Strongly Path-dependent Options

In this lecture. . .

- strong path dependence
- pricing many strongly path-dependent contracts in the Black–Scholes partial differential equation framework
- how to handle both continuously-sampled and discretely-sampled paths
- jump conditions for differential equations

σ + Math + Numerics

Introduction

To be able to turn the valuation of a derivative contract into the solution of a partial differential equation is a big step forward.

The partial differential equation approach is one of the best ways to price a contract because of its flexibility and because of the large body of knowledge that has grown up around the fast and accurate numerical solution of these problems.

In this lecture we see how to generalize the Black–Scholes analysis, delta hedging and no arbitrage, to the pricing of many more derivative contracts, specifically contracts that are strongly path dependent.

Path-dependent quantities represented by an integral

We start by assuming that the underlying asset follows the log-normal random walk

$$dS = \mu S \, dt + \sigma S \, dX.$$

Imagine a contract that pays off at expiry, T, an amount that is a function of the path taken by the asset between time zero and expiry.

- Let us suppose that this path-dependent quantity can be represented by an integral of some function of the asset over the period zero to T:

$$I(T) = \int_0^T f(S, \tau)\, d\tau.$$

This is not such a strong assumption, in particular most of the path-dependent quantities in exotic derivative contracts, such as averages, can be written in this form with a suitable choice of $f(S, t)$.

Prior to expiry we have information about the possible final value of S (at time T) in the present value of S (at time t). For example, the higher S is today, the higher it will probably end up at expiry. Similarly, we have information about the possible final value of I in the value of the integral to date:

$$I(t) = \int_0^t f(S,\tau)d\tau. \tag{1}$$

As we get closer to expiry, so we become more confident about the final value of I.

- One can imagine that the value of the option is therefore not only a function of S and t, but also a function of I; I will be our new independent variable, called a **state variable**.

We see in the next section how this observation leads to a pricing equation.

In anticipation of an argument that will use Itô's lemma, we need to know the stochastic differential equation satisfied by I. This could not be simpler.

- Incrementing t by dt in (1) we find that

$$\boxed{dI = f(S,t)\,dt.} + 0.dX \tag{2}$$

Observe that I is a smooth function (except at discontinuities of f) and from (2) we can see that its stochastic differential equation contains no stochastic terms.

Examples

An Asian option has a payoff that depends on the average of the asset price over some period. If that period is from time zero to expiry and the average is arithmetic then we write

$$I = \int_0^t S \, d\tau.$$

The payoff may then be, for example,

$$\max\left(\frac{I}{T} - S, 0\right).$$

This would be an average strike put, of which more later.

If the average is geometric then we write

$$I = \int_0^t \log(S) \, d\tau.$$

As another example, imagine a contract that pays off a function of the square of the underlying asset, but only counts those times for which the asset is below S_u. We write

$$I = \int_0^t S^2 \mathcal{H}(S_u - S) d\tau,$$

where $\mathcal{H}(\cdot)$ is the Heaviside function.

We are now ready to price some options.

Continuous sampling: The pricing equation

We will derive the pricing partial differential equation for a contract that pays some function of our new variable I.

- The value of the contract is now a function of the three variables, $V(S, I, t)$.

Set up a portfolio containing one of the path-dependent option and short a number Δ of the underlying asset:

$$\Pi = V(S, I, t) - \Delta S.$$

The change in the value of this portfolio is given by

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \frac{\partial V}{\partial I} dI + \left(\frac{\partial V}{\partial S} - \Delta \right) dS.$$

- Choosing

$$\Delta = \frac{\partial V}{\partial S}$$

to hedge the risk, and using (2), we find that

$$d\Pi = \left(\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + f(S,t)\frac{\partial V}{\partial I} \right) dt.$$

This change is risk free, and thus earns the risk-free rate of interest r, leading to the pricing equation...

3D

$$\frac{\partial V}{\partial t}+\tfrac{1}{2}\sigma^2S^2\frac{\partial^2 V}{\partial S^2}+f(S,t)\frac{\partial V}{\partial I}+rS\frac{\partial V}{\partial S}-rV=0$$

This is to be solved subject to

$$V(S,I,T)=P(S,I).$$

This completes the formulation of the valuation problem. The obvious changes can be made to accommodate dividends on the underlying.

Example

Continuing with the arithmetic Asian example, we have

$$I = \int_0^t S^2 \, d\tau,$$

so that the equation to be solved is

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + S^2 \frac{\partial V}{\partial I} + rS \frac{\partial V}{\partial S} - rV = 0.$$

Path-dependent quantities represented by an updating rule

For practical and legal reasons path-dependent quantities are never measured continuously. There is minimum timestep between sampling of the path-dependent quantity. This timestep may be small, one day, say, or much longer.

From a practical viewpoint it is difficult to incorporate every single traded price into an average, for example. Data can be unreliable and the exact time of a trade may not be known accurately.

If the time between samples is small we can confidently use a continuous-sampling model, the error will be small. If the time betwen samples is long, or the time to expiry itself is short we must build this into our model. This is the goal of this section.

We introduce the idea of an **updating rule**, an algorithm for defining the path-dependent quantity in terms of the current 'state of the world.'

- The path-dependent quantity is measured on the **sampling dates** t_i, and takes the value I_i for $t_i \leq t < t_{i+1}$.

- At the sampling date t_i the quantity I_{i-1} is updated according to a rule such as

  $$I_i = F(S(t_i), I_{i-1}, i).$$

  Note how, in this simplest example (which can be generalized), the new value of I is determined by only the old value of I and the value of the underlying on the sampling date, and the sampling date.

Examples

We saw how to use the continuous running integral in the valuation of Asian options. But what if that integral is replaced by a discrete sum? In practice, the payoff for an Asian option depends on the quantity

$$I_M = \sum_{k=1}^{M} S(t_k),$$

where M is the total number of sampling dates. This is the discretely sampled sum. A more natural quantity to consider is

$$A_M = \frac{I_M}{M} = \frac{1}{M} \sum_{k=1}^{M} S(t_k), \tag{3}$$

because then the payoff for the discretely-sampled arithmetic average strike put is then

Can we write (3) in terms of an updating rule? Yes, easily: if we write

$$A_i = \frac{1}{i}\sum_{k=1}^{i} S(t_k)$$

then we have

$$\begin{aligned} A_1 &= S(t_1), \quad A_2 = \frac{S(t_1)+S(t_2)}{2} = \tfrac{1}{2}A_1 + \tfrac{1}{2}S(t_2), \\ A_3 &= \frac{S(t_1)+S(t_2)+S(t_3)}{3} = \tfrac{2}{3}A_2 + \tfrac{1}{3}S(t_3), \cdots. \end{aligned}$$

- The updating rule is therefore

  $$A_i = \frac{1}{i}S(t_i) + \frac{i-1}{i}A_{i-1}.$$

  We will see how to use this for pricing in the next section. But first, another example.

The lookback option has a payoff that depends on the maximum or minimum of the realized asset price. If the payoff depends on the maximum sampled at times t_i then we have

$$I_1 = S(t_1), \quad I_2 = \max(S(t_2), I_1), \quad I_3 = \max(S(t_3), I_2) \cdots .$$

- The updating rule is therefore simply

$$I_i = \max(S(t_i), I_{i-1}).$$

How do we use these updating rules in the pricing of derivatives?

Discrete sampling: The pricing equation

- We anticipate that the option value will be a function of three variables, $V(S, I, t)$.

The first step in the derivation is the observation that the stochastic differential equation for I is degenerate:

$$dI = 0.$$

This is because the variable I can only change at the discrete set of dates t_i. This is true if $t \neq t_i$ for any i.

- So provided we are not *on* a sampling date the quantity I is constant, the stochastic differential equation for I reflects this, and the pricing equation is simply the basic Black–Scholes equation:

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

How does the equation know about the path dependency?

- Across a sampling date the option value is continuous.

As we get closer and closer to the sampling date we become more and more sure about the value that I will take according to the updating rule. Since the outcome on the sampling date is known and since *no money changes hands* there cannot be any jump in the value of the option. This is a simple application of the no arbitrage principle.

If we introduce the notation t_i^- to mean the time infinitesimally before the sampling date t_i and t_i^+ to mean infitesimally just after the sampling date, then continuity of the option value is represented mathematically by

$$V(S, I_{i-1}, t_i^-) = V(S, I_i, t_i^+).$$

- In terms of the updating rule, we have

$$V(S, I, t_i^-) = V(S, F(S(t_i), I, i), t_i^-)$$

  This is called a **jump condition**.

We call this a jump condition even though there is no jump in this case.

If we follow the path of S in time we see that it is continuous. However, the path for I is discontinuous. There is a deterministic jump in I across the sampling date.

If we were to plot V as a function of S and I just before and just after the sampling date we would see that *for fixed S and I* the option price would be discontinuous. But this plot would have to be interpreted correctly; $V(S, I, t)$ may be discontinuous as a function of S and I but V is continuous along each *realized* path of S and I.

Examples

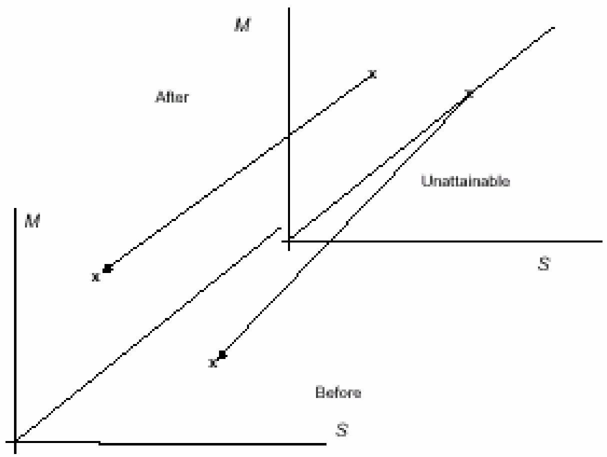
- To price an arithmetic Asian option with the average sampled at times t_i solve the Black–Scholes equation for $V(S, A, t)$ with

$$V(S, A, t_i^-) = V\left(S, \frac{i-1}{i}A + \frac{1}{i}S, t_i^+\right).$$

- To price a lookback depending on the maximum sampled at times t_i solve the Black–Scholes equation for $V(S, M, t)$ with

$$V(S, M, t_i^-) = V\left(S, \max(S, M), t_i^+\right).$$

How this particular jump condition works is shown below. The top right-hand plot is the S, M plane just after the sample of the maximum has been taken. Because the sample has just been taken the region $S > M$ cannot be reached, it is the region labelled 'Unattainable.' This means that the option value at time t_i^- for $S < M$ is the same as the t_i^+ value. *However*, for $S > M$ the option value comes from the $S = M$ line at time t_i^+ for the same S value.



The algorithm for discrete sampling

The path-dependent quantity, I, is updated discretely and so the partial differential equation for the option value between sampling dates is the Black–Scholes equation. The algorithm for valuing an option on a discretely-sampled quantity is as follows.

- Working backwards from expiry, solve

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

  between sampling dates. Stop when you get to the timestep on which the sampling takes place.

- Then apply the appropriate jump condition across the current sampling date to deduce the option value immediately before the present sampling date using the calculated value of the option just after. Use this as your final condition for further timestepping of the Black–Scholes equation.

- Repeat this process to arrive at the current value of the option.

Higher dimensions

The methods outlined above are not restricted to a single path-dependent quantity. Any finite number of path-dependent variables can be accommodated, theoretically. Imagine a contract that pays off the difference between a continuous geometric and a continuous arithmetic average. To price this one would need to introduce I_g and I_a, defined by

$$I_g = \int_0^t \log(S)d\tau \quad \text{and} \quad I_a = \int_0^t S d\tau.$$

The solution would then be a function of four variables, $V(S, I_g, I_a, t)$. However, this is at the limit of practicality for a numerical solution of a partial differential equation. Unless there is a similarity solution, reducing the dimensionality of the problem, it may be better to consider Monte Carlo simulation.

The same thoughts apply to discrete sampling or a combination of discrete and continuous.

Pricing via expectations

We can value options in the Black–Scholes world by taking the present value of the expected payoff under a risk-neutral random walk. Simply simulate the random walk

$$dS = rS\,dt + \sigma S\,dX$$

for many paths, calculate the payoff for each path—and this means calculating the value of the path-dependent quantity which is usually very simple to do—take the average payoff over all the paths and then take the present value of that average. That is the option fair value.

- This is a very general and powerful technique, useful for path-dependent contracts for which either a partial differential equation approach is impossible or too high dimensional. The only disadvantage is that it is hard to value American options in this framework.

Early exercise

If you have found a partial differential equation formulation of the option problem then it is simple to incorporate the early exercise feature of American and Bermudan options.

- Simply apply the constraint

$$V(S, I, t) \geq P(S, I),$$

  together with continuity of the delta of the option, where $P(S, I)$ is the payoff function (and it can also be time dependent). This condition is to be applied at any time that early exercise is allowed.

If you have found a partial differential equation formulation of the problem and it is in sufficiently low dimension then incorporating early exercise in the numerical scheme is a matter of adding a couple of lines of code.