Martingales, martingales ... and more martingales!!!

3 concepts:

① The process!

A martingale is a stochastic process M_t such that

$$\begin{cases} \mathbb{E}[\,|M_t|\,] < \infty & \rightarrow \text{integrability} \\ \mathbb{E}[M_t \mid F_s] = M_s & \rightarrow \text{martingale condition} \end{cases}$$

② - Change of Measures

RN, $\mathbb{P} \sim \mathbb{Q}$

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \Lambda$$

$$\boxed{\mathbb{Q}(A) = \int_A \Lambda \, d\mathbb{P}}$$

$$= \int_A \frac{d\mathbb{Q}}{\cancel{d\mathbb{P}}} \cancel{d\mathbb{P}}$$

$$= \int_A d\mathbb{Q}$$

Stochastic processes

Λ = exponential martingale

$$\Lambda_t = \Lambda_0 \exp\left\{ +/- \int_0^t \theta(s)\, dX(s) - \frac{1}{2}\int_0^t \theta^2(s)\, ds \right\}$$

③ Equivalent Martingale Measure:

$\mathbb{Q}$ such that S_t is a martingale under $\mathbb{Q}$.

S_t under $\mathbb{P}$

$\mathbb{Q} \sim \mathbb{P}$

$c(t) = V(t, S_t)$

stochastic

O.DE

Mart

BM

$$dc(t) = \cancel{f(t, S_t)\,dt} + \cancel{g(t, S_t)\,dX(t)}$$

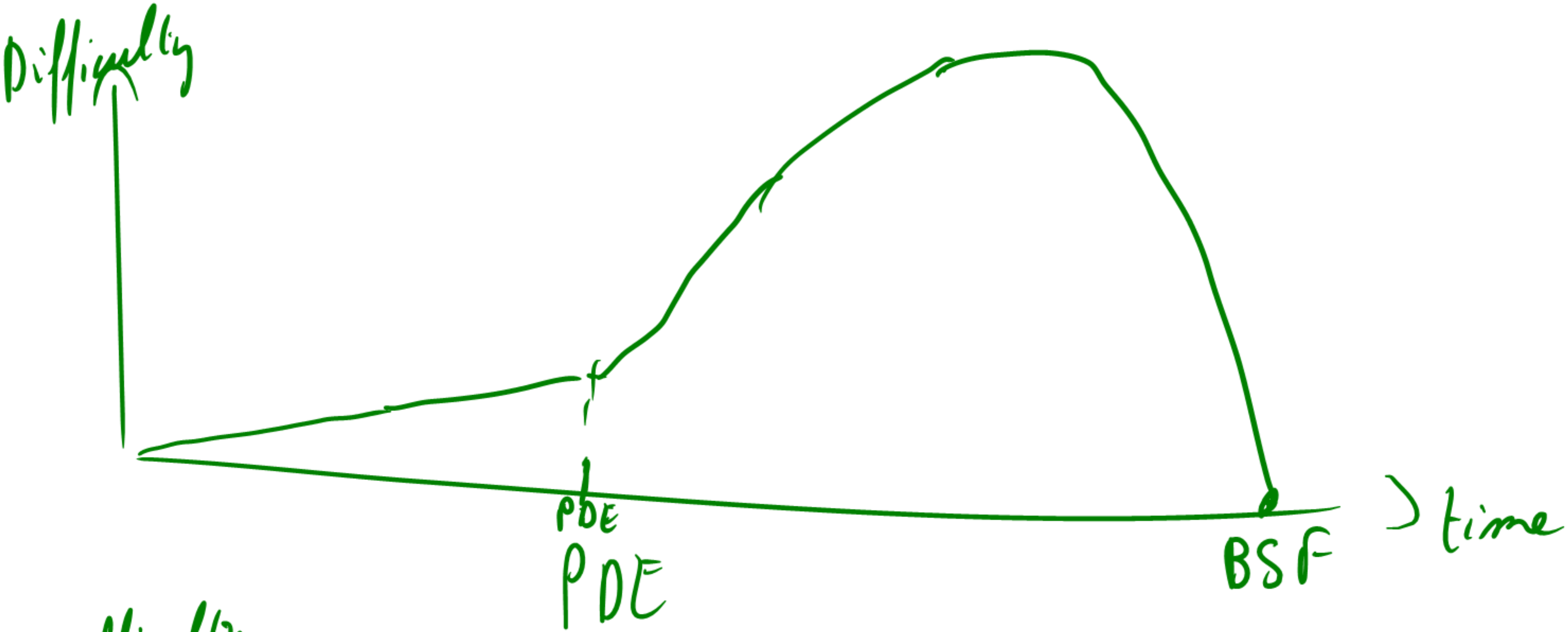
2 function

drift
deterministic
scales with dt

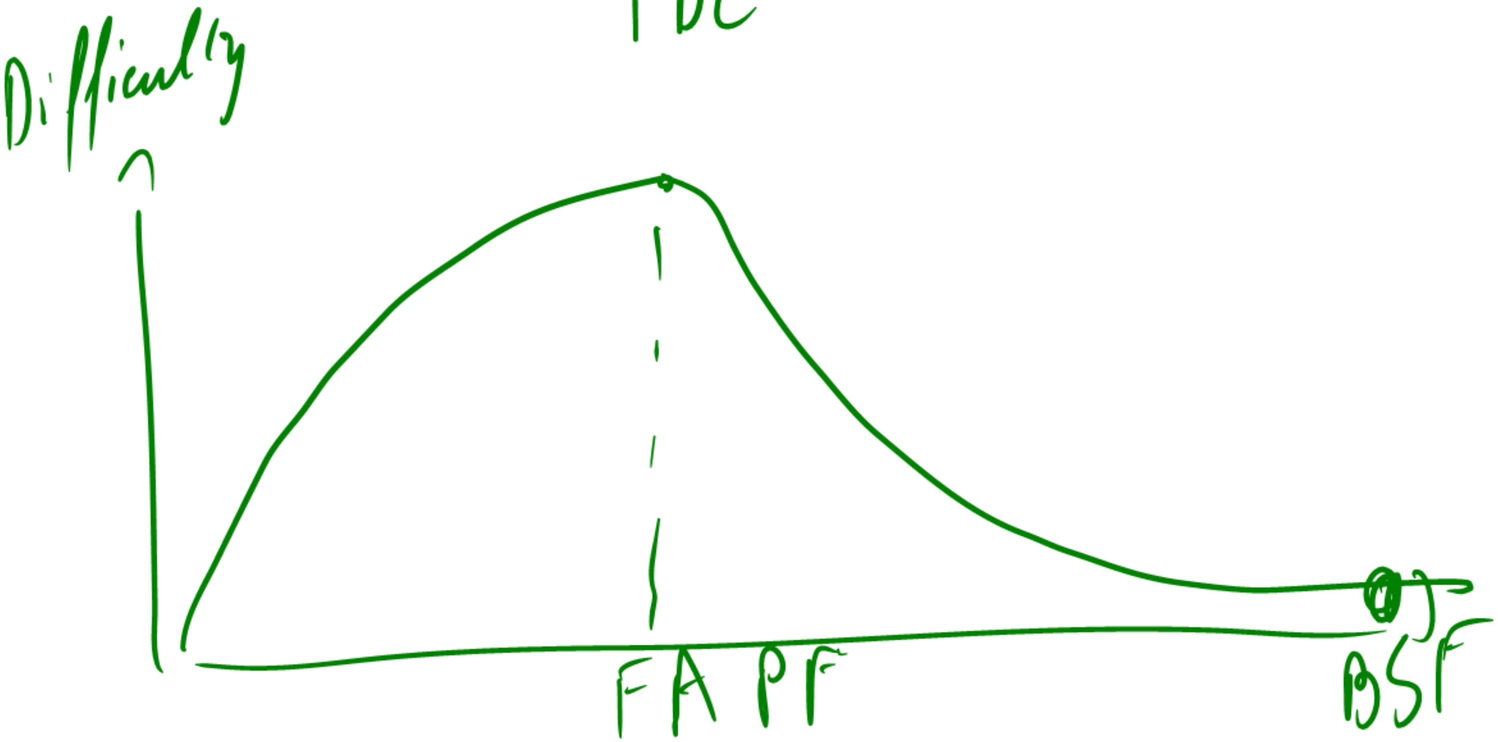
diffusion
random
scales with $\sqrt{dt}$

1. PDE

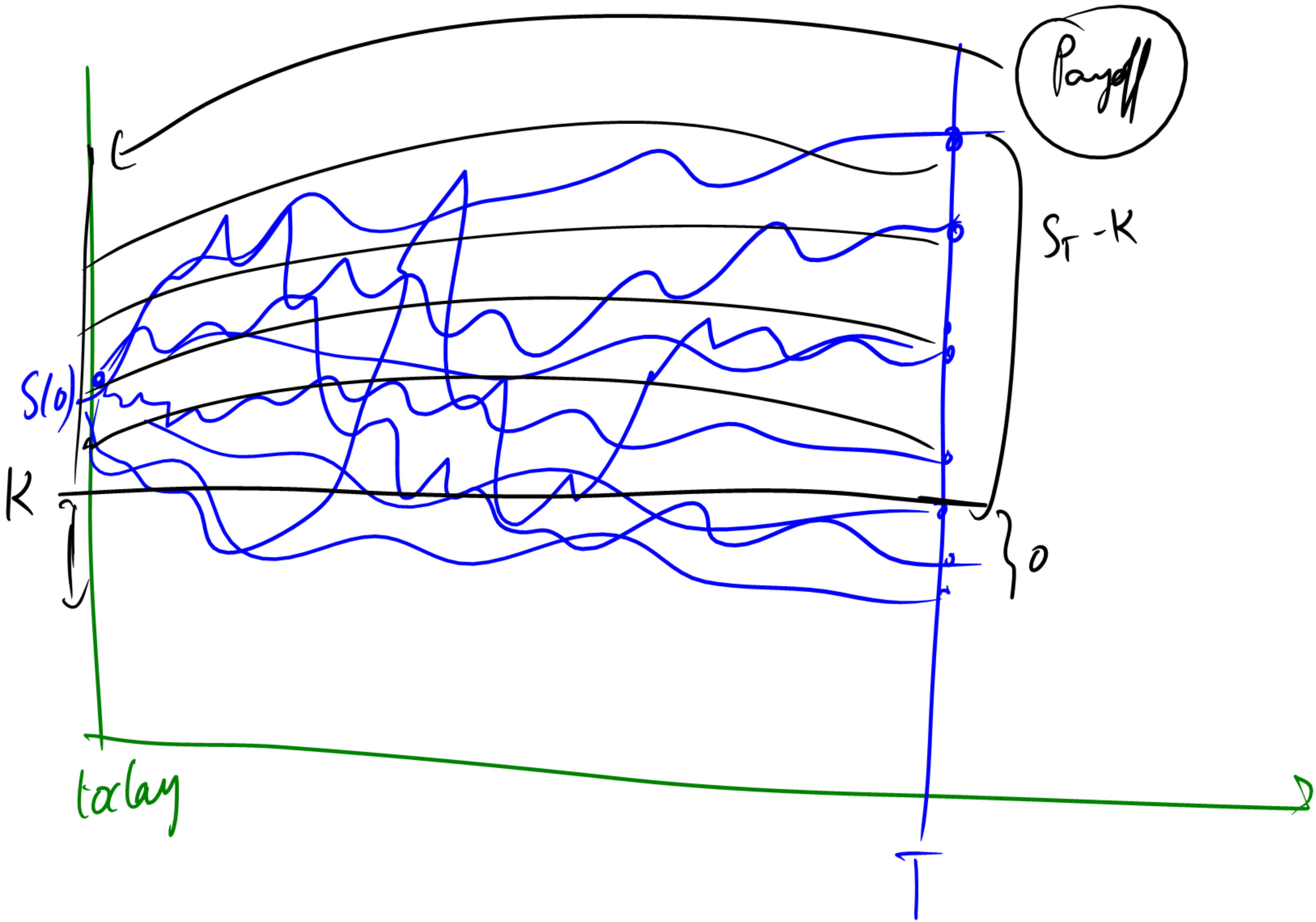
2. Martingale problem



Difficulty
PDE
PDE
BSF
time

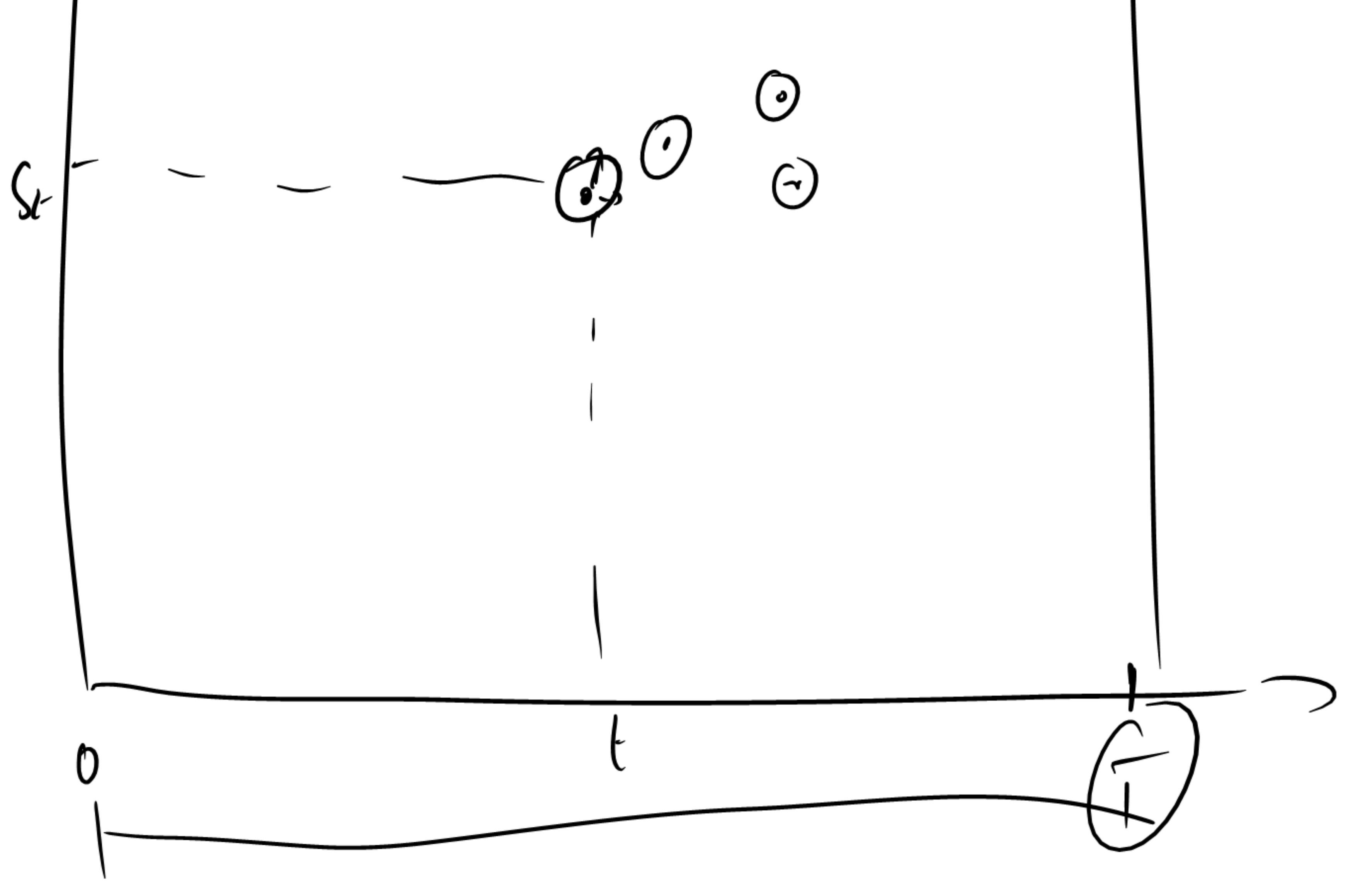


Difficulty
FAPF
BSF

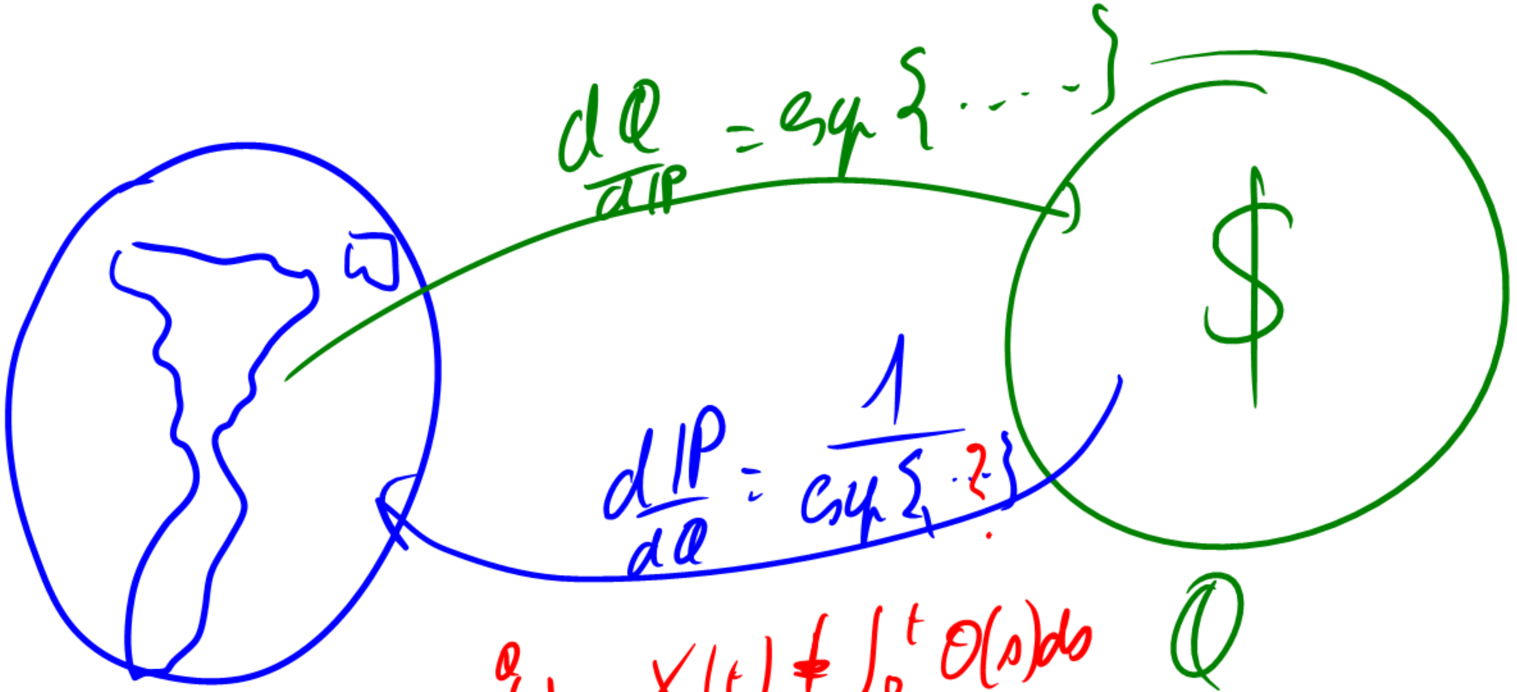

Payoff
C
$S_T - K$
S(0)
K
0
today
T

- Stochastic Processes are global!
- PDEs are local!

Expiry



$$dS^*(t) = (\cancel{\mu - r})\, S^*(t)\, dt + \sigma S^*(t)\, dX(t)$$



$\overset{Q}{X}(t) = X(t) + \int_0^t \theta(s)\,ds$

$\mathbb{P}$ $\qquad$ $\mathbb{Q}$ $\qquad$ θ ?

$X(t)$ $\qquad$ $\overset{Q}{X}(t)$

Risk Management
Portfolio Management
Trading $\rightarrow$ Pricing

$S^*(t)$ is a martingale

Under $\mathbb{P}$: $dS^*(t) = (\mu - r) S^*(t)dt + \sigma S^*(t) dX(t)$

$\mathbb{Q}$

$$X^{\mathbb{Q}}(t) = X(t) + \int_0^t \theta(s)ds$$

$$\Leftrightarrow \boxed{dX^{\mathbb{Q}}(t)} = dX(t) + \theta(t)dt$$

$$\Leftrightarrow \boxed{dX(t) = dX^{\mathbb{Q}}(t) - \theta(t)dt}$$

The dynamics of $S^*(t)$ under the measure $\mathbb{Q}$:

$$dS^*(t) = (\mu - r) S^*(t)dt + \sigma S^*(t) \left[dX^{\mathbb{Q}}(t) - \theta(t)dt\right]$$

$$dS^*(t) = \boxed{\left(\mu - r - \theta(t)\sigma\right)} S^*(t)dt + \sigma S^*(t) \cdot dX^{\mathbb{Q}}(t)$$

$\mathbb{Q}$ is an EMM $\Rightarrow$ $S^*(t)$ is a mart

$\Rightarrow$ $S^*(t)$ is driftless

$$\mu - r - \theta(t)\sigma = 0 \Leftrightarrow \theta(t) = \boxed{\frac{\mu - r}{\sigma}}$$

Q-measure:

$$dV^*(t) = d\left(\frac{V(t)}{B(t)}\right) = d\left(\overbrace{\underbrace{V(t)}_{1} \cdot \underbrace{B^{-1}(t)}_{2}}\right)$$

Itô Product Rule!

1- $V(t)$ -> trading strategy

$$V(t) = \phi^S(t)\, S(t) + \phi^B_t\, B(t)$$

Self financing

$$dV(t) = \phi^S_t\, dS(t) + \phi^B_t\, dB(t)$$

2 $$B^{-1}(t) = \frac{1}{B(t)} = \boxed{e^{-rt}}$$

$$dB^{-1}(t) = -r B^{-1}(t)\,dt$$

3- $$dV^*(t) = d\left(V(t) \cdot B^{-1}(t)\right) = dV(t) \cdot B^{-1}(t) + V(t) \cdot dB^{-1}(t)$$

$$dV^*(t) = d\left(V(t) \cdot B^{-1}(t)\right)$$

$$= dV(t) \cdot B^{-1}(t) + dB^{-1}(t) \cdot V(t)$$

$$= \left(\phi_t^S dS(t) + \phi_t^B dB_t\right) \cdot B^{-1}(t) - r B^{-1}(t)\,dt \left(\phi_t^S S_t + \phi_t^B B_t\right)$$

$$= \phi_t^S \left[B^{-1}(t) \cdot dS(t) + dB^{-1}(t) S_t \right]$$

$$+ \phi_t^B \left[\cancel{r B(t)\,dt \cdot B^{-1}(t) - r B^{-1}(t)\,dt\,B(t)} \right]$$

$$0$$

$$= \phi_t^S \left[B^{-1}(t)\,dS(t) + dB^{-1}(t)\,S(t) \right]$$

$$= \phi_t^S\, d\left(S(t) \cdot B^{-1}(t) \right) = \phi_t^S\, dS^*(t)$$

$$dV^*(t) = \phi_t^S \cdot \underbrace{dS^*(t)}_{\text{Martingale}} \qquad \mathbb{Q}$$

$V^*(t)$ is a mart!!!...

$$dV^*(t) = \boxed{\phi_t^S \cdot \sigma S^*(t)} \, dX^{\mathbb{Q}}(t) + 0 \, dt$$

is a martingale

$$\boxed{\underbrace{\mathbb{E}\left[V^*(T) \mid F_t\right]}_{\text{Best estimate}} = V^*(t)}$$

today!!!

Martingale

$$\frac{\chi(t,S_t)}{B(t)} \overset{(3)}{=} V^{\$}(t) = \mathbb{E}^{\mathbb{Q}}\left[V^{\$}(T) \mid \mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[\frac{G(S_T)}{B(T)} \mid \mathcal{F}_t\right]$$

$$\chi(t,S_t) = B(t)\,\mathbb{E}^{\mathbb{Q}}\left[\frac{G(S_T)}{B(T)} \mid \mathcal{F}_t\right]$$

$$\chi(t,S_t) = \mathbb{E}^{\mathbb{Q}}\left[\frac{B(t)}{B(T)} \times G(S_T) \mid \mathcal{F}_t\right]$$

Discounting! Payoff

$t=0$

$$C(t) = \mathbb{E}^{Q}\left[\frac{1}{B(T)} \max(S_T - K, 0)\right]$$

$$= \mathbb{E}^{Q}\left[\frac{1}{B(T)} \times (S_T - K)\,\mathbb{1}_{\{S_T > K\}}\right] = \mathbb{E}^{Q}\left(\frac{1}{B_T} \times S_T\,\mathbb{1}_{\{\dots\}}\right) - \mathbb{E}^{Q}\left[\frac{1}{B_T} \times K \times \mathbb{1}_{\{\dots\}}\right]$$

$$\mathbb{1}_{\{S_T > K\}} = \begin{cases} 1 & \text{if } S_T > K \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{1}_{\{x \in A\}} = \begin{cases} 1 & \text{if } x \in A \\ 0 & \end{cases}$$

$$\mathbb{E}^{Q}\left[B^{-1}(T)\max[S_T - K, 0]\right]$$

$$= \underbrace{\mathbb{E}^{Q}\left[B^{-1}(T)\, S_T\, \mathbb{1}_{\{S_T > K\}}\right]}_{1} - \underbrace{\mathbb{E}^{Q}\left[B^{-1}(T)\, K\, \mathbb{1}_{\{S_T > K\}}\right]}_{2}$$

$$V = \mathbb{E}^{\mathbb{Q}}\left[\beta^{-1}(T) K \, \mathbb{1}_{\{S_T > K\}} \right]$$

$$\mathbb{P} \longrightarrow \mathbb{Q}$$

$$= K e^{-rT} \, \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{S_T > K\}} \right]$$

$$= K e^{-rT} \, \mathbb{P}^{\mathbb{Q}}\left[S_T > K \right] \Rightarrow N(d_2)$$

$$\mathbb{E}\left[\mathbb{1}_{\{X \in A\}}\right]$$

X is a RV.

$$= \int_\Omega \mathbb{1}_{\{X \in A\}} \, d\mathbb{P}$$

$$= \int_A 1 \cdot d\mathbb{P} + \underbrace{\int_{\Omega \setminus A} 0 \cdot d\mathbb{P}}_{0}$$

$$= \mathbb{P}(A)$$

Proba that X is in A !!!

$$P^{Q}\left[S_T > K\right]$$

$$= P^{Q}\left[S_0 \exp\left\{\left(r - \tfrac{1}{2}\sigma^2\right)t + \sigma X(t)\right\} > K\right]$$

$$= P^{Q}\left[\frac{S_0}{K} \cdot \exp\left\{\left(r - \tfrac{1}{2}\sigma^2\right)t + \sigma X(t)\right\} > 1\right]$$

$$= P^{Q}\left[\ln(S_0/K) + \left(r - \tfrac{1}{2}\sigma^2\right)t > -\sigma X(t)\right]$$

$$= \sigma\left(-X(t)\right)$$

$$\{\sqrt{t}$$

$$\{\sim N(0,1)$$

$$= P^{Q}\left[\ln(S_0/K) + \left(r - \tfrac{1}{2}\sigma^2\right)t > \sigma\sqrt{t}\,\{\right]$$

$$= P^{Q}\left[\frac{\ln(S_0/K) + \left(r - \tfrac{1}{2}\sigma^2\right)t}{\sigma\sqrt{t}} > \{\right]$$

$$= N(d_2)$$

$$(1) = \mathbb{E}^{Q}\left[B^{-1}(T)\, S(T)\, \mathbb{1}_{\{S_T > K\}} \right]$$

$$= \mathbb{E}^{Q}\left[S^{*}(T)\ \mathbb{1}_{\{S_T > K\}} \right]$$

$$= \mathbb{E}^{Q}\left[S_0 \exp\left\{ -\frac{1}{2}\sigma^2 T + \sigma X(T) \right\} \mathbb{1}_{\{S_T > K\}} \right]$$

$$= S_0\ \mathbb{E}^{Q}\left[\underbrace{\exp\left\{ -\frac{1}{2}\sigma^2 T + \sigma X(T) \right\}}_{\exp\left\{ -\frac{1}{2}\int_0^T \sigma^2 dt + \int_0^T \sigma\, dX(t) \right\}} \mathbb{1}_{\{S_T > K\}} \right]$$

$$T$$

$$S_0\, \mathbb{E}^{\mathbb{Q}}\left[\underbrace{\exp\left(\int_0^T \sigma\, dX(t) - \frac{1}{2}\int_0^T \sigma^2\, dt \right)}_{\mathcal{E}(\ ) \to \frac{d\bar{\mathbb{Q}}}{d\mathbb{P}}} \mathbb{1}_{\{S_T > K\}} \right]$$

$$= S_0 \int_\Omega \frac{d\bar{\mathbb{Q}}}{\cancel{d\mathbb{Q}}} \times \mathbb{1}_{\{S_T > K\}}\ \cancel{d\mathbb{Q}}$$

$$= S_0 \int_\Omega \mathbb{1}_{\{S_T > K\}}\ d\bar{\mathbb{Q}}$$

$$= S_0\, \mathbb{E}^{\bar{\mathbb{Q}}}\left[\mathbb{1}_{\{S_T > K\}} \right]$$

$$= \boxed{S_0\, P^{\bar{\mathbb{Q}}}(S_T > K)} \longrightarrow S_0 N(d_1)$$

$$P(S_T > K)$$

RW Proba

$$= P\left(S_0 \exp\left\{\left(\mu - \tfrac{1}{2}\sigma^2\right)T + \sigma X(T)\right\} > K\right)$$

$$= P\left(\ln(S_0/K) + \left(\mu - \tfrac{1}{2}\sigma^2\right)T > \sigma(-X_T)\right)$$

$$\xi \sim N(0,1)$$

$$-X(T) = \xi\sqrt{T}$$

$$= P\left(\frac{\ln(S_0/K) + \left(\mu - \tfrac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} > \xi\right)$$

$$= N(\text{"}d_0\text{"})$$