CQF Lecture 5.3 Intensity Models

Exercises

1. Default event modelling relies on the concept of *hazard rate* or *default intensity* λ_t. Denoting the lifetime of a firm as τ (time until default arrival), the conditional survival probability for the firm between between times t and T is expressed as:

$$S(t,T) = \Pr(\tau > T | \tau > t) \qquad \forall\, T > t$$

   (a) The mathematical definition of default intensity is

$$\lambda_t = \lim_{h \to 0^+} \frac{\Pr(t < \tau \le t+h | \tau > t)}{h} = -\frac{d \log S(h)}{dh} = -\frac{S'}{S}$$

   Interpret the meaning of λ_t in plain English.

   (b) Show that survival probability follows multiplication rule over time increments $S(t, T+s) = S(t,T) \times S(T, T+s) \quad \forall\, s > 0$.

   (c) Use the result from the previous step to show that for a small time period $s \equiv h$,

$$S(t,T) = \exp\left\{ - \int_t^T \lambda_s ds \right\}.$$

   Assuming constant λ, what distribution of τ does the result suggest?

   (d) Conditional probability of an instantaneous default at time T is expressed as

$$\mathrm{PD}(t,T) = \Pr(\tau \le T | \tau \ge T)$$

   The conditional part indicates that once defaulted, the firm remains in default state after time T. Given that $\mathrm{PD} = 1 - S$ show

$$\mathrm{PD}(t,T) = \int_t^T \lambda_s S(t,s)\, ds.$$

2. Consider a risky (corporate) bond $V(r,p,t)$ with *pdt* reflecting default probability and the short-term rate evolves as

$$dr = u(r,t)dt + w(r,t)dX$$

   By constructing a hedged portfolio, where $Z(r,t)$ is risk-free zero coupon bond

$$\Pi = V(r,p,t) - \Delta Z(r,t)$$

   show that BPE that incorporates default is

$$\frac{\partial V}{\partial t} + \tfrac{1}{2} w^2 \frac{\partial^2 V}{\partial r^2} + (u - \lambda w) \frac{\partial V}{\partial r} - (r+p)V = 0$$

   Note: in this equation, λ has a meaning of the market price of risk (see Stochastic Interest Rates Modelling lecture).

The exercises have been edited by CQF Faculty, Richard Diamond, R.Diamond@fitchlearning.com

3. To model migration (change) of credit ratings, transition matrices are used. A transition matrix is a discrete equivalent to a probability density function. It represents probabilities of change in rating over finite horizon.

   All transition probabilities must be reflected in a fair price of a claim, for example, a bond re-rated from BBB to A should experience noticeable appreciation. The highest probability is usually of no migration.

   Consider transition matrix $\mathbf{P}_{dt}$ of probabilities for a change of rating over small dt.

$$\mathbf{P} = \begin{pmatrix} 1 - pdt & pdt \\ 0 & 1 \end{pmatrix}$$

   The transition matrix represents a two-state Markov Chain for a simple rating model with 'default' and 'no default' states for one reference name. Given that $pdt = \lambda dt$ is constant and small, the top left probability of no default is close to 1. However, once default occurs the probability that a firm stays in default is equal to 1 (the lower right corner). Default is 'an absorbing state'.

   (a) Given that $\mathbf{Q}\, dt = \mathbf{I} - \mathbf{P}$ find intensity matrix $\mathbf{Q}$.

   (b) For constant interest rates, we can set up an equivalent of transition density function

$$\frac{d\mathbf{V}}{dt} = (r\mathbf{I} + \mathbf{Q})\, \mathbf{V}$$

   where vector $\mathbf{V}^T = (V,\, V_1)$ has bond values for each of states as its entiries.
   Find solutions for V_1 and V by substituting the matricies $\mathbf{V},\ \mathbf{Q}$ defined in previous steps and solving a system of two linear ODEs. The final condition for the bond value in default is $V_1(T;T) = \theta$ where θ is a recovery rate.

   (c) Interpret the role of probability of default p in the pricing solution for bond V.

   Note: for a detailed explanation see Chapter 40 (Credit Risk) in Volume 2, PWOQF that outlines the more complex relationship between $\mathbf{P},\ \mathbf{Q}$ as $\mathbf{P}(0, T) = e^{-\mathbf{Q}T}$. Compared to the textbook, the sign for $\mathbf{Q}$ is inverted here, without the loss of meaning, in order to match classical expression for survival probability as a function of hazard rate $e^{-\lambda T}$ where $\lambda(s) \geq 0$.