted as 1% p.a. we do not receive 1% on our capital at maturity, we receive 1/52 (assuming 52 weeks in a year) of 1%.

In fact, 1/52 is too general. This fraction, known as the **day-count fraction**, takes slightly different forms depending on the currency:

- For USD, EUR and JPY it takes the form:

$$\frac{\text{Actual no. days in the period}}{360}, \text{or } \frac{\text{Actual}}{360}$$

- whilst for GBP it is:

$$\frac{\text{Actual}}{365}$$

We say that USD, EUR and JPY use 360 **day-count basis**, whereas GBP uses 365.

Illustration

EUR 10,000,000 on deposit for 1 week at 1% p.a.:

$$10{,}000{,}000 \times \left(1 + \left(0.01 * \frac{7}{360}\right)\right) = 1{,}000{,}194.44$$

The term (0.01*7/360) is the 1 week **periodic rate**. Convention dictates that with **nominal** rates we obtain the periodic rate by using the appropriate day count fraction.

In Excel we can use the FV function, or not.

	A	B
1		
2	PV	1000000
3	Basis	360
4	Period	7
5	Rate	1%
6	FV (function)	-£1,000,194.44
7	FV	1,000,194.44

=FV(B5*(B4/B3),1,0,B2). Notice the automatic formatting to currency and the minus sign. Also there is one 7 day period at a periodic rate of 1% * 7/360

It may be as well to do the calculation without the FV function: =B2*(1+B5*(B4/B3))

Discounting

Discounting single sums

Discounting is the process for determining the present value of a future cash flow. Discounting requires:

- a cash flow sitting in the future (FV, or future value);
- a maturity date – a number of periods away (n); and
- a periodic interest rate (r).

These together give us the value of the investment (FV) on the maturity date.

Illustration (single future cash flow)

If we invested 100,000 for 1 year at a rate of 2% p.a., we would have:

- 2,000 interest (PV x r); or
- a total maturity amount of 102,000: PV x (1+ r).

This maturity amount is the future value. If we determined the future value by multiplying the present value by 1+ r then to calculate the PV from FV we divide FV by 1+ r:

$$\frac{102{,}000}{1.02} = 100{,}000$$

If we invested the 100,000 for a *further* year at 2% p.a. we would see

- Interest of 2,040 (102,000 x 2%); and
- A total maturity amount of 104,040: 102,000 x (1+ 2%).

For the 2 year investment we simply multiplied the PV amount by (1 + 2%) for the first year then (1+2%) for the second year. In other words:

- 100,000 x $(1+2\%)^2$

Again, if we divide the future value by $(1+r)^n$. We have the present value:

$$\frac{104{,}040}{(1.02)^2} = 100{,}000$$

In Excel we use the PV function.

Notice the signs: Excel needs to differentiate between the direction of the future-value cash flow and the present value cash flow:

E7 =PV(E4,E5,E6,E8,E9)

	B	C	E
1	Compounding and Discounting Single Sums		
2			
3	**Parameter**	**Excel**	**PV**
4	Periodic Interest rate (must be 0.08 or 8%)	rate	2%
5	No. of periods	nper	2
6	Interim payments (if any)	pmt	0
7	Amount invested	PV	£100,000.00
8	Amount of principle + interest at the end	FV	-£104,040.00
9	Interim / Interest payment at the beginning (1) or end (0)	type	0
10			

We can formerly express this process as:

$$PV = \frac{FV}{(1+r)^n}$$

Where FV is the future value of an amount PV invested today for n periods at an interest rate of r per period.

Discount factors

We could also write the discounting formula as:

$$PV = FV \times \frac{1}{(1+r)^n}$$

Where the $\frac{1}{(1+r)^n}$ element is known as the **discount factor**.
Discount factors are the reducing factor for that period at that interest rate.

Illustration

The 2 year rate is 2%, the two year discount factor would therefore be:

$$\frac{1}{(1.02)^2} = 0.96116878$$

Using the discount factor to determine the present value of 104,040:

104,040 * 0.96116878 = 100,000

Discounting where payment frequency is continuous

There are certain circumstances where life is easier if we discount using the assumption that interest is paid continuously.

This would mean that the calculation for the PV of a cash flow of 102,020.13 paid in a year's time at 2% p.a. paid continuously would look like:

$$\frac{102{,}020.13}{\left(1+\frac{0.02}{\text{INFINITY}}\right)^{1\times\text{INFINITY}}}$$

To get round this we use a constant, e, which has the value 2.71828. so if r is the continuously compounded periodic rate:

- $PV = FVe^{-rn}$

So:

$$102{,}020.13 \times 2.71828^{-0.02 \times 1} = 100{,}000$$

Certificate in Quantitative Finance

The function for e^x in excel is EXP(x):

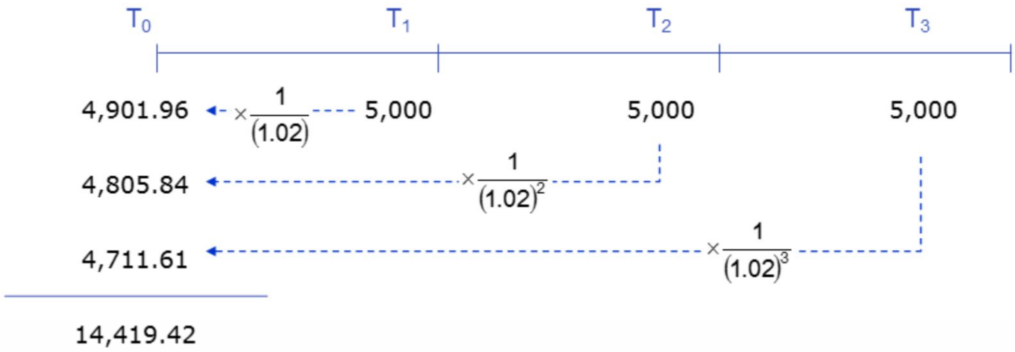
J7 | f_x | =J8*EXP(-J4*J5)

	B	C	J
1	Compounding and Discounting Single Sums		
2			PV
3	**Parameter**	**Excel**	
4	Periodic Interest rate (must be 0.08 or 8%)	rate	2%
5	No. of periods	nper	1
6	Interim payments (if any)	pmt	
7	Amount invested	PV	£100,000.00
8	Amount of principle + interest at the end	FV	£102,020.13
9	Interim / Interest payment at the beginning (1) or end (0)	type	

Discounting multiple uniform sums at a uniform rate

A series of uniform sums at equal intervals is called an annuity. This might be the regular contributions made to a pension fund for example. We can discount these using a uniform interest rate as follows:

- 3-year 5,000 annuity
- rate is 2% p.a.



To calculate the present value of an annuity in Excel we use the PV function. But here there is no FV amount – all we have are uniform interim payments. If we do add aFV amount Excel will treat it as a cash flow:

E19 =PV(E16,E17,E18,0)

	A	B	C	E
13		Annuity - multiple, uniform cash flows		
14				
15		Parameter	Excel	PV
16		Periodic Interest rate (must be 0.08 or 8%)	rate	0.02
17		No. of periods	nper	3
18		Interim payments (if any)	pmt	5,000.00
19		Amount invested	PV	-£14,419.42
20		Amount of principle + interest at the end	FV	£0.00
21		Interim / Interest payment att he beginning (1) or end (0)	type	0

Discounting non-uniform cash flows: net present value (NPV)

If we face a series of uneven cash flows we PV each one at the chosen discount rate. The result is the net-present-value or NPV of the series. If we assume the cash flows below are annual and we use a discount rate of 2% pa:

If we present value these cash flows at 2% their net present value comes to 456.28

Period	Cash Flow	PV at Rate 2%
0	-500	-500.00
1	-1000	-980.39
2	0	0.00
3	600	565.39
4	700	646.69
5	800	724.58
	NPV	456.28

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This is the first time we have had investment and return (cash flows changing signs). This leads us from PV to NPV

CF
2nd CE/C
CF0 (T_0) = 500 [+/-] ENTER ↓
C01 (T_1) = −1000
→ F01 (Freq) = 1 → so ↓
IGNORE C02 (T_2) = ∅ → so ↓
→ F02 = ↓
C03 (T_3) = +600
C04 (T_4) = +700
C05 (T_5) = +800

NPV
I = 6 ENTER
↓
NPV = CPT
+212.65

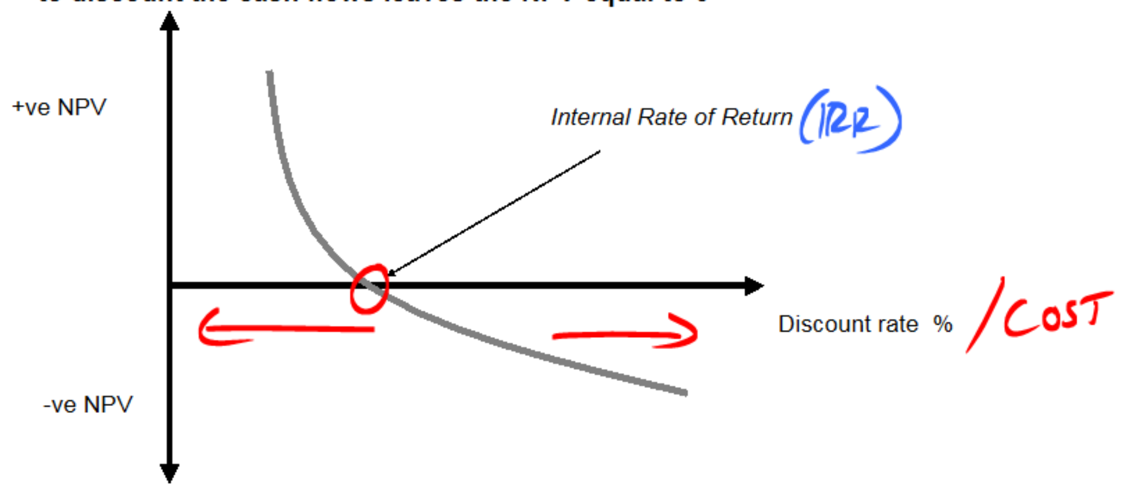
Internal rate of return

If we calculate the NPV for different discount rates we can see NPV move from positive to negative:

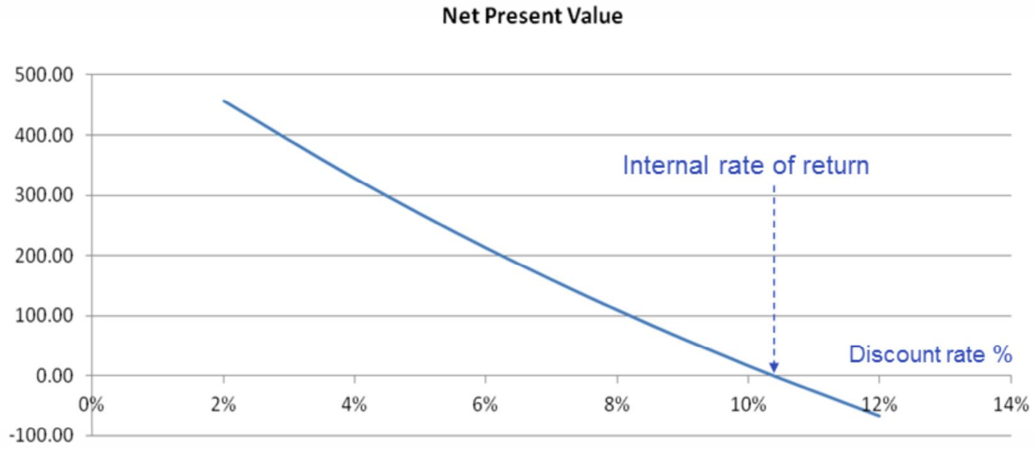
		PV at Rate					
Period	**Cash Flow**	2%	4%	6%	8%	10%	12%
0	-500	-500.00	-500.00	-500.00	-500.00	-500.00	-500.00
1	-1000	-980.39	-961.54	-943.40	-925.93	-909.09	-892.86
2	0	0.00	0.00	0.00	0.00	0.00	0.00
3	600	565.39	533.40	503.77	476.30	450.79	427.07
4	700	646.69	598.36	554.47	514.52	478.11	444.86
5	800	724.58	657.54	597.81	544.47	496.74	453.94
	NPV	456.28	327.76	212.65	109.36	16.54	-66.98

MORTGAGE = 6% (DISC), RENTAL = 6% (IRR), NPV = Ø

Given a set of cash flows and an NPV, the IRR is the rate of return that when used to discount the cash flows leaves the NPV equal to 0



The discount rate where NPV is zero is referred to as the internal rate of return:



Certificate in Quantitative Finance

Excel has an IRR function. It assumes that the first cash flow is now, at T_0, and that all the cash flows are 1 period apart. Excel gives us the periodic IRR:

I3 =IRR(B3:B8)

	A	B	I
1			
2	**Period**	**Cash Flow**	**IRR**
3	0	-500	10.38%
4	1	-1000	
5	2	0	
6	3	600	
7	4	700	
8	5	800	

Discounting for money market periods

EUR 1,000,194.44 paid in 1 week. The 1 week rate is presently at 1% p.a.:

$$\frac{1{,}000{,}194.44}{\left(1+\left(0.01*\frac{7}{360}\right)\right)}=10{,}000{,}000$$

In Excel we can use the FV function, or not.

D6

	A	B
1		
2	FV	1,000,194.44
3	Basis	360
4	Period	7
5	Rate (p.a.)	1%
6	PV (function)	-£1,000,000.00
7	PV	1,000,000.00

=PV(B5*B4/B3,1,0,B2). Notice the automatic formatting to currency and the minus sign. Also there is one 7 day period at a periodic rate of 1% * 7/360

It may be as well to do the calculation without the FV function: =B2/(1+(B5*B4/B3))

Non-uniform rates of return

If we have an investment where the returns are uneven we can calculate the **average annual compound rate (average rate of return)** using the **geometric mean**:

Illustration

Here are the prices of an asset over 5 periods. We are going to calculate the average rate of return. The first step is to determine each periods return. We might do this using discrete or continuous returns.

Period (or date)	Price	Return (discrete)
0	47	
1	50	6.38%
2	54	8.00%
3	48	-11.11%
4	45	-6.25%
5	51	13.33%

$$\frac{(50-47)}{47}$$

Geometric mean

- With the arithmetic mean we *add* the values and *divide* by the number of observations.
- With the geometric mean we *multiply* the values together and take the *nth root* of this product.

The geometric mean, however does not accommodate negative numbers (we have two negative returns in our illustration). The solution to this is to add 1 to each of the returns (giving us something called the **price relative**) and subtracting it later:

Period (or date)	Price	Return (discrete)	1+ return (price relative)
0	47		
1	50	6.38%	1.0638
2	54	8.00%	1.0800
3	48	-11.11%	0.8889
4	45	-6.25%	0.9375
5	51	13.33%	1.1333
			1.0165

$$\sqrt[5]{1.0638 \times 1.08 \times 0.8889 \times 0.9375 \times 1.1333}$$

Excel has a geometric mean function – GEOMEAN:

E8 =GEOMEAN(E3:E7)

	A	B	C	D	E
1		**Period (or date)**	**Price**	**Return (discrete)**	**1+ return (price relative)**
2		0	47		
3		1	50	6.38%	1.0638
4		2	54	8.00%	1.0800
5		3	48	-11.11%	0.8889
6		4	45	-6.25%	0.9375
7		5	51	13.33%	1.1333
8					1.0165

This answer of 1.0165 is telling us that the average price relative is 1.0165 per period.

If we subtract the 1 we added to each return at the start (during the calculations these 1s were multiplied together – making 1) we get the average return for each period, namely 1.65%.

We can check this.

If we take the original price and compound it for 5 periods using this average return we should arrive at the period 5 price:

$$PV \times (1 + r)^n = FV$$

- PV is 47
- r is 1.65%
- n is 5

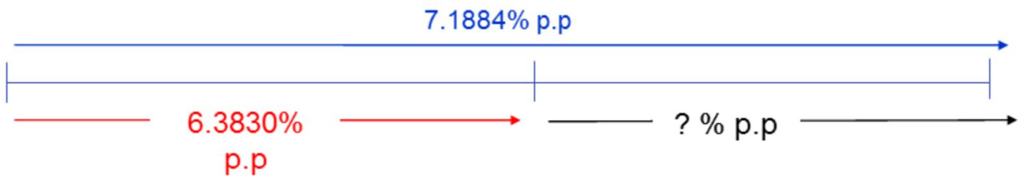
$$47 \times (1.0165)^5 = 51$$

Determining a single period rate form an average compound return

If we know the two period rate (i.e. the per period rate, or average) and the one period rate we can calculate the second one period rate (this is a forward rate).

Illustration

- The two period rate is 7.1884% p.p.
- The one period rate is 6.3830% p.p.



$$\frac{(1.071884)^2}{(1.06383)} = 1.0800$$

- These two rates give us a forward rate of 8% p.p.

An informal way of looking at this would be:

$$\frac{(1+r_{\text{LONGERPERIOD}})^{\text{LONGERPERIODS}}}{(1+r_{\text{SHORTERPERIOD}})^{\text{SHORTERPERIODS}}} = (1+r_{\text{FORWARDPERIOD}})^{\text{FORWARDPERIODS}}$$

Which becomes:

$$\sqrt[\text{FORWARDPERIODS}]{\frac{(1+r_{\text{LONGERPERIOD}})^{\text{LONGERPERIODS}}}{(1+r_{\text{SHORTERPERIOD}})^{\text{SHORTERPERIODS}}}} - 1 = r_{\text{FORWARDPERIOD}}$$

In most cases, though, the forward rate is for 1 period so we tend to use:

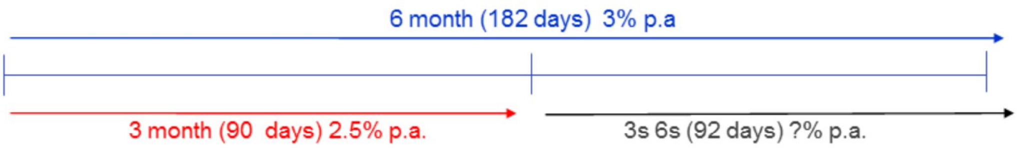
$$\frac{(1+r_{\text{LONGERPERIOD}})^{\text{LONGERPERIODS}}}{(1+r_{\text{SHORTERPERIOD}})^{\text{SHORTERPERIODS}}} - 1 = r_{\text{FORWARDPERIOD}}$$

Forward period rate form an average compound return (money markets)

The same principle apples to money markets, although we do not use powers. Instead we use day count fractions.

Illustration

- The 0 6s rate is 3% p.a.
- The 0 3s rate is 2.5% p.a.



- Assuming that money invested by either route (6 months or 3 months then three months again) must accrue to the same future value (rs here are annual rates):

$$\left(1+\left(r_{0\times 6}*\frac{182}{360}\right)\right)=\left(1+\left(r_{0\times 3}*\frac{90}{360}\right)\right)\times\left(1+\left(r_{3\times 6}*\frac{92}{360}\right)\right)$$

Re-arranging we get:

$$\frac{\left(1+\left(r_{0\times 6}*\frac{182}{360}\right)\right)}{\left(1+\left(r_{0\times 3}*\frac{90}{360}\right)\right)}=1+\left(r_{3\times 6}*\frac{92}{360}\right)$$

Then:

$$\left[\frac{\left(1+\left(r_{0\times 6}*\frac{182}{360}\right)\right)}{\left(1+\left(r_{0\times 3}*\frac{90}{360}\right)\right)}-1\right]*\frac{360}{92}=r_{3\times 6}$$