Combinatorics

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Contents

Lecture 33: Cut Norm Proofs

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Let A be a $m \times n$ matrix with $\vec{x} \in \mathbb{R}^n$, $\vec{y} \in \mathbb{R}^m$ and $|\vec{x}|_\infty \leq 1$ and $|\vec{y}|_\infty \leq 1$. Then, we consider $\max |\langle A\vec{x}, \vec{y}\rangle| = \|A\|_\pi$.

Proposition 0.1. We claim

$$\|A\|_\square \leq \|A\|_\pi.$$

Proof. If S, T are submatrices inducing $\|A\|_\square$. That is

$$\left| \sum_{i \in T, j \in S} a_{i,j} \right| = \|A\|_\square.$$

Letting $\vec{x}, \vec{y}$ be indicator vectors for S, T respectively, we see this sum is simply

$$\left| \sum_{i \in T, j \in S} a_{ij} \right| = |\langle Ax, y\rangle| \leq \max |\langle Ax, y\rangle| .$$

It is also possible to set an upper bound, $\|A\|_\pi \leq 4\|A\|_\square$. Let x, y be vectors such that $|\langle Ax, y\rangle| = \|A\|_\pi$. Then, we see we can fix k and perform a sort of division such that $\left|\sum_{i=1}^m \sum_{j=1}^n a_{ij} x_j y_i\right| = |Px_k + Q|$ for some matrices P, Q. Then, as this is a linear with $x_k \in [-1, 1]$ we see the maximum modulus must be achieved on an endpoint, hence we can restrict $x \in \{-1, 1\}^n$, $y \in \{-1, 1\}^m$. Then dividing the columns of the matrix into two pieces, those for which $x_i = 1$ and those for which $x_i = -1$, and denoting them T^+, T^- and similairly dividing the rows into S^+, S^- according to the sign of y_i, we see

$$\|A\|_\pi = \left| \sum_{i,j} a_{ij} x_j y_i \right|$$

.

Then, we can split this sum into four pieces according to the elements belonging to T^+, S^+ , T^+, S^- and so on, we see each piece is less than $\|A\|_\Box$ so triganle inequality yields the upper bound. $\square$

Lecture 34

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Notation. We will begin denoting a matrix of size $n \times m$ with n, m being the number of indices in the rows, columns respectively as an $R \times C$ matrix with set R of row indices and C of column indices. It is of note that this definition allows us to consider R, C to be unordered and hence we can imagine them in any convenient order we want.

Definition 0.1. Given an $R \times C$ matrix , then, for subsets $S \subseteq R, T \subseteq C, d \in R'$, we denote the new $R \times C$ matrix

$$\text{Cut}\,(S, T, d) = (c_{ij})\,; c_{ij} = \begin{cases} d, & i \in S, j \in T \\ 0, & i \notin S \text{ or } j \notin T \end{cases}$$

We see this matrix is simply a scaled copy of $J_{|S|,|T|}$ embedded in the zero matrix of size $R \times C$.

First, we examine an ε, regular pair (R, C) of density $d = d\,(R, C)$. Denote A to be the biadjacency matrix $A(R, C)$. Applying ε-regularity yields the following result,

Proposition 0.2. $A\,(R, C) = dJ_{|R|,|C|} + W$ for some sufficiently exceptional matrix having $\|W\|_\Box \le \varepsilon\,|R|\,|C|$ if and only if (R, C) is an ε-regular pair.

Proof. First the forward implication. Then, denote $B = A - dJ_{|R|,|C|}$. Then, $|b_{ij}| \le 1$ for all i, j. Moreover, $b_{ij} = \begin{cases} -d, & a_{ij} = 0 \\ 1-d, & a_{ij} = 1 \end{cases}$.
Then, suppose $S \subseteq R$, $T \subseteq C$. If $|S| \le \varepsilon\,|R|$ or $|T| \le \varepsilon\,|C|$, then

$$\left|\sum_{S,T} b_{ij}\right| \le |S|\,|T| \le \varepsilon\,|R|\,|C|\,.$$

In this case (R, C) is ε-regular.
Otherwise, if $|S| > \varepsilon\,|R|$ and $|T| > \varepsilon\,|C|$, then $|d\,(S, T) - d| < \varepsilon$. Expanding terms yields

$$\begin{aligned} |d\,(S, T) - d| &= \left|\frac{e\,(S, T)}{|S|\,|T|} - d\right| \\ &= |e\,(S, T) - d\,|S|\,|T|| \\ &< \varepsilon\,|S|\,|T| \\ &< \varepsilon\,|R|\,|C|\,. \end{aligned}$$

Then, note that $e\,(S, T) - d\,|S|\,|T| = \sum_{i \in S, j \in T} b_{ij}$ and the ε-regularity immediately follows. $\square$

Now, we generalize this concept. Suppose A is an $R \times C$ matrix. Then, we wish to construct

$$A = D^{(1)} + \ldots + D^{(s)} + w$$

for some $D^{(t)} = \mathrm{Cut}\,(R_t, C_t, d_t)$ for sets R_t, C_t and densities d_t and an exceptional set W with the following conditions holding,

- S is bounded,
- $|d_t|$ is bounded,
- and $\|W\|_{\square}$ is small.

More precisely,

Proposition 0.3. There are real $c_1 > 0$, $c_2 > 0$ so that for every $\varepsilon \in (0, 1)$ with A being an $R \times C$ matrix having $\|A\|_{\infty} \leq 1$ we find

$$A = D^{(1)} + \ldots + D^{(s)} + w$$

having

- $\|W\|_{\square} \leq \varepsilon\,|R|\,|C|$,
- $S < \frac{c_1}{\varepsilon^2}$,
- $\sup\{d_t : 1 \leq t \leq s\} \leq 2$.