

# Analysis I

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October 7, 2021

## Contents

### 1 Measurable Functions

3

### Lecture 13: Negative Results (2) and Measurable Functions

Tue 05 Oct 2021 13:02

We construct a cantor set.

First, suppose the interval  $[0, 1]$  and a series of sets  $C_0, C_1, \dots$  where  $C_i = C_{i-1} \setminus D_i$  where  $D_i$  is just the set consisting of the middle thirds of each interval of  $C_{i-1}$ . Then, we let  $C = \bigcap_{k \in \mathbb{N}} C_k$ . We then define the  $n$ th partition of  $[0, 1] \setminus C_k$  to be  $J_{k,n}$ . We define  $\mathcal{O} = \bigcup_{k,n \in \mathbb{N}} J_{k,n}$  and  $\xi : \mathcal{O} \rightarrow \mathbb{R}$ ,  $x \in J_{k,n} \mapsto \frac{n}{2^k}$ . We see this is well defined by an inductive argument.

**Definition 0.1** (Cantor-Lebesgue Function). We define

$$\varphi : [0, 1] \longrightarrow \mathbb{R}$$
$$x \longmapsto \varphi(x) = \begin{cases} 0, & x = 0 \\ \xi(x), & x \in \mathcal{O} \\ \sup\{\xi(y) : y \in \mathcal{O} \cap [0, x)\}, & x \in C \setminus \{0\} \end{cases}$$

to be the **Cantor-Lebesgue Function**

**Proposition 0.1.**  $\varphi$  is a continuous increasing function such that  $\varphi([0, 1]) = [0, 1]$ .

*Proof.* It is clear  $\xi$  is and this guarantees  $\varphi$  to be increasing.

Next, note  $\varphi(0) = 0$  and  $\varphi(1) = 1$ . Hence, we have the intermediate value theorem guaranteeing the image is  $[0, 1]$  if  $\varphi$  is continuous.

We see  $\varphi$  is continuous on  $\mathcal{O}$  since it is constant on each interval  $J_{k,n}$ . Now, we consider  $x \in C \setminus \{0, 1\}$ . For a given  $\varepsilon$ , let  $k \in \mathbb{N}$  such that  $\frac{1}{2^k} < \varepsilon$ . Then, there is  $n \in \mathbb{N}$  such that  $1 \leq n \leq 2^k - 2$  such that for all  $u \in J_{k,n}$ ,  $v \in J_{k,n+1}$  such that for all  $u, v$  we find  $u < x < v$ . Let  $a_k \in J_{k,n}$   $b_k \in J_{k,n+1}$  then by monotonicity

of  $\varphi$ , for all  $y \in [0, 1]$  with  $|x - y| < \delta = \min\{x - a_k, x + b_k\}$  we find

$$\begin{aligned} |\varphi(x) - \varphi(y)| &\leq \varphi(b_k) - \varphi(a_k) \\ &= \frac{n+1}{2^k} - \frac{n}{2^k} \\ &= \frac{1}{2^k} \\ &< \varepsilon. \end{aligned}$$

Finally, given  $\varepsilon > 0$ , we take  $k \in \mathbb{N}$  such that  $\frac{1}{2^k} < \varepsilon$  and let  $c_k \in I_{k,1}$ ,  $d_k \in I_{k,2^k-1}$ . Then, for  $0 \leq y \leq c_k$ , we find

$$\begin{aligned} |\varphi(0) - \varphi(y)| &= |\varphi(y)| \\ &\leq \varphi(c_k) \\ &= \frac{1}{2^k} \\ &< \varepsilon. \end{aligned}$$

Similarly, for  $d_k < y \leq 1$ , we find

$$\begin{aligned} |\varphi(1) - \varphi(y)| &\leq |1 - \varphi(d_k)| \\ &= 1 - \frac{2^k - 1}{2^k} \\ &= \frac{1}{2^k} \\ &< \varepsilon. \end{aligned}$$

□

**Definition 0.2** (Modified Cantor-Lebesgue Function). Let  $\psi = x + \varphi(x)$  be the **modified Cantor-Lebesgue Function**. It is clear  $\psi$  is continuous, strictly increasing and has  $\psi([0, 2]) = [0, 2]$ .

**Proposition 0.2.** The function  $\psi$  has the following properties

1.  $\psi(C)$  is measurable with  $\mu(\psi(C)) = 1$ .
2. There is a measurable set  $S \subseteq C$  such that  $\psi(S)$  is not measurable.

*Proof.* • Note that  $[0, 1] = C \cup \mathcal{O}$  and  $\psi$  is injective and continuous. Hence, we have  $[0, 2] = \psi(C) \cup \psi(\mathcal{O})$  with  $\psi(C) \cap \psi(\mathcal{O}) = \emptyset$ . Since  $\psi$  is strictly increasing, we know  $\psi^{-1}$  is well-defined and continuous. Hence,  $\psi$  is an open map and we see  $\psi(\mathcal{O})$  is open in  $[0, 2]$ , hence  $\psi(C)$  is closed. Hence, both sets are measurable. We see  $\psi(\mathcal{O})$  is the union of a countable collection of open disjoint intervals,  $\{I_i : i \in \mathbb{N}\}$  such that  $\varphi|_{J_i}$  is constant by construction. Hence, we have for each  $i \in \mathbb{N}$  we find  $\psi(I_i) = x_i + I_i$  where  $x_i \in [0, 1]$  is a constant. Since  $\psi$  is injective, we find it preserves

disjointness, hence the collection  $\{\psi(I_i) : i \in \mathbb{N}\}$  is disjoint. Then, by countable additivity and translation invariance of  $\mu$  we find

$$\begin{aligned} \mu(\psi(\mathcal{O})) &= \mu\left(\bigcup_{i \in \mathbb{N}} I_i\right) \\ &= \sum_{i \in \mathbb{N}} \mu(\psi(I_i)) \\ &= \sum_{i=1}^{\infty} \mu(\psi(I_i)) \\ &= \sum_{i=1}^{\infty} \ell(x_i + I_i) \\ &= \sum_{i=1}^{\infty} \ell(I_i) \\ &= \mu(\mathcal{O}). \end{aligned}$$

Since,  $\mu(C) = 0$ , we find

$$\mu(\mathcal{O}) = \mu([0, 1] \setminus C) = \mu([0, 1]) = 1.$$

Consequently,  $\mu(\psi(\mathcal{O})) = 1 = \mu(\mathcal{O})$ . Hence, we find  $\mu(\psi(C)) = 1$ .

Since  $\psi(C)$  has positive measure, it contains a nonmeasurable subset  $T$ , however, we see  $S = \psi^{-1}(T)$  is measurable as  $S \subseteq C$  and  $\mu(C) = 0$ .  $\square$

**Corollary 1.** There is a measurable set  $S \subseteq C$  such that  $S$  is not borel.

*Proof.* Since  $\psi$  has a continuous inverse, we see it maps borel sets to borel sets. Let  $S$  be a subset of  $C$  such that  $\psi(S)$  is not measurable. Since  $\psi(S)$  is not measurable, it is not a borel set. Hence  $S$  is not borel, but it was measurable with measure 0.  $\square$

## 1 Measurable Functions

**Definition 1.1** (Measurable Functions). A function  $f : S \rightarrow \overline{\mathbb{R}}$  is **Lebesgue-measurable** on  $S$  if  $S \subseteq \mathbb{R}$  is measurable and  $f^{-1}((c, \infty])$  is a measurable set for every  $c \in \mathbb{R}$ . This is equivalent to the condition that  $f^{-1}(B)$  is measurable for all  $B \in \overline{\mathcal{B}}$ , the extended borel  $\sigma$ -algebra.

**Proposition 1.1.** Let  $S \subseteq \mathbb{R}$  be measurable, then a function  $f : S \rightarrow \overline{\mathbb{R}}$  is measurable if and only if one of the following holds for all  $c \in \mathbb{R}$ :

- $f^{-1}([c, \infty])$  is measurable,
- $f^{-1}([-\infty, c])$  is measurable,
- $f^{-1}([-\infty, c))$  is measurable.

**Definition 1.2.** The extended Borel  $\sigma$ -algebra,  $\overline{\mathcal{B}}$  consists of all subsets  $B \subseteq \overline{\mathbb{R}}$  such that  $B \setminus \{-\infty, \infty\} \in \mathcal{B}$ .

**Remark.** It is clear  $\overline{\mathcal{B}}$  is the smallest  $\sigma$ -algebra containing all open subsets of  $\overline{\mathbb{R}}$ .

## Lecture 14: Measurable Functions (2)

Thu 07 Oct 2021 12:58

**Recall.** A function  $f : S \rightarrow \mathbb{R}$  was measurable if  $S$  is measurable and  $f^{-1}((c, \infty])$  is measurable for all  $c \in \mathbb{R}$ . There was an equivalent definition using the extended borel  $\sigma$ -algebra that we will use occasionally.

**Proposition 1.2.** Suppose  $f : S \rightarrow \overline{\mathbb{R}}$  is continuous on the measurable set  $S$ , then  $f$  is measurable.

*Proof.* Let  $H$  be an extending function, then we must show  $H \circ f$  is continuous. We see any subray,  $f(X_0) = (c, \infty]$  will have  $(H \circ f)(X_0) = (\hat{c}, 1]$ . We know the preimage of this to be open in  $S$ , hence measurable.  $\square$

**Proposition 1.3.** Let  $S \subseteq \mathbb{R}$ . Suppose  $f : S \rightarrow \mathbb{R}$  is measurable. and let  $g : B \rightarrow \mathbb{R}$  with  $B \in \overline{\mathcal{B}}$  and  $f(S) \subseteq B$ . Then,  $g \circ f : S \rightarrow \mathbb{R}$  is measurable.

*Proof.* For  $c \in \mathbb{R}$ , we note that  $(g \circ f)^{-1}((c, \infty]) = f^{-1}(g^{-1}((c, \infty]))$ . By continuity of  $g$ , we know  $g^{-1}((c, \infty]) \in \overline{\mathcal{B}}$ . And, since  $f$  is measurable, we find  $f^{-1}(g^{-1}((c, \infty]))$ .  $\square$

**Corollary 2.** Let  $S \subseteq \mathbb{R}$  and  $f : S \rightarrow \mathbb{R}$  to be a measurable function. Then, for every  $\alpha \in \mathbb{R}$  and  $0 < \rho < \infty$ , we find  $\alpha f$  and  $|f|^\rho$  are measurable.

*Proof.* We see the functions  $g(u) = \alpha u$  on  $\overline{\mathbb{R}}$  and  $h(u) = |u|^\rho$  on  $\overline{\mathbb{R}}$  to be the corresponding functions. We see the case  $h$  is clearly continuous and well defined. On the other hand  $g$  may be poorly defined if  $\alpha = 0$  and  $f(x) = \infty$ . Recall, however, we had  $0 \cdot \pm\infty = 0$  so  $g$  is just the zero functions and we see continuity holds.  $\square$

**Definition 1.3** (Almost-everywhere). Let  $S$  be measurable, then a property is said to hold true **almost everywhere** on  $S$  or **for almost all**  $x \in S$  if there is a set  $T$  with  $\mu(T) = 0$  and the property holds on all of  $S \setminus T$ .

**Proposition 1.4.** Let  $S \subseteq \mathbb{R}$  and suppose  $f, g : S \rightarrow \overline{\mathbb{R}}$  such that  $f$  is measurable and  $g = f$  almost everywhere on  $S$ , then  $g$  is measurable.

*Proof.* Let  $T = \{x \in S : f(x) \neq g(x)\}$ . Fix  $c \in \mathbb{R}$  and let  $F = f^{-1}((c, \infty]) \setminus T$  and  $G = f^{-1}((c, \infty]) \cup T$ . Clearly, both  $F$  and  $G$  are measurable. Furthermore,  $F \subseteq G$  and  $\mu(G \setminus F) = \mu(T) = 0$ . Since,  $F \subseteq g^{-1}((c, \infty]) \subseteq G$ . And, by an earlier characterization we recall that a set  $X$  is measurable if and only if there were nested sets around it with a difference of measure 0. Hence,  $g$  is measurable.  $\square$

**Remark.** Suppose  $f : S \rightarrow \overline{\mathbb{R}}$  is a measurable set and  $S \subseteq X \subseteq \mathbb{R}$ . If  $\mu(X \setminus S) = 0$  and  $h : X \rightarrow \overline{\mathbb{R}}$  is any extension of  $f$ , then  $h$  is measurable since  $h^{-1}((c, \infty]) = f^{-1}((c, \infty]) \cup \{x \in X \setminus S : h(x) \in (c, \infty]\}$ . This is the union of a measurable set with a set of measure 0, so we see  $h$  is measurable.