CQF Module 2 Examination

June 2016 Cohort

Instructions

Answers to all questions are required. Complete mathematical and computational workings must be provided to obtain maximum credit. Submission must include Excel file(s) and code if used. Books and lecture notes may be referred to. Questions to Richard.Diamond@fitchlearning.com.

Portfolio computational tasks are best solved by matrix manipulation on a spreadsheet. Use Excel functions *MMULT()*, *MINV() and TRANSPOSE()*. If familiar, can use Python, Matlab or R.

A. Optimal Portfolio Allocations [56%]

Consider an investment universe composed of the following risky assets:

Asset	$\boldsymbol{\mu}$	$\boldsymbol{\sigma}$
A	0.04	0.07
B	0.08	0.12
C	0.12	0.18
D	0.15	0.26

with a correlation structure

$$\boldsymbol{R} = \begin{pmatrix} 1 & 0.2 & 0.5 & 0.3 \\ 0.2 & 1 & 0.7 & 0.4 \\ 0.5 & 0.7 & 1 & 0.9 \\ 0.3 & 0.4 & 0.9 & 1 \end{pmatrix}$$

1. The global minimum variance portfolio is obtained by optimising *s.t.* the budget constraint

$$\underset{\boldsymbol{w}}{\operatorname{argmin}} \frac{1}{2} \boldsymbol{w'}\, \boldsymbol{\Sigma}\, \boldsymbol{w} \qquad \text{s.t. } \boldsymbol{w'1} = 1$$

   - Obtain analytical solution for optimal allocations $\mathbf{w}^*$. Provide full workings, analytical solution for the Lagrangian multiplier. No computation necessary.

2. Consider the following optimization task for a targeted return $m = 10\%$, for which the net of allocations invested (borrowed) in a risk-free asset:

$$\underset{\boldsymbol{w}}{\operatorname{argmin}} \frac{1}{2} \boldsymbol{w'\Sigma w}, \qquad \text{s.t. } r + (\boldsymbol{\mu} - r\mathbf{1})'\mathbf{w} = 0.1$$

   - Obtain analytical solution for the the Lagrangian multiplier and optimal allocations $\mathbf{w}^*$.
   - For the target return of 3% and risk-free rate of 0.5% calculate optimal allocations $\mathbf{w}^*$ and portfolio standard deviation $\sigma_\Pi = \sqrt{\mathbf{w}^{*\prime}\boldsymbol{\Sigma}\mathbf{w}^*}$ using asset data and correlation above.

3. Provide the definition of *a tangency portfolio* and calculate the slope of Capital Market Line. Briefly explain its role in evaluating investments. To calculate μ_T, σ_T for the risk-free rate 0.5% use ready formulae for $\mathbf{w_T}$ from the lecture (no derivation necessary here, only computation).

4. For the tangency portfolio $\mathbf{w_T}$ and confidence level $c = 99\%$, calculate the risk measures:

   - Analytical VaR (1D or 10D depends on the time scale of original returns/standard deviation data and is not defined here)

   $$\text{VaR}_c(X) = \boldsymbol{w'\mu} + \text{Factor} \times \sqrt{\boldsymbol{w'\Sigma w}}$$

   where the Factor is a standardised percentile drawn from the inverse **CDF** function for the Normal distribution Factor $= \Phi^{-1}(1-c)$.

   - Marginal Contributions to Risk measured by VaR diag$(\boldsymbol{w}) \times \dfrac{dVaR}{d\boldsymbol{w}}$, where the derivative is

   $$\frac{dVaR}{d\boldsymbol{w}} = \boldsymbol{\mu} + \frac{\boldsymbol{\Sigma w}}{\sqrt{\boldsymbol{w'\Sigma w}}} \times \text{Factor}$$

   - Expected Shortfall according to

   $$\text{ES}_c(X) = \mu_\Pi + \sigma_\Pi \frac{\phi(\text{VaR}_c)}{1-c}$$

   where portfolio return is $\mu_\Pi = \boldsymbol{w'\mu}$ and portfolio risk $\sigma_\Pi = \sqrt{\boldsymbol{w'\Sigma w}}$ and $\phi()$ is Normal *pdf.*

B. Analytical Risk [21%, 7% each]

X refers to a random variable, while $X(t)$ represents the Brownian Motion.

1. Assume that P&L of an investment portfolio over time τ follows Normal distribution $N(\mu, \sigma^2\tau)$. Begin with the definition of VaR as a percentile $Pr(X \geq x) \leq c$ to obtain the known formula for Analytical VaR

   $$\text{VaR}(X) = \mu + \sigma\sqrt{\tau} \times \Phi^{-1}(1-c).$$

   For two correlated random variables K and M, the bivariate Normal distribution is defined as $K \sim N(\mu_k, \sigma_k)$, $M \sim N(\mu_m, \sigma_m)$ with ρ_{km}.

   - Find the mean and variance for this joint Normal distribution (analytically).
   - Prove the sub-additivity of Value at Risk, $\text{VaR}(K+M) \leq \text{VaR}(K) + \text{VaR}(M)$.

2. Covariance matrix can be decomposed as $\boldsymbol{\Sigma} = \boldsymbol{AA'}$ by Cholesky method. The resulting lower triangular matrix $\boldsymbol{A}$ is used for imposing correlation on a vector of independent Normal $\boldsymbol{X}$.

   $$\boldsymbol{A} = \begin{pmatrix} \sigma_1 & 0 \\ \rho\sigma_2 & \sqrt{1-\rho^2}\sigma_2 \end{pmatrix} \quad \text{and} \quad \boldsymbol{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

   - For the two-variate case, show analytically what $\boldsymbol{\Sigma} = \boldsymbol{AA'}$ is equal to.
   - Write down the results for correlated $Y_1(t)$ and $Y_2(t)$ from $\boldsymbol{Y} = \boldsymbol{AX}$.
   - Does $Y_2(t)$ keep the properties of the Brownian Motion if $X_1(t), X_2(t)$ are Standard Normal? **Note:** consider the distribution of the $Y_2(t)$ increment, its variance.

3. Let's redefine the Cholesky result as a one-factor risk model, where $X_1(t) \equiv Z_t$ represents a common factor such as market, while $X_2(t) \equiv \epsilon_t$ for an idiosyncratic residual, $\epsilon_t \sim N(0,1)$ and $\sigma_1 = \sigma_2 = 1$.

$$Y_2(t) = \rho Z_t + \sqrt{1-\rho^2}\epsilon_t$$

Let's specify the probability of some event as $\Pr(Y_2 \leq -2.33)$. Provide all derivation steps to show that the probability conditional on market factor is calculated using the Normal *cdf* $\Phi()$ as

$$\Pr(\text{Event}|Z_t) = \Phi\left(\frac{-2.33 - \rho Z_t}{\sqrt{1-\rho^2}}\right)$$

Hint: simply rearrange around the idiosyncratic factor $\epsilon_t \sim N(0,1)$.

C. Value at Risk on FTSE 100 [23%]

Imagine that each morning you calculate 99%/10day VaR from available prior data only. Once ten days pass you compare that VaR number to the realised return and check if your prediction about the worst loss was breached. You are given a dataset of FTSE 100 index levels, continue in Excel.

B.1 Calculate the rolling 99%/10day Value at Risk for an investment in the market index using a sample standard deviation of log-returns, as follows:

- The rolling standard deviation for a sample of 21 is computed for days 1-21, 2-22, ..., there must be 21 observations in the sample. So, you have a time series of σ_t.
- Scale standard deviation to reflect a ten days move $\sigma_{10D} = \sqrt{10 \times \sigma^2}$ (we can add variances) and scale an average daily return as $\mu_{10D} = \mu \times 10$ where μ is a mean return of all data given.
- Calculate Value at Risk for each day t (starting on Day 21) as follows:

$$\text{VaR} = \mu_{10D} + \text{Factor} \times \sigma_{10D} \quad \dagger$$

where Factor is a percentile of the Standard Normal Distribution that 'cuts' 1% on the tail.

In Excel, you will have a final column with VaR_t as a percentage since calculation is done on returns.

B.2 Calculate two numbers: (a) the percentage of VaR breaches and (b) conditional probability of breach in VaR, given that a breach was observed for the previous period.

- VaR is fixed at time t and compared to the realised return at time $t + 10$. A breach occurs when a realised 10-day return $r_{10D,t} = \ln(S_{t+10}/S_t)$ is below the VaR_t quantity (negative scale).
- 20/08/2009 is the first day on which VaR_t computation is available. Number of breaches divided by number of observations will give the percentage of breaches.
- Plot time series of VaR_t and indicate breaches. Briefly discuss, are the breaches independent?

In Excel, you will add two columns for $r_{10D,t}$ and $0, 1$ indicator, where 1 means a breach.

B.3 Now that you know all the data in the end, construct the histogram and Q-Q plot for each, 1D and 10D log-returns. Briefly discuss if the Normal distribution of returns is a reasonable assumption.