Analysis I

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Contents

Lecture 11: Measure Theory (3)

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We prove the final theorem from last lecture.

Proof.

- $(1 \Rightarrow 2)$. There are 2 cases, S being bounded and S being unbounded.

  If S is bounded, there is an interval $(a,b) \supseteq S$, $a,b \in \mathbb{R}$. Then for any given $\varepsilon > 0$, we find $\{I_k : k \in \mathbb{N}\} \in J(S)$ and $\{J_k : k \in \mathbb{N}\} \in J([a,b] \setminus S)$ such that $\mu(s) \geq \sum_{k=1}^{\infty} \ell(I_k) - \frac{\varepsilon}{3}$ and $\mu([a,b] \setminus S) \geq \sum_{k=1}^{\infty} \ell(J_k) - \frac{\varepsilon}{3}$. Let $O = \bigcup_{k \in \mathbb{N}} I_k$, $U = \bigcup_{k \in \mathbb{N}} J_k$ and $C = [a,b] \setminus U$. Then, $C \subseteq S \subseteq O$. Note that O, U are open and C is closed. Then,

$$\begin{aligned}\mu(S) &\geq \mu(O) - \frac{\varepsilon}{3}\\ \mu([a,b] \setminus S) &\geq \mu(U) - \frac{\varepsilon}{3}..\end{aligned}$$

  Furthermore, U, C are disjoint and $\mu(U) < \infty$ (as it is an interval minus a measurable set) and $[a,b] \subseteq U \cup C$. Hence,

$$\begin{aligned}\mu(C) &\geq \mu([a,b]) \setminus \mu(U)\\ &= b - a - \mu(U).\end{aligned}$$

  Then, since $\mu(C) \leq \mu(S) < \infty$, we have

$$\begin{aligned}\mu(O \setminus C) &= \mu(O) - \mu(C)\\ &\leq \frac{\varepsilon}{3} + \underbrace{\mu(S) - (b-a)}_{=-\mu([a,b\setminus S])} + \mu(U)\\ &= \frac{\varepsilon}{3} - \mu([a,b] \setminus S) + \mu(U)\\ &\leq \frac{2\varepsilon}{3}\\ &< \varepsilon.\end{aligned}$$

  For a general S, let $S_n = S \cap [n, n+1]$, $n \in \mathbb{Z}$. Then, there are open O_n and closed C_n such that $C_n \subseteq S_n \subseteq O_n$ and $\mu(O_n \setminus C_n) < \frac{\varepsilon}{3 \cdot 2^{|n|}}$.

Let $O = \bigcup_{n\in\mathbb{Z}} O_n$ and $C = \bigcap_{n\in\mathbb{Z}} C_n$. Then, O is open and C is closed by definition and we see $O \setminus C = \bigcup_{n\in\mathbb{Z}} (O_n \setminus C_n)$ by demorgen and we have $C \subseteq S \subseteq O$. Then,

$$\begin{aligned}\mu(O \setminus C) &\leq \sum_{n\in\mathbb{Z}} \mu(O_n \setminus C_n)\\ &< \sum_{n\in\mathbb{Z}} \frac{\varepsilon}{3 \cdot 2^{|n|}}\\ &= \varepsilon \text{ by geometric summation.}\end{aligned}$$

- $(2 \Rightarrow 3)$. For each $n \in \mathbb{N}$, there are closed C_n and open O_n such that $C_n \subseteq S \subseteq O_n$ and $\mu(O_n \setminus C_n) < \frac{1}{n}$. Let $F = \bigcup_{n\in\mathbb{N}} C_n$ and $G = \bigcap_{O_n}$. Then, F is a F_σ set and G is a G_δ set. Then, we have $F \subseteq S \subseteq G$ and $\mu(G \setminus F) \leq \mu(O_n\ C_n) < \frac{1}{n}$ for all $n \in \mathbb{N}$. Hence, $\mu(G \setminus F) = 0$.
- $(3 \Rightarrow 4)$. This is immediately obvious as F_σ and G_δ sets are measurable.
- $(4 \Rightarrow 1)$. Let $A \subseteq \mathbb{R}$ and $\varepsilon > 0$. Then $S^c \subseteq G \cup (G \cap F^c)$. Then, $A \cap S^c \subseteq (A \cap G^c) \cup (G \cap F^c)$.
  Hence,

$$\begin{aligned}\mu^*(A \cap S^c) &\leq \mu^*(A \cap G^c) + \underbrace{\mu^*(G \cap F^c)}_{<\varepsilon}\\ &\leq \mu^*(A \cap G^c) + \varepsilon.\end{aligned}$$

  And, as G is measurable, we have

$$\mu^*(A) = \mu^*(A \cap G) + \mu^*(A \cap G^c) \geq \mu^*(A \cap S) + \mu^*(A \cap S^c) - \varepsilon$$

  . Hence, in the infimum we have

$$\mu^*(A) \geq \mu^*(A \cap S) + \mu^*(A \cap S^c).$$

  So, S is measurable.

□

Definition 0.1 (Nested Sets)**.** A countable collection of sets $\{S_k : k \in \mathbb{N}\}$ is called

1. **ascending** if $S_k \subseteq S_{k+1}$ for all k.
2. **descending** if $S_{k+1} \subseteq S_k$ for all k.

Lemma 0.1.

1. If $\{S_k : k \in \mathbb{N}\}$ is an ascending collection of measurable sets, then $\mu\left(\bigcup_{k\in\mathbb{N}} S_k\right) = \lim_{k\to\infty} \mu(S_k)$.
2. If $\{S_k : k \in \mathbb{N}\}$ is a descending collection of measurable sets and $\mu(S_1) < \infty$. Then, $\mu\left(\bigcap_{k\in\mathbb{N}} S_k\right) = \lim_{k\to\infty} \mu(S_k)$.

Proof. 1. It suffices to consider the case $\mu(S_k) < \infty$ for all k, else the union and limit both trivially have measure ∞. Define $S_0 = \varnothing$, $X_n = S_n \setminus S_{n-1}$. Then, $\{X_k : k \in \mathbb{N}\}$ is a disjoint collection of measurable sets such that $\bigcup_{k\in\mathbb{N}} X_k = \bigcup_{k\in\mathbb{N}} S_k$. Hence, as we know the lebesque measure to be countably additive, we have

$$\begin{aligned}
\mu\left(\bigcup_{k\in\mathbb{N}} X_k\right) &= \sum_{k=1}^{\infty} \mu(X_k) \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \mu(X_k) \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \left(\mu(S_k) - \mu(S_{k-1})\right) \\
&= \lim_{k\to\infty} \mu(S_k).
\end{aligned}$$

2. Let $X_n = S_1 \ S_n$. Then, $\{X_k : k \in \mathbb{N}\}$ is an ascending collection of measurable sets such that $\bigcup_{k\in\mathbb{N}} X_k = S_1 \setminus \left(\bigcap_{k\in\mathbb{N}} S_k\right)$. Since $S_k \subseteq S_1$ and $\mu(S_1) < \infty$ we have by the first lemma that

$$\begin{aligned}
\mu(S_1) - \mu\left(\bigcap_{k\in\mathbb{N}} S_k\right) &= \mu\left(\bigcup_{k\in\mathbb{N}} X_k\right) \\
&= \lim_{k\to\infty} \mu(X_k) \\
&= \mu(S_1) - \lim_{k\to\infty} \mu(S_k).
\end{aligned}$$

As $\mu(S_1)$ is finite we know this to be well defined, hence

$$\mu\left(\bigcap_{k\in\mathbb{N}} S_k\right) = \lim_{k\to\infty} \mu(S_k).$$

□

Theorem 0.1 (Borel-Cantelli Lemma)**.** Suppose $\{S_k : k \in \mathbb{N}\}$ is a countable collection of measurable sets such that $\sum_{k=1}^{\infty} \mu(S_k) < \infty$. Then, the set of all $x \in \mathbb{R}$ which belong to an infinite subcollection of $\{S_k : k \in \mathbb{N}\}$ has measure 0.

Proof. Note that x belongs to an infinite subcollection of $\{S_k : k \in \mathbb{N}\}$ if and only if $x \in \bigcap_{k\in\mathbb{N}} \bigcup_{n=k}^{\infty} S_n$.
Then, the collection $\{\bigcup_{n=k}^{\infty} S_n : k \in \mathbb{N}\}$ is descending and $\mu\left(\bigcup_{n\in\mathbb{N}} S_n\right) \leq$

$\sum_{n=1}^{\infty} \mu(S_n) < \infty$. Hence, by the preceding lemma, we have

$$\begin{aligned}\mu\left(\bigcap_{k\in\mathbb{N}} \bigcup_{n=k}^{\infty} S_n\right) &= \lim_{k\to\infty} \mu\left(\bigcup_{n=k}^{\infty} S_n\right) \\ &\leq \lim_{k\to\infty} \sum_{n=k}^{\infty} \mu(S_n) \\ &= 0\end{aligned}$$

.

This final equality is because for all $\varepsilon > 0$ there is a $K \in \mathbb{N}$ such that for $k \geq K$ we have

$$\left|\sum_{i=1}^{\infty} \mu(S_i) - \sum_{i=1}^{k-1} \mu(S_i)\right| < \varepsilon.$$

□

Problem.

1. Is every set measurable?
2. Is every set of measure 0 countable?
3. Is every measurable set Borel?

Lecture 12

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