Algebraic Theory I

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Lecture 38: Polynomials (4)

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Recall. We found the content of a polynomial over a UFD, R, and its quotient field K,essentially being its generalized gcd in order to reduce polynomials in K to polynomials in R.
Moreover, for $f, g \in K[x]$, then $\text{Cont}(f) \cdot \text{Cont}(g) = \text{Cont}(fg)$.

Now, let $f \in R[x]$ with $f = gh$ for $g, h \in K[x]$, K being the quotient field of R. Then, denote $c_g = \text{Cont}(g)$ and $c_h = \text{Cont}(h)$. Then, we find $f = (c_g c_h) g_1 h_1$ for some $h_1, g_1 \in R[x]$.
Then, we see $\text{Cont}(f) = \text{Cont}(h) = c_g c_h$. Since $f \in R[x]$, we see $\text{Cont}(f) \in R$. This implies all factorizations over K admit a factorization over R.

Now, if $f, g \in R[x]$ with $h \in K[x]$ and $f = gh$, then the same argument shows $\text{Cont}(f) = \text{Cont}(g)\,\text{Cont}(h)$. Hence if f, g are primitive, we find $\text{Cont}(h) \in R$, so $h \in R[x]$.

Theorem 0.1. Let R be a UFD with quotient field K. Let $f \in R[x]$ (we will prove the case f primitive for simplicity, though the non-primitive case is completely analogous). Then, we find f is irreducible in $R[x]$ if and only if f is irreducible in $K[x]$.

Proof. Suppose f irreducible in $K[x]$ but not in $R[x]$. Denote $f = gh$ with $g, h \in R[x]$ being non-units (in $R[x]$).
We know $\text{Cont}(f) = \text{Cont}(g)\,\text{Cont}(h) = 1$. $f = gh$ is a factorization in K unless g or h is a unit. So, assume WLOG g is a unit in $K[x]$, hence g is constant and $\text{Cont}(g) = g$ hence $g^{-1} = \text{Cont}(h)$. So g is a unit in R $\lightning$.
Now, assume f irreducible in $R[x]$ but not in $K[x]$.
Then $f = gh$ for some $g, h \in K[x]$ being non-units in $K[x]$. Hence, we find g, h are nonconstant polynomials in K. Denote $c_g = \text{Cont}(g), c_h = \text{Cont}(h)$ with $g = c_g g_1$ and $h = g_h h_1$ for $g_1, h_1 \in R[x]$ being primitive, Thus, $f = (c_g c_h) g_1 h_1$ with $c_g, c_h = \text{Cont}(f) \in R[x]$ by hypothesis. Since g, h are nonconstant, g_1, h_1 are nonconstant, hence nonunits and nonzero, so this is a factorization of f over $R[x]$ $\lightning$. So the claim is shown. $\square$

Theorem 0.2. A ring R is a UFD if and only if $R[x]$ is a UFD. Moreover if R be a UFD with quotient field K then $f \in R[x]$ is prime if and only if one of the following hold

1. $f = p \in R$ is a constant with p being prime in R, or
2. f is irreducible over $K[x]$ with $\mathrm{Cont}(f) = 1$.

Proof. We begin by examining the prime elements of $R[x]$. First, we show constant polynomials with prime content are prime in $R[x]$.
Let $f = p \in R[x]$ with $p \in R$ being a prime in R. To show f is prime in $R[x]$, suppose $p \mid gh$ with $g, h \in R[x]$. Then let $c_g = \mathrm{Cont}(g)$ and $c_h = \mathrm{Cont}(h)$ so $g = c_g g_1$ nad $h = c_h h_1$ for primitive $g_1, h_1 \in R[x]$. So, $p \mid (c_g c_h) g_1 h_1$, so $p \mid c_g c_h$. So $p \mid c_g$ or c_h, WLOG suppose the case c_g. Then, $p \mid g$, so p is prime in $R[x]$.
Now, suppose $f \in R[x]$ with f primitive and f irreducible over $K[x]$. Since K is a field, $K[x]$ is a PID, hence UFD, so primes are irreducible, hence f is prime in $K[x]$. Suppose $f \mid gh$ (over R), sometimes denoted $f \mid_R gh$, with $g, h \in R[x]$. Then, $f \mid_{K[x]} gh$, so $f \mid_{K[x]} g$ or h. Assume WLOG the case g and suppose $f = gt$ for some $t \in K[x]$. Since $\mathrm{Cont}(g), \mathrm{Cont}(f) \in R$ we see $\mathrm{Cont}(t) \in R$, hence $t \in [x]$, so $f \mid_{R[x]} g$, hence f is prime.
Now, let $f \in R[x]$ be prime. First, suppose $f = p \in R$ is a constant polynomial which is prime in $R[x]$. If $p \mid_{R[x]} ab$ with $ab \in R$, then we see $p \mid_R ab$. So, $pq = ab \in R$ for a polynomial q implies $\deg(q) \leq 1$. That is, $p \mid_{R[x]} ab$ and since p is prime in $R[x]$ we find WLOG $p \mid_{R[x]} a$. So, $p \mid_R a$ by a similar argument, and we see $p \in R$ is prime.
Otherwise, suppose the prime $f \in R[x]$ has $\deg(f) \geq 1$. We wish to show $\mathrm{Cont}(f) = 1$ and f irreducible over $R[x]$. But, $f = \mathrm{Cont}(f) f_1$ with $f_1 \in R[x]$ being primitive and $\deg(f) = \deg(f_1) \geq 1$ implies f_1 is a nonunit (in $R[x]$ and $K[x]$). If $\mathrm{Cont}(f) = 1$ this is a contradiction as f is prime (hence irreducible) over $R[x]$. So, $\mathrm{Cont}(f) = 1$.
Finally, we must show f irreducible over $K[x]$ but the preceding lemma handles precisely this case.
Next class we show the final piece of the theorem, that R is a UFD if and only if $R[x]$ is a UFD. □

Lecture 39: Polynomials (5)

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Recall. We characterized the prime elements of $R[x]$ for some UFD R . Next, we show the final part of the theorem, that R is a UFD implies $R[x]$ is a UFD.

Proof. Let $f \in R[x]$ be nonzero. Clearly, $f \in K[x]$ with $f = \mathrm{Cont}(f)\left(\prod_{i=1}^{n} g_i\right)$ where $g_i \in K[x]$ are irreducible polynomials. But, since R is a UFD, we can factor $\mathrm{Cont}(f)$ into primes from R. We know this factorization to also be primes in $R[x]$. Hence f can be factorized as the factorization of its content times a product of irreducible polynomials inn $K[x]$ which are also prime.
Lastly, we need to show this factorization unique. This is essentially trivial as $\mathrm{Cont}(f) \in R$ and $\prod_{i=1}^{n} g_i \in K[x]$, a UFD, so we see any factorization in $R[x]$ is the product of these unique factorizations, so it is unique. □

The converse can be proved directly by examining only constant polynomials. Unfortunately, this conclusion does not extend to PIDs as we have already shown. However, we can extend this to multivariate polynomial rings to yield the following generalization.

Corollary 1. If R is a UFD, then $R[x_1, \ldots, x_n]$ is a UFD.