Stochastic Calculus and Martingales

A Practitioner's Toolbox

CQF

In this lecture...

... we talk about stochastic calculus and about martingales:

- What is a martingale in continuous time?
- Martingales and Itô calculus
- Martingale unmasked: how do I know if my stochastic process is a martingale?
- Exponential martingales, Girsanov and change of measure
- The martingale zoology in a hurry: martingale, local martingale, supermartingale, submartingales, semimartingales...
- Itô in higher dimensions
- The Itô product rule

The term 'martingale' may refer to very different ideas...

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... a piece of equipment used to control the head carriage of horses.

The term 'martingale' may refer to very different ideas...

... a once popular gambling strategy that can have disastrous consequences.

The martingale betting system is thought to have originated in XVIIIth century France.

The martingale strategy for a coin toss gain is very simple: whenever you lose, double your bets!

The idea is that a single win is enough to erase all of your previous losses.

However, it is a bad idea because you have no idea when the next 'win' will occur. Unless you have unlimited funding, the actual probability of ruin is very high.

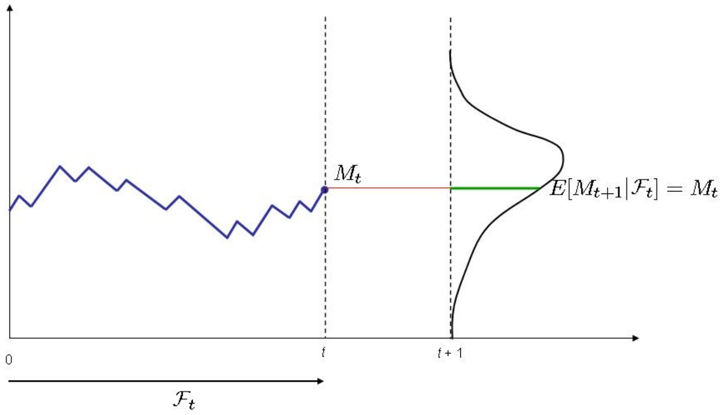
Figure : 'The card players' by Theodoor Rombouts (1597-1637)

The term 'martingale' may refer to very different ideas...

... a stochastic process that has no drift.

Essentially, this is the idea of a fair (random) game.

This is the concept we explore in this lecture.



1 Martingales Is A Driftless Stochastic Process

Martingales are a key concept in probability and in mathematical financial.

We encounter them through three distinct, but closely connected ideas:

1. *Martingales* as a class of stochastic process;
2. *Exponential martingales*, which are a specific and extremely useful example of a martingale;
3. *Equivalent martingale measures*, where we look for a probability measure $\mathbb{Q}$ such that a given stochastic process $S(t)$ is a martingale under $\mathbb{Q}$ regardless of its nature under $\mathbb{P}$. The correspondence between the measures $\mathbb{P}$ and $\mathbb{Q}$ is done through a change of measure.

Most of this lecture focuses on the first concept, that is, martingales as a class of stochastic processes.

We explore exponential martingales and describe their relation with change of measures towards the end of the lecture.

Equivalent martingale measures are central to derivatives pricing. We will see this idea in detail in Modules 3 and 4.

1.1 Reminder: Discrete Time Martingales

Definition (Discrete Time Martingale)

A discrete time stochastic process

$$\{M_t : t = 0, \dots, T\}$$

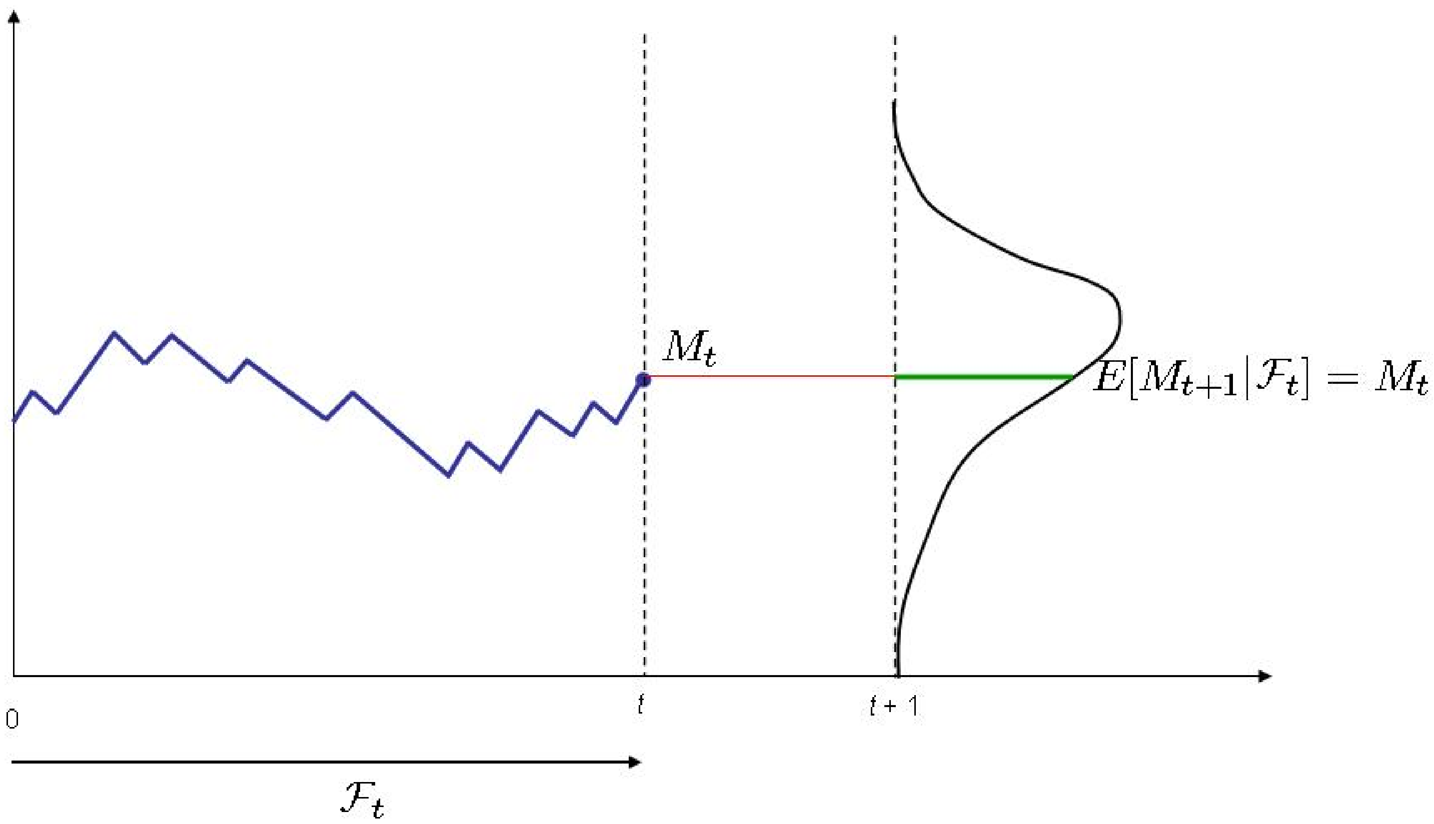
such that M_t is $\mathcal{F}_t$-measurable for $t = 0, \dots, T$ is a **martingale** if

$$\mathbb{E}\,|M_t| < \infty$$

and

$$\mathbb{E}\left[M_{t+1}|\mathcal{F}_t\right] = M_t \tag{1}$$

Figure : Discrete Time Martingale



1.2 Continuous Time Martingales

Next, we generalize our definitions to continuous time:

Definition (Continuous Time Martingale)

A continuous time stochastic process

$$\left\{ M_t : t \in \mathbb{R}^+ \right\}$$

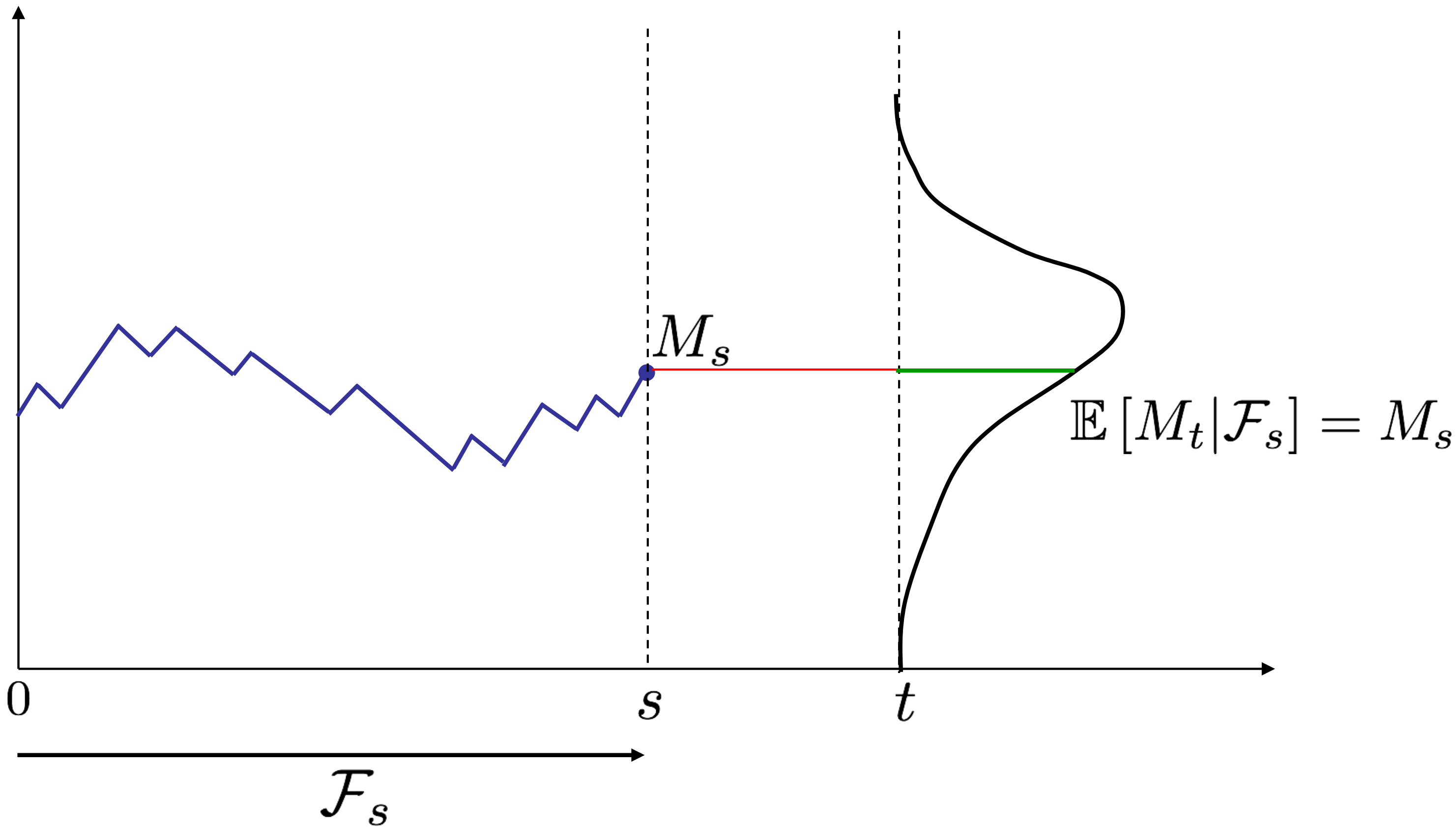
such that M_t is $\mathcal{F}_t$-measurable for $t \in \mathbb{R}^+$ is a **martingale** if

$$\mathbb{E}\left|M_t\right| < \infty$$

and

$$\mathbb{E}\left[M_t \middle| \mathcal{F}_s\right] = M_s, \quad 0 \leq s \leq t$$

Figure : Continuous Time Martingale



2 A Few More Things About Martingales

1. Brownian motions are martingales;
2. Itô integrals are martingales;
3. Markov processes are not necessarily martingales (and vice versa);

2.1 A Brownian Motion is a Martingale

In the Stochastic Calculus lecture we saw that Brownian motions are martingales. In fact, not only are Brownian motion martingales, but they can be characterized in terms of martingales.

Key Fact (Levy's Martingale Characterization)

Let $X(t), t > 0$ be a stochastic process and let $\mathcal{F}_t$ be the filtration generated by it. $X(t)$ is a Brownian motion iff all of the following conditions are satisfied:

1. $X(0) = 0$ *a.s.;*
2. *the sample paths* $t \mapsto X(t)$ *are continuous a.s.;*
3. X_t *is a martingale with respect to the filtration* $\mathcal{F}_t$*;*
4. $|X_t|^2 - t$ *is a martingale with respect to the filtration* $\mathcal{F}_t$*.*

The Lévy characterization can be contrasted with the classical definition of a Brownian motion as a stochastic process $X(t)$ satisfying:

1. $X(0) = 0$ a.s.;
2. the sample paths $t \mapsto X(t)$ are continuous a.s.;
3. **independent increments**: for $t_1 < t_2 < t_3 < t_4$ the increments $X_{t_4} - X_{t_3}$ $X_{t_2} - X_{t_1}$ are independent;
4. **normally distributed increments**: $X_t - X_s \sim N(0, |t - s|)$.

Lévy's characterization neither mentions independent increments nor normally distributed increments.

Instead, Lévy introduces two easily verifiable martingale conditions.

2.2 Itô Integrals are Martingales

Next, we explore the link between Itô integration and martingale.

Consider the stochastic process $Y(t) = X^2(t)$. By Itô, we have

$$X^2(T) = T + \int_0^T 2X(t)dX(t)$$

Taking the expectation, we get

$$E[X^2(T)] = T + E\left[\int_0^T 2X(t)dX(t)\right]$$

Now, the quadratic variation property of Brownian motions implies that

$$E[X^2(T)] = T$$

and hence

$$E\left[\int_0^T 2X(t)dX(t)\right] = 0$$

Therefore, the Itô integral

$$\int_0^T 2X(t)dX(t)$$

is a martingale.

In fact, this property is shared by all Itô integrals.

Key Fact

Itô Integrals are Martingales.

Let $g(t, X_t)$ be a function on [0,T] and satisfying technical condition. Then the Itô integral

$$\int_0^T g(t, X_t) dX_t$$

is a martingale.

So, Itô integrals are martingales.

But does the converse hold? Can we represent any martingale as an Itô integral?

The answer is yes!

Key Fact (Martingale Representation Theorem)

If M is a martingale, then there exists a function $g(t, X_t)$ satisfying technical condition such that

$$M_T = M_0 + \int_0^T g(t, X_t) dX_t$$

Example

We will show that

$$\mathbb{E}\left[X^2(T)\right] = T$$

using only Itô and the fact that Itô integrals are martingales.

Consider the function $F(t, X_t) = X_T^2$, then by Itô's lemma,

$$\begin{aligned} X_T^2 &= X_0^2 + \frac{1}{2}\int_0^T 2dt + \int_0^T 2X_t dX_t \\ &= \int_0^T dt + 2\int_0^T X_t dX_t \end{aligned}$$

since $X_0 = 0$

Taking the expectation,

$$\mathbb{E}\left[X_T^2\right] = \mathbb{E}\left[\int_0^T dt\right] + 2\mathbb{E}\left[\int_0^T X_t dX_t\right]$$

Now, $\int_0^T X_t dX_t$ is an Itô integral and as a result $\mathbb{E}\left[\int_0^T X_t dX_t\right] = 0$

Moreover,

$$\mathbb{E}\left[\int_0^T dt\right] = \mathbb{E}[T] = T$$

We can conclude that

$$\mathbb{E}\left[X^2(T)\right] = T$$

As an aside, we can usually exchange the order of integration between the time integral and the expectation so that

$$\mathbb{E}\left[\int_0^T f(X_t)dt\right] = \int_0^T \mathbb{E}\left[f(X_t)\right] dt$$

This is due to an analysis result known as **Fubini's Theorem**.

(end of example)

Aside: Properties of Itô Integrals:

Key Fact

1. **Linearity**:

$$\int_0^T (\alpha f(t) + \beta g(t))\, dX_t = \int_0^T \alpha f(t) dX_t + \int_0^T \beta g(t) dX_t$$

2. **Itô isometry**:

$$\mathbb{E}\left[\left|\int_0^T f(t) dX_t\right|^2\right] = \mathbb{E}\left[\int_0^T |f(t)|^2\, dt\right]$$

3. **Martingale**:

$$\mathbb{E}\left[\int_0^T f(t) dX_t \,|\mathcal{F}_s\right] = \int_0^s f(t) dX_t$$

The **linearity property** is carried over from the general definition of integrals as the limit of a sum.

The **isometry property** is used to extend the definition of the Itô integral to a very general class of functions. As a result, it is often mentioned as one of the key properties.

(end of aside)

2.3 Markov vs. Martingale

Markov processes and martingales are not synonymous.

The martingale property states that the expected value at time t is the value realized at time s, with $s < t$. The process is **driftless**.

The Markov property implies that the expected value at time t does not depend on the history (trajectory) up to time s, but it may depend on the value at time s. The process is **memoryless** (no path dependence).

3 Proving that a Continuous Time Stochastic Process is a Martingale

Let's consider a stochastic process $Y(t)$ solving the following SDE:

$$dY(t) = f(t)dt + g(t)dX(t), \qquad Y(0) = Y_0$$

How can we tell whether $Y(t)$ is a martingale?

The answer has to do with the fact that Itô integrals are martingales.

$Y(t)$ is a martingale if and only if it satisfies the martingale condition

$$\mathbb{E}\left[Y_t | \mathcal{F}_s\right] = Y_s, \quad 0 \leq s \leq t$$

Let's start by integrating the SDE between s and t to get an exact form for $Y(t)$:

$$Y(t) = Y(s) + \int_s^t f(u)du + \int_s^t g(u)dX(u)$$

Taking the expectation conditional on the filtration at time s, we get

$$\begin{aligned} \mathbb{E}\left[Y_t|\mathcal{F}_s\right] &= \mathbb{E}\left[Y(s) + \int_s^t f(u)du + \int_s^t g(u)dX(u)|\mathcal{F}_s\right] \\ &= Y(s) + \mathbb{E}\left[\int_s^t f(u)du|\mathcal{F}_s\right] \end{aligned}$$

where the last line follows from the fact that a Itô integral is a martingale and therefore

$$\mathbb{E}\left[\int_s^t g(u)dX(u)|\mathcal{F}_s\right] = \int_s^s g(u)dX(u) = 0$$

.

So, $Y(t)$ is a martingale if and only if

$$\mathbb{E}\left[\int_s^t f(u)du|\mathcal{F}_s\right] = 0$$

This condition is satisfied only if $f(t) = 0$ for all t.

Going back to our SDE, we conclude that $Y(t)$ is a martingale if and only if it is of the form

$$dY(t) = g(t)dX(t), \qquad Y(0) = Y_0$$

This is why we say that martingales are “driftless processes”!

Example

Are the following processes martingales?

(i) $Y(t) = X(t) + 4t$

(ii) $Y(t) = X^2(t) + k$, where k is a given constant. Does the answer depend on the value of k?

(iii) $Y(t) = t^2 X(t) - 2\int_0^t sX(s)ds$

(iv) $Y(t) = X_1(t)X_2(t)$ where $X_1(t)$ and $X_2(t)$ are two standard Brownian motions with correlation ρ so that $dX_1(t)dX_2(t) \to \rho dt$. Does the answer depend on the value of ρ?

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Intuitively, this cannot be a Martingale since $X(t)$ is a martingale and $4t$ adds some drift.

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Mathematically, the SDE for $Y(t)$ is:

$$dY(t) = 4dt + dX(t)$$

$Y(t)$ is a Brownian motion with drift. Hence, $Y(t)$ is **not a martingale**.

(ii) $Y(t) = X^2(t) + k$

Intuitively, this cannot be a Martingale since $X^2(t) - t$ is a martingale (recalling the *quadratic variation* property of Brownian motions!).

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Mathematically, by Itô applied to the function $f(x) = x^2 + k$, the SDE for the process $Y(t) = f(X(t))$ is given by

$$dY(t) = dt + 2X(t)dX(t)$$

The dynamics of $Y(t)$ has a drift: $Y(t)$ is **not a martingale** and this result is independent from the specific value of k.

(iii) $Y(t) = t^2 X(t) - 2\int_0^t sX(s)ds$

The easiest way to tackle this problem is by defining a new stochastic process $Z(t) = t^2 X(t)$. To check that $Y(t)$ is a martingale it is enough to check that the drift of $Z(t)$ is equal to $2\int_0^t sX(s)ds$.

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By Itô applied to the function $f(s,x) = s^2x$, the SDE for the process $Z(t) = f(t, X(t))$ is

$$dZ(t) = 2tX(t)dt + t^2dX(t)$$

with $Z(0) = 0$.

Integrating over $[0, t]$,

$$Z(t) = 2\int_0^t sX(s)ds + \int_0^t s^2 dX(s)$$

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As required, the drift of $Z(t)$ is equal to $2\int_0^t sX(s)ds$. Hence

$$Y(t) = \int_0^t s^2 dX(s)$$

and $Y(t)$ **is a martingale**.

(iv) $Y(t) = X_1(t)X_2(t)$ where $X_1(t)$ and $X_2(t)$ are two independent standard Brownian motions

By the *Itô product rule*,

$$dY(t) = X_1(t)dX_2(t) + X_2(t)dX_1(t) + \rho dt$$

For $Y(t)$ to be a martingale, its dynamics must be driftless, i.e. we must have $\rho dt = 0$.

This is only the case when $\rho = 0$ and the two Brownian motions $X_1(t)$ and $X_2(t)$ are independent.

In the general case, when $\rho \neq 0$, $Y(t)$ is **not a martingale**.

4 Exponential Martingales

Let's start with a motivating example.

Consider the stochastic process $Y(t)$ satisfying the SDE

$$dY(t) = f(t)dt + g(t)dX(t), \qquad Y(0) = Y_0 \tag{2}$$

where $f(t)$ and $g(t)$ are two time-dependent functions and $X(t)$ is a standard Brownian motion.

Define a new process $Z(t) = e^{Y(t)}$

How should we choose $f(t)$ if we want the process $Z(t)$ to be a martingale?

Consider the process $Z(t) = e^{Y(t)}$. Applying Itô to the the function $F(y) = e^y$ and the process $Y(t)$ given in (2), we obtain:

$$
\begin{aligned}
dZ(t) &= de^{Y(t)} \\
&= \frac{dF}{dy}\left(f(t)dt + g(t)dX(t)\right) + \frac{1}{2}\frac{d^2F}{dy^2}g^2(t)dt \\
&= e^{Y(t)}\left(f(t) + \frac{1}{2}g^2(t)\right)dt + e^{Y(t)}g(t)dX(t) \\
&= Z(t)\left[\left(f(t) + \frac{1}{2}g^2(t)\right)dt + g(t)dX(t)\right]
\end{aligned}
$$

$Z(t)$ is a martingale if and only if it is a driftless process.

Therefore for $Z(t)$ to be a martingale we must have

$$f(t) + \frac{1}{2}g^2(t) = 0$$

This is only possible if

$$f(t) = -\frac{1}{2}g^2(t)$$

Going back to the process $Y(t)$, we must have

$$dY(t) = -\frac{1}{2}g^2(t)dt + g(t)dX(t), \qquad Y(0) = Y_0$$

implying that

$$Y(T) = Y_0 - \frac{1}{2}\int_0^T g^2(t)dt + \int_0^T g(t)dX(t)$$

Hence, in terms of Z(t):

$$dZ(t) \;=\; Z(t)g(t)dX(t)$$

and

$$Z(t) \;=\; Z_0 \exp\left\{-\frac{1}{2}\int_0^T g^2(t)dt + \int_0^T g(t)dX(t)\right\}$$

Because the stochastic process $Z(t)$ is the exponential of another process (namely $Y(t)$) and because it is a martingale, we call $Z(t)$ an **exponential martingale**.

We have actually just stumbled upon a much more general and very important result.

Key Fact (Novikov Condition)

A stochastic process $Y(t)$ satisfies the **Novikov condition** *if*

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \theta_s^2 ds\right)\right] < \infty$$

Key Fact

If a stochastic process $\theta(t)$ satisfies the Novikov condition, then the process M^θ defined as

$$M_t^\theta \;=\; \exp\left(-\int_0^t \theta_s dX_s - \frac{1}{2}\int_0^t \theta_s^2 ds\right), \quad t \in [0, T]$$

is a martingale.

Example

In our earlier example,

- $\theta_t = -g(t)$;
- $M_t^\theta = Z(t)$.

Exponential martingales are extremely important in probability theory because they can be used as a **Radon Nikodým derivative** Λ to define a new probability measure $\mathbb{P}$ via

$$\frac{d\mathbb{Q}}{d\mathbb{P}}_t = \Lambda_t$$

This idea is central to **Girsanov's Theorem**. Girsanov's Theorem is an essential result in mathematical finance. We will manipulate it extensively in Modules 3 and 4. For now, we simply state the main result.

Reminder: The Radon-Nikodym Theorem

Key Fact (The Radon-Nikodym Theorem)

If the measures $\mathbb{P}$ *and* $\mathbb{Q}$ *share the same null sets, then, there exists a random variable* Λ *such that for all subsets* $A \subset \Omega$

$$\mathbb{Q}(A) = \int_A \Lambda d\mathbb{P}$$

where

$$\Lambda = \frac{d\mathbb{Q}}{d\mathbb{P}}$$

is called the **Radon-Nikodym derivative**.

Key Fact (Girsanov's Theorem)

Given a process θ satisfying the Novikov condition, we can define the probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ equivalent to $\mathbb{P}$ through the Radon Nicodým derivative

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\int_0^t \theta_s dX_s - \frac{1}{2}\int_0^t \theta_s^2 ds\right), \quad t \in [0, T]$$

In this case, the process $X^{\mathbb{Q}}$ defined as

$$X_t^{\mathbb{Q}} = X_t + \int_0^t \theta(s) ds, \quad t \in [0, T] \tag{3}$$

is a standard Brownian Motion on $(\Omega, \mathcal{F}, \mathbb{Q})$.

5 The Martingale Zoology in a Hurry

While reading articles and books, you may encounter the terms

- supermartingale;
- submartingale;
- local martingale;
- semimaringales;
- quasimartingale.

How do these concepts relate to the martingales we have discussed so far?

Supermartingales and Submartingales

Definition (Discrete Time Supermartingale (Submartingale))

A discrete time stochastic process

$$\{M_t : t = 0, \ldots, T\}$$

such that M_t is $\mathcal{F}_t$-measurable for $t = 0, \ldots, T$ is a **supermartingale (submartingale)** if

$$\mathbb{E}\,|M_t| < \infty$$

and

$$\mathbb{E}\,[M_{t+1}|\mathcal{F}_t] \leq (\geq) M_t$$

Figure : Discrete Time SuperMartingale

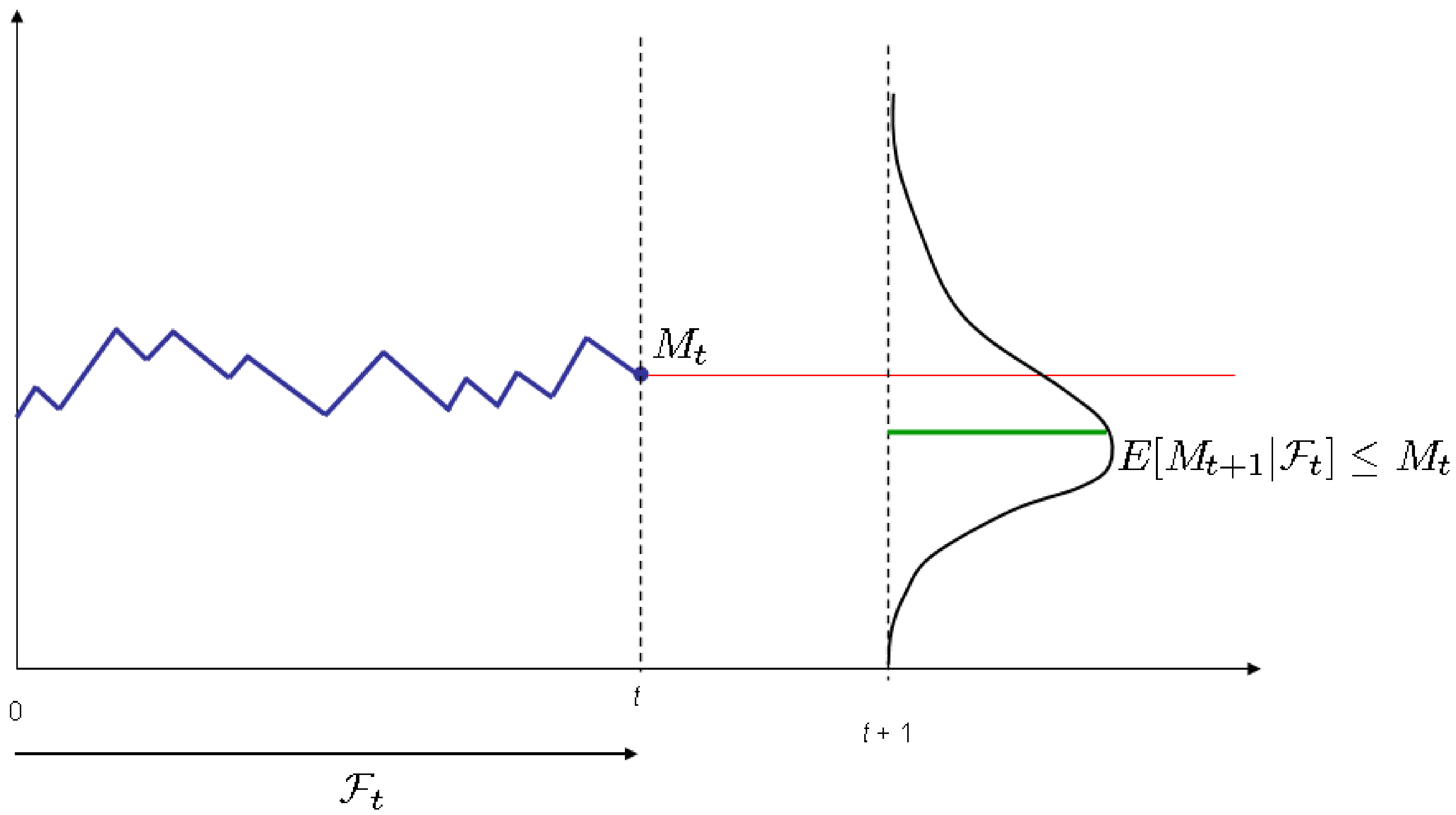
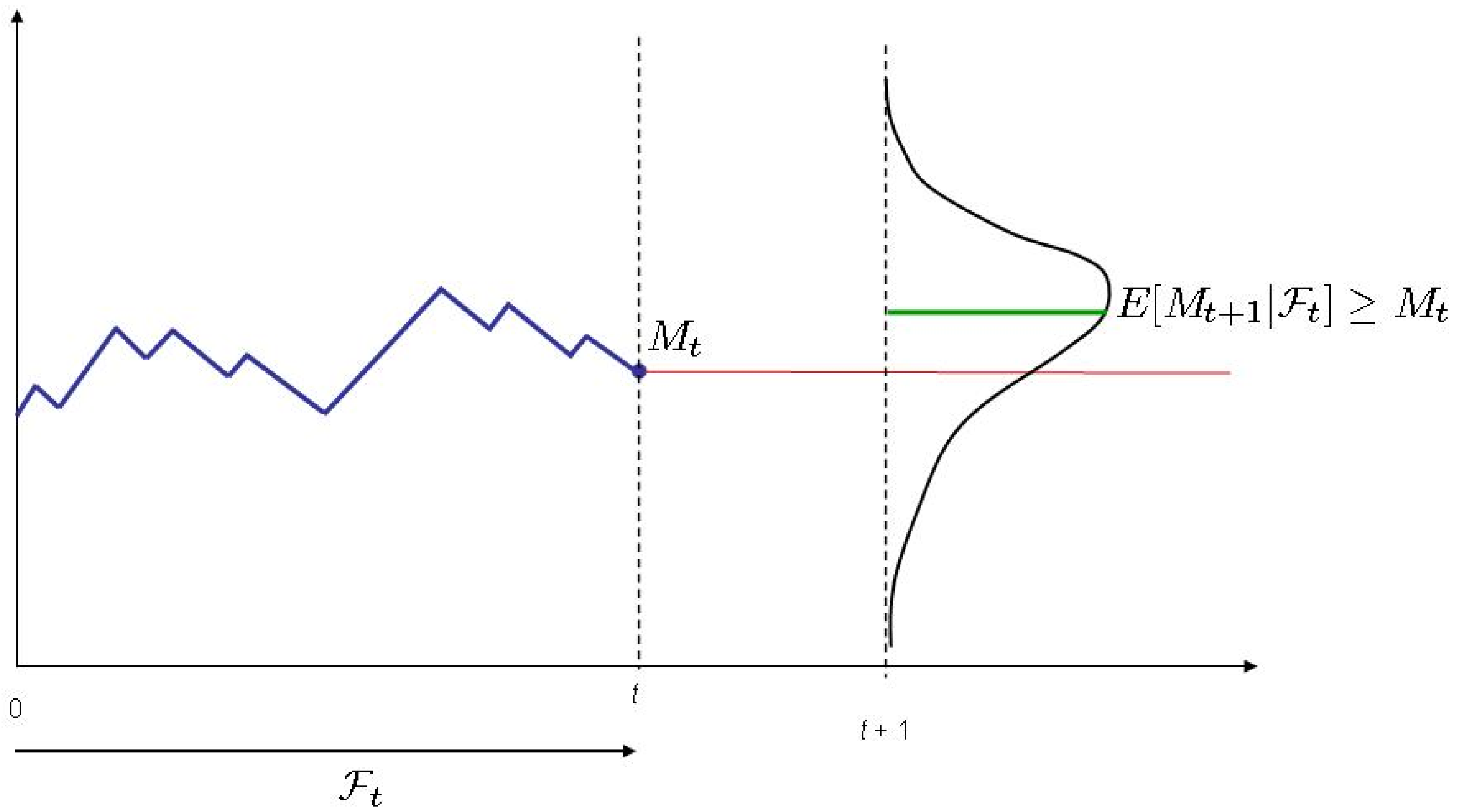


Figure : Discrete Time SubMartingale



Definition (Continuous Time Supermartingale (Submartingale))

A continuous time stochastic process

$$\{M_t : t \in \mathbb{R}^+\}$$

such that M_t is $\mathcal{F}_t$-measurable for $t \in \mathbb{R}^+$ is a **supermartingale (submartingale)** if

$$\mathbb{E}\,|M_t| < \infty$$

and

$$\mathbb{E}\,[M_t|\mathcal{F}_s] \leq (\geq) M_s \quad \forall 0 \leq s \leq t$$

Figure : Continuous Time SuperMartingale

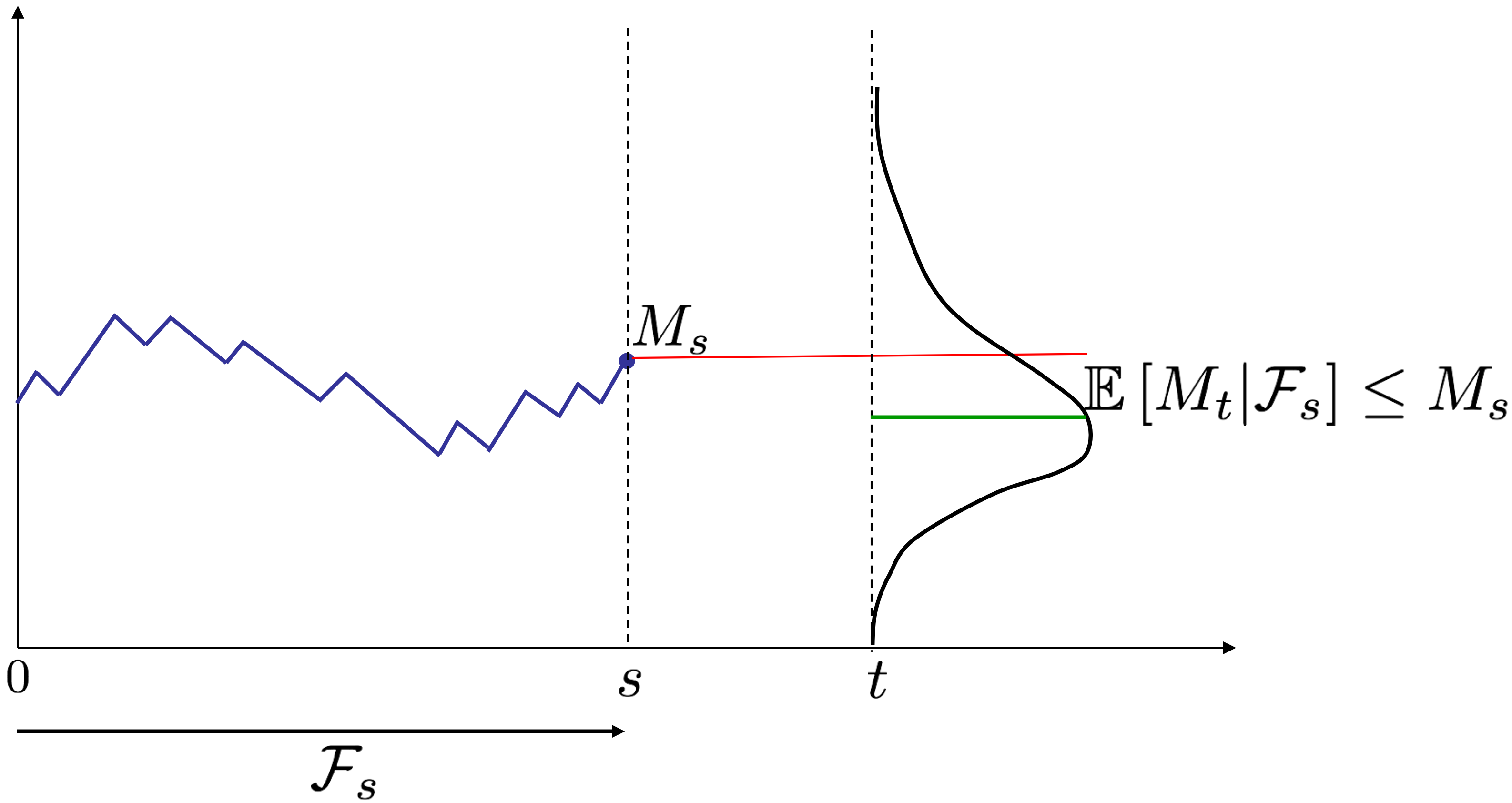
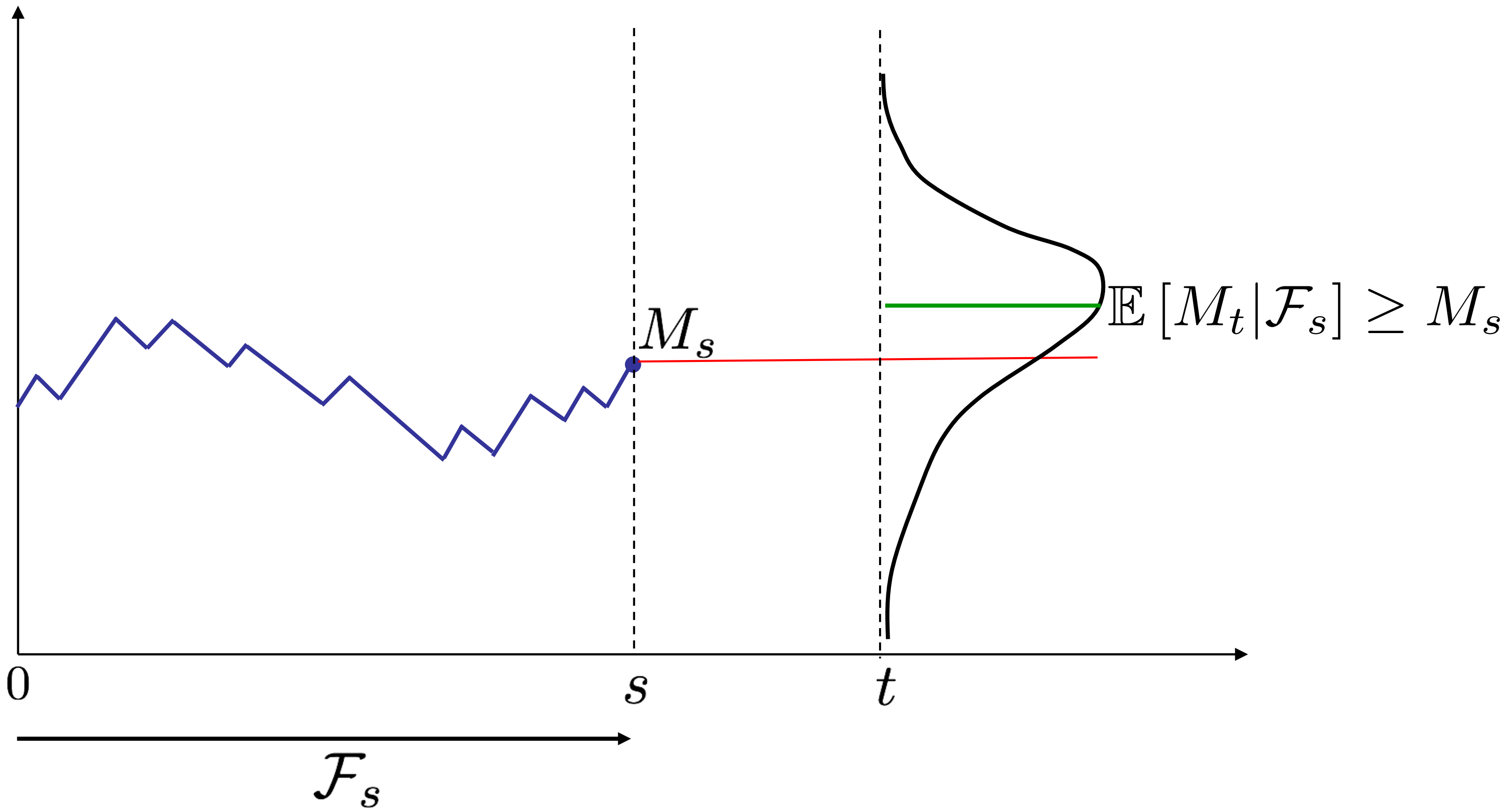


Figure : Continuous Time SubMartingale



Local Martingale

To define what a **local martingale** is, we need to understand the concept of **stopping time**.

Loosely speaking, a **stopping time** is a random variable τ which gives you a random point in time.

In finance, we often use stopping times to model the time of default, bankruptcy or the time at which a barrier is crossed.

The Appendix to this lecture contains a more detailed presentation of stopping times.

Definition (Local martingale)

A **local martingale** is a process which is locally a martingale.

More specifically, a $\mathcal{F}_t$-adapted process $M(t)$ is called a **local martingale** with respect to $\mathcal{F}_t$ if we can find an increasing sequence of $\mathcal{F}_t$ stopping times $(\tau_k)_{k=1,\dots}$ such that

1. $\tau_k \to \infty$ as $k \to \infty$;
2. $M(t \wedge \tau_k)$ is a martingale for all k.

Math Remark: we use almost sure convergence in condition 1.

Semimartingales

Semimartingales are an advanced topic in probability theory.

Definition (Semimartingales)

A process $Z(t)$ is a **semimartingales** if we can write it as

$$Z(t) = A(t) + M(t)$$

where

- $A(t)$ is a predictable process, with a 'nice' variation and $A(0) = 0$;
- $M(t)$ is a local martingale.

Math Remark: $A(t)$ need to have **locally bounded variation**. Practically, this condition rules out the infinite variation we get with Brownian motions.

Semimartingales are very important in stochastic analysis because they are 'good integrators' meaning that the Itô and Stratanovich integrals are well defined.

Practically, all of the processes that you see in the CQF are **semimartingales**. For example,

- Brownian motions;
- Itô diffusion processes;
- jump-diffusion or Lévy processes;

are all semimartingales.

Quasimartingale

Quasimartingales are an advanced topic in probability theory.

Rao's Theorem provides a useful characterization of quasimartingales.

Definition (Quasimartingale (Rao's Theorem))

A process $Z(t)$ is a **quasimartingale** if we can write it as

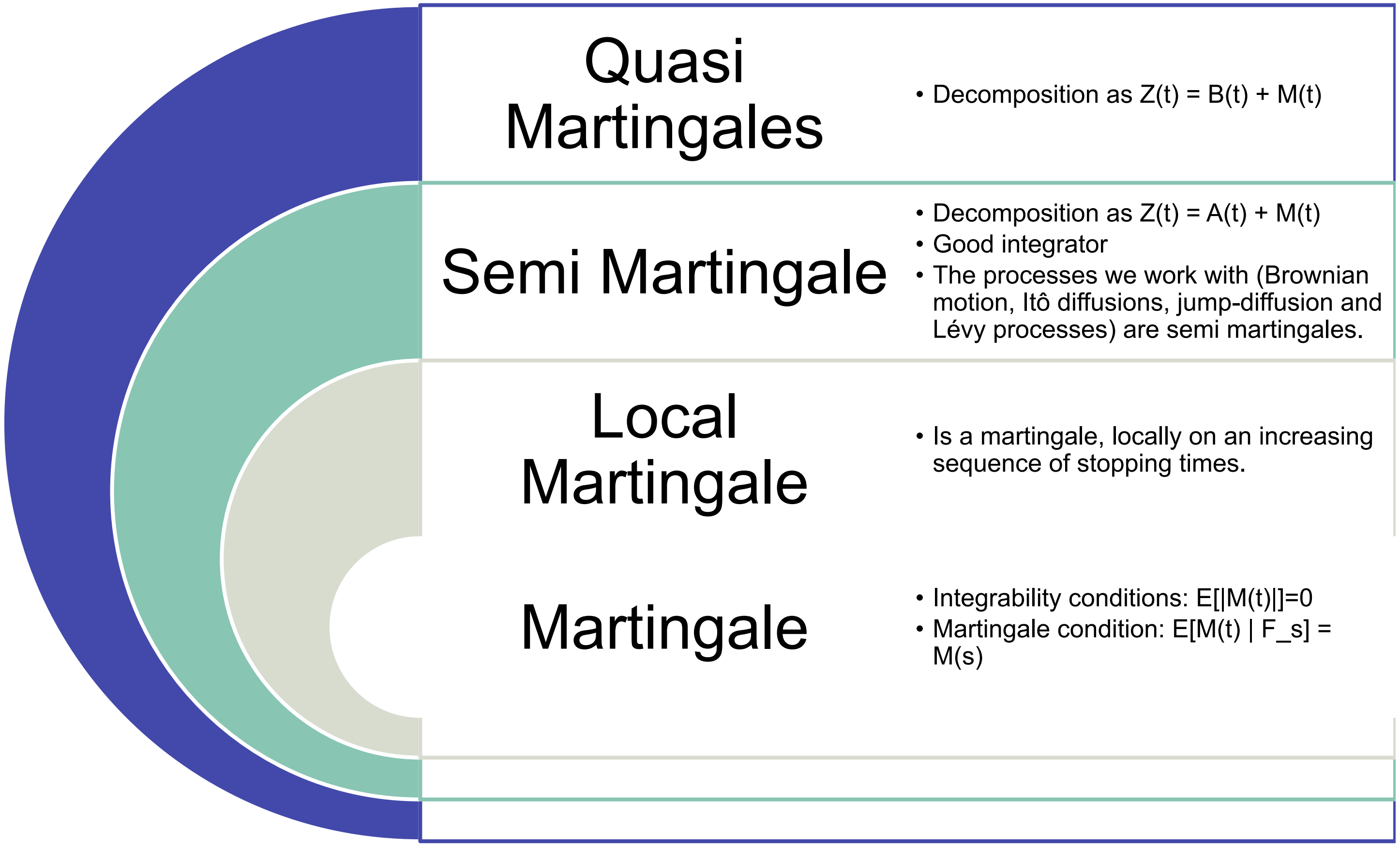
$$Z(t) = B(t) + M(t)$$

where

- $B(t)$ is a predictable process, with 'manageable' variation and $B(0) = 0$;
- $M(t)$ is a local martingale.

Math Remark: $B(t)$ need to have path of **locally integrable variation**. Practically, this condition is less strict than locally bounded variation, while still ruling out the infinite variation we get with Brownian motions.

Figure : The Martingale Zoology



6 Further Itô Clalculus

In this section, we take a look at the Itô product rule, a key application of the Itô formula.

6.1 Reminder: Multidimensional Itô

We can easily extend the Itô formula to functions of several stochastic processes.

This extension will prove useful to:

- price option on several underlying assets (basket options) ;
- price fixed income instruments using multi-factor interest rate models;
- price convertible bonds and structured products;
- include stochastic volatility in a model;
- simulate the risk on a portfolio;
- optimize a portfolio.

Let start with 2 dimensions.

Let's say we want to price a given financial instrument deriving its value $V(t, S_1, S_2)$ from 2 stochastic processes S_1, S_2, where

$$\begin{aligned} dS_i(t) &= f_i(t, S_k, k=1,2)dt + g_i(t, S_k, k=1,2)dX_i(t), \\ &\quad i = 1, 2 \end{aligned}$$

and where

$$dX_1(t)dX_2(t) \to \rho dt$$

For convenience, we will simplify the notation as

$$dS_i(t) = f_i dt + g_i dX_i(t), \quad i = 1, 2$$

Writing a naive 2-dimensional Taylor expansion, we see that

$$\begin{aligned} dV(t) &= \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S_1}dS_1(t) + \frac{\partial V}{\partial S_2}dS_2(t) \\ &\quad + \frac{1}{2}\frac{\partial^2 V}{\partial S_1^2}dS_1^2(t) + \frac{\partial^2 V}{\partial S_1 \partial S_2}dS_1(t)dS_2(t) + \frac{1}{2}\frac{\partial^2 V}{\partial S_2^2}dS_2^2(t) + \ldots \end{aligned}$$

Since dt is already very small, we will ignore all the terms of order $(dt)^\alpha, \alpha > 1$.

Since $dX_i^2(t) \to dt$ in the mean square limit, we see that

$$dS_i^2(t) \to g_i^2 dt, \quad i = 1, 2$$

in the mean square limit.

Also, since $dX_1(t)dX_2(t) \to \rho dt$, we see that

$$dS_1(t)dS_2(t) \to \rho g_1 g_2 dt$$

in the mean square limit.

Substituting into the Taylor expansion, we get the 2-dimensional Itô formula:

$$\begin{aligned} dV(t) &= \Bigg(\frac{\partial V}{\partial t} + f_1 \frac{\partial V}{\partial S_1} + f_2 \frac{\partial V}{\partial S_2} + \frac{1}{2} g_1^2 \frac{\partial^2 V}{\partial S_1^2} + \rho g_1 g_2 \frac{\partial^2 V}{\partial S_1 \partial S_2} \\ &\quad + \frac{1}{2} g_2^2 \frac{\partial^2 V}{\partial S_2^2} \Bigg) dt + g_1 \frac{\partial V}{\partial S_1} dX_1(t) + g_2 \frac{\partial V}{\partial S_2} dX_2(t) \end{aligned}$$

And now for the full n-dimensional formula!

Let's say we want to price a given financial instrument deriving its value $V(t, S_1, \ldots, S_n)$ from n stochastic processes $S_1, \ldots, S_n$, where

$$\begin{aligned} dS_i(t) &= f_i(t, S_k, k = 1, \ldots, n)dt + g_i(t, S_k, k = 1, \ldots, n)dX_i(t), \\ & \quad\; i = 1, \ldots, n \end{aligned}$$

We have n different Brownian motions $(X_1, \ldots, X_n)$ at work. We are also given the pairwise correlations of the Brownian increments:

$$dX_i(t)dX_j(t) \to \rho_{ij}dt, \quad i,j = 1, \ldots, n, \; i \neq j$$

Note that $\rho_{ii} = 1$ (i.e. a Brownian motion is perfectly correlated with itself) and we get the usual:

$$(dX_i)^2(t) = dX_i(t)dX_i(t) \to \rho_{ii}dt = dt$$

For convenience, we simplify the notation as

$$dS_i(t) = f_i dt + g_i dX_i(t), \quad i = 1, \ldots, n$$

Now, applying the same logic as in the 2-dimensional case, we obtain the n-dimensional version of Itô's formula

$$
\begin{aligned}
dV(t) &= \left(\frac{\partial V}{\partial t} + \sum_{i=1}^{n} f_i \frac{\partial V}{\partial S_i} + \frac{1}{2} \sum_{i=1}^{n} g_i^2 \frac{\partial^2 V}{\partial S_i^2} + \sum_{\substack{i=1 \\ j>i}}^{n} \rho_{ij} g_i g_j \frac{\partial^2 V}{\partial S_i S_j} \right) dt \\
&\quad + \sum_{i=1}^{n} g_i \frac{\partial V}{\partial S_i} dX_i(t)
\end{aligned}
$$

6.2 Itô Formula for Stochastic Integrals

Given a function $V(t, S(t))$ defined over $[0, T]$ and with $V(0, S(0)) = V_0$, where $S(t)$ is a stochastic process evolving according to $dS(t) = f(t, S(t))dt + g(t, S(t))dX(t)$, with f and g satisfying technical condition, we have:

$$\begin{aligned} V(T, S_T) &= V_0 + \int_0^T \left\{ \frac{\partial V}{\partial t} + f(t, S(t))\frac{\partial V}{\partial S} + \frac{1}{2}g^2(t, S(t))\frac{\partial^2 V}{\partial S^2} \right\} dt \\ &\quad + \int_0^T g(t, S(t))\frac{\partial V}{\partial S} dX(t) \end{aligned}$$

How did we get to this formula?

Simple. All we did was to define an integration range on which our functional relationship V is valid, namely $[0, T]$, and then to integrate our trusted Itô for SDEs over the integration range.

Implementation Tip

When you need to use Itô for stochastic integral, for all practical purpose:

- *Start from the differential form, which is easier to manipulate, and use Itô or Taylor (substituting $dX^2(t)$ for dt);*
- *Integrate over the required range;*
- *Done!*

We can obtain the integral version of the Itô's formula for n-dimensional processes in the same way:

$$
\begin{aligned}
V(T) \quad = \quad & V_0 + \int_0^T \left(\frac{\partial V}{\partial t} + \sum_{i=1}^n f_i \frac{\partial V}{\partial S_i} + \frac{1}{2} \sum_{i=1}^n g_i^2 \frac{\partial^2 V}{\partial S_i^2} \right. \\
& \left. + \sum_{\substack{i=1 \\ j>i}}^n \rho_{ij} g_i g_j \frac{\partial^2 V}{\partial S_i S_j} \right) dt + \sum_{i=1}^n \int_0^T g_i \frac{\partial V}{\partial S_i} dX_i(t)
\end{aligned}
$$

Exercise: What Are We Correlating, Exactly?

Consider a pair of correlated arithmetic Brownian motions :

$$dX(t) = adt + b \cdot dW_1(t), \quad X(0) = 0$$
$$dY(t) = cdt + d \cdot dW_2(t), \quad Y(0) = 0$$

where $W_1(t)$ and $W_2(t)$ are two Brownian motions with $\mathbb{E}\left[dW_1(t)dW_2(t)\right] = \rho dt$.

Using a Euler scheme, simulate the evolution of the two arithmetic Brownian motions using the following sets of parameters:

Short time horizon:

Parameter	Set 1	Set 2	Set 3	Set 4
a	0.1	0.1	0.1	0.1
b	0.2	0.2	0.2	0.2
c	0.1	0.1	-0.1	-0.1
d	0.2	0.2	0.2	0.2
ρ	0.9	-0.9	0.9	-0.9
time interval	$[0, 1]$	$[0, 1]$	$[0, 1]$	$[0, 1]$
discretization δt	0.01	0.01	0.01	0.01

Long time horizon:

Parameter	Set 5	Set 6	Set 7	Set 8
a	0.1	0.1	0.1	0.1
b	0.2	0.2	0.2	0.2
c	0.1	0.1	-0.1	-0.1
d	0.2	0.2	0.2	0.2
ρ	0.9	-0.9	0.9	-0.9
time interval	$[0, 10]$	$[0, 10]$	$[0, 10]$	$[0, 10]$
discretization δt	0.1	0.1	0.1	0.1

What do you observe?

6.3 Itô Product Rule

The **Itô product rule** tells you how to deal with a process $V(t)$ defined as the the product of another two processes $S_1(t)$ and $S_2(t)$, that is:

$$V(t) = S_1(t) \times S_2(t)$$

This is in fact a direct application of a more general result: **Itô's Lemma in 2 dimensions**.

The Itô product rule is an application of Iô's lemma in 2 dimensions with $V(t) = S_1(t) \times S_2(t)$.

Then,

$$
\begin{aligned}
\frac{\partial V}{\partial S_1} &= S_2 \\
\frac{\partial V}{\partial S_2} &= S_1 \\
\frac{\partial^2 V}{\partial S_1^2} &= 0 \\
\frac{\partial^2 V}{\partial S_2^2} &= 0 \\
\frac{\partial^2 V}{\partial S_1 \partial S_2} &= 1
\end{aligned}
$$

and

$$
\begin{aligned}
dV(t) &= \left(f_1 S_2(t) + f_2 S_1(t) + \rho g_1 g_2\right) dt \\
&\quad + g_1 S_2(t) dX_1(t) + g_2 S_1(t) dX_2(t)
\end{aligned}
$$

In integral form,

$$
\begin{aligned}
V(T) &= V(0) + \int_0^T \left(f_1 S_2(t) + f_2 S_1(t) + \rho g_1 g_2\right) dt \\
&\quad + \int_0^T g_1 S_2(t) dX_1(t) + \int_0^T g_2 S_1(t) dX_2(t)
\end{aligned}
$$

In fact, the Itô product rule is very close to the product rule from ordinary calculus.

To see this, notice that we can express the Itô product rule as:

$$
\begin{aligned}
& d\left(S_1(t) \times S_2(t)\right) \\
= \quad & \underbrace{S_1(t) \times dS_2(t) + dS_1(t) \times S_2(t)}_{\text{ordinary product rule}} + \underbrace{dS_1(t) \times dS_2(t)}_{\text{cross variation adjustment}}
\end{aligned}
$$

This expression is actually the most useful formulation of the Itô product rule.

In financial mathematics, we often need to use the Itô product rule to deal with discounting and change of measures.

Next, we look at two simple and very useful applications of the Itô product rule. We will need these two applications when we price derivatives in Modules 3 and 4!

Application 1

Apply the Itô product rule to compute $V(t) = S_1(t) \times S_2(t)$, where

- $S_1(t)$ is stochastic:

$$dS_1(t) = f_1(t, S_1)dt + g_1(t, S_1)dX(t)$$

- $S_2(t)$ is deterministic:

$$dS_2(t) = f_2(t, S_2)dt$$

What do we need this for?

We need this result to price a stock options in Module 3.

Take

- $S_1(t)$ as the stock price;
- $S_2(t)$ as the (deterministic) discount factor.

Applying the Itô product rule (or deriving the result from scratch), we get:

$$dV(t) \;=\; (f_1 S_2(t) + f_2 S_1(t))\, dt + g_1 S_2(t) dX(t)$$

The integral form is:

$$V(T) \;=\; V(0) + \int_0^T (f_1 S_2(t) + f_2 S_1(t))\, dt + \int_0^T g_1 S_2(t) dX(t)$$

Application 2

Apply the Itô product rule to compute $V(S_1, S_2) = S_1(t) \times S_2(t)$ where

- $S_1(t)$ and $S_2(t)$ are both stochastic;
- but they depend on <u>the same</u> Brownian motion $X(t)$.

Here,

$$
\begin{aligned}
dS_i(t) \quad &= \quad f_i(t, S_k, k = 1, 2)dt + g_i(t, S_k, k = 1, 2)dX(t), \\
&\qquad i = 1, 2
\end{aligned}
$$

What do we need this for?

We need this result to price a Zero Coupon Bond in Module 4.

We will need to manipulate the product of two processes:

- the bond price, and;
- the Radon-Nikodym derivative of a change of measures.

Applying the Itô product rule (or deriving the result from scratch), we get:

$$
\begin{aligned}
dV(t) &= (f_1 S_2(t) + f_2 S_1(t) + g_1 g_2)\, dt \\
&\quad + (g_1 S_2(t) + g_2 S_1(t))\, dX(t)
\end{aligned}
$$

The integral version is:

$$\begin{aligned} V(T) &= V(0) + \int_0^T \left(f_1 S_2(t) + f_2 S_1(t) + g_1 g_2\right) dt \\ &\quad + \int_0^T \left(g_1 S_2(t) + g_2 S_1(t)\right) dX(t) \end{aligned}$$

In this lecture, we have seen...

- What is a martingale in continuous time?
- Martingales and It calculus
- Martingale unmasked: how do I know if my stochastic process is a martingale?
- Girsanov, exponential martingales and change of measure
- The martingale zoology in a hurry: martingale, local martingale, supermartingale, submartingales, semimartingales...
- Itô in higher dimensions
- The Itô product rule

To go a bit further...

There is a plethora of stochastic calculus and stochastic analysis books. Most are good, some are better than others, but the great majority share a common feature: they are rather arcane!

Both of Steven Shreve's books ([6] and [7]) are invaluable. They are clearly written and contain all you need to know on stochastic calculus applied to finance... and then some more.

Baxter and Rennie [1] provides a good overview of the key techniques. The book is intuitive: Baxter and Rennie tend to explain the important concepts in plain old English and then show how this translates in math.

Chin, Nel and Ólafsson [2] contains a large number of worked out exercises on probability and stochastic calculus. It is a very good companion book.

Hull [3] is a *tour de force*. It has an overview of pretty much every topic, but the stochastics are covered more from a finance perspective than a quantitative finance perspective. majority share a common feature: they are rather arcane!

Øksendal [5] is the next step. It gives you all of the results you will ever need to manipulate Itô processes, and their proofs.

Neftci [4] is OK, but a bit too much of a bird's eye view and not quite rigorous enough.

Appendix: A Primer on Stopping Times

Take a sequence of gambles $n = 1, 2, \ldots$ with

- accumulated winnings $W(n)$ and;
- marginal winnings $X(n)$ at round n.

Hence,

$$W(n) = X(1) + X(2) + \ldots + X(n), \quad W(0) = w$$

Let $\mathcal{F}_n$ be the filtration generated by the rounds, that is $\mathcal{F}_n := \sigma(X(i)), 0 \leq i \leq n)$. It is easy to see that $W(n)$ is $\mathcal{F}_n$-measurable.

We could stop the game whenever we want, let's say at time τ.
We could stop:

1. *after a given number of rounds*, say N. In this case $\tau = N$ is deterministic;
2. *when we reach a given wealth objective* W^T:

$$\tau := \min\left\{n : W(n) = W^T\right\}$$

3. *when we are ruined*:

$$\tau := \min\{n : W(n) = 0\}$$

In the last two cases, $\tau \in \{1, 2, \ldots\} \cup \{\infty\}$ is a random variable called a **stopping time** and it is such that $\{\tau = n\} \in \mathcal{F}_n$: τ is $\mathcal{F}_n$-measurable.

Objectives such as "keep playing until either we reach a given wealth objective W^T or we are ruined", that is

$$\tau = \min\left\{n : W(n) = W^T \text{or } W(n) = 0\right\}$$

are known as **hitting times**.

Hitting times are a particularly important concept for structural default models as well as for barrier options.

Notation: we denote by $a \wedge b := min(a, b)$.

Consider a sequence of random variables $\xi_1, \xi_2, \ldots, \xi_n$ and a stopping time τ. We call $\xi_n^\tau = \xi_{\tau \wedge n}$ the sequence **stopped** at τ.

We state without proof the following proposition:

Key Fact

Let τ be a stopping time:

1. *If ξ_n is a martingale, then so is ξ_n^τ;*
2. *If ξ_n is a supermartingale, then so is ξ_n^τ;*
3. *If ξ_n is a submartingale, then so is ξ_n^τ.*

Appendix B: A Short History of Modern Probability

The story of Brownian motions is intertwined with the lives and work of four men: Robert Brown, Louis Bachelier, Albert Einstein and Norbert Wiener.

In 1827, Scottish botanist Robert Brown (1773-1858) observed the seemingly random movement of pollen particles in a fluid. He correctly deduced that this motion was actually a result of the pollen colliding with molecules of water. The concept of Brownian motion was born.

In 1880, Thorvald N. Thiele provided the first mathematical description of the Brownian motion in a paper on least square.

In 1900, Louis Bachelier (1870-1946) used the Brownian motion as a mathematical object to model the dynamics of asset prices. Using as much heuristics as mathematics, Bachelier derived a formula to price options.

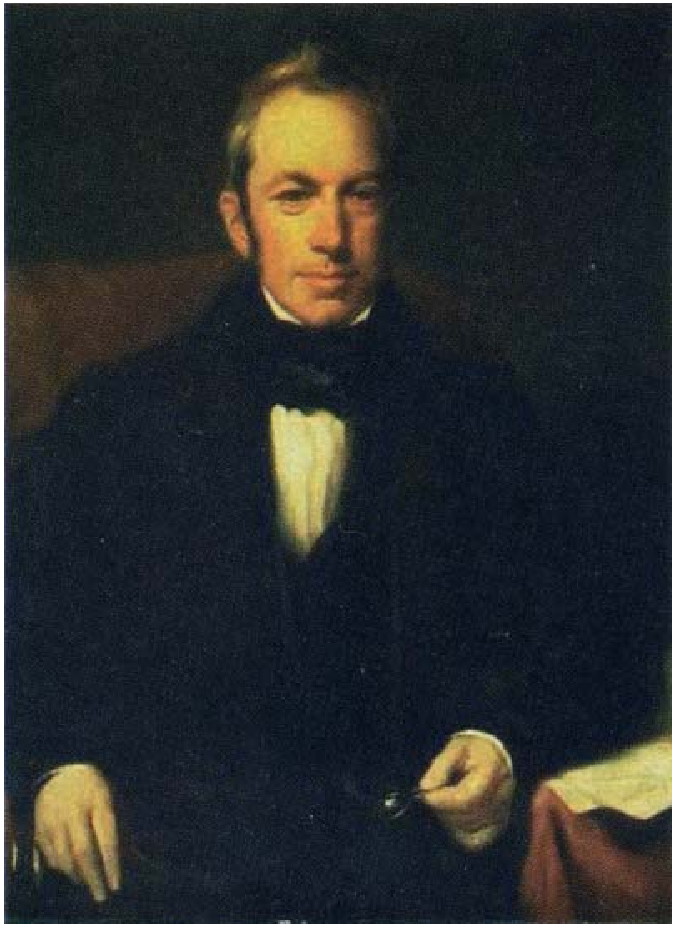


Figure : R. Brown (1773-1858)

Figure : L. Bachelier (1870-1946)

1905 was Albert Einstein's (1879-1955) *annus mirabilis*. One of the four groundbreaking papers he published on that year focused on Brownian motions and drew the attention of the physicists community to stochastic processes.

Einstein did not solve all the mystery of Brownian motions. In fact he even predicted that it might not be possible to prove that Brownian motions are continuous. But he cracked some of their difficulties and applied them to solve interesting problems in physics. As a result, Brownian motions gained a quick recognition. In fact, stochastic processes are a key component of quantum mechanics.

The late 1920s and 1930s were a Golden Age for probability theory.

Lifting the last corner of the veil did not take too long. American mathematician Norbert Wiener (1894-1964) proved that the Brownian motion has a continuous path in 1924, less than 2 decades after Einstein conjectured that this result may never be proved. Almost 90 years later, Wiener's proof remains challenging even for trained mathematician.

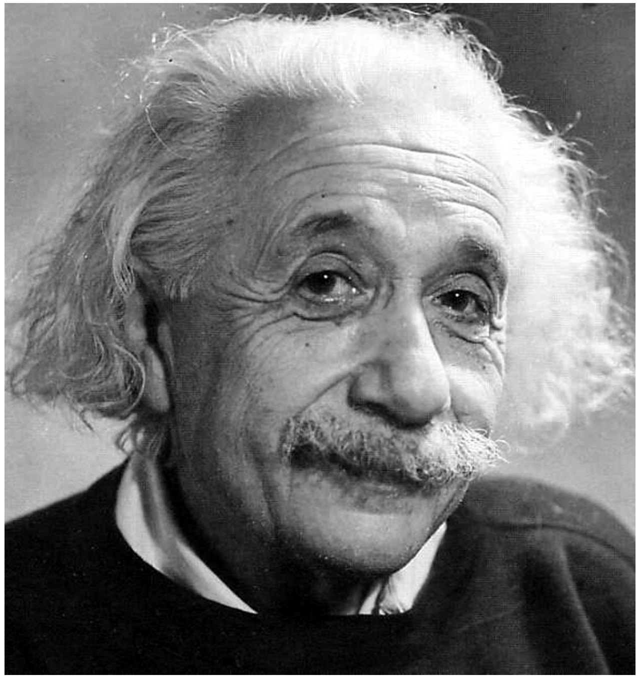


Figure : A. Einstein (1879-1955)

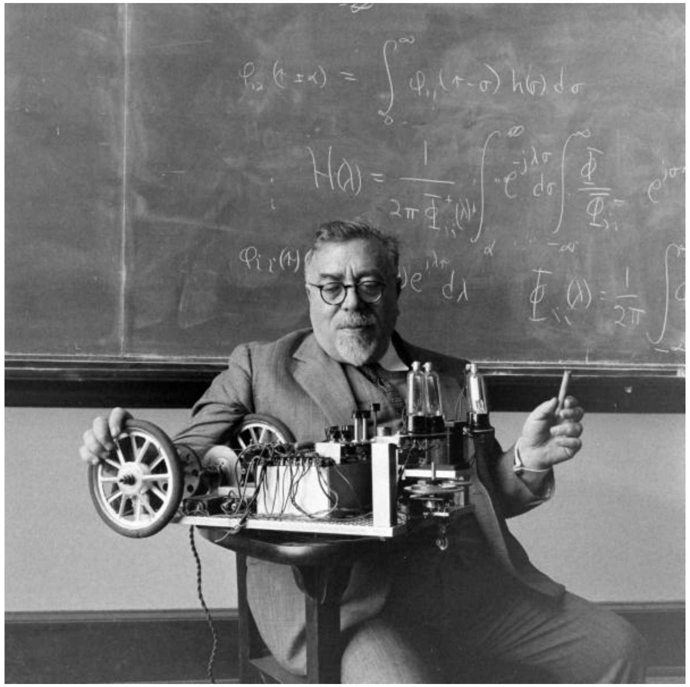


Figure : N. Wiener (1884-1964)

The two towering figures of the era where influential French mathematician Paul Levy (1886-1971) and young Russian mathematician Andrei Kolmogorov (1903-1987).

While Levy worked to extend the current theory of probabilities by introducing a number of new concepts, techniques and methods, Kolmogorov set about to rewrite probability theory using measure theory to give it the most solid mathematical foundations.

Kolmogorov published a short monograph in German detailing his results in 1933 and the Moscow school of probability reached it zenith, remaining influential to this day.

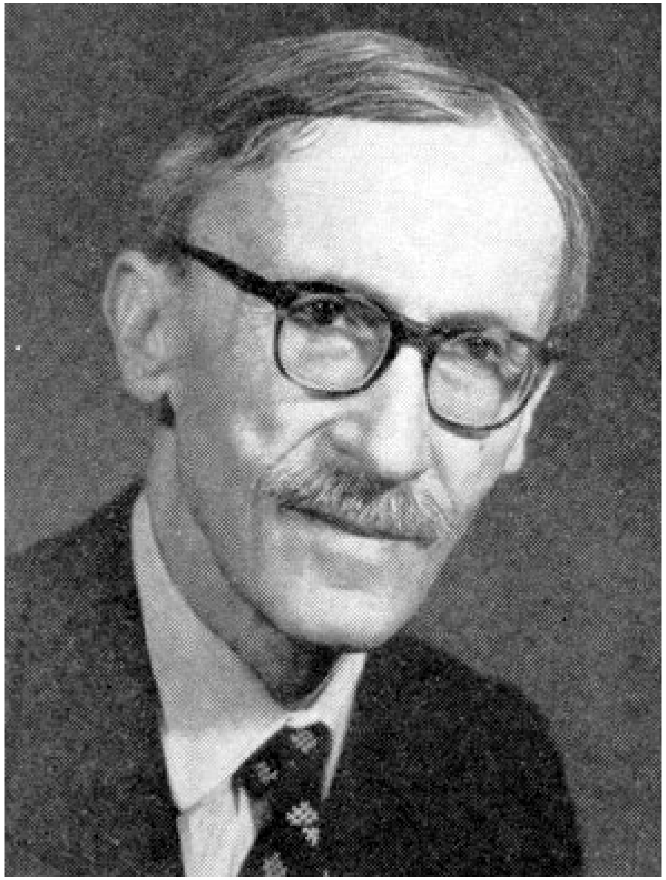


Figure : P. Lévy (1886-1971)

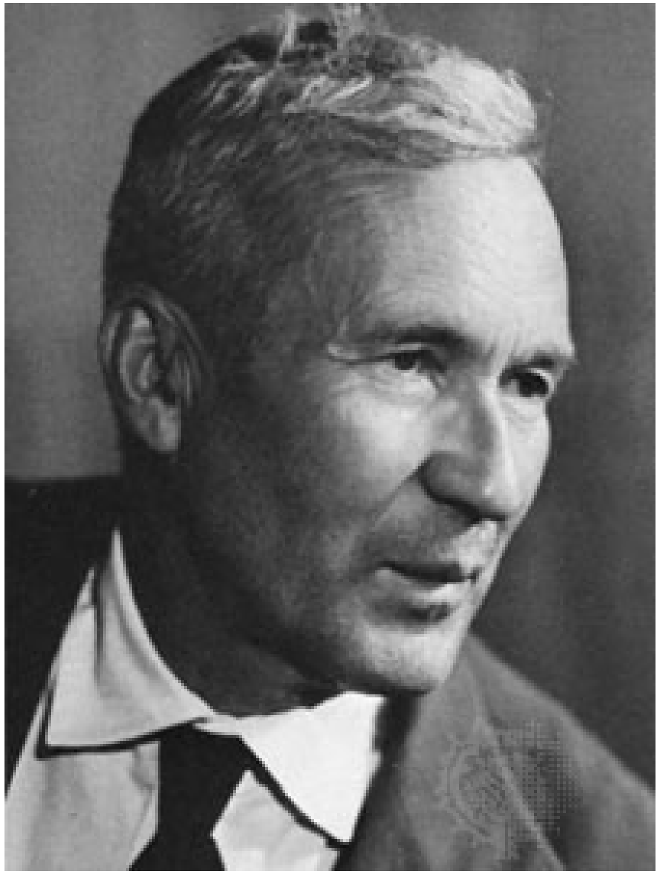


Figure : A. Kolmogorov (1903-1971)

In parallel, in the West, research into advanced topics continued. Starting in the 1950s Joseph Doob (1910-2004) in the USA and later Paul Meyer (1934-2003) in France developed the theory of martingales based on Paul Levy's earlier work.

Although not specifically designed with Kolmogorov's reformulation of Probability Theory in mind, Martingale theory has now been fully integrated into the newly consolidated body of Probability Theory and form an important and often used component.

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