QUASI-POLYNOMIAL GROWTH OF BETTI SEQUENCES OVER COMPLETE INTERSECTION RINGS

by

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A DISSERTATION

Presented to the Faculty of

The Graduate College at the University of Nebraska

In Partial Fulfilment of Requirements

For the Degree of Doctor of Philosophy

Major: Mathematics

Under the Supervision of Professors Luchezar L. Avramov and Mark E. Walker

Lincoln, Nebraska

July, 2019

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This thesis is separated into 3 chapters. Chapter 1 is an introduction which provides standard definitions and notations that shall be used throughout the remainder of the document. In particular, it outlines the classical notions of complete intersection rings and Betti-sequences, as well as summarizes some related classical results and recent previous work done towards the main result of the thesis.

Chapter two formalizes the definition of an invariant, called the quadratic codimension of a commutative ring, which has appeared in the literature previously under various notations, but not this nomenclature. Here we investigate the behavior of this invariant and collect several results relating the invariant to other properties of commutative rings.

In Chapter three, homological methods are introduced, and the homological perspective through which the proof of the main result will take place is presented. In particular, we discuss the functorality of the homotopy Lie algebra $\pi(R)$ of a commutative ring and the maps induced on these objects by homomorphisms of local rings. The chapter contains a proof of the existence of a graded subalgbera $\mathcal{L}$ of the Yoneda algebra $\operatorname{Ext}_R^*(k,k)$ in a local commutative ring with finite global dimension and with the graded piece $\mathcal{L}^1$ of dimension related to quadratic codimension. The chapter closes with an application to Betti sequences over complete intersection rings; a new bound on the maximum degree of the difference of the polynomials governing the even and odd subsequences of the Betti sequence for every finitely generated module is presented.

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DEDICATION

To my parents, my friends, and my cats.

ACKNOWLEDGMENTS

I would first and foremost like to extend my extreme gratitude to my advisors, Professor Luchezar Avramov and Mark Walker. Lucho's insight and teaching has been an inspiration to me, and he has taught me more than I could have imagined. Mark has been incredibly supportive, responsive, patient, and an excellent teacher throughout my studies. Our joint meetings have been thrilling, informative, fun, and inspirational. I hope to provide for future students as they have provided for me.

I would further like to thank the members of my supervisory committee, Tom Marley and Alexandra Seceleanu, and Elizabeth Theiss-Morse.

The commutative algebra faculty at UNL has been incredibly supportive and helpful. Throughout my years they have organized a multitude of lecture classes, reading courses, seminars, and social gatherings, without which would have been a much less stimulating experience. I would in particular like to thank Roger Weigand for interesting conversations and excellent social events, and Alessandro De Stefani for advice and guidance.

I would like to extend thanks to all of the graduate students at UNL, especially those in commutative algebra for being good friends and colleagues. I would specifically like to thank Josh Pollitz, Luigi Ferraro, Seth Lindokken, and Mohsen Gheibi with whom it was enjoyable to talk to about mathematics. Thank you also to the regular CARS attendees for listening to many of my presentations and giving me advice and feedback.

I extend my deepest gratitude to the commutative algebra community as a whole for everything they've done to inspire me to succeed. From all the excellent conferences I had the joy of attending, to the guidance and support structure they've provided, and opportunities to learn that have been presented to me. In particular, I'd like to thank Daniel Katz for influencing me to study commutative algebra in the first place, and Craig Huneke for his excellent teaching and support during my time at University of Kansas.

My friends and family have given me a lot of support over the last few years, and without them I don't think this thesis would have come to fruition. I'd like to specifically thank my great aunt Anna Poulos for believing in me, my friend Lucas Underwood for his uncanny ability to keep it positive , and my partner Caroline Matson for her love, insight and advice. I'd also like to thank my trivia team for helping me relax and have fun at least once a week.

Lastly, I would like to thank the Department of Mathematics at the University of Nebraska-Lincoln as a whole for creating such a supportive environment in which to learn and grow. In particular, I'd like to thank Allan Donsig and Susan Hermiller for their advice in teaching.

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Chapter 1

Introduction

A ring, such as the integers, real numbers, functions, or the set of $n \times n$ matrices, is a form of algebraic structure that are ubiquitous across mathematics. These algebraic objects are sets in which one can add, subtract and multiply, but not necessarily divide the elements. In general, multiplication may not be commutative such as in the set of matrices. We call rings in which order of multiplication can be interchanged *commutative rings* and the study of such rings and the sets on which they act is called *commutative algebra*.

There is a hierarchy which classifies how well-behaved commutative rings are. The nicest of such are called fields, and these are precisely the commutative rings in which one can divide by every nonzero element. There are many different ways in which a ring can be well-behaved, and these properties can often be classified via different criteria. One way to study a ring is to look at the *ideals* of the ring; namely the subsets of the ring which are subgroups and absorb multiplication by ring elements. A main class of interest in commutative in commutative algebra are *Noetherian* rings, which are the rings for which every ideal is generated by a finite number of elements. This means one can find a finite set of elements in the ring in which every element of the ideal is a linear combination of these elements with coefficients in the ambient ring. Another important class of rings are *local*, this means that the ring has a unique proper ideal that is maximal with respect to set inclusion. This thesis is primarily about commutative Noetherian local rings.

Another way to study rings is, as opposed to observing the structure of the ring itself, to study the algebraic objects upon which it admits an action. These are called *modules* over the ring, and they themselves are closed under addition and scaling by elements of the ring. This is a ring-theoretic analog of a vector space over a field, as is studied in classical linear algebra. However, unlike in linear algebra, not every module over a ring admits a basis. However, there are a certain class of modules that *do* admit bases; these are called *free modules*. In fact, another way to classify whether or not a ring is a field is to classify its free modules. Namely, if every module over a commutative ring admits a basis, then that ring is necessarily a field.

If the ring admits a module M that is not free, we can measure how far away it is from being so. To accomplish this, we can map a free module surjectively to M by choosing a set of generators for M. This map will necessarily admit a non-zero kernel, called the first *syzygy* of M. If this syzygy is also not free, we may repeat the process, and the resulting kernel is called the second syzygy of M. We can iterated this process, (and in the case of finite generation, do so minimally) which may terminate or continue ad infinitum, resulting in a *free resolution* of M. This method of approximating an arbitrary module by free modules was introduced by Hilbert. In the 1890s he proved that for a polynomial ring in finitely many variables the process will always terminate; a result which is now referred to as the Hilbert Syzygy Theorem.

The types of (minimal) free resolutions that a ring admits can be used to classify the ring itself. In modern language, a local ring is called *regular* if every minimal free resolution of a finitely generated module eventually terminates. Examples of such rings are fields and power series rings over a field. If there is a single module for which the ring admits an infinite minimal free resolution, then the ring cannot be regular. However, one can study the types of infinite resolutions that may occur. One method of studying infinite free resolutions of finitely generated modules is to observe the sequence of the ranks of the free modules that appear in the resolution, called the *Betti sequence* of a module. Specifically, one can study the growth of the Betti sequences of modules over a ring. If every Betti sequence displays polynomial growth, then the ring is called a *complete intersection*, which is in someways the most well-behaved class beyond regular rings. In particular, the completion of a complete intersection ring is isomorphic to a regular ring factored out by a regular sequence of elements, called its *Cohen presentation*.

This thesis is about relating the types of growth over complete intersection rings to the types of elements which appear in its Cohen presentation. Although the growth of every Betti sequence of a module over a complete intersection ring is on the order of a polynomial, it is, in general, eventually a quasi-polynomial of period two. This means that, after a certain point, there is a specific polynomial which gives the sequence of even entries in the Betti sequence and another polynomial governing the odd entries. This thesis will put a global bound on how similar these two polynomials are forced to be based upon an invariant of the complete intersection, which is related to the elements in the Cohen presentation.

1.1 Preliminaries

The goal of this thesis will be to analyze the types of growth of Betti sequences that can occur for every module over specific classes of local or graded rings. We begin by setting notation and recalling some standard definitions which will be relevant to our cause. By $(R, \mathfrak{m}, k)$ we denote a local commutative Noetherian ring with unique maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$. Recall that the $\mathfrak{m}$-adic completion of R, denoted $\widehat{R}$, is defined by $\varprojlim R/\mathfrak{m}^i$.

For a field k, a k-vector space V is *graded* if there is a decomposition $V = \bigoplus_{i\in\mathbb{Z}} V_i$ where each V_i is a vector space over k. If V is a graded k-vector space such that $\dim_k V_i < \infty$ for each i and $V_i = 0$ for $i \ll 0$, we denote its *Hilbert Series* by $H_V(t) = \sum_{i\in\mathbb{Z}} \dim_k V_i t^i \in \mathbb{Z}[\![t]\!][t^{-1}]$. A k-algebra A is called *standard graded* over k if $A = \bigoplus_{i=0}^{\infty} A_i$, $A_iA_j \subseteq A_{i+j}$, $k = A_0$,

$\dim_k A_i < \infty$ and A is generated by A_1 as a k-algebra. The *homogeneous maximal ideal* of a standard graded k algebra A is $A_+ = \bigoplus_{i>0} A_i$.

We will refer to the following invariants of local or graded rings frequently throughout the thesis.

Definition 1.1.1. Let R be a local (or graded) ring with (homogenous) maximal ideal $\mathfrak{m}$, and set $k = R/\mathfrak{m}$. The **embedding dimension** of R, denoted $\operatorname{edim} R$, is the minimal number of generators of $\mathfrak{m}$, or, equivalently, $\operatorname{edim} R := \dim_k \mathfrak{m}/\mathfrak{m}^2$. The **codimension** of R, denoted $\operatorname{cdim} R$, is defined as $\dim_k \mathfrak{m}/\mathfrak{m}^2 - \dim R = \operatorname{edim} R - \dim R$ where $\dim R$ is the Krull dimension of R.

It is often useful to view these invariants as the size of ideals in an ambient regular local ring of which R is a quotient, which we now make precise.

Definition 1.1.2. Let $(R, \mathfrak{m}, k)$ be a local ring. A **minimal Cohen presentation** of R is an isomorphism $\widehat{R} \cong Q/I$ where $(Q, \mathfrak{n}, k)$ is a regular local ring and $I \subseteq \mathfrak{n}^2$.

Minimal Cohen presentations exist by the Cohen Structure Theorem (see, for example, [8, Sec. 17.4]). This presentation is minimal in the sense that Q is a regular local ring with smallest possible embedding dimension for which $\widehat{R}$ can be a quotient. A local ring R and its completion $\widehat{R}$ have the same Krull dimension, embedding dimension and codimension. Thus, in particular, the condition $I \subseteq \mathfrak{n}^2$ guarantees that $\operatorname{edim} R = \dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim_k \mathfrak{n}/\mathfrak{n}^2 = \operatorname{edim} Q = \dim Q$. The last equality is because Q is regular. Further we have $\operatorname{cdim} R = \dim Q - \dim R$.

1.2 Complete Intersections

Let R be a local or graded ring. Recall that for an R-module M, $f_1, \ldots, f_r \in R$ form a *M-regular sequence* (or a *regular sequence on M*) if f_1 is a nonzero divisor on M, f_i is a nonzero divsor on $M/(f_1, \ldots, f_{i-1})M$ for $2 \leq i \leq c$, and $(f_1, \ldots, f_c)M \neq M$.

Definition 1.2.1. A local ring $(R, \mathfrak{m}, k)$ is a **complete intersection** if $\widehat{R} \cong Q/I$, where Q is a regular local ring and I is generated by a regular sequence in Q.

Definition 1.2.2. A standard graded k algebra A is called a **graded complete intersection** if $A \cong P/I$, where P a polynomial ring in $\dim_k A_1$ many elements and I is generated by a homogeneous regular sequence of elements in P.

In a local or graded complete intersection, the defining ideal is minimally generated by a regular sequence of length equal to its codimension.

1.3 Associated Graded objects

We can pass from the local to the graded setting using the following construction.

Definition 1.3.1. Given a ring R and an ideal $I \subseteq R$, the **associated graded ring** of R with respect to I is the graded ring $\mathrm{gr}_I R := \bigoplus_{n \geq 0} I^i/I^{i+1}$. If $(R, \mathfrak{m}, k)$ is local, we denote $\mathrm{gr}_{\mathfrak{m}} R$ simply as $\mathrm{gr}\ R$.

Note that if $\overline{x} \in I^i/I^{i+i}$ and $\overline{y} \in I^j/I^{j+1}$, we may lift these elements to $x \in I^i$ and $y \in I^j$ so that $xy \in I^{i+j}$, and define $\overline{x} \cdot \overline{y} \in I^{i+j}/I^{i+j+1}$ to be the image of xy which is well-defined modulo I^{i+j+1}.

The ring $\mathrm{gr}\ R$ is a standard graded k-algebra. Note that as $\mathfrak{m}^i/\mathfrak{m}^{i+1} \cong \widehat{\mathfrak{m}}^i/\widehat{\mathfrak{m}}^{i+1}$, there is a k-algebra isomorphism $\mathrm{gr}\ R \cong \mathrm{gr}\ \widehat{R}$.

Definition 1.3.2. Let $f \in R$ be a nonzero element and let $n = \sup\{i : f \in \mathfrak{m}^i\}$. Then the **inital form** of f, denoted f^*, is defined to be the image of f in $\mathfrak{m}^n/\mathfrak{m}^{n+1}$ which is nonzero. By convention, we define 0^* to be the zero element of $\mathrm{gr}\ R$. Note that f^* is a homogeneous element of $\mathrm{gr}\ R$. By the initial ideal of an ideal I, we mean $I^* := (f^* : f \in I)$, which is a homogeneous ideal in $\mathrm{gr}\ R$.

1.4 Free Resolutions and Betti Numbers

Definition 1.4.1. Let $(R, \mathfrak{m}, k)$ be a regular ring and M a finitely generated R module. A **free resolution** of M over R is a complex

$$F_\bullet : \cdots \to F_i \xrightarrow{\partial_i} \cdots \to F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \to 0$$

where each F_i is a finitely generated free module over R, $H_0(F_\bullet) \cong M$, and $H_i(F_\bullet) = 0$ for $i > 0$. In addition, $F_\bullet$ is called **minimal** if $\partial_i(F_i) \subseteq \mathfrak{m}F_{i-1}$ for each $i \geq 1$.

Upon choosing a basis for each F_i and viewing each ∂_i as a matrix with entries in R, we have $F_\bullet$ is minimal if and only if the entries in each ∂_i are elements of $\mathfrak{m}$, and minimal free resolutions are unique up to isomorphismof complexes. Note that, for a finitely generated module M, $\operatorname{Ext}^i_R(M, k) := H^i(\operatorname{Hom}_R(F_\bullet, k))$ is finite dimensional vector space over k.

Definition 1.4.2. For a finitely generated R-module M, the i**th Betti number** of M over R is $\beta_i^R(M) := \dim_k \operatorname{Ext}^i_R(M, k)$. The **Poincaré series** $P_M^R(t)$ of M over R is the generating function of the sequence of Betti numbers; that is

$$P_M^R(t) := \sum_{i \geq 0} \beta_i^R(M) t^i.$$

If $F_\bullet$ is a minimal free resolution of M, then $\beta_i^R(M) = \operatorname{rank}_R F_i$. Thus, since $\widehat{R}$ is flat over R, if $F_\bullet$ is a minimal free resolution of M, then $\widehat{F_\bullet} := F_\bullet \otimes_R \widehat{R}$ is a minimal free resolution of $\widehat{M}$ over $\widehat{R}$, and $\beta_i^R(M) = \beta_i^{\widehat{R}}(\widehat{M})$.

1.5 Betti Sequences

By the process of adjoining variables, one can form an R-free resolution of k (and in fact, any finite generated R-module) that is a DG-algebra over R, (see, for instance, [5, Chapter 6]).

When $(R, \mathfrak{m}, k)$ is a local complete intersection ring, Tate [17] uses the process to construct a minimal free resolution k over R and proves the following.

Theorem 1.5.1. *[17, Thm. 6] Suppose R is a local complete intersection ring of embedding dimension e and codimension c. Then*

$$P_k^R(t) = \frac{(1+t)^e}{(1-t^2)^c} = \frac{(1+t)^{e-c}}{(1-t)^c}.$$

An application of the method of partial fraction decomposition on the above formula shows that there is a polynomial $b^k(n) \in \mathbb{Q}[n]$ such that $\beta_n^R(k) = b^k(n)$ for $n \gg 0$. Gulliksen extended and modified the result for finitely generated modules, we present a special case of [10, Cor. 4.1].

Theorem 1.5.2. *[10, Cor. 4.1] Let R be a complete intersection ring of codimension c, and M a finitely generated R-module. Then there exists $h_M(t) \in \mathbb{Z}[t]$ such that*

$$P_M^R(t) = \frac{h_M(t)}{(1-t^2)^c}.$$

Once again using partial fraction decomposition, one finds that there are polynomials $p_+^M, p_-^M \in \mathbb{Q}[x]$ such that for $n \gg 0$

$$\beta_n^R(M) = \begin{cases} p_+^M(n) & n \text{ is even} \\ p_-^M(n) & n \text{ is odd.} \end{cases}$$

Thus, the Betti numbers of any finitely generated M over a complete intersection ring display quasi-polynomial growth. Notice that in Tate's result, we see that the Betti numbers of k are eventually given by a polynomial, and one can reduce the denominator of the rational Poincaré series to eliminate all factors of $1+t$. There is a relationship between the order of the pole of at $t = 1$ and the similarity of p_-^M and p_+^M, which is made precise in the following proposition:

Proposition 1.5.3. *Let R be a local ring and suppose that, for some finitely generated R-module M, we have*

$$P_M^R(t) = \frac{h(t)}{(1-t)^c(1+t)^b}$$

with $b \leq c$ and $h(t) \in \mathbb{Q}(t)$. Then $\deg(p_+^M - p_-^M) < b$*. Here,* $\deg(0) = -\infty$.

Proof. If $c = 0$, then $b = 0$, and $p_-^M = p_+^M = 0$ and the statement holds. Assume $c > 0$. Using the method of partial fraction decomposition, we have

$$P_M^R(t) = \frac{\ell(t)}{(1-t)^c} + \frac{m(t)}{(1+t)^b}$$

Now, for large n, the nth term of the power series representation of $\frac{1}{(1-t)^c}$ has coefficient $(n+1)(n+2)\dots(n+c-1)/(c-1)!$, a polynomial in n of degree $c-1$. Multiplying by $\ell(t)$ sums finitely shifted copies of this power series, hence for large n the coefficient of t^n in $\frac{\ell(t)}{(1-t)^c}$ is also a polynomial $g_M(n)$ in n of degree at most $c-1$.

Now, if $b = 0$, then $p_-^M = p_+^M = g_M$ and the statement holds. Suppose $b \geq 1$. Then the coefficient of the nth term of the power series expansion of $\frac{1}{(1+t)^b}$ is $(-1)^{n+b-1}(n+1)\dots(n+b-1)/(b-1)!$, which depends on the parity of n. However, the even and the odd terms are each given by a polynomial in n of degree at most $b-1$. Once again, multiplying by $m(t)$ doesn't affect the degree of these polynomials, hence we get two polynomials $h_+(t)$ and $h_-(t)$ of degree at most $b-1$, one giving the even coefficients and one giving the odd coefficients. Thus we have $p_+^M(t) = g_M(t) + h_+(t)$ and $p_-^M(t) = g_M(t) + h_-(t)$ so that

$$\deg(p_+^M - p_-^M) = \deg(g_M(t) + h_+(t) - (g_M(t) + h_-(t)) = \deg(h_+(t) - h_-(t)) \leq b - 1.$$

$\square$

The main result of this thesis addresses the similarity of p_-^M and p_+^M, though much work precedes what is shown in Chapter 3. Avramov [2] showed that, for any finitely generated module M, p_-^M and p_+^M have the same leading term, whose degree is bounded above by $c-1$. Later, Avramov [4] showed that a sufficient condition for the equality of p_+^M and p_-^M is that gr R is a graded complete intersection of quadrics. More recently, Avramov, Seceleanu and Yang [3] proved a result concerning the equality of the two polynomials in the form of the following theorem.

Theorem 1.5.4. *[3, 4.2] Assume that R is a complete intersection of codimension $c \geq 0$. All finite R-modules have eventually polynomial Betti sequences (i.e. $p_-^M = p_+^M$ for all M) if*

$$gr\ R \cong P/(g_1, \ldots, g_c) \text{ with } \deg g_1 = \deg g_2 = \ldots \deg g_{c-1} = 2 \leq \deg g_c.$$

where $g_1, \ldots, g_c$ is a regular sequence in P.

The converse is true for codimension at most three, and is open in general. at In this thesis, we investigate how the number of quadratic relations in a complete intersection ring affects the similarity of p_-^M and p_+^M. In particular, we put a new global bound on the degree of $p_-^M - p_+^M$ for complete intersection rings of arbitrary codimension which is based on the *quadratic codimension*, introduced in the next chapter.

Chapter 2

Quadratic Codimension

Let $(R, \mathfrak{m}, k)$ denote a local commutative Noetherian ring R with maximal ideal $\mathfrak{m}$ and residue field k. We first recall the definitions of some relevant invariants which are intrinsic to R.

In the case that R is a (graded) complete intersection with presentation $R \cong P/I$, the codimension of R is precisely the minimal number of generators of I. We wish to describe an invariant that encapsulates the size of the "quadratic part" of I. In the graded case this is straightforward, this invariant is $\dim I_2$ as an P_0 module, i.e. the dimension of the quadratic part of I in P, and if R is a complete intersection this is precisely the number of minimal homogeneous generators of I that are quadratic forms.

To generalize this definition to the case where R is local, we use a certain symmetric algebra. We note that $S_k^{\bullet}(\mathfrak{m}/\mathfrak{m}^2) \cong k[x_1, \ldots, x_e]$ the symmetric algebra of the vector space $\mathfrak{m}/\mathfrak{m}^2$ over k where $e = \operatorname{edim} R$. Then there is a map $\mu : S_k^2(\mathfrak{m}/\mathfrak{m}^2) \to \mathfrak{m}^2/\mathfrak{m}^3$ given by multiplication.

Definition 2.0.1. Let $(R, \mathfrak{m}, k)$ be a local ring. Let $P = S_k^{\bullet}(\mathfrak{m}/\mathfrak{m}^2)$. Let $\mu : S_k^2(\mathfrak{m}/\mathfrak{m}^2) \to \mathfrak{m}^2/\mathfrak{m}^3$ be the multiplication map, and set $J_2^{\square} := \ker \mu$. Let $J^{\square}$ be the ideal generated by $J_2^{\square}$ in P, and set $R^{\square} := P/J^{\square}$. Then the **quadratic codimension** of R, denoted $\operatorname{qdim} R$, is the codimension of $R^{\square}$, which is the height of $J^{\square}$ in P.

Note that $\operatorname{qdim} R$ is indeed an invariant of R, as it only depends on the symmetric algebra and multiplication map which are isomorphism invariant. There is an alternative way to view this invariant using a minimal Cohen presentation.

Proposition 2.0.2. *Let $(R, \mathfrak{m}, k)$ be a local ring with minimal Cohen presentation $\widehat{R} \cong Q/I$ where $(Q, \mathfrak{n}, k)$ is a local ring with $I \subseteq \mathfrak{n}^2$. Set $I_2^\square$ to be the image of I in $\mathfrak{n}^2/\mathfrak{n}^3$, and $I^\square$ to be the ideal generated by $I_2^\square$ in gr Q. Then $R^\square \cong$ gr $Q/I^\square$, and* $\operatorname{qdim} R = $ *ht* $I^\square$.

Proof. Let $e = \operatorname{edim} R$. Since Q is a regular local ring,

$$\text{gr } Q \cong S_k^\bullet(\mathfrak{n}/\mathfrak{n}^2) \cong S_k^\bullet(\mathfrak{m}/\mathfrak{m}^2).$$

Thus there is a map

$$\pi : (\text{gr } Q)_2 \to (\mathfrak{n}^2/I)/(\mathfrak{n}^3/I) \cong \widehat{\mathfrak{m}}^2/\widehat{\mathfrak{m}}^3 = \mathfrak{m}^2/\mathfrak{m}^3$$

given by multiplication. The kernel of this map are the degree 2 forms of gr Q which are initial forms of elements in I, which is exacty $I_2^\square$. Under the isomorphism gr $Q \cong S_k^\bullet(\mathfrak{m}/\mathfrak{m}^2)$, we have $I_2^\square = \ker \pi \cong \ker \mu = J_2^\square$ following the notation in Definition 2.0.1. Thus the isomorphism gr $Q \to S_k^\bullet(\mathfrak{m}/\mathfrak{m}^2)$ maps $I^\square$ bijectively onto $J^\square$, hence $R^\square \cong (\text{gr } Q)/I^\square$. As height is isomorphism invariant, $\operatorname{qdim} R = \text{ht } I^\square$. $\square$

Lemma 2.0.3. *Let R be a local ring with minimal Cohen presentation $\widehat{R} \cong Q/I$ and let $f_1, \ldots, f_c$ minimally generate I. Then $I^\square = (\{f_i^* : f_i \notin \mathfrak{n}^3\})$, and hence $\mu_{gr\,Q}(I^\square) \leq \mu_Q(I)$. In particular, if R is a complete intersection, then* $\operatorname{qdim} R \leq \operatorname{cdim} R$.

Proof. By definition $I^\square = (f^* : f \in I \setminus \mathfrak{n}^3)$. Let $\overline{g}$ in $I^\square$ with $g \in I \setminus \mathfrak{n}^3$. We can write

$g = \sum_{i=1}^{c} a_i f_i$. Thus

$$g^* = \sum_{i=1}^{c}(a_i f_i + \mathfrak{n}^3) = \sum_{\substack{i=1\\ a_i f_i \notin \mathfrak{n}^3}}^{c} (a_i f_i + \mathfrak{n}^3) = \sum_{\substack{i=1\\ a_i f_i \notin \mathfrak{n}^3}}^{c} (a_i f_i)^*.$$

Now, since $f_i \in \mathfrak{n}^2$ for each i, we have $a_i f_i \notin \mathfrak{n}^3$ if and only if $a_i \notin \mathfrak{n}$ and $f_i \notin \mathfrak{n}^2$. In this case $(a_i f_i)^* = a_i^* f_i^*$. Therefore

$$g^* = \sum_{\substack{i=1,\\ a_i \notin \mathfrak{n},\\ f_i \notin \mathfrak{n}^3}}^{c} a_i^* f_i^*.$$

Each g^* belongs to $(\{f_i^* : f_i \notin \mathfrak{n}^3\})$. Thus $I^\square \subseteq (\{f_i^* : f_i \notin \mathfrak{n}^3\})$. However, the other containment is evident, so we have equality. As $I^\square$ is generated by at most c elements, we have $\operatorname{qdim} R \leq c$.

For the last statement, we note that if R is a complete intersection, then $\operatorname{cdim} R = c$. $\square$

Example 2.0.4. Let $R = k[\![x, y]\!]/(xy, x^2 + y^3)$, which is a complete intersection. Then $\operatorname{cdim} R = \operatorname{ht}(xy, x^2 + y^3) = 2$. However, $I^\square = (\overline{xy}, \overline{x}^2)$, which has height 1 in $\operatorname{gr} Q \cong k[\overline{x}, \overline{y}]$. Thus, $\operatorname{qdim} R = 1$, yet $\operatorname{cdim} R = 2$.

Quadratic codimension behaves well when factoring out by general elements. Given a complete intersection ring of positive dimension and infinite residue field, one can find an Artinian quotient of R with the same quadratic codimension as the following two lemmas show.

Lemma 2.0.5. *Let $(R, \mathfrak{m}, k)$ be a local ring with $R \cong Q/I$ where $(Q, \mathfrak{n})$ is a regular local ring. If $|k| > \infty$, depth$(R) > 0$, and $\mu(I) < \operatorname{edim} Q$, then there exists an element $z \in \mathfrak{n} \setminus \mathfrak{n}^2$ such that the following hold:*

1. *z is a non-zerodivisor on R and*

2. the element $z^* := z + \mathfrak{n}^2$ in gr Q is not contained in any minimal prime over the ideal $I^\square \subseteq$ gr Q.

Proof. Since R has positive depth, $\mathfrak{n}$ is not in an associated prime of I. Let $\mathrm{Ass}_Q(I) := \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$. Then, using prime avoidance, we have $(\mathfrak{p}_i + \mathfrak{n}^2)/\mathfrak{n}^2$ is a proper k-subspace of $\mathfrak{n}/\mathfrak{n}^2$ for each $i = 1, \ldots, n$. Now, as k is an infinite field, $\bigcup_{i=1}^n \frac{\mathfrak{p}_i+\mathfrak{n}^2}{\mathfrak{n}^2} \subsetneq \mathfrak{n}/\mathfrak{n}^2$. Set $U := \frac{\mathfrak{n}}{\mathfrak{n}^2} \setminus \left(\bigcup_{i=1}^n \frac{\mathfrak{p}_i+\mathfrak{n}^2}{\mathfrak{n}^2}\right)$. Then U is a non-empty Zariski-open subset of Spec gr $Q =$ Spec $S_k^\bullet(\mathfrak{n}/\mathfrak{n}^2)$. Further, if $\overline{z} \in U$, then

$$\overline{z} \notin \bigcup_{i=1}^n \frac{\mathfrak{p}_i + \mathfrak{n}^2}{\mathfrak{n}^2} \implies z \notin \bigcup_{i=1}^n \mathfrak{p}_i$$

where $z \in \mathfrak{n}$ is any lift of $\overline{z}$. Thus z is not in any associated prime of I, which implies that z is a non-zero divisor on R.

Now, let $\mathrm{Min}_{\mathrm{gr}\,Q}(I^\square) = \{\mathfrak{q}_1, \ldots, \mathfrak{q}_r\}$. Since $I^\square$ is homogeneous, so is each $\mathfrak{q}_j$. Let $(\mathrm{gr}\ Q)_+$ denote the homogeneous maximal ideal of gr Q. Suppose that $\mathfrak{q}_j = (\mathrm{gr}\ Q)_+$ for some j. Then $\mathfrak{q}_i \subseteq \mathfrak{q}_j$ for all i, hence the only associated prime of $I^\square$ is $(\mathrm{gr}\ Q)_+$. However, by Krull's Height Theorem this means that $I^\square$ is minimally generated by at least $\mathrm{edim}\,\mathrm{gr}\ Q = \mathrm{edim}\, Q$ elements. However, by Lemma 2.0.3, $\mu_{grQ}(I^\square) \leq \mu_Q(I) < \mathrm{edim}\, Q$, which is a contradiction. Hence $\mathfrak{q}_i \subsetneq (\mathrm{gr}\ Q)_+$ for all i. Thus, using prime avoidance, $\mathfrak{q}_i \cap \mathfrak{n}/\mathfrak{n}^2$ is a proper subspace of $\mathfrak{n}/\mathfrak{n}^2$ for all i, and as k is infinite, $\bigcup_{i=1}^r(\mathfrak{q}_i \cap \mathfrak{n}/\mathfrak{n}^2) \subsetneq \mathfrak{n}/\mathfrak{n}^2$. Set $V := \frac{\mathfrak{n}}{\mathfrak{n}^2} \setminus (\bigcup_{i=1}^r(\mathfrak{q}_i \cap \mathfrak{n}/\mathfrak{n}^2))$. Then V is a nonempty Zariski open subset of Spec gr Q. Now if $z^* \in V$, then

$$z^* \notin \bigcup_{i=1}^r (\mathfrak{q}_i \cap \mathfrak{n}/\mathfrak{n}^2) \implies z^* \notin \bigcup_{i=1}^r \mathfrak{q}_i.$$

Thus, as U, V are nonempty, there exists z such that $z^* \in U \cap V$, which satisfies both desired conditions. □

Lemma 2.0.6. *For a complete intersection ring R with an infinite residue field there exists an Artinian quotient $\overline{R}$ of $\widehat{R}$ by a regular sequence such that* $\mathrm{cdim}\,\overline{R} = \mathrm{cdim}\, R$ *and* $\mathrm{qdim}\,\overline{R} =$

qdim R.

Proof. If $\operatorname{depth}(R) = 0$, then R is Artinian to begin with, so take $\overline{R} = \widehat{R}$. Suppose $\operatorname{depth}(R) \geq 1$. Let Q, R, z as in Lemma 2.0.5, and set $\overline{R} = Q/(I, z)$. Note that as $z \in \mathfrak{n} \setminus \mathfrak{n}^2$ is a nonzero divisor, $\operatorname{edim} \overline{R} = \operatorname{edim} R - 1$ and $\dim(\overline{R}) = \dim(R) - 1$, hence $\operatorname{cdim} \overline{R} = \operatorname{cdim} R$. Next, set $\overline{Q} = Q/(z)$, so that $\overline{Q}$ is a regular local ring and $\overline{I} = (I, z)/(z) \subseteq \overline{Q}$. Hence $\overline{I} \subseteq (\mathfrak{n}/(z))^2$ so that $\overline{R} \cong \overline{Q}/\overline{I}$ is a minimal Cohen presentation of $\overline{R}$. Set gr $Q = P$, and $\overline{P} = P/(z^*)$, which is isomorphic to gr $\overline{Q}$. The surjection $\pi : P \to \overline{P}$ is such that $\pi(I_2^\square) \supseteq \overline{I}_2^\square$, thus $\pi(I^\square) \supseteq \overline{I}^\square$, thus, the induced map $\tilde{\pi} : R^\square \to \overline{R}^\square$ is a surjection. The kernel of this map is generated by $\tilde{z}$, the image of z^* in $R^\square$, that is, $R^\square/(\tilde{z}) \cong (\overline{R})^\square$. However, this element is not contained in any of the minimal primes of $R^\square$, so we have $\dim \overline{P} = \dim P - 1$, and $\dim \overline{R}^\square = \dim R^\square - 1$. Thus

$$\operatorname{cdim} R^\square = \dim P - \dim R^\square = \dim \overline{P} - 1 - (\dim \overline{R}^\square - 1) = \operatorname{cdim} \overline{R}^\square,$$

hence $\operatorname{qdim} \overline{R} = \operatorname{qdim} R$. Iteration of this process yields a quotient of R of depth 0, and since R and each successive quotient thereof is a complete intersection and therefore Cohen-Macaulay, the process terminates with an Artinian ring. □

We will see in section that quadratic codimension in a complete intersection ring affects the possible behavior in the growth of Betti sequences over finitely generated modules.

2.1 Multiplicity

We next want to relate quadratic codimension to a well-studied invariant for local rings, called its multiplicity and denoted $e(R)$. Recall that for a finitely generated R-module M setting gr $M = \bigoplus_{n\in\mathbb{Z}} \frac{\mathfrak{m}^n M}{\mathfrak{m}^{n+1} M}$ yields a graded module over gr R. Now, as gr R is a standard graded k-algebra and gr M is generated in degree 0, there are polynomials $h_M^R(t), \overline{h}_M^R(t) \in \mathbb{Z}[t]$ with

$\overline{h}^R_M(1) \neq 0$, such that the Hilbert series for gr M can be written

$$H_{\text{gr M}}(t) = \frac{h^R_M(t)}{(1-t)^{\dim(R)}} = \frac{\overline{h}^R_M(t)}{(1-t)^{\dim(M)}}.$$

Definition 2.1.1. Let $(R, \mathfrak{m}, k)$ be a local ring and M a finitely generate R-module. Then the **multiplicity** of M is $\overline{h}^R_M(1)$ and is denoted by $e(M)$.

Note that gr R $\cong$ gr $\widehat{R}$ implies that $e(R) = e(\widehat{R})$.

Proposition 2.1.2. *[7, VIII.7, Prop 4] Let $(Q, \mathfrak{n}, k)$ be a local and $f_1, \ldots, f_c$ elements of Q with $f_i \in \mathfrak{n}^{d_i} \setminus \mathfrak{n}^{d_i-1}$ for integers $d_1, \ldots, d_c$. If $f_1, \ldots, f_c$ are part of a system of parameters for Q, then*

$$e(Q/(f_1, \ldots, f_c)) \geq d_1 \cdots d_c e(Q). \tag{2.1}$$

This is an equality if $f_1^*, \ldots, f_c^*$ form a regular sequence in gr Q.

Corollary 2.1.3. *If R is a complete intersection of codimension c, then $e(R) \geq 2^c$.*

Proof. Apply Proposition 2.1.2 to a complete intersection R with minimal Cohen presentation $\widehat{R} = Q/(f_1, \ldots, f_c)$. We have that each $f_i \in \mathfrak{n}^2$, hence $d_i \geq 2$ for each i. Further, Q is regular, so $e(Q) = 1$ which implies $e(R) = e(\widehat{R}) \geq 2^c$. $\square$

Definition 2.1.4. A complete intersection R is said to be of **minimal multiplicity** if $e(R) = 2^{\operatorname{cdim} R}$.

What follows are a pair of results relating multiplicity and associated graded rings which will be useful going forward.

Proposition 2.1.5. *(Valabrega-Valla [19, 2.7 and 1.1]) Let R and Q be local rings with $R = Q/(f_1, \ldots, f_c)$. Then $f_1^*, \ldots, f_c^*$ is a regular sequence in gr Q if and only if $f_1, \ldots, f_c$ is a regular sequence in Q and gr $R \cong$ gr $Q/(f_1^*, \ldots, f_c^*)$.*

Proposition 2.1.6. *(Rossi-Valla [15, 1.8]) Let $(Q, \mathfrak{n}, k)$ be a local ring, with gr Q Cohen-Macaulay, and let $f_1, \ldots, f_c \in Q$ be a regular sequence. Let d_i be positive integers so that* $\deg(f_i^*) \geq d_i$ *for each i. The following are equivalent:*

1. *$e(Q/(f_1, \ldots, f_c)) = d_1 d_2 \cdots d_c e(Q)$*

2. *$f_1^*, \ldots, f_c^*$ is a regular sequence in gr Q and* $\deg(f_i^*) = d_i$ *for each i.*

2.2 Koszul Algebras

Definition 2.2.1. Let A be a standard graded k-algebra. An A-module M is called **Koszul** if M has a linear resolution over A, that is, there exists an exact sequence

$$\cdots \to A(-i)^{\beta_i} \to \cdots \to A(-2)^{\beta_2} \to A(-1)^{\beta_1} \to A^{\beta_0} \to M \to 0.$$

We say that A is a **Koszul algebra** if k is a Koszul A-module. If $(R, \mathfrak{m}, k)$ is a local ring, then a finitely generated R-module M is **Koszul** if gr M has a linear resolution over gr R. We say that R is a **Koszul ring** if k is a Koszul R-module.

Herzog and Iyengar [12, 1.5] show that if M is Koszul over a ring R, then

$$P_M^R(t) = \frac{h_M^R(-t)}{h_R(-t)}. \tag{2.2}$$

The following theorem and its proof is a rephrasing of 2.7 in [3].

Theorem 2.2.2. *(Avramov-Seceleanu-Yang) Let $(R, \mathfrak{m}, k)$ be a local ring. The following are equivalent:*

(i) R is a complete intersection and $\operatorname{qdim} R = \operatorname{cdim} R$.

(ii) R is a complete intersection of minimal multiplicity.

(iii) R is a complete intersection and is Koszul.

(iv) R is a complete intersection and gr $R \cong R^{\square}$ as graded k-algebras.

(v) gr R is a graded complete intersection of quadrics with $\operatorname{cdim} R^* = \operatorname{cdim} R$.

Proof. We may assume R is complete. Let $R \cong Q/I$ be a minimal Cohen presentation with $(Q, \mathfrak{n}, k)$ a regular local ring and $f_1, \ldots, f_c$ be a minimal generating set for I.

(i) $\implies$ (v): We have $c = \operatorname{cdim} R$ and that the height of $I^{\square}$ is c, so it must be generated by at least c elements. By Lemma 2.0.3, $I^{\square} = (\{f_i : f_i \in \mathfrak{m}^2 \setminus \mathfrak{m}^3\})$. Thus each f_i belongs to $\mathfrak{n}^2 \setminus \mathfrak{n}^3$. Now, as this ideal is generated by exactly c elements, and gr Q is a polynomial ring and hence Cohen-Macaulay, $f_1^*, \ldots, f_c^*$ form a regular sequence of quadrics. Now, Proposition 2.1.5 implies gr R $\cong$ gr $Q/(f_1^*, \ldots, f_c^*)$, so R^* is a graded complete intersection of codimension c.

(v) $\implies$ (iii): R^* has codimension c, so we can pick $g_1, \ldots, g_c \in I$ so that $g_1^*, \ldots, g_c^*$ minimally generate I^*. As gr $Q/I^* =$ gr R, by assumption $g_1^*, \ldots, g_c^*$ is a regular sequence of quadrics in gr Q. By Proposition 2.1.5, $g_1, \ldots, g_c$ is a regular sequence and $I = (g_1, \ldots, g_b)$, so that R is a complete intersection. Applying the construction of a minimal free resolution due to Tate [17] to a graded complete intersection of quadrics yields a linear minimal graded free resolution of k, implying that R is Koszul.

(iii) $\implies$ (ii): If R is Koszul, then by (2.2), $P_k^R(t) = \frac{h_k^R(-t)}{h_R(-t)} = \frac{(1-t)^{\dim(R)}}{h_R(-t)}$ as $H_k^R(t) = 1$. Thus we find

$$H_R(t) = \frac{1}{P_k^R(-t)} = \frac{(1+t)^c}{(1-t)^{\dim(R)}}.$$

Thus $h_R(t) = (1+t)^c$, so evaluating at $t = 1$ gives $e(R) = 2^c$.

(ii) $\implies$ (v): We have $e(R) = 2^c$ and the $f_1, \ldots, f_c$ are regular by assumption. Thus by Proposition 2.1.6, and Proposition 2.1.5, the elements $f_1^*, \ldots, f_c^*$ are a regular sequence in $(\text{gr } Q)_2$, hence gr R is a graded complete intersection of quadrics of codimension c.

(v) $\implies$ (iv): We have $I^* = I^{\square}$, so gr $R = R^{\square}$, and the proof above of (v) $\implies$ (iii) shows that R is a complete intersection.

(iv) $\implies$ (i): Note that $\dim \operatorname{gr} R = \dim R$, so we have

$$\operatorname{qdim} R = \operatorname{cdim} R^{\square} = \operatorname{cdim} \operatorname{gr} R = \operatorname{edim} \operatorname{gr} R - \dim(\operatorname{gr} R) = \operatorname{edim} R - \dim(R) = \operatorname{cdim} R.$$

$\square$

Chapter 3

Homological Methods

3.1 Yoneda Algebras

Let $(R, \mathfrak{m}, k)$ be a local ring and $F_\bullet$ be a free resolution of k over R. Then $\mathrm{Hom}_R(F_\bullet, F_\bullet)$ is an associative DG-R-algebra under composition. Passing to cohomology, we have $H^*(\mathrm{Hom}_R(F_\bullet, F_\bullet)) \cong \mathrm{Ext}^*_R(k, k)$ is an associative graded R-algebra with product induced by composition. For an R-module M, $\mathrm{Hom}_R(M, F_\bullet)$ is a left $\mathrm{Hom}_R(F_\bullet, F_\bullet)$-module via post-composition. Thus passing to cohomology, we have that $\mathrm{Ext}^*_R(M, k)$ is a graded left $\mathrm{Ext}^*_R(k, k)$-module.

Proposition 3.1.1. *Let $(R, \mathfrak{m}, k)$ be a local ring, and $\widehat{R}$ the $\mathfrak{m}$-adic completion of R. Then $Ext^*_R(k, k) \cong Ext^*_{\widehat{R}}(k, k)$ as associative graded k-algebras.*

Proof. Let $F_\bullet$ be a minimal free resolution of k over R. Then $\widehat{F_\bullet} = F_\bullet \otimes_R \widehat{R}$ is a minimal free resolution of k over $\widehat{R}$. Notice that the cohomology of $\mathrm{Hom}_R(F_\bullet, F_\bullet)$ is a finite dimensional k-vector space and hence already complete. Thus, there is a quasi-isomorphism $\mathrm{Hom}_R(F_\bullet, F_\bullet) \to \mathrm{Hom}_{\widehat{R}}(\widehat{F_\bullet}, \widehat{F_\bullet})$ given by extension of scalars, using $\mathrm{Hom}_{\widehat{R}}(\widehat{F_\bullet}, \widehat{F_\bullet}) \cong \mathrm{Hom}_R(F_\bullet, F_\bullet) \otimes_R \widehat{R}$. Since this is functorial, it is a map of algebras. As the map is a quasi-isomorphism, we have the result. $\square$

Proposition 3.1.2. *Let $(R, \mathfrak{m}, k)$ be a complete intersection ring with* $\operatorname{cdim} R = \operatorname{qdim} R$.

Then

$$Ext^*_{Ext^*_R(k,k)}(k,k) \cong gr\ R.$$

Proof. By Proposition 2.2.2, R (and hence gr R) is Koszul. By Proposition 1.5 in [12], we have $\operatorname{Ext}^*_R(k,k) \cong \operatorname{Ext}^*_{\mathrm{gr}\ R}(k,k)$. Classical Koszul duality then gives the result. $\square$

3.2 Homotopy Lie Algebras

The Yoneda algebra $\operatorname{Ext}^*_R(k,k)$ is a Hopf Algebra, and as such it is the universal enveloping algebra of a graded Lie algebra by Milnor-Moore [14] in characteristic 0, André [1] in characteristic $p \neq 2$, and Sjödin [16] in the remaining case (with minor adjustments). This graded Lie algebra is called the *homotopy Lie algebra* of R, denoted $\pi^*(R)$. Thus, we write $\operatorname{Ext}^*_R(k,k) \cong U(\pi^*(R))$. Recall the definition of a graded Lie algebra and some basic facts. For a more complete discussion, see [5, Ch. 10].

Definition 3.2.1. A **graded Lie algebra** over k is a graded k-module $\mathfrak{g} = \{\mathfrak{g}^n\}_{\{n\in\mathbb{Z}\}}$ equipped with a k-bilinear pairing

$$[\ ,\ ] : \mathfrak{g}^i \times \mathfrak{g}^j \to \mathfrak{g}^{i+j} \qquad \text{for } i,j \in \mathbb{Z}, \qquad (\vartheta,\xi) \mapsto [\vartheta,\xi]$$

and a reduced square

$$\mathfrak{g}^{2h+1} \to \mathfrak{g}^{4h+2} \qquad \text{for } h \in \mathbb{Z}, \qquad \upsilon \mapsto \upsilon^{[2]}$$

such that for $\vartheta,\xi,\zeta,\upsilon,\omega \in \mathfrak{g}$ the following conditions hold:

1. $[\vartheta,\xi] = -(-1)^{|\vartheta||\xi|}[\xi,\vartheta]$,

   and $[\vartheta,\vartheta] = 0$ for $\vartheta \in \mathfrak{g}^{\mathrm{even}}$.

2. $[\vartheta, [\xi, \zeta]] = [[\vartheta, \xi], \zeta] + (-1)^{|\vartheta||\xi|}[\xi, [\vartheta, \zeta]]$
   and $[\xi, [\xi, \xi]] = 0$ for $\xi \in \mathfrak{g}^{\text{odd}}$.

3. $(\upsilon + \omega)^{[2]} = \upsilon^{[2]} + \omega^{[2]} + [\upsilon, \omega]$ for $\upsilon, \omega \in \mathfrak{g}^{\text{odd}}$ with $|\upsilon| = |\omega|$.

4. $(a\upsilon)^{[2]} = a^2\upsilon^{[2]}$ for $a \in k$ and $\upsilon \in \mathfrak{g}^{\text{odd}}$.

5. $[\upsilon^{[2]}, \vartheta] = [\upsilon, [\upsilon, \vartheta]]$ for $\upsilon \in \mathfrak{g}^{\text{odd}}$ and $\vartheta \in \mathfrak{g}$.

The second line of (1) and the second line of (2) are only necessary in characteristic 2 and 3 respectively. If the characteristic of k is not 2, then $\upsilon^{[2]} = \frac{1}{2}[\upsilon, \upsilon]$, and (3)-(5) follow from (1) and (2). Any graded associative algebra A has the structure of a graded Lie algebra by setting the bracket to be the graded commutator, that is $[a, b] = ab - (-1)^{|a||b|}ba$ and defining $a^{[2]} = a^2$. A graded Lie subalgebra is a subset of a lie algebra which is closed under the bracket and reduced square. To every graded Lie algebra $\mathfrak{g}$ one can associate a graded associative algebra, $U(\mathfrak{g})$ called its universal enveloping algebra Let $\mathfrak{g}$ be a graded Lie algebra such that $\mathfrak{g}^n = 0$ for $n < 0$. Let $\{\vartheta_i\}_{i\in\mathbb{N}}$ be a total ordering of a basis of $\mathfrak{g}$ with $|\vartheta_i| \leq |\vartheta_j|$ for $i < j$. Let $(i_1, i_2, \ldots)$ be an indexing sequence of integers, with $i_j \leq 1$ if $|\vartheta_j|$ is odd and $i_j = 0$ for $j \gg 0$. Then one can form a *normal monomial* in $U(\mathfrak{g})$ from I via $\vartheta^I = \vartheta_p^{i_p} \ldots \vartheta_1^{i_1}$. One consequence of the Poincaré-Birkhoff-Witt theorem (see [5, 10.1.3] or [16]) is that a k-basis for $U(\mathfrak{g})$ is given by the set of normal monomials.

In defining $\pi^*(R)$, we follow the lead of Avramov-Yang [6, Sec. 2]. For a complete discussion, see Gulliksen-Levin [11, Ch. 2 and 3] and Milnor-Moore[14]. Let $(R, \mathfrak{m}, k)$ be a local ring. First we note that Tate [17] showed that there is a free resolution X of k over R which is a differential graded R-algebra. As $\operatorname{Tor}_*^R(k, k) \cong H(X \otimes_R k)$, it is canonically a graded k-algebra with divided powers defined for elements of positive even degree. ([11, 2.3.4]). For each $i \in \mathbb{Z}$, define $\pi_*(R)$ to be the quotient of $\operatorname{Tor}_*^R(k, k)$ by its decomposable elements D, the k-subspace generated by elements of the form xy and $z^{(n)}$ where x and y have positive degree and z has positive even degree with $n \geq 2$. Note that since the assignment $R \mapsto \operatorname{Tor}_*^R(k,k)$ is covariant, the functor $R \mapsto \pi_*(R)$ from the category of local rings to the category of graded vector spaces over k.

There is a canonical isomorphism $\operatorname{Ext}_R^i(k,k) \cong \operatorname{Hom}_k(\operatorname{Tor}_i^R(k,k), k)$. Set $\pi^*(R)$ to be the graded k-space $\operatorname{Hom}_R(\pi_i(R), k)$, which is a graded subspace of $\operatorname{Ext}_R^*(k,k)$ under the canonical identification above. Under this identification, $\pi^*(R)$ is closed under commutation of its elements and the squares of all odd elements, so that $\pi^*(R)$ is a graded Lie algebra over k, and $\operatorname{Ext}_R^*(k,k) \cong U(\pi^*(R))$ (see [16]).

There is a canonical isomorphism of vector spaces

$$\operatorname{Hom}_R(\mathfrak{m}/\mathfrak{m}^2, k) \cong \pi^1(R),$$

and if $R \cong P/I$ where $(P, \mathfrak{n})$ is a regular local ring and $I \subseteq \mathfrak{n}^2$, there is an isomorphism

$$\operatorname{Hom}_k(I/\mathfrak{n}I, k) \cong \pi^2(R).$$

As $\pi^*(R)$ is the k-dual of $\pi_*(R)$, the assignment $R \mapsto \pi^*(R)$ is a contravariant functor from the category of local rings with residue field k to the category of graded Lie algebras over k.

Let $R \cong Q/J$ where $(Q, \mathfrak{n}, k)$ is a (not necessarily regular) local ring and J is generated by a regular sequence (not necessarily in $\mathfrak{n}^2$). Let $\psi : Q \to R$ be the canonical projection and π^ψ the induced map $\pi_*(Q) \to \pi_*(R)$. It is shown in the proof of [11, 3.4.1] that π_i^ψ is bijective for $i \geq 3$ and that there is a natural sequence of k-vector spaces

$$0 \to \pi_2(Q) \xrightarrow{\pi_2^\psi} \pi_2(R) \xrightarrow{\alpha} J/\mathfrak{n}J \xrightarrow{\beta} \pi_1(Q) \xrightarrow{\pi^\psi} \pi_1(R) \to 0. \tag{3.1}$$

Lemma 3.2.2. *Let $(Q, \mathfrak{n}, k)$ be a local ring, and set $R = Q/(h)$ where $h \in \mathfrak{n}$ is a nonzerodi-*

visor on Q.

1. *If h belongs to $\mathfrak{n} \setminus \mathfrak{n}^2$, then $\pi^*(R)$ injects into $\pi^*(Q)$ and*

$$\dim_k \pi^i(R) = \begin{cases} \dim_k \pi^i(Q) - 1 & i = 1 \\ \dim_k \pi^i(Q) & i \neq 1. \end{cases}$$

2. *If h belongs to $\mathfrak{n}^2$ then $\pi^*(R)$ surjects onto $\pi^*(Q)$ and*

$$\dim_k \pi^i(R) = \begin{cases} \dim_k \pi^i(Q) + 1 & i = 2 \\ \dim_k \pi^i(Q) & i \neq 2. \end{cases}$$

Proof. Let $\psi : Q \to R$ be the canonical projection.

1. We have $\pi_1(Q) \cong \mathfrak{n}/\mathfrak{n}^2$, and $\pi_1(R) = \mathfrak{m}/\mathfrak{m}^2 = \mathfrak{n}/((h) + \mathfrak{n}^2)$. As h is not an element of $\mathfrak{n}^2$, the induced map π_1^ψ has nonzero kernel. However, this kernel is a subspace of $(h)/\mathfrak{n}(h)$ and $\dim_k (h)/(h)\mathfrak{n} = 1$, thus turning to the sequence (3.1), we see that $\ker \pi_1^\psi = (h)/\mathfrak{n}(h)$, $\alpha = 0$. This simultaneously shows that $\pi_2(Q) \to \pi_2(R)$ is an isomorphism and $\dim_k \pi_1(R) = \dim_k \pi_1(Q) - 1$. Taking k-linear duals then gives the result.

2. As $h \in \mathfrak{n}^2$, we have $\pi_1(Q) = \mathfrak{n}/\mathfrak{n}^2 = \mathfrak{n}/((h) + \mathfrak{n}^2) = \mathfrak{m}/\mathfrak{m}^2 = \pi_1(R)$. Considering (3.1), we have that π_1^ψ is surjective, so it must be an isomorphism and $\beta = 0$. Now, $\dim_k \frac{(h)}{\mathfrak{n}(h)} = 1$, hence $\dim_k \pi_2(R) = \dim_k \pi_2(Q) + 1$. Taking k-linear duals then gives the desired result.

$\square$

Theorem 3.2.3. *Let $(R, \mathfrak{m}, k)$ be a local ring that admits an isomorphism $R \cong Q/I$ where $(Q, \mathfrak{n}, k)$ is a local ring and $I \subseteq \mathfrak{n}^2$ is generated by a regular sequence of length c. Suppose Q*

admits an artinian, Koszul quotient $\overline{Q} = Q/J$ such that J is generated by a regular sequence contained in $\mathfrak{n}^2$ that maps to a linearly independent set in $\mathfrak{n} \setminus \mathfrak{n}^2$. Then there exists an associative subalgebra $\mathcal{L} \subseteq Ext^*_R(k,k)$ with the following properties:

1. *$Ext^*_R(k,k)$ is free of finite rank as a left $\mathcal{L}$-module.*

2. *$\mathcal{L}$ has finite global dimension.*

3. *$\mathcal{L}$ is the universal enveloping algebra of a graded Lie-algebra λ with*

$$\dim_k \lambda^i = \begin{cases} \dim_k \pi^i(\overline{Q}) & i \neq 2 \\ \dim_k \pi^2(R) & i = 2. \end{cases}$$

Proof. As stated, we'll construct $\mathcal{L}$ as the universal enveloping algebra of a graded Lie algebra λ. The diagram of local rings

$$\begin{tikzcd} & \overline{Q} \\ R & Q \arrow[l, two heads] \arrow[u, two heads] \end{tikzcd}$$

Induces a diagram of homotopy Lie algebras (see Lemma 3.2.2)

$$\begin{tikzcd} & \pi^*(\overline{Q}) \arrow[d, hook] \\ \pi^*(R) \arrow[r, two heads] & \pi^*(Q) \end{tikzcd}$$

Let λ^* be the pullback $\pi^*(R) \times_{\pi^*(Q)} \pi^*(\overline{Q})$, so that λ^* fits in the following commutative diagram of graded Lie algebras with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \ker(p_1)^* \arrow[r, hook] \arrow[d, "\cong"] & \lambda^* \arrow[r, two heads, "p_1"] \arrow[d, hook, "p_2"] & \pi^*\left(\overline{Q}\right) \arrow[r] \arrow[d, hook, "\alpha"] & 0 \\
0 \arrow[r] & \ker(\beta)^* \arrow[r, hook] & \pi^*(R) \arrow[r, two heads, "\beta"] & \pi^*(Q) \arrow[r] & 0
\end{tikzcd}$$

Since β^i is an isomorphism for $i \neq 2$ so is p_1. By Lemma 3.2.2 α^2 is an isomorphism, hence so is p_2^2, which gives

$$\dim_k \lambda^i = \begin{cases} \dim_k \pi^i(\overline{Q}) & i \neq 2 \\ \dim_k \pi^i(R) & i = 2. \end{cases}$$

Thus

$$\dim_k (\ker p_1)^i = \begin{cases} 0 & i \neq 2 \\ c & i = 2. \end{cases}$$

Denote the universal enveloping algebras of these graded Lie algebras by $\mathcal{K} = U(\ker(p_1)^*)$, $\mathcal{L} = U(\lambda^*)$, $\mathcal{Q} = U(\pi^*(\overline{Q})) = \operatorname{Ext}^*_{\overline{Q}}(k,k)$, and $\mathcal{E} = U(\pi^*(R)) = \operatorname{Ext}^*_R(k,k)$.

The Hochschild-Serre spectral sequnce in this context (see [9, p.311]) is

$$\operatorname{Ext}^i_{\mathcal{Q}}\left(k, \operatorname{Ext}^j_{\mathcal{K}}(k,k)\right) \implies \operatorname{Ext}^{i+j}_{\mathcal{L}}(k,k). \tag{3.2}$$

Since $\ker(p_1)$ is concentrated in degree 2, $\mathcal{K}$ is a polynomial ring, and so $\operatorname{Ext}^j_{\mathcal{K}}(k,k) = 0$ for large values of j. Since $\overline{Q}$ is Koszul, we have by Lemma 3.1.2

$$\operatorname{Ext}^*_{\operatorname{Ext}^*_{\overline{Q}}(k,k)}(k,k) = \operatorname{Ext}^*_{\mathcal{Q}}(k,k) \cong \operatorname{gr}_{\mathfrak{m}_{\overline{Q}}} \overline{Q}.$$

So, as $\overline{Q}$ is artinian and hence finite dimensional as a k-vector space, $\mathcal{Q}$ has finite global dimension. The spectral sequence (3.2) implies that $\operatorname{Ext}^n_{\mathcal{L}}(k,k) = 0$ in high degrees. Therefore, by [18, 35.11], $\mathcal{L}$ has finite global dimension.

Now, by Poincaré-Birkhoff-Witt, $\operatorname{Ext}_R^*(k,k)$ is spanned by the normal monomials formed from a basis for for $\pi^*(R)$ (for details, see [5, 10.1.3]). Any normal monomial γ from $\pi^*(R)$ can be written as $\gamma = \alpha\beta$ where α is a (potentially trivial) normal monomial consisting of elements from $\pi^{\geq 2}(R)$ and β is a normal monomial consisting of elements from $\pi^1(R)$. As there are finitely many normal monomials from $\pi^1(R)$, we conclude that $\operatorname{Ext}_R^*(k,k)$ is finitely generated as a left module over $U(\pi^{\geq 2}(R))$. However, by the diagram above $\lambda^i \cong \pi^i(R)$ for $i \geq 2$ and $\lambda^1 \subseteq \pi^1(R)$, hence $\lambda^* \subseteq \pi^*(R)$, and we have that $\mathcal{L} \subseteq \mathcal{E}$.

□

3.3 Homotopy Lie Algebras over Complete Intersections

In the case where R is a complete intersection ring of embedding dimension e and codimension c, $\pi^*(R)$ is concentrated in degrees one and two, with

$$\dim_k \pi^i(R) = \begin{cases} e & i = 1 \\ c & i = 2. \end{cases}$$

Theorem 3.3.1. *Let $(R, \mathfrak{m}, k)$ be a complete intersection ring of codimension c and quadratic codimension q. Then there is an associative k subalgebra $\mathcal{L} \subseteq \operatorname{Ext}_R^*(k,k)$ such that the following hold:*

1. *$\operatorname{Ext}_R^*(k,k)$ is free of finite rank as a left $\mathcal{L}$-module.*

2. *$\mathcal{L}$ has finite global dimension.*

3. *The Hilbert series of $\mathcal{L}$ has the form*

$$H_{\mathcal{L}}(t) = \frac{1}{(1+t)^{c-q}(1-t)^c}.$$

Proof. We may assume R is complete by Proposition 3.1. Let $R \cong P/I$ be a minimal Cohen presentation for R, that is, $(P, \mathfrak{n}, k)$ is a regular local ring and $I \subseteq \mathfrak{n}^2$. As $I^{\square}$ has height q in a polynomial ring, it contains a regular sequence of length q that is part of a generating set. Following 2.0.3, we know $I^{\square}$ is generated by images of generators for I that are not contained in $\mathfrak{n}^3$. Choose generators $f_1^*, \dots, f_q^*, f_{q+1}, \dots f_s^* \in I_2^{\square}$ so that $f_1^*, \dots f_q^*$ are a regular sequence in $\operatorname{gr} Q$. Now, note there is a surjection $I/\mathfrak{n}I \to (I + \mathfrak{n}^3)/\mathfrak{n}^3 = I_2^{\square}$. Lift the generating set via this map to $\overline{f_1}, \dots, \overline{f_q}, \overline{f_{q+1}} \dots, \overline{f_s}$, which necessarily are part of a basis for $I/\mathfrak{n}I$. Now extend this to a full basis $\overline{f_1}, \dots, \overline{f_q}, \overline{f_{q+1}} \dots, \overline{f_s}, \overline{f_{s+1}}, \dots, \overline{f_c}$. Finally, lift these elements to Q. Thus, we have found a regular sequence of generators $f_1, \dots, f_q, f_{q+1}, \dots f_s, f_{s+1}, \dots, f_c$ for I with $f_1, \dots, f_s \notin \mathfrak{n}^3$, so that $I^{\square} = (f_1^*, \dots, f_q^*, f_{q+1}^*, \dots, f_s^*)$ and that $f_1^*, \dots, f_q^*$ is a regular sequence in $\operatorname{gr} P$.

Let $e = \dim_k \mathfrak{n}/\mathfrak{n}^2$ be the embedding dimension of R. Set $Q_0 = P/(f_1, \dots, f_q)$. If $e > q$, then Q_0 is not artinian, hence $\mathfrak{m}_{Q_0} \notin \operatorname{Ass}_{Q_0}(0)$. Therefore, by prime avoidance $\mathfrak{m}_{Q_0} \not\subseteq \mathfrak{m}_{Q_0}^2 \cup \bigcup_{\mathfrak{p} \in \operatorname{Ass}(0)} \mathfrak{p}$. Thus, there exists some $\overline{h_{q+1}}$ in Q_0 which is Q_0-regular, and in $\mathfrak{m}_{Q_0} \setminus \mathfrak{m}_{Q_0}^2$, lift this element to h_{q+1} in P. Hence, h_{q+1} is regular in P and part of basis for $\mathfrak{n}/\mathfrak{n}^2$. Setting $Q_i = P/(f_1, \dots, f_q, h_{q+1}, \dots, h_{q+i})$ and iterating this argument, we find a regular sequence $h_{q+1}, \dots, h_e$ in P so that $h_{q+1}^*, \dots, h_e^*$ are linearly independent in $\mathfrak{n}/\mathfrak{n}^2$, and $f_1, \dots, f_q, h_{q+1}, \dots, h_e$ form a maximal regular sequence in P.

Set $Q = P/(f_1, \dots, f_q)$, and $\overline{Q} = Q/(h_{q+1}, \dots, h_e)$. Thus, we have 4 complete intersection rings, $P, Q, \overline{Q}$, and R with the following invariants:

	Embedding Dimension	Codimension
P	e	0
Q	e	q
$\overline{Q}$	q	q
R	e	c

We now have R, Q, and $\overline{Q}$ as in Theorem 3.2.3. Thus, there is a subalgebra $\mathcal{L}$ of $\operatorname{Ext}_R^*(k,k)$ with finite global dimension, such that $\operatorname{Ext}_R^*(k,k)$ is finitely generated over $\mathcal{L}$ as a left module. Further,

$$\dim_k \lambda^i = \begin{cases} q & i = 1 \\ c & i = 2 \\ 0 & i \neq 1, 2. \end{cases}$$

By the graded version of the Poincaré-Birkhoff-Witt Theorem (see for example, [5, Ch. 10]), we have that

$$H_{\mathcal{L}} = \frac{(1+t)^q}{(1-t^2)^c} = \frac{1}{(1-t)^c(1+t)^{c-q}}.$$

□

3.4 Growth of Betti Sequences

This section is devoted to proving the following theorem:

Theorem 3.4.1. *Let R be a complete intersection of codimension c and quadratic codimension q. Then for any finitely generated R-module M, there exists a polynomial $g_M(t) \in \mathbb{Z}[t]$ such that the Poincaré series of M has the form*

$$P_M^R(t) = \frac{g_M(t)}{(1-t)^c(1+t)^b} \tag{3.3}$$

where $b = \max\{0, c-q-1\}$.

The proof will be portioned into several lemmas.

Lemma 3.4.2. *Let R be a complete intersection of codimension c and quadratic codimension q. Then for any finitely generated R-module M, there exists a polynomial $g_M(t) \in \mathbb{Z}[t]$ such that the Poincaré series has the form*

$$P_M^R(t) = \frac{g_M(t)}{(1-t)^c(1+t)^{c-q}}. \tag{3.4}$$

Proof. By Theorem 3.3.1, there exists a finite extension $\mathcal{L} \subseteq \operatorname{Ext}_R^*(k,k)$ such that $\mathcal{L}$ has finite global dimension and $\operatorname{Ext}_R^*(k,k)$ is module finite over $\mathcal{L}$ and the Hilbert series of $\mathcal{L}$ is

$$H_{\mathcal{L}}(t) = \frac{1}{(1-t)^c(1+t)^{c-q}}.$$

Thus, as $\mathcal{M} := \operatorname{Ext}_R^*(M,k)$ is finitely generated over $\operatorname{Ext}_R^*(k,k)$, $\mathcal{M}$ is a finitely generated $\mathcal{L}$ module, and there exists a finite graded free resolution of $\mathcal{M}$

$$0 \leftarrow \mathcal{F}_0 \leftarrow \mathcal{F}_1 \leftarrow \cdots \leftarrow \mathcal{F}_n \leftarrow 0.$$

Each $\mathcal{F}_i$ is a finite direct sum of shifts of $\mathcal{L}$. Thus

$$\begin{aligned} H_{\mathcal{M}}(t) &= \sum_{i=0}^{n}(-1)^i H_{F_i}(t) \\ &= \sum_{i=0}^{n}(-1)^i g_i(t) H_{\mathcal{L}} \\ &= g_M(t) H_{\mathcal{L}}(t) \end{aligned}$$

where each $g_i(t) \in \mathbb{Z}[t]$, hence $g_M(t) \in \mathbb{Z}[t]$. However, using the Hilbert series for $\mathcal{L}$ as shown in Theorem 3.3.1 and noting that $P_M^R(t) = H_{\mathcal{M}}(t)$, we have

$$P_M^R(t) = \frac{g_M(t)}{(1-t)^c(1+t)^{c-q}}. \tag{3.5}$$

□

Lemma 3.4.3. *Let R be a complete intersection of codimension c and quadratic codimension*

q. Let

$$0 \to N' \to N \to N'' \to 0$$

be a short exact sequence of finitely generated modules, and let $g_{N'}(t), g_N(t), g_{N''}(t)$ be as in Lemma 3.4.2. Then

$$g_N(-1) = g_{N'}(-1) + g_{N''}(-1).$$

Proof. Set $M = \operatorname{Ext}^*_R(N, k)$, $M' := \operatorname{Ext}^*_R(N', k)$ and $M'' := \operatorname{Ext}^*_R(N'', k)$. Apply $\operatorname{Ext}^*_R(-, k)$ to get an exact sequence sequence of $\mathcal{L}$-modules

$$\cdots \to M'[-1] \xrightarrow{\beta} M'' \to M \to M' \xrightarrow{\alpha} M''[1] \to \dots$$

where $[-]$ denotes a suspension. Set $U := \ker(\alpha)$ and $V := \operatorname{coker}(\beta)$, so that

$$0 \to V \to M \to U \to 0$$

and

$$0 \to U \to M' \to M''[1] \to V[1] \to 0$$

are exact sequences of finitely generated $\mathcal{L}$-modules. Then, using the additivity of the Hilbert series on these sequences, we have

$$\begin{aligned} H^{\mathcal{L}}_M(t) &= H^{\mathcal{L}}_U(t) + H^{\mathcal{L}}_V(t) \\ &= H^{\mathcal{L}}_{M'}(t) - H^{\mathcal{L}}_{M''[1]}(t) + H^{\mathcal{L}}_{V[1]}(t) + H^{\mathcal{L}}_V(t) \\ &= H^{\mathcal{L}}_{M'}(t) - tH^{\mathcal{L}}_{M''}(t) + (1+t)H^{\mathcal{L}}_V(t) \end{aligned}$$

Thus

$$\frac{g_M(t)}{(1-t)^c(1+t)^{c-q}} = \frac{g_{M'}(t) - tg_{M''}(t) + (1+t)g_V(t)}{(1-t)^c(1+t)^{c-q}}$$

Clearing denominators and evaluating at $t = -1$ gives

$$g_M(-1) = g_{M'}(-1) + g_{M''}(-1).$$

$\square$

Lemma 3.4.4. *Let $(R, \mathfrak{m}, k)$ be a local ring and set $S = R[x]_{\mathfrak{m}[x]}$. Then*

1. *$(S, \mathfrak{m}_S)$ is a local faithfully flat extension of R,*
2. *S has an infinite residue field,*
3. *if R is complete and a complete intersection ring, then S is also a complete intersection ring with $\operatorname{qdim} S = \operatorname{qdim} R$ and $\operatorname{cdim} S = \operatorname{cdim} R$.*

Proof. 1. The fact that the construction yields a local faithfully flat extension is classical (see for example, Section 8.4 in [13]).

2. Note that $S/\mathfrak{m}_S \cong \left(\frac{R[x]}{\mathfrak{m}R[x]}\right)_{\mathfrak{m}R[x]}$ which is the fraction field of $k[x]$, and is therefore infinite.

3. Assume R is complete, and $R \cong Q/I$ with $(Q, \mathfrak{n})$ regular. Set $P = Q[x]_{\mathfrak{n}[x]}$. Then $P/IP \cong R[x]_{\mathfrak{m}[x]}$. By Huneke-Swanson [13, Lemma 8.4.2], P is regular, IP is generated by a regular sequence, and $\mu(IP) = \mu(I)$, hence S is a complete intersection of codimension c. Now, for any $\mathfrak{m}$-primary ideal $\mathfrak{p}$, $\lambda(R/\mathfrak{p}) = \lambda(S/\mathfrak{p}S)$ (see [13, Lemma 4.8.2]). Thus $\dim_k \mathfrak{m}^2/\mathfrak{m}^3 = \dim_{k(x)} \mathfrak{m}_S^2/\mathfrak{m}_S^3$, That is, $\mathfrak{m}^2/\mathfrak{m}^3 \otimes_k k(x) \cong \mathfrak{m}_S^2/\mathfrak{m}_S^3$. Following the notation of Definition 2.0.1, we have

$$0 \to J_2^{\square} \to S^2(\mathfrak{m}/\mathfrak{m}^2) \to \mathfrak{m}^2/\mathfrak{m}^3 \to 0.$$

By extending scalars, we find $J_S^{\square} \cong J_S^{\square} \otimes_k k(x)$. Thus, $S^{\square} \cong R^{\square} \otimes_k k(x)$, hence R and S have the same quadratic codimension.

$\square$

Lemma 3.4.5. *Theorem 3.4.1 holds for all complete intersections provided that it holds for Artinian complete intersections with infinite residue fields.*

Proof. We may assume R is complete. Set $R' = R[x]_{\mathfrak{m}[x]}$ and $M' = M \otimes_R R'$. By Lemma 3.4.4 $R \subseteq R'$ is a flat extension, so we have $P_{M'}^{R'}(t) = P_M^R(t)$, and R' is a complete intersection ring with the same codimension and quadratic codimension. By passing to R' we may assume an infinite residue field.

Let R be a complete intersection and N a finitely generated R-module. Then for some $\ell \geq 0$ the syzygy $M := \Omega^{\ell}(N)$ is a maximal Cohen Macaulay module. Let z be an element as in Lemmas 2.0.5 and 2.0.6. Then since z is regular on R, it is also regular on M. Let $F_\bullet$ be a minimal free resolution of M over R. Then $F_\bullet \otimes_R R/(z)$ is a minimal free resolution of $\overline{M}$, so $P_{M/zM}^{R/(z)}(t) = P_M^R(t)$. Now, by Lemma 2.0.6, we iterate the process to find an Artinian quotient $\overline{R}$ of R with $\operatorname{cdim} R = \operatorname{cdim} \overline{R}$ and $\operatorname{qdim} R = \operatorname{qdim} \overline{R}$ and a finitely generated $\overline{R}$ module $\overline{M}$ so that $P_{\overline{M}}^{\overline{R}}(t) = P_M^R(t)$. Thus, if the theorem holds for $\overline{R}$, we have

$$P_M^R(t) = \frac{g_M}{(1-t)^c(1+t)^b}$$

and noting that

$$P_N^R(t) = p_N(t) + t^{\ell} P_M^R(t),$$

for some $p_N \in \mathbb{Z}[t]$, we can write the Poincaré series for N in the desired form. $\square$

We now prove Theorem 3.4.1.

Proof. By Lemma 3.4.5, we may also assume R is Artinian with infinite residue field. Let M be a finitely generated R-module. By Lemma 3.4.2 we have that

$$P_R^M(t) = \frac{g_M(t)}{(1-t)^c(1+t)^{c-q}}.$$

If $c = q$, then we are done. If $c > q$, it suffices to show that $g_M(-1) = 0$. To this end, we induce on the length of M. If $\ell(M) = 1$, then $M \cong k$. By Tate's Theorem 1.5.1, we know that

$$P_k^R(t) = \frac{g_k(t)}{(1-t)^c(1+t)^{c-q}} = \frac{1}{(1-t)^c}.$$

Thus, since $q < c$, $g_k(-1) = 0$. Now, suppose the length of M is at least two. Then there is a short exact sequence of nonzero modules

$$0 \to M \to M \to M'' \to 0$$

with $\ell(M') < \ell(M)$ and $\ell(M'') < M$. Then by induction, $g_{M'}(-1) = 0$ and $g_{M''}(-1) = 0$, hence by Lemma 3.4.3 $g_M(-1) = 0$. □

Applying Proposition 1.5.3, one can reinterpret Theorem 3.4.1 as follows, using the notation of Section 1.5. One can find a different proof in [3].

Corollary 3.4.6. *Let R be a complete intersection ring of codimension c and quadratic codimension q, and M be a finitely generated R-module. Then*

$$\deg(p_+^M - p_-^M) \leq c - q - 1$$

where $\deg 0$ *is defined to be* $-\infty$*.*

Example 3.4.7. Let $R = k[x, y, z]/(x^2, y^3, z^3)$. Then R is a graded complete intersection of codimension 3 and quadratic codimension 1. The corollary states that for any module M, if $p_-^M \neq p_+^M$, then they can only differ in their constant term. Setting $M = (x, y, z)^2$, we find an example of such a module. For the field $k = \mathbb{Z}/101\mathbb{Z}$, MACAULAY2 gives

$$P_M^R(t) = \frac{1+3t}{(1-t)^3(1+t)},$$

and for $n > 3$ one has

$$\beta_n(M) = \begin{cases} n^2 + \frac{5}{2}n + \frac{3}{2} & n \text{ is even} \\ n^2 + \frac{5}{2}n + 1 & n \text{ is odd.} \end{cases}$$

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