3 Groups

Learning Objectives:

- Determine whether binary structures are groups
- Prove that groups satisfy certain properties, use these to solve equations
- Recognize whether a table represents a group, determine properties of groups from Cayley Tables

Thinking back to the very first day of class, we said that [ ] was

[ ]

Now that we have stripped algebraic manipulation down to its fundamentals, it is time to build up the kind of structures in which it make sense to "do algebra". Consider the following algebraic structures:

$*$	a	b	c
a	c	a	b
b	a	b	c
c	b	c	a

$*$	a	b	c
a	a	a	a
b	b	b	b
c	b	b	b

More concretely, consider $\langle \mathbb{Z}, + \rangle$ or $\langle \mathbb{Q}^*, \cdot \rangle$. Then one can solve any equation that resembles those above:

There are only three properties that make this possible – [ ], the existence of an [ ] element [ ], and the ability to solve the equation [ ] and [ ] for all [ ].

Definition 1. Let ☐ be a ☐ with operation ☐. Then ☐ is a ☐ if it satisfies the following three ☐:

1. ☐ For all ☐ we have ☐
2. ☐ there is an element ☐ in ☐ such that ☐
3. ☐ for wach ☐ there is an element ☐ such that ☐

Example 1. We now turn our heads to our motivating examples, namely $\langle U, \cdot\rangle$ and $\langle U_n, \cdot\rangle$.

Remark 1. When speaking about a ☐ ☐, one usually drops the ☐ and simply writes ☐. If we need to specify what the ☐ is, we say "the group ☐ under ☐", for example " the group ☐ under ☐."

Definition 2. A group ☐ is called ☐ if its binary operation is ☐

Remark 2. This is in honor of Norwegian mathematician Niels Henrik Abel (1802-1829).

Example 2. We now present several examples and non-examples.

1.

2.

3.

4.

5.

6.

Example 3. We now present several examples related to Linear algebra.

1. Let V be a vector space.

2. Let $M_{m\times n}(\mathbb{R})$ be the set of $m \times n$ matrices with real entries.

3. Let $M_n(\mathbb{R})$ be the set of $n \times n$ matrices with real entries.

4. Claim: The subset of $M_n(\mathbb{R})$ consisting of all *invertible* matrices forms a group under multiplication.

   Proof. ∎

Definition 3. The subset ______ of ______ of ______ matrices is called the ______.

Remark 3. Note that ______ is the set of all invertible linear transformations from ______ to ______. More generally, for any set ______ you can interpret the set of invertible functions ______ as a group under ______.

Exercise 1. Prove that $\mathbb{Q}^+$ under $*$ is a group, where $*$ is defind by $a * b = \dfrac{ab}{3}$.

Proof.

$\square$

3.1 Properties of Groups

We now make precise what it means "cancel" something in algebra.

Theorem 1. (Cancellation Laws)

If $\boxed{}$ is a group with binary operation $\boxed{}$, then the **left and right cancellation laws** hold in $\boxed{}$. That is, for all $\boxed{}$ we have

Proof.

$\square$

The next result is related to solutions "linear" equations. For example

Theorem 2. If ____ is a group with binary operation ____, and if ____ then the linear equations ____ and ____ have unique solutions ____ and ____ in ____.

Proof.

∎

Remark 4. Note that in the above theorem, the solutions ____ and ____ need not be the same unless ____ is ____.

Theorem 3. (Uniqueness of Identity and Inverse Elements)
In a group ____ with operation ____, there is only one identity element. That is, there is only one element ____ such that

Given ____ there is only one ____ of ____. That is, there is only one element ____ in ____ such that

Proof. □

Corollary 1. (Inverse Formula)

Suppose that [ ] is a group and [ ]. Then

Proof. □

3.2 Finite Groups and Cayley Tables

We will now examine all the possible groups of low cardinality. A general theme of group theory is to *classify* all groups – that is, describe every possible group structure on a given set.

Let G be a group with $|G| < \infty$. Let's examine the possible group structures for low values of $|G$—.

$|G| = 0$:

$|G| = 1$:

$|G| = 2$:

$|G| = 3$:

Remark 5. Some things to notice:

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