Another way:

$$\mathbb{E}^{\mathbb{P}}_{n}\left[M_{n+m} \mid \mathcal{F}_{n}\right] = M_{n}$$

$$\forall \; n, m \geq 0$$

Stoch. Integral,

$$\int_0^T f(t, X_t)\, dX_t \qquad [0, T] \qquad dt = \frac{T}{N} \qquad t_i = i\,dt$$

① $$\lim_{N \to \infty} \sum_{0}^{N-1} f(t_i, X_{t_i})(X_{t_{i+1}} - X_{t_i})$$

② $$\lim_{N \to \infty} \sum_{0}^{N-1} f(t_{i+1}, X_{t_{i+1}})(X_{t_{i+1}} - X_{t_i})$$

③ $$\lim_{N\to\infty}\sum_{0}^{N-1} f\left(t_{i+\frac{1}{2}}, X_{t_{i+\frac{1}{2}}}\right)\left(X_{i+1}-X_{i}\right)$$

$$t_{i+\frac{1}{2}}=\frac{1}{2}\left(t_{i}+t_{i+1}\right)$$

$$X_{t_{i+\frac{1}{2}}}=X\left(t_{i+\frac{1}{2}}\right)$$

$$\mathbb{E}\left[\lim_{N\to\infty}\sum_{0}^{N-1} f(t_i, X_{t_i})(X_{i+1}-X_i)\right]$$

$$\lim_{N\to\infty}\sum_{0}^{N-1} f(t_i, X_{t_i})\,\mathbb{E}\left[X_{i+1}-X_i\right]$$

$$=0$$

F

$$\int_0^T \frac{dF}{dX}\, dX = F(X) - F(X_0) - \frac{1}{2}\int_0^T \frac{d^2F}{dX^2}\, dt$$

$$d(X^2) = \frac{dF}{dX}\, dX + \frac{1}{2}\frac{d^2F}{dX^2}\, dt$$

$$\mathbb{E}[X^2] = T$$

Itô on $F = X^2$ $\quad \dfrac{dF}{dX} = 2X$

$$\frac{d^2F}{dX^2} = 2$$

$$d(X^2) = 2X\,dX + \frac{1}{2} \cdot 2 \cdot dt$$

$$\int_0^T d(X^2) = X_T^2 - \cancelto{0}{X_0^2} = \int_0^T 2X\,dX + \cancelto{T}{\int_0^T dt}$$

$$X^2(T) = T + \int_0^T 2X \, dX$$

$$\mathbb{E}\left[X^2(T)\right] = T + \underbrace{\mathbb{E}\left[\int_0^T 2X \, dX\right]}_{\therefore \text{ Itô integral, are Martingale}}$$

$$dr = -\gamma(r - \bar{r})\,dt + \sigma\,dX$$

$$du = -\gamma u\,dt + \sigma\,dX$$

$$u_t = u_0 e^{-\gamma t} + \sigma \int_0^t e^{+\gamma(s-t)}\,dX$$

$$\mathbb{E}(u_t) = \mathbb{E}\left[u_0 e^{-\gamma t}\right]^0 + \sigma\,\underbrace{\mathbb{E}\left[\int_0^t e^{\gamma(s-t)}\,dX\right]}_{=0}$$

$$= u_0 e^{-\gamma t}$$

$$V[u] = \underbrace{E[u^2]}_{} - \underbrace{E^2[u]}_{u_0^2 e^{-2\gamma t}}$$

$$dS = \mu S\, dt + \sigma S\, dX$$

Using Itô with $V = \log S$

$$S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma X_t}$$

$$\mathbb{E}[S_t \mid S_0] = \mathbb{E}\left[S_0 e^{(\mu-\frac{1}{2}\sigma^2)t + \sigma X_t} \Big| S_0\right]$$

$$= S_0 e^{(\mu - \frac{1}{2}\sigma^2)t}\, \mathbb{E}\left[e^{\sigma X_t}\right]$$

Focus on $\mathbb{E}\left[e^{\sigma X_t}\right]$

$$p(x,t) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2}x^2/t}$$

$$\mathbb{E}\left[e^{\sigma X_t}\right] = \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} e^{\sigma X_t} e^{-\frac{1}{2}x^2/t}\, dx$$

$$= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} e^{-\frac{1}{2t}\left[x^2 - 2t\sigma x + \sigma^2 t - \sigma^2 t^2\right]} dx$$

$$= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} e^{-\frac{1}{2t}\left[(x-\sigma t)^2 - \sigma^2 t^2\right)} dx$$

$$= \underbrace{e^{+\frac{1}{2}\sigma^2 t} \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} e^{-\frac{1}{2t}(x-\sigma t)^2} \, dx}$$

Put $u = \dfrac{x - \sigma t}{\sqrt{t}} \Rightarrow \sqrt{t}\, du = dx$

$$= e^{\frac{1}{2}\sigma^2 t} \times \underbrace{\frac{1}{\sqrt{2\pi \cancel{t}}} \int_{\mathbb{R}} e^{-u^2} \cancel{\sqrt{t}} \, du}_{=1} = e^{\frac{1}{2}\sigma^2 t}$$

$$\mathbb{E}\left[S_t \mid S_0\right] = S_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)t} \cdot e^{\frac{1}{2}\sigma^2 t}$$

$$= \boxed{S_0 e^{\mu t}}$$

$\mathbb{P}$ $\leftarrow$ Prob measure / Prob Dist^n $\rightarrow$ $\mathbb{Q}$

change of measure

$\Lambda = \frac{d\mathbb{Q}}{d\mathbb{P}}$

Risk-neutral finance / pricing

$\Lambda^{-1} = \frac{d\mathbb{P}}{d\mathbb{Q}}$

$X_t^{\mathbb{P}}$ $\rightarrow$ $X_t^{\mathbb{Q}}$

Process with drift (Not a martingale)

Martingale ("hide drift")

$$
\begin{tikzcd}
 & V = S_1 S_2 \arrow[dl] \arrow[dr] & \\
X_1 & & X_2
\end{tikzcd}
$$

$$f_1 = f_2 = 0$$

$$g_1 = g_2 = 1$$

$$\boxed{d(X_1 X_2) = X_2\, dX_1 + X_1\, dX_2 + \rho\, dt}$$