Algebraic Theory I

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October 6, 2021

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Lecture 18: Solvable Groups (2) and Free Groups

Mon 04 Oct 2021 11:28

Recall. A group is solvable if there exists a chain of subgroups

$$\{1\} \trianglelefteq H_0 \trianglelefteq H_1 \trianglelefteq \ldots \trianglelefteq H_n = G$$

such that H_i/H_{i-1} is abelian.
We had that this is equivalent to the condition that $G^{(n)} = \{1\}$ where $G^{(0)} = G$ and $G^{(i)} = \left[G^{i-1}, G^{i-1}\right]$ for some $n \geq 0$. We showed the forward implication, so now we show the reverse implication.

Proof. Suppose $G^{(n)} = 1$ for some $n \geq 0$. Then, we have a chain

$$G = G^{(0)} \trianglelefteq G^{(1)} \trianglelefteq \ldots \trianglelefteq G^{(n)} = \{1\}.$$

So, we have

$$\{1\} = G^{(n)} \trianglerighteq G^{(n-1)} \trianglerighteq \ldots \trianglerighteq G^{(0)} = G.$$

Furthermore, we know the commutator of $G^{(i)}$is a characteristic subgroup, hence it is normal.
Then, define $H_i = G^{(n-i)}$ for $0 \leq i \leq n$. We need only show the quotients to be abelian. We see $H_i/H_{i-1} = G^{(n-i)}/G^{(n-i+1)}$. But, $G^{(n-i+1)} = \left[G^{(n-i)}, G^{(n-i)}\right]$ by definition. Hence, $G^{(n-i)}/G^{(n-i+1)}$ is abelian by the lemma from last class. So, the chain condition holds and G is solvable. □

Theorem 0.1. Let G be a solvable group with H being a subgroup. Then, H is solvable.

Proof. We simply show $H^{(n)} \leq G^{(n)}$ for all n by induction. For the base case we know $H = H^{(0)} \leq G^{(0)} = G$. Then, we note $H^{(n)} = \left[H^{(n-1)}, H^{(n-1)}\right] \subseteq \left[G^{(n-1)}, G^{(n-1)}\right] = G^{(n)}$ by inductive hypothesis. Since G is solvable, we find a $n \geq 0$ such that $G^{(n)} = \{1\}$. Then, $H^{(n)} \leq G^{(n)} = \{1\}$, so $H^{(n)} = \{1\}$ hence H is solvable. □

Theorem 0.2. If G is solvable and $\varphi : G \to G'$ is a homomorphism, then $\varphi(G)$ is also solvable.

Proof. We see $\varphi\left(G^{(0)}\right) = \varphi(G)^{(0)}$. So, $\varphi\left(G^{(0)}\right) = \varphi(G)^{(0)}$. We induce on n. We see

$$\begin{aligned}
\varphi\left(G^{(n)}\right) &= \varphi\left(\left[G^{(n-1)}, G^{(n-1)}\right]\right) \\
&= \varphi\left(\left\langle x^{-1}y^{-1}xy : x, y \in G^{(n-1)}\right\rangle\right) \\
&= \left\langle \varphi\left(x^{-1}y^{-1}xy : x, y \in G^{(n-1)}\right)\right\rangle \\
&= \left\langle \varphi(x)^{-1}\varphi(y)^{-1}\varphi(x)\varphi(y) : x, y \in G^{(n-1)}\right\rangle \\
&= \left\langle \overline{x}^{-1}\overline{y}^{-1}\overline{xy} : \overline{x}, \overline{y} \in \varphi\left(G^{(n-1)}\right)\right\rangle \\
&= \left\langle \overline{x}^{-1}\overline{y}^{-1}\overline{xy} : \overline{x}, \overline{y} \in \varphi(G)^{(n-1)}\right\rangle \text{ by the inductive hypothesis.} \\
&= \left[\varphi(G)^{(n-1)}, \varphi(G)^{(n-1)}\right] \\
&= \varphi(G)^{(n)}.
\end{aligned}$$

Since G is solvable, we find an $n \geq 0$ such that $G^{(n)} = \{1\}$. Hence, $\varphi\left(G^{(n)}\right) = \varphi(\{1\}) = \{1\} = \varphi(G)^{(n)}$, so $\varphi(G)$ is solvable. □

Theorem 0.3. If G is a group with $H \trianglelefteq G$, then G is solvable if and only if H and G/H are solvable.

Proof. ($\Rightarrow$). We know all subgroups and homomorphic images to be solvable, hence this direction is already proven.
($\Leftarrow$). Assume H and G/H are solvable. As H is solvable it has a normal chain

$$H_0 \trianglelefteq H_1 \trianglelefteq \ldots \trianglelefteq H_n = H$$

with H_i/H_{i-1} is abelian for all $1 \leq i \leq n$. Similarly, since G/H is solvable there is a normal chain

$$\{1\} = K_{n+0} \trianglelefteq K_{n+1} \trianglelefteq \ldots K_{n+s} = G/H$$

With K_{n+i}/K_{n+i-1} being abelian for all $i \geq 1$. We know by the lattice theorem that there are groups H_{n+i} such that $K_{n+i} = H_{n+i}/H$ for some $H_{n+i} \leq G$ and $H \leq H_{n+i}$. Then, we have

$$\{1\} = H/H \trianglelefteq H_{n+1}/H \trianglelefteq \ldots \trianglelefteq H_{n+s}/H = G/H.$$

Then, we have $H_n = H$ and $H_{n+s} = G$ and, as each contains the kernel, this correspondance preserves normality, hence we have

$$H_n = H \trianglelefteq H_{n+1} \trianglelefteq H_{n+2} \trianglelefteq \ldots H_{n+s} = G.$$

Then, note that $H_{n+i}/H_{n+i-1} = (H_{n+i}/H)\,/\,(H_{n+i-1}/H) = K_{n+i}/K_{n+i-1}$ which we know to be abelian. Hence all successive quotients are abelian. So,

$$\{1\} = H_0 \trianglelefteq H_1 \trianglelefteq \ldots \trianglelefteq H_n \trianglelefteq H_{n+1} \trianglelefteq H_{n+2} \trianglelefteq \ldots H_{n+s} = G.$$

with H_i/H_{i-1} being abelian, so G is solvable. $\square$

Remark. Subgroups and quotients of nilpotent groups are nilpotent, but this converse does not hold in general for nilpotent groups.

1 Free Groups

Recall. $\langle \alpha, \tau : \alpha^n = 1, \tau^2 = 1, \tau\alpha\tau = \alpha^{-1} \rangle = D_{2n}$ is the dihedral group of order $2n$. This is technically ill defined. In general, we have generators α, τ and a set of relations that allow us to say when products of generators are equal. Similairly, we find $\langle \alpha : \alpha^n = 1, \alpha^{n+1} = 1 \rangle = \{1\}$. We have not, however, ensured that these form groups. This problem motivates the definition of free groups.

If S is a set, then we let S^{-1} be a disjoint set of formal symbols with $x \mapsto x^{-1}$, so $S = \{a, b, c\}$ and $S^{-1} = \{a^{-1}, b^{-1}, c^{-1}\}$. Then, let $F(S)$ to be the set of all formal products of elements from $S \cup S^{-1} \cup \{1\}$. Next class we will define an equivalence relation which takes these products into a group.

Lecture 19: Free Groups (2)

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Recall we had a set of letters $X = \{a, b, c, \ldots, a^{-1}, b^{-1}, c^{-1}, \ldots, 1\}$. Then, we define a word on the alphabet X to be a string $\omega = x_1^{\varepsilon_1} x_2^{\varepsilon_2} \ldots, x_s^{\varepsilon_s}$ where $x_1, x_2, \ldots, x_s \in X$ and $\varepsilon_i = \pm 1$. For example with $X = \{x_1, x_2, x_3\}$ we have a word $x_1 x_1 x_2 x_1^{-1} x_1 x_3$ for example. Then, define 1 to be the empty product, that being a string with no symbols. Now, we define an equivalence relation on the words to induce a group.
We say two words $\omega_1 \sim \omega_2$ if we can transform ω_1 into ω_2 with a finite sequence of the following operations

- Remove a sequential pair xx^{-1} or $x^{-1}x$ from the string.
- Insert a substring xx^{-1} or $x^{-1}x$ into the string.

So, we see $x_1 x_2 x_3^{-1} x_4 \sim x_1 x_2 x_3^{-1} x_2 x_2^{-1} x_1^{-1} x_1 x_4$ and so on. It is trivial to verify this to be an equivalence relation, so we omit the proof. Henceforth, we will denote the equivalence class of a word ω by $[\omega]$. So, we see if $\omega_1 \sim \omega_2$, we have $[\omega_1] = [\omega_2]$.
Now, let $F(X)$ be the set of all equivalence classes on X and define $[\omega_1][\omega_2] := [\omega_1\omega_2]$ with $\omega_1\omega_2$ simply being the concatenation of the two words. First, we verify this to be well-defined. Suppose $w' \sim w$ and $v' \sim v$ are 4 words. Hence, there is a simple sequence taking $v \mapsto v'$ and $w \mapsto w'$. It is easy to see then, that the same operations applied to their respective parts will take $vw \mapsto v'w'$ and $wv \mapsto w'v'$, hence $[vw] = [v'w']$.
Next, we show this forms a group. We see $[w][1] = [w \cdot 1] = [w]$ and likewise $[1][w] = [w]$, so 1 is the identity.

Next,

$$\begin{aligned}
[w]\left([u]\,[v]\right) &= [w]\,[uv] \\
&= [w(uv)] \\
&= [(wu)\,v] \\
&= [wu]\,[v] \\
&= \left([w]\,[u]\right)[v]
\end{aligned}$$
.

Hence, $F(X)$ is associative. Lastly, we show inverses exist. Let $w = x_1^{\varepsilon_1} \dots x_s^{\varepsilon_s}$, then let $w^{-1} = x_s^{-\varepsilon_s} \dots x_1^{-\varepsilon_1}$ and we see $ww^{-1} \sim 1$, so $F(X)$ has inverses.

Definition 1.1 (Free Group)**.** For an alphabet X, we define $F(X)$ to be the **Free Group on** X. More generally, the free group F on X is a group F together with an injection $\sigma : X \hookrightarrow F$ such that any $\alpha : X \to G$, with G being an arbitrary group, extends to a unique homomorphism $\beta : F \to G$ such that $\beta \circ \sigma = \alpha$.

$$\begin{tikzcd}
 & F(X) \arrow[dr, dotted, "\beta"] & \\
X \arrow[ur, "{\sigma, x \mapsto [x]}"] \arrow[rr, "\alpha"] & & G
\end{tikzcd}$$

Figure 1: In this commutative diagram solid lines represent given maps and dotted lines represent maps that must then exist

Next, recall a homomprhism $\varphi : H \to G$ is determined by the images of generators of H. Let $H = \langle X \rangle$. Then for an arbitrary $h \in H$ with $h = x_1^{\varepsilon_1} \dots x_n^{\varepsilon_n}$ we find $\varphi(h) = \varphi(x_1)^{\varepsilon_1} \dots \varphi(x_n)^{\varepsilon_n}$ with $x_i \in X$ and $\varepsilon_i = \pm 1$.
Now, let G be a group with $\alpha : X \to G$ being a map and $\sigma : X \hookrightarrow F$ be the inclusion map. Let $w = x_1^{\varepsilon_1} \dots x_n^{\varepsilon_n}$ and let $(w) = \alpha(x_1)^{\varepsilon_1} \dots \alpha(x_n)^{\varepsilon_n}$ with $x_i \in X$ and $\varepsilon_i = \pm 1$. Then, we define $\beta([w]) = [\beta(w)]$. It is simple to check this is well defined as we may always insert or delete substrings of the form $\alpha(x_i)^{\varepsilon_i}\, \alpha(x_i)^{-\varepsilon_i}$

in order to induce an equivalence. We see β is also a homomoprhism as

$$\begin{aligned}\beta\left([w]\,[v]\right) &= \beta\left([wv]\right)\\ &= \beta\left(wv\right)\\ &= \beta\left(w\right)\beta\left(v\right)\\ &= \beta\left([w]\right)\beta\left([v]\right).\end{aligned}$$

Lastly, we see the map β is unique as a homomoprhism is completely characterized by where it sends the generators.