Mod 3.7 exercises

Computational Methods

1. Consider the one-dimensional heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}; \quad t > 0, \quad 0 < x < L \tag{1.1}$$

where the unknown function $u = u(x,t)$; c^2 is a constant. To discretize the equation, take N and M steps for x and t respectively, so

$$\begin{aligned} x &= n\delta x \qquad 0 \leq n \leq N \\ t &= m\delta t \qquad 0 \leq m \leq M, \end{aligned}$$

where $\delta x = \frac{L}{N}$; $\delta t = \frac{T}{M}$.
By using the following approximations

$$\begin{aligned} \frac{\partial u}{\partial t}(n\delta x, m\delta t) &\sim \frac{u_n^{m+1} - u_n^m}{\delta t}, \\ \frac{\partial^2 u}{\partial x^2}(n\delta x, m\delta t) &\sim \frac{u_{n-1}^m - 2u_n^m + u_{n+1}^m}{\delta x^2} \end{aligned}$$

and writing $r = c^2 \dfrac{\delta t}{\delta x^2}$, derive the following **forward marching scheme** for (1.1)

$$u_n^{m+1} = A u_{n-1}^m + B u_n^m + C u_{n+1}^m, \tag{1.2}$$

where A, B, C should be stated.
Assume an initial disturbance E_n^m given by

$$E_n^m = \overline{a}^m e^{in\omega}, \tag{1.3}$$

which is oscillatory of amplitude $\overline{a}$ and frequency ω; $i = \sqrt{-1}$. By substituting (1.3) into (1.2), show that

$$\overline{a} = 1 + 2r\left(1 - 2\sin^2 \frac{\omega}{2}\right).$$

2. Consider the following linear system

$$\begin{aligned} 3x + 4y &= 1 \\ -5x + 2y &= 2 \end{aligned} \tag{2.1}$$

Solve this using <u>*LU* decomposition</u> where L has a unit diagonal, to show that the solution is given by

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3/13 \\ 11/26 \end{pmatrix}.$$

How can system (2.1) be modified to allow the Gauss-Seidel Method to be used?
Given $\mathbf{x}^{(0)} = (0,0)^{\mathrm{T}}$, calculate two iterations $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$; and use these to evaluate

$$\frac{\left\|\mathbf{x}^{(2)} - \mathbf{x}^{(1)}\right\|_\infty}{\left\|\mathbf{x}^{(1)}\right\|_\infty}.$$