Analysis I: Homework II

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Problem (9)**.** Let $\mathscr{F} = \{\{3\}, \{\{3\}, \mathbb{N}\}, \{\mathbb{Z}, \{\{3\}, \mathbb{N}\}\}\}$. Show all possible choice functions on $\mathscr{F}$.

Solution. Let $f_1, f_2, f_3, f_4 : \mathscr{F} \to \{\{3\}, \mathbb{N}, \mathbb{Z}, \{\{3\}, \mathbb{N}\}\}$ and $F \in \mathscr{F}$. We list all possible function rules which are a choice function.

1.
$$f_1(F) = \begin{cases} 3, & F = \{3\} \\ \{3\}, & F = \{\{3\}, \mathbb{N}\} \\ \mathbb{Z}, & F = \{\mathbb{Z}, \{\{3\}, \mathbb{N}\}\} \end{cases}.$$

2.
$$f_2(F) = \begin{cases} 3, & F = \{3\} \\ \{3\}, & F = \{\{3\}, \mathbb{N}\} \\ \{\{3\}, \mathbb{N}\}, & F = \{\mathbb{Z}, \{\{3\}, \mathbb{N}\}\} \end{cases}.$$

3.
$$f_3(F) = \begin{cases} 3, & F = \{3\} \\ \mathbb{N}, & F = \{\{3\}, \mathbb{N}\} \\ \mathbb{Z}, & F = \{\mathbb{Z}, \{\{3\}, \mathbb{N}\}\} \end{cases}.$$

4.
$$f_4(F) = \begin{cases} 3, & F = \{3\} \\ \mathbb{N}, & F = \{\{3\}, \mathbb{N}\} \\ \{\{3\}, \mathbb{N}\}, & F = \{\mathbb{Z}, \{\{3\}, \mathbb{N}\}\} \end{cases}.$$

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Problem (10)**.** Let $(a_k), (b_k) \in \mathrm{CS}(\mathbb{Q})$.

1. Show that $(a_k b_k) \in \mathrm{CS}(\mathbb{Q})$.
2. Show that $\left(\frac{a_k}{b_k}\right) \in \mathrm{CS}(\mathbb{Q})$ if there is $N \in \mathbb{N}$ and rational $\varepsilon > 0$ such that $|b_n| \geq \varepsilon$ for $n \geq N$.

Lemma 1. All $(x_k) \in \mathrm{CS}(\mathbb{Q})$ are bounded by some $M \in \mathbb{N}$.

Proof of lemma. Let $(x_k) \in \mathrm{CS}(\mathbb{Q})$ and suppose for all $M \in \mathbb{N}$ there is $k \in \mathbb{N}$ such that $|x_k| \geq M$. Then, let $\varepsilon > 0$ and $N \in \mathbb{N}$ such that for all $m, n \geq N$, we have $|x_n - x_m| < \varepsilon$ and set $m \geq n$. Fix $M \in \mathbb{N}$ such that $M \geq |x_N|$ and $M \geq \varepsilon$ and let m be sufficiently large such that $|x_m| \geq 2M$. Then, $|x_m - x_N| \geq |2M - M| = M \geq \varepsilon$. $\lightning$. Hence, there must be $M \in \mathbb{N}$, such that $|x_k| < M$ for all $k \in \mathbb{N}$. $\square$

Solution.

1. We wish to show that for all $\varepsilon > 0$ there is a $N \in \mathbb{N}$ such that $|a_n b_n - a_m b_m| < \varepsilon$ for $n, m \geq N$. Let $0 \neq M \geq |a_n|, |b_n|$ for all $n \in \mathbb{N}$ to be an upper bound by the lemma. Then, as (a_k) and $(b_k) \in \mathrm{CS}(\mathbb{Q})$ we see for all $\frac{\varepsilon}{2M} > 0$, there is $N_a, N_b \in \mathbb{N}$ such that $|a_n - a_m| < \frac{\varepsilon}{2M}$, $n, m \geq N_a$, and $|b_n - b_m| < \frac{\varepsilon}{2M}$, $n, m \geq N_b$. Lastly, let $N = \max\{N_a, N_b\}$. Then, we see, for all $m, n \leq N$ we have

$$\begin{aligned}
|a_n b_n - a_m b_m| &= |a_n b_n - a_n b_m + a_n b_m + a_m b_m| \\
&= |a_n (b_n - b_m) - b_m (a_n - a_m)| \\
&\leq |a_n| |b_n - b_m| + |b_m| |a_n - a_m| \\
&< |a_n| \frac{\varepsilon}{2M} + |b_m| \frac{\varepsilon}{2M} \text{ by } (a_k), (b_k) \in \mathrm{CS}(\mathbb{Q}) \\
&= \frac{\varepsilon}{2M} (|a_n| + |b_m|) \\
&\leq 2M \frac{\varepsilon}{2M} = \varepsilon \text{ by boundedness.}
\end{aligned}$$

   Hence $(a_k b_k) \in \mathrm{CS}(\mathbb{Q})$.

2. Denote $|b_n| \geq \varepsilon_B$ for $n \geq N_B$. Furthermore, as $(a_k), (b_k) \in \mathrm{CS}(\mathbb{Q})$ we know for all $\varepsilon > 0$ there are $N_a, N_b \in \mathbb{N}$ such that $|a_n - a_m| < \varepsilon$ for $n, m \geq N_a$ and $|b_n - b_m| < \varepsilon$ for $n, m \geq N_b$. Let $\frac{M\varepsilon}{2} > 0$ and $N = \max\{N_B, N_a, N_b\}$. Then, we see for $n, m \geq N \geq N_B$ (hence division

will be well defined) we have

$$\begin{aligned}
\left|\frac{a_n}{b_n} - \frac{a_m}{b_m}\right| &= \left|\frac{a_n b_m - a_m b_n}{b_n b_m}\right| \\
&= |a_n b_m - a_m b_n| \frac{1}{|b_m b_n|} \\
&= |a_n b_m - a_n b_n - a_m b_n + a_n b_n| \frac{1}{|b_n b_m|} \\
&\leq (|a_n| |b_m - b_n| + |b_n| |a_n - a_m|) \frac{1}{|b_n b_m|} \\
&< \frac{M\varepsilon}{2} (|a_n| + |b_n|) \frac{1}{|b_n| |b_m|} \\
&\leq \frac{2M^2\varepsilon}{2} \cdot \frac{1}{M^2} \text{ by boundedness} \\
&= \varepsilon.
\end{aligned}$$

Hence, $\left(\frac{a_k}{b_k}\right) \in \mathrm{CS}(\mathbb{Q})$.

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Problem (11)**.** Let (x_k) be a rational sequence such that there is $M \in \mathbb{Z}$ such that $|x_n| \leq M$ for all n and $x_{n+1} \geq x_n$ for all n.

1. Without resorting to real numbers, show that $(x_k) \in \mathrm{CS}(\mathbb{Q})$.
2. Let $s = \sup\{x_n : n \in \mathbb{N}\}$. Use the Least upper bound property to show $(x_k) \in \mathrm{CS}(\mathbb{Q})$.

Solution.

1. Suppose (x_k) has the ascribed properties and $(x_k) \notin \mathrm{CS}(\mathbb{Q})$. That is, there is a rational $\varepsilon > 0$ such that for all $N \in \mathbb{N}$, $|x_n - x_m| \geq \varepsilon$ for some $n, m \geq N$. Moreover, there are infinitely many pairs, $n, m \geq N$ such that $|x_n - x_m| \geq \varepsilon$ (else we could set $N = \max\{n, m\}$).
   We will show that this contradicts the boundedness assumption. Note that for any pair $n, m \in \mathbb{N}$ such that $|x_n - x_m| \geq \varepsilon$ we can find a pair $p, q \geq \max\{n, m\}$ such that $|x_p - x_q| \geq \varepsilon$ (setting $N = \max\{n, m\}$). Let (p_i, q_i) be a sequence of such pairs with $p_i \geq q_i$. That is, $q_1 \leq p_1 \leq q_2 \leq p_2 \leq \ldots \leq q_i \leq p_i \leq q_{i+1} \leq p_{i+1} \leq \ldots$. Then, we have $|x_{p_i} - x_{q_i}| \geq \varepsilon$ and as $x_{p_i} \geq x_{q_i}$ by the increasing hypothesis, we see $x_{p_i} \geq x_{q_i} + \varepsilon$ for all $i \in \mathbb{N}$. Furthermore the increasing hypothesis guarantees $x_{q_i} \geq x_{p_{i-1}}$. Now, we induce on i to show $x_{p_i} \geq i\varepsilon + x_{q_1}$. For the base case we see
   $$\begin{aligned} x_{p_2} &\geq x_{q_2} + \varepsilon \\ &\geq x_{p_1} + \varepsilon \\ &\geq x_{q_1} + 2\varepsilon. \end{aligned}$$
   Now, let us assume $x_{p_{i-1}} \geq (i-1)\,\varepsilon + x_{q_1}$. Lastly, we see
   $$\begin{aligned} x_{p_i} &\geq x_{q_i} + \varepsilon \\ &\geq x_{p_{i-1}} + \varepsilon \\ &\geq x_{q_1} + (i-1)\,\varepsilon + \varepsilon = x_{q_1} + i\varepsilon. \end{aligned}$$
   Finally, as we know for all positive $p, q \in \mathbb{Q}$, there is $m \in \mathbb{N}$ such that $mp > q$ (the archemedian property on rationals), we see there is $n \in \mathbb{N}$ such that $n\varepsilon > M$ (the upper bound) and a $m \in \mathbb{N}$ such that $m\varepsilon > |x_{q_1}|$, hence $m\varepsilon + x_{q_1} > 0$ (even in the case $x_{q_1} < 0$). Then we see,
   $$\begin{aligned} x_{p_{(n+m)}} &\geq \underbrace{x_{q_1} + m\varepsilon}_{>0} + n\varepsilon \\ &> n\varepsilon \\ &> M > 0. \end{aligned}$$
   As $|x_{p_n}| = x_{p_n} > M$ this contradicts the boundedness assumption. $\lightning$. So, we must have that $(x_k) \in \mathrm{CS}(\mathbb{Q})$.
2. As M is an upper bound, we see $s \leq M$ is well defined. Hence, $|x_n| \leq s \leq M$ for all $n \in N$. Furthermore, we see for any rational $\varepsilon > 0$, we have $s - \varepsilon < x_j \leq x_{j+1} \leq \ldots \leq s$ for sufficiently large j else $s - \varepsilon$ would be an upper bound. Hence, for $n, m \geq j$, we have $|x_n - x_m| < |s - (s - \varepsilon)| = \varepsilon$. Thus, $(x_k) \in \mathrm{CS}(\mathbb{Q})$.

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Problem (12)**.** Show that the extension of the total ordering $\leq$ from $\mathbb{R}$ to $\mathbb{C}$ does not yield a total ordering on $\mathbb{C}$.

Solution. Were the total ordering $\leq$ to extend to $\mathbb{C}$, then we would have $\mathbb{C}$ is an ordered field (under addition and multiplication). Recall that in an ordered field $0 \leq a^2$ for all a in the field. But, we note that $i^2 = -1 < 0$, a contradiction. Hence, $\mathbb{C}$ is not an ordered field and, as we know $\mathbb{C}$ to be a field, there must not exist a total ordering. Hence, $\leq$ is not a total ordering on $\mathbb{C}$.

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Problem (13)**.** Let X be the collection of all sets A for which $A \notin A$. Prove $X \in X \Leftrightarrow X \notin X$.

Solution. Suppose $X \in X$. Then, $X \in X$ is a set containing itself, hence $X \notin X$ by construction.
Conversely, suppose $X \notin X$. Then, X is a set for which $X \notin X$, so $X \in X$ by construction. $\blacksquare$