Algebraic Theory I: Homework I

Thomas Fleming

Mon 06 Sep 2021 18:53

Problem (1)**.** Let G be a group and $H, K \trianglelefteq G$ with $H \cap K = \{1\}$. Show that $hk = kh$ for $h \in H$, $k \in K$.

Solution. Note that as K is normal, we have $h^{-1}kh = n \in K$ and $khk^{-1} = m \in H$ for $k \in K$, $h \in H$. Then, note that $h^{-1}khk^{-1} = nk^{-1} = h^{-1}m$. But as $h^{-1}, m \in H$, we see $h^{-1}m \in H$ by closure. Similarly, $nk^{-1} \in K$. Hence $1 = h^{-1}khk^{-1} \in H \cap K = \{1\}$. Now, multiplying by k from the right and h from the left yields $kh = hk$.

Problem (2)**.** Let G be a nontrivial group and H to be a maximal normal subgroup of G. Show that G/H has no proper nontrivial normal subgroups.

Solution. Suppose $K \trianglelefteq (G/H)$ is a nontrivial subgroup. Then, the lattice theorem guarantees $K = T/H$ for some $H \leq T \leq G$ with $T \trianglelefteq G$. As K is nontrivial, we see there is a $t \in T \setminus H$ else K would be trivial. Hence, $H < T \leq HT \leq G$ and, as $H, T \trianglelefteq G$, we see $HT \trianglelefteq G$ (as $xHT = HxT = HTx$). Hence, $H < HT \trianglelefteq G$, so H is not the maximal normal subgroup of G . $\lightning$

Problem (3)**.** Let G be a group action acting transitively on the set Ω and let $\alpha : G \to \operatorname{Perm}(\Omega)$ be the corresponding homomorphism given by $\alpha(g)(x) = x^g$ for $g \in G$ and $x \in \Omega$. For any $x \in \Omega$ show that $\ker(\alpha) = \bigcap_{g \in G} gG_x g^{-1}$

Solution. Let $x \in \Omega$, and for each $g \in G$, define $x^g = a_g \in \Omega$ and note that by transitivity, $\{a_g : g \in G\} = \Omega$.

$$\begin{aligned}
\bigcap_{g \in G} gG_x g^{-1} &= \bigcap_{g \in G} G_{x^g} \\
&= \bigcap_{g \in G} \{h \in G : (x^g)^h = x^g\} \\
&= \{h \in G : (x^g)^h = x^g \ \forall\ g \in G\} \\
&= \{h \in G : a_g^h = a_g \ \forall\ a_g \in \Omega\} \\
&= \{h \in G : \alpha(h)(a_g) = \alpha(1_G)(a_g) = 1_{\operatorname{Perm}(\Omega)}(a_g), a_g \in \Omega\} \text{ as } \alpha \text{ is a homomorphism} \\
&= \ker(\alpha).
\end{aligned}$$

Problem (4)**.** Let G be a group acting transitively on a finite set Ω, and let $H \trianglelefteq G$. Consider the action of H on Ω inherited from G and let $\mathscr{O}_1, \ldots, \mathscr{O}_r$ be the distinct orbits of this action.

1. Show that there is a well defined action of G on $\{\mathscr{O}_1, \ldots, \mathscr{O}_r\}$ defined by $\mathscr{O}^g = \{x^g : x \in \mathscr{O}_i\}$, with this action being transitive.
2. Show that $|\mathscr{O}_i| = |\mathscr{O}_j|$ for all i, j.
3. For $x \in \mathscr{O}_1$ show that $|\mathscr{O}_1| = |H : H \cap G_x|$ and $r = |G : HG_x|$.

Lemma 0.1. $|\mathscr{O}_i^g| \leq \mathscr{O}_i$. Suppose every $x_j \in \mathscr{O}_i$ mapped to a unique x_k by g, that is $x_j^g = x_n^g$ implies $x_j = x_n$, This is clearly the case of maximal size. Then, $|\mathscr{O}_i^g| = |\mathscr{O}_i|$. If $x_j^g = x_n^g = x_k$ for some $j \neq n$, then $|\mathscr{O}_i^g| < |\mathscr{O}_i|$. Hence the inequality holds regardless.

Solution. Let $\{\mathscr{O}_1, \mathscr{O}_2, \ldots, \mathscr{O}_r\} = \mathscr{O}$.

1. First we show the action is well defined. First, note that $\mathscr{O}_i^1 = \{x^1 = x \in \mathscr{O}_i\} = \mathscr{O}_i$. Furthermore let x_i to be the generating element for each respective $\mathscr{O}_i$. Then,

$$\begin{aligned}
(\mathscr{O}_i^g)^h &= \{(x^g)^h : g, x \in \mathscr{O}_i\} \\
&= \{x^{hg} : h, g \in \mathscr{O}_i\} \\
&= \mathscr{O}_i^{hg}.
\end{aligned}$$

   Next, note that for each pair $1 \leq i, j \leq r$ and each $g, g^{-1} \in H$, there is a $h_{i;j} \in g$ and $\hat{g}h_{i;j} \in h_{i;j}G = Gh_{i;j}$ such that $h_{i;j}\hat{g} = \hat{g}h_{i;j}$ and $x_i^{\hat{h}_{i;j}} = x_j$. Hence

$$\begin{aligned}
\mathscr{O}_i^{h_{i,j}} &= \{(x_i^g)^{h_{i;j}} : g \in H\} \\
&= \{x_i^{h_{i;j}g} : g \in H\} \\
&= \{x_i^{h_{i;j}g} : h_{i;j}g \in h_{i:j}H\} \\
&= \{x_i^{g\hat{h}_{i;j}} : g\hat{h}_{i;j} \in Hh_{i;j}\} \\
&= \{\left(x_i^{\hat{h}_{i;j}}\right)^g : g \in H\} \\
&= \{x_j^g : g \in H\} \\
&= \mathscr{O}_j.
\end{aligned}$$

   So, the action is transitive.

2. First, let $g \in G$ and $1 \leq i, k \leq r$ such that $\mathscr{O}_i^g = \mathscr{O}_k$. Then, we note that $|\mathscr{O}_k| = |\mathscr{O}_i^g| \leq |\mathscr{O}_k|$. Now, let $h \in G$ such that $\mathscr{O}_k^h = \mathscr{O}_i$. Then, $|\mathscr{O}_i| = |\mathscr{O}_k^g| \leq |\mathscr{O}_k|$. Hence, $|\mathscr{O}_k| \leq |\mathscr{O}_i| \leq |\mathscr{O}_k|$, so $|\mathscr{O}_i| = |\mathscr{O}_k|$.

3. First let $x \in \mathscr{O}_i$ and denote the stabilizer of x within H to be H_x. Then, note that

$$G_x \cap H = \{g \in G : x^g = x\} \cap H = \{g \in G \cap H : x^g = x\} = \{g \in H : x^g = x\} = H_x.$$

Then, point-stabilizer theorem shows $|\mathscr{O}_1| = |H : H_x| = |H : G_x \cap H|$. Now, note that normalcy ($G_x \cap H = H \trianglelefteq H$, $G_x \trianglelefteq HG_x$, and $H \leq N_G(G_x)$)and the 3rd isomorphism theorem guarantees $|H : G_x \cap H| = |HG_x : G_x|$. Lastly, note that as g is transitive, we have $\mathscr{O}_{G;x} = G$ (the orbit of x in G). Then, orbit-stabilizer guarantees $|\mathscr{O}_{G;x}| = |G : G_x| = |G|$. Finally, the 2nd isomorphism theorem says $(G/G_x) \simeq (G/HG_x) / (HG_x/G_x)$, hence $|G : G_x| = |G : HG_x| \cdot |HG_x : G_x|$. Lastly, note that as all orbits were of equal cardinality and G acts transitively, we must have $||$, we may construct our equality.

$$\begin{aligned}
|G| &= |G : G_x| \\
&= |G : HG_x| \cdot |HG_x : G_x| \\
&= |G : HG_x| \cdot |O_i| \\
.
\end{aligned}$$

But, as $G = r\,|\mathscr{O}_i|$, we see $|G : HG_x| = r$.

Problem (5)**.** Let G be a group acting transitively on a finite Ω. Define a block to be a nonempty subset $B \subseteq \Omega$ such that for every $g \in G$, B and $B^g = \{x^g : g \in G, x \in B\}$ have either $B = B^g$ or $B \cap B^g = \varnothing$.

1. Show that the definition for a block B and $g \in G$ gives a well defined group action of G on the set $\Omega_B := \{B^g : g \in G\}$.
2. If B is a block with $x \in B$, then $G_x \leq G_B = \{g \in G : B^g = B\} \leq G$.
3. Show that there does not exist a block B with $1 < |B| < |\Omega|$ if and only if for every $x \in \Omega$ the only subgroups of G containing G_x are G and G_x itself.

Solution.

1. First, note that $(B^g)^1 = B^{1g} = B^g$.
   Next,

$$\begin{aligned}\left((B^g)^h\right)^k &= \{\left((x^g)^h\right)^k : x \in B\} \\ &= \{(x^g)^{kh} : x \in B\} \\ &= (B^g)^{kh}\end{aligned}$$

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2. Let B be a block and $x \in B$. Then,

$$G_B = \{g \in G : B^g = B\} = \{g \in G : \{x^g : x \in B\} = B\}.$$

   Suppose there is a $h \in G_x \backslash G_B$. That is, there is an h such that $x^h = x$ but $B^h \neq B$, implying $B^h \cap B = \varnothing$. But, we know $x \in B$ and $x^h = x \in B^h$, hence $B^h \cap B \neq \varnothing$. $\lightning$. Hence for all $g \in G_x$, $g \in G_B$, so $G_x \leq G_B$.

3. Suppose for all $x \in \Omega$, the only subgroups of G containing G_x are G and G_x. Then, as $G_x \leq G_B \leq G$, we must have $G_B = G_x$ for all $x \in \Omega$ or $G_B = G$. Suppose $G_B = G$ and $|B| < |\Omega|$, let $y \in \Omega \setminus B$ then $\{x^g : g \in G, x \in B\} = B$. But as the action is transisitve, for each $x \in B$ there is a $h \in G = G_B$ such that $x^h = y \in B^h = B$. $\lightning$ as $y \notin B$. Hence $|B| = |\Omega|$ in this case.
   Now, consider the case $G_B = G_x$ for all $x \in \Omega$ where $|B| > 1$. Then, let $x, y \in B$ be distinct elements and note that $G_B = G_x = G_y$. Let $g \in G$ such that $x^g = y$. Then, as $x^g = y \in B^g$, we see $B^g = B$ hence $g \in G_B$, but as $x^g = y \neq x$, then $g \notin G_x$ hence $G_x \neq G_B$. $\lightning$. So $|B| = 1$ in this case.
   The other direction eludes me.