COHEN-MACAULAYNESS AND THE FIRST HILBERT COEFFICIENT FOR A PARAMETER IDEAL

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Abstract. This goal of this paper is to provide the background necessary to understand a recently solved conjecture of Vasconcelos, and present a proof that uses more elementary techniques, therefore being more accessible to those less-versed in high-level commutative algebra.

1. Introduction

In 2010, L. Ghezzi, S. Goto, J. Hong, K. Ozi, T. T. Phuong, and W. Vasconcelos affirmatively solved a previous conjecture by Vasconcelos on the vanishing of the first Hilbert coefficient, for the case of a parameter ideal in a Noetherian local ring [1]. The goal of this paper is to present the background knowledge necessary to understand the conjecture, as well as a proof employing more elementary techniques when possible than were presented in the original paper. Before the conjecture is presented, recall the following definition:

Definition 1.1. For a local ring R of dimensions d with maximal ideal $\mathfrak{m}$, a set of elements $x_1, \ldots, x_d$ is called a **system of parameters** if they satisfy any of the following equivalent conditions:
(i) $\mathfrak{m}$ is a minimal prime of $(x_1, \ldots, x_d)$.
(ii) The radical of $(x_1, \ldots, x_d)$ is $\mathfrak{m}$.
(ii) $\mathfrak{m}^n \subseteq (x_1, \ldots, x_d)$ for some $n \in \mathbb{N}$.
(iv) $(x_1, \ldots, x_d)$ is $\mathfrak{m}$-primary.

The conjecture was originally posed by Vasconcelos in a lecture at a conference in Yokohama in 2008. The conjecture was in [1], and stated as follows:

Conjecture 1.2. *Assume that a Noetherian local ring R is unmixed. Then R is a Cohen-Macaulay local ring, once the first Hilbert coefficient, $e_1(I) = 0$ for some parameter ideal I of R.*

In many sources, a parameter ideal refers to an ideal generated by a subset of a system of parameters. For the purposes of this paper, by parameter ideal we will always mean an ideal generated by a system of parameters, as it does in the conjecture. Before the conjecture was solved affirmatively, some work by Ghezzi, Hong, and Vasconcelos had solved the conjecture for an integral domain which is the homomorphic image of a Cohen-Macaulay ring. Furthermore, Mandal, Singh, and Verma proved that $e_1(I) \leq 0$ for every parameter ideal I of an arbitrary Noetherian local ring. We begin the discussion by defining the Hilbert Polynomial for any commutative Noetherian ring with identity. We will also outline what it means for a ring to be Cohen-Macaulay. Next, we will show that we need only work in the complete case. Some basic results concerning superficial elements and local cohomology will follow. Lastly, we will address the necessity of the unmixed condition, and then the main proof.

2. The Hilbert Polynomial

In order to begin our discussion, we start by defining the Hilbert *function*. The general purpose of the Hilbert function is to measure the growth of the dimension of the homogeneous components of a graded commutative algebra. Specifically, it records the length of a composition series of each layer of a graded module. In sufficiently high inputs, the function agrees with a polynomial, which allows us to view information about infinite layers of such a module in a finite object.

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Much work has been done concerning the multiplicity, or the leading coefficient of the Hilbert polynomial with respect to the maximal ideal in a local ring. Notably, it has been shown that for a regular local ring, the multiplicity is 1, and, in the unmixed case, a Theorem by Nagata has shown the converse to be true.

Unless otherwise stated, throughout the paper we will let $(R, \mathfrak{m}, k)$ denote a Noetherian local ring R with maximal ideal $\mathfrak{m}$ and residue field k. By a parameter ideal, we mean an ideal generated by a system of parameters. For an R-module M, we use the notation $\lambda(M)$ to mean the length of a composition series of M. Originally, Hilbert worked with graded rings generated over a field, that is a ring $R = \bigoplus_{n=0}^{\infty} R_n$ where R_0 is a field. When working over an arbitrary Noetherian ring, one works with the associated graded ring $\mathrm{gr}_I R$ of a ring with respect to some ideal I, which will be defined below. In 1951, Samuel showed that in the local case, it is useful to study more general polynomial functions associated to a local ring and primary ideals. In this case, it can be shown that the classical definition of the Hilbert polynomial of the associated graded ring coincides with the definition of Hilbert-Samuel polynomial. However, we begin in the general case.

Definition 2.1. Let $R = \bigoplus_{n=0}^{\infty} R_n$ be a graded Noetherian ring with R_0 Artinian and let $M = \bigoplus_{n=0}^{\infty} M_n$ be a graded R-module, with $\lambda(M_n) < \infty$. The **Hilbert function** $HF_M(n) : \mathbb{Z} \to \mathbb{Z}$ is defined as

$$HF_M(n) := \lambda(M_n)$$

The **Hilbert-Poincare series** with respect to M, denoted $H_M(t)$, is the generating function of $\lambda(M_n)$, that is

$$H_M(t) = \sum_{n=0}^{\infty} \lambda(M_n) t^n \quad \in \mathbb{Z}[|t|]$$

Where $\mathbb{Z}[|t|]$ is the power series ring with integer coefficients. The following theorem moves us further down the road in pursuit of the Hilbert polynomial.

Theorem 2.2. *Let R be a graded Noetherian ring with R_0 Artinian, and M a finitely generated graded R module. If we write $R = R_0[z_1, \ldots, z_d]$, with deg$(z_i) = k_i$, then*

$$H_M(t) = \frac{f_M(t)}{\prod_{i=1}^{d} (1 - t^{k_i})}$$

where $f_M(t) \in \mathbb{Z}[t]$.

In order to prove this, we will need the following propositions.

Proposition 2.3. *Let R be a graded ring. The following are equivalent:*
(i) R is Noetherian;
(ii) R_0 is Noetherian and $R_+ = \bigoplus_{i \geq 1} R_i$ is a finitely generated ideal;
(iii) R_0 is Noetherian and $R \cong R_0[x_1, \ldots, x_n]/I$, Where $R_0[x_1, \ldots, x_n]$ is the polynomial ring with variables $x_1, \ldots, x_n$ over R_0, and deg$(x_i) = k_i$ where I is a homogeneous ideal.

Proof. (iii) $\implies$ (i): This is a consequence of the Hilbert basis theorem.
(i) $\implies$ (ii): The object R_+ is an ideal, and hence is finitely generated as R is noetherian. Thus $R_0 \simeq R/R_+$ is Noetherian as well.
(ii) $\implies$ (iii): Choose $z_1, \ldots, z_n \in R_+$ with deg $(z_i) = k_i$ such that $R_+ = Rz_1 + \cdots + Rz_n$. We claim that $R = R_0[z_1, \ldots, z_n]$. As it is clear $R \supseteq R_0[z_1, \ldots, z_n]$, it is enough to show $R_i \subseteq R_0[z_1, \ldots, z_n]$ for all i. We show this by induction on i. The $i = 0$ case is trivial. Let $i > 0$ and suppose the claim is true for all R_j, $j < i$. Let $f \in R_j$ and write it as

$$f = \sum_{j=1}^{n} s_j z_j$$

where $s_j \in R$. If we restrict to the degree i component,

$$f = \sum_{j=1}^{n} s'_j z_j$$

where $s'_j \in R_{i-k_j}$. By induction, $s'_j \in R_0[z_1, \ldots, z_n]$, and the claim holds. □

Proposition 2.4. *Let R be a Noetherian graded ring and M a finitely generated graded R-module. Then for all $n \in \mathbb{Z}$, M_n is a finitely generated R_0-module. In particular, if R_0 is Artinian, then the length of M_n is finite.*

Proof. Consider the submodule of M defined by $M_{\geq n} := \bigoplus_{i \geq n} M_i$. Since M is noetherian, $M_{\geq n}$ is a finitely generated graded R-module. Therefore, we have that $M_n := \frac{M_{\geq n}}{M_{\geq n+1}}$ is also a finitely generated graded R-module. Note that M_n is naturally a finitely generated graded $R/\text{ann}(M_n)$-module. But $R_+ \subseteq \text{ann}(M_n)$, hence M_n is a finitely generated $R_0 = R/R_+$ -module. □

We are now ready to prove Theorem 2.2.

Proof. By Proposition 2.4, the length of M_n is finite for all n. Induct on d. For the $d = 0$ case, let $R = R_0$ and assume that M is finitely generated over R_0. For n sufficiently large, we have $M_n = 0$. Thus for some r, we have $M = \bigoplus_{i=0}^{r} M_i$ with $\lambda(M_i) < \infty$. Hence

$$H_M(t) = \sum_{i=0}^{r} \lambda(M_i)t^i \in \mathbb{Z}[t].$$

Now assume that $d > 0$. Multiplication by z_d defines an R_0-linear map $M_n \to M_{n+k_d}$. Consider the exact sequence

$$0 \longrightarrow K_n \longrightarrow M_n \xrightarrow{\cdot z_d} M_{n+k_d} \longrightarrow C_{n+k_d} \longrightarrow 0$$

Where K_n and C_{n+k_d} are the kernel and cokernel of the map defined by multiplication by z_d, respectively. Write $K = \bigoplus K_n$ and $C = \bigoplus C_n$. K is a submodule of M and $C = M/z_rM$, so both K and C are finitely generated over R. We note that $z_dK = 0$ and $z_dC = 0$, so that K and C are modules over $R_0[z_1', \ldots, z_{d-1}'] = R/z_dR$. So that we can apply the induction hypotheses to $H_K(t)$ and $H_C(t)$. Now, by additivity of length,

$$\lambda(M_{n+k_d}) + \lambda(K_n) = \lambda(M_n) + \lambda(C_{n+k_d})$$

Multiply by t^{n+k_d} to get

$$\lambda(M_{n+k_d})t^{n+k_d} + t^{k_d}\left(\lambda(K_n)t^n\right) = t^{k_d}\left(\lambda(M_n)t^n\right) + \lambda(C_{n+k_d})t^{n+k_d}$$

Now sum over n to get

$$H_M(t) + t^{k_d}H_K(t) = t^{k_d}H_M + H_C(t)$$

Solving for $H_M(t)$, we see

$$H_M(t) = \frac{H_C(t) - t^{k_d}H_K(t)}{1 - t^{k_d}}$$

Now, by induction, we have

$$H_C(t) = \frac{f_C(t)}{\prod_{i=1}^{d-1}(1 - t^{k_i}} \quad \text{and} \quad H_K = \frac{f_K(t)}{\prod_{i=1}^{d-1}(1 - t^{k_i})}$$

So that

$$H_M(t) = \frac{f_C(t) - t^{k_d}f_K(t)}{\prod_{i=1}^{d}(1 - t^{k_d})}$$

As $f_C(t)$ and $f_K(t)$ are both in $\mathbb{Z}[t]$, this completes the proof. □

Corollary 2.5. *For R as in Theorem 2.2, if R is generated in degree 1 (that is, $deg(z_i) = 1$ for all $1 \leq i \leq d$), Then for a fixed M, the function $\lambda(M_n) = H_M(n)$ is a polynomial with rational coefficients if n is sufficiently large. Moreover, the degree of $H_M(n)$ is at most $d - 1$.*

Proof. Note that for some $f(t) \in \mathbb{Z}[t]$,

$$\begin{aligned}\sum_{i \geq 0} \lambda(M_i)t^i &= \frac{f(t)}{(1-t)^d} \\ &= f(t)\sum_{i \geq 0}\binom{d+i-1}{d-1}t^i\end{aligned}$$

If $\deg(f) = N$, one may write
$$f(t) = a_N t^N + \cdots + a_0$$
where $a_i \in \mathbb{Z}$. The coefficient of t^n in $H_M(t)$ is a polynomial in n of degree $d-1$ with coefficients in $\mathbb{Q}$. In particular, we may write
$$\lambda(M_n) = \sum_{j=0}^{N} a_j \binom{d+n-j-1}{d-1}.$$
We have that
$$\binom{d+n-j-1}{d-1} = \frac{\overbrace{(d+n-j-1)(d+n-j-2)\dots(n-j+1)}^{d-1}}{(d-1)!}$$
$$= \frac{n^{d-1}}{(d-1)!} + \text{ lower terms.}$$
We now set
$$P_M(n) = \sum_{j=0}^{N} a_j \binom{d+n-j-1}{d-1}$$
and note that $\lambda(M_n) = P_M(n)$ for $n \geq N$. $\square$

Definition 2.6. The polynomial $P_M(n)$ presented in corollary 2.5 is called the **Hilbert polynomial** of M.

We may simplify some of these definitions while working in the local case. For $(R, \mathfrak{m}, k)$ and I an $\mathfrak{m}$-primary ideal, we define
$$\text{grad}_I R := G = \bigoplus_{n=0}^{\infty} \frac{I^n}{I^{n+1}}$$
where $I^0 = R$. For convenience, we label $G_i = I^i/I^{i+1}$. This is called the **associated graded ring** of R with respect to I. It is clearly a graded ring. As R is Noetherian and I is finitely generated, $G = G_0[G_1]$ is a Noetherian graded ring, and the ideal $G_+ = \bigoplus_{n\geq 1} G_n$ is finitely generated (Proposition 2.3). Now, let M be an R-module. For $I \subseteq R$, as above, we define the graded module
$$\mathfrak{M}(I) = \bigoplus_{n=0}^{\infty} \frac{I^n M}{I^{n+1}M}$$
This is a graded G-module generated in degree zero, so it is finitely generated. By Corollary 2.5 we may conclude
$$H_{\mathfrak{M}(I)}(n) = \lambda\left(\frac{I^n M}{I^{n+1}M}\right) \in \mathbb{Q}[n]$$
On the left hand side, we are technically computing length as a $G_0 = R/I$-module, but as $\mathfrak{M}(I)$ is annihilated by I, this is equivalent to computing its length as an R-module.

Definition 2.7. Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring and $I \subseteq R$ be an $\mathfrak{m}$-primary ideal. For an R-module M we define the **Hilbert-Samuel polynomial** $P_{I,M}$ of M with respect to I as follows:
$$P_{I,M}(n) = \lambda\left(\frac{M}{I^n M}\right) \quad \text{for } n >> 0$$

The polynomial has a positive leading coefficient, and degree at most $\lambda(I/\mathfrak{m}I)$, the minimal number of generators of I. If $M = R$, we write $P_{I,R}(n) = P_I(n)$.

To see this, consider the sequence
$$0 \longrightarrow \frac{I^n M}{I^{n+1}M} \longrightarrow \frac{M}{I^{n+1}M} \longrightarrow \frac{M}{I^n M} \longrightarrow 0$$
By additivity of length, we can see that
$$P_{I,M}(n+1) = H_{\mathfrak{M}(I)}(n) + P_{I,M}(n)$$

Expanding this expression inductively, it is clear that

$$P_{I,M}(n+1) = \sum_{i=0}^{n} H_{\mathfrak{M}(I)}(i) = \sum_{i=0}^{n} \lambda\left(\frac{I^iM}{I^{i+1}M}\right)$$

To continue, we will need a few standard facts about polynomials with rational coefficients, and manipulating binomial coefficients, which are presented below. Some are presented without proof, but one can find a more detailed discussion in [6].

Fact 2.8. *Recall that*

$$\binom{x+i}{i} = \frac{(x+i)(x+i-1)\dots(x+1)}{i!}$$

So that $\{\binom{x+i}{i} : i \geq 0\}$ *are a* $\mathbb{Q}$ *basis of* $\mathbb{Q}[x]$.

Fact 2.9. *A standard result:*

$$\sum_{j=0}^{n} \binom{j+k}{k} = \binom{n+k+1}{k+1}$$

We will employ the following identity frequently throughout the paper.

Fact 2.10. Pascal's Recurrence: *For* $n, d \in \mathbb{N}$,

$$\binom{n-1}{d-1} + \binom{n-1}{d} = \binom{n}{d}$$

Remark 2.11. *If* $f(x) \in \mathbb{Q}[x]$ *and we write*

$$f(x) = \sum_{j=0}^{d-1} b_j \binom{x+j}{j}$$

with the assumption $b_{d-1} \neq 0$ *then* $f(x)$ *us a polynomial of degree* $d-1$ *with leading coefficient* $\frac{b_{d-1}}{(d-1)!}$. *Moreover, if we set*

$$g(n) := \sum_{i=0}^{n} f(i)$$

then $g(n)$ *is a polynomial of degree* d *with leading coefficient* $\frac{b_{d-1}}{d!}$.

Proof. It is easy to see that deg $f = d-1$ and the leading coefficient is $\frac{b_{d-1}}{(d-1)!}$. Thus

$$g(n) = \sum_{i=0}^{n} f(i) = \sum_{i=0}^{n}\sum_{j=0}^{d-1} b_j \binom{i+j}{j} = \sum_{j=0}^{d-1} b_j \sum_{i=0}^{n} \binom{i+j}{j} = \sum_{j=0}^{d-1} b_j \binom{n+j+1}{j+1}$$

which is now a polynomial of degree d and leading coefficent $\frac{b_{d-1}}{d!}$. □

Remark 2.12. *Let* $f(x) \in \mathbb{Q}[x]$ *be a polynomial of degree* d. *Write*

$$f(x) = \sum_{j=0}^{d} b_j \binom{x+j}{j}$$

If $f(n) \in \mathbb{N}$ *for all* $n >> 0$, *then* $b_j \in \mathbb{Z}$ *for all* j *and* $b_d > 0$.

Making use of these facts, if we assume I is generated by d elements, then each $H_{\mathfrak{M}(I)}(i)$ is a polynomial of degree $d-1$ and we may write

$$H_{\mathfrak{M}(I)}(i) = \sum_{j=0}^{d-1} b_j \binom{i+j}{j}$$

and since $H_{\mathfrak{M}(I)}(n) \in \mathbb{N}$ for $n >> 0$, the leading coefficient of each $H_{\mathfrak{M}(I)}(i)$ is positive, thus it follows that

$$P_{I,M}(n+1) = \sum_{i=0}^{n} H_{\mathfrak{M}(I)}(i)$$

is a polynomial of degree d with positive leading coefficient.

Theorem 2.13. *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring. If I is an $\mathfrak{m}$-primary ideal, then deg($P_{I,R}$) $= \dim R$.*

Employing the remarks once more, it is possible to write $\lambda(R/I^{n+1})$ as a polynomial in such a way where its coefficients provide information about the ring R. These are called the Hilbert coefficients, defined below.

Definition 2.14. Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring of dimension d. Then for each $\mathfrak{m}$-primary ideal $I \subseteq R$ and $n >> 0$, we can write

$$\lambda\left(\frac{R}{I^{n+1}}\right) = e_0(I)\binom{n+d}{d} - e_1(I)\binom{n+d-1}{d-1} + \cdots + (-1)^d e_d(I)$$

Where each $e_i(I) \in \mathbb{Z}$. We name $e_i(I)$ the ith **Hilbert Coefficient** of R with respect to I. In particular, if $I = \mathfrak{m}$, then $e_0(\mathfrak{m})$ is clled the **multiplicity** of R.

To restate Conjecture 1.2 with the language that is now at our disposal, we write the main conjecture of the paper as follows.

Conjecture 2.15. *Assume that R is unmixed. Then R is a Cohen-Macaulay ring if and only if $e_1(I) = 0$ for some parameter ideal I of R.*

There are, of course, two terms of in the conjecture that have yet to be discussed, namely *unmixed* and *Cohen-Macaulay.* The latter will be discussed in detail in Section 3, and the definition of the former will appear in Section 4 and its importance will be discussed later in Section 7. Before we move on, we will show that it is possible to get a bound on the multiplicity without any further machinery. The techniques used in the following proof will be used throughout much of the paper.

Proposition 2.16. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d and $I = (x_1, \ldots, x_d)$ be a parameter ideal. Then*

$$e_0(I) \leq \lambda\left(\frac{R}{I}\right)$$

Proof. We proceed by induction on the dimension of the ring. The dimension 0 case is vacuous, since $I = 0$, we have

$$\lambda(R) = \lambda\left(\frac{R}{I}\right) = \lambda\left(\frac{R}{I^{n+1}}\right) = e_0(I)\binom{n}{0} = e_0(I)$$

Let's consider the case where $d = 1$, say $I = (x)$. Start with the map $R/(x^n) \to R/(x^{n+1})$ defined by multiplication by x, and extend this to the exact sequence

$$0 \longrightarrow \frac{(x^{n+1}:x)}{(x^n)} \longrightarrow \frac{R}{(x^n)} \xrightarrow{\cdot x} \frac{R}{(x^{n+1})} \longrightarrow \frac{R}{(x, x^{n+1})} \longrightarrow 0$$

First we note that $R/(x, x^{n+1}) \cong R/(x)$. By the additivity of length, we get

$$\lambda\left(\frac{R}{(x^{n+1})}\right) - \lambda\left(\frac{R}{(x^n)}\right) = \lambda\left(\frac{R}{(x)}\right) - \lambda\left(\frac{(x^{n+1}:x)}{(x^n)}\right)$$

Since length is always non-negative, we simplify this to

$$\lambda\left(\frac{R}{(x^{n+1})}\right) - \lambda\left(\frac{R}{(x^n)}\right) \leq \lambda\left(\frac{R}{(x)}\right)$$

Thus, passing to the Hilbert polynomial, by definition 2.14 we write for $n >> 0$,

$$\lambda\left(\frac{R}{(x^{n+1})}\right) = e_0(I)\binom{n+1}{1} - e_1(I)\binom{n}{0} = e_0(I)(n+1) - e_1(I)$$

$$\lambda\left(\frac{R}{(x^n)}\right) = e_0(I)\binom{n}{1} - e_1(I)\binom{n-1}{0} = e_0(I)(n) - e_1(I)$$

Thus

$$\lambda\left(\frac{R}{(x^{n+1})}\right) - \lambda\left(\frac{R}{(x^n)}\right) = (e_0(I)(n+1) - e_1(I)) - (e_0(I)(n) - e_1(I))$$
$$= e_0(I) \leq \lambda\left(\frac{R}{(x)}\right)$$

Assume this statement hold up to dimension $d-1$. Thus for a d dimensional ring with parameter ideal, $I = (x_1, \ldots, x_d)$ we consider the map $R/I^n \to R/I^{n+1}$ defined by multiplication by x_1, and extend this to the exact sequence

$$0 \longrightarrow \frac{(I^{n+1} : x_1)}{I^n} \longrightarrow \frac{R}{I^n} \xrightarrow{\cdot x_1} \frac{R}{I^{n+1}} \longrightarrow \frac{R}{(x_1, I^{n+1})} \longrightarrow 0$$

Write $\overline{R} = R/(x_1)$ and $\overline{I} = I/(x_1)$ so that $R/(x_1, I^{n+1}) \cong \overline{R}/\overline{I}^{n+1}$. Note that $\dim \overline{R} = d-1$. By additivity of length, we get

$$\lambda\left(\frac{R}{I^{n+1}}\right) - \lambda\left(\frac{R}{I^n}\right) = \lambda\left(\frac{\overline{R}}{\overline{I}^{n+1}}\right) - \lambda\left(\frac{(I^{n+1} : x_1)}{I^n}\right)$$

Again, since length is non-negative, we drop the last term and write

$$\lambda\left(\frac{R}{I^{n+1}}\right) - \lambda\left(\frac{R}{I^n}\right) \leq \lambda\left(\frac{\overline{R}}{\overline{I}^{n+1}}\right)$$

Assume $n >> 0$. Then the Hilbert Polynomials are

$$\lambda\left(\frac{R}{I^{n+1}}\right) = e_0(I)\binom{n+d}{d} - e_1(I)\binom{n+d-1}{d-1} + \cdots + (-1)^d e_d(I)$$
$$\lambda\left(\frac{R}{I^n}\right) = e_0(I)\binom{n+d-1}{d} - e_1(I)\binom{n+d-2}{d-1} + \cdots + (-1)^d e_d(I)$$
$$\lambda\left(\frac{\overline{R}}{\overline{I}^{n+1}}\right) = e_0(\overline{I})\binom{n+d-1}{d-1} - e_1(\overline{I})\binom{n+d-2}{d-2} + \cdots + (-1)^{d-1} e_{d-1}(\overline{I})$$

After applying Pascal's Recurrence (Fact 2.10), we may write the difference as

$$\lambda\left(\frac{R}{I^{n+1}}\right) - \lambda\left(\frac{R}{I^n}\right) = e_0(I)\left(\binom{n+d}{d} - \binom{n+d-1}{d}\right) - e_1(I)\left(\binom{n+d-1}{d-1} - \binom{n+d-2}{d-2}\right) + \ldots$$
$$= e_0(I)\binom{n+d-1}{d-1} - e_1(I)\binom{n+d-2}{d-2} + \cdots + (-1)^{d-1} e_{d-1}(I)$$
$$\leq e_0(\overline{I})\binom{n+d-1}{d-1} - e_1(\overline{I})\binom{n+d-2}{d-2} + \cdots + (-1)^{d-1} e_{d-1}(\overline{I}) = \lambda\left(\frac{\overline{R}}{\overline{I}^{n+1}}\right)$$

Since these are both polynomials of the same degree, and $n >> 0$, we need only compare the leading terms of the polynomials:

$$e_0(I)\frac{n^{d-1}}{(d-1)!} \leq e_0(\overline{I})\frac{n^{d-1}}{(d-1)!}$$

Hence,

$$e_0(I) \leq e_0(\overline{I})$$

By induction, $e_0\left(\overline{I}\right) \leq \lambda\left(\overline{R}/\overline{I}\right)$, but $\overline{R}/\overline{I} \cong R/I$, thus

$$e_0(I) \leq e_0\left(\overline{I}\right) \leq \lambda\left(\frac{R}{I}\right)$$

Which completes the proof. $\square$

We will expand on this result in the next section for the case of a Cohen-Macaulay ring.

3. The Cohen-Macaulay Property

A Cohen-Macaulay ring refers to a particular class of commutative rings possessing certain qualities deemed to be rather nice by algebraists. They are named after Francis Macaulay and for Irvin Cohen, both of whom proved a form of what is called the unmixedness theorem, where *unmixed* in this case refers to a property possessed by the primary composition of certain ideals in Cohen-Macaulay rings possess.

There are many examples of Cohen-Macaulay rings, in particular all regular local rings are Cohen-Macaulay, as are all Artinian rings, Gorenstein rings and complete intersection rings. In addition, the Cohen-Macaulay property is closed under localization, completion, and the adjoining of polynomial variables to a ring, allowing one a multitude of techniques to work with when studying these objects.

In order to define a Cohen-Macaulay ring, we need to develop some theory about regular sequences of rings and modules, which we will use to define a standard and useful invariant of the object in question, and relate this to other standard invariants in commutative algebra. Throughout the following, we let R be a Noetherian commutative ring.

Definition 3.1. Let M be an R-module. The elements $\underline{x} = \{x_1, \dots, x_n\} \subseteq R$ are said to form a **regular sequence** on M if
(i) x_1 is a non-zero divisor on M and for all $2 \leq i \leq n$, x_i is a non-zero divisor on $M/(x_1, \dots, x_{i-1})M$.
(ii) $(x_1, \dots, x_n)M \neq M$.

It is clear that an element x is a zero-divisor on a module M if and only if there exists an associated prime $\mathfrak{p}$ of M such that $x \in \mathfrak{p}$. Thus, x_{i+1} is a non-zero-divisor on $M/(x_1, \dots, x_i)M$ if and only if $x_{i+1} \notin \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Ass}(M/(x_1, \dots, x_i)M)$. Furthermore, if M is a finitely generated R-module, then $M/(x_1, \dots, x_i)M$ is also finitely generated, and $\operatorname{Ass}(M/(x_1, \dots, x_i)M$ is a finite set. Therefore, x_{i+1} is a non-zero-divisor on $M/(x_1, \dots, x_i)M$ if and only if $x_{i+1} \notin \bigcup\{\mathfrak{p} : \mathfrak{p} \in \operatorname{Ass}(M/(x_1, \dots, x_i)M)\}$.

Eventually, we would like to study maximal regular sequences. If we have a regular sequence of a certain length that we would like to extend to a longer sequence, the next remark is useful to narrow down which ideals contain such elements.

Remark 3.2. *If $I \subseteq R$ is an ideal and we wish to take $x_i \in I$ a non-zero divisor on $M' = M/(x_1, \dots, x_{i-1})M$ then we can find such an $x_i \iff \forall\, \mathfrak{p} \in Ass(M'), I \not\subseteq \mathfrak{p}$.*

As mentioned in the introduction to this section, the Cohen-Macaulay property is closed under localization. This relies on the fact that, given a regular sequence and the right conditions, we can find a natural regular sequence upon localization.

Proposition 3.3. *Let $\underline{x} = \{x_1, \dots, x_n\} \subseteq R$ be a regular sequence on M. Let $S \subseteq R$ be a multiplicatively closed set. If $(\underline{x})_S \cdot M_S \neq M_S$, then*

$$\underline{x}' = \left\{\frac{x_1}{1}, \frac{x_2}{1}, \dots, \frac{x_n}{1}\right\}$$

is a regular sequence on M_S.

Proof. Suppose $\frac{b}{s} \cdot \frac{x_i}{1} \in (\frac{x_1}{1}, \dots, \frac{x_{i-1}}{1})M_S$

$$\frac{b}{s} \cdot \frac{x_i}{1} = \frac{x_1}{1} \cdot \frac{a_1}{s_1} + \cdots + \frac{x_{i-1}}{1} \cdot \frac{a_{i-1}}{s_{i-1}}$$

With each $a_j \in M$, $b \in M$, $s_j \in S$. Then

$$\frac{bx_i}{s} = \frac{x_1 a_1' + \cdots + x_{i-1} a_{i-1}'}{s_0}$$

Thus, there exists $s' \in S$ such that

$$s' \cdot (s_0 b x_i - s(x_1 a_1' + \cdots + x_{i-1} a_{i-1}')) = 0$$

$$
\begin{aligned}
\implies x_i \cdot s's_0 b \in (x_1,\dots,x_{i-1})M \implies & ss_0 b \in (x_1,\dots,x_{i-1})M \\
\implies & \frac{b}{1} \in (x_1,\dots,x_{i-1})M \\
\implies & \frac{b}{s} \in (x_1,\dots,x_{i-1})M \\
\implies & \frac{x_i}{1} \text{ is a non-zero divisor on } (x_1,\dots,x_{i-1})M_S
\end{aligned}
$$

□

A natural question arises: is it possible to permute the elements of a regular sequence? We will show that this is possible, if the sequence of elements is taken from an appropriate location within the ring, namely the Jacobson radical.

Proposition 3.4. *Let $\overline{x} = x_1 \dots, x_n$ be a regular sequence on M. Then $x_1, \dots, x_{i+1}, x_i, \dots, x_n$ is a regular sequence on $M \iff x_{i+1}$ is a non-zero divisor on $(x_1, \dots, x_{i-1})M$.*

Proof. ($\implies$) Is clear from the definition. For the converse, by replacing M with $M/(x_1,\dots,x_{i-1})M$, we may assume without loss of generality that $i = 1$.

Suppose x_2 is a non-zero divisor on M, and $x_1 \cdot \overline{a} = 0 \in M/x_2M$ for some $a \in M$. Then

$$x_1 a \in x_2 M \implies x_1 a = x_2 b \text{ for some } b \in M$$

Then we have $x_2 \cdot \overline{b} = 0 \in M/x_1M$. But by assumption x_2 is not a zero divisor on M/x_1M, so $\overline{b} \equiv 0 \in M/x_1M \implies b = x_1 c$ for some $c \in M$. Now,

$$
\begin{aligned}
x_1 a = x_2 x_1 c \implies & x_1(a - x_2 c) = 0 \in M \\
\implies & a - x_2 c = 0 \in M \\
\implies & a \equiv 0 \in M/x_2M \\
\therefore\ & x_2, x_1 \text{ is a regular sequence on } M
\end{aligned}
$$

By assumption $x_3 \dots, x_n$ is a regular sequence on $M/(x_1, x_2)M$, thus we have proved the proposition. □

We are ready to prove a major result concerning when one can reorder the elements of a regular sequence, while maintaining regularity, but first a remark about the Jacobson radical.

Remark 3.5. *Suppose $x_1, \dots, x_n \in Jac(R)$ and M is a finitely generated R-module. By Nakayama's lemma,*

$$(x_1, \dots, x_n)M \neq M$$

Proposition 3.6. *Let M be a finitely generated R-module. Suppose*
(i) *$x_1, \dots, x_n \in Jac(R)$.*
(ii) *$x_1, \dots, x_n$ is a regular sequence on M.*

Then for any permutation $\sigma \in S_n$, $x_{\sigma(1)}, \dots, x_{\sigma(n)}$ is a regular sequence on M.

Proof. As our elements come from the Jacobson radical, reordering the elements will not violate the second part of the definition of regular sequence ($M \neq (x_1, \dots, x_n)M$).

Since any permutation can be obtained by a finite number of transpositions on adjacent elements, it is enough to show $x_1, \dots, x_{i+1}, x_i, \dots, x_n$ is a regular sequence on M for all i.

As in Proposition 3.4, we may assume $i = 1$, and it suffices to show $x_2, x_1, x_3, \dots x_n$ is a regular sequence, and by the preceding proposition, all that's left to show is that x_2 is regular on M. Supose $x_2 \cdot a = 0 \in M$ for $a \in M$. Since x_1, x_2 is a regular sequence, $a \in x_1 M$. Thus

$$
\begin{aligned}
a = x_1 a' \text{ and } x_2 x_1 a' = 0 \in M \implies & x_1(x_2 a') = 0 \\
\implies & x_2 a' = 0 \\
\implies & a' = x_1 a_2 \\
\implies & a = x_1^2 a_2
\end{aligned}
$$

By induction, there exists $a_n \in M$ for all n such that $a = x_1^n a_n$. But then, since $x_i \in Jac(R)$,

$$a \in \bigcap_{n=1}^{\infty} x_1^n M = 0$$

Therefore, x_2 is a non-zero divisor on M. $\square$

Now that a little ground work has been laid concerning regular sequences and the order thereof, we focus now on the length of maximal regular sequences. A **maximal regular sequence** from I is a regular sequence of maximal length, that is, if $x_1, \ldots, x_n$ is a regular sequence in I, we cannot fix some $x_{n+1} \in I$ such that $x_1, \ldots, x_n, x_{n+1}$ is a regular sequence. From the work above, we know that given a maximal regular sequence of length n whose elements are contained in the Jacobson radical, we can find other maximal regular sequences of the same length by permuting the elements.

We now address whether it is possible to find maximal regular sequences of varying length whose elements come from the same ideal. A theorem of Rees shows that this is not the case.

Theorem 3.7. (Rees) *If $I \subseteq R$ is an ideal and M is a finitely generated R-module, then every maximal M-regular sequence from I has the same length.*

Proof. We will show the following by induction: If $x_1, \ldots, x_n \in I$ is maximally M-regular and $y_1, \ldots, y_n \in I$ is a regular sequence on M, then it is maximal on I.

The $n = 0$ case is vacuous. For $n = 1$, assume $x_1 \in I$ is a maximal regular sequence on M and $y_1 \in I$. We want to show $I \subseteq \mathfrak{p}$, $\mathfrak{p} \in \operatorname{Ass}(M/y_1M)$. We know that $I \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Ass}(M/x_1M)$. Write $\mathfrak{p} = (0 : \overline{a})$ for some $\overline{a} \in M/x_1M$. Then by Remark 3.2,

$$y_1\overline{a} \equiv 0 \text{ in } M/x_1M \implies y_1 a = x_1 b \text{ in } M$$

We claim that $\mathfrak{p} = (0 :_{M/y_1M} \tilde{b})$. Suppose $r \in (0 : \tilde{b})$. Then

$$\begin{aligned} r \cdot \tilde{b} = 0 \text{ in } M/y_1M &\implies rb = y_1 c \\ &\implies rx_1 b = x_1 y_1 c \\ &= ry_1 a = y_1 x_1 c \\ &\implies ra = x_1 c \end{aligned}$$

The last implication follows since y_1 is a non zero divisor. Hence $r\overline{a} \equiv 0$ in M/x_1M, thus $r \in \mathfrak{p}$. Thus, $(0 : \tilde{b}) \subseteq \mathfrak{p}$.

Now, let $\{\mathfrak{p}_\alpha\}_\alpha$ be the union of all the associated primes for the following modules:

$$\begin{aligned} &M, \\ &M/x_1M, \ \ldots \ M/(x_1, \ldots, x_{n-1})M, \\ &M/y_1M, \ \ldots \ M/(y_1, \ldots, y_{n-1})M \end{aligned}$$

As each module has finitely many, there are finitely many in total. Note that $I \not\subseteq \mathfrak{p}_\alpha$ for all α, thus by the prime avoidance lemma, $I \not\subseteq \bigcup_\alpha \mathfrak{p}_\alpha$. Thus there exists $z \in I$ that is not a zero divisor in any of the modules listed above.

Thus we have $x_1, \ldots, x_{n-1}, z$ is a regular sequence from I, and $y_1, \ldots, y_{n-1}, z$ another regular sequence from I. Now we claim $x_1, \ldots, x_{n-1}, z$ is a maximal regular sequence from I. This is because x_n is a maximal regular sequence on $M/(x_1, \ldots, x_{n-1})M$ and by the $n = 1$ case, so is z.

By Proposition 3.6, we can rearrange the sequences so that $z, x_1, \ldots, x_{n-1}$ and $z, y_1, \ldots, y_n$ are regular sequences from I. Now, $x_1, \ldots, x_{n-1}$ is a maximal regular sequence from I on M/zM and $y_1, \ldots, y_{n-1}$ is another regular sequence from I on M/zM. By induction, both are maximal regular sequences from I on M/zM.

Therefore, $z, y_1, \ldots, y_{n-1}$ is a maximal regular sequence from I on M, and by rearranging once more, $y_1 \ldots, y_{n-1}, z$ is a maximal regular sequence from I on M. Thus, z is a maximal regular sequence from I on $M/(y_1, \ldots, y_{n-1})M$. However, $y_n \in I$ is regular on $M/(y_1, \ldots, y_{n-1})M$, and by the $n = 1$ case, y_n is maximal on $M/(y_1, \ldots, y_{n-1})M$, thus $y_1, \ldots, y_n$ is a maximal regular sequence on I.

$\square$

We may finally conclude that the length of an M-regular sequence from a particular ideal is an invariant.

Definition 3.8. Let $I \subseteq R$ be an ideal and M be a finitely generated R-module.
(i) The **grade** of I on M, denoted $\operatorname{grad}(I, M)$ is the length of any (and hence all) maximal M-regular sequences from I.
(ii) If $M = R$, we write $\operatorname{grad}(I) = \operatorname{grad}(I, R)$ and call this the grade of I.
(iii) If $(R, \mathfrak{m})$ is local and M is finitely generated, then $\operatorname{grad}(\mathfrak{m}, M)$ is called the **depth** of M, the length of the longest regular sequence on M. If $M = R$ we say the depth of R.

An alternate but equivalent definition of the length of a maximal regular sequence can be given in terms of the Ext functor, which is at times useful when proving certain results. For a proof, see [2].

Proposition 3.9. *Let R be Noetherian. Suppose $I \subseteq R$ is an ideal, and M is a finitely generated R-module. Also, suppose $x_1, \ldots, x_n$ is a maximal M-regular sequence from I. Then*

$$Ext^i\left(\frac{R}{I}, M\right) = 0 \quad \forall\, i < n$$

and

$$Ext^n\left(\frac{R}{I}, M\right) \neq 0$$

Thus, grad$(I, M) = n$, where the first non-vanishing $Ext^i(R/I, M)$ module occurs when $i = n$.

The next easy remark will be useful for inductive proofs that will follow.

Remark 3.10. *Let M be a finitely generated R-module, and $I \subseteq R$ an ideal with $IM \neq M$. Then if $x \in I$ is a non-zero-divisor,*

$$grad(I, M/xM) = grad(I, M) - 1$$

Now that the length of a maximal regular sequence has been shown to be an invariant of rings and modules, we seek to relate this invariant to others, specifically Krull dimension and projective dimension. A general result relating dimension and depth is easy to see.

Proposition 3.11. *Let M be a finitely generated R-module and $I \subseteq R$ an ideal with $IM \neq M$. Then*

$$grad(I, M) \leq \ dim\ M$$

Proof. Let $x_1, \ldots, x_n$ be a maximal M-regular sequence from I. Then by Krull's principal ideal theorem

$$\dim\ \left(\frac{M}{(x_1, \ldots, x_n)M}\right) = \ \dim\ (M) - n$$

Thus $n \leq \ \dim\ M$. □

For the particular case that M is an ideal, we have the following corollary.

Corollary 3.12. *Let R be Noetherian. If $I \subseteq R$ is an ideal, then grad$(I) \leq$ ht (I).*

Although this relationship between dimension and grade is nice, is nice, it doesn't tell us much about what $\operatorname{grad}(I, M)$ might actually be. A useful relationship can be established between grade of a module, its projective dimension, and the projective dimension of the underlying ring. This is called the Auslander-Buchsbaum formula, and is proved below. First, recall the definition of projective dimension.

Definition 3.13. Given an R-module M, a **projective resolution** is an infinite exact sequence of modules

$$\ldots \longrightarrow P_n \longrightarrow \ldots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

with each P_i a projective R module. If the resolution is finite, and if n is the least possible value with $P_n \neq 0$ and $P_i = 0$ for $i > n$, with then we say M has **projective dimension** n.

Remark 3.14. *Over a local ring, a projective module is a free module.*

Proposition 3.15. **(Rees)** *If T is finitely generated and we have an ideal $J \subseteq R$ such that J annihilates T, then grad$(J) \leq pd_R(T)$.*

Proof. Let $\underline{x} = x_1, \ldots, x_n \in J$ be a maximal regular sequence. Then

$$\operatorname{Ext}^n_R(T, R) \cong \operatorname{Hom}_R\left(T, \frac{R}{(\underline{x})R}\right)$$

We claim these are nonzero. If so, then $n \leq pd\, T$ as $\operatorname{Ext}^i_R(T, R) = 0$ for all $i > pd\, T$.

For the claim, take $\mathfrak{p} \in \operatorname{Ass}(R/(\underline{x}))$ such that $J \subseteq \mathfrak{p}$, and localize at $\mathfrak{p}$. If $\operatorname{Hom}_R(T, R/(\underline{x})R) = 0$, then $\operatorname{Hom}_R(T, R/(\underline{x})R)_{\mathfrak{p}}$ is also 0.

Without loss of generality we work in the case $(R, \mathfrak{p})$ is a local ring. (Note that $T_{\mathfrak{p}} \neq 0$, since $ann(T) \subseteq \mathfrak{p}$). Thus $T/\mathfrak{p}T \neq 0$, so there exists a homomorphism

$$f : T/\mathfrak{p}T \to R/\mathfrak{p}$$

Such that $f(\tilde{1}) \neq 0$. Where $\tilde{1} \in T$. We then have a chain

$$T \xrightarrow{\pi} T/\mathfrak{p}T \xrightarrow{f} R/\mathfrak{p}$$

Where π is the natural projection. Thus, since $\pi(1) = \tilde{1}$, it follows that $f(\pi(1)) \neq 0$, we have a nonzero homomorphism from T to $R/(\underline{x})R$, which gives the claim. □

The following is a well-known result relating projective dimension and depth of an R-module to the depth of R. For a proof, see Chapter 19 of [2] .

Theorem 3.16. (Auslander-Buchsbaum Formula) *Let $(R, \mathfrak{m})$ be local and M a finitely generated R-module. Assume the projective dimension of M is finite. Then:*

$$depth(R) = depth(M) + pd\ M$$

We are ready to define what it means for a ring to be Cohen-Macaulay.

Proposition 3.17. *Let R be a Noetherian ring. The following are equivalent:*
(i) *$grad(I) = ht\ I$ for all ideals $I \subseteq R$*
(ii) *$grad(\mathfrak{p}) = ht\ \mathfrak{p}$ for all prime ideal $\mathfrak{p} \subseteq R$.*
(iii) *$grad(\mathfrak{m}) = ht\ \mathfrak{m}$ for all maximal ideals $\mathfrak{m} \subseteq R$.*

*If R satisfies any (and hence all) of these conditions, R is said to be **Cohen-Macaulay**.*

In the case that the ring in question is the homomorphic image of a regular local ring, we can characterize whether or not it is Cohen-Macaulay by the projective dimension of the regular local ring and the kernel of the homomorphism as follows:

Proposition 3.18. *Suppose that R is the homomorphic image of a regular local ring. That is, $R = S/L$ for a regular local ring S and some ideal $L \subseteq S$. Suppose R has the finite free resolution*

$$0 \longrightarrow F_r \longrightarrow \ldots \longrightarrow F_1 \longrightarrow R \longrightarrow 0$$

Then $grad(L) = pd_S(R) \iff R$ is Cohen-Macaulay .

Proof. Assume $\operatorname{grad}(L) = \operatorname{pd}_S(R)$. By the Auslander-Buchsbaum formula (Proposition 3.16), we know that

$$\operatorname{depth}(R) + \operatorname{pd}_S(R) = \dim S$$

Thus we have

$$\begin{aligned} \dim R &= \dim S - \operatorname{ht}\ L \\ &= \dim S - \operatorname{grad}(L) \\ &= \dim S - \operatorname{pd}_S\ (R) \\ &= \operatorname{depth}(R) \end{aligned}$$

Hence R is Cohen-Macaulay .

Conversely, suppose that R is Cohen-Macaulay . First we note that $\text{depth}_R(R) = \text{depth}_S(R)$, as the action is the same. Thus

$$\begin{aligned}\text{depth}(R) &= \dim(S) - \text{pd}_S(R)\\ &\leq \dim S - \text{grad}(L)\\ &= \dim S - \text{ht } L\\ &= \dim(R)\end{aligned}$$

Since R is Cohen-Macaulay , we have equality throughout. Thus, $\text{grad}(L) = \text{pd}(R)$. ☐

We can alternately define the Cohen-Macaulay property for local rings with the following proposition.

Proposition 3.19. *Let $(R, \mathfrak{m})$ be a local ring. The following are equivalent:*
(i) *R is Cohen-Macaulay*
(ii) *Some system of parameters is a regular sequence*
(iii) *Every system of parameters is a regular sequence.*

From these definitions, it is clear that given a Cohen-Macaulay ring R, we can form new Cohen-Macaulay rings from it. It is not hard to show that if R is Cohen-Macaulay, then the polynomial ring $R[x]$ is Cohen-Macaulay as well. Also if I is a *perfect* ideal, that is, grad $I = pd\ I$, then an easy application of the Auslander-Buchsbaum formula shows that R/I is also Cohen-Macaulay. The property is also closed under localization, made formal in the following proposition.

Proposition 3.20. *Let S be a multiplicatively closed set. If R is Cohen-Macaulay then R_S is also Cohen-Macaulay.*

Proof. Let $\mathfrak{p}_S \subseteq R_S$ be a prime ideal. Then $\mathfrak{p} \subseteq R$ and $\mathfrak{p} \cap S \neq 0$. Then,

$$\text{ht } \mathfrak{p} = \text{grad}(\mathfrak{p}) \leq \text{grad}(\mathfrak{p}_S) \leq \text{ht } \mathfrak{p}_S = \text{ht } \mathfrak{p}$$

Therefore, $\text{grad}(\mathfrak{p}_S) = \text{ht } (\mathfrak{p}_S)$, as this is true for all primes in R_S, we conclude that R_S must be Cohen-Macaulay. ☐

We can also classify when a homomorphic image of a regular local ring is Cohen-Macaulay by examining its projective dimension over the regular local ring and the grade of the kernel of the map, as the following proposition shows.

A well-known aspect of all Cohen-Macaulay rings is that they are *catenary*, that is for all primes $\mathfrak{p} \subseteq \mathfrak{q}$, all maximal chains of primes between $\mathfrak{p}$ and $\mathfrak{q}$ have the same length. This is related to a defining quality that all Cohen-Macaulay rings share, namely that their ideals are *unmixed*.

Definition 3.21. An ideal $I \subseteq R$ is **unmixed** if the associated primes of R/I are all of equal height.

The original definition of Cohen-Macaulay came from the study of unmixed ideals. Macaulay showed that an ideal generated by n elements of height n is unmixed in a polynomial ring with coefficients in a field, and Cohen showed this fact for regular local rings. Today, these facts are stated in the *unmixedness theorem* as a characterization for Cohen-Macaulay rings, which is stated below without proof.

Theorem 3.22. (Unmixedness Theorem) *A Noetherian ring R is Cohen-Macaulay if and only if every ideal I generated by height I elements is unmixed.*

The theorem presented above is not to be confused with the definition of an *unmixed ring*, which we will present in the next section, and heavily focus on in section 7.

Let's attempt to refine the bound on multiplicity which was shown at the end of the last section for the case of a Cohen-Macaulay ring (Proposition 2.16). First, a necessary theorem presented without proof. This can be found in [5].

Theorem 3.23. *If $I \subseteq R$ is generated by a regular sequence of length d, then*

$$gr_I\ R \cong \frac{R}{I}[T_1, \ldots, T_d]$$

If R is local, then the converse holds as well.

Theorem 3.24. *Let $(R, \mathfrak{m})$ be a local ring of dimension d. The following are equivalent:*
(i) *R is Cohen-Macaulay*
(ii) *$e_0(I) = \lambda(R/I)$ for some parameter ideal I.*
(iii) *$e_0(I) = \lambda(R/I)$ for every parameter ideal I.*

Proof. **(i)** $\implies$ **(ii)**: By Theorem 3.23, since R is Cohen-Macaulay, I is generated by a regular sequence, thus we have $gr_I(R) = R/I[T_1, \ldots, T_d]$. Thus, for any k, $\lambda\left(\frac{I^k}{I^{k+1}}\right)$ is the the number of degree k monomials multiplied by the length of R/I, so that

$$\lambda(R/I^{n+1}) = \sum_{i=0}^{n} \lambda(R/I)\binom{i+d}{i} = \lambda(R/I)\binom{n+d}{d}$$

Which forces $e_0(I) = \lambda(R/I)$. **(ii)** $\implies$ **(iii)** is clear from Theorem 3.23.
(iii) $\implies$ **(i):** We will use Theorem 3.23. Suppose $I = (x_1, \ldots, x_d)$ be generated by a system of parameters. Let $S = (R/I)[Y_1, \ldots, Y_d]$, so that there exists a surjection $S \to gr_I\, R$. It suffices to show that this is an isomorphism. Suppose that it isn't, and let J be the kernel of the surjection so that

$$0 \longrightarrow J \longrightarrow S \longrightarrow gr_I(R) \longrightarrow 0$$

Each of these objects are graded, so we may examine the sequence at the kth degree:

$$0 \longrightarrow J_k \longrightarrow S_k \longrightarrow \frac{I^k}{I^{k+1}} \longrightarrow 0$$

Where $S_k = \bigoplus_{f \in \underline{Y}_k} f \cdot \frac{R}{I}$ and $\underline{Y}_k$ is the set of monomials in $Y_1, \ldots, Y_d$ of degree k. Thus

$$\lambda\left(\frac{I^k}{I^{k+1}}\right) = \lambda(S_k) - \lambda(J_k)$$

There are $\binom{k+d-1}{d-1}$ monomials of degree k, so

$$\lambda\left(\frac{I^k}{I^{k+1}}\right) = \lambda\left(\frac{R}{I}\right)\binom{k+d-1}{d-1} - \lambda(J_k)$$

Sum these lengths from $k = 0$ to $k = n$ so get

$$\lambda\left(\frac{R}{I^n}\right) = \lambda\left(\frac{R}{I}\right)\binom{n+d}{d} - \sum_{k=0}^{n} \lambda(J_k)$$

The leading coefficient on the left hand side is $\lambda(R/I)\binom{n+d}{d}$, Thus, after subtracting these terms, we conclude $\sum_{k=0}^{n} \lambda(J_k)$ is a polynomial in n of degree at most $d-1$.

Suppose that $J \neq 0$, so that $\exists\, f \in J$ with $f \neq 0$. As J is an ideal of S, the elements of J are polynomials in $Y_1 \ldots Y_d$ with coefficients in R/I. Since I is generated by a system of parameters, $\mathfrak{m}^r \subseteq I$ for $r >> 0$, so that if $\overline{\mathfrak{m}}$ is the image of $\mathfrak{m}$ in R/I, then $\overline{\mathfrak{m}}^r = (0)$ in S. Thus, there exists $t \in \overline{\mathfrak{m}} \subseteq S$ in a high enough power of $\overline{\mathfrak{m}}$ so that $tf \neq 0$, but $tf \cdot \overline{\mathfrak{m}} = 0$. We have $tf \in J$. Thus

$$J \supseteq tf \cdot S \cong \frac{S}{(0 : tf)} \cong \frac{S}{\overline{\mathfrak{m}}} \cong \frac{R}{\mathfrak{m}}[Y_1, \ldots, Y_d]$$

But then $\dim J \geq d$, so that $\lambda(J_n)$ is degree at least $d-1$ which means that $\sum_{k=0}^{n} \lambda(J_n)$ has degree at least d, a contradiction, hence $J = 0$. Therefore $S \cong gr_I\, R$, so I is generated by a regular sequence. But I was generated by a system of parameters, therefore some system of parameters of R is a regular sequence, and we conclude that R is Cohen-Macaulay. ☐

This proposition is very important for showing a ring is Cohen-Macaulay and will be used in the proof of the main theorem.

4. Completion

A standard technique among proofs in commutative algebra is to pass to the complete case, that is, to show that proving a desired result for a complete ring suffices as a proof for the general case. In this section, we recall the definition of completion, and present some standard results that allow much of our later work to be done in the case of a complete ring. Furthermore, a major condition in the main conjecture is the unmixed property for a ring, the definition of which references the completion of the ring.

Definition 4.1. Let $I \subseteq R$ be an ideal with $\bigcap_n I^n = 0$. Define a metric on the elements of R as follows: For $r_1, r_2 \in R$,

$$d(r_1, r_2) = \frac{1}{2^n} \iff r_1 - r_2 \in I^n \backslash I^{n+1}$$

The topology induced by this metric is called the **I-adic topology** on R. A sequence of elements in R, $\{r_i\}_{i\geq 1}$ is a **Cauchy sequence** in this topology if for any integer k, there exists an integer N such that $n, m \geq N$ implies $r_m - r_n \in I^k$. A ring is **complete** with respect the I-adic topology if every Cauchy sequence converges to an element in R.

The author wishes to emphasize here that the metric itself is rarely used in practice, and the choice of the integer 2 in its definition is arbitrary. One could use any positive number greater than 1 in the definition and describe the same topology.

Example 4.2. A straightforward example of a complete ring is $k[[x_1, \dots, x_n]]$, the power series ring over a field k. This complete in the $I = (x_1, \dots, x_n)$-adic topology.

In general, a Noetherian ring R does not have to be complete, but we can build a complete ring from it. In order to do this, we must ensure each Cauchy sequence converges. We say two Cauchy sequences $\{r_i\}$ and $\{s_i\}$ in R are *equivalent* if and only if $\lim_{n\to\infty} r_n - s_n = 0$.

Definition 4.3. Let $I \subseteq R$ be an ideal. The set of of all equivalence classes of Cauchy sequences in the I-adic topology is called the **I-adic completion** of R denoted $\hat{R}_I$. If R is a local ring with maximal ideal $\mathfrak{m}$, then the $\mathfrak{m}$-adic completion is simply denoted $\hat{R}$.

An alternative, but equivalent definition of the I-adic completion of a ring can be stated using inverse limits. A basis for the open neighborhoods of 0 is given by the powers of I which form a descending filtration

$$R \supseteq I \supseteq I^2 \supseteq \ldots \supseteq I^n$$

And the completion is the inverse limit of the factor rings, that is

$$\hat{R}_I = \varprojlim R/I^n$$

This definition allows us to easily generalize the I-adic topology to modules. A basis for the open neighborhood in an R-module M is given by the set $\{x + I^n M : x \in M, n \geq 0\}$, and I-adic completion is defined as $\hat{M}_I = \varprojlim (M/I^n M)$.

Example 4.4. If $R = k[x_1, \dots, x_n]$ and $I = (x_1, \dots, x_n)$, then the completion of R in the I-adic topology is $k[[x_1, \dots, x_n]]$.

There are a few standard results which we will make use of to come.

Theorem 4.5. *Let R be a Noetherian ring, I an ideal of R.*
(i) *$\hat{R}_I$ is Noetherian.*
(ii) *$\hat{R}_I / I^n \hat{R}_I \cong R/I^n$ Thus $\hat{R}_I$ is complete with respect to $I\hat{R}_I$.*
(iii) *If M is a finitely generated R module, then*

$$\hat{R}_I \otimes_R M \cong \hat{M}_I$$

In particular, if S is a ring that is also a finite R-module, then $\hat{R}_I \otimes_R S$ is the completion of S in the IS-adic topology.
(iv) *$\hat{R}_I$ is flat as an R-module.*

For a proof, see [2].

A very useful result is presented below, which was proved by Cohen in 1946. In words, Eisenbud states this result roughly as "any complete local Noetherian ring is a homomorphic image of a power series ring in finitely many variables over a 'nice' ring". We state the result formally below in the *equicharacteristic* case ("equicharacteristic" means the ring contains a field).

Theorem 4.6. (Cohen Structure Theorem) *Let $(R, \mathfrak{m}, k)$ be a local Noetherian ring that is complete in the $\mathfrak{m}$-adic topology. If R contains a field, then we may write*

$$R \cong k[[x_1, \ldots x_n]]/I \quad \text{for some } n \in \mathbb{N} \text{ and some ideal } I \subseteq k[[x_1, \ldots, x_n]]$$

An outline of the proof can be found in [2]. In fact, it is true that *any complete Noetherian local ring is the homomorphic image of a complete Noetherian regular local ring.* This fact shows just how well-behaved complete local Noetherian rings are, and why one often works in the complete case whenever possible. Fortunately, this technique can be employed in a variety of situations; a local ring and its completion share many properties. A few standard results are especially useful, the most pertinent to this discussion is the relationship between the a ring, its completion, and their respective Hilbert polynomials.

Proposition 4.7. *Let $(R\mathfrak{m})$ be a local ring and $I \subseteq R$ be an ideal, and $\hat{R}$ its I-adic completion. with $\hat{I} = I\hat{R}$. Then*

$$P_{I,R}(n) = P_{\hat{I},\hat{R}}(n)$$

Proof. Since $\hat{R}_I/I^n\hat{R}_I = R/I^n$, the relation is immediate by

$$P_{I,R}(n) = \lambda(R/I^n) = \lambda(\hat{R}/\hat{I}^n) = P_{\hat{I},\hat{R}}(n)$$

□

Proposition 4.8. *If R is a local ring with maximal ideal $\mathfrak{m}$, then $\hat{R}$ is a local ring with maximal ideal $\hat{\mathfrak{m}} = \mathfrak{m}\hat{R}$.*

Proof. Since $R/\mathfrak{m} \cong \hat{R}/\mathfrak{m}\hat{R}$ and the former is a field, $\mathfrak{m}\hat{R}$ must be a maximal ideal in $\hat{R}$. Furthermore, if $x \in \hat{\mathfrak{m}}$, then $(1-x)^{-1} = 1 + x + x^2 + \ldots$, which converges in $\hat{R}$. Thus, $1 - x$ is a unit for all $x \in \hat{\mathfrak{m}}$, so that $\mathfrak{m}$ is contained in the Jacobson radical, therefore it must be the Jacobson radical, hence $\hat{\mathfrak{m}}$ is the only maximal ideal in $\hat{R}$. □

Proposition 4.9. *If $(R, \mathfrak{m}, k)$ is a local ring and $\hat{R}$ its $\mathfrak{m}$-adic completion, then*
(i) *dim $R =$ dim $\hat{R}$*
(i) *depth $R =$ depth $\hat{R}$.*

Proof. **(i)** Since $P_{\hat{\mathfrak{m}},\hat{R}}(n) = P_{\mathfrak{m},R}(n)$, and these polynomials have the same degree, which implies they have the same Krull dimension.
(ii) This is most easily seen using the Ext definition of depth (Proposition 3.9) and using some homological methods. In particular, for all i, we have

$$\mathrm{Ext}^i_R(R/\mathfrak{m}, R) \otimes \hat{R} \cong \mathrm{Ext}^i_R(\hat{R}/\hat{\mathfrak{m}}, \hat{R})$$

Since depth is the first nonvanishing Ext module, the depth of R is the depth of $\hat{R}$. □

Of course, if we put both of these facts together, we arrive at an extremely nice result:

Proposition 4.10. *If $(R, \mathfrak{m}, k)$ is a local ring, and $\hat{R}$ its $\mathfrak{m}$-adic completion, then R is Cohen-Macaulay if and only if $\hat{R}$ is Cohen-Macaulay.*

These facts allow much of the analysis leading up to and including the proof of the main theorem to take place in the complete local case. Furthermore, we can (finally) define what it means for a ring to be *unmixed*, and the relationship of this condition to the Cohen-Macaulay property.

Definition 4.11. Let $(R, \mathfrak{m})$ be a local Noetherian ring of dimension d, and $\hat{R}$ be the $\mathfrak{m}$-adic completion of R. Then we say that R is **unmixed** if for all $\mathfrak{p} \in \mathrm{Ass}(\hat{R})$,

$$\dim \hat{R}/\mathfrak{p} = d$$

In the case that R is Cohen-Macaulay, we can state an immediate corollary to the Unmixedness Thoerem:

Corollary 4.12. *If $(R, \mathfrak{m}, k)$ is Cohen-Macaulay , then it is* ***unmixed****, that is, for any associated prime $\mathfrak{p}$ of $\hat{R}$,*

$$dim\ (\hat{R}/\mathfrak{p}) = \ dim\ (R)$$

Proof. Since R is Cohen-Macaulay, so is $\hat{R}$. Thus by the unmixedness theorem, all the associated primes of $\hat{R}$ have the same height. Thus for all $\mathfrak{p}$ in $Ass(\hat{R})$, we have

$$\dim \hat{R}/\mathfrak{p} = \ \dim \hat{R} = \ \dim R$$

Therefore, R is unmixed. □

We end the section with a couple of propositions about the completion which will be used in the sections to come. As they do not require any additional machinery then has already been outlined.

Proposition 4.13. *If a local ring R is complete in the $\mathfrak{m}$-adic topology, and $I \subseteq R$ is any ideal, then R is complete in the I-adic topology.*

Proof. Let $\{x_n\}$ be a Cauchy sequence in the I-adic topology. Then for a fixed k there exists N such that $m, n > N \implies x_m - x_n \in I^k \subseteq \mathfrak{m}^k$. Since the sequence is also Cauchy in the $\mathfrak{m}$-adic topology, there exists a limit in this topology, x. Thus, for any i, there exists j large enough such that $x_j - x \in \mathfrak{m}^i$ and $x_n - x_j \in I^k$ Hence

$$x_n - x = x_n - x_j + x_j - x \in I^n + \mathfrak{m}^i$$

As we can choose some k for all i, we may write

$$x_n - x \in \bigcap_i (I^k + \mathfrak{m}^i)$$

But, by Krull's Intersection Theorem, $\bigcap\limits_i \mathfrak{m}^i = 0$, hence $x_n - x \in I^k$. Therefore, x is the I-adic limit of the sequence, so R is complete in the I-adic topology as well. □

Proposition 4.14. *Let M be an R-module. If R is complete in the I-adic topology, M/IM is finitely generated as an R-module and $\bigcap\limits_n I^n M = 0$, then M is also finitely generated as an R-module.*

Proof. Suppose $\overline{x_1}, \ldots, \overline{x_n}$ generate M/IM, and let x_i be an inverse image of $\overline{x_i}$. Then for $x \in M$, we may write

$$x = r_1x_1 + \cdots + r_nx_n + y_1 \quad y_1 \in IM$$

Similarly, we may write

$$y_1 = i_{1,1}x_1 + \cdots + i_{1,n}x_n + y_2 \quad i_{1,j} \in I, \ \ y_2 \in I^2M$$

so that

$$x = (r_1 + i_{1,1})x_1 + \cdots + (r_n + i_{1,n})x_n + y_2$$

Continuing inductively, for any m, we can write

$$x = (r_1 + \sum_{j=1}^{m} i_{j,1})x_1 + \cdots + (r_n + \sum_{j=1}^{m} i_{j.n})x_n + y_{m+1} \quad y_{m+1} \in I^{m+1}M$$

Now, as R is complete in the I-adic topology, each coefficient $r_k + \sum_{j=1}^{m} i_{j,k}$ has a limit, say s_k. Hence,

$$x - (s_1x_1 + \cdots + s_nx_n) \in \bigcap_m I^m M = 0$$

Therefore, $(x_1, \ldots, x_n) = M$. □

5. Superficial Elements

Up to this point, we have covered topics that are well-known to a student who has a basic background in commutative algebra, and we are ready to proceed on to topics that are less elementary, but nonetheless crucial to our proof of the main theorem. Our proof will be an induction argument where we induce on the Krull dimension. Due to the nature of the inductive proof, we will use induction hypothesis on a ring of dimension less than the ring in question, namely the factor ring $R/(x)$ where $x \in R$. For our induction to work, we will need to choose the element x with some very specific properties. The most obvious of which is that we need to the dimension and depth of $R/(x)$ to each be one less then R. This is immediately attained if we choose x to be a regular element in R. Unfortunately, this is not quite enough.

In order to apply the induction hypothesis, not only will the dimension (and depth) need to decrease in $R/(x)$, but we also wish to preserve that if $e_1(I) = 0$, then $e_1(\overline{I}) = 0$ where $\overline{I}$ is the image of I in $R/(x)$. We will also need to retain the unmixed property in our inductive step, which will take a bit more work. To ensure all of these conditions, we will require that x be a *superficial element*. This alone is enough to preserve that $e_1(I) = e_1(\overline{I})$. In order to preserve unmixedness, the element x will also need to avoid some particular prime ideals which contain the *non-Cohen-Macaulay locus*, and we will need to pass to a homomorphic image of $R/(x)$, which will be discussed in more detail in section 7.

The goal of this section is to define a superficial element, and prove that, given the proper conditions, it is possible to find a superficial element $x \in R$ with all of the properties that are necessary for the main proof.

Definition 5.1. Let $(R, \mathfrak{m})$ be a local Noetherian ring, and $I \subset R$ an ideal. Then $x \in I \backslash I^2$ is said to be **superficial** for I (of degree 1) if there exists $c \in \mathbb{N}$ such that for all $n \geq c$, $(I^{n+1} : x) \cap I^c = I^n$.

Remark 5.2. *If $x \in I$ is a superficial element and a non-zero divisor, then for $n >> 0$,*

$$(I^{n+1} : x) = I^n$$

Proof. By Artin-Rees, there exists k such that

$$(I^{n+1} : x) = (0 : x) + I^{n-k}(I^k : x)$$

If x is a non-zero divisor, then $(I^{n+1} : x) = I^{n-k}(I^k : x) = I^n$. So, if for large n, we have $(I^{n+1} : x) \cap I^c = I^n \cap I^c = I^n$, thus, we don't need to intersect with I^c to ensure that x is superficial. $\square$

The interested reader may find many more facts about superficial elements in [4]. However, for the purposes of the main theorem, we need only the following general proposition.

Proposition 5.3. *Let $(R, \mathfrak{m}, k)$ with k infinite be a local ring of dimension d and depth greater than 0. Suppose $I = (x_1, \ldots, x_d)$ is a parameter ideal. Then there exists a superficial element $x \in I$. Furthermore, we can choose the element so that is a non-zero divisor, x is part of a minimal generating set of I, and x avoids finitely many prime ideals.*

Proof. Let $\mathcal{R}$ be the Rees ring with respect to I. That is,

$$\mathcal{R} = R[It] = R \oplus It \oplus I^2t^2 \oplus \ldots \oplus I^nt^n \oplus \ldots$$

Then

$$I\mathcal{R} = IR \oplus I^2t \oplus I^3t^2 \oplus \ldots \oplus I^{n+1}t^n \oplus \ldots$$

Let $I\mathcal{R}$ have the primary decomposition

$$I\mathcal{R} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r \cap \mathfrak{q}_{r+1} \cap \cdots \cap \mathfrak{q}_s$$

Let these primary ideals be arranged so that

$$\begin{cases} \mathcal{R}_+ \not\subseteq \sqrt{\mathfrak{q}_i} & i \leq i \leq r \\ \mathcal{R}_+ \subseteq \sqrt{\mathfrak{q}_j} & r < j \leq s \end{cases}$$

Then $\exists\, c$ such that

$$(\flat) \qquad (\mathcal{R}_+)^c \subseteq \mathfrak{q}_{r+1} \cap \cdots \cap \mathfrak{q}_s$$

Set

$$J_i = \{x : xt \in \sqrt{\mathfrak{q}_i}\}, \quad 1 \leq i \leq r$$

Then for each $1 \leq i \leq r$, we have

$$J_i \subsetneq I,$$

and

$$\mathfrak{m}I \subsetneq I,$$
$$(x_2, \ldots, x_d) + \mathfrak{m}I \subsetneq I$$

and

$$C_i = \{\mathfrak{p}_i \cap I : \mathfrak{p}_i \in \operatorname{Ass}(R)\} \subsetneq I \ \ \forall i$$

Say there are t of these, $C_1, \ldots, C_t$. Set: $A = \mathfrak{m}I$ and $B = (x_2, \ldots, x_d) + \mathfrak{m}I$. Finally, Set

$$L := J_1 \cup \cdots \cup J_r \cup C_1 \cup \cdots \cup C_t \cup A \cup B \tag{$\sharp$}$$

We claim that $I \not\subseteq L$. Suppose that $I \subseteq L$. Then the vector space $I/\mathfrak{m}I$ is the union of its subspaces, that is

$$I/\mathfrak{m}I = J_1/\mathfrak{m}J_1 \cup \cdots \cup J_r/\mathfrak{m}J_r \cup C_1/\mathfrak{m}C_1 \cup \cdots \cup C_t/\mathfrak{m}C_t \cup A/\mathfrak{m}A \cup B/\mathfrak{m}B$$

But, since each $J_i, C_j, A, B \subsetneq I$, the subspaces $J_i/\mathfrak{m}J_i$, $C_j/\mathfrak{m}C_j$, $A/\mathfrak{m}A$ and $B/\mathfrak{m}B$ are proper subspaces of $I/\mathfrak{m}I$. As the vector space cannot be a union of its proper subspaces, we conclude $I \not\subseteq L$.

Thus, $\exists\, x \in I$ but $x \notin L$. Since x is not in any associated prime of R, x must be a non-zero divisor. We claim that $(x, x_2, \ldots, x_d)$ is a minimal generating set for I. To see this, since $x \in I$ we can write

$$x = r_1 x_1 + \cdots + r_d x_d$$

Since $x \notin (x_2, \ldots, x_d) + \mathfrak{m}I$, we must have $r_1 \notin \mathfrak{m}$. Thus, r_1 is a unit in R, so we can write

$$x_1 = r_1^{-1} x - r_2 x_2 + \cdots + r_d x_d$$

Thus $I = (x, x_2, \ldots, x_d)$.

Next, we claim that x is a superficial element for I. Let c be as in ($\flat$), and suppose that x is not superficial. Then for $n > c$, there exists α such that

$$\alpha \in \big((I^{n+1} : x) \cap I^c\big) \setminus I^n$$

We know that $\alpha \in I^c$, let u be the greatest value such that $\alpha \in I^u$, so that $c \leq u < n$. Then

$$\alpha x \in I^{n+1} \implies \alpha t^u x t \in I^{n+1} t^{u+1} \subseteq I\mathcal{R}$$
$$\implies \alpha t^u x t \in \mathfrak{q}_i \ \ \forall i$$

Since each $\mathfrak{q}_i$ is primary, we have either $\alpha t^u \in \mathfrak{q}_i$ or $(xt)^m \in \mathfrak{q}_i$ for some i. However, if $1 \leq i \leq r$, $(xt)^n \notin \mathfrak{q}_i$, else we have $xt \in \sqrt{\mathfrak{q}_i}$ which would imply $x \in J_i$, which is a contradiction. Thus $\alpha t^u \in \mathfrak{q}_i$ for $1 \leq i \leq r$. Furthermore, since $u \geq c$,

$$\alpha t^u \in I^u t^u \subseteq I^c t^c \subseteq \mathcal{R}_+^c \subseteq \mathfrak{q}_{r+1} \cap \cdots \cap \mathfrak{q}_s$$

Hence $\alpha t^u \in \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_s = I\mathcal{R}$. But $\alpha t^u \in I\mathcal{R} \implies \alpha \in I^{u+1}$. Thus we have arrived at a contradiction, since u was the greatest value such that $\alpha \in I^u$. This completes the proof. $\square$

An important part of this proof was that the superficial element x was found inside I, and was done so in such a way that allows us to avoid finitely many prime ideals. That is, if we wanted to impose more conditions, such as ensuring that $x \in I$ but $x \notin \mathfrak{p}$ for some prime ideal $\mathfrak{p}$ that does not contain I, we can do so by adding $I \cap \mathfrak{p}$ to the list of objects that define L in the proof (see ($\sharp$)). We will eventually make use of this fact so that if R is unmixed, we can choose a superficial element x so that

$$\operatorname{Ass}(R/xR) = \{\mathfrak{p} : \mathfrak{p} \text{ is a ht 1 prime containing } x\} \cup \mathfrak{m}$$

To do so, we must first show that this is a finite set, which will be accomplished in section 7.

As mentioned previously, once we have a superficial element $x \in I$ that is a non-zero-divisor, we can study the Hilbert coefficients $e_j(I)$ by calculating the coefficients for $R/(x)$. This works exceptionally well, as shown in the following proposition.

Proposition 5.4. *Let $(R, \mathfrak{m})$ be a d dimensional local ring and $I \subseteq R$ a parameter ideal. Suppose that $x \in I$ is a superficial element in I which is a non-zero-divisor. Set $\overline{R} = R/(x)$ and $\overline{I} = I/(x)$. Then $e_i(I) = e_i(\overline{I})$ for $0 \leq i \leq d-1$.*

Proof. Let $n >> 0$ and consider the exact sequence

$$0 \longrightarrow R/I^n \xrightarrow{\cdot x} R/I^{n+1} \longrightarrow R/(I^{n+1}, x) \longrightarrow 0$$

with multiplication by x injective, as $(I^{n+1} : x) = I^n$. Let $\overline{R} = R/(x)$ and $\overline{I}$ be the image of I in R.

By additivity of length, we have,

$$\lambda\left(\frac{R}{I^n}\right) - \lambda\left(\frac{R}{I^{n-1}}\right) = \lambda\left(\frac{\overline{R}}{\overline{I}^n}\right)$$

Thus, we have, up to a constant,

$$\sum_{j=0}^{d}(-1)^j e_j(I)\binom{n-j+d}{d} - \sum_{j=0}^{d}(-1)^j e_j(I)\binom{n-1-j+d}{d} = \sum_{j=0}^{d-1}(-1)^j e_j(\overline{I})\binom{n-1-j+d}{d-1}$$

After applying Pascal's Recurrence (Fact 2.10), we conclude that $e_j(I) = e_j(\overline{I})$ except possibly at the constant term, when $j = d$. □

In our main proof, we will need this result for a different but similar factor ring, but the result will follow partially from this proposition.

6. Local Cohomology

In the paper which positively solved the conjecture, the main technique of the proof is to use high-level results of local cohomology [1]. Indeed, this makes the proof rather concise and elegant, but hard for the initiate in commutative algebra to access. The intended goal of this paper was to prove the theorem using results as elementary as possible, nonetheless, it will be necessary to make use of local cohomology. The point of this section is to introduce only as much local cohomology as is needed for the main proof, and do so using as little machinery as possible. Local cohomolgy was invented by Alexander Grothendieck and introduced in his lectures in the early 1960s. It was first recorded in text by Robin Hartshorne. Grothendieck introduced the concept to prove Lefschetz theorems in algebraic geometry, which used sheaf cohomolgy. However the concepts can be restricted to the case of modules for the purpose of this paper. In doing so, the definitions and results presented below will arise from purely commutative algebraic beginnings.

Given a module M, we compute its local cohomology modules from its unique minimal injective resolution. In order to begin, it is necessary to recall some facts about injective modules over a Noetherian commutative ring.

Definition 6.1. An R-module Q is said to be an **injective** R-module if it satisfies the following property: For all modules N, M and R-module homomorphisms $i : N \to M$ and $f : N \to Q$, with i an injection, there exists an R-module homomorphism $\rho : M \to Q$ satisfying $i \circ \rho = f$. That is, in the following diagram, if we have the solid arrows, we can find the dotted arrow.

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r, hook, "i"] \arrow[d, "f"] & M \arrow[dl, dotted, "\exists\rho"] \\ & Q & \end{tikzcd}$$

Note that defintion states that maps into Q can always be extended. A more useful characterization of injective module states that we only need to be able to complete such diagrams for which we have an ideal injecting into a ring, rather than all diagrams.

After this definition, one could construct an injective resolution of any module in general, but this resolution would not necessarily be minimal. In order to find a minimal resolution, it is necessary ensure that the injective modules in our resolution are essential extensions.

Definition 6.2. Let R be a ring and $M \subseteq E$ be R-modules. we say that E is an **essential extension** of M if every non-zero submodule of E intersects M nontrivially.

Proposition 6.3. *Let R be a ring. Then*
(i) *Given any R-modules $M \subseteq F$, there is a maximal submodule $E \subseteq F$ such that $M \subseteq E$ is an essential extension.*
(ii) *If F is injective, then so is E.*
(iii) *There is, up to isomorphism, a unique essential extension E of M that is an injective R-module. This E is called the* ***injective hull*** *of M, written $E_R(M)$ and is unique up to isomorphism.*

Definition 6.4. Let M be an R-module. A sequence of R-modules and R-module homomorphisms

$$\mathcal{Q}: \quad 0 \longrightarrow M \xrightarrow{i} Q_0 \xrightarrow{\psi_0} Q_1 \xrightarrow{\psi_1} Q_2 \xrightarrow{\psi_3} \dots$$

is called an injective resolution of M if for all $i \geq 0$, Q_i is an injective R-module and $im(\psi_{i-1}) = ker(\psi_i)$. The resolution is **minimal** if setting $M_i = \text{coker}(\psi_{i-1})$ we have $Q_{i+1} = E(M_i)$ and ψ_i is the composition of the maps

$$Q_i \longrightarrow M_i \longrightarrow E(M_i) = Q_{i+1}$$

Not only is this resolution guaranteed to exist, it is unique up to isomorphism.

More can be said about the structure of the injective hull of a given module.

Proposition 6.5. *If R is Noetherian and Q is an injective R-module, then*

$$Q = \bigoplus_{\alpha} E_R(R/\mathfrak{p}_\alpha)$$

where each $\mathfrak{p}_\alpha$ is prime. Furthermore $\mathfrak{p} \in Ass(Q) \iff \mathfrak{p} = \mathfrak{p}_\alpha$ for some α.

Note that in the proposition, it is often the case that a summand appears multiple times.

The following is a basic definition upon which the definition of local cohomology is based, much of the remaining section is presented in "Lectures on Local Cohomology" by Craig Huneke [10].

Definition 6.6. Let R be a commutative Noetherian ring, $I \subseteq R$ an ideal, and M an R-module. Set

$$\Gamma_I(M) = \{x \in M : \text{ there exists } n \in \mathbb{N} \text{ such that } I^n x = 0\}$$

This is called the **torsion functor** with respect to I.

Note that the torsion functor is left exact and covariant.

Definition 6.7. Let $I \subseteq R$ be an ideal then the ith **local cohomology module** $H^i_I(\_\_)$ is the ith right derived functor of Γ_I.

To make this more clear, to find $H^i_I(M)$, we first take a minimal injective resolution of M, apply Γ_I, and take the cohomology at the ith position in the resolution. Note that this construction is independent of the resolution. Furthermore, since Γ_I is left exact,

$$\Gamma_I(M) = H^0_I(M)$$

There are other ways to think about local cohomology, but will not be used in this paper. Namely, one can view the ith local cohomology module as the direct limit of $\mathrm{Ext}^i_R(R/I_n, M)$ where I_n is a nested system of ideals cofinal to the powers of I. One may also compute the local cohomology modules from Koszul cohomology, which for the sake of brevity will not be presented in this text.

The following is a very important fact concerning the local cohomology modules, which is immediate from the definition of Γ_I.

Proposition 6.8. *If $z \in H^i_I(M)$ then $\exists\, n$ such that $I^n z = 0$.*

Proof. This is immediate from the definition. If we apply Γ_I to an injective resolution of M, then every element in the objects of complex is annihilated by a power of I, and passing to cohomology does not alter this. ☐

Remark 6.9. *If I, J are ideals of R with $\sqrt{I} = \sqrt{J}$, then Γ_I and Γ_J are the same functor, so that for any R-module M, $H^i_I(M) = H^i_J(M)$ for all i.*

An important property of the local cohomology functors which will be exploited in the main proof is that given short exact sequence of R-modules, there is an induced long exact sequence in local cohomology.

Proposition 6.10. *Given an ideal I and short exact sequence of R-modules*

$$0 \longrightarrow N \longrightarrow M \longrightarrow L \longrightarrow 0$$

There is induced long exact sequence on local cohomology:

$$0 \longrightarrow H^0_I(N) \longrightarrow H^0_I(M) \longrightarrow H^0_I(L) \longrightarrow H^1_I(N) \longrightarrow H^1_I(M) \longrightarrow H^1_I(L) \longrightarrow \dots$$

Proposition 6.11. *Let R be a noetherian ring, $I \subseteq R$ an ideal and M and R-module. Let $\phi : R \to S$ be a homomorphism, and N an S-module.*
(i) *If ϕ is flat, then $H^i_I(M) \otimes_R S \cong H^i_{IS}(M \otimes_R S)$. In particular, local cohomology commutes with localization and completion.*
(ii) (**Independence of Base**) *$H^i_I(N) \cong H^i_{IS}(N)$ where the first local cohomology is computed over the base ring R.*

In the case of a local ring R with maximal ideal $\mathfrak{m}$, these properties lead to an important fact, namely that the local cohomology modules of R with respect to $\mathfrak{m}$ are the same as those of its $\mathfrak{m}$-adic completion, $\hat{R}$. That is,

Proposition 6.12. *Let $(R, \mathfrak{m}, k)$ be a local ring, and M be a finitely generated R-module. Then for all i:*

$$H^i_{\mathfrak{m}}(M) \cong H^i_{\hat{\mathfrak{m}}}(\hat{M})$$

Now that we know some facts about the local cohomology of an R-module, the natural question arises as to what information about the module can be extracted from local cohomology. There is a natural relationship between the depth of a ring and its local cohomology modules. Indeed, one could use the modules themselves to define depth as follows:

Proposition 6.13. *Let R be a Noetherian ring, with $I \subseteq R$ an ideal and M a finitely generated R-module such that $IM \neq M$. Then $grad_I(M)$ is the greatest integer i such that $H^j_I(M) = 0$ for all $j < i$. In particular, the depth of M is i if the first non-vanishing is $H^i_{\mathfrak{m}}(M)$ is the first non-vanishing local cohomology module.*

Proof. This follows from Proposition 3.9 and the Ext definition of local cohomology in [10]. ☐

Clearly, we can use this fact to determine the depth of a ring without having to explicitly construct a regular sequence, instead we need only compute local cohomology. The resulting process will be integral of our proof, as we can now relate the Cohen-Macaulay property to local cohomology.

Remark 6.14. *A local ring $(R, \mathfrak{m})$ is Cohen-Macaulay if $H^j_{\mathfrak{m}}(R) = 0$ for all $j <$ dim R.*

As depth is the first non-vanishing module, if this happens at precisely the dimension of the ring in question, then depth and dimension are the same and the ring is Cohen-Macaulay.

We now turn our attention away from general results of local cohomology and instead focus on more specific facts that are essential to the task at hand.

Proposition 6.15. *If $(R, \mathfrak{m})$ is a local ring, and M is an R-module, then*

$$H^i_{\mathfrak{m}}(H^0_{\mathfrak{m}}(M)) = 0$$

for all $i > 0$.

Proof. Since $H^0_{\mathfrak{m}}(M)$ are all the elements of M annihilated by some power of $\mathfrak{m}$, if we localize at any $\mathfrak{p} \subsetneq \mathfrak{m}$, $\left(H^0_{\mathfrak{m}}(M)\right)_{\mathfrak{p}} = 0$. If $H^0_{\mathfrak{m}}(M)$ has a mnimal injective resolution

$$0 \longrightarrow H^0_{\mathfrak{m}}(M) \longrightarrow Q_0 \longrightarrow Q_1 \longrightarrow \dots$$

Then as Q_0 is an essential extension of $H^0_{\mathfrak{m}}(M)$, $(Q_0)_{\mathfrak{p}} = 0$. Thus, the only associated prime of each Q_i is precisely $\mathfrak{m}$, as this is the only associated prime of $H^0_{\mathfrak{m}}(M)$. This implies that $Q_i = \bigoplus E(R/\mathfrak{m})$. Thus, each Q_i is annihilated by a power of $\mathfrak{m}$. When we apply $\Gamma_{\mathfrak{m}}$ to the resolution, we have $\Gamma_{\mathfrak{m}}(Q_i) = Q_i$ for all i. Taking cohomology, since the sequence is exact, we get $H^i_{\mathfrak{m}}(H^0_{\mathfrak{m}}(M)) = 0$, except in the first position, in which case we have $H^0_{\mathfrak{m}}(H^0_{\mathfrak{m}}(M)) = H^0_{\mathfrak{m}}(M)$. ☐

An essential argument of our proof relies on the finiteness of certain local cohomology modules. In particular, we wish to show that for a complete local unmixed ring $(R, \mathfrak{m})$, $H^1_{\mathfrak{m}}(R)$ is finitely generated as an R-module. In order to show this, we make use of the following construction.

Definition 6.16. Let R be a commutative ring, and S be the multiplicatively closed set of non-zero-divisors on R. The **total quotient ring K of R** is the ring obtained from inverting all elements of S, that is, $K = R_S$.

Definition 6.17. Let $(R, \mathfrak{m})$ be a Noetherian, local ring. Then the **maximal ideal transform** T is the set of all elements of the total quotient ring K of R whose conductor to R contains a power of the maximal ideal $\mathfrak{m}$. That is.

$$T = \left\{ \frac{a}{b} \in K : \mathfrak{m}^n \cdot \frac{a}{b} \subseteq R \text{ for some n} \right\}$$

The following is a generalization of the Krull-Akizuki Theorem. This result is presented in a more general context in a paper by Jacob Matijevic in 1976 [3], but for this paper, we present a proof in a more applicable context to the task at hand.

Proposition 6.18. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, T be the maximal ideal transform of R, and $x \in T$ be a non-zero-divisor. Then T/xT is finitely generated as an R-module.*

Proof. It suffices to show that for some $n > 0$, $T \subseteq Rx^{-n} + xT$. First we show that for any $t \in T$, there exists some k such that $t \in Rx^{-k} + xT$, then we will show that for a large enough k, this will work for all $t \in T$.

Given $t \in T$, let $J = \{r \in R : rt \in R\}$. By the definition of T, some power of the maximal ideal $m^n \subseteq J$. Thus R/J is an Artinian. Thus in R/J the descending chain

$$(x) + J \supseteq (x^2) + J \supseteq \ldots \supseteq (x^n) + J \supseteq \ldots$$

stabilizes, that is, for some k, $(x^k) + J = (x^{k+1}) + J$. Therefore, for some $r \in R, j \in J$, we can write

$$x^k = rx^{k+1} + j$$

Multiplying by t,

$$tx^k = rtx^{k+1} + jt$$

$$\implies t = jtx^{-k} + rtx \in Rx^{-k} + xT$$

as desired, since $jt \in R$, by definition.

Next we define $I_j = (x^jT \cap R, xR)$ and claim that $I_j = I_{j+1}$ for some value of j. To see this, we note that $(xT \cap R)/xR$ is a module of finite length. We know this as $(xT \cap R)/xR$ is a finite R-module generated by elements of the form xt_i with $t_i \in T$. Since for each t_i, there is some k_i such that $\mathfrak{m}^{k_i}t_i \subseteq R$ there is a large enough power k that works for all i. Thus, $(xT \cap R)/xR$ is a finite $R/\mathfrak{m}^k$ module for some k. Thus, the descending chain

$$(xT \cap R, xR) \supseteq (x^2T \cap R, xR) \supseteq \ldots$$

stabilizes, so that there exists n such that $(x^nT \cap R, xR) = (x^{n+1}T \cap R, xR)$.

Suppose that there is some $t \in T$ such that $t \notin Rx^{-n} + Tx$. Choose m minimal so that

$$t \in Rx^{-m} + xT$$

We know that such a value exists by above. Thus, for some $t_1 \in T$, $r_1 \in R$

$$t = r_1x^{-m} + xt_1$$

$$\implies x^m t = r_1 + x^{m+1}t_1$$

$$\implies x^m(t - xt_1) = r_1 \in I_m = I_{m+1}$$

since $m > n$. Now, for some $t_2 \in T$, $r_2 \in R$ we may write

$$x^m(t - xt_1) = x^{m+1}t_2 + xr_2$$

$$\implies t = x(t_2 + t_1) + r_2x^{-(m-1)} \in Rx^{-(m-1)} + xT$$

which contradicts the minimality of m.

Since $T \subseteq Rx^{-n} + xT$, $x^nt \in R + x^{n+1}T$, thus

$$T/xT \cong x^nT/x^{n+1}T \subseteq \frac{x^{n+1}T + R}{x^{n+1}T} \cong R/(x^{n+1}T \cap R)$$

This shows that T/xT is contained in a finitely generated module, therefore it must be finitely generated. $\square$

We are ready to prove the result that is necessary for the main theorem.

Proposition 6.19. *Let $(R, \mathfrak{m})$ be a local, Noetherian ring complete in the $\mathfrak{m}$-adic topology. Suppose further that R is unmixed and that $\dim R \geq 2$. Then $H^1_{\mathfrak{m}}(R)$ is finitely generated as an R-module.*

Proof. Let K be the total quotient ring of R. Consider the exact sequence

$$0 \longrightarrow R \longrightarrow K \longrightarrow K/R \longrightarrow 0$$

This leads to the exact sequence in Local Cohomology:

$$0 \longrightarrow H^0_{\mathfrak{m}}(R) \longrightarrow H^0_{\mathfrak{m}}(K) \longrightarrow H^0_{\mathfrak{m}}(K/R) \longrightarrow H^1_{\mathfrak{m}}(R) \longrightarrow H^1_{\mathfrak{m}}(K) \longrightarrow \ldots$$

We claim that $H^0(R) = H^0(K) = 0$. Since the depth$(R) > 0$, there exists a non-zero-divisor x in $\mathfrak{m}$. As $H^0_{\mathfrak{m}}(K)$ are all the elements killed by a power of $\mathfrak{m}$, if $y \in H^0_{\mathfrak{m}}(K)$ then for some r, $x^ry = 0 \implies y = 0$. Thus $H^0_{\mathfrak{m}}(K) = 0$, from which $H^0_{\mathfrak{m}}(R) = 0$ follows immediately.

Next, we claim that $H^1_{\mathfrak{m}}(K) = 0$. Since $x \in \mathfrak{m}$ is a non-zero-divisor, $\frac{1}{x} \in K$, so that $\mathfrak{m}K = K$. But by independence of base (Proposition 6.11).

$$H^1_{\mathfrak{m}}(K) \cong H^1_{\mathfrak{m}K}(K) \cong H^1_{(1)}(K) \cong 0$$

thus the claim holds.

Therefore, the first five terms long exact sequence reduce to an isomorphism $H^0_{\mathfrak{m}}(K/R) \cong H^1_{\mathfrak{m}}(R)$. Therefore it suffices to show that $H^0_{\mathfrak{m}}(K/R)$ is finitely generated.

It is easy to see that $H^0_{\mathfrak{m}}(K/R) = \{\frac{a}{b} \in K : \mathfrak{m}^n \cdot \frac{a}{b} \in R \text{ for some } n\}$ is the maximal ideal transform T of R. Thus, we need only to show that T is finitely generated as an R-module. However, by Proposition 6.18, T/xT is finitely generated. But, by 4.13, R is also complete with respect to the (x)-adic topology. It sufficed to show that $(x)^nT = 0$, as by Proposition 4.14 we have that T is finitely generated. Suppose that $(x)^nT \neq 0$. Then $\exists a \in R$, with $a \in \bigcap_n x^nT$. Thus, for all n, there exists $t_n \in T$ such that $a = x^nt_n$. But then for each n, there is some r_n with $\mathfrak{m}^{r_n} \cdot a \subseteq x^nR$, so that $\mathfrak{m}^{r_n} \subseteq (x^nR : a)$. Let $r \in (x^nR : a)$. Then we have

$$ra \in x^nR \implies ra \in x^nR \bigcap (a) = x^{n-k}(x^kR \cap (a)) = x^{nj-k}(ta)$$

By Artin- Rees. Now, we may write

$$(r - tx^{n-k})a = 0$$

so that

$$r - tx^{n-k} \in (0 : a) \implies r \in (0 : a) + x^{n-k}R$$

And we conclude

$$(x^nR : a) \subseteq (0 : a) + x^{n-k}R$$

Take $\mathfrak{p} \in Ass(R)$ with $(0 : a) \subseteq \mathfrak{p}$. Then we have

$$\mathfrak{m}^{r_n} \subseteq \mathfrak{p} + x^{n-k}R \implies \dim R/\mathfrak{p} = 1$$

However, this contradicts the unmixed hypothesis. We conclude that $H^1_{\mathfrak{m}}(R)$ is finitely generated as an R-module. $\square$

7. Unmixed Property and the Non-Cohen-Macaulay Locus

This section will be dedicated to illuminating the importance of the unmixed condition in the main theorem, and in particular that we can find a homomorphic image of a ring that can be used in the inductive argument of the main proof. Although every Cohen-Macaulay ring is unmixed, the converse is not true in general. One important aspect of the main theorem is that it provides a criterion for when an unmixed ring is Cohen-Macaulay. Recall the definition of unmixed for a local ring:

Definition 7.1. A local ring $(R, \mathfrak{m})$ **unmixed** if $\dim \hat{R}/\mathfrak{p} = \dim R$ for every $\mathfrak{p} \in \mathrm{Ass}(\hat{R})$, where $\hat{R}$ is the $\mathfrak{m}$-adic completion of R.

It has been shown in Section 4 that we may work largely in the complete case. We need to see what would prevent a complete local ring from being Cohen-Macaulay. The Cohen Structure Theorem (4.6) goes a long way in giving a comfortable way to examine the structure of such a ring. Recall that the structure theorem states that every complete Noetherian local ring is the homomorphic image of a complete Noetherian *regular* local ring. We would like to be able to characterize what happens when such a ring fails to be Cohen-Macaulay, and in particular, what we can say when an unmixed complete noetherian local ring fails to be Cohen-Macaulay. In a sense, we would like to pinpoint the "misbehaved" prime ideals in the ring at which localization does not produce a Cohen-Macaulay ring. We first analyze this by means of a projective resolution.

Proposition 7.2. *Let $R = S/L$ where S is a regular local ring, with ht $L = g$. Suppose further that R is unmixed. Then for $L \subseteq \mathfrak{q} \subseteq S$, $R_\mathfrak{q}$ is not Cohen-Macaulay $\iff$ $pd_S(R_\mathfrak{q}) > g$. In particular, if R has the following free resolution over S*

$$\ldots \longrightarrow F_{g+1} \xrightarrow{\phi_{g+1}} F_g \xrightarrow{\phi_g} F_{g-1} \longrightarrow \ldots \longrightarrow R \longrightarrow 0$$

Then (Im $\phi_g)_\mathfrak{q}$ is not free.

Proof. Suppose that $R_\mathfrak{q}$ is not Cohen-Macaulay. Since $R_\mathfrak{q} \cong (S/L)_\mathfrak{q} \cong S_\mathfrak{q}/L_\mathfrak{q}$, by the proposition (3.18) we may write

$$\mathrm{pd}(R_\mathfrak{q}) = \mathrm{pd}((S/L)_\mathfrak{q}) > \mathrm{grad}(L_\mathfrak{q}) \geq \mathrm{grad}(L)$$

Since S is Cohen-Macaulay, $\mathrm{grad}(L) = \mathrm{ht}\ (L) = g$. Then $\mathrm{pd}(R_\mathfrak{q}) > \mathrm{grad}(L_\mathfrak{q}) \geq g$, and in particular, $(\mathrm{Im}\ \phi_g)_\mathfrak{q}$ is not free.

Conversely, suppose that $\mathrm{pd}_S(R_\mathfrak{q}) > g$ so that $(\mathrm{Im}\ \phi_g)_\mathfrak{q})$ is not free. It suffices to show that ht $L_\mathfrak{q}$ = ht $L = g$, as then we have

$$\begin{aligned} \mathrm{pd}(R_\mathfrak{q}) &> g = \mathrm{ht}\ L_\mathfrak{q} \\ \dim\ (S_\mathfrak{q}) - \mathrm{depth}(R_\mathfrak{q}) &> \mathrm{ht}\ L_\mathfrak{q} \\ \dim\ (S_\mathfrak{q}) - \mathrm{ht}\ (L_\mathfrak{q}) &> \mathrm{depth}(R_\mathfrak{q}) \\ \dim\ (R_\mathfrak{q}) &> \mathrm{depth}(R_\mathfrak{q}) \end{aligned}$$

Therefore, $R_\mathfrak{q}$ is not Cohen-Macaulay. To show that ht $L_\mathfrak{q}$ = ht $L = g$, let $\mathfrak{p} \in \mathrm{Ass}(R)$. Then, since R is unmixed and complete, we have $\dim R/\mathfrak{p} = \dim R \implies \dim S/\mathfrak{p} = \dim R$. But then $\dim S/\mathfrak{p} = \dim R = \dim S - \mathrm{ht}\ L$ for all $\mathfrak{p}$ minimal over L. WHich implies that ht $\mathfrak{p}$ = ht L for all $\mathfrak{p}$ minimal over L. Thus, when we localize at $\mathfrak{q}$, the height of L remains unchanged, as there exists some prime $\mathfrak{p} \subseteq \mathfrak{q}$ minimal over L whose hight is g. Therefore, ht $L_\mathfrak{q}$ = ht $L = g$. □

Now that we know what happens in a free resolution upon localization at a "bad" prime, we seek to be able to say something specific about this prime. For brevity, let's restrict our focus to a finite presentation of the image of the map at the gth step.

Proposition 7.3. *Let S be a domain, and C a torsion-free S-module. Suppose that the rank of C is r, that is $\dim_K(C \otimes K) = r$, where K is the quotient field of S. Suppose that C is finitely presented:*

$$S^n \xrightarrow{\psi} S^t \xrightarrow{\phi} C \longrightarrow 0$$

where ϕ is given by a matrix with elements in S. Let $I_{t-r}(\psi)$ be the ideal generated by the $t - r$ minors of ψ. Then for a prime ideal $\mathfrak{q}$, $C_\mathfrak{q}$ is a free module $\iff$ $I_{t-r}(\psi) \not\subseteq \mathfrak{q}$.

Proof. Suppose that $C_\mathfrak{q}$ is a free module. Then for some $r > 0$, $C_\mathfrak{q} \cong S_\mathfrak{q}^r$, and since localization is flat, the presentation becomes

$$S_\mathfrak{q}^n \xrightarrow{\psi_\mathfrak{q}} S_\mathfrak{q}^t \longrightarrow S_\mathfrak{q}^r \longrightarrow 0$$

where $\phi_\mathfrak{q}$ is now a matrix with elements in $S_\mathfrak{q}$. Let L be the image of $\phi_\mathfrak{q}$, then $S_\mathfrak{q}^t = L \oplus V$ for some module V tensoring with $S_\mathfrak{q}/\mathfrak{q}S_\mathfrak{q}$ gives

$$k^n \xrightarrow{\phi_\mathfrak{q}} \frac{S_\mathfrak{q}^t}{\mathfrak{q}S_\mathfrak{q}} = \frac{L \oplus V}{\mathfrak{q}L \oplus \mathfrak{q}V} = k^t \longrightarrow k^r \longrightarrow 0$$

We conclude that the image of $\psi_\mathfrak{q}$ is rank $t - r$. Thus $\psi_\mathfrak{q}$ is given by a matrix with $t-r$ linearly independent rows. Hence, $\psi_\mathfrak{q}$ has a submatrix of size $t - r$ which is invertible, so there must be some $t - r$ minor which lies outside of $\mathfrak{q}$. Hence, $I_{t-r}(\psi) \not\subseteq \mathfrak{q}$.

Now suppose that $I_{t-r}(\psi) \not\subseteq \mathfrak{q}$, and localize the presentation at $\mathfrak{q}$. This means there exist some $t - r$ matrix that is invertible, and therefore at least $t-r$ linearly independent columns. Without loss of generality, say that the matrix has columns $\gamma_1, \ldots, \gamma_t$, where the first $t - r$ columns are linearly independent. Say that $L = \langle \gamma_1, \ldots, \gamma_{t-r} \rangle$ that is, the image of these columns, which is of rank $t - r$. Extend this to a basis for $S_\mathfrak{q}^t$ and with V being generated by the r elements necessary to extend to the basis, so that

$$S_\mathfrak{q}^t = L \oplus V$$

Note that since S^t is a free module over a local ring, V is necessarily free. Set $T = \text{Im } \psi_\mathfrak{q}$. Then, since $L \subseteq T$, we get

$$T = S_\mathfrak{q}^t \cap T = (L \cap T) \oplus (V \cap T) = L \oplus V'$$

For some V'. Let J be the nonzero elements of $S_\mathfrak{q}$. Then upon inverting the elements of J we have

$$T_J = L_J \oplus V'_J$$

Where T_J, L_J, and V'_J are all vector spaces. But now both T_J and L_J have dimension $t - r$, which means $V'_J = 0 \implies V' = 0$. Thus $L = T$. So that

$$C_\mathfrak{q} \cong \frac{S_\mathfrak{q}^t}{\text{im } \psi_\mathfrak{q}} = \frac{L \oplus V}{L} \cong V$$

However, we have already concluded that V is free, thus $C_\mathfrak{q}$ is free as well. $\square$

Note that in our case, since S is a regular local ring, it is a domain, and since the Im ϕ_g is a submodule of a free module, it is torsion free. Thus, returning to the general case where $R = S/L$, with ht $L = g$, we see that if R has the free resolution

$$\ldots \longrightarrow F_{g+1} \xrightarrow{\phi_{g+1}} F_g \xrightarrow{\phi_g} F_{g-1} \longrightarrow \ldots \longrightarrow R \longrightarrow 0$$

Then R_Q is not Cohen-Macaulay $\iff$ $(\text{Im } \phi_g)_Q$ is not free $\iff$ $Q \supseteq I(\phi_{g+1})$, Where $I(\phi_{g+1})$ is the ideal of minors for ϕ_{g+1}. We refer to the collection of primes that contain the image of this ideal in R as the **Non-Cohen-Macaulay Locus**. In the main proof we will use induction, and, as we want to be able to work in depth and dimension one less than the given ring, we don't need to avoid every prime containing the non-Cohen-Macaulay locus, but only those that are grade 1. However, if a ring is unmixed, this is possible to do. First, one needs the following proposition.

Proposition 7.4. *For a complete, unmixed ring R, there are only finitely many primes $\mathfrak{p}$ such that depth$(R_\mathfrak{p}) = 1$ but ht $\mathfrak{p} > 1$.*

Proof. Since R is complete, every $\mathfrak{p} \subseteq R$ for which $R_\mathfrak{p}$ is not Cohen-Macaulay contains the non-Cohen-Macaulay locus, i.e. some fixed $J \subseteq R$. Now, since R is unmixed, for any $\mathfrak{q} \in \text{Ass}(R)$ we have $\dim(R/\mathfrak{q}) = \dim R$, thus, J is not contained in any associated prime of R.

Suppose we have $\mathfrak{p}$ such that depth$(R_\mathfrak{p}) = 1$ but ht $\mathfrak{p} > 1$. Then $R_\mathfrak{p}$ is not Cohen-Macaulay , which forces $J \subseteq \mathfrak{p}$. Let $x \in J$ be a non-zero divisor. We know that such an x exists, otherwise J would be contained in an associated prime. Then $\mathfrak{p} \in \text{Ass}(R/xR)$, since in $R_\mathfrak{p}$, the image of x forms a maximal regular sequence. As R/xR as only finitely many associated primes, there are only finitely many such $\mathfrak{p}$. $\square$

By way of superficial elements, we may find a regular element that avoids these primes that can be used in our inductive argument in the main proof. Specifically, we want to find a regular element R so that associated primes of R/xR are minimal, except for possibly the maximal ideal.

Remark 7.5. *If we include primes of the form $\{\mathfrak{p} : depth(R_{\mathfrak{p}}) = 1, ht\ \mathfrak{p} > 1\}$ in the list of primes we avoid when locating our superficial element x (See the proof of Proposition 5.3, equation $\sharp$), then x is a non-zero divisor, and*

$$Ass(R/xR) = \{\mathfrak{p} : x \in \mathfrak{p}, ht\ \mathfrak{p} = 1\} \cup \{\mathfrak{m}\}$$

Proposition 7.6. *For a superficial element x that meets the criteria of the remark, Set $A = R/xR$. Then we can create a homomorphic image of A with the same dimension that is unmixed. Namely, if we set $H = H^0_{\mathfrak{m}A}(A)$. Then A/H is unmixed, and $\dim A/H = \dim A$.*

Proof. We have $\dim A = \dim R-1$, and for all $\mathfrak{p} \in \operatorname{Ass}(A)$ where $\mathfrak{p} \neq \mathfrak{m}$, we have $\dim (A/\mathfrak{p}) = \dim R-1 = \dim A$. We conclude that if the maximal ideal was not an associated prime of A, then A would be unmixed. Although it is not possible to remove $\mathfrak{m}$ from the list without altering A, we can create a homomorphic image of A for which $\mathfrak{m}$ is not an associated prime. Set $H = H^0_{\mathfrak{m}A}(A)$, so that $H = \{a \in A : \mathfrak{m}^n Aa = 0$ for some $n\}$. Then, H is an ideal in A, and $\mathfrak{m}^n \cdot A \neq 0$ for any n, so that H is a proper ideal of A. Therefore, $\operatorname{Ass}(A/H) = \{\mathfrak{p} \subseteq R : x \in \mathfrak{p}, \operatorname{ht}_R \mathfrak{p} = 1\}$. Hence, A/H is unmixed, and furthermore, $\dim A/H = \dim A$. □

Proposition 7.7. *Let $(R, \mathfrak{m})$ a local ring of dimension $d \geq 3$, I a parameter ideal, x a superficial element that is a non-zero divisor, part of a minimal generating set for I and avoids finitely many primes as in Remark 7.5. Let $A = R/xR$, and $H = H^0_{\mathfrak{m}A}(A)$. If $e_1(I) = 0$, then $e_1(\overline{I}) = 0$, where $\overline{I}$ is the image of I in A/H.*

Proof. We have already shown (Proposition 5.4) that for a superficial element $x \in I$, $e_1(I) = 0 \implies e_1(\tilde{I}) = 0$ where $\tilde{I}$ is the image of I in $R/xR = A$. Now we note

$$\lambda\left(\frac{A}{\tilde{I}^{n+1}}\right) = \lambda\left(\frac{(A/H)}{\overline{I}^{n+1}}\right) + \lambda\left(\frac{H, \overline{I}^{n+1}}{\overline{I}^{n+1}}\right)$$

However,

$$\frac{(H, \overline{I}^{n+1})}{\overline{I}^{n+1}} = \frac{H}{(H \cap \overline{I}^{n+1})} = \frac{H}{\overline{I}^{n+1-k}(\overline{I}^k \cap H)}$$

With the last equality by Artin-Rees. If we choose n large, since H is everything killed by a power of the maximal ideal, H will be annihilated by I^{n-k}. Thus

$$\frac{H}{\overline{I}^{n+1-k}(\overline{I}^k \cap H)} = H$$

Therefore

$$\lambda\left(\frac{A}{\tilde{I}^{n+1}}\right) = \lambda\left(\frac{(A/H)}{\overline{I}^{n+1}}\right) + \lambda(H)$$

and

$$e_0(\tilde{I})\binom{n+d-1}{d-1} - e_1(\tilde{I})\binom{n+d-2}{d-2} + \cdots = e_0(\overline{I})\binom{n+d-1}{d-1} - e_1(\overline{I})\binom{n+d-2}{d-2} + \cdots + \lambda(H)$$

So that for $d \geq 3$, $e_1(\tilde{I}) = 0 \implies e_1(\overline{I}) = 0$. □

In the case that our ring A is one dimensional, we can say a little more about the relationship between $e_1(I)$ and the length of $H^0_{\mathfrak{m}A}(A)$.

Proposition 7.8. *Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension 1. Set $H = H^0_{\mathfrak{m}A}(A)$. Then for a parameter ideal $I = (z) \subseteq A$,*

$$e_1(I) = -\lambda(H)$$

Proof. We have

$$\lambda\left(\frac{A}{(z)^{n+1}}\right) = e_0(I)(n+1) - e_1(I)$$

Consider the chain of ideals

$$(0:\mathfrak{m}) \subseteq (0:\mathfrak{m}^2) \subseteq \ldots \subseteq (0:\mathfrak{m}^r) \subseteq \ldots$$

Since A is noetherian, this chain stabilizes, say at r, and we conclude $H = (0:\mathfrak{m}^r)$. Now consider the chain

$$(0:z) \subseteq (0:z^2) \subseteq \ldots \subseteq (0:z^t)\ldots$$

This chain stabilizes as well, that is, there exists t such that $(0:z^t) = (0:z^{t+1})$. Since $\mathfrak{m}^s \subseteq (z)$ for some s, these chains are cofinal and we may write $H = (0:z^t)$. Next we note $(z^{n+1}) \subseteq (H, z^{n+1})$, so for some $n >> 0$,

$$\lambda\left(\frac{A}{(z^{n+1})}\right) = \lambda\left(\frac{A}{(H, z^{n+1})}\right) + \lambda\left(\frac{(H, z^{n+1})}{(z^{n+1})}\right)$$

Now we see that

$$\frac{(H, z^{n+1})}{(z^{n+1})} = \frac{H}{(z^{n+1}\cap H)} = \frac{H}{(z^{n-k}((z^k \cap H))}$$

With the last equality following from Artin-Rees. If we choose n large, H will annihilate z^{n-k} so that $(z^{n-k}(z^k \cap H)) = 0$. Therefore may write

$$\lambda\left(\frac{A}{(z^{n+1})}\right) = \lambda\left(\frac{A}{(H, z^{n+1})}\right) + \lambda(H) = \lambda\left(\frac{A/H}{(\overline{z}^{n+1})}\right) + \lambda(H)$$

Where $\overline{z}$ is the image of z in A/H. However, since A/H is unmixed (Proposition 7.6) and dimension 1, it is Cohen-Macaulay. Therefore, $\overline{z}$ is a regular element and by Proposition 3.23, $gr_{(z)}(A/H) \cong \frac{A/H}{(\overline{z})}[T]$ so that

$$\lambda\left(\frac{(A/H)}{(\overline{z}^{n+1})}\right) = \lambda\left(\frac{A/H}{(\overline{z})}\right)(n+1)$$

Thus, altogether, we have

$$\lambda\left(\frac{A}{z^{n+1}}\right) = e_0(I)(n+1) - e_1(I) = \lambda\left(\frac{A/H}{(\overline{z})}\right)(n+1) + \lambda(H)$$

We conclude that

$$e_0(I) = \lambda\left(\frac{A/H}{(\overline{z})}\right)$$

And thus

$$e_1(I) = -\lambda(H)$$

$\square$

8. The Main Theorem

The tools to prove the main theorem are now in place. It is time to prove the main conjecture.

Theorem 8.1. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d, and $I \subseteq R$ be a parameter ideal. Then R is Cohen-Macaulay if and only if R is unmixed and $e_1(I) = 0$.*

Proof. Throughout the proof, we may assume without loss of generality that R is complete with respect to the $\mathfrak{m}$-adic topology, as $\hat{R}_{\mathfrak{m}}$ is Cohen-Macaulay if and only if R is Cohen-Macaulay, and $e_i(I) = e_i(I\hat{R}_{\mathfrak{m}})$ for all i (Proposition 4.7). We may also assume that R contains an infinite residue field, as, if it doesn't then $R \subseteq R[x]_{\mathfrak{m}R[x]}$ is flat extension, and $R[x]_{\mathfrak{m}R[x]}$ has an infinite residue field.

First we present the easier implication. Assume that R is Cohen-Macaulay. Then R is necessarily unmixed. Induct on dimension. First assume that dimension of R is $d = 0$. Then $I = 0$ and

$$\lambda(R/I^n) = \lambda(R) = e_0(I)\binom{n}{0} = e_0(I) = \lambda(R/I) = \lambda(R)$$

Which of course implies that $e_1(I) = 0$.

Suppose the claim holds up to dimension $d-1$. Then for dim $R = d$, let $x \in I$ be a regular element, that is superficial and part of generating set for I (This is possible by Proposition 5.3). Let $n >> 0$ and consider the exact sequence

$$0 \longrightarrow R/I^n \xrightarrow{\cdot x} R/I^{n+1} \longrightarrow R/(x, I^{n+1}) \longrightarrow 0$$

It is clear that R/xR is Cohen-Macaulay and dimension $d-1$. Let $\overline{R} = R/xR$ and $\overline{I}$ denotes the image of I in $\overline{R}$, So that $R/(x, I^{n+1}) \cong \overline{R}/\overline{I}^{n+1}$. Then, by additivity of length we have

$$(\star) \qquad \lambda(R/I^{n+1}) - \lambda(R/I^n) = \lambda(\overline{R}/\overline{I}^{n+1})$$

Write

$$\lambda(R/I^{n+1}) = e_0(I)\binom{n+d}{d} - e_1(I)\binom{n+d-1}{d-1} + \text{ lower terms}$$
$$\lambda(R/I^{n}) = e_0(I)\binom{n+d-1}{d} - e_1(I)\binom{n+d-2}{d-2} + \text{ lower terms}$$

Invoke Pascal's Recurrence (Fact 2.10) so that the difference is written as

$$\lambda(R/I^{n+1}) - \lambda(R/I^n) = e_0(I)\binom{n+d-1}{d-1} - e_1(I)\binom{n+d-2}{d-2} + \text{ lower terms}$$

On the other hand

$$\lambda(\overline{R}/\overline{I}^n) = e_0(\overline{I})\binom{n+d-1}{d-1} - e_1(\overline{I})\binom{n+d-2}{d-2} + \text{lower terms}$$

As R and $\overline{R}$ are both Cohen-Macaulay , and $R/I \cong \overline{R}/\overline{I}$, we conclude $e_0(I) = \lambda(R/I) = \lambda(\overline{R}/\overline{I}) = e_0(\overline{I})$ by Proposition 3.24. Thus we may write the equation $(\star)$ as

$$\lambda(R/I)\binom{n+d-1}{d-1} - e_1(I)\binom{n+d-2}{d-2} + \text{ lower terms} = \lambda(R/I)\binom{n+d-1}{d-1} - e_1(\overline{I})\binom{n+d-2}{d-2} + \text{ lower terms}$$

Subtracting $\lambda(R/I)\binom{n+d-1}{d-1}$ from either side of the equation leaves us with

$$-e_1(I)\binom{n+d-2}{d-2} + \text{ lower terms} = -e_1(\overline{I})\binom{n+d-2}{d-2} + \text{ lower terms}$$

As these are both polynomials in n, we may equate the leading terms:

$$\frac{e_1(I)n^{d-2}}{(d-2)!} = \frac{e_1(\overline{I})n^{d-2}}{(d-2)!}$$

However, by induction $e_1(\overline{I}) = 0$, thus $e_1(I)$ is also 0.

For the other implication, suppose that R is unmixed, $I \subseteq R$ is a parameter ideal, and $e_1(I) = 0$. Induct on the dimension. For the base case, let $d = 0$, so that $I = (0)$ and

$$\lambda(R/I^{n+1}) = \lambda(R/I) = \lambda(R) = e_0(I)\binom{n+0}{0} = e_0(I)$$

Since $\lambda(R/I) = e_0(I)$, R must be Cohen-Macaulay.

Suppose that the dimension $d = 1$. Since R is unmixed, the maximal ideal is not an associated prime. Therefore $\operatorname{depth}(R) > 0$, so R is Cohen-Macaulay.

Let $d = 2$. Then since R is unmixed, $\operatorname{depth}(R) > 0$. Let $I = (x, z)$ where x is a superficial element satisfying the properties of Proposition 7.5, in particular, x is a non-zero-divisor. Set $A = R/xR$, and $\tilde{I}$ to be the image of I in A. We have $e_1(I) = e_1(\tilde{I}) = 0$ by Proposition 5.4. It suffices to show that A is Cohen-Macaulay. Set $H = H^0_{\mathfrak{m}A}(A)$. We have by Proposition 7.8, since $e_1(\tilde{I}) = -\lambda(H)$, we have $\lambda(H) = 0$. But then $A/H = A$ is one-dimensional and unmixed, so it is Cohen-Macaulay, hence R must also be Cohen-Macaulay.

Assume the theorem holds up to dimension $d-1$. Let x be a superficial element in I which is a non-zerodivisor and for which $\operatorname{Ass}(R/xR) = \{\mathfrak{p} \subseteq R : x \in \mathfrak{p}, \operatorname{ht} \mathfrak{p} = 1\} \cup \{\mathfrak{m}\}$. We have shown that it is possible to do so (Proposition 7.5), as R has an infinite residue field, and since R is unmixed, $\operatorname{depth} R > 0$.

Set $A = R/xR$, and $H = H^0_{\mathfrak{m}A}(A)$, so that A/H is unmixed (Proposition 7.6). Then $\dim A/H = d-1$, and $e_1(\overline{I}) = 0$ (Proposition 7.7) so that, by induction A/H is Cohen-Macaulay. It suffices to show that A is Cohen-Macaulay. In order to do so, we show that $H^i_{\mathfrak{m}A}(A) = 0$ for all $i < d-1$, which forces $\operatorname{depth}(A) = d-1$, since $\dim A = d-1$, and depth of A can be at most the dimension of A.

To show this, first consider the short exact sequence

$$0 \longrightarrow H \longrightarrow A \longrightarrow A/H \longrightarrow 0$$

This leads to the long exact sequence of local cohomology (Proposition 6.10).

$$0 \longrightarrow H^0_{\mathfrak{m}A}(H) \longrightarrow H^0_{\mathfrak{m}A}(A) \longrightarrow H^0_{\mathfrak{m}A}(A/H) \longrightarrow H^1_{\mathfrak{m}A}(H) \longrightarrow \dots$$

Since A/H is Cohen-Macaulay and dimension $d-1$, $H^i_{\mathfrak{m}A}(A/H) = 0$ for all $i < d-1$. (Proposition 6.14). Furthermore, we have shown $H^i_{\mathfrak{m}A}(H) = H^i_{\mathfrak{m}A}(H^0_{\mathfrak{m}A}(A)) = 0$ for all $i > 0$ (Proposition 6.15). Thus we have $H^i_{\mathfrak{m}A}(A) = 0$ for all $0 < i < d-1$. All that remains is to prove that $H^0_{\mathfrak{m}A}(A) = 0$.

To do so, consider the exact sequence

$$0 \longrightarrow R \xrightarrow{\cdot x} R \longrightarrow A \longrightarrow 0$$

Which leads to the long exact sequence

$$0 \longrightarrow H^0_{\mathfrak{m}}(R) \xrightarrow{\cdot x} H^0_{\mathfrak{m}}(R) \longrightarrow H^0_{\mathfrak{m}}(A) \longrightarrow H^1_{\mathfrak{m}}(R) \xrightarrow{\cdot x} H^1_{\mathfrak{m}}(R) \longrightarrow H^1_{\mathfrak{m}}(A) \longrightarrow \dots$$

By our prior analysis, we have concluded that $H^i_{\mathfrak{m}}(A) = 0$ for $0 < i < d-1$, so that for every $1 < i < d-1$ we have

$$0 \longrightarrow H^i_{\mathfrak{m}}(R) \xrightarrow{\cdot x} H^i_{\mathfrak{m}}(R) \longrightarrow 0$$

At the beginning of the sequence, all that remains is

$$((\star)) \qquad 0 \longrightarrow H^0_{\mathfrak{m}}(R) \xrightarrow{\cdot x} H^0_{\mathfrak{m}}(R) \longrightarrow H^0_{\mathfrak{m}}(A) \longrightarrow H^1_{\mathfrak{m}}(R) \xrightarrow{\cdot x} H^1_{\mathfrak{m}}(R) \longrightarrow 0$$

and at the dth level,

$$(\star\star) \qquad 0 \longrightarrow H^{d-1}_{\mathfrak{m}}(R) \xrightarrow{\cdot x} H^{d-1}_{\mathfrak{m}}(R) \longrightarrow H^{d-1}_{\mathfrak{m}}(A) \longrightarrow H^d_{\mathfrak{m}}(R) \xrightarrow{\cdot x} H^d_{\mathfrak{m}}(R) \longrightarrow 0$$

We conclude that every map in the long exact sequence which is defined by multiplication by x is injective, except for the map from $H^1_{\mathfrak{m}}(R)$ to itself (in $\star$) and $H^d_{\mathfrak{m}}(R)$ to itself (in $\star\star$).

In the maps where multiplication by x is injective, so is multiplication by a power of x. Let $z \in H^i_{\mathfrak{m}}(R)$, then $xz \in H^i_{\mathfrak{m}}(R)$. By definition, xz is killed by a power of $\mathfrak{m}$. But $\mathfrak{m}^n z = 0 \implies x^n z = 0$. However, multiplication by x^n is also injective, so it must be the case that $z = 0$. We find that each $H^i_{\mathfrak{m}}(R) = 0$ except when $i = 1$ and $i = d$. The non-vanishing parts of our long exact sequence are

$$0 \longrightarrow H^0_{\mathfrak{m}}(A) \longrightarrow H^1_{\mathfrak{m}}(R) \xrightarrow{\cdot x} H^1_{\mathfrak{m}}(R) \longrightarrow 0$$

and

$$0 \longrightarrow H^{d-1}_{\mathfrak{m}}(A) \longrightarrow H^d_{\mathfrak{m}}(R) \xrightarrow{\cdot x} H^d_{\mathfrak{m}}(R) \longrightarrow 0$$

It has already been shown that $H^1_{\mathfrak{m}}(R)$ is finitely generated (Proposition 6.19). Since multiplication by x is surjective, $H^1_{\mathfrak{m}}(R) = xH^1_{\mathfrak{m}}(R) \implies H^1_{\mathfrak{m}}(R) = 0$ by Nakayama's Lemma. Therefore $H^0_{\mathfrak{m}}(A) = 0$. We conclude that for all $i < d-1$, $H^i_{\mathfrak{m}}(A) = 0$. Therefore, $\operatorname{depth}(A) = d - 1 = \dim A \implies A$ is a Cohen-Macaulay ring. However, since x is a non-zerodivisor and $A = R/xR$ is Cohen-Macaulay, R must also be Cohen-Macaulay.

$\square$

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