Analysis I: Homework 8 and 9 Corrections

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Problem (39).

- Assume (f_n) does not converge to f in measure. That is, there is an $\varepsilon > 0$ so that for all $N \in \mathbb{N}$

$$m\left(\{x \in \mathbb{R} : |f_{n_N}(x) - f(x)| > \varepsilon\}\right) > \varepsilon$$

  for some $n_N \geq N$. Denote this set A_N. Then, we see

$$\int |f_{n_N} - f| \geq \int_{A_N} |f_{n_N} - f| \geq \int \varepsilon \chi_{A_N} = \varepsilon m(A_N) \geq \varepsilon^2.$$

  That is, for some $\varepsilon' = \varepsilon^2 > 0$, and all $N \in \mathbb{N}$ we find an $n_N \geq N$, so that $\int |f_n - f| \geq \varepsilon'$, so f_n does not converge to f in mean.

- First, note that if $x = 0$ or 1, then $f_n(x) = x$ for all $n \in \mathbb{N}$. Then, if $x \in (0,1)$, the ratio test proves $\sum_{i=1}^{\infty} nx^n < \infty$, hence $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} nx^n = 0$.
  To see that f_n converges to 0 in measure denote $E_{\varepsilon;n} = \{x \in [0,1] : nx^n < \varepsilon\}$. Then, suppose $c \in E_{\varepsilon;n}$, then either $c = 1$ or $\lim_{n\to\infty} f_n(c) = 0$. We can exclude the first case as this happens only on a set of measure 0. Hence, fixing $\varepsilon > 0$ and assuming $c \in \left[0, 1 - \frac{\varepsilon}{2}\right)$ we see there is a $N \in \mathbb{N}$ so that $f_n(c) < \varepsilon$ for all $n \geq N$. So, we have $m(E_{\varepsilon;n}) \leq m\left(\left[1 - \frac{\varepsilon}{2}, 1\right]\right) < \varepsilon$ for all $n \geq N$, so f_n converges to 0 in measure.

- Finally, to show that f does not converge in measure take $\varepsilon = \frac{1}{100}$. Then, we define $a_n = 1 - \left(\frac{1}{100}\right)^{\frac{1}{n+1}}$ we define $s_n = f_n(a_n)\chi_{[a_n,1]}$. Then, we find f_n dominates s_n for every n, hence

$$\int f_n \geq \int s_n = n\left(\frac{1}{100}^{n+1} - \frac{1}{100}\right) \geq n\left(100^{-2} - \frac{1}{100}\right) = n\delta$$

  for all $n \geq 1$. Since this grows linearly with n, we find for sufficiently large n, $n\delta > \varepsilon$. Hence, it is shown.

Problem (42). 1. First, we prove the case $s < \infty$. Let $f \in L^s(S)$. Then, we define r so that $\frac{1}{s} + \frac{1}{r} = \frac{1}{p}$ (hence $\frac{s}{p}$ and $\frac{r}{p}$ are conjugate). Then, as we aim to show $\|f\|_p$ finite, we see it suffices to show $\|f\|_p^p = \int_S |f|^p = \|f^p\|_1$ finite. We see

$$\begin{aligned}
\|f\|_p^p &= \|1f\|_p^p \\
&= \|1^p f^p\|_1 \\
&\leq \|1\|_{\frac{r}{p}} \|f^p\|_{\frac{s}{p}} \\
&= (\int_S 1^{\frac{r}{p}})^{\frac{p}{r}} \left(\int_S |f^p|^{\frac{s}{p}} \right)^{\frac{p}{s}} \\
&= \|1\|_p^{\frac{1}{r}} \|f\|_s^p \\
&= m(S)^{\frac{1}{r}} \|f\|_s^p \\
&< \infty.
\end{aligned}$$

We find this finite by assumption, hence $f \in L^p(S)$, so the claim is shown. Next, we show the case $s = \infty$. In this case $f \in L^p(S)$ is bounded almost everywhere (else its esssup would be infinite). Then, we see for $p < \infty$ $\int_S |f|^p \leq \int_S \operatorname{esssup}(f)^p = S\|f\|_\infty^p < \infty$ by assumption so the claim holds. It is clear that if $m(S) = \infty$ this does not hold. For an example, sake $S = [0, \infty]$ and $f = \frac{1}{x}$, we see $\|f\|_1 = \int_{[0,\infty]} \frac{1}{x} = \infty$, however $\|f\|_2 = (\int_{[0,\infty]} \frac{1}{x^2})^{\frac{1}{2}}$. As $\frac{1}{x^2}$ is integrable on $[0, \infty]$ we find its root to be finite, hence $f \in L_2([0, \infty])$ but $f \notin L_1([0, \infty])$.

2. Let $f \in L^r(S) \cap L^s(S)$. Denote the following sets, $A = \{x : x \in S, |f(x)| < 1\}$ and $B = \{x : x \in S, |f(x)| > 1\}$. It is clear $A \cup B = S$, with A, B being disjoint. Then, we see if $s \neq \infty$, we have

$$\begin{aligned}
\|f\|_p^p &= \int_S |f|^p \\
&= \int_A |f|^p + \int_B |f|^p \\
&\leq \int_A |f|^r + \int_B |f|^s \\
&\leq \int_S |f|^r + \int_S |f|^s \\
&= \|f\|_r^r + \|f\|_s^s \\
&< \infty.
\end{aligned}$$

In the other case where $s = \infty$ we apply the same logic as in 41, that being $|f| \leq \operatorname{esssup}(f)$ on all but a set of measure 0, hence they may be interchanged in the integral:

$$\begin{aligned}
\|f\|_r^r &= \int_S |f|^r \\
&= \int_S |f|^p \, |f|^{r-p} \\
&\leq \int_S |f|^p \, \underbrace{[\operatorname{esssup}(f)]^{r-p}}_{\text{constant}} \\
&= \|f\|_\infty^{r-p} \|f\|_p^p < \infty \text{ by assumption.}
\end{aligned}$$

Problem (43).

- First, note that $\int_I \cos(nx) = \int_I \cos^+(nx) + \int_I \cos^-(nx)$. Since I is a bounded interval, we see for all but a set of measure 0 on its boundary, if $x \in I$, then there is an $\varepsilon > 0$ so that $(x-\varepsilon, x+\varepsilon) \in I$. Then, $\cos^-(nx) = \cos^+\left(n\left(x + \frac{\pi}{2n}\right)\right)$, so for almost every x, we find there is an $N \in \mathbb{N}$ so that $x + \frac{\pi}{2n} \in I$ for all $n \geq N$. Moreover it is bounded by $g = 1$ everywhere, so DCT proves it integrable. Then,

$$\begin{aligned}
\lim_{n\to\infty} \int_I \cos(nx) &= \lim_{n\to\infty} \int_I \cos^+(nx) - \lim_{n\to\infty} \int_I \cos^+\left(n\left(x + \frac{\pi}{2n}\right)\right) \\
&= \lim_{n\to\infty} \int_I \cos^+(nx) - \int_{I + \frac{\pi}{2n}} \cos^+(nx) \\
&= \lim_{n\to\infty} - \int_{\left(I + \frac{\pi}{2n}\right)\setminus I} \cos^+(nx) \\
&\geq - \int_{\left(I + \frac{\pi}{2n}\right)\setminus I} 1 \\
&= - \lim_{n\to\infty} \frac{\pi}{2n} \\
&= 0.
\end{aligned}$$

The same argument shows $\lim\limits_{n\to\infty} \int_I \cos(nx) \leq 0$ taking $\cos^-$ instead. Hence, $\lim\limits_{n\to\infty} \int_I \cos(nx) = 0$.

- First, note that since $\cos(nx) \in [-1, 1]$ we have $\int |f \cos(nx)| \leq \int |f| < \infty$, hence it is integrable. Then, we see

$$\begin{aligned}
\int f \cos(nx) &= \int_{\mathbb{R}} f \cos^+(nx) - \int_{\mathbb{R}} f \cos^-(nx) \\
&= \int_{\mathbb{R}} f \cos^+(nx) - \int_{\mathbb{R} + \frac{\pi}{2n}} f \cos^+(nx) \\
&= \int_{\mathbb{R}} f \cos^+(nx) - \int_{\mathbb{R}} f \cos^+(nx) \\
&= 0.
\end{aligned}$$