The Collector: Know Your Weapon—Part 1*

Espen Gaarder Haug

Trading options is War! For an option trader a pricing or hedging formula is just like a weapon. A soldier who has perfected her pistol shooting[1] can beat a guy with a machine gun that doesn't know how to handle it. Similarly, an option trader knowing the ins and outs of the Black–Scholes–Merton (BSM) formula[13] can beat a trader using a state-of-the-art stochastic volatility model. It comes down to two rules, just as in war. Rule number one: Know your weapon. Rule number two: Don't forget rule number one. In my fifteen+ years as a trader I have seen many a BSD[2] option trader getting confused with what the computer was spitting out. They often thought something was wrong with their computer system/implementation. Nothing was wrong, however, except their knowledge of their weapon. Before you move on to a more complex weapon (like a stochastic volatility model) you should make sure you know conventional equipment inside-out. In this bonus chapter I will not show the nerdy quants how to come up with the BSM formula using some new fancy mathematics—you don't need to know how to melt metal to use a gun. Neither is it a guideline on how to trade. It is meant rather like a short manual of how your weapon works in extreme situations. Real war (trading)—the pain, the pleasure, the adrenaline of winning and losing millions of dollars—can only be learned through real action. Now, the manual:

BSD trader Soldier, welcome to our trading team, this is your first day and I will instruct you about the Black–Scholes weapon.

New hired trader Hah, my Professor taught me probability theory, Itô calculus, and Malliavin calculus! I know everything about stochastic calculus and how to come up with the Black–Scholes formula.

BSD trader Soldier, you may know how to construct it, but that doesn't mean you know a shit about how it operates!

New hired trader I have used it for real trading. Before my Ph.D. I was a market maker in stock options for a year. Besides, why do you call me soldier? I was hired as an option trader.

BSD trader Soldier, you have not been in real war. In real war you often end up in extreme situations. That's when you need to know your weapon.

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*For this chapter I got a lot of ideas from the Wilmott forum. Thanks! And especially thanks to Alexander Adamchuk, Jørgen Haug, Hicham Mouline and James Ward for useful comments.

New hired trader I have read *Liar's Poker*, Hull's book, *Wilmott on Wilmott*, Taleb's *Dynamic Hedging*, Haug's formula collection. I know about Delta Bleed and all that stuff. I don't think you can tell me much more. I have even read *Fooled by Ran*...

BSD trader SHUT UP, SOLDIER! If you want to survive the first six months on this trading floor you better listen to me. On this team we don't allow any mistakes. We are warriors, trained in war!

New hired trader Yes, Sir!

BSD trader Good, let's move on to our business. Today I will teach you the basics of the Black–Scholes weapon.

1 Background on the BSM formula

Let me refresh your memory of the BSM formula

$$c = Se^{(b-r)T}N(d_1) - Xe^{-rT}N(d_2)$$
$$p = Xe^{-rT}N(-d_2) - Se^{(b-r)T}N(-d_1),$$

where

$$d_1 = \frac{\ln(S/X) + (b + \sigma^2/2)T}{\sigma\sqrt{T}},$$
$$d_2 = d_1 - \sigma\sqrt{T},$$

and

S	=	stock price
X	=	strike price of option
r	=	risk-free interest rate
b	=	cost-of-carry rate of holding the underlying security
T	=	time to expiration in years
σ	=	volatility of the relative price change of the underlying stock price
N(x)	=	the cumulative normal distribution function

2 Delta Greeks

2.1 Delta

As you know, the delta is the option's sensitivity to small movements in the underlying asset price.

$$\Delta_{call} = \frac{\partial c}{\partial S} = e^{(b-r)T}N(d_1) > 0$$
$$\Delta_{put} = \frac{\partial p}{\partial S} = -e^{(b-r)T}N(-d_1) < 0$$

Delta higher than unity I have many times over the years been contacted by confused commodity traders claiming something is wrong with their BSM implementation. What they observed was a spot delta higher than one.

As we get deep-in-the-money $N(d_1)$ approaches one, but it never gets higher than one (since it's a cumulative probability function). For a European call option on a non-dividend-paying stock the delta is equal to $N(d_1)$, so the delta can never go higher than one. For other options the delta term will be multiplied by $e^{(b-r)T}$. If this term is larger than one and we are deep-in-the-money we can get deltas considerably higher than one. This occurs if the cost-of-carry is larger than the interest rate, or if interest rates are negative. Figure 1 illustrates the delta of a call option. As expected the delta reaches above unity when time to maturity is large and the option is deep-in-the-money.

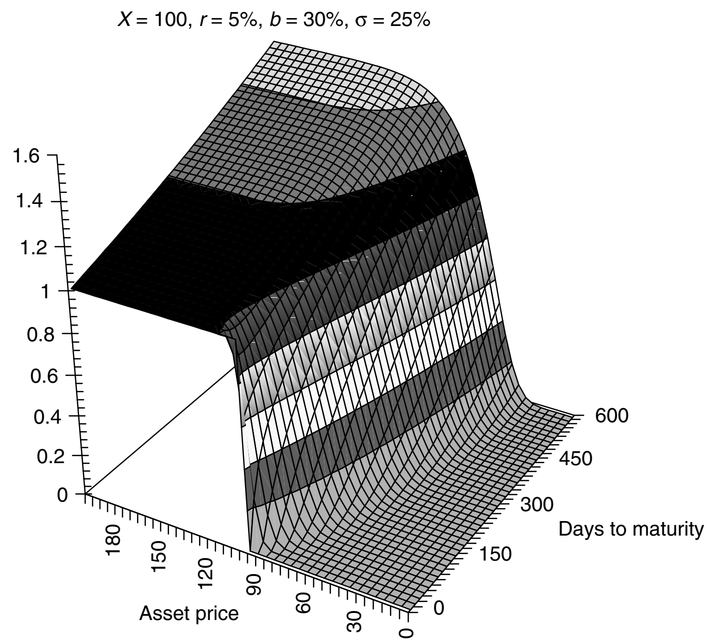

Figure 1: Spot delta

2.2 Delta mirror strikes and asset

For a put and call to have the same absolute delta value we can find the delta symmetric strikes as

$$X_p = \frac{S^2}{X_c} e^{(2b+\sigma^2)T}, \qquad X_c = \frac{S^2}{X_p} e^{(2b+\sigma^2)T}.$$

That is

$$\Delta_c(S, X_c, T, r, b, \sigma) = -\Delta_p\left(S, \frac{S^2}{X_c} e^{(2b+\sigma^2)T}, T, r, b, \sigma\right)$$

where X_c is the strike of the call and X_p is the strike of a put. These relationships are useful to determine strikes for delta neutral option strategies, especially for strangles, straddles, and butterflies. The weakness of this approach is that it works only for a symmetric volatility smile. In practice, however, you often only need an approximately delta neutral strangle. Moreover, volatility smiles are often more or less symmetric in the currency markets.

In the special case of a straddle-symmetric-delta-strike, described by Wystrup (1999), the formulas above can be simplified further to

$$X_c = X_p = Se^{(b+\sigma^2/2)T}.$$

Related to this relationship is the straddle-symmetric-asset-price. Given the identical strikes for a put and call, for what asset price will they have the same absolute delta value? The answer is

$$S = Xe^{(-b-\sigma^2/2)T}.$$

At this strike and delta-symmetric-asset-price the delta is $\frac{e^{(b-r)T}}{2}$ for a call, and $-\frac{e^{(b-r)T}}{2}$ for a put. Only for options on non-dividend paying stocks[3] ($b = r$) can we simultaneously have an absolute delta of 0.5 (50%) for a put and a call. Interestingly, the delta symmetric strike is also the strike given the asset price where the gamma and vega are at their maximums, *ceteris paribus*. The maximal gamma and vega, as well as the delta neutral strikes, are not at-the-money forward as I have noticed has been assumed by many traders. Moreover, an in-the-money put can naturally have absolute delta lower than 50% while an out-of-the-money call can have delta higher than 50%.

For an option that is at the straddle-symmetric-delta-strike the generalized BSM formula can be simplified to

$$c = \frac{Se^{(b-r)T}}{2} - Xe^{-rT}N(-\sigma\sqrt{T}),$$

and

$$p = Xe^{-rT}N(\sigma\sqrt{T}) - \frac{Se^{(b-r)T}}{2}.$$

At this point the option value will not change based on changes in cost of carry (dividend yield etc.). This is as expected as we have to adjust the strike accordingly.

2.3 Strike from delta

In several OTC (over-the-counter) markets options are quoted by delta rather than strike. This is a common quotation method in, for example, the OTC currency options market, where one typically asks for a delta and expects the sales person to return a price (in terms of volatility or pips) as well as the strike, given a spot reference. In these cases one needs to find the strike that corresponds to a given delta. Several option software systems solve this numerically using Newton–Raphson or bisection. This is actually not necessary, however. Using an inverted cumulative normal distribution $N^{-1}(\cdot)$ the strike can be derived from the delta analytically as described by Wystrup (1999). For a call option

$$X_c = S\exp[-N^{-1}(\Delta_c e^{(r-b)T})\sigma\sqrt{T} + (b + \sigma^2/2)T],$$

and for a put we have

$$X_p = S\exp[N^{-1}(-\Delta_p e^{(r-b)T})\sigma\sqrt{T} + (b + \sigma^2/2)T].$$

To get a robust and accurate implementation of this formula it is necessary to use an accurate approximation of the inverse cumulative normal distribution. I have used the algorithm of Moro (1995) with good results.

2.4 DdeltaDvol and DvegaDspot

DdeltaDvol: $\frac{\partial\Delta}{\partial\sigma}$ which mathematically is the same as DvegaDspot: $\frac{\partial \text{vega}}{\partial S}$, a.k.a. Vanna,[4] shows approximately how much your delta will change for a small change in the volatility, as well as how much your vega will change with a small change in the asset price:

$$\begin{aligned}\text{DdeltaDvol} &= \frac{\partial c}{\partial S\partial\sigma} = \frac{-e^{(b-r)T}d_2}{\sigma}n(d_1)\\ &= \frac{\partial p}{\partial S\partial\sigma} = \frac{e^{(b-r)T}d_2}{\sigma}n(d_1),\end{aligned}$$

where $n(x)$ is the standard normal density

$$n(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}.$$

One fine day in the dealing room my risk manager asked me to get into his office. He asked me why I had a big outright position in some stock index futures—I was supposed to do 'arbitrage trading'. That was strange as I believed I was delta neutral: long call options hedged with short index futures. I knew the options I had were far out-of-the-money and that their DdeltaDvol was very high. So I immediately asked what volatility the risk management used to calculate their delta. As expected, the volatility in the risk-management-system was considerably below the market and again was leading to a very low delta for the options. This example is just to illustrate how a feeling of your DdeltaDvol can be useful. If you have a high DdeltaDvol the volatility you use to compute your deltas becomes very important.[5]

Figure 2 illustrates the DdeltaDvol. As we can see the DdeltaDvol can assume positive and negative values. DdeltaDvol attains its maximal value at

$$S_L = Xe^{-bT-\sigma\sqrt{T}\sqrt{4+T\sigma^2}/2},$$

and attains its minimal value when

$$S_U = Xe^{-bT+\sigma\sqrt{T}\sqrt{4+T\sigma^2}/2}.$$

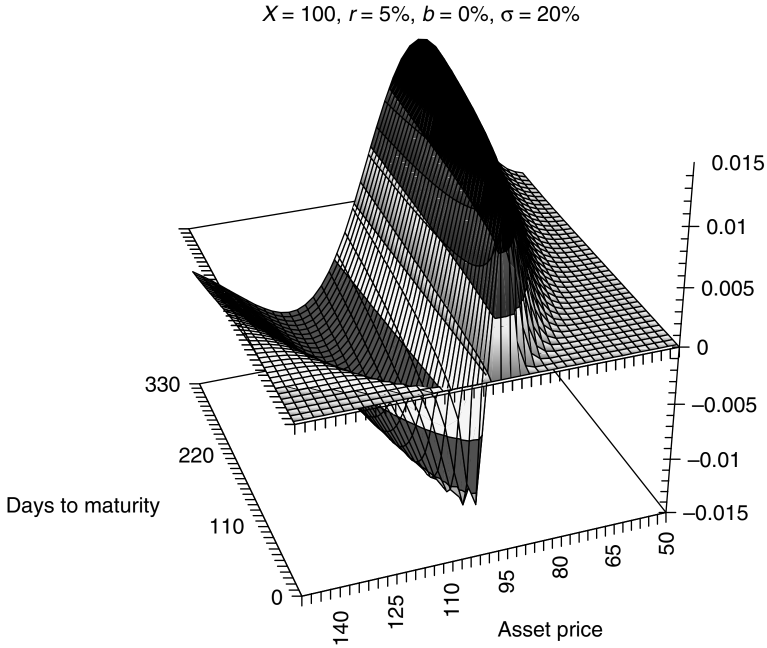

Figure 2: DdeltaDvol

Similarly, given the asset price, options with strikes X_L have maximum negative DdeltaDvol at

$$X_L = Se^{bT - \sigma\sqrt{T}\sqrt{4+T\sigma^2}/2},$$

and options with strike X_U have maximum positive DdeltaDvol when

$$X_U = Se^{bT + \sigma\sqrt{T}\sqrt{4+T\sigma^2}/2}.$$

One naturally can ask if these measures have any meaning. Black and Scholes assumed constant volatility, or at most deterministic volatility. Despite being theoretically inconsistent it might well be a good approximation. How good an approximation it is I leave up to you to find out or discuss at the Wilmott forum, www.wilmott.com. For more practical information about DvegaDspot or Vanna see Webb (1999).

2.5 DdeltaDtime, Charm

DdeltatDtime, a.k.a. Charm (Garman 1992) or Delta Bleed (a term used in the excellent book by Taleb 1997), is delta's sensitivity to changes in time,

$$-\frac{\partial \Delta_c}{\partial T} = -e^{(b-r)T}\left[n(d_1)\left(\frac{b}{\sigma\sqrt{T}} - \frac{d_2}{2T}\right) + (b-r)N(d_1)\right] \leq\geq 0,$$

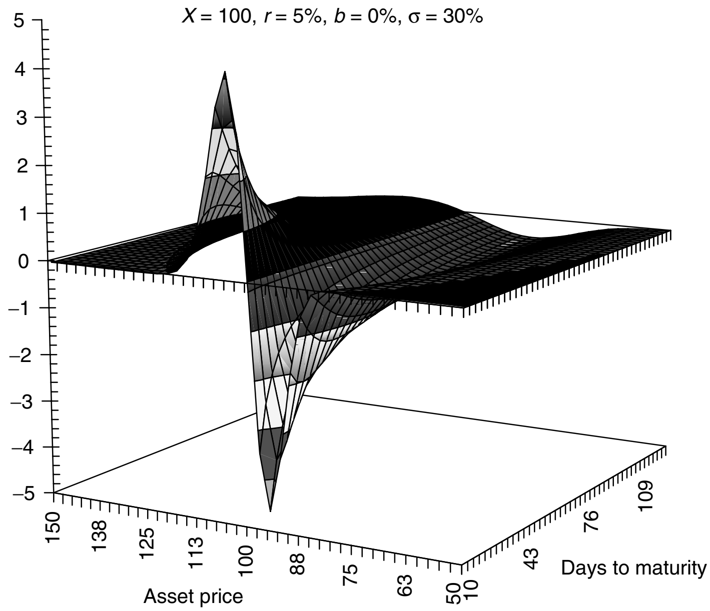

Figure 3: Charm

and

$$-\frac{\partial \Delta_p}{\partial T} = -e^{(b-r)T}\left[n(d_1)\left(\frac{b}{\sigma\sqrt{T}} - \frac{d_2}{2T}\right)\right.$$
$$\left. - (b-r)N(-d_1)\right] \leq\geq 0.$$

This Greek gives an indication of what happens with delta when we move closer to maturity. Figure 3 illustrates the Charm value for different values of the underlying asset and different times to maturity.

As Nassim Taleb points out one can have both forward and backward bleed. He also points out the importance of taking into account how expected changes in volatility over the given time period will affect delta. I am sure most readers already have his book in their collection (if not, order it now!). I will therefore not repeat all his excellent points here.

All partial derivatives with respect to time have the advantage over other Greeks in that we know which direction time will move. Moreover, we know that time moves at a constant rate. This is in contrast, for example, to the spot price, volatility, or interest rate.[6]

2.6 Elasticity

The elasticity of an option, a.k.a. the option leverage, omega, or lambda, is the sensitivity in percent to a percent movement in the underlying asset price. It is given by

$$\Lambda_{call} = \Delta_{call}\frac{S}{call} = e^{(b-r)T}N(d_1)\frac{S}{call} > 1$$

$$\Lambda_{put} = \Delta_{put}\frac{S}{put} = -e^{(b-r)T}N(-d_1)\frac{S}{put} < 0$$

The option's elasticity is a useful measure on its own, as well as to estimate the volatility, beta, and expected return from an option.

Option volatility The option volatility σ_o can be approximated using the option elasticity. The volatility of an option over a short period of time is approximately equal to the elasticity of the option multiplied by the stock volatility σ.[7]

$$\sigma_o \approx \sigma|\Lambda|.$$

Option beta The elasticity is also useful to compute the option's beta. If asset prices follow geometric Brownian motions the continuous-time capital asset pricing model of Merton (1971) holds. Expected asset returns then satisfy the CAPM equation

$$E[\text{return}] = r + E[r_m - r]\beta_i$$

where r is the risk-free rate, r_m is the return on the market portfolio, and β_i is the beta of the asset. To determine the expected return of an option we need the option's beta. The beta of a call is given by see for instance Black and Scholes (1973)

$$\beta_c = \frac{S}{call}\Delta_c\beta_S,$$

where β_S is the underlying stock beta. For a put the beta is

$$\beta_p = \frac{S}{put}\Delta_p\beta_S.$$

For a beta neutral option strategy the expected return should be the same as the risk-free rate (at least in theory).

Option Sharpe ratios As the leverage does not change the Sharpe (1966) ratio, the Sharpe ratio of an option will be the same as that of the underlying stock,

$$\frac{\mu_o - r}{\sigma_o} = \frac{\mu_S - r}{\sigma}$$

where μ_o is the return of the option, and μ_S is the return of the underlying stock. This relationship indicates the limited usefulness of the Sharpe ratio as a risk-return measure for options (?). Shorting a lot of deep out-of-the-money options will likely give you a 'nice' Sharpe ratio, but you are almost guaranteed to blow up one day (with probability one if you live long enough). An interesting question here is should you use the same volatility for all strikes? For instance deep-out-of-the-money stock options typically trade for much higher implied volatility than at-the-money options. Using the volatility smile when computing Sharpe ratios for deep out-of-the-money options can also possibly make the Sharpe ratio work better for options. McDonald (2002) offers a more detailed discussion of option Sharpe ratios.

3 Gamma Greeks

3.1 Gamma

Gamma is the delta's sensitivity to small movements in the underlying asset price. Gamma is identical for put and call options, *ceteris paribus*, and is given by

$$\Gamma_{call,put} = \frac{\partial^2 c}{\partial S^2} = \frac{\partial^2 p}{\partial S^2} = \frac{n(d_1)e^{(b-r)T}}{S\sigma\sqrt{T}} > 0$$

This is the standard gamma measure given in most textbooks (Haug 1997, Hull 2000, Wilmott 2000).

3.2 Maximal gamma and the illusions of risk

One day in the trading room of a former employer of mine, one of the BSD traders suddenly got worried over his gamma. He had a long dated deep-out-of-the-money call. The stock price had been falling, and the further the out-of-the-money the option went the lower the gamma he expected. As with many option traders he believed the gamma was largest approximately at-the-money-forward. Looking at his Bloomberg screen, however, the further out-of-the-money the call went the higher his gamma got. Another BSD was coming over, and they both tried to come up with an explanation for this. Was there something wrong with Bloomberg?

In my own home-built system I was often playing around with three- and four-dimensional charts of the option Greeks, and I already knew that gamma doesn't attain its maximum at-the-money-forward (four dimensions? a dynamic three-dimensional graph). I didn't know exactly where it attained its maximum, however. Instead of joining the BSD discussion, I did a few computations in Mathematica. A few minutes later, after double checking my calculations, I handed over an equation to the BSD traders showing exactly where the BSM gamma would be at its maximum.

How good is the rule of thumb that gamma is largest for at-the-money or at-the-money-forward options? Given a strike price and time to maturity, the gamma is at maximum when the asset price is[8]

$$S_{\overline{\Gamma}} = Xe^{(-b-3\sigma^2/2)T}.$$

Given the asset price and time to maturity, gamma is maximal when the strike is

$$X_{\overline{\Gamma}} = Se^{(b+\sigma^2/2)T}.$$

Confused option traders are bad enough, confused risk management is a pain in the behind. Several large investment firms impose risk limits on how much gamma you can have. In the equity market it is common to use the standard textbook approach to compute gamma, as shown above. Putting on a long-term call (put) option that later is deep-out-of-the-money (in-the-money) can blow up the gamma risk limits, even if you actually have close to zero gamma risk. The high gamma risk for long dated deep-out-of-the-money options typically is only an illusion. This illusion of risk can be avoided by looking at percentage changes in the underlying asset (gammaP), as is typically done for FX options.

Saddle gamma Alexander (Sasha) Adamchuk was the first to make me aware of the fact that gamma has a saddle point.[9] The saddle point is attained for the time[10]

$$T_S = \frac{1}{2(\sigma^2 + 2b - r)},$$

and at asset price

$$S_{\overline{\Gamma}} = Xe^{(-b-3\sigma^2/2)T_S}.$$

The gamma at this point is given by

$$\Gamma_S = \Gamma(S_{\overline{\Gamma}}, T_S) = \frac{\sqrt{\frac{e}{\pi}}\sqrt{\frac{2b-r}{\sigma^2}+1}}{X}$$

Many traders get surprised by this feature of gamma—that gamma is not necessarily decreasing with longer time to maturity. The maximum gamma for a given strike price is first decreasing until the saddle gamma point, then increasing again, given that we follow the edge of the maximal gamma asset price.

Figure 4 shows the saddle gamma. The saddle point is between the two gamma 'mountain' tops. This graph also illustrates one of the big limitations in the textbook gamma definition, which is actually in use by many option systems and traders. The gamma increases dramatically when we have long time to maturity and the asset price is close to zero. How can the gamma be larger than for an option closer to at-the-money? Is the real gamma risk that big? No, this is in most

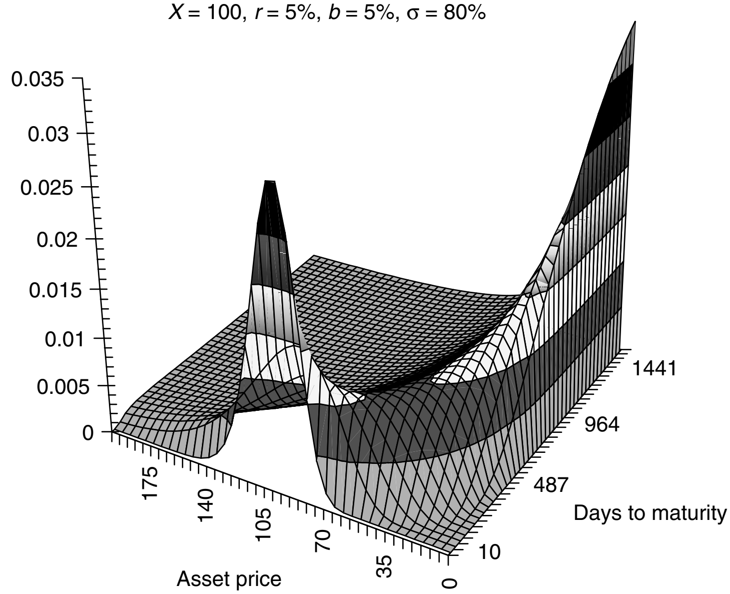

Figure 4: Saddle gamma

cases simply an illusion, due to the above unmotivated definition of gamma. Gamma is typically defined as the change in delta for a one unit change in the asset price. When the asset price is close to zero a one unit change is naturally enormous in percent of the asset price. In this case it is also highly unlikely that the asset price will increase by one dollar in an instant. In other words, the gamma measurement should be reformulated, as many option systems already have done. It makes far more sense to look at percentage moves in the underlying than unit moves. To compare gamma risk from different underlyings one should also adjust for the volatility in the underlying.

3.3 GammaP

As already mentioned, there are several problems with the traditional gamma definition. A better measure is to look at percentage changes in delta for percentage changes in the underlying,[11] for example a 1% point change in underlying. With this definition we get for both puts and calls (gamma percent)

$$\Gamma_P = \frac{S\Gamma}{100} > 0. \tag{1}$$

GammaP attains a maximum at an asset price of

$$S_{\overline{\Gamma}_P} = Xe^{(-b-\sigma^2/2)T}.$$

Alternatively, given the asset price the maximal Γ_P occurs at strike

$$X_{\overline{\Gamma}_P} = Se^{(b+\sigma^2/2)T}.$$

Interestingly, this is also where we have a straddle symmetric asset price as well as maximal gamma. This implies that a delta neutral straddle has maximal Γ_P. In most circumstances going from measuring the gamma risk as Γ_P instead of gamma we avoid the illusion of a high gamma risk when the option is far out-of-the-money and the asset price is low. Figure 5 is an illustration of this, using the same parameters as in Figure 4.

If the cost-of-carry is very high it is still possible to experience high Γ_P for deep-out-of-the-money call options with a low asset price and a long time to maturity. This is because a high cost-of-carry can make the ratio of a deep-out-of-the money call to the spot close to the at-the-money-forward. At this point the spot-delta will be close to 50% and so the Γ_P will be large. This is not an illusion of gamma risk, but a reality. Figure 6 shows Γ_P with the same parameters as in Figure 5, with cost-of-carry of 60%.

To makes things even more complicated, the high Γ_P we can have for deep-out-of-the-money calls (in-the-money puts) is the only case when we are dealing with spot gammaP (change in spot delta). We can avoid this by looking at future/forward gammaP. However, if you hedge with spot, then spot gammaP is the relevant metric. Only if you hedge with the future/forward the forward gammaP is the relevant metric. The forward gammaP we have when the cost-of-carry is set to zero, and the underlying asset is the futures price.

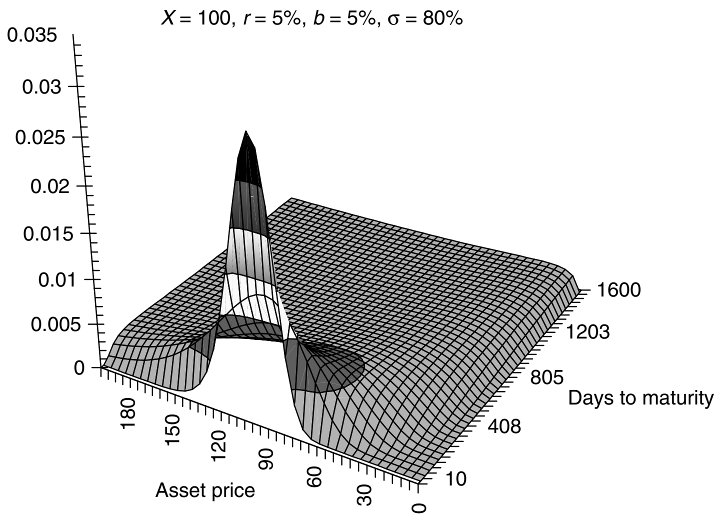

Figure 5: GammaP

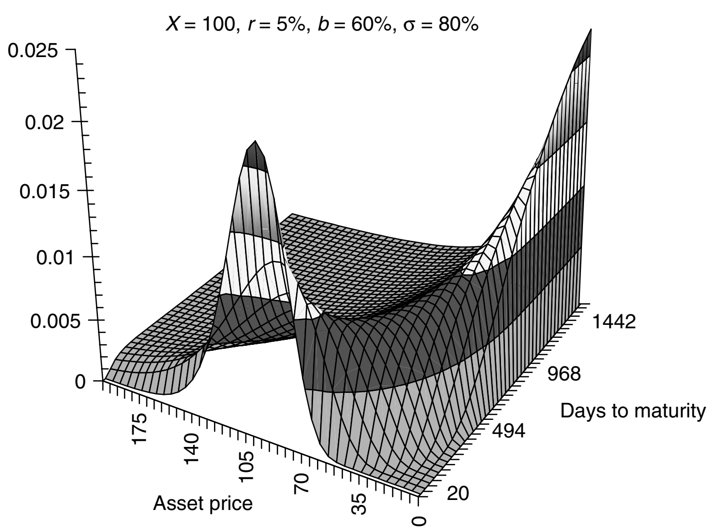

Figure 6: Saddle gammaP

3.4 Gamma symmetry

Given the same strike the gamma is identical for both put and call options. Although this equality breaks down when the strikes differ, there is a useful put and call gamma symmetry. The put-call symmetry of Bates (1991) and Carr and Bowie (1994) is given by

$$c(S, X, T, r, b, \sigma) = \frac{X}{Se^{bT}} p\left(S, \frac{(Se^{bT})^2}{X}, T, r, b, \sigma\right).$$

This put-call value symmetry yields the gamma symmetry; however, the gamma symmetry is more general as it is independent of whether the option is a put or call, for example it could be two calls, two puts, or a put and a call.

$$\Gamma(S, X, T, r, b, \sigma) = \frac{X}{Se^{bT}}\Gamma(S, \frac{(Se^{bT})^2}{X}, T, r, b, \sigma).$$

Interestingly, the put-call symmetry also gives us vega and cost-of-carry symmetries, and in the case of zero cost-of-carry also theta and rho symmetry. Delta symmetry, however, is not obtained.

3.5 DgammaDvol, Zomma

DgammaDvol, a.k.a. Zomma, is the sensitivity of gamma with respect to changes in implied volatility. In my view, DgammaDvol is one of the more important Greeks for options trading. It is given by

$$\begin{aligned}\text{DgammaDvol}_{call,put} &= \frac{\partial \Gamma}{\partial \sigma} \\ &= \Gamma\left(\frac{d_1 d_2 - 1}{\sigma}\right) \leq\geq 0.\end{aligned}$$

For the gammaP we have DgammaPDvol

$$\text{DgammaPDvol}_{call,put} = \Gamma_P\left(\frac{d_1 d_2 - 1}{\sigma}\right) \leq\geq 0$$

where Γ is the textbook Gamma of the option.

For practical purposes, where one typically wants to look at DgammaDvol for a one unit volatility change, for example from 30% to 31%, one should divide the DGammaDVol by 100. Moreover, DgammaDvol and DgammaPDvol are negative for asset prices between S_L and S_U and positive outside this interval, where

$$\begin{aligned}S_L &= Xe^{-bT-\sigma\sqrt{T}\sqrt{4+T\sigma^2}/2}, \\ S_U &= Xe^{-bT+\sigma\sqrt{T}\sqrt{4+T\sigma^2}/2}.\end{aligned}$$

For a given asset price the DgammaDvol and DgammaPDvol are negative for strikes between

$$X_L = Se^{bT-\sigma\sqrt{T}\sqrt{4+T\sigma^2}/2}$$

and

$$X_U = Se^{bT+\sigma\sqrt{T}\sqrt{4+T\sigma^2}/2},$$

and positive for strikes above X_U or below X_L, *ceteris paribus*. In practice, these points will change with other variables and parameters. These levels should, therefore, be considered good approximations at best.

In general you want positive DgammaDvol—especially if you don't need to pay for it (flat volatility smile). In this respect DgammaDvol actually offers a lot of intuition for how stochastic volatility should affect the BSM values (?). Figure 7 illustrates this point. The DgammaDvol is positive for deep-out-of-the-money options, outside the S_L and S_U interval. For at-the-money options and slightly in- or out-of-the-money options the DgammaDvol is negative. If the volatility is stochastic and uncorrelated with the asset price then this offers a good indication for which strikes you should use higher/lower volatility when deciding on your volatility smile. In the case of volatility correlated with the asset price this naturally becomes more complicated.

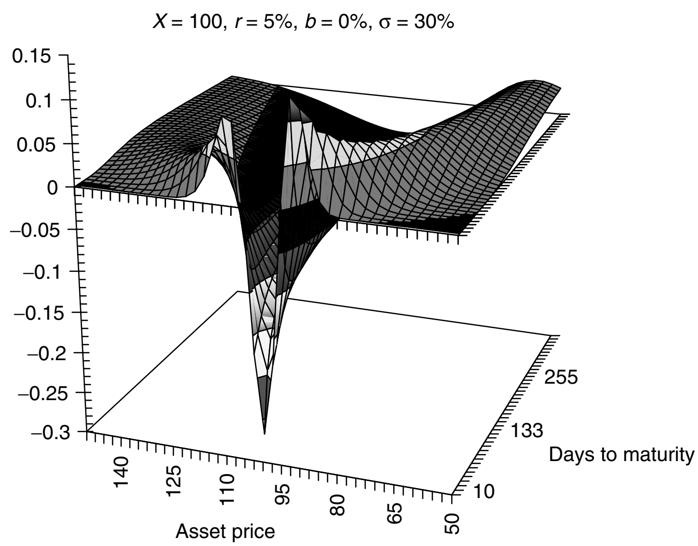

Figure 7: DgammaDvol

3.6 DgammaDspot, Speed

I have heard rumors about how being on speed can help see higher dimensions that are ignored or hidden for most people. It should be of little surprise that in the world of options the third derivative of the option price with respect to spot, known as Speed, is ignored by most people. Judging from his book, Nassim Taleb is also a fan of higher-order Greeks. There he mentions Greeks of up to seventh order.

Speed was probably first mentioned by Garman (1992),[12] for the generalized BSM formula we get

$$\frac{\partial^3 c}{\partial S^3} = -\frac{\Gamma\left(1 + \frac{d_1}{\sigma\sqrt{T}}\right)}{S}.$$

A high Speed value indicates that the gamma is very sensitive to moves in the underlying asset. Academics typically claim that third- or higher-order 'Greeks' are of no use. For an option trader, on the other hand, it can definitely make sense to have a sense of an option's Speed. Interestingly, Speed is used by Fouque *et al*. (2000) as a part of a stochastic volatility model adjustment. More to the point, Speed is useful when gamma is at its maximum with respect to the asset price. Figure 8 shows the graph of Speed with respect to the asset price and time to maturity.

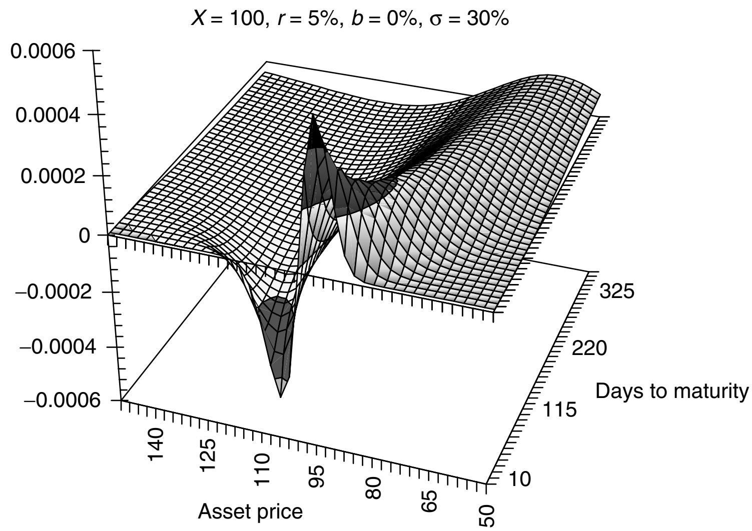

Figure 8: Speed

For Γ_P we have an even simpler expression for Speed, that is SpeedP (Speed for percentage gamma)

$$\text{SpeedP} = -\Gamma \frac{d_1}{100\sigma\sqrt{T}}.$$

3.7 DgammaDtime, Colour

The change in gamma with respect to small changes in time to maturity, DGammaDtime, a.k.a. GammaTheta or Colour (Garman 1992), is given by (assuming we get closer to maturity):

$$\begin{aligned}-\frac{\partial \Gamma}{\partial T} &= \frac{e^{(b-r)T} n(d_1)}{S\sigma\sqrt{T}}\left(r - b + \frac{bd_1}{\sigma\sqrt{T}} + \frac{1 - d_1 d_2}{2T}\right)\\ &= \Gamma\left(r - b + \frac{bd_1}{\sigma\sqrt{T}} + \frac{1 - d_1 d_2}{2T}\right) \leq\geq 0.\end{aligned}$$

Divide by 365 to get the sensitivity for a one day move. In practice one typically also takes into account the expected change in volatility with respect to time. If you, for example on Friday are wondering how your gamma will be on Monday you typically will also assume a higher implied volatility on Monday morning. For Γ_P we have DgammaPDtime

$$-\frac{\partial \Gamma_P}{\partial T} = \Gamma_P\left(r - b + \frac{bd_1}{\sigma\sqrt{T}} + \frac{1 - d_1 d_2}{2T}\right) \leq\geq 0.$$

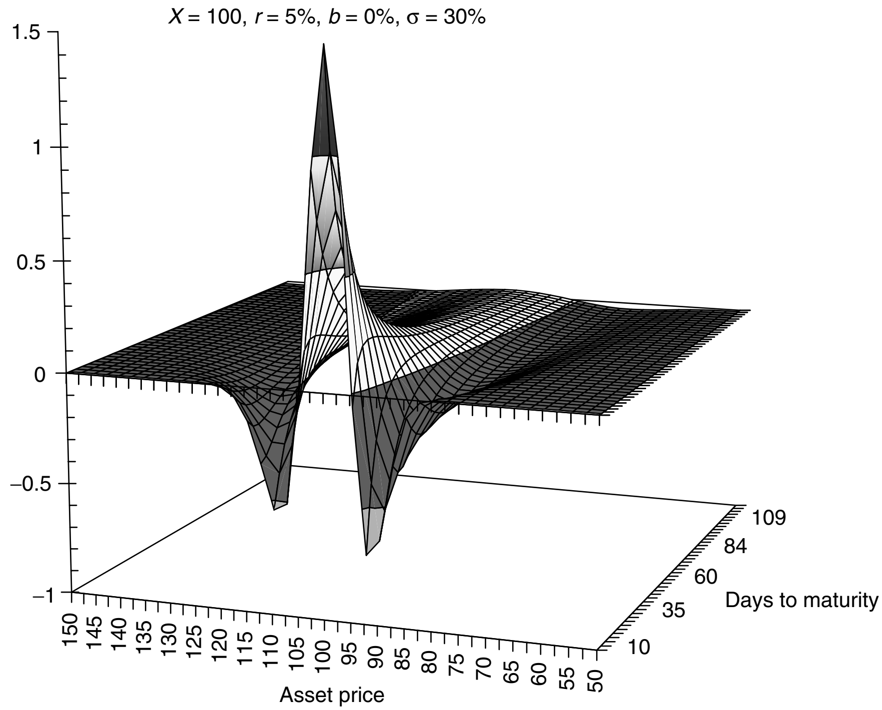

Figure 9: DgammaDtime

Figure 9 illustrates the DgammaDtime of an option with respect to varying asset price and time to maturity.

4 Numerical Greeks

So far we have looked only at analytical Greeks. A frequently used alternative is to use numerical Greeks. Most first-order partial derivatives can be computed by the two-sided finite difference method

$$\frac{c(S + \Delta S, X, T, r, b, \sigma) - c(S - \Delta S, X, T, r, b, \sigma)}{2\Delta S}.$$

In the case of derivatives with respect to time, we know what direction time will move and it is more accurate (for what is happening in the 'real' world) to use a backward derivative

$$\Theta \approx \frac{c(S, X, T, r, b, \sigma) - c(S, X, T - \Delta T, r, b, \sigma)}{\Delta T}.$$

Numerical Greeks have several advantages over analytical ones. If for instance we have a sticky delta volatility smile then we can also change the volatilities accordingly when calculating the numerical delta. (We have a sticky delta volatility smile when the shape of the volatility smile sticks to the deltas but not to the strike; in other words the volatility for a given strike will move as the underlying moves.)

$$\Delta_c \approx \frac{c(S + \Delta S, X, T, r, b, \sigma_1) - c(S - \Delta S, X, T, r, b, \sigma_2)}{2\Delta S}.$$

Numerical Greeks are moreover model independent, while the analytical Greeks presented above are specific to the BSM model.

For gamma and other second derivatives, $\frac{\partial^2 f}{\partial x^2}$, for example DvegaDvol, we can use the central finite difference method

$$\Gamma \approx \frac{c(S + \Delta S, \ldots) - 2c(S, \ldots) + c(S - \Delta S, \ldots)}{\Delta S^2}.$$

If you are very close to maturity (a few hours) and you are approximately at-the-money the analytical gamma can approach infinity, which is naturally an illusion of your real risk. The reason is simply that analytical partial derivatives are accurate only for infinite small changes, while in practice one sees only discrete changes. The numerical gamma solves this problem and offers a more accurate gamma in these cases. This is particularly true when it comes to barrier options (Taleb 1997).

For Speed and other third-order derivatives, $\frac{\partial^3 f}{\partial x^3}$, we can, for example, use the following approximation

$$\begin{aligned}\text{Speed} \approx \frac{1}{\Delta S^3}&[c(S + 2\Delta S, \ldots) - 3c(S + \Delta S, \ldots)\\ &+3c(S, \ldots) - c(S - \Delta S, \ldots)].\end{aligned}$$

What about mixed derivatives, $\frac{\partial f}{\partial x \partial y}$, for example DdeltaDvol and Charm. This can be calculated numerical by DdeltaDvol

$$\begin{aligned}&\approx \frac{1}{4\Delta S \Delta\sigma}[c(S + \Delta S, \ldots, \sigma + \Delta\sigma)\\ &-c(S + \Delta S, \ldots, \sigma - \Delta\sigma) - c(S - \Delta S, \ldots, \sigma + \Delta\sigma)\\ &+c(S - \Delta S, \ldots, \sigma - \Delta\sigma)].\end{aligned}$$

In the case of DdeltaDvol one would ‘typically’ divide it by 100 to get the ‘right’ notation.

End Part 1

BSD trader That is enough for today soldier.

New hired trader Sir, I learned a few things today. Can I start trading now?

BSD trader We don’t let fresh soldiers play around with ammunition (capital) before they know the basics of a conventional weapon like the Black–Scholes formula.

New hired trader Understood, Sir!

BSD trader Next time I will tell you about Vega-kappa, probability Greeks and some other stuff. Until then you are dismissed! Now bring me a double cheeseburger with a lot of fries!

New hired trader Yes, Sir!

FOOTNOTES & REFERENCES

1. The author was among the best pistol shooters in Norway.
2. If you don't know the meaning of this expression, BSD, then it's high time you read Michael Lewis' *Liar's Poker*.
3. And naturally also for commodity options in the special case where cost-of-carry equals r.
4. I wrote about the importance of this Greek variable back in 1992. It was my second paper about options, and my first written in English. Well, it got rejected. What could I expect? Most people totally ignored DdeltaDvol at that time and the paper has collected dust since then.
5. An important question naturally is what volatility you should use to compute your deltas. I will not give you an answer to that here, but there has been discussions on this topic at www.wilmott.com.
6. This is true only because everybody trading options at Mother Earth moves at about the same speed, and is affected by approximately the same gravity. In the future, with huge space stations moving with speeds significant to that of the speed of light, this will no longer hold true. See Haug (2003a and b) for some possible consequences.
7. This approximation is used by Bensoussan *et al.* (1995) for an approximate valuation of compound options.
8. Rubinstein (1990) indicates in a footnote that this maximum curvature point possibly can explain why the greatest demand for calls tend to be just slightly out-of-the-money.
9. Described by Adamchuk at the Wilmott forum www.wilmott.com February 6, 2002, http://www.wilmott.com/310/messageview.cfm?catid=4&threadid=664&highlight_key=y&keyword1=vanna and even earlier on his page http://finmath.com/Chicago/NAFTCORP/Saddle_Gamma.html
10. It is worth mentioning that T_S must be larger than zero for the gamma to have a saddle point, that means b must be larger then $\frac{r-\sigma^2}{2}$, and r must be smaller than $\sigma^2 + 2b$.
11. Wystrup (1999) also describes how this redefinition of gamma removes the dependence on the spot level S. He calls it 'traders gamma'. This measure of gamma has for a long time been popular, particularly in the FX market, but is still absent in options textbooks.
12. However, he was too 'lazy' to give us the formula so I had to do the boring derivation myself.
13. In the light of Know Your Weapon III it is actually not the Black–Scholes–Merton formula traders use, but the description of many Greeks in this chapter is probably even more important for the Market Formula.

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