Analysis I: Homework III

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Solution (17). ■

Solution (18)**.** As $[a, b]$ is compact, we see f is uniformly continuous. Hence, there is a $\delta > 0$ such that for all $\varepsilon > 0$ and $x, y \in [a, b]$ we find $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$.
Fix $\varepsilon > 0$, and define the following sequence. Let $y_0 = a$ and $y_i = \max\{a+\delta\cdot i, b\}$ for $i \geq 0$. Then, we see $\{[y_{i-1}, y_i] : i \in \mathbb{N}\}$ is a cover and there is a $n \geq 0$ such that $y_n = b$, hence $y_m = b$ for $m \geq n$ and we see $\{[y_{i-1}, y_i] : 1 \leq i \leq n\}$ is a finite subcover. Define

$$\begin{aligned} g : [a, b] &\longrightarrow \mathbb{R} \\ x &\longmapsto g(x) = \frac{f(y_i) - f(y_{i-1})}{y_i - y_{i-1}} (x - y_i) + f(y_i), \qquad \text{for } x \in [y_{i-1}, y_i]. \end{aligned}$$

We see g is simply the piecewise linear interpolation of f on the y_i's and it is well defined (the endpoints agree for each closed interval). Hence, for all $x \in [a, b]$ there is an $i \geq 1$ such that $x \in [y_{i-1}, y_i] = [y_{i-1}, y_{i-1} + \delta] = [y_i - \delta, y_i]$, hence $|y_{i-1} - x| < \delta$ and $|y_i - x| < \delta$ so we see $|f(y_{i-1}) - f(x)| < \frac{\varepsilon}{3}$ and $|f(y_i) - f(x)| < \frac{\varepsilon}{3}$. Then, either $f(y_{i-1}) \leq g(x) \leq f(y_i)$ or $f(y_i) \leq g(x) \leq f(y_{i-1})$ as g is the linear interpolation between these two points. Then, we see $|f(y_i) - g(x)| \leq |f(y_i) - f(y_{i-1})|$. Hence, we find

$$\begin{aligned} |g(x) - f(x)| &\leq |f(y_i) - g(x)| + |f(x) - f(y_i)| \\ &\leq |f(y_i) - f(y_{i-1})| + \frac{\varepsilon}{3} \\ &\leq |f(y_i) - f(x)| + |f(x) - f(y_{i-1})| + \frac{\varepsilon}{3} \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \varepsilon. \end{aligned}$$

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Solution (19)**.** First, we prove the second inequality.
If one of $\limsup_{n\to\infty} x_n, \limsup_{n\to\infty} y_n = \infty$ (and the other is not $-\infty$), we have

$$\limsup_{n\to\infty} (x_n + y_n) \leq \infty.$$

Hence, we may assume neither limit superior to be ∞. Similarly, if $\liminf_{n\to\infty} x_n = -\infty$ we see $-\infty \leq \limsup_{n\to\infty} (x_n + y_n)$. Hence, we can assume the limit inferior to not take on $-\infty$. Then, we know

$$\inf\{x_n : n \geq K\} + \sup\{y_n : n \geq K\} \leq \sup\{x_n + y_n : n \geq K\} \leq \sup\{x_n : n \geq K\} + \sup\{y_n : n \geq K\}.$$

Hence, we have

$$\begin{aligned}
\lim_{K\to\infty} \sup\{x_n + y_n : n \geq K\} &\leq \lim_{K\to\infty} (\sup\{x_n : n \geq K\} + \sup\{y_n : n \geq K\}) \\
&= \lim_{K\to\infty} \sup\{x_n : n \geq K\} + \lim_{K\to\infty} \sup\{y_n : n \geq K\} \\
&= \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\liminf_{n\to\infty} x_n + \limsup_{n\to\infty} y_n &= \lim_{K\to\infty} (\inf\{x_n : n \geq K\} + \sup\{y_n : n \geq K\}) \\
&\leq \lim_{K\to\infty} (\sup\{x_n + y_n : n \geq K\}) \\
&= \limsup_{n\to\infty} (x_n + y_n).
\end{aligned}$$

Consider the following two sequences

$$\begin{aligned}
x_n &= \begin{cases} 1, & n \equiv 1 \pmod 2 \\ -1, & n \equiv 0 \pmod 2 \end{cases} \\
y_n &= \begin{cases} -1, & n \equiv 1 \pmod 2 \\ 1, & n \equiv 0 \pmod 2 \end{cases}
\end{aligned}$$

.

Obviously $\sup\{x_n : n \geq K\} = \sup\{y_n : n \geq K\} = 1$ for all K. On the other hand, we see $x_n + y_n = 0$ for every $n \in \mathbb{N}$, hence $\sup\{x_n + y_n : n \geq K\} = 0$ for all K. As these values hold for all K, we see the limit has no effect hence

$$\begin{aligned}
\limsup_{n\to\infty} x_n + y_n &= \lim_{k\to\infty} (\sup\{x_n + y_n : n \geq K\}) \\
&= \lim_{K\to\infty} 0 \\
&= 0 \\
&< 1 \\
&= \lim_{K\to\infty} (\sup\{x_n : n \geq K\} + \sup\{y_n : n \geq K\}) \\
&= \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n
\end{aligned}$$

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Similairly, define x_n to be the same and $y_n = 0$ for all n. Hence, $\sup\{y_n : n \geq k\} = 0$ and $\inf\{x_n : n \geq K\} = -1$ for all K with $\sup\{x_n + y_n : n \geq K\} = 1$ for all K. Hence as these hold for all K, we find

$$\begin{aligned}
\liminf_{n\to\infty} x_n + \limsup_{n\to\infty} y_n &= -1 + 0 \\
&= -1 \\
&\leq \limsup_{n\to\infty} (x_n + y_n) \\
&= 1
\end{aligned}$$

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Solution (20).

- If $x \in A \triangle B$, then WLOG let $x \in A \setminus B$. So, $x \in A \subseteq C$ and $a \notin B$. Hence, $x \in C \setminus B$ and $x \notin C \setminus A$, so $x \in (C \setminus B) \setminus (C \setminus A)$. So, $x \in (C \setminus A) \triangle (C \setminus B)$.
- If $x \in (C \setminus A) \triangle (C \setminus B)$, then WLOG let $x \in ((C \setminus A) \setminus (C \setminus B)$. Then, note if $x \in C$ and $x \notin C \setminus B$, then $x \in B$. So, we see

$$\begin{aligned} x \in \{x \in C \setminus A : x \notin C \setminus B\} &= \{x \in C : x \notin A, x \notin C \setminus B\} \\ &= \{x \in C : x \notin A, x \in B\} \\ &= \{x \in C : x \in B \setminus A\} \\ &= C \cap (B \setminus A) \\ &= B \setminus A \end{aligned}$$
.

  Hence, $x \in B \setminus A$, so $x \in A \triangle B$.

If $C = \mathbb{R}$, we see $A \triangle B = (\mathbb{R} \setminus A) \triangle (R \setminus B) = A^c \triangle B^c$. ∎

Solution (21)**.** As S is measurable and finite, there is an open O of finite measure such that $S \subseteq O$ and for all $\varepsilon > 0$, we find $m(O \setminus S) < \frac{\varepsilon}{4}$. As O is the countable disjoint union of intervals $\{I_j : j \in \mathbb{N}\}$, we see $m(O) = \sum_{i=1}^{\infty} m(I_j)$, by countable additivity. As this series is finite we see for all $\varepsilon > 0$, there is a K such that

$$\left| \sum_{j=1}^{\infty} m(I_j) - \sum_{k=1}^{K} m(I_k) \right| = \left| m(O) - \sum_{k=1}^{K} m(I_k) \right| < \frac{\varepsilon}{4}.$$

Denote $U = \bigcup_{i=1}^{K} I_j$. Clearly, U is measurable and of finite measure and

$$|m(O) - m(U)| = m(O \setminus U) < \frac{\varepsilon}{4}.$$

Hence as $U, S \subseteq O$, we find

$$S \triangle U = (O \setminus S) \triangle (O \setminus U).$$

So, as $(O \setminus S) \setminus (O \setminus U)$ is disjoint from $(O \setminus U) \setminus (O \setminus S)$ and all measures are finite, we see

$$\begin{aligned} m(S \triangle U) &= m((O \setminus S) \setminus (O \setminus U)) + |m((O \setminus U) \setminus (O \setminus S))| \\ &\leq |m(O \setminus S) - m(O \setminus U)| + |m(O \setminus U) - m(O \setminus S)| \\ &\leq 2m(O \setminus S) + 2m(O \setminus U) \\ &< \frac{2\varepsilon}{4} + \frac{2\varepsilon}{4} = \varepsilon \end{aligned}$$

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Solution (22). 1. It suffices to assume $m(S) < \infty$, because for all sets of infinite measure, we can choose a subset of finite measure $S' \subseteq S$ and $S \cap (a,b) \supseteq S' \cap (a,b)$, so $m(S \cap (a,b)) \geq m(S' \cap (a,b))$.

Then assuming $m(S)$ finite, for $\varepsilon = \frac{1}{3}m(S)$, we find an open U with $S \subseteq U$ and $m(U \setminus S) < \varepsilon = \frac{1}{3}m(S)$. Hence, $m(U) < \frac{4}{3}m(S)$. As U is open it is the countable union of disjoint intervals (a_i, b_i) and $m(U) = \sum_{i=1}^{\infty}(b_i - a_i) < \frac{4}{3}m(S)$. Hence,

$$\sum_{i=1}^{\infty} \frac{3}{4}(b_i - a_i) < m(S).$$

Suppose $m(S \cap (a_i, b_i)) \leq \frac{3}{4}(b_i - a_i)$ for all the intervals (a_i, b_i) . Then,

$$\begin{aligned} m(S) &= \sum_{i=1}^{\infty} m(S \cap (a_i, b_i)) \\ &\leq \sum_{i=1}^{\infty} \frac{3}{4} m(a_i, b_i) \\ &= \sum_{i=1}^{\infty} \frac{3}{4}(b_i - a_i) \\ &< m(S) \lightning. \end{aligned}$$

Hence, we have atleast one (a_i, b_i) such that $m(S \cap (a_i, b_i)) > \frac{3}{4}(b_i - a_i)$.

2. First, note that $S \cap (r+S) = \{s - r \in S : s \in S\}$, and suppose $S \cap (r+S) \cap (a,b) = \varnothing$.
   That is, for all $s \in S \cap (a,b)$, we have $s + r \notin S \cap (a,b) \subseteq (a,b)$. Hence, $s \in (b-r, b) \subseteq \left(b - \frac{1}{4}(b-a), b\right) = \left(\frac{1}{4}a + \frac{3}{4}b, b\right)$. But, we see $m\left(\left(\frac{1}{4}a + \frac{3}{4}b, b\right)\right) = \frac{1}{4}(b-a) < \frac{3}{4}(b-a)$.
   So, we have $S \cap (a,b) \subseteq \left(\frac{1}{4}a + \frac{3}{4}b, b\right)$, $\lightning$.
   Hence there is a $s \in S \cap \left(a, a + \frac{3}{4}(b-a)\right)$, so $s + r \in (a,b)$, so

   $$S \cap (r+S) \cap (a,b) \neq \varnothing.$$

   For each $x \in \left[-\frac{1}{4}(b-a), \frac{1}{4}(b-a)\right]$, note that we have some $s \in S$ such that $s+x \in S$ or $s-x \in S$ since $S \cap (r+S)$ is nonempty, $0 \leq r \leq \frac{1}{4}(b-a)$. Denote $s+x = \overline{s}$ and $s-x = \hat{s}$. If $\overline{s} \in S$, then $\overline{s} - s = x \in S - S$. Otherwise, if $\hat{s} \in S$, then $s - \hat{s} = x \in S - S$. Hence, $\left[-\frac{1}{4}(b-a), \frac{1}{4}(b-a)\right] \subseteq S - S$.

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Solution (23). ■

Solution (24). ■

Solution (25). ■

Solution (26)**.** Let $S_i = (i, \infty)$ for each $i \in \mathbb{N}$. Clearly, each s_i is measurable and $\bigcap_{n\in\mathbb{N}} S_n = \varnothing$. However, $m(S_i) = \infty - i = \infty$ for all i, so we find $m\left(\bigcap_{n\in\mathbb{N}} S_n\right) = 0 \neq \infty = \lim\limits_{n\to\infty} m(S_n)$.
For the second claim consider $M = \mathbb{Q}$. Then, recall by the density of $\mathbb{Q}$ in $\mathbb{R}$, we have that for all $r \in \mathbb{R}$ and some fixed $\varepsilon = \frac{1}{n} > 0$, we have a $x \in Q$ so that $x - \frac{1}{n} < r \leq x < x + \frac{1}{n}$, hence $r \in \bigcup_{x\in\mathbb{Q}} \left(x - \frac{1}{n}, x + \frac{1}{n}\right)$, so this union is simply $\mathbb{R}$. Then, we have

$$\begin{aligned}\bigcap_{n\in\mathbb{N}} \bigcup_{x\in\mathbb{Q}} \left(x - \frac{1}{n}, x + \frac{1}{n}\right) &= \bigcap_{n\in\mathbb{N}} \mathbb{R} \\ &= \mathbb{R} \\ &\neq \mathbb{Q}.\end{aligned}$$

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Solution (27).

- Consider the following construction. Let $A_i = (a_i, b_i)$ and $C_i = (c_i, d_i)$, where $(a_1, b_1) = (0, 1)$, $(c_1, d_1) = (1, 2)$ and $a_i = d_{i-1}$, $b_i = a_i + \frac{1}{i^2}$, $c_i = b_i$, $d_i = c_i + \frac{1}{i}$. Define $A = \bigcup_{i \in \mathbb{N}} A_i$ and $C = \bigcup_{i \in \mathbb{N}} C_i$. Then, note all A_i are disjoint and all C_i are disjoint. Then, we see $m(C) = \sum_{i=1}^{\infty} m(C_i) = \sum_{i=1}^{\infty} \frac{1}{i} = \infty$ and $m(A) = \sum_{i=1}^{\infty} m(A_i) = \sum_{i=1}^{\infty} \frac{1}{n^2} < \infty$, hence A is of finite measure. However, we have $a_i = d_{i-1} = c_{i-1} + \frac{1}{i}$ and $c_i = b_i = a_{i-1} + \frac{1}{i^2}$. Hence, $a_i = a_{i-1} + \frac{1}{i^2} + \frac{1}{i} = \sum_{j=1}^{i} \frac{1}{j} + \frac{1}{j^2} + a_1$, so for any bounded interval $I \subseteq [-M, M]$ and bound M, we see there is a n such that $\sum_{i=1}^{n} \frac{1}{i} > M$, hence $a_n = \sum_{i=1}^{n} \frac{1}{i^2} + \frac{1}{i} + a_1 > \sum_{i=1}^{n} \frac{1}{i} > M$, so $A \nsubseteq I$.

- Recall for a measurable E there is a finite collection of open intervals $\{I_k : 1 \leq k \leq K\}$ such that for $\varepsilon > 0$, and $U = \bigcup_{k=1}^{K} I_k$ $m(E \triangle U) < \varepsilon$. Moreover, every I_k is bounded, as if one was of the form (a, ∞) or $(-\infty, a)$ we would find $m(U) = \infty$ and $m(E \triangle U) = m(E \setminus U) + m(U \setminus E) = m(E \setminus U) + \infty = \infty$ as $m(E \setminus U)$ is finite by assumption. Hence, a finite union of bounded intervals is bounded, so U is a bounded set with $m(E \triangle U) < \varepsilon$, but $m(E \triangle U) = m(E \setminus U) + m(U \setminus E) \geq m(E \setminus U)$. Hence, $m(E \setminus U) < \varepsilon$.

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Solution (28)**.** As $\inf\{|x-y| : x \in A, y \in B\} = p > 0$, we see $A \cap B = \varnothing$. Then, we see for any coverings $\{A_k\}$, $\{B_k\}$ of A and B by open disjoint intervals, with $A_k \cap B_j \neq \varnothing$, for some $k, j \in \mathbb{N}$, we find $m^*\left(A_k \setminus B_j\right) = m^*\left(A_k \cap B_j^c\right) \leq m^*\left(A_k\right)$. Moreover, denoting $A_k \setminus B_j = C_{k,j}$, we see each $C_{k,j}$ is either empty, an interval, or the union of 2 intervals (depending on how A_k and B_j intersect). If $C_{k,j}$ is an interval or empty denote $C_{k,j,1} = C_{k,j}$, otherwise if $C_{k,j}$ is the union of two intervals denote the one with a smaller left endpoint $C_{k,j,1}$ and the one with larger left endpoint $C_{k,j,2}$. Similairly, we see $m^*\left(B_j \setminus A_k\right) \leq m^*\left(B_j\right)$ and we can denote new intervals $D_{k,j,1} = B_j \setminus A_k$ if the difference is a single interval or $D_{k,j,1}, D_{k,j,2}$ if the difference is the union of two intervals. All together, this yields two new coverings $\{C_{k,j,n} : k, j \in \mathbb{N}, n \in \{1,2\}\}$ and $\{D_{k,j,n} : k, j \in \mathbb{N}, n \in \{1,2\}\}$. Now, as these collections are still countable we may reindex them to yield equivalent covers $\{\overline{A}_k : k \in \mathbb{N}\}$, $\{\overline{B}_k : k \in \mathbb{N}\}$ of A and B respectively with $\sum_{i=1}^{\infty} m^*(\overline{A_k}) \leq \sum_{i=1}^{\infty} m^*(A_k)$ and $\sum_{i=1}^{\infty} m^*\left(\overline{B}_k\right) \leq \sum_{i=1}^{\infty} m^*\left(B_k\right)$. Hence, every pair of coverings with intersection admits a disjoint pair of smaller cumulative measure, so we can assume all pairs of coverings are disjoint when passing to the infimum .
Now, we see

$$\begin{aligned}
m^*\left(A \cup B\right) &= \inf\{\sum_{k=1}^{\infty} \ell\left(J_k\right) : \{J_k : k \in \mathbb{N}\} \in J\left(A \cup B\right)\} \\
&= \inf\{\sum_{k=1}^{\infty} \ell\left(A_k \cup B_k\right) : \{A_k : k \in \mathbb{N}\} \in J\left(A\right), \{B_k : k \in \mathbb{N}\} \in J\left(B\right), A_k \cap B_j = \varnothing, j, k \in \mathbb{N}\} \\
&= \inf\{\sum_{k=1}^{\infty} \ell\left(A_k\right) + \ell\left(B_k\right) : \{A_k : k \in \mathbb{N}\} \in J\left(A\right), \{B_k : k \in \mathbb{N}\} \in J(B), A_k \cap B_j = \varnothing, k, j \in \mathbb{N}\} \\
&\geq \inf\{\sum_{k=1}^{\infty} \ell\left(A_k\right) : \{A_k : k \in \mathbb{N}\} \in J\left(A\right)\} + \inf\{\sum_{k=1}^{\infty} \ell\left(B_k\right) : \{B_k : k \in \mathbb{N}\} \in J\left(B\right)\} \\
&= m^*\left(A\right) + m^*\left(B\right).
\end{aligned}$$

But, applying subadditivity implies $m^*\left(A \cup B\right) \leq m^*\left(A\right) + m^*\left(B\right)$ for A, B disjoint. Hence $m^*\left(A \cup B\right) = m^*\left(A\right) + m^*\left(B\right)$. ■

Solution (29). ■

Solution (30)**.** Let

$$\begin{aligned} f:[0,1] &\longrightarrow \mathbb{R} \\ x &\longmapsto f(x) = \left\{ \begin{array}{cc} x, & x \in C \\ x-2, & x \notin C \end{array} \right. . \end{aligned}$$

Where $C \subseteq \mathbb{R}$ is a nonmeasurable set. We see f is injective, so $f^{-1}(\{c\}) = \{\hat{c}\}$ for some $\hat{c} \in [-2,1]$, hence as all singeltons are measurable, we see all singleton preimages are measurable. However, $f^{-1}([0,\infty]) = C$ and C is not measurable, so f is not measurable. ■