MATH 7237/8237

LECTURE Feb. 3, 2021

CLIQUES IN GRAPHS

Let G be graph.

Definition A subgraph of G that is complete is called a **clique** of G.

Often the word *clique* is preceded by a number or a parameter indicating the order of the clique.

For example, we say "the 4-cliques of G" or "the k-cliques of G".

Note that the 2-cliques of a graph are simply its edges, and 3-cliques are usually called "triangles".

Let us make clear that "clique" is not synonymous with "complete graph" as cliques are always thought as subgraphs of a larger graph.

Definition A **maximal clique** in a graph is a clique that is not contained in a larger clique.

Definition A **maximum clique** in a graph is a clique of largest order in the graph.

Clearly, a maximum clique in a graph is also maximal, but the converse may fail (e.g., the union of K_3 and K_4.)

Note also that there may be more than one maximum cliques in a graph (e.g., the union of two triangles.)

Definition The order of the largest clique in a graph G is called the **clique number** of G and is denoted by $\omega(G)$.

Problem *Determine* $\omega\left(K_{m,n}\right)$, $\omega\left(P_n\right)$, *and* $\omega\left(C_n\right)$.

THE MAXIMUM CLIQUE PROBLEM

The problem of finding $\omega(G)$ of a given graph G is one of the fundamental hard algorithmic problems and is known as the **maximum clique problem.**

It belongs to a large class of $\mathcal{NP}$-complete problems, all equivalent to each other and possessing the same algorithmic hardness.

The open question of paramount importance can be stated in terms of graphs and cliques as follows:

Problem *Is there an algorithm that can determine $\omega\left(G\right)$ for any graph G of order n in a number of steps that is polynomial in n?*

The difficulty comes from the fact that the number of steps of the current algorithms grows as $n^{\omega(G)}$.

INDEPENDENT SETS

Let G be graph.

Definition A set of vertices $X \subset V(G)$ is called an **independent set** of G if no two vertices of X are joined in G.

Sometimes independent sets are also called **stable sets**.

Definition A **maximal independent set** in a graph is an independent set that is not contained in a larger independent set.

Definition A **maximum independent set** in a graph is an independent set of largest cardinality in the graph.

Clearly, a maximum independent set in a graph is also maximal, but the converse may fail (e.g., $K_{3,4}$.)

INDEPENDENCE NUMBER

Definition The order of a maximum independent set in a graph G is called the **independence number** of G and is denoted by $\alpha(G)$.

Note that every clique in a graph G determines an independent set in the complement $\overline{G}$.

In particular, one notices that $\alpha(G) = \omega(\overline{G})$.

Problem *Determine* $\alpha(K_{m,n})$, $\alpha(P_n)$, *and* $\alpha(C_n)$.

Answers: $\max(m,n)$, $\lceil n/2 \rceil$, and $\lfloor n/2 \rfloor$

Problem *Prove that complete graphs are the only graphs with independence number* 1.

Problem *Prove that if G is a graph of order n, then $\alpha(G) + \delta(G) \leq n$.*

Let I be a maximum independent set of G, that is, $|I| = \alpha(G)$. Take a vertex $v \in I$.

Note that none of the neighbors of v belongs to I, as I is independent.

Hence, G has at least $|I| + d(v)$ vertices, and therefore,

$$n \geq |I| + d(v) \geq \alpha(G) + \delta(G).$$

Problem *Let $n \geq 2\delta \geq 2$. Construct a graph G of order n such that*

$$\alpha(G) + \delta(G) = n.$$

The graph $K_{\delta,n-\delta}$ satisfies the requirement.

MATCHINGS

Definition A set of vertex disjoint edges in a graph G is called a **matching** in G.

Often the word *matching* is preceded by a number or a parameter indicating the cardinality of the matching (i.e., the number of its edges.)

For example, we say "the 2-matchings of G" or "the k-matchings of G".

Definition If a vertex v belongs to an edge of a matching M, we say that M **covers** v.

Note that the vertices covered by a matching may induce a lot of other edges, but they do not count in the matching.

For example, each pair of opposite edges in C_4 determines a 2-matching, and there are two such pairs.

Problem *Check that K_4 has three 2-matchings.*

Problem *Prove that the number of 2-matchings of C_n is $n(n-3)/2$.*

The number of all pairs of edges of C_n is $\binom{n}{2}$.

Each pair is either a 2-matching of C_n or a pair of edges that share a vertex.

There are exactly n pairs of edges sharing a vertex, so the number of 2-matchings of C_n is equal to

$$\binom{n}{2} - n = n(n-3)/2.$$

Problem *Let $n \geq 4$. Prove that the number of 2-matchings of K_n is* $3\binom{n}{4}$.

Proof. Partition the set of all 2-matchings of K_n into $\binom{n}{4}$ groups, according to the set of four vertices they cover.

Each such group is a 4-clique of K_n, and therefore determines precisely three distinct 2-matchings.

MATCHING NUMBER

Definition A **maximal matching** in a graph is a matching that is not contained in a larger matching (as a set of edges).

Definition A **maximum matching** in a graph is a matching of largest cardinality in the graph.

Definition The **matching number** of a graph G is the number of edges in a maximum matching of G. The matching number is denoted by $\beta(G)$.

Problem *What are the matching numbers of* P_n, C_n, K_n, $K_{m,n}$?

Answers: $\lfloor n/2 \rfloor$, $\lfloor n/2 \rfloor$, $\lfloor n/2 \rfloor$, $\min(m, n)$.

Problem *Prove that if G is a graph, then*

$$2\beta(G) + \alpha(G) \geq n.$$

Let M be a maximum matching in G, that is, $|M| = \beta(G)$.

Write U for the set of vertices covered by the edges of M. Clearly,

$$|U| = 2|M| = 2\beta(G).$$

Note that the set $V(G) \setminus U$ is independent. Otherwise there are two adjacent vertices u and v in $V(G) \setminus U$, and so the set $M \cup \{u, v\}$ is a matching, contrary to our assumption that M is a maximum matching.

Hence $V(G) \setminus U$ is an independent set, and therefore

$$n - 2\beta(G) = |V(G) \setminus U| \leq \alpha(G),$$

yielding the desired inequality.

PERFECT MATCHINGS

Definition A matching in a graph G is called **perfect** if it covers every vertex of G.

Since every vertex of a matching in G belongs to $V(G)$, it turns out that if M is a perfect matching in G, then $v(G) = 2|M|$.

A matching covers an even number of vertices, so perfect matchings may exist only if $v(G)$ is even.

Problem *Every path of even order has a perfect matching.*

Indeed, if

$$v_1, \ldots, v_{2n}$$

is a path, the edges $\{v_{2k-1}, v_{2k}\}$, where $k = 1, \ldots, n$, are disjoint, so they form a matching.

Problem *Prove that $K_{m,n}$ has a perfect matching if and only if $m = n$.*

Problem *If M is a perfect matching in G, then for every edge $\{u, v\} \in M$, the set $M \setminus \{u, v\}$ is a perfect matching in $G - u - v$.*

Problem *Prove that the number of perfect matchings of K_{2n} is equal to*

$$\frac{\binom{2n}{2}\binom{2n-2}{2}\ldots\binom{2}{2}}{n!}.$$

The number of ordered sequences of n vertex-disjoint edges in K_{2n} is

$$\binom{2n}{2}\binom{2n-2}{2}\ldots\binom{2}{2}.$$

To get the number of perfect matchings, we have to divide this number by $n!$.

ALTERNATING PATHS

Let M be a matching in a graph G.

Definition A path in G is called M-**alternating** if every other edge of the path belongs to M.

Example If

$$P = v_1, \ldots, v_{2n}$$

is a path, then the edges $\{v_2, v_3\}, \{v_4, v_5\}, \ldots, \{v_{2n-2}, v_{2n-1}\}$ form a matching, say M. Now P is an M-alternating path.

An M-alternating path may or may not start or end with an edge of M.

Definition An M-alternating path is called M-**augmenting** if its first and last vertices are not covered by M.

BERGE'S THEOREM

Theorem (Berge) A matching M in G is maximum if G contains no M-augmenting path.

Let M be a matching in G and $P = v_1, \ldots, v_{2n}$ be an M-augmenting path.

Then the edges

$$\{v_2, v_3\}, \{v_4, v_5\}, \ldots, \{v_{2n-2}, v_{2n-1}\}$$

belong to M, and the edges

$$\{v_1, v_2\}, \{v_3, v_4\}, \ldots, \{v_{2n-1}, v_{2n}\}$$

do not belong to M.

Replacing the former with the latter ones we obtain a matching with more edges than M. So M is not maximum.

BERGE'S THEOREM

Now, let M be a matching that is not maximum and M' be a maximum matching. We must show that G contains an M-augmentng path.

Let H be the graph formed by the edges that belong to exactly one of the sets M and M'.

The degree of any vertex of H is one or two, hence the components of H are either even cycles or paths, and their edges alternate between M and M'.

However $|M'| > |M|$. Hence, one of the components of H is a path, say P, and M' covers the ends of P.

Therefore, P is an M-augmenting path.

MATCHINGS IN BIPARTITE GRAPHS

Let us write $G[A,B]$ to denote a bipartite graph with vertex classes A and B.

In many applications of graphs it is necessary to find if there is a matching in a bipartite graph $G[A,B]$ that covers all vertices of A.

This problem was resolved by Hall in 1935.

For a set S of vertices in a graph G, write $N(S)$ for the set of all neighbors of vertices of S.

Theorem (Hall) Let $G[A,B]$ be a bipartite graph. There is a matching covering all vertices of A if and only if $|N(S)| \geq |S|$ for every set $S \subset A$.

Clearly, if there is a matching covering all vertices of A, then $|N(S)| \geq |S|$ for every set $S \subset A$.

The proof of the converse is more involved and we shall skip it.

Instead we shall show a few applications of Hall's theorem.

For a start, let us mention that Hall's theorem is also known as the *Marriage Theorem* stated as:

If every group of girls in a village collectively like at least as many boys as there are girls in the group, then each girl can marry a boy she likes.

Here are three more mathematical corollaries of Hall's theorem.

Corollary A bipartite graph $G[A, B]$ has a perfect matching if and only if $|A| = |B|$ and $|N(S)| \geq |S|$ for every set $S \subset A$.

Corollary Every regular bipartite graph has a perfect matching as long as it is not edgeless.

Corollary If $r \geq 1$, then every r-regular bipartite graph is a union of r disjoint perfect matchings.

THANK YOU