Algebraic Theory I

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Lecture 30: Ring Theory (5)

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Again, we suppose R to be commutative unless otherwise stated.

Proposition 0.1. If R is an integral domain with $p \in R$ being prime, then p is irreducible.

Proof. We know p is nonzero and a non-unit. Then, suppose $p = xy$ $x, y \in R$. Since p prime, we see $p \mid xy$ implies $p \mid x$ or $p \mid y$. WLOG, suppose $p \mid x$, then $x \in (p)$, so $x = rp$ for an $r \in R$. Then, we see

$$p = xy = (rp)\, y = (ry)\, p.$$

Canceling p yields $1 = ry$, so y is a unit. Hence, p is irreducible. $\square$

Remark. Here are a few basic facts about principal ideals, prime ideals, etc. we have shown, compiled together:

- $x \mid y \Leftrightarrow y \in (x) = Rx$.
- $x \mid y$ and $y \mid x \Leftrightarrow (x) = (y)$.
- If R is an integral domain with $x \neq 0$ then $(x) = (y) \Leftrightarrow ux = y$ for a unit u.
- $(x) = R \Leftrightarrow x$ is a unit.
- $p \in R$ is prime implies (p) is a prime ideal.
- (p) is a prime ideal and $p \neq 0$ implies $p \in R$ is prime.
- $p \in R$ irreducible implies (p) is maximal among all proper principal ideals.
- If R is an integral domain and $p \neq 0$, then $(p) \subset R$ is maximal among principal ideals $\Leftrightarrow p \in R$ is irreducible.
- If R is an integral domain with $p \in R$ being prime then p is also irreducible.

Definition 0.1 (Factorization)**.** If R is a commutative ring, a **factorization** of an element $x \in R$ is an expression

$$x = u \prod_{i=1}^{n} y_i$$

where u is a unit and $y_1, \ldots, y_n$ are irreducibles.
The factorization is a **unique factorization** if for a second factorization

$$x = u' \prod_{i=1}^{n'} y_i'$$

we find $n = n'$ and there exists a permutation π of $\{1, \ldots, n\}$ such that $y_{\pi(i)} = y_i'$ up to units for all $1 \leq y \leq n$.

Definition 0.2 (Unique Factorization Domain)**.** A commutative ring R that is an integral domain in which every nonzero $x \in R$ has a unique factorization is called a **Unique Factorization Domain (UFD)**.

Theorem 0.1. If R is a UFD, then $p \in R$ is prime if and only if p is irreducible.

Proof. Since R is a UFD, it is an integral domain, hence a prime is irreducible. Now, let p be irreducible, so $p \neq 0$ and p is a non-unit. Suppose $p \mid xy$ for some $x, y \in R$. Then, we see $xy = rp$ for some $r \in R$, hence letting

$$x = u_1 \prod_{i=1}^{n} x_i$$
$$y = u_2 \prod_{i=1}^{m} y_i$$

be the unique factorizations for x and y respectively yields a factorization

$$xy = u_3 \prod_{i=1}^{n} x_i \prod_{i=1}^{m} y_i.$$

Hence,

$$rp = rxy = u_3 \prod_{i=1}^{n} x_i \prod_{i=1}^{m} y_i \cdot r.$$

Hence, we find

$$u_3 \prod_{i=1}^{n} x_i \prod_{i=1}^{m} y_i \cdot r = r \cdot p.$$

Hence, cancelling r, we must have $p = x_j$ or y_k for some $1 \leq j \leq n$ or $1 \leq k \leq m$ as it is irreducible. So, $p \mid x$ or $p \mid y$, hence p is prime. $\square$

It is of note that a factorization can contain multiple copies of a particular irreducible. Hence, we can also represent a factorization as a multi-set. That is, if $x = up_1^{\alpha_1} \dots p_n^{\alpha_n}$, we can represent this as the multi-set

$$\text{Fac}(x) = \{\underbrace{p_1, \dots, p_1}_{\alpha_1 \text{times}}, \underbrace{p_2, \dots, p_2}_{\alpha_2 \text{times}}, \dots, \underbrace{p_n, \dots, p_n}_{\alpha_n \text{times}}\}.$$

Then, we can view the factorization of a product xy as the union of their respective factorization multisets, $\text{Fac}(x) \cup \text{Fac}(y) = \text{Fac}(xy)$.

Definition 0.3 (Finitely Generated)**.** An ideal I is finitely generated if $I = (x_1, x_2, \dots, x_n)$ for a finite set $\{x_1, x_2, \dots, x_n\}$.

Definition 0.4 (Noetherian Ring)**.** A commutative ring is **Noetherian** if it satisfies the **ascending chain condition (a.c.c.)** on ideals. That is, if $I_1 \subseteq I_2 \subseteq I_3 \subseteq \dots$ is an ascending chain for some ideals $I_1, I_2, \dots$, then there exists a $m \geq 1$ such that $I_i = I_m$ for all $i \geq m$.
More simply, a ring is Noetherian if all properly ascending chains of ideals are finite in lengths.

This definition is rather clunky, so the following characterization is the more standard use case:

Theorem 0.2. *R is a noetherian ring if and only if all ideals in R are finitely generated.*

Remark. A Noetherian ring which is also an integral domain is sometimes called a **Noetherian Domain**.

Noetherian domains are a weaker class of rings than principal ideal domains, but they are more "resiliant" to algebraic operations. That is, most algebraic operations preserve Noetherian-ness even if they do not preserve the PID property.

Lecture 31: Noetherian Rings

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1 Noetherian Rings

Recall. A commutative ring is noetherian if it satisfies the ascending chain condition on ideals. We claimed this to be equivalent to the property that all ideals are finitely generated.

Proof. First, we assume R to be noetherian. Suppose there is an ideal I which is not finitely generated. Then, let $x_1 \in I$ be a nonzero element of I. Hence, we have $(0) \subset (x_1)$ with $(x_1) \neq I$ by assumption. Moreover, there is an $x_2 \neq x_1$ which is also nonzero such that $(0) \subset (x_1) \subset (x_1, x_2)$ and $(x_1, x_2) \neq I$ by assumption. Recursing, we see there are $x_1, x_2, \dots \in I$ such that $(x_1, x_2, \dots, x_n) \subset (x_1, x_2, \dots, x_n, x_{n+1}) \subset I$ for all n. Hence, letting

$I_n = (x_1, \dots, x_n)$ we obtain an infinite strictly ascending chain of ideals $\lightning$. Hence, $I_n = I$ for some n, so I is finitely generated.
Now, assume all ideals are finitely generated. Suppose there is an infinite proper chain of ideals

$$I_0 \subset I_1 \subset \dots$$

with each containment being proper. Then, we see $\bigcup_{k \in N_0} I_k = I$ is an ideal. Moreover since I is finitely generated there are $y_1, y_2, \dots, y_n \in I$ such that $I = (x_1, x_2, \dots, x_n)$. Then, since $y_1, y_2, \dots, y_n \in \bigcup_{k \in N_0} I_k$, we see each one is in I_k for some k. Since each $I_k \subset I_{k+1}$, let I_m be an ideal containing all $y_1, y_2, \dots, y_n$. Then, we see $I \subset I_m$, but this is a contradiction as $I \neq I_m$ by the proper containment assumption and $I \nsubseteq I_m$ as I_m is within the union. $\lightning$. Hence, the chain cannot be strictly ascending. □

Proposition 1.1. Let R be a commutative ring. If R satisfies the ascending chain condition on all principal ideals, then every nonzero element in R has a factorization.

Proof. Let $x \in R$ be a nonzero, nonunit. If x is irreducible, $x = x$ is a factorization. Hence, we can assume $x = x_1 x_2$ with x_1, x_2 being nonzero, nonunits. Similarly, we see x_1, x_2 cannot both be irreducible else this would be a factorization. Hence define $x_1 = x_{11}x_{12}$ and $x_2 = x_{21}x_{22}$ with atleast 3 of $x_{11}x_{12}x_{21}x_{22}$ being non-units. Hence, $x_1 = x_{11}x_{12}x_{21}x_{22}$. Recursing n times yields

$$x = \prod_{i=1}^{2^n} x_i$$

with atleast 2^{n-1} elements being nonunits. If for some n, we find all x_i, $1 \leq i \leq 2^n$ to be irreducible (or units), then x has been factored. Hence, we may assume atleast one x_i to be not an irreducible for all n. Then, we see there must be a sequence k_i such that $(x) \subset (x_1) \subset (x_{k_1}) \subset (x_{k_2}) \subset \dots$ as each x_{k_i} splits into a product of elements which are not both irreducble or units. Moreover, each containment must be proper, so letting n grow yields $\lightning$, as such a chain will continue indefinitely unless all x_i are irreducble or units at some step. Hence we must have at some point all x_i to be irreducibles, hence x is factorable. □

Theorem 1.1. If R is a noetherian domain then R is a unique factorization domain if and only if all irreducible elements are prime.

Proof. Note, we have already shown all primes to be irreducible in an integral domain (hence noetherian domain) and we know UFD implies primes are irreducibles. Hence, only one implication remains to be shown, that all irreducible being prime implies UFD.
Since R is a noetherian domain, factorizations exist. Hence, we need only show

these factorizations are unique. Suppose

$$\begin{aligned} x &= ux_1x_2\ldots x_n \\ &= u'y_1y_2\ldots y \end{aligned}$$

with u, u' being units and x_i, y_i being irreducibles for each i. We proceed by induction on $|\mathrm{Fac}(x)|$. If $|\mathrm{Fac}(x)| = 1$, then x is irreducible and the claim is obviously true. Of course the case $|\mathrm{Fac}(x)| = 0$ implies x a unit, hence not factorable, so the claim is vacuously true in this case.
Now, assuming the case $n-1$, if $|\mathrm{Fac}(x)| = n$ (as is the case in the original x), we see $x_1 \mid x$ with x_1 being irreducible, hence prime. Supposing the claim false, we see $x_1 \mid u'y_1y_2\ldots y_t$, so WLOG, $x_1 \mid y_1$ up to units. As y_1 is irreducible and divided by x_1 , we see $y_1 = x_1r_1$ with r_1 being a unit, hence $x_1 = y_1$ up to units. Repeating yields for each $1 \leq i \leq n$, $x_i = y_j$ for some $1 \leq j \leq t$ (up to permutation of the y_i's) up to units, hence

$$\begin{aligned} x &= ux_1x_2\ldots x_n \\ &= \hat{u}x_1x_2\ldots x_ny_s\ldots y_t \text{ for a unit } \hat{u} \text{ and some } s \leq t\ . \end{aligned}$$

This yields, $y_1y_2\ldots y_t = 1$ up to units, $\lightning$ as the y_i's were assumed nonunits. $\square$