Analysis I

Thomas Fleming

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Contents

Lecture 9: Extended $\mathbb{R}$ (2) and Intro to Measure Theory

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Definition 0.1. Let $S \subseteq \mathbb{R}$ and $f : S \to \overline{\mathbb{R}}$. Then, we say f is continuous at $x_0 \in S$ if $H \circ f$ is continuous at x_0 on S for any extending function H. Similarity, we say f is continuous on S if $H \circ f$ is continuous on S for any extending function H.
Furthermore, we say f is (strictly) increasing/decreasing/monotone if $H \circ f$ is (strictly) increasing/decreasing/monotone.
Again, if (f_n) is a series of functions $f_n : S \to \overline{\mathbb{R}}$, we say (f_n) converges pointwise/uniformly to $f : S \to \overline{\mathbb{R}}$ if $(H \circ f_n)$ converges pointwise/uniformly to $H \circ f$ for any extending function H.

Definition 0.2. Let $S \subseteq \overline{\mathbb{R}}$ and suppose $a \in \overline{R}$ is an accumulation point of a sequence taking values in $S \setminus \{a\}$.
Then, a function $f : S \setminus \{a\} \to \overline{R}$ is said to have the limit $L \in \overline{\mathbb{R}}$ (relative to S) if for any extending function H and for each $\varepsilon > 0$ we have an $\delta > 0$ such that

$$|H(f(x)) - H(L)| < \varepsilon \text{ for all } x \in S \setminus \{a\} \text{ with } |H(x) - H(a)| < \delta.$$

We denote this by $\lim\limits_{x \to a} f(x) = L$ or $\lim\limits_{\substack{x \to a \\ S}} f(x) = L$

1 Measure Theory

Definition 1.1 (Length)**.** Let $I = (a, b)$ be an interval, then we define the measure function $\ell : \mathscr{P}(\mathbb{R}) \to \mathbb{R}_0^+$ with the following properties:

$$\ell(\varnothing) = 0$$
$$\ell(I) = b - a, a, b \in \mathbb{R}.$$

In all other cases $\ell(I = \infty)$.

We would like to generalize this notion by constructing a set function λ such that

$$\lambda : \mathscr{P}(\mathbb{R}) \to [0, \infty]$$
$$\lambda(I) = \ell(I) \text{ for intervals } I \subseteq \mathbb{R}$$
$$\lambda(x + S) = \lambda(S) \text{ for } x \in \mathbb{R}, S \subseteq \mathbb{R}, x + S = \{x + s : s \in S\}$$
$$\text{if } \{S_m : m \in \mathbb{N}\} \text{ is a countable disjoint collection of sets in } \mathbb{R}, \text{ then}$$
$$\lambda\left(\bigcup_{n=1}^{\infty} S_m\right) = \sum_{n=1}^{\infty} \lambda(S_n)$$

.

It turns out such a function produces contradictions, hence it is poorly posed. Hence, we must alter or remove one of these constraints and as all of the properties are very straight forward it is best to alter the domain of λ itself.

Definition 1.2 (Measure)**.** Let $\mathscr{A}$ be a σ-algebra.

1. A set function $\mu : \mathscr{A} \to [0, \infty]$ is called **countably additive** if for every countable disjoint collection $\{S_n \in \mathscr{A} : n \in \mathbb{N}\}$ we have
$$\mu\left(\bigcup_{n \in \mathbb{N}} S_n\right) = \sum_{i=1}^{\infty} \mu(S_i).$$
2. A countable additive set function $\mu : \mathscr{A} \to [0, \infty]$ such that $\mu(\varnothing) = 0$ is called a **measure**.

Proposition 1.1. Let $\mu : \mathscr{A} \to [0, \infty]$. Then, μ is monotone in the sense that if $A, B \in \mathscr{A}$ with $A \subseteq B$, then we have $\mu(A) \leq \mu(B)$.

Proof. Since $B = A \cup (B \setminus A)$ and since μ is countably additive, then

$$\mu(B) = \mu(A) + \mu(B \setminus A) \geq \mu(A).$$

□

Now, we wish to extend our notion to arbitrary subsets of $\mathbb{R}$.

Notation. For $A \in \mathscr{P}(\mathbb{R})$, then $J(A)$ is defined to be the collection of all countable covers $\{I_n : n \in \mathbb{N}\}$ of A consisting of open, bounded intervals I_n.

Definition 1.3 (Lebesque Outer Measure)**.** Let $A \in \mathscr{P}(\mathbb{R})$, then the quantity $\mu^*(A) \in [0, \infty]$ is defined by

$$\mu^*(A) = \inf\{\sum_{i=1}^{\infty} \ell(J_i) : \{J_i : i \in \mathbb{N}\} \in J(A)\}.$$

This function $\mu^* : \mathscr{P}(\mathbb{R}) \to [0, \infty]$ is called the **Lebesque outer measure**.

Lemma 1.1.

1. The outer measure is monotone
2. The outer measure is translation invariant.
3. The outer measure is countable subadditive, that being for $\{S_n : n \in \mathbb{N}\}$ is a countable collection of sets, then $\mu^*\left(\bigcup_{n\in\mathbb{N}} S_n\right) \leq \sum_{n=1}^{\infty} \mu^*(S_n)$.

Proof.

1. Note that $J(A) \subseteq J(B)$, hence $\mu^*(A) \leq \mu^*(B)$.
2. Similarly, as each $\ell(J_i)$ is translationally invariant, we see μ^* is translationally invariant.
3. Let $\varepsilon > 0$. Then for each $n \in \mathbb{N}$, let $\{I_{n,k} : k \in \mathbb{N}\} \in J(S_n)$ be a collection of intervals such that $\sum_{k=1}^{\infty} \ell(J_{n,k}) \leq \mu^*(S_n) + \frac{\varepsilon}{2^n}$.
   Since, $\{I_{n,k} : n, k \in \mathbb{N}\} \in J\left(\bigcup_{n\in\mathbb{N}} S_n\right)$, we must have that

$$\begin{aligned} \mu^*\left(\bigcup_{n\in\mathbb{N}} S_n\right) &\leq \sum_{n\in\mathbb{N}} \sum_{k\in\mathbb{N}} \ell(I_{n,k}) \\ &\leq \sum_{n=1}^{\infty} \mu^*(S_n) + \varepsilon. \end{aligned}$$

   Since this holds for all $\varepsilon > 0$, this completes the proof.

□

Lemma 1.2. For every interval $I \subseteq \mathbb{R}$, the outer measure is $\mu^*(I) = \ell(I)$.

Proof. Let $I \subseteq \mathbb{R}$ be nonempty (if $I = \varnothing$ it is trivial that $\mu^*(\varnothing) = 0$). First, assume $I = [a, b]$ with $a \leq b \in \mathbb{R}$. Let $\{J_n : n \in \mathbb{N}\} \in J(I)$, then by Heine-Borel there is a finite subcovering $\{I_n : 1 \leq n \leq N\}$ such that no $I_n = \varnothing$. Note that as we will be taking the infimum, then any infinite collection containing this finite collection will be larger (or equal) hence will not matter in the infimum. Furthermore, we can assume that no interval J_n has $J_n \subseteq J_m$ for some $n \neq m$, and we can assume $I_n = (a_n, b_n)$ to be ordered such that $a_n < a_{n+1}$ for $1 \leq n \leq N-1$. Consequently, $b_n > a_{n+1}$ for $1 \leq n \leq N-1$ as otherwise their would be a gap in the covering, and $b_n > b$, as $b \in I$, and $a_1 < a$ by the same reasoning. Hence, we have an overlapping covering of $[a, b]$ by open bounded intervals (a_n, b_n). Hence,

$$\begin{aligned}
\ell(J) &= b - a \\
&\leq b_N - a_1 \\
&\leq \sum_{i=1}^{N} (b_i - a_i) \\
&= \sum_{i=1}^{N} \ell(I_i) \\
&\leq \sum_{i=1}^{\infty} \ell(I_i) \\
&\Rightarrow \ell(I) \leq \mu^*(I).
\end{aligned}$$

Now, we look to obtain the opposite inequality. Let $\varepsilon > 0$, then $\{(a - \varepsilon, b + \varepsilon)\} \in J(I)$, hence

$$\begin{aligned}
\mu^*(I) &\leq b - a + 2\varepsilon \\
&= \ell(I) + 2\varepsilon
\end{aligned}$$

as ε is arbitrary, we then have

$$\mu^*(I) \leq \ell(I).$$

Hence $\mu^*(I) = \ell(I)$ for this case.
Now, assume $I \in \{(a, b) : [a, b), (a, b]\}$ is any bounded interval with $a < b$. By monotonicity, for every $\varepsilon > 0$, we have

$$\begin{aligned}
\ell(I) - 2\varepsilon &= b - a - 2\varepsilon \\
&\leq \ell([a + \varepsilon, b - \varepsilon]) \\
&= \mu^*([a + \varepsilon, b - \varepsilon]) \\
&\leq \mu^*(I) \\
&\leq \mu^*(I) \\
&\leq \mu^*([a, b]) \\
&= b - a \\
&= \ell(I).
\end{aligned}$$

Hence, for every $\varepsilon > 0$, $\ell(I) - 2\varepsilon \leq \mu^*(I) \leq \ell(I)$, hence as ε is arbitrary $\mu^*(I) = \ell(I) = b - a$. This covers all the bounded cases, hence only the unbounded case remains.
If I is unbounded and $a \in I$, then $[a, a + n] \subseteq J$ for all $n \in \mathbb{N}$ or $[a - n, a] \subseteq J$ for all $n \in \mathbb{N}$. In either case, by the monotonicity of the outer measure, $\mu^*(I) \geq n$ for all $n \in \mathbb{N}$, hence $\mu^*(I) = \infty = \ell(I)$. This completes the proof. □

Hence, we have that μ^* conforms to all of our desired properties with the notable exception of countable additivity. This, of course, means μ^* is not in fact a measure, so we will again modify our measure function in order to induce a countably additive measure. This construction will come next lecture and will consist of again restricting the domain to a subset of $\mathscr{P}(\mathbb{R})$, the Lebesque measurable sets, a collection which will be introduced and formalized next lecture.

Lecture 10

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Definition 1.4. A set $S \subseteq R$ is **measureable/Lebesque mesaurable** if for every $A \subseteq \mathbb{R}$,

$$\mu^* (A) = \mu^* (A \cap S) + \mu^* (A \cap S^c) .$$

It actually suffices to show only

$$\mu^* (A) \geq \mu^* (A \cap S) + \mu^* (A \cap S^c) .$$

Proposition 1.2. Every set $S \subseteq \mathbb{R}$ with $\mu^* (S) = 0$ is measurable.

Proof. For every $A \subseteq \mathbb{R}$, $\mu^* (A \cap S) \leq \mu^* (S) = 0$. Similairly, $\mu^* (A \cap S^c) = 0$. $\square$

Definition 1.5. A set $S \subseteq \mathbb{R}$ with $\mu^* (S) = 0$ is said to have measure 0.

Lemma 1.3. For each $a \in \mathbb{R}$, (a, ∞) is measurable.

. Given $A \subseteq \mathbb{R}$ and $\varepsilon > 0$, we fine $\{I_n : n \in N\} \in J (A)$ such that

$$\mu^* (A) \geq \sum_{n=1}^{\infty} \ell (I_n) - \varepsilon.$$

Since $A \cap (a, \infty) \subseteq \bigcup_{n \in \mathbb{N}} (I_n \cap (a, \infty))$ and

$$A \cap (a, \infty)^c \subseteq \left(\bigcup_{n \in \mathbb{N}} (I_n \cap (-\infty, a)) \right) \cup (a - \varepsilon, a + \varepsilon) .$$

It follows that $\mu^* (A \cap (a, \infty)) \leq \sum_{n=1}^{\infty} \ell (I_n \cap (a, \infty))$ and $\mu^* (A \cap (a, \infty)^c) \leq \sum_{n=1}^{\infty} \ell (I_n \cap (-\infty, a)) + 2\varepsilon$. As $\ell (I_n) = \ell (I_n \cap (a, \infty)) + \ell (I_n \cap (-\infty, a))$ as the singular point a will not change the length. Hence,

$$\begin{aligned} \mu^* (A) &\geq \sum_{n=1}^{\infty} \ell (I_n \cap (a, \infty)) + \sum_{n=1}^{\infty} \ell (I_n \cap (-\infty) , a) - \varepsilon \\ &\geq \mu^* (A \cap (a, \infty)) + \mu^* (A \cap (a, \infty)^c) - 3\varepsilon. \end{aligned}$$

$\square$

Proposition 1.3. The collection of Lebesque measurable sets in $\mathbb{R}$ is a σ-algebra $\mathscr{L}$ containing all Borel sets.

Proof. If the measurable sets form of σ-algebra $\mathscr{L}$, then $\mathscr{L}$ must contain all open and closed subsets of $\mathbb{R}$, since it contains all intervals of the form (a, ∞). To show that the measurable sets form a σ-algebra $\mathscr{L}$ we first note that $(a, a) = \varnothing$ and the complement of each measurable set are both measurable sets. This is due to the symmetry in the definition of measurbility

$$\mu^*(A) \geq \mu^*(A \cap S) + \mu^*(A \cap S^c).$$

Now, suppose $\{S_n : n \in \cap\}$ is a countable collection of measurable sets. Let $S = \bigcup_{n\in\mathbb{N}} S_n$, then we need only show S is measurable.
Given $A \subseteq \mathbb{R}$, we define a sequence with $A_1 = A$, $A_{n+1} = A \cap (\bigcap_{k=1}^{n} S_k^c)$. Hence, $A_2 = A \cap S_1^c$, $A_3 = A \cap (S_1^c \cap S_2^c)$. Now, note that $A_{n+1} = A_n \cap S_n^c$, hence the sequence is decreasing in size. And $A \cap S = \bigcup_{k\in\mathbb{N}} (A_k \cap S_k)$. We present a short proof of this claim.
Note that for $x \in A \cap S$, there is a smallest positive integer k such that $x \in S_k$. If $k = 1$, then $x \in A_1 \cap S_1$, if $k > 1$, then $x \notin S_n$ for any $n < k$, consequently $x \in A_k$ by construction. Hence, $x \in A_k \cap S_k$, so $A \cap S \subseteq \bigcup_{k\in\mathbb{N}} (A_k \cap S_k)$.
Now, $\bigcup_{k\in\mathbb{N}} (A_k \cap S_k) \subseteq A \cap S$, as each $A_k \in A$ and $S_k \in S$, hence their intersection and subsequent union are also contained. Hence the equality is shown

$$A \cap S = \bigcup_{k\in\mathbb{N}} (A_k \cap S_k).$$

By measurability of S_n, we know any set A has

$$\mu^*(A_n) = \mu^*(A_n \cap S_n) + \underbrace{\mu^*(A_n \cap S_n^c)}_{A_{n+1}}$$

.
Hence, by induction, we have $\mu^*(A) = \mu^*(A_1) = \sum_{k=1}^{n} \mu^*(A_k \cap S_k) + \mu^*(A_{n+1})$. Since $A \cap \left(\bigcap_{k\in\mathbb{N}} S_k^c\right) = A \cap S^c \subseteq A_{n+1}$ for any n.
Hence,

$$\mu^*(A) \geq \sum_{k=1}^{n} \mu^*(A_i \cap S_i) + \mu^*(A \cap S^c).$$

Finally, as $\bigcup_{k\in\mathbb{N}} (A_k \cap S_k) = A \cap S$ and since μ^* is contably subadditive, we obtain

$$\begin{aligned}
\mu^*(A) &\geq \sum_{k=1}^{\infty} \mu^*(A_k \cap S_k) + \mu^*(A \cap S^c) \\
&\geq \mu^*\left(\bigcup_{k\in\mathbb{N}} (A_k \cap S_k)\right) + \mu^*(A \cap S^c) \\
&= \mu^*(A \cap S) + \mu^*(A \cap S^c).
\end{aligned}$$

□

Definition 1.6 (Lebesque Measure)**.** The **Lebseque Measure** of a measurable set $S \subseteq \mathbb{R}$, denoted by $\mu^*(S)$ is defined by $\mu(S) = \mu^*(S)$. The set function $\mu : \mathscr{L} \to [0, \infty]$ is called the **Lebesque Measure**.

Theorem 1.1. The Lebesque measure μ is a measure on $\mathscr{L}$ such that

- $\mu(I) = \ell(I)$ for every interval $I \subseteq \mathbb{R}$.
- μ is translation invariant.
- μ is countablly additive.

Proof.

1. As μ^* has the interval property, μ trivially inherits this,
2. Similarly, as μ^* was translationaly invariant, we see μ inherits this.
3. Let $\{S_k : k \in \mathbb{N}\}$ be a countable, disjoint collection of measurable sets and define $T_n = \bigcup_{k=n}^{\infty} S_k$ for $n \in \mathbb{N}$.
   Since, $T_{n+1} = T_n \cap S_n^c$ we have

$$\mu(T_n) = \mu(T_n \cap S_n) + \mu\left(\underbrace{T_n + S_n^c}_{=T_{n+1}}\right)$$

   by measurability of S_n.
   Consequently, $\mu(T_1) = \sum_{k=1}^{n} \mu\left(\underbrace{T_k \cap S_k}_{=S_n}\right) + \mu(T_{n+1}) \geq \sum_{k=1}^{n} \mu(S_k)$ for every $n \in \mathbb{N}$. Thus $T_1 = \bigcup_{k\in\mathbb{N}} S_k$ gives $\mu\left(\bigcup_{k\in\mathbb{N}} S_k\right) \geq \sum_{k=1}^{n} \mu(S_k)$. And as we already know the inequality goes in the other direction by subadditivity of μ^*, we see equality holds.

□

Corollary 1. Every countable set of real numbers is measurable with measure 0.

Proof. Let C be our countable sets and note that $C = \bigcup_{k\in\mathbb{N}} \{x_k\}$ with $x_k \neq x_m$ for $k \neq m$. Then, we see that

$$\mu\left(\bigcup_{k\in\mathbb{N}} \{x_k\}\right) = \sum_{k=1}^{\infty} \mu(\{x_k\}) = 0.$$

□

Theorem 1.2 (Properties of Lebesque Measure)**.** Let $S \subseteq \mathbb{R}$, the following are equivalent

1. S is measurable.
2. For each $\varepsilon > 0$, there is an open set O and a closed set C such that $C \subseteq S \subseteq O$ and $\mu(O \setminus C) < \varepsilon$.
3. There is a G_δ set G and a F_σ set F such that $F \subseteq S \subseteq G$ and $\mu(G \setminus F) = 0$.
4. For each $\varepsilon > 0$, there are measurable sets G and F such that $F \subseteq S \subseteq G$ and $\mu(G \setminus F) < \varepsilon$.

We will prove this result next time, though it is completely trivial that $3 \geq 4$, so we will primarily focus on proving $1 \Rightarrow 2$ and $4 \Rightarrow 1$.