5.1 Eigenvectors and Eigenvalues

Learning Objectives:

- Determine if a vector is an eigenvector for a matrix
- Find the eigenspace for given eigenvalues
- Relate distinct eigenvalues to linear independence

We will study certain simple ways that matrices "act on" vectors. From an alternative perspective, certain vectors for which the transformation $\mathbf{x} \to A\mathbf{x}$ is fairly simple.

Example 1. Let $A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$. Find the images of $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$ under multiplication by A. Compare the vectors to their images.

Solution:

Definition 1. An ________ of an ________ matrix ____ is a ________ vector ____ such that ________ for some ________ ____. A scalar ____ is called an ________ of ____ if there is a nontrivial solution of ____ of ____. Such an ____ is called an ________________

Example 2. Let $A = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$. Determine if $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ are eigenvectors for A. If so, find their corresponding eigenvalues.

Solution:

Example 3. Show that 5 is an eigenvalue of the matrix $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$, then find the corresponding eigenvectors.

Solution:

Warning: Although we can use ______ to find ______ if we know ______, we can not, in general, ______ a matrix in order to find its ______. We wll outline a strategy for this in the next section.

Remark 1. Looking back at the last example, we see that ______ is an ______ for ______ if and only if the equation

has a ______ solution. Any such ______ will be an ______ that corresponds to ______.

Definition 2. . The set of all ______ that correspond to ______ is called the ______ of ______ corresponding to ______. Since it is the ______ of ______, it is a ______ of ______.

Example 4. Find a basis for the eigenspace corresponding to $\lambda = 3$ of the matrix $A = \begin{bmatrix} 4 & 2 & -1 \\ 2 & 7 & -2 \\ -1 & -2 & 4 \end{bmatrix}$.

Then describe the eigenspace.

Solution:

Exercise 1.

1. Is $\lambda = 2$ an eigenvalue of $\begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix}$? Why or why not?

2. Is $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ an eigenvector for $\begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 1 \\ -3 & 4 & 5 \end{bmatrix}$? If so, find its eigenvalue.

3. Find a basis for the eigenspace of A corresponding to $\lambda = 3$ if $A = \begin{bmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{bmatrix}$.

Now that we know the basic computational tools regarding eigenvectors and eigenvalues, we shall start to expand the theory. We begin with a situation where eigenvalue are easily obtained.

Theorem 1. The ____ of a ____ matrix are the ____ on the ____.

Proof. $\square$

Example 5. Find the eigenvalues of the following matrices.

$$A = \begin{bmatrix} 5 & 7 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} 4 & 0 & 0 \\ 5 & -7 & 0 \\ -5 & 2 & 9 \end{bmatrix}.$$

Solution:

Remark 2. If ____ is an ____ for a matrix ____, it means that ____ has a nontrivial solution, which means the ____ equation ____ has a nontrivial solution. That is, ____ is an ____ for ____ if and only if ____ is not ____! In this case, the ____ for ____ is the same as ____.

To finish this section, we present a theorem relating eigenvalues to linear independence of eigenvectors which will be used later.

Theorem 2. If ______ are ______ that correspond to ______ ______ ______ of an ______ matrix ______, then the set ______ is ______

Proof.

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Exercise 2. 1. Find the eigenvalues of the matrix $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 5 \\ 0 & 0 & -1 \end{bmatrix}$.

2. Find the eigenvalues of the matrix $\begin{bmatrix} 4 & 0 & 0 \\ 7 & 0 & 0 \\ 1 & 0 & -3 \end{bmatrix}$

3. Find one eigenvalue of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$ without any computation. Explain your answer.

4. Construct a 3×3 matrix with only two distinct eigenvalues.