Understanding Volatility

In this lecture. . .

- The several kinds of volatility
- Vega, and why it is dangerous
- What the market prices of options tell us about volatility
- Introduction to term structure and smile

$\sigma_i \quad \sigma_a$

- Volatility Arbitrage: should you hedge using implied or actual volatility?

$\Delta = N(d_1)$

By the end of this lecture you will

- know about all of the most important kinds of volatility
- be able to relate implied volatility to actual volatility
- understand calibration of the term structure
- know about volatility skew (beyond 'fear' and 'greed')
- see how to utilise delta-replication for arbitraging options

Introduction

Black-Scholes is a convenient valuation model that matches a market price of option to volatility, under the risk-neutral condition. Exposure to asset price direction is delta-hedged.

That does not mean "there is no risk". It means that we 'earn' from asset movements up and down, which is volatility.

Volatility is the most important parameter determining the value of an option, yet it is also the hardest to measure.

$$V(K, T, r, S, \sigma)$$

The different types of volatility

- Actual/Local σ_a
- Historical/Realized σ_{hist}
- Implied σ_i or σ_{BS} σ_{imp}
- Forward

Local Vol.

$\sigma(S,t)$

calibration

$\sigma_i(K,T)$

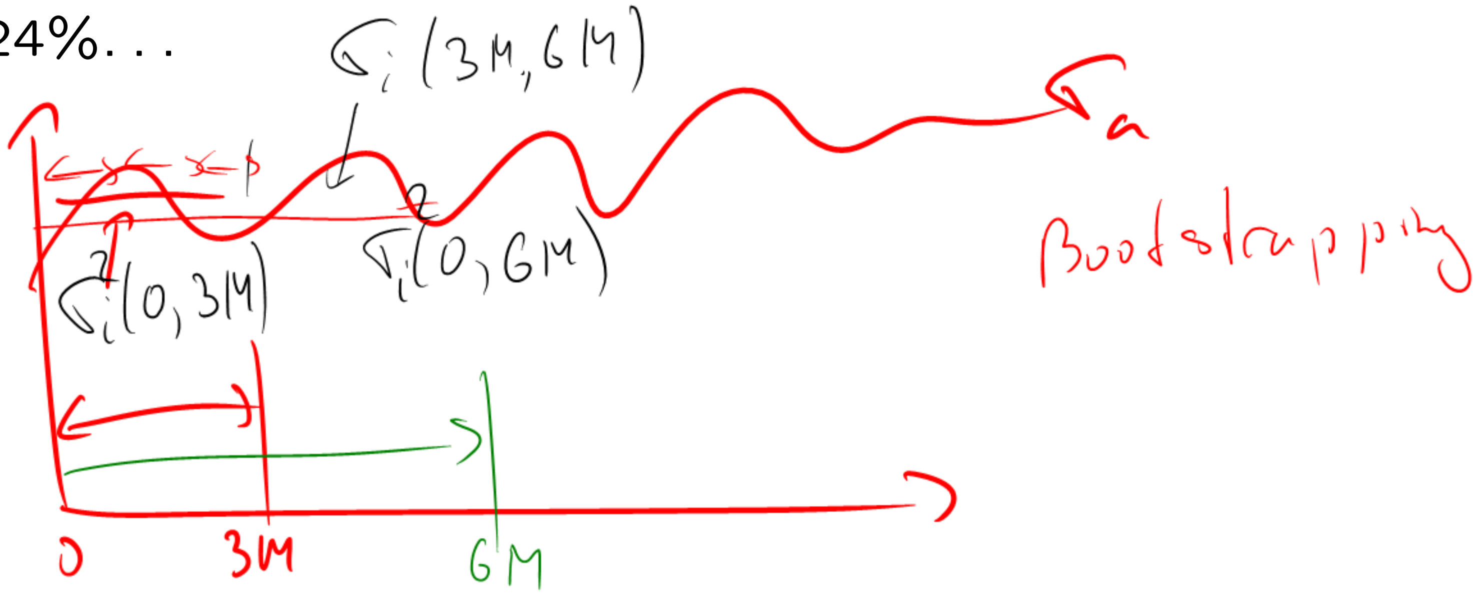
$\sigma(t) = F(\sigma_i(T))$

Actual/Local Volatility

This is the measure of the amount of randomness in an asset return at any instant of time.

- There is no 'timescale' associated with actual volatility.

Example: The actual volatility is now 20%. . . now it is 22%. . . now it is 24%. . .



The actual or 'local' volatility $\sigma(S,t)$ goes into the Black–Scholes equation and its Greeks.

- We will have to consider the implications for Black-Scholes if we use $\sigma(t)$ and then $\sigma(S,t)$.

Actual volatility input is as an instantaneous quantity, that is very difficult to measure.

$$dS = \mu S dt + \sigma_a S dX$$

$$d\sigma = \ldots$$

Historical/Realized Volatility

A measure of the amount of randomness over some period in the past.

The period is always specified, and so is the mathematical method for estimation, such as MAD, std. deviation, EWMA, GARCH. (1,1)

Volatility forecasting is best done using 'filtering'. EGARCH

- For historical volatility, we treat short timescale (day to day) and long-term horizons differently.

$$\frac{\sum_{t=1}^{N} (u_t-\mu)^2}{N-1} = \sigma^2$$

N_{obs}

Example 1: The 60-day volatility using daily returns.

Perhaps of interest if you are pricing a 60-day option, which you are hedging daily.

Example 2: I sold a 30-day call option for a 25% implied volatility, I hedged it every day.

Did I make money?

(The answer depends on how much volatility will realise and how you hedged.)

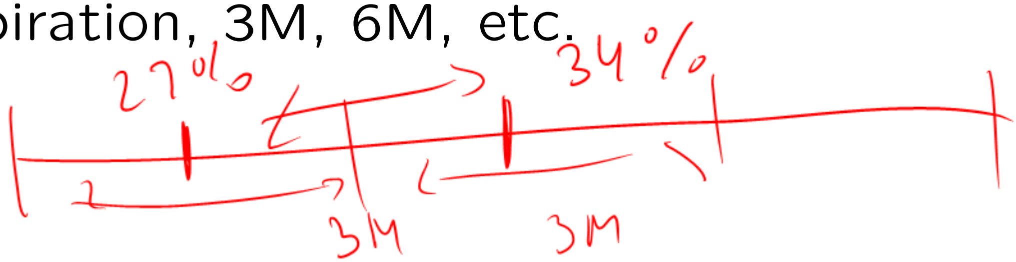
Implied Volatility

The implied volatility is a number. When used in the Black-Scholes formulae it outputs the market price of the option.

It is often described as **the market's view of the volatility** to realise over the lifetime of the particular option.

That is a naive view. Option prices are influenced by model factors (eg, CVA) and real-life factors (eg, supply and demand).

- There is one 'timescale' associated with implied volatility: expiration, 3M, 6M, etc.



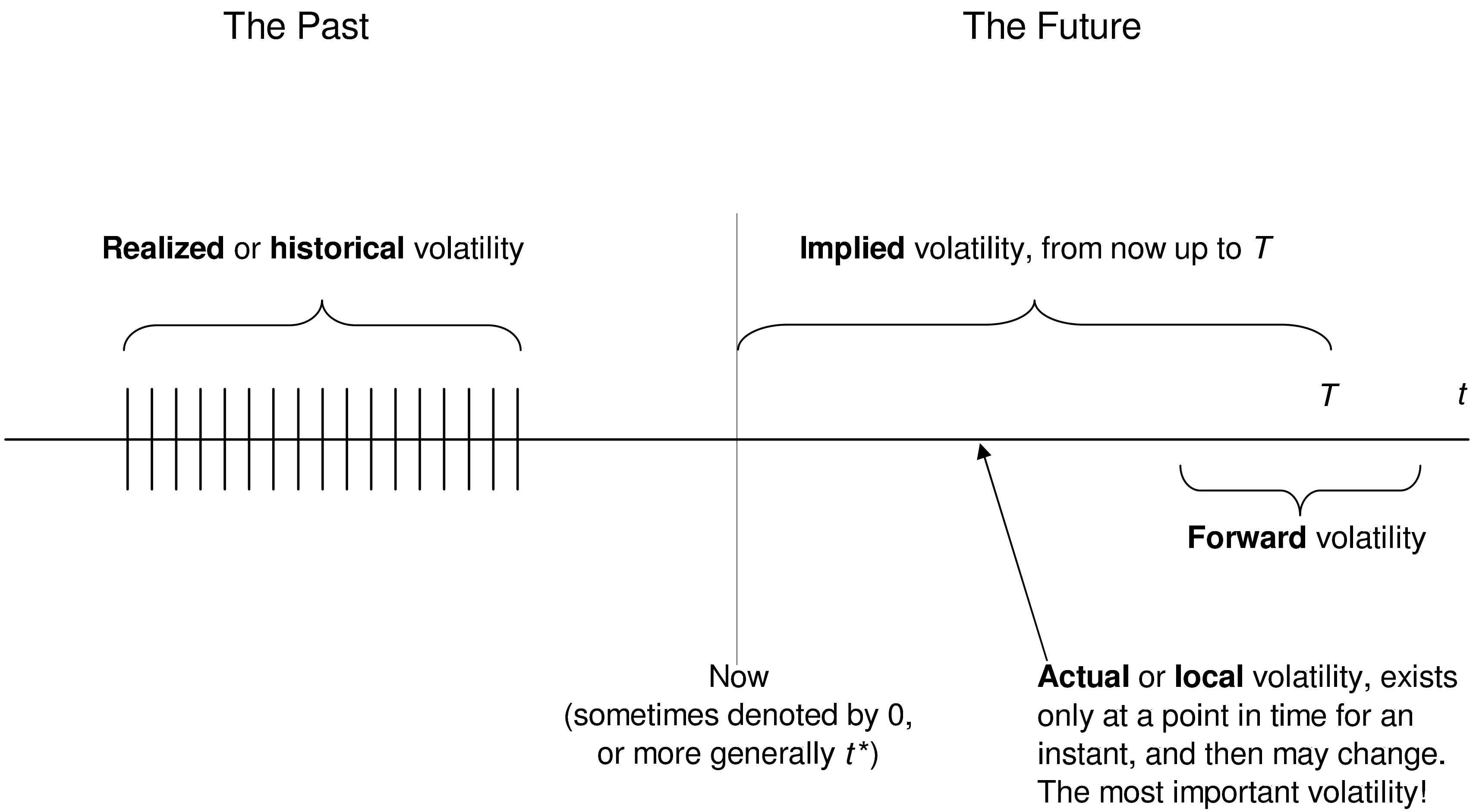
(Forward Volatility)

The adjective *forward* refers to the volatility (whether actual or implied) *over some period in the future*.

$\sigma_i(3M, 6M)$

- Forward volatility is associated with either a time period, or a future instant.

Future volatility expected to realise over a period of time can be traded in the form of *a variance swap*.


The Past
The Future
Realized or **historical** volatility
Implied volatility, from now up to *T*
T
t
Forward volatility
Now
(sometimes denoted by 0,
or more generally *t**)
Actual or **local** volatility, exists
only at a point in time for an
instant, and then may change.
The most important volatility!

Model Risk

Please . . . always know which volatility you, and others, are talking about.

In the second part, we are going to introduce the models for actual volatility, each coming with some strong assumptions.

- deterministic volatility $\sigma(t)$
- local volatility $\sigma(S,t)$

price

$$\frac{\delta V}{\delta K} = \frac{\delta C}{\delta K}$$

strike

We will need models to uncover the information the market signals, assuming it exists.

More about implied volatility

"What volatility must I use to get the correct market price?"

This is called the implied volatility. Substituted into the Black-Scholes formulæ, it gives *a theoretical price*, which is equal to the observed market price.

Black-Scholes formulæ

In the Black–Scholes world of **constant volatility**, the value of a European call option is simply

$$V_{BS}(S, t; \sigma, r; E, T) = SN(d_1) - Ee^{-r(T-t)}N(d_2)$$

where

$$d_1 = \frac{\log(S/E) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = \frac{\log(S/E) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

V is a function of six arguments, all but σ are easy to agree on.

How do we know which volatility to put into the formulæ?

Turn the relationship between volatility and an option price on its head.

A trader sees on his screen that a call option with four months until expiry and a strike of 100 is trading at $6.51 with the underlying at 101.5 and a short-term interest rate of 6%.

To obtain the implied volatility number, we have to numerically solve **a root-finding problem**.

$$V_{BS}(\sigma_i) - 6.51 = 0$$

- Bisection
- Newton Raphson

(The answer is $\sigma_i \approx 20.2\%$. What about a put option?)

σ_i is **a parameter** of the Black-Scholes which we treat as a variable.

The implied volatility is attributed to the underlying. However, the actual volatility exists even in the absence of options.

The proper measure of randomness in the asset is σ_a, which turns out to be an instantaneous quantity. We can't measure that quantity directly.

If we think that implied volatility is a market's view of volatility over the life of the option, then

$$\textit{Future} \quad \sigma_{Hist} \quad \textit{would turn out as} \quad \sigma_i$$

In practice, the option hedgers look if

$$\sigma_i \quad \textit{similar to Past} \quad \sigma_{Hist}$$

The smarter way would be to model the process σ_a and avoid the perils of historical estimation ← Statistical Estimates

$$\sigma_i \quad \textit{turns out as an average over} \quad \sigma_a$$

Local Vol.

Implied volatility is the only volatility we can **'see'**.

Uses of implied volatility

Implied volatility has many uses, e.g., local volatility, market's view, market-making, hedging exotic options, etc.

$\sigma_i < \sigma_r$

Buy side/Hedge funds: make their own volatility forecasts. If it is different from implied, then arbitrage opportunities exist! They buy/sell options and make a profit if they are right.

Sell side/Investment banks: use implied volatility to tell them how to price exotics. They sell exotics with profit margin on top and hedge to lock in gains.

Useful sold exotic options with known analytical solutions are:

- Binaries (digitals)
- Autocalls (worst of) Hedged with Barrier Options
- Passport options (on long/short P&L)

Sell-side business relies on information contained in implied volatility, subject to **calibration** (fitting of the volatility process).

More about calibration later.

Vega

What happens to the value of an option when *perceptions* about volatility of the underlying change?

An option value can change even when the underlying doesn't move.

A market that is panicking sells puts and calls more expensively.

Because implied volatility can change on its own we would like to know how sensitive option value is *wrt* that change.

Vega is the sensitivity of an option value to volatility (whether implied or actual that is vague)

$\sigma(t, S)$

$$\frac{\partial V}{\partial \sigma}$$

$\sigma_i(T, K)$

Example: If an option ($\times 100$ shares) has a Vega of 37.5 it means that if implied volatility goes up by 1 unit (e.g., from 20 to 21%), the cash option price will change by $37.5.

Vega is a sensitivity to a parameter, not a variable. Makes it different from legitimate greeks: Delta, Gamma and Theta.

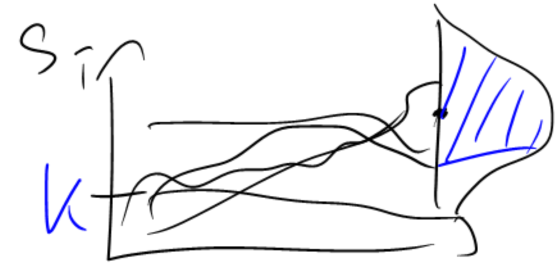
Vega is a **bastard greek**.

Bastard greeks are illegitimate because they involve differentiating with respect to a parameter, *which has been assumed constant in the derivation of the formulae* (Black-Scholes).

Such sensitivities are inherently inconsistent and lead to errors in risk estimation.

Just because the model Vega is small, it is simply dangerous to think there is no exposure to the implied volatility (going up or down) risk factor.

Vega for an exotic



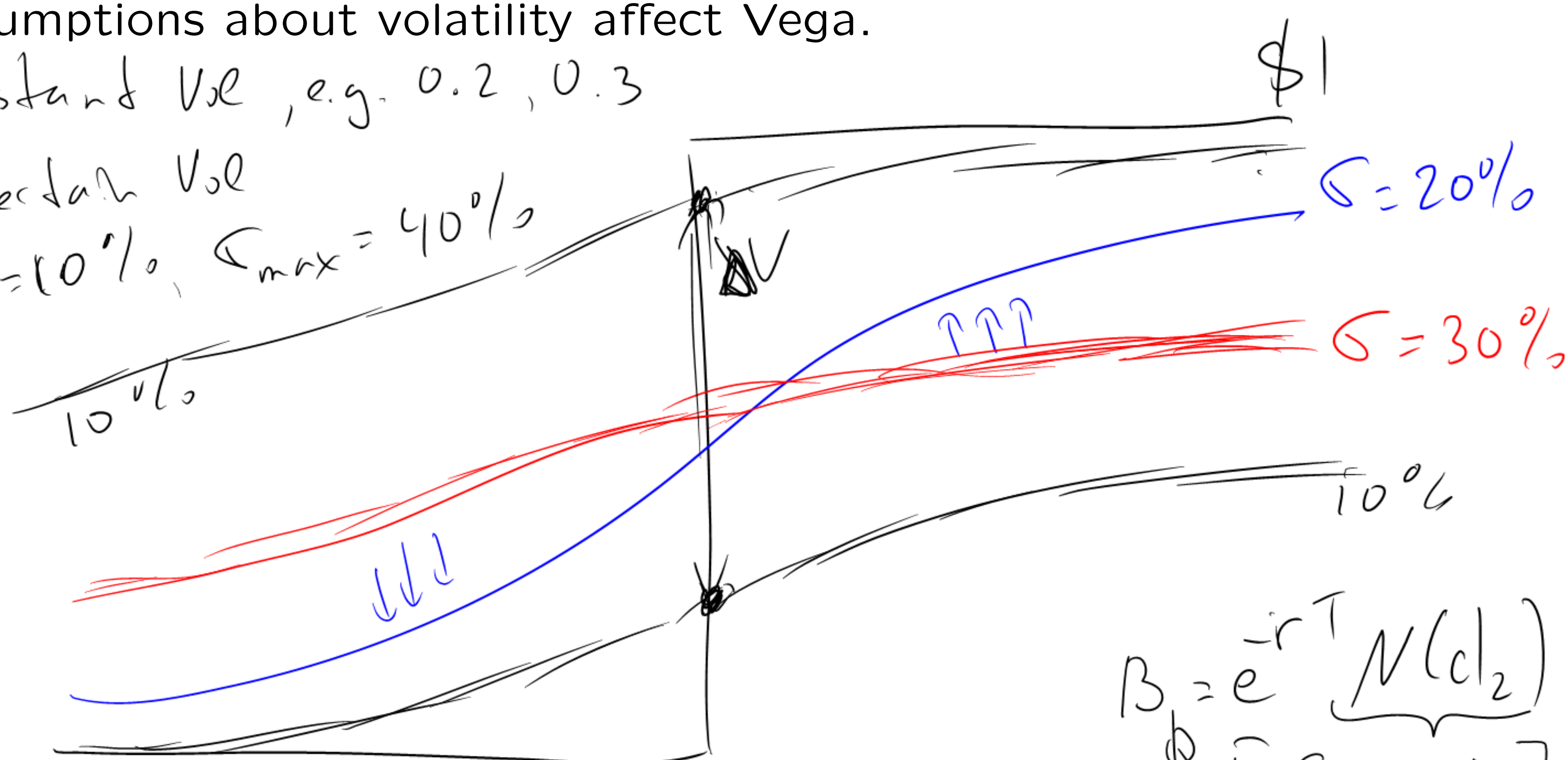
A Vega can be calculated as zero at precisely the point, at which there is a great deal of volatility risk. Let's see how modelling assumptions about volatility affect Vega.

1) Constant Vol, e.g. 0.2, 0.3

2) Uncertain Vol

$\sigma_{min} = 10\%$, $\sigma_{max} = 40\%$

$\cancel{\frac{dV}{d\sigma}}$



$B_b = e^{-rT} \underbrace{N(d_2)}$

$\Pr[S_T > K]$

$\Delta = N(d_1)$

Why is Vega important?

- **Hedging:** you can statically hedge one option with another.

  If option C_1 has a Vega of 37.5 and option C_2 has a Vega of 75 then, the spread $2C_1 - C_2$ will be *Vega-neutral*.

  The sell-side uses this to reduce risk when selling exotics and managing their option books.

- **Risk management:** if you buy an option to speculate on price movement, the level of implied volatility is a risk factor.

  The option value will be quoted as implied volatility and you want to know your exposure, e.g., $\partial V/\partial\sigma = 37.5$.

Hedging with Greeks:

Suppose the following contracts are available to us:

	Delta	**Gamma**	**Vega**
Stock	1	0	0
Option A	0.4	0.026	27
Option B	0.6	0.018	36

Consider two strategies:

1. Construct a portfolio that is Delta and Gamma-neutral but positive Vega. (Why need one?)

2. Option B is an OTC exotic, you are going to sell it for more than it is worth. Construct a portfolio that is Delta and Vega-neutral.

Strategy 1: If we expect implied volatility to rise in short run.

Buy a quantity A of option A, B of option B, and C of the stock.

Delta neutral: $C + 0.4A + 0.6B = 0$

Gamma neutral: $0.026A + 0.018B = 0$

Positive vega: $27A + 36B = 1$

I have arbitrarily set vega to 1. In practice, that would be constrained by risk or cash limits.

More importantly, you would make the vega positive (negative) if you expect the implied volatilities to rise (fall).

The solution of this set of simultaneous equations is

$\times 100$

$$C = -0.01867, \qquad A = -0.04 \quad \text{and} \quad B = 0.05778.$$

To achieve positive vega (exposure to implied volatility), you must short stock and contract A and go long contract B.

Opposite positions will work for a negative vega exposure.

Strategy 2: Sell OTC Option B to make a profit, but be delta and vega neutral.

Vega = 24 Vega = 36

Buy a quantity A of option A, -1 of option B, and C of the stock.

Delta neutral: $C + 0.4A + 0.6 \times (-1) = 0$

Vega neutral: $27A + 36 \times (-1) = 0$

The solution is

$$C = 0.06667, \quad \text{and} \quad A = 1.3333.$$ $\times 100$

Please remember that you can add up Greeks arithmetically in order to calculate the total exposure.

Greeks are 'risk factors'.

Some Rules of Thumb:
How the Greeks change with varying volatility

Why is this important?

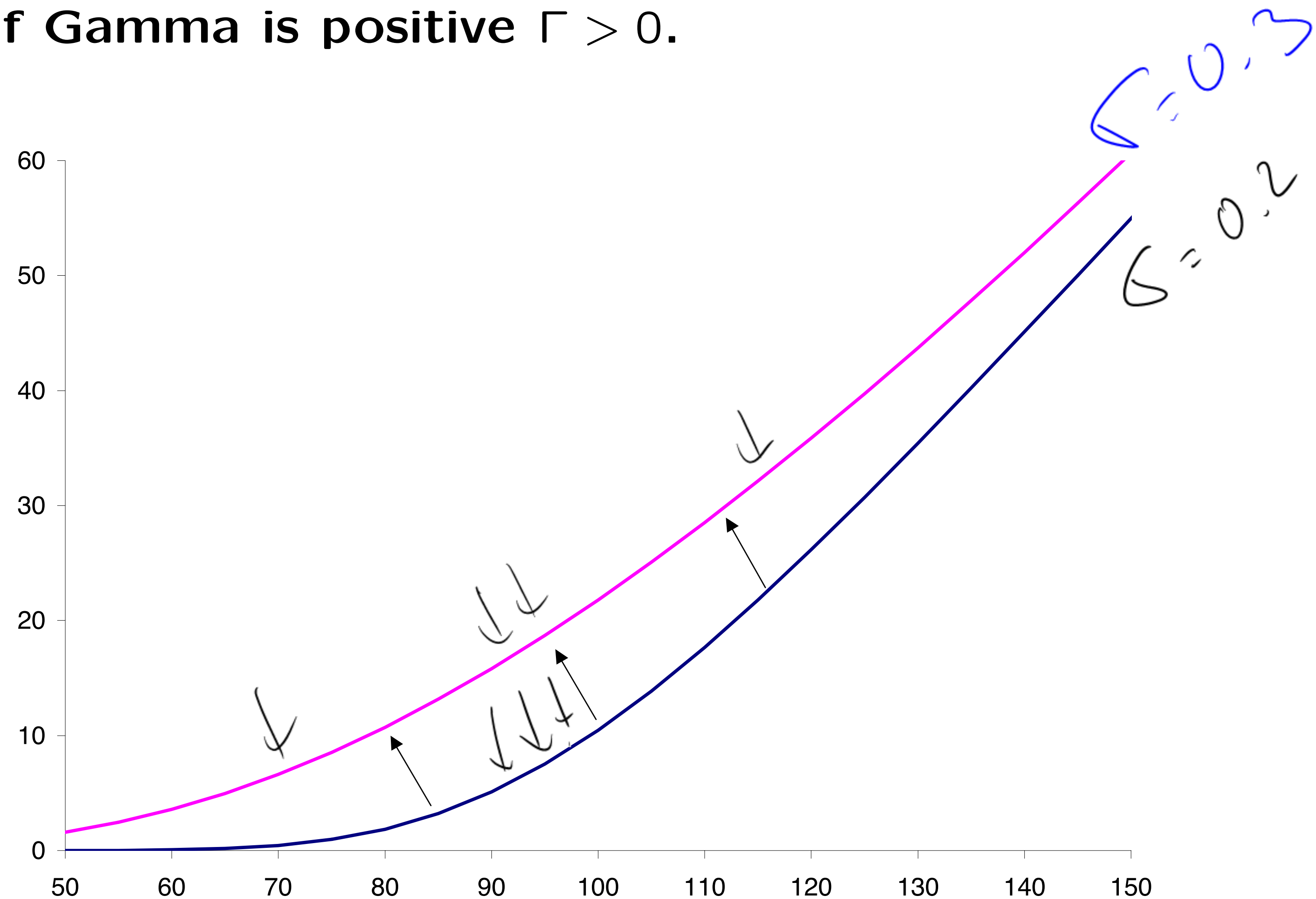
- Need to know how the option value changes when the market's perception of future volatility changes

- Hedge ratios may have to be updated to maintain the desired exposure (i.e., gamma or vega-neutral)

- As part of risk management

Call

Rule of Thumb 1: Increasing volatility will increase option value if Gamma is positive $\Gamma > 0$.

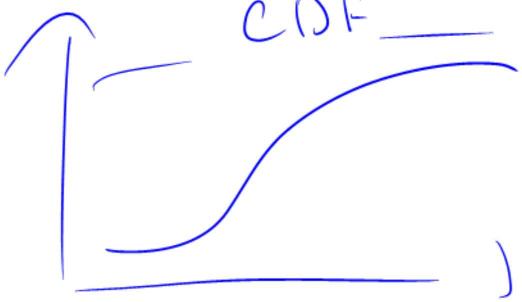


Rule of Thumb 2: Increasing volatility 'smoothes' out curvature in the asset direction.

This is true for option price, also the Greeks. For example, Delta versus underlying asset.

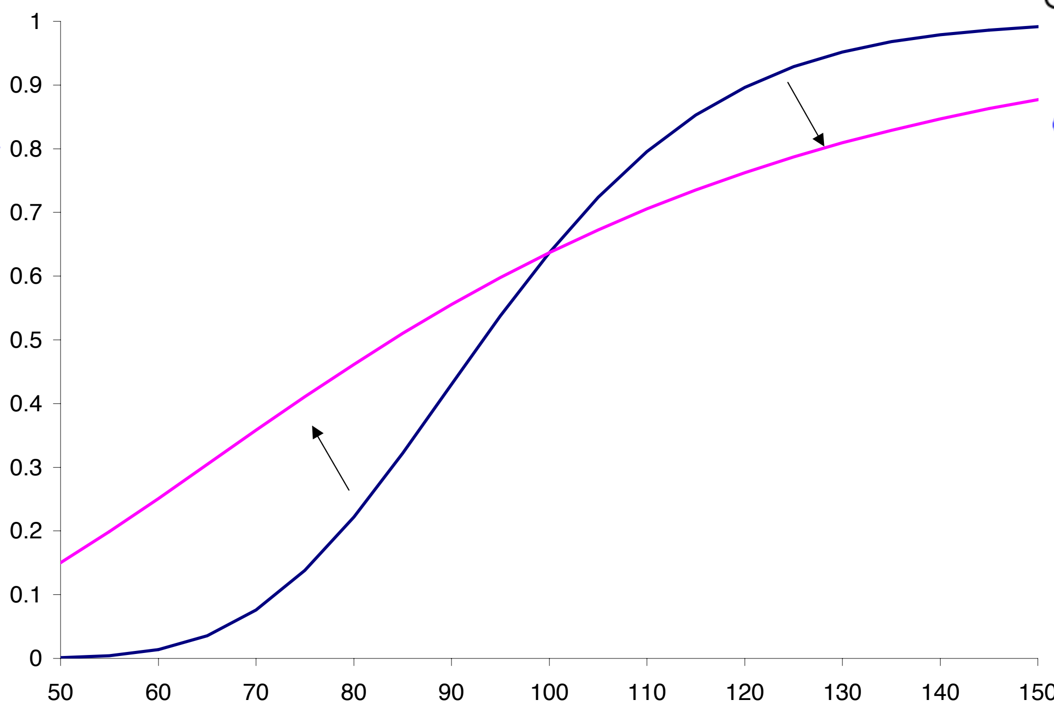
$\Delta = N(d_1)$

CDF



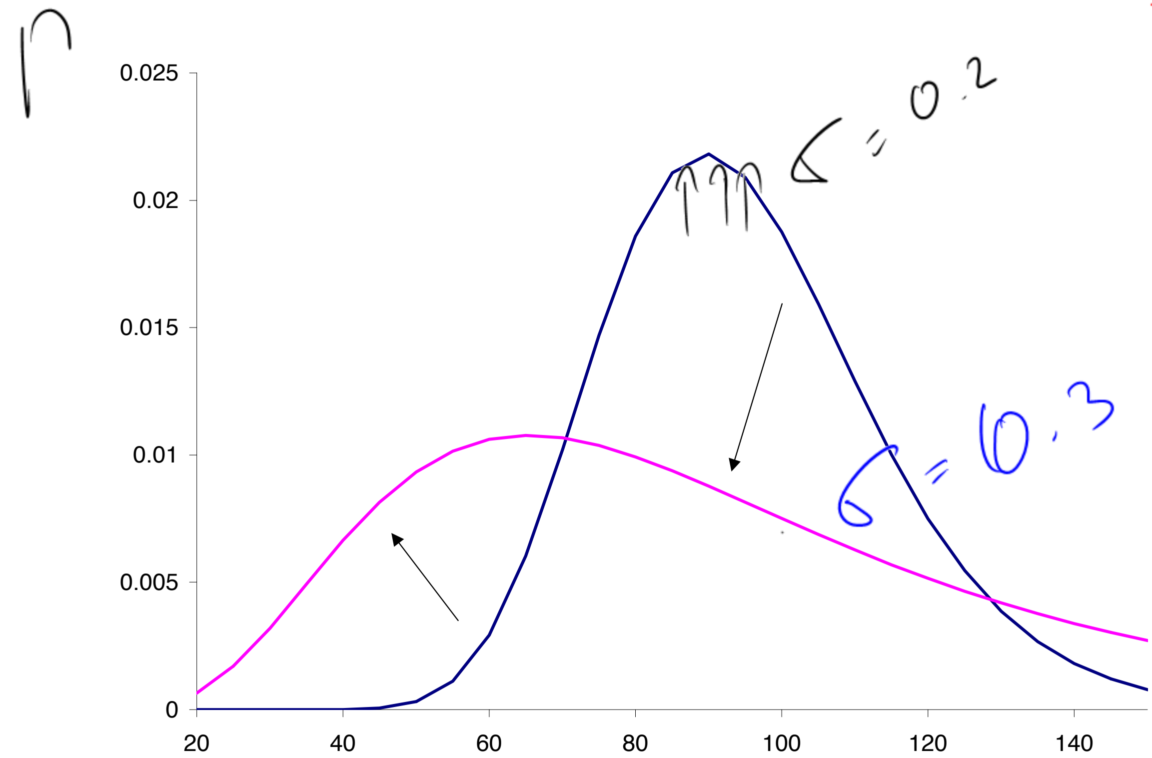
$\sigma = 0.2$

$\sigma = 0.3$



Rule of Thumb 3: Increasing volatility makes Gamma less 'severe'

When volatility is low, Gamma can get very large.



Rule of Thumb 4: Doubling implied volatility is like quadrupling time to expiry.

Volatility and time to expiry always appear in this combination:

$$\sigma\sqrt{T}$$

$$2\sigma\sqrt{T} = \sigma\sqrt{4T}.$$

We are now going to move away from the world of known, constant volatility.

Problems with Black–Scholes model

- Effects of *discrete hedging*   PW & RA CQF Extra
- Transaction costs on the underlying. Illiquidity
- **Uncertainty in volatility**
- Market jumps, crashes, discontinuous asset paths
- **Default risk** of the underlying
- Uncertainty in dividends
- Discrete price effect (especially in the teenies)
- Feedback, market manipulation
- **Supply and demand**

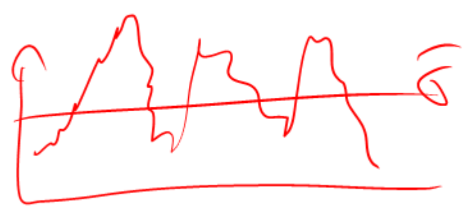
Heston

$$dS/S = \mu\, dt + \sigma_t\, dX_1$$

$$d\sigma_t = \underbrace{\theta(\sigma_t - \bar{\sigma})}\, dt + \underbrace{\nu\, dX_2}_{\text{Vol of Vol}} \qquad \perp$$

$\bar{\sigma}$ – long term avg.

θ – speed of reversion to $\bar{\sigma}$ level.



Out of the money puts are a popular form of insurance.

When someone sells a deep OTM put, the premium has to cover *replication costs*: discrete delta hedging, probability of a crash (aka fat tail), market frictions. . . business costs (CVA, funding, collateral).

Oh, and Profit!

Black–Scholes has only **one parameter** measuring replication costs: the volatility.

So we can't really back out just perfectly meaningful option value/asset volatility from option price. Why?

Counterparty issues

Models the sell-side uses in order to add their charges address

- Liquidity – bid/ask spreads of all instruments used affect an exotic structure or hedging solution

- Counterparty exposure / credit risk

- Collateralisation – mitigates credit risk but introduces valuation of collateral and usually FX considerations ('quanto')

- Cost of funding

Costs reflected in adjustments

Adjustments are taken against Mark-to-Market value (option price) to recognise credit risk, funding costs and operations with collateral – also discrete hedging and model-specific risk.

Risk-free Derivative = Risky Derivative + Adjustments

The major adjustment is CVA – the cost of buying protection on the counterparty that pays the portfolio value in case of default (Cesari, 2011; updated sources of market practice available).

The requirement to hedge against credit risk affects business costs of every regulated financial entity.

Back to the issue of information offered by the implied volatility...

One is usually taught to think of the implied volatility as the market's view of the future value of volatility.

People often say the nonsensical 'the market is always right.'

Yes and no.

If there is any useful information to have about the future behaviour of actual volatility, then one will engage in calibration.

Two different approaches

We are going to map two opposite directions of thinking,

First, we will assume the market is right! The implied volatility offers a close to perfect information actual volatility. We just need some maths to relate the two (e.g., local volatility model).

Second, we will assume the market is wrong! If we have a better idea of volatility than the market, the opportunity for **volatility arbitrage** opens. If we are right we will make money.

Calibration

Implied volatilities have a term structure: 3M, 6M, 9M, 12M.

- If the market knows what actual volatility is going to be then our task is to obtain this information, to have σ_a for pricing any options and exotics.

This is **calibration** or **fitting**: making our theoretical option values to match the market prices.

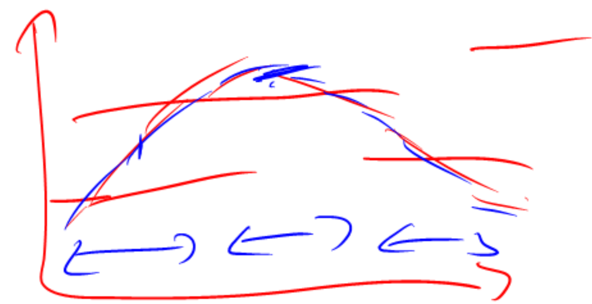
Some maths will be required.

Term-dependent implied volatility

Below are the market prices of European call options with three, six, nine and 12 months until expiry.

All have strike prices of 110 and the underlying asset is currently 108.5. The short-term interest rate over this period is 4%.

Expiry	Value	IV
3 months	4.74$	22.8%
6 months	6.72$	20.9%
9 months	8.22$	19.7%
12 months	9.63$	19.1%



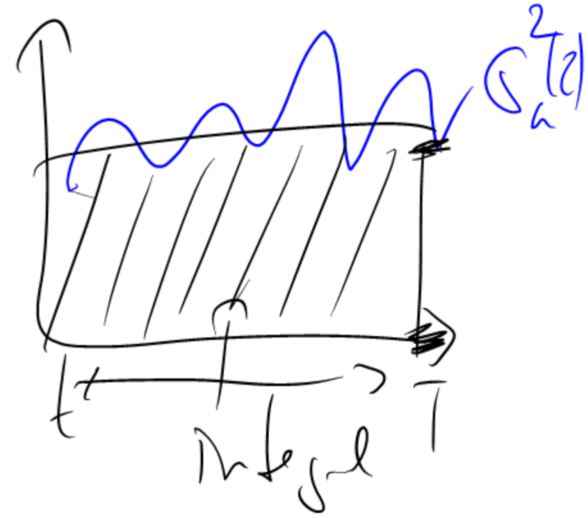
Clearly these prices cannot be correct if actual volatility is constant for the whole year. What is to be done?

We aim to reconcile a term-dependent implied volatility with the Black–Scholes model.

The simplest assumption we can introduce without disturbing the Black–Scholes world (of constant volatility) is to model the actual volatility as

- a *time*-dependent, deterministic 'variable' $\sigma(t)$

The Black–Scholes formulæ are still valid when volatility is time-dependent provided we use



$$\underbrace{\sqrt{\frac{1}{T-t}\int_t^T \sigma(\tau)^2 d\tau}}_{\text{volatility}}$$

in place of σ, i.e. now

$$d_1 = \frac{\log(S/E) + r(T-t) + \frac{1}{2}\int_t^T \sigma(\tau)^2 d\tau}{\sqrt{\int_t^T \sigma(\tau)^2 d\tau}}$$

We are putting integrals into the Black-Scholes formulæ!

Obvious?

$\sigma(t,S)$

If the actual volatility varies with time $\sigma_a \to \sigma(t)$, the value of a vanilla option is

A function of 'average' volatility between now and expiry.

You can't average volatilities but you can add variances!

The sum of variances over the life of the option, from now t to expiration T, is an integral over the instantaneous process

$$\int_t^T \sigma^2(\tau)d\tau$$

and therefore, the equivalent 'average' volatility is

$$\sqrt{\frac{1}{T-t}\int_t^T \sigma^2(\tau)d\tau}.$$

Let's step back a bit...

$\sigma, \sigma(t), \sigma(S,t)$

$\int \sigma(\tau)\, d\tau$

If σ is constant then given derivation assumptions we have the Black–Scholes **pricing equation**:

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

PDE

And for vanilla calls and puts this has nice, **closed-form solutions**, known as the Black–Scholes formulæ.

$$C(S,t) = SN(d_1) - Ee^{-r(T-t)}N(d_2)$$

$$P(S,t) = C(S,t) + Ee^{-r(T-t)} - S$$

If $\sigma(t)$ is a function of time then we *still* have the Black–Scholes equation:

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2(t)S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

And for vanilla calls and puts the Black–Scholes formulæ still work with the special expression that relies on $\int_t^T \sigma^2(\tau)d\tau$.

Adding an asset price to volatility function $\sigma(S,t)$ is allowable for Black–Scholes equation but the **the formulæ stop being valid**!

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2(S,t)S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

The reasoning above is **not a mathematical proof** that using the Black–Scholes formulæ with $\sigma(t)$ is a correct move!

It is not often explained where the Black-Sholes works and where it breaks down.

Two ways of such proof would be:

1. Transform the Black–Scholes equation into the simpler heat equation (as we saw before) and change the time variable.

2. Substitution!

Substitution for implied volatility

If $\sigma(t)$ tells us how the actual volatility varies with calendar time, then the following tells us how implied volatility should behave

$$\sqrt{\frac{1}{T-t}\int_t^T \sigma^2(\tau)d\tau} = \text{ implied volatilities } \sigma(T)$$

We see the implied volatilities but not the actual. . . so the formula works the other way around.

Remember that we want a function of time $\sigma(t)$.

Let's consider one 'technical thing'.

We would expect to know the parameter $\sigma(t)$ to the Black-Scholes equation and have to find an answer, effectively the option price

$$\sigma(t) \quad \Rightarrow \quad V(\sigma_{imp}(T))$$

However, we do know the answer, the market option price V and have to identify the parameter, $\sigma(t)$.

$$\sigma(t) \quad \Leftarrow \quad \sigma_{imp}(T)$$

By calibrating a model for $\sigma(t)$, we 'parametrise' the Black-Scholes equation with time-dependent volatility.

This is an **inverse problem**.

Mathematically, inverse problem leads to **an integral equation**.

When we attempt to fit $\sigma(t)$ to European option implied volatility $\sigma_{imp}(T;t^*)$ measured at time t^*, the solution is

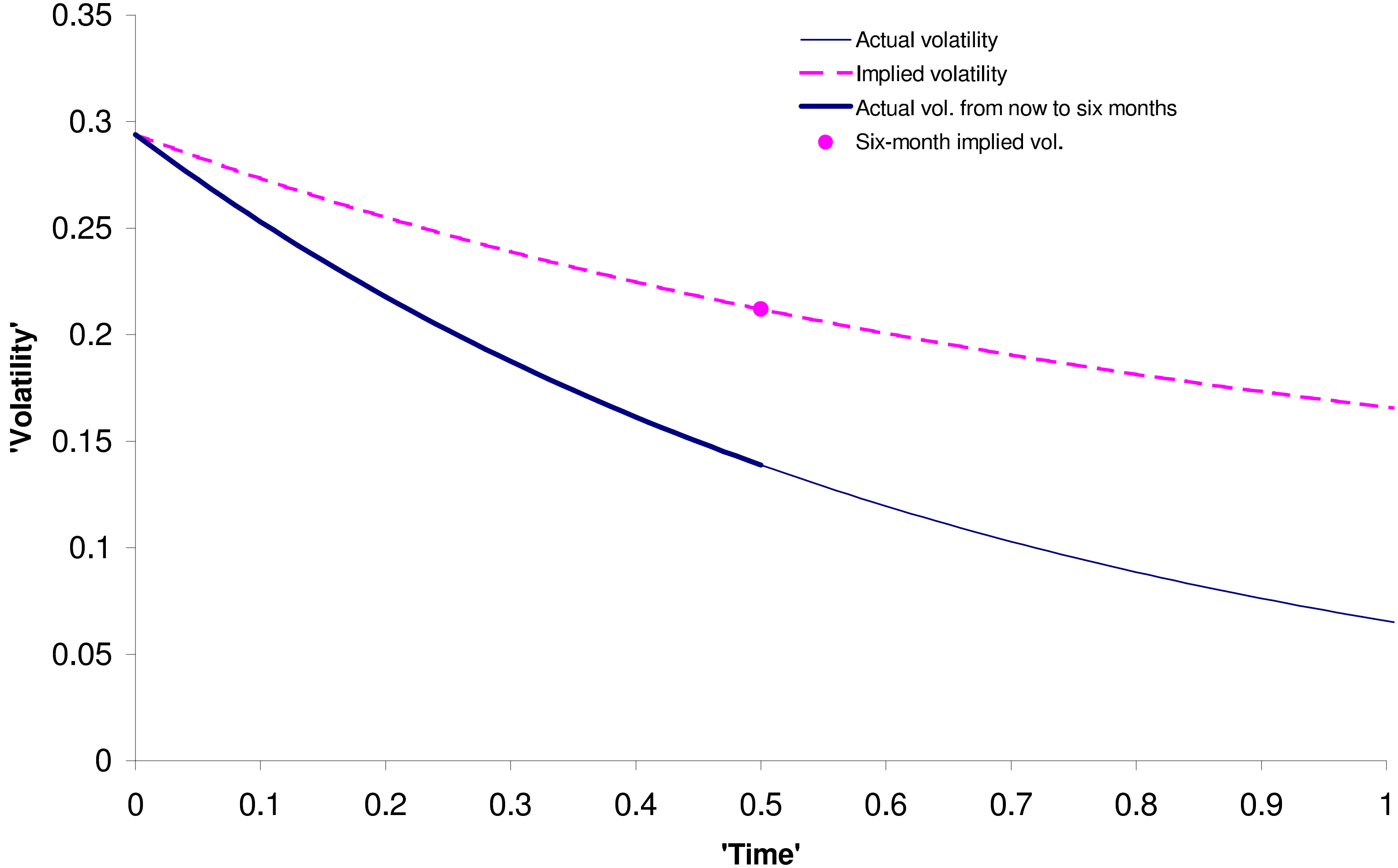
$$\sigma(t) = \sqrt{\sigma_{imp}^2(T;t^*) + 2(T-t^*)\,\sigma_{imp}(T;t^*)\,\frac{\partial \sigma_{imp}(T;t^*)}{\partial T}} \qquad (1)$$

For multiple expiries $T_1 < T_2 < T_3 < \ldots < T_n \to t$, so interpolate over $\sigma_{imp}(T)$ to make a continuous function $\sigma_{imp}(t)$

$$\sigma(t) = \sqrt{\sigma_{imp}^2(t;t^*) + 2(t-t^*)\,\sigma_{imp}(t;t^*)\,\frac{\partial \sigma_{imp}(t;t^*)}{\partial t}}. \qquad (2)$$

Whoa!

Potential results for implied $\sigma_{imp}(t)$ and actual $\sigma(t)$



In practice implied volatilities are only known at a finite number of expiration points.

Example:

One-month implied volatility is 30%.

Two-month implied volatility is 25%.

Three-month implied volatility is 26%.

Fitting: construct an actual volatility function, which is consistent with this term structure of implied volatilities.

There is no unique solution to this fitting problem.

- The most simple assumption is to assume that $\sigma(t)$ is **piecewise constant** rather than any particular function.

- In absence of data, we can make assumptions such as

$$\sigma(t+\tau) = \sigma(t)$$

or

$$\sigma(t_1 - t_0) = \sigma(t_2 - t_1)$$

For t less than one month the answer is simple:

$$\sigma(t) = 0.3.$$

Use σ_1 to denote this constant value over the first period.

For t greater than one month but less than two months, it is harder to get an answer.

It must be consistent with the two-month implied volatility 25% (annualised).

But two-month implied volatility depends on the actual $\sigma(t)$ from $t = 0$ up to two months.

The implied variance is a time-weighted average of the actual variance:

$$\frac{2}{12} \times 0.25^2 = \frac{1}{12} \times 0.3^2 + \frac{1}{12} \times \sigma_2^2.$$

Below is consistent with general result for piecewise volatility (**??**)

$$\sigma_2 = \sqrt{2 \times 0.25^2 - 1 \times 0.3^2}$$

The solution is

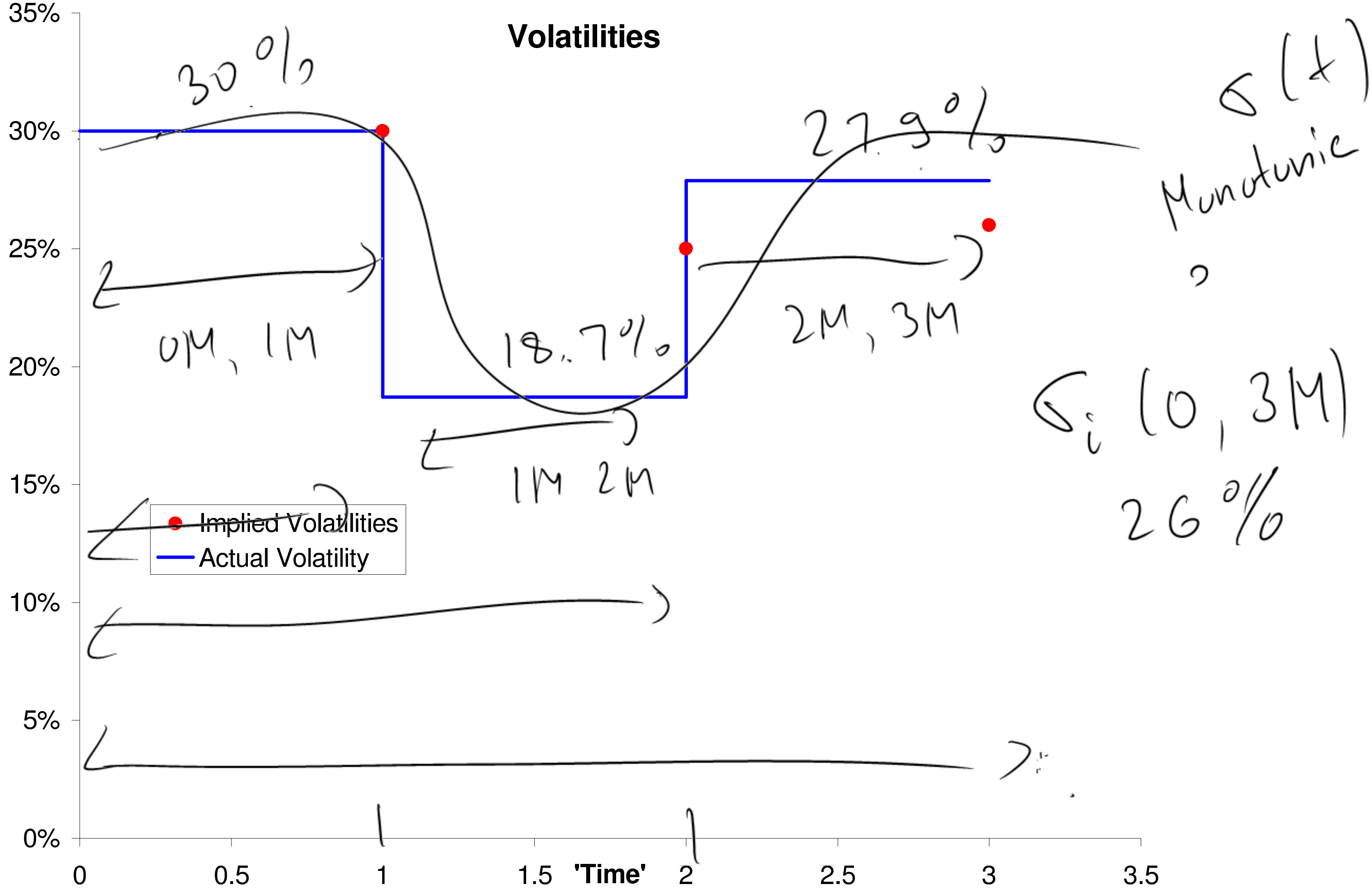
$$\sigma_2 = 0.187.$$

Finally, to get σ_3, which is the actual volatility during the third month (assumed constant), we must have

$$\frac{3}{12} \times 0.26^2 = \frac{1}{12} \times 0.3^2 + \frac{1}{12} \times 0.187^2 + \frac{1}{12} \times \sigma_3^2.$$

The solution is

$$\sigma_3 = 0.279.$$


Volatilities
35%
30%
25%
20%
15%
10%
5%
0%
0
0.5
1
1.5
'Time'
2
2.5
3
3.5
Implied Volatilities
Actual Volatility
30%
27.9%
18.7%
0M, 1M
1M 2M
2M, 3M
σ(t)
Monotonic ?
σ_i (0, 3M)
26%

If we have implied volatility for expiries T_i and assume the actual volatility to be piecewise constant then for $T_{i-1} < \mathbf{t} < T_i$,

$$\sigma(\mathbf{t}) = \sqrt{\frac{T_i - t^*}{T_i - T_{i-1}}\,\sigma^2_{imp}(T_i; t^*) - \frac{T_{i-1} - t^*}{T_i - T_{i-1}}\,\sigma^2_{imp}(T_{i-1}; t^*)} \qquad (3)$$

where t^* is earlier time of calibration $t^* < T_{i-1} < T_i$.

So, assuming for the first period $\sigma(t_{i-1}) = \sigma_{imp}(T_{i-1}; t^*)$

$$\sqrt{\frac{T_{i-1} - t^*}{T_i - t^*}\sigma^2(t_{i-1}) + \frac{T_i - T_{i-1}}{T_i - t^*}\sigma^2(t)} = \sigma^2_{imp}(0, T_i) \qquad (4)$$

The boostrapping is consistent with the discretised solution to the inverse problem.

Volatility Skew

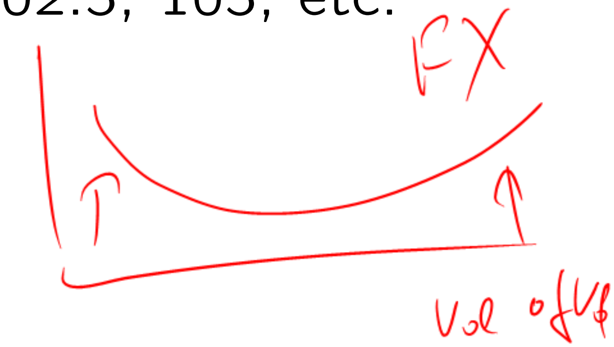
That's term structure dealt with. . . now 'strike structure' better known as the volatility skew or smile.

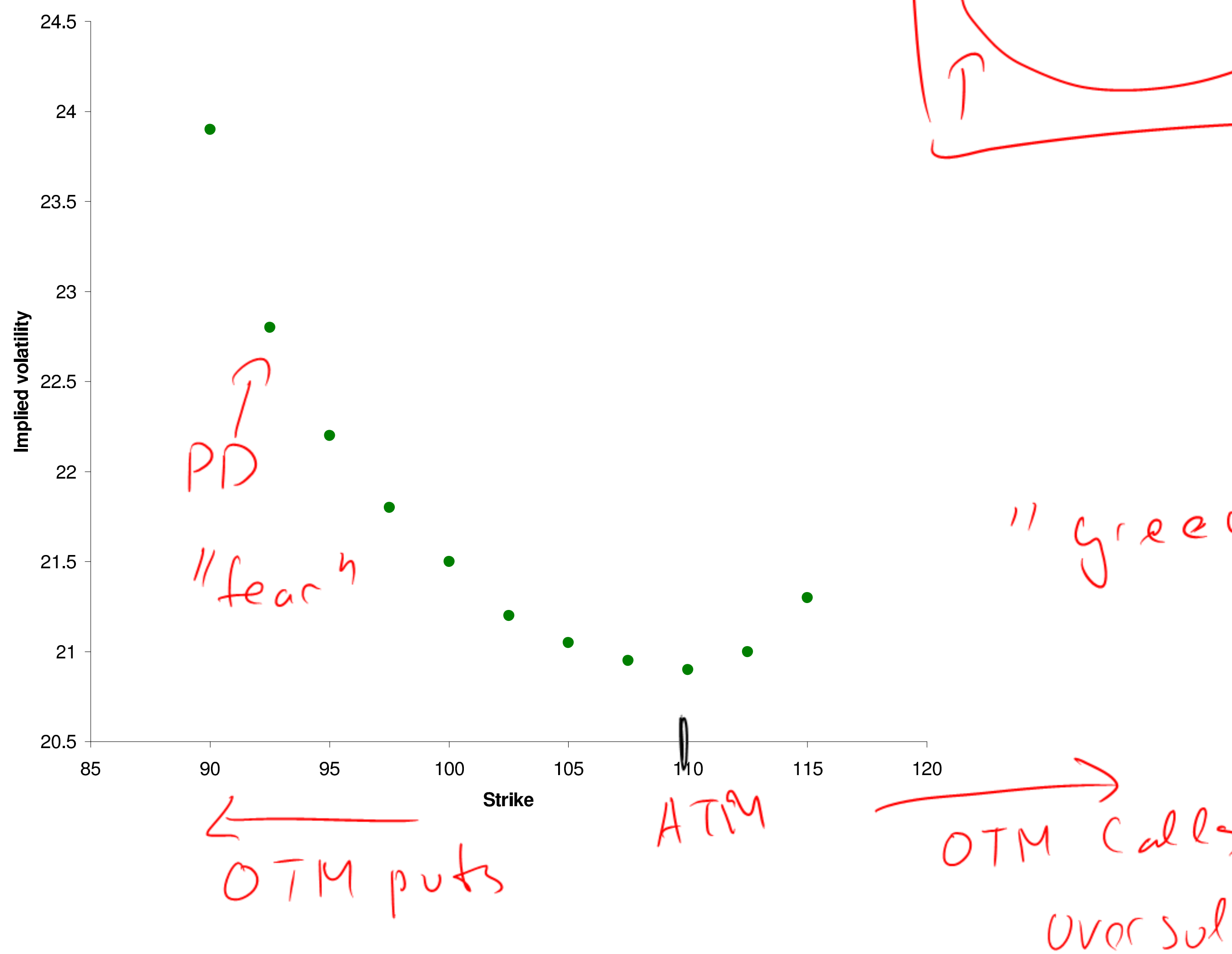
Now, we have two European call options with different strikes, both expiring in six months

Strike	Value	IV
100	12.84$	**21.5%**
110	6.72$	**20.9%**

These are two conflicting views on volatilities for the same expiry.

Concentrating on the same example, suppose call options are traded with strikes of 90, 92.5, 95, 97.5, 100, 102.5, 105, etc.





The shape of this curve of implied volatility *versus* strike is called **the volatility smile**.

In some markets, such as equity options, it shows considerable asymmetry, a **skew**, due to expensive OTM puts.

Sometimes, the shape is upside down in a **frown**.

Calibrating volatility that varies with expiration and strike

We now have implied volatility being a function of two variables, strike and expiration:

$$\sigma_i(K, T)$$

We ask the question again:

What is the simplest change to the Black–Scholes model that will accommodate such implied volatility?

Just as we *assumed* actual volatility is a function of t to match implied volatility being a function of T, we now make actual volatility a function of S and t!

$$\sigma(S,t) \quad \Rightarrow \quad \sigma_i(K,T)$$

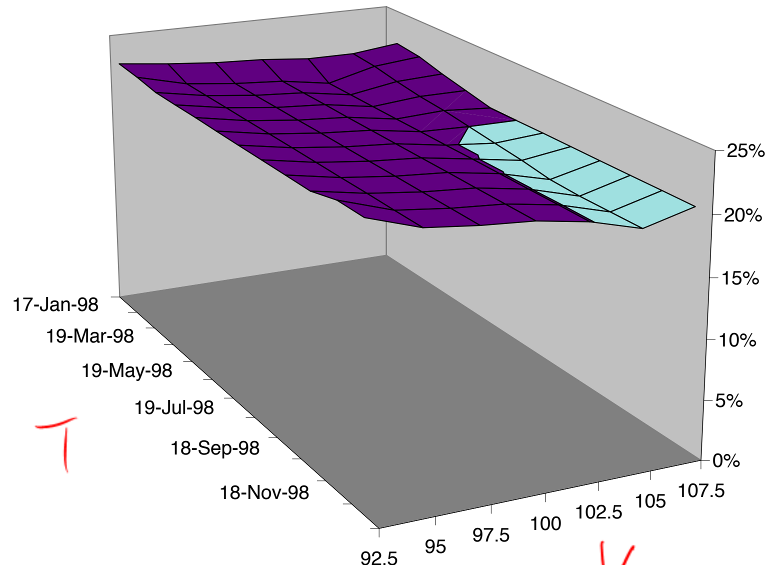
Again, in this problem we know the answer (market prices) and now have to find the question (the parameter for the Black-Scholes model). The calibration task becomes

$$\sigma(S,t) \quad \Leftarrow \quad \sigma_i(K,T)$$

This is another **inverse problem**.

Volatility Surface

Implied volatility plotted against strike and maturity in 3D.



T K $\sigma_i(K,T)$

This **implied volatility surface** represents the constant value of volatility that gives each traded option a theoretical (Black-Scholes) value equal to the market price.

We saw how T *the time dependence* of implied vol. is turned into a volatility of the underlying that was time dependent t.

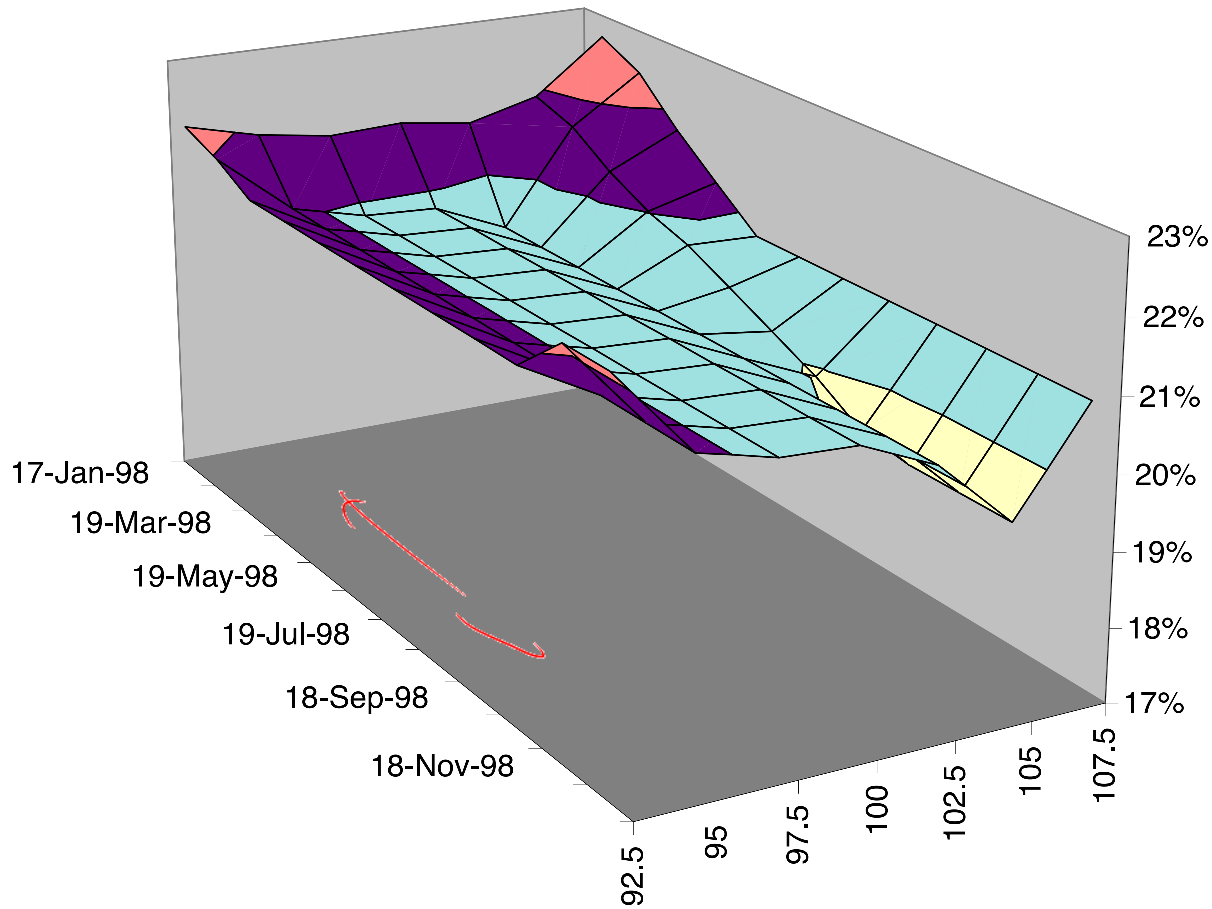
We considered integral equation solution to the inverse problem linking $\sigma(t)$ to $\sigma_i(T; t^*)$.

- Can we similarly deduce $\sigma(S, t)$ from $\sigma_i(K, T; t^*)$ – the implied volatility at time t^*?

- Yes we can, and the result is called **a local volatility surface** $\sigma(S, t)$.

This surface can be thought of as the market's view of the forward volatility when the asset price is S at time t.

A lot of maths omitted!



$\sigma(t, S)$

Local volatility surface calibrated from European call prices.

Rule of Thumb: Local Volatility

Local volatility $\sigma(S, t)$ changes approximately **twice as fast** with stock price as implied volatility $\sigma_i(K, T)$ changes with strike.

This makes the local volatility surface to appear 'wilder' in comparison to the implied volatility surface.

How do I use this local volatility surface?

If the surface is the market's view of forward volatility, and that these predictions will come to pass –

then we price other, more complex, products using this asset- and time-dependent volatility $\sigma(S, t)$.

This can work but also is a very naive.

- Not only do the predictions not come true but even if we come back a few days later to look at the 'prediction,' i.e. to refit the surface, we see that $\sigma(S, t)$ has changed throughout.

As long as we price our exotic contract consistently with the vanillas, *and simultaneously hedge with these vanillas*, then we are reducing our exposure to the model risk of $\sigma(S,t)$.

That's enough about calibration, *for now*!

Delta-hedging and what we can do with it

If we have a better forecast for actual volatility, than a local $\sigma(S,t)$ that a trader uses in a bank, it is possible to make money.

Scenario: Implied or calibrated volatility for an option is 20% σ_i but we believe that actual volatility will realise over time as 30%. σ_a

- Black–Scholes tells you all about how to delta hedge when there is just one constant volatility, now there are **two**!

Cash: $-V_{BS}(\sigma_i) + V_{BS}(\sigma_a)$

So you believe an option is mispriced... **how can you profit from this?** How can we make money if our forecast is correct?

Buy an option and delta hedge. The formula for delta is well-known,

$$\Delta = N(d_1)$$

where

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{s^2}{2}} ds$$

$$d_1 = \frac{\ln(S/E) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

Let's call

$$\sigma_a = \text{ actual volatility, 30\%}$$

$$\sigma_i = \text{ implied volatility, 20\%.}$$

Market participants can easily agree on S, E, $T-t$ but not σ.

So, should we use σ_a or σ_i to calculate Δ?

This is one of those questions that no one seems to know the full answer to.

Case 1: Hedge with actual volatility, σ_a

Set up a portfolio by *buying* the option for V^i and selling Δ^a quantity of the stock. Cash position is

$$-V^i + \Delta^a S$$

By selling Δ^a shares we are replicating a short position in a correctly priced option V^a.

Eventually, we shall have a pile of money equal to V^a.

The profit to make is exactly the difference in the Black–Scholes theoretical prices

$$V(S,t;\sigma_a) - V(S,t;\sigma_i)$$

or simply

$$V^a - V^i$$

How do we know this guaranteed profit?

How is the profit realised on MtM basis? (P&L over dt)

Let's do the maths **on the mark-to-market basis**, by which we mean to consider P&L over each time step.

'Today' at time t:

Option	V^i
Stock	$-\Delta^a\ S$
Cash	$-V^i + \Delta^a\ S$

t

'Tomorrow' at time $t + dt$:

Option	$V^i + dV^i$
Stock	$-\Delta^a\ S - \Delta^a\ dS$
Cash	$(-V^i + \Delta^a\ S)(1 + r\ dt)$

$t + dt$

Therefore we have made marked to market,

$$dV^i - \Delta^a \, dS - (V^i - \Delta^a \, S) \, rdt \; \dagger \tag{5}$$

Because the option would be correctly valued at V^a then we have

$$dV^a - \Delta^a \, dS - (V^a - \Delta^a \, S) \, rdt = 0$$

So the mark-to-market profit over a time step is $\dagger - 0$,

$$dV^i - dV^a + r(V^a - \Delta^a \, S) \, dt - r(V^i - \Delta^a \, S) \, dt$$

$$= dV^i - dV^a - r(V^i - V^a) \, dt \quad \Rightarrow \quad \boxed{\underbrace{e^{rt} \, d\left(e^{-rt}(V^i - V^a)\right)}}$$

over dt

This is profit from time t to $t + dt$ (see Solutions).

PV-ing that increment of profit to t_0 gives

$$e^{-r(t-t_0)}\,\underbrace{e^{rt}\,d\left(e^{-rt}(V^i - V^a)\right)} = e^{rt_0}\,d\left(e^{-rt}(V^i - V^a)\right)$$

So **the total profit** from t_0 to expiration is

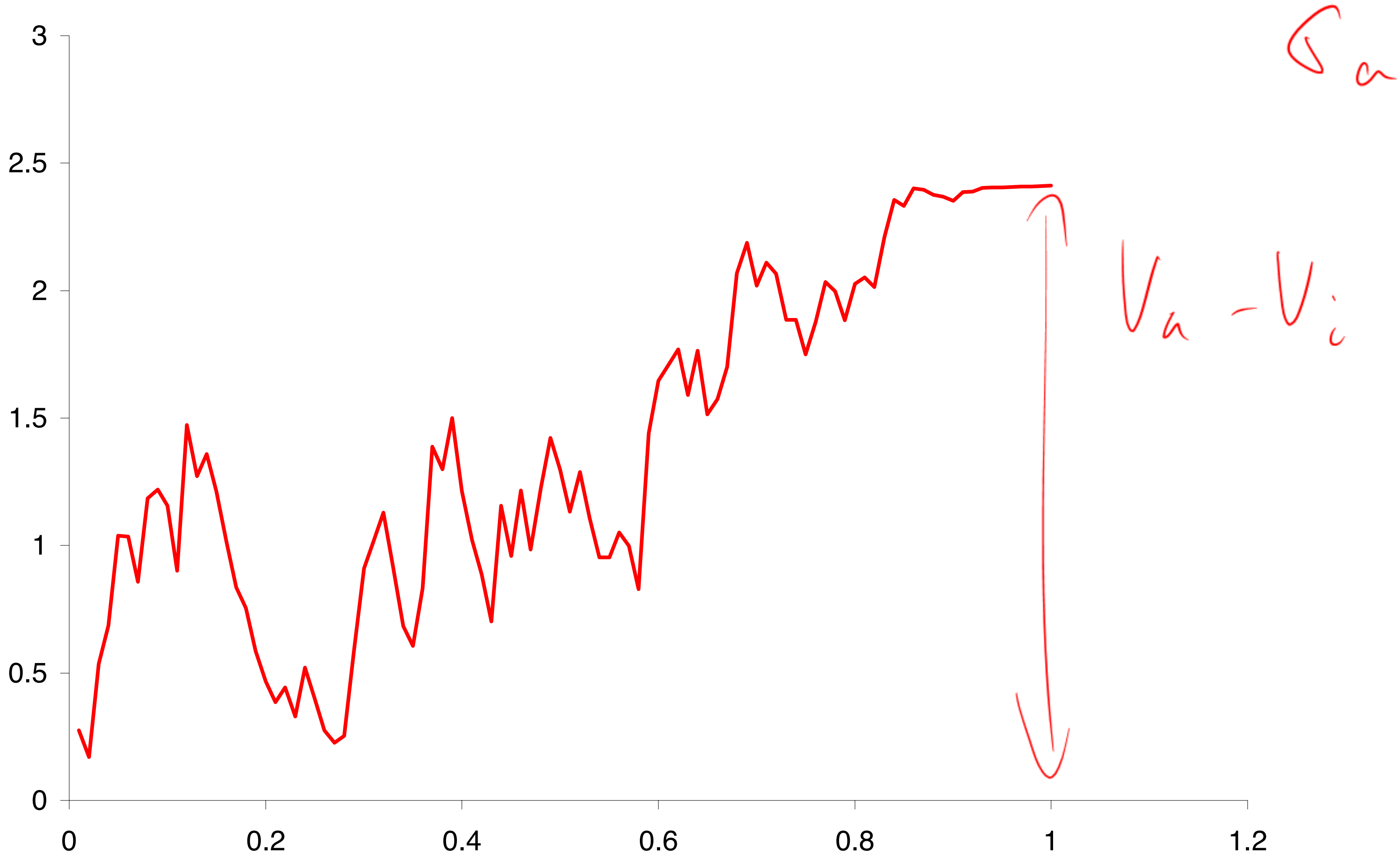
$$e^{rt_0}\int_{t_0}^{T} d\left(e^{-rt}(V^i - V^a)\right) = \boxed{V^a - V^i}$$

over

t_0 T

As we said.

The total profit is a known quantity.



An analogy is a bond: outcome is guaranteed but may lose on a mark-to-market basis in the meantime.

OK, what if we want to study the P&L?

We can write **mark-to-market** from t to $t + dt$ by *invoking Itô lemma* to expand $\underline{dV^i}$ in expression (5)

$$\underline{\Theta^i \, dt + \tfrac{1}{2}\sigma_a^2 S^2 \Gamma^i \, dt + \Delta^i \, dS} - \Delta^a \, dS - r(V^i - \Delta^a S) \, dt$$

$$= \Theta^i \, dt + \mu S(\Delta^i - \Delta^a) \, dt + \tfrac{1}{2}\sigma_a^2 S^2 \Gamma^i \, dt - r(V^i - \Delta^a S) \, dt + (\Delta^i - \Delta^a)\sigma_a S \, dX$$

Using the fact that V^i satisfies Black-Scholes, can substitute $(\Theta^i - rV^i)dt$ with $(-r\Delta^i S - \frac{1}{2}\sigma_i^2 S^2 \Gamma^i)dt$.

$$= (\Delta^i - \Delta^a)\sigma_a S \, dX + (\mu - r)S(\Delta^i - \Delta^a) \, dt + \tfrac{1}{2}\left(\sigma_a^2 - \sigma_i^2\right) S^2 \Gamma^i \, dt$$

Over dt

$$= \frac{1}{2}\left(\sigma_a^2 - \sigma_i^2\right) S^2 \Gamma^i \, dt + \left(\Delta^i - \Delta^a\right)\left((\mu - r) S \, dt + \sigma_a S \, dX\right).$$

Conclusion: The final profit is guaranteed but how that profit is achieved is random.

When S changes, so will V, but these changes do not cancel each other out.

The fluctuation in the portfolio mark-to-market value is random and may even go negative. You could lose before you gain.

Case 2: Hedge with implied volatility, σ_i

That means we are balancing the random fluctuations in the mark-to-market option value dV^i **just** with the fluctuations in the stock price.

We are saying that by delta-hedging we offset the entire P&L of an option (linear and non-linear parts).

Such model is inconsistent.

$$\underbrace{-V^i + \Delta^i S}$$

Portfolio cash does not depend on actual volatility!

Buy the option today for V^i, hedge using Δ^i of the stock.

Let's see how that works out mathematically.

The mark-to-market profit over dt looks similar but $\underline{\Delta^a \to \Delta^i}$

$$\underline{dV^i - \Delta^i \, dS} - r(V^i - \Delta^i S) \, dt$$

$$= \underline{\Theta^i \, dt + \tfrac{1}{2}\sigma_a^2 S^2 \Gamma^i \, dt} - r(V^i - \Delta^i S) \, dt$$

$$= \underbrace{\tfrac{1}{2}\left(\sigma_a^2 - \sigma_i^2\right) S^2 \Gamma^i \, dt \ \ddagger}$$

$\sigma_a > \sigma_i$

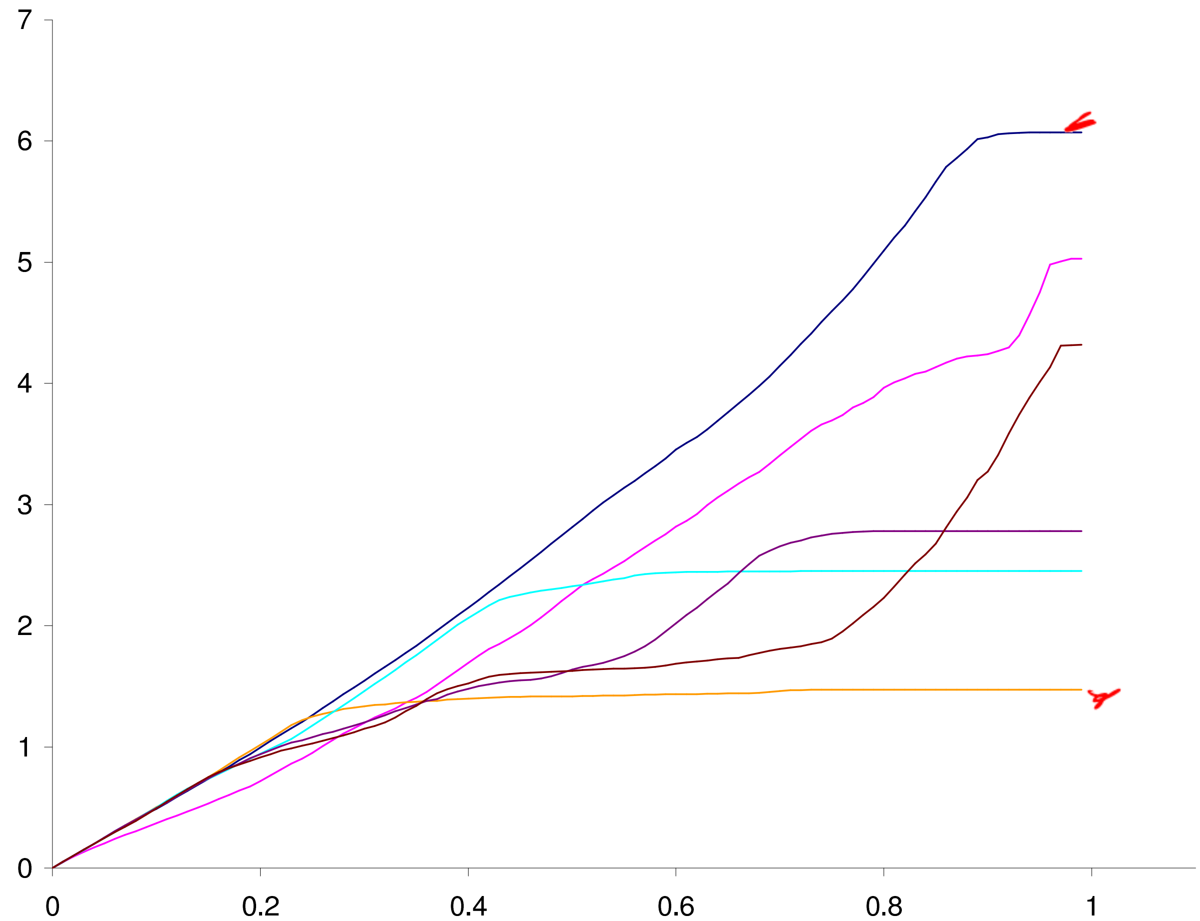
To get **the total profit**, add up the present value of all of those moves over dt (result on the left)

σ_i $\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ σ_a

$\frac{1}{2}\left(\sigma_a^2-\sigma_i^2\right)\int_{t_0}^{T} e^{-r(t-t_0)} S^2 \Gamma^i \, dt \ \ddagger$	vs. $\int_{t_0}^{T} d\left(e^{-r(t-t_0)}\left(V^i - V^a\right)\right)$
	$\ldots$
Gamma P&L	$V^i - V^a$

Gamma P&L is always positive but not a guaranteed amount.

Why? $\sigma_a > \sigma_i$ but there is a source of randomness in $\ddagger$.



Using implied volatility σ_i as a prediction, while asset evolves according to its own actual volatility σ_a. **End result is uncertain.**

An analogy, money market/bank account: Gamma P&L is positive, always increasing in value but unlikely to match the liability. Hedge portfolio has to appreciate at the risk-free rate.

$$rV\,dt$$

By delta hedging with σ_i, we don't know which option V we replicated!

Summary

Please take away the following important ideas:

- there are different kinds of volatility to which people refer
- you can recover the actual volatility if you make big assumptions. This is calibration $\sigma(t, S)$
- the volatility smile violates Black-Scholes in all markets
- to hedge with other options requires a model that tells you the future smile
- options can be used for making a profit from volatility models