May 10, 2021 MAT 370: Exam #4 Name: ____________________

Instructions: Read the entire statement of each problem. Solve each problem carefully and organize your work. Be sure to write your answers in complete sentences where appropriate. The exam is worth 60 points.

Part One: True or False. Decide whether the following statements are true or false. You do not need to justify your answer.

1. (2 Points) For any group G, there is a subgroup $H \leq G$ with $|H| = 2$.

2. (2 Points) Right coset multiplication is well-defined for any normal subgroup H of a group G.

3. (2 Points) If $H \leq G$ and G is finite, then $(G : H) = |G|/|H|$.

4. (2 Points) Every coset of a group G is a subgroup of G.

5. (2 Points) Every subgroup of an abelian group is normal.

Part Two: Short Answer and Computation Problems

1. (3 Points) Find the order of the element $(14, 30, 30)$ in the group $\mathbb{Z}_{21} \times \mathbb{Z}_{60} \times \mathbb{Z}_{90}$. Explain your reasoning.

2. (3 Points) Find all left cosets of the subgroup $\langle 20 \rangle$ in $\mathbb{Z}_{25}$.

3. (3 Points) Find every abelian group of order 450 up to isomorphism.

4. (6 Points) Let $\phi : \mathbb{Z}_9 \to \mathbb{Z}_3$ be a homomorphism of groups.

   (a) According to Lagrange's Theorem, what are the possible orders for a subgroup of $\mathbb{Z}_9$?

   (b) If ϕ is onto, what is the order of $\ker \phi$?

5. (7 Points) Let $\varphi : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ be a group homomorphism and $\varphi(1,0) = 6$ and $\varphi(0,1) = -2$.

   (a) Find $\varphi(-5,7)$.

   (b) Find the kernel of φ. [Hint: the kernel is cyclic.]

6. (4 Points) There is a homomorphism $\varphi : \mathbb{Z} \to S_7$ with $\varphi(1) = (1, 4, 6, 7)(2, 3, 5)$ Find $\varphi(32)$.

7. (4 Points) Set $G = \mathbb{Z}$ and $H = \langle 9, 15 \rangle$.

   (a) What are the cosets of $\langle 9, 15 \rangle$?

   (b) What familiar group is G/H isomorphic to? Explain.

Part Three: Proofs Choose two of the following theorems and prove them. If you do not clearly select which proofs are to be graded, I will only grade the first two.

1. Let $\varphi : G_1 \to G_2$ be a homomorphism of groups.

    (a) Prove that if φ is one-to-one, then $\ker \varphi$ is the trivial subgroup.

    (b) Prove that if $\ker \varphi$ is the trivial subgroup, then φ is one-to-one.

2. Let G be a group with identity element e and suppose $|G| = n$. Show that $a^n = e$ for any element $a \in G$.

3. Let R be a group and $P \leq R$. Show that if $rpr^{-1} = p$ for every $r \in R$ and $p \in P$, then P is a normal subgroup. **Note: R may not be abelian.**