Stochastic Calculus and Itô's lemma

Throughout this problem sheet, you may assume that X_t is a Brownian Motion (Wiener Process) and dX_t is its increment; $X_0 = 0$.

1. Let ϕ be a random variable which follows a standardised normal distribution, i.e. $\phi \sim N(0,1)$. Calculate the expected value and variance given by $\mathbb{E}[\psi]$ and $\mathbb{V}[\psi]$, in turn, where $\psi = \sqrt{dt}\phi$. dt is a small time-step. **Note: No integration is required.**
   Firstly we know $\mathbb{V}[\phi] = \mathbb{E}\left[\phi^2\right] = 1$ from the definition of $N(0,1)$.
   $\mathbb{E}[\psi] = \mathbb{E}\left[\sqrt{dt}\phi\right] = \sqrt{dt}\mathbb{E}[\phi]$, because dt is not a RV and we also know that $\mathbb{E}[\phi] = 0, \therefore \mathbb{E}[\psi] = 0$. $\mathbb{V}[\psi] = \mathbb{E}\left[\psi^2\right] - \mathbb{E}[\psi]^2 \to \mathbb{E}\left[dt\phi^2\right] \Rightarrow \mathbb{V}[\psi] = dt\mathbb{E}\left[\phi^2\right] = dt.$

2. Consider the following examples of SDEs for a diffusion process G. Write these in standard form, i.e.

$$dG = A(G,t)dt + B(G,t)dX_t.$$

   Give the drift and diffusion for each case.

   a. $df + dX_t - dt + 2\mu t f dt + 2\sqrt{f}dX_t = 0$

$$df = (1 - 2\mu t f)\,dt + \left(-1 - 2\sqrt{f}\right)dX_t$$

   b. $\dfrac{dy}{y} = (A + By)\,dt + (Cy)\,dX_t$

$$dy = \left(Ay + By^2\right)dt + \left(Cy^2\right)dX_t$$

   c. $dS = (\nu - \mu S)dt + \sigma dX_t + 4dS$

$$\begin{aligned} dS - 4dS &= (\nu - \mu S)dt + \sigma dX_t \\ dS &= -\tfrac{1}{3}(\nu - \mu S)dt - \tfrac{1}{3}\sigma dX_t \end{aligned}$$

3. Use Itô's lemma to obtain a SDE for each of the following functions:

a. $f(X_t) = (X_t)^n$

$$df = nX_t^{n-1}dX_t + \frac{1}{2}n(n-1)X_t^{n-2}dt$$

b. $y(X_t) = \exp(X_t)$

$$\begin{aligned} dy &= \exp(X_t)\,dX_t + \frac{1}{2}\exp(X_t)\,dt \text{ or} \\ \frac{df}{y} &= \frac{1}{2}dt + dX_t \end{aligned}$$

c. $g(X_t) = \ln X_t$

$$dg = -\frac{1}{2X_t^2}dt + \frac{1}{X_t}dX_t$$

d. $h(X_t) = \sin X_t + \cos X_t$

$$dh = (\cos X_t - \sin X_t)\,dX_t - \frac{1}{2}(\sin X_t + \cos X_t)\,dt$$

e. $f(X_t) = a^{X_t}$, where the constant $a > 1$

$$\begin{aligned} f(X_t) &= a^{X_t} \Rightarrow \ln f = X_t \ln a \Rightarrow \frac{1}{f}f'(X) = \ln a \Rightarrow f'(X_t) = (\ln a)\,f \\ \text{therefore } \; f'(X_t) &= (\ln a)\,a^{X_t} \;\text{ and hence }\; f''(X_t) = (\ln a)^2\,a^{X_t} \\ df &= (\ln a)\,a^{X_t}dX_t + \frac{1}{2}(\ln a)^2\,a^{X_t}dt \\ \text{or } \frac{df}{f} &= \frac{1}{2}(\ln a)^2\,dt + (\ln a)\,dX_t \end{aligned}$$

4. Using the formula below for stochastic integrals, for a function $F(X_t, t)$,

$$\int_0^t \frac{\partial F}{\partial X_t}dX_t = F(X_t,t) - F(X_0,0) - \int_0^t \left(\frac{\partial F}{\partial \tau} + \frac{1}{2}\frac{\partial^2 F}{\partial X_t^2}\right)d\tau$$

show that we can write

a. $\int_0^t X_\tau^3 dX_\tau = \frac{1}{4}X_t^4 - \frac{3}{2}\int_0^t X_\tau^2 d\tau$. Here we have ordinary derivatives and no $\frac{\partial F}{\partial t}$

$$\frac{dF}{dX_t} = X_t^3 \longrightarrow F(X_t) = \frac{1}{4}X_t^4(t) \longrightarrow \frac{d^2F}{dX_t^2} = 3X_t^2(t)$$

which substituted into the formula gives the result

b. $\int_0^t \tau dX_\tau = tX_t - \int_0^t X_\tau d\tau$

$$\frac{\partial F}{\partial X_t} = t \longrightarrow F(X_t,t) = tX_t \Rightarrow \frac{\partial^2 F}{\partial X_t^2} = 0 \text{ and } \frac{\partial F}{\partial t} = X_t$$

substituting all of these terms in to the formula

$$\int_0^t \tau dX_\tau = tX_t - 0 - \int_0^t \left(X_\tau + \frac{1}{2}\times 0\right)d\tau = tX_t - \int_0^t X_\tau d\tau$$

c. $\int_0^t (X_\tau + \tau)\, dX_\tau = \frac{1}{2}X_t^2 + tX_t - \int_0^t \left(X_t + \frac{1}{2}\right) d\tau$

$$\frac{\partial F}{\partial X_t} = X_t + t \longrightarrow F(X_t) = \frac{1}{2}X_t^2 + tX_t \longrightarrow \frac{\partial F}{\partial t} = X_t$$

and $\frac{\partial^2 F}{\partial X^2} = 1$, therefore leading to the required result.

5. Consider a diffusion process S_t which follows Geometric Brownian Motion

$$\frac{dS_t}{S_t} = \mu dt + \sigma dX_t.$$

Use Itô's Lemma to show that the SDE dV for $V = \log(tS)$ is given by

$$dV = \left(\frac{1}{t} + \mu - \frac{1}{2}\sigma^2\right) dt + \sigma dX_t.$$

2D Itô's lemma gives

$$dV = \left(V_t + \mu S V_S + \frac{1}{2}\sigma^2 S^2 V_{SS}\right) dt + (\sigma S V_S)\, dX_t$$

$$V(S,t) = \log(tS) \rightarrow V_t = \frac{1}{t}; \quad V_S = \frac{1}{S} \text{ and } V_{SS} = -\frac{1}{S^2}$$

substituting into expression for dV gives

$$\begin{aligned} dV &= \left(\frac{1}{t} + \mu S\left(\frac{1}{S}\right) + \frac{1}{2}\sigma^2 S^2\left(-\frac{1}{S^2}\right)\right) dt + \left(\sigma S.\frac{1}{S}\right) dX_t \\ dV &= \left(\frac{1}{t} + \mu - \frac{1}{2}\sigma^2\right) dt + \sigma dX_t. \end{aligned}$$

6. Consider a function $V(t, S_t, r_t)$ where the two stochastic processes S_t and r_t evolve according to a two factor model given by

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dX_t^{(1)} \\ dr_t &= \gamma(m - r_t)\, dt + c dX_t^{(2)}, \end{aligned}$$

in turn. and where $dX_t^{(1)} dX_t^{(2)} = \rho dt$. The parameters μ, σ, γ, m and c are constant. Let $V(t, S_t, r_t)$ be a function on $[0, T]$ with $V(0, S_0, r_0) = v$. Using Itô, deduce the integral form for $V(T, S_T, r_T)$.

$$\begin{aligned} V_T &= v + \int_0^T \left(\frac{\partial V}{\partial t} + \mu S_t \frac{\partial V}{\partial S_t} + \gamma(m - r_t)\frac{\partial V}{\partial r_t}\right. \\ &\quad \left. + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S_t^2} + \frac{1}{2}c^2 \frac{\partial^2 V}{\partial r_t^2} + \rho\sigma c S_t \frac{\partial^2 V}{\partial S_t \partial r_t}\right) dt \\ &\quad + \int_0^T \sigma S_t \frac{\partial V}{\partial S_t} dX_t^{(1)} + \int_0^T c \frac{\partial V}{\partial r_t} dX_t^{(2)}. \end{aligned}$$

Start by writing

$$V(t+dt, S_t+dS_t, r_t+dr_t) = V(t, S_t, r_t) + \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S_t}dS_t + \frac{\partial V}{\partial r_t}dr_t + \frac{1}{2}\frac{\partial^2 V}{\partial S_t^2}dS_t^2 + \frac{1}{2}\frac{\partial^2 V}{\partial r_t^2}dr_t^2 + \frac{\partial^2 V}{\partial S_t \partial r_t}dS_t dr_t.$$

We know

$$
\begin{aligned}
dS_t^2 &= \sigma^2 S_t^2 dt; \\
dr_t^2 &= c^2 dt; \\
dS_t dr_t &= \rho\sigma c S_t dt
\end{aligned}
$$

substituting relevant terms in the above TSE and rearranging

$$
\begin{aligned}
dV &= \left(\frac{\partial V}{\partial t} + \mu S_t \frac{\partial V}{\partial S_t} + \gamma\left(m - r_t\right)\frac{\partial V}{\partial r_t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S_t^2} + \frac{1}{2}c^2 \frac{\partial^2 V}{\partial r_t^2} + \rho\sigma c S_t \frac{\partial^2 V}{\partial S_t \partial r_t}\right) dt \\
&\quad + \sigma S_t \frac{\partial V}{\partial S_t} dX_t^{(1)} + c\frac{\partial V}{\partial r_t} dX_t^{(2)}
\end{aligned}
$$

Now integrate over 0 and T

$$
\begin{aligned}
\int_0^T dV &= \int_0^T \left(\frac{\partial V}{\partial t} + \mu S_t \frac{\partial V}{\partial S_t} + \gamma\left(m - r_t\right)\frac{\partial V}{\partial r_t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S_t^2} + \frac{1}{2}c^2 \frac{\partial^2 V}{\partial r_t^2} + \rho\sigma c S_t \frac{\partial^2 V}{\partial S_t \partial r_t}\right) dt \\
&\quad + \int_0^T \sigma S_t \frac{\partial V}{\partial S_t} dX_t^{(1)} + \int_0^T c\frac{\partial V}{\partial r_t} dX_t^{(2)}.
\end{aligned}
$$

The right hand side cannot be further simplified. The left hand side becomes

$$
V(t, S_t, r_t) - V(0, S_0, r_0)
$$

where $V(0, S_0, r_0) = v$. Taking this to the other side gives the result.