Analysis I: Homework 7

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Problem (36)**.** Our function will be φ, the cantor-lebesque function. We have already shown it to be continuous and increasing with $\varphi(1) = 1, \varphi(0) = 0$. Moreover, letting C be the cantor set, we see $[0,1] \setminus C \coloneqq C^c$ is open in $[0,1]$ so for all $x \in C^c$, there is an $\varepsilon > 0$ so that $(x - \varepsilon, x + \varepsilon) \subseteq C^c$. Then, since for all intervals I in the $[0,1]$ complement of the cantor set, we find $I \subseteq J_{n,k}$ for some $n, k \in \mathbb{N}$, we have $\xi(I) = \{\frac{n}{2^k}\}$, so

$$\overline{D}\left(\varphi\left(x\right)\right) = \lim_{r \to 0} \sup\{\frac{\varphi\left(x+h\right) - \varphi\left(x\right)}{h} : 0 < |h| < r\} = \lim_{r \to 0} \sup\{\frac{0}{h} : 0 < |h| < r\} = 0.$$

Similarly, we find $\underline{D}\left(\varphi\left(x\right)\right) = 0$. Hence, φ is differentiable at x and since $\varphi' = 0$ almost everywhere, yet φ is not constant by the initial claim, we find φ is not absolutely continuous.