Combinatorics

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September 17, 2021

Contents

Lecture 10: Hadamard Matrices (3)

Wed 15 Sep 2021 10:20

Recall. The tensor product of matrices A, B is $A \otimes B$ and this preserves hadamardness.

Example.

$$\underbrace{\begin{bmatrix} + & + \\ + & - \end{bmatrix}}_{=H} \otimes \begin{bmatrix} + & + \\ + & - \end{bmatrix} = \begin{bmatrix} + & + & + & + \\ + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{bmatrix} = H \otimes H.$$

Furthermore, $H \otimes H \otimes H$ will be an 8×8 hadamard matrix. And the arbitrary $\bigotimes_{i=1}^n H$ yields a hadamard matrix of order 2^n. ◇

A natural question arises, what are the singular values of an arbitrary hadamard H?

Recall. Singular values are the square roots of the eigenvalues of AA^*.

Other definitions also arise, for example the largest singular value of A, denoted σ_1 is equal to the operator norm on A. Similarly, we can change the matrix slightly to remove singular value σ_1 and this yields σ_2 is the operator norm on the modified $\hat{A}$.

For now, we return to the original definition, and we note that as $HH^* = nI$, we have eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ with $\lambda_i = n$ and corresponding eigenvactor $e_i = \begin{pmatrix} 0 \\ \vdots \\ \underbrace{1}_{\text{position } i} \\ \vdots \\ 0 \end{pmatrix}$. Hence the singular values of A are all $\sqrt{n}$.

Proposition 0.1. Let $H = [h_{i,j}]$ with $|h_{i,j}| = 1$ for all i, j. Then, the following are equivalent

- $HH^* = nI$
- All singular values are equal to $\sqrt{n}$
- For singular values σ_i, $1 \leq i \leq n$, the sum $\sum_{i=1}^{n} \sigma_i = n\sqrt{n}$.

Definition 0.1 (Nuclear Norm)**.** For a matrix A, we define the **nuclear norm or trace norm** to be $\|A\|_* = \sum_{i=1}^{n} \sigma_i (A)$.

Remark. If A is $n \times n$ with $|a_{i,j}| \leq 1$ then $\|A\|_* \leq n\sqrt{n}$. Furthermore, equality holds if and only if A is hadamard.

Now, let A be $m \times n$ with $m \leq n$. Then, $\|A\|_* \leq m\sqrt{n}$. Equality holds if and only if A is a **partial hadamard matrix** meaining $AA^* = nI_m$.

Definition 0.2 (Regular Matrix)**.** For a matrix A we say A is **regular** if all row sums are equal.

We examine the properties of a regular hadamard matrix.
It is clear, as we may switch rows and columns and multiply by ± 1 for each row, that these row sums are fragile, and occasionally we may even induce a regular hadamard matrix from a nonregular one this way.

Example. $\begin{bmatrix} + & + & + & - \\ + & + & - & + \\ + & - & + & + \\ - & + & + & + \end{bmatrix}$ is a regular hadamard matrix induced by the 4×4 hadamard from earlier. $\diamond$

Remark. A regular matrix need not be symmetric. For example $\begin{bmatrix} + & + & + & - \\ + & - & + & + \\ - & + & + & + \\ + & + & - & + \end{bmatrix}$ is regular and nonsymmetric.

Note that a real symmetric hadamard matrix has real eigenvalues.

Proposition 0.2. Suppose H is a $n \times n$ symmetric and regular (row sum d). Then, $n = d^2$.

Proof. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of H and note that $|\lambda_i| = \sqrt{n}$ for $1 \leq i \leq n$ as the eigenvalues of a real symmetric matrix are precisely the singular values.
Next, we note that there is atleast one $\lambda_i = \sqrt{n}$ and one $\lambda_j = -\sqrt{n}$. Otherwise, suppose WLOG all $\lambda_i = \sqrt{n}$, then $\sum_{i=1}^{n} \lambda_i = n\sqrt{n} = (H)$, but the trace can be

atmost n by an earlier theorem. Hence, $\lambda_1 = \sqrt{n}$ and $\lambda_n = -\sqrt{n}$. Then, note that $Hj = dj$ for $j = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$. Hence, d is an eigenvalue with j as its eigenvector. Hence $d = -\sqrt{n}$ or $d = \sqrt{n}$. Hence either case yields $n = d^2$. ☐

Definition 0.3 (Constant Diagonal)**.** A hadamard matrix H is said to have a **constant diagonal** if $h_{1,1} = h_{2,2} = \ldots = h_{n,n}$.

This property can always be ensured for a hadamard matrix with just elementary transformations. Furthermore, if H is a $n \times n$ constant diagonal hadamard matrix with $\delta = h_{1,1}$. Then, δH has a constant diagonal of 1 and we define $A = \frac{1}{2}(J_n - \delta H)$, hence the diagonal of A is constant 0. Next, note that δH is a hadamard matrix and for an element $h_{i,j} = 1$, we see $\delta h_{i,j} = \delta$. Similairly if $h_{i,j} = -1$ we have $\delta h_{i,j} = -\delta$. Hence the entries of A are $a_{i,j} = 0$ if $\delta h_{i,j} = 1$ and $a_{i,j} = 1$ if $\delta h_{i,j} = -1$. So, this matrix has all entries 0 and 1, something we call a **digraph matrix**. Furthermore, if H is regular, the graph induced by A is a strongly regular graph.

Lecture 11: Hadamard Matrices (4)

Fri 17 Sep 2021 10:19

Recall. A matrix was regular if all row sums are equal.

As it turns out, for regular real hadamard matrices regular also implies equal column sums.

Proof. Let H be hadamard regular and $n \times n$ with $\sum_{i=1}^{n} h_{i,j} = d$ for all j. Then, note that $Hj = dj$ with $j = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$. Hence, d is an eigenvalue and as $H^*H = HH^*$, then we have that $H^*Hj = H^*dj$. Hence

$$nIj = dH^*j \text{ by hadamardness}$$

and as $Ij = j$ we have that $H^*j = \frac{n}{d}j$, hence $\frac{n}{d}$ is an eigenvalue of H^*, hence the row sums of H^* are all $\frac{n}{d}$, and as $H^* = H^T$ for real H, we see the column sums of H are $\frac{n}{d}$.
Additionally, if $d \neq 0$, then $\sum_{i=1}^{n} r_i(H) = \sum_{i=1}^{n} c_i(H)$, implying $nd = n \cdot \frac{n}{d}$, hence $n = d^2$ as we have proven earlier.
We have, of course, neglected the case where $d = 0$. In this case we have that $nj = \vec{0}$, but as $n \neq 0$ by assumption, and $cj \neq \vec{0}$ for $c \neq 0$, we have a contradiction. Hence $d \neq 0$. It is also true that the independence requirement of hadamard matrices implies this row sum cannot be 0. ☐

Proposition 0.3. Suppose H is a $n \times n$ matrix with entries $|h_{i,j}| = 1$ and singular values $\sigma_1 = \sigma_2 = \ldots = \sigma_n = \sqrt{n}$. Then, H is hadamard.

Proof. Recall from an earlier proposition, we know $\sum_{i=1}^{n} \sigma_i^2 = n^2$. Recall that a diagonal element of HH^* is $b_{i,i} = \sum_{k=1}^{n} a_{i,k} \cdot \overline{a_{i,k}} = \sum_{k=1}^{n} |a_{i,k}|^2 = n$ by construction. Hence, the diagonals are all $b_{i,i} = n$ for all $1 \leq i \leq n$. Next, we wish to see if there are any 0 entries in HH^*. Next, we take a principal submatrix $A_{i,j} = \begin{bmatrix} n & \overline{b_{i,j}} \\ b_{i,j} & n \end{bmatrix}$ (note this is as HH^* will is hermitian, so we know opposing entries will be complex conjugates) Then, we see $\lambda_1(A_{i,j}) = n + |b_{i,j}|$ and $\lambda_2(A_{i,j}) = n - |b_{i,j}|$.
Now, we examine how the eigenvalues of a matrix and its principal sumbatrices are related. Let A be a $n \times n$ hermitian matrix and A' to be A with the i'th row and j'th column removed. Denoted the eigenvalues of A to be $\lambda_1, \lambda_2, \ldots, \lambda_n$ in decreasing order and eigenvalues of A' to be $\lambda_1', \lambda_2', \ldots, \lambda_{n-1}'$. Then, it is a theorem of Cauchy that $\lambda_1 \geq \lambda_1' \geq \lambda_2 \geq \lambda_2' \geq \ldots \geq \lambda_{n-1}' \geq \lambda_n$. Applying this again yields a matrix A'' with eigenvalues $\lambda_1 \geq \lambda_1' \geq \lambda_1'{}'$ and $\lambda_{n-2}'' \geq \lambda_{n-1}' \geq \lambda_n$. Returning to our original construction yields $\lambda_1(HH^*) \geq \lambda_1(A_{i,j}) \geq \lambda_2(A_{i,j}) \geq \lambda_n(HH^*)$ and as $\lambda_1(HH^*) = \sigma_1^2 = n$ and similarly, $\lambda_n(HH^*) = \sigma_n^2 = n$, hence $\lambda_1(A_{i,j}) = \lambda_2(A_{i,j}) = n$ implying $b_{i,j} = 0$ for all $j \neq i$ and $b_{i,i} = n$ so $HH^* = nI$. □

Recall. For a matrix H which is hadamard and has entries $h_{i,i} = \delta$ for all i, then the matrix $A = \frac{1}{n}(J - \delta H)$ is a square matrix with entries $0, 1$ and all 0s along the diagonal.

Proposition 0.4. If H is symmetric, then A is the adjacency matrix of a simple graph. If H is also regular with row sum d, then A is the adjacency matrix of a SRG with

$$\begin{aligned} n &= n \\ k &= \frac{n - \varepsilon\sqrt{n}}{2} \\ \lambda &= \frac{n - 2\varepsilon\sqrt{n}}{4} \\ \mu &= \frac{n - 2\varepsilon\sqrt{n}}{4} \end{aligned}$$

where $\varepsilon = \begin{cases} -1, & \delta d < 0 \\ 1, & \delta d > 0 \end{cases}$. It is of note that $\delta d \neq 0$ as $\delta = \pm 1$ and $d \neq 0$ by the earlier proof. Hence, $\varepsilon\sqrt{n} = \delta d$

Proof. First, we examine a few matrix products. Note that as $Hj = d$, we have $HJ = dJ$. Similarly, $JH = dJ$ and of course $H^2 = nI$.

Next, we examine A^2. By definition

$$\begin{aligned}
A^2 &= \frac{1}{4}\left(J-\delta H\right)^2 \\
&= \frac{1}{4}\left(J^2 - 2J\delta J + \delta^2 H^2\right) \\
&= \frac{1}{4}\left(nJ - 2\delta dJ + nI\right) \\
&= \frac{1}{4}\left(n - 2\delta d\right)J + \frac{1}{4}nI \\
&= \frac{1}{4}\left(n - 2\delta d\right)\left(J - I\right) + \frac{1}{4}\left(n - 2\delta d\right)I + \frac{1}{4}nI \\
&= \frac{1}{4}\left(n - 2\delta d\right)\left(J - I\right) + \frac{n-\delta d}{2}I.
\end{aligned}$$

Recalling our equation for the square of the adjacency matrix of a graph,

$$A^2 = \left(\lambda - \mu\right)A + \mu\left(J - I\right) + kI$$

yields $\lambda = \mu$, $\mu = \frac{n-2\delta d}{4} = \frac{n-2\varepsilon\sqrt{n}}{4} = \lambda$ and $k = \frac{n-\delta d}{2} = \frac{n-\varepsilon\sqrt{n}}{2}$. □