# Analysis I

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## Contents



## Lecture 19: End of Convergence, Functions of Bounded Variation, and Derivatives

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Recall we had the dominated convergence theorem. A similair version of the theorem makes use of convergence in measure as follows.

**Theorem 0.1** (Dominated Convergence - Convergence in Measure)**.** Let $(f_n)$ be a sequence of measurable functions $f_n : \mathbb{R} \to \overline{\mathbb{R}}$ and suppose there is an integrable function $g : \mathbb{R} \to \overline{\mathbb{R}}$ so that $|f_n| \leq g$ for all $n \in \mathbb{N}$. If $(f_n) \to f : \mathbb{R} \to \overline{\mathbb{R}}$ in measure, (with $f$ measurable), then $f$ is integrable and $\lim\limits_{n\to\infty} \int |f_n - f| = 0$ and $\lim\limits_{n\to\infty} \int f_n = f$.

*Proof.* First, note a subsequence of $(f_n)$ converges to $f$ pointwise almost everywhere. Hence, we find $|f| \leq g$ almost everywhere, so $f$ is integrable. We cam assume $|f_n - f| \leq 2g$ (almost) everywhere. Then, we find a subsequence $(g_n) = (f_{n_k})$ such that $\limsup_{n\to\infty} |f_n - f| = \lim\limits_{n\to\infty} |g_k - f|$. Then, as $(g_k) \to f$ in measure, we find another subsequence $(h_j) = (g_{k_j}) = \left(f_{n_{k_j}}\right)$ which converges pointwise to $f$ almost everywhere.
Applying dominated convergence theorem yields

$$\lim_{n\to\infty} \int |h_j - f| = 0.$$

Then, we find

$$\begin{aligned}\limsup_{n\to\infty} \int |f_n - f| &= \lim_{n\to\infty} \int |g_k - f| \\ &= \lim_{n\to\infty} |h_j - f| \\ &= 0.\end{aligned}$$

This completes the proof. □

# 1 Functions of Bounded Variation and Absolutely Continuous Functions

**Remark.** For this chapter $[a, b] \subseteq R$ will always denote a compact interval on $\mathbb{R}$.

**Definition 1.1** (Partition)**.** A finite sequence $P = (x_k)_{k=n}^{N}$ with $n, N \in \mathbb{Z}$ and $n < N$ is called a **partition** of $[a, b]$ if $x_n = a$, $x_N = b$ and $x_{k-1} \leq x_k$ for $n < k \leq N$. We denote the collection of all partitions of $[a, b]$ to be $\mathscr{P}([a, b])$.

**Definition 1.2.** Let $f : [a, b] \to \mathbb{R}$ be a function. Then,

- For a partition $P = (x_k)_{k=n}^{N}$, we denote
$$V(f, P) = \sum_{k=n+1}^{N} \left| f(x_k) - f_{(x_{k-1})} \right|$$
to be the **variation of $f$ with respect to** $P$.
- We define the quantity $\mathrm{TV}(f) = \sup\{V(f, P) : P \in \mathscr{P}([a, b])\}$ to be the **total variation of $f$** .

**Remark.** If $f : [a, b] \to \mathbb{R}$ and $c \in [a, b]$ with partitions $P_1 = (x_k)_{k=n}^{N}$ of $[a, c]$ and $P_2 = (x_k)_{k=N}^{K}$ of $[c, b]$ . Then denote, $P = (x_k)_{k=n}^{K}$ to be a partition of $[a, b]$ and we find
$$V(f, P) = V\left(f\mid_{[a,c],P_1}\right) + V\left(f\mid_{[c,b]}, P_2\right).$$

Moreover,
$$\mathrm{TV}(f) = \mathrm{TV}\left(f\mid_{[a,c]}\right) + \mathrm{TV}\left(f\mid_{[c,b]}\right).$$

**Definition 1.3** (Bounded Variation)**.** A function $f : \mathbb{R} \to \overline{\mathbb{R}}$ has **bounded variation** if $\mathrm{TV}(f) < \infty$.

**Theorem 1.1** (Jordan's Theorem)**.** A function $f : [a, b] \to \mathbb{R}$ is of bounded variation if and only if there are increasing functions $g, h : [a, b] \to \mathbb{R}$ so that $f = g - h$.

*Proof.* Suppose $\mathrm{TV}(f) < \infty$ and let $x, y \in [a,b]$ with $x < y$. Then, we find

$$\begin{aligned}\mathrm{TV}\left(f\mid_{[a,y]}\right) &= \mathrm{TV}\left(f\mid_{[a,x]}\right) + \mathrm{TV}\left(f\mid_{[x,y]}\right)\\ &\geq \mathrm{TV}\left(f\mid_{[a,x]}\right) + |f(y) - f(x)|\\ &\geq \mathrm{TV}\left(f\mid_{[a,x]}\right) + f(x) - f(y).\end{aligned}$$

Furtheromre, $h : x \mapsto \mathrm{TV}\left(f\mid_{[a,x]}\right)$ and $g : x \mapsto \mathrm{TV}\left(f\mid_{[a,x]}\right) + f(x)$ are increasing. This fact is trivial for $h$ and we find , adding $f(y)$ to both sides of the former inequality yields $g(y) \geq g(x)$ for arbitrary $y \geq x$, so this claim holds as well.
Taking the difference, $g - h = f$.
Conversely, suppose $f = g - h$ for increasing $g, h : [a,b] \to \mathbb{R}$. Then, let $x, y \in [a,b]$ with $y \geq x$. Then, we find

$$\begin{aligned}|f(y) - f(x)| &= |g(y) - g(x) + h(x) - h(y)|\\ &\leq |g(y) - g(x)| + |h(x) - h(y)|\\ &= g(y) - g(x) + h(y) - h(x).\end{aligned}$$

Hence, for a partition $P = (x_k)_{k=n}^{N}$, we find

$$\begin{aligned}V(f,P) &= \sum_{k=n+1}^{N} |f(x_k) - f(x_{k-1})|\\ &\leq \sum_{k=n+1}^{N} \left(g(x_k) - g(x_{k-1}) + h(x_k) - h(x_{k-1})\right) \quad = g(b) - g(a) + h(b) - h(a)\\ &< \infty.\end{aligned}$$

□

**Definition 1.4** (Absolute Continuity)**.** A function $f : [a,b] \to \mathbb{R}$ is **absolutely continuous** if for each $\varepsilon > 0$ we find a $\delta > 0$ such that for every finite disjoint collection of nonempty intervals $\{(a_k, b_k) \subseteq [a,b] : 1 \leq k \leq K\}$ with $\sum_{k=1}^{K}(b_k - a_k) < \delta$, we have $\sum_{k=1}^{K} |f(a_k) - f(b_k)| < \varepsilon$.

**Remark.** Absolute continuity is stronger than uniform continuity, but weaker than lipschitz continuity.

**Theorem 1.2.** If a function $f : [a,b] \to \mathbb{R} \to$ is absolutely continuous, then $f$ is continuous and $f$ has bounded variation.

*Proof.* $f$ is trivially continuous, taking a finite disjoint collection consisting only of 1 interval $\{(x,y)\}$ yields the definition of continuity.
Now we show bounded variation. For $\varepsilon = 1$, let $\delta > 0$ be the number such that the definition of absolute continuity holds for $f$.
Now fix $(x_k)_{k=n}^{N} \in \mathscr{P}([a,b])$ so that $x_k - x_{k-1} < \delta$ for all $n < k \leq N$. Then, if $P \in \mathscr{P}([x_{k-1}, x_k])$, we see $V\left(f\mid_{[x_{k-1},x_k]}, P\right) < 1$ by definition of absolute

continuity.
So, we have $\text{TV}([x_{k-1}, x_k]) \leq 1$, so $\text{TV}(f) = \sum_{k=n+1}^{N} \text{TV}\left(f \mid_{[x_{k-1}, x_k]}\right) \leq N - n$ by the $\varepsilon$ assumption. $\square$

As it turns out, absolutely continuous functions have a relation to integrable functions, particularly, an integrable function $f$ is simply the anti-integral of an absolutely continuous one.

**Proposition 1.1.** If $f : [a, b] \to \overline{\mathbb{R}}$ is integrable, then,

$$F : [a, b] \to \mathbb{R}, \; x \mapsto \int_{[a,x]} f$$

is absolutely continuous.

This claim can be generalized into a sort of fundamental theorem of calculus for the lebesque integrals to characterize integrals and derivatives. For now, we only prove the weak version.

*Proof.* For $\varepsilon > 0$ there is a $\delta > 0$ such that $\int_S |f| < \varepsilon$ for every measurable set $S \subseteq [a, b]$ with $m(S) < \delta$.
Now, let $\{(a_k, b_k) : 1 \leq k \leq K\}$ be a disjoint collection of intervals in $[a, b]$ with $\sum_{k=1}^{K} (b_k - a_k) < \delta$. Fix $S = \bigcup_{k=1}^{K} (a_k, b_k)$. Then, since $m(S) < \delta$ and

$$\begin{aligned}
\sum_{k=1}^{K} |F(b_k) - F(a_k)| &= \sum_{k=1}^{K} \left| \int_{[a_k, b_k]} f \right| \\
&\leq \sum_{k=1}^{K} \int_{[a_k, b_k]} |f| \\
&= \int_S |f| \\
&< \varepsilon \text{ by assumption.}
\end{aligned}$$

Hence, absolute continuity holds. $\square$

# 2 Derivatives and Fundamental Theorem of Calculus

**Proposition 2.1.** Let $f : (a, b) \to \overline{\mathbb{R}}$ be monotone on $(a, b) \subseteq \mathbb{R}$ with $a, b \in \overline{\mathbb{R}}$ and $a < b$. Then,

$$\lim_{x \to a} f(x) = \inf\{f(x) : x \in (a, b)\}, \lim_{x \to b} f(x) = \sup\{f(x) : x \in (a, b)\}$$

are both well defined.

## Lecture 20: Derivatives (2)

Tue 02 Nov 2021 12:58

**Recall.** A monotone function on an interval has well defined limits at both its endpoints.

**Definition 2.1** (Upper/Lower Derivatives)**.** Let $S \subseteq \mathbb{R}$, $f : S \to \mathbb{R}$

- We define $\overline{D}f(x) = \lim_{\tau \to 0} \sup\{\frac{f(x+h)-f(x)}{h} : 0 < |h| < \tau\}$ to be the **upper derivative**.
- We define $\underline{D}f(x) = \lim_{\tau \to 0} \inf\{\frac{f(x+h)-f(x)}{h} : 0 < |h| < \tau\}$ to be the **lower derivative**.
- If, for some $x \in \overset{\circ}{S}$, we find $\overline{D}f(x), \underline{D}f(x) \in \mathbb{R}$, with the upper and lower derivatives being equal, we say $f$ is **differentiable** at $x$.
  We denote $f'(x) = \overline{D}f(x) = \underline{D}f(x)$.

We know, the limits of the upper and lower derivatives to be well defined as the supremum and infimum are monotone functions with respect to $\tau$.

**Proposition 2.2.** Let $f : S \to \mathbb{R}$ and let $x \in \overset{\circ}{S}$. Then, $f$ is differentiable at $x$ if and only if

$$\lim_{y \to x} \frac{f(y) - f(x)}{y - x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \in \mathbb{R}.$$

That is, the classical derivative is equivalent to the lebesque derivative, so we will use the new definition for most proofs, but the old for most computations.

**Theorem 2.1** (Mean-Value Theorem)**.** Let $f : [a, b] \to \mathbb{R}$ be continuous and differentiable at every $x \in (a, b)$ . Then, there exists $\xi \in (a, b)$ so that $f(b) - f(a) = f'(\xi)(b - a)$.

**Lemma 2.1.** Let $f : [a, b] \to \mathbb{R}$ be increasing and suppose $\overline{D}f(x) = \underline{D}f(x)$ for almost every $x \in [a, b]$. Then, $\overline{D}f(x)$ and $\underline{D}f(x)$ are finite almost everywhere. Moreover, $f$ is differentiable almost everywhere (on $[a, b]$). Furthermore, $f'$ is an integrable function and

$$\int_{[a,b]} f' \leq f(b) - f(a).$$

*Proof.* Extend $f$ to $[a, \infty)$ by letting $f(c) = f(b)$ for all $c \geq b$. Define a sequence $(g_n)$, $g_n : [a, b] \to \overline{\mathbb{R}}$ with

$$x \mapsto n\left(f\left(x + \frac{1}{n}\right) - f(x)\right).$$

Then, b assumption, we know $(g_n(x))$ to be convergent in $\overline{\mathbb{R}}$ with limit $f'(x)$ for almost every $x \in (a, b)$. Each $g_n$ is measurable, hence $\lim_{n\to\infty} g_n$ is increasing,

we see $g(n) \geq 0$, hence $\overline{D}f \geq 0$.
Applying Fatou's lemma yields

$$\begin{aligned}
\int_{[a,b]} \overline{D}f &= \int_{[a,b]} \liminf_{n\to\infty} f_n \\
&\leq \liminf_{n\to\infty} \int_{[a,b]} g_n \\
&= \liminf_{n\to\infty} n \left( \int_{\left[a+\frac{1}{n}, b+\frac{1}{n}\right]} f - \int_{[a,b]} f \right) \\
&= \liminf_{n\to\infty} \left( \underbrace{n \int_{\left[b, b+\frac{1}{n}\right]} f}_{=f(b)} - \underbrace{n \int_{\left[a, a+\frac{1}{n}\right]} f}_{\leq f(a)} \right) \\
&\leq f(b) - f(a).
\end{aligned}$$

We know the final inequality holds because $f$ is constant on $\left[b, b+\frac{1}{n}\right]$ and though $f$ is not constant, it is increasing on $\left[a, a+\frac{1}{n}\right]$ hence the upper bound of their difference is attained by $f(a)$.
Consequently, $\overline{D}f$ is integrable (so finite almost everywhere). And, since $\overline{D}f = \underline{D}f$ , we find $f'(x)$ exists and equals $\overline{D}f(x)$ for almost every $x \in [a,b]$. ☐

Later, we will prove equality holds precisely in the case of absolute continuity.

**Definition 2.2** (Vitali Covering)**.** Let $S \subseteq \mathbb{R}$. We call a collection of closed, bounded intervals (denoted $\mathscr{C}$) of positive length a **Vitali covering** of $S \subseteq \mathbb{R}$ if for every $x \in S$ and $\varepsilon > 0$ we find an $I \in \mathscr{C}$ such that $x \in I$ and $l(I) < \varepsilon$.

**Example.** A vitali covering of $S = [0,1]$ goes as follows. Let $H = \mathbb{Q} \cap [0,1]$, then $\mathscr{C} = \{[x, x+h] : h \in H, x \in [0,1]\}$. ◇

**Theorem 2.2** (Vitali Covering Lemma)**.** Let $\mathscr{C}$ be a Vitali covering of the set $S \subseteq \mathbb{R}$ with $m^*(S) < \infty$. Then, for every $\varepsilon > 0$ there is a finite, disjoint collection of intervals $\{I_k \in \mathscr{C} : 1 \leq k \leq n\}$ such that

$$m^* \left( S \setminus \bigcup_{k=1}^{n} I_k \right) < \varepsilon.$$

**Theorem 2.3** (Lebesque's Theorem)**.** *Let $f : I \to \mathbb{R}$ be a monotone function on an interval $I \subseteq \mathbb{R}$. Then, $f$ is differentiable at almost every $x \in I$ and $f'$ is integrable on every interval $[a, b] \subseteq I$. In particular, if $f$ is increasing, then*

$$\int_{[a,b]} f' \leq f(b) - f(a).$$

*Proof.* It suffices to show $I$ is open and bounded, else we could replace $I$ by $\overset{\circ}{I} \cap (-n, n)$ for $n \in N$ and we find $\overset{\circ}{I} = \bigcup_{n \in \mathbb{N}} \overset{\circ}{I} \cap (-n, n)$ . Similarly, we can assume $f$ to be increasing. Hence, for all $x \in I$, we have $0 \leq \underline{D} f(x) \leq \overline{D} f(x) \leq \infty$. So, we need only show $\overline{D} f(x) = \underline{D} f(x)$ with this quantity being finite for almost every $x \in I$.
For $p, q \in \mathbb{Q}$ and $p > q > 0$, define $E_{p,q} = \{x \in I : \underline{D} f(x) < q < p < \overline{D} f(x) < \infty\}$. Then,

$$\{x \in I : \underline{D} f(x) < \overline{D} f(x) < \infty\} = \bigcup_{p,q \in Q^+} E_{p,q}.$$

If $f$ fails to be differentiable at $x \in I$, then either $x \in E_{p,q}$ for some $p, q \in \mathbb{Q}$ or $\overline{D} f(x) = \infty$. We know $\overline{D} f$ to be finite almost everywhere, so by subadditivity, we need only show the other component, $E_{p,q}$, has measure 0.
Fix $p, q \in \mathbb{Q}$ and suppose $m^* (E_{p,q} = m_0)$. Then, $m_0 \in [0, \infty)$ by the boundedness assumption. Given $\varepsilon > 0$ there is a nonempty open $U$ such that $E_{p,q} \subseteq U$ and $m(U) < m_0 + \varepsilon$. Suppose $x \in E_{p,q}$ . Since $\underline{D} f(x) < q$ by definition of $E_{p,q}$; for every $\delta > 0$ we find a $0 < h < \delta$ such that $[x, x+h] \subseteq U$ and $f(x+h) - f(x) < qh$ or $[x-h, x] \subseteq U$ and $f(x) - f(x-h) \leq qh$.
The collection $\mathscr{L}$ of all such intervals $[x, x+h]$ or $[x-h, x]$ for a fixed $\delta > 0$ and $x \in E_{p,q}$ forms a Vitali covering of $E_{p,q}$. We find all intervals $[a, b] \in \mathscr{L}$ have the property $f(b) - f(a) < q(b-a)$ by the earlier observation. Then, by the Vitali covering lemma, there is a finite, disjoint collection of intervals $\{I_n \in \mathscr{L} : 1 \leq n \leq N\}$ such that for $V = \bigcup_{n=1}^{N} I_n$, we have $m^* (E_{p,q} \setminus V) < \varepsilon$. Note that $m(V) < m_0 + \varepsilon$ since $V \subseteq U$. Since $m^* (E_{p,q} \setminus V) + m^* (E_{p,q} \cap V) \geq m_0$ since the two sets together contain $E_{p,q}$ , we have $m^* (E_{p,q} \cap V) \geq m_0 - \varepsilon$.
Now, we follow a similair construction. If $x \in E_{p,q} \cap V$ , then $p < \overline{D} f(x)$ implies for all $\delta > 0$ there is an $0 < h < \delta$ such that $[x, x+h] \subseteq V$ and $f(x+h) - f(x) \geq ph$ or $[x-h, x] \subseteq V$ and $f(x) - f(x-h) \geq ph$ . The collection $\mathscr{U}$ of all such intervals $[x, x+h]$ or $[x-h, x]$ for a fixed $\delta > 0$ and $x \in E_{p,q} \cap V$ is a vitali covering of $E_{p,q} \cap V$. Moreover, if $[c, d] \in \mathscr{U}$ , then $f(d) - f(c) \geq p(d-c)$. Applying Vitali Covering lemma yields a finite disjoint collection of intervals $\{I_k \in \mathscr{U} : 1 \leq k \leq K\}$ such that for $W = \bigcup_{k=1}^{K} J_k$ , we have $m^* ((E_{p,q} \cap V) \setminus W) < \varepsilon$. Since

$$m^* ((E_{p,q} \cap V) \setminus W) + m(W) \geq m^* (E_{p,q} \cap V)$$

we have that $m(W) \geq m_0 - 2\varepsilon$.
We know each interval $J_k = [c_k, d_k]$ from $W$ must be contained in $V$, furthermore it is contained in an interval $I_n = [a_n, b_n]$ of $V$. As each interval is disjoint and monotonic, we must have that

$$\sum_{k=1}^{K} (f(d_k) - f(c_k)) \leq \sum_{n=1}^{N} (f(b_n) - f(a_n)).$$

Now, since $I_n \in \mathscr{L}$ and $J_k \in \mathscr{U}$ , we have

$$\begin{aligned} p\sum_{k=1}^{K}(d_k - c_k) &= pm(w) \\ &\le qm(V) \\ &= q\sum_{n=1}^{N}(b_n - a_n) \end{aligned}$$

.

Hence, $p(m_0 - 2\varepsilon) \le q(m_0 + \varepsilon)$ for each $\varepsilon > 0$, so $pm_0 \le qm_0$ and as $p > q$, we must have $m_0 = 0$, so $f$ is differentiable on all but sets of measure 0, so it is differentiable almost everywhere. □

**Corollary 1.** If the function $f : [a,b] \to \mathbb{R}$ is of bounded variation on the interval $[a,b] \subseteq \mathbb{R}$, then it is differentiable at almost every $x \in [a,b]$. Consequently, if $f$ is absolutely continuous on $[a,b]$, then it is differentiable at almost every $x \in [a,b]$.

*Proof.* Bounded variation implies $f = g - h$ for increasing functions $g, h$. Applying lebesque's theorem yields $g, h$ are differentiable almost everywhere, hence $f$ is differentiable almost everywhere. □