MATH 8237

LECTURE Mar. 12, 2021

PLANAR GRAPHS

In this lecture we shall discuss planar and non-planar graphs.

Roughly speaking, **planar** graphs are those that can be drawn in the plane without any two edges intersecting except at the vertices.

Let us recall that graphs are abstract objects, while their drawings are point sets with some particular structure.

Studying planar graphs is not an easy task.

In particular, this study requires some nontrivial knowledge of topology.

For a start, let us denote the plane as $\mathbb{R}^2$, known also as *the two-dimensional Euclidean space*.

A **planar embedding** of a graph G is a mapping of the vertices of G into distinct points in the plane, together with a mapping of each edge uv of G into a line joining the points corresponding to u and v, with the condition that no two lines have points in common except the vertex points.

The above definition raises new questions, like "what exactly a line is?"

Later, we shall see that we can require that lines representing edges to be indeed straight line segments, but for the time being, it is convenient to think of lines as homeomorphic images of the unit interval, that is, images of $[0, 1]$ under continuous one-to-one mappings into $\mathbb{R}^2$.

When one tries to investigate which graphs are planar, it turns out that all graphs of order up to four are planar. Note that the "Mercedes" graphic sign is a planar embedding of K_4.

CLOSED CURVES

One may also try to draw K_5 in the plane, but after a few minutes one realizes that this is not possible.

However a formal proof of the fact that K_5 is not planar requires a considerable detour.

Definition A **closed curve** is a homeomorphic image of the circle.

More intuitively, a closed curve is a line that can be drawn in the plane without self-intersections and such that the start and end points are the same.

With this loose definition of a closed curve, one can come up with some extremely intricate shapes, which still can be considered closed curves.

PATH–CONNECTED SETS

Definition A set of points in the plane is called **path-connected** if any two points of the set can be joined by a line entirely within the set.

Some simple examples of path-connected sets are the disks, the interior of rectangles and polygons, the straight lines, etc.

In general, path-connected sets may have very complicated fractal structure, like snowflakes.

Moreover, it should be noted that there are sets that are connected (in topological sense,) but not path-connected.

A drawing of a connected graph is always path connected.

JORDAN'S THEOREM

Looking at the circle, we see that it splits the plane into two disjoint path connected sets, one bounded and one unbounded.

Theorem (Jordan) *Any closed curve C in $\mathbb{R}^2$ partitions $\mathbb{R}^2 \backslash C$ into two disjoint path connected sets.*

Jordan's theorem seems obvious for simple closed curves, but in fact its proof is not at all elementary, and it took decades to be completed.

Let C be a closed curve in $\mathbb{R}^2$ and suppose that U and V are the two path connected parts of $\mathbb{R}^2 \backslash C$.

One of the sets U and V contains all the points that are sufficiently far from C. This particular set is denoted by $Ext\,(C)$ and the other one by $Int\,(C)$.

NON-PLANAR GRAPHS

An immediate consequence of Jordan's theorem is that if C is a closed curve in $\mathbb{R}^2$, and a line joins a point from $Ext\,(C)$ to a point from $Int\,(C)$, then this line intersects C at some point.

With some case analysis one can use this fact to prove:

Theorem K_5 *is not planar.*

Definition Graphs that are not planar are called **non-planar** graphs.

Note that Jordan's theorem is a necessary tool to prove that K_5 is non-planar, and no proof seems possible without it.

Along the same lines, a similar statement can be deduced for $K_{3,3}$:

Theorem $K_{3,3}$ *is non-planar.*

It is clear that any graph that contains K_5 or $K_{3,3}$ is non-planar as well (Convince yourself.)

In addition, we can significantly expand the class of non-planar graphs by introducing the operation *edge subdivision*.

SUBDIVISION GRAPHS

Let G be a graph and $\{u, v\}$ be an edge of G.

Let us introduce a new vertex w and replace the edge $\{u, v\}$ with the edges $\{u, w\}$ and $\{w, v\}$. Call this operation an **edge subdivision** of $\{u, v\}$.

Edge subdivisions can be iterated indefinitely and can be carried out both on the original edges of G and on the newly appearing edges.

Definition A graph H is called a **subdivision** of a graph G, if H can be obtained from G by a sequence of edge subdivisions.

A FEW QUESTIONS ON SUBDIVISIONS

Problem *Any subdivision of a path is a path.*

Problem *Any subdivision of a cycle is a cycle.*

Problem *Any subdivision of a connected graph is connected.*

KURATOWSKI'S THEOREM

It is not hard to see that a graph is planar if and only if any subdivision of it is planar.

Using this observation, we come up with a large class of non-planar graphs:

Theorem *Any subdivision of K_5 or $K_{3,3}$ is non-planar.*

Amazingly, the converse of this statement is also true, as proved by the topologist Kuratowski.

Theorem (Kuratowski) *A graph is planar if and only if it does not contain subdivisions of K_5 or $K_{3,3}$.*

WAGNER'S THEOREM

Trying to draw large planar graphs, one may think that the ability to use lines of any shape to represent edges is crucial for successful drawing.

This turns out not to be the case, as proved by Wagner in 1936:

Theorem (Wagner) *If G is a planar graph, then it can be drawn in the plane so that each edge is represented by a segment of straight line.*

Moreover, it turns out that if we are allowed to draw in the three-dimensional space $\mathbb{R}^3$, then every graph can be drawn with each edge represented by a segment of straight line.

POLYHEDRAL GRAPHS

However, if we are allowed to draw graphs only on a two-dimensional sphere (the surface of a solid ball,) we gain nothing compared to the plane.

Theorem *A graph is planar if and only if it can be drawn on a two-dimensional sphere.*

Graphs drawn on a sphere are closely related to convex polyhedra.

The solid cubes, the tetrahedron, the prisms are examples of convex polyhedra.

The three basic parameters of a convex polyhedron are:

- the number of its vertices v,
- the number of its edges e,
- the number of its faces f.

EULER'S FORMULA

- There is a fundamental relation between the parameters v, e, and f discovered by Euler:

- **Theorem** Let v, e, and f be the number of vertices, edges, and faces of a convex polyhedron. Then

$$v - e + f = 2.$$

- It turns out that the same relation can be extended to planar graphs with appropriate definition of face, but we shall not dwell on this topic.

THE FOUR COLOR THEOREM

A long standing open question in graph theory asked what is the chromatic number of planar graphs.

It is not hard to see that it is at least 4 and at most 5.

After more that hundred years of significant efforts by many mathematicians, in 1977, Appel and Haken answered the question:

Theorem *Every planar graph is 4-colorable.*

Their original proof, in addition of being extremely long, required 2000 hours of computer computations to check a large number of small cases.

THANK YOU