Mod.1 Lecture 4 - Solutions to the Exercises

Stochastic Differential Equations

X_t is a Brownian Motion (Wiener Process) and dX_t or $dX(t)$ is its increment. $X_0 = 0$.

1. The change in a share price $S(t)$ satisfies

$$dS = A(S,t)\,dX_t + B(S,t)\,dt,$$

for some functions A and B. If $f = f(S,t)$, then Itô's lemma gives the following SDE

$$df = \left(\frac{\partial f}{\partial t} + B\frac{\partial f}{\partial S} + \frac{1}{2}A^2\frac{\partial^2 f}{\partial S^2}\right)dt + A\frac{\partial f}{\partial S}dX_t.$$

Can (non-zero) A and B be chosen so that a function $g = g(S)$ has a change which has zero drift, but non-zero diffusion? State any appropriate conditions.
A function $g(S)$ will satisfy the shorter SDE

$$dg = \left(B\frac{dg}{dS} + \frac{1}{2}A^2\frac{d^2 g}{dS^2}\right)dt + A\frac{dg}{dS}dX_t.$$

For $g(S)$ to have a zero drift but non-zero diffusion, we require the condition

$$B\frac{dg}{dS} + \frac{1}{2}A^2\frac{d^2 g}{dS^2} = 0.$$

i.e.

$$\frac{dg}{dS} + \frac{1}{2}\frac{A^2}{B}\frac{d^2 g}{dS^2} = 0$$

We can find a solution to this problem if $\frac{A^2}{B}$ is independent of time.

2. Show that $F(X_t) = \arcsin(2aX_t + \sin F_0)$ is a solution of the SDE

$$dF = 2a^2(\tan F)\left(\sec^2 F\right)dt + 2a(\sec F)\,dX_t,$$

where F_0 and a is a constant. The following standard result may be used

$$\frac{d}{dx}\sin^{-1} ax = \frac{a}{\sqrt{1-a^2x^2}}$$

$F = \arcsin(2aX_t + \sin F_0)$ implies $\sin F = 2aX_t + \sin F_0$ hence

$$\frac{dF}{dX_t} = \frac{2a}{\sqrt{1-(2aX_t+\sin F_0)^2}} = 2a\left\{1-(2aX_t+\sin F_0)^2\right\}^{-1/2}$$

$$\frac{d^2F}{dX_t^2} = \frac{(2a)^2(2aX_t+\sin F_0)}{\left\{1-(2aX_t+\sin F_0)^2\right\}^{3/2}}$$

So Itô gives

$$dF = \frac{2a}{\sqrt{1-(2aX_t+\sin F_0)^2}}dX + \frac{1}{2}\frac{(2a)^2(2aX_t+\sin F_0)}{\left\{1-(2aX_t+\sin F_0)^2\right\}^{3/2}}dt$$

We know $\cos^2 F + \sin^2 F = 1 \Longrightarrow \cos F = \sqrt{1-\sin^2 F} = \sqrt{1-(2aX_t+\sin F_0)^2}$. Some more trig.

$$\sec F=\frac{1}{\cos F}=\frac{1}{\sqrt{1-(2aX_t+\sin F_0)^2}}$$

and

$$(\tan F)\left(\sec^2 F\right)=\frac{\sin F}{\cos F}\frac{1}{\cos^2 F}=\frac{\sin F}{\cos^3 F}=\frac{2aX_t+\sin F_0}{\left\{1-(2aX_t+\sin F_0)^2\right\}^{3/2}}$$

which gives

$$dF=2a^2(\tan F)\left(\sec^2 F\right)dt+2a(\sec F)\,dX_t.$$

3. Show that

$$\int_0^t X_\tau\left(1-e^{-X_\tau^2}\right)dX_\tau=\overline{F}(X_t)+\int_0^t G(X_\tau)\,d\tau.$$

where the functions $\overline{F}$ and G should be determined.
We can do this two ways. Perform Itô's lemma and then integrate or use the stochastic integral version as given in the first question. Comparing

$$\int_0^t X(\tau)\left(1-e^{-X^2(\tau)}\right)dX(\tau)=\overline{F}(X(t))+\int_0^t G(X(t))\,d\tau$$

with

$$\int_0^t\frac{\partial F}{\partial X}dX(\tau)=F(X(t),t)-F(X(0),0)+\int_0^t-\left(\frac{\partial F}{\partial\tau}+\tfrac{1}{2}\frac{\partial^2F}{\partial X^2}\right)d\tau$$

suggests that

$$\frac{\partial F}{\partial X}=X(\tau)\left(1-e^{-X^2(\tau)}\right)$$

so integrating over $[0,t]$ gives $\overline{F}(X(t),t)$, which we will do by substitution, i.e. put $u=X^2$ which gives

$$F(X(t),t)-F(X(0),0)=\tfrac{1}{2}X^2(t)+\tfrac{1}{2}e^{-X^2(t)}-\tfrac{1}{2}.$$

Also knowing $\frac{\partial F}{\partial X}$ allows us to easily obtain $\frac{\partial^2F}{\partial X^2}=2X^2(t)\,e^{-X^2(t)}-e^{-X^2(t)}+1$. Hence

$$G(X(t))=-\tfrac{1}{2}\frac{\partial^2F}{\partial X^2}=-\tfrac{1}{2}\left(1-e^{-X^2(t)}\right)-X^2(t)\,e^{-X^2(t)}$$

and we have shown

$$\int_0^t X(\tau)\left(1-e^{-X^2(\tau)}\right)dX(\tau)=\overline{F}(X(t))+\int_0^t G(X(t))\,d\tau$$

where

$$\begin{aligned}\overline{F}(X(t),t)&=\tfrac{1}{2}X^2(t)+\tfrac{1}{2}e^{-X^2(t)}-\tfrac{1}{2}\\ G(X(t))&=-\tfrac{1}{2}\left(1-e^{-X^2(t)}\right)-X^2(t)\,e^{-X^2(t)}.\end{aligned}$$

4. Consider the process

$$d(\log y)=(\alpha-\beta\log y)\,dt+\delta dX_t.$$

The parameters $\alpha,\ \beta,\ \delta$ are constant. Show that y satisfies

$$\frac{dy}{y}=\left(\alpha-\beta\log y+\frac{1}{2}\delta^2\right)dt+\delta dX_t.$$

By Ito's lemma if $dZ = a(Z,t)dt + b(Z,t)dX_t$ and $Y = f(Z)$ then

$$dY = \left(a\frac{\partial Y}{\partial Z} + \frac{1}{2}b^2\frac{\partial^2 Y}{\partial Z^2} \right) dt + b\frac{\partial Y}{\partial Z}dX_t$$

here $Z \equiv \log y_t,\ a \equiv (\alpha - \beta Z),\ b \equiv \delta,\ ,\ Y = e^Z = y,\ \frac{\partial Y}{\partial Z} = e^Z = \frac{\partial^2 f}{\partial Z^2}$, putting all these in Ito's lemma we have

$$dY \equiv dy_t = \left((\alpha - \beta \log y_t)\, y_t + \frac{1}{2}\delta^2 y_t \right) dt + \delta y_t dX_t$$

hence

$$\frac{dy_t}{y_t} = \left(\alpha - \beta \log y_t + \frac{1}{2}\delta^2 \right) dt + \delta dX_t$$

5. Show that

$$G = e^{t + ae^{X_t}}$$

is a solution of the stochastic differential equation

$$dG\,(t) = G\left(1 + \tfrac{1}{2}(\ln G - t) + \tfrac{1}{2}(\ln G - t)^2\right) dt + G\,(\ln G - t)\, dX_t,$$

where a is a constant.

$$\frac{\partial G}{\partial t} = G, \quad \frac{\partial G}{\partial X_t} = aGe^{X_t}, \quad \frac{\partial^2 G}{\partial X_t^2} = ae^{X_t}G + ae^{X_t}\frac{\partial G}{\partial X_t} = ae^{X_t}G + a^2e^{2X_t}G$$

In Itô, i.e.

$$\begin{aligned} dG &= \left(\frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial X_t^2} \right) dt + \frac{\partial G}{\partial X_t}dX_t \\ &= \left(G + \tfrac{1}{2}ae^{X_t}G + \tfrac{1}{2}a^2e^{2X_t}G\right) dt + ae^{X_t}GdX_t \end{aligned}$$

From $G = e^{t + ae^{X_t}}$ we have

$$ae^{X_t} + t = \ln G \Longrightarrow ae^{X_t} = \ln G - t$$

so we can write the SDE in terms of the process G

$$dG = G\left(1 + \tfrac{1}{2}ae^{X_t} + \tfrac{1}{2}a^2e^{2X_t}\right) dt + ae^{X_t}GdX_t$$

So

$$dG = G\left(1 + \tfrac{1}{2}(\ln G - t) + \tfrac{1}{2}(\ln G - t)^2\right) dt + G\,(\ln G - t)\, dX_t.$$

6. The Ornstein-Uhlenbeck process satisfies the spot rate SDE given by

$$dr_t = \kappa\,(\theta - r_t)\, dt + \sigma dX_t,\ r_0 = u,$$

where κ, θ and σ are constants. Solve this SDE by setting $Y_t = e^{\kappa t}r_t$ and using Itô's lemma to show that

$$r_t = \theta + (x - \theta)\, e^{-\kappa t} + \sigma \int_0^t e^{-\kappa(t-s)}dX_s.$$

First write Itô for Y_t given $dr_t = A(r_t, t)\,dt + B(r_t, t)\,dX_t$

$$
\begin{aligned}
dY_t &= \left(\frac{\partial Y_t}{\partial t} + A(r_t, t)\frac{\partial Y_t}{\partial r_t} + \tfrac{1}{2}B^2(r_t, t)\frac{\partial^2 Y_t}{\partial r_t^2}\right)dt + B(r_t, t)\frac{\partial Y_t}{\partial r_t}dX_t \\
&= \left(\frac{\partial Y_t}{\partial t} + \kappa(\theta - r_t)\frac{\partial Y_t}{\partial r_t} + \tfrac{1}{2}\sigma^2\frac{\partial^2 Y_t}{\partial r_t^2}\right)dt + \sigma\frac{\partial Y_t}{\partial r_t}dX_t
\end{aligned}
$$

$$
\frac{\partial Y_t}{\partial t} = \kappa e^{\kappa t} r_t;\ \ \frac{\partial Y_t}{\partial r_t} = e^{\kappa t};\ \ \frac{\partial^2 Y_t}{\partial r_t^2} = 0.
$$

$$
\begin{aligned}
d\left(e^{\kappa t} r_t\right) &= \left(\kappa e^{\kappa t} r_t + \kappa(\theta - r_t)e^{\kappa t}\right)dt + \sigma e^{\kappa t}dX_t \\
&= \kappa\theta e^{\kappa t}dt + \sigma e^{\kappa t}dX_t
\end{aligned}
$$

$$
\begin{aligned}
\int_0^t d\left(e^{\kappa s} r_s\right) &= \kappa\theta\int_0^t e^{\kappa s}ds + \sigma\int_0^t e^{\kappa s}dX_s \\
e^{\kappa t} r_t - u &= \theta e^{\kappa t} - \theta + \sigma\int_0^t e^{\kappa s}dX_s \\
r_t &= u e^{-\kappa t} + \theta - \theta e^{-\kappa t} + \sigma e^{-\kappa t}\int_0^t e^{\kappa s}dX_s \\
r_t &= \theta + (u - \theta)e^{-\kappa t} + \sigma\int_0^t e^{-\kappa(t-s)}dX_s.
\end{aligned}
$$