CQF Module 2: Martingales I
Solutions

CQF

1. Use Itô's formula to determine whether the following are martingales:

   (i) $Y(t) = e^{1/2t} \cos X(t)$;

   (ii) $Y(t) = e^{\alpha t} \sin X(t)$ for some constant α with $0 < \alpha < 1$. Does the answer depend on the value of α?

   (iii) $Y(t) = (X(t) + t) \exp\left\{-\frac{1}{2}t - X(t)\right\}$.

   A diffusion process $Y(t)$ is a martingale if the drift coefficient of its SDE is identically 0.

   (i) Consider the function $F(s, x) = e^{1/2s} \cos x$.

$$\begin{aligned} \frac{\partial F}{\partial s}(s,x) &= \frac{1}{2} e^{1/2s} \cos x \\ \frac{\partial F}{\partial x}(s,x) &= -e^{1/2s} \sin x \\ \frac{\partial^2 F}{\partial x^2}(s,x) &= -e^{1/2s} \cos x \end{aligned}$$

   By Itô, the dynamics of the process $Y(t) = F(t, X(t))$ is given by

$$\begin{aligned} dY(t) &= \left(\frac{\partial F}{\partial t}(t, X(t)) + \frac{1}{2} \frac{\partial^2 F}{\partial x^2}(s,x) \right) dt - e^{1/2s} \sin X(t) dX(t) \\ &= -e^{1/2s} \sin X(t) dX(t) \end{aligned}$$

   Hence $Y(t)$ **is a martingale**.

   (ii) Consider the function $F(s, x) = e^{\alpha s} \sin x$ with α in $(0, 1)$.

$$\begin{aligned} \frac{\partial F}{\partial s}(s,x) &= \alpha e^{\alpha s} \sin x \\ \frac{\partial F}{\partial x}(s,x) &= e^{\alpha s} \cos x \\ \frac{\partial^2 F}{\partial x^2}(s,x) &= -e^{\alpha s} \sin x \end{aligned}$$

By Itô, the dynamics of the process $Y(t) = F(t, X(t))$ is given by

$$\begin{aligned} dY(t) &= \left(\frac{\partial F}{\partial t}(t, X(t)) + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(s,x)\right) dt + e^{\alpha s}\cos X(t) dX(t) \\ &= \left(\alpha - \frac{1}{2}\right) e^{\alpha t}\sin X(t) dt + e^{\alpha s}\cos X(t) dX(t) \end{aligned}$$

For $Y(t)$ to be a martingale we must have $\alpha = \frac{1}{2}$. With any other choice of value for α, $Y(t)$ is not a martingale.

(iii) Consider the function $F(s,x) = (x+s)\exp\left\{-\frac{1}{2}s - x\right\}$.

$$\begin{aligned} \frac{\partial F}{\partial s}(s,x) &= \left(1 - \frac{1}{2}(x+s)\right)\exp\left\{-\frac{1}{2}s - x\right\} \\ \frac{\partial F}{\partial x}(s,x) &= (1 - x - s)\exp\left\{-\frac{1}{2}s - x\right\} \\ \frac{\partial^2 F}{\partial x^2}(s,x) &= (-2 + x + s)\exp\left\{-\frac{1}{2}s - x\right\} \end{aligned}$$

By Itô, the dynamics of the process $Y(t) = F(t, X(t))$ is given by

$$\begin{aligned} dY(t) &= \left(\frac{\partial F}{\partial t}(t, X(t)) + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(s,x)\right) dt + (1 - x - s)\exp\left\{-\frac{1}{2}s - x\right\} dX(t) \\ &= \left(1 - \frac{1}{2}(x+s) + \frac{1}{2}[-2 + x + s]\right)\exp\left\{-\frac{1}{2}s - x\right\} dt \\ &\quad + (1 - x - s)\exp\left\{-\frac{1}{2}s - x\right\} dX(t) \\ &= \left(1 - \frac{1}{2}(x+s) + \frac{1}{2}[-2 + x + s]\right)\exp\left\{-\frac{1}{2}s - x\right\} dt \\ &\quad + (1 - x - s)\exp\left\{-\frac{1}{2}s - x\right\} dX(t) \\ &= (1 - x - s)\exp\left\{-\frac{1}{2}s - x\right\} dX(t) \end{aligned}$$

Hence $Y(t)$ **is a martingale**.

2. ***Moments of the Brownian Motion X(t)*** - *Consider the function $m_n(t)$ defined as*

$$m_n(t) = \mathbf{E}[X^n(t)], \qquad n = 1, 2, \ldots \tag{1}$$

where $X(t)$ is a standard Brownian motion.

Applying Itô's formula, show that:

$$m_n(t) = \frac{1}{2}n(n-1)\int_0^t m_{n-2}(s)ds \tag{2}$$

for $n = 2, 3, \ldots$

Deduce from (2) *that*

$$m_4(t) = 3t^2 \tag{3}$$

compute $m_6(t)$

Answer: Because of the expectation, we cannot tackle expression (1) upfront.

Consider instead the auxiliary function $g_n(t, x) = x^n$ for $n \geq 2$. Note the relation between $g_n(t, x)$ and $m_n(t)$:

$$m_n(t) = \mathbf{E}[g_n(t, X(t))]$$

Applying Itô's lemma to the function g_n and the standard Brownian motion, we get

$$\begin{aligned} g_n(t) &= g_n(0) + \int_0^t \frac{\partial g_n}{\partial s}ds + \int_0^t \frac{\partial g_n}{\partial x}dX(s) + \frac{1}{2}\int_0^t \frac{\partial^2 g_n}{\partial x^2}ds \\ &= = n\int_0^t X^{n-1}(s)dX(s) + \frac{1}{2}n(n-1)\int_0^t X^{n-2}(s)ds \end{aligned}$$

since $g_n(0) = 0$

Take expectation on both sides to get:

$$\begin{aligned} m_n(t) &= \mathbf{E}\left[g_n(t, X(t))\right] \\ &= = n\mathbf{E}\left[\int_0^t X^{n-1}(s)dX(s) + \frac{1}{2}n(n-1)\int_0^t X^{n-2}(s)ds\right] \end{aligned}$$

By linearity of expectation,

$$m_n(t) = n\mathbf{E}\left[\int_0^t X^{n-1}(s)dX(s)\right] + \frac{1}{2}n(n-1)\mathbf{E}\left[\int_0^t X^{n-2}(s)ds\right]$$

Recall that $\int_0^t X^{n-1}(s)dX(s)$ is an Itô integral and it is therefore a martingale, so $\mathbf{E}[\int_0^t X^{n-1}(s)dX(s)] = 0$.

Interchanging the order of integration to take the expectation inside the integral (that's Fubini's theorem), we finally get

$$\begin{aligned} m_n(t) &= \frac{1}{2}n(n-1)\int_0^t \mathbf{E}\left[X^{n-2}(s)\right] ds \\ &= \frac{1}{2}n(n-1)\int_0^t m_{n-2}(s)ds \end{aligned}$$

Now let's apply this formula for $n = 4$:

$$\begin{aligned} m_4(t) &= 6\int_0^t \mathbf{E}\left[X^2(s)\right] ds \\ &= 6\int_0^t sds \\ &= 3t^2 \end{aligned}$$

What about $n = 6$?

$$\begin{aligned} m_6(t) &= 15\int_0^t \mathbf{E}\left[X^4(s)\right] ds \\ &= 45\int_0^t s^2ds \\ &= 15t^3 \end{aligned}$$

3. *Let $Y_t = X_t^4$ where X_t is a Brownian motion. Using Itô's lemma, express the SDE for Y_t. Then, deduce the stochastic integral for Y_t over $[0, T]$. Finally, deduce from the stochastic integral an expression for $\mathbf{E}[Y_t]$.*

First, note that $Y_t = f(X_t)$ where $f(x) = x^4$. Hence,

$$\begin{aligned} \frac{\partial f}{\partial t} &= 0 \\ \frac{\partial f}{\partial x} &= 4x^3 \\ \frac{\partial f}{\partial x} &= 12x^2 \end{aligned}$$

By Itô's lemma,

$$dY_t = 6X_t^2 dt + 4X_t^3 dX_t$$

Since X_t is a Brownian motion then $X_0 = 0$ and therefore $Y_0 = 0$. Thus, integrating the SDE over $[0, T]$, we get

$$Y_T = 6\int_0^T X_t^2 dt + 4\int_0^T X_t^3 dX_t$$

taking the expectation and by linearity of the expectation operator,

$$\mathbf{E}[Y_T] = 6\mathbf{E}\left[\int_0^T X_t^2 dt\right] + 4\mathbf{E}\left[\int_0^T X_t^3 dX_t\right]$$

Now, the Itô integral $\int_0^T X_t^3 dX_t$ is a martingale and hence $\mathbf{E}\left[\int_0^T X_t^3 dX_t\right] = 0$. Also, by Fubini's Theorem, we can change the order of integration and therefore slide the expectation inside $\int_0^T X_t^2 dt$. Hence,

$$\mathbf{E}[Y_T] = 6\int_0^T \mathbf{E}\left[X_t^2\right] dt$$

Now, $\mathbf{E}[X_t^2] = \mathbf{E}[(X_t - X_0)^2] = t - 0 = t$. Therefore,

$$\mathbf{E}[Y_T] = 6\int_0^T t dt = 3T^2$$

4. *Let $X_n, n = 1, \ldots$ be i.i.d random variables where $P(X_n = 1) = p$ and $P(X_n = -1) = 1 - p$. You can think of X_n as being the nth coin toss in a sequence. Let $S_n, n = 1, \ldots$ be the associated random walk, defined as*

$$S_n = X_1 + X_2 + \ldots + X_n \tag{4}$$

S_n can be viewed as the P&L of the entire coin toss game. We also introduce the filtration $\mathcal{F}_n$ generated by the X_n and such that X_n is $\mathcal{F}_n$-adapted.

Find conditions under which the random walk is (a) a martingale, (b) a submartingale (c) a supermartingale.

Answer:

Let's start from the conditional expectation

$$\mathbf{E}\left[S_n|\mathcal{F}_{n-1}\right] = \mathbf{E}\left[S_{n-1} + X_n|\mathcal{F}_{n-1}\right]$$

Now, S_{n-1} is $\mathcal{F}_{n-1}$ adapted because X_{n-1} and therefore $X_{n-2}, \ldots, X_1$ are $\mathcal{F}_{n-1}$ adapted. As a result,

$$\mathbf{E}\left[S_{n-1} + X_n | \mathcal{F}_{n-1}\right] = S_{n-1} + \mathbf{E}\left[X_n | \mathcal{F}_{n-1}\right]$$

Because the X_n are i.i.d. random variables,then X_n is independent from $\mathcal{F}_{n-1}$ and as a result

$$\begin{aligned} \mathbf{E}\left[X_n | \mathcal{F}_{n-1}\right] &= \mathbf{E}\left[X_n\right] \\ &= 2p - 1 \end{aligned}$$

Thus,

$$\mathbf{E}\left[S_n | \mathcal{F}_{n-1}\right] = S_{n-1} + 2p - 1$$

We can now solve the question:

(a) S_n is *martingale* iff

$$\mathbf{E}\left[S_n | \mathcal{F}_{n-1}\right] = S_n$$

which only occurs if $p = \frac{1}{2}$;

(b) S_n is *submartingale* iff

$$\mathbf{E}\left[S_n | \mathcal{F}_{n-1}\right] > S_n$$

which only occurs if $p > \frac{1}{2}$;

(c) S_n is *supermartingale* iff

$$\mathbf{E}\left[S_n | \mathcal{F}_{n-1}\right] < S_n$$

which only occurs if $p < \frac{1}{2}$.

5. **Discrete Time Martingale**: *Let $Y_1, \ldots, Y_n$ be a sequence of independent random variables such that $\mathbf{E}[Y_i] = 0$ for $i = 1, \ldots, n$. Let $\mathcal{F}_n$ be the filtration generated by the sequence $Y_1, \ldots, Y_n$. Consider the random variable $S_n = \sum_{i=1}^{n} Y_i$. Prove that S_n is a martingale for all n.*

Reminder - *proving that a process S_n is a martingale involves proving that $\mathbf{E}[|S_n|] < \infty$ and that $\mathbf{E}[S_{n+1}|\mathcal{F}_n] = S_n$*

First,

$$\begin{aligned} \mathbf{E}[|S_n|] &= \mathbf{E}[|Y_1 + Y_2 + \ldots + Y_n|] \\ &\leq \mathbf{E}[|Y_1| + |Y_2| + \ldots + |Y_n|] \\ &= \mathbf{E}[|Y_1|] + \mathbf{E}[|Y_2|] + \ldots + \mathbf{E}[|Y_n|] \\ &< \infty \end{aligned}$$

since we have a finite sum of finite finite numbers.

Second,

$$\begin{aligned}\mathbf{E}[S_{n+1}|\mathcal{F}_n] &= \mathbf{E}[S_n + Y_{n+1}|\mathcal{F}_n] \\ &= \mathbf{E}[S_n|\mathcal{F}_n] + \mathbf{E}[Y_{n+1}|\mathcal{F}_n]\end{aligned}$$

by linearity of the expectation operator.

Now, since S_n is $\mathcal{F}_n$-measurable (i.e. if we have the filtration $\mathcal{F}_n$ we know what S_n is), then $\mathbf{E}[S_n|\mathcal{F}_n] = S_n$.

Also, since $Y_1, \ldots, Y_n, Y_{n+1}$ are independent, then Y_{n+1} is independent from $\mathcal{F}_n$ and hence $\mathbf{E}[Y_{n+1}|\mathcal{F}_n] = \mathbf{E}[Y_{n+1}] = 0$.

Therefore,

$$\mathbf{E}[S_{n+1}|\mathcal{F}_n] = \mathbf{E}[S_n]$$

and we can conclude that S_n is a martingale for all n.