Algebraic Theory I

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Contents

1 Ring Localization

Lecture 33: Localization of Rings

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Recall. Recall R denotes a commutative ing. If $S \subseteq R$ is a multiplicative subset, we see $x, y \in S$ implies $xy \in S$ and $0 \notin S$ but $1 \in S$.
Then, we define $S^{-1}R = \{X/s : x \in R, s \in S\}$. Then, we see $\frac{x_1}{s_1} = \frac{x_2}{s_2}$ if and only if there is an $s \in S$ so that $s\,(s_2x_1 - s_1x_2) = 0$. Of course, if R is an integral domain we see this iplies $s_2x_1 - s_1x_2 = 0$, the normal definition of fraction equality.

Now, we turn this set into a ring. We define $\frac{x_1}{s_1} \cdot \frac{x_2}{s_2} := \frac{x_1x_2}{s_1s_2}$ and $\frac{x_1}{s_2} + \frac{x_2}{s_2} = \frac{s_2x_1}{s_1s_2} + \frac{s_1x_2}{s_1s_2} = \frac{s_2x_1+s_1x_2}{s_1s_2}$. Now, we need to show that $+, \cdot$ are well defined (meaining they do not vary for different representatives of a given equivalence class). This fact is easily checked by symbolic manipulation so we omit the proof. For the addition case suppose $\frac{x_1}{s_1} = \frac{x_1'}{s_1'}$ and similairly for $\frac{x_2}{s_2}$ then take the multiplicative representation of the fraction and multiply the $\frac{x_1}{s_1}$ representation by $-s_2s_2'ts$ and the $\frac{x_2}{s_2}$ representation by $-s_1s_1'st$ and by adding together these representations we see terms cancel and we obtain that addition is in fact well defined. Moreover, it is trivial to check that the ring axioms hold.

Definition 1.1 (Ring Localization)**.** We denote this new fraction ring $S^{-1}R$ to be the **localization of** R with additive identity $\frac{0}{1}$, multiplicative identity $\frac{1}{1}$ and $\frac{tx}{ts} = \frac{x}{s}$ for all $t \in S$.

Note that $s \in S$ is nonzero by definition, so $\frac{1}{s} \cdot \frac{s}{1} = \frac{1}{1} = 1_{S^{-1}R}$, so every element has an inverse.

Proposition 1.1. If R is a commutative ring with $S \subseteq R$ being a multiplicative subset. Then the map

$$\pi : R \longrightarrow S^{-1}R$$
$$x \longmapsto \pi(x) = \frac{x}{1}$$

is a ring homomorphism. Moreover, if S has no zero-divisors, then π is an injection.

Proof. If $x, y \in R$ then $\pi(x \pm y) = \frac{x \pm y}{1} = \frac{x}{1} \pm \frac{y}{1} = \pi(x) \pm \pi(y)$. Furthermore $\pi(1) = \frac{1}{1} = 1$.
Lastly, $\pi(xy) = \frac{xy}{1} = \frac{x}{1}\frac{y}{1} = \pi(x)\pi(y)$. Hence, π is a ring homomorphism. Now consider $\ker(\pi) = \{x \in R : \frac{x}{1} = \frac{0}{1}\}$. We see this implies an $s \in S$ so that $s(1x - 1 \cdot 0) = sx = 0$, hence s is a zero divisor if $x \neq 0$ $\lightning$. So, the kernel is trivial. $\square$

Example. If R is a commutative ring and $P \subseteq R$ is a prime ideal, then $S := R \setminus P$ is a multiplicative set. Moreover, $0 \in P$ so $0 \notin S$ and $P \subset R$ is proper, so $1 \in S$.
If $x, y \in S$ with $xy \notin S$, then $xy \in P$ so $x \in P$ or $y \in P$ $\lightning$. So, S is closed under multiplication. Then the localization $S^{-1}R$ is often denoted R_P. This is the canonical example of localization which we will study more next class. $\diamond$

The use of this construction is that it allows us to embed an integral domain R in a field $R_{(0)}$ called the **field of fractions**.

2 Chinese Remainder Theorem

Lecture 34: Chinese Remainder Theorem

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Theorem 2.1 (Classical Chinese Remainder Theorem)**.** If $m_1, \ldots, m_r$ are relatively prime integers, then for $a_1, \ldots, a_r$ we find an $x \in \mathbb{Z}$ so that $x \equiv a_i \mod m_i$ for each $1 \leq i \leq r$.

Theorem 2.2 (Generalized Chinese Remainder Theorem)**.** Let R be a commutative ring with $I_1, \ldots, I_n \subseteq R$ being ideals so that $I_i + I_j = R$ for all $i \neq j$. That is, the I_is are pairwise co-maximal. Then for any $x_1, \ldots, x_n \in R$ we find an $x \in R$ so that $x \equiv x_i \mod I_i$ for all $1 \leq i \leq n$.

Recall. $x \equiv x_i \mod I_i$ if $x - x_i \in I_i$.

Proof. If $n = 1$ this is trivial. Of course, $x = x$.
For the case $n = 2$ we have $I_1 + I_2 = R$, hence $1 \in R = I_1 + I_2$. Hence, $1 = a_1 + a_2$ with $a_1 \in I_1, a_2 \in I_2$. Then, let $x = x_1a_1 + x_2a_2$, and we see $a_1 + a_2 = 1$ but $a_1 \equiv 0 \mod I_1$ and likewise $a_2 \equiv 0 \mod I_2$, hence $a_1 \equiv 1$

$\mod I_2$ and $a_2 \equiv 1 \mod I_1$. Hence,

$$\begin{aligned} x &= x_1a_2 + x_2a_1 \\ &\equiv x_1a_2 \mod I_1 \\ &\equiv x_1 \mod I_1 \\ \text{and } x &\equiv x_2a_1 \\ &\equiv x_2 \mod I_2. \end{aligned}$$

Hence, the claim holds for $n = 2$. Now, we induce on n.
Let $n \geq 3$ and suppose the case $n-1$ to be true. Then, we find Then, we see $I_1 + I_i = R$ for all $i \geq 2$ by hypothesis. Hence, $1 = a_i + b_i$ with $a_i \in I_1$, $b_i \in I_i$. Then, we find

$$1 = \underbrace{1 \cdot \ldots \cdot 1}_{n \text{ times}} = \prod_{i=1}^{n}(a_i + b_i) \in \prod_{i=1}^{n}(I_1 + I_i) \subseteq I_1 + \prod_{i=2}^{n} I_i.$$

Moreover, we know $I_1 + \prod_{i=2}^{n} I_i$ to be an ideal as the product and sum of ideals are still ideals.
Then applying the case $n = 2$, we find a $y \in R$ so that $y_1 \equiv 1 \mod I_1$ and $y_1 \equiv 0 \mod \prod_{i=2}^{n} I_i$. Repeating for each $1 \leq i \leq n$ yields a $y_j \in R$ so that $y_j \equiv 1 \mod I_j$ and $y_j \equiv 0 \mod \prod_{1\leq i\leq n; i\neq j} I_i$. Now, define $x = \prod_{i=1}^{n} x_i y_i$. We see $y_j \in I_i$ for all $i \neq j$, hence $y_j x_j \equiv 0 \mod I_i$ for all $i \neq j$. Hence $x \equiv x_i y_i \equiv x_i \mod I_i$. $\square$

Note that in the preceding proof $\prod I_i$ denotes the ideal product as defined in the homework. In the next theorem we will use this symbol for the cartesian product, so ideal products will be written without product notation when the context is not necessarily clear.

Corollary 1 (Alternative Statement of the Chinese Remainder Theorem)**.** Let R be a commutative ring with $I_1, \ldots, I_n \subseteq R$ being pairwise comaximal distinct ideals of R. Then the map

$$\begin{aligned} f : R &\longrightarrow \prod_{i=1}^{n} R/I_i \\ x &\longmapsto (x \mod I_i)_{1\leq i\leq n} \end{aligned}$$

is a surjective ring homomorphism with kernel $\ker(f) = \bigcap_1^n I_i$. Specifically,

$$R/\left(\bigcap_{i=1}^{n} I_i\right) \simeq \prod_{i=1}^{n}(R/I_i).$$

Proof. It is easily confirmed that f is a ring homomorphism with the prescribed kernel. Hence, the only claim that remains to be shown is the surjectivity. For f to be surjective, we need to take an arbitrary congruence system $\hat{x} = (x_1 \mod I_1, x_2 \mod I_2, \ldots, x_n \mod I_n)$ in the codomain of f and find a solution $x \in R$ so that $x \equiv x_i \mod I_i$ for all $1 \leq i \leq n$ (that is $f(x) = \hat{x}$). We see the generalized remainder theorem yields such an x, so f is surjective. $\square$