Parcial Final

1. Dada la información en el enunciado todo se reduce a Probar que $\vec{\nabla} \times (\vec{\nabla} \times \vec{U}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{U}) - \nabla^2 \vec{U}$

Dem:

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{U}) = \vec{\nabla} \times \left(\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} \right) \hat{\imath} + \left(\frac{\partial V_x}{\partial V_z} - \frac{\partial V_z}{\partial x} \right) \hat{\jmath} + \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right) \hat{k} \right)$$

$$= \left(\frac{\partial}{\partial y}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right) - \frac{\partial}{\partial z}\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial V_x} \right) \right) \hat{\imath}$$

$$+ \left(\frac{\partial}{\partial z}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} \right) - \frac{\partial}{\partial x}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right) \right) \hat{\jmath}$$

$$+ \left(\frac{\partial}{\partial x}\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x} \right) - \frac{\partial}{\partial y}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} \right) \right) \hat{k}$$

$$= \left(\frac{\partial^2 V_y}{\partial y \partial x} - \frac{\partial^2 V_x}{\partial y^2} - \frac{\partial^2 V_x}{\partial z^2} + \frac{\partial^2 V_z}{\partial z \partial x} \right) \hat{\imath}$$

$$+ \left(\frac{\partial^2 V_z}{\partial z \partial y} - \frac{\partial^2 V_y}{\partial z^2} - \frac{\partial^2 V_y}{\partial x^2} + \frac{\partial^2 V_x}{\partial x \partial y} \right) \hat{\jmath}$$

$$+ \left(\frac{\partial^2 V_x}{\partial x \partial z} - \frac{\partial^2 V_z}{\partial x^2} - \frac{\partial^2 V_z}{\partial y^2} + \frac{\partial^2 V_y}{\partial y \partial z} \right) \hat{k}$$

$$= \left(\left(\frac{\partial^2 V_y}{\partial y \partial x} + \frac{\partial^2 V_z}{\partial z \partial x} \right) - \left(\frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2} \right) \right) \hat{\imath}$$

$$+ \left(\left(\frac{\partial^2 V_x}{\partial x \partial y} + \frac{\partial^2 V_z}{\partial z \partial y} \right) - \left(\frac{\partial^2 V_y}{\partial x^2} + \frac{\partial^2 V_y}{\partial z^2} \right) \right) \hat{\jmath}$$

$$+ \left(\left(\frac{\partial^2 V_x}{\partial x \partial z} + \frac{\partial^2 V_y}{\partial y \partial z} \right) - \left(\frac{\partial^2 V_z}{\partial x^2} + \frac{\partial^2 V_z}{\partial y^2} \right) \right) \hat{k}$$

$$= \left(\left(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_y}{\partial y \partial x} + \frac{\partial^2 V_z}{\partial z \partial x}\right) - \left(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2}\right)\right)\hat{\imath}$$

$$+ \left(\left(\frac{\partial^2 V_x}{\partial x \partial y} + \frac{\partial^2 V_y}{\partial y^2} + \frac{\partial^2 V_z}{\partial z \partial y}\right) - \left(\frac{\partial^2 V_y}{\partial x^2} + \frac{\partial^2 V_y}{\partial y^2} + \frac{\partial^2 V_y}{\partial z^2}\right)\right)\hat{\jmath}$$

$$+ \left(\left(\frac{\partial^2 V_x}{\partial x \partial z} + \frac{\partial^2 V_y}{\partial y \partial z} + \frac{\partial^2 V_z}{\partial z^2}\right) - \left(\frac{\partial^2 V_z}{\partial x^2} + \frac{\partial^2 V_z}{\partial y^2} + \frac{\partial^2 V_z}{\partial z^2}\right)\right)\hat{k}$$

$$= \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right) - \left(\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(V_x\hat{\imath} + V_y\hat{\jmath} + V_z\hat{k})\right)$$

$$= \vec{\nabla}(\vec{\nabla} \cdot \vec{U}) - \nabla^2 \vec{U} \quad \boxtimes$$

c.

$$\nabla^2 U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \approx \frac{(U_{i+1,j}) + (U_{i-1,j}) + (U_{i,j+1}) + (U_{i,j-1}) - 4U_{i,j}}{h^2}$$

$$\Rightarrow U_{i,j} = \frac{1}{4}\left[(U_{i+1,j}) + (U_{i-1,j}) + (U_{i,j+1}) + (U_{i,j-1}) - h^2 \nabla^2 U\right]$$

como $\nabla^2 U = -\omega$

$$\Rightarrow U_{i,j} = \frac{1}{4}\left[(U_{i+1,j}) + (U_{i-1,j}) + (U_{i,j+1}) + (U_{i,j-1}) + h^2 \omega_{ij}\right]$$

Ahora se hace lo siguiente

$$\nu \nabla^2 \omega \approx \nu \frac{(\omega_{i+1,j}) + (\omega_{i-1,j}) + (\omega_{i,j+1}) + (\omega_{i,j-1}) - 4\omega_{ij}}{h^2}$$

$$\frac{\partial U}{\partial y}\frac{\partial \omega}{\partial x} \approx \left(\frac{(U_{i,j+1}) - (U_{i,j-1})}{2h}\right)\left(\frac{(\omega_{i+1,j}) - (\omega_{i-1,j})}{2h}\right)$$

$$\frac{\partial U}{\partial x}\frac{\partial \omega}{\partial y} \approx \left(\frac{(U_{i+1,j}) - (U_{i-1,j})}{2h}\right)\left(\frac{(\omega_{i,j+1}) - (\omega_{i,j-1})}{2h}\right)$$

Entonces como

$$\nu \nabla^2 \omega = \frac{\partial \omega}{\partial x}\frac{\partial U}{\partial y} - \frac{\partial U}{\partial x}\frac{\partial \omega}{\partial y}$$

$$\frac{v}{h^2}\left((w_{i+1,j})+(w_{i-1,j})+(w_{i,j+1})+(w_{i,j-1})-4w_{ij}\right)$$

$$\| \rightarrow \text{Igualdad}$$

$$\left(\frac{(v_{i,j+1})-(v_{i,j-1})}{2h}\right)\left(\frac{(w_{i+1,j})-(w_{i-1,j})}{2h}\right)-\left(\frac{(v_{i+1,j})-(v_{i-1,j})}{2h}\right)\left(\frac{(w_{i,j+1})-(w_{i,j-1})}{2h}\right)$$

Entonces

$$(w_{i+1,j})+(w_{i-1,j})+(w_{i,j+1})+(w_{i,j-1})-4w_{ij}$$

$$\|$$

$$\overbrace{\frac{h^2}{4h^2 v}}^{d}\overbrace{\left[\left((v_{i,j+1})-(v_{i,j-1})\right)\left((w_{i+1,j})-(w_{i-1,j})\right)-\left((v_{i+1,j})-(v_{i-1,j})\right)\left((w_{i,j+1})-(w_{i,j-1})\right)\right]}^{A}$$

$$w_{ij}=-\frac{1}{4}\left(-\left((w_{i+1,j})+(w_{i-1,j})+(w_{i,j+1})+(w_{i,j-1})\right)+dA\right)$$

$$w_{ij}=\frac{1}{4}\left((w_{i+1,j})+(w_{i-1,j})+(w_{i,j+1})+(w_{i,j-1})\right)-\frac{1}{16v}\left[A\right]$$

$$w_{ij}=\frac{1}{4}\left((w_{i+1,j})+(w_{i-1,j})+(w_{i,j+1})+(w_{i,j-1})\right)$$

$$-\frac{R}{16}\left(\left((v_{i,j+1})-(v_{i,j-1})\right)\left((w_{i+1,j})-(w_{i-1,j})\right)\right)$$

$$+\frac{R}{16}\left(\left((v_{i+1,j})-(v_{i-1,j})\right)\left((w_{i,j+1})-(w_{i,j-1})\right)\right)$$

2.

Sea $U(x,y)$ la función de corriente. Entonces haciendo una expansión de Taylor:

$$U(x,y+h) = U(x,y) + \frac{\partial U}{\partial y}(x,y)\,h + \frac{\partial^2 U}{\partial y^2}(x,y)\,\frac{h^2}{2} + \dots$$

Debido a las condiciones del problema se tiene que

$$\omega = \omega_z = \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}$$

Adicionalmente debido a la viscosidad del fluido, la velocidad es constante a lo largo de la viga.

$$V_x = \frac{\partial U}{\partial y} = 0$$

Además como la corriente fluye suavemente a lo largo de la viga superior se tiene que

$$\frac{\partial V_y}{\partial x} = 0 \implies \omega = -\frac{\partial V_x}{\partial y} = -\frac{\partial^2 U}{\partial y^2}$$

Entonces

$$U(x,y+h) = U(x,y) + \cancelto{0}{\frac{\partial U}{\partial y}}(x,y)\,h + \frac{\partial^2 U}{\partial y^2}(x,y)\,\frac{h^2}{2} + \dots$$

$$U(x,y+h) \approx U(x,y) - \frac{\omega h^2}{2}$$

right

$$\omega \approx \frac{-2\left(U(x,y+h) - U(x,y)\right)}{h^2} \implies \omega_{ij} = \frac{-2}{h^2}\left[(U_{i,j+1}) - (U_{i,j})\right]$$

left (Proceso analogo)

$$\hookrightarrow \omega \approx \frac{-2}{h^2}\left(U(x,y) - U(x,y+h)\right) \implies \omega_{ij} = \frac{-2}{h^2}\left[(U_{i,j-1}) - (U_{i,j})\right]$$