Combinatorics

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Contents

Lecture 11: Hadamard Matrices (4)

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Recall. A matrix was regular if all row sums are equal.

As it turns out, for regular real hadamard matrices regular also implies equal column sums.

Proof. Let H be hadamard regular and $n \times n$ with $\sum_{i=1}^{n} h_{i,j} = d$ for all j.

Then, note that $Hj = dj$ with $j = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$. Hence, d is an eigenvalue and as $H^*H = HH^*$, then we have that $H^*Hj = H^*dj$. Hence

$$nIj = dH^*j \text{ by hadamardness}$$

and as $Ij = j$ we have that $H^*j = \frac{n}{d}j$, hence $\frac{n}{d}$ is an eigenvalue of H^*, hence the row sums of H^* are all $\frac{n}{d}$, and as $H^* = H^T$ for real H, we see the column sums of H are $\frac{n}{d}$.
Additionally, if $d \neq 0$, then $\sum_{i=1}^{n} r_i(H) = \sum_{i=1}^{n} c_i(H)$, implying $nd = n \cdot \frac{n}{d}$, hence $n = d^2$ as we have proven earlier.
We have, of course, neglected the case where $d = 0$. In this case we have that $nj = \vec{0}$, but as $n \neq 0$ by assumption, and $cj \neq \vec{0}$ for $c \neq 0$, we have a contradiction. Hence $d \neq 0$. It is also true that the independence requirement of hadamard matrices implies this row sum cannot be 0. $\square$

Proposition 0.1. Suppose H is a $n \times n$ matrix with entries $|h_{i,j}| = 1$ and singular values $\sigma_1 = \sigma_2 = \ldots = \sigma_n = \sqrt{n}$. Then, H is hadamard.

Proof. Recall from an earlier proposition, we know $\sum_{i=1}^{n} \sigma_i^2 = n^2$. Recall that a diagonal element of HH^* is $b_{i,i} = \sum_{k=1}^{n} a_{i,k} \cdot \overline{a_{i,k}} = \sum_{k=1}^{n} |a_{i,k}|^2 = n$ by construction. Hence, the diagonals are all $b_{i,i} = n$ for all $1 \leq i \leq n$. Next, we wish to see if there are any 0 entries in HH^*. Next, we take a principal submatrix $A_{i,j} = \begin{bmatrix} n & \overline{b_{i,j}} \\ b_{i,j} & n \end{bmatrix}$ (note this is as HH^* will is hermitian, so we know

opposing entries will be complex conjugates) Then, we see $\lambda_1(A_{i,j}) = n + |b_{i,j}|$ and $\lambda_2(A_{i,j}) = n - |b_{i,j}|$.
Now, we examine how the eigenvalues of a matrix and its principal sumbatrices are related. Let A be a $n \times n$ hermitian matrix and A' to be A with the i'th row and j'th column removed. Denoted the eigenvalues of A to be $\lambda_1, \lambda_2, \ldots, \lambda_n$ in decreasing order and eigenvalues of A' to be $\lambda'_1, \lambda'_2, \ldots, \lambda'_{n-1}$. Then, it is a theorem of Cauchy that $\lambda_1 \geq \lambda'_1 \geq \lambda_2 \geq \lambda'_2 \geq \ldots \geq \lambda'_{n-1} \geq \lambda_n$. Applying this again yields a matrix A'' with eigenvalues $\lambda_1 \geq \lambda'_1 \geq \lambda'_1{}'$ and $\lambda''_{n-2} \geq \lambda'_{n-1} \geq \lambda_n$. Returning to our original construction yields $\lambda_1(HH^*) \geq \lambda_1(A_{i,j}) \geq \lambda_2(A_{i,j}) \geq \lambda_n(HH^*)$ and as $\lambda_1(HH^*) = \sigma_1^2 = n$ and similarly, $\lambda_n(HH^*) = \sigma_n^2 = n$, hence $\lambda_1(A_{i,j}) = \lambda_2(A_{i,j}) = n$ implying $b_{i,j} = 0$ for all $j \neq i$ and $b_{i,i} = n$ so $HH^* = nI$. □

Recall. For a matrix H which is hadamard and has entries $h_{i,i} = \delta$ for all i, then the matrix $A = \frac{1}{n}(J - \delta H)$ is a square matrix with entries $0, 1$ and all 0s along the diagonal.

Proposition 0.2. If H is symmetric, then A is the adjacency matrix of a simple graph. If H is also regular with row sum d, then A is the adjacency matrix of a SRG with

$$\begin{aligned} n &= n \\ k &= \frac{n - \varepsilon\sqrt{n}}{2} \\ \lambda &= \frac{n - 2\varepsilon\sqrt{n}}{4} \\ \mu &= \frac{n - 2\varepsilon\sqrt{n}}{4} \end{aligned}$$

where $\varepsilon = \begin{cases} -1, & \delta d < 0 \\ 1, & \delta d > 0 \end{cases}$. It is of note that $\delta d \neq 0$ as $\delta = \pm 1$ and $d \neq 0$ by the earlier proof. Hence, $\varepsilon\sqrt{n} = \delta d$

Proof. First, we examine a few matrix products. Note that as $Hj = d$, we have $HJ = dJ$. Similarly, $JH = dJ$ and of course $H^2 = nI$.

Next, we examine A^2. By definition

$$\begin{aligned}
A^2 &= \frac{1}{4}\left(J - \delta H\right)^2 \\
&= \frac{1}{4}\left(J^2 - 2J\delta J + \delta^2 H^2\right) \\
&= \frac{1}{4}\left(nJ - 2\delta dJ + nI\right) \\
&= \frac{1}{4}\left(n - 2\delta d\right) J + \frac{1}{4} nI \\
&= \frac{1}{4}\left(n - 2\delta d\right)\left(J - I\right) + \frac{1}{4}\left(n - 2\delta d\right) I + \frac{1}{4} nI \\
&= \frac{1}{4}\left(n - 2\delta d\right)\left(J - I\right) + \frac{n - \delta d}{2} I.
\end{aligned}$$

Recalling our equation for the square of the adjacency matrix of a graph,

$$A^2 = \left(\lambda - \mu\right) A + \mu\left(J - I\right) + kI$$

yields $\lambda = \mu$, $\mu = \frac{n-2\delta d}{4} = \frac{n-2\varepsilon\sqrt{n}}{4} = \lambda$ and $k = \frac{n-\delta d}{2} = \frac{n-\varepsilon\sqrt{n}}{2}$. □

Lecture 12

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