# Combinatorics

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## Contents

### Lecture 13

Fri 24 Sep 2021 10:20

I originally missed this lecture, so this is transcribed from a classmates notes.

### Lecture 14: Quasi-Random Graphs

Fri 24 Sep 2021 10:21

**Remark.** Random graphs have applications in ramsey theory. For instance, if $p \geq 2$, $q \geq 2$, then there is a ramsey number $R(p,q)$ such that for a graph $G$ of order at least $R(p,q)$, then either $G$ constains a $p$-clique or an independent set of $q$ vertices.
It is a well known result that $R(3,3) = 6$.
A counterexample to a graph of order 5 is $K_5$.

We can even obtain an upper bound $R(p,q) \leq \binom{p+q-2}{q-1}$. This is obtained from the trivial fact that $R(p,q) \leq R(p,q-1) + R(p-1,q)$.
Now, we examine the diagonal case, $R(k,k) \leq \binom{2k-2}{k-1} \leq \frac{4^{k-1}}{\sqrt{k}}$. From this we obtain, $\sqrt[k]{R(k,k)} \leq 4$, and a probabalistic argument from erdos yields

$$\sqrt{2} \leq \sqrt[k]{R(k,k)} \leq 4$$

**Proposition 0.1.** For all $k$ and $n$ with $n \leq \sqrt{2}^k$, there is a graph of order $n$ such that $G$ has no $k$-clique and no independent set of order $k$.

*Halmos.* Fix $n$ vertices, $1, 2, \ldots, n$ and consider all labeled graphs , denoted $LG$. Now, recall there are $2^{\binom{n}{2}}$ labeled graphs of order $n$. Next, denote $k_k(G)$ to be the number of $k$-cliques in a graph $G$ and we see an independent set is simply a clique of $\overline{G}$, so we see we need only consider $k_k(G) + k_k(\overline{G})$, hence the total number of graphs with either a $k$-clique or $k$-independent set of order $n$ are $S = \sum_{g \in LG(n)} k_k(G) + k_k(\overline{G}) = 2 \cdot \binom{n}{k} 2^{\binom{n}{2} - \binom{k}{2}}$.
The leading $\binom{n}{k}$ is due to the fact that there are $\binom{n}{k}$ subsets of order $k$ in a

set of order $n$ and the exponent comes from the total amount of possible edges outside of the $k$-clique.
Now we construct a bipartite graph $G$ with $A = LG(n)$ and $B$ being the set of all possible $k$-cliques. We see each $a \in A$ is a labeled graph, so it may have differing numbers of $k$-cliques, each $b \in B$ is a $k$-clique, and all $k$-cliques participate in the same number of labeled graphs of order $n$ hence $B$ is regular to $A$.
Taking our earlier definition of $S$ and manipulating yields

$$S \leq 2^{\binom{n}{2}} \left( \frac{2\binom{n}{k}}{2^{\binom{k}{2}}} \right).$$

Hence,

$$\frac{S}{2^{\binom{n}{2}}} \leq \frac{2\binom{n}{k}}{2^{\binom{k}{2}}} < 1.$$

Assuming $k \geq 3$ and applying definitions yields

$$\overline{2}$$

and n$\left(_{k = \frac{n(n-1)\dots(n-k+1)}{k!} \leq \frac{2 \cdot 2^{\frac{k^2}{2}}}{k! 2^{\binom{k(k-1)}{2}}}}\right.$ taking $k = \sqrt{2}^k$ yields $\frac{2(\sqrt{2})k}{k!}$.$\Big)$

$$\frac{2\binom{n}{k}}{2^{\binom{k}{2}}} < \frac{2n^k}{k! 2^{kk-1}}$$

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2 and $\binom{n}{k} = \frac{n(n-1)\dots(n-k+1)}{k!} \leq \frac{2 \cdot 2^{\frac{k^2}{2}}}{k! 2^{\binom{k(k-1)}{2}}}$ taking $k = \sqrt{2}^k$ yields $\frac{2(\sqrt{2})k}{k!}$.

**Remark.** Note that after $\binom{n}{2}$ flips of a fair coin, one obtains a graph in $LG(n)$. Take a subset $M$ of cardinality $k$ in the set of all such graphs and note that there is a $\frac{1}{2^{\binom{k}{2}}}$ probability this will be a $k$-clique. Hence the total probability summed over all subsets $M$ is $\binom{n}{k} \frac{2}{2^{\binom{k}{2}}}$. Applying the subbadditivity of probability yields that this is strictly less than 1. Hence, there is such a graph not containing a $k$-clique or independent set of order 20.

$\square$