Asian Options

In this lecture. . .

- many types of Asian option with a payoff that depends on an average

- the different types of averaging of asset prices that are used in defining the payoff

- how to price these contracts in the partial differential equation framework

Introduction

Asian options give the holder a payoff that depends on the average price of the underlying over some prescribed period.

This averaging of the underlying can significantly reduce the price of an Asian option compared with a similar vanilla contract.

Anything that reduces the up-front premium in an option contract tends to make them more popular.

Payoff types

Assuming for the moment that we have defined our average A, what sort of payoffs are common?

- As well as calls, puts etc. there is also the classification of **strike** and **rate**.

S, A, t

These classifications work as follows. Take the payoff for a vanilla option, a vanilla call, say,

$$\max(S - E, 0).$$

Replace the strike price E with an average and you have an **average strike call**. This has payoff

$$\max(S - A, 0).$$

$S - \cancel{E} A$

An **average strike put** thus has payoff

$$\max(A - S, 0).$$

Now take the vanilla payoff and instead replace the asset with its average, what you get is a rate option. For example, an **average rate call** has payoff

$$\max(A - E, 0)$$

$\cancel{S}\ A$

and an **average rate put** has payoff

$$\max(E - A, 0).$$

- The average rate options can be used to lock in the price of a commodity or an exchange rate for those who have a continual and fairly predictable exposure to one of these over extended periods.

The difference between calls and puts is simple from a pricing point of view.

Types of averaging

- The precise definition of the average used in an Asian contract depends on two elements: how the data points are combined to form an average and which data points are used.

The former means whether we have an arithmetic or geometric average or something more complicated. The latter means how many data points do we use in the average, all quoted prices, or just a subset, and over what time period.

$$A_{i+1} = \frac{i}{i+1} A_i + \frac{\ln S_i}{i+1}$$

$$e^{A}$$

Arithmetic or geometric

The two simplest and obvious types of average are the

- **arithmetic average**: the sum of all the constituent prices, equally weighted, divided by the total number of prices used

- **geometric average**: the *exponential* of the sum of all the *logarithms* of the constituent prices, equally weighted, divided by the total number of prices used

Another popular choice is the exponentially weighted average, meaning instead of having an equal weighting to each price in the average, the recent prices are weighted more than past prices in an exponentially decreasing fashion.

Discrete or continuous

How much data do we use in the calculation of the average? Do we take every traded price or just a subset?

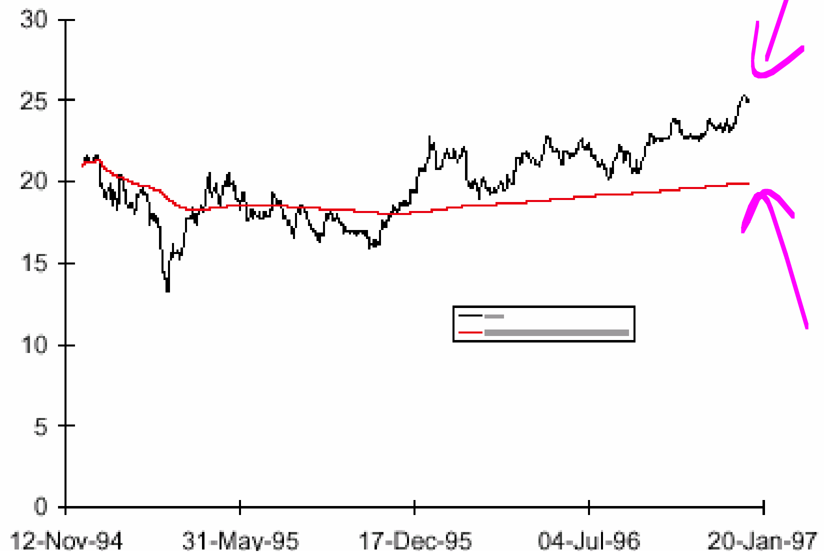
- If we take closely-spaced prices over a finite time then the sums that we calculate in the average become integrals of the asset (or some function of it) over the averaging period. This would give us a **continuously-sampled average**.

- More commonly, we only take data points that are reliable, using closing prices, a smaller set of data. This is called **discrete sampling**.

Continuously-sampled averages

The continuously-sampled running arithmetic average is defined as

$$\frac{1}{t}\int_0^t S(\tau)\,d\tau.$$



If we introduce the new state variable

$$I = \int_0^t S(\tau)\, d\tau$$

then the partial differential equation for the value of an option contingent on this average is

$$\frac{\partial V}{\partial t} + S\frac{\partial V}{\partial I} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

The continuously-sampled geometric average is defined to be

$$\exp\left(\frac{1}{t}\int_0^t \log S(\tau)\, d\tau\right).$$

To value an option contingent on this average we define

$$I = \int_0^t \log S(\tau)\, d\tau$$

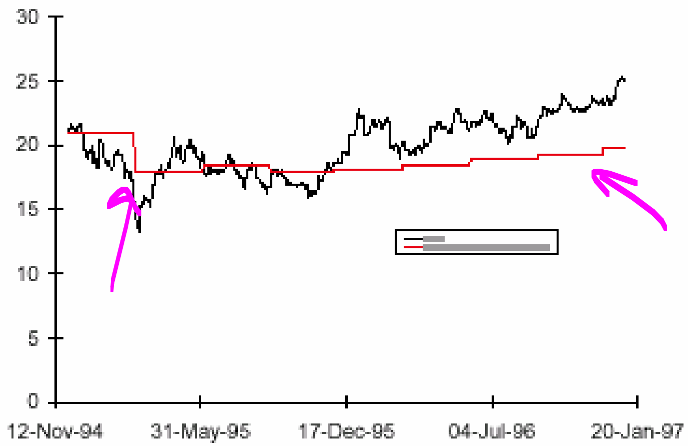
and the partial differential equation for the value of the option is

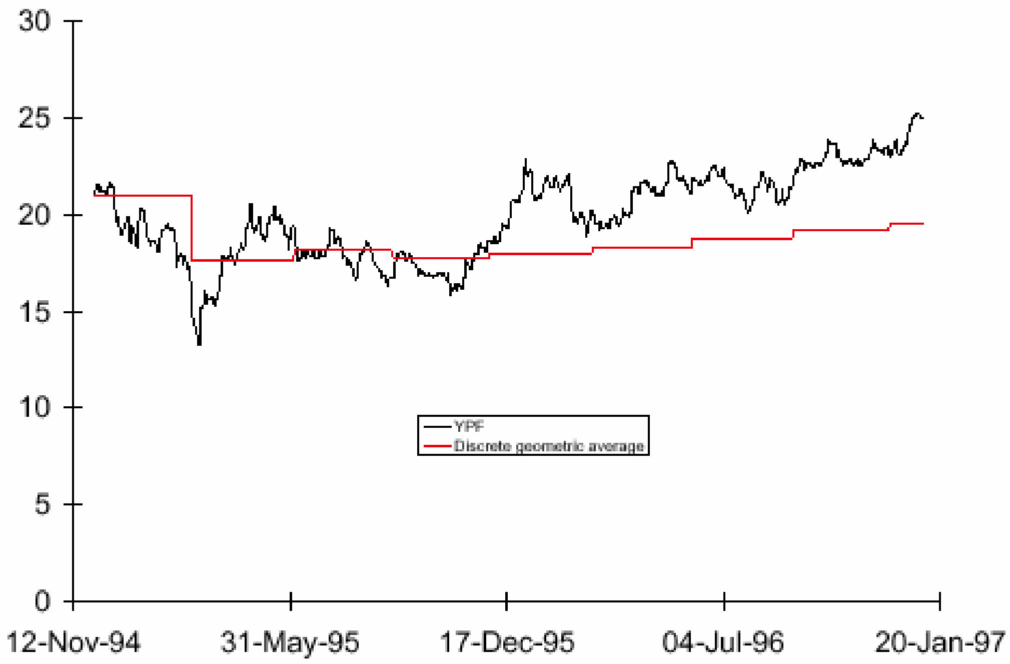
$$\frac{\partial V}{\partial t} + \log S \frac{\partial V}{\partial I} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

Discretely-sampled averages

If the sampling dates are t_i, $i = 1, \ldots$ then the discretely-sampled arithmetic average is defined by

$$A_i = \frac{1}{i} \sum_{k=1}^{i} S(t_k).$$





An asset and its discretely-sampled geometric average.

Earlier, we modeled the continuously-sampled average as an integral. By a discretely-sampled average we mean the sum, rather than the integral, of a finite number of values of the asset during the life of the option:

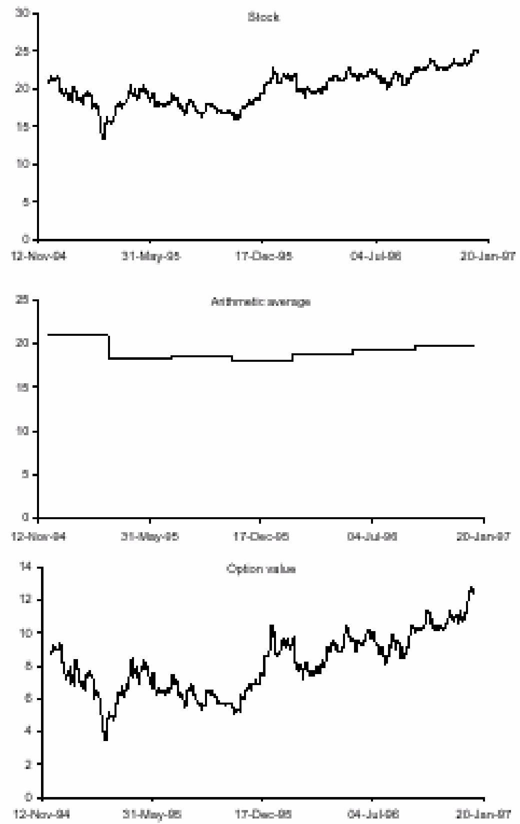
$$A_i = \frac{1}{i}\sum_{k=1}^{i} S(t_k).$$

In particular

$$\begin{aligned} A_1 &= S(t_1), \quad A_2 = \frac{S(t_1)+S(t_2)}{2} = \tfrac{1}{2}A_1 + \tfrac{1}{2}S(t_2), \\ A_3 &= \frac{S(t_1)+S(t_2)+S(t_3)}{3} = \tfrac{2}{3}A_2 + \tfrac{1}{3}S(t_3), \cdots . \end{aligned}$$

It is easy to see that these are equivalent to

$$A_i = \frac{i-1}{i}A_{i-1} + \frac{1}{i}S(t_i).$$


Stock
30
25
20
15
10
5
0
12-Nov-94
31-May-95
17-Dec-95
04-Jul-96
20-Jan-97
Arithmetic average
25
20
15
10
5
0
12-Nov-94
31-May-95
17-Dec-95
04-Jul-96
20-Jan-97
Option value
14
12
10
8
6
4
2
0
12-Nov-94
31-May-95
17-Dec-95
04-Jul-96
20-Jan-97

At the top of the figure is shown a realized asset price path. Below that is the discretely-sampled average. This is necessarily piecewise constant. At the bottom of the figure is the value of some option (it doesn't matter which).

- The option value must be continuous to eliminate arbitrage opportunities.

The jump condition for an Asian option with discrete arithmetic averaging is then simply

$$V(S, A, t_i^-) = V\left(S, \frac{i-1}{i}A + \frac{1}{i}S, t_i^+\right).$$

This is a result of the continuity of the option price across a sampling date i.e. no arbitrage.

Similarly the discretely-sampled geometric average has the jump condition

$$V(S, A, t_i^-) = V\left(S, \exp\left(\frac{i-1}{i}\log(A) + \frac{1}{i}\log(S)\right), t_i^+\right)$$

where

$$A_i = \exp\left(\frac{i-1}{i}\log(A_{i-1}) + \frac{1}{i}\log(S(t_i))\right).$$

Exponentially-weighted and other averages

Simple modifications that are easily handled in the partial differential equation framework are the exponential average and the average up to a fixed time.

In the exponential continuously-sampled arithmetic average just introduce the new variable

$$I = \lambda \int_{-\infty}^{t} e^{-\lambda(t-\tau)} S(\tau)\, d\tau$$

which satisfies

$$dI = \lambda(S - I)dt.$$

From this, the governing partial differential equation is obvious. The geometric equivalent is dealt with similarly.

When the average is only taken up to a fixed point, so that, for example, the payoff depends on

$$I = \int_0^{T_0} S(\tau)\, d\tau \quad \text{with} \quad T_0 < T,$$

then the new term in the partial differential equation (the derivative with respect to I) disappears for times greater that T_0. That is,

$$\frac{\partial V}{\partial t} + S\mathcal{H}(T_0 - t)\frac{\partial V}{\partial I} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0,$$

where $\mathcal{H}$ is the Heaviside function.

The Asian tail

Often the averaging is confined to only a part of the life of the option.

- For example, if the averaging of the underlying is only over the final part of the option's life it is referred to as an **Asian tail**.

Such a contract would reduce the exposure of the option to sudden moves in the underlying just before the payoff is received. A feature like this is also common in pension awards.

Early exercise

The only point to mention is that the details of the payoff on early exercise have to be well defined. The payoff at expiry depends on the value of the average up to expiry, this will, of course, not be known until expiry. Typically, on early exercise it is the average to date that is used. For example, in an American arithmetic average strike put the early payoff would be

$$\max\left(\frac{1}{t}\int_0^t S(\tau)d\tau - S, 0\right).$$

$S - A$

Similarity reductions

As long as the stochastic differential equation or updating rule for the path-dependent quantity only contains references to S, t and the path-dependent quantity itself then the value of the option depends on three variables. Unless we are very lucky, the value of the option must be calculated numerically.

Some options have a particular structure that permits a reduction in the dimensionality of the problem by use of a similarity variable. The dimensionality of the continuously-sampled arithmetic average strike option can be reduced from three to two.

The payoff for the call option is

$$\max\left(S - \frac{1}{T}\int_0^T S(\tau)\,d\tau,\ 0\right).$$

S, A

$$\frac{S}{A}$$

We can write the running payoff for the call option as

$$I \max\left(R - \frac{1}{t}, 0\right),$$

where

$$I = \int_0^t S(\tau)\, d\tau$$

and

$$R = \frac{S}{\int_0^t S(\tau)\, d\tau}.$$

The payoff at expiry may then be written as

$$I \max\left(R - \frac{1}{T}, 0\right).$$

In view of the form of the payoff function, it seems plausible that the option value takes the form

$$V(S, I, t) = I\,W(R, t), \quad \text{with} \quad R = \frac{S}{I}.$$

We find that W satisfies

$$\frac{\partial W}{\partial t} + \frac{1}{2}\sigma^2 R^2 \frac{\partial^2 W}{\partial R^2} + R(r - R)\frac{\partial W}{\partial R} - (r - R)W = 0. \qquad (1)$$

Put-call parity for the European average strike

The payoff at expiry for a portfolio of one European average strike call held long and one put held short is

$$I \max(R - 1/T, 0) - I \max(1/T - R, 0).$$

Whether R is greater or less than T at expiry, this payoff is simply

$$S - \frac{I}{T}.$$

The value of this portfolio is identical to one consisting of one asset and a financial product whose payoff is

$$-\frac{I}{T}.$$

In order to value this product find a solution of the average strike equation of the form

$$W(R,t) = b(t) + a(t)R \tag{2}$$

and with $a(T) = 0$ and $b(T) = -1/T$; such a solution would have the required payoff of $-I/T$.

Substituting (2) into (1) and satisfying the boundary conditions, we find that

$$a(t) = -\frac{1}{rT}\left(1 - e^{-r(T-t)}\right), \quad b(t) = -\frac{1}{T}e^{-r(T-t)}.$$

We conclude that

$$V_C - V_P = S - \frac{S}{rT}\left(1 - e^{-r(T-t)}\right) - \frac{1}{T}e^{-r(T-t)} \int_0^t S(\tau)\, d\tau,$$

where V_C and V_P are the values of the European arithmetic average strike call and put. This is put-call parity for the European average strike option.