Due: February 21 **Mat 370: Written Assignment #4**

Instructions: Solve each problem carefully on separate paper. To receive full credit, you must show all work and justify your answers. In addition, your work must be organized, legible, and include units and complete sentences where appropriate. Please staple your work if you use multiple pages.

1. Decide whether the given map φ is an isomorphism of the first binary structure with the second. If it is not an isomorphism, explain why.

   (a) $\langle \mathbb{Z}, + \rangle$ with $\langle \mathbb{Z}, + \rangle$ where $\varphi(n) = -n$.

   (b) $\langle \mathbb{Q}, \cdot \rangle$ with $\langle \mathbb{Q}, \cdot \rangle$ where $\varphi(x) = x^2$.

   (c) $\langle \mathbb{Z}, + \rangle$ with $\langle \mathbb{Z}, + \rangle$ where $\varphi(n) = n + 1$.

   (d) $\langle \mathbb{R}, \cdot \rangle$ with $\langle \mathbb{R}, \cdot \rangle$ where $\varphi(x) = x^3$.

2. Prove that if $\varphi : S \to S'$ is an isomorphism of $\langle S, * \rangle$ with $\langle S', *' \rangle$ and $\psi : S' \to S''$ is an isomorphism of $\langle S', *' \rangle$ with $\langle S'', *'' \rangle$ then the composite function $\psi \circ \varphi$ is an isomorphism of $\langle S, * \rangle$ with $\langle S'', *'' \rangle$.

3. For the following, determine whether the binary operation $*$ determines a group structure on the given sets. If no group results, give the first group axiom $\mathscr{G}_1, \mathscr{G}_2, \mathscr{G}_3$ in the definition of a group that does not hold.

   (a) Let $*$ be defined on $\mathbb{Z}$ by $a * b = ab$.

   (b) Let $*$ be defined on $\mathbb{R}^+$ by letting $a * b = \sqrt{ab}$.

   (c) Let $*$ be defined on $\mathbb{Q}$ by setting $a * b = ab$.

4. Let n be a positive integer, and let $n\mathbb{Z} = \{nm \mid m \in \mathbb{Z}\}$

   (a) Show that $\langle n\mathbb{Z}, + \rangle$ is a group.

   (b) Show that $\langle n\mathbb{Z}, + \rangle \cong \langle \mathbb{Z}, + \rangle$.

5. Let G be an abelian group and let $c^n = c * c * \cdots * c$ for n factors c, where $c \in G$ and $n \in \mathbb{Z}^+$. Give a mathematical induction proof that $(a * b)^n = (a^n) * (b^n)$ for all $a, b \in G$.