# Algebraic Theory I: Homework IV

Thomas Fleming

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**Problem** (1)**.** Let $(X, \subseteq)$ be the set of all ideals not containing $I$ partially ordered by inclusion.
It suffices to show that for a totally ordered set $(I_\alpha)_{\alpha\in\Omega}$ with ordered set $\Omega$, and ideals $I_\alpha \in X$ there is an upper bound. Take $U = \bigcup_{\alpha\in\Omega} I_\alpha$. $U$ is the union of ideals so it is clearly an ideal. Suppose $U \notin X$, that is $U \supseteq I$. Then, there is a subsequence $\alpha_1, \ldots, \alpha_n$ and a permutation $\pi$ so that

$$\left(x_{\pi(1)}\right) \subseteq I_{\alpha_1}, \left(x_{\pi(1)}, x_{\pi(2)}\right) \subseteq I_{\alpha_2}, \ldots, \left(x_{\pi(1)}, \ldots, x_{\pi(n)}\right) = I \subseteq I_{\alpha_n} \ \lightning.$$

Hence, $U \in X$, and $U$ contains all $I_\alpha$ so it is an upper bound. Hence, there is a maximal element $M \in X$ by Zorn's Lemma.

**Problem** (2)**.**

First note that $2, \sqrt{-D}, 1+\sqrt{-D}$ are all non-units in $R$ as their respective inverses in $\mathbb{C}$ all have noninteger coefficients.
Then define

$$\begin{aligned} N : \mathbb{Z}\left[\sqrt{-D}\right] &\longrightarrow \mathbb{Z} \\ (a+bi) = x &\longmapsto x\overline{x} = a^2+b^2. \end{aligned}$$

Then

$$\begin{aligned} N\left((a+bi)(c+di)\right) &= \left[(ac-bd)+(bc+ad)\,i\right]\left[(ac-bd)-(bc+ad)\,i\right] \\ &= (ac-bd)^2+(bc+ad)^2 \\ \text{and } N(a+bi)\,N(c+di) &= \left(a^2+b^2\right)\left(c^2+d^2\right) \\ &= (ac)^2-2acbd+(bd)^2+(ad)^2+2adbc+(bc)^2 \\ &= (ac-bd)^2+(bc+ad)^2 \\ &= N\left((a+bi)(c+di)\right) \end{aligned}$$

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In particular $N$ is a ring homomorphism of $R$. Next, suppose 2 is not irreducible in $R$. Then, there are non-units $x = a+b\sqrt{-D}, y = c+d\sqrt{-D} \in R$ so that $\left(a+b\sqrt{-D}\right)\left(c+d\sqrt{-D}\right) = 2$. Passing to $N$,

$$\begin{aligned} N(2) = 4 &= N(x)\,N(y) \\ &= \left(a^2+b^2D\right)\left(c^2+d^2D\right) \in \mathbb{Z} \end{aligned}$$

.

Since units pull back to units under homomorphisms, we can assume both of these quantities to be non-units. Hence the only possibility is

$$a^2+b^2D = c^2+d^2D = 2 \text{ (up to units)}.$$

In this case $D > 2$ so we see $b = d = 0$, hence either $a = 2$, $c = 1$ or $a = 1$, $c = 2$. In either case $\lightning$, as $x, y$ were assumed nonunits. Hence 2 is irreducible in $R$.
Now assume $\sqrt{-D}$ non-irreducible in $R$. Then, we find non-units $x = a+b\sqrt{-D}, y = c+d\sqrt{-D} \in R$ so that $\sqrt{-D} = xy$. Passing to $N$, we find

$$N\left(\sqrt{-D}\right) = D = \left(a^2+b^2D\right)\left(c^2+d^2D\right).$$

If WLOG $b = 1$ , then we see $a = d = 0$ and $c = 1$ $\lightning$ as $y$ is not a unit. If $b > 1$ or $d > 1$ , then $b^2D > D$ so $\lightning$. Hence $b = d = 0$. Hence, $D = a^2c^2$, but $D$ was square-free $\lightning$.
Lastly, suppose $1+\sqrt{-D}$ is irreducible in $R$. Then, we find non-units $x = a+b\sqrt{-D}, y = c+d\sqrt{-D} \in R$ so that $xy = 1+\sqrt{-D}$. Hence

$$N\left(1+\sqrt{-D}\right) = 1+D = \left(a^2+b^2D\right)\left(c^2+d^2D\right).$$

If WLOG $b = 1$, then $d = 0$ otherwise $1+D > 2D^2$ $\lightning$, and similarly $c = 1$. Hence $y$ is a unit $\lightning$. So, $1+\sqrt{-D}$ is an irreducible.

Now, note that the element $D^2 + D$ has two distinct factorizations. First, it is again clear that $D \pm \sqrt{-D}$ is a non-unit as its complex inverse has nonintegral coefficients. Then, we note $D\left(D+1\right) = D^2 + D = \left(D + \sqrt{-D}\right)\left(D - \sqrt{-D}\right)$. We see $D, \left(D+1\right)$ are not units and likewise for $\left(D \pm \sqrt{-D}\right)$. Moreover, the factorizations are not pairwise associate, hence there are two factorizations for $D^2 + D$, so $Z\left[\sqrt{-D}\right]$ is not a UFD.

**Problem** (3)**.** Let $I, J \subseteq R$ be ideals of a commutative ring $R$. Then, let $x, y \in IJ$ with $x = \sum_{i=1}^{n} a_i b_i$ and $y = \sum_{i=n+1}^{m} a_i b_i$ for $a_i \in I$ $b_i \in J$, $1 \leq i \leq m$. Then, we see $x + y = \sum_{i=1}^{m} a_i b_i \in IJ$.
Next, if $r \in R$, and $x \in I$ with $x = \sum_{i=1}^{n} a_i b_i$ for some $a_i \in I$ $b_i \in J$ , then $rx = r\sum_{i=1}^{n} a_i b_i = \sum_{i=1}^{n} r a_i b_i$ with $ra_i \in I$ by absorption property and $b_i \in J$ by assumption for $1 \leq i \leq n$ . Hence $rx \in IJ$, so $IJ$ is an ideal.

**Problem** (4)**.** Let $I$ be an ideal in $R$ and $I_i = \{x_i : x \in I\} \subseteq R_i$ for each $1 \leq i \leq n$. First, fix $i$ and let $r_i \in R_i$. Then, there is an $\mathbf{r} \in R$ so that $\mathbf{r}$ has $r_i$ in its $i$'th coordinate. Hence, we see $\mathbf{rx} \in I$ for all $\mathbf{x} \in I$, so $r_i x_i \in I_i$ for all $x_i \in I_i$ by the pointwise multiplication. Similairly, fix $i$ and let $x_i, y_i \in I_i$. Then there are $\mathbf{x}, \mathbf{y} \in I$ having $x_i, y_i$ in their $i$'th coordinates respectively and $\mathbf{x} + \mathbf{y} \in I$. Hence, $x_i + y_i \in I_i$. So, each $I_i$ is an ideal. Now, we show $I$ to be the product of the $I_i$'s.
As each $I_i$ is simply the projection of $I$ into its $i$'th coordinate it is clear $I \subseteq \prod_{i=1}^n I_i$. Hence, let $\mathbf{x} = (x_1, \ldots, x_n) \in \prod_{i=1}^n I_i$. Then, we see there are vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n \in I$ each having $x_i$ in their $i$'th coordinates respectively and $\mathbf{x}_i \cdot j_i = \left(0, \ldots, \underbrace{x_i}_{\text{position } i}, \ldots, 0\right) \in I$ for $j_i \in R$ being the indicator vector in the $i$'th coordinate. Hence the sum $\mathbf{x} = \sum_{i=1}^n \mathbf{x}_i j_i \in I$ by closure of addition. So, equality holds.

**Problem** (5). 1. It is trivial that $I \subseteq \sqrt{I}$ (taking $n = 1$ for all $x \in I$). To show $\sqrt{I}$ an ideal, let $x_1, x_2 \in \sqrt{I}$ with $x_1^p \in I$ and $x_2^{p_2} \in I$. Then, there are $a_0, a_1, \dots, a_{p+q} \in R$ so that

$$(x_1 + x_2)^{p+q} = a_{p+q}x_1^{p+q} + a_{p+q-1}x_1^{p+q-1}x_2^1 + \dots + a_p x_1^p x_2^q + \dots a_1 x_1^1 x_2^{p+q-1} + a_0 x_2^{p+q}.$$

We know each term of this sum to be in $I$ by the absorbition property of $x_1^p$ and $x_2^q$, hence the sum is in $I$, so $x_1 + x_2 \in \sqrt{I}$. Next, let $x \in R, a \in \sqrt{I}$ with $a^n \in I$ , then $(xa)^n = x^n a^n \in I$ by absorption, so $xa \in \sqrt{I}$, so $\sqrt{I}$ is an ideal.

2. Suppose $\sqrt{I} = R$. Then, $1 \in \sqrt{I}$, hence $1^n = 1 \in I$, so $I = R$. Conversely, $I = R \subseteq \sqrt{I}$ so the claim holds.

3. Let $M$ be a maximal ideal among inclusion and $n \geq 1$. Then $M \subseteq \sqrt{M}$ with $\sqrt{M}$ being an ideal so either $\sqrt{M} = R$ or $\sqrt{M} = M$ if $\sqrt{M} = R$, $\lightning$ by previous part, so $\sqrt{M} = M$. Moreover, as $M^n \subseteq M$ , we see $\sqrt{M^n} \subseteq \sqrt{M}$. Hence, we need only show the reverse inclusion. Let $x \in \sqrt{M}$. Then, $x^m \in M$ for some $m \geq 1$. Then, we see $x^{mn} = \underbrace{x^m \cdot x^m \cdot \ldots \cdot x^m}_{n \text{ times}} \in M^n$, so $x \in \sqrt{M^n}$. Hence equality holds.