# Analysis I

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## Contents



### Lecture 16: Conclusion of Measure Theory and Lebesque Integration

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**Recall.** We stated the theorems behind littlewood's 3 principles, now we prove them.

*Proof.* 1. (2.2). Let $J$ be the collection of all open intervals $(a, b)$ with $a, b \in \mathbb{Q}$ and $a < b$. Since $J$ is countable we can order the intervals $J = \{J_k : k \in \mathbb{N}\}$. Let $\varepsilon > 0$ and first we do the case $S$ is bounded. For each $n \in \mathbb{N}$, there is a closed set $C_n \subseteq f^{-1}(J_n)$ and a $D_n = S \setminus f^{-1}(I_n)$ such that $\mu(S \setminus (C_n \cup D_n)) < \frac{\varepsilon}{2^n}$. Since $S$ is bounded, $C_n$ and $D_n$ are compact. Let $K = \bigcap_{n\in\mathbb{N}} (C_n \cup D_n)$ and as $C_n, D_n \subseteq S$ , we see $K \subseteq S$. Furthermore, $K$ is compact and we find $\mu(S \setminus K) \leq \sum_{i=1}^{\infty} \mu(S \setminus (C_n \cup D_n)) < \varepsilon$. Now, we show the restriction is continuous. Let $\varepsilon > 0$, then for $x \in K$ we find $a, b \in \mathbb{Q}$ such that $a < f(x) < b$ and $b - a < \varepsilon$. Hence, there is $n \in \mathbb{N}$ such that $I_n = (a, b)$. Consequently, $x \in f^{-1}(I_n)$ and $x \notin S \setminus f^{-1}(I_n)$. So, $x \in \left(S \setminus f^{-1}(I_n)\right)^c \subseteq D_n^c$. As $D_n$ is closed, $D_n^c$ is open, hence there is a $\delta > 0$ so that $(x - \delta, x + \delta) \subseteq D_n^c$. If $y \in K \cap D_n^c$, then $y \in C_n$, thus $y \in f^{-1}(I_n)$ , hence $a < f(y) < b$. So, $|f(x) - f(y)| < b - a = \varepsilon$ for $y \in (x - \delta, x + \delta)$.
   Now, we do the unbounded case. As $S$ is unbounded and $\varepsilon > 0$ , we find $N \in N$ so that $S' = S \cap [-N, N]$ has the property $\mu(S \setminus S') < \frac{\varepsilon}{2}$, that is $S$ is approximated by a bounded function arbitrarily well. Since $S'$ is bounded, there is a compact set $K \subseteq S' \subset S$ so that $f \mid K$ is continuous and $\mu(S' \setminus K) < \frac{\varepsilon}{2}$. Then, $\mu(S \setminus K) = \mu(S \setminus S') + \mu(S' \setminus K) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

2. (2.4). Let $E^*$ be the set of all $x \in S$ such that $(f_n(x))$ does not converge. By assumption, $\mu(E^*) = 0$. Since $f(x) = \lim_{n\to\infty} f_n(x) = \limsup_{n\to\infty} f_n(x)$ for all $x \in S \setminus E^*$, then $f$ is measurable. For $k, \ell \in \mathbb{N}$ , let $E_{k,\ell} = \{x \in S : |f_\ell(x) - f(x)| \geq \frac{1}{k}\}$. Then, $E_{k,\ell}$ is measurable. Fix $k$. If for each $n \in \mathbb{N}$ there is a $\ell \geq n$ so that $|f_\ell(x) - f(x)| \geq \frac{1}{k}$ , then $x \in E^*$ as $f$ does not converge at that point. Hence, $\bigcap_{n\in\mathbb{N}} \bigcup_{\ell=n}^{\infty} E_{k,\ell} \subseteq E^*$. Since $\mu(\bigcup_{\ell=1}^{\infty} E_{k,\ell}) \leq \mu(S) \leq \infty$ , and the collection $\{\bigcup_{\ell=n}^{\infty} E_{k,\ell}\}$ is clearly descending. Hence, $\mu\left(\bigcap_{n\in\mathbb{N}} \bigcup_{k=n}^{\infty} E_{k,\ell}\right) = \lim_{n\to\infty} \mu(\bigcup_{\ell=n}^{\infty} E_{k,l}) \leq \mu(E^*) = 0$.

This holds for all $k \in \mathbb{N}$. So, for $\varepsilon > 0$ and $k \in \mathbb{N}$, we have a $n_k \in \mathbb{N}$ such that $\mu\left(\bigcup_{\ell=n_k}^{\infty} E_{k,\ell}\right) < \frac{\varepsilon}{2^k}$. Thus, $E = \bigcup_{k\in\mathbb{N}} \bigcup_{\ell=n_k}^{\infty} E_{k,\ell}$ is measurable and $\mu(E) < \sum_{k=1}^{\infty} \bigcup_{\ell=n_k}^{\infty} E_{k,l} = \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon$. If $x \in S \setminus E$, then $|f_n(x) - f(x)| < \frac{1}{k}$ for $k \in \mathbb{N}$ if $n \geq n_k$. So, $(f_n)$ converges uniformly on $S \setminus E$.

□

This concludes measure theory.

# 1 Lebesque Integration

**Definition 1.1** (Lebesque Integral: Nonnegative Simple Functions)**.** Let $s$ be a nonnegative simple function of the form $s = \sum_{k=1}^{K} a_k \chi_{S_k}$ where $\{S_k : 1 \leq k \leq K\}$ is a disjoint collection of measurable sets. Then, the **Lebesque Integral** of $s$ is defined to be

$$\int s = \int s(x)\,dx = \int s d\mu = \sum_{k=1}^{K} a_k \mu(S_k).$$

**Proposition 1.1.** If $s$ is nonnegative and simple with two representations, $s = \sum_{k=1}^{K} a_k \chi_{S_k} = \sum_{j=1}^{J} b_j \chi_{T_j}$ for disjoint collections of measurable sets $\{S_k : 1 \leq k \leq K\}$ and $\{T_j : 1 \leq j \leq J\}$. Then

$$\sum_{k=1}^{K} a_k \mu(S_k) = \sum_{j=1}^{J} b_j \mu(T_j).$$

In particular, $\int s$ is well defined.

The proof of this is trivial.

**Lemma 1.1.** Let $s, t$ be nonnegative and simple and $\alpha \geq 0$. Then

$$\alpha \cdot \int s = \int \alpha \cdot s \text{ and } \int (s+t) = \int s + \int t$$

.

*Proof.* Clearly, multiplying the sum times $\alpha$ yields $\alpha \sum_{k=1}^{K} a_k \mu(S_k) = \sum_{k=1}^{K} \alpha a_k \mu(S_k)$. For the second claim. Suppose $s = \sum_{k=1}^{K} a_k \chi_{S_k}$ and $g = \sum_{j=1}^{J} b_j \chi_{T_j}$ are canonical representations. Then, $s + t = \sum_{k=1}^{K} \sum_{j=1}^{J} (a_k + b_j) \chi_{S_k \cap T_j}$ with

$\{S_k \cap T_j : 1 \leq k \leq K, 1 \leq j \leq J\}$ is a disjoint collection and

$$\begin{aligned}
\int (s+t) &= \sum_{k=1}^{K} \sum_{j=1}^{J} (a_k + b_j)\, \mu(S_k \cap T_j) \\
&= \sum_{k=1}^{K} a_k \sum_{j=1}^{J} \mu(S_k \cap T_j) + \sum_{j=1}^{J} b_j \sum_{k=1}^{K} \mu(S_k \cap T_j) \\
&= \sum_{k=1}^{K} a_k \mu(S_k) + \sum_{j=1}^{J} b_j \mu(T_j) \\
&= \int s + \int t.
\end{aligned}$$

$\square$

**Lemma 1.2.** Let $s, t$ be nonnegative and simple such that $s \leq t$. Then, $\int s \leq \int t$.

*Proof.*

$$\begin{aligned}
\int t &= \int (t - s + s) \\
&= \int \underbrace{(t-s)}_{\geq 0} + \int s \\
&\geq \int s.
\end{aligned}$$

$\square$

**Definition 1.2.** Let $f : S \to \overline{\mathbb{R}}$, then the **zero extension** of $f$ to $\mathbb{R}$ is

$$\begin{aligned}
f^* : \mathbb{R} &\longrightarrow \overline{\mathbb{R}} \\
x &\longmapsto f^*(x) = \begin{cases} f(x), & x \in S \\ 0, & x \notin S \end{cases}.
\end{aligned}$$

Moreover, this function preserves measurability.

**Definition 1.3** (Lebesque Integral of a General Nonnegative Function)**.** Let $f : \mathbb{R} \to \overline{\mathbb{R}}$ be a nonnegative measurable function and $\mathscr{S}(f)$ be the collection of all nonnegative simple functions, $s$, such that $s \leq f$. Then, the **Lebesque Integral** of $f$ over $\mathbb{R}$ is defined to be

$$\int f = \int_{\mathbb{R}} f(x)\, dx = \sup\{\int s : s \in \mathscr{S}(f)\}$$

.

If $f : S \to \overline{\mathbb{R}}$ is nonnegative and measurable, then

$$\int_S f = \int_S f(x)\, dx = \int_{\mathbb{R}} f^*$$

.

**Theorem 1.1** (Chebyshev's Inequality)**.** Let $f : \mathbb{R} \to \overline{\mathbb{R}}$ be nonnegative and measurable. Then, for any $\lambda \in (0, \infty)$ , then

$$\mu(\{x \in \mathbb{R} : f(x) \geq \lambda\}) \leq \frac{1}{\lambda} \int f.$$

*Proof.* Let $E = \{x \in \mathbb{R} : f(x) \geq \lambda\}$. This is the preimage of an extended borel set, hence measurable. Let $s = \lambda \chi_E$. Then, $s \in \mathscr{S}(f)$. Hence, $\int s = \lambda \mu(E) \leq \int f$. Hence the inequality holds. □

**Theorem 1.2.** Let $f : \mathbb{R} \to \overline{\mathbb{R}}$ be nonnegative ad measurable. Then $\int f = 0$ if and only if $f(x) = 0$ for almost every $x \in \mathbb{R}$.

*Proof.* Suppose $\int f = 0$ , then by chebyshev

$$\begin{aligned}
\mu(\{x \in \mathbb{R} : f(x) > 0\}) &= \mu(\bigcup_{n \in \mathbb{N}} \{x \in \mathbb{R} : f(x \geq n)\}) \\
&\leq \sum_{n=1}^{\infty} \mu\left(\{x \in \mathbb{R} : f(x) \geq \frac{1}{n}\}\right) \\
&= \sum_{i=1}^{\infty} \int f \\
&= 0.
\end{aligned}$$

Conversely, if $f(x) = 0$ almost everywhere, then for every $s \in \mathscr{S}(f)$ , we see $s$ is zero almost everywhere, hence $\int s = 0$ , so $\int f = \sup\{0 : s \in \mathscr{S}(f)\} = 0$. □

## Lecture 17: General Lebesque Integral

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