‘Advanced’ Volatility Modeling in Complete Markets

In this lecture. . .

- the relationship between implied volatility and actual volatility in a deterministic world
- the local volatility surface
- the difference between ‘random’ and ‘uncertain’
- non-linear pricing equations
- optimal static hedging with traded options
- how non-linear equations make a mockery of calibration

By the end of this lecture you will

- understand the mathematics behind deterministic volatility

- be able to use uncertainty, instead of randomness, for modeling

- be able to optimally statically hedge a portfolio

Introduction

In this lecture we will see two extremes to modeling volatility.

- From deterministic to hardly modeled at all.
- From one value to an infinite number of values.
- From assuming that the market is correct to assuming that it knows nothing.

The first subject is a return to deterministic volatility and volatility surfaces.

The second subject is that of uncertain parameters.

Volatility surfaces, smiles and skews revisited

Recall that we earlier looked at how implied volatility varies with strike and expiration in practice.

Volatility matrices

Here is an example of price versus strike and expiry. . .

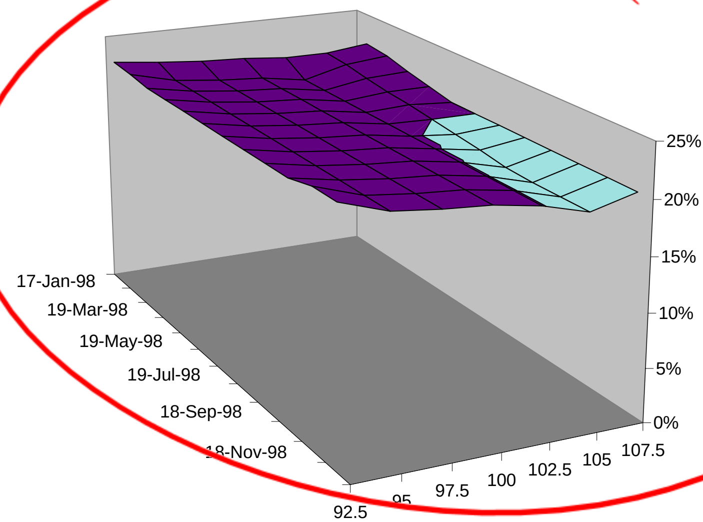
	95	**100**	**105**
1 month	11.80	7.24	3.50
3 months	13.33	9.23	5.76
7 months	16.02	12.13	7.97

This can be reinterpreted in terms of implied volatility versus strike and expiry. . .

	95	**100**	**105**
1 month	24.1%	22.9%	21.2%
3 months	22.7%	21.5%	20.5%
7 months	21.8%	20.5%	19.4%

Volatility surfaces

We can show implied volatility against both maturity and strike in a three-dimensional plot.



This implied volatility surface represents the constant value of volatility that gives each traded option a theoretical value equal to the market value.

There are various ways of interpreting this.

One of the key points is to ask whether we believe that the option market is right or wrong in its assessment of implied volatility.

In other words, is implied volatility a good forecast of actual volatility?

For this half of the lecture we assume that

- the market is correct, implied volatility is a perfect predictor of future actual volatility

- actual volatility is a deterministic function of stock price and time

The last assumption is the simplest that is potentially consistent with varying implied volatility.

$$dS = \cancel{\mu} S\,dt + \underline{\sigma(S,t)}\,S\,dX$$

Notation

Actual volatility:

$$\sigma(S,t)$$

Implied volatility:

$$\sigma_{\text{i}}(E,T)$$

$$dS = \dots + \sigma\cancel{(E,T)}\, S\, dX$$

Our task is to find $\sigma(S,t)$ given market prices of options i.e. knowing $\sigma_i(E,T)$.

$F = kx$

This is working backwards!

Normally we say that we know the parameters (here the 'actual' volatility) and then calculate the solution (here the 'implied' volatility).

$\sigma(S,t)$ is the **local volatility surface**.

$\sigma \rightarrow V \rightarrow \sigma_i$

$V \rightarrow \sigma$

Calibration

τ, T

S, t

Backing out the local volatility surface from European call option prices

Market prices of traded vanilla options are never consistent with the constant volatility assumed by Black & Scholes.

To match the theoretical prices of traded options to their market prices always requires a volatility structure that is a function of both the asset price, S, and time, t i.e. $\sigma(S, t)$.

To back out the local volatility surface from the prices of market traded instruments we are going to assume that we have a distribution of European call prices of all strikes and maturities. These prices will be denoted by $V(E, T)$.

t^* S^*

$\frac{\partial}{\partial E}, \frac{\partial}{\partial T}$

This notation is vastly different from before.

Previously, we had the option value as a function of the underlying and time.

Now the asset and time are fixed at S^* and t^*, today's values.

We will use the dependence of the market prices on strike and expiry to calculate the volatility structure.

Assume that the risk-neutral random walk for S is

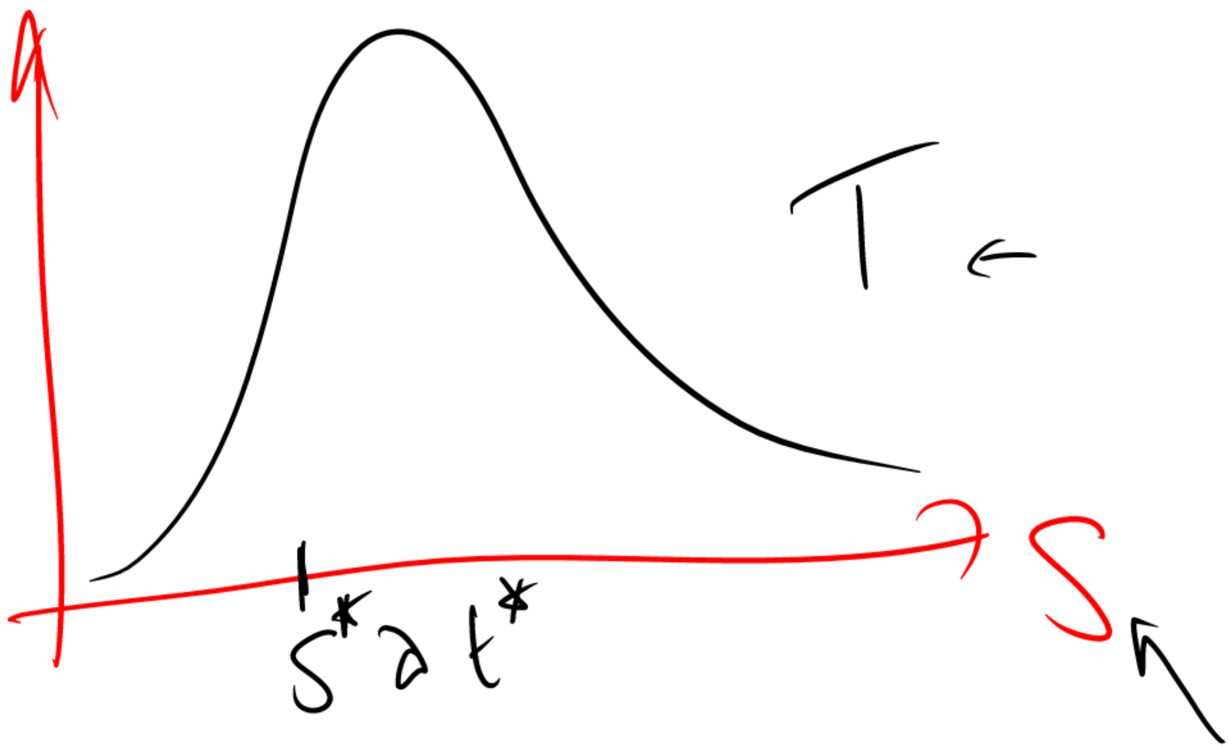
$$dS = rS\,dt + \sigma(S,t)S\,dX.$$

This is our usual one-factor model for which all the building blocks of delta hedging and arbitrage-free pricing hold.

The only novelty is that the volatility is dependent on the level of the asset and time.

In the following, we are going to rely heavily on the transition probability density function $p(S^*, t^*; S, T)$ for the risk-neutral random walk.

Note that the backward variables are fixed at today's values and the forward time variable is T.



Remember that one of the interpretations of the Binomial/Black–Scholes model is that value of an option is the present value of the expected payoff.

So for a call option we can write

?

$$V(E,T) = e^{-r(T-t^*)} \int_0^\infty \max(S-E,0)\,p(S^*,t^*;S,T)\,dS.$$

Integral Eqn.

$$V = e^{-r(T-t^*)}\int_E^\infty (S-E)p(S^*,t^*;S,T)\,dS. \qquad (1)$$

We are *very* lucky that the payoff is the maximum function so that after differentiating with respect to E we get

$$\frac{\partial V}{\partial E} = -e^{-r(T-t^*)}\int_E^\infty p(S^*,t^*;S,T)\,dS.$$

And after another differentiation, we arrive at

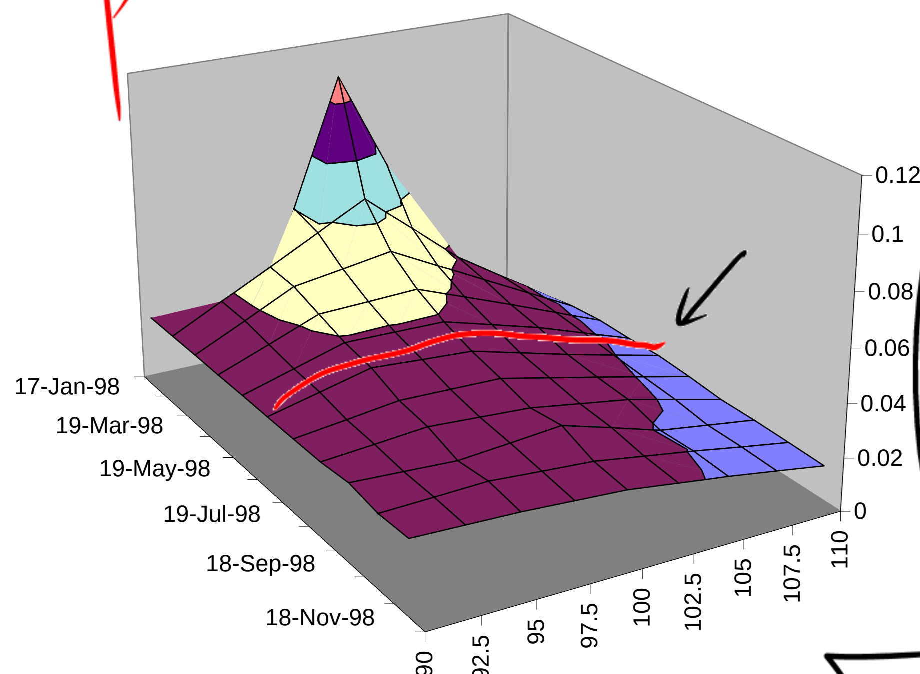
$$\frac{\partial^2 V}{\partial E^2} = e^{-r(T-t^*)}p(S^*,t^*;E,T).$$

There is therefore a surprisingly simple relationship between the second derivative of the option value with respect to strike and the risk-neutral probability density function:

$$p(S^*, t^*; E, T) = e^{r(T-t^*)} \frac{\partial^2 V}{\partial E^2}. \tag{2}$$

Before even calculating volatilities we can find the transition probability density function. In a sense, this is the market's view of the future distribution.

But it's the market view of the risk-neutral distribution and not the real one.



Our transition density function satisfies two equations, the forward and the backward.

We are going to exploit the forward equation for the transition probability density function, the Fokker–Planck equation,

$\sigma(S,t)$

$$\boxed{\frac{\partial p}{\partial T} = \frac{1}{2}\frac{\partial^2}{\partial S^2}(\sigma^2 S^2 p) - \frac{\partial}{\partial S}(rSp).} \qquad (3)$$

Here σ is our, still unknown, function of S and t. However, *in this equation* $\sigma(S,t)$ *is evaluated at* $t = T$.

$$dS = rS\,dt + \sigma(S,t)\,S\,dX$$

?

$V = e^{-r(T-t)} \int_E^\infty \quad T \, dS$

From (1) we have

$$\frac{\partial V}{\partial T} = -rV + e^{-r(T-t^*)} \int_E^\infty (S-E) \frac{\partial p}{\partial T} \, dS.$$

This can be written as

$$\frac{\partial V}{\partial T} = -rV + e^{-r(T-t^*)} \int_E^\infty \left(\frac{1}{2} \frac{\partial^2 (\sigma^2 S^2 p)}{\partial S^2} - \frac{\partial (rSp)}{\partial S} \right) (S-E) \, dS$$

using the forward equation (3).

Integration by parts.

Integrating this by parts twice, assuming that p and its first S derivative tend to zero sufficiently fast as S goes to infinity we get

$$\frac{\partial V}{\partial T} = -rV + \tfrac{1}{2}e^{-r(T-t^*)}\sigma^2 E^2 p + re^{-r(T-t^*)}\int_E^\infty Sp\,dS.$$

In this expression $\sigma(S,t)$ has $S=E$ and $t=T$.

Writing

$$\int_E^\infty Sp\,dS = \int_E^\infty (S-E)p\,dS + E\int_E^\infty p\,dS$$

and collecting terms, we get

$$\frac{\partial V}{\partial T} = \tfrac{1}{2}\sigma^2 E^2 \frac{\partial^2 V}{\partial E^2} - rE\frac{\partial V}{\partial E}.$$

Rearranging this we find that

$$\sigma = \sqrt{\frac{\frac{\partial V}{\partial T} + rE\frac{\partial V}{\partial E}}{\frac{1}{2}E^2\frac{\partial^2 V}{\partial E^2}}}.$$

This is the answer we have been seeking.

$\xi = S$
$\tau = t$

There is one subtle point... In this σ is a function of E and T. We must relabel the variables to get $\sigma(S,t)$.

$$\left.\frac{\partial V}{\partial \xi}\right|_{\xi = S} \neq \frac{\partial V}{\partial S}$$

This calculation of the volatility surface from option prices worked because of the particular form of the payoff, the call payoff, which allowed us to derive the very simple relationship between derivatives of the option price and the transition probability density function.

When there is a constant and continuous dividend yield on the underlying the relationship between call prices and the local volatility is

$$\sigma = \sqrt{\frac{\frac{\partial V}{\partial T} + (r - D)E\frac{\partial V}{\partial E} + DV}{\frac{1}{2}E^2\frac{\partial^2 V}{\partial E^2}}}. \tag{4}$$

One of the problems with this expression concerns data far in or far out of the money.

Unless we are close to at the money both the numerator and denominator of (4) are small, leading to inaccuracies when we divide one small number by another.

One way of avoiding this is to relate the local volatility surface to the implied volatility surface.

In the same way that we found a relationship between the local volatility and the implied volatility in the purely time-dependent case we can find a relationship in the general case of asset- and time-dependent local volatility.

The result is

$V = SN(\quad)$ $\sigma_i(E,T)$

$$\sigma(E,T)^2 = \frac{\sigma_{\mathrm{i}}^2 + 2(T-t^*)\sigma_{\mathrm{i}}\frac{\partial \sigma_{\mathrm{i}}}{\partial T} + 2(r-D)E(T-t^*)\sigma_{\mathrm{i}}\frac{\partial \sigma_{\mathrm{i}}}{\partial E}}{\left(1 + Ed_1\sqrt{T-t^*}\frac{\partial \sigma_{\mathrm{i}}}{\partial E}\right)^2 + E^2(T-t^*)\sigma_{\mathrm{i}}\left(\frac{\partial^2 \sigma_{\mathrm{i}}}{\partial E^2} - d_1\left(\frac{\partial \sigma_{\mathrm{i}}}{\partial E}\right)^2\sqrt{T-t^*}\right)} \tag{5}$$

$E \to S$

$T \to t$

where $\sigma(S,t)$

$$d_1 = \frac{\log(S^*/E) + (r - D + \frac{1}{2}\sigma_{\mathrm{i}}^2)(T-t^*)}{\sigma_{\mathrm{i}}\sqrt{T-t^*}}.$$

Again, we must relabel the variables to get $\sigma(S,t)$.

In terms of the implied volatility the implied risk-neutral probability density function is

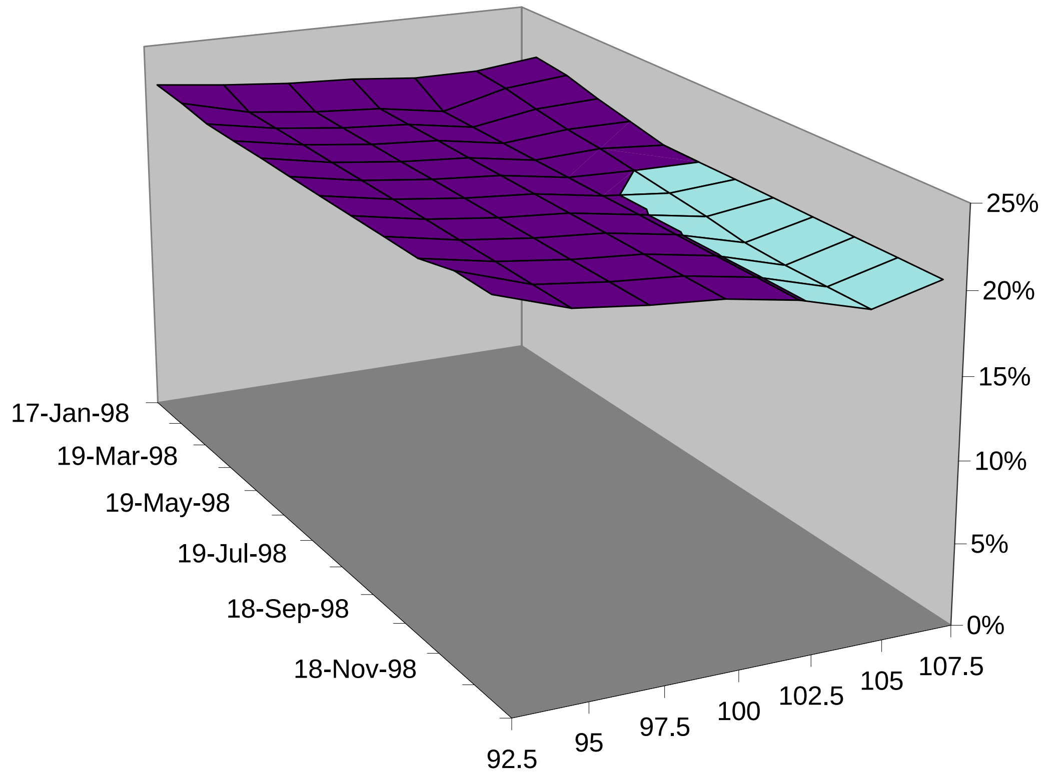
$$p(S^*, t^*; E, T) = \frac{1}{E\sigma_{\text{i}}\sqrt{2\pi(T - t^*)}} e^{-\frac{1}{2}d_2^2}$$

$$\left(\left(1 + Ed_1\sqrt{T - t^*}\frac{\partial \sigma_{\text{i}}}{\partial E}\right)^2 + E^2(T - t^*)\sigma_{\text{i}}\left(\frac{\partial^2 \sigma_{\text{i}}}{\partial E^2} - d_1\left(\frac{\partial \sigma_{\text{i}}}{\partial E}\right)^2\sqrt{T - t^*}\right)\right).$$

And again, we must relabel the variables to get $p(S^*, t^*; E, T)$ in terms of its natural variables S and t instead of E and T.

One of the advantages of writing the local volatility and probability density function in terms of the implied volatility surface is that if you put in a flat implied volatility surface you get out a flat local surface and a lognormal distribution.

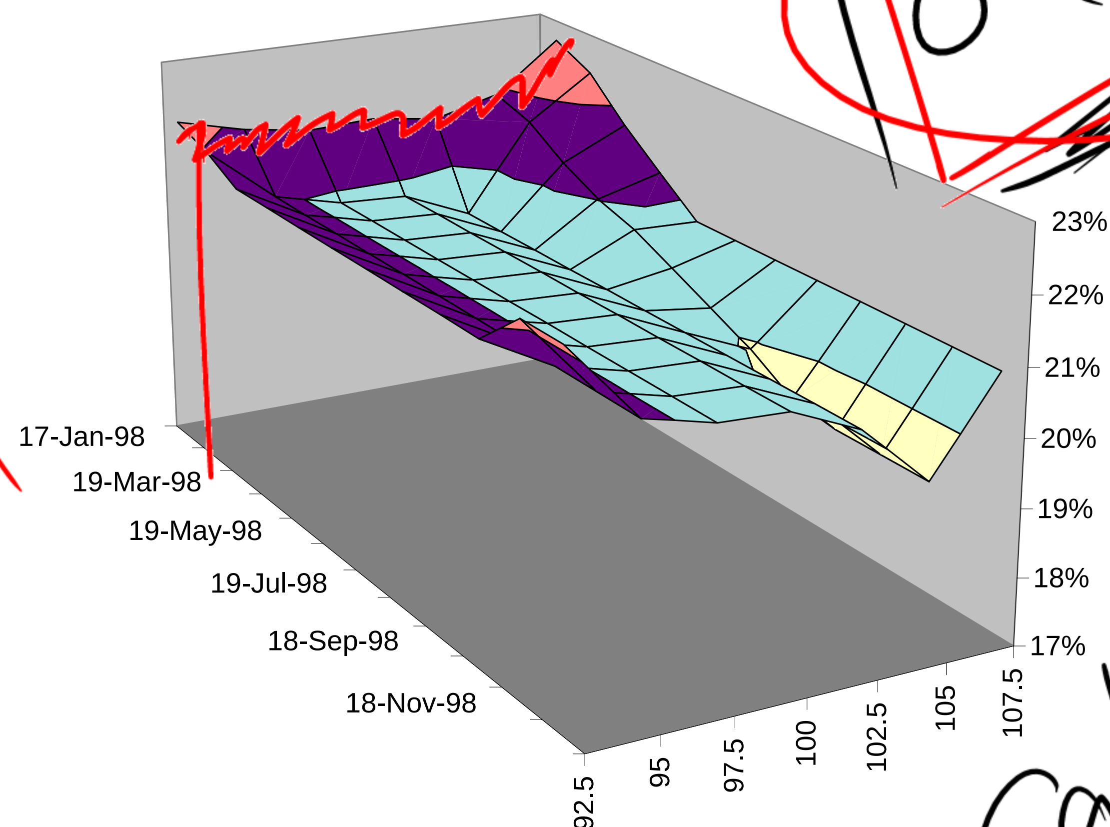
Recall that we started with this as the *implied* volatility surface...



Now we have the *local* volatility surface. . .

$\sigma(S,t)$

$\sigma(S,t)$



Re-Calibrated

Local volatility surface calculated from European call prices.

Much less well behaved!

In practice there only exist a finite, discretely-spaced set of call prices.

To deduce a local volatility surface from this data requires some interpolation and extrapolation.

This can be done in a number of ways and there is no correct way.

One of the problems with these approaches is that the final result depends sensitively on the form of the interpolation.

The problem is actually 'ill-posed,' meaning that a small change in the input can lead to a large change in the output.

There are many ways to get around this ill-posedness, coming under the general heading of 'regularization.'

What do we do with the local volatility surface?

Once we've found $\sigma(S, t)$ we can now use it to find the values of other (non-vanilla) products!

This can be done, for example, by Monte Carlo simulation or by finite difference solution of the relevant PDE.

You can then say that the value of other derivatives "is consistent with the traded options."

Pros and cons of the deterministic volatility ~~model~~ Dupire

Dermant Kani

Pros:

1993

Rubinstein

- Famous people invented it
- It is easy to fool people that it is a good model

Cons:

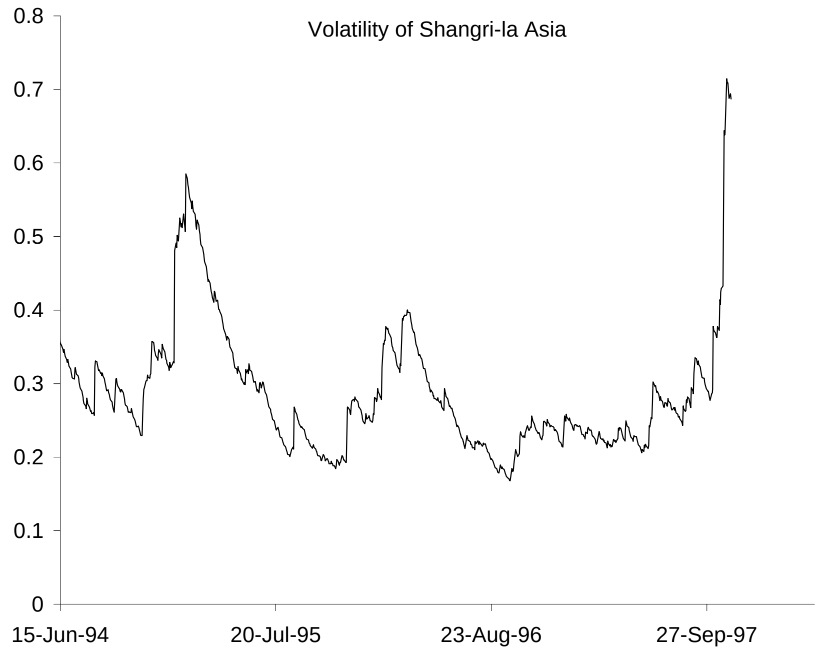
- The assumption $\sigma(S, t)$ is not sensible
- Even if it were sensible, the problem is an inverse problem with serious numerical instability difficulties (the lesson of CSI Miami)
- The results are unstable (the lesson of the fortune teller), proving that the model does not capture correct dynamics

Uncertain Parameters

In the Black–Scholes model which variables and parameters are easily measurable?

- Asset price
- Time to expiry
- Risk-free interest rate (to some extent)
- Dividends (to some extent)

- Volatility: Volatility is certainly not constant as assumed in the simple Black–Scholes formulæ. Here is an example of a volatility time series.



A typical time series for historical volatility; an implied volatility time series would look similar.

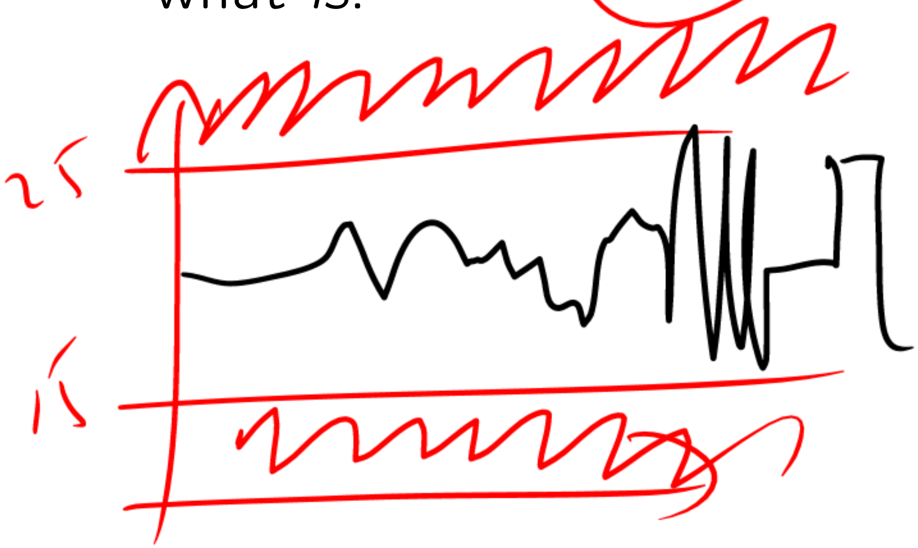
How far can we get by not even attempting to ‘model’ volatility?!

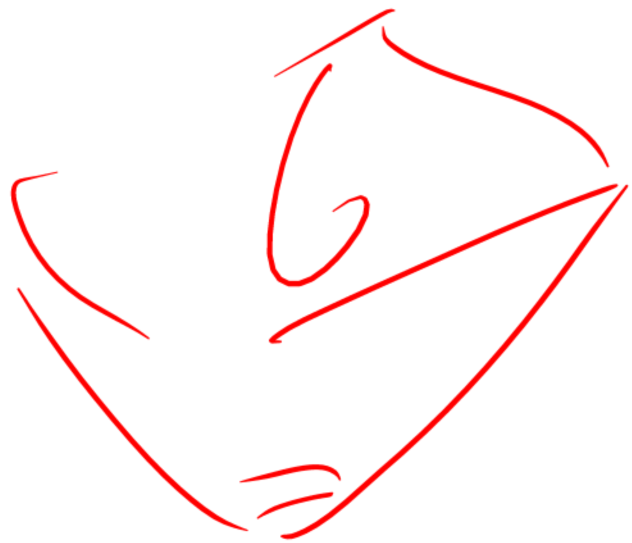
What is ‘uncertainty’?

‘Uncertainty’ is when you do not have either a deterministic or probabilistic description.

You have neither a formula nor a probability density function.

You say what is *not* allowed to happen, with no description of what *is*!





We will address the problem of how to value options when parameter values are *uncertain*.

- We assume that all we know about the parameters is that *they lie within specified ranges*.

- We do not find a *single* value for an option, instead we find that the option's value can also lie within a range: there is no such thing as *the* value.

There are many possible values any of which *might* turn out to be correct.

We will see that this problem is non linear, and thus an option valued in isolation has a different range of values from an option valued as part of a portfolio.

If we put other options into the portfolio this will change the value of the original portfolio.

This leads to the idea of incorporating traded options into an OTC portfolio in such a way as to maximize its value. This is called optimal static hedging.

Best and worst cases

$15 \leq \sigma \leq 25$

The first step in valuing options with uncertain parameters is to acknowledge that we can do no better than give ranges for the future values of the parameters.

For volatility, for example, this range may be the range of past historical volatility, or implied volatilities, or encompass both of these. Then again, it may just be an educated guess.

The range we choose represents our estimate of the upper and lower bounds for the parameter value for the life of the option or portfolio in question. These ranges for parameters lead to ranges for the option's value.

Thus it is natural to think in terms of a lowest and highest possible option value; if you are long the option, then we can also call the lowest value the *worst* value and the highest the *best*.

Uncertain volatility: the model of Avellaneda, Levy & Parás and Lyons (1995)

Suppose that the volatility lies within the band

$$\sigma^- < \sigma < \sigma^+.$$

Follow the Black–Scholes hedging and no-arbitrage argument as far as we can and see where it leads us.

Construct a portfolio of one option, with value $V(S,t)$, and hedge it with $-\Delta$ of the underlying asset.

The value of this portfolio is thus

$$\Pi = V - \Delta S.$$

We still have

$$dS = \mu S\, dt + \sigma S\, dX,$$

even though σ is unknown.

The change in the value of this portfolio is

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \cancel{\left(\frac{\partial V}{\partial S} - \Delta \right) dS}.$$

Even with the volatility unknown, the choice of $\Delta = \partial V / \partial S$ eliminates the risk:

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt.$$

At this stage we would normally say that if we know V then we know $d\Pi$. This is no longer the case since we do not know σ.

The argument now deviates subtly from Black–Scholes.

- We will be pessimistic: the volatility over the next time step is such that the portfolio increases by the least amount.

If we have a long position in a call option, for example, we assume that the volatility is at the lower bound σ^-; for a short call we assume that the volatility is high.

But let's do the analysis in full generality. (It's not much of a model if it will only tell us how to price single calls or puts.)

The return on this worst-case portfolio is then set equal to the risk-free rate:

$$\min_{\sigma^- < \sigma < \sigma^+} (d\Pi) = r\Pi \, dt.$$

Thus we set

$$\min_{\sigma^- < \sigma < \sigma^+} \left(\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt = r \left(V - S \frac{\partial V}{\partial S} \right) dt.$$

Observe that the volatility term is multiplied by the option's gamma.

Therefore the value of σ that will give this its minimum value depends on the sign of the gamma.

When the gamma is positive we choose σ to be the lowest value σ^- and when it is negative we choose σ to be its highest value σ^+.

The worst-case value V^- satisfies

$$\frac{\partial V^-}{\partial t} + \frac{1}{2}\sigma(\Gamma)^2 S^2 \frac{\partial^2 V^-}{\partial S^2} + rS\frac{\partial V^-}{\partial S} - rV^- = 0 \qquad (6)$$

where $\Gamma = \frac{\partial^2 V^-}{\partial S^2}$ and

$$\sigma(\Gamma) = \begin{cases} \sigma^+ & \text{if } \Gamma < 0 \\ \sigma^- & \text{if } \Gamma > 0. \end{cases}$$

VΛ = VΛmin

If Gamma < 0 then VΛ = VΛmax

We can find the best option value V^+, and hence the range of possible values by solving

$$\frac{\partial V^+}{\partial t} + \frac{1}{2}\sigma(\Gamma)^2 S^2 \frac{\partial^2 V^+}{\partial S^2} + rS\frac{\partial V^+}{\partial S} - rV^+ = 0$$

where $\Gamma = \frac{\partial^2 V^+}{\partial S^2}$ but this time

$$\sigma(\Gamma) = \begin{cases} \sigma^+ & \text{if } \Gamma > 0 \\ \sigma^- & \text{if } \Gamma < 0. \end{cases}$$

Two observations about the best case:

1. The best case for a long position is the same as the worst case for a short position. (To see this just change a few signs in the PDE.)

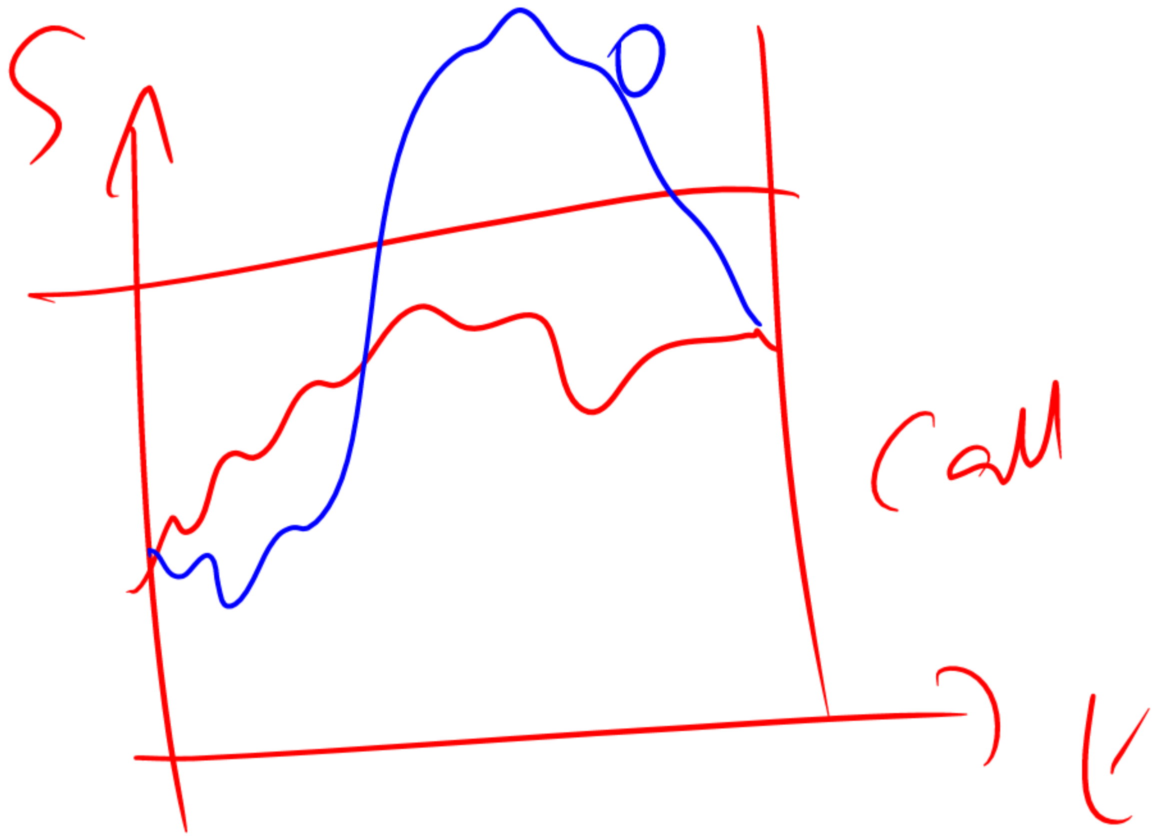
2. In practice it would be madness to go around assuming the best-case outcome for anything!

Equation (6) (Avellaneda, Levy & Parás and Lyons) is the same as the Hoggard–Whalley–Wilmott transaction cost model.

The equation must in general be solved numerically, because it is non linear.

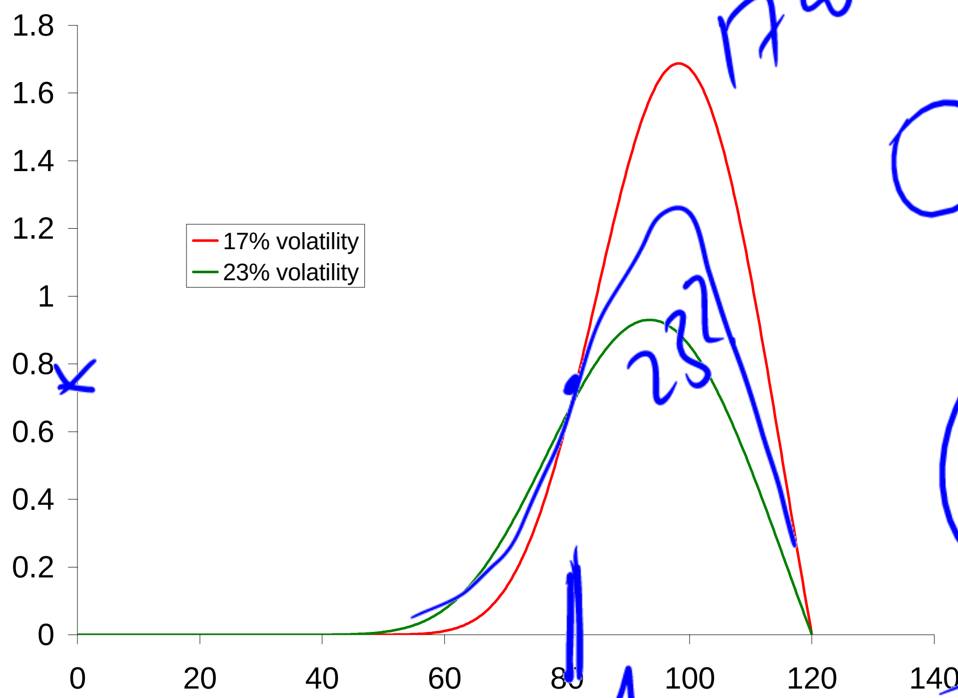
Example: An up-and-out call

Value an up-and-out call. Volatility lies in the range 17% to 23%.



Naive approach: assume volatility is constant. . .

If we naively priced the option using first a 17% volatility and then a 23% volatility we would get two curves looking like those in the following figure.



$O = \frac{\partial}{\partial \sigma}$ = vega

Bastard Greek

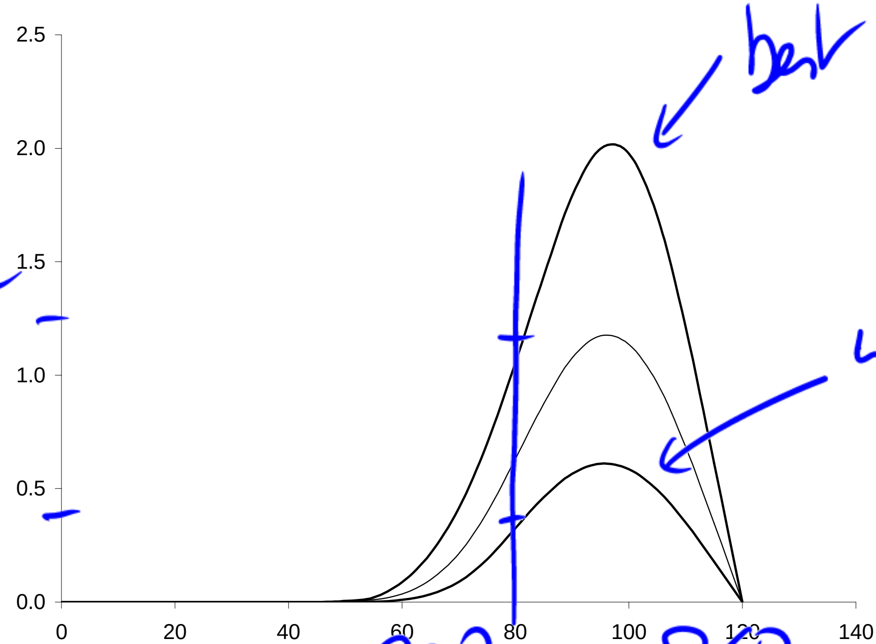
This figure suggests that there is a point at which the value is insensitive to the volatility. Vega is zero.

Actually, the value is very sensitive to volatility as we shall now see (and in the process also see why vega can be a poor measure of sensitivity).

In the next figure are shown the best and worst prices for an up-and-out call option *using the non-linear model*.

In the figure is a Black–Scholes value, the middle line, using a volatility of 20%. The other two bold lines give the worst-case long and short values assuming a volatility ranging over 17% to 23%.

17-23



Note:

- We must solve the non-linear equation numerically because the gamma for this contract is not single signed.

- The problem is genuinely non linear, and we cannot just substitute each of 17% and 23% into a Black–Scholes formula.

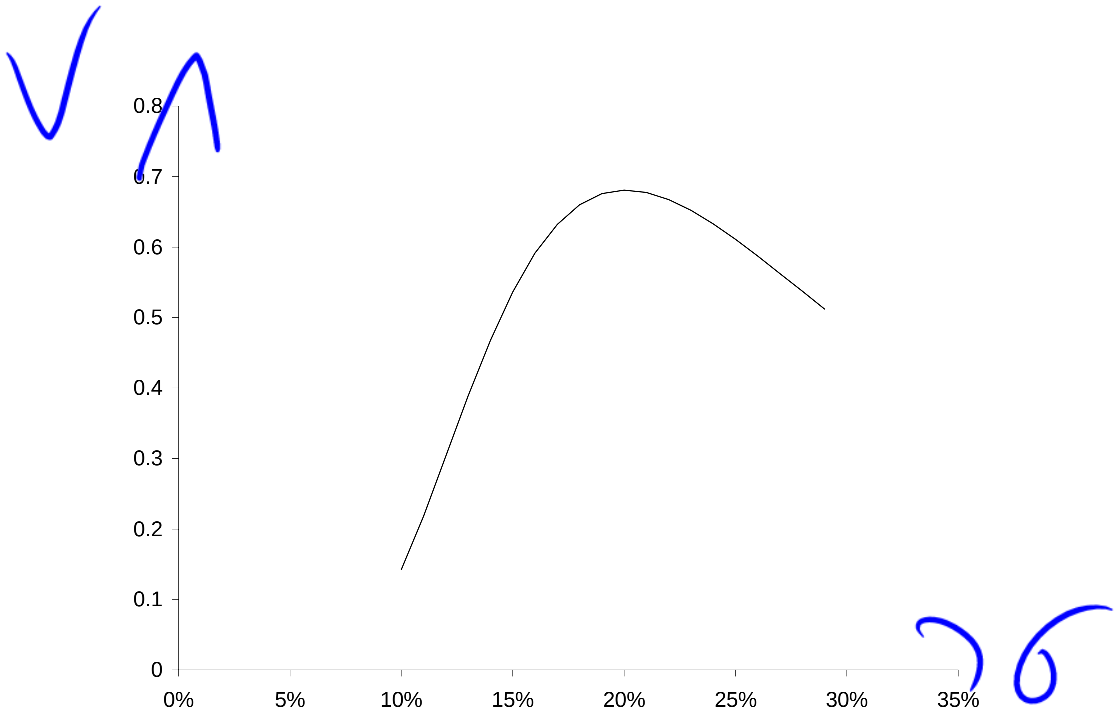
- Observe that the best/worst of these two curves (the 'envelope') is not the same as the best/worst of the previous figure.

- Where is the price most/least sensitive to the volatility?

It can be extremely dangerous to calculate a contract's vega when the contract has a gamma that changes sign.

Continuing with this barrier option example, let us look at implied volatilities.

In the figure below is shown the Black–Scholes value of an up-and-out call option as a function of the volatility.



This contract has a gamma that changes sign, and a price that is not monotonic in the volatility.

This figure shows that there is a maximum option value of 0.68 when the volatility is about 20%.

Suppose that the market is pricing this contract at 0.55. From the figure we can see that there are *two* volatilities that correspond to this market price. Which, if either, is correct?

The question is probably meaningless because of the non-single-signed gamma of this contract.

Take this example further. What if the market price is 0.72? This value cannot be reached by any single volatility.

Does this mean that there are arbitrage opportunities? Not necessarily.

This could be due to the market pricing with a non-constant volatility, either with a volatility surface, stochastic volatility or a volatility range.

As we have seen from the best/worst prices for this contract, the uncertainty in the option value may be large enough to cover the market price of the option, and there may be no guaranteed arbitrage at all.

Nonlinearity

The uncertain parameter partial differential equations that we have derived are non linear.

Because of this nonlinearity, we must distinguish between long and short positions.

For example for a long call we have

$$V^-(S,T) = \max(S-E,0)$$

and for a short call

$$V^-(S,T) = -\max(S-E,0).$$

Because Equation (6) is non linear the value of a portfolio of options is not necessarily the same as the sum of the values of the individual components.

Long and short positions have different values.

For example, a long call position has a lower value than a short call. In both cases we are being pessimistic: if we own the call we assume that it has a low value, if we are short the call and thus may have to pay out at expiry, we assume that the value of the option to its holder is higher.

Note that here we mean a long (or short) position valued *in isolation*.

Obviously, if we hold one of each simultaneously then they will cancel each other regardless of the behavior of any parameters.

This is a very important point to understand: *the value of a contract depends on what else is in the portfolio*.

A problem with the model . . . which will be sorted out

Unfortunately, the model as it stands predicts very wide spreads on options.

For example, suppose that we have a European call, strike price \$100, today's asset price is \$100, there are six months to expiry, no dividends but a spot interest rate that we expect to lie between 5% and 6% and a volatility between 20% and 30%.

We can calculate the values for long and short calls assuming these ranges for the parameters directly from the Black–Scholes formulæ. *This is because the gamma and the portfolio value are single-signed for a call.*

A long call position is worth **\$6.89** (the Black–Scholes value using a volatility of 20% and an interest rate of 5%) and a short call is worth **\$9.88** (the Black–Scholes value using a volatility of 30% and an interest rate of 6%).

This spread is much larger than that in the market. The market prices may, for example, be based on an interest rate of 5.5% with a volatility between 24% and 26%.

Unless the model can produce narrower spreads the model will be useless in practice.

The spreads *can* be tightened by 'static hedging.'

This means the purchase and sale of traded option contracts so as to improve the marginal value of our original position.

This only works because we have a *non-linear* governing equation: the price of a contract depends on what else is in the portfolio.

This static hedge can be optimized so as to give the original contract its best value, we can squeeze even more value out of our contract with the best hedge.

Static Hedging

Delta hedging is a wonderful concept. It leads to preference-free pricing (risk neutrality) and a risk-elimination strategy that can be used in practice. There are quite a few problems, though, on both the practical and the theoretical side.

- In practice, hedging must be done at discrete times and is costly. Sometimes one has to buy or sell a prohibitively large number of the underlying in order to follow the theory.

- On the theoretical side, we have to accept that the model for the underlying is not perfect, at the very least we do not know parameter values accurately.

Many of these problems can be reduced or eliminated if we follow a strategy of static hedging as well as delta hedging: buy or sell more liquid contracts to reduce the cashflows in the original contract.

Static hedging: non-linear governing equation

Many pricing models are non linear:

- Transaction costs: Purchase and sale of the underlying for delta hedging when there are costs leads to non-linear equations for the option value.

- Uncertain parameters: When parameters, such as volatility, dividend rate, interest rate, are permitted to lie in a range, options can be valued in a worst-case scenario.

Non-linear equations

Nonlinearity has many important consequences.

Because of the nonlinearity,

- the value of a portfolio of options is not necessarily the same as the sum of the values of the individual components

- the value of a contract depends on what else is in the portfolio

These two points are key to the importance of non-linear pricing equations: they give us a bid-offer spread on option prices, and they allow *optimal* static hedging.

But the most important point about the non-linear model we have here is that the value of a portfolio of options is at least as valuable as the sum of the individual components:

$$\text{Value}(A + B + C + D + \cdots)$$

$$\geq \text{Value}(A) + \text{Value}(B) + \text{Value}(C) + \text{Value}(D) + \cdots.$$

$$\text{Value}(J + P + G + R) \geq \text{Value}(J) + \text{Value}(P) + \text{Value}(G) + \text{Value}(R)$$

With these models always price a portfolio rather than option by option.

This way you will get cancelation of gamma and so less exposure to volatility, and so better prices!

For the rest of this lecture we discuss the pricing and hedging of options when the governing equation is non linear.

The ideas are applicable to any non-linear models.

We use the notation $V_{NL}(S,t)$ to mean the solution of the model in question, whichever model it may be.

Pricing with a non-linear equation

One of the interesting points about non-linear models is the prediction of a spread between long and short prices.

If the model gives different values for long and short then this is in effect a spread on option prices.

This can be seen as either a good or a bad point.

- It is good because it is realistic, spreads exist in practice.

- It only becomes bad when this spread is too large to make the model useful.

Motivation

Suppose that we want to sell an option with some payoff that does not exist as a traded contract, i.e. an exotic or OTC contract.

We want to determine how low a price can we sell it for, with the constraint that we guarantee that we will not lose money as long as our range for volatility is not breached.

Using the uncertain parameter model we will expect to get best-worst ranges that are too big. This means that we will lose the deal.

We will now see how to use traded options to decrease that bid-offer spread.

To make the example as simple to follow as possible, we are going to ‘pretend’ that a vanilla call is ‘exotic’ i.e. not traded.

This is easier to visualize, to gain intuition.

Example:

- European call
- strike price \$100
- today's asset price is \$100
- there are six months to expiry
- no dividends
- interest rate of 5%
- volatility lies between 20% and 30%

Long call position is worth **$6.89**.

Short call is worth **$9.63**.

This spread is too large and we will be unable to either buy or sell the contract!

Static hedging

Suppose that call options on this particular stock are traded with strikes of \$90 and \$110 and with six months to expiry.

Strike	Expiry	Bid	Ask	Quantity
90	180 days	14.42	14.42	?
110	180 days	4.22	4.22	?

Can we take advantage of these contracts for pricing and/or hedging our 100 call?

Suppose that the market prices the 90 and 110 calls with an implied volatility of 25%.

The market prices, i.e. the Black–Scholes prices, are therefore 14.42 and 4.22 respectively. These numbers are shown in the table.

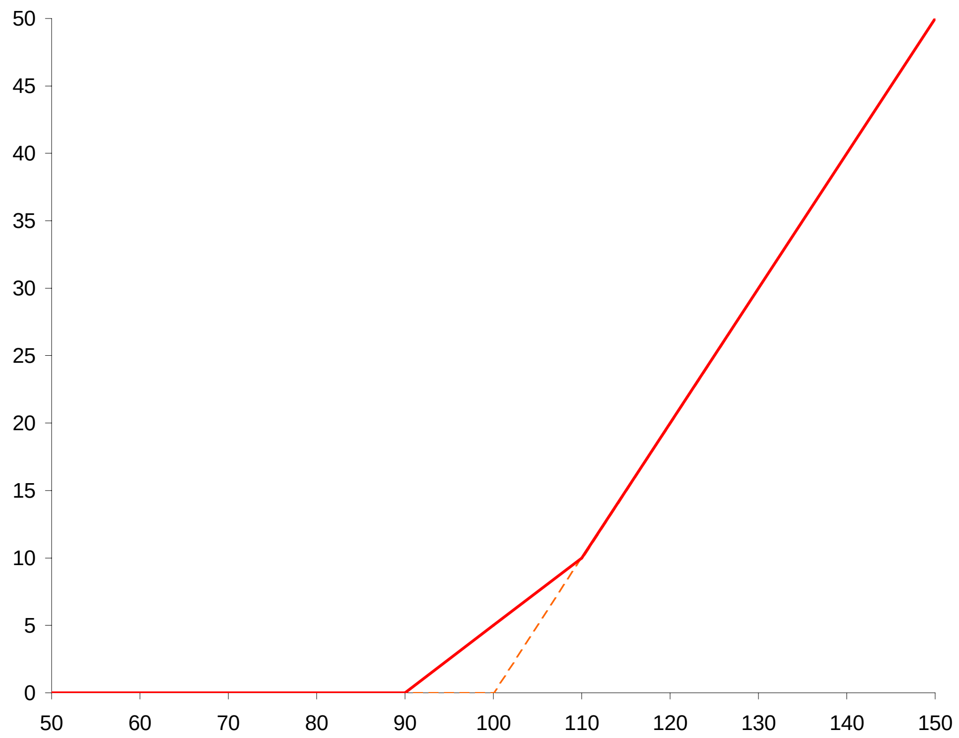
The question marks are to emphasize that we can buy or sell as many of these contracts as we want, but in a fashion which will be made clear shortly.

Shouldn't our quoted prices for the 100 call reflect the availability of contracts with which we can hedge?

Let's try hedging the 100 call with the traded options and see what happens. . .

Let's buy 0.5 of the 90 calls and 0.5 of the 110s.

This 'hedging portfolio' has payoff as shown:



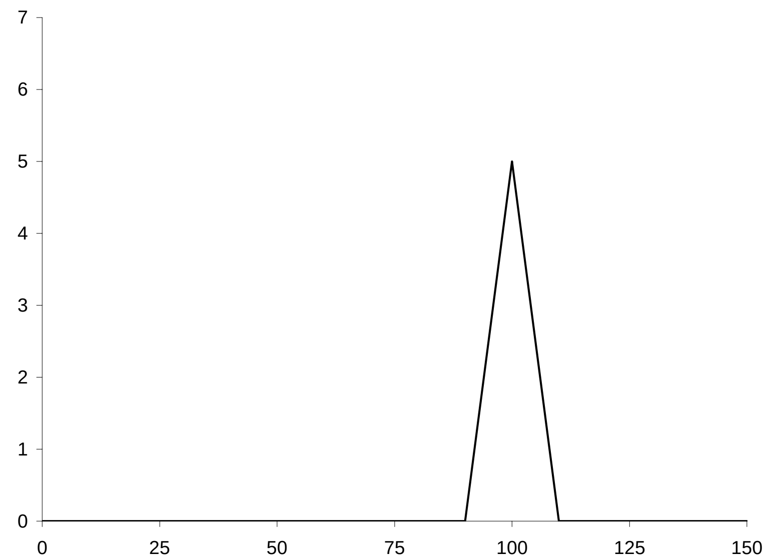
This payoff is very close to the payoff for our 100 call, so the values should be 'similar.'

And we know the value of the hedge from the market!

Hold that thought...

The values won't be the same, however, because of the difference in payoffs.

If we sell the 100, and ‘statically hedge’ it by buying 0.5 of the 90 and 0.5 of the 110, then we have a residual payoff as shown in the figure.



We call this a static hedge because we put it in place now and do not expect to change it. This contrasts with the delta hedge, for which we expect to hedge frequently.

The statically-hedged portfolio has a much smaller payoff than the original unhedged call.

It is this *new portfolio* that we value by solving the non-linear pricing equation, and that we must delta hedge; this last point must be emphasized, *the residual payoff must be delta hedged*.

And because the payoff is much smaller, the bid-offer or worst/best spread will be smaller.

To value the residual payoff in our uncertain parameter framework we solve the non-linear pricing equation with final condition

$$V_{NL}(S,T) = -\max(S-100,0)+$$

$$\frac{1}{2}(\max(S-90,0)+\max(S-110,0)).$$

Let us see what effect this has on the price at which we would *sell* the 100 call.

First of all, observe that we have paid

$$0.5 \times \$14.42 + 0.5 \times \$4.22 = \$9.32$$

for the static hedge, 0.5 of each of the 90 and 110 calls.

Now, solve the equation using the residual payoff as the final condition. The solution gives a value for the residual contract today of \$0.61.

The net value of the call is therefore

$$\$9.32 - \$0.61 = \$8.71.$$

To determine how much we should pay to *buy* the 100 call we take as our starting point the sale of 0.5 of each of the 90 and 110 calls.

This nets us \$9.32.

But now we solve the equation using the *negative* of the previous residual payoff as the final condition; the effect of the nonlinearity is different from the previous case because $\sigma(\Gamma)$ takes different values in different places in S, t-space.

We get a value of \$1.65 for the hedged position.

Thus we find that we would pay

$$\$9.32 - \$1.65 = \$7.67.$$

Note how the use of a very simple static hedge has reduced the spread from \$6.89–9.63 to \$7.67–8.71.

This is a substantial improvement, as it represents a volatility range of 23–27%; while our initial estimate for the volatility range was 20–30%

The reason for the smaller spread is that the residual portfolio has a smaller absolute spread, only \$0.61–1.65 and this is because it has a much smaller payoff than the unhedged 100 call.

In the above example we decided to hedge the 100 call using 0.5 of each of the 90 and 110 calls.

What prompted this choice?

There was no reason why we should not choose other numbers. Since our problem is non linear *the value of our OTC option depends on the combination of options with which we hedge*.

So, generally speaking, we expect a different OTC option value if we choose a different static hedge portfolio.

Of course, we now ask 'If we get different values for an option depending on what other contracts we hedge it with then is there a *best* static hedge?'

Optimal static hedging

Continuing with this example, what are the optimal static hedges for long and short positions?

Buying 0.5 each of the two calls to hedge the short 100 call we find a marginal value of \$8.71 for the 100 call.

Slightly better, and *optimal*, is to buy 0.62 of the 90 call and 0.53 of the 110 call giving a marginal value of **\$8.66**.

The optimal hedge for a long position is different. We should sell 0.67 of the 90 call and sell 0.55 of the 110 call. The marginal value of the 100 call is then **\$7.76**.

With symbols

Here is the full definition of 'value' in this model, in words first:

$$\text{Value(Exotic)} =$$

$$\max_{q_i}\left(V_{NL}\left(\text{Exotic}+\sum_{i=1}^{N} q_i\ \text{Vanilla}_i\right)-\sum_{i=1}^{N} q_i\ \text{CostVanilla}_i\right).$$

The value of an exotic depends on what it is hedge with. . . and after it has been optimally hedged.

Let's do the maths.

Suppose that we want to find the lowest price for which we can sell a particular OTC or 'exotic' option with payoff $\Lambda(S)$.

Suppose that we can hedge our exotic with a variety of traded options, of which there are n.

These options will have payoffs (at the same date as our exotic to keep things simple for the moment) which we call

$$\Lambda_i(S).$$

At this point we can introduce bid and offer prices for the traded options: C_i^+ is the ask price of the ith option and C_i^- the bid, with $C_i^- < C_i^+$.

Now we set up our statically hedged portfolio: we will have λ_i of each option in our hedged portfolio.

The cost of setting up this static hedge is

$$\sum_i \lambda_i C_i(\lambda_i),$$

where

$$C_i(\lambda_i) = \begin{cases} C_i^+ & \text{if } \lambda_i > 0 \\ C_i^- & \text{if } \lambda_i < 0 \end{cases} .$$

If $\lambda_i > 0$ then we have a positive quantity of option i at the offer price in the market, C_i^+; if $\lambda_i < 0$ then we have a negative quantity of option i at the bid price in the market, C_i^-.

We let $V^{-}(S,t)$ be the pessimistic value of our *hedged* position.

The residual payoff for our statically hedged option is

$$V^{-}(S,T) = \Lambda(S) + \sum_i \lambda_i \Lambda_i(S).$$

Now we solve the pricing equation with this as final data, to find the *net* value of our position (today, at time $t = 0$, say) as

$$V^{-}(S(0),0) - \sum_i \lambda_i C_i(\lambda_i) = F(\lambda_1, \ldots, \lambda_n).$$

This is a mathematical representation of the type of problem we solved in our first hedging example.

Our goal now is to choose the λ_i to minimize $F(\cdots)$ if we are selling the exotic, and maximize if buying. (Thus the best hedge in the two cases will usually be different.) This is what we mean by 'optimal' static hedging.

Calibration?

What does all this mean for calibration?

Do our theoretical prices look anything like market prices?

In the trivial case where the option on which we are quoting is also traded then we would find that our quoted price was the same as the market price.

This is because we would hedge one for one and the residual payoff, which we would normally delta hedge, would be identically zero.

What does this say about calibration?

Calibration is perfect. . . by definition!

What about calibrating to both bid and offer? And liquidity?

Pros and cons of the uncertain volatility model

Pros:

- It is robust
- It values, and dynamically hedges, and statically hedges at the same level, consistently within the model
- Portfolios can be optimized
- Calibration is automatic

Cons:

- Everything has to be solved numerically, by finite-difference methods
- Cannot be used by the buy side

Summary

Please take away the following important ideas

- The most parsimonious volatility model that can be made consistent with market prices is that of deterministic volatility, $\sigma(S,t)$
- 'Uncertainty' is when you do not have either a deterministic or probabilistic description
- Volatility can be modeled as uncertain, which leads to a non-linear pricing equation and the possibility of static hedging