Analysis I

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Contents

Lecture 8: Continuity (2) and Extended $\mathbb{R}$

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We begin with some results on continuity over intervals and inverses.

Recall. If $f : I \to \mathbb{R}$ is monotone with I being an interval. Then, f is continuous if and only if $f(I)$ is an interval.

Corollary 1. A continuous strictly monotone function $f : I \to \mathbb{R}$, with I being an interval, has a continuous inverse $f^{-1} : f(I) \to \mathbb{R}$.

Proposition 0.1. A strictly monotone function $f : I \to \mathbb{R}$, with I being an interval, has a continuous inverse.

Theorem 0.1 (Heine's Theorem)**.** A continuous function $F : S \to \mathbb{R}$ with S being compact is uniformly continuous.

Proof. For $\varepsilon > 0$ and $x \in S$, there is a $\delta_x > 0$ such that $|f(x) - f(y)| < \frac{\varepsilon}{2}$ if $|x - y| < \delta_x$. Let $U_x = \left(x - \frac{\delta_x}{2}, x + \frac{\delta_x}{2}\right)$. Since $\{U_x : x \in S\}$ is an open cover of S, there are $x_1, x_2, \ldots, x_n$ such that $\{U_{x_k} : 1 \leq k \leq n\}$ is a finite open subcover of S. Let $\delta = \min\{\frac{1}{2}\delta_{x_1}, \frac{1}{2}\delta_{x_2}, \ldots, \frac{1}{2}\delta_{x_n}\}$ and suppose $x, y \in S$ such that $|x - y| < \delta$. Then there is x_k for some $1 \leq k \leq n$ such that $x \in U_{x_k}$ and $|x_k - y| \leq |x - y| + |x_k - x| < \delta + \frac{\delta_{x_k}}{2} \leq \delta_{x_k}$.
Consequently

$$|f(x) - f(y)| \leq |f(x) - f(x_k)| + |f(x_k) - f(y)| < \varepsilon.$$

$\square$

Justification for continuity

There is an equivalence between open sets/continuity and measurable sets/measurableness.

Definition 0.1 (Convergence of functions)**.** Let (f_n) be a sequence of functions $f_n : S \to \mathbb{R}$. Then

1. (f_n) **converges pointwise** if $(f_n(x))$ is convergent for every $x \in S$. The limit is defined pointwise for every $x \in S$ with $f(x) = \lim_{n\to\infty} f_n(x)$ with $f : S \to \mathbb{R}$ being a function.
2. (f_n) **converges uniformly** to the function $f : S \to \mathbb{R}$ if for each $\varepsilon > 0$, there $N \in \mathbb{N}$ such that for all $x \in S$, $|f(x) - f_n(s)| < \varepsilon$ if $n \geq N$.

Theorem 0.2. Suppose (f_n) is a sequence of continuous functions $f_n : S \to \mathbb{R}$ which converges uniformly to $f : S \to \mathbb{R}$. Then, f is continuous.

Proof. Let $x \in S$ and $\varepsilon > 0$. Then, there is $k \in \mathbb{N}$ such that $|f(y) - f_k(y)| < \frac{\varepsilon}{3}$ for all $y \in S$. Consequently, for any $y \in S$

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_k(x)| + |f_k(x) - f_k(y)| + |f(y) - f_k(y)| \\ &< \frac{2\varepsilon}{3} + |f_k(x) - f_k(y)|. \end{aligned}$$

Since f_k is continuous, we can pick a sufficient $\delta > 0$ such that this completes the proof. $\square$

1 Extended $\mathbb{R}$

Recall that many objects such as the lim sup and lim inf required a boundedness assumption. We wish to discard this assumption when possible. Hence we introduce the following system.

Definition 1.1 (Extending Functions)**.** A function $h : \mathbb{R} \to \mathbb{R}$ is **extending** if h is strictly inreasing and $h(\mathbb{R}) = (-1, 1)$. Note that every extending function is continuous by these assumptions and has a continuous inverse.

Now, we introduce two external elements $-\infty, +\infty$ and we define $\infty = H^{-1}(1)$ and $-\infty = H^{-1}(-1)$ and we extend the ordering $\leq$ such that $-\infty < \infty$ and $-\infty < x < \infty$ for every $x \in \mathbb{R}$.

Definition 1.2 (Extended Real Numbers)**.** We denote $R \cup \{-\infty, \infty\} = \overline{\mathbb{R}} = [-\infty, \infty]$ to be the **extended real numbers** for use with extending functions.

In this way, the extending function h extends from R to $\overline{R}$ and it retains its strictly increasing and the image requirement $h\left(\overline{\mathbb{R}}\right) = [-1, 1]$.

Notation.
- $(a, \infty] = \{x \in \overline{\mathbb{R}} : x > a\}$
- $[-\infty, a] = \{x \in \overline{\mathbb{R}} : x \leq a\}$
- $(a, \infty)_{\overline{\mathbb{R}}} = \{x \in \overline{\mathbb{R}} : a < x < \infty\}$
- and so on.

It is of note that the interval (a, ∞) is $\mathbb{R}$ is still defined as normal, it is only when it is chosen as part of $\overline{\mathbb{R}}$.
Now we examine the topology on $\overline{\mathbb{R}}$.

Definition 1.3 (Topology on $\overline{\mathbb{R}}$)**.**
1. $S \subseteq \overline{\mathbb{R}}$ is open/closed if $H(S)$ is relatively open/closed in $[-1, 1]$ for any extending function H.
2. $S \subseteq \overline{\mathbb{R}}$ has $\sup(S) = H^{-1}(\sup(H(S)))$.
3. A sequence (x_n) in $\overline{\mathbb{R}}$ is convergent (in $\overline{\mathbb{R}}$) if for any extending function H, $(H(x_n))$ isconvergent. In this case we define
$$\lim_{n\to\infty} x_n = H^{-1}\left(\lim_{n\to\infty} H(x_n)\right)$$
.
4. A point $x_0 \in \overline{\mathbb{R}}$ is an accumulation or cluster point of the sequence (x_n) in $\overline{\mathbb{R}}$ if for any extending function H we have $H(x_0)$ is an accumulation point of $(H(x_n))$.
5. Let (x_n) in $\overline{\mathbb{R}}$. Then,
$$\limsup_{n\to\infty} x_n = H^{-1}\left(\limsup_{n\to\infty} H(x_n)\right)$$
$$\liminf_{n\to\infty} x_n = H^{-1}\left(\liminf_{n\to\infty} H(x_n)\right)$$

Example.
- $\overline{\mathbb{R}}$ is open and closed in $\overline{\mathbb{R}}$.
- $\mathbb{R}$ is open but not closed in $\overline{\mathbb{R}}$.
- $(7, \infty] \mapsto (H(7), 1]$, hence it is open.

◇

Proposition 1.1. If (x_n) is a sequence with $x_n \in \overline{\mathbb{R}}$. Then

$$\limsup_{n\to\infty} x_n, \liminf_{x\to\infty} x_n \in \overline{\mathbb{R}}$$

with

$$\begin{aligned}\limsup_{n\to\infty} x_n &= \inf\left(\sup\{x_k : k\in\mathbb{N}, k\geq n\} : n\in\mathbb{N}\right)\\ &= \lim_{n\to\infty} \sup\{x_k : k\in\mathbb{N}, k\geq n\} \text{ and}\\ \liminf_{n\to\infty} x_n &= \sup\left(\inf\{x_k : k\in\mathbb{N}, k\leq n\} : n\in\mathbb{N}\right)\\ &= \lim_{n\to\infty} \inf\{x_k k\in\mathbb{N}, k\leq n\}\end{aligned}$$

Remark. A sequence (x_n) in $\mathbb{R}$ is said to converge to ∞ if it is convergent in $\overline{R}$ with $\lim\limits_{n\to\infty} x_n = \infty$.

Definition 1.4.

1. If $a \in (-\infty, \infty]$, then $a + \infty = \infty + a = \infty$.
2. If $a \in [-\infty, \infty)$ then $a + (-\infty) = (-\infty) + a = -\infty$.
3. If $a \in (0, \infty]$ then $a \cdot \infty = \infty \cdot a = \infty$.
4. If $a \in [-\infty, a)$ then $a \cdot \infty = \infty \cdot a = -\infty$.
5. If $a \in (-\infty, \infty) \setminus \{0\}$ then $\frac{\infty}{a} = \frac{1}{a} \cdot \infty$.
6. If $a \in (-\infty, \infty)$ then $\frac{a}{\infty} = \frac{a}{-\infty} = 0$.
7. If $a \in [-\infty, \infty] \setminus \{0\}$ then $\left|\frac{a}{0}\right| = \infty$ (though $\frac{a}{0}$ is left undefined).
8. $|\infty| = |-\infty| = \infty$ and $\infty^p = \infty$, $\infty^{-p} = 0$ for $p > 0$.
9. $0 \cdot \infty = \infty \cdot 0 = 0 \cdot (-\infty) = (-\infty) \cdot 0 \coloneqq 0$.
10. $\frac{\infty}{\infty} = \frac{-\infty}{\infty} = \frac{\infty}{-\infty} = \frac{-\infty}{-\infty} \coloneqq 0$

These last definitions go against our conventional logic involving ∞, but they are simply definitions which will be useful for measure theoretic results later on.

These conventions do have the unfortunate consequence that $\lim\limits_{n\to\infty} \frac{x_n}{y_n} \neq \frac{\lim_{n\to\infty} x_n}{\lim_{n\to\infty} y_n}$ in general for sequences $(x_n), (y_n), \left(\frac{x_n}{y_n}\right)$ in $\overline{\mathbb{R}}$. These facts still hold in sequences which converge in $\mathbb{R}$ (in $\overline{\mathbb{R}}$), it is simply when a sequence converges only in $\overline{\mathbb{R}}$ for which we have issues.

Remark. We left undefined $\infty - \infty$, $-\infty + \infty$, and $\frac{x}{0}$ for $x \in \overline{\mathbb{R}}$. Furthermore, we have $\frac{x}{y} = x \cdot \frac{1}{y}$ only if $x \in \overline{R}$, $y \in \overline{\mathbb{R}} \setminus \{0\}$.

Lecture 9: Extended $\mathbb{R}$ (2) and Intro to Measure Theory

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Definition 1.5. Let $S \subseteq \mathbb{R}$ and $f : S \to \overline{\mathbb{R}}$. Then, we say f is continuous at $x_0 \in S$ if $H \circ f$ is continuous at x_0 on S for any extending function H. Similarity, we say f is continuous on S if $H \circ f$ is continuous on S for any extending function H.
Furthermore, we say f is (strictly) increasing/decreasing/monotone if $H \circ f$ is (strictly) increasing/decreasing/monotone.
Again, if (f_n) is a series of functions $f_n : S \to \overline{\mathbb{R}}$, we say (f_n) converges pointwise/uniformly to $f : S \to \overline{\mathbb{R}}$ if $(H \circ f_n)$ converges pointwise/uniformly to $H \circ f$ for any extending function H.

Definition 1.6. Let $S \subseteq \overline{\mathbb{R}}$ and suppose $a \in \overline{R}$ is an accumulation point of a sequence taking values in $S \setminus \{a\}$.
Then, a function $f : S \setminus \{a\} \to \overline{R}$ is said to have the limit $L \in \overline{\mathbb{R}}$ (relative to S) if for any extending function H and for each $\varepsilon > 0$ we have an $\delta > 0$ such that

$$|H(f(x)) - H(L)| < \varepsilon \text{ for all } x \in S \setminus \{a\} \text{ with } |H(x) - H(a)| < \delta.$$

We denote this by $\lim\limits_{x \to a} f(x) = L$ or $\lim\limits_{\substack{x \to a \\ S}} f(x) = L$

2 Measure Theory

Definition 2.1 (Length)**.** Let $I = (a, b)$ be an interval, then we define the measure function $\ell : \mathscr{P}(\mathbb{R}) \to \mathbb{R}_0^+$ with the following properties:

$$\begin{aligned} \ell(\varnothing) &= 0 \\ \ell(I) &= b - a, a, b \in \mathbb{R}. \end{aligned}$$

In all other cases $\ell(I = \infty)$.

We would like to generalize this notion by constructing a set function λ such that

$$\begin{aligned} &\lambda : \mathscr{P}(\mathbb{R}) \to [0, \infty] \\ &\lambda(I) = \ell(I) \text{ for intervals } I \subseteq \mathbb{R} \\ &\lambda(x + S) = \lambda(S) \text{ for } x \in \mathbb{R}, S \subseteq \mathbb{R}, x + S = \{x + s : s \in S\} \\ &\text{if } \{S_m : m \in \mathbb{N}\} \text{ is a countable disjoint collection of sets in } \mathbb{R}, \text{ then} \\ &\lambda\left(\bigcup_{n=1}^{\infty} S_m\right) = \sum_{n=1}^{\infty} \lambda(S_n) \\ &. \end{aligned}$$

It turns out such a function produces contradictions, hence it is poorly posed. Hence, we must alter or remove one of these constraints and as all of the properties are very straight forward it is best to alter the domain of λ itself.

Definition 2.2 (Measure)**.** Let $\mathscr{A}$ be a σ-algebra.

1. A set function $\mu : \mathscr{A} \to [0, \infty]$ is called **countably additive** if for every countable disjoint collection $\{S_n \in \mathscr{A} : n \in \mathbb{N}\}$ we have
$$\mu\left(\bigcup_{n\in\mathbb{N}} S_n\right) = \sum_{i=1}^{\infty} \mu\left(S_i\right).$$
2. A countable additive set function $\mu : \mathscr{A} \to [0, \infty]$ such that $\mu\left(\varnothing\right) = 0$ is called a **measure**.

Proposition 2.1. Let $\mu : \mathscr{A} \to [0, \infty]$. Then, μ is monotone in the sense that if $A, B \in \mathscr{A}$ with $A \subseteq B$, then we have $\mu\left(A\right) \leq \mu\left(B\right)$.

Proof. Since $B = A \cup (B \setminus A)$ and since μ is countably additive, then
$$\mu\left(B\right) = \mu\left(A\right) + \mu\left(B \setminus A\right) \geq \mu\left(A\right).$$
$\square$

Now, we wish to extend our notion to arbitrary subsets of $\mathbb{R}$. For $A \in \mathscr{P}\left(\mathbb{R}\right)$, then $J\left(A\right)$ is defined to be the collection of all countable covers $\{I_n : n \in \mathbb{N}\}$ of A consisting of open, bounded intervals I_n.

Definition 2.3 (Lebesque Outer Measure)**.** Let $A \in \mathscr{P}\left(\mathbb{R}\right)$, then the quantity $\mu^*\left(A\right) \in [0, \infty]$ is defined by
$$\mu^*\left(A\right) = \inf\{\sum_{i=1}^{\infty} \ell\left(J_i\right) : \{J_i : i \in \mathbb{N}\} \in J\left(A\right)\}.$$
This function $\mu^* : \mathscr{P}\left(\mathbb{R}\right) \to [0, \infty]$ is called the **Lebesque outer measure**.

Lemma 2.1.
1. The outer measure is monotone
2. The outer measure is translation invariant.
3. The outer measure is countable subadditive, that being for $\{S_n : n \in \mathbb{N}\}$ is a countable collection of sets, then $\mu^*\left(\bigcup_{n\in\mathbb{N}} S_n\right) \leq \sum_{n=1}^{\infty} \mu^*\left(S_n\right)$.

Proof.
1. Note that $J\left(A\right) \subseteq J\left(B\right)$, hence $\mu^*\left(A\right) \leq \mu^*\left(B\right)$.
2. Similarly, as each $\ell\left(J_i\right)$ is translationally invariant, we see μ^* is translationally invariant.
3. Let $\varepsilon > 0$. Then for each $n \in \mathbb{N}$, let $\{I_{n,k} : k \in \mathbb{N}\} \in J\left(S_n\right)$ be a collection of intervals such that $\sum_{k=1}^{\infty} \ell\left(J_{n,k}\right) \leq \mu^*\left(S_n\right) + \frac{\varepsilon}{2^n}$.

Since, $\{I_{n,k} : n, k \in \mathbb{N}\} \in J\left(\bigcup_{n\in\mathbb{N}} S_n\right)$, we must have that

$$\mu^*\left(\bigcup_{n\in\mathbb{N}} S_n\right) \leq \sum_{n\in\mathbb{N}}\sum_{k\in\mathbb{N}} \ell\left(I_{n,k}\right) \\ \leq \sum_{n=1}^{\infty} \mu^*\left(S_n\right) + \varepsilon.$$

Since this holds for all $\varepsilon > 0$, this completes the proof.

□

Lemma 2.2. For every interval $I \subseteq \mathbb{R}$, the outer measure is $\mu^*\left(I\right) = \ell\left(I\right)$.

Proof. Let $I \subseteq \mathbb{R}$ be nonempty (if $I = \varnothing$ it is trivial that $\mu^*\left(\varnothing\right) = 0$). First, assume $I = [a, b]$ with $a \leq b \in \mathbb{R}$. Let $\{J_n : n \in \mathbb{N}\} \in J\left(I\right)$, then by Heine-Borel there is a finite subcovering $\{I_n : 1 \leq n \leq N\}$ such that no $I_n = \varnothing$. Note that as we will be taking the infimum, then any infinite collection containing this finite collection will be larger (or equal) hence will not matter in the infimum. Furthermore, we can assume that no interval J_n has $J_n \subseteq J_m$ for some $n \neq m$, and we can assume $I_n = (a_n, b_n)$ to be ordered such that $a_n < a_{n+1}$ for $1 \leq n \leq N-1$. Consequently, $b_n > a_{n+1}$ for $1 \leq n \leq N-1$ as otherwise their would be a gap in the covering, and $b_n > b$, as $b \in I$, and $a_1 < a$ by the same reasoning. Hence, we have an overlapping covering of $[a, b]$ by open bounded intervals (a_n, b_n). Hence,

$$\begin{aligned}
\ell\left(J\right) &= b - a \\
&\leq b_N - a_1 \\
&\leq \sum_{i=1}^{N} \left(b_i - a_i\right) \\
&= \sum_{i=1}^{N} \ell\left(I_i\right) \\
&\leq \sum_{i=1}^{\infty} \ell\left(I_i\right) \\
&\Rightarrow \ell\left(I\right) \leq \mu^*\left(I\right).
\end{aligned}$$

Now, we look to obtain the opposite inequality. Let $\varepsilon > 0$, then $\{(a-\varepsilon, b+\varepsilon)\} \in J\left(I\right)$, hence

$$\begin{aligned}
\mu^*\left(I\right) &\leq b - a + 2\varepsilon \\
&= \ell\left(I\right) + 2\varepsilon
\end{aligned}$$

as ε is arbitrary, we then have

$$\mu^*\left(I\right) \leq \ell\left(I\right).$$

Hence $\mu^*\left(I\right) = \ell\left(I\right)$ for this case.
Now, assume $I \in \{(a, b) : [a, b), (a, b]\}$ is any bounded interval with $a < b$. By

monotonicity, for every $\varepsilon > 0$, we have

$$\begin{aligned}
\ell(I) - 2\varepsilon &= b - a - 2\varepsilon \\
&\leq \ell([a+\varepsilon, b-\varepsilon]) \\
&= \mu^*([a+\varepsilon, b-\varepsilon]) \\
&\leq \mu^*(I) \\
&\leq \mu^*(I) \\
&\leq \mu^*([a,b]) && = b - a \\
&= \ell(I).
\end{aligned}$$

Hence, for every $\varepsilon > 0$, $\ell(I) - 2\varepsilon \leq \mu^*(I) \leq \ell(I)$, hence as ε is arbitrary $\mu^*(I) = \ell(I) = b - a$. This covers all the bounded cases, hence only the unbounded case remains.
If I is unbounded and $a \in I$, then $[a, a+n] \subseteq J$ for all $n \in \mathbb{N}$ or $[a-n, a] \subseteq J$ for all $n \in \mathbb{N}$. In either case, by the monotonicity of the outer measure, $\mu^*(I) \geq n$ for all $n \in \mathbb{N}$, hence $\mu^*(I) = \infty = \ell(I)$. This completes the proof. $\square$

Hence, we have that μ^* conforms to all of our desired properties with the notable exception of countable additivity. This, of course, means μ^* is not in fact a measure, so we will again modify our measure function in order to induce a countably additive measure. This construction will come next lecture and will consist of again restricting the domain to a subset of $\mathscr{P}(\mathbb{R})$, the Lebesque measurable sets, a collection which will be introduced and formalized next lecture.