Analysis I: Homework 7

Thomas Fleming

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Problem (31 (Collaborated with Andrea))**.** Let $f = \frac{\sin(x)}{x}$. We aim to show $\int f^+ = \infty$. First, note that $\int f^+ \geq \int_{[0,\infty]} f^+$ so it suffices to show this quantity infinite. Moreover, in this interval we find f positive for $x \in [2n\pi, (2n+1)\pi]$ for $n \in \mathbb{Z}_0^+$. Hence, defining $f_n = f^+ \chi_{[0,(2n+1)\pi]}$ we see each f_n is measurable (it is continuous) and non-negative with $\lim\limits_{n\to\infty} f_n(x) = f^+(x)$ for all $x \geq 0$. Moreover, since $[0, (2n+1)\pi] \subseteq [0, (2(n+1)+1)\pi]$ we see $f_n \leq f_{n+1}$ for all $x \in [0, \infty)$. Hence, applying dominated convergence yields

$$
\begin{aligned}
\int f^+ &\geq \int_{[0,\infty)} f^+ \\
&= \int_{[0,\infty]} \lim_{n\to\infty} f_n \\
&= \lim_{n\to\infty} \int_{[0,\infty]} f_n \\
&= \lim_{n\to\infty} \sum_{i=0}^{n} \int_{[2i\pi,(2i+1)\pi]} f_n \\
&\geq \lim_{n\to\infty} \sum_{i=0}^{n} \int_{[2i\pi,(2i+1)\pi]} \frac{\left(\sin^+(x)\,|_{[0,2n\pi]}\right)^*}{(2i+1)\pi} \\
&= \lim_{n\to\infty} \sum_{i=0}^{n} \frac{1}{(2i+1)\pi} \int_{[2i\pi,(2i+1)\pi]} \left(\sin^+(x)\,|_{[0,2n\pi]}\right)^* \\
&\geq \lim_{n\to\infty} \sum_{i=0}^{n} \frac{1}{(2i+1)\pi} \text{ since } \int_{[2i\pi,(2i+1)\pi]} \sin^+(x) = \int_{[0,\pi]} \sin(x) = 2. \\
&= \frac{1}{\pi} \lim_{n\to\infty} \sum_{i=0}^{n} \frac{1}{(2i+1)} \\
&= \frac{1}{\pi} \sum_{i=0}^{\infty} \frac{1}{2i+1} \\
&= \infty.
\end{aligned}
$$

Hence, $\int f^+$ is not finite, so f is nonintegrable.

Problem (32)**.** First, note that $f \coloneqq \frac{1}{\sqrt{x}}$ is measurable (preimage of an interval is an interval) and finite almost everywhere. Then, we define $A_n = \left[\frac{1}{(n-1)^2}, \frac{1}{n^2}\right]$ and the simple functions $s_n = \sum_{i=1}^{n} i\chi_{A_i}$. As each term is positive, we see s_n is increasing for fixed x. Moreover, $s = \lim\limits_{n\to\infty} s_n$ is integrable by applying DCT

$$\begin{aligned}
\int s &= \lim_{n\to\infty} \int s_n \\
&= \lim_{n\to\infty} \sum_{i=1}^{\infty} \left(\frac{1}{(i-1)^2} - \frac{1}{i^2}\right) i \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{(i-1)^2} + \frac{1}{i\,(i-1)} \\
&\leq \lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{i^2} \\
&= \frac{\pi^2}{3}.
\end{aligned}$$

Then, for any $t \in \mathscr{S}(f)$, we see $t \leq s_n \leq s$ for some $n = \left\lceil \frac{1}{\sqrt{\inf\{x : x \in \text{support}(t)\}}} \right\rceil$. Hence, $\int s$ is an upper bound of $\sup\{\int t : t \in \mathscr{S}(f)\} = \int f$, so $\int f$ is bounded and hence f is integrable.

Problem (33)**.** First, basic limits show $\lim_{n\to\infty} h_n(x) = \begin{cases} 3, & x \in (-1,1) \\ 2, & x = -1 \text{ or } x = 1 \\ 1, & x \in (-\infty,-1) \cup (1,\infty) \end{cases}$

Moreover, $h_n(x)$ is continuous for every $n \in \mathbb{N}$, hence measurable. So, we see $h_n \cdot f$ is measurable for every $n \in \mathbb{N}$. Then, $\lim_{n\to\infty} (h_n \cdot f)(x) = \begin{cases} 3f(x), & x \in (-1,1) \\ 2f(x), & x = \pm 1 \\ f(x), & x \in (-\infty,-1) \cup (1,\infty) \end{cases}$.

Hence, we see $|h_n \cdot f| \leq 3|f|$ with $3|f|$ being integrable (since f is integrable). Applying dominated convergence yields

$$\lim_{n\to\infty} \int h_n \cdot f = \int \lim_{n\to\infty} h_n \cdot f = \int_{[-\infty,-1]} f + \int_{[-1,1]} 3f + \int_{[1,\infty]} f = \int f \, \mathrm{dx} + 2\int_{[-1,1]} f \, \mathrm{dx}\,.$$

Problem (34)**.** First, basic limits again show $\lim_{n\to\infty} e^{-\frac{x}{n}} = 1$. Moreover, fixing x, we see $e^{-\frac{x}{n}} < e^{-\frac{x}{n+1}}$, so we see $e^{-\frac{x}{n}} |f| \leq e^{-\frac{x}{n+1}} |f|$. Then, denoting $e^{-\frac{x}{n}} |f| = f_n$, we see $\lim_{n\to\infty} f_n = \lim_{n\to\infty} e^{-\frac{x}{n}} \lim_{n\to\infty} |f| = \lim_{n\to\infty} |f|$ with each f_n being measurable (as it is the product of continous functions) and increasing, hence passing to the 0-extension and applying monotone convergence yields

$$1 \geq \lim_{n\to\infty} \int_{(0,\infty)} f_n = \lim_{n\to\infty} \int f_n^* = \int \lim_{n\to\infty} f_n^* = \int (|f|)^* = \int_{(0,\infty)} |f| \,.$$

Since f is continuous, we see it is measurable, and since it is absolutely integrable on $(0,\infty)$, we have f being integrable on $(0,\infty)$.

Problem (35)**.** First, recall $\sum_{i=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$.
Then, define $g_n = \sum_{i=1}^{n} f_i^2$ and note that $g_n \leq g_{n+1}$ as each term is finite. Moreover g_n is the sum of measurable functions, so it is measurable. Lastly, define $\lim_{n\to\infty} g_n(x) = g(x) = \sum_{i=1}^{\infty} f_n^2(x)$ Then, monotone convergence and zero extensions yield

$$\begin{aligned}
\int_{[0,1]} g = \lim_{n\to\infty} \int_{[0,1]} g_n &= \lim_{n\to\infty} \int g^* \\
&= \lim_{n\to\infty} \int (\sum_{i=1}^{n} f_n^2)^* \\
&= \lim_{n\to\infty} \int_{[0,1]} \sum_{i=1}^{n} f_n^2 \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \int_{[0,1]} f_n^2 \\
&\leq \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n^4} \\
&= \frac{\pi^4}{90}
\end{aligned}$$

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Moreover, $0 \leq \int_{[0,1]} f_n^2$ as the integrand is always non-negative. Hence, as the sum is bounded and strictly increasing, we see the terms tend to 0. That is $\lim_{n\to\infty} \int_{[0,1]} f_n^2 = 0$.