

Algebraic Theory I

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1 Nilpotent Groups 1

Lecture 13

Fri 24 Sep 2021 11:30

I originally missed this lecture so it is transcribed from a classmates notes.

1 Nilpotent Groups

Lecture 14: Nilpotent Groups

Fri 24 Sep 2021 11:30

Let G be a group, and $Z_0(G) = \{1\}$ with $Z_1(G) = Z(G)$. Thus, $G/Z_1(G)$ is a group which has $Z(G/Z_1(G)) = \frac{Z_2(G)}{Z_1(G)}$ where $Z_2(G)$ is the preimage of $Z(G/Z_1(G))$, that being the subgroup of G containing $Z_1(G)$. We see we may continue

$$\begin{aligned} Z_2(G)/Z_1(G) &= Z(G/Z_1(G)) \\ \text{then, } (G/Z_1(G)) / (Z_2(G)/Z_1(G)) &\simeq G/Z_2(G) \\ \text{which has a center } Z(G/Z_2(G)) &= Z_3(G)/Z_2(G). \end{aligned}$$

Definition 1.1 (Nilpotence). We recursively define $Z_i(G)$ to be the subgroup such that $Z(G/Z_i(G)) = Z_i(G)/Z_{i-1}(G)$. This yields a growing sequence $Z_0(G) \leq Z_1(G) \leq Z_2(G) \leq \dots$. We say a group G is **nilpotent** if $G = Z_n(G)$ for some $n \geq 0$. The minimal $n \geq 0$ for which this is the case is called the **nilpotent class** of G .

Example. The trivial group $\{1\}$ is nilpotent with class $c = 0$. A nontrivial abelian group is nilpotent with class $c = 1$.

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Theorem 1.1. Every finite p -group is nilpotent.

Proof. We know the center of a nontrivial p -group to be nontrivial and its subgroups and quotient groups will also be p -groups. Hence $Z_1(G)$ is nontrivial except in the case G is trivial. Hence we have that $Z_2(G)/Z_1(G)$ is nontrivial unless $Z_2(G) = G$. Hence either $Z_1 < Z_2$ or $Z_2 = G$. Now, denote $|G| = n$. Then either $1 = |Z_0| < |Z_1| < \dots < |Z_n|$ hence $Z_n = G$ or $Z_i = G$ for some $i < n$, so $Z_n = G$. Hence, G is nilpotent. \square

Definition 1.2. A subgroup $H \leq G$ is **characteristic** if for every automorphism of G , we have $\alpha(H) = H$. This is equivalent to $\alpha(H) \leq H$ for all automorphisms as $\alpha^{-1} : G \rightarrow G$ is also an automorphism, hence $H \leq \alpha(H)$, so equality holds. Since conjugation is always an automorphism, being characteristic implies normality.

Proving vs. Using Characteristicness

This means that in order to show that something is characteristic we need only show $\alpha(H) \leq H$, but when we use that something is characteristic we will often use the full equality.

Lemma 1.1. As we know $K \trianglelefteq H$ and $H \trianglelefteq G$ does not imply $K \trianglelefteq G$. On the other hand, K being characteristic in H and $H \trianglelefteq G$ does yield $K \trianglelefteq G$.

Proof. Let $\alpha_x : G \rightarrow G$ be the conjugation by x map. We know this to be an automorphism of G , hence as H is normal, we have $\alpha_x|_H : H \rightarrow H$ is an automorphism of H , and since K is characteristic in H , we see an automorphism of H fixed K , hence $\alpha_x(K) = xKx^{-1} = K$ for all $x \in G$, hence $K \trianglelefteq G$. \square

Lemma 1.2. Let G be a finite group with p being prime and P being a sylow p -group in G . Then, the following are equivalent

1. P is the unique sylow p -group in G .
2. $P \trianglelefteq G$.
3. P is characteristic in G .
4. Any subgroup generated by elements whose orders are each powers of p is itself a p -group.

Proof. 1. We have already shown $1 \Leftrightarrow 2$.

2. As conjugation is always an automorphism, we see $2 \Leftrightarrow 3$ is trivial.

3. We show $1 \Rightarrow 3$. Let $\alpha : G \rightarrow G$ be an arbitrary automorphism of G . Then, $\alpha(P) \leq G$ and $|\alpha(P)| = |P|$. As P is the unique sylow p -group, we see there is no distinct group of cardinality $|P|$, hence $\alpha(P) = P$.

4. Now we show $1 \Rightarrow 4$. Let X be a set satisfying $\text{ord}(x) = p^n$ for each $x \in X$. Then each $\langle x \rangle$ is contained in a p -group, and as there is a unique

maximal p -group, we have that $\langle x \rangle \subseteq P$ for each $x \in X$. Hence, $\langle X \rangle \subseteq P$ and as X is a p -group we have that $X = P$.

5. $4 \Rightarrow 1$. Let X to be the union of all sylow p -groups in G . By hypothesis, $\langle X \rangle$ is a p -group and thus it is contained in some sylow p -group so WLOG, we have $\langle X \rangle \subseteq P$. But if there were distinct p -groups, $P' \neq P$ then $P' \subseteq X$ and $P \subset \langle P' \cup P \rangle \subseteq X \subseteq P$. \nmid . Hence P is the unique sylow p -group.

□