$f(x)$ x is a n-dim vector. $(x \in \mathbb{R}^n)$ $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

1st order

$$\nabla f(x) = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix}$$

$$\underbrace{\frac{\partial f}{\partial x} \qquad \underbrace{Df}}_{PDE}$$

$$\frac{df}{dx} \qquad \frac{d^2 f}{dx^2}$$

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$$

2nd order:

$$Hf = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} & & \\ \frac{\partial^2 f}{\partial x_1 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & & \\ \frac{\partial^2 f}{\partial x_1 \partial x_3} & \frac{\partial^2 f}{\partial x_2 \partial x_3} & \ddots & \\ & & & \frac{\partial^2 f}{\partial x_n^2} \end{pmatrix}$$

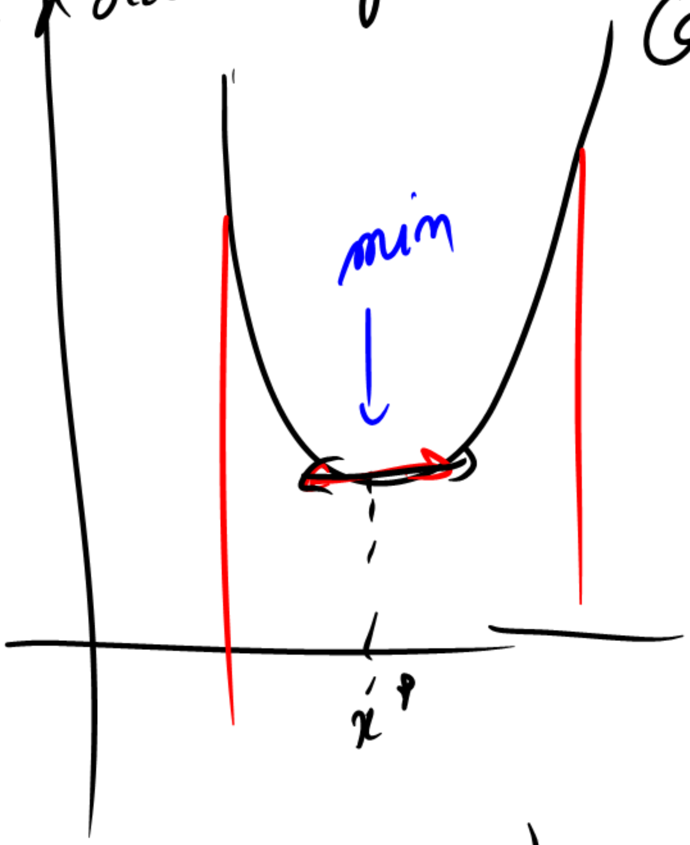
$n \times n$ matrix

3 types of stationary points ;

1st order $\nabla f(x^*) = 0$
necessary

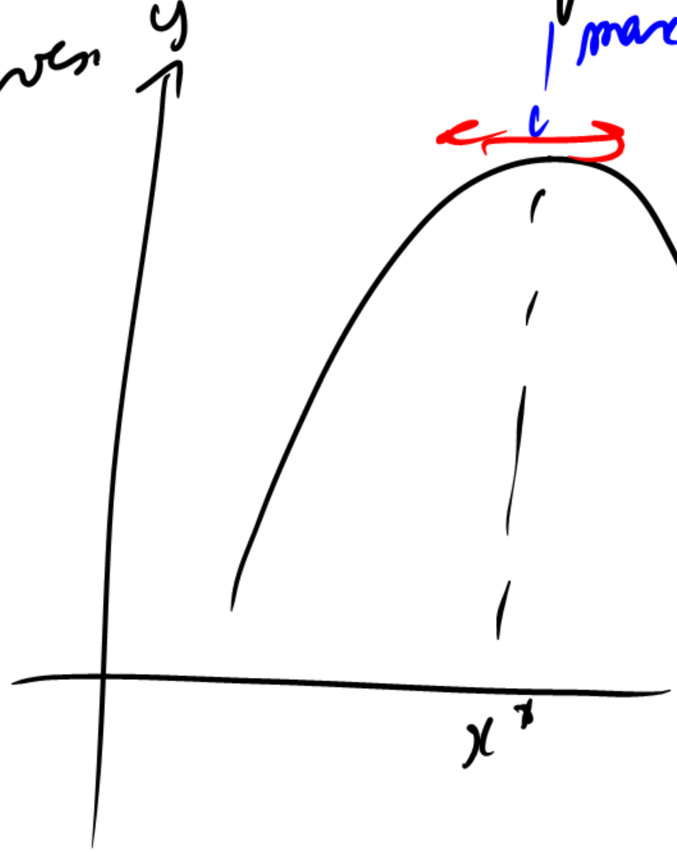
2nd order $Hf > 0$

Convex



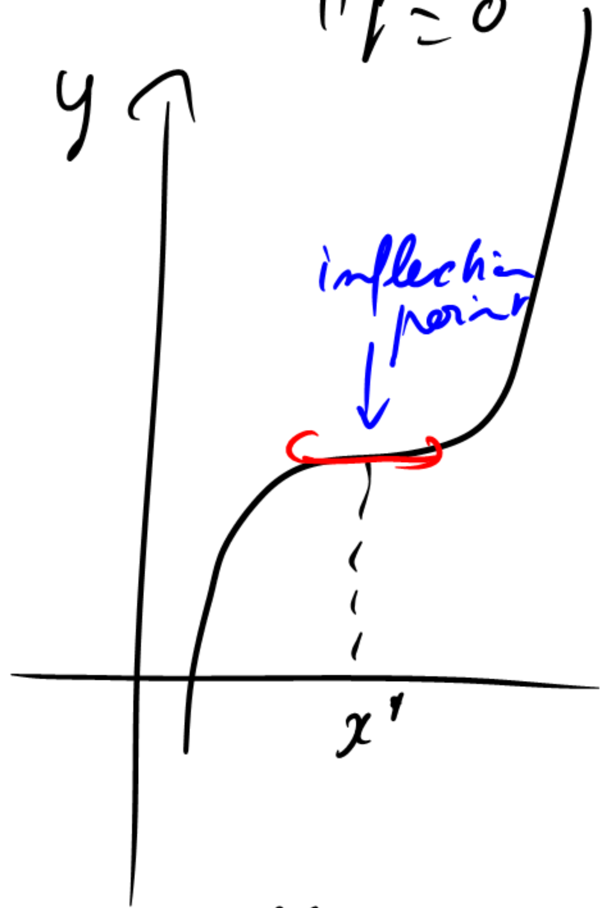
$\nabla f(x^*) = 0$

$Hf < 0$



$\nabla f(x^*) = 0$

$Hf = 0$



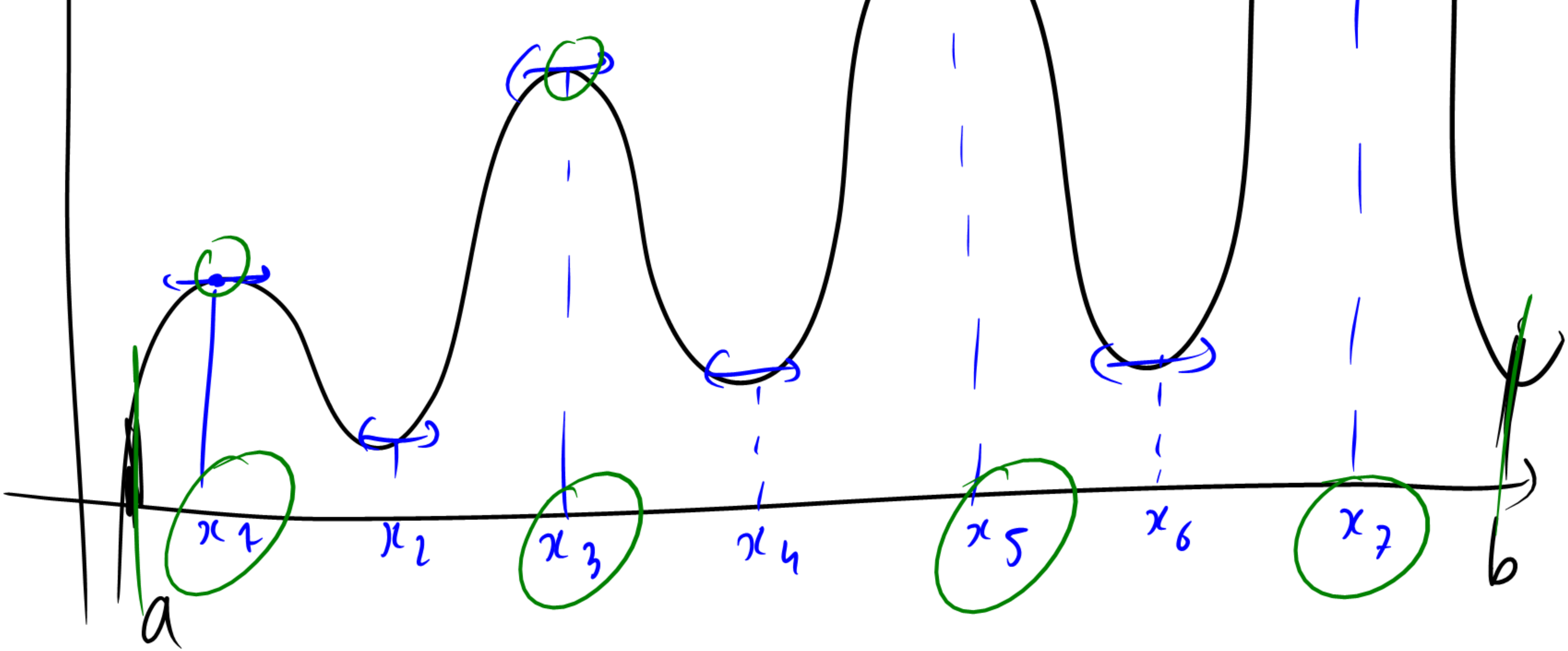
$$\frac{df}{dx}(x^*)$$

$$\frac{d^2 f}{dx^2}(x^*)$$

$$\frac{d^2 f}{dx^2} = 0$$

Max

- 1st order condition $\frac{df}{dx}(x^*) = 0$
- 2nd order condition $\frac{d^2f}{dx^2} < 0$
- compute the value of f at a, b, x_1, x_3, x_5 and x_7



Positive Definite :

$M > 0$

for all the vectors $v \in \mathbb{R}^{n}$

M is a $\boxed{n \times n}$ matrix

square matrix

$n \times n$

$$\underbrace{v' M v}_{\text{scalar}} > 0$$

Neg Definite

$P < 0$

$$\underbrace{v' P v}_{\text{scalar}} < 0$$

v is a n-element column vector

$\to$ v is a $n \times 1$ matrix

M is n by n

$$\underbrace{v' \quad M}_{1 \times n \quad n \times n} \quad \underset{n \times 1}{v}$$

$$1 \times n \quad n \times 1$$

$$1 \times 1 \to \text{scalar}$$

* Objective function

$$V(w) = \mu_\pi(w) - \frac{\lambda}{2}\sigma_\pi^2(w)$$

$$= r + w'(\mu - r\mathbb{1}) - \frac{\lambda}{2} w'\Sigma w$$

$$\frac{d}{dw}\left(r + w(\mu - r) - \frac{\lambda}{2}w^2\Sigma\right) \rightarrow n = 1$$

$$= \mu - r - \lambda w \Sigma$$

* $\max_w V(w)$

1st order condition

$$\nabla V(w^*) = 0$$

$$\nabla V = 0 + \underbrace{(\mu - r\mathbb{1})}_{n \times 1} - \cancel{\frac{\lambda}{2}} \times \cancel{2}\,\underbrace{\Sigma w}_{n \times 1} \rightarrow$$

$n \times n \quad n \times 1$

1st order condition:

$$\nabla V(\omega^*) = 0$$

$$\Leftrightarrow \quad (\mu - r\mathbb{1}) - \lambda \Sigma \omega^* = 0$$

$$\Leftrightarrow \quad (\mu - r\mathbb{1}) = \lambda \Sigma \omega^*$$

$$\Leftrightarrow \quad \frac{1}{\lambda} \Sigma^{-1} (\mu - r\mathbb{1}) = \Sigma^{-1}\Sigma \omega^*$$

$$\Leftrightarrow \quad \boxed{\omega^* = \frac{1}{\lambda} \Sigma^{-1} (\mu - r\mathbb{1})}$$

$$\mathbb{1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

$$\mu - r\mathbb{1} = \begin{pmatrix} \mu_1 - r \\ \mu_2 - r \\ \vdots \\ \mu_n - r \end{pmatrix}$$

scalar

2nd order condition: $\frac{\partial}{\partial \omega}\left[(\underbrace{\cancel{\mu - r\mathbb{1}}}_{n \times 1}) - \lambda \Sigma \cancel{\omega^*} \right] = \underbrace{- \lambda \underbrace{\Sigma}_{> 0}}_{< 0}$

$$L(w,\lambda,\gamma) = \frac{1}{2} w' \Sigma w + \lambda(m - w'\mu) + \gamma(1 - w'\mathbb{1})$$

1st order condition

$$\left.\frac{\partial L}{\partial w}\right|_{w^*} = \frac{1}{\cancel{2}} \times \cancel{2}\,\Sigma w^* - \lambda\mu - \gamma\mathbb{1} = 0$$

$$\Leftrightarrow \quad w^* - \lambda\Sigma^{-1}\mu - \gamma\Sigma^{-1}\mathbb{1} = 0$$

$$\Leftrightarrow \quad \boxed{w^* = \Sigma^{-1}(\lambda\mu + \gamma\mathbb{1})}$$ candidate

$$\frac{\partial L}{\partial \lambda} = 0 \quad \Leftrightarrow \quad w^{*\prime}\mu = m$$

$$\frac{\partial L}{\partial \gamma} = 0 \quad \Leftrightarrow \quad w^{*\prime}\mathbb{1} = 0$$

$\Leftrightarrow$

$$\begin{cases} w'\mu = m \\ w'\mathbb{1} = 1 \end{cases} \qquad @\ w^* = \Sigma^{-1}(\lambda\mu + \gamma\mathbb{1})$$

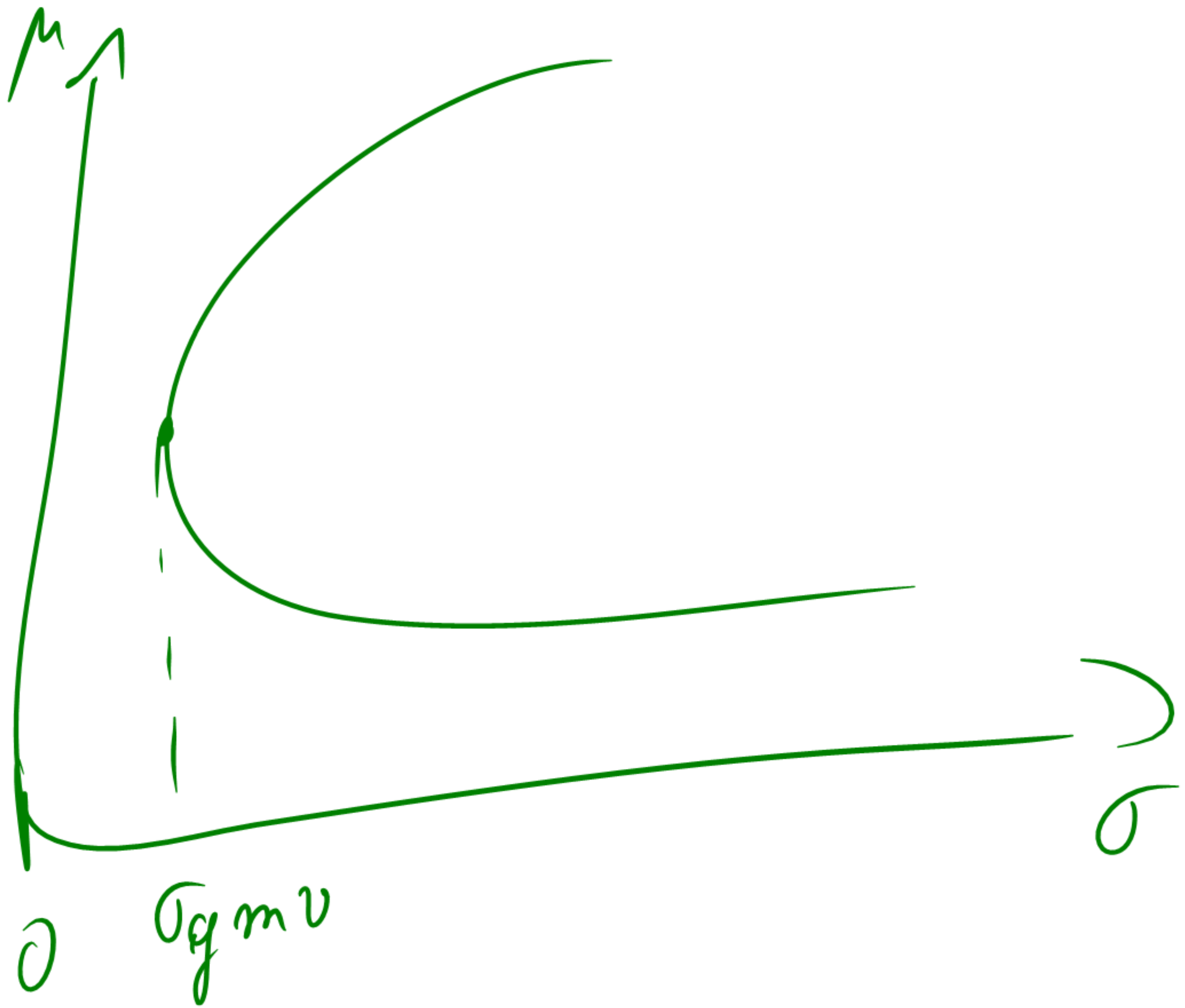
$$\begin{cases} \mu' w^* = m \\ \mathbb{1}' w^* = 1 \end{cases}$$

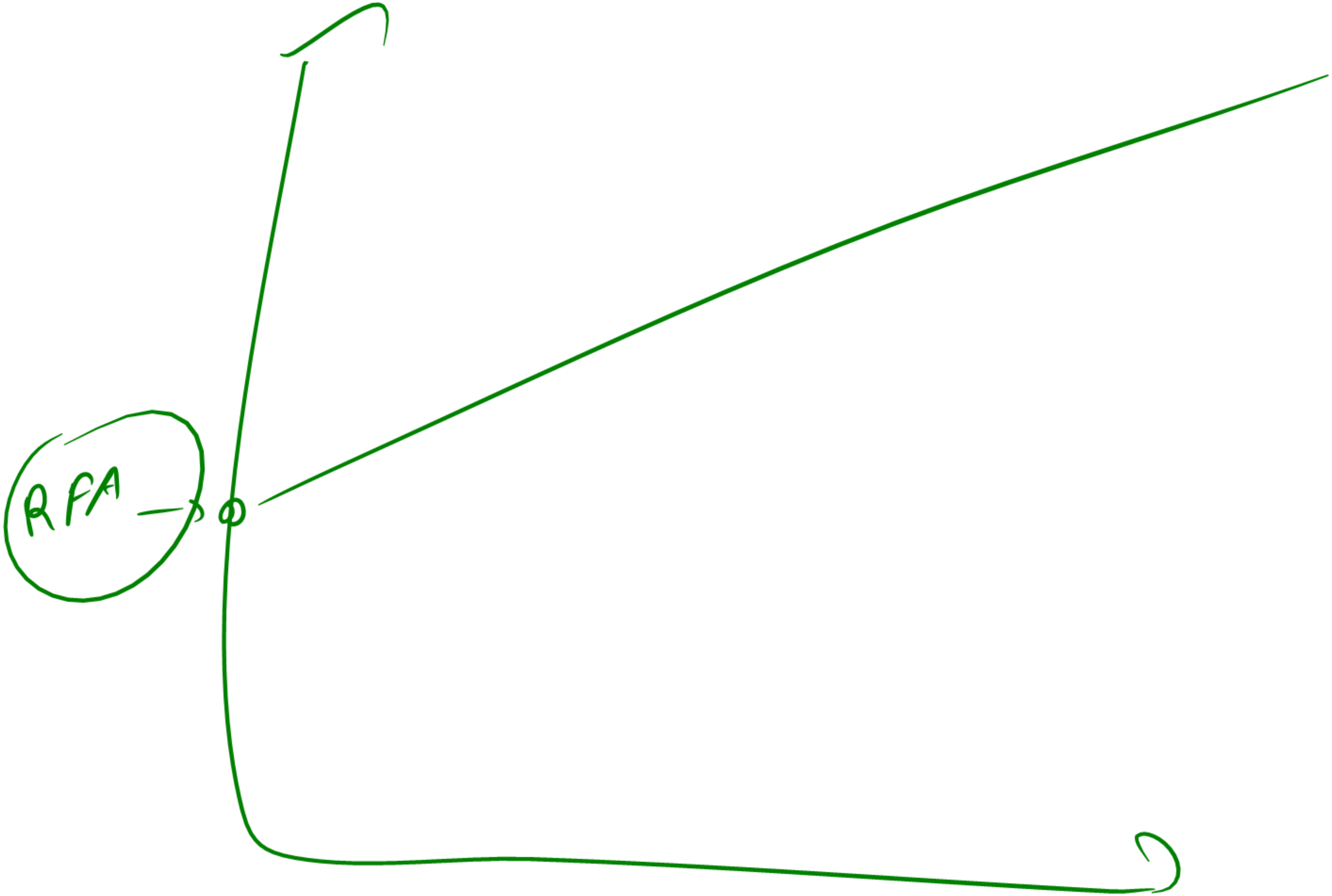
scalars

$$\begin{cases} \lambda \mu' \Sigma^{-1} \mu + \gamma \mu' \Sigma^{-1} \mathbb{1} = m \\ \lambda \mathbb{1}' \Sigma^{-1} \mu + \gamma \mathbb{1}' \Sigma^{-1} \mathbb{1} = 1 \end{cases}$$

scalars as well

Minimum variance → n risky and No RFA




RFA

$\ln W$ $\hat{=}$ log return on your portfolio

↳ log utility