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9 Calibration Algorithms to Caps and Swaptions Based on Optimization Techniques

9.1 INTRODUCTION

The previous chapter was dedicated to calibration algorithms to swaptions. We have presented a separated approach, a Longstaff-Schwartz-Santa Clara approach, a locally single factor method, a calibration with historical correlations and finally a calibration to co-terminal swaptions. Some of the algorithms should be used very carefully in practice. The locally single factor may lead to negative variances and the calibration with historical correlations may lead to negative correlations.

This chapter describes simultaneous calibration approaches to caps and swaptions. The first algorithm is non-parametric, whilst the second parametric. The non-parametric calibration is based on the Rebonato approach. At the beginning we derive annual caplet volatilities driven by the dynamics of annual forward rates from the dynamics of quarterly forward rates. It is very important to notice that some market participants often forget about such a transformation. The consequence may be wrong calibration results and mispriced derivative instruments.

Next we compute forward swap rates and present an approximation of the swaption formula for LIBOR Market Model as a linear combination of forward LIBOR rates. In the calibration we use also piecewise – constant instantaneous volatilities described in Chapter 7. Finally we constitute an optimization function minimizing the difference between theoretical and market swaption volatilities. The result of the optimization is a matrix of annualized instantaneous volatilities and also instantaneous correlations. At the end we compare results of the theoretical and market caplet volatilities and the theoretical and market swaption volatilities.

The second part of the chapter is dedicated to parametric calibration to caps and swaptions. First we use caplet volatilities derived in Chapter 7 and based on that constitute some appropriate parametric functions of caplet volatilities. Next we define optimization functions minimizing differences between theoretical and market caplet prices. After that we move into calibration to swaptions. For that reason we constitute another set parametric functions and run optimizations but now to minimize the differences between the theoretical and market quotations of swaptions.

After presenting all the algorithms we move to analyse the results of such computations. Taking real market data (LIBOR rates, FRA, IRS, caps and swapations, historical correlations) from a particular working day we present in detail how the algorithms should be implemented in practice.

9.2 NON PARAMETRIC CALIBRATION TO CAPS AND SWAPTIONS

The first step before a simultaneous calibration to cap and swaptions is deriving annual caplet volatilities driven by the dynamic of annual forward rates from the dynamics of the quarterly forward rates.

Preliminary computations

We concentrate our computations on a particular case of four quarterly interest forward rates covering one particular one year period. If we do that, a generalization for all one year periods for all swaptions is straightforward.

The first step is to compute the ratios of discount factors as functions of quarterly forward rates.

$$L(t, T_{k+i}, T_{k+i+1}) = \frac{1}{\delta_{k+i,k+i+1}}\left[\frac{B(t, T_{k+i})}{B(t, T_{k+i+1})} - 1\right] \Rightarrow$$
$$\frac{B(t, T_{k+i})}{B(t, T_{k+i+1})} = \delta_{k+i,k+i+1} L(t, T_{k+i}, T_{k+i+1}) + 1$$

for $i = 0, 1, 2, 3$.

Now we can express a one year forward rate covering the period $T_k \div T_{k+4}$ as functions of previously computed discount factor ratios

$$\begin{aligned} L(t, T_k, T_{k+4}) &= \frac{1}{\delta_{k,k+4}}\left[\frac{B(t, T_k)}{B(t, T_{k+4})} - 1\right] \\ &= \frac{1}{\delta_{k,k+4}}\left[\frac{B(t, T_k)}{B(t, T_{k+1})}\frac{B(t, T_{k+1})}{B(t, T_{k+2})}\frac{B(t, T_{k+2})}{B(t, T_{k+3})}\frac{B(t, T_{k+3})}{B(t, T_{k+4})} - 1\right]. \end{aligned}$$

Substituting the expression for fractions of discount factors from above expressions we obtain:

$$\begin{aligned} L(t, T_k, T_{k+4}) = \frac{1}{\delta_{k,k+4}} \cdot \\ \left\{\left[\delta_{k,k+1} L(t, T_k, T_{k+1}) + 1\right]\left[\delta_{k+1,k+2} L(t, T_{k+1}, T_{k+2}) + 1\right]\left[\delta_{k+2,k+3} L(t, T_{k+2}, T_{k+3}) + 1\right]\right. \\ \left. \times \left[\delta_{k+3,k+4} L(t, T_{k+3}, T_{k+4}) + 1\right]\right\} \end{aligned}$$

$$L\left(t, T_k, T_{k+4}\right) = \tfrac{1}{\delta_{k,k+4}} \cdot \tag{9.1}$$

$$\left\{\begin{array}{l}
1 + \delta_{k,k+1} L\left(t, T_k, T_{k+1}\right) + \delta_{k+1,k+2} L\left(t, T_{k+1}, T_{k+2}\right) + \delta_{k+2,k+3} L\left(t, T_{k+2}, T_{k+3}\right) + \delta_{k+3,k+4} L\left(t, T_{k+3}, T_{k+4}\right) \\
\quad + \delta_{k,k+1} L\left(t, T_k, T_{k+1}\right) \delta_{k+1,k+2} L\left(t, T_{k+1}, T_{k+2}\right) + \delta_{k,k+1} L\left(t, T_k, T_{k+1}\right) \delta_{k+2,k+3} L\left(t, T_{k+2}, T_{k+3}\right) \\
\quad + \delta_{k,k+1} L\left(t, T_k, T_{k+1}\right) \delta_{k+3,k+4} L\left(t, T_{k+3}, T_{k+4}\right) \\
\quad + \delta_{k+1,k+2} L\left(t, T_{k+1}, T_{k+2}\right) \delta_{k+2,k+3} L\left(t, T_{k+2}, T_{k+3}\right) + \delta_{k+1,k+2} L\left(t, T_{k+1}, T_{k+2}\right) \delta_{k+3,k+4} L\left(t, T_{k+3}, T_{k+4}\right) \\
\quad + \delta_{k+2,k+3} L\left(t, T_{k+2}, T_{k+3}\right) \delta_{k+3,k+4} L\left(t, T_{k+3}, T_{k+4}\right) \\
\quad + \delta_{k,k+1} L\left(t, T_k, T_{k+1}\right) \delta_{k+1,k+2} L\left(t, T_{k+1}, T_{k+2}\right) \delta_{k+2,k+3} L\left(t, T_{k+2}, T_{k+3}\right) \\
\quad + \delta_{k,k+1} L\left(t, T_k, T_{k+1}\right) \delta_{k+1,k+2} L\left(t, T_{k+1}, T_{k+2}\right) \delta_{k+3,k+4} L\left(t, T_{k+3}, T_{k+4}\right) \\
\quad + \delta_{k,k+1} L\left(t, T_k, T_{k+1}\right) \delta_{k+2,k+3} L\left(t, T_{k+2}, T_{k+3}\right) \delta_{k+3,k+4} L\left(t, T_{k+3}, T_{k+4}\right) \\
\quad + \delta_{k+1,k+2} L\left(t, T_{k+1}, T_{k+2}\right) \delta_{k+2,k+3} L\left(t, T_{k+2}, T_{k+3}\right) \delta_{k+3,k+4} L\left(t, T_{k+3}, T_{k+4}\right) \\
\quad + \delta_{k,k+1} L\left(t, T_k, T_{k+1}\right) \delta_{k+1,k+2} L\left(t, T_{k+1}, T_{k+2}\right) \delta_{k+2,k+3} L\left(t, T_{k+2}, T_{k+3}\right) \delta_{k+3,k+4} L\left(t, T_{k+3}, T_{k+4}\right)
\end{array}\right\}$$

The dynamics of quarterly forward rates can be expressed under the measure Q^{k+4} as:

$$\begin{aligned}
&dL\left(t, T_k, T_{k+1}\right) = \\
&\quad -\sigma_{k+i+1}^{inst}(t) F\left(t, T_{k+i}, T_{k+i+1}\right) \sum_{j=k+i+2}^{k+i+4} \frac{\rho_{k+i+1,j} \delta_{j-1,j} \sigma_j^{inst}(t) L\left(t, T_{j-1}, T_j\right)}{1 + \delta_{j-1,j} F\left(t, T_{j-1}, T_j\right)} dt \\
&\quad + \sigma_{k+i+1}^{inst}(t) F\left(t, T_{k+i}, T_{k+i+1}\right) dW_{k+i+1}^{k+4}(t)
\end{aligned} \tag{9.2}$$

for $i = 0, 1, 2, 3$

where $\rho_{k.j}$ denotes the instantaneous correlation between the forward rates $L(t, T_{k-1}, T_k)$ and $L(t, T_{j-1}, T_j)$, and $\sigma_{k+i}^{inst}(t) = \sigma^{inst}(t, T_{k+i}, T_{k+i+1})$, $i = 0, 1, 2, 3$ is the instantaneous volatility of the quarterly forward rate $L(t, T_{k+i}, T_{k+i+1})$, $i = 0, 1, 2, 3$.

The dynamic of annual the forward rate under the measure Q^{k+4} can be written:

$$dF\left(t, T_k, T_{k+4}\right) = \sigma(t) L\left(t, T_k, T_{k+4}\right) dW_{k+4}^{k+4}(t) \tag{9.3}$$

where $\sigma^{inst}(t) = \sigma^{inst}(t, T_k, T_{k+4})$ is the instantaneous volatility of the annual forward rate $L(t, T_k, T_{k+4})$.

Our goal is to derive an annual caplet volatility driven by the dynamic of the annual forward rate from the dynamics of quarterly forward rates. We present the algorithm based on the example for period $T_k \div T_{k+4}$ under the measure Q^{k+4}.

First we assume, that correlations between quarterly forward rates are equal to one.

Let us take logarithms of the dynamics presented by equations (9.2) and (9.3). Using Ito's lemma we have:

$$\begin{aligned}
d \ln L\left(t, T_{k+i}, T_{k+i+1}\right) = &-\sigma_{k+i+1}^{inst}(t) \sum_{j=k+i+2}^{k+i+4} \frac{\rho_{k+i+1,j} \delta_{j-1,j} \sigma_j^{inst}(t) L\left(t, T_{j-1}, T_j\right)}{1 + \delta_{j-1,j} L\left(t, T_{j-1}, T_j\right)} dt \\
&- \frac{1}{2} \sigma_{k+i+1}^{inst}(t)^2 dt + \sigma_{k+i+1}^{inst}(t) dW_{k+i+1}^{k+4}(t)
\end{aligned}$$

for $i = 0, 1, 2, 3$ and

$$d \ln L(t, T_k, T_{k+4}) = -\frac{1}{2}\sigma^{inst}(t)^2\, dt + \sigma^{inst}(t)\, dW_{k+4}^{k+4}(t)$$

For further computations we need the multi-dimensional Ito Lemma. In general if we have a function:

$$V(S_1, S_2, \ldots, S_n, t)$$

we can write

$$dV = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_i\sigma_j\rho_{ij}S_iS_j\frac{\partial^2 V}{\partial S_i \partial S_j}\right)dt + \sum_{i=1}^{n}\frac{\partial V}{\partial S_i}dS_i. \tag{9.4}$$

In our case we need a four-dimensional version of the Ito lemma. For simplicity we can write:

$$\begin{aligned} &L(t, T_k, T_{k+4}) = L_4^A(t) \\ &L(t, T_{k+i}, T_{k+i+1}) = L_{i+1}^Q(t)\,, \text{ for } i = 0, 1, 2, 3 \end{aligned}$$

and

$$\begin{aligned} &\delta_{k,k+4} = \delta_4^A,\;\; \delta_{k+i,k+i+1} = \delta_{i+1}^Q,\; \text{ for } i = 0, 1, 2, 3 \\ &\sigma_i^Q(t) = \sigma_{k+i}^{inst}(t)\,, \text{ for } i = 1, 2, 3, 4. \end{aligned}$$

So using four dimensional Ito we have:

$$L_4^A(t) = \frac{1}{\delta_4^A}\cdot$$

$$\left\{\begin{array}{l} \delta_1^Q L_1^Q(t) + \delta_2^Q L_2^Q(t) + \delta_3^Q L_3^Q(t) + \delta_4^Q L_4^Q(t) + \delta_1^Q L_1^Q(t)\,\delta_2^Q L_2^Q(t) + \delta_1^Q L_1^Q(t)\,\delta_3^Q L_3^Q(t) + \\ \delta_1^Q L_1^Q(t)\,\delta_4^Q L_4^Q(t) + \delta_2^Q L_2^Q(t)\,\delta_3^Q L_3^Q(t) + \delta_2^Q L_2^Q(t)\,\delta_4^Q L_4^Q(t) + \delta_3^Q L_3^Q(t)\,\delta_4^Q L_4^Q(t) + \\ \delta_1^Q L_1^Q(t)\,\delta_2^Q L_2^Q(t)\,\delta_3^Q L_3^Q(t) + \delta_1^Q L_1^Q(t)\,\delta_2^Q L_2^Q(t)\,\delta_4^Q L_4^Q(t) + \delta_1^Q L_1^Q(t)\,\delta_3^Q L_3^Q(t)\,\delta_4^Q L_4^Q(t) + \\ \delta_2^Q L_2^Q(t)\,\delta_3^Q L_3^Q(t)\,\delta_4^Q L_4^Q(t) + \delta_1^Q L_1^Q(t)\,\delta_2^Q L_2^Q(t)\,\delta_3^Q L_3^Q(t)\,\delta_4^Q L_4^Q(t) \end{array}\right\}$$

and

$$\frac{\partial F_4^A(t)}{\partial F_i^Q(t)} = \frac{1}{\delta_4^A}\left[\delta_i^Q + \delta_i^Q\sum_{\substack{j=1\\ j\neq i}}^{4}\delta_j^Q L_j^Q(t) + \delta_i^Q\sum_{\substack{j=1\\ j\neq i}}^{3}\sum_{\substack{k=j+1\\ k\neq i}}^{4}\delta_j^Q\delta_k^Q L_j^Q(t)\,L_k^Q(t) + \delta_4^A\prod_{\substack{j=1\\ j\neq i}}^{4}L_j^Q(t)\right]$$

for $i = 0, 1, 2, 3$.

We do not need in our computations the term dt. So we can write:

$$dL_4^A(t) = (\ldots)\,dt + \frac{\partial L_4^A(t)}{\partial L_1^Q(t)} dL_1^Q(t) + \frac{\partial L_4^A(t)}{\partial L_2^Q(t)} dL_2^Q(t) + \frac{\partial L_4^A(t)}{\partial L_3^Q(t)} dL_3^Q(t) + \frac{\partial L_4^A(t)}{\partial L_4^Q(t)} dL_4^Q(t)$$

Then under the measure Q^4

$$dL_i^Q(t) = (\ldots)\,dt + \sigma_i^Q(t)\,L_i^Q(t)\,dW_i^4(t), \text{ for } i = 1, 2, 3, 4$$

where:

$$dW_i^4(t)\,dW_j^4(t) = \rho_{i,j}dt, \text{ for } i = 0, 1, 2, 3 \text{ and } i \neq j$$

we obtain after performing straightforward calculations:

$$\begin{aligned}
dL_4^A(t) = (\ldots)\,dt &+ \sigma_1^Q(t)\left(\frac{\delta_1^Q}{\delta_4^A} L_1^Q(t) + \frac{\delta_1^Q\delta_2^Q}{\delta_4^A} L_1^Q(t)\,L_2^Q(t) + \frac{\delta_1^Q\delta_3^Q}{\delta_4^A} L_1^Q(t)\,L_3^Q(t)\right.\\
&+ \frac{\delta_1^Q\delta_4^Q}{\delta_4^A} L_1^Q(t)\,L_4^Q(t) + \frac{\delta_1^Q\delta_2^Q\delta_3^Q}{\delta_4^A} L_1^Q(t)\,L_2^Q(t)\,L_3^Q(t) + \frac{\delta_1^Q\delta_2^Q\delta_4^Q}{\delta_4^A} L_1^Q(t)\,L_2^Q(t)\,L_4^Q(t)\\
&\left.+ \frac{\delta_1^Q\delta_3^Q\delta_4^Q}{\delta_4^A} L_1^Q(t)\,L_3^Q(t)\,L_4^Q(t) + \frac{\delta_1^Q\delta_2^Q\delta_3^Q\delta_4^Q}{\delta_4^A} L_1^Q(t)\,L_2^Q(t)\,L_3^Q(t)\,L_4^Q(t)\right) dW_1^4(t)\\
&+ \sigma_2^Q(t)\left(\frac{\delta_2^Q}{\delta_4^A} L_2^Q(t) + \frac{\delta_1^Q\delta_2^Q}{\delta_4^A} L_1^Q(t)\,L_2^Q(t) + \frac{\delta_2^Q\delta_3^Q}{\delta_4^A} L_2^Q(t)\,L_3^Q(t) + \frac{\delta_2^Q\delta_4^Q}{\delta_4^A} L_2^Q(t)\,L_4^Q(t)\right.\\
&+ \frac{\delta_1^Q\delta_2^Q\delta_3^Q}{\delta_4^A} L_1^Q(t)\,L_2^Q(t)\,L_3^Q(t) + \frac{\delta_1^Q\delta_2^Q\delta_4^Q}{\delta_4^A} L_1^Q(t)\,L_2^Q(t)\,L_4^Q(t) + \frac{\delta_2^Q\delta_3^Q\delta_4^Q}{\delta_4^A} L_2^Q(t)\\
&\left.\times L_3^Q(t)\,L_4^Q(t) + \frac{\delta_1^Q\delta_2^Q\delta_3^Q\delta_4^Q}{\delta_4^A} L_1^Q(t)\,L_2^Q(t)\,L_3^Q(t)\,L_4^Q(t)\right) dW_2^4(t) \qquad (9.5)\\
&+ \sigma_3^Q(t)\left(\frac{\delta_3^Q}{\delta_4^A} L_3^Q(t) + \frac{\delta_1^Q\delta_3^Q}{\delta_4^A} L_1^Q(t)\,L_3^Q(t) + \frac{\delta_2^Q\delta_3^Q}{\delta_4^A} L_2^Q(t)\,L_3^Q(t) + \frac{\delta_3^Q\delta_4^Q}{\delta_4^A} L_3^Q(t)\,L_4^Q(t)\right.\\
&+ \frac{\delta_1^Q\delta_2^Q\delta_3^Q}{\delta_4^A} L_1^Q(t)\,L_2^Q(t)\,L_3^Q(t) + \frac{\delta_1^Q\delta_3^Q\delta_4^Q}{\delta_4^A} L_1^Q(t)\,L_3^Q(t)\,L_4^Q(t)\\
&\left.+ \frac{\delta_2^Q\delta_3^Q\delta_4^Q}{\delta_4^A} L_2^Q(t)\,L_3^Q(t)\,L_4^Q(t) + \frac{\delta_1^Q\delta_2^Q\delta_3^Q\delta_4^Q}{\delta_4^A} L_1^Q(t)\,L_2^Q(t)\,L_3^Q(t)\,L_4^Q(t)\right) dW_3^4(t)\\
&+ \sigma_4^Q(t)\left(\frac{\delta_4^Q}{\delta_4^A} L_4^Q(t) + \frac{\delta_1^Q\delta_4^Q}{\delta_4^A} L_1^Q(t)\,L_4^Q(t) + \frac{\delta_2^Q\delta_4^Q}{\delta_4^A} L_2^Q(t)\,L_4^Q(t) + \frac{\delta_3^Q\delta_4^Q}{\delta_4^A} L_3^Q(t)\,L_4^Q(t)\right.\\
&+ \frac{\delta_1^Q\delta_2^Q\delta_4^Q}{\delta_4^A} L_1^Q(t)\,L_2^Q(t)\,L_4^Q(t) + \frac{\delta_1^Q\delta_3^Q\delta_4^Q}{\delta_4^A} L_1^Q(t)\,L_3^Q(t)\,L_4^Q(t)\\
&\left.+ \frac{\delta_2^Q\delta_3^Q\delta_4^Q}{\delta_4^A} L_2^Q(t)\,L_3^Q(t)\,L_4^Q(t) + \frac{\delta_1^Q\delta_2^Q\delta_3^Q\delta_4^Q}{\delta_4^A} L_1^Q(t)\,L_2^Q(t)\,L_3^Q(t)\,L_4^Q(t)\right) dW_4^4(t)
\end{aligned}$$

Let us set:

$$a_i(t) = \frac{1}{F_4^A(t)}\left[\frac{\delta_i^Q}{\delta_4^A}F_i^Q(t) + \frac{\delta_i^Q F_i^Q(t)}{\delta_4^A}\sum_{\substack{j=1\\j\neq i}}^{4}\delta_j^Q F_j^Q(t) + \frac{\delta_i^Q F_i^Q(t)}{\delta_4^A}\sum_{\substack{j=1\\j\neq i}}^{3}\sum_{\substack{k=j+1\\k\neq i}}^{4}\delta_j^Q\delta_k^Q F_j^Q(t)F_k^Q(t) + \prod_{\substack{j=1\\j\neq i}}^{4}F_j^Q(t)\right]$$

for $i = 1, 2, 3, 4$.

Taking variance on both sides which is conditional on time t we can write:

$$\sigma^{inst}(t)^2 = \sum_{i=1}^{4} a_i(t)^2\sigma_i^Q(t)^2 + 2\sum_{i=1}^{4}\sum_{j=i}^{4}\rho_{i,j}\sigma_i^Q(t)\sigma_j^Q(t)a_i(t)a_j(t) \tag{9.6}$$

We can freeze all F at zero time value to obtain

$$\sigma_{appr}^{inst}(t)^2 = \sum_{i=1}^{4} a_i(0)^2\sigma_i^Q(t)^2 + 2\sum_{i=1}^{4}\sum_{j=i}^{4}\rho_{i,j}\sigma_i^Q(t)\sigma_j^Q(t)a_i(0)a_j(0) \tag{9.7}$$

We can consider F as a swap rate being the underlying of the $T_k \times 1$ swaption. In such case the squared Black's swaption volatility can be written as:

$$\sigma_{Black}^2 \approx \frac{1}{\delta_{t,T_k}}\int_0^{T_k}\sigma_{appr}(t)^2\,dt = \frac{1}{\delta_{t,T_k}}\left(\sum_{i=1}^{4}a_i(0)^2\int_0^{T_k}\sigma_i^Q(t)^2\,dt + 2\sum_{i=1}^{4}\sum_{j=i+1}^{4}\rho_{i,j}a_i(0)a_j(0)\int_0^{T_k}\sigma_i^Q(t)\sigma_j^Q(t)\,dt\right) \tag{9.8}$$

Considering the first integral:

$$\frac{1}{\delta_{t,T_k}}\int_0^{T_k}\sigma_1^Q(t)^2\,dt = \sigma^{cpl}(t, T_k, T_{k+1})^2$$

If we assume that the forward rates have constant volatilities, we can write:

$$\frac{1}{\delta_{t,T_k}}\int_0^{T_k}\sigma_2^Q(t)^2\,dt = \frac{1}{\delta_{t,T_k}}\int_0^{T_k}\sigma^{cpl}(t, T_{k+1}, T_{k+2})^2\,dt = \sigma^{cpl}(t, T_{k+1}, T_{k+2})^2$$

$$\frac{1}{\delta_{t,T_k}}\int_0^{T_k}\sigma_3^Q(t)^2\,dt = \frac{1}{\delta_{t,T_k}}\int_0^{T_k}\sigma^{cpl}(t, T_{k+2}, T_{k+3})^2\,dt = \sigma^{cpl}(t, T_{k+2}, T_{k+3})^2$$

$$\frac{1}{\delta_{t,T_k}}\int_0^{T_k}\sigma_4^Q(t)^2\,dt = \frac{1}{\delta_{t,T_k}}\int_0^{T_k}\sigma^{cpl}(t, T_{k+3}, T_{k+4})^2\,dt = \sigma^{cpl}(t, T_{k+3}, T_{k+4})^2$$

and

$$\frac{1}{\delta_{t,T_k}}\int_0^{T_k}\sigma_i^Q(t)\sigma_j^Q(t)\,dt = \frac{1}{\delta_{t,T_k}}\int_0^{T_k}\sigma^{cpl}\left(t, T_k, T_{k+i}\right)\sigma^{cpl}\left(t, T_{k+i}, T_{k+j}\right)dt = \sigma^{cpl}\left(t, T_k, T_{k+i}\right)\sigma^{cpl}\left(t, T_{k+i}, T_{k+j}\right)$$

for $i = 1, 2, 3, 4$ and $j = i + 1$.

Finally we obtain

$$\sigma_{Black}^2 \approx \left(\sum_{i=1}^{4} a_i(0)^2 \sigma^{cpl}\left(t, T_k, T_{k+1}\right)^2 + 2\sum_{i=1}^{4} \sum_{j=i+1}^{4} \rho_{i,j} a_i(0) a_j(0) \sigma^{cpl}\left(t, T_k, T_{k+i}\right) \sigma^{cpl}\left(t, T_{k+i}, T_{k+j}\right) \right). \quad (9.9)$$

The result is very similar to result presented in Brigo and Mercurio (2001) but here our example is for quarterly forward rates instead of annual rates. Let us present a practical example.

Example 9.1 Annualization of volatility

We will compute the annual market volatilities generated from quarterly caplet volatilities. In our computations we use market data taken from 21/01/2005 (with value date equal to 25/01/2005). We set intra-correlation of quarterly rates to one.

Let us consider period 1Y – 2Y. Table 9.1 presents data for interval with start date 25/01/2006 and end date 25/01/2007. Four quarterly sub-intervals have expiry dates 24/04/2006, 25/07/2006, 25/10/2006 and 25/01/2007. For each sub-interval we have the caplet volatility, quarterly year fraction and forward rate.

Table 9.1 Quarterly initial data

Tenor	Date	Caplet volatility	Year fraction	Forward rate (quarterly)	Discount factor	Annual forward rate
1Y	25/01/2006					
1.25Y	25/04/2006	20.15 %	0.25000	2.5440 %	0.971064	
1.50Y	25/07/2006	21.89 %	0.25278	2.6591 %	0.964580	
1.75Y	25/10/2006	23.65 %	0.25556	2.7755 %	0.957787	
2Y	25/01/2007	25.50 %	0.25556	2.8925 %	0.950759	2.7471 %

Taking data from Table 9.1 we compute the annual forward rate covering period 1Y – 2Y

$$F(0, T_{1Y}, T_{2Y}) = \left[\frac{B(0, T_{1Y})}{B(0, T_{2Y})} - 1 \right] \frac{360}{(T_{2Y} - T_{1Y})} = 2.7471\%.$$

Now we have to compute parameters: $a_1(0), a_2(0), a_3(0), a_4(0)$:

$$a_1(0) = 0.233223$$
$$a_2(0) = 0.246394$$
$$a_3(0) = 0.259912$$
$$a_4(0) = 0.270786$$

Having computed parameters $a_1(0), a_2(0), a_3(0), a_4(0)$ and assuming $\rho_{ij} = 1$ we can compute the annual volatility:

$$\sigma_{Black}^2(0, T_{1Y}, T_{2Y}) \approx \sum_{i=1Y}^{2Y} a_i(0)^2 \sigma^{cpl}(t, T_k, T_{k+1})^2$$
$$+ 2\sum_{i=1Y}^{2Y} \sum_{j=i+0.25Y}^{2Y} \rho_{i,j} a_i(0) a_j(0) \sigma^{cpl}(0, T_k, T_{k+i}) \sigma^{cpl}\left(0, T_{k+i}, T_{k+j}\right)$$

$$\begin{aligned}\sigma^2_{Black}(0, T_{1Y}, T_{2Y}) \approx\ & a_1(0)^2 \sigma^{cpl}(t, T_{1Y}, T_{1.25Y})^2 + a_2(0)^2 \sigma^{cpl}(t, T_{1.25Y}, T_{1.5Y})^2 \\ & + a_3(0)^2 \sigma^{cpl}(t, T_{1.5Y}, T_{1.75Y})^2 + a_4(0)^2 \sigma^{cpl}(t, T_{1.75Y}, T_{2Y})^2 \\ & + 2 \cdot a_1(0) \cdot a_2(0) \sigma^{cpl}(t, T_{1Y}, T_{1.25Y}) \sigma^{cpl}(t, T_{1.25Y}, T_{1.5Y}) \\ & + 2 \cdot a_1(0) \cdot a_3(0) \sigma^{cpl}(t, T_{1Y}, T_{1.25Y}) \sigma^{cpl}(t, T_{1.5Y}, T_{1.75Y}) \\ & + 2 \cdot a_1(0) \cdot a_4(0) \sigma^{cpl}(t, T_{1Y}, T_{1.25Y}) \sigma^{cpl}(t, T_{1.75Y}, T_{2Y}) \\ & + 2 \cdot a_2(0) \cdot a_3(0) \sigma^{cpl}(t, T_{1.25Y}, T_{1.5Y}) \sigma^{cpl}(t, T_{1.5Y}, T_{1.75Y}) \\ & + 2 \cdot a_2(0) \cdot a_4(0) \sigma^{cpl}(t, T_{1.25Y}, T_{1.5Y}) \sigma^{cpl}(t, T_{1.75Y}, T_{2Y}) \\ & + 2 \cdot a_3(0) \cdot a_4(0) \sigma^{cpl}(t, T_{1.5Y}, T_{1.75Y}) \sigma^{cpl}(t, T_{1.75Y}, T_{2Y})\end{aligned}$$

Using the real values we obtain:

$$\upsilon^2_{Black}(0, T_{1Y}, T_{2Y}) \approx 23.14\,\%$$

End of example 9.1

Table 9.2 presents the results of the computations of all annual forward volatilities:

Table 9.2 Annual forward volatilities

Tenor	Date	Caplet vol	Year fraction quarterly	Year fraction annual	Discount factor	Forward rate quarterly	Forward rate annual	a1(0) a2(0) a3(0) a4(0)	Annual volatility
1Y	25/01/2006	16.41 %	0.25556	1.0139	0.97724	2.4150 %	2.2972 %		
1.25Y	25/04/2006	20.15 %	0.25000		0.971064	2.5440 %			
1.50Y	25/07/2006	21.89 %	0.25278		0.96458	2.6591 %			
1.75Y	25/10/2006	23.65 %	0.25556		0.957787	2.7755 %			
2Y	25/01/2007	25.50 %	0.25556	1.0139	0.950759	2.8925 %	2.7471 %	0.233223 0.246394 0.259912 0.270786	**23.14** %
2.25Y	25/04/2007	22.12 %	0.25000		0.943868	2.9201 %			
2.50Y	25/07/2007	22.55 %	0.25278		0.936727	3.0161 %			
2.75Y	25/10/2007	22.98 %	0.25556		0.929333	3.1131 %			
3Y	25/01/2008	23.41 %	0.25556	1.0139	0.92177	3.2105 %	3.1018 %	0.2377 0.248157 0.258869 0.266908	**23.05 %**
3.25Y	25/04/2008	20.97 %	0.25278		0.914291	3.2361 %			
3.50Y	25/07/2008	20.83 %	0.25278		0.906679	3.3216 %			
3.75Y	27/10/2008	20.77 %	0.26111		0.898851	3.3352 %			
4Y	26/01/2009	20.51 %	0.25278	1.0194	0.890895	3.5326 %	3.3995 %	0.242234 0.248582 0.257753 0.264238	**21.03 %**
5Y	25/01/2010	19.38 %	0.25278	1.0111	0.858974	3.7707 %	3.6754 %	0.242123 0.249997 0.258101 0.263498	**19.98 %**

10Y	26/01/2015	15.70 %	0.25278	1.0111	0.694285	4.5640 %	4.5342 %	0.247076 0.252105 0.257512 0.260177	**16.28 %**
20Y	27/01/2025	11.31 %	0.26111	1.0222	0.434558	4.8409 %	4.9805 %	0.2517 0.252753 0.256584 0.25766	**11.67 %**

Our next step in simultaneous calibration is computing forward swap rates which can be expressed as:

$$S_{sN}(t) = \frac{B(t, T_n) - B(t, T_N)}{\sum\limits_{i=n+1}^{N} \delta_{i-1,i} B(t, T_i)} \tag{9.10}$$

The forward swap rates computed using equation (9.10) are presented below in Table 9.3.

Table 9.3 Forward swap rates

	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y
1Y	2.92 %	3.08 %	3.22 %	3.35 %	3.47 %	3.58 %	3.68 %	3.76 %	3.81 %
2Y	3.25 %	3.39 %	3.51 %	3.63 %	3.73 %	3.83 %	3.91 %	3.96 %	4.01 %
3Y	3.54 %	3.65 %	3.77 %	3.87 %	3.97 %	4.04 %	4.08 %	4.13 %	4.17 %
4Y	3.79 %	3.90 %	4.00 %	4.10 %	4.16 %	4.20 %	4.24 %	4.27 %	4.30 %
5Y	4.03 %	4.12 %	4.21 %	4.27 %	4.30 %	4.33 %	4.36 %	4.39 %	4.42 %
7Y	4.41 %	4.45 %	4.45 %	4.47 %	4.49 %	4.51 %	4.54 %	4.56 %	4.57 %
10Y	4.51 %	4.53 %	4.57 %	4.61 %	4.62 %	4.64 %	4.66 %	4.68 %	4.71 %

The tenors in the rows represents the start dates of the forward swap rate, tenors in columns represents the end dates of the forward swap rate.

Having the forward swap rates and the ATM volatilities we can use the Black formula for swaptions to compute values of all swaptions from the tables.

$$Swaption_{nN}(0) = \sum_{i=n+1}^{N} \delta_{i-1,i} B(0, T_i) \left(S_{nN}(0) N(d_1) - K N(d_2)\right), \tag{9.11}$$

where

$$d_1 = \frac{\ln(S_{nN}(0)/K) + \delta_{0,n}\sigma_{nN}^2/2}{\sigma_{nN}\sqrt{\delta_{0,n}}}, \; d_2 = d_1 - \sigma_{nN}\sqrt{\delta_{0,n}},$$

Our goal is to find such instantaneous correlations of the LIBOR rates, which together with the instantaneous volatilities obtained from caplet prices will give possibly negligible difference between Black and LFM swaption prices.

The detailed algorithm of the calibration to swaption and cap prices is based on the Rebonato approach.

The LFM Black squared swaption volatility for swaptions can be approximated by

$$\left(\sigma_{n,N}^{LFM}\right)^2 = \sum_{i,j=n+1}^{N} \frac{w_i(0)\, w_j(0)\, F(0, T_{i-1}, T_i)\, F\left(0, T_{j-1}, T_j\right) \rho_{i,j}}{S_{n,N}(0)^2} \int_0^{T_n} \sigma_i(t)\, \sigma_j(t)\, dt$$

where the swap rates are expressed as a linear combination of the forward rates.

$$S_{n,N}(t) = \sum_{i=n+1}^{N} w_i(t)\, F(t, T_{i-1}, T_i)$$

$$S_{n,N}(0) \overset{assumption}{=} \sum_{i=n+1}^{N} w_i(0)\, F(0, T_{i-1}, T_i)$$

All $w_i(t)$ and $F(t, T_{i-1}, T_i)$ are frozen at the time 0 value.

In the calibration we consider piecewise-constant instantaneous volatilities according to specification in Table 9.4.

Table 9.4 Piecewise – constant instantaneous volatilities

Time / Forward rate	$t \in (0, T_0)$	$t \in (T_0, T_1)$	$t \in (T_1, T_2)$	$t \in (T_{M-2}, T_{M-1})$
$F(t, T_0, T_1)$	$\sigma_{1,1} = \Phi_1 \Psi_1$			
$F(t, T_1, T_2)$	$\sigma_{2,1} = \Phi_2 \Psi_2$	$\sigma_{2,2} = \Phi_2 \Psi_1$		
$F(t, T_{M-1}, T_M)$	$\sigma_{M,1} = \Phi_M \Psi_M$	$\sigma_{M,2} = \Phi_M \Psi_{M-1}$	$\sigma_{M,3} = \Phi_M \Psi_{M-2}$	$\sigma_{M,M} = \Phi_M \Psi_1$

We assume that caplet volatilities multiplied by time $\sigma^{cpl}(0, T_{i-1}, T_i)$ are read from the market. These volatilities should be annualized (an example can you find in the previous section). Then the parameters Φ can be given in terms of parameters Ψ as:

$$\Phi_i^2 = \frac{\left(\sigma^{cpl}(0, T_{i-1}, T_i)\right)^2}{\sum_{i=1}^{j} \delta_{j-2,j-1} \Psi_{i-j+1}^2}$$

The last approximation is

$$\rho_{i,j} = \cos\left(\theta_i - \theta_j\right).$$

Calibration is based on the algorithm of minimization for finding the best fitting parameters Ψ and θ starting from certain initial guesses and with restriction which implies that all $\rho_{i,j} > 0$.

Now we can move into some examples clarifying the theory. Let us define the Number Of Grid Points as number of annual forward rates plus one equal to the number of annual caplet rates. In our case of calibration we take Number of Grid Points equal to 21. This is because we start our calibration from period 2Y – 3Y.

Let us define vectors ε and θ. The initial values of these vectors are presented in Table 9.5 (same as in Brigo and Mercurio (2001)).

Table 9.5 Initial values of vectors ε, θ

i	$\varepsilon(i)$	$\theta(i)$
1	1.0000	$\pi/2 = 1.5708$
2	1.0000	$\pi/2 = 1.5708$
3	1.0000	$\pi/2 = 1.5708$
4	1.0000	$\pi/2 = 1.5708$
5	1.0000	$\pi/2 = 1.5708$
6	1.0000	$\pi/2 = 1.5708$
7	1.0000	$\pi/2 = 1.5708$
8	1.0000	$\pi/2 = 1.5708$
9	1.0000	$\pi/2 = 1.5708$
10	1.0000	$\pi/2 = 1.5708$
11	1.0000	$\pi/2 = 1.5708$
12	1.0000	$\pi/2 = 1.5708$
13	1.0000	$\pi/2 = 1.5708$
14	1.0000	$\pi/2 = 1.5708$
15	1.0000	$\pi/2 = 1.5708$
16	1.0000	$\pi/2 = 1.5708$
17	1.0000	$\pi/2 = 1.5708$
18	1.0000	$\pi/2 = 1.5708$
19	1.0000	$\pi/2 = 1.5708$
20	1.0000	$\pi/2 = 1.5708$
21	1.0000	$\pi/2 = 1.5708$

Now we have to generate the vector Ψ whose values are functions of parameters of vector ε. For technical reasons the first two rows of the vector Ψ will take zero values.

Algorithm 9.1 Calculation of vector Ψ

We start our loop from i = 3 up to [Number Of Grid Points] = 21

We take initial values

$SumTemp = 0$

$Sum = 0$

For each i we have to compute *Sum* in the following way:

Start from j = 3 up to current value of i and compute *Sum* using the loop procedure:

$SumTemp = \delta\ (j-2, j-1)\,\varepsilon\,(i-j+3)^2$

$Sum = SumTemp + Sum$

Then vector Ψ will have parameters:

$$\Psi\,(i) = sqrt\left\lfloor \mathbf{\Sigma\_cplann}\,(i)^2\,\delta\,(j-2, j-1)\,/Sum \right\rfloor$$

End of algorithm 9.1

In the algorithm above **Σ_cplann** is a vector of annualized caplet volatilities.

Initial values of Ψ vector for initial values of vector ε are presented in Table 9.6.

Table 9.6 Initial values of vector Ψ

i	$\Psi(i)$
1	0.0000
2	0.0000
3	0.2314
4	0.2305
5	0.2105
6	0.1997
7	0.1918
8	0.1841
9	0.1761
10	0.1676
11	0.1627
12	0.1696
13	0.1648
14	0.1499
15	0.1428
16	0.1358
17	0.1397
18	0.1342
19	0.1284
20	0.1225
21	0.1167

The next step is the generation of the matrix of weights W that will be used in further computations (e.g. for forward swap rates). This is presented by algorithm 9.2 below.

Algorithm 9.2 Generation of weights

```
For i = 1 To NumberOfSwaptionMaturities
If i <= 5 Then n = i
If i = 6 Then n = 7
  //due to the fact, that in our case we have no market volatilities for period 6Y,
    alternatively we may interpolate this value and skip the assignment
If i = 7 Then n = 10
  //due to the fact, that in our case we have no market volatilities
    for periods 8Y and 9Y,
    alternatively we may interpolate this values and skip the assignment
For j = 2 To NumberOfSwaptionUnderlyings
    Sum = 0
    For k = i + 1 To i + j
      Sum_1 = 0
      For 1 = i + 1 To i + j
        Sum_Temp = δ(l − 1, l) · B(l + 1)
        Sum = Sum + Sum_Temp
```

Next l

$W(i, j, k-i) = \delta(k-1, k)B(k+1)/Sum$ // Elements of matrix of weights

Next j

Next i

End of algorithm 9.2

The structure of the matrix of weights W is thus as is presented below.

$$
\mathbf{R} = \begin{bmatrix}
\begin{bmatrix} W(1,2,1) \\ W(1,2,2) \end{bmatrix} & \begin{bmatrix} W(1,3,1) \\ W(1,3,2) \\ W(1,3,3) \end{bmatrix} & \begin{bmatrix} W(1,4,1) \\ W(1,4,2) \\ W(1,4,3) \\ W(1,4,4) \end{bmatrix} & \ldots & \begin{bmatrix} W(1,10,1) \\ W(1,10,2) \\ W(1,10,3) \\ \ldots \\ W(1,10,10) \end{bmatrix} \\
\begin{bmatrix} W(2,2,1) \\ W(2,2,2) \end{bmatrix} & \begin{bmatrix} W(2,3,1) \\ W(2,3,2) \\ W(2,3,3) \end{bmatrix} & \begin{bmatrix} W(2,4,1) \\ W(2,4,2) \\ W(2,4,3) \\ W(2,4,4) \end{bmatrix} & \ldots & \begin{bmatrix} W(2,10,1) \\ W(2,10,2) \\ W(2,10,3) \\ \ldots \\ W(2,10,10) \end{bmatrix} \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
\begin{bmatrix} W(7,2,1) \\ W(7,2,2) \end{bmatrix} & \begin{bmatrix} W(7,3,1) \\ W(7,3,2) \\ W(7,3,3) \end{bmatrix} & \begin{bmatrix} W(7,4,1) \\ W(7,4,2) \\ W(7,4,3) \\ W(7,4,4) \end{bmatrix} & \ldots & \begin{bmatrix} W(7,10,1) \\ W(7,10,2) \\ W(7,10,3) \\ \ldots \\ W(7,10,10) \end{bmatrix}
\end{bmatrix}
$$

Figure 9.1 Structure of matrix W.

We need to create the instantaneous volatility matrix whose values are functions of the parameter ε.

Algorithm 9.3 Instantaneous volatility matrix

For i = 1 To NumberOfGridPoints-2

For j = 1 To i

$\mathbf{\Sigma\_inst}(i, j) = \boldsymbol{\Psi}(i+1) \cdot \varepsilon(i-j+3)$

```
Next j
Next i
```

End of algorithm 9.3

The algorithm for the theoretical LFM swaption prices is presented below. Additionally a minimization function is attached for the purpose of decreasing differences as much as possible between the theoretical and market swaption volatilities.

Algorithm 9.4 LFM theoretical swaption prices

```
MininizationFunction = 0
For i = 1 to NumberOfSwaptionMaturities
  if i = 6 then n = 7
    // this condition resulting from the fact
      that in our case we have no volatilities for periods 6Y, 8Y and 9Y
        elseif i = 7 then n = 10
        else n = i
  end
  for j = 2 to NumberOfSwaptionUnderlyings
      Sum_2 = 0
      for k = 1 to j
        for 1 = 1 to j
          Sum_1 = 0;
          for z = 1 to n
            Sum_1Temp =
            Σ_inst(n + k − 1, z) · Σ_inst(n + 1 − 1, z) · Σ(z, z + 1)
            Sum_1 = Sum_1 + Sum_1Temp
          Next z
            Sum_2Temp =
        Sum_1 · W(i, j, k) · W(i, j, l) · F(k + n + 1) · F(1 + n + 1)·
            cos(θ(k + 2) − θ(1 + 2))/S(i, j)^2 · δ(1, n + 1)
        Sum_2 = Sum_2 + Sum_2Temp
        Next l
      Next k
      Σ_swaption_LFM(i, j) = sort(Sum_2)
      // elements of matrix of LFM swaption volatilities
      Temp = Σ_swaption_LFM(i, j) − Σ_swaption_mkt(i, j)
      // difference between LFM and market swaption volatilities
    MinimizationFunction = MinimizationFunction + Temp;
    // creation of minimization function
  Next j
Next i
```

End of algorithm 9.4

In algorithm 9.4 **F** denotes a vector of forward rates with values $\mathbf{F}(i)$

Table 9.7 Forward rates

i	$\mathbf{F}(i)$	Forward rate $F(0, T_k, T_{k+1y})$ for k
1	0	0
2	0.022972	1Y
3	0.027471	2Y
4	0.031018	3Y
5	0.033995	4Y
6	0.036754	5Y
7	0.039064	6Y
8	0.041521	7Y
9	0.043117	8Y
10	0.045201	9Y
11	0.045342	10Y
12	0.044500	11Y
13	0.045794	12Y
14	0.045739	13Y
15	0.046933	14Y
16	0.047866	15Y
17	0.046906	16Y
18	0.047492	17Y
19	0.048352	18Y
20	0.049212	19Y
21	0.049805	20Y

We can write algorithm 9.4 as a function: MinimizationFunction and together with constraints for $\theta(i)$ we can define an optimization algorithm.

Target function: MinimizationFunction $\rightarrow$ MIN

$$\text{Constraints}: -\frac{\pi}{x} \leq \theta(i) - \theta(i-1) < \frac{\pi}{x}, \text{ where } x \text{ are positive numbers}$$

We now present the results of our computations for the market data from 21/01/2005. We set our restrictions for $\theta(i)$ as

$$-\frac{\pi}{2} \leq \theta(i) - \theta(i-1) < \frac{\pi}{2}.$$

Presented below are the results for vectors $\boldsymbol{\Psi}(i), \varphi(i), \varepsilon(i)$ after running the optimization algorithm.

Table 9.9 presents the theoretical and market volatilities of caps.

We show in detail how the LFM caplet volatilities were computed for periods $1Y \div 2Y$, $2Y \div 3Y, 3Y \div 4Y$.

Table 9.8 Results of optimization for vectors $\mathbf{\Psi}(i), \varphi(i), \varepsilon(i)$

i	$\varepsilon(i)$	$\mathbf{\Psi}(i)$	$\theta(i)$
3	1.0617	0.2180	1.4224
4	1.3808	0.1872	2.1771
5	1.6176	0.1534	1.1391
6	1.2381	0.1491	1.4950
7	0.9211	0.1513	1.5943
8	1.1882	0.1467	1.5815
9	0.4533	0.1500	1.5823
10	0.9412	0.1461	1.5712
11	1.1465	0.1418	1.5740
12	0.7785	0.1520	1.5607
13	0.5637	0.1528	1.5708
14	0.9090	0.1408	1.5708
15	0.9048	0.1357	1.5708
16	0.9387	0.1299	1.5708
17	0.9810	0.1342	1.5708
18	1.0081	0.1291	1.5708
19	1.0125	0.1237	1.5708
20	1.0162	0.1182	1.5708
21	1.0094	0.1128	1.5708

Table 9.9 Theoretical and market volatilities of caps

Tenor	Market caplet volatility	LFM caplet volatility
1Y–2Y	23.1450 %	23.1450 %
2Y–3Y	23.0510 %	23.0510 %
3Y–4Y	21.0307 %	21.0499 %
4Y–5Y	19.9772 %	19.9754 %
5Y–6Y	19.1815 %	19.1780 %
6Y–7Y	18.4074 %	18.4028 %
7Y–8Y	17.6072 %	17.6130 %
8Y–9Y	16.7570 %	16.7657 %
9Y–10Y	16.2764 %	16.2723 %
10Y–11Y	16.9667 %	16.9634 %
11Y–12Y	16.4732 %	16.4700 %
12Y–13Y	14.9914 %	14.9931 %
13Y–14Y	14.2830 %	14.2865 %
14Y–15Y	13.5744 %	13.5770 %
15Y–16Y	13.9757 %	13.9759 %
16Y–17Y	13.4165 %	13.4149 %
17Y–18Y	12.8406 %	12.8372 %
18Y–19Y	12.2463 %	12.2491 %
19Y–20Y	11.6680 %	11.6726 %

Example 9.2 LFM caplet volatilities

For the period $1Y \div 2Y$ we have:

$$\sigma_{LFM}^{cpl}(0, T_{1Y}, T_{2Y}) = \sqrt{\frac{\psi(3)^2\,\varepsilon(3)^2}{\delta_{0,1Y}}} = \sqrt{\frac{0.2180^2 \cdot 1.0617^2}{1}} = 23.1450\,\%$$

For the period $2Y \div 3Y$ we have:

$$\sigma_{LFM}^{cpl}(0, T_{2Y}, T_{3Y}) = \sqrt{\frac{\psi(4)^2\left[\varepsilon(3)^2 + \varepsilon(4)^2\right]}{\delta_{0,2Y}}} = \sqrt{\frac{0.1872^2 \cdot [1.0617^2 + 1.3808^2]}{2}} = 23.0520\,\%$$

For the period $3Y \div 4Y$ we have:

$$\sigma_{LFM}^{cpl}(0, T_{3Y}, T_{4Y}) = \sqrt{\frac{\psi(5)^2\left[\varepsilon(3)^2 + \varepsilon(4)^2 + \varepsilon(5)^2\right]}{\delta_{0,3Y}}}$$

$$= \sqrt{\frac{0.1534^2 \cdot [1.0617^2 + 1.3808^2 + 1.6176^2]}{3}} = 21.0499\,\%$$

End of example 9.2

Table 9.10 presents the results for matrix of weights **W**

Table 9.10 Matrix of Weights

W(1,i,j)	I = 1	2	3	4	5	6	7	8	9	10
j = 2	0.507635	0.492365								
3	0.343916	0.333571	0.322513							
4	0.262115	0.254231	0.245803	0.237850						
5	0.213516	0.207094	0.200228	0.193750	0.185412					
6	0.181174	0.175725	0.169899	0.164402	0.157328	0.151472				
7	0.158206	0.153447	0.148360	0.143560	0.137382	0.132269	0.126777			
8	0.141090	0.136846	0.132309	0.128029	0.122519	0.117959	0.113061	0.108187		
9	0.127793	0.123949	0.119840	0.115962	0.110972	0.106842	0.102405	0.097990	0.094247	
10	0.117294	0.113766	0.109994	0.106436	0.101856	0.098064	0.093993	0.089940	0.086505	0.082152
W(2,i,j)	I = 1	2	3	4	5	6	7	8	9	10
j = 2	0.508428	0.491572								
3	0.344540	0.333118	0.322341							
4	0.263316	0.254586	0.246350	0.235748						
5	0.214606	0.207491	0.200778	0.192138	0.184987					
6	0.182285	0.176242	0.170540	0.163202	0.157127	0.150603				
7	0.159325	0.154043	0.149060	0.142645	0.137336	0.131633	0.125958			
8	0.142109	0.137398	0.132953	0.127231	0.122496	0.117410	0.112347	0.108056		
9	0.128883	0.124611	0.120579	0.115390	0.111095	0.106482	0.101891	0.097999	0.093069	
10	0.118353	0.114429	0.110727	0.105962	0.102018	0.097782	0.093566	0.089992	0.085464	0.081706
...										
W(7,i,j)	I = 1	2	3	4	5	6	7	8	9	10
j = 2	0.511016	0.488984								
3	0.347558	0.332573	0.319869							
4	0.266578	0.255085	0.245340	0.232997						
5	0.218015	0.208615	0.200646	0.190552	0.182172					

Table 9.10 Continued

W(7,i,j)	I = 1	2	3	4	5	6	7	8	9	10
6	0.185584	0.177583	0.170800	0.162206	0.155073	0.148754				
7	0.162500	0.155494	0.149554	0.142030	0.135784	0.130250	0.124388			
8	0.145261	0.138998	0.133688	0.126962	0.121379	0.116432	0.111192	0.106087		
9	0.131828	0.126145	0.121326	0.115222	0.110155	0.105666	0.100910	0.096277	0.092471	
10	0.121238	0.116011	0.111579	0.105966	0.101306	0.097177	0.092804	0.088543	0.085042	0.080334

The matrix of annualized instantaneous volatilities is presented by Table 9.11:

Table 9.11 Matrix of annualized instantaneous volatility

	1Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y	11Y	12Y	13Y	14Y	15Y	16Y	17Y	18Y	19Y
F(t,1Y,2Y)	0.23																		
F(t,2Y,3Y)	0.26	0.20																	
F(t,3Y,4Y)	0.25	0.21	0.16																
F(t,4Y,5Y)	0.18	0.24	0.21	0.16															
F(t,5Y,6Y)	0.14	0.19	0.24	0.21	0.16														
F(t,6Y,7Y)	0.17	0.14	0.18	0.24	0.20	0.16													
F(t,7Y,8Y)	0.07	0.18	0.14	0.19	0.24	0.21	0.16												
F(t,8Y,9Y)	0.14	0.07	0.17	0.13	0.18	0.24	0.20	0.16											
F(t,9Y,10Y)	0.16	0.13	0.06	0.17	0.13	0.18	0.23	0.20	0.15										
F(t,10Y,11Y)	0.12	0.17	0.14	0.07	0.18	0.14	0.19	0.25	0.21	0.16									
F(t,11Y,12Y)	0.09	0.12	0.18	0.14	0.07	0.18	0.14	0.19	0.25	0.21	0.16								
F(t,12Y,13Y)	0.13	0.08	0.11	0.16	0.13	0.06	0.17	0.13	0.17	0.23	0.19	0.15							
F(t,13Y,14Y)	0.12	0.12	0.08	0.11	0.16	0.13	0.06	0.16	0.12	0.17	0.22	0.19	0.14						
F(t,14Y,15Y)	0.12	0.12	0.12	0.07	0.10	0.15	0.12	0.06	0.15	0.12	0.16	0.21	0.18	0.14					
F(t,15Y,16Y)	0.13	0.13	0.12	0.12	0.08	0.10	0.15	0.13	0.06	0.16	0.12	0.17	0.22	0.19	0.14				
F(t,16Y,17Y)	0.13	0.13	0.12	0.12	0.12	0.07	0.10	0.15	0.12	0.06	0.15	0.12	0.16	0.21	0.18	0.14			
F(t,17Y,18Y)	0.13	0.12	0.12	0.12	0.11	0.11	0.07	0.10	0.14	0.12	0.06	0.15	0.11	0.15	0.20	0.17	0.13		
F(t,18Y,19Y)	0.12	0.12	0.12	0.12	0.11	0.11	0.11	0.07	0.09	0.14	0.11	0.05	0.14	0.11	0.15	0.19	0.16	0.13	
F(t,19Y,20Y)	0.11	0.11	0.11	0.11	0.11	0.11	0.10	0.10	0.06	0.09	0.13	0.11	0.05	0.13	0.10	0.14	0.18	0.16	0.12

Let us show an example of the computation for the first six elements of the matrix represented by Table 9.9.

Example 9.3 Annualized instantaneous volatility

(a) Piecewise constant instantaneous volatility $\sigma^{inst}(0, T_{1Y,2Y}, T_{0,1Y})$ of the forward rate $F(t, T_{1Y}, T_{2Y})$ at an interval $0 \div T_{1Y}$ will be computed as:

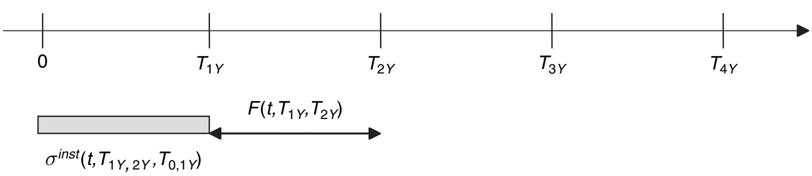

Figure 9.2 Instantaneous volatility of forward rate $F(t, T_{1Y}, T_{2Y})$ at interval $0 \div T_{1Y}$.

$$\mathbf{\Sigma\_inst}\,(1,1) = \mathbf{\Sigma}^{inst}\left(0, T_{1Y,2Y}, T_{0,1Y}\right) = \psi\,(3)\,\varepsilon\,(3) = 0.2180 \cdot 1.0617 = 0.2314$$

(b) Piecewise constant instantaneous volatility $\mathbf{\Sigma}^{inst}(0, T_{2Y,3Y}, T_{0,1Y})$ of the forward rate $F(t, T_{2Y}, T_{3Y})$ at an interval $0 \div T_{1Y}$ will be computed as:

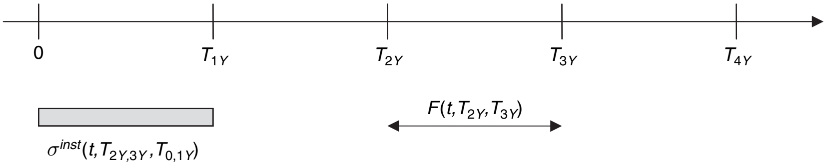

Figure 9.3 Instantaneous volatility of forward rate $F(t, T_{2Y}, T_{3Y})$ at interval $0 \div T_{1Y}$.

$$\mathbf{\Sigma\_inst}\,(2, 1) = \mathbf{\Sigma}^{inst}\left(0, T_{2Y,3Y}, T_{0,1Y}\right) = \psi\,(4)\,\varepsilon\,(4) = 0.1872 \cdot 1.3808 = 0.2584$$

(c) Piecewise constant instantaneous volatility $\boldsymbol{\sigma}^{inst}(0, T_{2Y,3Y}, T_{1Y,2Y})$ of the forward rate $F(t, T_{2Y}, T_{3Y})$ at an interval $T_{1Y} \div T_{2Y}$ will be computed as:

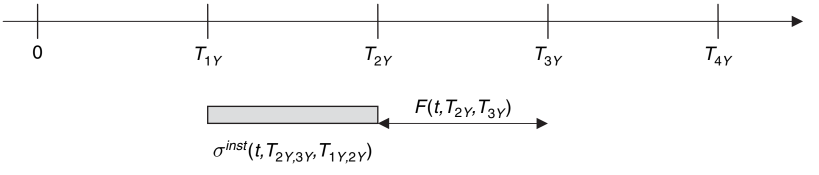

Figure 9.4 Instantaneous volatility of forward rate $F(t, T_{2Y}, T_{3Y})$ at interval $T_{1Y} \div T_{2Y}$.

$$\mathbf{\Sigma\_inst}\,(2, 2) = \mathbf{\Sigma}^{inst}\left(0, T_{2Y,3Y}, T_{1Y,2Y}\right) = \psi\,(4)\,\varepsilon\,(3) = 0.1872 \cdot 1.0617 = 0.1987$$

(d) Piecewise constant instantaneous volatility $\boldsymbol{\sigma}^{inst}(0, T_{3Y,4Y}, T_{0,1Y})$ of the forward rate $F(t, T_{3Y}, T_{4Y})$ at an interval $0 \div T_{1Y}$ will be computed as:

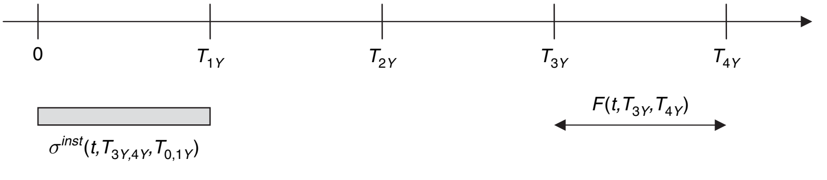

Figure 9.5 Instantaneous volatility of forward rate $F(t, T_{3Y}, T_{4Y})$ at interval $0 \div T_{1Y}$

$$\mathbf{\Sigma\_inst}\,(3, 1) = \mathbf{\Sigma}^{inst}\left(0, T_{3Y,4Y}, T_{0,1Y}\right) = \psi\,(5)\,\varepsilon\,(5) = 0.1534 \cdot 1.6176 = 0.2481$$

(e) Piecewise constant instantaneous volatility $\sigma^{inst}(0, T_{3Y,4Y}, T_{1Y,2Y})$ of the forward rate $F(t, T_{3Y}, T_{4Y})$ at an interval $T_{1Y} \div T_{2Y}$ will be computed as:

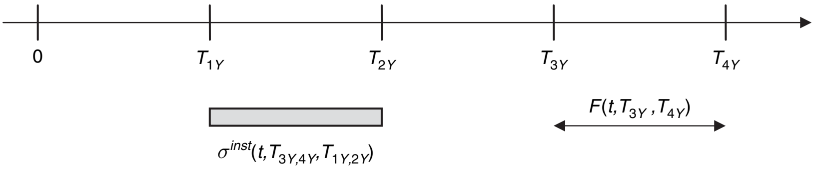

Figure 9.6 Instantaneous volatility of forward rate $F(t, T_{3Y}, T_{4Y})$ at interval $T_{1Y} \div T_{2Y}$.

$$\mathbf{\Sigma\_inst}\,(3,2) = \sigma^{inst}\left(0, T_{3Y,4Y}, T_{1Y,2Y}\right) = \psi\,(5)\,\varepsilon\,(4) = 0.1534 \cdot 1.3808 = 0.2118$$

(f) Piecewise constant instantaneous volatility $\sigma^{inst}(0, T_{3Y,4Y}, T_{2Y,3Y})$ of the forward rate $F(t, T_{3Y}, T_{4Y})$ at an interval $T_{2Y} \div T_{3Y}$ will be computed as:

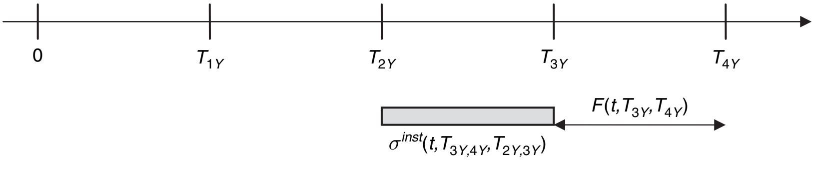

Figure 9.7 Instantaneous volatility of forward rate $F(t, T_{3Y}, T_{4Y})$ at interval $T_{2Y} \div T_{3Y}$.

$$\mathbf{\Sigma\_inst}\,(3,3) = \sigma^{inst}\left(0, T_{3Y,4Y}, T_{2Y,3Y}\right) = \psi\,(5)\,\varepsilon\,(3) = 0.1534 \cdot 1.0617 = 0.1628$$

The other volatilities would be computed in similar manner.

End of example 9.3

Let us move to present the results of the instantaneous correlations. The matrix of instantaneous correlation is presented by Table 9.12 below.

We present an example of the computation for the first three elements of the matrix presented by Table 9.10.

Example 9.4 Instantaneous correlations

(a) Instantaneous correlation $\rho^{inst}(0, T_{1Y,2Y}, T_{2Y,3Y})$ between the forward rates $F(t, T_{1Y}, T_{2Y})$ and $F(t, T_{2Y}, T_{3Y})$ will be computed as:

$$\rho^{inst}\left(t, T_{1Y,2Y}, T_{2Y,3Y}\right) = \cos\left[\theta\,(3) - \theta\,(4)\right] = \cos\left[1.4224 - 2.1771\right] = 0.728$$

Table 9.12 Matrix of instantaneous correlation

	1Y 2Y	2Y 3Y	3Y 4Y	4Y 5Y	5Y 6Y	6Y 7Y	7Y 8Y	8Y 9Y	9Y 10Y	10Y 11Y	11Y 12Y	12Y 13Y	13Y 14Y	14Y 15Y	15Y 16Y	16Y 17Y	17Y 18Y	18Y 19Y	19Y 20Y
F(t,1Y,2Y)	1.000																		
F(t,2Y,3Y)	0.728	1.000																	
F(t,3Y,4Y)	0.960	0.508	1.000																
F(t,4Y,5Y)	0.997	0.776	0.937	1.000															
F(t,5Y,6Y)	0.985	0.835	0.898	0.995	1.000														
F(t,6Y,7Y)	0.987	0.828	0.904	0.996	1.000	1.000													
F(t,7Y,8Y)	0.987	0.828	0.903	0.996	1.000	1.000	1.000												
F(t,8Y,9Y)	0.989	0.822	0.908	0.997	1.000	1.000	1.000	1.000											
F(t,9Y,10Y)	0.989	0.824	0.907	0.997	1.000	1.000	1.000	1.000	1.000										
F(t,10Y,11Y)	0.990	0.816	0.912	0.998	0.999	1.000	1.000	1.000	1.000	1.000									
F(t,11Y,12Y)	0.989	0.822	0.908	0.997	1.000	1.000	1.000	1.000	1.000	1.000	1.000								
F(t,12Y,13Y)	0.989	0.822	0.908	0.997	1.000	1,000	1,000	1,000	1,000	1,000	1,000	1,000							
F(t,13Y,14Y)	0.989	0.822	0.908	0.997	1.000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000						
F(t,14Y,15Y)	0.989	0.822	0.908	0.997	1.000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000					
F(t,15Y,16Y)	0.989	0.822	0.908	0.997	1.000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000				
F(t,16Y,17Y)	0.989	0.822	0.908	0.997	1.000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000			
F(t,17Y,18Y)	0.989	0.822	0.908	0.997	1.000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000		
F(t,18Y,19Y)	0.989	0.822	0.908	0.997	1.000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	
F(t,19Y,20Y)	0.989	0.822	0.908	0.997	1.000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000

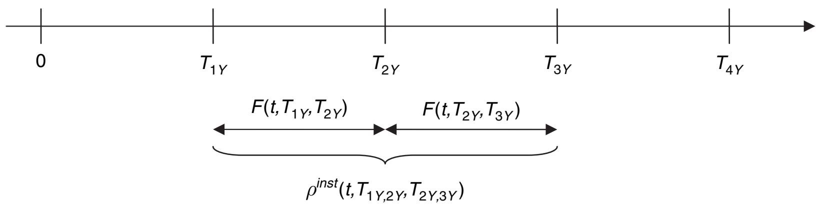

Figure 9.8 Instantaneous correlation between forward rates $F(t, T_{1Y}, T_{2Y})$ and $F(t, T_{2Y}, T_{3Y})$.

(b) Instantaneous correlation $\rho^{inst}(0, T_{1Y,2Y}, T_{3Y,4Y})$ between the forward rates $F(t, T_{1Y}, T_{2Y})$ and $F(t, T_{3Y}, T_{4Y})$ will be computed as:

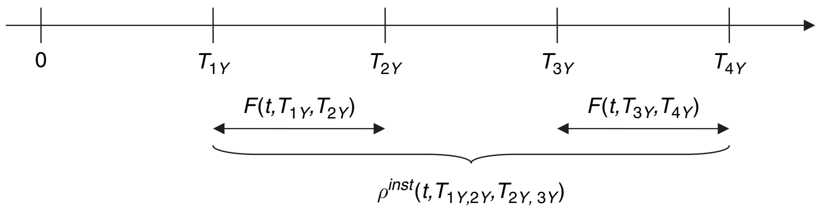

Figure 9.9 Instantaneous correlation between forward rates $F(t, T_{1Y}, T_{2Y})$ and $F(t, T_{3Y}, T_{4Y})$.

$$\rho^{inst}\left(t, T_{1Y,2Y}, T_{3Y,4Y}\right) = \cos\left[\theta(3) - \theta(5)\right] = \cos\left[1.4224 - 1.1391\right] = 0.960$$

(c) Instantaneous correlation $\rho^{inst}(0, T_{2Y,3Y}, T_{3Y,4Y})$ between the forward rates $F(t, T_{2Y}, T_{3Y})$ and $F(t, T_{3Y}, T_{4Y})$ will be computed as:

$$\rho^{inst}\left(t, T_{2Y,3Y}, T_{3Y,4Y}\right) = \cos\left[\theta(4) - \theta(5)\right] = \cos\left[2.1771 - 1.1391\right] = 0.508$$

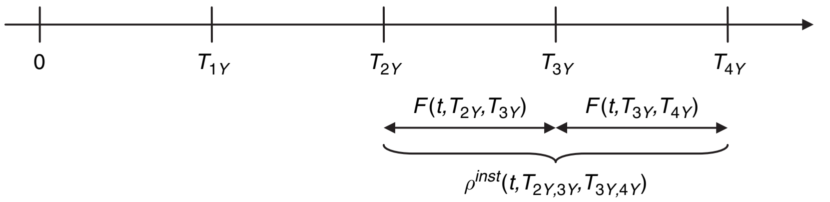

Figure 9.10 Instantaneous correlation between forward rates $F(t, T_{2Y}, T_{3Y})$ and $F(t, T_{3Y}, T_{4Y})$.

End of example 9.4

Table 9.13 presents the theoretical and market volatilities of swaptions together with differences between them after calibration has been done. The results can be regarded as a test of quality of the calibration.

Table 9.13 Theoretical and market volatilities of swaptions

Market	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y
1Y	23.0000 %	22.1000 %	20.9000 %	19.6000 %	18.6000 %	17.6000 %	16.9000 %	16.3000 %	15.9000 %
2Y	21.5000 %	20.5000 %	19.4000 %	18.3000 %	17.4000 %	16.7000 %	16.2000 %	15.8000 %	15.4000 %
3Y	20.1000 %	19.0000 %	18.0000 %	17.0000 %	16.3000 %	15.8000 %	15.5000 %	15.2000 %	15.0000 %
4Y	18.7000 %	17.7000 %	16.8000 %	16.0000 %	15.5000 %	15.1000 %	14.8000 %	14.7000 %	14.5000 %
5Y	17.4000 %	16.5000 %	15.8000 %	15.1000 %	14.8000 %	14.5000 %	14.3000 %	14.2000 %	14.0000 %
7Y	16.0800 %	15.3000 %	14.6800 %	14.1400 %	13.9200 %	13.7000 %	13.5800 %	13.4800 %	13.3600 %
10Y	14.1000 %	13.5000 %	13.0000 %	12.7000 %	12.6000 %	12.5000 %	12.5000 %	12.4000 %	12.4000 %
Theoretical	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y
1Y	22.8456 %	22.2082 %	21.1750 %	19.6068 %	19.2116 %	17.3138 %	16.8400 %	16.7739 %	16.3004 %
2Y	21.4321 %	20.2366 %	19.1965 %	18.4057 %	17.3602 %	16.2795 %	16.0939 %	15.9419 %	15.4138 %
3Y	19.5268 %	18.3651 %	17.8254 %	16.8311 %	16.1050 %	15.5008 %	15.3907 %	15.1239 %	14.7108 %
4Y	18.4133 %	17.4195 %	16.6899 %	15.8987 %	15.4312 %	15.0399 %	14.8372 %	14.5533 %	14.2019 %
5Y	17.6008 %	16.4634 %	15.8282 %	15.2690 %	15.0064 %	14.5916 %	14.3339 %	14.0665 %	13.7521 %
7Y	15.8337 %	14.9355 %	14.8172 %	14.4852 %	14.0973 %	13.6810 %	13.4305 %	13.2774 %	13.0863 %
10Y	15.3569 %	14.2214 %	13.7411 %	13.2651 %	13.0214 %	12.7576 %	12.5968 %	12.4248 %	12.2293 %
Difference	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y
1Y	0.6713 %	−0.4897 %	−1.3158 %	−0.0349 %	−3.2882 %	1.6260 %	0.3549 %	−2.9076 %	−2.5183 %
2Y	0.3158 %	1.2848 %	1.0491 %	−0.5774 %	0.2289 %	2.5177 %	0.6551 %	−0.8981 %	−0.0894 %
3Y	2.8517 %	3.3415 %	0.9702 %	0.9935 %	1.1962 %	1.8937 %	0.7052 %	0.5007 %	1.9279 %
4Y	1.5331 %	1.5849 %	0.6556 %	0.6330 %	0.4437 %	0.3977 %	−0.2516 %	0.9976 %	2.0558 %
5Y	−1.1543 %	0.2218 %	−0.1786 %	−1.1190 %	−1.3944 %	−0.6319 %	−0.2368 %	0.9403 %	1.7708 %
7Y	1.5315 %	2.3825 %	−0.9349 %	−2.4417 %	−1.2738 %	0.1385 %	1.1012 %	1.5027 %	2.0486 %
10Y	−8.9138 %	−5.3435 %	−5.7007 %	−4.4495 %	−3.3444 %	−2.0610 %	−0.7740 %	−0.1998 %	1.3769 %

We present some examples of the theoretical swaption results. We will concentrate on four swaptions: $1Y \times 2Y$, $1Y \times 3Y$, $2Y \times 2Y$, $2Y \times 3Y$.

Example 9.5 LFM swaption volatilities

(a) Swaption $1Y \times 2Y$

Figure 9.11 presents graphically all components we need to compute the LFM swaption volatility.

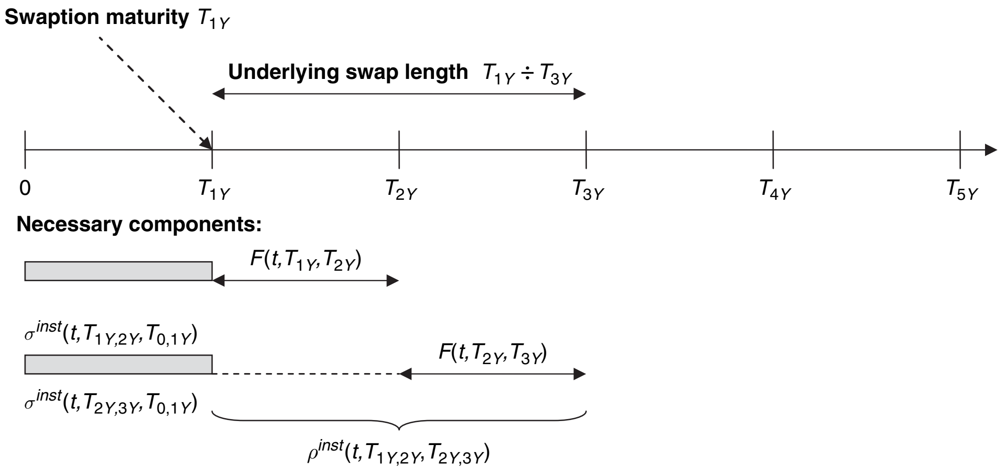

Figure 9.11 LFM swaption volatility for $T_{1Y} \div T_{3Y}$ (swaption 1Y × 2Y).

Thus the swaption volatility will be computed as:

$$\begin{aligned}\sigma_{LFM}^{swpt}\left(S_{1Y\times 2Y}(0)\right) = \Big\{ \big\lfloor & \mathbf{W}(1,2,1)^2 F(0,T_{1Y},T_{2Y})^2\, \mathbf{\Sigma\_inst}(1,1)^2 \\ & + \mathbf{W}(1,2,2)^2 F(0,T_{2Y},T_{3Y})^2\, \mathbf{\Sigma\_inst}(2,1)^2 \\ & + 2\mathbf{W}(1,2,1)\,\mathbf{W}(1,2,2)\,F(0,T_{1Y},T_{2Y})\,F(0,T_{2Y},T_{3Y}) \\ & \cos\left[\theta(3)-\theta(4)\right]\mathbf{\Sigma\_inst}(1,1)\,\mathbf{\Sigma\_inst}(2,1)\big]/\left[\delta_{0,1Y} S_{1Y\times 2Y}(0)\right]\Big\}^{\frac{1}{2}}\end{aligned}$$

Using real market data we obtain:

$$\begin{aligned}\sigma_{LFM}^{swpt}\left(S_{1Y\times 2Y}(0)\right) = \{ \lfloor & 0.507635^2 \cdot 0.027471^2 \cdot 0.2314^2 + 0.492365^2 \cdot 0.031018^2 \cdot 0.2584^2 \\ & + 2 \cdot 0.507635 \cdot 0.492365 \cdot 0.027471 \cdot 0.031018 \\ & \cdot \cos\left[1.4224 - 2.1771\right] \cdot 0.2314 \cdot 0.2584\right]/[1 \cdot 0.0292]\}^{\frac{1}{2}} = 22.8456\,\%\end{aligned}$$

(b) Swaption $1Y \times 3Y$

Figure 9.12 presents graphically all components we need to compute the LFM swaption volatility.

The swaption volatility will be computed as:

$$\begin{aligned}\sigma_{LFM}^{swpt}\left(S_{1Y\times 3Y}(0)\right) = \Big\{ \big\lfloor & \mathbf{W}(1,3,1)^2 F(0,T_{1Y},T_{2Y})^2\, \mathbf{\Sigma\_inst}(1,1)^2 \\ & + \mathbf{W}(1,3,2)^2 F(0,T_{2Y},T_{3Y})^2\, \mathbf{\Sigma\_inst}(2,1)^2 \\ & + \mathbf{W}(1,3,3)^2 F(0,T_{3Y},T_{4Y})^2\, \mathbf{\Sigma\_inst}(3,1)^2 \\ & + 2\mathbf{W}(1,3,1)\,\mathbf{W}(1,3,2)\,F(0,T_{1Y},T_{2Y})\,F(0,T_{2Y},T_{3Y}) \\ & \times \cos\left[\theta(3)-\theta(4)\right]\mathbf{\Sigma\_inst}(1,1)\,\mathbf{\Sigma\_inst}(2,1)\end{aligned}$$

$$
\begin{aligned}
&+2\mathbf{W}(1,3,1)\,\mathbf{W}(1,3,3)\,F(0,T_{1Y},T_{2Y})\,F(0,T_{3Y},T_{4Y})\\
&\times\cos[\theta(3)-\theta(5)]\,\boldsymbol{\Sigma}\_\mathbf{inst}(1,1)\,\boldsymbol{\Sigma}\_\mathbf{inst}(3,1)\\
&+2\mathbf{W}(1,3,2)\,\mathbf{W}(1,3,3)\,F(0,T_{2Y},T_{3Y})\,F(0,T_{3Y},T_{4Y})\\
&\times\cos[\theta(4)-\theta(5)]\,\boldsymbol{\Sigma}\_\mathbf{inst}(2,1)\,\boldsymbol{\Sigma}\_\mathbf{inst}(3,1)]/\left[\delta_{0,1Y}S_{1Y\times 3Y}(0)\right]\Big\}^{\frac{1}{2}}
\end{aligned}
$$

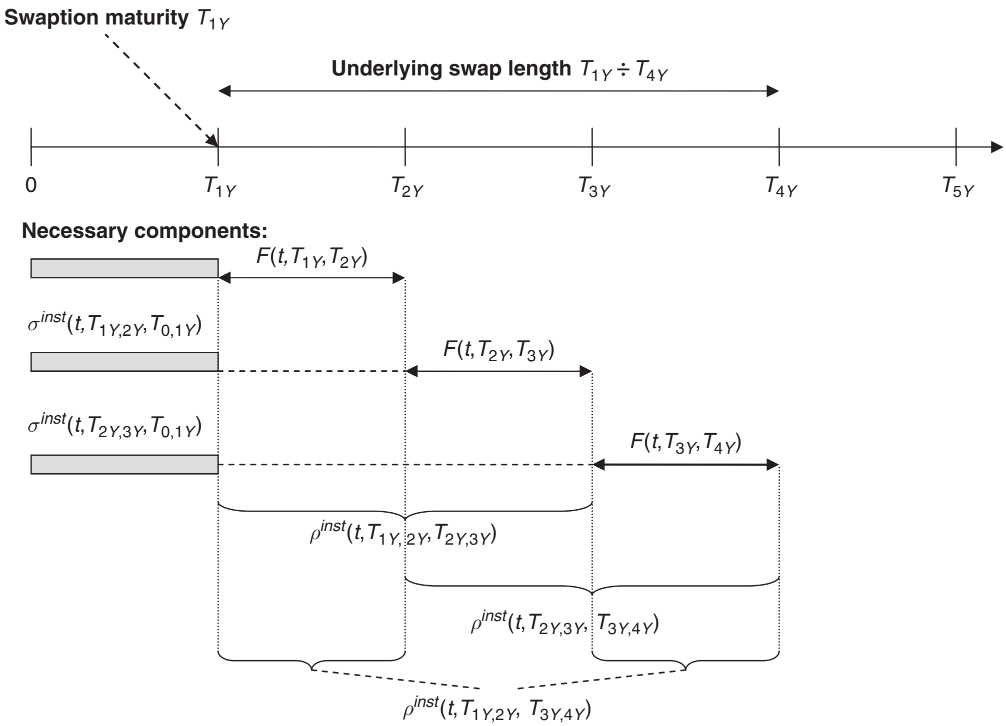

Figure 9.12 LFM swaption volatility for $T_{1Y} \div T_{4Y}$ (swaption $1Y \times 3Y$).

Using the real market data we obtain:

$$
\begin{aligned}
\sigma_{LFM}^{swpt}(S_{1Y\times 3Y}(0)) = \Big\{ \lfloor & 0.343916^2\cdot 0.027471^2\cdot 0.2314^2 + 0.333571^2\cdot 0.031018^2\\
&\cdot 0.2584^2 + 0.322513^2\cdot 0.033995^2\cdot 0.2481^2 + 2\cdot 0.343916\\
&\cdot 0.333571\cdot 0.027471\cdot 0.031018\cdot\cos[1.4224-2.1771]\cdot 0.2314\\
&\cdot 0.2584 + 2\cdot 0.343916\cdot 0.322513\cdot 0.027471\cdot 0.033995\\
&\cdot\cos[1.4224-1.1391]\cdot 0.2314\cdot 0.2481 + 2\cdot 0.333571\cdot 0.322513\\
&\cdot 0.031018\cdot 0.033995\cdot\cos[2.1771-1.1391]\cdot 0.2584\cdot 0.2481]/\\
&[1\cdot 0.0308]\Big\}^{\frac{1}{2}} = 22.2082\,\%
\end{aligned}
$$

(c) Swaption $2Y \times 2Y$

Figure 9.13 presents graphically all components we need to compute the third LFM swaption volatility.

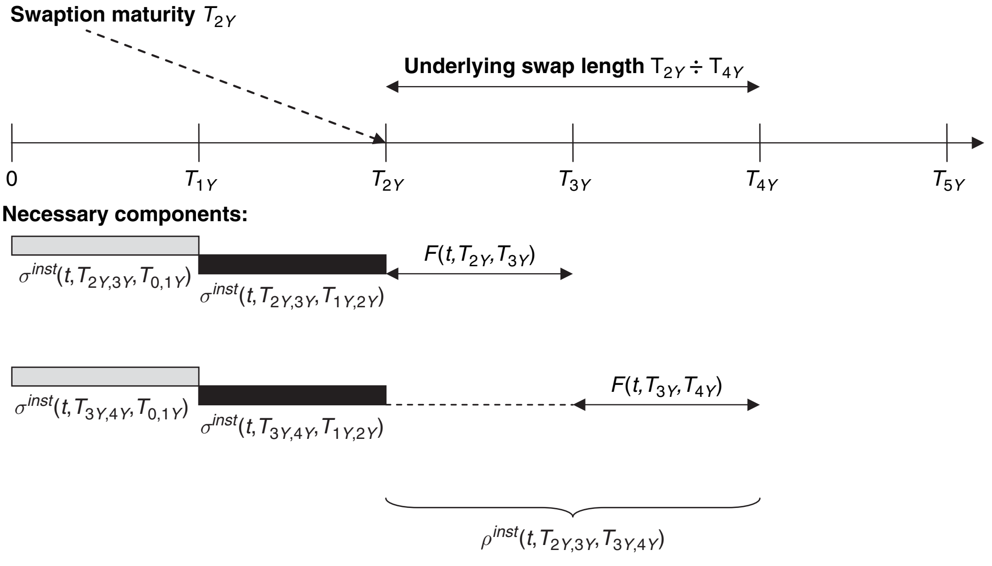

Figure 9.13 LFM swaption volatility for $T_{2Y} \div T_{4Y}$ (swaption $2Y \times 2Y$).

Thus we can compute the swaption volatility by:

$$\begin{aligned}\sigma_{LFM}^{swpt}(S_{2Y\times 2Y}(0)) = \Big\{ \Big\lfloor & \mathbf{W}(1,2,1)^2 F(0,T_{1Y},T_{2Y})^2 \Big\lfloor \mathbf{\Sigma\_inst}(2,1)^2 + \mathbf{\Sigma\_inst}(2,2)^2 \Big\rfloor \\ & + \mathbf{W}(2,2,2)^2 F(0,T_{3Y},T_{4Y})^2 \Big[\mathbf{\Sigma\_inst}(3,1)^2 + \mathbf{\Sigma\_inst}(3,2)^2 \Big] \\ & + 2\mathbf{W}(2,2,1)\,\mathbf{W}(2,2,2)\,F(0,T_{2Y},T_{3Y})\,F(0,T_{3Y},T_{4Y}) \\ & \times \cos[\theta(3) - \theta(4)] \cdot (\mathbf{\Sigma\_inst}(2,1)\,\mathbf{\Sigma\_inst}(3,1) \\ & + \mathbf{\Sigma\_inst}(2,2)\,\mathbf{\Sigma\_inst}(3,2))] / \big[\delta_{0,2Y} S_{2Y\times 2Y}(0)\big] \Big\}^{\frac{1}{2}}\end{aligned}$$

Finally, using the real market data we obtain:

$$\begin{aligned}\sigma_{LFM}^{swpt}(S_{2Y\times 2Y}(0)) = \Big\{ \big[& 0.508428^2 \cdot 0.031018^2 \cdot \big[0.2584^2 + 0.1987^2\big] + 0.491572^2 \\ & \cdot 0.033995^2 \cdot \big[0.2481^2 + 0.2118^2\big] + 2 \cdot 0.508428 \cdot 0.491572 \\ & \cdot 0.031018 \cdot 0.033995 \cdot \cos[1.4224 - 2.1771] \\ & \cdot (0.2584 \cdot 0.2481 + 0.1987 \cdot 0.2118)\big] / [2 \cdot 0.0325] \Big\}^{\frac{1}{2}} = 21.4321\,\%\end{aligned}$$

(d) Swaption $2Y \times 3Y$

Figure 9.14 presents graphically all components we need to compute our final LFM swaption volatility example.

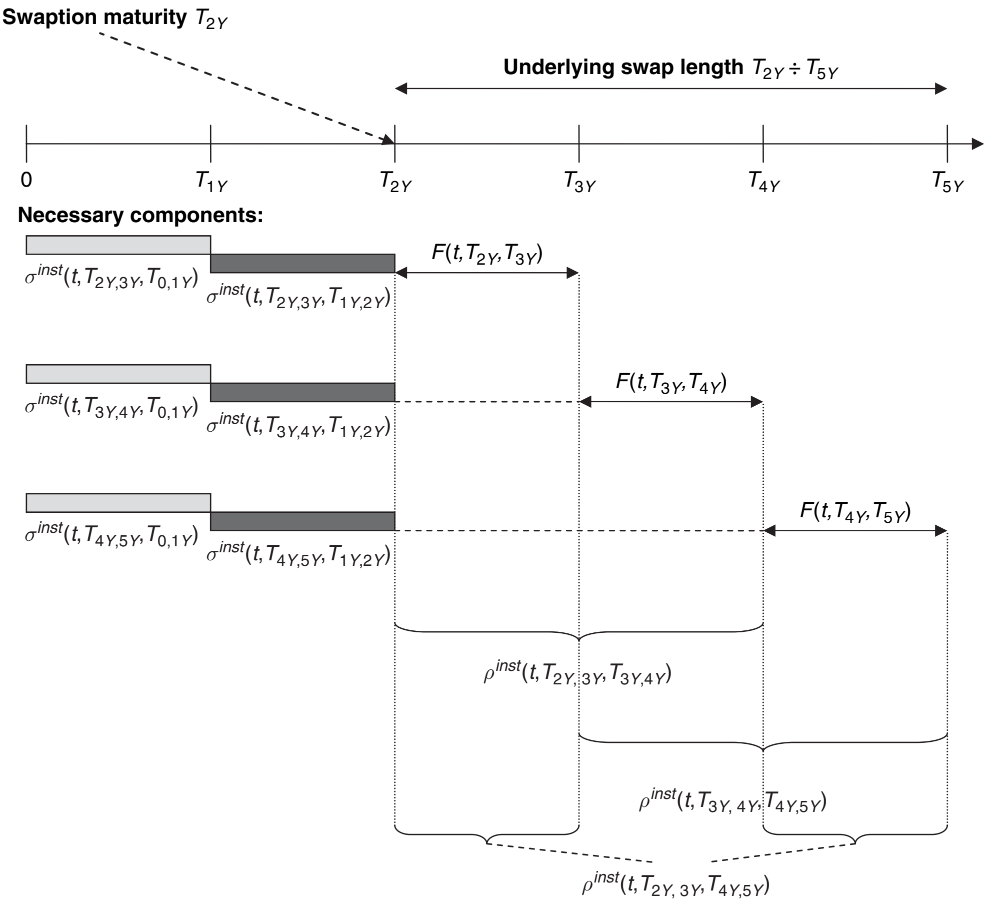

Figure 9.14 LFM swaption volatility for $T_{2Y} \div T_{5Y}$ (swaption $2Y \times 3Y$).

The final swaption volatility is given by:

$$
\begin{aligned}
\sigma_{LFM}^{swpt}(S_{2Y\times 3Y}(0)) = \Big\{ \Big\lfloor & \mathbf{W}(2,3,1)^2 F(0,T_{2Y},T_{3Y})^2 \lfloor \mathbf{\Sigma\_inst}(2,1)^2 + \mathbf{\Sigma\_inst}(2,2)^2 \rfloor \\
& + \mathbf{W}(2,3,2)^2 F(0,T_{3Y},T_{4Y})^2 \left[\mathbf{\Sigma\_inst}(3,1)^2 + \mathbf{\Sigma\_inst}(3,2)^2 \right] \\
& + \mathbf{W}(2,3,3)^2 F(0,T_{4Y},T_{5Y})^2 \left[\mathbf{\Sigma\_inst}(4,1)^2 + \mathbf{\Sigma\_inst}(4,2)^2 \right] \\
& + 2\mathbf{W}(2,3,1)\,\mathbf{W}(2,3,2)\, F(0,T_{2Y},T_{3Y})\, F(0,T_{3Y},T_{4Y}) \cos[\theta(3) - \theta(4)] \\
& \cdot (\mathbf{\Sigma\_inst}(2,1)\,\mathbf{\Sigma\_inst}(3,1) + \mathbf{\Sigma\_inst}(2,2)\,\mathbf{\Sigma\_inst}(3,2))
\end{aligned}
$$

$$+2\mathbf{W}(2,3,1)\,\mathbf{W}(2,3,3)\,F(0,T_{2Y},T_{3Y})\,F(0,T_{4Y},T_{5Y})\cos[\theta(3)-\theta(5)]$$
$$\cdot(\mathbf{\Sigma\_inst}(2,1)\,\mathbf{\Sigma\_inst}(4,1)+\mathbf{\Sigma\_inst}(2,2)\,\mathbf{\Sigma\_inst}(4,2))$$
$$+2\mathbf{W}(2,3,2)\,\mathbf{W}(2,3,3)\,F(0,T_{3Y},T_{4Y})\,F(0,T_{4Y},T_{5Y})\cos[\theta(4)-\theta(5)]$$
$$\cdot(\mathbf{\Sigma\_inst}(3,1)\,\mathbf{\Sigma\_inst}(4,1)$$
$$+\mathbf{\Sigma\_inst}(3,2)\,\mathbf{\Sigma\_inst}\;(4,2))]/\left[\delta_{0,2Y}S_{2Y\times 3Y}(0)\right]\Big\}^{\frac{1}{2}}$$

Thus we obtain:

$$\sigma^{swpt}_{LFM}(S_{2Y\times 3Y}(0))=\big\{\big[0.344540^2\cdot 0.031018^2\cdot\left[0.2584^2+0.1987^2\right]$$
$$+0.333118^2\cdot 0.033995^2\cdot\left[0.2481^2+0.2118^2\right]$$
$$+0.322341^2\cdot 0.036754^2\cdot\left[0.1846^2+0.2411^2\right]$$
$$+2\cdot 0.344540\cdot 0.333118\cdot 0.031018\cdot 0.033995\cdot\cos[1.4224-2.1771]$$
$$\cdot(0.2584\cdot 0.2481+0.1987\cdot 0.2118)$$
$$+2\cdot 0.344540\cdot 0.322341\cdot 0.031018\cdot 0.036754\cdot\cos[1.4224-1.1391]$$
$$\cdot(0.2584\cdot 0.1846+0.1987\cdot 0.2411)$$
$$+2\cdot 0.333118\cdot 0.322341\cdot 0.033995\cdot 0.036754\cdot\cos[2.1771-1.1391]$$
$$\cdot\;(0.2481\cdot 1.1846+0.2118\cdot 0.2411)\big]/[2\cdot 0.0339]\big\}^{\frac{1}{2}}$$

End of example 9.5

The next part of the chapter will describe the different parametric methods for calibration to caps and swaptions.

9.3 PARAMETRIC METHOD OF CALIBRATION

This section is in two parts. The first describes the parametric calibration to caps. The second will use the results and apply them for the parametric calibration to swaptions.

9.3.1 Parametric calibration to cap prices

The purpose of this chapter is to describe a detailed algorithm allowing a calibration of the LIBOR Market Model parametric calibration to cap prices.

Step 1

Derivation of caplet prices from cap prices.

Such a derivation was done in previous sections. Thus it is not necessary to present algorithm once again. The table below presents the quarterly caplet volatilities $\boldsymbol{\sigma}^{cpl}(T_0, T_{i-3M}, T_i)$ for different maturities. All maturities are expressed using day count fractions using Act/360 base convention.

Table 9.14 Caplet implied volatilities

T_i	$\frac{(T_i - T_0)}{360}$	$\sigma^{caplet}(T_0, T_{i-3M}, T_i)$	T_i	$\frac{(T_i - T_0)}{360}$	$\Sigma^{caplet}(T_0, T_{i-3M}, T_i)$
6M	0.5028	0.1641	5Y 6M	5.5778	0.1902
9M	0.7583	0.1641	5Y 9M	5.8306	0.1879
1Y	1.0139	0.1641	6Y	6.0861	0.1859
1Y 3M	1.2639	0.2015	6Y 3M	6.3361	0.1844
1Y 6M	1.5167	0.2189	6Y 6M	6.5889	0.1824
1Y 9M	1.7722	0.2365	6Y 9M	6.8444	0.1804
2Y	2.0278	0.2550	7Y	7.1000	0.1781
2Y 3M	2.2778	0.2212	7Y 3M	7.3528	0.1766
2Y 6M	2.5306	0.2255	7Y 6M	7.6056	0.1743
2Y 9M	2.7861	0.2298	7Y 9M	7.8611	0.1724
3Y	3.0417	0.2341	8Y	8.1167	0.1700
3Y 3M	3.2944	0.2097	8Y 3M	8.3667	0.1677
3Y 6M	3.5472	0.2083	8Y 6M	8.6194	0.1657
3Y 9M	3.8083	0.2077	8Y 9M	8.8750	0.1637
4Y	4.0611	0.2051	9Y	9.1361	0.1622
4Y 3M	4.3139	0.2007	9Y 3M	9.3806	0.1623
4Y 6M	4.5667	0.1982	9Y 6M	9.6333	0.1612
4Y 9M	4.8194	0.1959	9Y 9M	9.8944	0.1599
5Y	5.0722	0.1938	10Y	10.1472	0.1570
5Y 3M	5.3250	0.1925			

Step 2

Having the caplet volatilities $\sigma^{caplet}(T_0, T_{i-3M}, T_i)$ we need to multiply the time to maturities (expresses as day count fractions) by the squared implied caplet volatility. Our results are presented in Table 9.15.

Step 3

We need to find parameters v_1, v_2, v_3, v_4 using optimization algorithms. First let us define the set of functions:

$$f(T_i - t) = \left| v_1 + \left[v_2 + v_3 \frac{T_i - t}{360} \right] e^{-v_4 \frac{T_i - t}{360}} \right| \tag{9.12}$$

for $T_i = T_{3M}, T_{9M}, T_{1Y}, \ldots, T_{10Y}$ respectively.

Having that we will compute integrals of $f(T_i - t)^2$ as:

$$I(T_i - t)^2 = \int_0^{T_i} \left| v_1 + \left[v_2 + v_3 \frac{T_i - t}{360} \right] e^{-v_4 \frac{T_i - t}{360}} \right|^2 dt - \int_0^{T_{i-1}} \left| v_1 + \left[v_2 + v_3 \frac{T_i - t}{360} \right] e^{-v_4 \frac{T_i - t}{360}} \right|^2 dt \tag{9.13}$$

for $T_i = T_{6M}, T_{9M}, T_{1Y}, \ldots, T_{10Y}$ respectively.

Next let us define:

$$f_{FO}(T_i) = \sum_{T_i} I(T_i - t)^2 \tag{9.14}$$

Table 9.15 Squared caplet implied volatilities multiplied by time to maturity

T_i	$\frac{(T_i - T_0)}{360}\sigma^{caplet}(T_0, T_{i-3M}, T_i)^2$	T_i	$\frac{T_i - T_0}{360}\Sigma^{caplet}(T_0, T_{i-3M}, T_i)^2$
6M	0.0135392	5Y 6M	0.2017819
9M	0.0204210	5Y 9M	0.2058560
1Y	0.0273028	6Y	0.2103288
1Y 3M	0.0513167	6Y 3M	0.2154491
1Y 6M	0.0726744	6Y 6M	0.2192108
1Y 9M	0.0991244	6Y 9M	0.2227467
2Y	0.1318563	7Y	0.2252092
2Y 3M	0.1114504	7Y 3M	0.2293152
2Y 6M	0.1286794	7Y 6M	0.2310605
2Y 9M	0.1471291	7Y 9M	0.2336461
3Y	0.1666919	8Y	0.2345717
3Y 3M	0.1448702	8Y 3M	0.2352982
3Y 6M	0.1539100	8Y 6M	0.2366597
3Y 9M	0.1642888	8Y 9M	0.2378295
4Y	0.1708347	9Y	0.2403605
4Y 3M	0.1737656	9Y 3M	0.2470959
4Y 6M	0.1793935	9Y 6M	0.2503264
4Y 9M	0.1849549	9Y 9M	0.2529813
5Y	0.1905048	10Y	0.2501189
5Y 3M	0.1973245		

Having that our minimization function will be:

$$f_{\min} = \sqrt{\sum_{T_i}\left(\left[\frac{T_i - T_0}{360}\sigma^{caplet}(T_0, T_{i-3M}, T_i)^2\right] - [f_{FO}(T_i)]\right)^2} \to \min \tag{9.15}$$

Running the optimization starting from initial values: $v_1, v_2, v_3, v_4 = 0.1$ we obtain the values: $v_1 = 0.112346$, $v_2 = -0.441811$, $v_3 = 0.971559$, $v_4 = 1.223058$, $f_{\min} = 0.0436646$. Table 9.16 presents the results of our computations.

Step 4

Let us define the function

$$\varepsilon(T_i - T_0) = g_1 + g_2\cos\left[g_3\frac{T_i - T_0}{360}\right] \tag{9.16}$$

Next let us define correction factor

$$corr(T_i - T_0) = [1 + \varepsilon(T_i - T_0)]\, I(T_i - t)^2 \tag{9.17}$$

And

$$\begin{aligned} f_{SO}(T_i) &= \sum_{T_i}[1 + \varepsilon(T_i - T_0)]\, I(T_i - t)^2 = \\ &\sum_{T_i}\left\{[1 + g_1 + g_2\cos(g_3(T_i - T_0))]\, I(T_i - t)^2\right\} \end{aligned} \tag{9.18}$$

Table 9.16 Results for step 3 of parametric calibration

T_i	$f(T_i - T_0)$	$\int_0^{T_i} f(T_i - t)^2\,dt$	$f_{FO}(T_i)$	T_i	$f(T_i - T_0)$	$\int_0^{T_i} f(T_i - t)^2\,dt$	$f_{FO}(T_i)$
6M	0.137578	0.010484	0.010484	5Y 6M	0.117769	0.003552	0.199307
9M	0.229015	0.009274	0.019758	5Y 9M	0.116523	0.003467	0.202775
1Y	0.269545	0.016331	0.036088	6Y	0.115547	0.003440	0.206214
1Y 3M	0.279902	0.019118	0.055207	6Y 3M	0.114809	0.003316	0.209530
1Y 6M	0.273766	0.019513	0.074720	6Y 6M	0.114231	0.003314	0.212844
1Y 9M	0.258856	0.018194	0.092914	6Y 9M	0.113783	0.003321	0.216165
2Y	0.240319	0.015946	0.108860	7Y	0.113439	0.003298	0.219463
2Y 3M	0.221586	0.013338	0.122198	7Y 3M	0.113179	0.003245	0.222708
2Y 6M	0.203654	0.011421	0.133619	7Y 6M	0.112980	0.003232	0.225940
2Y 9M	0.187368	0.009757	0.143376	7Y 9M	0.112826	0.003257	0.229198
3Y	0.173246	0.008296	0.151671	8Y	0.112709	0.003250	0.232448
3Y 3M	0.161419	0.007066	0.158737	8Y 3M	0.112622	0.003173	0.235621
3Y 6M	0.151575	0.006180	0.164918	8Y 6M	0.112555	0.003204	0.238825
3Y 9M	0.143257	0.005665	0.170583	8Y 9M	0.112504	0.003236	0.242061
4Y	0.136747	0.004947	0.175530	9Y	0.112464	0.003304	0.245365
4Y 3M	0.131513	0.004542	0.180072	9Y 3M	0.112436	0.003091	0.248456
4Y 6M	0.127338	0.004229	0.184301	9Y 6M	0.112414	0.003195	0.251651
4Y 9M	0.124027	0.003989	0.188290	9Y 9M	0.112397	0.003299	0.254950
5Y	0.121417	0.003804	0.192094	10Y	0.112384	0.003193	0.258143
5Y 3M	0.119369	0.003661	0.195755				

Having that our minimization function will then be:

$$\tilde{f}_{\min} = \sqrt{\sum_{T_i}\left(\left[\frac{T_i - T_0}{360}\sigma^{caplet}(T_0, T_{i-3M}, T_i)^2\right] - \left[\sum_{T_i}[1 + g_1 + g_2\cos(g_3(T_i - T_0))]\,I(T_i - t)^2\right]\right)^2} \rightarrow \min \tag{9.19}$$

Running the optimization starting from initial values: $g_1, g_2, g_3 = 0.1$ we obtain values: $g_1 = -7.02054$, $g_2 = 7.027038$, $g_3 = 0.012987$, $\tilde{f}_{\min} = 0.043502$. Table 9.17 below displays the results of our computations.

Table 9.17 Results for step 4 of parametric calibration

T_i	$\varepsilon(T_i - T_0)$	$corr(T_i - T_0)$	$f_{SO}(T_i)$	T_i	$\varepsilon(T_i - T_0)$	$corr(T_i - T_0)$	$f_{SO}(T_i)$
6M	0.006343	0.010551	0.010551	5Y 6M	−0.011935	0.003509	0.199764
9M	0.006152	0.009331	0.019881	5Y 9M	−0.013643	0.003420	0.203184
1Y	0.005884	0.016427	0.036308	6Y	−0.015445	0.003387	0.206570
1Y 3M	0.005546	0.019225	0.055533	6Y 3M	−0.017284	0.003258	0.209829
1Y 6M	0.005130	0.019613	0.075145	6Y 6M	−0.019218	0.003251	0.213079
1Y 9M	0.004632	0.018278	0.093424	6Y 9M	−0.021249	0.003250	0.216330
2Y	0.004057	0.016011	0.109435	7Y	−0.023358	0.003221	0.219551
2Y 3M	0.003419	0.013384	0.122819	7Y 3M	−0.025520	0.003162	0.222713
2Y 6M	0.002699	0.011451	0.134270	7Y 6M	−0.027757	0.003142	0.225856
2Y 9M	0.001894	0.009775	0.144045	7Y 9M	−0.030095	0.003159	0.229015
3Y	0.001011	0.008304	0.152349	8Y	−0.032511	0.003144	0.232159
3Y 3M	0.000062	0.007067	0.159416	8Y 3M	−0.034948	0.003062	0.235221
3Y 6M	−0.000962	0.006175	0.165590	8Y 6M	−0.037487	0.003084	0.238305
3Y 9M	−0.002100	0.005653	0.171243	8Y 9M	−0.040131	0.003106	0.241412
4Y	−0.003278	0.004931	0.176174	9Y	−0.042911	0.003162	0.244573
4Y 3M	−0.004532	0.004521	0.180695	9Y 3M	−0.045587	0.002950	0.247524
4Y 6M	−0.005861	0.004205	0.184900	9Y 6M	−0.048428	0.003040	0.250564
4Y 9M	−0.007267	0.003960	0.188860	9Y 9M	−0.051442	0.003129	0.253693
5Y	−0.008747	0.003771	0.192631	10Y	−0.054435	0.003019	0.256712
5Y 3M	−0.010303	0.003624	0.196254				

Step 5

In the calibration to swaptions the delta function will be used. The function will be defined as:

$$\Delta\left(T_i\right)=\frac{\frac{T_i-T_0}{360}\sigma^{caplet}\left(T_0,T_{i-3M},T_i\right)^2}{f_{SO}\left(T_i\right)}-1 \tag{9.20}$$

Table 9.18 presents the results of our computations of step 5.

Table 9.18 Results for step 5 of parametric calibration

T_i	$\Delta(T_i)$	T_i	$\Delta(T_i)$
6M	0.283266	5Y 6M	0.010102
9M	0.027149	5Y 9M	0.013151
1Y	−0.248024	6Y	0.018194
1Y 3M	−0.075917	6Y 3M	0.026785
1Y 6M	−0.032884	6Y 6M	0.028775
1Y 9M	0.061017	6Y 9M	0.029662
2Y	0.204885	7Y	0.025772
2Y 3M	−0.092562	7Y 3M	0.029643
2Y 6M	−0.041638	7Y 6M	0.023045
2Y 9M	0.021408	7Y 9M	0.020222
3Y	0.094142	8Y	0.010393
3Y 3M	−0.091244	8Y 3M	0.000327
3Y 6M	−0.070538	8Y 6M	−0.006906
3Y 9M	−0.040611	8Y 9M	−0.014838
4Y	−0.030308	9Y	−0.017226
4Y 3M	−0.038351	9Y 3M	−0.001728
4Y 6M	−0.029781	9Y 6M	−0.000947
4Y 9M	−0.020677	9Y 9M	−0.002806
5Y	−0.011036	10Y	−0.025684
5Y 3M	0.005453		

Now we can extend the calibration scheme to swaptions.

9.3.2 Parametric calibration to swaptions

Parametric calibration to swaptions will be done using the previously computed parameters.

Algorithm 9.5 presents our parametric calibration to swaptions.

Algorithm 9.5 Parametric calibration to swaptions

SwaptionImpliedVolatility=0; Counter=0 // Setting initial values
$\theta_1=\theta_2=\theta_3=\theta_4=1$
// Setting the initial values of parameters used in parametric swaption calibration
$\phi_1=\phi_2=\phi_3=\phi_4=1$
// Setting the initial values of parameters used in parametric swaption calibration
$\nu_i=$ Algorithmx.x $\to \nu_i$, $i=1,2,3,4$
// Taking values computed in caplet calibration

Step 1 of calibration

For i = 1 to NumberOfCapletPeriods

$\delta_0(i) = (T_i - T_0)/basis$, $(T_i = T_{3M}, T_{6M}, \ldots, T_{10y})$ // basis = 365
$\Theta(i) = \theta_1 + (\theta_2 + \theta_3\delta_0(i))e^{-\theta_4\delta_0(i)}$
$\Phi(i) = \phi_1 + (\phi_2 + \phi_2\delta_0(i))e^{-\phi_1\delta_0(i)}$
$f(i) = |v_1 + (v_2 + v_2\delta_0(i))e^{-v_1\delta_0(i)}|$ // values taken form caplet calibration
$\Delta(i) =$ Algorithmx.x $\rightarrow \Delta(i)$, $i = 1, 2, \ldots,$ Number of caplet periods
// values taken form caplet calibration
$\varepsilon(i) =$ Algorithmx.x $\rightarrow \varepsilon(i)$, $i = 1, 2, \ldots,$ Number of caplet periods
// values taken form caplet calibration
$\delta_{3M}(i) = (T_i - T_{i-3M})/basis$
$\psi(i) = \delta_{3M}(i)L(T_0, T_{i-3M}, T_{3M})^2$
// where $L(T_0, T_{i-3M}, T_i)$ is a forward LIBOR for period $T_{i-3M} \div T_i$

Next i

Step 2 of calibration

For i = 1 to NumberOfCaplet Periods
  For j = 1 to NumberOfCaplet Periods
    $\rho(i, j) = \cos[\Phi(i) - \Phi(j)] - \sin[\Phi(i)]\sin[\Phi(j)]\{1 - \cos[\Theta(i) - \Theta(j)]\}$
    // instantaneous correlations
  Next j
Next i

For a_1 = 1 to 6
  For b_1 = 1 to 6
    $i = 1 + 4x$, $x = 1, 2, 3, 4, 5, 7 \Rightarrow i = 5, 9, 13, 17, 21, 29$
    // Index allowing to choose time to maturity for swaptions
    $j = 4x$, $x = 1, 2, 3, 4, 5, 7 \Rightarrow j = 4, 8, 12, 16, 20, 28$
    // Index allowing to choose length of underlying swap
    $j = i + j - 1 \Rightarrow j = 8, 16, 24, 32, 40, 56$
    If j <=NumberOfCapletPeriods // NumberOfCapletPeriods = 40 in our case
      counter = counter +1
      $\Psi = 0$
      For i_1 = i to j
        $\Psi = \Psi + \psi(i\_1)$
    Next i
    $\sigma^{swaption} = 0$ // Setting initial swaption volatility to zero
    For k = i to j
      For l = i to j
      $\psi\_k = \psi(k)/\Psi$, $\psi\_l = \psi(l)/\Psi$
      $\rho\_kl = 0$
      For i_1 = 1 to i − 1
        $f\_k = f(k - i\_1)[1 + \varepsilon(i\_1)][1 + \Delta(k)]$
        $f\_l = f(l - i\_1)[1 + \varepsilon(i\_1)][1 + \Delta(l)]$

$\rho\_kl = \rho\_kl + f\_k \cdot f\_l \cdot \delta(i\_1)$

Next i_1

$\sigma^{swaption} = \sigma^{swaption} + \psi\_k \cdot \psi\_l \cdot \rho(k, l) \cdot \rho\_kl$

Next l

Next k

$T\_i = \delta_0(i)$

$\sigma^{swaption} = \sqrt{\sigma^{swaption}/T\_i}$

$err = \sigma^{swaption} - \sigma^{swaption}()$

// difference between theoretical and market swaption volatilities

$rsme = rsme + err^2$ // root mean squared error

End If

Next b_1

Next a_1

End of algorithm 9.5

Results of the calibration

Step 1

Preliminary computations

Table 9.19 presents the preliminary computation results for swaption calibration (step 1)

Table 9.19 Results for step 1 of parametric swaption calibration

T_i	$\delta_0(i)$	$\delta_{3M}(i)$	$L(T_0, T_{i-3M}, T_i)$	$\delta_{3M}(i) \cdot L(T_0, T_{i-3M}, T_i)^2$
3M	0.2466	0.2528		
6M	0.4959	0.2556	0.02194	0.00012173
9M	0.7479	0.2556	0.02294	0.00013453
1Y	1.0000	0.2500	0.02415	0.00014905
1Y 3M	1.2466	0.2528	0.02544	0.00016180
1Y 6M	1.4959	0.2556	0.02659	0.00017873
1Y 9M	1.7479	0.2556	0.02775	0.00019686
2Y	2.0000	0.2500	0.02892	0.00021381
2Y 3M	2.2466	0.2528	0.02920	0.00021318
2Y 6M	2.4959	0.2556	0.03016	0.00022994
2Y 9M	2.7479	0.2556	0.03113	0.00024767
3Y	3.0000	0.2528	0.03211	0.00026342
3Y 3M	3.2493	0.2528	0.03236	0.00026471
3Y 6M	3.4986	0.2611	0.03322	0.00027889
3Y 9M	3.7562	0.2528	0.03335	0.00029046
4Y	4.0055	0.2528	0.03533	0.00031546
4Y 3M	4.2548	0.2528	0.03462	0.00030299
4Y 6M	4.5041	02528	0.03576	0.00032320
4Y 9M	4.7534	0.2528	0.03693	0.00034470
5Y	5.0027	0.2528	0.03771	0.00035940
5Y 3M	5.2521	0.3302	0.03702	0.00034643
5Y 6M	5.5778	0.1753	0.02919	0.00028135

Table 9.19 Continued

T_i	$\delta_0(i)$	$\delta_{3M}(i)$	$L(T_0, T_{i-3M}, T_i)$	$\delta_{3M}(i) \cdot L(T_0, T_{i-3M}, T_i)^2$
5Y 9M	5.7507	0.2556	0.05662	0.00056206
6Y	6.0027	0.2500	0.03955	0.00039983
6Y 3M	6.2493	0.2528	0.03986	0.00039724
6Y 6M	6.4986	0.2556	0.04053	0.00041530
6Y 9M	6.7507	0.2556	0.04121	0.00043405
7Y	7.0027	0.2528	0.04189	0.00044850
7Y 3M	7.2521	0.2528	0.04152	0.00043569
7Y 6M	7.5014	0.2556	0.04212	0.00044839
7Y 9M	7.7534	0.2556	0.04272	0.00046647
8Y	8.0055	0.2500	0.04333	0.00047984
8Y 3M	8.2521	0.2528	0.04378	0.00047927
8Y 6M	8.5014	0.2556	0.04438	0.00049779
8Y 9M	8.7534	0.2611	0.04498	0.00051693
9Y	9.0110	0.2444	0.04461	0.00051951
9Y 3M	9.2521	0.2528	0.04479	0.00049047
9Y 6M	9.5014	0.2611	0.04421	0.00049402
9Y 9M	9.7589	0.2528	0.04373	0.00049923
10Y	10.008	0.2493	0.04564	0.00052654

Step 2 of the algorithm is the running an optimization process where the goal is to minimize the difference between the theoretical and market prices of swaptions where the optimization is over the correlation parameters.

Starting from initial values $\theta_1 = \theta_2 = \theta_3 = \theta_4 = 1$ and $\phi_1 = \phi_2 = \phi_3 = \phi_4 = 1$ after the optimization we obtain the following values:

Table 9.20 Final values of parameters θ, ϕ

θ_1	0.1000000	ϕ_1	0.0404450
θ_2	0.1354474	ϕ_2	−0.0404486
θ_3	−0.0796023	ϕ_3	−0.0000264
θ_4	0.2899784	ϕ_4	0.0006514

We can now calculate the following parameters of $\Theta(i) = \theta_1 + (\theta_2 + \theta_3\delta_0(i))e^{-\theta_4\delta_0(i)}$ and $\Phi(i) = \phi_1 + (\phi_2 + \phi_2\delta_0(i))e^{-\phi_1\delta_0(i)}$ which are then presented in Table 9.21.

We can compare the theoretical and market swaption volatilities. Table 9.22 presents these results.

The instantaneous correlations are almost equal to one. The results seem to be much worse than when obtained via non-parametric calibration algorithms. There is the possibility to improve the results by manipulating the functional forms of functions $\Theta(i) = \theta_1 + (\theta_2 + \theta_3\delta_0(i))e^{-\theta_4\delta_0(i)}$ and $\Phi(i) = \phi_1 + (\phi_2 + \phi_2\delta_0(i))e^{-\phi_1\delta_0(i)}$. Another way to improve results may be by the use of more effective optimization algorithms allowing the minimization of the differences between theoretical and market swaption volatilities further.

Table 9.21 Final values of parameters Θ, Φ

T_i	$\Theta(i)$	$\Phi(i)$	T_i	$\Theta(i)$	$\Phi(i)$
3M	0.20782734	−0.000003618	5Y 3M	0.03836996	−0.000003828
6M	0.18311905	−0.000003639	5Y 6M	0.03878027	−0.000003827
9M	0.16110851	−0.000003659	5Y 9M	0.03917682	−0.000003825
1Y	0.14178780	−0.000003678	6Y	0.03994451	−0.000003822
1Y 3M	0.12523052	−0.000003695	6Y 3M	0.04088341	−0.000003818
1Y 6M	0.11060945	−0.000003711	6Y 6M	0.04199164	−0.000003813
1Y 9M	0.09777542	−0.000003727	6Y 9M	0.04324713	−0.000003807
2Y	0.08669785	−0.000003741	7Y	0.04461381	−0.000003800
2Y 3M	0.07738401	−0.000003754	7Y 3M	0.04605355	−0.000003791
2Y 6M	0.06933744	−0.000003766	7Y 6M	0.04756188	−0.000003782
2Y 9M	0.06245475	−0.000003777	7Y 9M	0.04913960	−0.000003771
3Y	0.05669461	−0.000003787	8Y	0.05075542	−0.000003760
3Y 3M	0.05197979	−0.000003796	8Y 3M	0.05236029	−0.000003747
3Y 6M	0.04813327	−0.000003804	8Y 6M	0.05399617	−0.000003734
3Y 9M	0.04496754	−0.000003811	8Y 9M	0.05565359	−0.000003719
4Y	0.04259363	−0.000003816	9Y	0.05734173	−0.000003702
4Y 3M	0.04081855	−0.000003821	9Y 3M	0.05891017	−0.000003686
4Y 6M	0.03957031	−0.000003824	9Y 6M	0.06051381	−0.000003668
4Y 9M	0.03878407	−0.000003826	9Y 9M	0.06214526	−0.000003649
5Y	0.03840144	−0.000003828	10Y	0.06369579	−0.000003628

Table 9.22 Theoretical and market swaption volatilities

Market	1Y	2Y	3Y	4Y	5Y	7Y
1Y	22.7000 %	23.0000 %	22.1000 %	20.9000 %	19.6000 %	17.6000 %
2Y	22.4000 %	21.5000 %	20.5000 %	19.4000 %	18.3000 %	16.7000 %
3Y	20.9000 %	20.1000 %	19.0000 %	18.0000 %	17.0000 %	15.8000 %
4Y	19.5000 %	18.7000 %	17.7000 %	16.8000 %	16.0000 %	
5Y	18.2000 %	17.4000 %	16.5000 %	15.8000 %	15.1000 %	
7Y	16.7200 %	16.0800 %	15.3000 %			
Theoretical	**1Y**	**2Y**	**3Y**	**4Y**	**5Y**	**7Y**
1Y	22.8165 %	22.0415 %	19.2960 %	17.1058 %	15.5043 %	13.6844 %
2Y	22.8386 %	20.6389 %	18.4044 %	16.5967 %	15.3733 %	13.7711 %
3Y	20.1999 %	18.9490 %	17.2257 %	15.8731 %	14.8469 %	13.6224 %
4Y	19.4248 %	18.2455 %	16.7513 %	15.4790 %	14.4750 %	
5Y	18.9177 %	17.7192 %	16.2348 %	14.9735 %	14.3294 %	
7Y	17.1656 %	15.8917 %	15.0285 %			
Difference	**1Y**	**2Y**	**3Y**	**4Y**	**5Y**	**7Y**
1Y	0.1165 %	−0.9585 %	−2.8040 %	−3.7942 %	−4.0957 %	−3.9156 %
2Y	0.4386 %	−0.8611 %	−2.0956 %	−2.8033 %	−2.9267 %	−2.9289 %
3Y	−0.7001 %	−1.1510 %	−1.7743 %	−2.1269 %	−2.1531 %	−2.1776 %
4Y	−0.0752 %	−0.4545 %	−0.9487 %	−1.3210 %	−1.5250 %	
5Y	0.7177 %	0.3192 %	−0.2652 %	−0.8265 %	−0.7706 %	
7Y	0.4456 %	−0.1883 %	−0.2715 %			

9.4 CONCLUSIONS

In this chapter we have presented two expanded calibration algorithms to caps and swaptions simultaneously. Both algorithms have used optimization techniques. The first algorithm, based on the Rebonato approach, resulted in some quite good results. The market and theoretical caplet volatilities are practically the same – all the differences are negligible. The obtained piecewise constant matrix of instantaneous volatilities also seems to be logical with values in the matrix close to untransformed market data. The outstanding values are rare and have small impact in real derivatives valuation.

The matrix of instantaneous correlations is also quite good. Correlations are close to one and there are no outstanding values as in the case of historical correlations. Based on instantaneous volatilities and implied correlations the table comparing differences between the theoretical and market swaption volatilities seems to be correct. Differences are relatively small and thus such results may be used in practice for valuation purposes.

The second algorithm was based on parametric functions approximating the market quotations of caps and swaptions. First we have to compute the parametric approximations of the caplet prices. Next we have to run an optimization to minimize the differences between the theoretical and market quotations of caps. We see that the results of calibration are acceptable but definitely worse than those obtained during non parametric calibration.

Having calibrated the approximating functions to caps we have moved into the calibration to swaptions. As a result we have obtained a matrix of instantaneous correlations. However the results are not as good as for non-parametric calibration. Almost all instantaneous correlations are equal to one. The reason for that may be in the wrong form of the widely used approximation functions. The other reason for wrong results may by using ineffective optimization algorithms when minimizing differences between the theoretical and market swaption volatilities. Definitely there is a place for further research in order to find more accurate optimization routines.

It is worth noting that in this chapter there are many of intermediate results. The reason for that is to help the reader fully understand the theory of calibration and the practical algorithms for implementation of the techniques in practice.

Both algorithms are frequently used in the market. However as we have seen in some market circumstances the results may be not satisfactory. This was especially evident in the case for parametric calibration. For that reason there is a clear visible trend in the market to use non-parametric calibration algorithms. Most of them are based on the piecewise constant volatility assumption which helps to price exotic interest derivatives in practice.