Algebraic Theory I

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Lecture 16

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Lecture 15: Nilpotent Groups (2)

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Lemma 0.1. If H, K are groups, then $Z(H \times K) = Z(H) \times Z(K)$.

Proof. Let $(x, y) \in H \times K$. If $(x, y) \in Z(H \times K)$ then

$$\underbrace{(a,1)(x,y)(a,1)^{-1}}_{=(axa^{-1},1)} = (x,y).$$

Hence, $x \in Z(H)$ and similarly, $y \in Z(K)$. Hence, $Z(H \times K) \subseteq Z(H) \times Z(K)$. The other direction of inclusion is trivial and left as an exercise. □

Lemma 0.2. Let $\varphi : G \to G'$ be a homomorphism with $\ker(\varphi) = K$ and $H \leq G$ such that $K \leq H$. Then, $N_G(H) = f^{-1}(N_{G'}(\varphi(H)))$.

Proof. Let $x \in N_G(H)$, so $xHx^{-1} = H$. Hence,

$$\varphi(H) = \varphi\left(xHx^{-1}\right) = \varphi(x)\varphi(H)\varphi(x)^{-1}.$$

Thus,

$$\begin{aligned}
\varphi(x) &\in N_{G'}(\varphi(H)) \\
\Rightarrow x &\in \varphi^{-1}(N_{G'}(\varphi(H))) \\
\Rightarrow N_G(H) &\subseteq \varphi^{-1}(N_{G'}(\varphi(H))).
\end{aligned}$$

Conversely, let $x \in \varphi^{-1}(N_{G'}(\varphi(H)))$, hence $\varphi(x) \in N_{G'}(\varphi(H))$.
Then, we see

$$\begin{aligned}
\varphi(H) &= \varphi(x)\varphi(H)\varphi\left(x^{-1}\right) \\
&= \varphi\left(xHx^{-1}\right) \\
\Rightarrow xHx^{-1} &\subseteq \varphi^{-1}(\varphi(H)) \\
&= \langle H, \ker(\varphi)\rangle \\
&= H \text{ as } \ker(\varphi) \subseteq H.
\end{aligned}$$

Hence, $xHx^{-1} \subseteq H$, so $x \in N_G(H)$. This concludes the proof. $\square$

Now, recall that if G is a finite group with P being a sylow p-group, then TFAE

1. P is unique.
2. $P \trianglelefteq G$.
3. P is characteristic.
4. Any subgroup generated by elements whose orders are powers of p is itself a p-group.

Theorem 0.1. If G is a finite group, then the following are equivalent:

1. G is nilpotent.
2. $H < G \Rightarrow H < N_G(H)$.
3. All sylow p-groups are normal.
4. G is the direct product of its sylow p-groups.

Proof.
- $(2 \Rightarrow 3)$. Let P be a sylow p-group of G. Assume P is not normal, then denote $N = N_G(P) \subset G$. Hence, by the preceding lemma, P is characteristic in N. Then, as $N \trianglelefteq N_G(N)$, we see $P \trianglelefteq N_G(N)$. But $N = N_G(P)$ was the largest subgroup in which P was normal, hence $N_G(P) = N_G(N)$. So, by contrapositive of the assumption, (2), we have $N = N_G(N)$, so $N = G$, hence $P \trianglelefteq G$.
- $(3 \Rightarrow 4)$.
- $(1 \Rightarrow 2)$. Let G be nilpotent. If G is abelian, then $N_G(A) = G$ for all $A \leq G$, hence any proper subgroup $H < G$ has $H < N_G(H) = G$. Hence, assume G is non-abelian and proceed by induction on $|G|$ with base case $|G| = p$ being already completed p-prime. Suppose indirectly that there is an $H < G$ such that $H = N_G(H)$.
  Now, we note that $Z(G) \leq N_G(H) = H$ by definition of $Z(G)$. That is, $Z(G) \leq H$. Let $\varphi : G \to G/Z(G)\,,\ x \mapsto \varphi(x) = xZ(G)$. Since G is nilpotent, $Z(G) = 1 \Leftrightarrow G = 1$, but we assumed G to be nonabelian, so this is not the case. Hence, we can assume $Z(G) = \{1\}$, hence $|G/Z(G)| < |G|$. As we know, G being nilpotent implies $G/Z(G)$ is nilpotent. Lastly, we note that $Z(G) \leq H < G$, so by the lattice theorem, we have $H/Z(G) < G/Z(G)$. Applying the induction hypothesis yields $H/Z(G) < N_{G/Z(G)}(H/Z(G))$. Recalling the lemma from last class, $\varphi^{-1}\left(N_{G/Z(G)}(H/Z(G))\right) = N_G(H)$. Then, we note
  $$\varphi^{-1}(\varphi(H)) < \varphi^{-1}\left(N_{\varphi(G)}(\varphi(H))\right) = N_G(H).$$
  And as $\ker(\varphi) = Z(G) \leq H$, we have $H < N_G(H)$.

$\square$