Algebraic Theory I

Thomas Fleming

September 28, 2021

Contents

1 Nilpotent Groups

Lecture 14: Nilpotent Groups

Fri 24 Sep 2021 11:30

Let G be a group, and $Z_0(G) = \{1\}$ with $Z_1(G) = Z(G)$. Thus, $G/Z_1(G)$ is a group which has $Z(G/Z_1(G)) = \frac{Z_2(G)}{Z_1(G)}$ where $Z_2(G)$ is the preimage of $Z(G/Z_1(G))$, that being the subgroup of G containing $Z_1(G)$. We see we may continue

$$\begin{aligned} Z_2(G)/Z_1(G) &= Z(G/Z_1(G)) \\ \text{then, } (G/Z_1(G))/(Z_2(G)/Z_1(G)) &\simeq G/Z_2(G) \\ \text{which has a center } Z(G/Z_2(G)) &= Z_3(G)/Z_2(G). \end{aligned}$$

Definition 1.1 (Nilpotence)**.** We recursively define $Z_i(G)$ to be the subgroup such that $Z(G/Z_i(G)) = Z_i(G)/Z_{i-1}(G)$. This yields a growing sequence $Z_0(G) \trianglelefteq Z_1(G) \trianglelefteq Z_2(G) \trianglelefteq \ldots$. We say a group G is **nilpotent** if $G = Z_n(G)$ for some $n \geq 0$. The minimal $n \geq 0$ for which this is the case is called the **nilpotent class** of G.

Example. The trivial group $\{1\}$ is nilpotent with class $c = 0$.
A nontrivial abelian group is nilpotent with class $c = 1$.

$\diamond$

Theorem 1.1. Every finite p-group is nilpotent.

Proof. We know the center of a nontrivial p-group to be nontrivial and its subgroups and quotient groups will also be p-groups. Hence $Z_1(G)$ is nontrivial except in the case G is trivial. Hence we have that $Z_2(G)/Z_1(G)$ is nontrivial unless $Z_2(G) = G$. Hence either $Z_1 < Z_2$ or $Z_2 = G$. Now, denote $|G| = n$.

Then either $1 = |Z_0| < |Z_1| < \ldots < |Z_n|$ hence $Z_n = G$ or $Z_i = G$ for some $i < n$, so $Z_n = G$. Hence, G is nilpotent. $\square$

Definition 1.2. A subgroup $H \leq G$ is **characteristic** if for every automorphism of G, we have $\alpha(H) = H$. This is equivalent to $\alpha(H) \leq H$ for all automorphisms as $\alpha^{-1} : G \to G$ is also an automorphism, hence $H \leq \alpha(H)$, so equality holds. Since conjugation is always an automorphism, being characteristic implies normality.

Proving vs. Using Characteristicness

This means that in order to show that something is characteristic we need only show $\alpha(H) \leq H$, but when we use that something is characteristic we will often use the full equality.

Lemma 1.1. As we know $K \trianglelefteq H$ and $H \trianglelefteq G$ does not imply $K \trianglelefteq G$. On the other hand, K being characteristic in H and $H \trianglelefteq G$ does yield $K \trianglelefteq G$.

Proof. Let $\alpha_x : G \to G$ be the conjugation by x map. We know this to be an automorphism of G, hence as H is normal, we have $\alpha_x\,|_H\colon H \to H$ is an automorphism of H, and since K is characteristic in H, we see an automorphism of H fixed K, hence $\alpha_x(K) = xKx^{-1} = K$ for all $x \in G$, hence $K \trianglelefteq G$. $\square$

Lemma 1.2. Let G be a finite group with p being prime and P being a sylow p-group in G. Then, the following are equivalent

1. P is the unique sylow p-group in G.
2. $P \trianglelefteq G$.
3. P is characteristic in G.
4. Any subgroup generated by elements whose orders are each powers of p is itself a p-group.

Proof.

1. We have already shown $1 \Leftrightarrow 2$.
2. As conjugation is always an automorphism, we see $2 \Leftarrow 3$ is trivial.
3. We show $1 \Rightarrow 3$. Let $\alpha : G \to G$ be an arbitrary automorphism of G. Then, $\alpha(P) \leq G$ and $|P| = |\alpha(P)|$. As P is the unique sylow p-group, we see there is no distinct group of cardinality $|P|$, hence $\alpha(P) = P$.
4. Now we show $1 \Rightarrow 4$. Let X be a set satisfying $\operatorname{ord}(x) = p^n$ for each $x \in X$. Then each $\langle x \rangle$ is contained in a p-group, and as there is a unique maximal p-group, we have that $\langle x \rangle \subseteq P$ for each $x \in X$. Hence, $\langle X \rangle \subseteq P$ and as X is a p-group we have that $X = P$.
5. $4 \Rightarrow 1$. Let X to be the union of all sylow p-groups in G. By hypothesis, $\langle X \rangle$ is a p-group and thus it is contained in some sylow p-group so WLOG,

we have $\langle X \rangle \subseteq P$. But if there were distinct p-groups, $P' \neq P$ then $P' \subseteq X$ and $P \subset \langle P' \cup P \rangle \subseteq X \subseteq P$. $\lightning$. Hence P is the unique sylow p-group.

□

Lecture 15: Nilpotent Groups (2)

Tue 28 Sep 2021 17:46

Lemma 1.3. If H, K are groups, then $Z(H \times K) = Z(H) \times Z(K)$.

Proof. Let $(x, y) \in H \times K$. If $(x, y) \in Z(H \times K)$ then

$$\underbrace{(a,1)(x,y)(a,1)^{-1}}_{=(axa^{-1},1)} = (x,y).$$

Hence, $x \in Z(H)$ and similarly, $y \in Z(K)$. Hence, $Z(H \times K) \subseteq Z(H) \times Z(K)$. The other direction of inclusion is trivial and left as an exercise. □

Lemma 1.4. Let $\varphi : G \to G'$ be a homomorphism with $\ker(\varphi) = K$ and $H \leq G$ such that $K \leq H$. Then, $N_G(H) = f^{-1}(N_{G'}(\varphi(H)))$.

Proof. Let $x \in N_G(H)$, so $xHx^{-1} = H$. Hence,

$$\varphi(H) = \varphi\left(xHx^{-1}\right) = \varphi(x)\varphi(H)\varphi(x)^{-1}.$$

Thus,

$$\begin{aligned} \varphi(x) &\in N_{G'}(\varphi(H)) \\ \Rightarrow x &\in \varphi^{-1}(N_{G'}(\varphi(H))) \\ \Rightarrow N_G(H) &\subseteq \varphi^{-1}(N_{G'}(\varphi(H))). \end{aligned}$$

Conversely, let $x \in \varphi^{-1}(N_{G'}(\varphi(H)))$, hence $\varphi(x) \in N_{G'}(\varphi(H))$.
Then, we see

$$\begin{aligned} \varphi(H) &= \varphi(x)\varphi(H)\varphi\left(x^{-1}\right) \\ &= \varphi\left(xHx^{-1}\right) \\ \Rightarrow xHx^{-1} &\subseteq \varphi^{-1}(\varphi(H)) \\ &= \langle H, \ker(\varphi) \rangle \\ &= H \text{ as } \ker(\varphi) \subseteq H. \end{aligned}$$

Hence, $xHx^{-1} \subseteq H$, so $x \in N_G(H)$. This concludes the proof. □

Now, recall that if G is a finite group with P being a sylow p-group, then TFAE

1. P is unique.
2. $P \trianglelefteq G$.
3. P is characteristic.
4. Any subgroup generated by elements whose orders are powers of p is itself a p-group.

Theorem 1.2. If G is a finite group, then the following are equivalent:

1. G is nilpotent.
2. $H < G \Rightarrow H < N_G(H)$.
3. All sylow p-groups are normal.
4. G is the direct product of its sylow p-groups.

Proof.

- $(2 \Rightarrow 3)$. Let P be a sylow p-group of G. Assume P is not normal, then denote $N = N_G(P) \subset G$. Hence, by the preceding lemma, P is characteristic in N. Then, as $N \trianglelefteq N_G(N)$, we see $P \trianglelefteq N_G(N)$. But $N = N_G(P)$ was the largest subgroup in which P was normal, hence $N_G(P) = N_G(N)$. So, by contrapositive of the assumption, (2), we have $N = N_G(N)$, so $N = G$, hence $P \trianglelefteq G$.
- $(3 \Rightarrow 4)$.

$\square$