Analysis I

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Lecture 21: Fundamental Theorem of Calculus

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For the duration of this lecture, $[a, b]$ will denote a compact interval in $\mathbb{R}$, principally, it is not in $\overline{\mathbb{R}}$.

Lemma 0.1. Suppose $f : [a, b] \to \overline{\mathbb{R}}$ is integrable. Then, $f = 0$ almost everywhere if and only if $\int_{[a,x]} f = 0$ for all $x \in [a, b]$.

Proof. If $f = 0$ almost everywhere, then the integral must be 0 for all $x \in [a, b]$ so the forward implication holds.
Conversely, assume $\int_{[a,x]} f = 0$ for all $x \in [a, b]$. Then, let $E = \{x \in [a, b] : f(x) > 0\}$ and assume $m(E) > 0$. Then, there is a closed set $C \subset E$ so that $m(C) > 0$. Letting $O = (a, b) \setminus C$ (an open set) we see $\int_{[a,b]} f = \int_C f + \int_O f$ and as $\int_C f > 0$ as $C \subseteq E$ with $m(C) > 0$. Hence, we find $\int_O f \neq 0$. Hence, $m(O) > 0$, and there is an interval $(c, d) \subseteq O$ so that $\int_{[c,d]} \neq 0$. Since $\int_{[a,d]=0}$ by assumption, then we find $\int_{[a,d]} f = \int_{[a,c]} f + \int_{[c,d]} f$, hence $\int_{[a,c]} f \neq 0$ $\lightning$. $\square$

Proposition 0.1. Syppose $g : [a, b] \to \mathbb{R}$ is continuous. For every $x \in [a, b)$ and $\varepsilon > 0$ there is a δ with $0 < \delta < b - x$ such that

$$\left| \frac{1}{h} \int_{x,x+h} (g - g(x)) \right| < \varepsilon \text{ for } 0 < h < \delta \, .$$

Proof. Write $g(x) = g(x) \chi_{[x,x+h]}$. Then the claim immediately follows. $\square$

Theorem 0.1 (Fundamental Theorem of Calculus I)**.** Suppose $f : [a, b] \to \overline{\mathbb{R}}$ is integrable. Then the function

$$\begin{aligned} F : [a, b] &\longrightarrow \mathbb{R} \\ x &\longmapsto F(x) = \int_{[a,x]} f \end{aligned}$$

is absolutely continuous and differentiable almost everywhere with $F' = f$ almost everywhere.

Proof. It is clear that F is absolutely continuous and differentiable almost everywhere by a result from last lecture and the fact that absolute continuity $\Rightarrow$ bounded variation $\Rightarrow$ differentiable a.e.
Moreover, we can assume $f \geq 0$, otherwise replacing f by f^+ or f^-. We can temporarily assume f is bounded (though we will later remove this requirement). Let $f(x) \leq M$ for all $x \in [a, b]$. Then, extend f, F to functions on $[a, \infty)$ by letting $f(x) = f(b)$ for all $x \geq b$. Define the following sequence of continuous functions (g_n)

$$\begin{aligned} g_n : [a, b] &\longrightarrow \overline{\mathbb{R}} \\ x &\longmapsto g_n(x) = n\left(F\left(x + \frac{1}{n}\right) - F(x)\right) = n\left(\int_{a,x+\frac{1}{n}} f - \int_{a,x} f\right) \\ &\qquad = n \int_{\left[x,x+\frac{1}{n}\right]} f. \end{aligned}$$

Then, we find the sequence is pointwise convergent with limit $F'(x)$ for almost every $x \in [a, b]$. Furthermore, F' is measurable and $0 \leq g_m \leq M$ for all $x \in [a, b]$. So,applying dominated convergence and the previous proposition yields g_m is dominated by M with pointwise limit F', so $F' \leq M$ almost everywhere. So, F' is integrable and for all $x \in [a, b]$ we find

$$\begin{aligned} \int_{[a,x]} F' &= \lim_{n\to\infty} \int_{[a,x]} g_n \\ &= \lim_{n\to\infty} n\left(\int_{\left[a+\frac{1}{n},x+\frac{1}{n}\right]} F - \int_{[a,x]} F\right) \\ &= \lim_{n\to\infty} n\left(\int_{\left[x,x+\frac{1}{n}\right]} F - \int_{\left[a,a+\frac{1}{n}\right]} F\right) \\ &= F(x) - F(a) \\ &= F(x). \end{aligned}$$

Now, if f was unbounded, then define the sequences (f_n) and (F_n) with

$$\begin{aligned} f_n : [a, b] &\longrightarrow \overline{\mathbb{R}} \\ x &\longmapsto f_n(x) = \inf\{f(x), n\} \\ F_n : [a, b] &\longrightarrow \overline{\mathbb{R}} \\ x &\longmapsto F_n(x) = \int_{[a,x]} f_n. \end{aligned}$$

Since $f - f_n \geq 0$, we see $F - F_n$ is increasing for each n. Hence, $F - F_n$ is differentiable almost everywhere with $(F - F_n)' \geq 0$ almost everywhere. Consequently for $x \in [a, b]$ we see

$$\int_{[a,x]} F' \geq \int_{[a,x]} F_n'$$

for all $x \in [a, b]$. Since F_n is bounded for all n , we see $\int_{[a,x]} F_n' = F_n(x)$ by the bounded case. Thus, $\int_{[a,x]} F' \geq F_n(x)$ for all $x \in [a, b]$.
Now, applying MCT, we see (f_n) is a pointwise convergent sequence of functions which are increasing the F_ns also converge pointwise to F on $[a, b]$. Hence, $\int_{[a,x]} F' \geq F(x)$ for ever $x \in [a, b]$ by passing the earlier inequality to the limit. Since f is nonnegative, we see F is increasing, so we also have $\int_{[a,x]} F' \leq F(x) - F(a) = F(x)$. Hence $\int_{[a,x]} F' = F(x)$ since

$$\int_{[a,x]} (F' - f) = \int_{[a,x]} F' - \int_{[a,x]} f = \int_{[a,x]} F' - F(x) = 0 \text{ for a.e. } x \in [a, b].$$

□

In order to prove the other part of the fundamental theorem of calculus, we will need the following lemma:

Lemma 0.2. If the function $f : [a, b] \to \mathbb{R}$ is absolutely continuous with $f' = 0$ almost everywhere then f is a constant function.

Proof. We will show $f(c) = f(a)$ for all $c \in (a, b]$. Fix $c \in (a, b]$ and let $E = \{x \in (a, c) : f' \text{ exists at } x, f'(x) = 0\}$.
By assumption, $m(E) = c - a > 0$, hence for $\varepsilon > 0$ choose $\delta > 0$ such that absolute continuity holds. For each $x \in E$ and $k > 0$, we see there is an $h \in (0, k)$ with either $[x, x + h] \subseteq [a, c]$ and $|f(x + h) - f(x)| < \varepsilon h$ or $[x - h, x] \subseteq [a, c]$ and $|f(x - h) - f(x)| < \varepsilon h$ (or both). Then, the collection $\mathscr{C}$ of these intervals for all $k > 0$ and $x \in E$ is a vitali covering of E. By the Vitali covering lemma, we find a finite disjoint collection $\{[x_k, y_k] \in \mathscr{C} : 1 \leq k \leq n\}$ so that $V = \bigcup_{k=1}^{N} [x_k, y_k]$ has $m(E \setminus V) < \delta$. Reindex these intervals such that $x_k < x_{k+1}$ for all k and let $y_0 = a$, $x_{n+1} = c$. Then, we see

$$a = y_0 \leq x_1 < y_1 < x_2 < y_2 < \ldots < x_n < y_n \leq x_{n+1} = c.$$

Hence, the set $P = \{x_i : 1 \leq i \leq n + 1\} \cup \{y_i : 1 \leq i \leq n + 1\}$ is a partition of $[a, c]$. Since

$$\sum_{k=1}^{n} (y_k - x_k) = m(V) > m(E) = c - a - \delta$$

we see the leftover pieces

$$\sum_{k=0}^{n} (x_{k+1} - y_k) \leq m(E \setminus V) < \delta.$$

Since f is absolutely continuous, we see $\sum_{k=0}^{n} |f(x_{k+1}) - f(y_k)| < \varepsilon$. Consequently,

$$\begin{aligned} |f(c) - f(a)| &\leq \sum_{k=1}^{n} |f(y_k) - f(x_k)| + \sum_{k=0}^{n} |f(x_{k+1} - f(y_k))| \\ &< \sum_{k=1}^{n} \varepsilon (y_k - x_k) + \varepsilon \\ &\leq \varepsilon (c - a) + \varepsilon \end{aligned}$$

for all $\varepsilon > 0$, so we see $f(c) - f(a) = 0$ for all $c \in (a, b]$ and the claim follows. $\square$

Theorem 0.2 (Fundamental Theorem of Calculus II)**.** Suppose the function $F : [a, b] \to \mathbb{R}$ is absolutely continuous. Then, F is differentiable almost everywhere and its derivative, F', is integrable with

$$\int_{[a,x]} F' = F(x) - F(a)$$

for all $x \in [a, b]$.

Proof. Since F is absolutely continuous, it is of bounded variation, so there are two increasing functions, $T, S : [a, b] \to \mathbb{R}$ with $F = T - S$. Moreover, the derivatives T', S' exist almost everywhere and are integrable.
Hence, F' exists almost everywhere and $F' = T' - S'$ almost everywhere, so it is integrable as well.
Then, letting $G(x) = \int_{[a,x]} F'$. We see G is absolutely continuous, so $F - G$ must be absolutely continuous. Then, by the FTC part 1, we see $(F - G)'$ exists almost everywhere and $(F - G)'(x) = 0$ for almost every $x \in [a, b]$. Hence $F - G$ is a constant function. So, we see $F(x) - G(x) = F(x) - \int_{[a,x]} F' = F(a)$ by letting $x = a$. $\square$

1 Intro to Functional Analysis

Lecture 22: L^p spaces

Thu 11 Nov 2021 19:29

I skipped a chapter on supporting lines and Jensen's inequality because the material was rather simple and well explained in Hagen's notes.

Definition 1.1 (Essential Supremum)**.** Let $f : S \to \overline{\mathbb{R}}$ be measurable. Then, we denote the quantity

$$\operatorname{esssup} f = \inf\{M \in \overline{\mathbb{R}} : m(\{x \in S : f(x) > M\}) = 0\}$$

is called the **essentail supremum** of f. Note that $f \leq \operatorname{esssup} f$ almost everywhere.

Definition 1.2 (Lp space)**.** Let $f : S \to \overline{\mathbb{R}}$ be measurable ,then

- For $1 \leq p \leq$ we define $\|f\|_p = \left(\int_S |f|^p\right)^{\frac{1}{p}}$ to be the L^p **norm** of f.
- $\|f\|_\infty = \text{esssup}\,|f|$ is the L^∞ **norm** of f.

Definition 1.3 (Equivalent functions)**.** For $1 \leq p \leq \infty$ let $V_p(s)$ be the set of all measurable functions $f : S \to \overline{\mathbb{R}}$ so that $\|_p < \infty$. Then, functions $f, g \in V_p(S)$ are **equivalent**, denoted $f \sim g$, if $f = g$ almost everywhere in S.
The set of all equivalence classes $V_p(S)/\sim$ is denoted $L^p(S)$ and called the **Lebesque space**.

Remark. If $f \sim g$ in $L_P(S)$, then $f = g$ almost everywhere (on S) hence $\|f - g\|_p = 0$. Hence the L^p norm can be extended to norms on equivalence classes by simply denoting $\|\,[f]\,\|_p = \|f\|$ for some equivalence class $[f] \in L^p(S)$
.

Theorem 1.1 (Minkowski's Inequality)**.** Suppose $f, g \in L^p(S)$ for a $1 \leq p \leq \infty$. Then, $\|f + g\|_p \leq \|f\|_p + \|g\|_p$.
Moreover, if $1 < p < \infty$, then $\|f + g\|_p = \|f\|_p + \|g\|_p$ if and only if there is a $c \geq 0$ so that $f = cg$ almost everywhere.

Proof. Let $x = \|f\|_p$, $s = \|g\|_p$. Then, we see the claim is trivial true if $r = 0$, $s = 0$, or $p = \infty$. Hence, define $\lambda = \frac{r}{r+s}$ and we may assume f, g are finite by definition of L^p space. Since $t \mapsto |t|^p$ is convex on $\mathbb{R}$ and $\lambda \in (0, 1)$, we see

$$\begin{aligned}
|f + g|^p &= \left|\lambda \frac{f}{\lambda} + (1 - \lambda)\frac{g}{1 - \lambda}\right|^p \\
&\leq \lambda \left|\frac{f}{\lambda}\right|^p + (1 - \lambda)\left|\frac{g}{1 - \lambda}\right|^p \\
\Rightarrow \|f + g\|_p &\leq \lambda \|\frac{f}{\lambda}\|_p^p + (1 - \lambda)\|\frac{g}{1 - \lambda}\|_p^p \\
&= \lambda (r + s)^p + (1 - \lambda)(r + s)^p \\
&= (\|f\|_p + \|g\|_p)^p
\end{aligned}$$

.

Note that this last step comes from appealing to the definition of lambda and noting $r^p = \int |f|^p$ and similarly for g. Now, we note that $t \mapsto |t|^p$ is strictly convex for $1 < p < \infty$, so equality occurs if and only if $\frac{f}{\lambda} = \frac{g}{1-\lambda}$ (almost everywhere if f, g are functions and not equivalence classes) hence f is a multiple of g. □

Remark. Note that this implies $L^p(S)$ is closed under addition, and constant multiplication (this part is trivial), so it is a linear space.

Definition 1.4 (Normed Linear Space)**.** A linear space V is a **normed linear space** if there is a function $\|.\| : V \to \mathbb{R}$ called the **norm of** V so that the following hold

- $\|v\| \geq 0$ for all $v \in V$,
- $\|v\| = 0$ if and only if $v = 0$,
- $\|\lambda v\| = |\lambda| \|v\|$ for all $\lambda \in R$, $v \in V$,
- $\|v + w\| \leq \|v\| + \|w\|$ for all $v, w \in V$.

Remark. $V_p(S)$ is not itself a normed linear space as the function $f(x) = \begin{cases} 0, & x \notin \mathbb{Q} \\ 1, & x \in Q \end{cases}$ has $\|f\| = 0$ even though f is not the zero function. We rule out this possibility by considering only the equivalence classes, in which case $f \sim 0$, so $L^p(S)$ is in fact a normed metric space.

Definition 1.5 (Conjugate)**.** For $p \in [1, \infty]$ we define the **conjugate** of p to be the extended real number $q \in [1, \infty]$ so that $\frac{1}{p} + \frac{1}{q} = 1$.

Lemma 1.1 (Young's Inequality)**.** Suppose $p \in (1, \infty)$ with q its conjugate and $a, b \in \mathbb{R}$ with $a, b \geq 0$. Then, $ab \leq \frac{a^p}{p} + \frac{q^p}{p}$. Moreover equality holds if and only if $a^p = b^q$.
Specifically $\sqrt{ab} \leq \frac{a+b}{2}$, that is the geometric mean is at most the arithmetic mean.

Proof. It suffices to assume a, b positive as the 0 case is trivial. Then, define $F(t) = a^{p(1-t)} b^{qt} = a^p \left(\frac{b^q}{a^p}\right)^t$. We see F is convex on $\mathbb{R}$ as it is exponential. Hence,

$$\begin{aligned} ab &= F\left(\frac{1}{p} \cdot 0 + \left(1 - \frac{1}{p}\right) q\right) \\ &\leq \frac{1}{p} F(0) + \left(1 - \frac{1}{p}\right) F(1) \\ &= \frac{a^p}{p} + \frac{b^q}{q}. \end{aligned}$$

As F is strictly convex (except in the case $\frac{b^q}{a^p} = 1$), we see equality will not arrive except in this exceptional case. $\square$