February 21, 2020 — MAT 370: Exam #1 — Name: ____________________

Instructions: Read the entire statement of each problem. Solve each problem carefully and organize your work. Be sure to write your answers in complete sentences where appropriate. The exam is worth 60 points.

Part One: True or False. Decide whether the following statements are true or false. You do not need to justify your answer.

1. (2 Points) If two binary algebraic structures are isomorphic then they must have the same cardinality.

2. (2 Points) A set must contain at least one element.

3. (2 Points) Every function φ from a set S to a set T is a relation between S and T.

4. (2 Points) Let $\langle S, * \rangle$ be a binary structure. The property "S contains the element 1" is a structural property of S.

5. (2 Points) For $a, b \in \mathbb{Z}$ the rule $a \star b = a/b$ defines a binary operation on $\mathbb{Z}$.

Part Two: Short Aswer and Computation Problems

1. (4 Points) Explain why the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is **not** one-to-one.

2. (5 Points) There is an isomorphism $\varphi : U_6 \to \mathbb{Z}_6$ such that $\varphi(\zeta) = 5$ where $\zeta = e^{i\frac{\pi}{3}}$. Find $\varphi(\zeta^i)$ for $i = 0, 2, 3, 4$.

3. (4 Points) Explain why the binary structure $\langle \mathbb{Z}, \cdot \rangle$ is not a group.

4. (6 Points) Let $S = \{a, b, c\}$ and $T = \{\#, 2, c\}$. List the elements of $S \times T$. What is the cardinality of $S \times T$?

5. (5 Points) Suppose that $G = \{e, a, b, c, d\}$ is an abelian group with e the identity element for $*$. Fill in the gaps in the table below.

$*$	e	a	b	c	d
e					
a		c		d	
b		e			
c			a		e
d		b	c		

6. Define a binary operation $*$ on $\mathbb{Z}$ by $a * b = \min\{a, b\} + 4$.

(a) (3 Points) Is $*$ commutative? Explain or give a counterexample.

(b) (3 Points) Is $*$ associative? Explain or give a counterexample.

Part Three: Proofs. For the following three problems, pick **only two** of them to submit for a grade. You may attempt all three problems, but indicate which two you'd like to submit. If you do not clearly indicate which two you select, only the first two shall be graded.

1. (10 Points) Show that the relation $\mathcal{R}$ on $\mathbb{Z}$ defined by "$x\mathcal{R}y$ if and only if $x - y$ is even" is an equivalence relation. (Here, the number 0 counts as being even.)

2. (10 Points) Show that $\langle \mathbb{Q}, * \rangle$ is a group if $*$ is defined by $a * b = 5ab$.

3. (10 Points) Show that the binary structure $\langle \mathbb{Z}, + \rangle$ is isomorphic to $\langle 5\mathbb{Z}, + \rangle$, where $5\mathbb{Z} = \{0, \pm 5, \pm 10, \pm 15, ...\}$.