Taller 4

1. Demostrar que la transformada de Fourier del operador derivada esta dado por: $\mathcal{F}\left(\frac{dF(x)}{dx}\right) = i\omega \mathcal{F}(F(x))$.

se utilizara la siguiente definición de la transformada de Fourier.

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(x) e^{-i\omega x} dx$$

Dem:

$$\mathcal{F}(F'(x)) = i\omega \mathcal{F}(F(x))$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F'(x) e^{-i\omega x} dx = \frac{i\omega}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(x) e^{-i\omega x} dx$$

$$u = e^{-i\omega x} \qquad dv = F'(x)$$

$$du = -i\omega e^{-i\omega x} \qquad v = F(x)$$

$$\Rightarrow \quad F(x) e^{-i\omega x}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} (-i\omega) F(x) e^{-i\omega x} dx = i\omega \int_{-\infty}^{\infty} F(x) e^{-i\omega x} dx$$

$$\Rightarrow \quad F(x) e^{-i\omega x}\Big|_{-\infty}^{\infty} = 0 \qquad \Rightarrow \quad 0 = 0 \qquad \boxtimes$$

Dado que $F(x)$ se asume absolutamente integrable en $\mathbb{R}$ se tiene que el termino que queda es igual a cero.

2. Dado $F(x) = e^{-x^2/25} \cos x$ se sabe que $F'(x) = -\frac{1}{25} e^{-x^2/25} (25 \sin x + 2x \cos x)$

Ahora dado que $F(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega$

$$\Rightarrow \quad F'(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} i\omega F(\omega) e^{i\omega t} d\omega$$

Entonces, como $F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \cos x \, e^{-x^2/25} e^{-i\omega x} dx$

$$\Rightarrow F'(t) = \frac{i\omega}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} \cos t\, e^{-t^2/25}\, e^{-i\omega t}\, dt\right) e^{i\omega t}\, d\omega$$

$$F'(t) = \frac{i\omega}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} \frac{e^{it}+e^{-it}}{2}\, e^{-t^2/25}\, e^{-i\omega t}\, dt\right) e^{i\omega t}\, d\omega$$

$$= \frac{i\omega}{2\pi}\int_{-\infty}^{\infty}\left(\int_{\mathbb{R}} \frac{1}{2}\left(e^{it-i\omega t-t^2/25} + e^{-it-i\omega t-t^2/25}\, dt\right)\right) e^{i\omega t}\, d\omega$$

$$= \frac{i\omega}{4\pi}\int_{R}\left(\int_{R} e^{it-i\omega t-t^2/25}\, dt + \int_{R} e^{-it-i\omega t-t^2/25}\, dt\right) e^{i\omega t}\, d\omega$$

$$= \frac{i\omega}{4\pi}\int_{R}\left(5\sqrt{\pi}\, e^{-25/4(\omega-1)^2} + 5\sqrt{\pi}\, e^{-25/4(\omega+1)^2}\right) e^{i\omega t}\, d\omega$$

$$= \frac{i\omega}{4\pi}\left[5\sqrt{\pi}\int_{R} e^{-25/4(\omega-1)^2}\, e^{i\omega t}\, d\omega + 5\sqrt{\pi}\int_{R} e^{-25/4(\omega+1)^2}\, e^{i\omega t}\, d\omega\right]$$

$$= -\frac{1}{25}\, e^{-t^2/25}\,(25 \sin x + 2x \cos x)$$

3. a) Sean $x[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-2]$ y $h[n] = \delta[n] + \delta[n-1]$ donde δ está dado por:

$$\delta[n] = \begin{cases} 1 & \text{si } n = 0 \\ 0 & \text{si } n \neq 0. \end{cases}$$

Partiendo de la definición de convolución discreta tenemos que:

$$y[n] = (x * h)[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

$$y[n] = \sum_{m=-\infty}^{\infty} \left[\delta[m] + \delta[m-1] + 2\delta[m-2]\right]\left[\delta[n-m] + \delta[n-m-1]\right]$$

$$y[n] = \sum_{m=-\infty}^{\infty} \Big[\delta[m]\delta[n-m] + \delta[m]\delta[n-m-1] + \delta[m-1]\delta[n-m] + \delta[m-1]\delta[n-m-1] + 2\delta[m-2]\delta[n-m] + 2\delta[m-2]\delta[n-m-1]\Big]$$

$$y[n] = \delta[n] + \delta[n-1] + \delta[n-1] + \delta[n-2] + 2\delta[n-2] + 2\delta[n-3]$$

$$y[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2] + 2\delta[n-3]$$

Así, vemos que la secuencia $y[n]$ está dada por:

$$y[n] = \begin{cases} 1 & \text{si } n = 0\,; \\ 2 & \text{si } n = 1 \vee n = 3\,; \\ 3 & \text{si } n = 2\,; \\ 0 & \text{si } n < 0 \vee n > 3 \end{cases}$$

//

2. Resolver la siguiente ecuación diferencial

$$\frac{d}{dt} u(t) = (u(t))^q \quad , \quad t \in [0, 10]$$

solución

Esta ecuacion es de la forma

$$\frac{d}{dt} u(t) = g(t)\, h(u) \qquad \text{donde } g(t) = 1 \ \text{ y } h(u) = (u(t))^q$$

Entonces $\displaystyle\int \frac{du}{h(u)} = \int g(t)\, dt \Rightarrow \int \frac{du}{u^q} = \int dt \Rightarrow \int u^{-q}\, du = t + c$

caso 1: $q = 1$

$$\int \frac{du}{u} = t + c \Rightarrow \ln|u| = t + c \Rightarrow u(t) = c e^t$$

caso 2: $q > 1$ y $0 < q < 1$ y $q < 0$

$$\int u^{-q} du = t + c \Rightarrow \frac{u^{-q+1}}{1-q} = t + c \Rightarrow u^{1-q} = (t+c)(1-q) \Rightarrow u^{1-q} = t - tq + c - cq$$

$$\Rightarrow u = \left(t(1-q) + c(1-q)\right)^{1/(1-q)}$$

caso 3: $q = 0$

$$\int du = t + c \Rightarrow u(t) = t + c$$

Escaneado con CamScanner

c. Ecuación diferencial de Riccati:

$x^3 y' = x^4 y^2 - 2x^2 y - 1$. solución particular $y_1 = x^{-2}$

Entonces $y' = x y^2 - 2x^{-1} y - x^{-3} \rightarrow y' + 2x^{-1} y - x y^2 = -x^{-3}$

Esta ecuación es de la forma $\frac{dy}{dx} + P(x) y + q(x) y^2 = F(x)$

Dado que se conoce la solución particular $y_1 = x^{-2}$ se hace el siguiente cambio de variable $y = y_1(x) + \frac{1}{z(x)}$

Entonces

$$\frac{dy}{dx} = -P(x)y - q(x)y^2 + F(x) = \frac{dy_1}{dx} - \frac{1}{z^2}\frac{dz}{dx}$$

$$\Rightarrow -P(x)y - q(x)y^2 + F(x) = -\frac{1}{z^2}\frac{dz}{dx} + \left(-P(x)y_1 - q(x)y_1^2 + F(x)\right)$$

$$\Rightarrow \frac{1}{z^2}\frac{dz}{dx} = P(x)(y - y_1) + q(x)(y^2 - y_1^2)$$

$$\Rightarrow \frac{1}{z}\frac{dz}{dx} = \frac{P(x)}{z} + q(x)\left(\frac{1}{z^2} + \frac{2y_1}{z}\right) \text{, como } y = y_1(x) + \frac{1}{z(x)}$$

$$\Rightarrow \frac{dz}{dx} - \left(P(x) + 2q(x)y_1\right)z = q(x)$$

Entonces como $P(x) = 2x^{-1}$; $q(x) = -x$; $y_1 = x^{-2}$

$$\Rightarrow \frac{dz}{dx} - \left(2x^{-1} + 2(-x)(x^{-2})\right)z = -x$$

$$\Rightarrow \frac{dz}{dx} - \left(\frac{2}{x} - \frac{2}{x}\right)z = -x \quad \Rightarrow \int dz = \int -x\,dx = z = \frac{-x^2}{2} + c$$

$$\Rightarrow y = x^{-2} + \left(\frac{-x^2}{2} + c\right)^{-1} \quad \text{como } y(\sqrt{2}) = 0 = \frac{1}{2} + \left(-\frac{2}{2} + c\right)^{-1} \Rightarrow c = -1$$

$$\Rightarrow y = x^{-2} + \left(\frac{-x^2}{2} - 1\right)^{-1}$$

3. $\frac{dU}{dt} = \alpha U \quad , \; U(0) = U_0$

$$\int \frac{du}{U} = \int \alpha \, dt$$

$$\ln|U| = \alpha t + c$$

$$U = e^{\alpha t + c}$$

$$U = A e^{\alpha t}$$

$$U_0 = A e^{0}$$

$$U_0 = A \longrightarrow \boxed{U(t) = U_0 e^{\alpha t}}$$

Ahora

$$F'(x) = \left.\frac{dY}{dx}\right|_{P_0} = F(x_0, Y_0) = \frac{Y_1 - Y_0}{x_1 - x_0} \Rightarrow Y_1 = Y_0 + (x_1 - x_0) F(x_0, Y_0) = Y_0 + h F(x_0, Y_0)$$

Entonces

$$Y_1 = Y_0 + h F(x_0, Y_0)$$

$$Y_2 = Y_1 + h F(x_1, Y_1)$$

$$\vdots$$

$$Y_{i+1} = Y_i + h F(x_i, Y_i)$$

$$\vdots$$

$$Y_n = Y_{n-1} + h F(x_{n-1}, Y_{n-1})$$

Entonces

$x_0 = 0$ y $Y_0 = U_0$

$$x_1 = x_0 + h = h$$

$\frac{dU}{dt} = \alpha U$ y $\frac{dU}{dt} \approx \frac{\Delta U}{\Delta t}$

$$\frac{\Delta U}{U} \approx \alpha \Delta t$$

$$U_1 - U_0 \approx U_0 \alpha \Delta t$$

$$U_1 \approx U_0 (1 + \alpha \Delta t)$$

$$U_2 - U_1 \approx U_1 \alpha \Delta t$$

$$U_2 \approx U_1 (1 + \alpha \Delta t)$$

$$U_2 \approx U_0 (1 + \alpha \Delta t)^2$$

$$U_K = U_0 (1 + \alpha \Delta t)^K$$

se generaliza por inducción

4.

$F(q_0, p_0) = 0$

$\dot{q} = F(q, p)$

$\dot{p} = g(q, p)$

se considera

$$\frac{dq}{dt} = a_1 q + b_1 p \qquad y \ \det\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$$

$$\frac{dp}{dt} = a_2 q + b_2 p$$

Este sistema de ecuaciones diferenciales es sencillo y tiene una solución conocida

$q = A e^{mt}$ siempre que m sea una raiz de la ecuación auxiliar $m^2 - (a_1 + b_2)m + (a_1 b_2 - a_2 b_1) = 0$

$p = B e^{mt}$ Es evidente que $m \neq 0$ para cumplir la condición del determinante

Entonces $E = (q, p) \rightarrow \frac{dE}{dt} = (\dot{q}, \dot{p})$

$$\frac{dE}{dt} = ME$$

$$\begin{vmatrix} \dot{q} \\ \dot{p} \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \begin{vmatrix} p \\ q \end{vmatrix}$$

matriz de estabilidad de

$x' = 2x - y$

$y' = x + 2y$

$$\frac{\partial x'}{\partial x} = 2 \qquad \frac{\partial y'}{\partial x} = 1 \qquad \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix}$$

$$\frac{\partial x'}{\partial y} = -1 \qquad \frac{\partial y'}{\partial y} = 2$$

$$\frac{dF}{dt} = \frac{\partial F}{\partial q}\frac{dq}{dt} + \frac{\partial F}{\partial p}\frac{dp}{dt}$$

$$\frac{dg}{dt} = \frac{\partial g}{\partial q}\frac{dq}{dt} + \frac{\partial g}{\partial p}\frac{dp}{dt}$$

$$a_1 = \left.\frac{\partial F}{\partial q}\right|_{q_0, p_0} \qquad a_2 = \left.\frac{\partial g}{\partial q}\right|_{q_0, p_0}$$

$$b_1 = \left.\frac{\partial F}{\partial p}\right|_{q_0, p_0} \qquad b_2 = \left.\frac{\partial g}{\partial p}\right|_{q_0, p_0}$$

5.

a. Demostrar la formula de iteración para el metodo de Adams-Bashfort de 3 Puntos.

Un metodo multipaso de m puntos se usa para resolver problemas de la forma $y' = F(t,y)$, $a \leq t \leq b$, $y(a) = \alpha$. Este tiene una ecuación de diferencia para encontrar una aproximación de w_{i+1} en el punto t_{i+1} dada por:

$$w_{i+1} = a_{m-1} w_i + a_{m-2} w_{i-1} + \cdots + a_0 w_{i+1-m}$$
$$+ h \left(b_m F(t_{i+1}, w_{i+1}) + b_{m-1} F(t_i, w_i) + \cdots + b_0 F(t_{i+1-m}, w_{i+1-m}) \right)$$

donde m es un entero mayor a 1.

$i = m-1, m, \ldots, N-1$; $h = \frac{(b-a)}{N}$; $a_0, \ldots, a_{m-1}, b_0, \ldots, b_m$ son constantes

y los valores vienen especificados por $w_0 = \alpha$, $w_1 = \alpha_1$, $w_2 = \alpha_2, \ldots, w_{m-1} = \alpha_{m-1}$

Por definición se tiene que cuando $b_m = 0$ el metodo es explicito y cuando $b_m \neq 0$ es implicito

Ahora para demostrar el metodo multipaso:

Dado $y' = F(t,y)$, $a \leq t \leq b$, $y(a) = \alpha$

$$\Rightarrow y(t_{i+1}) - y(t_i) = \int_{t_i}^{t_{i+1}} y'(t)\, dt = \int_{t_i}^{t_{i+1}} F(t, y(t))\, dt$$

$$\Rightarrow y(t_{i+1}) = y(t_i) + \int_{t_i}^{t_{i+1}} F(y,t)\, dt$$

Dado que no se conoce $y(t)$ es necesario integrar un Polinomio interpolador $P(t)$. Este se escoge de manera apropiada usando puntos con información obtenida anteriormente, i.e., $(t_0, w_0), \ldots, (t_i, w_i)$

Entonces usando el hecho de que $Y(t_i) \approx w_i$

$$\Rightarrow Y(t_{i+1}) \approx w_i + \int_{t_i}^{t_{i+1}} P(t)\,dt$$

Ahora para formar el polinomio $P_{m-1}(t)$ usando

$(t_i, F(t_i, Y(t_i)))$, $(t_{i-1}, F(t_{i-1}, Y(t_{i-1})))$, ..., $(t_{i+1-m}, F(t_{i+1-m}, Y(t_{i+1-m})))$

como $P_{m-1}(t)$ es un polinomio interpolador de grado $m-1$, se tiene que $\exists\ \beta_i$ en (t_{i+1-m}, t_i) tal que

$$F(t, Y(t)) = P_{m-1}(t) + \frac{F^{(m)}(\beta_i, Y(\beta_i))}{m!}(t-t_i)(t-t_{i-1})\cdots(t-t_{i+1-m})$$

Entonces definiendo $t = t_i + sh \to dt = h\,ds$ en $P_{m-1}(t)$

$$\Rightarrow \int_{t_i}^{t_{i+1}} F(t, Y(t))\,dt = \int_{t_i}^{t_{i+1}} \sum_{k=0}^{m-1} (-1)^k \binom{-s}{k} \nabla^k F(t_i, Y(t_i))\,dt$$

$$+ \int_{t_i}^{t_{i+1}} \frac{F^{(m)}(\beta_i, Y(\beta_i))}{m!}(t-t_i)(t-t_{i-1})\cdots(t-t_{i+1-m})\,dt$$

$$= \sum_{k=0}^{m-1} \nabla^k F(t_i, Y(t_i))\, h\, (-1)^k \int_0^1 \binom{-s}{k} ds$$

$$+ \frac{h^{m+1}}{m!} \int_0^1 s(s+1)\cdots(s+m-1)\, F^{(m)}(\beta_i, Y(\beta_i))\,ds$$

Entonces $\displaystyle\int_{t_i}^{t_{i+1}} F(t, Y(t))\,dt = h\left(F(t_i, Y(t_i)) + \frac{1}{2}\nabla F(t_i, Y(t_i)) + \frac{5}{12}\nabla^2 F(t_i, Y(t_i)) + \cdots\right)$

$$+ \frac{h^{m+1}}{m!}\int_0^1 s(s+1)\cdots(s+m-1)\, F^{(m)}(\beta_i, Y(\beta_i))\,ds$$

Debido a que $s(s+1)\cdots(s+m-1)$ no cambia de signo en $[0,1]$ se sabe por el teorema del valor intermedio $\exists \mu_i$ tal que $t_{i+1-m} < \mu_i < t_{i+1}$ y por lo tanto el termino del error se convierte en

$$\frac{h^{m+1}}{m!}\int_0^1 s(s+1)\cdots(s+m-1)\ F^{(m)}(\beta_i, Y(\beta_i))\,ds = \frac{h^{m+1}F^{(m)}(\mu_i, Y(\mu_i))}{m!}\int_0^1 s(s+1)\cdots(s+m-1)\,ds$$

$$= h^{m+1}F^{(m)}(\mu_i, Y(\mu_i))\,(-1)^m\int_0^1 \binom{-s}{m}\,ds$$

Pero como $Y(t_{i+1}) - Y(t_i) = \int_{t_i}^{t_{i+1}} F(t, Y(t))\,dt$

Entonces

$$Y(t_{i+1}) - Y(t_i) = \int_{t_i}^{t_{i+1}} F(t, Y(t))$$

$\Rightarrow$

$$\boxed{Y(t_{i+1}) = Y(t_i) + h\left(F(t_i, Y(t_i)) + \frac{1}{2}\nabla F(t_i, Y(t_i)) + \frac{5}{12}\nabla^2 F(t_i, Y(t_i)) + \cdots\right] + h^{m+1}F^{(m)}(\mu_i, Y(\mu_i))\,(-1)^m\int_0^{-1}\binom{-s}{m}\,ds} \rightarrow 1$$

Finalmente para probar el metodo de Adams-Bashforth de 3 pasos en 1 se reemplaza $m=3$.

$$Y(t_{i+1}) \approx Y(t_i) + h\left(F(t_i, Y(t_i)) + \frac{1}{2}\nabla F(t_i, Y(t_i)) + \frac{5}{12}\nabla^2 F(t_i, Y(t_i))\right)$$

$$= Y(t_i) + h\left(F(t_i, Y(t_i)) + \frac{1}{2}\left[F(t_i, Y(t_i)) - F(t_{i-1}, Y(t_{i-1}))\right]\right.$$
$$\left. + \frac{5}{2}\left[F(t_i, Y(t_i)) - 2F(t_{i-1}, Y(t_{i-1})) + F(t_{i-2}, Y(t_{i-2}))\right]\right)$$

$$Y(t_{i+1}) = Y(t_i) + \frac{h}{12}\left[23F(t_i, Y(t_i)) - 16F(t_{i-1}, Y(t_{i-1})) + 5F(t_{i-2}, Y(t_{i-2}))\right]$$

Entonces el metodo consta de

$w_0 = \alpha_0$, $w_1 = \alpha_1$, $w_2 = \alpha_2$

$$w_{i+1} = w_i + \frac{h}{12}\left[23\,F(t_i, w_i) - 16\,F(t_{i-1}, w_{i-1}) + 5\,F(t_{i-2}, w_{i-2})\right]$$

Para $i = 2, 3, \ldots, N-1$

b. Adams Moulton:

se utilizara la siguiente aproximación de Taylor

$$\int_0^h \phi(x)\,dx \approx A\,\phi(-h) + B\phi(0) + C\phi(h)$$

Entonces

$$\phi(x) = 1 \Rightarrow A + B + C = \int_0^h dx = h \Rightarrow A + B + C = h \quad (1)$$

$$\phi(x) = x \Rightarrow A(-h) + C(h) = \int_0^h x\,dx = \frac{h^2}{2} \Rightarrow -A + C = \frac{h}{2} \quad (2)$$

$$\phi(x) = x^2 \Rightarrow Ah^2 + Ch^2 = \int_0^h x^2\,dx = \frac{h^3}{3} \Rightarrow A + C = \frac{h}{3} \quad (3)$$

Entonces resolviendo el sistema de ecuaciones

$A = \frac{-1}{12}h$, $B = \frac{8}{12}h$, $C = \frac{5}{12}h$

Entonces

$$\int_0^h \phi(x)\,dx \approx -\frac{h}{12}\phi(-h) + \frac{8h}{12}\phi(0) + \frac{5h}{12}\phi(h)$$

Entonces como $Y(x_{i+1}) - Y(x_i) = \int_{x_i}^{x_{i+1}} Y'(x)\,dx$

$$\Rightarrow \int_0^h Y'(t + x_i)\,dt \approx \frac{h}{12}\left[-Y'(-h + x_i) + 8Y'(x_i) + 5Y'(h + x_i)\right]$$

$$= \frac{h}{12}\left[5F(x_{i+1}, Y(x_{i+1})) + 8F(x_i, Y(x_i)) - F(x_{i-1}, Y(x_{i-1}))\right] \Rightarrow Y_{n+1} = Y_n + \frac{h}{12}(5F_{n+1} + 8F_n - F_{n-1}$$