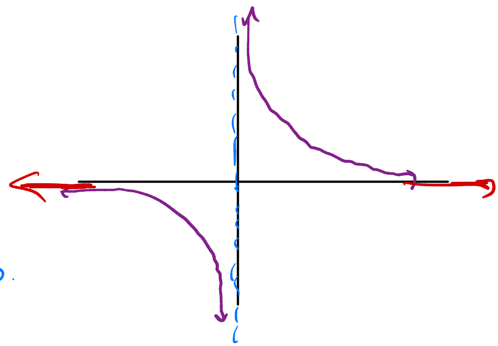
3/15

Reminders: (1) MyMathLab #6 } Due Friday @ 9AM
(2) Written Assignment #6 }
(3) Exam 2 – Next week

- Practice exam posted soon
- Covers Everything since last exam up through wednesday's class
- Review session? Details to come.

Recall: A function $f(x)$ is called rational if it can be written in the form $f(x) = \frac{N(x)}{D(x)}$ where $N(x)$ and $D(x)$ are polynomial functions.

Prototypical example: $f(x) = \frac{1}{x}$



As $x \to 0^+$, $f(x) \to +\infty$, as $x \to 0^-$, $f(x) \to -\infty$

the graph has a vertical asymptote at $x=0$.

End behavior: As $x \to \infty$, $f(x) \to 0$ and as $x \to -\infty$, $f(x) \to 0$

The graph has a horizontal asymptote at $y=0$.

Goal: Graph rational functions

One method: Realize the functions are transformations of $f(x)=\frac{1}{x}$ if possible.

Ex: $h(x)=\dfrac{3x-2}{x-1}$ use long division

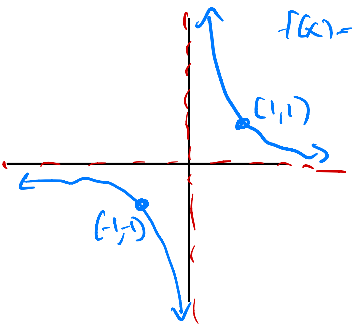
$$\begin{array}{r} 3 \leftarrow \text{quotient} \\ x-1\,\overline{)\,3x-2} \\ -(3x-3) \\ \hline 1 \leftarrow \text{remainder} \end{array}$$

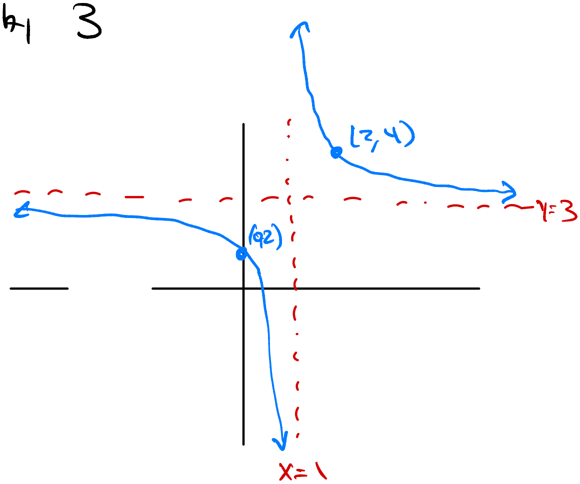
$$h(x) = Q(x) + \frac{R(x)}{D(x)}$$

$$= 3 + \frac{1}{x-1}$$

$h(x) = f(x-1) + 3$ where $f(x) = \dfrac{1}{x}$

So, the graph of h is the graph of $f(x)=\dfrac{1}{x}$ shifted to the right by 1 and up by 3





Finding Vertical Asymptotes

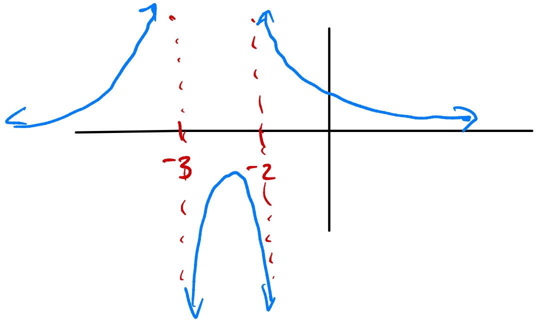
If $f(x)=\dfrac{N(x)}{D(x)}$ is a rational function in lowest terms ($N(x)$ and $D(x)$ do not have any common factors or zeros) and a is a real zero of $D(x)$ then the vertical line $x=a$ is a vertical asymptote on the graph of f.

- If the function is not in lowest terms, there could instead be a hole at $x=a$. (see example from last class)

E.g. $f(x) = \frac{1}{x^2+5x+6} = \frac{1}{(x+2)(x+3)}$

- f is in lowest terms
- Zeros of $D(x)$ are $x=-2,-3$

Conclusion: f has 2 vertical asymptotes, one at $x=-2$, the other at $x=-3$.

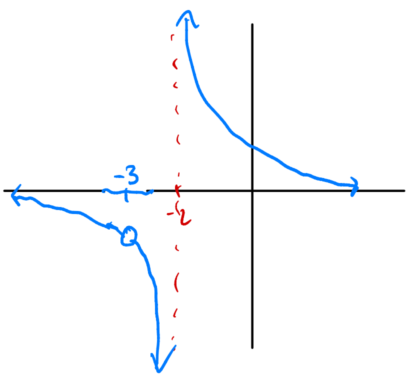


E.g. $g(x) = \frac{x+3}{x^2+5x+6} = \frac{\cancel{(x+3)}}{(x+2)\cancel{(x+3)}}$

$= \frac{1}{x+2} \quad (x \neq -3)$

- f is not in lowest terms
- $-2,-3$ are the zeros of $D(x)$
- once reduced, the only zero in the denominator is at $x=-2$

Conclusion: $g(x)$ has a vertical asymptote at $x=-2$ and a hole at $x=-3$.



To find vertical asymptotes:

1. Completely factor $N(x)$ & $D(x)$
2. Reduce, keeping track of any factors that cancel out
3. Any remaining factors in the denominator correspond to vertical asymptotes

and any canceled out factors correspond to holes.

Horizontal Asymptotes

Eg What are the horizontal asymptotes (if any) of the function

$$f(x) = \frac{3x^2+2x+1}{x^2+4} \quad ?$$

i.e. what happens as $x \to \infty$ or $x \to -\infty$ $(|x| \to \infty)$?

as long as $x \neq 0$ we can rewrite f as follows:

$$f(x) = \frac{3x^2+2x+1}{x^2+4} \cdot \frac{1/x^2}{1/x^2} = \frac{3 + \frac{2}{x} + \frac{1}{x^2}}{1 + 4/x^2}$$

$\frac{2}{x} \to 0$, $\frac{1}{x^2} \to 0$, $4/x^2 \to 0$

Note: as $x \to \infty$ or $x \to -\infty$, $\frac{1}{x} \to 0$

For large values of $|x|$, $f(x) \approx 3/1 = 3$

This shows that as $x \to \pm\infty$, $f(x) \to 3$

Conclusion: $f(x)$ has a horizontal asymptote at $y=3$.

i.e For large x, $f(x) \approx \frac{3x^2}{x^2} = 3$

Rules for locating horizontal asymptotes:

Let $f(x) = \frac{N(x)}{D(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + \cdots + b_1 x + b_0}$ $(a_n \neq 0$ & $b_m \neq 0)$

Then the end behavior of $f(x)$ is the same as the end behavior of $\frac{a_n x^n}{b_m x^m}$

so

1. If $n < m$ then the x-axis $(y=0)$ is a horizontal asymptote
2. If $n = m$ then the line $y = a_n/b_m$ is a horizontal asymptote
3. If $n > m$ then there is no horizontal asymptote and $f(x)$ has the same end behavior as $\frac{a_n}{b_m} x^{n-m}$ (look at chart from before)

E.g. Find the horizontal asymptotes (if any) of the following functions:

① $f(x) = \dfrac{5x+2}{1-2x} = \dfrac{5x+2}{-2x+1}$ here $n=m=1$, $a_n=5$, $b_m=-2$

so $y = \dfrac{a_n}{b_m} = -\dfrac{5}{2}$ is the horizontal asymptote.

② $g(x) = \dfrac{2x-5}{3x^2+2x+1}$, here $n=1$, $m=2$, so $n<m$

So $y=0$ is a horizontal asymptote.

③ $h(x) = \dfrac{4x^2-1}{x+1}$, here $n=2$, $m=1$ so $n>m$

and there is no horizontal asymptote

End behaviour of $h(x)$ is the same as $\dfrac{4x^2}{x} = 4x$

meaning as $x \to \infty$, $h(x) \to \infty$ and as $x \to -\infty$, $h(x) \to -\infty$.

Graphing Rational Functions

Suppose $f(x) = \dfrac{N(x)}{D(x)}$ is in lowest terms. (If you have to reduce, keep track of any holes)

① Find the intercepts

↳ solve $N(x)=0$ to find x-intercepts (if any)

↳ find $f(0)$ to find the y-intercept.

② Find vertical asymptotes (if any)

↳ Solve $D(x)=0$

③ Find the horizontal asymptote (if any)

↳ compare degree of $N(x)$ to degree of $D(x)$.

④ Plug in extra values to determine if the graph is above or below the horizontal asymptote.

E.g. Sketch a graph of $f(x) = \dfrac{3x+1}{x-2}$ (Note: no common factors so no holes)

① Find intercepts:

x-intercepts $\Rightarrow$ solve $f(x)=0$ $\quad \cancel{(x-2)}\dfrac{3x+1}{\cancel{x-2}} = 0 \cdot (x-2)$

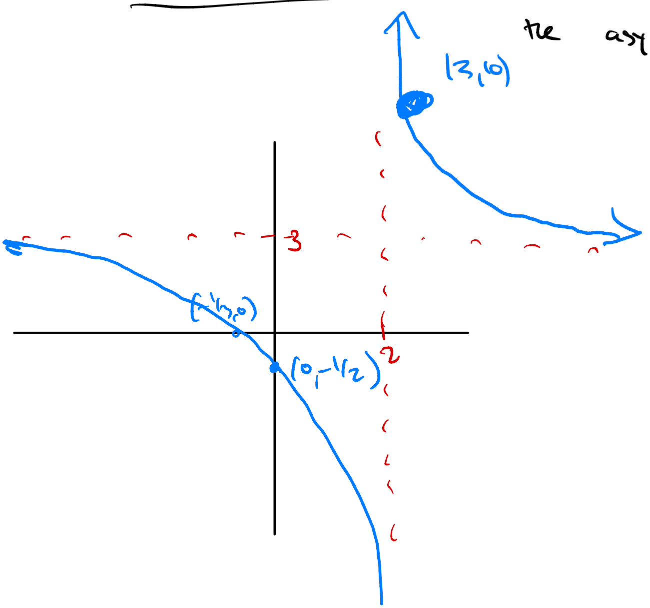
$$3x+1=0 \Rightarrow \boxed{x = -1/3}$$

so $(-1/3, 0)$ is on the graph.

Y-int: $f(0) = \dfrac{3(0)+1}{0-2} = -1/2$ so $(0, -1/2)$ is on the graph.

② Find vertical asymptotes: Since $D(x) = x-2$, the only vertical asymptote is at $\boxed{x=2}$

③ Find horizontal asymptote: Since $N(x)$ and $D(x)$ have the same degree the asymptote is $y = \frac{3}{1} = 3$



④ $f(3) = \dfrac{3(3)+1}{3-2}$

$= 10$