1 Linear Equations in Linear Algebra

1.1 Systems of Linear Equations

Learning Objectives:

- Identify linear equations and linear systems
- Use matrices and row operations to solve linear systems
- Identify if a system is consistent and if solutions are unique

As is suggested in the name, linear algebra is a branch of mathematics concerning linear equations and linear functions. For the beginning of this course, we will first learn some useful techniques for representing systems of linear equations and investigate the possible scenarios that can occur in finding solutions to such systems.

Definition 1. Linear Equation

Example 1. An example of a linear equation is

$$3x_1 + 2x_2 = 6$$

Another example is

$$4(x_1 - x_4) = \sqrt{2}x_2 + 1$$

Example 2. Some examples of equations that are **not** linear are

Note 1. You are familiar with the linear equation

$$y = mx + b$$

However, renaming the variables x and y to x_1 and x_2 respectively, we get

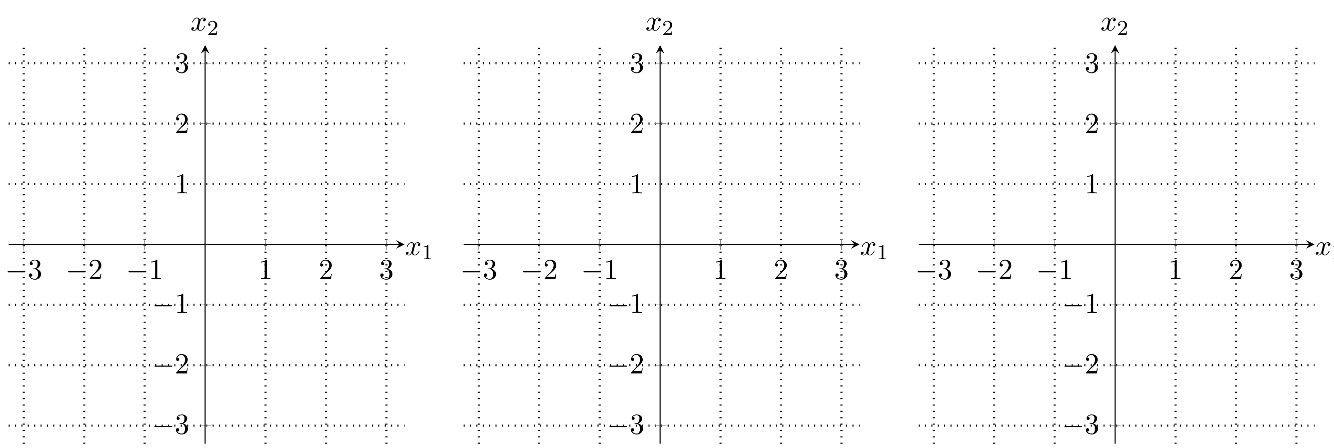
Definition 2. System of Linear Equations

Example 3. An example of a system of linear equations is

Definition 3. Solution, Solution Set, Equivalent Systems

In two variables, we can represent linear systems graphically in two dimensional space, by making the identification $(x, y) = (x_1, x_2)$. For example, one can rewrite the following systems and graph them:

$$\begin{aligned} x_1 - x_2 &= 1 \\ x_1 - 2x_2 &= 1 \end{aligned} \qquad \begin{aligned} x_1 - x_2 &= 1 \\ x_1 - x_2 &= 0 \end{aligned} \qquad \begin{aligned} x_1 - x_2 &= 1 \\ -2x_1 + 2x_2 &= -2 \end{aligned}$$



Fact 1. A system of linear equations either has

1. $\boxed{}$ or
2. $\boxed{}$ or
3. $\boxed{}$.

Definition 4. Consistent System

Matrix Notation

The crucial information in a linear system can be represented succinctly using a rectangular array of real numbers called a $\boxed{}$. For example, given the linear system

$$\begin{array}{rcrcrcr} 3x_1 & - & 2x_2 & + & 3x_3 & = & 4 \\ x_1 & & & + & x_3 & = & \pi \\ & & 2x_2 & - & \sqrt{2}x_3 & = & 100 \end{array}$$

We can record the the coefficients of each variable aligned in columns in the $\boxed{}$.

We can also include the values on the right hand side of the equation in the $\boxed{}$.

Often, a (dotted) line is drawn in to separate the coefficients from the values on the other side of the equations.

Definition 5. Size of a Matrix

The utility of matrices for our current purposes will be to simplify the solving of systems of equations. Whenever one solves a system of equations, the solution is produced by replacing the system with an equivalent system that is easier to solve. Instead of doing all our calculations by working with the equations themselves, we can instead write the system as an augmented matrix and work with the rows of this matrix, then translate the matrix back to a system of equations. However, we can only perform operations that don't change the solution of the system.

To summarize:

The three allowable operations are:

-
-
-

Example 4. Here, we will solve the system below by working both with the equations and with the augmented matrix side-by-side.

$$\begin{array}{rcrl} x_1 & + & x_2 & = 7 \\ -2x_1 & + & x_2 & = -2 \end{array}$$

Manipulating the equations using the three allowable options corresponds to the following operations on rows of the augmented matrix associated to the system.

Definition 6. Elementary Row Operations

1.

2.

3.

Note that each elementary row operation is ______. Further, since the solutions to a system are unchanged via row operations, we can conclude that

Existence and Uniqueness

Given a linear system, we know that there are three possibilities as to its solution set: it can have ______, ______, or ______ solutions. Thus, there are two fundamental questions we can ask about a system:

1.
2.

These two questions will appear frequently throughout the course, in many different situations.

Exercise 1. Determine if the system below is consistent by using augmented matrices:

$$\begin{array}{rcrcr} 3x_1 & - & x_2 & = & 10 \\ x_1 & - & 2x_2 & = & 0 \end{array}$$

Exercise 2. Determine if the following system is consistent by using augmented matrices:

$$\begin{array}{rcrcrcl} & & x_2 & - & 4x_3 & = & 8 \\ 2x_1 & - & 3x_2 & + & 2x_3 & = & 1 \\ 4x_1 & - & 8x_2 & + & 12x_3 & = & 1 \end{array}$$

1.2 Row Reduction and Echelon Forms

Learning Objectives

- Identify whether a matrix is in echelon form and find its pivot positions
- Utilize the row reduction algorithm to put a matrix in (reduced) echelon form.
- Determine the solution set of a linear system by using row reduction and its associated augmented matrix.

We have seen that row operations on augmented matrices can be used to put the matrix in a form where the corresponding solutions can be more easily discerned. Now, we will more rigorously say what this form is, and how it can be used to classify and identify the solution set of a system. In defining this form, we will refer to the ______ of a row, meaning the leftmost ______ entry (in a nonzero row).

Definition 7. Echelon Form, Reduced Echelon Form

1.

2.

3.

4.

5.

Example 5. Which form are the following matrices in?

$$A = \begin{bmatrix} 4 & 2 & 7 & 6 \\ 0 & 2 & 3 & 1 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 0 & 45 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -4 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & 1 & 5 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & 0 & 45 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & -4 \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$F = \begin{bmatrix} 1 & 0 & 0 & 45 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & -5 \end{bmatrix}$$

$$G = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix}$$

Note 2.

-

-

Theorem 1.

Proof. See Appendix A. □

While its true that a given matrix can have many different equivalent matrices in echelon form, each one of these possible matrices have something in common. The leading entries will always be in the same position. The locations of these leading terms are important, so we have special terminology to refer to them

Definition 8. Pivot Position, Pivot Column

-

-

.

Example 6. Suppose that a matrix A has been row reduced to the following:

$$A = \begin{bmatrix} 2 & -3 & 4 & 0 & 1 \\ 0 & -1 & 4 & 6 & 10 \\ 6 & -9 & 12 & 4 & 0 \\ -2 & 1 & 4 & 16 & 16 \end{bmatrix} \xrightarrow{\text{Row Reduce}} \begin{bmatrix} 2 & -3 & 4 & 0 & 1 \\ 0 & -1 & 4 & 6 & 10 \\ 0 & 0 & 0 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The Row Reduction Algorithm

Here, we will show by example the algorithm for producing a matrix in (reduced) echelon form which is row-equivalent to the matrix we started with.

Example 7. Let $A = \begin{bmatrix} 0 & 3 & 6 \\ 3 & -7 & -5 \\ 3 & -9 & -9 \end{bmatrix}$

Step 1:

Step 2:

Step 3:

Step 4:

The matrix is now in

Step 5:

The matrix is now in

Note 3.

Example 8. Use row reduction to solve the following system:

$$\begin{array}{rcrcr} & & 3x_2 & = & 6 \\ 3x_1 & - & 7x_2 & = & -5 \\ 3x_1 & - & 9x_2 & = & -9 \end{array}$$

Solution:

Solutions of Linear Systems

We can use row reduction to describe the solutions of linear systems, even if they have infinitely many solutions. Note that, given the linear system

$$\begin{array}{rcrcr} 2x_1 & + & 4x_2 & = & 6 \\ -3x_1 & - & 6x_2 & = & -9 \end{array}, \quad \text{its corresponding augmented matrix is } \left[\begin{array}{c} \\ \\ \end{array} \right]$$

Definition 9. Basic Variable, Free Variable

Returning to the system above, we can row reduce the augmented matrix to find:

Exercise 3. In the following system, determine which variables are free and which are basic. Then, describe the solution set in the form of the previous example.

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & & & = & 2 \\ -2x_1 & - & 2x_2 & + & x_3 & = & 0 \\ 3x_1 & + & 3x_2 & + & x_3 & = & 10 \end{array}$$

$$\begin{cases} x_1 \boxed{} \\ x_2 \boxed{} \\ x_3 \boxed{} \end{cases}$$

In the previous section, we mentioned that there are two fundamental questions we would like to be able to answer about a system of equations:

1.
2.

It turns out that just by **knowing the location of the pivot positions** of the augmented matrix, we can answer both of these questions.

Theorem 2. Existence and Uniqueness Theorem

i)

ii)

Example 9. Circle the pivot positions of the following augmented matrices and categorize them according to the Existnce and Uniqueness Theorem.

$$A = \begin{bmatrix} 4 & 2 & 7 & 6 \\ 0 & 2 & 3 & 1 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 0 & 45 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -4 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & 1 & 5 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Exercise 4. Row reduce the following matrix to reduced echelon form. Circle the pivot positions in both the final and original matrix, and list the pivot columns.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 \end{bmatrix}$$

Exercise 5. Find the **general solution** of the system below (that is, write the solutions like we did on pages 15 and 16):

$$\begin{array}{rcrcrcl} x_1 & - & 2x_2 & - & x_3 & = & 3 \\ 3x_1 & - & 6x_2 & - & 3x_2 & = & 9 \end{array}$$

$$\begin{cases} x_1 & \boxed{} \\ x_2 & \boxed{} \\ x_3 & \boxed{} \end{cases}$$

1.3 Vector Equations

Learning Objectives:

- Interpret vectors in $\mathbb{R}^2$ geometrically
- Add, subtract, and compute scalar multiples of vectors
- Use vectors to represent systems of linear equations and their solutions.

Vectors in $\mathbb{R}^2$

Definition 10. Vectors

Vectors will be denoted by a lowercase bold letter such as $\mathbf{v}$ or using an arrow above a lowercase letter, such as $\vec{v}$ or $\boxed{}$.

Some examples of vectors with real entries are

$$\mathbf{u} = \begin{bmatrix} \\ \\ \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} \\ \\ \end{bmatrix}, \quad \text{or more generally } \mathbf{w} = \begin{bmatrix} \\ \\ \end{bmatrix} \text{ for any } \boxed{}$$

Notation 1.

Equality of Vectors

Addition of Vectors

$$\begin{bmatrix} 2 \\ -5 \end{bmatrix} + \begin{bmatrix} 6 \\ 8 \end{bmatrix} = \begin{bmatrix} \boxed{} \\ \boxed{} \end{bmatrix} = \begin{bmatrix} \boxed{} \\ \boxed{} \end{bmatrix}$$

Scalar Multiplication

$$\text{if } \mathbf{u} = \begin{bmatrix} 2 \\ -5 \end{bmatrix} \text{ and } c = 3, \text{ then } c\mathbf{u} = 3 \begin{bmatrix} 2 \\ -5 \end{bmatrix} = \begin{bmatrix} \boxed{} \\ \boxed{} \end{bmatrix}$$

Note 4. To distinguish them from vectors, in this context, real numbers are referred to as $\boxed{}$

Exercise 6. Let $\mathbf{u} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$. Compute the following:

a) $\mathbf{u} + \mathbf{v}$

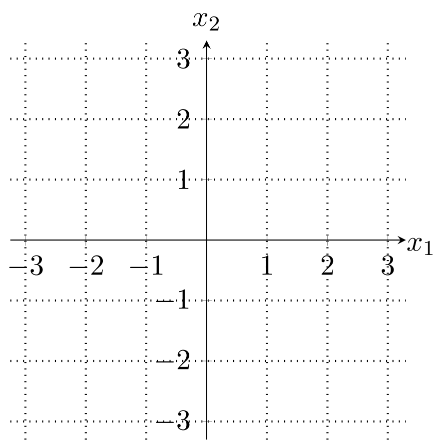
b) $3\mathbf{v}$

c) $4\mathbf{u} - \frac{1}{2}\mathbf{w}$

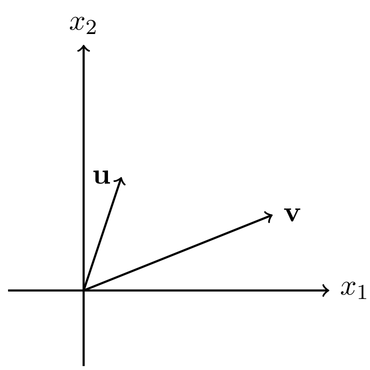
d) Find a vector $\mathbf{a}$ such that $3\mathbf{a} - 2\mathbf{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Geometric Description of $\mathbb{R}^2$

Consider the Cartesian plane. Since every point $\boxed{}$ in the plane consists of an ordered pair of numbers, we can identify the point with the vector $\begin{bmatrix} \boxed{} \\ \boxed{} \end{bmatrix}$. Thus we may regard $\boxed{}$ with the set of all points in the plane.

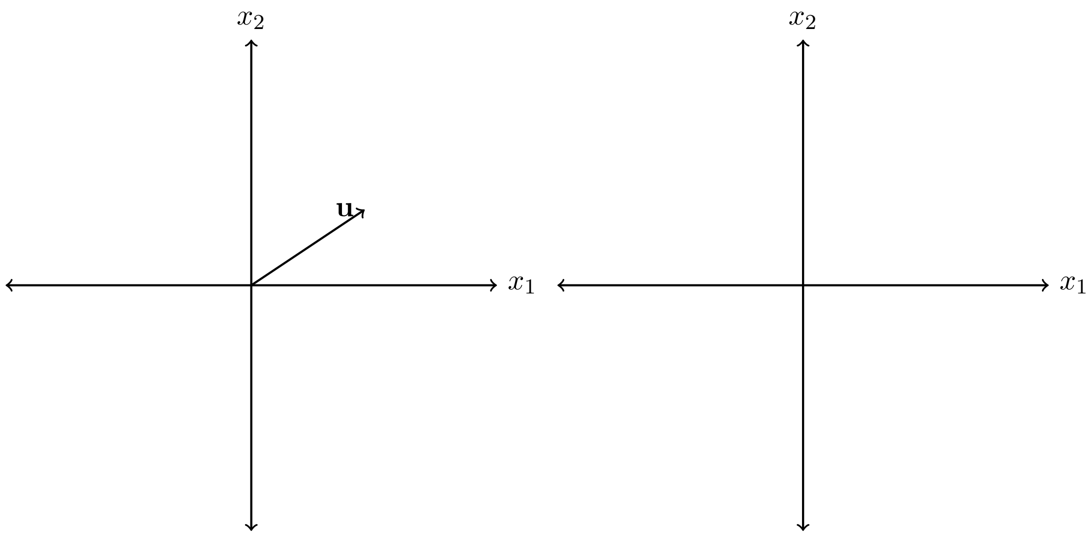


It is often useful to visualize a vector such as $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ as an $\boxed{}$ starting at the point $\boxed{}$ and ending at the point $\boxed{}$. In this situation, one can visualize the $\boxed{}$ of two vectors geometrically:



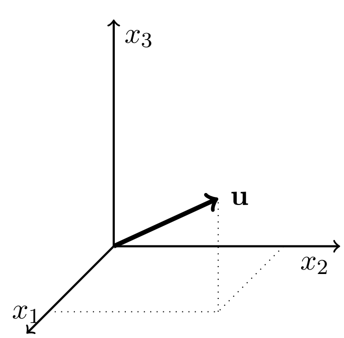
Parallelogram Rule for Addition:

Scalar Multiplication and Geometry



- If $c > 1$ then $c\mathbf{u}$ is $\boxed{}$ than $\mathbf{u}$ and points in the $\boxed{}$ direction.
- If $0 < c < 1$ then $c\mathbf{u}$ is $\boxed{}$ than $\mathbf{u}$ and points in the $\boxed{}$ direction.
- If c is $\boxed{}$ then $c\mathbf{u}$ points in the $\boxed{}$ direction as $\mathbf{u}$.
- The set of all multiples of $\mathbf{u}$ is a $\boxed{}$ through $\boxed{}$.

Note 5. There are 3-dimensional analogs of these geometric representations. Vectors in $\mathbb{R}^3$ are $\boxed{}$ matrices. This means the vector has with $\boxed{}$ entries. They correspond to points in a 3-dimensional coordinate space us x_1, x_2, and x_3 axes.



Vectors in $\mathbb{R}^n$

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

Algebraic Properties of Vectors in $\mathbb{R}^n$

Let $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ be vectors in $\mathbb{R}^n$ and c, d be scalars in $\mathbb{R}$.

1.
2.
3.
4.
5.
6.
7.
8.

We often write $\boxed{}$ instead of $\boxed{}$ for simplicity.

Exercise 7. Simplify the following:

a) $2\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} - \begin{bmatrix} 4 \\ 5 \\ 6 \\ 7 \end{bmatrix} =$

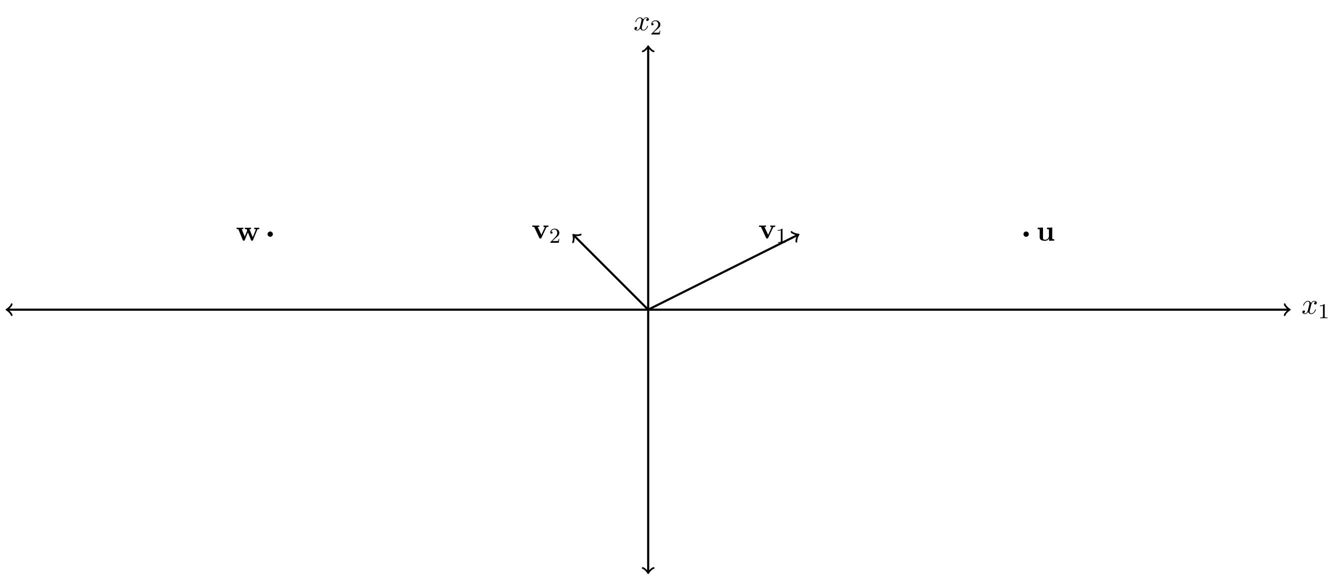
b) $4\left(\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} -4 \\ 5 \\ 10 \end{bmatrix}\right) =$

Linear Combinations

Definition 11. Linear Combination

Note: the scalars can be any real number including zero. For example the following are all possible linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$:

Example 10. Let $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ Estimate the linear combinations that generate the vectors $\mathbf{u}$ and $\mathbf{w}$.



$\mathbf{u} =$

$\mathbf{w} =$

Example 11. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$. Is $\mathbf{b}$ a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$? That is, determine if there are there weights x_1 and x_2 such that

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = \mathbf{b}$$

Solution: After rewriting, we shall see that this is the same as solving a system of linear equations.

To summarize our findings from the previous example, a vector equation such as

has the same solution set as the linear system whose augmented matrix is

In particular, **b** can be expressed as a $\boxed{}$ of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ if and only if there exists a solution to the linear system that corresponds to the matrix above. Thus, one can "translate" between the following:

$$\left\{\boxed{}\right\} \leftrightarrow \left\{\boxed{}\right\} \leftrightarrow \left\{\boxed{}\right\}$$

Span

A related important idea in linear algebra is to study the set of all possible linear combinations generated by a set of vectors.

Definition 12. Span

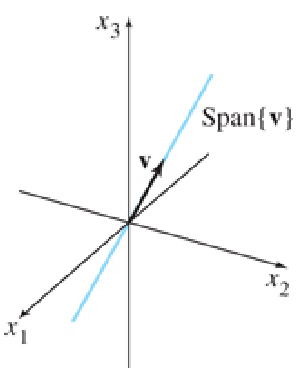
Note 6. Span and Consistency

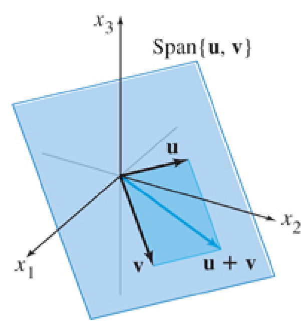
Remark 1. Span$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ contains the vector $\mathbf{v}_1$, because

The zero vector is also in Span$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$, since

Geometric Meaning of Span

If $\mathbf{u}$ and $\mathbf{v}$ are vectors $\mathbb{R}^3$, then Span$\{\mathbf{u}\}$ and Span$\{\mathbf{u}, \mathbf{v}\}$ have nice geometric representations.





As long as $\mathbf{u}$ and $\mathbf{v}$ are not zero then

- Span$\{\mathbf{v}\}$ is a
- Span$\{\mathbf{u}, \mathbf{v}\}$ is a

Exercise 8. Let $\mathbf{v}_1 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$

a. List 3 distinct vectors in Span$\{\mathbf{v}_1\}$.

b. List 3 distince vectors in Span$\{\mathbf{v}_1, \mathbf{v}_2\}$.

c. Determine if $\mathbf{b} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$ is in Span$\{\mathbf{v}_1, \mathbf{v}_2\}$.

Exercise 9. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}$. Then $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ is a plane through the origin in $\mathbb{R}^3$. Is $\mathbf{b}$ on that plane?

(Hint: you can determine this by solving a system of linear equations)

1.4 The Matrix Equation $A\mathbf{x} = \mathbf{b}$

Learning Objectives:

- Represent linear systems/vector equations using matrix equations.
- Use theorems to determine (with justification) the existence of solutions and related statements.
- Compute matrix-vector products

So far, we have seen that linear systems can be viewed from several perspectives, including augmented matrices and vector equations. In this section, we present yet another perspective: equations involving both matrices and vectors.

Definition 13. Matrix-Vector Product

Note 7.

Example 12. Compute the following products, if possible

1. $\begin{bmatrix} 2 & 0 & -1 \\ 3 & 4 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

2. $\begin{bmatrix} 2 & 0 \\ -1 & 4 \\ 7 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

Example 13. Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be vectors in $\mathbb{R}^m$. Write the linear combination $6\mathbf{v}_1 - 3\mathbf{v}_2 + \mathbf{v}_3$ as a matrix times a vector.

Example 14. Write the following system of equations first as a vector equation, then as a matrix equation.

$$\begin{array}{rcrcrcr} 2x_1 & + & 3x_2 & - & 5x_3 & = & 0 \\ & & x_2 & + & 7x_3 & = & 10 \end{array}$$

Theorem 3.

Existence of Solutions

Fact 2.

A harder existence question is whether $A\mathbf{x} = \mathbf{b}$ is consistent for *all possible* $\mathbf{b}$:

.

Example 15. Let $A = \begin{bmatrix} 1 & 2 \\ -2 & -4 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$. Is the equation $A\mathbf{x} = \mathbf{b}$ consistent for all possible b_1, b_2?

Note 8.

Remark 2. The word ______ can be used as noun, such as ______ or as a verb such as ______.

Theorem 4.

a.

b.

c.

d.

Warning: In the statement of Theorem 4, A refers to the ____ matrix ____, **not** the ____ matrix, which would be ____.

Computing $A\mathbf{x}$

Noticing a pattern when computng a matrix-vector product leads to a more efficient method for computing the entries of $A\mathbf{x}$

Example 16. Compute $A\mathbf{x}$ where $A = \begin{bmatrix} 2 & 3 \\ 5 & 0 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

The Row-Vector Rule for Computing $A\mathbf{x}$:

Exercise 10. Compute the following matrix-vector products:

a) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 7 \end{bmatrix} =$

b) $\begin{bmatrix} 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} =$

c) $\begin{bmatrix} 4 & -1 \\ -2 & 0 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} =$

d) $\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix} =$

Properties of the Matrix-Vector Product

Theorem 5.

1.

2.

Proof. □

Exercise 11.

1. Compute the following matrix operations. If a product is undefined, explain why.

(a) $\begin{bmatrix} 2 & 4 \\ 7 & 5 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 6 \\ -2 \end{bmatrix}$

(b) $\begin{bmatrix} 6 & 0 \\ -2 & 5 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 8 \\ -2 \\ 1 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 1 & 0 \\ 3 & 5 & -2 \\ -3 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

2. For the A and $\mathbf{b}$ below, write the augmented matrix that corresponds to the matrix $A\mathbf{x} = \mathbf{b}$, then solve the system and write the solution as a vector.

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 5 \\ -2 & -4 & -3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -2 \\ 2 \\ 9 \end{bmatrix}$$

3. Is $\mathbf{u} = \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}$ in the span of the collumns of $A = \begin{bmatrix} 3 & -5 \\ 2 & 6 \\ 1 & 1 \end{bmatrix}$? Why or why not?

1.5 Solutions of Linear Systems

Learning Objectives:

- Solve homogeneous and nonhomogeneous systems of equations
- Write solutions for linear systems in parametric vector form
- Visualize solution sets in $\mathbb{R}^2$ or $\mathbb{R}^3$

As we have seen in previous sections, a given system of linear equations can have either ____, ____, or ____ solutions. Describing the solution sets of the first two possibilities is straightforward. Today, we will develop a more explicit way to describe solutions set when there are infinitely many solutions.

Homogenous Linear Systems

Definition 14. Homogeneous Systems

Example 17. An example of a homogeneous system is

$$\begin{array}{rcrcl} 2x_1 & + & 4x_2 & = & 0 \\ -4x_1 & - & 8x_2 & = & 0 \end{array}$$

Notice that a homogeneous system $A\mathbf{x} = \mathbf{0}$ always has ____ solution, namely ____.

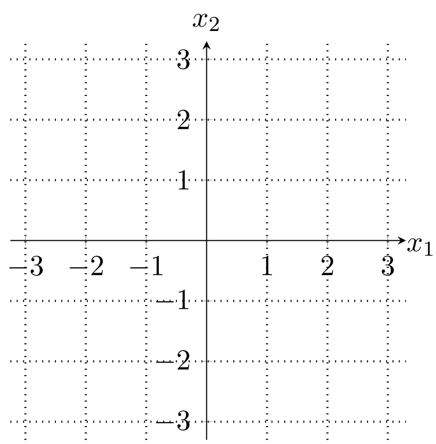
Definition 15. Trivial/Non-Trival Solutions

Thus, for a homogeneous system the ____ question is easy to answer, the more interesting question is the ____ question, that is, are there ____ solutions? Applying the ____ Theorem yields the following result:

Corollary 1.

Example 18. Determine if the following system has any ______ solutions, then describe the solution set algebraically in terms of vectors and geometrically by a graph.

$$\begin{array}{rcrcl} 2x_1 & + & 4x_2 & = & 0 \\ -4x_1 & - & 8x_2 & = & 0 \end{array}$$



Example 19. The following simple linear system has ______ solutions. Describe the solution set algebraically and geometrically.

$$10x_1 - 5x_2 - 3x_3 = 0$$

The original equation in the example above is an *implicit* description as a plane. The algebraic description that we came up with is called an *explicit* description of the plane, since it gives us every single point on the plane *explicitly* as a span of vectors.

An equation such as

is said to be a ______.

We can alternatively describe the solution set as:

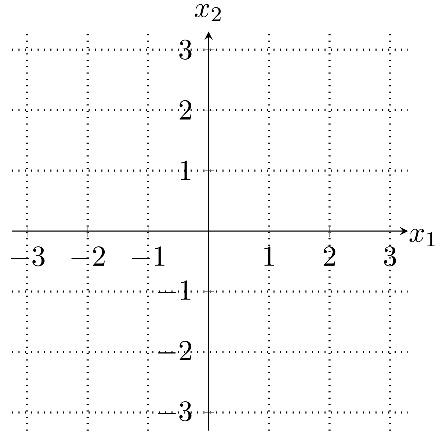
This is called ______ of a solution.

Solutions to Nonhomogeneous Systems

Writing the solution to a **nonhomogeneous** system in paramteric is similar to that of a homogenous system, but requires an extra step. A nonhomogeneous system will be in the form $\boxed{\qquad\qquad\qquad}$ where $\boxed{\qquad}$ is $\boxed{\qquad\qquad\qquad}$.

Example 20. Write the solutions to the following system in parametric vector form, and describe the solution set geometrically.

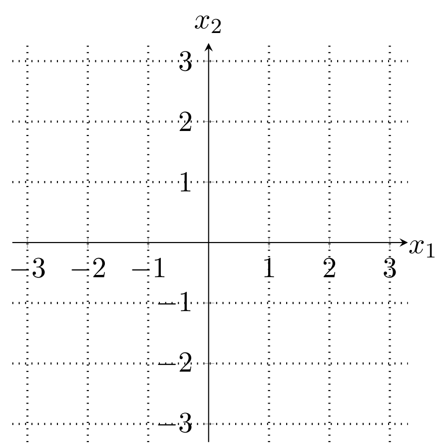
$$\begin{array}{rcrcr} 2x_1 & + & 4x_2 & = & 4 \\ -4x_1 & - & 8x_2 & = & -8 \end{array}$$



Note 9. In the previous example, we found the solutions to and before, we found the solutions to for the same A. Note that the solution set to the homogeneous equation was

while the solution set to the nonhomogeneous equation was

In other words, every solution to is a of a solution to .



The solution to the equation is the line through , while the solution to is the line through and to .

The next theorem formailizes the above discussion.

Theorem 6.

Warning: The theorem only applies when is consistent, meaning it has at least one . When has no solution, the solution set is .

How to Write the Solution Set of a (Consistent) System in Parametric Vector Form:

1.

2.

3.

4.

Exercise 12. . Describe all of the solution of $A\mathbf{x} = \mathbf{0}$ in parametric vector form if

$$A = \begin{bmatrix} 3 & -9 & 6 \\ -1 & 3 & -2 \end{bmatrix}$$

.

Exercise 13. Describe all of the solutions of $A\mathbf{x} = \mathbf{b}$ if

$$A = \begin{bmatrix} 3 & -9 & 6 \\ -1 & 3 & -2 \end{bmatrix} \quad \text{and } \mathbf{b} = \begin{bmatrix} 5 \\ -4 \end{bmatrix}$$

.

1.7 Linear Independence

Learning Objectives

- Determine if a set of vectors is linearly independent by solving a linear system
- Relate linear independence to the consistency of linear systems
- Use theorems to quickly determine linear dependence

In the previous section, we noted that a homogeneous matrix equation $A\mathbf{x} = \mathbf{0}$ always has at least one solution, the trivial solution $\mathbf{x} = \mathbf{0}$. The more interesting question is whether there are *nontrivial* solutions. Today, we expand on this topic from the perspective of vector equations.

Definition 16. Linearly Independent/Dependent Vectors, Linear Dependence Relation

Example 21. The set of vectors $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ is linearly $\boxed{}$ since

Here, the weights are $c_1 = \boxed{}$, $c_2 = \boxed{}$, $c_3 = \boxed{}$.

Example 22. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_3 = \begin{bmatrix} -7 \\ -8 \\ -9 \end{bmatrix}$.
Determine whether the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linear independent. If not, find a linear dependence relation among the vectors.

Linear Independence and Matrix Columns

Fact 3.

Exercise 14. Determine if the columns of the matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 3 & 4 \\ -2 & 5 & 14 \end{bmatrix}$ are linearly dependent.

There are several results that can help one determine if a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is linearly dependent without having to explicitly solve the equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n = \mathbf{0}\,.$$

Sets with One Vector:

Sets with Two Vectors:

Fact 4.

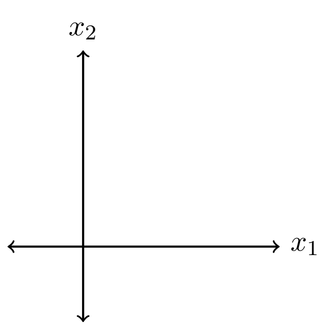
Example 23. Determine if the following sets of vectors are linearly independent or not.

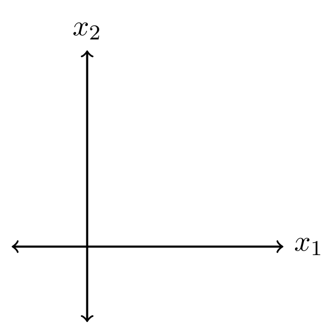
1. $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 6 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -6 \\ -12 \end{bmatrix}$

2. $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 6 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 3 \\ -12 \end{bmatrix}$

Warning:

There is a nice geometric interpretation for sets of two vectors. Two vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ are linearly indpendent if and only if they are not parallel, that is, the don't point in the same direction or exactly opposite directions.





Sets with Two or More Vectors:

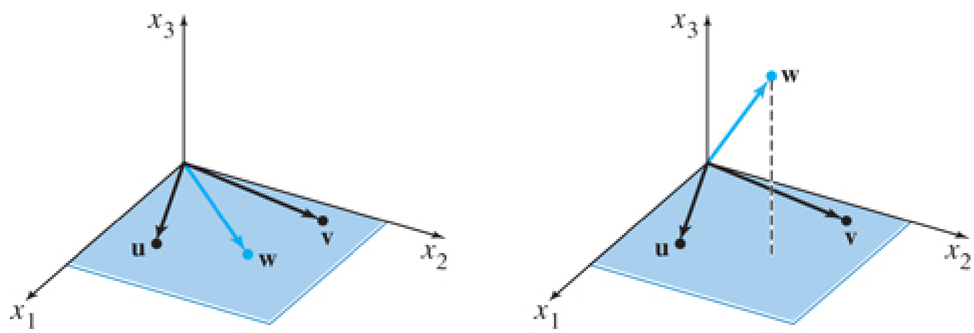
Theorem 7. Characterization of Linearly Dependent Sets

Proof.

□

Example 24. Let $\mathbf{u} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix}$. Describe the set spanned by $\mathbf{u}$ and $\mathbf{v}$. Explain why a vector $\mathbf{w}$ is in the span of $\mathbf{u}$ and $\mathbf{v}$ if and only if $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent.

Solution:



Linearly dependent, **w** in Span{**u**, **v**}

Linearly independent, **w** *not* in Span{**u**, **v**}

Note: this sort of analysis will always work for a set of vectors in $\mathbb{R}^3$. A span of vectors in $\mathbb{R}^3$ will always be either   , a   , a   or all of   . Thus you can tell if the set of vectors is linearly dependent by graphing its span.

An analogous statement is true for $\mathbb{R}^2$, and indeed $\mathbb{R}^n$ for $n > 3$ with some slight adjustments.

There are a couple criteria that can quickly answer questions of linear independence, but only in certain situations. Read the following theorems carefully, as their converses do not hold in general.

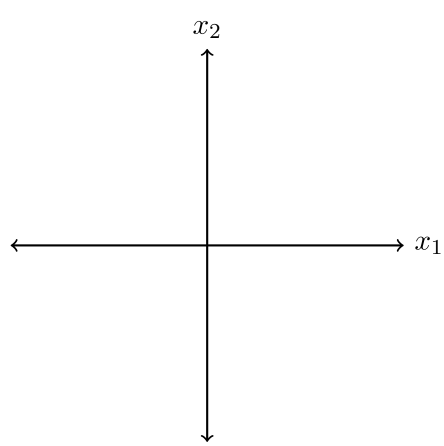
Theorem 8.

Proof.

□

Warning: The converse is NOT true in general. That is, if $\boxed{\quad\quad\quad\quad}$, you cannot determine whether or not a set is linearly dependent without doing more work.

Example 25. Consider the set of vectors $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \end{bmatrix} \right\}$. In this case $n = \boxed{\quad}$ and $p = \boxed{\quad}$. By the Theorem, the set is linearly $\boxed{\quad\quad\quad\quad}$, however, notice that none of the vectors are multiples of each other:



Theorem 9.

Proof. □

Example 26. Determine *by inspection* if the following sets of vectors are linearly independent.

a) $\begin{bmatrix} 4 \\ -1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -8 \\ 2 \\ -4 \\ -3 \end{bmatrix}$

b) $\begin{bmatrix} 4 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 6 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ -5 \\ 3 \end{bmatrix}.$

c) $\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$

Flowchart for Determining whether a set of vectors is linearly independent or linearly dependent:

Exercise 15. . Determine if the following sets of vectors are linearly independent. Justify your answer.

a) $\begin{bmatrix} 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \end{bmatrix}$

b) The columns of the matrix $A = \begin{bmatrix} 5 & 7 & 9 \\ 0 & 2 & 4 \\ 0 & -6 & -8 \end{bmatrix}$

c) $\begin{bmatrix} -4 \\ 6 \\ 10 \end{bmatrix}, \begin{bmatrix} 6 \\ -9 \\ -15 \end{bmatrix}$

d) $\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ -8 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}.$

1.8 Introduction to Linear Transformations

Learning Objectives:

- Compute images of vectors under matrix and linear transformations
- Determine if a vector is in the range of a matrix transformation
- Understand the axioms of a linear transformation and use them to compute images of vectors under such transformations

We have seen that linear systems, vector equations such as $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \dots x_n\mathbf{v}_n = \mathbf{b}$ and matrix equations such as $A\mathbf{x} = \mathbf{b}$ are essentially the same information repackaged differently. Today we change our perspective on this setting to one which uses *functions*.

General Idea:

Example 27. Let $A = \begin{bmatrix} 2 & 1 & 3 \\ -1 & -2 & -3 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$. Then

$$A\mathbf{x} = \begin{bmatrix} 2 & 1 & 3 \\ -1 & -2 & -3 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \boxed{} \\ \boxed{} \end{bmatrix} = \boxed{}$$

$$A\mathbf{v} = \begin{bmatrix} 2 & 1 & 3 \\ -1 & -2 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \boxed{} \\ \boxed{} \end{bmatrix} = \boxed{}.$$

One could picture the action of the matrix as follows:

Definition 17. Transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$

Some terminology and notation associated with transformations:

-
-
-
-
-

We can picture a transformation as follows:

Matrix Transformations

One way to define a transformation from $\boxed{\phantom{\mathbb{R}^n}}$ to $\boxed{\phantom{\mathbb{R}^n}}$ is by multiplication by an $\boxed{}$ matrix. Thus, we can define T by setting $T(\mathbf{x}) = A\mathbf{x}$. Another way to denote this is $\mathbf{x} \mapsto A\mathbf{x}$.

- Since A has $\boxed{}$ columns, the domain of T is $\boxed{\phantom{\mathbb{R}^n}}$.
- Since A has $\boxed{}$ rows, the codomain of T is $\boxed{\phantom{\mathbb{R}^m}}$
- The range of T is the set of $\boxed{\phantom{\text{linear combinations}}}$ of the columns A.

Example 28. Let $A = \begin{bmatrix} 1 & 3 \\ -1 & 1 \\ 4 & 2 \end{bmatrix}$ Then A defines a transformation $T : \boxed{\phantom{\mathbb{R}^2}} \to \boxed{\phantom{\mathbb{R}^3}}$ by $T(\mathbf{x}) = A\mathbf{x}$.

Thus,

$$T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 1 & 3 \\ -1 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \boxed{} \\ \boxed{} \\ \boxed{} \end{bmatrix}$$

(a) What is the image of $\mathbf{u} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ under T?

(b) Find an $\mathbf{x}$ in $\mathbb{R}^2$ whose image under T is $\mathbf{b} = \begin{bmatrix} 5 \\ -1 \\ 10 \end{bmatrix}$.

(c) Is there more than one possible $\mathbf{x}$ whose image under T is $\mathbf{b}$?

(d) Is the vector $\mathbf{c} = \begin{bmatrix} 0 \\ -2 \\ 5 \end{bmatrix}$ in the range of T?

Matrix transformations can come in many different geometric flavors.

Example 29. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ then the matrix transformation $T : \boxed{} \to \boxed{}$ defined by $\mathbf{x} \mapsto A\mathbf{x}$ is called a *projection onto the x-axis*. Let's find out why:

Example 30. Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$. The transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(\mathbf{x}) = A\mathbf{x}$ is called a *shear transformation.* Let's observe what it does to a square:

Linear Transformations

We have shown that if A is a matrix, then the transformation $T(\mathbf{x}) = A\mathbf{x}$ has the following algebraic properties:

Transformation that have properties like this make up the most important class of functions in linear algebra.

Definition 18. Linear Transformation

In other words, linear transformations preserve and
These properties lead to some useful facts when considering linear transformations.

Fact 5. If T is a linear transformation then

1.

2. If $\mathbf{u}$ and $\mathbf{v}$ are vectors and c and d are scalars, then

Proof.

1. This follows from the following:

2. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors and c, d scalars. Then combining the first and second properties we have:

$\square$

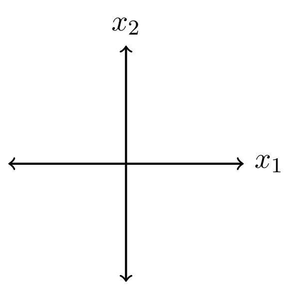
In fact, you can iterate the second fact to get the following useful identity. If T is a linear transformation, $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are vectors and $c_1, c_2, \ldots, c_n$ are scalars, then

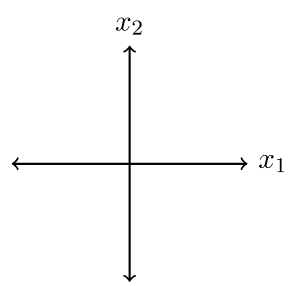
This means that T transforms linear combinations in $\mathbb{R}^n$ into linear combinations in $\mathbb{R}^m$! This phenomenon is called the *superposition principle* in physics. One implication of this is that if you know $T(\mathbf{u})$ and $T(\mathbf{v})$, then you an determine, for example, $T(4\mathbf{u} - 2\mathbf{v})$.

Remark 3.

Example 31. Let r be a scalar in $\mathbb{R}$ and define a transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ by $T(\mathbf{x}) = r\mathbf{x}$. This is called a **dilation** when $r > 1$ and a **contraction** when $0 < r < 1$. Suppose $r = 3$. Show that T is a linear transformation.

Example 32. Define a transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\mathbf{x}) = A\mathbf{x}$ where $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Find the images of $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. Use the linear property to compute $T(3\mathbf{u} - 6\mathbf{v})$. Finally, plot $\mathbf{u}, \mathbf{v}$ and their images on the axes below and describe the action of T geometrically.





Exercise 16. Let A be the matrix below, and define a linear transformation T by $T(\mathbf{x}) = A\mathbf{x}$.

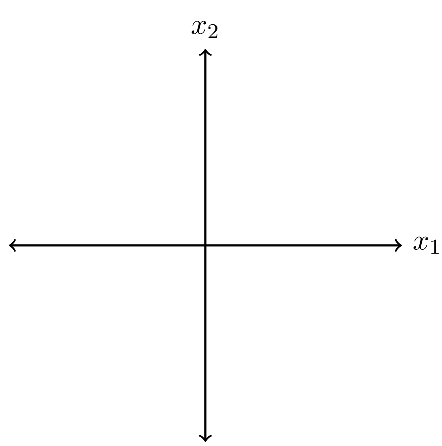
$$A = \begin{bmatrix} 2 & 0 & 4 \\ 2 & -3 & 1 \\ 4 & -6 & 7 \\ 8 & -9 & 12 \end{bmatrix}$$

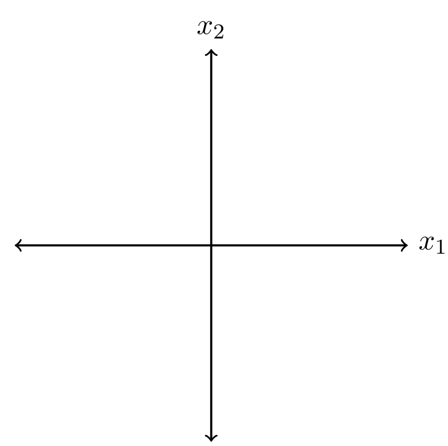
1. What is the domain of T?

2. What is the codomain of T?

3. Is the vector $\mathbf{b} = \begin{bmatrix} 6 \\ -3 \\ 4 \\ -5 \end{bmatrix}$ in the range of T?

Exercise 17. Define a transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\mathbf{x}) = A\mathbf{x}$ where $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. Find the images of $\mathbf{u} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$. Use the linear property to compute $T\left(\frac{1}{2}\mathbf{u} + 4\mathbf{v}\right)$. Finally, plot $\mathbf{u}, \mathbf{v}$ and their images on the axes below and describe the action of T geometrically.





1.9 The Matrix of a Linear Transformation

Learning Objectives:

- Use matrices to find formulas for linear transformations
- Become familiar with geometric descriptions of common transformations
- Determine if linear transformations are one-to-one or onto

Linear transformations can come in several different varieties. If a linear transformation is described geometrically or only in words, it can be useful to try to write down a formula for the transformation. We will learn that it is possible to do this using matrices. The key observation is to note that every vector in $\mathbb{R}^n$ can be written as a linear combination of the columns of a particular matrix.

Definition 19. The $n \times n$ Identity Matrix

Example 33. If $n = 3$, the identity matrix is

$$\boxed{} = \begin{bmatrix} \boxed{} & \boxed{} & \boxed{} \\ \boxed{} & \boxed{} & \boxed{} \\ \boxed{} & \boxed{} & \boxed{} \end{bmatrix}$$

Any vector in $\boxed{}$ can be written as a linear combination of the columns of $\boxed{}$. For example:

$$\begin{bmatrix} 4 \\ -5 \\ 30 \end{bmatrix} =$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} =$$

Thus if T is a linear transformation with domain $\mathbb{R}^3$, we have

Because of how often we will use I_n, we denoted its columns by $\mathbf{e}_1, \mathbf{e}_2, \dots \mathbf{e}_n$. That is,

Example 34. The columns of $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Thus, $\mathbf{e}_1 = \begin{bmatrix} \boxed{} \\ \boxed{} \end{bmatrix}$, $\mathbf{e}_2 = \begin{bmatrix} \boxed{} \\ \boxed{} \end{bmatrix}$. Suppose that T is a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ such that

$$T(\mathbf{e}_1) = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, T(\mathbf{e}_2) = \begin{bmatrix} -5 \\ 3 \\ 4 \end{bmatrix}$$

With no additional information, find a formula for $T(\mathbf{x})$ for $\mathbf{x}$ in $\mathbb{R}^2$ using matrices.

Theorem 10.

Proof.

□

Definition 20. Standard Matrix for a Linear Transformation

Example 35. Find the standard matrix A for the contraction transformation $T(\mathbf{x}) = \frac{1}{2}\mathbf{x}$ in $\mathbb{R}^2$.
Solution:

Example 36. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that rotates each point in $\mathbb{R}^2$ about the origin by an angle θ counterclockwise. Find the standard matrix of this transformation.
Solution:

Some of the types of linear transformations include:

-
-
-
-
-
-
-
-

Exercise 18. Find the standard matrix for the linear transfromation from $\mathbb{R}^2$ to $\mathbb{R}^2$ that vertically stretches vectors by a factor of 3 and reflects them over the y-axis.

Existence and Uniqueness in Transformations

Definition 21. Onto (or Surjective)

Another way to state this is that the [ ] of T is all of the [ ] of T.

The property of being onto is an "existence" property, since the question

is the same as the question

Thus a mapping T is **NOT** onto if

You can visualize onto transformations as follows:

Definition 22. One-to-One (or Injective)

Another way to say this is that for for each [ ] the equation [ ] has either a [ ] or [ ].

The property of being one-to-one is a uniqueness property, sinc the question

is the same as the question

Thus a mapping T is **NOT** one-to-one if

You can visualize one-to-one transformations as follows:

Example 37. Let T be a linear transformation whose standard matrix is

$$A = \begin{bmatrix} 1 & -4 & 8 & 6 \\ 3 & -10 & 23 & 6 \\ 2 & -8 & 16 & 7 \end{bmatrix}$$

Notice that $T : \boxed{} \to \boxed{}$. Does T map $\mathbb{R}^4$ onto $\mathbb{R}^3$? Is T a one-to-one transformation?

Solution:

Theorem 11.

Proof.

□

Theorem 12.

Proof. □

Corollary 2.

Exercise 19. Let T be the linear transformation defined by $T(\mathbf{x}) = A\mathbf{x}$ where

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & -1 & 2 \\ 5 & 0 & 1 \\ -3 & 6 & 2 \end{bmatrix}$$

(a) What is the domain of T? What is the codomain of T?

(b) Is T one-to-one? Is T onto? Explain your conclusions.

2 Matrix Algebra

2.1 Matrix Operations

Learning Objectives:

- Compute sums, products, and scalar multiples of matrices.
- Understand the properties of matrix addition and multiplication.
- Compute the transpose of a matrix and understand its properties.

Now that we have seen the utility of matrices in representing linear systems and linear transforamtions we will continue our study of linear algebra by viewing matrices as interesting objects in their own right. First, we shall set some common notation and terminology that we will make use of going forward.

Sums and Scalar Multiples

Example 38. Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} -3 & 0 & 7 \\ 4 & -2 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 9 & 8 \\ 7 & 6 \end{bmatrix}$$

Then

Theorem 13. Let A, B, and C be matrices of the same size and r and s be scalars

1.
2.
3.
4.
5.
6.

These are more or less easy to check by direct verification, as vector addition and multiplication satisfy a similar set of properties.

Matrix Multiplication

We have seen before the it is possible to multiply a $\boxed{}$ and a $\boxed{}$, to produce another $\boxed{}$ provided the sizes match appropriately. By iterating the general rule, we can multiply two $\boxed{}$ together to produce a new $\boxed{}$. Let $\mathbf{x}$ be a vector and A and B be matrices. We can think of the matices as transformations:

Thus $A(B\mathbf{x})$ is produced from $\mathbf{x}$ by a composition of transformations. We would like to be able to represent this composite map by a single matrix, AB. Let's formalize this discussion to find AB.

Definition 23. (Product of Matrices)

WARNING: As with matrix-vector products, $\boxed{}$ is undefined if the sizes do not match appropriately.

Example 39. Compute AB if

$$A = \begin{bmatrix} 3 & -1 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}.$$

Just like with matrix-vector products, there is a quicker way for calculating the products of matrices.

Fact 6. (Row-Column Rule For Matrix Products)

Example 40. Compute the following matrix products:

$$\begin{bmatrix} 2 & -6 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ -1 & 6 & 8 \end{bmatrix} =$$

$$\begin{bmatrix} 4 & 5 \\ -2 & 3 \\ 0 & 5 \\ 6 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 4 \\ -1 & 0 & -1 \end{bmatrix} =$$

Theorem 14.

1.

2.

3.

4.

5.

Proof. You can check these by direct varification. Property 1, is a bit tricky. For an easy proof, consider enrolling in MAT 370... ☐

There is one property that holds for "normal" multiplication that is conspicuously absent. Most of the time, $\boxed{}$. One reason for this is that A and B may not have the appropriate sizes. If it happens to be the case that $\boxed{}$ we say that A and B $\boxed{}$ with one another.

Example 41. Let $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & -3 \\ 4 & 2 \end{bmatrix}$. Compute the following:

$AB =$

$BA =$

Thus, in general, there is no reason that $\boxed{}$.

WARNING: Matrix multiplication is, in some ways, very different from "normal" multiplication.

1.

2.

3.

Remark 4.

The Transpose of a Matrix

Definition 24. (Transpose of a Matrix)

Example 42. Compute the transposes of the following matrices:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 3 \\ -1 & 0 \\ 4 & 6 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 2 & 2 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

The following theorem summarizes the properties of the transpose and describes its interaction with other matrix operations.

Theorem 15. (**Algebra with Transposed Matrices**)

1.

2.

3.

4.

Exercise 20. Given the following matrices, compute the expressions below.

$$A = \begin{bmatrix} 1 & -3 \\ 4 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 8 & -1 & 2 \\ 0 & 4 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 5 & -1 \\ 0 & 3 \\ 2 & 2 \end{bmatrix}$$

1. AB

2. BC

3. CA

4. C^T

5. A^3

6. $(A^T)A$

2.2 The Inverse of a Matrix

Learning Objectives:

- Use the determinant to find inverses of 2×2 matrices
- Relate inverses to systems of equations and matrix algebra
- Use row reduction to find inverses of larger matrices

We have seen that there are certain ways in which matrix multiplication is different from "normal multiplication" of real numbers. While there is no way to "divide" by a matrix, this section explores the matrix analog of the multiplicative inverse (or reciprocal) of a nonzero number.

Recall that

In certain, situations, we can find a similar relationship between two matrices.

Definition 25. (Invertible Matrix, Inverse of a Matrix, Singular/Nonsingular)

Example 43. If $A = \begin{bmatrix} 2 & 7 \\ -1 & -4 \end{bmatrix}$ and $C = \begin{bmatrix} 4 & 7 \\ -1 & -2 \end{bmatrix}$ then

There is a simple formula for the inverse of 2×2 matrices.

Theorem 16. (**Inverse Formula for 2 $\times$ 2 Matrices**)

Proof. Homework! $\square$

Definition 26. (**Determinant of a 2×2 Matrix**)

Thus, the theorem says that A is $\boxed{}$ if and only if $\boxed{}$.

Example 44. If possible, find the inverses of the matrices below.

$$A = \begin{bmatrix} 2 & 6 \\ 3 & 5 \end{bmatrix}$$

$$B = \begin{bmatrix} 6 & 15 \\ 4 & 10 \end{bmatrix}$$

Invertible matrices are central to linear algebra. They are used to preform calculations and algebraic manipulations and derive formulas. They can also be used to solve systems of equations.

Theorem 17.

Proof. $\square$

Although this gives us a new method to solve linear systems of the form $A\mathbf{x} = \mathbf{b}$, it doesn't work unless A is invertible, and row-reduction is almost always faster, with perhaps the exception of the 2×2 case.

Example 45. Use the inverse of a matrix to solve the following system.

$$\begin{array}{rcrcl} 2x_1 & + & 3x_2 & = & 4 \\ x_1 & - & x_2 & = & 2 \end{array}$$

Next, we show some more useful facts about invertible matrices:

Theorem 18.

a.

b.

c.

Proof. □

Remark 5.

Elementary Matrices

We will now develop a method to find the inverse of a matrix of size $n \times n$ for any $n \geq 1$.

Definition 27. Elementary Matrix

Example 46. Below is the identity matrix I_3 and a few 3×3 elementary matrices:

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Lets observe what multiplying by these matrices does to a general matrix:

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} =$$

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} =$$

Thus, we can conclude the performing an $\boxed{}$ is actually the same as multipliying by an $\boxed{}$!! Since we can always undo an elementary row operation, every elementary matrix is invertible. The inverse of an elementary matrix E is the same type of matrix that transforms E back to I_n.

Example 47. Find the inverses of the matrices below:

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad E_3 = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The following theorem provides a nice way to think of an invertible matrix which we can use to find the inverse of such a matrix.

Theorem 19.

Proof.

□

An Algorithm for Finding A^{-1}

By the previous theorem, we know that the same sequence of row operations that turns A into I_n will turn I_n into A^{-1}. Thus, we just need to row reduce A and keep track of the steps. Luckily, there is an efficient way to do this.

Algorithm for finding A^{-1}:

Example 48. Find the inverse of the matrix $A = \begin{bmatrix} 0 & -2 & 1 \\ 3 & 0 & -9 \\ 4 & 1 & 0 \end{bmatrix}$.

Solution:

Exercise 21.

1. Determine if the following matrices are invertible. Briefly eplain why. **You don't need to find the inverse.**

   (a) $\begin{bmatrix} 2 & -3 \\ 4 & -6 \end{bmatrix}$

   (b) $\begin{bmatrix} 1 & 4 & 3 & 2 \\ 2 & 3 & 1 & -10 \end{bmatrix}$

   (c) $\begin{bmatrix} 5 & 2 & 3 \\ 0 & 6 & -10 \\ 0 & 0 & 24 \end{bmatrix}$

   (d) $\begin{bmatrix} 6 & -2 & 0 & 4 \\ 8 & -1 & 0 & 1 \\ 2 & -3 & 0 & 2 \\ 7 & 8 & 0 & 0 \end{bmatrix}$

2. Find the inverses of the following matrices, if they exist.

(a) $\begin{bmatrix} 5 & 6 \\ 3 & 4 \end{bmatrix}$.

(b) $\begin{bmatrix} 1 & 1 & -2 \\ -3 & -2 & 5 \\ -6 & 4 & 4 \end{bmatrix}$.

2.3 Characterization of Invertible Matrices

Learning Objectives:

- Understand what it means for two statements to be logically equivalent and how to prove an equivalence
- Relate the invertibility of a matrix to properties of systems of equations, vectors, matrices, and linear transformations.

The previous two sections have mostly been about computational properties of matrices. Today we relate the invertibility of a matrix to other properties that we have studies in Chapter 1.

Theorem 20. (**The Invertible Matrix Theorem**) Let A be a square $n \times n$ matrix. The following 12 properties are logically equivalent, meaning that they are either all true or all false.

a.

b.

c.

d.

e.

f.

g.

h.

i.

j.

k.

l.

Before we prove the The Invertible Matrix Theorem, we need to discuss how one would go about proving something so daunting. If a logical statement $\boxed{}$ being true always implies that another statement $\boxed{}$ is true, we say that $\boxed{}$ **implies** $\boxed{}$, and write $\boxed{}$. If you want to prove that $\boxed{}$ and $\boxed{}$ are logically equivalent you would have

Now suppose you have three logical statements, $\boxed{}$, $\boxed{}$, and $\boxed{}$. There are multiple options for proving these statements are equivalent. You could prove

Or you could show

Thus, having a $\boxed{\phantom{\text{cycle}}}$ of implications means that if one statement is true, they all must be true. The structure of the implications that we will prove is the following:

Luckily, we have already proved several of these implications! It only remains to prove 5 more implications.

Exercise 22. In this exercise, we will finish the proof of the Invertible Matrix Theorem.

1. Prove $(c) \implies (b)$.

2. Prove $(d) \implies (c)$

3. Prove $(j) \implies (d)$

4. Prove $(g) \implies (b)$

5. Prove $(k) \implies (g)$ [Hint: Consider the equation $AD\mathbf{b} = \mathbf{b}$].

Invertible Linear Transformations

We have seen that one interpretation of an $n \times n$ matrix A is that it transforms a vector $\mathbf{x}$ in $\boxed{}$ to a vector $A\mathbf{x} = \mathbf{b}$ in $\boxed{}$. So if A is invertible, then there is a matrix A^{-1} that transforms the vector $A\mathbf{x} = \mathbf{b}$ back into $\mathbf{x}$.

Definition 28. Invertible Linear Transformation

Theorem 21.

Proof. $\square$

3 Determinants

3.1 Introduction to Determinants

Learning Objectives:

- Understand the definition of the determinant
- Use cofactor expansion to compute determinants of $n \times n$ matrices
- Apply a theorem to quickily compute determinants of triangular matrices.

Recall from Section 2.2, that a 2×2 matrix is invertible if and only if its determinant is nonzero. That is,

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \text{ is invertible if and only if } \boxed{}.$$

One way to see this is that if A is invertible, then It is equivalent to the identity matrix, so (possibly after switching rows) we have $\boxed{}$ and then

So, since A must have a pivot in the second row, it must be that $\boxed{}$. Is there a similarly easy test with a larger marix? It turns out that there is. Let A be a 3×3 matrix. Then if A is invertible, we would be able to row reduce as follows:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \sim \begin{bmatrix} \boxed{} & \boxed{} & \boxed{} \\ \boxed{} & \boxed{} & \boxed{} \\ \boxed{} & \boxed{} & \boxed{} \end{bmatrix}$$

where

$\Delta =$

Since A is invertible, $\boxed{}$ must be $\boxed{}$. This number $\boxed{}$ is the $\boxed{}$ of A.

Our goal is to define $\det(A)$ for any square matrix A.

1×1:

2×2:

3×3: From the previous page, we have

$$\det(A) = \Delta = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

For example, if

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 4 & 5 & 0 & 6 \\ 7 & 0 & 8 & 9 \\ 0 & 10 & 11 & 12 \end{bmatrix}, \text{ then } A_{11} = \begin{bmatrix} \boxed{} & \boxed{} & \boxed{} \\ \boxed{} & \boxed{} & \boxed{} \\ \boxed{} & \boxed{} & \boxed{} \end{bmatrix} \text{ and } A_{23} = \begin{bmatrix} \boxed{} & \boxed{} & \boxed{} \\ \boxed{} & \boxed{} & \boxed{} \\ \boxed{} & \boxed{} & \boxed{} \end{bmatrix}$$

Definition 29. Determinant of a Matrix)

Example 49. Compute the deteminant of

$$A = \begin{bmatrix} 2 & 1 & 0 \\ -4 & -1 & 2 \\ 1 & 5 & 0 \end{bmatrix}$$

Solution:

Exercise 23. Compute the determinant of the following matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 0 & 1 & 2 \end{bmatrix}$$

Solution:

As you can see, finding determinants are computationally intensive. The next theorem will allow us speed up computation by making strategic decisions.

Definition 30. (Cofactor, Cofactor Expansion)

The "tricky part" of cofactor expansion is remember where to put the signs.

Theorem 22.

The plus or minus sign in the (i, j)-cofactor depends on the position of a_{ij} in the matrix, regardless of the sign of a_{ij} istelf. This leads to the checkerboard pattern:

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \\ + & - & + & \\ \vdots & & & \ddots \end{bmatrix}$$

With this in mind, we can use zeros strategically to make computation easier.

General Strategy:

Remark 6. To make computation of determinants easier to write down, it is common to use straight lines to denote determinants. Thus

$$\det\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

Example 50. Compute the determinant of

$$A = \begin{bmatrix} 2 & 1 & 0 \\ -4 & -1 & 2 \\ 1 & 5 & 0 \end{bmatrix}$$

Solution:

Example 51. Compute the determinant of the 5×5 matrix

$$B = \begin{bmatrix} 2 & -1 & 3 & 0 & -6 \\ 0 & 1 & 4 & -7 & -2 \\ 0 & 0 & 3 & 6 & -4 \\ 0 & 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}$$

Solution:

The matrix in the previous example is nearly triangular, which led to an easy computation. First, let's define triangular:

Definition 31. (Upper/Lower Triangular Matrix)

It is not so hard to see that the following must hold true:

Theorem 23.

Example 52. Find the determinant of

$$C = \begin{bmatrix} 6 & 0 & 0 & 0 \\ -19 & 2 & 0 & 0 \\ \pi & 100 & \frac{1}{3} & 0 \\ 4 & 3 & 2 & -1 \end{bmatrix}$$

Solution:

WARNING:

Exercise 24. 1. Compute the determinant of $A = \begin{bmatrix} \sqrt{2} & 4 \\ \frac{3}{4} & -\sqrt{2} \end{bmatrix}$.

2. Consider the following matrix:

$$A = \begin{bmatrix} 2 & -2 & 3 \\ 7 & 1 & 0 \\ -1 & 3 & 0 \end{bmatrix}$$

   (a) Compute $\det(A)$ by using cofactor expansion along the first row.

   (b) Compute $\det(A)$ by using cofactor expansion down the third column.

3. Compute the determinant of the following matrix:

$$A = \begin{bmatrix} 4 & 0 & 0 & 5 \\ 1 & 7 & 2 & -5 \\ 3 & 0 & 0 & 0 \\ 8 & 3 & 1 & 7 \end{bmatrix}$$

4. Compute the determinant of the following matrix:

$$A = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 3 & 0 & 2 & 4 & -2 \\ 3 & -2 & 5 & 12 & -4 \\ 0 & 0 & 1 & 0 & 0 \\ 5 & 0 & 7 & 3 & 1 \end{bmatrix}$$

Hint: Choose rows and columns to expand along that make the computation easier.

5. Let $A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$.

   (a) Compute $\det(A)$.

   (b) Compute $\det(5A)$ and $5\det(A)$. What do you notice?

   (c) Now, multiply the 2nd row of A by 5 and then take the determinant. What do you notice?

6. Show that for a 2×2 matrix A, $\det(A) = \det(A^T)$.

7. Consider the matrix below.

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

   (a) Compute $\det(A)$.

   (b) Swap the first two rows of A and then compute the determinant of the new matrix. What do you notice?

3.2 Properties of Determinant

Learning Objectives:

- Use row operations to simplify the computations.of determinants
- Understand Properties of the determinant
- Calculate determinants using the multiplicative property

Given an $n \times n$ matrix A, we can simplify the computation of $\det(A)$ using row reduction, however we need to keep track of the operations according to the following theorem.

Theorem 24.

Example 53. Compute $\det(A)$ if

$$A = \begin{bmatrix} 1 & -2 & 3 \\ -1 & 2 & 1 \\ 2 & 5 & 6 \end{bmatrix}$$

Solution:

Example 54. Find the determinant of the following matrix:

$$B = \begin{bmatrix} 3 & 6 & 0 & 0 \\ 2 & 4 & 4 & 2 \\ 0 & 2 & -1 & 1 \\ 0 & 0 & 4 & 2 \end{bmatrix}$$

Theorem 25.

Proof. $\square$

Theorem 26.

Proof. □

Theorem 27. **(Multiplicative Property for Determinants)**

Proof. See the page 176 of the book. □

Example 55. Check the multiplicative property for the following matrices:

$$A = \begin{bmatrix} 2 & 5 \\ 3 & 6 \end{bmatrix}, \qquad B = \begin{bmatrix} -1 & 7 \\ 2 & -3 \end{bmatrix}$$

Exercise 25. Complete the following exercises.

1. Find the determinant of the following matrix using row reduction to echelon form.

$$\begin{bmatrix} 1 & 3 & 2 & 4 \\ 0 & 1 & 2 & -5 \\ 2 & 7 & 6 & -3 \\ -3 & -10 & -7 & 2 \end{bmatrix}$$

2. Combine the methods of row reduction and cofactor expansion to compute the determinant of the matrix below:

$$\begin{bmatrix} 1 & 5 & 4 & 1 \\ 0 & -2 & -4 & 0 \\ 3 & 5 & 4 & 1 \\ -6 & 5 & 5 & 0 \end{bmatrix}$$

3. Find the determinants below if

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7$$

(a) $\begin{vmatrix} a & b & c \\ d & e & f \\ 5g & 5h & 5i \end{vmatrix}$

(b) $\begin{vmatrix} d & e & f \\ a & b & c \\ g & h & i \end{vmatrix}$

(c) $\begin{vmatrix} g & h & i \\ 6d & 6e & 6f \\ 2a & 2b & 2c \end{vmatrix}$

4. Use determinants to decide if the following three vectors are linearly dependent.

$$\begin{bmatrix} 4 \\ 6 \\ 2 \end{bmatrix}, \begin{bmatrix} -7 \\ 0 \\ 7 \end{bmatrix}, \begin{bmatrix} -3 \\ -5 \\ -2 \end{bmatrix}.$$

5. In general, for two square matrices of the same size, $\det(A+B) \neq \det(A)+\det(B)$. Find an examples of 2×2 matrices that demonstrate this fact.

6. Consider the matrix

$$B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

(a) Compute $\det(B^5)$.

(b) Compute $\det(B^{-1})$.

7. Show that for two square matrices A and B, even though $AB \neq BA$ in general, it is always the case that $\det(AB) = \det(BA)$.

8. Let A and P be matrices with P invertible. Show that $\det(PAP^{-1}) = \det(A)$.

9. Find a formula for $\det(rA)$ where A is an $n \times n$ matrix.

3.3 Applications of the Determinant

Learning Objectives:

- Apply Cramer's rule to solve systems of equations
- Use determinants to find matrix inverses
- Use determinants to calculate area and volume

Cramer's Rule

Before we present a theoretically important result, we need to develop some notation. Let A be an $n \times n$ matrix and let $\mathbf{b}$ be a vector in $\mathbb{R}^n$. Let $A_i(\mathbf{b})$ be the matrix obtained from A by replacing column i with the vector $\mathbf{b}$. That is,

Theorem 28. (Cramer's Rule)

Proof.

□

Note: although this result has important theoretical implications, it is usually not computationally efficient to solve systems larger than the 3x3 case.

Example 56. Use Cramer's rule to solve the following system:

$$\begin{array}{rcrcr} 4x_1 & + & 3x_2 & = & -2 \\ 2x_1 & - & x_2 & = & 5 \end{array}$$

Solution:

Exercise 26. Find the solution the following system using Cramer's Rule:

$$\begin{array}{rcrcr} 5x_1 & + & 7x_2 & = & 3 \\ 2x_1 & + & 4x_2 & = & 1 \end{array}$$

A Formula for A^{-1}

We now present a formula for the inverse of a matrix that is a bit complicated to write down. First some notation:

Let A be an $n \times n$ invertible matrix.

Definition 32. (**Adjugate Matrix**)

Theorem 29. (**Inverse Matrix Formula**)

Example 57. Find the inverse of the matrix $A = \begin{bmatrix} 0 & -2 & -1 \\ 5 & 0 & 0 \\ -1 & 1 & 1 \end{bmatrix}$.

Solution:

Exercise 27. Compute the adjugate of the following matrices and use them to determine their inverses.

1. $A = \begin{bmatrix} 1 & 1 & 3 \\ -2 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

If done properly, you should get:

$$\operatorname{adj}(A) = \begin{bmatrix} 1 & 2 & -5 \\ 2 & 1 & -7 \\ -2 & -1 & 4 \end{bmatrix}, \quad \text{so} \quad A^{-1} = \begin{bmatrix} -1/3 & -2/3 & 5/3 \\ -2/3 & -1/3 & 7/3 \\ 2/3 & 1/3 & -4/3 \end{bmatrix}$$

2. $B = \begin{bmatrix} 1 & 2 & 4 \\ 0 & -3 & 1 \\ 0 & 0 & -2 \end{bmatrix}$

If done properly, you should get:

$$\operatorname{adj}(B) = \begin{bmatrix} 6 & 4 & 14 \\ 0 & -2 & -1 \\ 0 & 0 & -3 \end{bmatrix} \quad \text{so} \quad B^{-1} = \begin{bmatrix} 1 & 2/3 & 7/3 \\ 0 & -1/3 & -1/6 \\ 0 & 0 & -1/2 \end{bmatrix}.$$

Determinants as Area or Volume

You can think of the columns of a 2×2 matrix as describing a parallelogram in $\mathbb{R}^2$ as follows:

Let $A = [\mathbf{u}\ \ \mathbf{v}]$.

Similarly, you can view a 3×3 matrix as determining parallelepiped in $\mathbb{R}^3$ as follows:

Let $A = [\,\mathbf{u}\ \ \mathbf{v}\ \ \mathbf{w}\,]$

Theorem 30. (**Determinant as Area/Volume**)

Proof. We will prove the statement for 2×2 matrices.

$\square$

Example 58. Find the area of the parallelogram determined by the matrix $A = \begin{bmatrix} 2 & 0 \\ 1 & 4 \end{bmatrix}$.

Solution:

Example 59. Calculate the area of the parallelogram with vertices $(-3,-3), (0,2)$, $(-1,-1)$, and $(2,4)$.

Solution:

Linear Transformations

We will now see how area and volume behave under linear transformations. First, we note that the linear transformation of a parallelogram results in a parallelogram:

Example 60. Determine the image of the parallelogram with vertices $(0,0),(0,3),(2,1),(2,4)$ under the transfromation $T:\mathbb{R}^2\to\mathbb{R}^2$ with standard matrix $A=\begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix}$.

Solution:

Notation: If P is a parallelogram, then the image of P under the transformation T is the new parallelogram denoted by $T(P)$.

Theorem 31.

Proof.

□

Remark 7. The conclusions of the previous theorem hold for **any region** in $\mathbb{R}^2$ or $\mathbb{R}^3$ with area/volume! Thus, you can use the theorem to determine the area/volume of the images of circles, spheres, cylinders, cones, etc.

Example 61. Let U be the unit circle in $\mathbb{R}^2$. What is the area of the ellipse $T(U)$ if T is the transformation determined by $A = \begin{bmatrix} 2 & 1 \\ 0 & -3 \end{bmatrix}$.

Solution:

Exercise 28.

1. Find the area of the parallelograms whose vertices are listed below:

   (a) (0, 0), (5, 2), (6, 4), (11, 6)

(b) (0, -2), (5, -2), (-3, 1), (2, 1)

2. Find the volume of parallelepiped with one vertex at the origin and adjacent vertices at $(1,3,0)$, $(1,2,4)$, and $(5,1,0)$

3. Find a formula for the area of the triangle whose vertices are $\mathbf{0}, \mathbf{u}, \mathbf{v}$.

4. Use the concept of volume to explain why the determinant of a 3×3 matrix is zero if and only if A is not nvertible without referencing the theorem from Section 3.2. [Hint: Think about the columns of A].

5. Find the area of the right half of the unit circle under the transformation T with standard matrix $A = \begin{bmatrix} 7 & 2 \\ -1 & 1 \end{bmatrix}$.

4 Vector Spaces

4.1 Vector Spaces and Subspaces

Learning Objectives:

- Identify vector spaces
- Determine whether a subset of vector space is a subspace
- Relate subspaces to the span of a set of vectors

We have spent a long time studying $\mathbb{R}^n$ in Chapters 1 and 2. However, there are many other algebraic structures that have the same algebraic properties, and thus we can extend our study to these more general structures.

Definition 33. (**Vector Space**)

1.

2.

3.

4.

5.

6.

7.

8.

9.

10.

Remark 8.

We now present several examples.

Example 62. $\mathbb{R}^n$ **is a vector space.**

Example 63. Arrows in 3-space

Example 64. Infinite Sequences

Example 65. Polynomials of degree at most n

Exercise 29.

1. The set of all real-valued functions with domain $\mathbb{R}$ is a vector space. Explain why, and give an example of addition and scalar multiplication.

2. A line in $\mathbb{R}^2$ not through the origin is NOT a vector space. Show why.

Vector Subspaces

Vector spaces can have "smaller" vector spaces living inside of them. In this case, the smaller set only needs to satisfy three properties to be a vector space in its own right, since the others are automatically satisfied.

Definition 34. Subspace

a)

b)

c)

Since H is a subset of a vector space V, it automatically satisfies axioms 2,3, 7-10 of a vector space. The properties above guarantee H satisfies the remaining axioms. Let's consider several examples.

Example 66.

Example 67.

Example 68.

Exercise 30.

1. Let V be the first quadrant in the xy-plane, that is, let

$$V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \geq 0, y \geq 0 \right\}$$

   (a) If $\mathbf{u}$ and $\mathbf{v}$ are in V, is $\mathbf{u} + \mathbf{v}$ in V? Why or why not?

   (b) Show that V is not a subspace of $\mathbb{R}^2$ [Hint: find a property in the definition of subspace that V does not satisfy.]

2. Determine if the subsets described below are subspaces of $\mathbb{P}_n$ for an appropriate value of n. [Hint: to do this, you need to either show that it satisfies all three properties of a subspace, or find one that it does not satisfy.]

   (a) All polynomials of the form $\mathbf{p}(t) = at^2$ where a is in $\mathbb{R}$.

   (b) All polynomials of the form $\mathbf{p}(t) = a + t^2$ where a is in $\mathbb{R}$.

   (c) All polynomials of degree at most 3 with integers as coefficients.

   (d) All polynomials in $\mathbb{P}_n$ such that $\mathbf{p}(0) = 0$.

A Subspace Spanned by a Set

We start with a familiar definition in the new, more general context.

Definition 35. (**Span for general Vector Spaces**)

Example 69.

You can generalize this argument to get the following theorem:

Theorem 32.

Remark 9.

One way to use this theorem is to show a subset of a vector space is a subspace being showing it is the span of one or more vectors.

Example 70. Let H be the set of points of the form $(a + 3b, a - b, a, 4b)$ for a, b in $\mathbb{R}$. Then H is a subset of $\mathbb{R}^4$. Show that H is a subspace of $\mathbb{R}^4$. Is the point $(5, 1, 2, 4)$ in H?
Solution:

This example points out the utility of describing a set as a span of vectors. If we can do that, we automatically know it is a subspace, and it makes it easier to answer other questions about the subset.

Exercise 31.

1. Let H be the set of all vectors of the form $\begin{bmatrix} s \\ 2s \\ 3s \end{bmatrix}$. Find a vector $\mathbf{v}$ in $\mathbb{R}^3$ such that $H = \text{Span}\{\mathbf{v}\}$. Why does this show that H is a subspace of $\mathbb{R}^2$?

2. Let W be the set of all vectors of the form $\begin{bmatrix} a+2b+3c \\ 2a-b \\ b-c \\ 4a-5c \end{bmatrix}$. Find three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in $\mathbb{R}^4$ such that $W = \text{Span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$.

3. Let
$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

   (a) Is $\mathbf{w}$ in $\{\mathbf{v}_1, \mathbf{v}_2.\mathbf{v}_3\}$? How many elements are in $\{\mathbf{v}_1, \mathbf{v}_2.\mathbf{v}_3\}$?

   (b) How many elements are in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2.\mathbf{v}_3\}$?

   (c) Is $\mathbf{w}$ in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2.\mathbf{v}_3\}$?

4.2 Null Spaces, Column Spaces, and Linear Transformations

Learning Objectives:

- Describe null spaces and column spaces of matrices and their analogs for linear transformations as spans of vectors
- Determine whether a vector is an element of a null space or column space
- Compare and contrasts null spaces and column spaces and their properties

In this section we will study two examples of subspaces of $\mathbb{R}^n$ that arise from matrices and linear transformations.

The Null Space of a Matrix

The first space we will look at is something we have already seen in a different context. Recall that, given the ______ system

$$\begin{array}{rcrcrcl} 4x_1 & - & x_2 & + & 2x_3 & = & 0 \\ 3x_1 & + & 2x_2 & + & 7x_3 & = & 0 \end{array}$$

we can write it as the matrix equation $A\mathbf{x} = \mathbf{0}$ where

Recall that the set of all $\mathbf{x}$ that satisfy $A\mathbf{x} = \mathbf{0}$ is called the *solution set* of the system. However, we can view this set as connected of the matrix A instead of arising from the system.

Definition 36. (Null Space of a Matrix)

Example 71. For the matrix A above, determine if the following vectors are in Nul A.

1. $\mathbf{v} = \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}$ is in Nul A.

2. $\mathbf{w} = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}$

One way to view Nul A is that it is the set of vectors in $\mathbb{R}^n$ that are mapped to the zero vector in the transformation $\mathbf{x} \mapsto A\mathbf{x}$:

The word 'space' is used becase Nul A is a vector space.

Theorem 33.

Proof.

□

An Explicit Description of Nul A

The definition of the nullspace of a matrix A is *implicit* because there is no obvious connection between the vectors in Nul A and the entries of A. However, by solving the equation $A\mathbf{x} = \mathbf{0}$ we can find an *explicit* description of Nul A as a span of vectors.

Example 72. Find a spanning set for the null space of the matrix A if

$$A = \begin{bmatrix} 1 & -2 & 3 & 9 \\ -2 & 4 & 1 & 3 \\ -1 & 2 & 1 & 3 \end{bmatrix}.$$

Solution:

Remark 10. There are two things to notice here:

1.

2.

Exercise 32.

1. Determine if $\mathbf{w} = \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix}$ is in Nul A if $A = \begin{bmatrix} 3 & -5 & -3 \\ 6 & -2 & 0 \\ -8 & 4 & 1 \end{bmatrix}$

2. Find an explicit description of Nul A by listing vectors that span the null space if

   (a) $A = \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 4 & -2 \end{bmatrix}$

   (b) $A = \begin{bmatrix} 1 & -2 & 0 & 4 & 0 \\ 0 & 0 & 1 & -9 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

The Column Space of a Matrix

The other subspace we will associate with a matrix is defined *explicitly* as a span of vectors.

Definition 37. (Column Space of a Matrix)

We have proved that every span of vectors is a subspace of the vector space in which they live, so since each column of A is in $\mathbb{R}^n$, we get the following theorem:

Theorem 34.

Remark 11.

Example 73. Can the set W be realized as the column space of some matrix A?

$$W = \left\{ \begin{bmatrix} x - y \\ 2x + 3y \\ -10y \end{bmatrix} : x, y \text{ in } \mathbb{R} \right\}$$

Solution

The Differences between Nul A and Col A

We shall illustrate several differences between the two spaces by examining a specific example.

Example 74. Let

$$A = \begin{bmatrix} 1 & -2 & 3 & 9 \\ -2 & 4 & 1 & 3 \\ -1 & 2 & 1 & 4 \end{bmatrix}$$

1. If Nul A is a subspace of $\mathbb{R}^k$, what is k?

2. If Col A is a subspace of $\mathbb{R}^k$, what is k?

3. Find a nonzero vector in Col A and a nonzero vector in Nul A.

4. Let $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 4 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}$

   (a) Determine if $\mathbf{u}$ is in Nul A. Could $\mathbf{u}$ be in Col A?

   (b) Determine if $\mathbf{v}$ is in Col A. Could $\mathbf{v}$ be in Nul A?

For a long list of comparisons and contrasts between Nul A and Col A, see page 206.

Exercise 33. Let $A = \begin{bmatrix} 2 & -6 \\ -1 & 3 \\ -4 & 12 \\ 3 & -9 \end{bmatrix}$

1. Find k such that Nul A is in $\mathbb{R}^k$.

2. Find k such that Col A is in $\mathbb{R}^k$.

3. Find a nonzero vector in Nul A and a nonzero vector in Col A.

4. Is $\begin{bmatrix} 18 \\ 6 \end{bmatrix}$ in Nul A? Is $\begin{bmatrix} -8 \\ 4 \\ 16 \\ -12 \end{bmatrix}$ in Col A?

Kernel and Range of a Linear Transformation

First, we generalize the notion of a linear transformation to arbitrary vector space.

Definition 38. (Linear Transformation of Vector Spaces)

1.

2.

We can now associate some vector spaces to linear transformations of vector spaces.

Definition 39. (Kernel, Range)

1.

2.

You can visualize these space as follows:

To relate this back to the previous notions from this section, consider the following example:

Example 75.

Example 76. (**Derivatives**)

Exercise 34. Define $T : \mathbb{P}_2 \to \mathbb{R}^2$ by $T(\mathbf{p}(t)) = \begin{bmatrix} \mathbf{p}(0) \\ \mathbf{p}(1) \end{bmatrix}$. For instance, if $\mathbf{p}(t) = 3 + 5t + 7t^2$ then $T(\mathbf{p}(t)) = \begin{bmatrix} 3 + 5(0) + 7(0)^2 \\ 3 + 5(1) + 7(1)^2 \end{bmatrix} = \begin{bmatrix} 3 \\ 15 \end{bmatrix}$.

1. Show that T is a linear transformation. [Hint: to do this, you need to show that T satisfies both properties of a linear transformation]

2. Find a polynomial in $\mathbb{P}_2$ that spans the kernel of T, and describe the range of T.

4.3 Linearly Independent Sets, Bases

Learning Objectives:

- Determine if a set of vectors is linearly independent
- Find bases for vector spaces and subspaces
- Relate the concepts of linear independence and spanning sets to theorems

We will now revisit the concept of linear independence from Chapter 1 in the context of general vector spaces.

Definition 40. (**Linear Independence for General Vector Spaces**)

Many of the results from chapter one still carry over. For example, a set with one vector $\{\mathbf{v}\}$ is linearly dependent if and only if $\mathbf{v} = \mathbf{0}$. There is one theorem in particular one should keep in mind:

Theorem 35.

Example 77. Recall that $\mathbb{P}_2$ is the set of polynomials of degree at most two. Consider the following polynomials in $\mathbb{P}_2$:

$$\mathbf{p}_1(t) = t^2 + 1, \qquad \mathbf{p}_2(t) = 4t - 2, \qquad \mathbf{p}_3(t) = 2t^2 - 12t + 8.$$

Example 78. Recall that the set of real-valued functions on the interval $\mathbb{R}$ form a vector space. Then the set $\{\sin t, \cos t\}$ is $\boxed{}$ because

Definition 41. **(Basis)**

1.

2.

Remark 12.

Example 79.

Example 80. (**Standard Basis for $\mathbb{R}^n$**)

Example 81. Let $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}$. Determine if the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ forms a basis for $\mathbb{R}^3$.

Solution:

Example 82. (**Standard Basis for $\mathbb{P}_n$**)

Exercise 35. Determine if the following sets of vectors form bases for $\mathbb{R}^3$. If not, explain why.

1. $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

2. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix},$

3. $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}.$

4. $\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} -7 \\ 5 \\ 4 \end{bmatrix}.$

The Spanning Set Theorem

Example 83. Consider the vectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ and let $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Note that $\mathbf{v}_3 = 2\mathbf{v}_1 - \mathbf{v}_2$. Show that $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ and find a basis for H.

Solution:

Theorem 36. (The Spanning Set Theorem)

1.

2.

Proof.

□

Bases for Nul A and Col A

Let A be an $m \times n$ matrix. We will now outline techniques to compute bases for Nul A and Col A.

Example 84. Let $A = \begin{bmatrix} 1 & 2 & 0 & -4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ Find a basis for Nul A and Col A. *Solution*

What if the matrix is not in reduced echelon form? Recall that any linear dependence relation along the columns of A is the same as a solution to $A\mathbf{x} = \mathbf{0}$. So if A is row equivalent to B, $A\mathbf{x} = \mathbf{0}$ has the same solutions as $B\mathbf{x} = \mathbf{0}$. Thus, if B is the reduced echelon form of A, then the columns in the same position of B that form a basis for Col B will form a basis for Col A.

Example 85. Find a basis for Col B if $B = \begin{bmatrix} 1 & 2 & 7 & 17 \\ 4 & 8 & 1 & 3 \\ -3 & -6 & 2 & 6 \end{bmatrix}$.

Solution

Theorem 37.

Proof. $\square$

WARNING:

Remark 13.

Example 86. Consider the vector space $\mathbb{R}^3$. In order for a set of vectors to form a basis it can't be "too small", or "too large". For example in

Exercise 36.

1. Find bases for Nul A and Col A for the matrix

$$A = \begin{bmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{bmatrix}$$

2. Find a basis for the space spanned by the vectors

$$\begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} -3 \\ -4 \\ 1 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ -8 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -6 \\ 9 \end{bmatrix}$$

3. Suppose $\mathbb{R}^4 = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$. Explain why $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ must form a basis for $\mathbb{R}^4$.

4.4 Coordinate Systems

Learning Objectives:

- Find coordinates relative to bases for vectors
- Use coordinates to determine linear independence
- Use coordinates to define linear trasnformations

One property of $\mathbb{R}^n$ that simplifies understanding calculations and concepts is the fact that it can be visualized using coordinate axes. Using bases, we can attempt to do this for arbitrary vector spaces.

Theorem 38. Unique Representation Theorem

Proof. □

Definition 42. (Coordinates Relative to a Basis)

Notation:

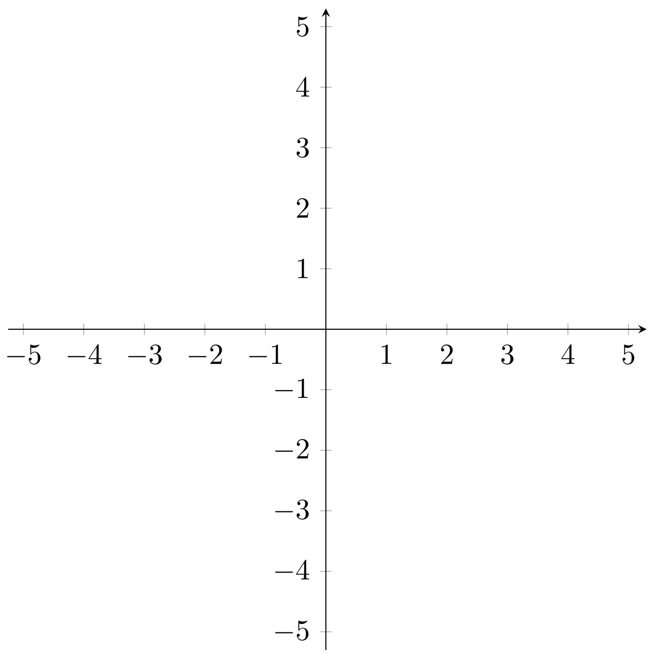
Example 87. The vectors $\mathbf{b}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ form a basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ for $\mathbb{R}^2$. Suppose that $\mathbf{x}$ in $\mathbb{R}^2$ has the coordinate vector $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$. What is $\mathbf{x}$?

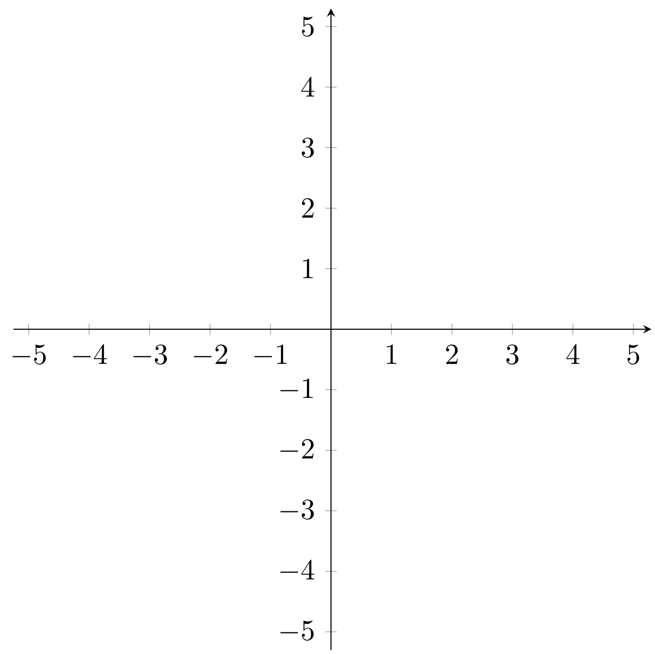
Solution:

Example 88. Let $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2\}$ be the standard basis for $\mathbb{R}^2$, and let $\mathbf{x} = \begin{bmatrix} 6 \\ -1 \end{bmatrix}$

Graphical Interpretation of Coordinates

Let's compare the bases $\mathcal{E}$ and $\mathcal{B}$. Each one corresponds to a "grid" in 2-dimensional space:





See the bottom of page 219 for a three dimensional variant of this example.

Exercise 37. Find the vector $\mathbf{x}$ determined by the given coordinate vector $[x]_{\mathcal{B}}$ relative to the given basis $\mathcal{B}$.

1. $\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ -5 \end{bmatrix}, \begin{bmatrix} -4 \\ 6 \end{bmatrix} \right\}$, $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$

2. $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 4 \\ 7 \\ 0 \end{bmatrix} \right\}$, $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$.

Coordinates in $\mathbb{R}^n$

Next we discuss how to find the $\mathcal{B}$-coordinate for a vector in $\mathbb{R}^n$.

Example 89. Let $\mathbf{b}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $\mathbf{b}_1 = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$, so that $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ forms a basis for $\mathbb{R}^2$. Let $\mathbf{x} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$. Find the coordinate vector $[\mathbf{x}]_{\mathcal{B}}$ for $\mathbf{x}$ relative to $\mathcal{B}$.

Solution:

Remark 14. This example shows a nice relationship between the vector $\mathbf{x}$, it $\mathcal{B}$-coordinates $[\mathbf{x}]_{\mathcal{B}}$, and the elements of the basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots \mathbf{b}_n\}$. Let's expand on this.

The Coordinate Map

Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \dots \mathbf{b}_n\}$ be a basis for a vector space V. This choice of basis produces a map $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ from V to $\mathbb{R}^n$, which can be visualized as follows:

Theorem 39.

Proof. $\square$

Remark 15.

Example 90. Let $\mathcal{B}$ be the standard basis for $\mathbb{P}_3$.

We can use this to our advantage to answer questions about polynomials using what we know about $\mathbb{R}^n$.

Example 91. Use coordinate vectors to determine if the polynomials $1 + 3t + 5t^2$, $1 - 3t^2$ and $1 - t$ are linearly independent.

Solution:

Exercise 38.

1. Find the coordinate vector $[\mathbf{x}]_{\mathcal{B}}$ relative to the basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ if $\mathbf{x} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, $\mathbf{b}_1 = \begin{bmatrix} -1 \\ -3 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$.

2. Find the change-of-coordinates matrix from $\mathcal{B}$ to the standard basis for $\mathbb{R}^2$ if $\mathcal{B} = \left\{ \begin{bmatrix} -1 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -5 \end{bmatrix} \right\}$.

3. Are the polynomials $1 + 2t^3$, $2 + t - 3t^2$, and $-t + 2t^2 - t^3$ linearly independent in $\mathbb{P}_3$?

4.5 The Dimension of a Vector Space

Learning Objectives:

- Use theorems to decide if a set is linearly independent
- Compute the dimension of subspaces of $\mathbb{R}^n$
- Classify subspaces of $\mathbb{R}^2$ or $\mathbb{R}^3$ geometrically

The previous section shows that if a vector space V has a basis $\mathcal{B}$ with n vectors, the coordinate map $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is an isomorphism from V to $\mathbb{R}^n$. This number n is a special invariant of the vectors space called its dimension. Before we present a formal definition, we need to present a couple results. First, the general version of the "more columns than entries in each column" theorem for linear dependence.

Theorem 40.

Proof. □

Theorem 41.

Proof. □

Since every basis has the same cardinality (number of vectors in the set), we can now state the following definition.

Definition 43. (Dimension of a Vector Space)

Example 92. Let's look at our favorite vector spaces: For $\mathbb{R}^n$:

For $\mathbb{P}_n$:

For $\mathbb{P}$:

Example 93. Let $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ where $\mathbf{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$. What is the dimension of the subspace H of $\mathbb{R}^3$?

Example 94. Find the dimension of the subspace

$$H = \left\{ \begin{bmatrix} a - b + 3c \\ 10b \\ \frac{1}{3}a + 2b + c \end{bmatrix} : a, b, c \text{ in } \mathbb{R} \right\}.$$

Solution:

Example 95. The subspaces of $\mathbb{R}^3$ can be classified by dimension:

Exercise 39. Find the dimension of the following subspaces:

1. $\left\{\left[\begin{array}{c} s-2t \\ s+t \\ 3t \end{array}\right] : s, t \text{ in } \mathbb{R}\right\}$

2. $\left\{\left[\begin{array}{c} a-4b-2c \\ 2a+5b-4c \\ -a+2c \\ -3a+7b+6c \end{array}\right] : a, b, c \text{ in } \mathbb{R}\right\}$

3. The subspace of all vectors in $\mathbb{R}^3$ whose first and thiird entries are equal.

4. The subspace of $\mathbb{R}^2$ spanned by $\left[\begin{array}{r} 2 \\ -5 \end{array}\right], \left[\begin{array}{r} -4 \\ 10 \end{array}\right], \left[\begin{array}{r} -3 \\ 6 \end{array}\right]$.

Subspaces of Finite-Dimensional Space

Theorem 42.

Proof. □

If we know the dimension of a vector space V, then we can find a basis for V by finding a set of the right size that either spans, or is linearly independent. Once it satisfies one condition, it will satisfy the other. The next theorem shows why.

Theorem 43. (The Basis Theorem)

Proof.

□

Dimension of Col$A$ and Nul$A$

Fact 7.

Example 96. Find the dimension of Nul A and Col A if

$$A = \begin{bmatrix} 1 & -2 & 3 & 9 \\ -2 & 4 & 1 & 3 \\ -1 & 2 & 1 & 4 \end{bmatrix}$$

Solution:

Exercise 40.

1. Find the dimension of Nul A and Col A if $A = \begin{bmatrix} 1 & 3 & 9 & -7 \\ 0 & 1 & 4 & -3 \\ 2 & 1 & -2 & 1 \end{bmatrix}$

2. Find the dimension of Nul A and Col A if $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 4 & 7 \\ 0 & 0 & 5 \end{bmatrix}$

3. Show that the polynomials $1, 1-t, 2-4t+t^2$, and $6-18t+9t^2-t^3$ form a basis for $\mathbb{P}_3$.

4.6 Rank

Learning Objectives:

- Relate the row space and the column space
- Determine the rank of a matrix
- Include column space, null space, and rank to the invertible matrix theorem

Definition 44. (**Row Space of a Matrix**)

Remark 16.

-
-

Example 97. Let

$$A = \begin{bmatrix} 1 & 2 & 3 & -1 & -4 \\ -3 & -6 & -5 & -5 & -8 \\ 1 & 2 & 0 & 5 & 19 \\ 2 & 4 & 0 & -8 & 30 \end{bmatrix}$$

As we can see above, it is straightforward to find a spanning set for Row A, but how does one find a *basis* for Row A?

Theorem 44.

Proof. □

Example 98. Find bases for Row A, Col A, and Nul A where A is the matrix from the previous example.

Exercise 41. Find bases for Row A, Col A, and Nul A if

$$A = \begin{bmatrix} 1 & -3 & 4 & -1 & 9 \\ -2 & 6 & -6 & -1 & -10 \\ -3 & 9 & -6 & -6 & -3 \\ 3 & -9 & 4 & 9 & 0 \end{bmatrix}$$

The Rank Theorem

We now describe a fundamental relationship between Row A, Col A, and Nul A.

Definition 45. (**Rank**)

Note: the book doesn't make use of this term, but frequently the dimension of Nul A is called the ____ of A.

Theorem 45. (The Rank Thereom)

Proof. □

Example 99.

1. If A is a 10×13 with a 4 dimensional null space, what is the rank of A?
2. Could a 8×13 matrix have a 4 dimensional null space?

Rank and the Invertible Matrix Theorem

Theorem 46. (The Inertible Matrix Theorem, continued) Let A be an $m \times n$ matrix. The following statements are equivalent to being invertible:

m.

n.

o.

p.

q.

r.

Exercise 42.

1. If A is a 3×8 matrix has rank 3, find dim Nul A, dim Col A, and dim Col A^T.

2. If the dimension of the null space of 5×6 matrix A is 4 dimensional, what is the dimension of the row space of A?

3. What is the the largest possible rank of a 7×5 matrix? What bout a 5×7 matrix?

4. . Is it possible for a 10×12 matrix to have a null space of dimension one? Why or why not?

5 Eigenvectors and Eigenvalues

5.1 Eigenvectors and Eigenvalues

Learning Objectives:

- Determine if a vector is an eigenvector for a matrix
- Find the eigenspace for given eigenvalues
- Relate distinct eigenvalues to linear independence

We will study certain simple ways that matrices "act on" vectors. From an alternative perspective, certain vectors for which the transformation $\mathbf{x} \to A\mathbf{x}$ is fairly simple.

Example 100. Let $A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$. Find the images of $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$ under multiplication by A. Compare the vectors to their images.
Solution:

Definition 46. (Eigenvector, Eigenvalue)

Example 101. Let $A = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$. Determine if $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ are eigenvectors for A. If so, find their corresponding eigenvalues.
Solution:

Example 102. Show that 5 is an eigenvalue of the matrix $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$, then find the corresponding eigenvectors.
Solution:

Warning:

Remark 17.

Definition 47. Eigenspace

Example 103. Find a basis for the eigenspace corresponding to $\lambda = 3$ of the matrix

$$A = \begin{bmatrix} 4 & 2 & -1 \\ 2 & 7 & -2 \\ -1 & -2 & 4 \end{bmatrix}$$

Describe the eigenspace geometrically.

Solution:

Exercise 43.

1. Is $\lambda = 2$ an eigenvalue of $\begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix}$? Why or why not?

2. Is $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ an eigenvector for $\begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 1 \\ -3 & 4 & 5 \end{bmatrix}$? If so, find its eigenvalue.

3. Find a basis for the eigenspace of A corresponding to $\lambda = 3$ if $A = \begin{bmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{bmatrix}$.

Now that we know the basic computational tools regarding eigenvectors and eigenvalues, we shall start to expand the theory. We begin with a situation where eigenvalue are easily obtained.

Theorem 47.

Proof. $\square$

Example 104. Find the eigenvalues of the following matrices.

$$A = \begin{bmatrix} 5 & 7 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} 4 & 0 & 0 \\ 5 & -7 & 0 \\ -5 & 2 & 9 \end{bmatrix}.$$

Solution:

Remark 18. If 0 is an eigenvalue for a matrix A if means that $(A - 0 \cdot I)\mathbf{x} = \mathbf{0}$ has a nontrivial solution, thus the homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution.

To finish this section, we present a theorem relating eigenvalues to linear independence of eigenvectors which will be used later.

Theorem 48.

Proof.

□

Exercise 44.

1. Find the eigenvalues of the matrix $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 5 \\ 0 & 0 & -1 \end{bmatrix}$.

2. Find the eigenvalues of the matrix $\begin{bmatrix} 4 & 0 & 0 \\ 7 & 0 & 0 \\ 1 & 0 & -3 \end{bmatrix}$

3. Find one eigenvalue of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$ without any computation. Explain your answer.

4. Construct a 3×3 matrix with only two distinct eigenvalues.

5.2 The Characteristic Equation

Learning Objectives:

- Use the characteristic equation to find eigenvalues
- Relate eigenvalues to similarity and invertibility
- Understand an application to dynamical systems

We will now outline a strategy to find eigenvalues for more general matrices.

Definition 48. (The Characteristic Equation)

Example 105. Find the characteristic equation for $A = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$. *Solution:*

Proposition 1.

Proof. □

Example 106. Find the eigenvalues for $A = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$.

Solution:

Remark 19. It can be shown that $\det(A - \lambda I)$ is always a polynomial, therefore it is often referred to as the *characteristic polynomial* for A.

Example 107. Find the characteristic polynomial and the eigenvalues for $A = \begin{bmatrix} 5 & 4 & 6 \\ 0 & 3 & 2 \\ 0 & 2 & 3 \end{bmatrix}$.

Solution:

Remark 20. .

Example 108. The characteristic equation for a certain 7×7 matrix is $\lambda^7 - 16\lambda^3$. Find the matrix's eigenvalues and their multiplicities.

Solution:

Remark 21. In general, solving polynomials can become difficult. In the case of degree two polynomials, You may need to use the $\boxed{}$. Above for degree 3 and 4, there is an analog but it is very complciated. There is no general method for solving polynomials of degree 5 or more. Also, roots to polynomials may not be real numbers. There is a way to make sense of *complex* eigenvalues, but we won't present it in this class.

Exercise 45. Find the eigenvalues and their multiplicities for the following matrices:

1. $\begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix}$

2. $\begin{bmatrix} 10 & 9 \\ -1 & 4 \end{bmatrix}$

3. $\begin{bmatrix} 0 & 3 & 1 \\ 3 & 0 & 2 \\ 1 & -2 & 0 \end{bmatrix}$

Similarity

Definition 49. (Similar Matrices)

Note 10.

Theorem 49.

Proof. $\square$

WARNINGS:

1. The converse of the theorem is not true. The matrices $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ and $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ have the same eigenvalues but are not similar.
2. The property of similarity is not the same as row equivalece . In general, performing row operations can change the eigenvalues.

An Application to Dynamical Systems

Eigenvalues and eigenvectors can be used to answer questions about long-term behavior of *dynamical systems*. An example of such a system for studying owl populations is given in the book. We will study a more contrived (and simple) example.

Example 109. Suppose that every month, 5% of the population that does not subscribe to Netflix signs up for a subscription, while 3% of Netflix subscribers cancel their subscription. If 40% of the total population are current subscribers, what percentage of the population can we expect to be subscribers in the long run?

Solution:

Exercise 46.

1. Show that if A and B are similar, then they have the same determinant.

2. Let $A = \begin{bmatrix} .6 & .3 \\ .4 & .7 \end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix} 3/7 \\ 4/7 \end{bmatrix}, \mathbf{x}_0 = \begin{bmatrix} .5 \\ .5 \end{bmatrix}$

   (a) Find a basis for $\mathbb{R}^2$ consisting of $\mathbf{v}_1$ and another eignevector $\mathbf{v}_2$ for A.

   (b) Verify that $\mathbf{x}_0$ can be written in the form $\mathbf{x}_0 = \mathbf{v}_1 + c\mathbf{v}_2$.

(c) For $k = 1, 2, \ldots$ define $\mathbf{x}_k = A^k\mathbf{x}_0$. Find $\mathbf{x}_1$ and $\mathbf{x}_2$. Then find a formula for $\mathbf{x}_k$ and show that $\mathbf{x}_k \to \mathbf{v}_1$ as $k \to \infty$.

5.3 Diagonalization

Learning Objectives:

- Determine if matrices are diagonalizable
- Diagonalize matrices
- Compute formulas for powers of diagonalizable matrices

We have seen that it is useful in certain applications to compute high powers of matrices. Today we will use similarity to show how this process can be made easier in some cases. Finding powers of diagonal matrix is particularly easy, as the next example demonstrates.

Example 110. Let $D = \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix}$. Compute D^2, D^3 and find a formula for D^k for $k \geq 1$.

Solution:

If a matrix A is *similar* to a diagonal matrix, the process is also not so hard.

Example 111. Let $A = \begin{bmatrix} -23 & 75 \\ 10 & 32 \end{bmatrix}$ find a formula for A^k for $k \geq 1$ given that $A = PDP^{-1}$ where

$$P = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}, \quad \text{and} \quad D = \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix}.$$

Solution:

Definition 50. (Diagonalizable Matrix)

Not every matrix is diagonalizable. The next theorem allows us to determine when this is the case.

Theorem 50. (The Diagonalization Theorem)

Proof.

□

Example 112. Diagonalize the following matrix, if possible.

$$A = \begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{bmatrix}.$$

Solution:

Exercise 47. Diagonalize the following matrix, if possible. Its eigenvalues are $\lambda = 3$ and $\lambda = 1$.

$$A = \begin{bmatrix} 7 & 4 & 16 \\ 2 & 5 & 8 \\ -2 & -2 & -5 \end{bmatrix}$$

Example 113. Diagonalize the following matrix, if possible.

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Solution:

Theorem 51.

Proof. $\square$

Example 114. Determine if the following matrix is diagonalizable:

$$A = \begin{bmatrix} 3 & 0 & 0 \\ -1 & 2 & 0 \\ 5 & 7 & 0 \end{bmatrix}$$

What if A does not have n distinct eigenvalues? The next theorem outlines this case.

Theorem 52. Let A be an $n \times n$ matrix with distinct eigenvalues $\lambda_1, \lambda_2 \ldots, \lambda_p$.

1.

2.

3.

Example 115. Diagonalize the following matrix, if possible.

$$A = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 1 & 4 & -3 & 0 \\ -1 & -2 & 0 & -3 \end{bmatrix}$$

Solution:

Exercise 48. Diagonalize the matrix $A = \begin{bmatrix} -8 & -6 \\ 15 & 11 \end{bmatrix}$, then use the diagonalization to compute A^5.