CQF Elective - Advanced Portfolio Management

Sébastien Lleo

Module 6

By the end of this course, you should be able to...

Black -Litterman

- Formulate and solve a dynamic portfolio optimization problem using stochastic control.
- Combine views with market data using filtering to determine the necessary parameters.
- Understand the importance of behavioural biases and be able to debias them
- Understand the Kelly portfolio and develop new insights into portfolio risk management.
- Implementation a dynamic portfolio optimization model.

-7

Portfolio Selection is a Two-Stage Process...

Figure : Does Harry Markowitz need any introduction?

At the very beginning of his seminal paper, *Portfolio Selection*, Markowitz (1952) states:

> *The process of selecting a portfolio may be divided into two stages. The first stage starts with observation and experience and ends with beliefs about the future performances of available securities. The second stage starts with the relevant beliefs about future performances and ends with the choice of portfolio. This paper is concerned with the second stage.*

The morning session is mostly about *Stage 2 - optimization* and mathematical modelling.

This afternoon, we will discuss *Stage 1 - Formulation* and implementation issues.

The Investment Decision

The central question for investment management practitioners is: where should I invest my money?

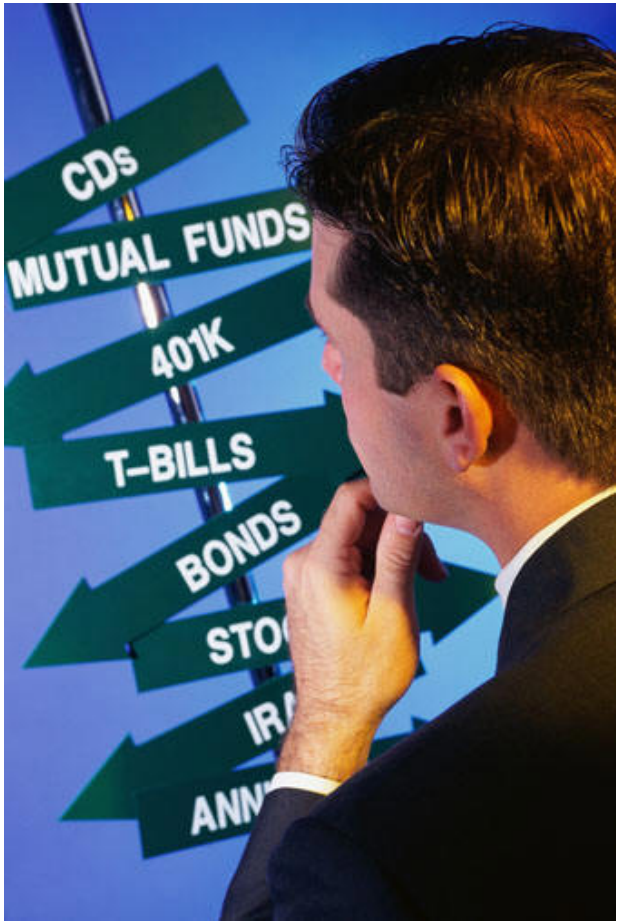
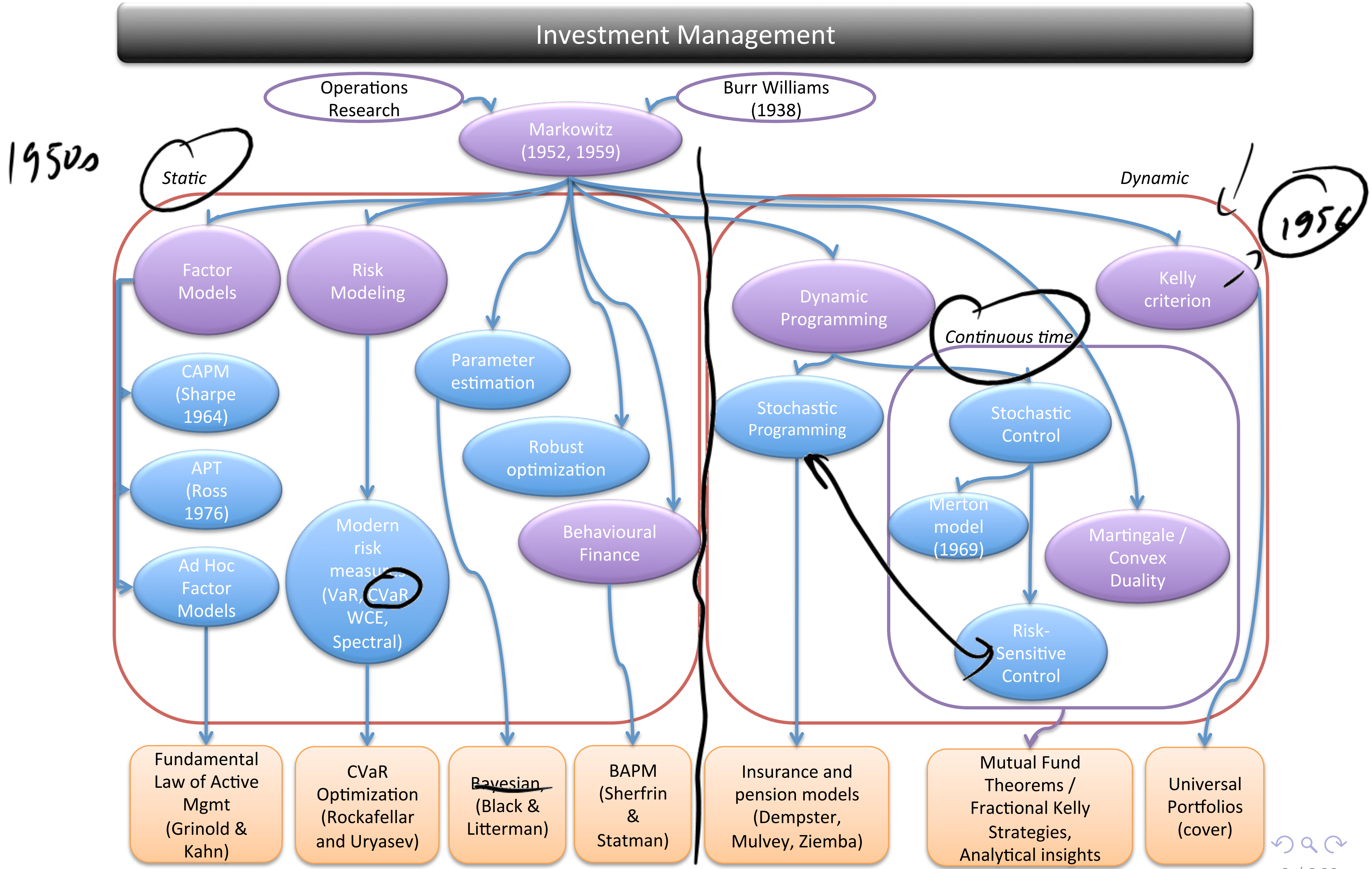


Figure : One BIG decision... Many models!



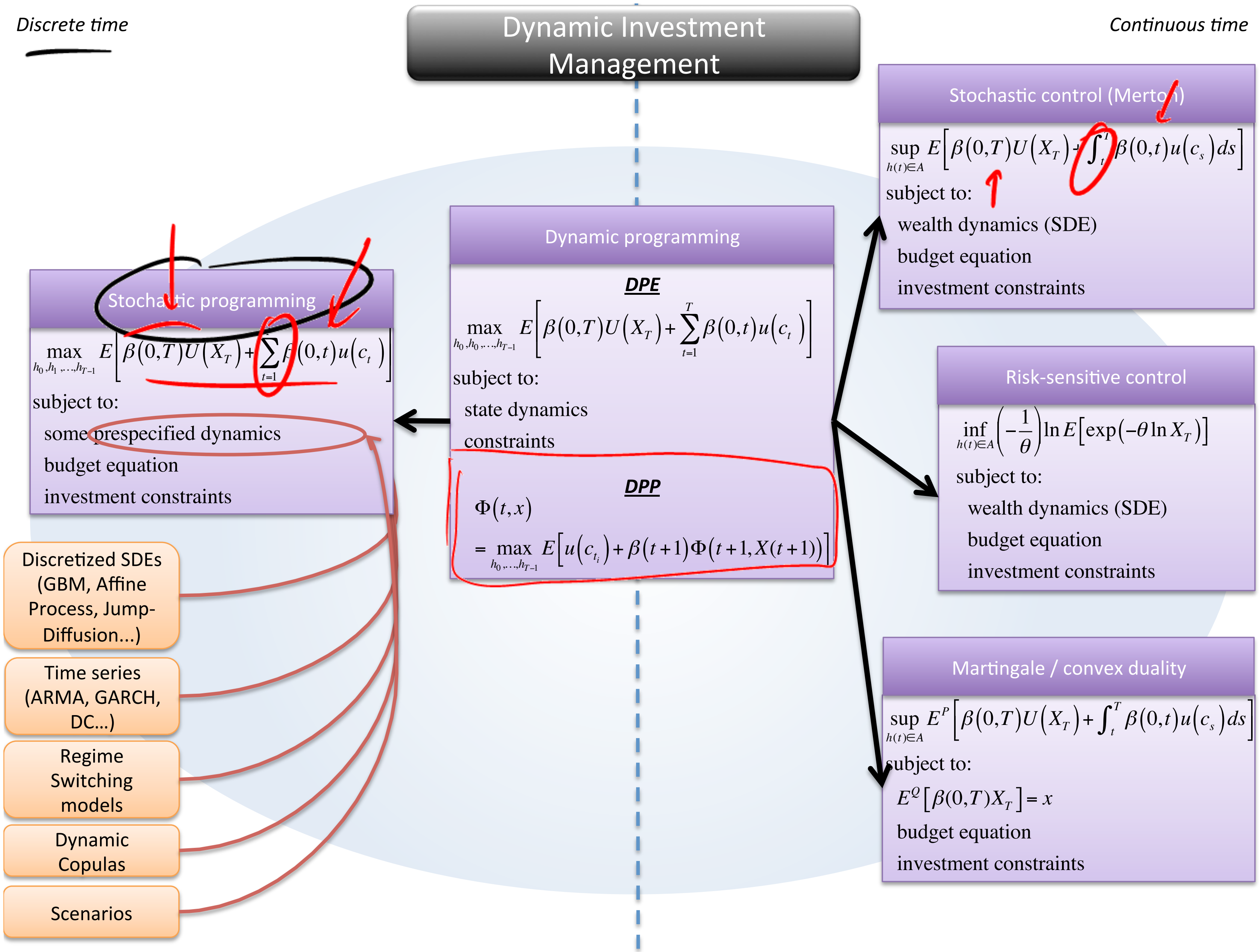
In Module 2, we saw the original static approach developed by Markowitz and Sharpe to solve the portfolio selection problem, as well as a number of its extensions.

One of the main drawbacks of the Markowitz approach is precisely that it is static. Several dynamic approaches have been developed through the years to address this weakness:

1. The most established approach calls for the application of an important technique called **stochastic control**, which stands at the crossroad of optimization, stochastic analysis and PDE theory.
2. The **stochastic programming** approach is often used to solve the type of long-term investment problems that insurance companies and pension funds face.
3. The continuous time **martingale approach** proves popular in academic circles.
4. In recent years, new techniques, such as **regime switching** models have emerged.

Stochastic Programming guys can do what the Stochastic Control guys do.

The inverse is not the same.



In this course, we build on our knowledge of stochastic processes to introduce **dynamic asset management** in continuous time via stochastic control.

Part 1: Solving The Investment Management Problem With Stochastic Control

In mathematics you don't understand things. You just get used to them.

- John von Neumann (1903-1957)

In this part, we will see...

CUTE — Toy Model. Doesn't really work!

- How to formulate an investment management problem in a continuous time setting;
- What stochastic control is about;
- How to use stochastic control to solve dynamic portfolio selection problems:
  - the Merton problem with 1 risky asset;
  - the Merton problem with m risky assets.

  1969
- What risk-sensitive control is about;
- What risk-sensitive investment management can do for you today.

1. The Investment Problem

You Do not have a one size fits all problem!

Our first objective of the day is to formulate an investment decision problem in a continuous time setting.

To achieve this objective, we need to understand the nature of the investment problem:

START HERE:

1. ▶ Who are the investors?
2. ▶ What are their objectives and degree of aversion to risk?
3. ▶ What are their time horizon?
4. ▶ What are their investment universe, that is the assets and asset classes the investors have access to?

We need to understand the nature of INVESTMENT DECISION PROBLEM - Continuous time setting
In order to achieve this objective, we need to understand the nature of the problem.
Very different goals & objectives from Hedge fund, Insurance co. w/ portfolio of assets/liabilities,
or if you're investing for 401k, pension fund. or Retirement planning. Different investment problems
and different mathematical problems

Key Insight

In investment management, you do not have a one-size-fit-all type of model that you can apply regardless of the portfolio or investor. Each investor is slightly different: you need to understand their needs, objectives and constraints.

For example, the challenges of a defined benefit pension plan in a software start up are very different from that of a defined benefit plan in a mature industrial firm.

Although the models will be different, the modelling principles are the same. They can be applied easily to design new models.

£ v

To keep the discussion simple at this stage, we consider an asset manager, Irene, who manages a fund with assets under management valued at v. She has access to an investment universe with $m \geq 1$ stocks and can also trade one money market instrument.

Irene's objective is to find the best possible investment strategy over a time horizon of T years[1].

T = 10 years

Irene does not wish to commit to a fixed investment strategy for the whole T years. Instead, she wants to maintain a day-to-day control and have the ability to change the investment strategy at any point in time.

Let's consider a hypothetical case:
We have an investment mgr, named Irene who has
ASSETS UNDER MANAGEMENT In (V \$) or V GPB - with access to investment universe of (m stocks), and can park all her money in REPO mkt if she wants. She has a time horizon of T= 10 (years). She wants to be active mgr. day to day control of the strategy.

that's why we need a DYNAMIC MODEL — and we're not dealing with benchmarks at moment.

[1] To keep the idea flowing, we will not discuss investment benchmarks in today's course. All the ideas and models presented today extend naturally to include investment benchmarks.

1.1 The Investment Universe

If you have a benchmark, instead of looking at returns — you look at excess returns. instead of looking at ABSOLUTE RISK. You look at RELATIVE RISK. You look for example, tracking error. You look at your ACTIVE RISK AVERSION, ACTIVE RISK PORTFOLIO Which will tell you how close or far-away from the benchmark you are.

As far as the investment universe is concerned, we could start with a ‘simple’ **Black-Scholes type model**.

W_j(t)

W_j' are independ B-M

The investor has access to

- $m \geq 1$ stock-like risky asset $S_i(t), i = 1, \ldots, n$ whose price is modelled as a geometric Brownian motion (GBM),

  We differ from Black Scholes, in the sense, we want to make sure we do not have any redundant assets like SPY, S&P 500, or options on S&P 500. because they're similar. Merton-Samuelson formula.

  asset i

$$\frac{dS_i(t)}{S_i(t)} = \mu_i dt + \sum_{j=1}^{m} \sigma_{ij} dW_j(t), \quad S_i(0) = s_i$$

  drift

  where the μ_i's and $\sigma_{i,j}$'s are constant, and the $W_j(t)$ are m independent Brownian motions on $(\Omega, \{\mathcal{F}_t\}, \mathcal{F}, \mathbb{P})$; Mu is constant, Sigma is constant. and you have W_j(t)

- A risk-free asset $S_0(t)$ which follows the dynamics given by

$$\frac{dS_0(t)}{S_0(t)} = rdt, \quad S_0(0) = s_0$$

  where $r \geq 0$ is the risk-free rate. r is constant

If you have a benchmark. Instead of looking at log return, you look at log EXCESS return and you get it.

We assume that:

Assumption

The matrix Σ is positive definite.

Remark: this assumption implies that the matrix Σ has rank m. Stated otherwise, the risk structure of any asset cannot be replicated by trading the other $m - 1$ assets as this would indicate that we either have a redundant asset or there is an arbitrage opportunity.

This is the investment universe that Robert Merton proposed when he introduced stochastic control to portfolio management in the late 1960s (Merton, 1969, 1971, 1992).

- Since then, the Merton model has become the dominent continuous-time model!

Figure : Robert Merton

W is m - dimensional

To Recap... Merton Model

- $m \geq 1$ stock-like risky asset $S_i(t), i = 1, \dots, n$ whose price is modelled as a geometric Brownian motion (GBM),

$$\frac{dS_i(t)}{S_i(t)} = \overset{d}{\mu_i} dt + \sum_{j=1}^{m} \sigma_{ij} dW_j(t), \quad S_i(0) = s_i$$

  where the μ_i's and $\sigma_{i,j}$'s are constant, and the $W_j(t)$ are m independent Brownian motions on $(\Omega, \{\mathcal{F}_t\}, \mathcal{F}, \mathbb{P})$;
- A risk-free asset $S_0(t)$ which follows the dynamics given by

$$\frac{dS_0(t)}{S_0(t)} = rdt, \quad S_0(0) = s_0$$

  where $r \geq 0$ is the risk-free rate.

μ_i = constant!

The biggest problem with this model is it assumes that the DRIFT (/mu) is constant.
EXPECTED RETURNS are extremely difficult to estimate! because it takes 100's of years of data
and that assumes that they're all drawn from the same distribution. but if you look at S&P 500 back to 1880s —
but they are not drawn from the same distribution. there are breaks in your data series.

The problem with this approach is that it assumes that the drift of the asset price process is constant:

AVOID GETTING EXPECTED RETURNS WRONG.
Not so much getting the ER right more so. You need to get your drift right!
covariance matrix is fairly stable — over 10 years.. but the Drift (/mu) is not
they're going to have bigger impact. And will drive your asset allocation most.

- This view fails to account for the uncertainty in our estimate of the expected returns.
  - Expected returns are very difficult to estimate accurately.
  - → Chopra and Ziemba (1993) showed that a misspecification of the expected return has a much larger and significant impact on the outcome of the portfolio selection process than a comparable misspecification of the covariance matrix.

- The model ignores the long-term co-movement of asset returns.
  - The diffusion term does not capture this type of long-term dependence.
  - Remember the exercise '*what are we correlating exactly?*' in Module 1?

ASSET 1

$$dS_t^1 = S_t^1\left(\mu^1 dt + \sigma^1 dX^1(t)\right)$$

$$dX_1 dX_2 = \rho\, dt$$

ASSET 2
correlated noise.
with correlated GBM

$$dS_t^2 = S_t^2\left(\mu^2 dt + \sigma^2 dX^2(t)\right)$$

$\mu_i(t) = a_i + A_i X(t)$
$dX(t) = (b + BX(t))dt + \Lambda dW(t)$

The solution is to treat the drift of asset prices as **stochastic**.

More specifically, we assume that the drift is driven by a stochastic process $X(t)$.

A good starting point would be to consider:

- an **affine function**, so

$$\mu_i(X(t)) = a_i + A_i X(t), \qquad i = 0, \ldots, m$$

  and;

- an **(affine) Ornstein-Uhlenbek process**: Vasicek or Ornstein Uhlenbeck process driving random drift

$$dX(t) = (b + BX(t))dt + \Lambda dW(t)$$

Gaussian
EASY to deal with
it's going to add the co-movement we need

This model is a slight extension of Merton's 1973 Intertemporal CAPM (Merton, 1973).

Concretely, what is this process $X(t)$ about?

The idea here is that process $X(t)$ tracks the state of financial markets and of the economy.

So, you could use $X(t)$ to model:

- *Macroeconomic variables*: GDP, inflation, interest rates;
- *Microeconomic variables*: revenue growth, profit margin;
- *Risk premia*: market risk premium, asset-specific risk premia;
- or *statistical factors* that you would get from a Principal Component Analysis (PCA), for example.

{PCA} not recommended
purely for statistics.

In our implementation section, we will use $X(t)$ to represent the evolution of the risk premia in a three factor Fama-French model.

What is X? It's not a scalar. It's more than one factor. It's an N-dimensional process.
A vector with N dimensions, — Fama French have 3-risk factors.

$X(t)$ should not be a single scalar process because this is just too limiting when we are dealing with m assets.

In fact, $X(t)$ should be a n-dimensional process, meaning a vector of $n \geq 0$ scalar processes solving:

$$dX(t) = (b + BX(t))dt + \Lambda dW(t),$$

where

- W(t) is now a $(n+m)$-dimensional Brownian motion, meaning a vector of $(n+m)$ independent Brownian motions,
- b is a n-element vector,
- B is a $n \times n$ matrix,
- Λ is a $n \times (n+m)$ matrix.

Modelling Tip:

Beware of adding too many new stochastic processes into your model!

Adding too many stochastic processes in your model:

- leads to unnecessary complication in your model, which will make you lose track of the most important insights;
- delays implementation and slows down running time;
- results in overfitting: the model will perform well in-sample but poorly out-of-sample.

$\boxed{n} \ll m$

Add new stochastic processes to model:

- new asset classes;
- risk factors that are essential to your trading strategy or to your risk control;
- important interactions.

To Recap... Factor-Driven Model

$(n+m)$ Brownian motions

The asset market we are considering has

- $m \geq 1$ stock-like risky asset $S_i(t), i = 1, \ldots, n$ with dynamics:

$$\frac{dS_i(t)}{S_i(t)} = \boxed{(a_i + A_i' X(t))} dt + \sum_{j=1}^{n} \sigma_{ij} dW_j(t), \quad S_i(0) = s_i$$

  where a_i is a scalar, A_i is a n-element vector and $\Sigma = (\sigma_{i,j})$ is a $m \times (n+m)$ matrix.

- one money market asset $S_0(t)$ which follows the dynamics given by

$$\frac{dS_0(t)}{S_0(t)} = \boxed{(a_0 + A_0' X(t))} dt, \quad S_0(0) = s_0$$

  where a_0 is a scalar, A_0 is a n-element vector.

Moreover,

- $W(t)$ is a $(m+n)$-dimensional Brownian motion on $(\Omega, \{\mathcal{F}_t\}, \mathcal{F}, \mathbb{P})$.
- The n-dimensional factor $X(t)$ solves:

$$dX(t) = (b + BX(t))dt + \Lambda dW(t)$$

$R_i(t)$ $\qquad$ $R(t) = \begin{pmatrix} R_1(t) \\ \vdots \\ R_m(t) \end{pmatrix}$

The investment problem is to decide a dynamic asset allocation between $m+1$ assets (m risky assets and one money market asset) over a time horizon T.

The next step is to write the dynamics of the investor's wealth in response to an **investment strategy**.

We define the **investment strategy** as the asset allocation, that is a m-dimensional stochastic process $h(t)$ representing the proportion or percentage of the investor's wealth allocated to the risky assets. This is exactly the same definition we used when we discussed Markowitz' mean-variance analysis, except that here the investment strategy is a stochastic process.

Assuming that the investor's wealth or the fund's assets under management cannot be negative as this would immediately result in bankruptcy,

- $h_i(t) > 0$ indicates a long position in asset i;
- $h_i(t) < 0$ represents a short position in asset i.

We do not need a decision variable to track the asset allocation to the bank account. Indeed, if we let $h_0(t)$ be the proportion of the investor's wealth allocated to the bank account we see that

allocation to the money market $\to$

100% of wealth

$$h_0(t) := 1 - \sum_{i=1}^{m} h_i(t) = 1 - \mathbf{1}' h(t)$$

where $\mathbf{1}$ is a m-element column vector with all entries equal to 1.

This leads naturally to the **budget equation**:

$$\boxed{\underbrace{h_0(t) + \sum_{i=1}^{m} h_i(t)}_{\text{scalars}} = \underbrace{h_0(t) + \mathbf{1}' h(t)}_{\text{vectors.}} = 100\%}$$

-> Factor model

The next step is to write the dynamics of the investor's wealth $V(t)$ in response to an investment strategy $h(t)$. In scalar notation,

$$
\begin{aligned}
dV(t) \quad = \quad & \underbrace{V(t)h_0(t)}_{\substack{\text{amount}\\\text{invested}\\\text{in the money}\\\text{market asset}}} \cdot \underbrace{(a_0 + A_0 X(t))dt}_{\substack{\text{return dynamics}\\\text{of the}\\\text{money market}\\\text{instrument}}} \\
& + \sum_{i=1}^{m} \underbrace{V(t)h_i(t)}_{\substack{\text{amount invested}\\\text{in risky asset } i}} \cdot \underbrace{\left\{ \left(a_i + A_i X(t))dt + \sum_{j}^{n+m} \sigma_{ij} dW(t) \right) \right\}}_{\substack{\text{return dynamics}\\\text{of risky assets } i}}
\end{aligned}
$$

with initial asset under management $V(0) = v$.

Applying the budget equation $h_0(t) = 1 - \sum_{i=1}^m \color{red}{h_i(t)}$ and factoring $V(t)$ out, we rewrite this expression as

$$dV(t) = V(t)\left\{\left[\overbrace{(a_0 + A_0X(t))}^{\text{money market}} + \underbrace{\sum_{i=1}^{m}(a_i - a_0) + (A_i - A_0)X(t)}_{\text{Risk Premium}}\right] dt + \sum_{j}^{n+m} \sigma_{ij}\, dW(t)\right\}$$

h(t)

with initial assets under management $V(0) = v$.

h(t)

In vector and matrix notation,

$$
\begin{aligned}
dV(t) \quad = \quad & \underbrace{V(t)h_0(t)}_{\substack{\text{amount}\\ \text{invested}\\ \text{in the money}\\ \text{market asset}}} \cdot \underbrace{(a_0 + A_0X(t))dt}_{\substack{\text{return dynamics}\\ \text{of the}\\ \text{money market}\\ \text{instrument}}} \\
& + \quad \underbrace{V(t){\color{red}h'(t)}}_{\substack{\text{amount invested}\\ \text{in each of the } m\\ \text{risky assets}\\ (m\text{-element vector})}} \quad \cdot \underbrace{\{(a + AX(t))dt + \Sigma dW(t))\}}_{\substack{\text{return dynamics}\\ \text{of the } n\\ \text{risky assets}\\ m\text{-element vector})}}
\end{aligned}
$$

Here,

- a is a m-element vector with ith entry equal to a_i;
- A is a $m \times n$-element matrix where the ith row is equal to A_i'.

Taking the budget equation $h_0(t) = 1 - \mathbf{1}' h(t)$ into account, ✓

$$
\begin{aligned}
dV(t) &= V(t)\left\{\left[\underbrace{(a_0 + A_0 X(t))} + h'(t)\underbrace{\left(\tilde{a} + \tilde{A}X(t)\right)}\right] dt + h'(t)\Sigma dW(t)\right\}, \\
&\quad V(0) = v
\end{aligned}
$$

$(R_1 \ldots R_m)\begin{pmatrix} \vdots \\ \vdots \end{pmatrix}$

$m \times (m+n)$

Here,

- $\tilde{a} = a - a_0 \mathbf{1}$;
- $\tilde{A} = A - \mathbf{1}A_0'$, that is, $\tilde{A}$ is a $m \times n$ matrix with row i given by $\tilde{A}_i = A_i - A_0'$.

$$\tilde{a} = a - a_0 \mathbb{1}$$

$$\tilde{A} = A - \mathbb{1} A_0'$$

Remark: To retrieve the same wealth SDE as we got in the Merton model, set:

- $n = 0$;
- $a_0 = r$;
- $A_0 = 0$;
- $a = \mu$;
- $A = 0$;

Then,

$$dV(t) \quad = \quad V(t)\left\{\left[r + h'(t)(\mu - r\mathbf{1})\right] dt + h'(t)\Sigma dW(t)\right\}, \quad V(0) = v$$

To Recap...

In the **factor-driven formulation**, the investor's wealth evolves according to the SDE:

$$\begin{aligned} dV(t) &= V(t)\left\{\left[(a_0 + A_0 X(t)) + h'(t)\left(\tilde{a} + \tilde{A}X(t)\right)\right]dt \right. \\ &\quad \left. + h'(t)\Sigma dW(t)\right\}, \qquad V(0) = v \end{aligned}$$

Here, $W(t)$ is a $(m+n)$-dimensional Brownian motion.

In the **Merton model**, the investor's wealth evolves according to the SDE:

$$\begin{aligned} dV(t) &= V(t)\left[rdt + h'(t)(\mu - r\mathbf{1})dt + h'(t)\Sigma dW(t)\right], \\ &\quad V(0) = v \end{aligned}$$

Here, $W(t)$ is a m-dimensional Brownian motion.

1.3 Utility

Standard economic theory tells us that wealth is generally not a good indicator of "economic satisfaction."

Instead, we need to consider a function of wealth, called a **utility function**, which tracks the "satisfaction" investors derive from their wealth based on her level of risk tolerance (see Appendix).

In our problem, we need a bequest function $U_T(V(T))$ to track the satisfaction derived from reaching a given level of wealth $V(T)$ at the end of the investment horizon T

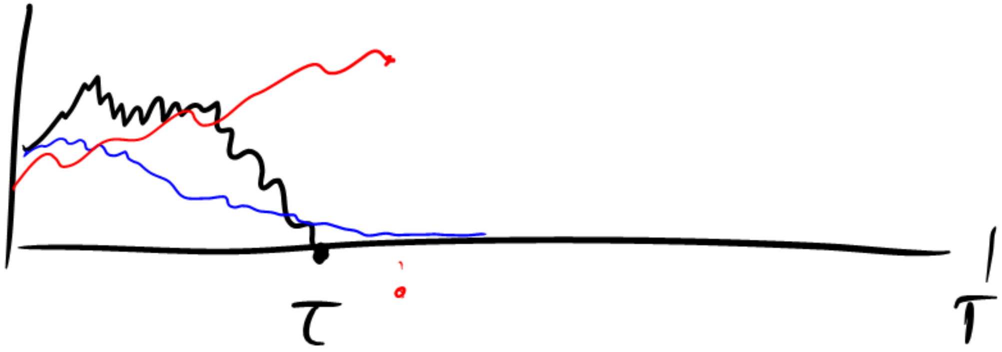
Broadly speaking, the investor's objective is to find an optimal asset allocation h_t to maximize the total utility derived from terminal wealth $U_T(V_T)$.

However, we cannot perform this optimization directly, because the wealth $V(t)$ is stochastic.

Instead, we do "the usual thing" and maximize the **expected value** of the utility derived from current consumption and terminal wealth:

$$\mathbb{E}\left[U_T(V_T)\right]$$

Bankruptcy



We must address one last condition: **bankruptcy**. Realistically, we cannot allow the investor's wealth to become negative. A non positive wealth should automatically result in bankruptcy, terminating our investment problem.

To model bankruptcy, we will introduce a random variable τ called a **stopping time**. The stopping time τ is defined as

$$\tau := \inf \{t > 0 | V_t = 0\} \wedge T$$

and it represents the earliest of either

1. the time at which the investor goes bankrupt; or
2. the end of the investment horizon.

Stochastic Optimization Problem

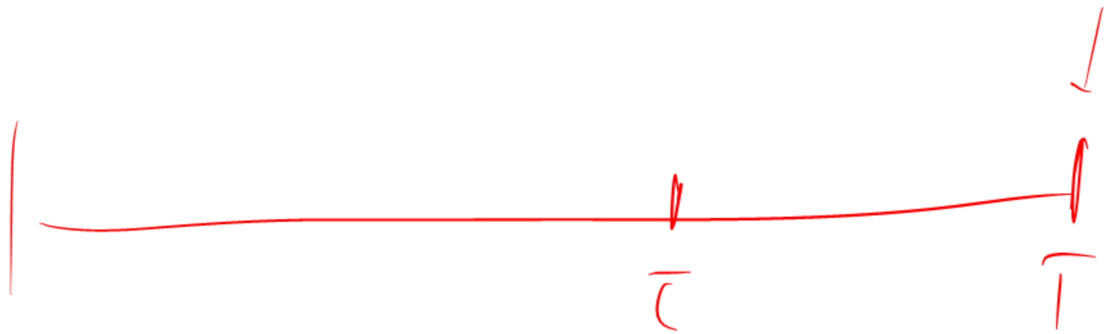
Utility / Reward function

To sum things up, we attempt to maximize the expectation

$$J(0, x_0) := \mathbb{E}\left[U_T(\tau, V_\tau)\right]$$

with

$$U_T(\tau, V_\tau) = 0 \qquad \text{if } \tau < T$$



We finish formalizing our problem by defining the value function $\Phi(t, V_T)$ as the solution to the optimization:

$$\Phi(t, V_T) := \text{``}\max_{h \in \mathbb{R}}\text{''}\, \mathbb{E}\left[U_T(V_T)\right]$$

This type of problems is called a **stochastic control** problem. There are two solution methods:

1. stochastic control theory;
2. equivalent martingale / stochastic duality.

In this presentation we focus on stochastic control theory, which is a far more applicable method.

Note that investment decision problems are initially set under the physical measure $\mathbb{P}$.

2. A Short Introduction to Stochastic Control

Stochastic control stands at the crossroad of stochastic analysis, PDE theory and optimization. As a result, the stochastic control toolbox is vast. It includes results from:

- *stochastic analysis*: Itô calculus, Girsanov Theorem and change of measure.
- *PDE theory*: existence and uniqueness of solutions to parabolic and elliptical PDEs, viscosity solutions to parabolic and elliptical PDEs, numerical techniques.
- *optimization*.
- *mathematical analysis*.

Feynman-Kac

It is used in chemical and electrical engineering, aeronautics and even finance and economics!

Historical Note

The field of optimal control originates in Richard Bellman's work on dynamic programming (see also Bellman, 1957). Although Bellman's approach was rather heuristics, it effectively sets the template for the multiple developments that have taken place over the last 55 years. Bellman's dynamic programming principle (DPP) is still at the heart of control theory: it is used to derive the Bellman equation in discrete time and the Hamilton-Jacobi-Bellman partial differential equation (HJB PDE) in continuous time.

Figure : Richard Bellman (1920-1984)

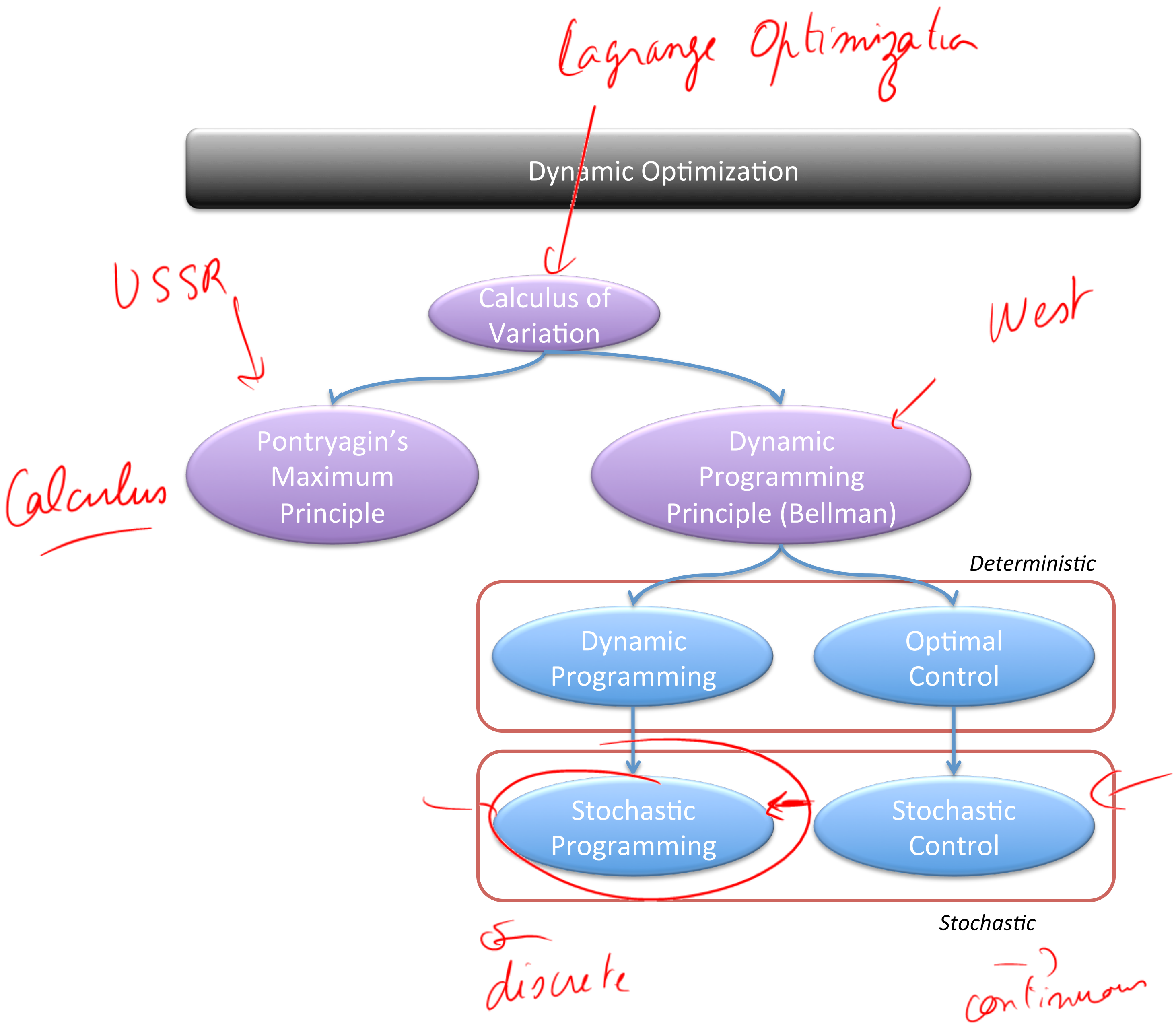


Figure : An overview of dynamic optimization

In the late 1960s, Merton (1969) (see also Merton, 1971, 1992) proposed a formulation of the investment problem as a stochastic control problem where the objective is to maximize the investor's utility of wealth.

The Merton model constitutes one of the few nonlinear stochastic control problems which can be solved analytically.

Figure : Robert Merton (born in 1944)

The Three Key Ingredients

To formulate a stochastic control problem, we need three ingredients:

1. *state process and its dynamics*: the **state process** models the dynamical system (a portfolio, spacecraft, airplane or chemical plant).
2. *class of control policies*: the **control policy** (also known as **policy** or **strategy**) represents the decision variables the decision maker, or controller, can use to stir the dynamical system.
3. *criterion to be optimized*: the **criterion** is the measure used by the controller. The controller will use the control policy to either
   - maximize the criterion: return or utility of a portfolio, range of an airplane; or
   - minimize the criterion: risk of a portfolio, fuel cost of putting a payload into orbit.

Let's look at how this all fits with our investment problem.

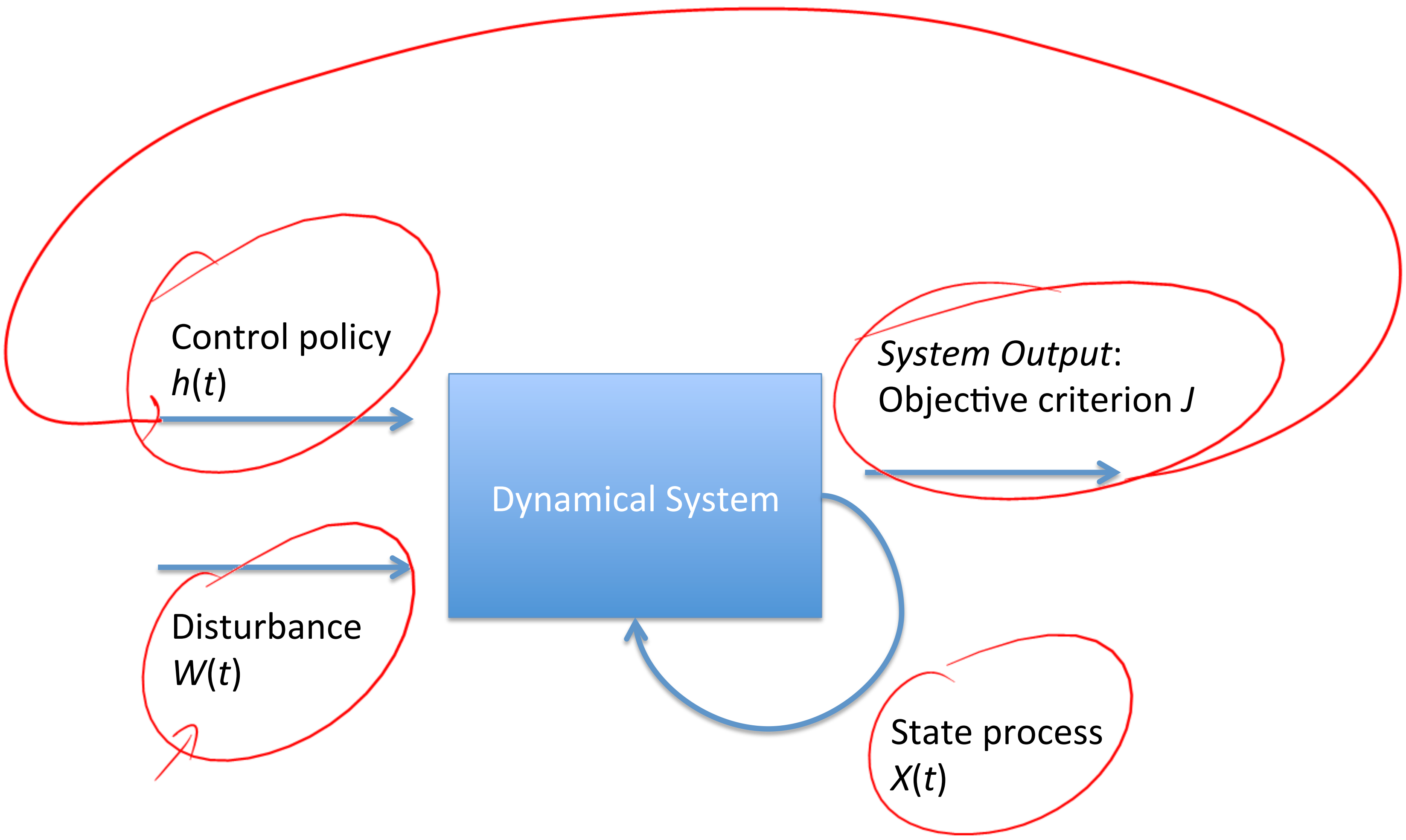


Figure : Schematical view of a simple controlled dynamical system

Ingredient 1: State Process

Let $(\Omega, \{\mathcal{F}_t\}, \mathcal{F}, \mathbb{P})$ be the underlying probability space and let $W(t) \in \mathbb{R}^{m+n}$ be a standard Brownian motion.

Technically, the state process $X(t)$ models the evolution of the dynamical system. $X(t)$ is a controlled Itô diffusion with dynamics

$$dX(t) = \mu\left(t, X(t), h(t)\right) dt + \Sigma(t, X(t), h(t)) dW(t) \qquad X(0) = x_0 \in \mathbb{R}^n. \quad (1)$$

where $h(t) \in U \subset \mathbb{R}^m$ is a m-dimensional control policy and U is closed.

$$dX(t) = \mu(t, X(t), h(t))\,dt + \Sigma(t, X(t), h(t))\,dW(t)$$

$$X(t) = \begin{pmatrix} X_1 \\ \vdots \\ X_m \end{pmatrix} \qquad \mu = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_m \end{pmatrix} \qquad \Sigma = \begin{array}{c} \overbrace{\qquad\qquad}^{m+m} \\ \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} \end{array}$$

$\Sigma_{,m}$

$$Df = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_m} \end{pmatrix} \quad m \qquad D^2 f = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \cdots & \frac{\partial^2 f}{\partial x_i \partial x_j} \\ \frac{\partial^2 f}{\partial x_j \partial x_i} & \ddots & \frac{\partial^2 f}{\partial x_m^2} \end{pmatrix} \quad m \times m$$

The backward evolution operator $\mathcal{L}$ associated with $X(t)$ is

$$\mathcal{L}^h f \quad = \quad f_t + \mu'(t,x,h)Df + \frac{1}{2}\mathrm{tr}\left(\Sigma\Sigma'(t,x,h)D^2 f\right)$$

for a 'nice' function[2] f.

Denote by $\mathcal{G}_t := \sigma(X(s), 0 \le s \le t)$ the sigma-field generated by the state process up to time t.

The key assumption is that $\boxed{\Sigma\Sigma' > 0}$. This assumption is called the **ellipticity condition**.

$$\boxed{\frac{\partial f}{\partial t} + \mu(.)\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2(.)\frac{\partial^2 f}{\partial x^2}}$$

drift of the process $Y(t) = f(t, X(t))$

[2] By 'nice' we mean a function on which we can safely apply Itô's lemma. Technically, this implies that $f \in C_0^{1,2}([0,T] \times \mathbb{R}^n)$.

Ingredient 2: Classes of Control Policies

From a very general perspective, we could view the control policy $h(t) \in \mathbb{R}$ as an $\mathcal{F}_t$-adapted stochastic process of the form $h(t) = u(t, \omega)$.

Based on the nature of the problem, we could define the control policy in a number of ways, such as:

1. *Deterministic or open loop control*: define $h(t) := u(t)$, where $u(t)$ is a deterministic function of time. In this case, the control does not depend on the state $X(t)$.

$h(t) = u(t)$

$h(t) = u(t, \omega)$ $\quad \omega \in \Omega$

time $\nearrow$ $\quad \nwarrow$ randomness

2. *Feedback or closed loop control*: define $h(t) := u(t, \omega)$ where $u(t, \omega)$ is adapted to $\mathcal{G}_t$, the filtration generated ~~by the state process~~.

3. *Markov control*: define $h(t) := u(t, X(t))$ for some function

$$u : [0, T] \times \mathbb{R}^n \to U$$

$h(t)$

$\downarrow$

$dX(t) = \mu(t, h(t), X(t))dt + \Sigma(\ldots)dW(t)$

   The main advantage of ~~Markov processes~~ is that the dynamics of $X(t)$ in response to a Markov control remains an Itô diffusion.

Because we assume that $X(t)$ is an Itô diffusion process, we can expect our stochastic control problem to be Markovian. This suggests that we could consider only **Markov control policies**.

Remark: further issues, such as the impact of the control on the parameters of the system (think about the impact that transactions coming from a very large portfolio such as PIMCO's or the foreign currency reserves of the People's Bank of China could have on the market) would also need to be considered in our definition of the control policy.

Progressively Measurable and Admissible Controls

Now that we have decided on Markov control policies, we need to place the control in an appropriate probabilistic framework: the framework of progressively measurable processes.

A process $h(\cdot)$ is $\mathcal{F}_t$ *progressively measurable* if the map

$$\begin{aligned} h \quad : \quad & [0, T] \times \Omega \to U \subset \mathbb{R}^m \\ & (s, \omega) \mapsto h(s, \omega) \end{aligned}$$

is $\mathcal{B}_t \times \mathcal{F}_t$ measurable for each $t \in [0, T]$, where $\mathcal{B}_t$ denotes the Borel σ-algebra on $[0, t]$.

a progressively measurable control is measurable (adapted) to both

1). Randomness $\to$ $\mathcal{F}_t$

2). Time $\to$ Borel σ-algera $\mathcal{B}_t$

Admissible Control

We say that a control process $h(t)$ is *admissible* or is *in class* $\mathcal{A}$ if the following conditions are satisfied:

1. $h(t)$ is progressively measurable;
2. $h(t)$ is such that

$\ell = 2$ → Moments of $X(t)$ will be bounded up to $\ell = 2$

$$\mathbb{E}\left[\int_0^T |h(t)|^{\ell} dt\right] < \infty, \quad \text{for } \ell = 1, 2, \dots \tag{2}$$

ℓ - integrable

Remark 1: it is enough for (2) to hold that we take $h \in U$ where U is compact. Indeed, in this case $h(t) \leq M$ for some $M < \infty$.

Remark 2: what is the purpose of the admissibility condition in (2)? From standard stochastic analysis theory, we know that for h fixed, the state process equation (1) admits a unique solution $X(t)$ and that for each $\ell = 1, 2, \ldots$, the ℓ^{th} order absolute moments $E_{0,x}\left[\|X(t)\|_\infty^\ell\right]$ are bounded for $t \in [0, T]$. Admissibility condition (2) ensures that these properties are still true for appropriately chosen Markov control policies.

An Important Result for Stochastic Control

Key Fact

Let $h \in \mathcal{A}$ be an admissible control, then the state process equation (1) *admits a unique solution $X(t)$ and for each $\ell = 1, 2, \ldots$, the ℓ^{th} order absolute moments $E_{0,x}\left[\|X(t)\|_\infty^\ell\right]$ are bounded for $t \in [0, T]$.*

The reference for this result is Appendix D in the book by Fleming and Soner (2006).

Ingredient 3: Criterion To Be Optimized

The criterion J represents an objective measure of performance (fuel, return on investment, utility of wealth, probability of shortfall) that the controller tries to minimize or maximize.

In this course, we adopt the convention that the controller's objective is to **maximize** the performance criterion because most portfolio applications lead to maximizations.

The criterion could be either over:

- a finite time horizon $[0, T]$;
- an infinite time horizon $[0, \infty)$ (outside of the scope of this course).

We could also consider additional issues related to

- stochastic discounting (i.e. *'what happens if our discounting rate is stochastic?'*);
- exit time (i.e. *'what happens if we go bankrupt?'*).

Elementary Finite Time Horizon Criterion

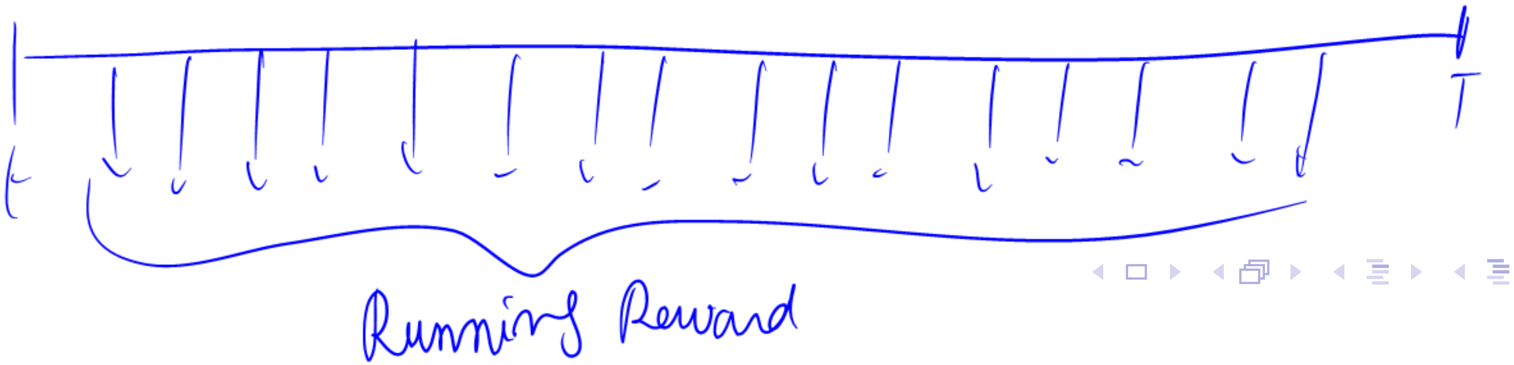
Over a **finite time horizon**, we could consider a criterion J of the form:

$$J(t,x,h) = \mathbb{E}_{t,x}\left[\int_t^T \underbrace{L(s,X(s),h(s))ds}_{\text{Running Reward}} + \Psi(T,X(T))\right] \tag{3}$$

where

- $L(\cdot)$ is the **running reward** function, which we use in finance to keep track of the utility derived from consumption, dividend/interest payment from the portfolio or to account for contributions to the portfolio;
- $\Psi(\cdot)$ is the **terminal reward** function, which we use in finance to track the utility of terminal wealth.

$\Psi(T, X(T))$



Introducing Stochastic Discounting

We can introduce stochastic discounting via a discount rate g that is a function of time t, the state $X(t)$ and even the control $h(t)$. This discounting rate g needs to be continuous in all its arguments and bounded from below, but it does not necessarily needs to be positive.

The criterion J takes the form:

$$J(t, x, h) = \mathbb{E}_{t,x}\left[\int_t^T \beta(t, s)L(s, X(s), h(s))ds + \beta(t, T)\Psi(T, X(T))\right]$$

Discount Factor

stochastic

control

where

$$\beta(t, s) := \exp\left\{-\int_t^s g(r, X_r, h_r)dr\right\} \tag{4}$$

discount rate

Remark 1: discounting is used to model either

- a loss of economic value due, for example, to the passage of time and inflation.
- the probability that the system will have abruptly stopped working by time $s \in [0, T]$. This approach to discounting is very close to the intensity-based models used in credit risk, and also to the exit times we will introduce next.

Remark 2: the discounting term plays a leading role **risk-sensitive control!**

Introducing Exit Times

The criterion we are considering so far cannot account for stopping times, such as the bankruptcy time we discussed earlier.

Consider a subset $O \subset \mathbb{R}^n$ of the state space, and denote by Q the 'cylinder' $Q := [0, T] \times O$.

Define the stopping time τ as the **exit time** of the cylinder Q, that is

$$\tau := \inf \{s : (s, X(s)) \notin Q\} \tag{5}$$

Because the state process $X(t)$ is an Itô diffusion, it cannot jump out of the 'cylinder' Q. It needs to drift and diffuse through the boundary first.

Now, the criterion J takes the form:

$$J(t,x,h) = \mathbb{E}_{t,x}\left[\int_t^{\tau} L(s,X(s),h(s))ds + \Psi(\tau,X(\tau))\right] \tag{6}$$

where $\tau := \inf\{s : (s,X(s)) \notin Q\}$.

Criterion To Be Optimized: General Form

Our objective is to find an admissible control policy $h(t)$ to maximize the criterion J

$$J(t,x,h) = \mathbb{E}_{t,x}\left[\int_t^{\tau} \beta(t,s)L(s,X(s),h(s))ds + \beta(t,\tau)\Psi(\tau,X(\tau))\right]$$

where the state $X(t)$ solves the SDE (1):

$$dX(t) = \mu\,(t,X(t),h(t))\,dt + \Sigma(t,X(t),h(t))dW(t) \qquad X(0) = x_0 \in \mathbb{R}^n.$$

At this stage the only element we are missing is a mathematical formalization of the optimization.

$+h(t)$

How Realistic Are All These Assumptions?

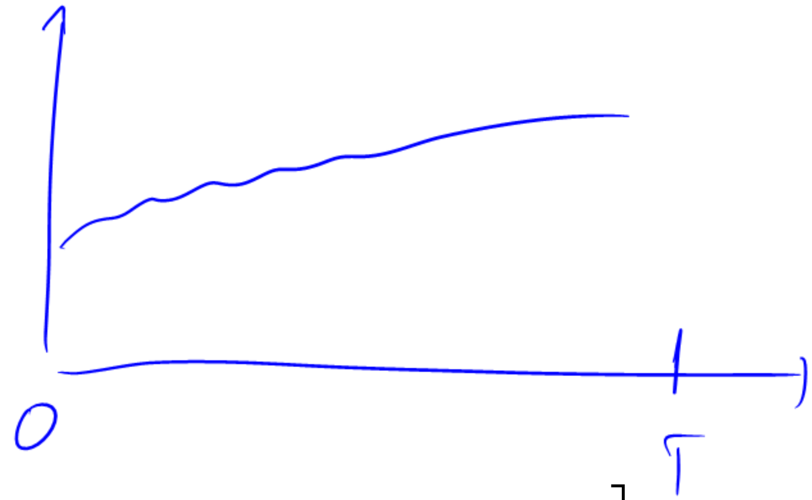
Before we formalize the stochastic control problem and design a plan to solve it, we take another glance at the assumptions we have made so far. After all, we are chiefly concerned with building a mathematical theory that will help us solve actual decision problems.

The classical assumptions presented above may not be satisfied in all problems. These assumptions are here to help us develop a general theory of stochastic control with minimum fuzz. From a practical perspective, the good news is that stochastic control is robust enough to enable us to relax most of these assumptions if we are willing to put some extra work and possibly look for a weak (viscosity) solution to the Hamilton-Bellman-Jacobi PDE.

Of these assumptions, the most important is undeniably the **ellipticity condition** $\Sigma\Sigma' > 0$. This condition ensures that the backward evolution operator $\mathcal{L}$ and the HJB PDE associated with the stochastic control problem are uniformly parabolic. In this case, we can use the literature on parabolic PDEs (such as Ladyzenskaja et al., 1968; Friedman, 1964) to try to prove existence and uniqueness of a smooth solution.

If the ellipticity condition fails, the HJB PDE will be degenerate parabolic and we cannot expect to find a smooth solution.

Solving The Stochastic Control Problem



We define the **value function** $\Phi(t,x)$ as

$$\begin{aligned}\Phi(t,x) &:= \sup_{h(t)\in\mathcal{A}} J(t,x,h) \\ &:= \sup_{h(t)\in\mathcal{A}} \mathbb{E}_{t,x}\left[\int_t^\tau \beta(t,s)L(s,X(s),h(s))ds + \beta(t,\tau)\Psi(\tau,X(\tau))\right]\end{aligned}$$

Here, 'sup' stands for *supremum*, which you can view as an extension of the concept of maximum.

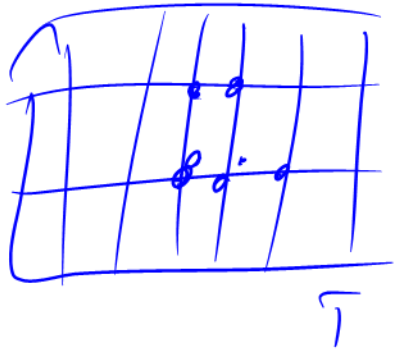
Hence, to solve a stochastic control problem, we need to find:

1. The optimal control policies $h^*(t)\in\mathcal{A}$;
2. The value function $\Phi(t,x)$.

Note that the value function should be smooth! Otherwise we cannot apply Itô...

It turns out that the best way of solving this optimization is by looking at an associated PDE called the **Hamilton-Jacobi-Bellman partial differential equation** (HJB PDE):

$$\frac{\partial \Phi}{\partial t} + H(t, x, \Phi, D\Phi, D^2\Phi) = 0 \tag{7}$$



with boundary condition

$$\Phi(\tau, x) = \Psi_\tau(\tau, x)$$

and where the **Hamiltonian**[3] H is given by:

$$\begin{aligned} & H(t, x, r, p, A) \\ = & \sup_{h \in \mathcal{U}} \left[\mu'(t, x, h)p + \frac{1}{2}\mathrm{tr}\left(\Sigma\Sigma'(t, x, h)A\right) - rg(t, x, h) + L(t, x, h) \right] \end{aligned} \tag{8}$$

Note that the HJB PDE is a nonlinear (because of the maximization) second-order parabolic PDE. In our case it is a semilinear PDE.

[3] The term "Hamiltonian" is a reference to classical mechanics

Where Does the HJB PDE Come From?

$\beta(t,s) = \exp\left\{-\int_t^s g(\ )du\right\}$

Intuitively, if we were to forget about the optimization for now and simply considered the criterion J for a fixed control $\bar{h}_t$, then the **Feynman-Kac formula** tells us that the expectation

Expectation Problem

$$
\begin{aligned}
\phi(0,x) &:= \mathbb{E}_{t,x}\left[\int_0^\tau \beta(t,s)L(s,X(s),h(s))ds \right.\\
&\qquad \left. +\beta(0,\tau)\boxed{\Psi(\tau,X(\tau))}\right] \\
dX(t) &= \mu\left(t,X(t),h(t)\right)dt + \Sigma(t,X(t),h(t))dW(t)
\end{aligned}
$$

Figure : Richard Feynman (1918-1988)

T

has an associated parabolic PDE expressed as:

$-rV$ in the BSDE

$$
\underbrace{\frac{\partial\phi}{\partial t} + \mu'(t,x,h)D\phi + \frac{1}{2}\mathrm{tr}\left(\Sigma\Sigma'(t,x,h)D^2\phi\right)}_{\text{Backward evolution operator of } X(t) \text{ applied to } \phi} - \underbrace{g(t,x,h)\phi}_{\text{Discounting term}} + \boxed{\underbrace{L(t,x,h)}_{\text{Running reward}}} = 0
$$

$\mathcal{L}\phi$

with boundary condition $\phi(\tau,x) = \boxed{\Psi(\tau,X(\tau))}$ equal to the terminal reward.

What About the Control? The Dynamic Programming Principle

The presence of a control process changes the nature of the problem and is the reason why the Feynman-Kac result is not enough to derive the HJB PDE.

Fortunately, we can rely on Bellman's Dynamic Programming Principle to fill the gap.

The **Dynamic Programming Principle (DPP)** is at the heart of dynamic programming and optimal control theory. We need the DPP to formally derive the HJB PDE associated with the control problem.

We also need the DPP to show that the value function is a viscosity solution of the HJB PDE in the event we cannot find a classical solution.

Definition (δ-optimal control)

Fix $\delta > 0$. A control policy $h^\delta \in \mathcal{A}$ is called δ-*optimal* if

$$
\begin{aligned}
J(t, x, h^\delta) &= \mathbb{E}_{t,x}\left[\int_t^\tau \beta(t,s) L(s, X(s), h(s)) ds + \beta(t,\tau)\Psi(\tau, X(\tau))\right] \\
&\geq \sup_{h(t)\in\mathcal{A}} J(t, x, h) - \delta \\
&= \Phi(t, x) - \delta
\end{aligned}
$$

Key Fact (Dynamic Programming Principle)

Let $(t, x) \in [0, T) \times \mathbb{R}^n$ *be given. Then for every stopping times* $\tau \in [0, T]$ *and* $\theta \in [t, \tau]$, *we have*

$$\Phi(t, x) = \sup_{h(t) \in \mathcal{A}} \mathbb{E}_{t,x} \left[\int_t^{\theta} \beta(t, s) L(s, X(s), h(s)) ds + \beta(t, \theta) \Phi(\theta, X(\theta)) \right]$$

Remark: Consider a discrete-time framework, set $\tau = T$ and take $\theta = t + 1$, then the DPP implies:

$$\begin{aligned}\Phi(t,x) &= \sup_{h(t)\in U} \mathbb{E}_{t,x}\left[L(t,X(t),h(t))ds\right.\\ &\quad \left.+\exp\left\{-g(t+1,X_{t+1},h_{t+1})\right\}\Phi(t+1,X(t+1))\right]\end{aligned} \tag{9}$$

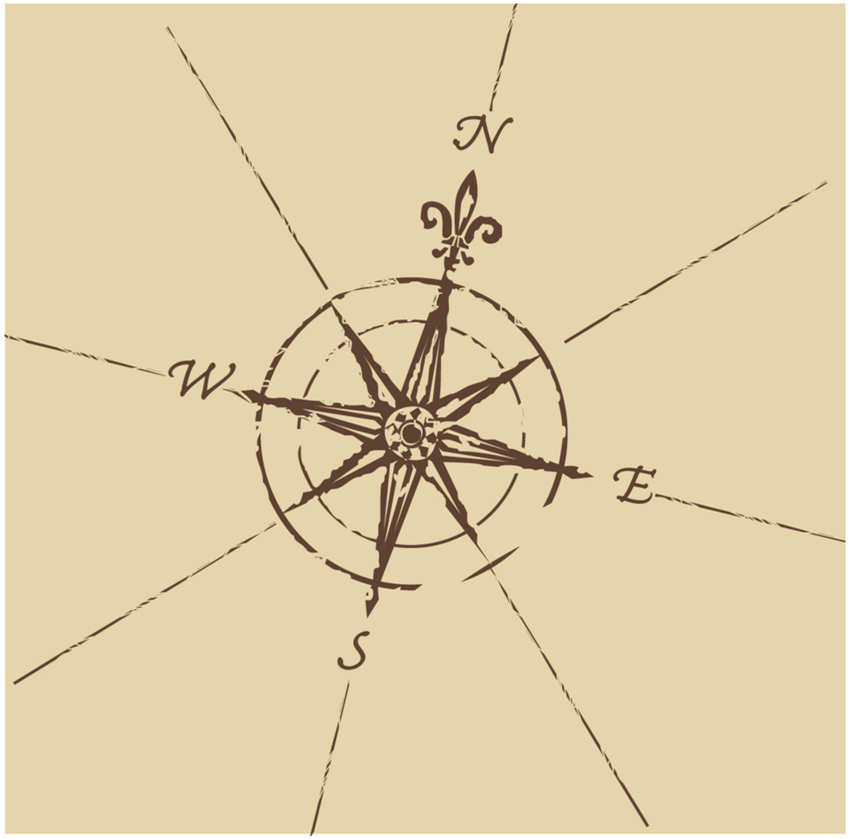
where the maximization takes place with respect to $u \in U$ and not with respect to the class of admissible controls $\mathcal{A}$.

As we know the boundary conditions (terminal conditions and lateral boundary conditions) for our problem, the DPP in equation (9) suggests the use of a backward induction scheme to solve the problem.

What Do We Still Need to Do?

Stochastic control provides a rigorous and robust theoretical framework to solve continuous-time stochastic optimization problems, but it does not directly give us the two most important results we seek:

1. The existence of a maximizing control $\hat{h}$;
2. The existence of a classical solution Φ or the HJB PDE.



The existence of a maximizing control $\hat{h}$ is purely problem-specific, and from a practical perspective it is the most important result.

The existence of a classical solution is also often problem-specific. Although some fairly general existence theorems exist, they are far from applying to the majority of problems. Rather, they give a template and a set of techniques and arguments that one may find useful to derive his/her own existence result.

Still, existence of a classical solution is crucial in so far as it ensures, via the verification theorem, that the minimizing control is optimal.

What should we do then?

In some instances, we may be fortunate enough (or have sufficiently rigged the problem!) that an explicit (closed-form) classical solution can be found.

To find an explicit solution, we often rely on ODE and PDE “folk results” and catalogue of solved problems to find the right *ansatz*.

Verification Theorem

Once we have found $\hat{h}$, we will need to prove that this is the optimal control for the problem. This is usually done through a **verification theorem**, unless we already know the functional form for Φ or have enough information about Φ to proceed more rapidly.

The verification theorem is essential because it ties up the loose ends by showing that if the HJB PDE has a smooth solution then that solution is equal to the value function Φ, and if there exists a Borel-measurable maximizer $\hat{h}(t, x)$ for the Hamiltonian, then $h^*(t) = \hat{h}(t, x)$ is optimal.

Proving it only requires Itô's lemma... which also explains why it only works if Φ is smooth!

Finding an explicit solution remains the exception. Over the last twenty years, the stochastic control community has increasingly turned to (weak) viscosity solutions to make it easier to show that the HJB PDE can be solved in some sense and then go on to to find the actual solution and the minimizing control numerically.

The advent of **viscosity solutions** has reinvigorating stochastic control by providing a new, and easier way of solving old problems and techniques to address a range of problems, such as jump-diffusion problems. However, there is no verification theorem for viscosity solutions: we have to rely on a comparison result to prove uniqueness and this result doe not tie in with the maximizing control.

Time for some applications to the Merton Problem!

3. The Merton Problem: Maximizing the Terminal Utility of Wealth with $m = 1$ Risky Asset

μ

$m = 1$

As a start, we solve the Merton problem with $m = 1$ risky asset.

In the Merton model, the dynamics of the wealth process $V(t)$ is geometric:

Wealth process

$$\frac{dV(t)}{V(t)} = \underbrace{[r + h_t(\mu - r)]\,dt + h_t \sigma dW_t}_{\text{geometric}}, \qquad X(0) = x$$

The reward criterion for this problem is

$$J(t, v, h) = \mathbb{E}\left[U_\tau(V_\tau)\right]$$

where the exit time τ is defined as

$$\tau := \inf\{t > 0 | V(t) = 0\} \wedge T$$

bankruptcy time

In fact, we do not need an exit time for this particular problem because the dynamics of the state is geometric and continuous.

Definition (Admissible Control - Merton Problem for Terminal Utility of Wealth)

A control process $h(t)$ is *admissible* or in class $\mathcal{A}$ if the following conditions are satisfied

1. $h(t)$ is progressively measurable;
2. $h(t)$ is such that

$$\mathbb{E}\left[\int_0^T |h(t)|^2 dt\right] < \infty \tag{10}$$

To summarize, the value function $\Phi(t, V(t))$ for the maximization of terminal utility is:

Stochastic control

$$\Phi(0, v) := \sup_{h \in \mathcal{A}} \mathbb{E}_{0,v} [U_T(V_T)]$$

given the following dynamics for the state process:

$$\frac{dV(t)}{V(t)} = [r + h_t(\mu - r)]\, dt + h(t)\sigma dW(t), \qquad V(0) = v$$

HJB PDE :

1). backward evolution operator of $V(t)$

2) ~~Discounting~~

3). ~~Running Reward~~

$$\sup_{h \in \mathcal{A}} \left[\frac{\partial \phi}{\partial t} + [r + h_t(\mu - r)]\, x \frac{\partial \phi}{\partial x} + \frac{1}{2}\sigma^2 h^2(t)\, x^2 \frac{\partial^2 \phi}{\partial x^2} \right] = 0$$

Terms in h

ϕ ???

The associated **Hamilton-Jacobi-Bellman PDE** is

$$\frac{\partial \Phi}{\partial t} + H\left(t, x, \frac{\partial \Phi}{\partial x}, \frac{\partial^2 \Phi}{\partial x^2}\right) = 0$$

with boundary condition

$$\frac{\partial \phi}{\partial x} \quad \frac{\partial^2 \phi}{\partial x^2} \qquad \Phi(T, x) = U_T(x)$$

and where

$$H(t, x, p, A) := \max_{h \in \mathbb{R}} \left[\underbrace{x\,[r + h(\mu - r)]\,p + \frac{1}{2}x^2 h^2 \sigma^2 A}_{F(h)}\right]$$

1st order condition $\frac{dF}{dh}\Big/\hat{h} = 0 \iff x(\mu - r)p + \hat{h}x^2\sigma^2 A = 0$

$$\iff \boxed{\hat{h} = \frac{\mu - r}{\sigma^2} \times \left(-\frac{p}{xA}\right)}$$

2nd order condition : $\frac{d^2F}{dh^2} = x^2\sigma^2 A < 0$

Define

$$F(t,x,h,p,A) := x\left[r + h(\mu - r)\right]p + \frac{1}{2}x^2h^2\sigma^2 A$$

The first order condition

$$\frac{\partial F(t,x,h,p,A)}{\partial h} = 0$$

gives us the candidate optimal asset allocation $\hat{h}$

$$\boxed{\hat{h} = \frac{\mu - r}{\sigma^2}\left(-\frac{p}{xA}\right)}$$

$\phi \to$ Optimal utility $\frac{\partial \phi}{\partial x} \to p$ $\frac{\partial^2 \phi}{\partial x^2} \to A$

$\to$ Concave

The expression for $\hat{h}$ works for any p and A, so we can just plug $\frac{\partial \Phi}{\partial x}$ into p and $\frac{\partial^2 \Phi}{\partial x^2}$ into A to get

depends on ϕ

$$
\begin{aligned}
\hat{h} &= \frac{\mu - r}{\sigma^2}\left(-\frac{\frac{\partial \Phi}{\partial x}}{x\frac{\partial^2 \Phi}{\partial x^2}}\right) \\
&= \frac{\mu - r}{\sigma^2}\frac{1}{xA(x)} \\
&= \frac{\mu - r}{\sigma^2}R(x)
\end{aligned}
$$

$$\frac{1}{xA(x)} = R(x)$$

Green return

variance

where

- $A(x)$ is the Arrow-Pratt absolute risk aversion function; and
- $R(x)$ is the relative risk-aversion function.

$\to$ utility function

A few observations:

- The asset allocation $\hat{h} = \frac{\mu - r}{\sigma^2(1-\gamma)}$ represents the allocation to the risky asset. The allocation to the risk-free asset is $\hat{h}_0 := 1 - \hat{h} = \frac{\mu - r}{\sigma^2(1-\gamma)}$.
- The asset allocation $\hat{h} = \frac{\mu - r}{\sigma^2(1-\gamma)}$ is constant. In particular, it is independent of t and T. Such strategy is called **myopic**.

Solving the Terminal utility of Wealth Problem with a CRRA Utility Function

For example, we could chose the constant relative risk aversion (CRRA) utility function called the **power utility function**[4] and defined as:

$$U_T(v) = \frac{v^\gamma}{\gamma} \qquad \lim_{\gamma \to 0} U_T(v) = \log v$$

where the risk-aversion coefficient γ is such that either $\gamma < 0$ or $0 < \gamma < 1$.

We could try the following functional form, or *ansatz*, for the value function:

$$\Phi(t, v) = f(t) \boxed{\frac{v^\gamma}{\gamma}}$$

$$\boxed{\phi(t, v) = f(t) \times \frac{v^\gamma}{\gamma}} \qquad \phi(T, v) = U_T(v)$$

separable

[4] In math finance, the function $v^\gamma, 0 < \gamma < 1$ is also often used. It is not the usual power utility function, but it is a CRRA utility function nonetheless.

$$\phi(t,v) = f(t)\cdot \frac{v^r}{r}$$

$$\hat{h} = \frac{\mu - r}{\sigma^2} \times \left(- \frac{\frac{\partial \phi}{\partial x}}{x \frac{\partial^2 \phi}{\partial x^2}} \right)$$

$$\frac{\partial \phi}{\partial x} = f(t) \times v^{r-1}$$

By substitution, we find that

$$\boxed{f(t) = \exp\{\lambda(T-t)\}}$$

$$\hat{h} = \frac{\mu - r}{\sigma^2} \times \frac{1}{1-\gamma}$$

where

$$\frac{\partial^2 \phi}{\partial x^2} = f(t) \times (r-1) v^{r-2}$$

$$\boxed{\lambda := r + \frac{1}{1-\gamma} \frac{(\mu - r)^2}{2\sigma^2}}$$

$\gamma \to \infty$

$\hat{h} \to 0$

Furthermore, the maximising allocation is given by

$$\boxed{\hat{h} = \frac{\mu - r}{\sigma^2(1-\gamma)}}$$

$\gamma \to 0$

Kelly Investment $\boxed{\hat{h} = \frac{\mu - r}{\sigma^2}}$

$\gamma = 1 \quad \hat{h} = \frac{1}{2} \frac{\mu - r}{\sigma^2}$

Plug ϕ and $\hat{h}$ into the PDE to find $f(t) \to$

Now that we have found

1. a Borel and admissible maximizer $\hat{h} = \frac{\mu - r}{\sigma^2(1-\gamma)}$;
2. a $C^{1,2}$ solution $\Phi(t, v) = f(t)\frac{v^\gamma}{\gamma}$ for the HJB PDE

we just need to apply a **verification theorem** to conclude that we have solved the problem.

To Recap... The Merton Model

The Merton model is one of the few nonlinear stochastic control problems that can be solved explicitly.

The optimal asset allocation is:

$$\boxed{\hat{h} = \frac{\mu - r}{\sigma^2(1-\gamma)}} = \underbrace{\frac{\mu - r}{\sigma^2}}_{\text{market opportunities}} \times \underbrace{\frac{1}{1-\gamma}}_{\text{Risk aversion.}}$$

4. A Reboot of the Merton Problem: Maximizing the Terminal Utility of Wealth with m Risky Asset

Solving investment problems with a single risky asset is a starting point, but it is far from satisfactory. Ideally we would like to solve problems involving an investment universe with an arbitrary number of assets m.

$\phi = f(t) \times \frac{v^r}{r}$

This extension is relatively easy:

- The optimization problem evolves from the maximization of a quadratic scalar form to the maximization a quadratic vector and matrix form;
- The state variable (the investor's wealth) is still one dimensional. This implies that the value function still takes only two scalar arguments: time and wealth.

As a result, the same strategy and ansatz should apply to the m-dimensional case.

$$\boxed{\hat{h} = (\Sigma \Sigma')^{-1}(\mu - r\mathbb{1})}$$

Rather than solving the m risky asset version of the terminal utility problem using the usual stochastic control approach, we will use a change of measure technique proposed independently by Kuroda and Nagai and by Øksendal (see Kuroda and Nagai Kuroda and Nagai (2002), exercise 8.18 in Øksendal Øksendal (2003) and also Fleming Fleming (1999)).

The interest of this method is that under an appropriate change of measure, the terminal utility problem can be solved via a pointwise maximization: no need for a HJB PDE!

Warning! Currently this method only works for terminal utility of wealth, not for consumption! But terminal utility of wealth is the interesting case anyway...

$$h(t) = \begin{pmatrix} h_1(t) \\ \vdots \\ h_m(t) \end{pmatrix} \qquad h_0(t) = 1 - h'(t)\,\underline{\mathbb{1}}$$

Let $\mathcal{G}_t := \sigma(S(s), 0 \leq s \leq t)$ be the sigma-field generated by the security process up to time t.

An *investment strategy* or *control process* is an $\mathbb{R}^m$-valued process with the interpretation that $h_i(t)$ is the fraction of current portfolio value invested in the ith asset, $i = 1, \ldots, m$. The fraction invested in the money market account is then $h_0(t) = 1 - \sum_{i=1}^m h_i(t)$.

Definition : Admissible

An $\mathbb{R}^m$-valued control process $h(t)$ is in class $\mathcal{A}$ if the following conditions are satisfied:

1. $h(t)$ is progressively measurable;
2. $\mathbb{E}\left(\int_0^T |h(s)|^2\, ds < +\infty\right) = 1, \quad \forall T > 0;$
3. the Doléans exponential χ_t^h, given by

Exponential

$$\chi_t^h := \exp\left\{\gamma \int_0^t h(s)'\Sigma dW_s - \frac{1}{2}\gamma^2 \int_0^t h(s)'\Sigma\Sigma' h(s) ds\right\} \quad (11)$$

is an $\boxed{\text{exponential martingale}}$ $t \in [0, T]$, i.e. $\mathbb{E}\left[\chi_T^h\right] = 1$.

Definition

We say that a control process $h(t)$ is *admissible* if $h(t) \in \mathcal{A}$.

The third condition in our definition of Admissibility is required to ensure that h does not wreck everything when we perform the change of measure.

Taking the budget equation into consideration, the wealth, $V(t)$, of the asset in response to an investment strategy $h \in \mathcal{A}$ follows the dynamics

$$\frac{dV(t)}{V(t)} = rdt + h'(t)\left(\mu - r\mathbf{1}\right)dt + h'(t)\Sigma dW_t \tag{12}$$

with initial endowment $V(0) = v$ and where $\mathbf{1} \in \mathbb{R}^m$ is the m-element unit column vector.

$$\mathbb{E}\left[\frac{V^{\gamma}(T)}{\gamma}\right] = \frac{1}{\gamma}\mathbb{E}\left[\boxed{e^{\gamma \ln V(T)}}\right]$$

The objective of an investor with a fixed time horizon T, and using the power utility function $U_T(x) = \frac{v^{\gamma}}{\gamma}$ is to maximize the expected utility of terminal wealth:

$$J(0,h;T,\gamma) \quad = \quad \mathbb{E}\left[U(V(T))\right] = \mathbb{E}\left[\frac{V(T)^{\gamma}}{\gamma}\right] = \mathbb{E}\left[\frac{e^{\gamma \ln V(T)}}{\gamma}\right]$$

with risk aversion coefficient $\gamma \in (-\infty, 0) \cup (0,1)$.

We define the value function Φ corresponding to the maximization of the auxiliary criterion function $J(t, h; T, \gamma)$ as

$$\Phi(t) = \sup_{h \in \mathcal{A}} J(t, h; T, \gamma) \tag{13}$$

By Itô's lemma,

$$\begin{aligned} \ln V(T) &= \ln v + \int_0^T \left\{ r + h'(t)\left(\mu - r\mathbf{1}\right) - \frac{1}{2} h'(t)\Sigma\Sigma' h(t) \right\} dt \\ &\quad + \int_0^T h'(t)\Sigma dW(t) \end{aligned}$$

Multiplying by γ and taking the exponential (no need for Itô here!!!), we get

$$\begin{aligned} e^{\gamma \ln V(T)} &= v^{\gamma} \exp\left\{\gamma \int_0^T \left\{r + h'(t)\left(\mu - r\mathbf{1}\right) - \frac{1}{2}h'(t)\Sigma\Sigma' h(t)\right\} dt \right. \\ &\quad \left. + \gamma \int_0^T h'(t)\Sigma dW(t)\right\} \end{aligned}$$

Where is the Doléans exponential we need to perform the change of measure?

We can see the diffusion part $\int_0^T h'(t)\Sigma dW(t)$ in the exponential but not the matching drift term $\int_0^T h'(t)\Sigma\Sigma' h(t)dt$...

The trick is to add and subtract the missing term $\int_0^T h'(t)\Sigma\Sigma' h(t)dt$ inside the exponential. After a bit of cleaning up, we get

$$e^{\gamma \ln V(T)} = v^{\gamma} \exp\left\{\gamma \int_0^T g(h(t);\gamma)dt\right\} \chi_t^h \tag{14}$$

under $\mathbb{P}$

E.M.

where

$$g(h;\gamma) := \boxed{-\frac{1}{2}(1-\gamma)\, h'\Sigma\Sigma' h + h'(\mu - r\mathbf{1}) + r}$$

and χ_t^h is the Doléans exponential defined in (11)

Here comes the change of measure! We define a measue $\mathbb{P}_h$ via the Radon-Nikodým derivative

$$\frac{d\mathbb{P}_h}{d\mathbb{P}} := \chi_T^h \tag{15}$$

Given an admissible control h, the Brownian motion W_t^h defined as

$$W_t^h = W_t - \gamma \int_0^t \Sigma' h(s)ds$$

is a standard Brownian motion under the measure $\mathbb{P}_h$.

$$\mathbb{E}\left[\beta(0,T) \times 1\right]$$

Moreover, the control criterion under this new measure is

$$I(0, h; T, \gamma) = \frac{v^\gamma}{\gamma}\mathbb{E}^h\left[\exp\left\{\int_0^T \boxed{\gamma g(h(s);\gamma)}\, ds\right\}\right] \tag{16}$$

where $\mathbb{E}^h[\cdot]$ denotes the expectation taken with respect to the measure $\mathbb{P}_h$.
The change of measure has removed the diffusion term!

$\gamma > 0 \quad \to$ maximize $g(h) = -\frac{1}{2}(1-\gamma)h'\Sigma\Sigma' h + h'(\mu - r\mathbf{1}) + r$

1^{st} order $\to \frac{dg}{dh}\Big|_{\hat{h}} = 0$

$(=)$

As a result, we can solve the control problem through a maximization of the function $\gamma g(h(s);\gamma)$ inside the integral!

The candidate optimal control $\hat{h}$ is simply the maximizer of the function $g(h;t,T)$ and is given by

$$\hat{h}(t) = \frac{1}{1-\gamma}(\Sigma\Sigma')^{-1}(\mu - r\mathbf{1}) \tag{17}$$

which represents a position of $\frac{1}{1-\gamma}$ in the Kelly criterion portfolio. The value function $\Phi(t)$, or optimal utility of wealth, equals

$$\Phi(t) = \frac{v^\gamma}{\gamma}\exp\left\{\gamma\left[r + \frac{1}{2(1-\gamma)}(\mu - r\mathbf{1})'(\Sigma\Sigma')^{-1}(\mu - r\mathbf{1})\right](T-t)\right\} \tag{18}$$

$$\frac{\gamma}{1-\gamma} \times (\mu - r\mathbf{1})'\Sigma^{-1}$$

Substituting (17) into (11), we obtain an exact form for the Doléans exponential $\hat{\chi}_t$ associated with the control $\hat{h}$:

$$\hat{\chi}_t := \exp\left\{\frac{\gamma}{1-\gamma}(\mu - r\mathbf{1})'\,\Sigma^{-1}W(t) - \frac{1}{2}\left(\frac{\gamma}{1-\gamma}\right)^2(\mu - r\mathbf{1})'\,(\Sigma\Sigma')^{-1}\,(\mu - r\mathbf{1})\,t\right\} \tag{19}$$

We can easily check that $\hat{\chi}_t$ is an exponential martingale. Therefore $\hat{h}$ is an admissible control and it is optimal, e.g. $h^*(t) = \hat{h}$ and therefore $\chi_t^* = \hat{\chi}_t$.

It turns out that there is also a nice interpretation for the change of measure: the term

$$\Sigma^{-1}(\mu - r\mathbf{1}) \tag{20}$$

inside the Doléans exponential $\hat{\chi}_t$ turns out to be the vector of asset Sharpe ratios!

The optimal wealth $V^*(T)$, that is the wealth resulting from the optimal investment strategy, is explicitly given by

$$\begin{aligned} & V^*(T) \\ = \quad & v \exp\left\{\left[r + \frac{1}{1-\gamma}\left(1 - \frac{1}{2}\frac{1}{1-\gamma}\right)(\mu - r\mathbf{1})'\left(\Sigma\Sigma'\right)^{-1}(\mu - r\mathbf{1})\right](T-t)\right. \\ & \left. + \frac{1}{1-\gamma}(\mu - r\mathbf{1})'\left(\Sigma\Sigma'\right)^{-1}\Sigma(W_T - W_t)\right\} \qquad (21) \end{aligned}$$

To Recap... The Merton Model with m Risky Assets

We used a change of measure to solve the Merton model with m risky assets without having to solve the associated HJP PDE.

The optimal asset allocation

$$\hat{h} = \frac{\mu - r}{\sigma^2(1-\gamma)}$$

is a straightforward generalization of the earlier case with a single risky asset.

5. Risk-Sensitive Control and Risk-Sensitive Asset Management

5.1 Risk Sensitive Control

Risk-Sensitive control uses the same three ingredients as standard stochastic control, namely

1. State process and its dynamics;.
2. Class of control policies;
3. Optimization criterion.

Terminal Reward

The difference is in the optimization criterion. In risk-sensitive control, the decision maker's objective is to select a control policy $h(t)$ to maximize the criterion

$$J(x,t,h;\theta) := -\frac{1}{\theta} \ln \mathbb{E}_{t,x}\left[e^{-\theta F(t,x,h)}\right] \qquad (22)$$

$= F$ if no randomness!

where

- t and x are the time and the state variable;
- F is a given reward function, and;
- the **risk sensitivity** $\theta \in (-1,0) \cup (0,\infty)$ represents the decision maker's degree of risk aversion.

$\boxed{\theta} = -\gamma$

We can view risk-sensitive control as a generalization of classical stochastic control in which the degree of risk aversion or risk tolerance of the optimizing agent is explicitly parameterized in the objective criterion and influences directly the outcome of the optimization.

Therefore, risk-sensitive control differs from traditional stochastic control in that it explicitly models the risk-aversion of the decision maker as an integral part of the control framework, rather than importing it in the problem via an externally defined utility function.

A Taylor expansion of the risk-sensitive criterion around $\boxed{\theta = 0}$ evidences the vital role played by the risk sensitivity parameter:

$$J(x, t, h; \theta) = \mathbb{E}\left[F(x, t, h)\right] - \frac{\theta}{2}\mathbf{Var}\left[F(x, t, h)\right] + O(\theta^2) \tag{23}$$

when

- $\theta \to 0$, we are in the "risk-null" case which corresponds to classical stochastic control;
- $\theta < 0$, we obtain the "risk-seeking" case corresponding to a maximization of the expectation of a convex decreasing function of $F(t, x, h)$;
- $\theta > 0$, we get the"risk-averse" case corresponding to a minimization of the expectation of a convex increasing function of $F(t, x, h)$.

We can immediately see the similarities with Markowitz' **Mean-Variance Analysis**.

- Contrary to the Mean-Variance criterion, the risk-sensitive criterion takes into account all the moments of the distribution.

Historical Note

The initial impetus in the development of a consistent risk-sensitive control theory is due to Jacobson Jacobson (1973) who solved the Linear-Exponential-of-Quadratic Regulator (LEQR) problem, which is the risk-sensitive equivalent of the traditional Linear Regulator problem found in classical control literature. Bensoussan and van Schuppen (1985) and Whittle (1990) greatly contributed to the development of the theory in the complete observation case while a succinct treatment of the partial observation LEQR case can be found in Bensoussan (2004).

Recent developments in the field of risk-sensitive control have been mostly concerned with exploring two avenues. The first is the formulation of a general theory for the resolution of the Hamilton-Jacobi-Bellman Partial Differential Equation associated with the risk-sensitive criterion. While the LEQR case admits an analytical solution and a number of other cases admit a classical $C^{1,2}$ solution, the formalization of a consistent theory has proved elusive. Indeed, proving the existence and uniqueness of a classical solution often requires assumptions on the behaviour of the reward function or bounds on the state space, restriction which would exclude the LEQR problem. Bensoussan et al. (1998) achieved an important step by developing a generalized theory of classical solutions which includes the LEQR problem as a subcase. However, we can still expect developments in this area as the number of applications of risk-sensitive control, and therefore the range of reward functions, grows.

The second avenue is concerned with the application of risk-sensitive control theory to solve specific scientific engineering, economics and financial problems. In the past decade, applications to the fields of finance and economics have strived. Risk-sensitive control was first introduced to finance theory by Lefebvre and Montulet (1994) in a corporate finance context and by Fleming (1995) in a portfolio selection context.

However, Bielecki and Pliska (1999) were the first to regard the continuous time risk-sensitive control as a practical tool that could be used to solve actual portfolio selection problems. The contribution that they brought to the subject is considerable. Kuroda and Nagai (2002), Nagai and Peng (2002) and Fleming and Sheu (2000, 2002)) also provided major contributions to the field.

Risk-Sensitive Asset Management (Virtual) Wall of Fame

Stanley Pliska

Hideo Nagai

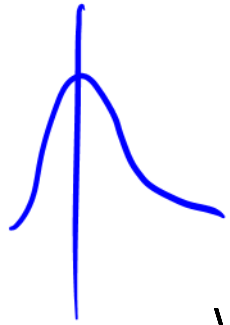
Mark Davis

Tomasz Bielecki

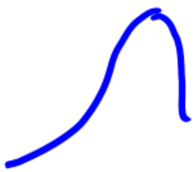
Wendell Fleming

Return from t to T = $\frac{\ln V(T)}{\ln v}$ $\frac{\ln V(T)}{-\ln v}$

In **risk-sensitive asset management model**, we take $\boxed{F_T = \ln V(T)}$, where $V(T)$ is the value of the investment portfolio corresponding to an asset allocation strategy h, then the risk-sensitive criterion becomes:



$$J_{\theta,T}(v,x;h) := \left(-\frac{1}{\theta}\right) \ln \mathbb{E}\, \boxed{e^{-\theta \ln V(T;h)}} \tag{24}$$



We like this criterion because it has two complementary interpretations:

1. **Dynamic "mean-variance" optimization**. A Taylor expansion of the risk sensitive criterion J around $\theta = 0$ yields:

$$J(x,t,h;\theta) = \mathbb{E}\underbrace{[\ln V(t,h)]}_{\ln \text{ return}} - \frac{\theta}{2} \operatorname{var}\underbrace{[\ln V(t,h)]}_{\ln \text{ Return}} + O(\theta^2) \tag{25}$$

odd moments
>0 +
<0 penalty
⊕ Skew
⊖ Kurtosis
is even

2. **Utility Maximization**:

$$\mathbb{E}\left[e^{-\theta \ln V(t)}\right] =: \boxed{\mathbb{E}\left[U(V(t))\right]}$$

defines the expected utility of wealth, under a Hyperbolic Absolute Risk Aversion (HARA) utility function.

Let $\mathcal{G}_t := \sigma((S(s), X(s)), 0 \le s \le t)$ be the sigma-field generated by the security and factor processes up to time t.

Definition

An $\mathbb{R}^m$-valued control process $h(t)$ is in class $\mathcal{H}(T)$ if the following conditions are satisfied:

1. $h(t)$ is progressively measurable with respect to $\{\mathcal{B}([0,t]) \otimes \mathcal{G}_t\}_{t \ge 0}$ and is càdlàg;
2. $P\left(\int_0^T |h(s)|^2 ds < +\infty\right) = 1, \quad \forall T > 0;$

Definition

An $\mathbb{R}^m$-valued control process $h(t)$ is in class $\mathcal{A}(T)$ if the following conditions are satisfied:

1. $h(t) \in \mathcal{H}(T)$;
2. the Doléans exponential χ_t^h, given by

$$\chi_t^h := \exp\left\{-\theta \int_0^t h(s)' \Sigma dW_s - \frac{1}{2}\theta^2 \int_0^t h(s)' \Sigma \Sigma' h(s) ds\right\} \tag{26}$$

   is an exponential martingale, i.e. $\mathbb{E}\left[\chi_T^h\right] = 1$.

The Risk-Sensitive Asset Management Criterion

The wealth, $V(t)$ of the investor in response to an investment strategy $h(t) \in \mathcal{H}(T)$, follows the dynamics

$$\frac{dV(t)}{V(t)} = \overbrace{\left(a_0 + A_0'X(t)\right)}^{\text{money market}} dt + h'(t)\left(\overbrace{\tilde{a} + \tilde{A}X(t)}^{\text{risk premium}}\right) dt + \underbrace{h'(t)\Sigma dW_t}_{\text{NO CHANGE HERE}}$$

where $\tilde{a} := a - a_0\mathbf{1}$, $\tilde{A} := A - \mathbf{1}A_0'$, $\mathbf{1} \in \mathbb{R}^m$ denotes the m-element unit column vector and with initial endowment $V(0) = v$.

The objective of the investor is to maximize the risk-sensitive asset management criterion $J(t, x, h; T, \theta)$

$$J(t, x, h; T, \theta) = -\frac{1}{\theta} \ln \mathbb{E}\left[e^{-\theta \ln V(T)}\right] = -\frac{1}{\theta} \ln \mathbb{E}\left[V(T)^{-\theta}\right]$$

with risk sensitivity $\theta \in (-\infty, 0) \cup (0, 1)$.

in terms of $X(t)$

Apply Itô to ln V(t)

Exponential Martingale ⟹ Change of Measure

By Itô's lemma,

$$\underbrace{e^{-\theta \ln V(t)}} \quad = \quad v^{-\theta} \underbrace{\exp\left\{\theta \int_0^t g(X_s, h(s); \theta) ds\right\}}_{\text{Discount factor}} \chi_t^h \qquad (27)$$

where

$$g(x, h; \theta) \quad = \quad \underbrace{\frac{1}{2}(\theta + 1)\, h' \Sigma \Sigma' h - h'(\tilde{a} + \tilde{A}x) - a_0 - A_0' x} \qquad (28)$$

and the exponential martingale χ_t^h is given by (26).

Solving the Control Problem

$$\frac{d\mathbb{P}_h}{d\mathbb{P}} = \chi_t^h = \exp\left\{-\theta\int_0^T h'(s)\Sigma' dW(s) - \frac{1}{2}\theta^2\int_0^T h'(s)\Sigma\Sigma' h(s)ds\right\}$$

We apply a change of measure argument to derive the control criterion under the measure $\mathbb{P}_h$

$$I(x,t,h;\theta) = -\frac{1}{\theta}\ln \mathbb{E}_{t,x}^h\left[\exp\left\{\theta\int_t^T g(X_s,h(s);\theta)ds\right\}\right] + \ln v \qquad (29)$$

where $\mathbb{E}_{t,x}^h[\cdot]$ denotes the expectation taken with respect to the measure $\mathbb{P}_h$ and with initial conditions (t,x).

Moreover, the dynamics of the state variable $X(t)$ under the new measure is

$$dX(t) = \left(b + BX(t) - \theta\Lambda\Sigma' h(t)\right)dt + \Lambda dW_t^h, \qquad t\in[0,T] \qquad (30)$$

Under $\mathbb{P}_h$, the state $X(t)$ is a controlled process.

Let Φ be the value function for the auxiliary criterion function $I(t, x; h; T, \theta)$. Then Φ is defined as

$$\Phi(t, x) = \sup_{h \in \mathcal{A}(T)} I(t, x; h; T, \theta) \tag{31}$$

$= \sup -\frac{1}{\theta} \ln \mathbb{E}\left[e^{-\theta \int_0^T g(\cdot)\,dt}\right]$

Now we can use the Feynman-Kač formula to write down the HJB PDE associated with the control problem:

$$\frac{\partial \Phi}{\partial t}(t, x) + \sup_{h \in \mathbb{R}^m} L_t^h(t, x, D\Phi, D^2\Phi) \quad = \quad 0 \tag{32}$$

where

$$\begin{aligned} L_t^h(s, x, p, M) \quad &= \quad \left(b + Bx - \theta\Lambda\Sigma' h\right)' \overset{D\phi}{p} + \frac{1}{2}\mathrm{tr}\left(\Lambda\Lambda' M\right) \\ &\quad - \frac{\theta}{2} p'\Lambda\Lambda' p + g(x, h; \theta) \end{aligned} \tag{33}$$

quadratic growth.

for $r \in \mathbb{R}$ and $p \in \mathbb{R}^n$ and subject to terminal condition

$$\Phi(T, x) = \ln v \tag{34}$$

Maximization → Hamiltonian to isolate the terms in R.

To isolate the optimization from the rest of the PDE, we can rewrite the supremum inside the HJB PDE as

$-f(\theta)R'\Sigma\Sigma'R + R'(\tilde{a}+\tilde{A}x)$

$$\begin{aligned}
&\sup_{h\in\mathbb{R}^m}\left\{\left(b+Bx-\theta\Lambda\Sigma' h(s)\right)' p + \frac{1}{2}\mathrm{tr}\left(\Lambda\Lambda' M\right) - \frac{\theta}{2}p'\Lambda\Lambda' p - g(x,h;\theta)\right\}\\
= \quad &\sup_{h\in\mathbb{R}^m}\left\{\left(b+Bx-\theta\Lambda\Sigma' h(s)\right)' p + \frac{1}{2}\mathrm{tr}\left(\Lambda\Lambda' M\right) - \frac{\theta}{2}p'\Lambda\Lambda' p\right.\\
&\left.-\frac{1}{2}\left(\theta+1\right)h'\Sigma\Sigma' h + h'(\tilde{a}+\tilde{A}x) + a_0 + A_0' x\right\}\\
= \quad &(b+Bx)' p + \frac{1}{2}\mathrm{tr}\left(\Lambda\Lambda' M\right) - \frac{\theta}{2}p'\Lambda\Lambda' p + a_0 + A_0' x\\
&+\sup_{h\in\mathbb{R}^m}\left\{-\theta h'\Sigma\Lambda' p - \frac{1}{2}\left(\theta+1\right)h'\Sigma\Sigma' h + h'(\tilde{a}+\tilde{A}x)\right\}
\end{aligned} \tag{35}$$

Merton model!

new term related to the covariance between asset risk and factor risk.

$$= \frac{1}{\theta+1}(\Sigma\Sigma')^{-1}\left(\underbrace{\tilde{a} + \tilde{A}X(t)}_{\text{Risk Premia}}\right) + \frac{\theta}{\theta+1}\left(-(\Sigma\Sigma')^{-1}\Sigma'\Lambda D\Phi\right)$$

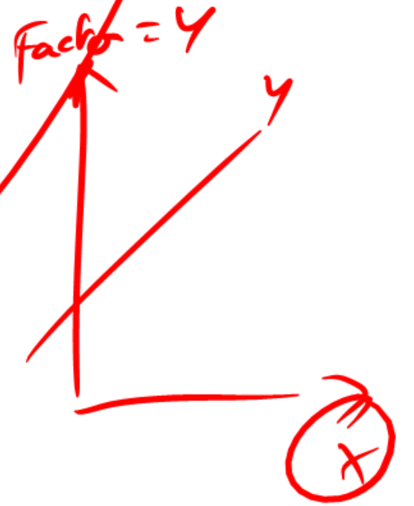
The term inside the sup is quadratic in h. It is globally convex, and as a result it admits a unique maximizer. This maximizer gives us the candidate optimal control:

$$\boxed{\hat{h}(t,x,p) = \frac{1}{\theta+1}\left(\Sigma\Sigma'\right)^{-1}\left[\tilde{a} + \tilde{A}X(t) - \theta\Sigma\Lambda' p\right]} \tag{36}$$

The maximizer depends explicitly on the term p standing in for the first order derivative of the value function Φ with respect to the state variable x.

In fact, this investment strategy is a very similar to the Merton allocation:

- the risk-sensitivity θ has replaced the risk aversion gamma, with the relation $\theta = -\gamma$;
- the drift is factor dependent: $(\tilde{a} + \tilde{A}X(t))$ has replaced $(\mu - r\mathbf{1})$;
- the factor $X(t)$ has generated a new term: $\Sigma\Lambda' p$.



$$(\Sigma\Sigma')^{-1}\Sigma'\Lambda D\Phi$$

asset risk

$Q(t)\,x^2 + q(t)\,x + k(t)$

We will now try to find an analytical solution for the PDE. From the form of the PDE, namely

- the quadratic discounting functional $g(s, h; \theta)$;
- the linear candidate control;

we could try a solution of the form

$$\boxed{\Phi(t,x) = \frac{1}{2}x' Q(t) x + x' q(t) + k(t)} \tag{37}$$

for the HJB PDE. With this choice, $\Phi(t, x)$ is obviously a $C^{1,2}$ function.

The terminal condition $\boxed{\Phi(T, x) = \ln v}$ give us the following terminal conditions for $Q(t)$, $q(t)$ and $k(t)$:

$$\begin{cases} Q(T) = 0 \\ q(T) = 0 \\ k(T) = \ln v \end{cases}$$

Substituting $\hat{h}$ and the functional form (37) in the HJB PDE, we find that:

- The $n \times n$ symmetric matrix Q solves a **Riccati equation** related to the coefficient of the quadratic term and used to determine the symmetric non-negative matrix $Q(t)$,

$$\dot{Q}(t) - Q(t)K_0Q(t) + K_1'Q(t) + Q(t)K_1 + \frac{1}{\theta+1}\tilde{A}'(\Sigma\Sigma')^{-1}\tilde{A} = 0 \quad (38)$$

for $t \in [0, T]$, with *terminal* condition $Q(T) = 0$ and with

$$K_0 = \theta\left[\Lambda\left(I - \frac{\theta}{\theta+1}\Sigma'(\Sigma\Sigma')^{-1}\Sigma\right)\Lambda'\right] \quad (39)$$

$$K_1 = B - \frac{\theta}{\theta+1}\Lambda\Sigma'(\Sigma\Sigma')^{-1}\tilde{A}$$

- a **linear ordinary differential equation** related to the coefficient of the linear term and used to determine the vector $q(t)$,

$$\dot{q}(t) + \left(K_1' - Q(t)K_0\right) q(t) + Q(t)b + \frac{1}{\theta + 1}\left(\tilde{A}' - \theta Q'(t)\Lambda\Sigma'\right)(\Sigma\Sigma')^{-1}\tilde{a}$$
$$= 0 \qquad (40)$$

with *terminal* condition $q(T) = 0$.

- an integral

$$k(s) = \ln v + \int_s^T l(t)dt \tag{41}$$

for $0 \leq s \leq T$ and where

$$\begin{aligned} l(t) \quad = \quad & \frac{1}{2}\mathrm{tr}\left(\Lambda\Lambda' Q(t)\right) - \frac{\theta}{2} q'(t)\Lambda\Lambda' q(t) + b' q(t) \\ & + \frac{1}{\theta+1}\tilde{a}'(\Sigma\Sigma')^{-1}\tilde{a} + \frac{1}{2}\frac{\theta^2}{\theta+1} q'(t)\Lambda\Sigma'(\Sigma\Sigma')^{-1}\Sigma\Lambda' q(t) \\ & - \frac{\theta}{\theta+1} q'(t)\Lambda\Sigma'(\Sigma\Sigma')^{-1}\tilde{a} \end{aligned} \tag{42}$$

Implementation Tip: Transforming a Terminal Value Problem into an Initial Value Problem

Most numerical solvers, including desolve in R, are designed to solve Initial Value Problems, while our two ODEs are Terminal Value problems.

Changing a *Terminal* Value Problem into an *Initial* Value Problem is straightforward: you only need to change variable.

Define

$$\boxed{\tau = T - t,}$$

then the Riccati equation becomes:

$$-\dot{Q}(\tau) - Q(\tau)K_0Q(\tau) + K_1'Q(\tau) + Q(\tau)K_1 + \frac{1}{\theta+1}\tilde{A}'(\Sigma\Sigma')^{-1}\tilde{A} = 0,$$

for $\tau \in [0, T]$ and with *initial* condition $\boxed{Q(0) = 0}$

The same goes for the linear ODE.

Implementation Tip: Dealing with Nested Differential Equations

Our model has nested ODEs: the integral for $I(t)$ depends on both $q(t)$ and $Q(t)$, while the ODE for $q(t)$ depends on $Q(T)$.

To speed up the computation, you can pre-compute $Q(t)$. For $K > 1$, split the interval into K steps of "length" δt with

$$\delta t = \frac{T}{K}$$

$Q(t)$

$n \times n$

and compute $Q(t)$ along the sequence.

Next, store the values of $Q(t)$ at time t by flattening the entries of the matrix into a vector with n^2 entries. This way, you will be able to store the entire sequence of values for $Q(t)$ in a $n^2 \times K$ matrix.

To compute $q(t)$, simply retrieve the appropriate value(s) of $Q(t)$ (you might need to interpolate between values!) from the matrix and use it in your calculation.

To conclude that

- $h^*(t) = \hat{h}(t, x, D\Phi(t, x))$ is the optimal control, and;
- $\tilde{\Phi}(t, x) = \frac{1}{2}x'Q(t)x + x'q(t) + k(t)$ is the value function.

we could either

- use a standard verification theorem on the exponentially transformed problem, or;
- develop a more direct verification argument *à la* Kuroda and Nagai ~~Kuroda and Nagai~~ (2002).

However, this argument would only be valid for the problem associated with the auxiliary criterion I and set under the measure $\mathbb{P}_{h^*}$. We still need to tie two lose ends:

- is the Doléans exponential $\chi_t^{h^*}$ an exponential martingale?
- is $h^*(t)$ optimal for the maximization of the criterion J under the statistical measure $\mathbb{P}$?

The answer to the first question is immediate: by standard argument (see Gihman and Skorokhod Gihman and Skorokhod (1972) for example), $\chi_t^{h^*}$ is an exponential martingale.

The second question is the object of the next proposition:

Proposition

The optimal control $h^*(t)$ *for the auxiliary problem* $\sup_{h \in \mathcal{A}(T)} I(v, x; h; t, T; \theta)$ *where* I *is defined in* (29) *is also optimal for the initial problem* $\sup_{h \in \mathcal{A}(T)} J(x, t, h; \theta)$ *where* J *is defined in* (27).

The optimal wealth $V^{\theta}(T)$ is

$$
\begin{aligned}
& V^{\theta}(T) \\
= \quad & v \exp \Big\{ \Big[\big(a_0 + A_0' X(t)\big) \\
& + \frac{1}{\theta+1} \left(1 - \frac{1}{2} \frac{1}{\theta+1}\right) \left[\tilde{a} + \tilde{A} X(t)\right]' (\Sigma\Sigma')^{-1} \left[\tilde{a} + \tilde{A} X(t)\right] \\
& - \left(\frac{\theta}{\theta+1}\right)^2 (Q(t)X(t) + q(t))' \Lambda\Sigma'(\Sigma\Sigma')^{-1} \left[\tilde{a} + \tilde{A} X(t)\right] \\
& - \frac{1}{2} \left(\frac{\theta}{\theta+1}\right)^2 (Q(t)X(t) + q(t))' \Lambda\Sigma'(\Sigma\Sigma')^{-1} \Sigma\Lambda' (Q(t)X(t) + q(t)) \Big] (T-t) \\
& + \frac{1}{\theta+1} \left[\tilde{a} + \tilde{A} X(t) - \theta\Sigma\Lambda' (Q(t)X(t) + q(t))\right]' (\Sigma\Sigma')^{-1} \Sigma (W_T - W_t) \Big\} \qquad (43)
\end{aligned}
$$

The growth rate of wealth, R_t, defined as

$$R_t^\theta = \ln \frac{V(T)^{-\theta}}{v} \tag{44}$$

follows the dynamics

$$\begin{aligned} dR_t^\theta = & \Big[\big(a_0 + A_0' X(t)\big) \\ & + \frac{1}{\theta+1}\left(1 - \frac{1}{2}\frac{1}{\theta+1}\right)\left[\tilde{a} + \tilde{A}X(t)\right]'(\Sigma\Sigma')^{-1}\left[\tilde{a} + \tilde{A}X(t)\right] \\ & - \left(\frac{-\theta}{\theta+1}\right)^2 (Q(t)X(t) + q(t))' \Lambda\Sigma'(\Sigma\Sigma')^{-1}\left[\tilde{a} + \tilde{A}X(t)\right] \\ & - \frac{1}{2}\left(\frac{-\theta}{\theta+1}\right)^2 (Q(t)X(t) + q(t))' \Lambda\Sigma'(\Sigma\Sigma')^{-1}\Sigma\Lambda'(Q(t)X(t) + q(t))\Big] dt \\ & + \frac{1}{\theta+1}\left[\tilde{a} + \tilde{A}X(t) - \theta\Sigma\Lambda'(Q(t)X(t) + q(t))\right]'(\Sigma\Sigma')^{-1}\Sigma dW_t \end{aligned} \tag{45}$$

To Recap... Risk-Sensitive Asset Management

Risk-sensitive asset management combines

1. Dynamic 'mean-variance' optimization, and;
2. Utility maximization

Following a change of measure, we can reformulate the optimal investment problem as a control problem. This control problem has a fully analytical solution (up to the resolution of a Riccati equation and an vector ODE).

The optimal asset allocation for an investor with risk-sensitivity θ is

$$h^{\theta}(t) = \frac{1}{\theta+1}\left(\Sigma\Sigma'\right)^{-1}\left[\tilde{a} + \tilde{A}X(t) - \theta\Sigma\Lambda'\left(Q(t)X(t) + q(t)\right)\right] \quad (46)$$

Part 2: The Kelly Portfolio and Fractional Kelly Strategies

Diversification is protection against ignorance. It makes little sense if you know what you are doing.

- Warren E. Buffett

1. Log Utility, the Kelly Criterion and the Kelly Portfolio

Even now, you might not be entirely convinced by an approach based on utility theory. You may be thinking that utility is too theoretical and that it does not apply to real world problems.

After all, shouldn't investment management aim at maximizing the return on their funds?

In continuous time, the (log) return is the logarithm of the price, so the criterion simplifies to

$$J(t, v, h) = \mathbb{E}_{t,v}\left[\ln V_T\right]$$

To check that our interpretation is correct, recall that the (log) return of the portfolio over a time horizon T equals

$$\boxed{\ln \frac{V(T)}{V_0} = \ln V(T) - \ln v}$$

where $\ln v$ is a constant, so we might as well ignore it in our optimization.

It turns out that our attempt to escape utility theory has just backfired. The utility of wealth is one of the oldest utility functions, dating back to Bernoulli in the 1738. Unsurprisingly, this utility function is called **log utility**.

The logarithm of wealth has another interpretation in the single processing and gambling literature. It gives rise to the **Kelly criterion**.

In this part of the course, we explore the interplay between economics and signal processing, gambling and investment, to gain insights into the optimal investment strategy.

All these ideas are also very important in practice because...

A number of "great" investors ranging from Keynes to Buffet, Gross and Thorp are or can be viewed as Kelly investors.

John Kelly

Going back to the (very) simple setting of the Merton model with a single risky asset, the dynamics of the wealth process $X(t)$ is given by

wealth

$$\frac{dV(t)}{V(t)} = [r + h_t(\mu - r)]\,dt + h_t\sigma\,dW_t, \qquad X(0) = x$$

Geometric

The solution to this SDE is

$$V(t) = v\exp\left\{\int_t^T \left[r + h_s(\mu - r) - \frac{1}{2}h_s^2\sigma^2\right] ds + \int_t^T h_s\sigma\,dW_s\right\}$$

$$\underbrace{\mathbb{E}\left[\ln V(t)\right] - \ln v}_{\mathbb{E}\ \ln\ \text{Return}} = \mathbb{E}\left[\int_t^T \left\{r + h_s(\mu - r) - \frac{1}{2}h_s^2\sigma^2\right\} ds\right] + \cancel{\mathbb{E}\left[\int_t^T h_s\sigma\,dW_s\right]}$$

0 Itô integral

Plugging into the criterion, we obtain

$$\begin{aligned} J(t,v,h) &= \mathbb{E}_{t,v}\left[\ln V_T\right] \\ &= \ln v + \mathbb{E}_{t,v}\left[\int_t^T \left(r + h_s(\mu - r) - \frac{1}{2}h_s^2\sigma^2\right) ds + \int_0^T h_s \sigma W_s\right] \\ &= \ln v + \mathbb{E}_{t,v}\left[\int_t^T \left(r + h_s(\mu - r) - \frac{1}{2}h_s^2\sigma^2\right) ds\right] \end{aligned}$$

for any admissible control $h \in \mathcal{A}$.

The key point here is that we can compute the value function $\Phi(t, X_t)$

$$\Phi(0,x) := \sup_{h\in\mathcal{A}} \mathbb{E}_{0,v}\left[\ln V(T)\right]$$

directly, through a pointwise maximization of the function $L(t,v,h)$ defined as

$$L(t,v,h) := r + h_t(\mu - r) - \frac{1}{2}h_t^2\sigma^2 = -\frac{1}{2}\sigma^2\left(h_t - \frac{\mu - r}{\sigma^2}\right)^2 + \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 + r$$

$\boxed{\frac{1}{1-\gamma} \cdot \frac{\mu - r}{\sigma^2}}$ $\lim_{\gamma \to 0}$ Power utility → log utility (a.k.a. Kelly ☺)

The unique maximizer of the function L is the constant control

$$\boxed{\hat{h} = \frac{\mu - r}{\sigma^2}} \leftarrow \text{Kelly allocation}$$

which is admissible. It is therefore the optimal control for the problem.

Substituting the maximizer into the criterion function, we conclude that

$$\begin{aligned} \Phi(t) &:= \sup_{h \in \mathcal{A}} \mathbb{E}\left[\ln V(T)\right] \\ &= \ln v + \left(r + \frac{1}{2} \left(\frac{\mu - r}{\sigma} \right)^2 \right) T \end{aligned}$$

The optimal portfolio $h^K = \frac{\mu - r}{\sigma^2}$ is generally called the **Kelly, log-optimal or growth-optimal portfolio**.

The maximization of the log of the state process is also known in the signal processing and gambling literature as the **Kelly criterion**. Early contributions to the theory and application of the Kelly criterion to gambling and investment include Kelly (1956), Latané (1959), Breiman (1961), Thorp (1971) and Markowitz Markowitz (1976). The main reference is undeniably the volume by MacLean, Thorp and Ziemba MacLean et al. (2010a). Readers interested in an historical account of the Kelly criterion and of its use at the gambling table and in the investment industry will refer to Poudstone Poundstone (2005).

Figure : John L. Kelly (1923-1965)

It turns out that several of the most successful investors, including Keynes, Buffett and Gross have used Kelly-style strategies in their funds (see Ziemba (2005), Thorp (2006) and Ziemba and Ziemba (2007) for details).

The Kelly criterion has a number of good as well as bad properties discussed by MacLean, Thorp and Ziemba (MacLean et al., 2010b).

Its 'good' properties extend beyond practical asset management and into asset pricing theory, as the Kelly portfolio is the **numéraire portfolio associated with the physical probability measure**. This observation forms the basis of the 'benchmark approach to finance' proposed by Platen (2006) and Platen and Heath (2006).

The 'bad' properties of the criterion are also well studied and understood. Samuelson, in particular, was a long time critique of the Kelly criterion. A main drawback of the Kelly criterion is that it is inherently a very risky investment.

Figure : Ed Thorp

Figure : William T. Ziemba

Reminder: Kelly As A Numéraire Portfolio

Currency EMM

The Fundamental Asset Pricing Formula states that, given a numéraire pair $(N_t, \mathbb{Q}^N)$, the value at time t of a derivative maturing at time T is the expected value under $\mathbb{Q}^N$ of the terminal cash flow of the contract discounted using the numéraire asset N_t:

$$\chi(t, S_t) \quad = \quad N_t \mathbb{E}^{\mathbb{Q}^N}\left[N_T^{-1} G(S_T) | \mathcal{F}_t\right], \quad t \in [0, T]$$

It turns out that the numéraire asset associated with the real world measure $\mathbb{P}$ is the Kelly portfolio!

So we could price any derivative under the real world measure $\mathbb{P}$ as:

$$\chi(t, S_t) \quad = \quad K_t \mathbb{E}^{\mathbb{P}}\left[K_T^{-1} G(S_T) | \mathcal{F}_t\right], \quad t \in [0, T]$$

where $K(t)$ is the \$ value at time t of an investment in the Kelly portfolio.

This also makes sense from a portfolio perspective! Remember that we were able to maximize the log utility of wealth <u>without</u> changing measure?

To understand what happened, take the limit $\theta \to 0$ in the derivation of the Risk-Sensitive Asset Management model to get the log utility / Kelly case. Then,

$\theta \to 0$

- $\boxed{h^*(t) = (\Sigma\Sigma')^{-1}\left(\tilde{a} + \tilde{A}X(t)\right)}$ as expected;
- $\boxed{\hat{\chi}_t = 1}$ meaning that $\mathbb{P}_h = \mathbb{P}$: we do not need to change measure at all, because we are already in the right numéraire pair.

$\chi_t = 1$

The Kelly criterion is really very special!

2. The Kelly Criterion in Gambling

We will briefly head to the casino to see another aspect of the Kelly criterion: its use in determining the optimal bet size in a gamble. For simplicity, we will focus here on purely random games, such as roulette.

Let's say that we enter the casino with EUR W in our pocket. We seat at a roulette table: the profit or loss on bet i can be viewed as a random variable X_i. We assume that the X_i's are IID and their distribution has a mean μ and variance σ^2.

What fraction f of our starting wealth should we bet on this gamble?

If we decide to place N bets, our terminal wealth will be

$$W \times \prod_{i=1}^{N} (1 + fX_i)$$

where X_i denotes the outcome of bet i.

One possible way of determining f would be to chose it to maximize the expected growth rate of wealth per bet, which is computed as

$$\frac{1}{N}\mathbb{E}\left[\ln\left(\overbrace{W \times \prod_{i=1}^{N}(1+fX_i)}^{\text{terminal wealth}}\right)\right] = \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left[\ln\left(1+fX_i\right)\right] + \frac{1}{N}\ln W$$

We can (and will) ignore the term $\frac{1}{N}\ln W$.

Since the bets are IID, the expected growth rate on N bets simplifies to

$$\mathbb{E}\left[\ln\left(1+fX\right)\right]$$

where X denotes the outcome of a single bet.

If we assume that the expected gain μ is small, a Taylor expansion of the expectation around $f = 0$ yields:

$$\mathbb{E}\left[\ln\left(1+fX\right)\right] \approx f\mu - \frac{1}{2}f^2\sigma^2$$

$(\Sigma^{-1}\Sigma^{-1})^{-1}\left(\mu - r\mathbf{1}\right)$

Maximizing this term yields the optimal fraction f^*

$$\boxed{f^* = \frac{\mu}{\sigma^2}} \leftarrow \frac{\mathbb{E}[R]}{\mathrm{Var}[R]}$$

and the expected growth rate per bet

$$\mathbb{E}\left[\ln\left(1 + f^* X\right)\right] = \frac{1}{2}\frac{\mu^2}{\sigma^2}$$

The optimal growth rate if you place N bets is

$$N\mathbb{E}\left[\ln\left(1 + f^* X\right)\right] = \boxed{\frac{1}{2}\frac{\mu^2}{\sigma^2}N} \checkmark$$

We can make three observations:

- *the sign of μ is critical*. If $\mu > 0$, $f^* > 0$ and we should accept the bet (go long). On the other hand, if $\mu < 0$, we should offer the bet (go short). If we cannot "short," we should refrain from betting.
- *the variance σ^2 simply acts as a scaling factor*. Overall, we should prefer less volatility to more, but volatility in itself does not turn a profitable bet into an unprofitable one.
- *the total expected growth rate of wealth increases with the number of bets N*. If we find profitable bets, we should therefore bet more often.

These conclusion are remarkably similar to Grinold and Khan's Fundamental Law of Active Management (see Grinold and Kahn (1999)).

3. Fund Separation Results

In the risk-sensitive asset management model, the optimal asset allocation is:

$$h^{\theta}(t) = \frac{1}{\theta+1}\left(\Sigma\Sigma'\right)^{-1}\left[\tilde{a} + \tilde{A}X(t) - \theta\Sigma\Lambda'\left(Q(t)X(t) + q(t)\right)\right]$$

$\theta \to 0$

Key Insight: Fund Separation

This optimal strategy is constituted of positions in two funds:

- An allocation of a fraction $\frac{1}{\theta+1}$ of wealth to the **Kelly portfolio**:

$$\boxed{h^{K}(t) = \left(\Sigma\Sigma'\right)^{-1}\left(\tilde{a} + \tilde{A}X(t)\right)} \tag{47}$$

- An allocation of a fraction $\frac{\theta}{\theta+1}$ of wealth to the **'intertemporal hedging portfolio'**: $\theta \to \infty$

$$h^{I}(t) = -\left(\Sigma\Sigma'\right)^{-1}\Sigma\Lambda'\left(Q(t)X(t) + q(t)\right] \tag{48}$$

$\frac{1}{\theta+1} + \frac{\theta}{\theta+1} = 100\%$

The term $\left(\Sigma\Sigma'\right)^{-1}\Sigma\Lambda'$ is a projection of the factor risk onto the subspace of asset risk. It is akin to an OLS estimation of non-tradable factor risk using tradable asset risk. The objective here is to use the assets to hedge the risk coming from the factors.

$h^{\infty} = \lim_{\theta \to \infty} h^{\theta}$

$\boxed{\frac{1}{1-\gamma}} \times h^K$ $\qquad h^K = (\Sigma_t \Sigma_t')^{-1} (\mu - r \mathbb{1})$

In the special case of the Merton model, we get Merton's Fund Separation Theorem as a corollary:

Key Insight: Merton's Fund Separation Theorem

Any portfolio can be expressed as a linear combination of investments in the Kelly (log-utility) portfolio

$$\boxed{h^K(t) = (\Sigma\Sigma')^{-1} (\mu - r\mathbf{1})} \tag{49}$$

and the risk-free rate. Moreover, if an investor has a risk sensitivity γ, the proportion of the Kelly portfolio will equal $\frac{1}{1-\gamma}$.

$\frac{1}{1-\gamma}$ % of your wealth → log utility (Kelly) portfolio

→ cash.

4. Fractional Kelly Strategies

The Kelly portfolio is an inherently risky strategy. To address this shortcoming, Blazenko and Ziemba (1987) propose the following *fractional* Kelly strategy:

- invest a fraction f of one's wealth in the Kelly portfolio, and;
- invest a proportion $1 - f$ in the risk-free asset.

MacLean, Sanegre, Zhao and Ziemba MacLean et al. (2004) MacLean, Ziemba and Li MacLean et al. (2005) pursued further research in this direction.

There are two key advantages to this definition:

1. A fractional Kelly strategy is much less risky than the full Kelly portfolio, while maintaining a significant part of the upside.

2. By choosing $f = \frac{1}{1-\gamma}$, fractional Kelly strategies appear as a corollary to the Fund Separation Theorem. In fact, fractional Kelly strategies are optimal in the continuous time setting of the Merton (1971) model.

Unfortunately, fractional Kelly strategies are no longer optimal when the basic assumptions of the Merton model, such as the lognormality of asset prices, are removed (see MacLean, Ziemba and Li MacLean et al. (2005)). This shortcoming can be addressed easily (see Davis and Lleo (2008) and Davis and Lleo (2012)).

For most practitioners and investors it might be easier, or at least more concrete, to think about risk in terms of fractional Kelly strategies than in terms of risk aversion coefficient and utility function.

In the Merton world, the Fund Separation Result leads naturally to an interpretation as a fractional Kelly strategy:

F S Result

$\gamma \to f$

Key Insight: Fractional Kelly Strategies in the Merton World

Any portfolio with risky allocation $h = f \times h^K$, $f \in (0, \infty)$ where h^K is the Kelly portfolio with risky allocation

$$h^K(t) \quad = \quad \frac{\mu - r}{\sigma^2}$$

is optimal. Moreover, the risk aversion coefficient γ of a CRRA investor is

$$\boxed{\gamma = \frac{f-1}{f}}$$

if $f \in \mathbb{R} \setminus \{1\}$. The case $f = 1$ corresponds to the log utility.

In its traditional definition, fractional Kelly strategies are only optimal in the Merton world. However, we can easily generalize this definition by observing that in the factor-based risk-sensitive asset management model, the lowest risk portfolio is not the money market instrument: it is the intertemporal hedging portfolio!

Key Insight: generalized Fractional Kelly Strategies

Any portfolio with a fraction $f > 0$ invested in the Kelly portfolio with risky allocation

$$h^K(t) = (\Sigma\Sigma')^{-1}\left(\tilde{a} + \tilde{A}X(t)\right) \tag{50}$$

and a fraction $1 - f$ invested in the interemporal hedging portfolio with risky allocation

$$h^I(t) = (\Sigma\Sigma')^{-1}\Sigma\Lambda'\left(q(t) + Q(t)X(t)\right) \tag{51}$$

is optimal. Moreover, the correspondance between Kelly fraction f and risk sensitivity θ is

$$f = \frac{1}{\theta + 1} \Leftrightarrow \boxed{\theta = \frac{1-f}{f}}$$

4. So, How Should We Invest Our Portfolio?

In the factor-based setting of the risk-sensitive asset management model, the optimal investment strategy for an investor with risk-sensitivity θ is

$$h^{\theta}(t) = \frac{1}{\theta+1}\left(\Sigma\Sigma'\right)^{-1}\left[\tilde{a} + \tilde{A}X(t) - \theta\Sigma\Lambda'\left(Q(t)X(t) + q(t)\right)\right] \tag{52}$$

and the Kelly portfolio is given by

$$h^{K}(t) = \left(\Sigma\Sigma'\right)^{-1}\left(\tilde{a} + \tilde{A}X(t)\right) \tag{53}$$

Kelly

The risk aversion coefficient θ has a profound influence over the optimal investment strategy and the optimal measure. In particular, the functions $Q(t)$, $q(t)$ and $k(t)$ depend on θ.

Next we look at three examples:

1. the Kelly portfolio;
2. full risk aversion with uncorrelated assets and factors; $\Sigma\Lambda' = 0$
3. overbetting.

$(\hat{a} + \hat{A}X)'(\Sigma' \Sigma)^{-1}(\mu - r1)$

The Kelly Portfolio

Our conclusions in the incomplete market setting of the factor model mirror that of the complete market setting of the initial Merton model. $\hat{a} + \hat{A}X$

The measures $\mathbb{P}_h$ is still the physical measure $\mathbb{P}$ in the limit as $\theta \to 0$, that is in the log utility or Kelly criterion case. $\frac{(\mu - r)^2}{\sigma^2}$

The wealth of an investment in the Kelly portfolio is

$$
\begin{aligned}
V^*(T) &= v \exp\Big\{\Big(\left[a_0 + A_0' X(t)\right] + \frac{1}{2}\left[\tilde{a} + \tilde{A}X(t)\right]' (\Sigma\Sigma')^{-1}\left[\tilde{a} + \tilde{A}X(t)\right]\Big)(T - t) \\
&\quad + \left[\tilde{a} + \tilde{A}X(t)\right]' (\Sigma\Sigma')^{-1}\Sigma(W_T - W_t)\Big\}
\end{aligned}
\tag{54}
$$

$\ln V^*(T) - \ln v$ money market SR^2

and the growth rate follows the dynamics

$$
\begin{aligned}
dR_t^* &= \Big(\left[a_0 + A_0' X(t)\right] + \frac{1}{2}\left[\tilde{a} + \tilde{A}X(t)\right]' (\Sigma\Sigma')^{-1}\left[\tilde{a} + \tilde{A}X(t)\right]\Big) dt \\
&\quad + \left[\tilde{a} + \tilde{A}X(t)\right]' (\Sigma\Sigma')^{-1}\Sigma\, dW_t
\end{aligned}
\tag{55}
$$

SR

In the ICAPM, the factor-dependent excess return $\tilde{a} + \tilde{A}X(t)$ has replaced the constant term $\mu - r\mathbf{1}$ found in the Merton model.

Full Risk Aversion with Uncorrelated Assets and Factors

$\theta \to \infty$

because $\Sigma' \Lambda' = 0$

When $\theta \to \infty$ and $\boxed{\Sigma\Lambda' = 0}$ the optimal strategy is to invest in the short term asset:

$$h^{\infty}(t) = \lim_{\theta \to \infty} \underbrace{\cancel{\frac{1}{\theta + 1}}}_{0 \text{ in the limit}} \left(\Sigma\Sigma'\right)^{-1} \left[\tilde{a} + \tilde{A}X(t) - \cancel{\theta\Sigma\Lambda'\left(Q(t)X(t) + q(t)\right)}\right] = 0 \tag{56}$$

As a result, the optimal wealth $V^{\theta}(T)$ is given by

$$V^{\theta}(T) \quad = \quad v \exp\left\{\int_0^T \boxed{a_0 + A_0' X(t)} dt\right\} \tag{57}$$

The value function $\Phi(t, x)$ is the price of a zero-coupon bond maturing at time T with a stochastic discount rate.

This observation is consistent with literature on affine processes such as Duffie and Kan (1996) and Duffie et al. (2003).

The Doléans exponential χ_t^∞ is:

$$\chi_t^\infty := \exp\Big\{ -\int_0^t \left(\tilde{a} + \tilde{A}X(s)\right)(\Sigma\Sigma')^{-1}\Sigma dW(s) \\ -\frac{1}{2}\int_0^t \left(\tilde{a} + \tilde{A}X(s)\right)' (\Sigma\Sigma')^{-1} \left(\tilde{a} + \tilde{A}X(s)\right)' ds \Big\} \tag{58}$$

In this case, $\mathbb{P}_\infty$ is the $\boxed{T\text{-bond}}$ measure.

The existence of a correlation structure between the assets and the factors, that is $\Sigma\Lambda' \neq 0$, leads to a more complicated case as it raises the possibility that the money market asset is not the least risky asset. Instead a low risk portfolio could be constituted by trading the risky assets. This idea generalizes the notion of 0-beta portfolio proposed by Black (1972).

Overbetting

$\theta = -\frac{1}{2}$

Investing in twice the Kelly portfolio is only optimal if the position is financed by shorting the intertemporal hedging portfolio. This corresponds to a risk aversion coefficient $\theta = -\frac{1}{2}$:

when $\Sigma'\Lambda' = 0$

$$h^{-1/2}(t) = \boxed{2\left(\Sigma\Sigma'\right)^{-1}\left[\tilde{a} + \tilde{A}X(t)\right.} + \cancel{\left.\frac{1}{2}\Sigma\Lambda'\left(Q(t)X(t) + q(t)\right)\right]} = 0 \qquad (59)$$

The corresponding optimal measure $\mathbb{P}_h$ is defined via the Doléans exponential χ_t^*:

$$\begin{aligned} \chi_t^{-\frac{1}{2}} \quad := \quad & \exp\left\{\int_0^t \left[\tilde{a} + \tilde{A}X(s) + \frac{1}{2}\Sigma\Lambda'\left(Q(s)X(s) + q(s)\right)\right]'\left(\Sigma\Sigma'\right)^{-1}\Sigma dW(s)\right. \\ & -\frac{1}{2}\int_0^t \left(\tilde{a} + \tilde{A}X(s) + \frac{1}{2}\Sigma\Lambda'\left(Q(s)X(s) + q(s)\right)\right)' \\ & \left.\times\left(\Sigma\Sigma'\right)^{-1}\left(\tilde{a} + \tilde{A}X(s) + \frac{1}{2}\Sigma\Lambda'\left(Q(s)X(s) + q(s)\right)\right)' ds\right\} \qquad (60) \end{aligned}$$

The optimal wealth of the Kelly portfolio is

$$
\begin{aligned}
V_{-\frac{1}{2}}(T) \quad = \quad & v \exp \Big\{ \Big[\big(a_0 + A_0' X(t) \big) \\
& - \left(Q(t)X(t) + q(t) \right)' \Lambda \Sigma' (\Sigma\Sigma')^{-1} \left[\tilde{a} + \tilde{A} X(t) \right] \\
& - \frac{1}{2} \left(Q(t)X(t) + q(t) \right)' \Lambda \Sigma' (\Sigma\Sigma')^{-1} \Sigma \Lambda' \left(Q(t)X(t) + q(t) \right) \Big] (T - t) \\
& + 2 \left[\tilde{a} + \tilde{A} X(t) + \frac{1}{2} \Sigma \Lambda' \left(Q(t)X(t) + q(t) \right) \right]' (\Sigma\Sigma')^{-1} \Sigma (W_T - W_t) \Big\}
\end{aligned}
\tag{61}
$$

...

...and the growth rate is

$$
\begin{aligned}
dR_{-\frac{1}{2}}(t) \quad = \quad & \Big[\big(a_0 + A_0' X(t)\big) \\
& - \left(Q(t)X(t) + q(t)\right)' \Lambda\Sigma'(\Sigma\Sigma')^{-1}\left[\tilde{a} + \tilde{A}X(t)\right] \\
& - \frac{1}{2}\left(Q(t)X(t) + q(t)\right)' \Lambda\Sigma'(\Sigma\Sigma')^{-1}\Sigma\Lambda'\left(Q(t)X(t) + q(t)\right)\Big] dt \\
& + 2\left[\tilde{a} + \tilde{A}X(t) + \frac{1}{2}\Sigma\Lambda'\left(Q(t)X(t) + q(t)\right)\right]' (\Sigma\Sigma')^{-1}\Sigma dW_t
\end{aligned}
\tag{62}
$$

When $\theta = -\frac{1}{2}$, the investor's expected return is the short term rate plus a term related to the intertemporal hedging portfolio.

$\Theta = -\frac{1}{2}$ $\Sigma \Lambda' = 0$

So, what is happening here?

$2\times$

To see more clearly, let's assume that the risk of the factors is unrelated to the securities risk, i.e. $\boxed{\Sigma\Lambda' = 0}$. In this case, the term related to the intertemporal hedging portfolio vanishes:

Risk of Kelly Portfolio

$$dR_{-\frac{1}{2}}(t) = \underbrace{\boxed{\left(a_0 + A_0' X(t)\right)}}_{\text{Money Market rate!}} dt + 2\left(\tilde{a} + \tilde{A}X(t)\right)' (\Sigma\Sigma')^{-1}\Sigma dW_t \tag{63}$$

- the expected growth rate of the portfolio equals the return on the money market account;
- the volatility of the growth rate equals twice that of the Kelly or log-utility portfolio.

The investor is giving up a compensation for the risk taken in order to double the volatility of his portfolio!

Do NOT Overbet!

Overbetting trades away the portfolio's expected return to get some extra volatility.

This is akin to selling 'skill' for 'luck' while leveraging: sooner or later this is a losing bet!

When you combine ignorance and leverage, you get some pretty interesting results.

- Warren E. Buffett

Skill vs. Luck

Identifying genuine skill is difficult because it requires a significant history.

Grinold and Kahn (1999) suggest a categorisation of managers around the two axes of skill and luck.

Skill is relatively stable over time, but luck may shift, so managers are likely to change category throughout their career. Keynes is a notable example.

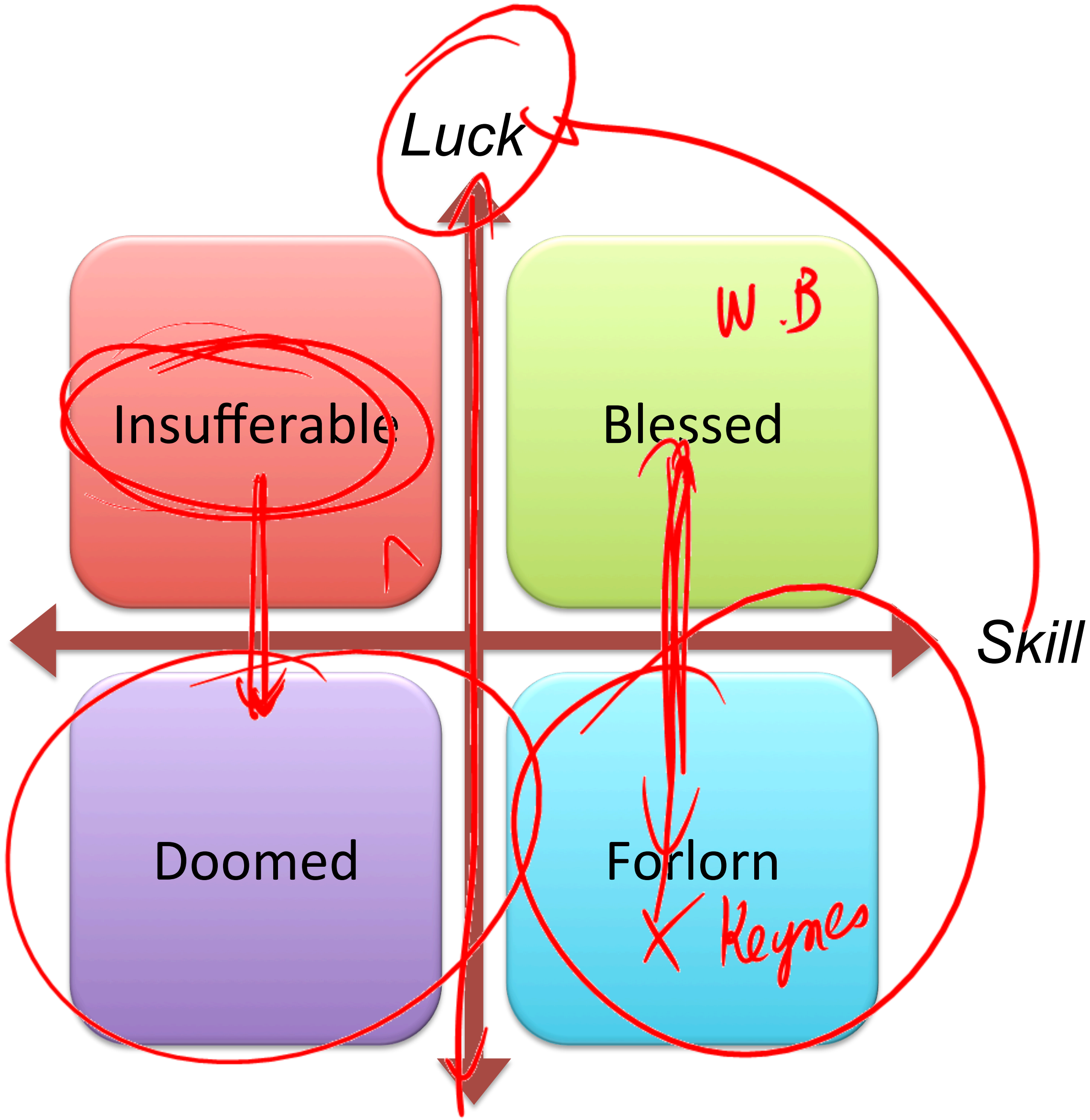
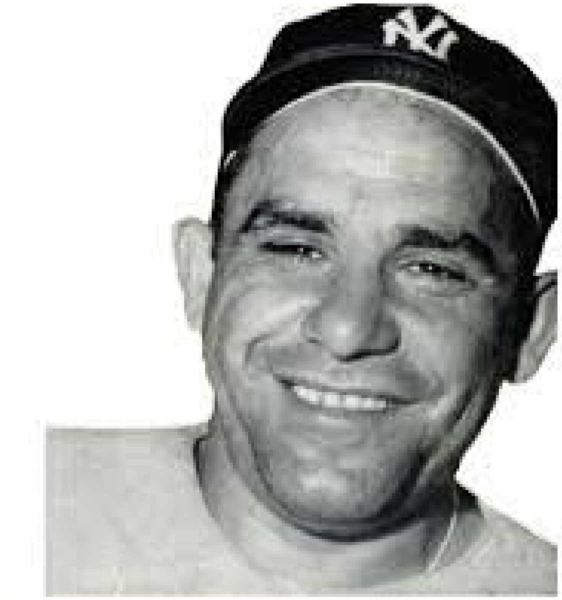


Figure : Distinguishing Skill from Luck

Part 3: Implementation



In theory, there is no difference between theory and practice. In practice, there is.

- Yogi Berra (1925-2015)

In the final part of the course, we look at four implementations of the Risk-Sensitive Asset Management model. By order of increasing sophistication,

1. Standard implementation with **full observation**;
2. Dynamic update with **partial observation**;
3. Dynamic Black-Litterman model to incorporate analyst views and expert optinions;
4. Behavioural Black-Litterman model, because analysts and experts may exhibit psychological biases.

Data

To show how to implement these models, we consider the case of Irene, an investor who manages a portfolio of $m = 11$ U.S. Exchange Traded Funds (ETFs).

Asset	Sector	ETF Ticker	Weight in the S&P 500
S_1	S&P500 Technology	XLK	21.77%
S_2	S&P500 Financials	XLF	16.44%
S_3	S&P500 Health Care	XLV	13.85%
S_4	S&P500 Consumer Discretionary	XLY	11.75%
S_5	S&P500 Industrial	XLI	10.34%
S_6	S&P500 Consumer Staples	XLP	9.89%
S_7	S&P500 Energy	XLE	9.34%
S_8	S&P500 Materials	XLB	3.39%
S_9	S&P500 Utilities	XLU	3.23%
S_{10}	Russell 2000	IWM	-
S_{11}	U.S. Real Estate Investment Trusts (REITs)	IYR	-

We collect $K = 739$ discounted weekly log returns from July 26, 2000 to August 31, 2014

The $n = 3$ factors follow the Fama-French model (see Fama and French, 1993):

- X_1 is the market risk premium;
- X_2 captures the difference in risk premia between small market capitalisation stocks and high market capitalisation stocks;
- X_3 represents the difference in risk premia between stocks with a high book-to-market ratio and stocks with a low book-to-market ratio.

The weekly data come from Kenneth French's online database.

The rate of return of the money market instrument is the weekly rate of a 1-month Treasury Bill as computed by Kenneth French based on rates provided by Ibbotson and Associates.

- We need this rate to discount the asset prices.

Implementation Tip: How much data is enough?!

Overall, you want to have as much data as you can get. This will provide a more accurate parameter estimation. As importantly (or even more importantly!), this will ensure that your data reflect a broader range of historical scenarios including some crashes and bubbles.

In terms of frequency, daily data tend to be too noisy while annual data give you a short time series. The best compromise is to use weekly or monthly data. Monthly data is appropriate for longer term investors such pension funds, endowment funds and insurance companies.

Before getting started, with the implementation, we rewrite the model in a more convenient way.

Implementation Tip: Use Discounted Assets!

> In portfolios selection, use discounted asset prices rather than nominal asset prices.

This trick dates back to at least Black and Litterman (1990, 1991, 1992). The point here is that the optimal investment policy ultimately depends on the excess return over the short-term rate, or **risk premia**, not on the nominal rates of return.

In our continuous time setting, the risk premium is the logarithm of the discounted prices while the nominal return equals the logarithm of the nominal price.

Discounted Asset Prices and Excess Returns

Considering discounted asset prices has three big advantages:

- it makes your model a bit easier to solve for;
- it reduces the number of parameter you need to estimate;
- the stochasticity of the money market / risk-free asset is absorbed inside the discounted asset price. You no longer need to worry about whether your money market asset is truly risk-free or only locally risk-free.

For the rest of the session, we consider discounted prices.

Denote by $\tilde{S}_i(t) = \frac{S_i(t)}{S_0(t)}$ **the discounted price of asset** $i = 1, \ldots, m$. The dynamics of $\tilde{S}_i(t)$ is given by the diffusion SDE

$$\frac{d\tilde{S}_i(t)}{\tilde{S}_i(t)} = (\tilde{a}(t) + \tilde{A}(t)X(t))_i dt + \sum_{k=1}^{d} \Sigma_{ik}(t) dW_k(t),$$

$$\tilde{S}_i(0) = \tilde{s}_i, \quad i = 1, \ldots, m, \tag{64}$$

We also define **excess returns** as

$$\mathfrak{s}_i(t) = \ln(\tilde{S}_i(t)), \quad i = 1, \ldots, m$$

The dynamics of the excess returns is given by

$$d\mathfrak{s}_i(t) \quad = \quad \left[(\tilde{a}(t) + \tilde{A}(t)X(t))_i - \frac{1}{2}\Sigma\Sigma_{ii}(t)'\right] dt + \boxed{\sum_{k=1}^{d} \Sigma_{ik}(t)dW_k(t)}$$

$$\mathfrak{s}_i(0) = \ln \tilde{s}_i, \quad i = 1, \ldots, M. \tag{65}$$

Quad Var

$\to \quad \Sigma_i \Sigma_i'(t$

The drift of the discounted asset prices is subject to the evolution of a n-dimensional vector of **factors** $X(t)$ modelled using the 'usual' affine dynamics:

$$dX(t) = (b(t) + B(t)X(t))dt + \Lambda(t)dW(t), \qquad X(0) = x. \tag{66}$$

Implementation 1: Standard Model with Full Observation

Our first implementation is a purely statistical exercise. The key assumptions are that:

1. Our model is a reasonable description of the dynamics of asset prices;
2. $X(t)$ contains all the factors that are relevant to describe the evolution of the asset price drift;
3. The assets are traded openly and their prices are observable;
4. The factors are observable;
5. Historical data contain all of the information we will ever need to estimate the parameters of our model.

Estimating the Diffusion Parameters $\Sigma\Sigma'$, $\Lambda\Lambda'$ and $\Sigma\Lambda'$

- We use the definition of the quadratic variation of $\mathfrak{s}(t) = \ln \tilde{S}(t)$, that is:

$$\langle \mathfrak{s}, \mathfrak{s} \rangle_t = \lim_{\Delta t_k \to 0} \sum_{t_k \le t} \underbrace{\left(\mathfrak{s}(t_{k+1}) - \mathfrak{s}(t_k)\right)}_{\text{variatio}} \left(\mathfrak{s}(t_{k+1}) - \mathfrak{s}(t_k)\right)' = \boxed{\Sigma\Sigma' t,} \qquad (67)$$

K=739

  to estimate the historical diffusion matrix $\Sigma\Sigma'$.
- Similarly for the diffusion matrix $\Lambda\Lambda'$ and for the cross variation term $\Sigma\Lambda'$.
- In our case,

$$\Sigma\Sigma' \approx \begin{pmatrix}
0.12 & 0.08 & 0.05 & 0.08 & 0.08 & 0.03 & 0.06 & 0.07 & 0.03 & 0.08 & 0.06 \\
0.08 & 0.19 & 0.07 & 0.11 & 0.11 & 0.05 & 0.09 & 0.10 & 0.05 & 0.11 & 0.12 \\
0.05 & 0.07 & 0.06 & 0.05 & 0.05 & 0.03 & 0.05 & 0.05 & 0.03 & 0.05 & 0.05 \\
0.08 & 0.11 & 0.05 & 0.11 & 0.08 & 0.04 & 0.07 & 0.09 & 0.04 & 0.09 & 0.08 \\
0.08 & 0.11 & 0.05 & 0.08 & 0.09 & 0.04 & 0.08 & 0.09 & 0.04 & 0.09 & 0.08 \\
0.03 & 0.05 & 0.03 & 0.04 & 0.04 & 0.04 & 0.04 & 0.04 & 0.03 & 0.04 & 0.04 \\
0.06 & 0.09 & 0.05 & 0.07 & 0.08 & 0.04 & 0.14 & 0.10 & 0.05 & 0.09 & 0.07 \\
0.07 & 0.10 & 0.05 & 0.09 & 0.09 & 0.04 & 0.10 & 0.12 & 0.04 & 0.09 & 0.08 \\
0.03 & 0.05 & 0.03 & 0.04 & 0.04 & 0.03 & 0.05 & 0.04 & 0.06 & 0.04 & 0.04 \\
0.08 & 0.11 & 0.05 & 0.09 & 0.09 & 0.04 & 0.09 & 0.09 & 0.04 & 0.11 & 0.09 \\
0.06 & 0.12 & 0.05 & 0.08 & 0.08 & 0.04 & 0.07 & 0.08 & 0.04 & 0.09 & 0.13
\end{pmatrix}$$

S&P500 ←

$$
\Sigma\Lambda' \approx
\begin{array}{c}
\begin{array}{ccc} x_1 & \approx 0 & ?\approx 0 \end{array} \\
\begin{pmatrix}
0.06 & 0.01 & 0.01 \\
0.09 & 0.00 & 0.03 \\
0.04 & -0.00 & 0.00 \\
0.07 & 0.01 & 0.01 \\
0.07 & 0.01 & 0.01 \\
0.04 & -0.00 & 0.00 \\
0.07 & 0.00 & 0.01 \\
0.07 & 0.01 & 0.01 \\
0.04 & -0.00 & 0.01 \\
0.07 & 0.02 & 0.01 \\
0.07 & 0.01 & 0.02
\end{pmatrix}
\end{array}, \qquad
\Lambda\Lambda' \approx
\begin{pmatrix}
0.0726 & 0.0062 & 0.0047 \\
0.0062 & 0.0147 & -0.0024 \\
0.0047 & -0.0024 & 0.0189
\end{pmatrix}.
$$

9

based on $K = 739$ discounted weekly log returns from July 26, 2000 to August 31, 2014.

$dX(t) = (b + BX(t))dt + \Lambda dW(t)$ $\quad X(t) \quad X_t$

Estimating the Drift Parameters b and B of $X(t)$

Discretize $X_{t+1} - X_t = (b + BX_t)\Delta t + \Lambda Z_t \sqrt{\Delta t}$

- To estimate the drift parameters b and B, we start by discretizing the SDE for $X(t)$, defined at (66), as:

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ Gaussian

$$AR(1):\ X_{t+1} = (b\Delta t + (B\Delta t + I_m)X_t) + \Lambda Z_t^X \sqrt{\Delta t},$$

where I_m is the $m \times m$ identity matrix and Z_t^X is a d-dimensional standard normal random variable for every t.

- Because Δt is fixed in our dataset, we can express the dynamics of X_t as a first order vector autoregressive VAR(1) process:

R

ar(.)

$$\boxed{X_{t+1} = b_1 + B_1 X_t + \Lambda Z_t \sqrt{\Delta t},} \tag{68}$$

where $b_1 := b\Delta t$ and $B_1 := I_m + B\Delta t$.

- Reversing the definition of b_1 and B_1, we obtain the following Maximum Likelihood estimates for b and B:

$$\hat{b} = \begin{pmatrix} -7.77E^{-04} \\ 6.79E^{-05} \\ -5.03E^{-05} \end{pmatrix}, \ \hat{B} = \begin{pmatrix} -56.00 & 9.84 & 3.32 \\ -0.40 & -53.32 & 3.40 \\ -2.16 & 2.94 & -50.64 \end{pmatrix}. \tag{69}$$

Estimating the Drift Parameters $\tilde{a}$ and $\tilde{A}$ of $\tilde{S}(t)$

lm(.)

We can obtain the drift parameters, a and A, via an $\boxed{\text{OLS regression}}$.

- Start by discretizing the process $\mathfrak{s}(t)$ given at (64) as:

$$\Delta\mathfrak{s}_{it} = \left(\tilde{a}_i - \frac{1}{2}\Sigma\Sigma'_{ii}\right)\Delta t + \left(\tilde{A}X_t\right)_i \Delta t + (\Sigma Z_t^{\mathfrak{s}})_i\sqrt{\Delta t}, \qquad (70)$$

  where $\Delta\mathfrak{s}_t = \mathfrak{s}_{t+1} - \mathfrak{s}_t$, and $Z_t^{\mathfrak{s}}$ is a d-dimensional standard normal variable for every t.

- Notice that X_t should be independent from the time discretization scheme. However, the values of X_t in the Fama-French database are weekly returns, and as such are a function of the discretization scheme.

- To address this inconsistency, we consider the following linear model instead:

$$\boxed{y_{it} = \alpha_i + \beta x_t + \epsilon_{it},} \qquad (71)$$

linear model (can be estimated using an OLS or GLS, ML)

where $y_{it} = \frac{\Delta\mathfrak{s}_t}{\Delta t}$, $\alpha_i = \tilde{a}_i - \frac{1}{2}\Sigma\Sigma'_{ii}$, $\beta_i = \tilde{A}[i,\cdot]'\Delta t$ is the ith row of matrix $\tilde{A}$ transposed, $x_t = \frac{X_t}{\Delta t}$ and ϵ_{it} is an error term.

Parameter	Estimate	Standard Error	t-Statistics	p-value
XLK				
Intercept	-0.06241	0.04530	-1.378	0.1687
X_1	0.92154	0.03479	26.486	$< 2e^{-16}$ ***
X_2	0.16212	0.07642	2.121	0.0342 *
X_3	-0.20861	0.06451	-3.234	0.0013 **
Adjusted R^2	$\boxed{0.5112}$			
Std. Error Resid.	1.224			
F-statistic	258.3			$< 2.2e^{-16}$ ***
XLF				
Intercept	-0.12349	0.03647	-3.386	0.0007 ***
X_1	1.20397	0.02801	42.983	$< 2e^{-16}$ ***
X_2	-0.09319	0.06153	-1.515	0.1303
X_3	1.09226	0.05193	21.032	$< 2e^{-16}$ ***
Adjusted R^2	0.7791			
Std. Error Resid.	0.9857			
F-statistic	868.7			$< 2.2e^{-16}$ ***
XLV				
Intercept	0.02707	0.03260	0.830	0.4070
X_1	0.67802	0.02504	27.077	$< 2e^{-16}$ ***
X_2	-0.25550	0.05500	-4.645	$4.02e^{-06}$ ***
X_3	-0.02012	0.04643	-0.433	0.6650
Adjusted R^2	0.5013			
Std. Error Resid.	0.8812			
F-statistic	248.3			$< 2.2e^{-16}$ ***

Parameter	Estimate	Standard Error	t-Statistics	p-value
XLY				
Intercept	-0.01003	0.03399	-0.295	0.7680
X_1	0.92990	0.02610	35.626	$< 2e^{-16}$ ***
X_2	0.28379	0.05733	4.950	$9.21e^{-07}$ ***
X_3	0.34332	0.04839	7.094	$3.07e^{-12}$ ***
Adjusted R^2	0.6795			
Std. Error Resid.	0.9185			
F-statistic	522.5			$< 2.2e^{-16}$ ***
XLI				
Intercept	-0.02413	0.02959	-0.815	0.4151
X_1	0.95666	0.02272	42.104	$< 2e^{-16}$ ***
X_2	0.13644	0.04991	2.734	0.00641 **
X_3	0.33969	0.04213	8.063	$3.01e^{-15}$ ***
Adjusted R^2	0.7393			
Std. Error Resid.	0.7996			
F-statistic	698.5			$< 2e^{-16}$ ***
XLP				
Intercept	0.02728	0.02600	1.049	0.2940
X_1	0.51915	0.01997	25.998	$< 2e^{-16}$ ***
X_2	-0.26492	0.04386	-6.040	2.45^{e-09} ***
X_3	0.05549	0.03702	1.499	0.1340
Adjusted R^2	0.4857			
Std. Error Resid.	0.7027			
F-statistic	233.3			$< 2.2e^{-16}$ ***

Parameter	Estimate	Standard Error	t-Statistics	p-value
XLE				
Intercept	0.006911	0.047031	0.147	0.8830
X_1	0.959066	0.036119	26.553	$< 2e^{-16}$ ***
X_2	-0.017043	0.079336	-0.215	0.8300
X_3	0.418873	0.066969	6.255	$6.75e^{-10}$ ***
Adjusted R^2	0.5989			
Std. Error Resid.	1.271			
F-statistic	365.8			$< 2.2e^{-16}$ ***
XLB				
Intercept	-0.009194	0.038474	-0.239	0.8110
X_1	0.973241	0.029548	32.938	$< 2e^{-16}$ ***
X_2	0.325417	0.064902	5.014	$6.69e^{-07}$ ***
X_3	0.372338	0.054784	6.796	2.22^{e-11} ***
Adjusted R^2	0.5279			
Std. Error Resid.	1.146			
F-statistic	276.1			$< 2.2e^{-16}$ ***
XLU				
Intercept	0.01177	0.03742	0.314	0.7530
X_1	0.56233	0.02874	19.568	$< 2e^{-16}$ ***
X_2	-0.34069	0.06312	-5.397	$9.13e^{-08}$ ***
X_3	0.27622	0.05328	5.184	2.81^{e-07} ***
Adjusted R^2	0.647			
Std. Error Resid.	1.04			
F-statistic	451.8			$< 2.2e^{-16}$ ***

Parameter	Estimate	Standard Error	t-Statistics	p-value
IWM				
Intercept	-0.04679	0.02418	-1.935	0.0533 †
X_1	0.93277	0.01857	50.235	$< 2e^{-16}$ ***
X_2	0.90894	0.04078	22.286	$< 2e^{-16}$ ***
X_3	0.44481	0.03443	12.921	$< 2e^{-16}$ ***
Adjusted R^2	0.375			
Std. Error Resid.	1.011			
F-statistic	148.6			$< 2.2e^{-16}$ ***
IYR				
Intercept	-0.02801	0.04241	-0.661	0.5090
X_1	0.84445	0.03257	25.929	$< 2e^{-16}$ ***
X_2	0.49090	0.07154	6.862	$1.44e^{-11}$ ***
X_3	0.81080	0.06038	13.427	$< 2e^{-16}$ ***
Adjusted R^2	0.5989			
Std. Error Resid.	1.146			
F-statistic	365.8			$< 2.2e^{-16}$ ***

Table : **Parameter estimated for the regression** (71) **and degree of significance**. This table reports the key statistics of regression (71) performed on all 11 ETFs. The levels of significance indicated in the table are as follows: *** indicates a significance level near 0, ** indicates a significance at the 0.001 level, * indicates a significance at the 0.01 level, † indicates a significance at the 0.05% level. The F-statistic is tested on 3 and 735 degrees of freedom.

Implementation 2: Dynamic Update with Partial Observation

Implementation got us off to a good start: we were able to estimate the parameters for our processes using historical data. Now let's try to improve our approach!

First, most factors are not directly observable in real time. To take but three examples:

- *The GDP* is only observed once a quarter and is subject to revisions over the following couple of months. The Atlanta Fed's GDPNow! is a great attempt at improving the timeliness of the information, but it is itself just an estimate.
- The historical *equity risk premium* can be computed with certainty, but we cannot know exactly what equity risk premium we will earn over the course of the day.
- *Consumer confidence indexes*, such as the University of Michigan Consumer Sentiment Index, are constructed based on surveys. There is a delay between the time the questionnaire is sent, answered, received, processed and aggregated into the index. In addition, confidence indexes are a proxy for consumer confidence, not an exact measure.

Second, adopting the view that ‘historical data contain all of the information we will ever need to estimate the parameters of our model’ is limiting.

- Financial markets can and will surprise you!

There might be some advantage to learning from current market condition.

In particular, we could look at current market conditions to get an estimate for our factors.

This is what we call a **partial observation problem**:

- We use a variable that we can observe in real time (asset prices) to construct a real-time estimate for an unobservable variable (the factor).

The key assumptions in our second implementation are:

1. Our model is a reasonable description of the dynamics of asset prices;
2. $X(t)$ contains all the factors that are relevant to describe the evolution of the asset price drift;
3. The assets are traded openly and their prices are observable;
4. The factors are not directly observable, but we can estimate them dynamically by observing asset prices;
5. Historical data do not contain all of the information to estimate the parameters of our model, but we can use them to understand the broad relations between assets and factors.

Mathematically, we will derive an estimate $\hat{X}(t)$ for the factor process $X(t)$ using **filtering**, and use this estimate in subsequent sections to solve a range of optimal investment problems.

Aside: Filtering

Filtering theory has developed considerably since the seminal work by Kalman (1960) and Kalman and Bucy (1961). Other references include

- Chapter 6 Øksendal (2003) contains a concise introduction to filtering;
- The book by Bain and Crisan (2009) contains an excellent treatment of filtering theory and applications.

The basic idea is that the state $X(t)$ is not directly observable in real time. It is hidden. On the other hand, the asset prices $S(t)$ are observable.

Applying a bit of Bayesian thinking, we can use these asset prices to estimate the current value of the state variable. This is what **filtering** is about.

Figure : Rudolf E. Kalman

When your (state,observation) system is linear, you get a closed form solution: the celebrated **Kalman** filter.

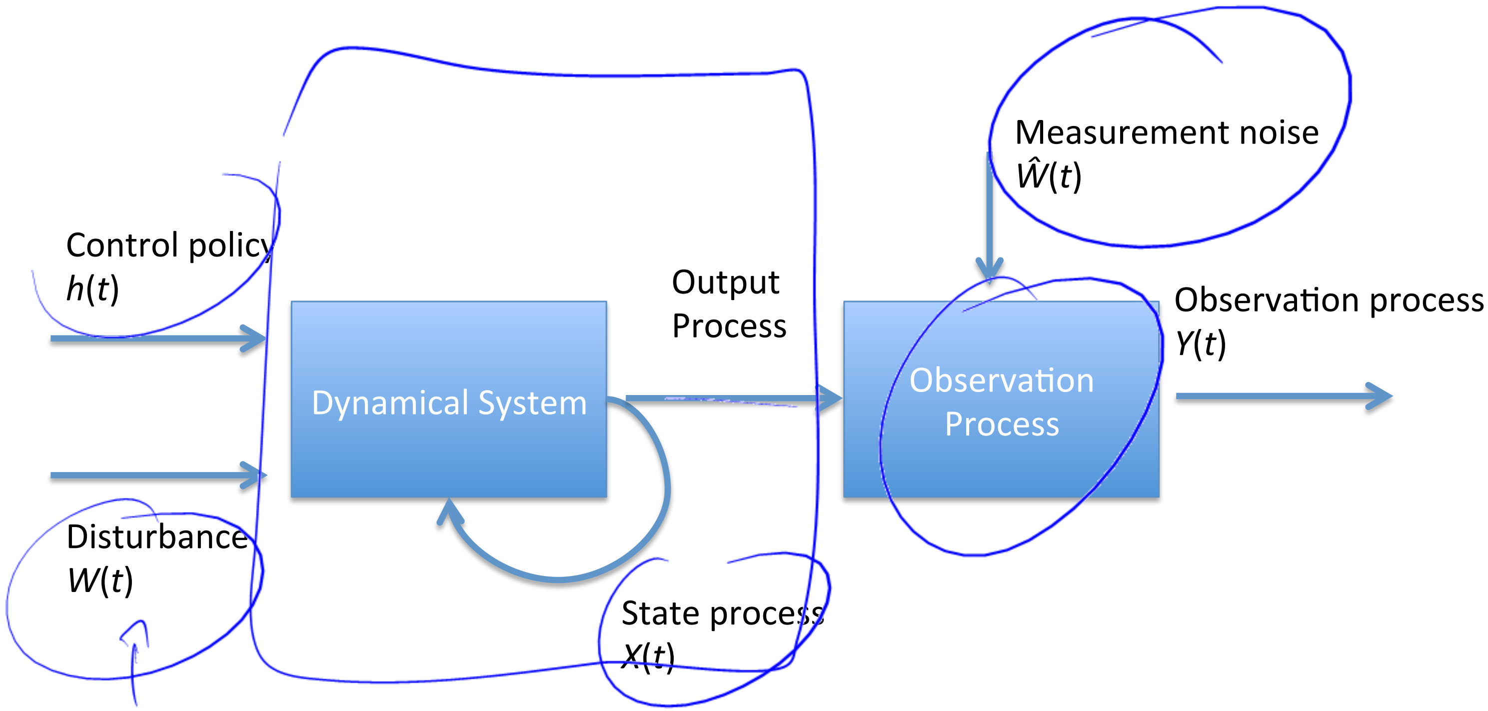


Figure : Schematical view of a simple controlled dynamical system with partial observation

Filtering techniques quickly gained acceptance in stochastic control. See:

- Bucy and Joseph (1987);
- Davis (1977);
- Bensoussan (2004)

In a portfolio management context,

- Brennan (1998) and Xia (2001) used filtering to estimate the parameters of their models;
- Nagai and Peng (2002) and Davis and Lleo (2011) also applied filtering techniques to risk-sensitive asset management models.

From a control perspective, the key is that the filtering problem and the stochastic control problem are effectively separable. We use this insight to incorporate analyst views and non-investable assets as observations in our filter even though they are not present in the portfolio optimization.

End of aside and back to the main development...

To apply the **Kalman filter**, we start by constructing the **observation vector** with the m investable risky assets $S_1(t), \ldots, S_m(t)$;

Prices are not directly suitable for use in a Kalman filter because of their geometric dynamics. On the other hand, excess returns, which we defined as

$$\mathfrak{s}_i(t) = \ln(\tilde{S}_i(t)), \quad i = 1, \ldots, m$$

are linear processes that can be used directly in the Kalman filter.

As a reminder, the dynamics of the excess returns is given by

$$\begin{aligned} d\mathfrak{s}_i(t) &= \left[(\tilde{a}(t) + \tilde{A}(t)X(t))_i - \frac{1}{2}\Sigma\Sigma_{ii}(t)'\right] dt + \sum_{k=1}^{d} \Sigma_{ik}(t) dW_k(t) \\ & \mathfrak{s}_i(0) = \ln \tilde{\mathfrak{s}}_i, \quad i = 1, \ldots, M. \end{aligned} \tag{72}$$

Factor (Signal) Excess Return. (Observation)

The pair of processes $\boxed{(X(t), \mathfrak{s}(t))}$ takes the form of the 'signal' and 'observation' processes in a Kalman filter system.

Consequently the conditional distribution of $X(t)$ is normal $N(\hat{X}(t), P(t))$ where $\boxed{\hat{X}(t) = \mathbb{E}[X(t)|\mathcal{F}_t^{\mathfrak{s}}]}$ satisfies the Kalman filter equation and $P(t)$ is a deterministic matrix-valued function.

$\hat{X}(t) = \mathbb{E}\big[X(t) \,|$ information kindly supplied by the stock market! $\big]$

Next, we define the filtration $\mathcal{F}_t^{\mathfrak{s}} = \sigma\{S(s), U(s), 0 \leq s \leq t\}$ generated by the observation process $\mathfrak{s}(t)$ alone.

Proposition

Define processes $\mathfrak{s}^1(t)$, $\mathfrak{s}^2(t) \in \mathbb{R}^m$ as follows.

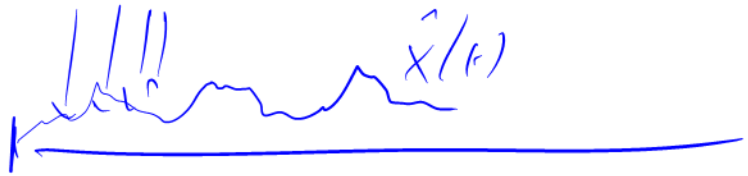
$$\mathfrak{s}^1(t) = \tilde{A}X(t)dt + \Sigma(t)dW(t), \qquad Y^1(0) = 0 \tag{73}$$

$$\mathfrak{s}^2(t) = \tilde{a} \cdot dt, \quad Y^2(0) = y_0 \tag{74}$$

so that $\mathfrak{s}(t) = \mathfrak{s}^1(t) + \mathfrak{s}^2(t)$. Also, define $\mathcal{S}_{it} = \sigma\{\mathfrak{s}^i(u), 0 \leq u \leq t\}$, $i = 1, 2$. Then
(i) The processes $\mathfrak{s}^1, \mathfrak{s}^2$ are each adapted to the filtration $\mathcal{F}_t^{\mathfrak{s}}$.
(ii) For any bounded measurable function f and $t \geq 0$,

$$\mathbb{E}[f(X(t))|\mathcal{F}_t^{\mathfrak{s}}] = \mathbb{E}[f(X(t))|\mathcal{S}_{1t}].$$

$X(t)$? $\longrightarrow$ $X(0) \sim \mathcal{N}(\mu_0, P_0)$



In the present case we need to assume that X_0 is a normal random vector $N(\mu_0, P_0)$ with known mean μ_0 and covariance P_0, and that X_0 is independent of the Brownian motion W.

- The estimation of μ_0 and P_0 depends closely dependent on the choice of factors.

The processes $(X(t), \mathfrak{s}^1(t))$ satisfying (66) and (94) and the filtering equations, which are standard, are stated in the following proposition.

$W(t)$ is $(n+m)$-dim.

Proposition (Kalman Filter:)

The conditional distribution of $X(t)$ given $\mathcal{F}_t^{\mathfrak{s}}$ is $N(\hat{X}(t), P(t))$, calculated as follows.

(i) The innovations process $\boxed{U(t) \in \mathbb{R}^m}$ defined by

$$\boxed{dU(t)} = (\boxed{\Sigma\Sigma'(t))^{-1/2}} (d\mathfrak{s}(t) - \tilde{A}\hat{X}(t)dt), \qquad U(0) = 0 \qquad (75)$$

is a vector Brownian motion.

(ii) $\hat{X}(t)$ is the unique solution of the SDE

$~~W(t)~~$ $U(t)$

$$d\hat{X}(t) = (b(t) + B(t)\hat{X}(t))dt + \boxed{\hat{\Lambda}(t) dU(t)}, \qquad \hat{X}(0) = \mu_0, \qquad (76)$$

where

$$\boxed{\hat{\Lambda}(t) = \boxed{(\Lambda(t)\Sigma'} + P(t)A_Y(t)') (\Sigma\Sigma'(t))^{-1/2}.}$$

(iii) $P(t)$ is the unique non-negative definite symmetric solution of the matrix Riccati equation

$$\boxed{\begin{aligned}
\dot{P}(t) \quad &= \quad \Lambda\Upsilon^{\perp}(\mathfrak{s}^{\perp})'\Lambda'(t) - P(t)\tilde{A}(t)'\left(\Sigma\Sigma'\right)^{-1}\tilde{A}P(t) \\
&\quad + \left(B(t) - \Lambda(t)\Sigma'\left(\Sigma\Sigma'\right)^{-1}\tilde{A}\right)P(t) \\
&\quad + P(t)\left(B(t)' - \tilde{A}(t)'\left(\Sigma\Sigma'\right)^{-1}\Sigma\Lambda'(t)\right), \qquad P(0) = P_0,
\end{aligned}}$$

where $\Upsilon^{\perp}(t) := I - \Sigma'\left(\Sigma'\Sigma\right)^{-1}\Sigma$.

Now the Kalman filter has replaced our initial state process $X(t)$ by an estimate $\hat{X}(t)$ with dynamics given in (76).

innovation

To recover the observation process, we use (75)

$$\begin{aligned} \boxed{d\mathfrak{s}(t)} &= d\mathfrak{s}_1(t) + d\mathfrak{s}_2(t) \\ &= (\tilde{a} + \tilde{A}(t)\hat{X}(t))dt + (\Sigma\Sigma'(t))^{1/2}\boxed{dU(t)}, \\ & \mathfrak{s}(0) = \mathfrak{s}. \end{aligned} \tag{77}$$

As a result, the SDEs for $\mathfrak{s}(t)$ and $\tilde{S}(t)$ are given by

$$
\begin{aligned}
d\mathfrak{s}_i(t) &= \left[(\tilde{a}(t) + \tilde{A}(t)\hat{X}(t))_i - \frac{1}{2}\Sigma\Sigma_{ii}(t)' \right] dt + \sum_{k=1}^{m} \Sigma_{ik}(t) dU_k(t), \\
& \quad \mathfrak{s}_i(0) = \ln \tilde{s}_i, \\
\frac{d\tilde{S}_i(t)}{\tilde{S}_i(t)} &= \left(\tilde{a}(t) + \tilde{A}(t)\hat{X}(t) \right)_i dt + \sum_{k=1}^{m} \hat{\Sigma}_{ik}(t) dU_k(t), \quad \tilde{S}_i(0) = \tilde{s}_i. \quad (78)
\end{aligned}
$$

Derive the Prior Expected Value of the Factor Process μ_0

Continuous time Fund Separation results identify the Kelly portfolio as the cornerstone of all investment strategies.

If we knew the composition of the Kelly portfolio *ex ante*, we could use a reverse optimization argument to derive the equilibrium risk premium.

A significant advantage of this approach is that it is preference-free: we do not need to know anything about the level of risk-sensitivity of an investor or group of investors.

The Kelly portfolio's asset allocation at time 0 is:

$$h^K(0) = \left(\Sigma\Sigma'\right)^{-1}\left(\tilde{a} + \tilde{A}X(0)\right)$$

The factor level $X(0)$ is not observable but we could back an estimate μ_0 out of the allocation of the Kelly portfolio, provided we know h^K and provided $A'A$ is invertible:

$$\mu_0 = (\tilde{A}'\tilde{A})^{-1}\tilde{A}'\left(\Sigma\Sigma' h^K(0) - \tilde{a}\right) \tag{79}$$

Typically $A'A$ will be invertible because $n < m$. The accuracy of the estimate μ_0 depends on the accuracy of the estimates for $\Sigma\Sigma'$, a and A. It also depends crucially on the allocation $h^K(0)$, which may not be directly observable.

We could construct a portfolio that approximates the Kelly strategy using Cover's 'universal portfolio' (Cover, 1991; Cover and Thomas, 2006, Chapter 16) which is shown to converge asymptotically to the Kelly strategy.

The major advantage of this method is that it does not make any assumption on the shape of the return distribution and does not imply any view about future performance.

Implementation 3: Dynamic Black Litterman

With implementation 2, our focus moved from using historical data to learning from the market. Who or what else could we learn from now?

From analyst reports to CNBC pundits, internet blogs and Google trends, we have access a larger numbers of opinions, views and data on financial markets and the economy.

Why not use them to formulate 'beliefs about the future performances of available securities'?

This was Black and Litterman's starting point and it is the next stop on our journey.

Figure : James Cramer, Trader turned CNBC Pundit

Collect Expert Opinions And Views

View = Likely scenario
+ Confidence level.

The next ingredient in our model consists in $k \geq 0$ **views** formulated by analysts and experts.

The analysts express today their views about the evolution of factors up to the end of the time horizon. When the factors represent risk premia, typical analyst statements would be:

- '*My research leads me to believe that the U.S. equity risk premium will slowly increase to 4% over the next two years. I am 90% confident that the risk premium will not go below 2% or go above 6% over the next two years.*', or;
- '*My research leads me to believe that the spread between 10-year Treasury Notes and 3-month Treasury Bills will remain low over the next year before gradually widening over the next 2 years to 200 basis points in response to improving macroeconomic conditions. I am 90% confident that the spread will remain in a 50 basis point to 300 basis point range.*'

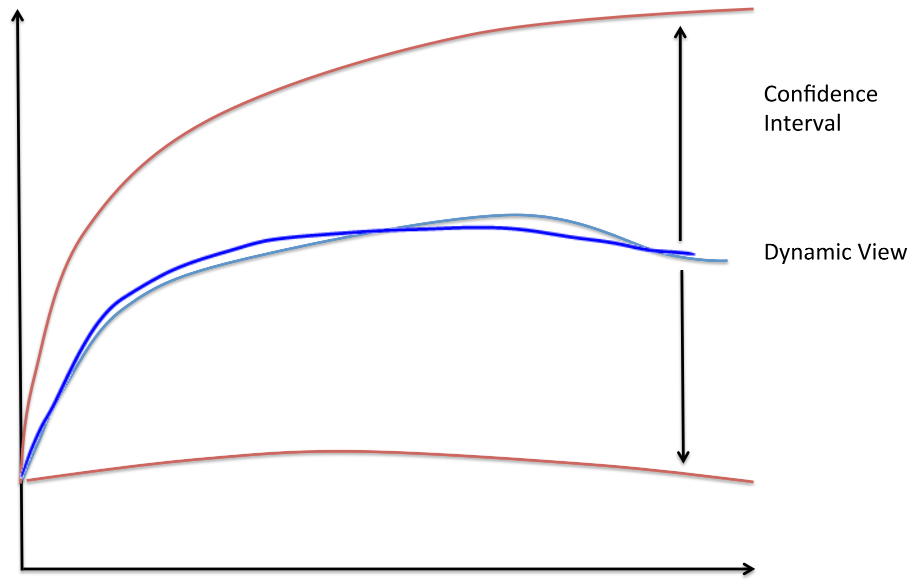
Mathematically, we can translate the system of views $Z(t)$ into a system of ordinary differential equations:

$$\boxed{\dot{Z}(t) = (a_Z(t) + A_Z(t)X(t))dt,} \qquad Z(0) = z \tag{80}$$

We introduce a white noise term to construct a dynamic confidence interval around the views:

$$\dot{Z}(t) = (a_Z(t) + A_Z(t)X(t))dt + \psi(t)\mathfrak{W}dt, \quad Z(0) = z \tag{81}$$

where $\mathfrak{W}$ is a k-dimensional Gaussian white noise process and ψ is a $k \times k$ matrix.



Finally, we express (81) as a stochastic differential equation driven by the $\mathcal{F}_t$-Brownian motion W:

$$
\begin{aligned}
dZ(t) &= (a_Z(t) + A_Z(t)X(t))dt + \Psi_Z(t)dW(t), \\
& Z(0) = z,
\end{aligned}
\tag{82}
$$

The mathematical modelled is presented in greater details in Davis and Lleo (2013).

Enter Adam and Beth

The next task is to collect and model analyst views.

Investor Irene works closely with two equity analysts: Adam and Beth. Both of them follow closely the equity risk premium and will provide a view on the future evolution of $X(t)$.

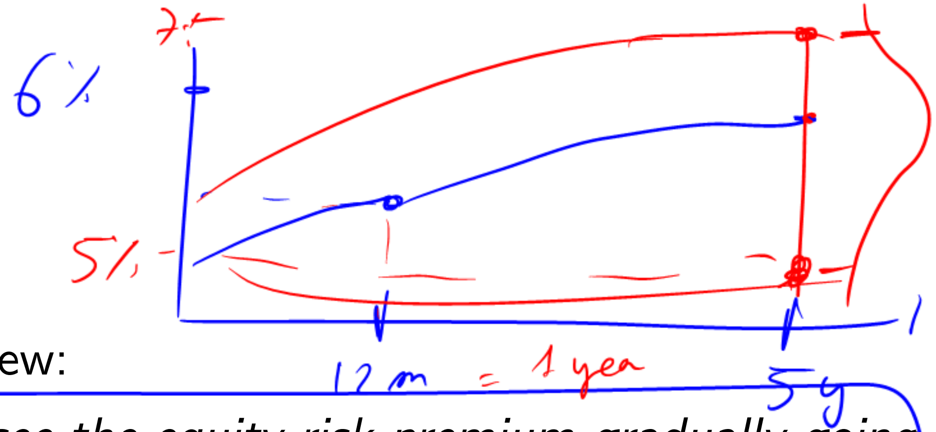
- Adam is a new analyst;
- Beth already has an extensive experience.

To make the computations simpler, we assume that

Assumption

The noise generated by the asset price process and the factor process are independent of the dynamic confidence interval around the views, implying that $\Lambda \Psi_Z' = 0$ *and* $\Sigma \Psi_Z' = 0$.

Adam's View



Adam expresses the following view:

> *Over the next five years, I see the equity risk premium gradually going back to 6%, which represents its long-term average in my opinion. Half of the move will take place in the next 12 month. I am 90% confident that the risk premium will not go below 5% or exceed 7% at the end of the five year horizon.*

We model Adam's view using an Ornstein-Uhlenbeck process:

$$dZ_1(t) = (\eta_1 - \kappa_1 Z_1(t))\, dt + \psi_1 W_{\ell+1}(t), \quad Z_1(0) = z_1, \tag{83}$$

where we have set $\ell := n + m$. The solution to this SDE is

$$Z_1(t) = z_1 e^{-\kappa_1 t} + \frac{\eta_1}{\kappa_1}(1 - e^{-bt}) + \psi_1 \int_0^t e^{-\kappa_1 (t-s)} dW_{\ell+1}(s). \tag{84}$$

$\frac{\eta_1}{\kappa_1} = 6\% \Rightarrow \eta_1 = 6\% \times \kappa_1$
$= 6\% \ln 2$

The mean and variance of $Z_1(t)$ are respectively:

$1\% \rightarrow 1.645$

$$
\begin{aligned}
\mathbb{E}\left[Z_1(t)\right] &= z_1 e^{-\kappa_1 t} + \boxed{\frac{\eta_1}{\kappa_1}}\left(1 - e^{-\kappa_1 t}\right) \\
\mathrm{Var}\left[Z_1(t)\right] &= \frac{\psi_1^2}{2\kappa_1}\left(1 - e^{-2\kappa_1 t}\right)
\end{aligned}
\qquad (85)
$$

confidence

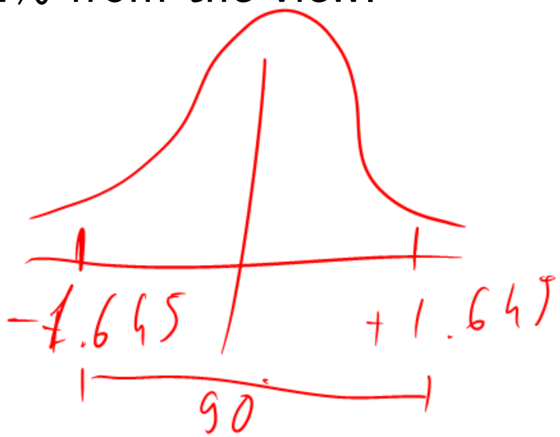
$\ln 2$

$7\% - 6\%$
$6\% - 5\%$

We can readily translate Adam's statement into a set of parameters for (84):

- $\kappa_1 = \ln 2$ because half of the move should take place over the next year and the half life of the Ornstein-Ulhenbek process is $\frac{\ln 2}{\kappa_1}$; $= 1$
- $\eta_1 = 0.06 \ln 2$ because the long term mean of the Ornstein-Uhlenbeck, $\frac{\eta_1}{\kappa_1}$, should equal 6%;
- $\psi_1 = \frac{0.01}{1.645}\sqrt{\frac{2\kappa_1}{1-e^{-2\kappa_1 T}}} = 0.716\%$ because Adam's 90% confidence interval is at -/+ 1% from the view.

$N(0,1)$



Beth's View

Beth's opinion is the following:

> *The equity risk premium will increase to 10% over the coming year, before declining back to 7% at the end of the end of the five year horizon, which represents its long term-average. I am 90% confident that the risk premium will not go below 6% or exceed 8% at the end of the five year horizon.*

We model this view with a generalized Ornstein-Uhlenbeck process (see Hull and White, 1994):

$$dZ_2(t) = (\eta_2(t) - \kappa_2 Z_2(t))\, dt + \psi_2 W_{\ell+2}(t) \tag{86}$$

The solution to this SDE is Polynomials

$$Z_2(t) = z_2 e^{-\kappa_2 t} + \int_0^t e^{-\kappa_2(t-u)}\eta_2(s)ds + \psi_2 \int_0^t e^{-\kappa_2(t-s)}\psi_2 dW_{\ell+2}(s).$$

$$\begin{aligned}
\mathbb{E}[Z_2(t)] &= z_2 e^{-\kappa_2 t} + \int_0^t e^{-\kappa_2(t-s)}\eta_2(s)ds \\
\mathrm{Var}[Z_2(t)] &= \frac{\psi_2^2}{2\kappa_2}\left(1 - e^{-2\kappa_2 t}\right)
\end{aligned}$$

Figure 215 suggests that the fifth order polynomial function

$$\mathcal{P}(t) = 3 \times 10^{-5} t^5 - 0.001 t^4 + 0.0112 t^3 - 0.0572 t^2 + 0.1169 t + 0.03 \qquad (87)$$

provides an adequate fit for Beth's views.

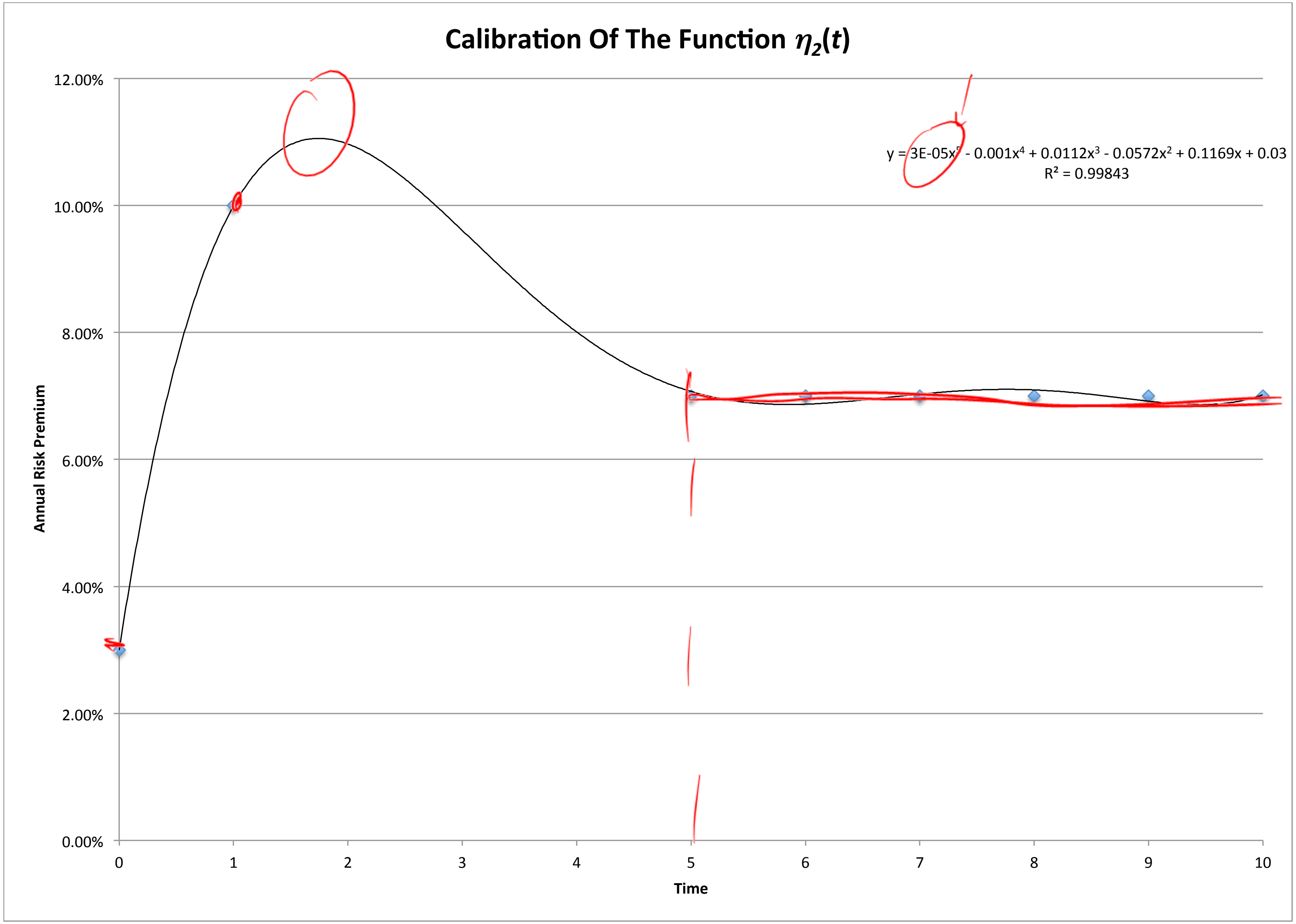


Figure : Polynomial Calibration Function For $\eta_2(t)$

To fit Beth's view of the overall evolution of the risk premium, we chose a polynomial function of order 4:

$$\eta_2(t) = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \alpha_3 t^3 + \alpha_4 t^4 \tag{88}$$

Then,

$$\begin{aligned}
\mathbb{E}\left[Z_2(t)\right] \quad = \quad & \frac{e^{-\kappa_2 t}}{\kappa_2^5}\left[-24\alpha_4 + \kappa_2\left(6\alpha_3 + \kappa_2\left(-2\alpha_2 + \alpha_1\kappa_2 - \alpha_0\kappa_2^2\right)\right)\right] \\
& + \frac{\alpha_4}{\kappa_2^5}\left[24 + \kappa_2 t\left(-24 + \kappa_2 t\left(12 + \kappa_2 t\left(-4 + \kappa_2 t\right)\right)\right)\right] \\
& + \frac{1}{\kappa_2^4}\left[\alpha_3\left(-6 + \kappa_2 t\left(6 + \kappa_2 t\left(-3 + \kappa_2 t\right)\right)\right)\right] \\
& + \frac{1}{\kappa_2^3}\left[\alpha_2(2 + \kappa_2 t(-2 + \kappa_2 t)) + \alpha_0\kappa_2 + \alpha_1(-1 + \kappa_2 t)\right] \\
& + z e^{-\kappa_2 t}
\end{aligned} \tag{89}$$

A Taylor expansion of this expression around $t = 0$ yields

$$
\begin{aligned}
\mathbb{E}\left[Z_2(t)\right] \quad = \quad & z + (\alpha_0 - \kappa_2 z)\, t + \frac{1}{2}\left(\alpha_1 - \alpha_0\kappa_2 + \kappa_2^2 z\right) t^2 \\
& + \frac{1}{6}\left(2\alpha_2 - \alpha_1\kappa_2 + \alpha_0\kappa_2^2 - \kappa_2^3 z\right) t^3 \\
& + \frac{1}{24}\left(6\alpha_3 - 2\alpha_2\kappa_2 + \alpha_1\kappa_2^2 - \alpha_0\kappa_2^3 + \kappa_2^4 z\right) t^4 \\
& + \frac{1}{120}\left(24\alpha_4 - 6\alpha_3\kappa_2 + 2\alpha_2\kappa_2^2 - \alpha_1\kappa_2^3 + \alpha_0\kappa_2^4 - \kappa_2^5 z\right) + O(t^6).
\end{aligned} \tag{90}
$$

Selecting $\kappa_2 = 2 \ln 2 = 1.3862$, which implies a half-life of 6 months, guarantees a rapid convergence back to Beth's view, and matching the terms in expression (90) with those in (87), we get:

- $z = 0.03$;
- $\alpha_0 = 0.158488831$;
- $\alpha_1 = 0.047657811$;
- $\alpha_2 = -0.045696037$;
- $\alpha_3 = 0.011526497$;
- $\alpha_4 = -0.001236294$.

diffusion

Finally, Beth's confidence interval implies that $\psi_1 = \frac{0.01}{1.645}\sqrt{\frac{2\kappa_2}{1-e^{-2\kappa_2 T}}} = 2.480\%$.

90% parameter

To summarize, we model the views that Adam and Beth expressed, as a stochastic differential equation:

$$dZ(t) = (a_Z(t) + A_Z(t)X(t))dt + \Psi_Z dW(t), \quad Z(0) = z$$

$n+m$

with

$$a_Z(t) = \begin{pmatrix} \eta_1 \\ \eta_2(t) \end{pmatrix}, \quad A_Z(t) = \begin{pmatrix} -\kappa_1 & 0 & 0 \\ -\kappa_2 & 0 & 0 \end{pmatrix}, \quad \Psi_Z = \left(\begin{array}{c|cc} \mathbf{0}_\ell & \psi_1 & 0 \\ \mathbf{0}_\ell & 0 & \psi_2 \end{array} \right).$$

where $\mathbf{0}_\ell$ is a ℓ-element zero vector.

Finalizing The Views

We model the views expressed by Adam and Beth using the stochastic differential equation

$$dZ(t) = (a_Z(t) + A_Z(t)X(t))dt + \Psi_Z dW(t), \quad Z(0) = z$$

with

$$a_Z(t) = \begin{pmatrix} \eta_1 \\ \eta_2(t) \end{pmatrix}, \quad A_Z(t) = \begin{pmatrix} -\kappa_1 & 0 & 0 \\ -\kappa_2 & 0 & 0 \end{pmatrix},$$
$$\Psi_Z = \begin{pmatrix} \mathbf{0}_\ell & \psi_1 & 0 \\ \mathbf{0}'_\ell & 0 & \psi_2 \end{pmatrix},$$

where $\mathbf{0}_\ell$ is a ℓ-element zero vector and $\ell = n + m = 14$.

Blending Data With Opinions, And Filter

We start by constructing the **observation vector** combining

1. m investable risky assets $S_1(t), \ldots, S_m(t)$;
2. k analyst views $Z_1(t), \ldots, Z_k(t)$, and;

Asset prices are not directly suitable for use in a Kalman filter because of their geometric dynamics. On the other hand, excess returns, which we defined as

$$\mathfrak{s}_i(t) = \ln(\tilde{S}_i(t)), \quad i = 1, \ldots, m$$

are linear processes that can be used directly in the Kalman filter.

The dynamics of the excess returns is given by

$$\begin{aligned} d\mathfrak{s}_i(t) &= \left[(\tilde{a}(t) + \tilde{A}(t)X(t))_i - \frac{1}{2}\Sigma\Sigma_{ii}(t)' \right] dt + \sum_{k=1}^{d} \Sigma_{ik}(t) dW_k(t) \\ \mathfrak{s}_i(0) &= \ln \tilde{\mathfrak{s}}_i, \quad i = 1, \ldots, M. \end{aligned} \tag{91}$$

The pair of processes $(X(t), Y(t))$, where

$$Y_i(t) = \begin{cases} \mathfrak{s}_i(t) = \ln \frac{S_i(t)}{S_0(t)}, & i = 1, \dots, m, \\ Z_{i-m}(t), & i = m+1, \dots, m+k \end{cases} \tag{92}$$

takes the form of the 'signal' and 'observation' processes in a Kalman filter system.

Consequently the conditional distribution of $X(t)$ is normal $N(\hat{X}(t), P(t))$ where $\hat{X}(t) = \mathbb{E}[X(t)|\mathcal{F}_t^Y]$ satisfies the Kalman filter equation and $P(t)$ is a deterministic matrix-valued function.

We express the dynamics of $Y(t)$ succinctly as

$$dY(t) \quad = \quad (a_Y(t) + A_Y(t)X(t))dt + \Gamma(t)dW(t), \quad Y(0) = y_0, \tag{93}$$

where

$$a_Y \quad = \quad \begin{pmatrix} \tilde{a} \\ a_Z \end{pmatrix}, \qquad A_Y = \begin{pmatrix} \tilde{A} \\ A_Z \end{pmatrix}, \qquad \Gamma = \begin{pmatrix} \Sigma \\ \Psi_Z \end{pmatrix}.$$

Next, we define the filtration $\mathcal{F}_t^Y = \sigma\{S(s), Z(s), U(s), 0 \le s \le t\}$ generated by the observation process $Y(t)$ alone.

Proposition

Define processes $Y^1(t)$, $Y^2(t) \in \mathbb{R}^m$ as follows.

$$Y^1(t) = A_Y(t)X(t)dt + \Gamma(t)dW(t), \qquad Y^1(0) = 0 \tag{94}$$

$$Y^2(t) = a_Y(t)dt, \quad Y^2(0) = y_0 \tag{95}$$

so that $Y(t) = Y^1(t) + Y^2(t)$. Also, define $\mathcal{Y}_{it} = \sigma\{Y^i(u), 0 \le u \le t\}$, $i = 1, 2$. Then
(i) The processes Y^1, Y^2 are each adapted to the filtration $\mathcal{F}_t^S$.
(ii) For any bounded measurable function f and $t \ge 0$,

$$\mathbb{E}[f(X(t))|\mathcal{F}_t^Y] = \mathbb{E}[f(X(t))|\mathcal{Y}_{1t}].$$

In the present case we need to assume that X_0 is a normal random vector $N(\mu_0, P_0)$ with known mean μ_0 and covariance P_0, and that X_0 is independent of the Brownian motion W.

- The estimation of μ_0 and P_0 depends closely dependent on the choice of factors: we will discuss estimation procedures for these parameters in relation to the applications considered in our paper.

The processes $(X(t), Y^1(t))$ satisfying (66) and (94) and the filtering equations, which are standard, are stated in the following proposition.

Proposition (Kalman Filter:)

assets views

The conditional distribution of $X(t)$ *given* $\mathcal{F}_t^Y$ *is* $N(\hat{X}(t), P(t))$*, calculated as follows.*

(i) The innovations process $U(t) \in \mathbb{R}^{m+k}$ *defined by*

$$dU(t) = \left(\Gamma\Gamma'(t)\right)^{-1/2} \left(dY(t) - A_Y(t)\hat{X}(t)dt\right), \qquad U(0) = 0 \qquad (96)$$

is a vector Brownian motion.

error $\sim ?\ (\Sigma_1 \Sigma_1')^{-1} \Sigma_1' \hat{\Lambda}(t) D \phi$

(ii) $\hat{X}(t)$ *is the unique solution of the SDE*

$$d\hat{X}(t) = \left(b(t) + B(t)\hat{X}(t)\right)dt + \hat{\Lambda}(t)dU(t), \qquad \hat{X}(0) = \mu_0, \qquad (97)$$

where

$$\hat{\Lambda}(t) = \left(\Lambda\Gamma'(t) + P(t)A_Y(t)'\right)\left(\Gamma\Gamma'(t)\right)^{-1/2}.$$

(iii) $P(t)$ *is the unique non-negative definite symmetric solution of the matrix Riccati equation*

$$\begin{aligned}
\dot{P}(t) &= \Lambda\Upsilon^{\perp}(\mathfrak{s}^{\perp})'\Lambda'(t) - P(t)A_Y(t)'\left(\Gamma\Gamma'(t)\right)^{-1}A_Y(t)P(t) \\
&\quad + \left(B(t) - \Lambda(t)\Gamma(t)'\left(\Gamma\Gamma'(t)\right)^{-1}A_Y(t)\right)P(t) \\
&\quad + P(t)\left(B(t)' - A_Y(t)'\left(\Gamma\Gamma'(t)\right)^{-1}\Gamma(t)\Lambda'(t)\right), \qquad P(0) = P_0,
\end{aligned}$$

where $\Upsilon^{\perp}(t) := I - \Gamma(t)'\left(\Gamma'(t)\Gamma(t)\right)^{-1}\Gamma(t)$*.*

Now the Kalman filter has replaced our initial state process $X(t)$ by an estimate $\hat{X}(t)$ with dynamics given in (97).

To recover the observation process, we use (96)

$$\begin{aligned} dY(t) &= dY_1(t) + dY_2(t) \\ &= (a_Y(t) + A_Y(t)\hat{X}(t))dt + (\Gamma\Gamma'(t))^{1/2} dU(t), \\ & \quad Y(0) = y_0. \end{aligned} \tag{98}$$

As a result, the SDEs for $\mathfrak{s}(t)$, $Z(t)$ and $\tilde{S}(t)$ are:

$$\begin{aligned}
d\mathfrak{s}_i(t) &= \left[\left(\tilde{a}(t) + \tilde{A}(t)\hat{X}(t)\right)_i - \frac{1}{2}\Sigma\Sigma_{ii}(t)'\right] dt + \sum_{k=1}^{m+k} \Sigma_{ik}(t) dU_k(t), \\
& \quad \mathfrak{s}_i(0) = \ln \tilde{s}_i, \\
dZ(t) &= (a_Z(t) + A_Z(t)\hat{X}(t))dt + \Psi_Z(t) dU(t), \quad Z(0) = z, \\
\frac{d\tilde{S}_i(t)}{\tilde{S}_i(t)} &= \left(\tilde{a}(t) + \tilde{A}(t)\hat{X}(t)\right)_i dt + \sum_{k=1}^{m+k} \Sigma_{ik}(t) dU_k(t), \quad \tilde{S}_i(0) = \tilde{s}_i.
\end{aligned}$$

Implementation 4: Behavioural Black Litterman

Warning! Behavioural finance has evidenced that psychological biases have an impact on the decision-making process of individuals and organisations:

- Hirschleifer (2001) classifies 22 different psychological biases into four categories: self-deception, heuristic simplification, emotion/affect and social;
- Shefrin (2005) identifies 12 main psychological pitfalls.

These biases will also impact the analyst views and opinions.

- How can we address these biases in a mathematical model?

Here, we consider four main psychological biases, examine their potential impact on the formulation and collection of analyst views, and propose general modelling principles to correct their impact on the model:

(i) *Overconfidence*;

(ii) *Excessive optimizm*;

(iii) *Conservatism*;

(iv) *'Groupthink'*.

For more detail, see Davis and Lleo (2016) for a non mathematical overview and Davis and Lleo (2015) for the technical paper.

Overconfidence:

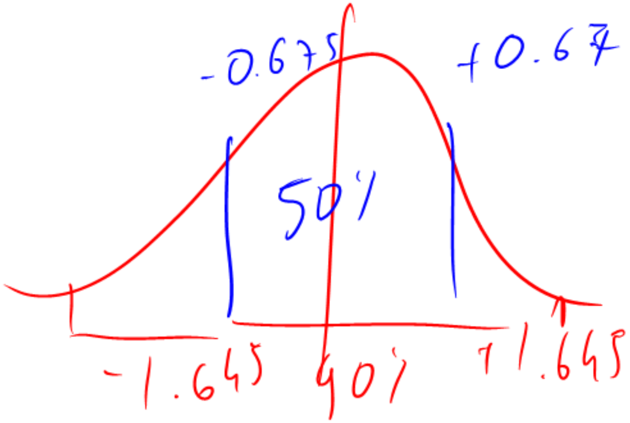
- *Definition*: tendency for individuals to be too confident in their beliefs.
- *Impact on views*: overconfidence may lead analysts to overestimate the accuracy of their views, resulting in confidence bounds that are too narrow.
- *How to address*: increase the magnitude of the diffusion term $\Psi_{\underline{z}}$ used in (82) to widen the confidence interval.

90% → 1.645

!

In our case,

- Adam has recently filled a survey designed to gauge his degree of overconfidence, such as the survey in Klayman et al. (1999). The survey reveals that Adam's 90% confidence interval corresponds in reality to a 50% confidence interval. We adjust the value of the diffusion parameter ψ_1 accordingly. The updated value is $\psi_1 = \frac{0.01}{0.675}\sqrt{\frac{2\kappa_1}{1-e^{-2\kappa_1 T}}} = 1.746\%$, where 0.675 is the parameter for a 50% confidence on a standard Normal distribution.
- Looking at Beth's extensive track record of views, the investor observes that actual realisations of the market risk premium fall 25% of the time outside of Beth' 90% confidence bounds. This suggests that the analyst exhibits a small degree of overconfidence, which we can account for by adjusting the parameter ψ_2 from 2.480% to 3.546%.



Excessive optimizm:

- *Definition*: tendency for individuals to see the world through 'rose-colored glasses.'
- *Impact on the views*: Excessively optimistic analysts will overestimate the probability of scenarios that they perceive as positive.
- *How to address*:
  - Increase the magnitude of the diffusion term Ψ_Z;
  - Not a perfect solution because Gaussian confidence bounds are symmetric;
  - Better solution: use jump-diffusion processes to model the asymmetry.

In our case, Beth and Adam may exhibit some degree of excessive optimizm when they forecast a 6% to 7% risk premium.

We could widen the standard deviation of the confidence interval by adjusting the parameters ψ_1 and ψ_2. However, this solution is not ideal because it increases the confidence bounds on both sides of the expert's view. This provides a strong motivation for including non-Gaussian confidence intervals in our model.

After correcting for overconfidence, the standard deviation around Beth's view is 0.869% implying a 39% probability that the risk premium will end up below 6% and a 29% probability that the risk premium will end up below 5%. Both of this probabilities are large enough to account for some degree of excessive optimism and investor Irene decides not to increase ψ_2.

After correcting for overconfidence, the standard deviation around Adam's view is 1.483% implying a 28% probability that the risk premium will end up below 5%, comparable to Beth's probability. Irene decides not to increase ψ_1.

Conservatism a.k.a anchoring-and-adjustment:

- *Definition*: tendency to overweight prior information relative to newly released information, often resulting in a failure to update one's beliefs in a Bayesian manner.
- *Impact on views*: affects the point estimate given by analysts as well as the confidence interval. It is a serious concern especially for multiperiod or continuous time models.
- *How to address*: our model does not require the analyst to update their views: analysts formulate their views at the initial stage when the model is parametrised and they are not asked to update them after. Hence, any effect of the conservatism bias remains confined to the initial set of views. After that, the treatment of the views as observations in the Kalman filter is Bayesian, so observations will be incorporated accurately into the model.

Groupthink:

- *Definition*: groupthink leads people in groups to act as if they value conformity over quality when making decisions (Shefrin, 2010).
- *Impact on views*: finance professionals are at a particular risk of exhibiting groupthink or of following the herd, for fear of falling behind the rest of their colleagues.
- *How to address*: to reduce the effects of groupthink,

model?

1. Add a correlation structure between the views via Ψ_Z;
2. Seek dissenting analysts whose views differ markedly from the majority;
3. Introduce historical and stress test scenarios.

Stress Test Scenarios

Adding stress test scenarios to the views is consistent with the ***guiding principle that it is more important to avoid large drawdowns in difficult times than to capture the highest returns in bullish markets***.

Stress test scenarios may also be helpful to mitigate the effect of behavioural biases such as narrow framing, excessive optimism or groupthink.

Stress test scenarios should have a wide, markedly skewed confidence interval, because the realised value of $X(t)$ is unlikely to be as extreme as suggested by the stress test scenario.

It is however possible that the realised value of $X(t)$ could be worse than the stress test scenario. For this reason, it is important to establish a two-sided confidence interval around the stress test scenario.

In our case, we model groupthink by adding both a correlation structure to the confidence interval around the views, and a historical scenario.

The diffusion matrix Ψ_Z, subject to a correlation ρ, becomes:

$$\Psi_Z = \begin{pmatrix} \mathbf{0}_\ell & \psi_1 & 0 \\ \mathbf{0}_\ell & \rho\psi_2 & \sqrt{1-\rho^2}\psi_2 \end{pmatrix}, \text{ so that } \Psi_Z\Psi_Z' = \begin{pmatrix} \psi_1^2 & \rho\psi_1\psi_2 \\ \rho\psi_1\psi_2 & \psi_2^2 \end{pmatrix}. \quad (99)$$

The difficulty here is in estimating ρ.

- A possibility would be to use the historical correlation of the forecasting error, defined as the difference between the forecast and the observation, across analysts.
- Investor Irene considers that groupthink is not prevalent among her analysts and picks $\rho = 0.5$.

As an additional measure, Irene decides to include a stress test scenario based on data from the 2008 financial crisis.

She models this scenario as a generalized Ornstein-Uhlenbeck process:

$$dZ_3(t) = (\eta_3(t) - \kappa_3 Z_3(t))\,dt + \psi_3 W_{\ell+3}(t) \tag{100}$$

The next figure suggests a fifth order polynomial function

$$\mathcal{Q}(t) = 0.0362t^5 - 0.437t^4 + 1.7867t^3 - 2.7691t^2 + 1.2301t - 0.0137$$

to model the stress test scenario, as higher order polynomial do not improve the fit significantly.

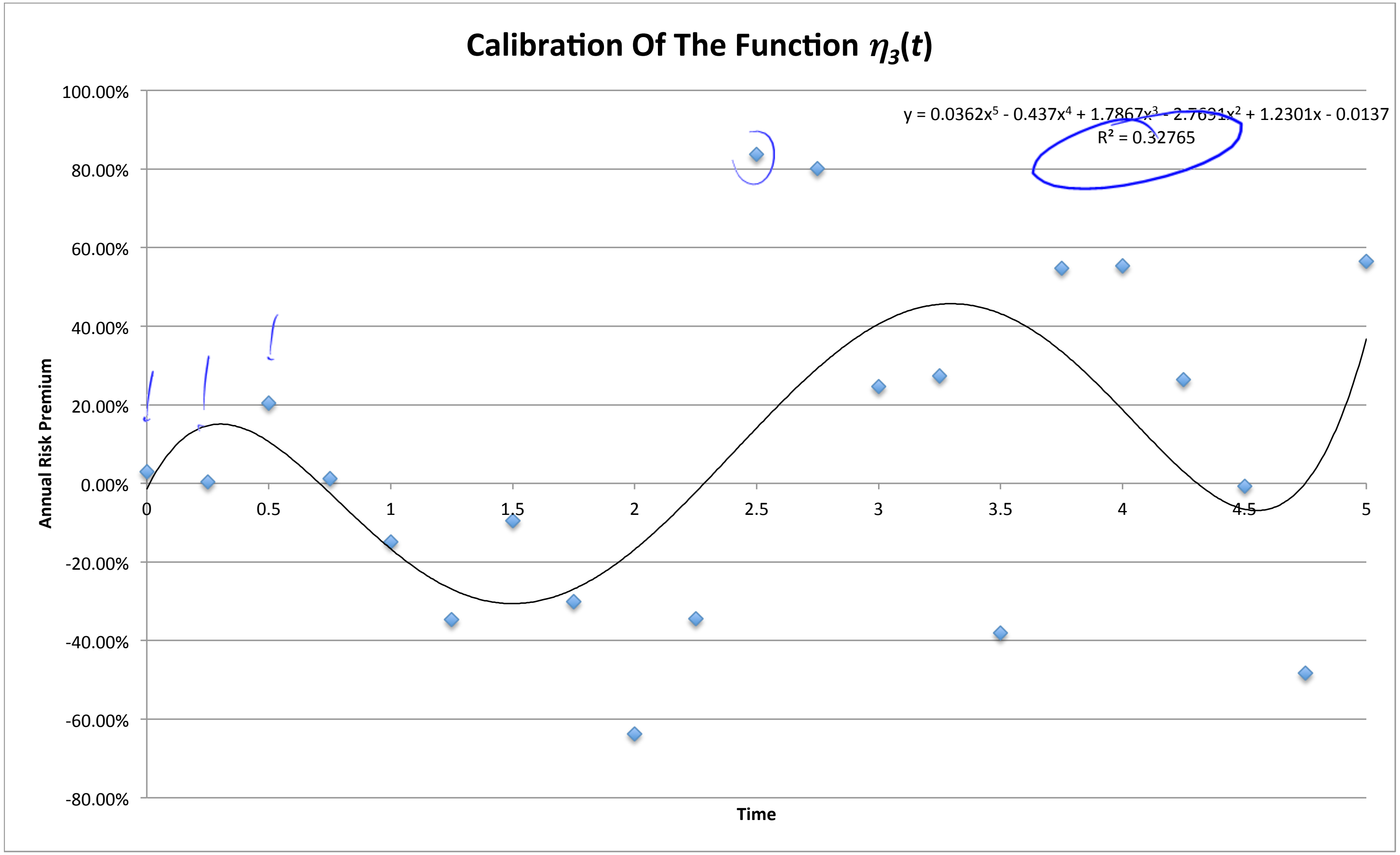


Figure : Polynomial Calibration Function For $\eta_3(t)$

To fit this polynomial function, we express the function η_3 as a fourth order polynomial:

$$\eta_3(t) = \beta_0 + \beta_1 t + \beta_2 t^2 + \beta_3 t^3 + \beta_4 t^4 \tag{101}$$

This yields an expectation and a Taylor expansion of the same form as (89)-(90). Selecting $\kappa_3 = \frac{1}{4} \ln 2 = 0.138629436$, implies a half life of 5 years. Matching the terms in expression (90) with those in (87), we get:

- $z = -0.0137$;
- $\beta_0 = 1.228200777$;
- $\beta_1 = -5.367671931$;
- $\beta_2 = 4.976221228$;
- $\beta_3 = -1.500310786$;
- $\beta_4 = 0.120418936$.

The confidence interval around the stress test scenario needs to be wide enough to include the analyst views.

Setting $\Psi_3 = 117\%$ guarantees that the 90% confidence interval around Adam and Beth's views is also in the confidence interval around the stress test scenario.

Now, we put everything together, starting with our last implementation.

Using the idea developed in this section, we express and solve a stochastic control problem in which $X(t)$ is replaced by $\hat{X}(t)$ and the dynamic equation (66) by the Kalman filter. This very old idea in stochastic control goes back at least to Wonham (1968).

Concretely, we can 'just' plug the estimations into our favorite stochastic investment management model, for us the **risk-sensitive asset management model** maximizing the criterion

$$J_{\theta,T}(v,x;h) := \left(-\frac{1}{\theta}\right) \ln \mathbb{E} e^{-\theta \ln V(T;h)} \tag{102}$$

where $V(t)$ is the value of the portfolio and $\theta \in (-1,0) \cup (0,\infty)$ is the risk sensitivity.

In our case,

- The value function Φ is the $C^{1,2}$ solution to associated HJB PDE. It has the form $\Phi(t,x) = e^{-\theta\tilde{\Phi}(t,x)}$, where

$$\tilde{\Phi}(t,x) = \frac{1}{2}x'Q(t)x + x'q(t) + k(t), \tag{103}$$

- There is a unique Borel measurable maximizer $\hat{h}(t,x,p)$ for $(t,x,p) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n$, given by:

$$\begin{aligned} \hat{h}(t,x,p) &= \frac{1}{1+\theta}\left(\hat{\Sigma}\hat{\Sigma}'\right)^{-1}\left[\hat{a} + \tilde{A}\hat{X}(t) - \theta\hat{\Sigma}\hat{\Lambda}'(t)p\right] \\ &= \frac{1}{1+\theta}\left(\Sigma\Sigma'\right)^{-1}\left[\hat{a} + \tilde{A}\hat{X}(t) - \theta\hat{\Sigma}\hat{\Lambda}'(t)p\right]. \end{aligned}$$

- The maximizer is optimal, meaning $h^*(t,\hat{X}(t)) = \hat{h}(t,\hat{X}(t),D\Phi)$.

The Kelly Portfolio and Fractional Strategies

Take the limit as $\theta \to 0$ to recover the **Kelly portfolio** :

$$h = \left(\hat{\Sigma}\hat{\Sigma}'\right)^{-1}\left(\tilde{a}(t) + \tilde{A}(t)\hat{X}(t)\right). \quad (104)$$

... and we even have a **Fractional Kelly Strategy**:

- invest a fraction $\frac{1}{\theta+1}$ in the Kelly portfolio;
- invest a fraction $\frac{\theta}{\theta+1}$ in an 'intertemporal hedging' portfolio

Note that we do not have a Fund Separation theorem in the classical sense because the 'intertemporal hedging' portfolio is not preference-free.

Optimizing Investor Irene's Portfolio

Investor Irene has a ten-year horizon and is willing to invest one third of her wealth in the Kelly portfolio.

The Fractional Kelly strategy implies that the investor's optimal investment strategy at time 0, $h^*(0)$, is an allocation of $1/3$ of initial wealth in the Kelly portfolio $h^K(0)$ and $2/3$ of initial wealth in an intertemporal hedging portfolio $h^I(0)$, which means that Irene's risk sensitivity is $\theta = 2$.

The Kelly allocation at time 0 given an initial state $\hat{X}(0) = \mu_0$ is:

$$h^K(0) = (\hat{\Sigma}\hat{\Sigma}')^{-1}\left(\tilde{a} + \tilde{A}\hat{X}(0)\right) =$$

$$\begin{pmatrix} -50.87\%, -140.39\%, 117.42\%, 116.25\%, -52.31\%, 58.45\%, 74.29\%, 22.9\%, 7.58\%, -72.86\%, 83.19\% \end{pmatrix}'.$$

The Kelly portfolio has net leverage funded by a short position in the money market instrument equal to 63.64% of the investor's wealth.

Irene's intertemporal hedging portfolio is

$$h^I(0) = (\hat{\Sigma}\hat{\Sigma}')^{-1}\hat{\Sigma}\hat{\Lambda}'(0)\left(q(0) + Q(0)\hat{X}(0)\right) =$$

$$(\ 5.73\%, 19.07\%, -7.07\%, 19.25\%, -15.31\%, 23.61\%, 14.71\%, 11.93\%, -6.78\%, -37.69\%, -7.87\%\ )'.$$

S&P 500

Russell.

The ETF positions may seem large when looked at individually. However, once these positions are netted, we see that the intertemporal hedging portfolio is still invested at 80.41% in the money market instrument.

$\theta = 2$

As a result, the overall portfolio allocation is

$$h^*(0) = \frac{1}{3} h^K(0) + \frac{2}{3} h^I(0) =$$

$$\begin{pmatrix} -13.13\%, -34.11\%, 34.43\%, 51.58\%, -27.65\%, 35.22\%, 34.56\%, 15.58\%, -2.00\%, -49.41\%, 22.48\% \end{pmatrix}$$

with 32.45% allocated to the money market instrument.

The next table presents Irene's asset allocation, including the Kelly portfolio and intertemporal hedging portfolio (IHP), for six models:

1. the *Universal Portfolio* due to Cover (1991) and already used to get the prior vector of factors $X(0)$;

Imp 1 →

2. the *Risk-Sensitive Asset Management Model* proposed by Bielecki and Pliska (1999) and Kuroda and Nagai (2002), with full observation.

Imp 2 →

3. the *Risk-Sensitive Asset Management Model with partial observation* due to Nagai and Peng (2002). Here, the asset prices supply the sole source of observations;

Imp 3 →

4. *Black-Litterman in Continuous Time*: the optimal investment model proposed in Davis and Lleo (2013). It includes Adam and Beth's views, but ignores behavioural biases and the stress test scenario;

Imp 4

5. the *Behavioural Black-Litterman* model addressing the biases in Adam and Beth's views, but without the stress test scenario;
6. the *Behavioural Black-Litterman* including the stress test scenario.

1/11 ≈ 9%

Imp 1

Imp

Asset Class	Universal Portfolio	Kelly Portfolio	Risk-Sensitive Asset Management with full observation		Risk-Sensitive Asset Management with partial observation	
			IHP	Optimal Portfolio	IHP	Optimal Portfolio
XLK	8.76%	-50.87%	-1.37%	-17.87%	2.65%	-15.19%
XLF	8.91%	-140.45%	3.41%	-44.55%	17.39%	-35.23%
XLV	9.11%	117.43%	-1.03%	38.46%	-1.20%	38.34%
XLY	9.16%	116.25%	0.12%	38.83%	17.66%	50.52%
XLI	9.04%	-52.34%	-0.24%	-17.61%	-2.25%	-18.94%
XLP	9.09%	58.43%	0.33%	19.70%	7.02%	24.16%
XLE	9.29%	74.27%	1.26%	25.60%	16.45%	35.72%
XLB	9.18%	22.89%	1.18%	8.42%	12.66%	16.07%
XLU	9.11%	7.54%	-0.21%	2.37%	-1.25%	1.68%
IWM	9.12%	-72.85%	6.72%	-19.81%	-39.64%	-50.71%
IYR	9.23%	83.18%	1.17%	28.51%	-12.36%	19.48%
Total risky allocation	100.00%	163.47%	11.34%	62.05%	17.14%	65.92%
Money Market	0.00%	-63.47%	88.66%	37.95%	82.86%	34.08%

-4%

Imp 3

Imp 4 w/o

Imp 4 with

Asset Class	Black-Litterman in Continuous Time		Behavioural Black-Litterman		Behavioural Black-Litterman (with stress test scenario)	
	IHP	Optimal Portfolio	IHP	Optimal Portfolio	IHP	Optimal Portfolio
XLK	5.64%	-13.20%	5.73%	-13.13%	5.73%	-13.13%
XLF	29.65%	-27.06%	19.07%	-34.11%	19.07%	-34.11%
XLV	-0.20%	39.01%	-7.07%	34.43%	-7.07%	34.43%
XLY	29.06%	58.12%	19.25%	51.58%	19.25%	51.58%
XLI	-3.70%	-19.91%	-15.31%	-27.65%	-15.31%	-27.65%
XLP	13.93%	28.77%	23.61%	35.22%	23.61%	35.22%
XLE	28.68%	43.88%	14.71%	34.56%	14.71%	34.56%
XLB	20.47%	21.28%	11.93%	15.58%	11.93%	15.58%
XLU	-1.54%	1.49%	-6.78%	-2.00%	-6.78%	-2.00%
IWM	-70.93%	-71.56%	-37.69%	-49.41%	-37.69%	-49.41%
IYR	-19.80%	14.53%	-7.87%	22.48%	-7.87%	22.48%
Total risky allocation	31.27%	75.34%	19.59%	67.55%	19.59%	67.55%
Money Market	68.73%	24.66%	80.41%	32.45%	80.41%	32.45%

Key Insight: Stress Test Scenarios

Stress test scenarios are invaluable to build an asset allocation because it is more important to avoid large drawdowns in difficult times than to capture the highest returns in bullish markets.

To construct a 'good' stress test scenario:

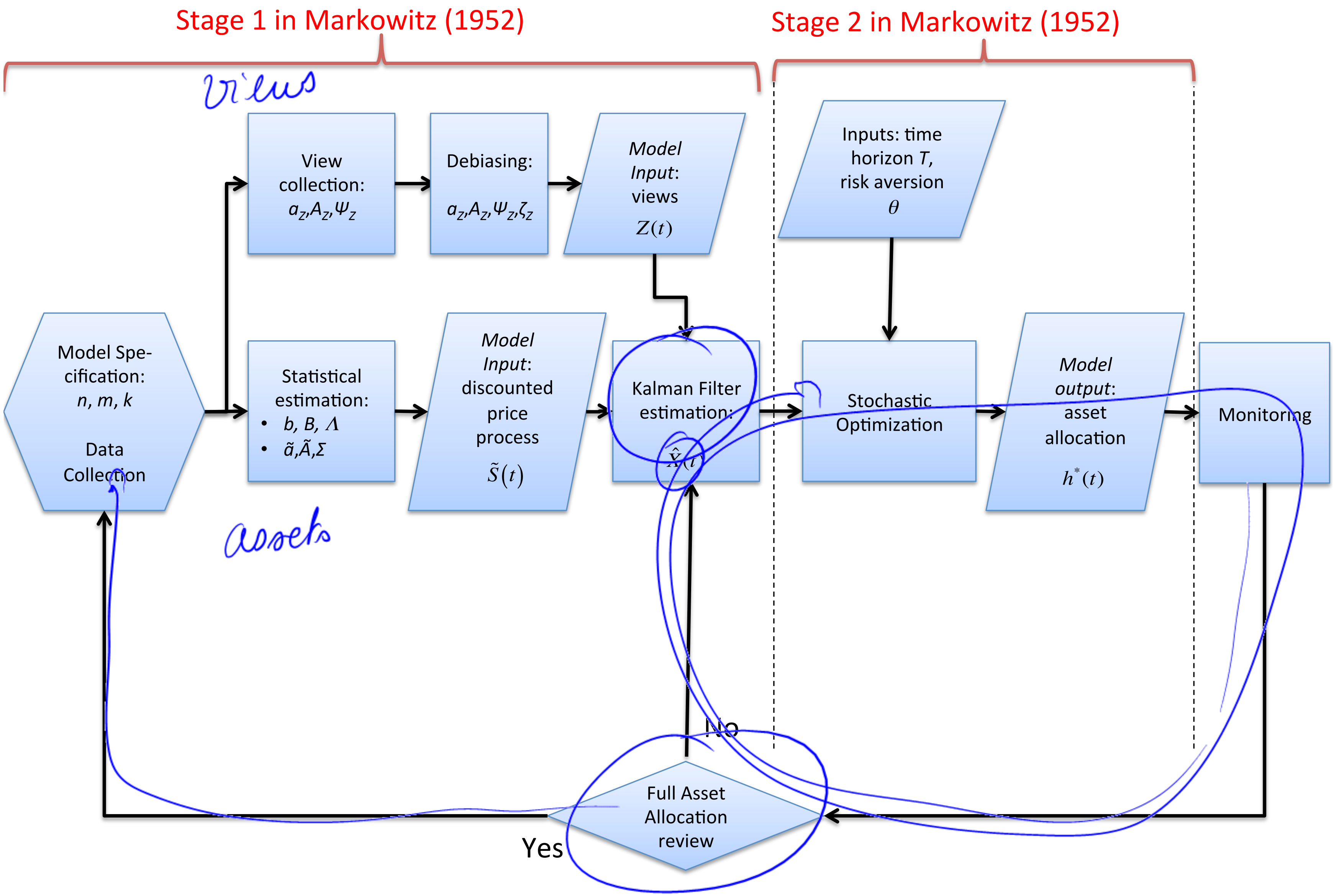
- DO NOT use in-sample historical data. They are already reflected in the parameters you estimated;
- DO use out-of-sample historical data;
- DO use expert opinions to construct a stress test scenario that will truly put your portfolio to the test.

Overview Of the Full Model

The full behavioural Black-Litterman model follows 6 steps:

- Step 1 — Parametrize the financial market
- Step 2 — Collect expert opinions and views
- Step 3 — Address the impact of behavioral biases
- Step 4 — Blend data with opinions and filter to estimate the current level of factors
- Step 5 — Optimize the portfolio
- Step 6 — Monitor

To illustrate the process, we consider the case of Irene, an investor who manages a portfolio of $m = 11$ U.S. Exchange Traded Funds (ETFs).



Stage 1 in Markowitz (1952)
Stage 2 in Markowitz (1952)
views
View collection: a_z, A_z, Ψ_z
Debiasing: $a_z, A_z, \Psi_z, \zeta_z$
Model Input: views $Z(t)$
Inputs: time horizon *T*, risk aversion θ
Model Specification: *n*, *m*, *k*
Data Collection
Statistical estimation:
• b, B, Λ
• $\tilde{a}, \tilde{A}, \Sigma$
Model Input: discounted price process $\tilde{S}(t)$
Kalman Filter estimation: $\hat{X}(t)$
Stochastic Optimization
Model output: asset allocation $h^*(t)$
Monitoring
assets
No
Yes
Full Asset Allocation review

Don't Forget to Monitor!

Monitoring begins once the initial portfolio allocation is set and implemented.

Typically, long-term investors conduct a full asset allocation review once a year or once every few years. A full review involves starting the portfolio selection process over: identifying the objectives and constraints, listing the investable assets, gathering financial market data and experts, before embarking on the full portfolio selection process.

In between full reviews, investors set up an ongoing monitoring of the portfolio against its target asset allocation. The filtering step in our procedure contributes to this ongoing monitoring. We can use the filter to incorporate new stock price data, measure the accuracy of the expert opinions and compare the performance of the portfolio against today's 'optimal' portfolio in real time.

What Next?

The scope of applications of the techniques we have seen in this course is broad.

Risk-sensitive control has found applications in:

1. Asset management;
2. Asset management with a performance benchmark;
3. Asset & Liability Management (ALM);

The investment model has evolved as well...

Jump-Diffusion RSAM II (2013)

$$\frac{dS_i(t)}{S_i(t^-)} = \left[a\left(t, X(t^-)\right)\right]_i dt + \sum_{k=1}^{N} \Sigma_{ik}(t, X(t^-))dW_k(t) + \int_{\mathbf{Z}} \gamma_i(t, X(t^-), z)\bar{N}(dt, dz), \quad i = 1, \ldots, m.$$

$$dX(t) = b\left(t, X(t^-)\right)dt + \Lambda(t, X(t))dW(t) + \int_{\mathbf{Z}} \xi\left(t, X(t^-), z\right)\bar{N}(dt, dz).$$

not for filtering

Jump-Diffusion RSAM (2011)

$$\frac{dS_i(t)}{S_i(t^-)} = (a + AX(t))_i dt + \sum_{k=1}^{N} \sigma_{ik} dW_k(t) + \int_{\mathbf{Z}} \gamma_i(z)\bar{N}(dt, dz), \quad i = 1, \ldots, m.$$

$$dX(t) = (b + BX(t))dt + \Lambda dW(t).$$

ICAPM (1973) / RSAM (1999)

$$\frac{dS_i(t)}{S_i(t)} = (a + AX(t))_i dt + \sum_{k=1}^{N} \sigma_{ik} dW_k(t), \quad i = 1, \ldots, m.$$

$$dX(t) = (b + BX(t))dt + \Lambda dW(t).$$

Merton Model (1969)

$$\frac{dS_i(t)}{S_i(t)} = \mu_i dt + \sum_{k=1}^{N} \sigma_{ik} dW_k(t),$$

$$S_i(0) = s_i, \quad i = 1, \ldots, m.$$

Wrap-Up

In this course, we have seen...

- Perform a dynamic portfolio optimization, using stochastic control.
- Combine views with market data using filtering to determine the necessary parameters.
- Understand the importance of behavioural biases and be able to address them
- Understand the implementation issues.
- Develop new insights into portfolio risk management.

Appendix A: Trace

The trace $\mathrm{tr}(M)$ of a $n \times n$ matrix $M = (m)_{ij}$ is the sum of its diagonal elements:

$$\mathrm{tr}(M) = \sum_{i=1}^{n} m_{ii}$$

The trace of a sum of two matrices A and B is

$$\mathrm{tr}(A + B) = \mathrm{tr}(A) + \mathrm{tr}(B)$$

The trace $\mathrm{tr}(A'B)$ of a product of two $n \times n$ matrices $A = (a_{ij})$ and $B = (b_{ij})$ has the following properties:

- $\mathrm{tr}(A'B) = \mathrm{tr}(AB') = \mathrm{tr}(B'A) = \mathrm{tr}(BA')$
- $\mathrm{tr}(A'B) = \sum_{i,j}^{n} a_{ij} b_{ij}$

Appendix B: Utility Theory

To effectively solve stochastic control problems in finance, it is helpful to know several stylised facts about utility functions. Chapter 1 in Ingersoll (1987) provides a good introduction to utility theory.

Although the concept of utility function dates back to D. Bernoulli in 1738 (see Bernoulli (1954)), the "modern" concept of utility function we use in finance is due to von Neumann and Morgenstern von Neumann and Morgenstern (1944). Utility is a general concept that can be quantified through a large number of functions. In fact, to be a utility function of wealth W, a function $U(W)$ must satisfy the following three properties:

1. the utility function is continuously differentiable and positive: $U(W) > 0$;
2. the utility is function increasing: $U'(W) > 0$ with $\lim_{W \to \infty} U'(W) = 0$;
3. the utility is function concave: $U''(W) < 0$.

Case 1- negative wealth is not allowed:

In the case when negative wealth is <u>not</u> allowed, $W \in \mathbb{R}^{+*}$ and we have the additional condition:

5. $\lim_{W \to 0^+} U(W) = -\infty$ and $\lim_{W \to 0} U'(W) = +\infty$

Conditions 1-5 are collectively known as the *Inada conditions*.

Case 2 - negative wealth is allowed:

In the case when negative wealth is <u>is</u> allowed, $W \in \mathbb{R}$ and the utility function will still satisfy conditions 1-4, plus the following condition:

6. $\lim_{W \to -\infty} U(W) = -\infty$ and $\lim_{W \to -\infty} U'(W) = +\infty$;

Economists use three related functions to assess the risk tolerance or risk aversion associated with a particular utility function:

1. *Arrow-Pratt absolute risk aversion function* $A(W)$ defined as

$$A(W) := -\frac{U''(W)}{U'(W)} \tag{105}$$

   and which measures infinitesimal risk aversion.

2. *Risk-tolerance function* $T(W)$:

$$T(W) := \frac{1}{A(W)} \tag{106}$$

3. *Relative risk-aversion function* $R(W)$:

$$R(W) := W \times A(W) \tag{107}$$

In finance, the most commonly used class of utility functions is the class of **HARA** (hyperbolic absolute risk aversion) or LRT (linear risk tolerance) utility functions, which are functions U of the form

$$U(W) = \frac{1-\gamma}{\gamma}\left(\frac{aW}{1-\gamma} + b\right)^{\gamma} \tag{108}$$

where $b > 0$.

Moreover,

- U is defined on the domain $\frac{aW}{1-\gamma} + b > 0$;
- the absolute risk-tolerance function T of a HARA utility function is:

$$T(W) = \frac{W}{1-\gamma} + \frac{b}{a}$$

Among the HARA class, five utility functions are of particular interest:

1. *Linear utility:* this is the case where $\gamma = 1$ and as a result,

$$U(W) = aW + b \tag{109}$$

2. *quadratic utility:* take $\gamma = 2$, then

$$U(W) = -\frac{1}{2}\left(-aW + b\right)^2 \tag{110}$$

3. *Power utility:* take $b = 0$, $\gamma \in (-\infty, 0) \cup (0, 1)$ and $a = (1-\gamma)^{\frac{\gamma-1}{\gamma}}$, then

$$U(W) = \frac{W^{\gamma}}{\gamma} \tag{111}$$

- The power utility function has constant relative risk aversion $R(W) = 1 - \gamma$ and therefore belongs to the class of constant relative risk aversion (CRRA) utility functions.
- As result, its absolute risk aversion $A(W)$ is decreasing in W;
- The power utility function does not allow negative wealth.

4. *Log utility:* take $b = \gamma = 0$, then in the limit

$$U(W) = \ln W \tag{112}$$

To see this consider the power utility function $\frac{W^\gamma}{\gamma}$ in the limit as $\gamma \to 0$. An application of L'Hospital's rule shows that $\lim_{\gamma \to 0} \frac{W^\gamma - 1}{\gamma} = \lim_{\gamma \to 0} \frac{W^\gamma \ln W}{1} = \ln W$.

- This function is the oldest utility function. It was first suggested by D. Bernoulli in 1738 (see Bernoulli (1954)).
- Another property of the log utility function is that it arises naturally as the limit of the power utility function as $\gamma \to 0$.
- The log utility function has a constant relative risk aversion $R(W) = 1$.
- The log utility function does not allow negative wealth.

5. *Negative exponential utility:* take $\gamma = -\infty$ and $b = 1$

$$U(W) = -e^{-aW} \tag{113}$$

- The exponential utility has constant absolute risk aversion $A(W) = a$. It belongs to the class of constant absolute risk aversion (CARA) utility functions.
- Note that the negative exponential utility function allows negative wealth.

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