3/9/2020 # Dihedral Groups Name: ____________________

Last class, we saw that S_3, the symmetric group on 3 letters, has elements

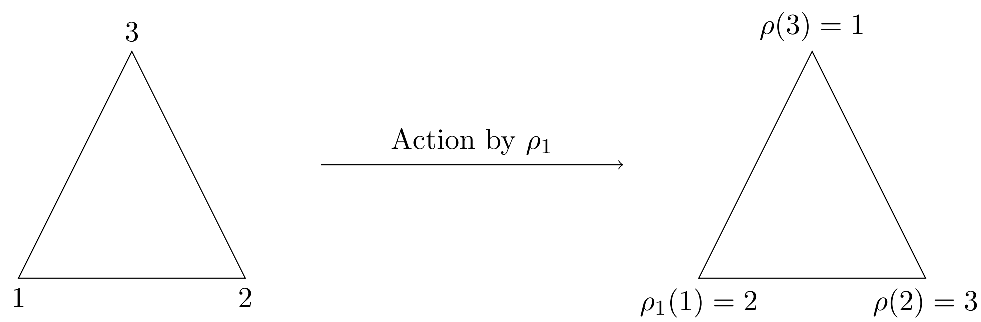
$$\rho_0 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \qquad \mu_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

$$\rho_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \qquad \mu_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

$$\rho_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \qquad \mu_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

We also noted that this group is NOT abelian, since $\rho_1\mu_1 = \mu_3$, but $\mu_1\rho_1 = \mu_2$. In fact S_3 is the smallest nonabelian group.

You may be wondering why we labeled the elements of the group in the way that we did. It is because the group S_3 can alternatively be viewed as the dihedral group D_3, the **group of symmetries of an equilateral triangle**! In this context, it is possible to interpret D_3 as "acting on" the vertices of the triangle, resulting in familiar geometric transformations.

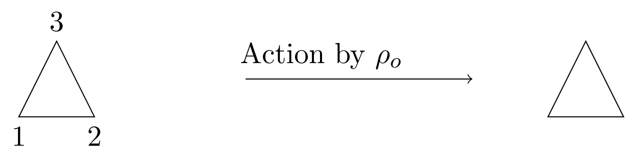


What familiar geometric transformation is this? Why does the use of the letter ρ ('r' in Greek) make sense?

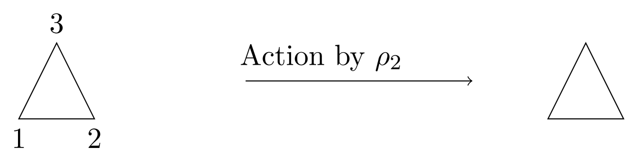
$$\rho_0 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \qquad \rho_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \qquad \rho_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

$$\mu_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \qquad \mu_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \qquad \mu_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

Compute the remaining transformations, and describe them in words.



Decription:



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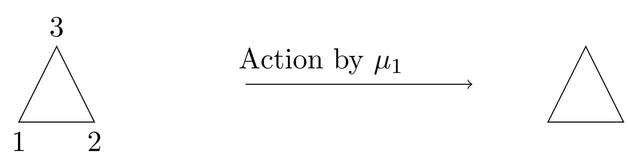

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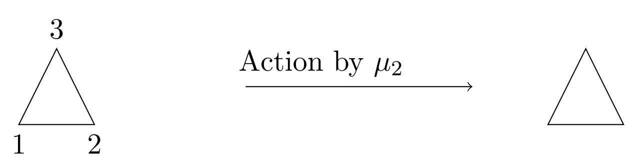

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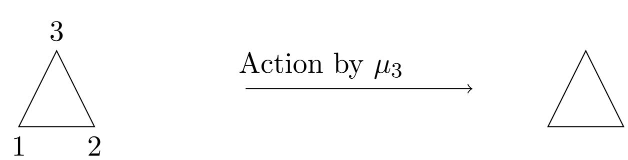
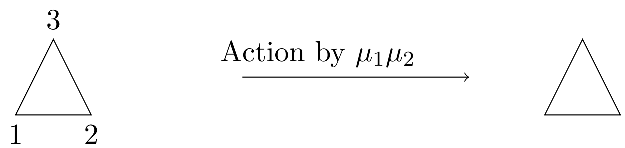
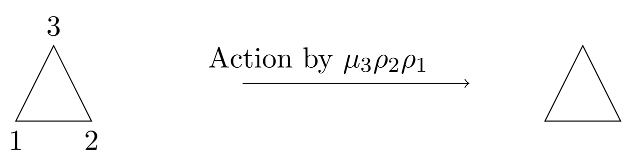


Description:

Compute the following actions, then check yourself algebraically. Remember, permutation multiplication is read from "right to left".



$\mu_1\mu_2 =$



$\mu_3\rho_2\rho_1 =$

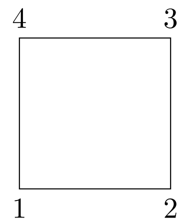
$$\rho_0 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \qquad \rho_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \qquad \rho_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

$$\mu_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \qquad \mu_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \qquad \mu_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

To complete our study of $S_3 = D_3$, find all of the subgroups of S_3 and draw its subgroup diagram.

The Dihedral Group D_4

Now that we have covered D_3 in laborious detail, let's consider the dihedral group D_4. This is the group of symmetries of an equilateral 4-sided polygon, that is, a square.



List the elements of D_4 below. Drawing pictures may help. Hint: D_4 has 8 elements in total.

The elements of D_4 have a similar standard labeling to D_3. There are four rotations, two mirror images, and two diagonal flips. The standard labeling is

$$\rho_0 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} \qquad \rho_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \qquad \rho_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} \qquad \rho_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$$

$$\mu_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \qquad \mu_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} \qquad \delta_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix} \qquad \delta_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}$$

The elements ρ_i are rotations, μ_i are mirror images, and δ_i are diagonal flips.

In the space below, find all of the **cyclic** subgroups of D_4. Thinking about the geometry will make this considerably less work than direct computation.

Below are some questions about D_4. Answer as specifically as you can with proper justification.

1. Is D_4 abelian? Why or why not?

2. Before we have found that the symmetries of the triangle D_3 is the same group as the symmetric group on 3 letters, S_3. Is $D_4 = S_4$? How can you tell?

3. Is the set $\{\rho_0, \rho_2, \delta_1\}$ a subgroup of D_4? If not, what elements must be added to the set to form a group?

4. Now that you know the order of D_3 and D_4, make a conjecture about the order of D_N, the symmetries of the N sided polygon.

5. There are two proper subgroups of D_4 that aren't cyclic. They are $\{\rho_0, \rho_2, \mu_1, \mu_2\}$ and $\{\rho_0, \rho_2, \delta_1, \delta_2\}$. Use this information and your earlier findings to draw a subgroup diagram for D_4.