October 23, 2020 MAT 370: Exam #2β Name: ____________________

Instructions: Read the entire statement of each problem. Solve each problem carefully and organize your work. Be sure to write your answers in complete sentences where appropriate. The exam is worth 60 points.

Part One: True or False. Decide whether the following statements are true or false. You do not need to justify your answer.

1. (2 Points) There is an infinite cyclic group with 2 distinct generators.

2. (2 Points) The group $\mathbb{Q}$ under $+$ is cyclic.

3. (2 Points) There are two non-isomorphic groups of order 3.

4. (2 Points) $\langle \mathbb{R}^{+}, \cdot \rangle$ is a subgroup of $\langle \mathbb{C}, \cdot \rangle$.

5. (2 Points) Every finitely generated group is abelian.

Part Two: Short Answer and Computation Problems

1. Let $G = \mathbb{Z}_{45}$.

   (a) (4 Points) Find the subgroups of G and draw a subgroup diagram.

   (b) (3 Points) Find the subgroup $\langle 9, 20 \rangle$ in $\mathbb{Z}_{45}$.

2. (5 Points) Suppose that $G = \{e, a, b, c, d\}$ is an abelian group with e the identity element for $*$. Fill in the gaps in the table below.

$*$	e	a	b	c	d
e					
a		c		d	
b		e			
c			a		e
d		b	c		

3. (6 Points) Let $G = \langle a, b, c \,|\, ac = b^{-1}, ab = ba \rangle$. Simplify the following words in G as much as possible:

(a) $acbc^4$

(b) $ab^{-2}cbc^{-2}a^{-2}b$

4. (12 Points) Determine, with justification, if the following binary structures are groups.

   (a) $\langle \mathbb{R}^*, \star \rangle$ where $a \star b = a^2 b$.

   (b) $\langle \mathbb{Z}[i], \cdot \rangle$

   (c) The set $\{0, \pm 3, \pm 9, \pm 27, \pm 81, \ldots, \pm 3^n, \ldots\}$ under multiplication.

Part Three: Proofs. For the following three problems, pick **only two** of them to submit for a grade. You may attempt all three problems, but indicate which two you'd like to submit. If you do not clearly indicate which two you select, only the first two shall be graded.

1. (10 Points) Let $\langle G, * \rangle$ be a group and $x \in G$. Show that there is exactly one element x^{-1} such that $x^{-1} * x = e$.

2. (10 Points) Let G be a group and fix an element $x \in G$. Set $C = \{y \in G \,|\, xy = yx\}$. Show that C is a subgroup of G.

3. (10 Points) Prove that $\langle \mathbb{C}^*, \star \rangle$ is a group if $a \star b = 10ab$.