Real Variables I: Homework I

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Problem (1)**.** Let $f : X \to Y$.

1. Show that for $A \subseteq X$, $B \subseteq Y$, $f\left(f^{-1}(B)\right) \subseteq B$ and $A \subseteq f^{-1}(f(A))$.
2. Give examples to show that the set inclusions can be proper.

Solution.

1. Let $b \in f\left(f^{-1}(B)\right)$ and note that, as b is in the image of $f^{-1}(B)$, there is $a \in f^{-1}(B)$ such that $f(a) = b$. As $a \in f^{-1}(B)$, we see $f(a) \in B$. As $f(a) = b \in B$ this completes the proof.

2. Now, let $a \in A$. We see $f(a) \in f(A)$ by definition, and as $f(a) \in f(A)$ we see that for all $b \in A$ such that $f(b) = f(a) \in f(A)$, we have $b \in f^{-1}(f(A))$. It is clear that a is one such element, so $a \in f^{-1}(f(A))$. This completes the proof.
3. Let $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto f(x) = x^2$ and denote $B = [-1, 1]$. We see $f^{-1}(B) = [-1, 1]$ and $f([-1, 1]) = [0, 1]$. Hence, $f\left(f^{-1}(B)\right) = [0, 1] \subset [-1, 1] = B$.
4. Now, let $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto f(x) = 0$ and denote $A = [0, 1]$. We see $f(A) = \{0\}$ and $f^{-1}(\{0\}) = \mathbb{R}$ as the function is zero everywhere. Hence $f^{-1}(f(A)) = \mathbb{R} \supset [0, 1] = A$.

Problem (2)**.** Let $A, B \subseteq X$. Prove or disprove

1. $A \triangle B = \varnothing \Leftrightarrow A = B$.
2. $A \triangle B = X \Leftrightarrow A = B^c$.

Solution. 1. Suppose $A \triangle B = \varnothing$ and let $a \in A$, $b \in B$. Then, we see $a \notin B \setminus A$ by definition. Furthermore, As $A \triangle B = (A \setminus B) \cup (B \setminus A) = \varnothing$, we see $a \notin A \setminus B$, but as $a \in A$ this implies $a \in B$. Hence $a \subseteq B$. Again, notice $b \notin A \setminus B$ by definition. Furthermore, $b \notin B \setminus A$ as this would make $A \triangle B$ nonempty, so $b \in A$. Hence, $A = B$.
Conversely, suppose $A = B$. Then,

$$A \triangle B = A \triangle A = (A \setminus A) \cup (A \setminus A) = \varnothing \cup \varnothing = \varnothing.$$

2. Suppose $A \triangle B = X$ and let $a \in A$. Then, we see $a \notin B \setminus A$ by definition, but $a \in X$, so $a \in A \setminus B$. Hence $a \notin B$. As every $a \in A$ has $a \notin B$, we see $A \subseteq B^c$. Now, let $b \in B^c$. We see $b \notin B$ by definition, hence $b \notin B \setminus A$. As $b \in X$, we must then have that $b \in A \setminus B$, hence $b \in A$. Thus, $B^c = A$.
Conversely, suppose $B^c = A$. Then,

$$A \triangle B = B^c \triangle B = (B^c \setminus B) \cup (B \setminus B^c) = B^c \cup B = X$$

by definition of complements.

Problem (3)**.** Suppose $f : X \to Y$ and $g : Y \to Z$ are functions.

1. Show that $f : X \to Y$ is injective if and only if there is a map $g : Y \to X$ such that $g \circ f$ is the identity on X. If such a map g exists is it necessarily unique, injective, or surjective.
2. Show that f is onto if and only if there is a map $g : Y \to X$ such that $f \circ g$ is the identity on Y.

Solution. 1. Let $g : X \to Y$ be a map such that $g \circ f$ is the identity on X. Then, suppose f is not injective. Let $x \neq y \in X$ such that $f(x) = f(y)$. Then $g(f(x)) = g(f(y)) = x$ or y or another such element. WLOG, suppose $g(f(x)) = g(f(y)) = x$. Then, $g(f(y)) = x$ contradicts the assumption that $g \circ f$ was the identity.
Now, suppose f is injective. Then, for each $x \in X$ there is a unique $f(x) \in Y$. Hence, let us define the map $g : Y \to X$ such that $g(f(x)) = x$ for all $x \in X$. We see this is a function as each $f(x) \in Y$ originates from only 1 $x \in X$ by injectivity. Hence, this implies $g \circ f$ is the identity by this definition. This completes the proof.

2.