Combinatorics

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Lecture 13

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I originally missed this lecture, so this is transcribed from a classmates notes.

Lecture 14: Quasi-Random Graphs

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Remark. Random graphs have applications in ramsey theory. For instance, if $p \geq 2$, $q \geq 2$, then there is a ramsey number $R(p,q)$ such that for a graph G of order at least $R(p,q)$, then either G constains a p-clique or an independent set of q vertices.
It is a well known result that $R(3,3) = 6$.
A counterexample to a graph of order 5 is K_5.

We can even obtain an upper bound $R(p,q) \leq \binom{p+q-2}{q-1}$. This is obtained from the trivial fact that $R(p,q) \leq R(p,q-1) + R(p-1,q)$.
Now, we examine the diagonal case, $R(k,k) \leq \binom{2k-2}{k-1} \leq \frac{4^{k-1}}{\sqrt{k}}$. From this we obtain, $\sqrt[k]{R(k,k)} \leq 4$, and a probabalistic argument from erdos yields

$$\sqrt{2} \leq \sqrt[k]{R(k,k)} \leq 4$$

Proposition 0.1. For all k and n with $n \leq \sqrt{2}^k$, there is a graph of order n such that G has no k-clique and no independent set of order k.

Halmos. Fix n vertices, $1, 2, \ldots, n$ and consider all labeled graphs , denoted LG. Now, recall there are $2^{\binom{n}{2}}$ labeled graphs of order n. Next, denote $k_k(G)$ to be the number of k-cliques in a graph G and we see an independent set is simply a clique of $\overline{G}$, so we see we need only consider $k_k(G) + k_k(\overline{G})$, hence the total number of graphs with either a k-clique or k-independent set of order n are $S = \sum_{g \in LG(n)} k_k(G) + k_k(\overline{G}) = 2 \cdot \binom{n}{k} 2^{\binom{n}{2} - \binom{k}{2}}$.
The leading $\binom{n}{k}$ is due to the fact that there are $\binom{n}{k}$ subsets of order k in a

set of order n and the exponent comes from the total amount of possible edges outside of the k-clique.
Now we construct a bipartite graph G with $A = LG(n)$ and B being the set of all possible k-cliques. We see each $a \in A$ is a labeled graph, so it may have differing numbers of k-cliques, each $b \in B$ is a k-clique, and all k-cliques participate in the same number of labeled graphs of order n hence B is regular to A.
Taking our earlier definition of S and manipulating yields

$$S \leq 2^{\binom{n}{2}} \left(\frac{2\binom{n}{k}}{2^{\binom{k}{2}}} \right).$$

Hence,

$$\frac{S}{2^{\binom{n}{2}}} \leq \frac{2\binom{n}{k}}{2^{\binom{k}{2}}} < 1.$$

Assuming $k \geq 3$ and applying definitions yields

$$\begin{aligned} \frac{2\binom{n}{k}}{2^{\binom{k}{2}}} &< \frac{2n^k}{k! 2^{k \frac{k-1}{2}}} \\ \binom{n}{k} &= \frac{n(n-1)\dots(n-k+1)}{k!} \\ &\leq \frac{2 \cdot 2^{\frac{k^2}{2}}}{k! 2^{\left(\frac{k(k-1)}{2}\right)}} \\ \text{taking } k = \sqrt{2}^k \text{ yields } & \frac{2\left(\sqrt{2}\right)^k}{k!}. \end{aligned}$$

Remark. Note that after $\binom{n}{2}$ flips of a fair coin, one obtains a graph in $LG(n)$. Take a subset M of cardinality k in the set of all such graphs and note that there is a $\frac{1}{2^{\binom{k}{2}}}$ probability this will be a k-clique. Hence the total probability summed over all subsets M is $\binom{n}{k}\frac{2}{2^{\binom{k}{2}}}$. Applying the subbadditivity of probability yields that this is strictly less than 1. Hence, there is such a graph not containing a k-clique or independent set of order 20.

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