# Combinatorics

Thomas Fleming

October 4, 2021

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### Lecture 17: Semi-circle Law

Fri 01 Oct 2021 10:20

Recall that for eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ we have $\lambda_1 = \frac{n}{2} + \sqrt{n \log(n)} = o(n)$. Additionally, we know $\sigma_1 = \lambda_1$ and $\sigma_2, \sigma_3, \ldots, \sigma_n$ correspond to $|\lambda_2|, |\lambda_3|, \ldots, |\lambda_n|$. Further, it is known by Furedi and Kowlos that $\sigma_2 = O(\sqrt{n})$.

**Theorem 0.1.** For a randomly chosen graph of order $n$, with eigenvalues $\lambda_2 \geq \lambda_3 \geq \ldots \geq \lambda_n$.Define $W_n(x) : \mathbb{R} \to \mathbb{Z}^+$ to be the number of eigenvalues $\lambda_i$, such that $\frac{\lambda_i}{\sqrt{n}} \leq x$, divided by $n$. Then, we find the function which $W_n(x)$ tends to pointwise, $W(x)$ has $W(x) = \begin{cases} \frac{2}{\pi}\sqrt{1-x^2}, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases}$

Here recall that $\sqrt{1-x^2}$ is an upper half semicircle of radius 1 and the factor $\frac{2}{\pi}$ compresses it into an ellipse. This fact essentially characterizes the distribution of eigenvalues of a random graph. That is, plurality of eigenvalues will be 0 and we find the number of eigenvalues of a given magnitude decreases as $\lambda \to \sqrt{n}$. We note that the leading $\frac{2}{\pi}$ is to normalize the area such that this is a probability density function. Then, we note $E\left[x^2 W(x)\right] = \int_{-1}^{1} \frac{2}{\pi} x^2 \sqrt{1-x^2} dx = \frac{1}{4}$. Hence, we find $\frac{1}{n^2} \sum_{i=2}^{n} \lambda_i^2 \approx \frac{1}{4}$.
It is a well known result that $\sum_{i=1}^{n} |\lambda_i| = \sum_{i=1}^{\infty} \sigma_i \leq \frac{1}{2} n^{\frac{3}{2}} \leq 2(n-1)$. Applying our integral formula from earlier yields $\sum_{i=1}^{\infty} |\lambda_i| = \int_{-1}^{1} |x| \sqrt{1-x^2} = 2\int_0^1 x\sqrt{1-x^2}$.
At this point, Runze found a contradiction in the argument and we ended class early.

### Lecture 18

Mon 04 Oct 2021 10:21