Analysis I

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Lecture 14: Measurable Functions (2)

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Recall. A function $f : S \to \mathbb{R}$ was measurable if S is measurable and $f^{-1}((c, \infty])$ is measurable for all $c \in \mathbb{R}$. There was an equivalent definition using the extended borel σ-algebra that we will use occasionally.

Proposition 0.1. Suppose $f : S \to \overline{\mathbb{R}}$ is continuous on the measurable set S, then f is measurable.

Proof. Let H be an extending function, then we must show $H \circ f$ is continuous. We see any subray , $f(X_0) = (c, \infty]$ will have $(H \circ f)(X_0) = (\hat{c}, 1]$. We know the preimage of this to be open in S, hence measurable. □

Proposition 0.2. Let $S \subseteq \mathbb{R}$. Suppose $f : S \to \mathbb{R}$ is measurable. and let $g : B \to \mathbb{R}$ with $B \in \overline{\mathscr{B}}$ and $f(S) \subseteq B$. Then, $g \circ f : S \to \mathbb{R}$ is measurable.

Proof. For $c \in \mathbb{R}$, we note that $(g \circ f)^{-1}((c, \infty]) = f^{-1}\left(g^{-1}((c, \infty])\right)$. By continuity of g, we know $g^{-1}((c, \infty]) \in \overline{\mathscr{B}}$. And, since f is measurable, we find $f^{-1}\left(g^{-1}((c, \infty])\right)$. □

Corollary 1. Let $S \subseteq \mathbb{R}$ and $f : S \to \mathbb{R}$ to be a measurable function. Then, for every $\alpha \in \mathbb{R}$ and $0 < \rho < \infty$, we find αf and $|f|^\rho$ are measurable.

Proof. We see the functions $g(u) = \alpha u$ on $\overline{\mathbb{R}}$ and $h(u) = |u|^\rho$ on $\overline{\mathbb{R}}$ to be the corresponding functions. We see the case h is clearly continuous and well defined. On the other hand g may be poorly defined if $\alpha = 0$ and $f(x) = \infty$. Recall, however, we had $0 \cdot \pm\infty = 0$ so g is just the zero functions and we see continuity holds. □

Definition 0.1 (Almost-everywhere)**.** Let S be measurable, then a property is said to hold true **almost everywhere** on S or **for almost all** $x \in S$ if there is a set T with $\mu(T) = 0$ and the property holds on all of $S \setminus T$.

Proposition 0.3. Let $S \subseteq \mathbb{R}$ and suppose $f, g : S \to \overline{\mathbb{R}}$ such that f is measurable and $g = f$ almost everywhere on S, then g is measurable.

Proof. Let $T = \{x \in S : f(x) \neq g(x)\}$. Fix $c \in \mathbb{R}$ and let $F = f^{-1}((c, \infty]) \setminus T$ and $G = f^{-1}((c, \infty]) \cup T$. Clearly, both F and G are measurable. Furthermore, $F \subseteq G$ and $\mu(G \setminus F) = \mu(T) = 0$. Since, $F \subseteq g^{-1}((c, \infty]) \subseteq G$. And, by an earlier characterization we recall that a set X is measurable if and only if there were nested sets around it with a difference of measure 0 . Hence, g is measurable. $\square$

Remark. Suppose $f : S \to \overline{\mathbb{R}}$ is a measurable set and $S \subseteq X \subseteq \mathbb{R}$. If $\mu(X \setminus S) = 0$ and $h : X \to \overline{\mathbb{R}}$ is any extension of f , then h is measurable since $h^{-1}((c, \infty]) = f^{-1}((c, \infty]) \cup \{x \in X \setminus S : h(x) \in (c, \infty]\}$. This is the union of a measurable set with a set of measure 0, so we see h is measurable.

Notation. Instead of saying that every extension of a measurable function $f : S \to \overline{\mathbb{R}}$ to a function $h : X \to \overline{\mathbb{R}}$, we often just say f is measurable on X as long as it is defined almost everywhere on X and is measurable on that set.

Proposition 0.4. Suppose $f : I \to \overline{\mathbb{R}}$ is monotone on $I \subseteq \mathbb{R}$. Then, the set of all points in I where f fails to be continuous is countable, hence measure 0. Another characterization is that f is continuous almost everywhere, hence f is measurable.

Proof. It suffices to consider the case f is increasing and I open. Let E be the set of all $x \in I$ where f fails to be continuous. For $x \in E$ let $\alpha_x = \sup(\{f(z) : z < x\} z \in J)$ and $\beta_x = \inf(\{f(z) : z > x\} z \in J)$. Since f is not continuous at x, we find the interval $(\alpha_x, \beta_x) = I_x$ to be nonempty. Also, if $x, y \in E$ are distinct with $x < y$ we find $\beta_x <= \alpha_y$. Hence, we find $I_x \cap I_y = \varnothing$. Since each interval I_x for $x \in E$ contains a rational number, we see E is countable. Hence, $\mu(E) = 0$ and we see $f|_{I \setminus E}$ is continuous on $I \setminus E$ which is measurable, hence the restriction is measurable and as f coincides with its restriction almost everywhere, we see f is measurable. $\square$

Definition 0.2 (Finite Functions)**.**
- Let $S \subseteq \mathbb{R}$. A function $f : S \to \overline{\mathbb{R}}$ is called **finite on** S if $|f(x)| < \infty$ for all $x \in S$.
- Let $f, g : S \to \overline{\mathbb{R}}$ Then we say $f < g$ if $f(x) < g(x)$ for all $x \in S$. Similarly for all other inequalities.
- f is called **nonnegative** if $f \geq 0$ and **positive** if the inequality is strict.

Proposition 0.5. Let $f, g : S \to \overline{\mathbb{R}}$ be measurable and finite almost everywhere. Then, $f+g, f-g, f \cdot g$ are measurable.
If $g(x) \neq 0$ for almost every $x \in S$, then $\frac{f}{g}$ is measurable.

Proof. 1. First, we prove addition. We may assume f, g are finite on S . Then, $h = f+g$ is well defined. Since for $x \in S$, we have $h(x) > q$ for $c \in R$ if and only if there is a $q \in \mathbb{Q}$ such that $f(x) > q$ and $g(x) > c - q$, we have

$$\begin{aligned} h^{-1}((c,\infty]) &= h^{-1}((c,\infty)) \text{ by finiteness.} \\ &= \bigcup_{q\in\mathbb{Q}} f^{-1}((q,\infty)) \cup g^{-1}(c-q,\infty). \end{aligned}$$

Hence, h as measurable as these are all measurable sets. If f, g are measurable, then so are $f, -g$, hence $f + (-g) = f - g$

2. With addition, subtraction is completely trivial,

3. Now multiplication, Let h be any measurable finite function on S. Consider $(h)^2$. If $c \geq 0$, we have

$$\left((h)^2\right)^{-1}((c,\infty)) = h^{-1}\left((-\infty,\sqrt{c})\right) \cup h^{-1}\left((\sqrt{c},\infty)\right).$$

If $c < o$, then

$$\left((h)^2\right)^{-1}((c,\infty)) = h^{-1}(\mathbb{R}) = S.$$

As in either case we had the preimage being measurable, we see $(h)^2$ is measurable. Since $f \cdot g = \frac{1}{2}(f+g)^2 - \frac{1}{2}(f)^2 - \frac{1}{2}\left(g^2\right)$ being the sum, constant multiple and square of measurable functions yields $f \cdot g$ to be measurable.

4. Lastly, let $h = \frac{1}{g}$, and note we can assume g is nonzero for all S , hence h is well defined on S and h is finite. If $c > 0$ we see $h^{-1}((c,\infty)) = g^{-1}(\left(0, \frac{1}{c}\right))$. As this interval is open and borel, we see $g^{-1}\left(\left(0,\frac{1}{c}\right)\right)$ is borel, hence $h^{-1}((c,\infty))$ is measurable.
Similairly, if $c = 0$, we see $h^{-1}((0,\infty)) = g^{-1}((0,\infty))$.
Lastly, if $c < 0$ we have $h^{-1}(c,\infty) = g^{-1}(\left(-\infty,\frac{1}{c}\right)) \cup g^{-1}((0,\infty)) = g^{-1}\left(\left[\frac{1}{c},0\right)^c\right)$ hence measurable. This completes the proof.

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Lecture 15: Measurable Functions (3) and Simple Functions

Thu 14 Oct 2021 13:01

Proposition 0.6. Let (f_n) be a sequence of measurable functions $f_n : S \to \overline{\mathbb{R}}$. Then, we define $f, g, F, G : S \to \overline{\mathbb{R}}$ with

- $f(x) = \sup\{f_n(x) : n \in \mathbb{N}\}$,
- $g(x) = \inf\{f_n(x) : n \in \mathbb{N}\}$,
- $F(x) = \limsup_{n\to\infty} f_n(x)$,
- $G(x) = \liminf_{n\to\infty} f_n(x)$

all being measurable.

Proof.
- Note that $f(x) > c$ if and only if there is an n such that $f_n(x) > c$. Hence, $f^{-1}((c, \infty]) = \bigcup_{n\in\mathbb{N}} f_n^{-1}((c, \infty)$ is measurable.
- It it clear $g(x) = -\sup\{-f_n(x) : n \in \mathbb{N}\}$.
- Next, note that $F(x) = \inf\{\sup\{f_k(x) : k \geq n\} : n \in \mathbb{N}\}$ and $G(x) = \sup\{\inf\{f_k(x) : k \geq n\} : n \in \mathbb{N}\}$, hence they are measurable by the first two theorems.

□

Remark. It is also true that for a measurable function $f : S \to \overline{\mathbb{R}}$ is measurable implies

$$f^+(x) = \sup\{f(x), 0\}$$
$$f^-(x) = \sup\{-f(x), 0\}$$

are also measurable.

1 Simple Functions

Definition 1.1. Let $S \subseteq \mathbb{R}$. Then,

$$\chi S : \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \longmapsto \chi_S(x) = \begin{cases} 1, & x \in S \\ 0, & x \notin S \end{cases}$$

is the **characteristic function of** S .
A measurable function $s : \mathbb{R} \to \overline{\mathbb{R}}$ is a **simple functions** if $s(\mathbb{R})$ is finite.

Proposition 1.1. If s is a simple function. Then, there exists a finite, disjoint collection of measurable sets $\{S_k : 1 \leq k \leq K\}$ and a finite sequence of distinct real numbers $(a_k)_{1\leq k\leq K}$ such that $\mathbb{R} = \bigcup_{k=1}^{K} S_k$ and $s = \sum_{k=1}^{K} a_k \chi_{S_k}$. Furthermore, this combination is unique up to permutation of the a_k, s_k. This representation is called the **canonical representation**.

Lemma 1.1. Let $f : \mathbb{R} \to \mathbb{R}$ be nonnegative and measurable with $f(\mathbb{R})$ being bounded, then for each $\varepsilon > 0$ there is a nonnegative simple function s such that $f \geq s$ and $f(x) - s(x) < \varepsilon$ for all $x \in \mathbb{R}$.

Proof. There is a $M > 0$ such that $f(\mathbb{R}) \subseteq [0, M)$. Given ε, let $y_k = k\varepsilon$ for $k \in \mathbb{N}_0$. Since, $y_k - y_{k-1} = \varepsilon$, there is $N \in \mathbb{N}$ such that $[0, M] \subseteq \bigcup_{k \in \mathbb{N}} [y_{k-1}, y_k)$. Let $S_k = f^{-1}([y_{k-1}, y_k))$ for $1 \leq k \leq N$. Define $s = \sum_{k=1}^{N} y_{k-1} \chi_{S_k}$. Then, $s \geq 0$ and s is simple. Furthermore, for each $x \in \mathbb{R}$, there is a unique k, with $1 \leq k \leq N$ such that $f(x) \in [y_{k-1}, y_k)$. Consequently, $s(x) = y_{k-1} \leq f(x) < y_k$. Hence, $f(x) - s(x) < y_k - y_{k-1} = \varepsilon$. $\square$

Theorem 1.1. $f : \mathbb{R} \to \overline{\mathbb{R}}$ is measurable if and only if there is a sequence of simple functions (s_n)a such that (s_n) converges pointwise to f and $|f| \geq |s_n|$ for all $n \in \mathbb{N}$.

Proof. Suppose the sequence (s_n). Then, f is measurable as

$$f = \lim_{n \to \infty} s_n = \limsup_{n \to \infty} s_n = \liminf_{n \to \infty} s_n.$$

Now, assume f is measurable. Then, $f = f^+ - f^-$. Both f^+ and f^- are measurable and nonnegative. Since the difference of two simple functions is simples, it suffices to assume $f \geq 0$, that is $f^- = 0$. Let $B_n = \{x \in \mathbb{R} : f(x) \leq n\}$ and $g_n = f\chi_{B_n}$ for all $n \in \mathbb{N}$. Since $g_n(x) = \inf\{f(x), n\chi_{B_n}\}$. Then, we see g_n is measurable as f and the simple function $n\chi_{B_n}$ are measurable. Furthermore, g_n is bounded. Hence, there is a measurable simple function r_n such that $g_n \geq r_n$ and $g_n(x) - r_n(x) < \frac{1}{n}$ for all x. Finally, define

$$s_n = r_n + n\chi_{B_n^c}.$$

Then, we find (s_n) is the sequence of functions desired. $\square$

Corollary 2. Let (f_n) be a sequence of nonnegative measurable functions $f_n : \mathbb{R} \to \overline{\mathbb{R}}$. Then, $x \mapsto \sum_{i=1}^{\infty} f_k(x)$ is measurable. In particular, if $f, g : \mathbb{R} \to \overline{\mathbb{R}}$ are nonnegative and measurable, then so is $f + g$.

Proof. For $N \in \mathbb{N}$, let $F_n = \sum_{k=1}^{N} f_k$. For each k there is sequence of simple functions $(s_{k,n})_n$ such that $(s_{k,n})_n$ converges pointwise to f_k and $f_k \geq s_{k,n} \geq 0$ for all n.
Hence, $\left(\sum_{k=1}^{N} s_{k,n}\right)_n$ is a sequence of nonnegative simple functions such that $F_N \leq \sum_{k=1}^{N} s_{k,n}$ for all n and

$$\lim_{n \to \infty} \sum_{k=1}^{N} s_{k,m}(x) = F_N(x)$$

for all $x \in \mathbb{R}$.
So, F_N is the limit of a sequence of measurable functions, so it is measurable. Furthermore, we have that for each $x \in \mathbb{R}$, $\left(F_{N(x)}\right)_N$ is increasing, we find

$$\sum_{k=1}^{\infty} f_k = \limsup_{N\to\infty} F_N = \lim_{N\to\infty} F_N.$$

□

2 Littlewood's 3 Principles

Remark.

1. Every measurable set is "nearly" the union of a finite collection of intervals.
2. Every measurable function is "nearly" continuous.
3. Every pointwise convergent sequence of measurable functions is "nearly" uniformly continuous.

We state these princeiples rigorously in the following way:

Theorem 2.1. If S is measurable, with $\mu(S) < \infty$, then for each $\varepsilon > 0$ there is a finite disjoint collection of open intervals $\{I_k : 1 \leq k \leq n\}$ such that for $U = \bigcup_{k=1}^{n} I_k$ we find

$$\mu(S \triangle U) < \varepsilon.$$

Theorem 2.2 (Lucin's Theorem)**.** Let $f : S \to \mathbb{R}$ be measurable with $\mu(S) < \infty$. Then, for each $\varepsilon > 0$ there is a compact $K \subseteq S$ such that $f|_K : K \to \mathbb{R}$ is continuous and $\mu(S \setminus K) < \varepsilon$.

Theorem 2.3 (Lucin's Theorem for functions on $\mathbb{R}$)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be measurable. Then, for all $\varepsilon > 0$ there is a continuous $g : \mathbb{R} \to \mathbb{R}$ and a closed set $E \subseteq \mathbb{R}$ such that $f = g$ on E and $\mu(E^c) < \varepsilon$. Moreover, $\sup\{|g(x)| : x \in \mathbb{R}\} \leq \sup\{|f(x)| : x \in \mathbb{R}\}$.

Theorem 2.4 (Egoroff's Theorem)**.** Let S be measurable with $\mu(S) < \infty$. Suppose (f_n) is a sequence of measurable functions $f_n : S \to \mathbb{R}$ which converges pointwise almost everywhere to $f : S \to \mathbb{R}$. Then, for all $\varepsilon > 0$, there is a measurable $E \subseteq S$ such that $\mu(E) < \varepsilon$ and (f_n) converges uniformly to f on $S \setminus E$.