Combinatorics

Thomas Fleming

November 12, 2021

Contents

Lecture 30: The (6,3) Problem

Wed 10 Nov 2021 16:24

Definition 0.1 (Hypergraphs)**.** A r **-uniform graph** or r**-graph**, G is a pair of two sets. V, being a vertex set and E, being a set of edges are subsets of cardinality r. That is, a graph where a single edge connects r (not necessarily distinct) vertices instead of 2 vertices as in a graph (with or without loops).

Proposition 0.1. Suppose G is a 3-uniform graph of order n such that every set of 6 vertices induces at most 2 edges. Then $e(G) = o\left(n^2\right)$.

Proof. First note that for a fixed vertex v, we find $d(v) \leq 4$ else the second assumption is violated. Hence, induce a graph from G by removing all vertices sharing a common pair. Denote this graph G' of order n'. Then, we see $e(G) - e(G') \leq 4(n - n')$ by the restriction on the degrees.
Next, induce a graph from G' by considering only the edges participating in a triple (that is pairs of two triangles sharing a single vertex) and "demote" it to a simple graph by convering every triple into a triangle. Call this G_2' and note that we convered a triple into 3 edges, so by the preceding lemma, we see $e(G_2') = o\left(n'^2\right)$ and $e(G') = \frac{1}{3}e(G') = o\left(n^2\right)$. Then, working backwards, we see $e(G) \leq 4(n - n') + o\left(n'^2\right) = n\left(n^2\right)$. □

Definition 0.2 (Super Regular Pair)**.** We define a graph $G(A, B)$ to be an (ε, δ)**-super regular pair** for some $\varepsilon, \delta \in (0, 1)$ if the following conditions hold

- For all $X \subseteq A$ and $Y \subseteq B$ with $|X| < \varepsilon|A|$ and $|Y| < \varepsilon|B|$ we have $e(X, Y) > \delta|X||Y|$.
- For every vertex $a \in A$, $d(a) > \delta|B|$ and similairly, for all $b \in B$, $d(b) > \delta|A|$.

Proposition 0.2. If (A, B) is an ε regular pair with $d(A, B) > 3\varepsilon$, then there are A', B' so that $|A'| > (1-\varepsilon)|A|$ and $|B'| > (1-\varepsilon)|B|$ such that (A', B') is an $(2\varepsilon, d-2\varepsilon)$-super regular pair.

Proof. Define $A_0 = \{a \in A : d(a) \leq (d-\varepsilon)|B|\}$. Then, we see $|A_0| < \varepsilon |A|$ by assumption. Define $A' = A \setminus A_0$. Then, $d_B(a) > (d-\varepsilon)|B|$, so once we have removed at most ε elements from B as well, we see $d_{B'}(a) > (d-2\varepsilon)|B|$. Similairly, we find $d_{A'}(b) > (d-2\varepsilon)|A|$ after removing an exceptional set B_0. Then, by the slicing lemma, we find an $\alpha > \varepsilon$ so that $|A'| > \alpha |A|$ and $|B'| > \alpha |B|$. Then, we see (A', B') is an ε' regular pair with $\varepsilon' = \sup\{\frac{\varepsilon}{\alpha}, 2\varepsilon\}$. If $\alpha = 1-\varepsilon$, we find $\varepsilon' = 2\varepsilon$. $\square$

We will complete this proof next time.

Lecture 31: Blowup Lemma

Fri 12 Nov 2021 16:22

Recall an ε-regular pair (A, B) admits an (ε, δ)-super-regular pair (A', B') with $A' \subseteq A$, $B' \subseteq B$.
Now, recall that for some $\varepsilon > 0$, if $n > \varepsilon^{-4}$, the the bipartite double of P_n is ε-regular. We construct a super-regular subpair from this bipartite double, denoted B_n with partite sets A, B. Applying the expander mixing lemma to two subsets $X \subseteq A$, $Y \subseteq B$ with $|X| > \varepsilon n$ $|Y| > \varepsilon n$, we find

$$\left(\frac{1}{2} - \varepsilon\right)|X||Y| < e(X, Y) < \left(\frac{1}{2} + \varepsilon\right)|X||Y|.$$

Then, inducing four subsets each of size $\sim \frac{n}{2}$, denoted A_1, A_2, B_1, B_2 of A and B respectively and completing the subgraphs (A_1, B_1) and (A_2, B_2) we see $d(A_1, B_1) = d(A_2, B_2) = 1$ and $d(A_1, B_2) \simeq d(A_2, B_1) \simeq \frac{1}{2}$. Collecting the densities, we find $d(A', B') = \frac{3}{4}$ where A', B' denote the sets A, B with the extra edges added between A_1, B_1 and A_2, B_2. From this, we can compute the new graph to be $\left(\varepsilon, \frac{1}{2+\kappa}\right)$-super regular for $\kappa > 0$.

Recall. We can obtain the blowup of a graph G on vertices $\{v_1, v_2, \ldots, v_r\}$ by replacing each vertex with a set $V_1, V_2, \ldots, V_r$ where each V_i is of equal cardinality. We construct the edges such that if $v_i \sim v_j$, then (V_i, V_j) is complete otherwise (V_i, V_j) is disconnected. Moreover if $|V_1| = \ldots = |V_r| = t$, then the blowup of this graph is $G \otimes J_t$.

Definition 0.3 (Generalized Blowup)**.** Let R be a graph with $V(R) = \{v_1, \ldots, v_r\}$. Then, we replace each vertex v_i with a set V_i of cardinality n_i and connect these sets in the same manner as a normal blowup. The induced graph is denoted $R(n_1, \ldots, n_r)$ and called the **generalized blowup**.

We modify this construction slightly. Let $\varepsilon, \delta \in (0, 1)$. Then we construct a new graph by applying the generalized blowup to G with numerical vector $(n_1, \ldots, n_r)$, but rather than each connected pair $v_i \sim v_j$ inducing a complete bipartite subgraph, we only connect sufficient edges in order for V_i, V_j to form an (ε, δ)-super regular pair. We denote this new graph $R_{\varepsilon,\delta}(n_1, \ldots, n_r)$.

Theorem 0.1 (Blowup Lemma)**.** Let R be a graph of order r with $\delta > 0$ and $\Delta \in \mathbb{N} \setminus \{1\}$. Then, there is $\varepsilon > 0$ so that if $H \subseteq R(n_1, n_2, \ldots, n_r)$ with $\Delta(H) \leq \Delta$, then $H \subseteq R_{\varepsilon,\delta}(n_1, \ldots, n_r)$

This lemma is espcially useful because it allows us to efficiently embed binary trees within these modified blowups. It is trivial to embed a binary tree into a complete generalized blowup, and $\Delta(T) = 3$ for a binary tree T, hence fixing a $\delta > 0$ we can find an ε so that the tree embeds in $R_{\varepsilon,\delta}(n_1, \ldots, n_r)$ as well.