MATH 7237/8237

LECTURE Jan. 29, 2021

TREES

Definition A graph is called **acyclic** if it contains no cycles.

Being acyclic is a **monotone property**, that is, every subgraph of an acyclic graph is acyclic.

Other monotone properties are "being bipartite" and "containing no triangles".

Definition A **tree** is a connected acyclic graph.

Definition A vertex of degree one in a tree is called a **leaf**.

Proposition *Every tree has at least two leafs.*

To see that, let $P = u_1, \ldots, u_k$ be a maximal path in T. We shall show that $d(u_1) = 1$ and $d(u_k) = 1$.

Indeed, if $d(u_1) \geq 2$, then u_1 has a neighbor $v \neq u_2$.

If $v \in \{u_3, \ldots, u_k\}$, say $v = u_l$, then $u_1, \ldots, u_{l-1}, v$ is a cycle it T, a contradiction.

If $v \notin \{u_1, \ldots, u_k\}$, then $v, u_1, \ldots, u_k$ is an extension of P, contradicting that P is maximal.

Examples of trees: Paths and stars are trees.

However, new examples of trees can be created easily by taking two disjoint trees and joining them by a single edge.

PROPERTIES OF TREES

Proposition *For every two vertices in a tree, there is a unique path joining them.*

Proposition *If a connected graph G has a cycle, then for any edge e from the cycle, the graph $G - e$ is connected .*

Indeed, let $C = u_1, \ldots, u_k$ be a cycle in G.

We shall show that every vertex v of G is joined to u_k by a path in $G - \{u_1, u_k\}$.

Since G is connected, there is a path $P = v_1, \ldots, v_l$ with $v = v_1$ and $u_k = v_l$ in G.

Choose j to be the smallest number such that $v_j \in \{u_1, \ldots, u_k\}$.

Let $v_j = u_i$. Clearly the path $P' = v_1, \ldots, v_j$ has no other vertices in common with C, and therefore the path

$$v_1, \ldots, v_{j-1}, u_i, \ldots, u_k$$

joins v to u_k in $G - \{u, v\}$.

Therefore, $G - \{u, v\}$ is connected.

Corollary *If G is a connected graph such that deleting any edge of G makes it disconnected, then G does not contains cycles, and hence G is a tree.*

Proposition *Every connected graph has a spanning tree.*

Keep removing edges belonging to a cycle as long as there are cycles. The graph remains connected. At the end we obtain a spanning tree.

Proposition *Deleting any edge from a tree makes it a disconnected graph.*

If $T - \{u, v\}$ were connected, there would be a path in $T - \{u, v\}$ that joins u and v.

Adding the edge $\{u, v\}$ to the path would give a cycle in T.

Proposition *The removal of a leaf from a tree yields a tree.*

Because a vertex of degree one cannot be an internal vertex of a path.

Proposition *For every tree T, we have $e(T) = v(T) - 1$.*
Remove a leaf and use induction on the order of the tree.

Proposition *Trees are bipartite graphs*.

CONNECTIVITY IN GRAPHS

Next we shall discuss measures of connectedness of graphs.

The necessity of such measures seems clear, because some graphs look to us more connected than others.

For example, complete graphs seem are more connected than cycles and cycles are more connected than paths.

Let $c(G)$ denote the number of components of a graph G.

Definition An edge e of a graph G is called a **cut edge** if

$$c(G - e) > c(G).$$

CUT VERTICES

Next, we extend the concept of cut edge to vertices.

Definition A vertex v of a graph G is called a **cut vertex** if

$$c\left(G-e\right)>c\left(G\right).$$

Examples of cut edges and cut vertices

Every edge in a tree is a cut edge;

Every non-leaf vertex of a tree is a cut vertex;

The path has the most cut vertices of all trees of the same order;

Cycles and complete graphs have no cut vertices;

Complete bipartite graphs other than stars have no cut vertices.

SEPARABLE AND NONSEPARABLE GRAPHS

The concept of cut vertex is very useful for structural characterization of graphs.

- **Definition** A connected graph G is called **nonseparable** if it has no cut vertices. Otherwise G is called **separable.**

- **Examples of separable and nonseparable graphs.**

  The only nonseparable graph of order two is K_2.

- Trees of order at least three are separable, since any non-leaf is a cut vertex.

- Cycles and complete graphs of any order are nonseparable.

- $K_{m,n}$ is separable if and only if $K_{m,n}$ is a star of order at least three.

Nonseparable graphs can withstand deletion of any vertex or any edge without getting disconnected. This property is guaranteed by the following characterization.

Theorem *A connected graph of order at least three is nonseparable if and only if every two vertices are joined by two internally disjoint paths.*

Two paths are **internally disjoint** if they have no vertices in common except possibly their ends.

The above theorem can be restated in numerous equivalent ways, like the following one:

Theorem *A connected graph is nonseparable if and only if any two of its edges lie on a common cycle.*

Problem *If G has a nonseparable spanning subgraph, then G is nonseparable.*

Problem *Let G be a connected graph of order at least three. If $\{u, v\}$ is a cut edge of G, then either u or v is a cut vertex of G.*

Note that $G - \{u, v\}$ has exactly two components. One component contains u and the other one contains v. Denote the components by C_u and C_v respectively.

Either $v(C_u) \geq 2$ or $v(C_v) \geq 2$, because $v(C_u) + v(C_v) \geq 3$. By symmetry, suppose that $v(C_v) \geq 2$.

The graph $G - v$ is disconnected because C_u and $C_v - v$ are components of $G - v$.

VERTEX CONNECTIVITY

So far, we can classify all graphs into three categories of increasing connectivity: disconnected, connected separable, and nonseparable.

For many purposes this rough classification is sufficient, but sometimes we need a finer numeric measure of connectivity, like the following one:

Definition The **vertex connectivity** of a noncomplete graph G is the minimum number of vertices that have to be removed to obtain a disconnected graph.

The vertex connectivity is denoted by $\kappa(G)$.

Note that no matter how many vertices we remove from a complete graph, the remaining graph is either a complete graph itself or has no vertices at all.

Therefore, we need a different approach to define $\kappa(K_n)$:

Definition The vertex connectivity of a complete graph K_n is defined to be $n-1$.

In addition to vertex connectivity, it is useful to have a slightly more relaxed concept:

Definition A graph G is said to be k-**connected** if $\kappa(G) \geq k$.

To get some idea of the concept k-*connected graph*, let us make a couple of observations:

1-connected graph is synonymous to connected graph.

2-connected graph is synonymous to nonseparable graph.

For $k \geq 3$ the concept *k-connected graph* brings new possibilities as in the following problem.

Problem *If $m \geq n$, then $K_{m,n}$ is n-connected.*

Write A and B for the vertex classes of $K_{m,n}$ and let $|A| = m$ and $|B| = n$.

Let $\kappa(K_{m,n}) = k$ and suppose that $x_1, \ldots, x_k$ are vertices such that $K_{m,n} - x_1 - \cdots - x_k$ is disconnected. Let $X = \{x_1, \ldots, x_k\}$.

If $k < n$, then $B \backslash X \neq \varnothing$ and $A \backslash X \neq \varnothing$.

Hence, $K_{m,n} - x_1 - \cdots - x_k$ is a complete bipartite graph, which is connected.

Thus, if $k < n$, then $K_{m,n} - x_1 - \cdots - x_k$ is not disconnected, and so $K_{m,n}$ *is n-connected*.

MENGER'S THEOREM

The following theorem is one of the most popular and useful theorems in graph theory.

Theorem (Menger) Let G be a graph. If u and v are nonadjacent vertices of G, then there are at least $\kappa(G)$ pairwise internally disjoint paths joining u and v.

Menger's theorem is, in fact, equivalent to a great number of fundamental statements in other areas of combinatorics.

THE FAN LEMMA

Applying Menger's theorem, one can deduce the following useful lemmas.

Lemma Let G be a k-connected graph, and let $x, y_1, \ldots, y_k$ be distinct vertices of G. Then there exist k pairwise internally disjoint paths joining x to $y_1, \ldots, y_k$.

The name Fan Lemma comes from the picture of many paths having one end in common.

A seemingly deeper, but in fact equivalent statement is the following version of the Fan Lemma:

Lemma Let G be a k-connected graph, and let X and Y be subsets of $V(G)$ of cardinality k. Then there exist k pairwise disjoint paths joining a vertex in X to a vertex in Y.

EDGE CONNECTIVITY

Vertex connectivity is not the only numeric parameter to characterize connected graphs.

There is a similar concept based on edges.

Definition The **edge connectivity** of a graph G is the minimum number of edges that have to be removed to obtain a disconnected graph.

The edge connectivity is denoted by $\kappa'(G)$.

Examples

For any tree T of order at least two, $\kappa'(T) = 1$;

For any cycle C, $\kappa'(C) = 2$;

$\kappa'(K_n) = n - 1$.

As seen from the above examples, edge connectivity and vertex connectivity are the same for many graphs.

However, in general, these parameters are distinct.

Indeed, let G be a graph formed by taking two cycles sharing a vertex. Clearly, $\kappa(G) = 1$, but $\kappa'(G) = 2$.

Here is a basic relation between $\kappa(G)$, $\kappa'(G)$, and $\delta(G)$:

Theorem For any graph G

$$\kappa(G) \leq \kappa'(G) \leq \delta(G).$$

For trees, cycles and complete graphs, $\kappa = \kappa' = \delta$.

THANK YOU