Analysis I

Thomas Fleming

October 19, 2021

Contents

Lecture 16: Conclusion of Measure Theory and Lebesque Integration

Tue 19 Oct 2021 13:02

Recall. We stated the theorems behind littlewood's 3 principles, now we prove them.

Proof.

1. (2.2). Let J be the collection of all open intervals (a, b) with $a, b \in \mathbb{Q}$ and $a < b$. Since J is countable we can order the intervals $J = \{J_k : k \in \mathbb{N}\}$. Let $\varepsilon > 0$ and first we do the case S is bounded. For each $n \in \mathbb{N}$, there is a closed set $C_n \subseteq f^{-1}(J_n)$ and a $D_n = S \setminus f^{-1}(I_n)$ such that $\mu(S \setminus (C_n \cup D_n)) < \frac{\varepsilon}{2^n}$. Since S is bounded, C_n and D_n are compact. Let $K = \bigcap_{n\in\mathbb{N}} (C_n \cup D_n)$ and as $C_n, D_n \subseteq S$, we see $K \subseteq S$. Furthermore, K is compact and we find $\mu(S \setminus K) \leq \sum_{i=1}^{\infty} \mu(S \setminus (C_n \cup D_n)) < \varepsilon$. Now, we show the restriction is continuous. Let $\varepsilon > 0$, then for $x \in K$ we find $a, b \in \mathbb{Q}$ such that $a < f(x) < b$ and $b - a < \varepsilon$. Hence, there is $n \in \mathbb{N}$ such that $I_n = (a, b)$. Consequently, $x \in f^{-1}(I_n)$ and $x \notin S \setminus f^{-1}(I_n)$. So, $x \in \left(S \setminus f^{-1}(I_n)\right)^c \subseteq D_n^c$. As D_n is closed, D_n^c is open, hence there is a $\delta > 0$ so that $(x - \delta, x + \delta) \subseteq D_n^c$. If $y \in K \cap D_n^c$, then $y \in C_n$, thus $y \in f^{-1}(I_n)$, hence $a < f(y) < b$. So, $|f(x) - f(y)| < b - a = \varepsilon$ for $y \in (x - \delta, x + \delta)$.
   Now, we do the unbounded case. As S is unbounded and $\varepsilon > 0$, we find $N \in N$ so that $S' = S \cap [-N, N]$ has the property $\mu(S \setminus S') < \frac{\varepsilon}{2}$, that is S is approximated by a bounded function arbitrarily well. Since S' is bounded, there is a compact set $K \subseteq S' \subset S$ so that $f \mid K$ is continuous and $\mu(S' \setminus K) < \frac{\varepsilon}{2}$. Then, $\mu(S \setminus K) = \mu(S \setminus S') + \mu(S' \setminus K) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

2. (2.4). Let E^* be the set of all $x \in S$ such that $(f_n(x))$ does not converge. By assumption, $\mu(E^*) = 0$. Since $f(x) = \lim_{n\to\infty} f_n(x) = \limsup_{n\to\infty} f_n(x)$ for all $x \in S \setminus E^*$, then f is measurable. For $k, \ell \in \mathbb{N}$, let $E_{k,\ell} = \{x \in S : |f_\ell(x) - f(x)| \geq \frac{1}{k}\}$. Then, $E_{k,\ell}$ is measurable. Fix k. If for each

$n \in \mathbb{N}$ there is a $\ell \geq n$ so that $|f_\ell(x) - f(x)| \geq \frac{1}{k}$, then $x \in E^*$ as f does not converge at that point. Hence, $\bigcap_{n\in\mathbb{N}} \bigcup_{\ell=n}^{\infty} E_{k,\ell} \subseteq E^*$. Since $\mu\left(\bigcup_{\ell=1}^{\infty} E_{k,\ell}\right) \leq \mu(S) \leq \infty$, and the collection $\{\bigcup_{\ell=n}^{\infty} E_{k,\ell}\}$ is clearly descending. Hence, $\mu\left(\bigcap_{n\in\mathbb{N}} \bigcup_{k=n}^{\infty} E_{k,\ell}\right) = \lim_{n\to\infty} \mu\left(\bigcup_{\ell=n}^{\infty} E_{k,l}\right) \leq \mu(E^*) = 0$. This holds for all $k \in \mathbb{N}$. So, for $\varepsilon > 0$ and $k \in \mathbb{N}$, we have a $n_k \in \mathbb{N}$ such that $\mu\left(\bigcup_{\ell=n_k}^{\infty} E_{k,\ell}\right) < \frac{\varepsilon}{2^k}$. Thus, $E = \bigcup_{k\in\mathbb{N}} \bigcup_{\ell=n_k}^{\infty} E_{k,\ell}$ is measurable and $\mu(E) < \sum_{k=1}^{\infty} \bigcup_{\ell=n_k}^{\infty} E_{k,l} = \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon$. If $x \in S \setminus E$, then $|f_n(x) - f(x)| < \frac{1}{k}$ for $k \in \mathbb{N}$ if $n \geq n_k$. So, (f_n) converges uniformly on $S \setminus E$.

□

This concludes measure theory.

1 Lebesque Integration

Definition 1.1 (Lebesque Integral: Nonnegative Simple Functions)**.** Let s be a nonnegative simple function of the form $s = \sum_{k=1}^{K} a_k \chi_{S_k}$ where $\{S_k : 1 \leq k \leq K\}$ is a disjoint collection of measurable sets. Then, the **Lebesque Integral** of s is defined to be

$$\int s = \int s(x)\,dx = \int s d\mu = \sum_{k=1}^{K} a_k \mu(S_k).$$

Proposition 1.1. If s is nonnegative and simple with two representations, $s = \sum_{k=1}^{K} a_k \chi_{S_k} = \sum_{j=1}^{J} b_j \chi_{T_j}$ for disjoint collections of measurable sets $\{S_k : 1 \leq k \leq K\}$ and $\{T_j : 1 \leq j \leq J\}$. Then

$$\sum_{k=1}^{K} a_k \mu(S_k) = \sum_{j=1}^{J} b_j \mu(T_j).$$

In particular, $\int s$ is well defined.

The proof of this is trivial.

Lemma 1.1. Let s, t be nonnegative and simple and $\alpha \geq 0$. Then

$$\alpha \cdot \int s = \int \alpha \cdot s \text{ and } \int (s+t) = \int s + \int t$$

.

Proof. Clearly, multiplying the sum times α yields $\alpha \sum_{k=1}^{K} a_k \mu(S_k) = \sum_{k=1}^{K} \alpha a_k \mu(S_k)$. For the second claim. Suppose $s = \sum_{k=1}^{K} a_k \chi_{S_k}$ and $g = \sum_{j=1}^{J} b_j \chi_{T_j}$ are canonical representations. Then, $s + t = \sum_{k=1}^{K} \sum_{j=1}^{J} (a_k + b_j) \chi_{S_k \cap T_j}$ with

$\{S_k \cap T_j : 1 \le k \le K, 1 \le j \le J\}$ is a disjoint collection and

$$\begin{aligned}
\int (s+t) &= \sum_{k=1}^{K} \sum_{j=1}^{J} (a_k + b_j)\, \mu\left(S_k \cap T_j\right) \\
&= \sum_{k=1}^{K} a_k \sum_{j=1}^{J} \mu\left(S_k \cap T_j\right) + \sum_{j=1}^{J} b_j \sum_{k=1}^{K} \mu\left(S_k \cap T_j\right) \\
&= \sum_{k=1}^{K} a_k \mu\left(S_k\right) + \sum_{j=1}^{J} b_j \mu\left(T_j\right) \\
&= \int s + \int t.
\end{aligned}$$

$\square$

Lemma 1.2. Let s, t be nonnegative and simple such that $s \le t$. Then, $\int s \le \int t$.

Proof.

$$\begin{aligned}
\int t &= \int (t - s + s) \\
&= \int \underbrace{(t-s)}_{\ge 0} + \int s \\
&\ge \int s.
\end{aligned}$$

$\square$

Definition 1.2. Let $f : S \to \overline{\mathbb{R}}$, then the **zero extension** of f to $\mathbb{R}$ is

$$\begin{aligned}
f^* : \mathbb{R} &\longrightarrow \overline{\mathbb{R}} \\
x &\longmapsto f^*(x) = \begin{cases} f(x), & x \in S \\ 0, & x \notin S \end{cases}.
\end{aligned}$$

Moreover, this function preserves measurability.

Definition 1.3 (Lebesque Integral of a General Nonnegative Function)**.** Let $f : \mathbb{R} \to \overline{\mathbb{R}}$ be a nonnegative measurable function and $\mathscr{S}(f)$ be the collection of all nonnegative simple functions, s, such that $s \leq f$. Then, the **Lebesque Integral** of f over $\mathbb{R}$ is defined to be

$$\int f = \int_{\mathbb{R}} f(x)\, dx = \sup\{\int s : s \in \mathscr{S}(f)\}$$

.

If $f : S \to \overline{\mathbb{R}}$ is nonnegative and measurable, then

$$\int_S f = \int_S f(x)\, dx = \int_{\mathbb{R}} f^*$$

.

Theorem 1.1 (Chebyshev's Inequality)**.** Let $f : \mathbb{R} \to \overline{\mathbb{R}}$ be nonnegative and measurable. Then, for any $\lambda \in (0, \infty)$, then

$$\mu(\{x \in \mathbb{R} : f(x) \geq \lambda\}) \leq \frac{1}{\lambda} \int f.$$

Proof. Let $E = \{x \in \mathbb{R} : f(x) \geq \lambda\}$. This is the preimage of an extended borel set, hence measurable. Let $s = \lambda$ □

Lecture 15: Measurable Functions (3) and Simple Functions

Thu 14 Oct 2021 13:01

Proposition 1.2. Let (f_n) be a sequence of measurable functions $f_n : S \to \overline{\mathbb{R}}$. Then, we define $f, g, F, G : S \to \overline{\mathbb{R}}$ with

- $f(x) = \sup\{f_n(x) : n \in \mathbb{N}\}$,
- $g(x) = \inf\{f_n(x) : n \in \mathbb{N}\}$,
- $F(x) = \limsup_{n\to\infty} f_n(x)$,
- $G(x) = \liminf_{n\to\infty} f_n(x)$

all being measurable.

Proof.

- Note that $f(x) > c$ if and only if there is an n such that $f_n(x) > c$. Hence, $f^{-1}((c, \infty]) = \bigcup_{n\in\mathbb{N}} f_n^{-1}((c, \infty)$ is measurable.
- It it clear $g(x) = -\sup\{-f_n(x) : n \in \mathbb{N}\}$.
- Next, note that $F(x) = \inf\{\sup\{f_k(x) : k \geq n\} : n \in \mathbb{N}\}$ and $G(x) = \sup\{\inf\{f_k(x) : k \geq n\} : n \in \mathbb{N}\}$, hence they are measurable by the first two theorems.

□

Remark. It is also true that for a measurable function $f : S \to \overline{\mathbb{R}}$ is measurable implies

$$f^{+}(x) = \sup\{f(x), 0\}$$
$$f^{-}(x) = \sup\{-f(x), 0\}$$

are also measurable.

2 Simple Functions

Definition 2.1. Let $S \subseteq \mathbb{R}$. Then,

$$\chi S : \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \longmapsto \chi_S(x) = \begin{cases} 1, & x \in S \\ 0, & x \notin S \end{cases}$$

is the **characteristic function of** S .
A measurable function $s : \mathbb{R} \to \overline{\mathbb{R}}$ is a **simple functions** if $s(\mathbb{R})$ is finite.

Proposition 2.1. If s is a simple function. Then, there exists a finite, disjoint collection of measurable sets $\{S_k : 1 \le k \le K\}$ and a finite sequence of distinct real numbers $(a_k)_{1\le k\le K}$ such that $\mathbb{R} = \bigcup_{k=1}^{K} S_k$ and $s = \sum_{k=1}^{K} a_k \chi_{S_k}$. Furthermore, this combination is unique up to permutation of the a_k, s_k. This representation is called the **canonical representation**.

Lemma 2.1. Let $f : \mathbb{R} \to \mathbb{R}$ be nonnegative and measurable with $f(\mathbb{R})$ being bounded, then for each $\varepsilon > 0$ there is a nonegative simple function s such that $f \ge s$ and $f(x) - s(x) < \varepsilon$ for all $x \in \mathbb{R}$.

Proof. There is a $M > 0$ such that $f(\mathbb{R}) \subseteq [0, M)$. Given ε, let $y_k = k\varepsilon$ for $k \in \mathbb{N}_0$. Since, $y_k - y_{k-1} = \varepsilon$, there is $N \in \mathbb{N}$ such that $[0, M] \subseteq \bigcup_{k\in\mathbb{N}} [y_{k-1}, y_k)$. Let $S_k = f^{-1}([y_{k-1}, y_k))$ for $1 \le k \le N$. Define $s = \sum_{k=1}^{N} y_{k-1}\chi_{S_k}$. Then, $s \ge 0$ and s is simple. Furthermore, for each $x \in \mathbb{R}$, there is a unique k, with $1 \le k \le N$ such that $f(x) \in [y_{k-1}, y_k)$. Consequently, $s(x) = y_{k-1} \le f(x) < y_k$. Hence, $f(x) - s(x) < y_k - y_{k-1} = \varepsilon$. □

Theorem 2.1. $f : \mathbb{R} \to \overline{\mathbb{R}}$ is measurable if and only if there is a sequence of simple functions (s_n)a such that (s_n) converges pointwise to f and $|f| \ge |s_n|$ for all $n \in \mathbb{N}$.

Proof. Suppose the sequence (s_n). Then, f is measurable as

$$f = \lim_{n\to\infty} s_n = \limsup_{n\to\infty} s_n = \liminf_{n\to\infty} s_n.$$

Now, assume f is measurable. Then, $f = f^+ - f^-$. Both f^+ and f^- are measurable and nonnegative. Since the difference of two simple functions is simples, it suffices to assume $f \geq 0$, that is $f^- = 0$. Let $B_n = \{x \in \mathbb{R} : f(x) \leq n\}$ and $g_n = f\chi_{B_n}$ for all $n \in \mathbb{N}$. Since $g_n(x) = \inf\{f(x), n\chi_{B_n}\}$. Then, we see g_n is measurable as f and the simple function $n\chi_{B_n}$ are measurable. Furthermore, g_n is bounded. Hence, there is a measurable simple function r_n such that $g_n \geq r_n$ and $g_n(x) - r_n(x) < \frac{1}{n}$ for all x. Finally, define

$$s_n = r_n + n\chi_{B_n^c}.$$

Then, we find (s_n) is the sequence of functions desired. □

Corollary 1. Let (f_n) be a sequence of nonnegative measurable functions $f_n : \mathbb{R} \to \overline{\mathbb{R}}$. Then, $x \mapsto \sum_{i=1}^{\infty} f_k(x)$ is measurable. In particular, if $f, g : \mathbb{R} \to \overline{\mathbb{R}}$ are nonnegative and measurable, then so is $f + g$.

Proof. For $N \in \mathbb{N}$, let $F_n = \sum_{k=1}^{N} f_k$. For each k there is sequence of simple functions $(s_{k,n})_n$ such that $(s_{k,n})_n$ converges pointwise to f_k and $f_k \geq s_{k,n} \geq 0$ for all n.
Hence, $\left(\sum_{k=1}^{N} s_{k,n}\right)_n$ is a sequence of nonnegative simple functions such that $F_N \leq \sum_{k=1}^{N} s_{k,n}$ for all n and

$$\lim_{n\to\infty} \sum_{k=1}^{N} s_{k,m}(x) = F_N(x)$$

for all $x \in \mathbb{R}$.
So, F_N is the limit of a sequence of measurable functions, so it is measurable. Furthermore, we have that for each $x \in \mathbb{R}$, $\left(F_{N(x)}\right)_N$ is increasing, we find

$$\sum_{k=1}^{\infty} f_k = \limsup_{N\to\infty} F_N = \lim_{N\to\infty} F_N.$$

□

3 Littlewood's 3 Principles

Remark.
1. Every measurable set is "nearly" the union of a finite collection of intervals.
2. Every measurable function is "nearly" continuous.
3. Every pointwise convergent sequence of measurable functions is "nearly" uniformly continuous.

We state these princeiples rigorously in the following way:

Theorem 3.1. If S is measurable, with $\mu(S) < \infty$, then for each $\varepsilon > 0$ there is a finite disjoint collection of open intervals $\{I_k : 1 \leq k \leq n\}$ such that for $U = \bigcup_{k=1}^{n} I_k$ we find

$$\mu(S \triangle U) < \varepsilon.$$

Theorem 3.2 (Lucin's Theorem)**.** Let $f : S \to \mathbb{R}$ be measurable with $\mu(S) < \infty$. Then, for each $\varepsilon > 0$ there is a compact $K \subseteq S$ such that $f|_K : K \to \mathbb{R}$ is continuous and $\mu(S \setminus K) < \varepsilon$.

Theorem 3.3 (Lucin's Theorem for functions on $\mathbb{R}$)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be measurable. Then, for all $\varepsilon > 0$ there is a continuous $g : \mathbb{R} \to \mathbb{R}$ and a closed set $E \subseteq \mathbb{R}$ such that $f = g$ on E and $\mu(E^c) < \varepsilon$. Moreover, $\sup\{|g(x)| : x \in \mathbb{R}\} \leq \sup\{|f(x)| : x \in \mathbb{R}\}$.

Theorem 3.4 (Egoroff's Theorem)**.** Let S be measurable with $\mu(S) < \infty$. Suppose (f_n) is a sequence of measurable functions $f_n : S \to \mathbb{R}$ which converges pointwise almost everywhere to $f : S \to \mathbb{R}$. Then, for all $\varepsilon > 0$, there is a measurable $E \subseteq S$ such that $\mu(E) < \varepsilon$ and (f_n) converges uniformly to f on $S \setminus E$.