1 Linear Equations in Linear Algebra

1.1 Systems of Linear Equations

Learning Objectives:

- Identify linear equations and linear systems
- Use matrices and row operations to solve linear systems
- Identify if a system is consistent and if solutions are unique

As is suggested in the name, linear algebra is a branch of mathematics concerning linear equations and linear functions. For the beginning of this course, we will first learn some useful techniques for representing systems of linear equations and investigate the possible scenarios that can occur in finding solutions to such systems.

Definition 1. A $\boxed{}$ in the variables $x_1, x_2, \ldots, x_n$ is an equation that can be written in the form

$$\boxed{}$$

where b and the $\boxed{}$ $a_1, a_2, \ldots, a_n$ are real numbers. The subscript n can be any positive integer.

Example 1. An example of a linear equation is

$$3x_1 + 2x_2 = 6$$

Another example is

$$4(x_1 - x_4) = \sqrt{2}x_2 + 1$$

since it can be written in the form:

$$\boxed{}$$

Here, $a_1 = \boxed{}$, $a_2 = \boxed{}$, $a_3 = \boxed{}$, $a_4 = \boxed{}$ and $b = \boxed{}$.
Some examples of equations that are **not** linear are

$$x_1x_2 = 5 \quad \text{and} \quad x_1^2 + 2x_2 = 1$$

since they cannot be rewritten in the form above.

Note 1. You are familiar with the linear equation

$$y = mx + b$$

However, renaming the variables x and y to x_1 and x_2 respectively, we get

and rearranging gives

so, in two variables, the familiar definition of a linear equation is equivalent to this "new" definition with $a_1 = \boxed{}$ and $a_2 = \boxed{}$.

Definition 2. A ______ is a collection of one or more linear equations using the same variables, say $x_1, \ldots, x_n$.

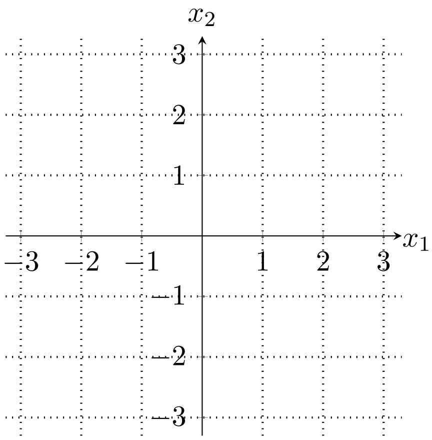
Example 2. An example of a system of linear equations is

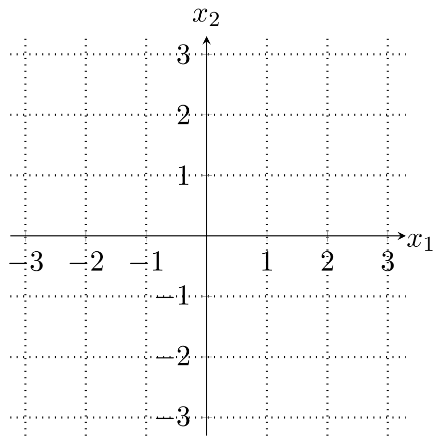
Definition 3. A ______ of a system is a list $(s_1, \ldots, s_n)$ of numbers that makes each equation a true statement when the values $s_1, \ldots, s_n$ are substituted for the variables $x_1, \ldots, x_n$ respectively. The set of all solutions is called the ______ of the linear system. Two systems are called ______ if they have the exact same solution set.

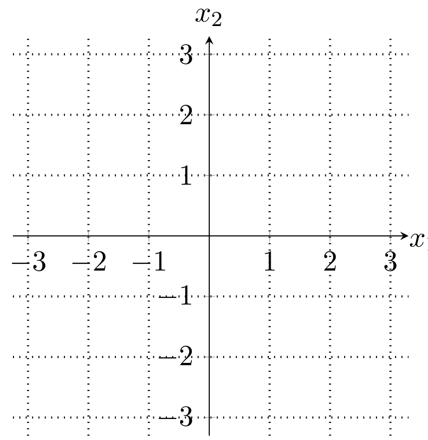
Example 3. In the system above, ______ is a solution since

In two variables, we can represent linear systems graphically, by making the identification $(x, y) = (x_1, x_2)$.

$$\begin{aligned} x_1 - x_2 &= 1 \\ x_1 - 2x_2 &= 1 \end{aligned} \qquad \begin{aligned} x_1 - x_2 &= 1 \\ x_1 - x_2 &= 0 \end{aligned} \qquad \begin{aligned} x_1 - x_2 &= 1 \\ -2x_1 + 2x_2 &= -2 \end{aligned}$$







Fact 1. A system of linear equations either has

1. or
2. or
3. .

Definition 4. A system of linear equations is said to be if it has either one solution or infinitely many solutions. If a system has no solutions it is .

Matrix Notation

The crucial information in a linear system can be represented succinctly using a rectangular array of real numbers called a .

Given the linear system

$$\begin{array}{rcrcrcr} 3x_1 & - & 2x_2 & + & 3x_3 & = & 4 \\ x_1 & & & + & x_3 & = & \pi \\ & & 2x_2 & - & \sqrt{2}x_3 & = & 100 \end{array}$$

We can record the the coefficients of each variable aligned in columns in the ______.

We can also include the values on the right hand side of the equation in the ______.

Often, a (dotted) line is drawn in to separate the coefficients from the other values.
The ______ of a matrix refers to how many rows and columns it has. A matrix with ______ rows and ______ columns is called a ______ matrix (read "3 by 4 matrix"). Thus, by an ______ matrix, we mean a rectangular array of real numbers with m rows and n columns. The utility of matrices for our current purposes will be to simplify the solving of systems of equations. Whenever one solves a system of equations, the solution is produced by replacing the system with an ______ system that is easier to solve. Instead of doing all our calculations by working with the equations themselves, we can instead write the system as an augmented matrix and work with the ______ of this matrix, then translate the matrix back to a system of equations. However, we can only perform operations that don't change the solution of the system.
The three allowable operations are:

- ______
- ______
- ______

Example 4. Here, we will solve the system below by working both with the equations and with the augmented matrix side-by-side.

$$\begin{aligned} x_1 &+ x_2 &= 7 \\ -2x_1 &+ x_2 &= -2 \end{aligned}$$

Manipulating the equations using the three allowable options corresponds to the following operations on rows of the augmented matrix associated to the system.

Definition 5. The that can be performed on a matrix are

1.
2.
3.

Two matrices are said to be if one of the matrices can be turned into the other one be a sequence of elementary row operations.

Note that each elementary row operation is . Further, since the solutions to a system are unchanged via row operations, we can conclude that

Existence and Uniqueness

Given a linear system, we know that there are three possibilities as to its solution set: it can have , , or solutions. Thus, there are two fundamental questions we can ask about a system:

1.
2.

These two questions will appear frequently throughout the course, in many different situations.

Exercise 1. Determine if the system below is consistent:

$$\begin{array}{rcrcr} 3x_1 & - & x_2 & = & 10 \\ x_1 & - & 2x_2 & = & 0 \end{array}$$

Exercise 2. Determine if the following system is consistent:

$$\begin{array}{rcrcrcr} & & x_2 & - & 4x_3 & = & 8 \\ 2x_1 & - & 3x_2 & + & 2x_3 & = & 1 \\ 4x_1 & - & 8x_2 & + & 12x_3 & = & 1 \end{array}$$