Rational Functions

Standard 2F

Def: A function of the form

$$f(x) = \frac{N(x)}{D(x)}$$

e.g. $f(x) = \dfrac{x^3 - 2x + 1}{x^2 - 4} = \dfrac{x^3 - 2x + 1}{(x+2)(x-2)}$

Domain: $(-\infty,-2)\cup(-2,2)\cup(2,\infty)$

where $N(x)$ and $D(x)$ are polynomials is called a rational function.

- The domain of such a function is all real numbers except for where $D(x)=0$.

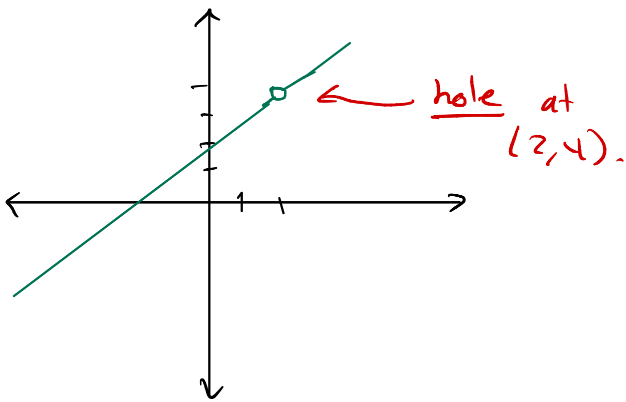
What happens in terms of the graph at these values of x?

E.g. $f(x) = \dfrac{x^2-4}{x-2}$ Domain: All real numbers except $x=2$
$(-\infty, 2)\cup(2,\infty)$.

What happens in terms of the graph when $x=2$?

$$f(x) = \frac{x^2-4}{x-2} = \frac{(x+2)\cancel{(x-2)}}{\cancel{(x-2)}} = x+2 \quad \left(\text{as long as } x \neq 2\right).$$

That is, the graph of $f(x)$ is the line $y = x+2$ with a point removed.



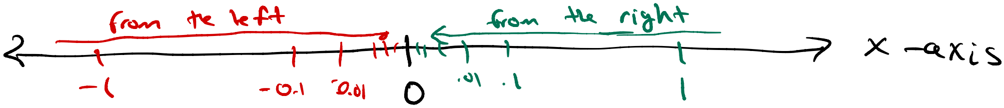
Takeaway: If you can completely cancel out a factor, that corresponds to a hole.

Vertical Asymptote

Consider the function $f(x) = 1/x$

we know $f(0) = 1/0$ is undefined, what happens on the graph as x gets "close" to 0?

I can approach zero from two diffent directions:



"As x approaches 0 from the right....

x	1	0.1	0.01	0.001	0.0001
$f(x)=\frac{1}{x}$	1	10	100	1000	10,000

... $f(x)$ increases without bound."

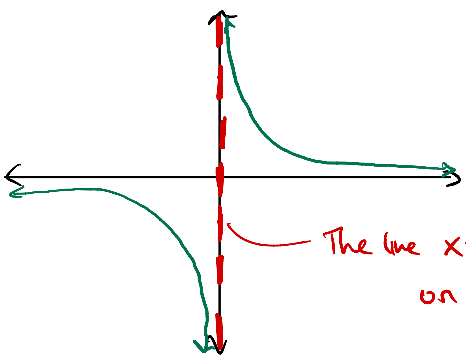
$$\boxed{\begin{array}{l} \text{As } x \to 0^+ \\ f(x) \to \infty \end{array}}$$

"As x approaches 0 from the left...

x	-1	-0.1	-0.01	-0.001	-0.0001
$f(x)=\frac{1}{x}$	-1	-10	-100	-1000	-10,000

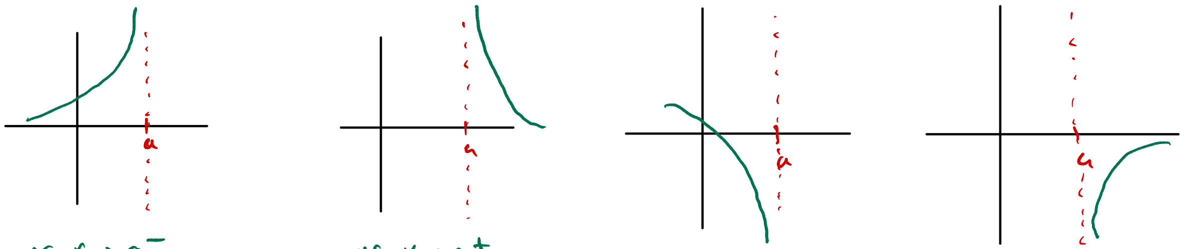
... $f(x)$ decreases without bound."

$$\boxed{\begin{array}{l} \text{As } x \to 0^- \\ f(x) \to -\infty \end{array}}$$



The line $x=0$ is a vertical asymptote on this graph.

Definition: The vertical line $x=a$ is a <u>vertical asymptote</u> for $f(x)$ if $f(x)\to\infty$ or $f(x)\to-\infty$ when $x\to a^+$ or $x\to a^-$.



as $x\to a^-$, $f(x)\to\infty$.

as $x\to a^+$, $f(x)\to\infty$.

as $x\to a^-$ $f(x)\to-\infty$

as $x\to a^+$ $f(x)\to-\infty$.

or some combination of two of them.

How to find zeros, holes, and vertical asymptotes:

Let $f(x)=\dfrac{N(x)}{D(x)}$. First, factor $N(x)$ and $D(x)$ completely and cancel any common factors.

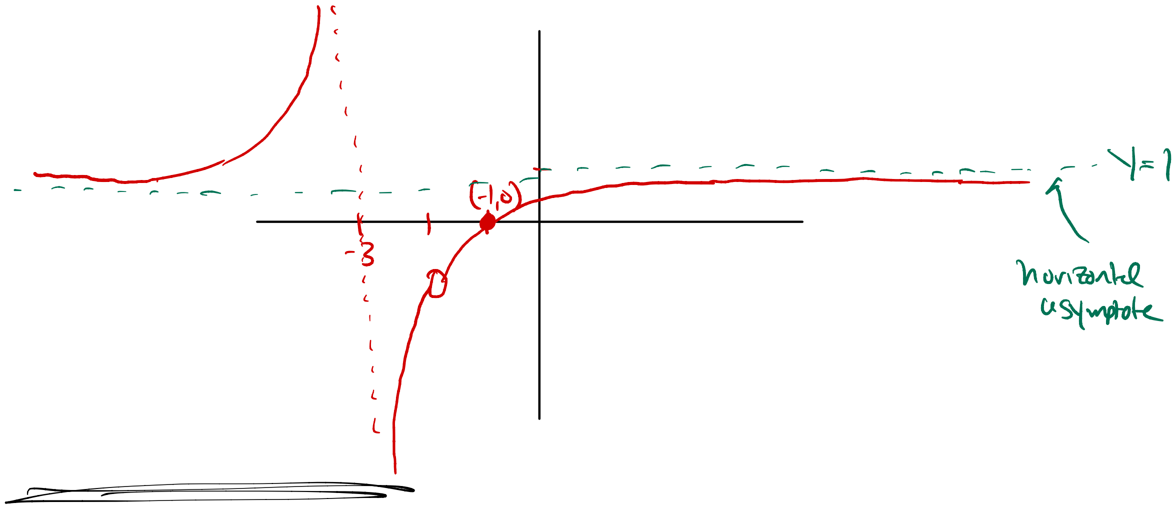
- A zero of $N(x)$ but not $D(x)$ corresponds to a horizontal intercept
- A zero of $D(x)$ but not $N(x)$ corresponds to a vertical asymptote.
- Any factors that canceled completely correspond to holes.

E.g. Find the horizontal intercepts, holes, and vertical asymptotes of

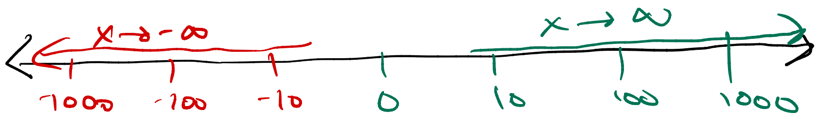
$$f(x)=\frac{x^2+3x+2}{x^2+5x+6}=\frac{\cancel{(x+2)}(x+1)}{\cancel{(x+2)}(x+3)}$$

Conclusion: • Since $x+2$ cancels completely, there's a hole at $x=-2$

- Since $x=-1$ is a zero of $N(x)$ but not $D(x)$, there's a horizontal intercept at $(-1,0)$.
- Since $x=-3$ is a zero of $D(x)$ but not $N(x)$, there's a vertical asymptote at $x=-3$.



Consider the function $f(x) = \frac{1}{x}$. What is its end behavior? That is, what happens as $x \to \infty$ and as $x \to -\infty$?



"As $\boxed{x \text{ increases (goes to right)}}$ . . .

x	10	100	1000	10,000
$f(x)=\frac{1}{x}$	0.1	0.01	0.001	0.0001

. . f(x) approaches 0."

$$\boxed{\begin{array}{l} \text{As } x \to \infty \\ f(x) \to 0 \end{array}}$$

"As x $\boxed{\text{decreases (goes to the left)}}$ . . .

x	-10	-100	-1000	-10,000
$f(x)=\frac{1}{x}$	-0.1	-0.01	-0.001	-0.0001

. . f(x) approaches 0."

$$\boxed{\begin{array}{l} \text{As } x \to -\infty \\ f(x) \to 0 \end{array}}$$

Def: The horizontal line $y=a$ is a <u>horizontal asymptote</u> for $f(x)$ if $f(x) \to a$ as $x \to \infty$ or as $x \to -\infty$ (or both).

Horizontal Asymptotes

E.g. What are the horizontal asymptotes (if any) of

$$f(x) = \frac{3x^2 + 2x + 1}{x^2 + 4} \;?$$

i.e. what happens as $x \to \infty$ or as $x \to -\infty$?

As long as $x \neq 0$, we can rewrite the function as follows:

$$f(x) = \frac{(3x^2 + 2x + 1)}{(x^2 + 4)} \cdot \frac{1/x^2}{1/x^2} = \frac{3 + \underbrace{2/x}_{\to 0} + \underbrace{1/x^2}_{\to 0}}{1 + \underbrace{4/x^2}_{\to 0}} \underset{\text{as } x \to \pm\infty}{\approx} \frac{3}{1} = 3$$

Note: As $x \to \infty$ or $-\infty$, $1/x \to 0$, so does $1/x^2$

Conclusion: As $x \to \infty$ and as $x \to -\infty$, $f(x) \to 3$.

So $f(x)$ has a horizontal asymptote at $\boxed{y=3}$.

Rules for finding Horizontal Asymptotes

Let $f(x) = \frac{N(x)}{D(x)} = \frac{\boxed{a_n x^n} + \cdots + a_0}{\boxed{b_m x^m} + \cdots + b_0}$ $\left(\begin{matrix} a_n \neq 0 \\ b_m \neq 0 \end{matrix}\right)$

The end behavior of $f(x)$ is the same as $\boxed{a_n x^n / b_m x^m}$

- If $n < m$, the x-axis $(y=0)$ is a horizontal asymptote
- If $n = m$ the line $y = a_n / b_m$ is a horizontal asymptote
- If $n > m$ $f(x)$ has the same end behavior as $\frac{a_n}{b_m} x^{n-m}$ according to the leading term test.