Quasi-Polynomial Growth of Betti Numbers Over Local Rings

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Let M be a finitely generated R-module. We will examine the growth of the Betti numbers $\beta_i^R(M) := \dim_k \operatorname{Ext}_R^i(M, k)$ of finitely generated modules over complete intersection rings.

Background

Theorem (Tate '57)

Let R be a complete intersection ring of codimension c and embedding dimension e. Then the Poincaré series of the residue field is given by

$$P_k^R(t) := \sum_{i \geq 0} \beta_i^R(k) t^i = \frac{(1+t)^e}{(1-t^2)^c} = \frac{(1+t)^{e-c}}{(1-t)^c}$$

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In particular, there exists a polynomial $b_k(n) \in \mathbb{Q}[n]$ of degree $c-1$ such that $\beta_n^R(k) = b_k(n)$ for large values of n.

Theorem (Gulliksen '74, Avramov '89)

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Let R be a complete intersection of codimension c and M be a finitely generated R-module. Then we can write

$$P_M^R(t) = \frac{p(t)}{(1-t^2)^c} = \frac{h(t)}{(1-t)^d(1+t)^b}$$

for some nonnegative integer $d \leq c$. polynomial $h(t)$ with $h(1) \neq 0$ and $b < d$.

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Equivalently, there exists two integer valued polynomials b_+^M, b_-^M with $\deg b_+^M = \deg b_-^M < d$ such that for $i \gg 0$

$$\beta_i^M(t) = \begin{cases} b_+^M(i) & i \text{ is even} \\ b_-^M(i) & i \text{ is odd.} \end{cases}$$

Further, b_-^M and b_+^M have the same leading term. The order of the pole at $t = -1$ determines how many of the terms (from leading term down) b_-^M and b_+^M have in common.

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- (Avramov '94) If R is a complete intersection of minimal multiplicity ($= 2^c$) then $b_-^M = b_+^M$ for all finitely generated modules M.
- (Avramov, Seceleanu, Yang '16) Let R be is a graded complete intersection of codimension c. Then $b_-^M = b_+^M$ for all finitely generated M if and only if at least $c-1$ of the minimal generators of the defining ideal are quadrics.

Quadratic Codimension

Let $(R, \mathfrak{m}, k)$ be a local ring, and $\hat{R} \cong Q/I$ a minimal Cohen presentation of R, so that $(Q, \mathfrak{n}, k)$ is a regular local ring. Set $I_2^\square$ to be the image of $I \subseteq \mathfrak{n}^2/\mathfrak{n}^3$, and let $I^\square \subseteq \operatorname{gr}_\mathfrak{n} Q$ be the ideal generated by $I_2^\square$.

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Definition

The **quadratic codimension** q of a local ring R is the height of the ideal $I^{\square}$ defined above.

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Example

Let $R = k[x, y]/(xy, x^2 + y^3)$. Then $I^{\square} = (\overline{xy}, \overline{x}^2)$ which has height one in $gr_{\mathfrak{n}} Q = k[\overline{x}, \overline{y}]$. Thus, for R, $c = 2$ and $q = 1$.

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In the graded setting, the quadratic codimension of a complete intersection ring measures the number of generators in a minimal generating set of I which are quadric.

The Main Result

Theorem (Avramov, P. , Walker)

Let $(R, \mathfrak{m}, k)$ be a complete intersection of codimension c and quadratic codimension q. Then for each finitely generated R-module M, we can write

$$P_M^R(t) = \frac{g_M(t)}{(1-t)^c(1+t)^b}$$

where $g_M(t) \in \mathbb{Z}[t]$, and $b = \max\{0, c-q-1\}$.

This puts an upper bound on the degree of the discrepancy between b_-^M and b_+^M, that is, $\deg b_-^M - b_+^M \leq c - q - 1$.

Homotopy Lie Algebras of Local Rings

Recall that $\mathrm{Ext}_R^*(M, k)$ is a (left) module over the associative algebra $\mathrm{Ext}_R^*(k, k)$ via the composition product. We can associate a graded Lie algebra over k to R, called its homotopy Lie algebra.

Homotopy Lie Algebras of Local Rings

Recall that $\operatorname{Ext}_R^*(M,k)$ is a (left) module over the associative algebra $\operatorname{Ext}_R^*(k,k)$ via the composition product. We can associate a graded Lie algebra over k to R, called its homotopy Lie algebra.

Theorem (Milnor-Moore '65, André '71, Sjödin '80)

For a local ring R, there exists a graded Lie algebra $\pi^*(R)$ with the following properties:

- $\dim_k \pi^i(R) = \varepsilon_i(R)$ *for each* $i \in \mathbb{Z}$.
- *There is an isomorphism of associative algebras*

$$U(\pi^*(R)) \cong Ext_R^*(k,k)$$

where $U(\pi^*(R))$ is the universal enveloping algebra of $\pi^*(R)$.

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Theorem (Sjödin '76)

Let $(R,\mathfrak{m},k)$ be a complete intersection of codimension c and embedding dimension e. Let $\hat{R} \cong Q/I$, with $\mathfrak{n} = (x_1,\dots,x_e)$, $I = (f_1,\dots,f_c)$ and write $f_h = \sum_{i,j} a_{ij}^h x_i x_j$ with $a_{ij}^h \in Q$.

- $\dim_k \pi^i(R) = \begin{cases} e & i = 1 \\ c & i = 2 \\ 0 & i \neq 1,2 \end{cases}$.
- *There are k bases $\xi_1,\dots,\xi_e$ and $\chi_1,\dots,\chi_c$ such that*

$$[\xi_i,\xi_j] = -\sum_{h=1}^{c} \overline{a}_{ij}^h \chi_h, \quad \xi^{[2]} = -\sum_{h=1}^{c} \overline{a}_{ii}^h \chi_h$$

Where $\overline{a}_{ij}^h$ is the image of a_{ij}^h in k.

The Special Subalgebra

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Theorem (Avramov, P., Walker)

Let R be a complete intersection of codimension c and quadratic codimension q. Then there is a Lie sub-algebra of $\lambda^* \subseteq \pi^*(R)$ with the following properties:

- $\dim_k \lambda^1 = q$
- $\lambda^i = \pi^i(R)$ *for* $i \neq 1$.
- $Ext_R^*(k, k)$ *is free of finite rank as a left* $U(\lambda^*)$*-module.*
- $U(\lambda^*)$ *has finite global dimension.*

Idea of the Proof

Theorem (Avramov, P. , Walker)

Let $(R, \mathfrak{m}, k)$ be a complete intersection of codimension c and quadratic codimension q. Then for each finitely generated R-module, we can write

$$P_M^R(t) = \frac{g_M(t)}{(1-t)^c(1+t)^b}$$

where $g_M(t) \in \mathbb{Z}[t]$, and $b = \max\{0, c-q-1\}$

Sketch of the Proof:
Set $\mathcal{L} := U(\lambda^*)$, Then $\mathcal{M} = \operatorname{Ext}_R^*(M, k)$ is a finite $\mathcal{L}$ module, and there is a finite free resolution

$$0 \leftarrow \operatorname{Ext}_R^*(M, k) \leftarrow \mathcal{F}_0 \leftarrow \cdots \leftarrow \mathcal{F}_r \leftarrow 0$$

Therefore

$$\begin{aligned} P_M^R(t) = H_{\mathcal{M}}(t) &= \sum_{i=0}^{r} (-1)^i H_{\mathcal{F}_i}(t) \\ &= \sum_{i=0}^{r} (-1)^i (\text{rank } \mathcal{F}_i) t^i H_{\mathcal{L}}(t) \\ &= h_{\mathcal{M}}(t) H_{\mathcal{L}}(t) \end{aligned}$$

For some polynomial $h_{\mathcal{M}}(t) \in \mathbb{Z}[t]$.

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For some polynomial $h_{\mathcal{M}}(t) \in \mathbb{Z}[t]$. Now, by the graded version of the Poincaré Birkhoff-Witt Theorem,

$$H_{\mathcal{L}}(t) = \frac{(1+t)^q}{(1-t^2)^c} = \frac{1}{(1-t)^c(1+t)^{c-q}}$$

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If $c = q$ was previously known (Avramov '94). If $c > q$, then one can reduce to case where R is Artinian, induce on the length of M, and use Tate's theorem for $P_k^R(t)$ to show that $h_{\mathcal{M}}(-1) = 0$. Hence, one may write

$$P_M^R(t) = \frac{g_M(t)}{(1-t)^c(1+t)^{c-q-1}}$$

with $g_M(t) \in \mathbb{Z}[t]$.

□

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$$P_M^R(t) = \frac{1+3t}{(1-t)^3(1+t)}.$$

For $n > 3$, one has

$$\beta_n(M) = \begin{cases} n^2 + \frac{5}{2}n + \frac{3}{2} & n \text{ is even} \\ n^2 + \frac{5}{2}n + 1 & n \text{ is odd} \end{cases}.$$

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