Real Variables I – MATH 7350 – Fall 2021
Homework 1

Department of Mathematical Sciences
Dr. Thomas Hagen

Instructions:
This homework is one part of the assignments which will be due on September 16 by 11.59pm. Begin each homework problem on a separate page. Typed or hand-written solutions are acceptable as long as the presentation is neat. Create a single PDF file of your work for this assignment and submit it in the online dropbox "Homework 1" on eCourseware. Make sure that all authors to be credited are listed on the first page, your solutions are legible and in correct and logical order, proofs are properly constructed, and submissions are easily accessible. Every reasonable and properly written solution attempt for a problem is worth 4 points. Each accepted solution is worth another 6 points. If your solution is not accepted, you have another week to resubmit a correction. Sloppy, late or illegible work will be rejected without points and without a chance to resubmit.

Problem 1

Let f be a map from X to Y.

(a) Show that for $A \subset X$, $B \subset Y$

$$f(f^{-1}(B)) \subset B \quad \text{and} \quad A \subset f^{-1}(f(A)).$$

(b) Give examples to show that the set inclusions in (a) can be proper.

Problem 2

Let A and B be subsets of X. Prove or disprove:

(a) $A \,\Delta\, B = \emptyset$ if and only if $A = B$

(b) $A \,\Delta\, B = X$ if and only if $A = B^c$

Problem 3

Suppose $f : X \to Y$ and $g : Y \to Z$ are two functions.

(a) Show that $f : X \to Y$ is one-to-one *if and only if* there is a map $g : Y \to X$ such that $g \circ f$ is the identity on X. If such a map g exists, is it necessarily unique, one-to-one or onto?

(b) Show that f is onto *if and only if* there is a map $g : Y \to X$ such that $f \circ g$ is the identity on Y.

Problem 4

Prove or disprove: If $\mathcal{A}$ is a σ-algebra of subsets of Y and $f : X \to Y$ is a function, then the collection $\{f^{-1}(A) \,|\, A \in \mathcal{A}\}$ is a σ-algebra of subsets of X.

Problem 5

Prove: The set of all polynomials with rational coefficients is countable.

Hint: Use the results from class.

Problem 6

Prove: The set of infinite sequences (x_k) with $x_k \in \{0, 1\}$ is not countable.

Hint: Assume to the contrary that there exists a bijection f from $\mathbb{N}$ to the set of all sequences (x_k) from $\{0, 1\}$. Then for each $n \in \mathbb{N}$, $f(n)$ is a sequence $(x_{n,k})_k$. Now construct a sequence (y_k) which differs from each of the sequences $f(n) = (x_{n,k})_k$.

Problem 7

Prove: Let A be a set and let $B = \{0, 1\}$. Then there exists a bijection from $\mathcal{P}(A)$ to the set of all functions from A to B.

Problem 8

Let X be the Cartesian product $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$. Define a relation $\sim$ on X to mean $(p, q) \sim (u, v)$ if and only if $p\,v = q\,u$.

(a) Show that $\sim$ is an equivalence relation on X.

(b) Show that there exists a bijective mapping $f : (X/\sim) \to \mathbb{Q}$.