Algebraic Theory I

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Contents

Lecture 11: Homework Review and Sylow Groups (4)

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Solution to Questions 4 and 5 From Homework I

1. For question 4 part 1 we needed to show $\mathscr{O}_i^g \in \mathscr{O}$ for all i and $g \in G$. We note that if $x \in \mathscr{O}_i$, then $\mathscr{O}_i = x^H$, hence $\mathscr{O}_i^g = x^{Hg} = x^{gH} = (x^g)^H = \mathscr{O}_j$ for whichever $\mathscr{O}_j \ni g$.
2. For question 5 part 3 we needed to show that G_x being a maximal subgroup for every $x \in G$ is equivalent to the existence of no trivial blocks $B \subseteq \Omega$. One direction was simple, so we only show the other. Assume there is a $x \in \Omega$ such that $G_x < H < G$ for some $H \leq G$, then we wish to find a nontrivial block B.
   Define $B = x^H = \{x^h : h \in H\}$. First, we show this is a block. Suppose $B \cap B^g \neq \varnothing$, then $\exists x^{h_1} \in B$ and $x^{gh_2} \in B^g$ for some $h_1, h_2 \in H$ with $x^{gh_2} = x^{h_1}$, implying $x^{h_1^{-1}gh_2} = x^{h_1^{-1}h_1} = x$. Hence, $h^{-1}gh_2 \in G_x \leq H$, so $g \in h_1Hh_2^{-1} = H$. But, if $g \in H$, we have $B^g = \left(x^H\right)^g = x^{gH} = x^H = B$, hence B is a block and furthermore, $G_B = H$.
   Now, if $B = \{x\}$, then $G_B = H = G_x$, $\lightning$. Furthermore, if $B = \Omega$, then $B_G = H = G$, $\lightning$. Hence B is a proper nontrivial block.

Proposition 0.1. Let G be a group of order $|G| = 7 \cdot 3^3$. Then, G is not simple.

Proof. Let n_3, n_7 be the number of sylow 3-groups and 7-groups respectively. Then, by Sylow's Theorems $n_7 | \frac{|G|}{7} = 3^3$, and $n_7 = 1 \pmod 7$. So, $n_7 = 1, 3, 9, 27$ by the first requirement, and the second requirement implies $n_7 = 1$. Hence there is a unique Sylow 7-group, hence it is normal by an earlier proposition. Thus, there is a normal subgroup of order 7, so G is not simple. Note that had we dried with n_3 instead of n_7, we would get $n_3 | 7$ and $n_3 = 1 \pmod 3$ implying that n_3 could be 7, hence only 1 direction worked. ☐

Example. We can show that no group of $|G| = 30$ is simple. Suppose $|G| = 2 \cdot 3 \cdot 5$, using n_2 yields essentially no results as all other primes are odd. Hence, we try with n_3, this yields possibilities $n_3 = 1$ or $n_3 = 10$. If $n_3 = 10$, we know G is not simple, so let us assume $n_3 = 10$.
Now, trying with n_5 yields $n_5 = 1$ or $n_5 = 6$. Again, we know if $n_5 = 1$, then G is not simple so let us assume $n_5 = 6$.
Let P_1, P_2 be 2 sylow 3-groups. Then, either $P_1 = P_2$ or $P_1 \cap P_2 = \varnothing$, as $|P_1| = |P_2| = 3$ is prime. Thus, the 3-groups may only intersect trivially as they are of prime order. Hence, there are at least $n_3 \cdot (3-1)$ elements of order 3 in G. Hence, there are at least 20 elements of order 3 in G.
Similairly, we see there must be atleast $n_5 \cdot (5-1)$ elements of order 5 in G hence there are 24 elements of order 5, but as no element can have order 3 and 5, and we have $|G| = 30 < 24 + 20 + 1$ (the 1 being the identity which we did not count yet), we see either n_3 or $n_5 = 1$. Hence, G cannot be simple as it must have either a normal 3-group or a normal 5-group. ◇

Lecture 12: Classification of Finite Groups

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Recall. We showed that for a finite group G we could exploit the number of sylow p-groups, n_p to set up a congruence system with the only solution being $n_p = 1$ for some p, hence G was not simple (as $n_p = 1$ guarantees the corersponding p-group to be normal). Failing this, we found we could assume a sylow p-group of order p had only trivial intersection to attain a lower bound on the size of the group which was larger than $|G|$, implying once again that $n_p = 1$ for a particular p, so G was not normal.

We wish to continue this example to classify all possible groups of $|G| = 30$.
We had that either a sylow 3-group, denoted P, or a sylow 5-group, denoted Q, must be normal, hence either $P \trianglelefteq G$ or $N \trianglelefteq G$ (with $Q_G(P) = G$ or $P \leq N_G(Q) = G$). Hence PQ is a group by the 2nd homomorphism theorem.
Hence as $P, Q \leq PQ$, we have $|P| = 3 \mid |PQ|$ and $|Q| = 5 \mid |PQ|$, so $15 \mid |PQ|$.
Furthermore, as $P \cap Q = \{1\}$ (all nonidentity elements of P have order 3, and all or Q have order 5). As $3 \mid 5-1$, then we know by an earlier theorem (a group of order pq with $p \not| q-1$ is abelian) we have an abelian group. Hence $PQ \simeq C_{15}$. Using cauchy's theorem yields an element t or order 2, then we have $t \notin PQ$ as PQ had no elements of even order. Hence, $\langle PQ, t\rangle = G$.
Let $H = \langle t\rangle \simeq C_2$ and let $N = PQ \simeq C_{15}$. Clearly, $N \trianglelefteq G$ and $H \cap N = \{1\}$.
By another theorem from class, we have that $G = HN \simeq N \rtimes_\alpha H$ by some automorphism $\alpha : C_2 \to \mathrm{Aut}(C_{15})$.It remains only to determine what automorphisms α are possible in this case. As $C_2 = \{1, x\}$ for some x of order 2, then we see α is completely characterized by the value of $\alpha(x)$ and as

$$\underbrace{\alpha\left(t^2\right)}_{=\alpha(1)=1} = (\alpha(t))^2$$

we see $\mathrm{ord}(\alpha(t)) \mid 2$.
Now note that

$$\begin{aligned}\mathrm{Aut}(C_{15}) &= \mathrm{Aut}(C_3 \times C_5)\\ &\simeq \mathrm{Aut}(C_3) \times \mathrm{Aut}(C_5)\\ &= C_2 \times C_4\end{aligned}$$

and as there are 4 elements in $C_2 \times C_4$ of order 1 or 2, we have at most 4 possible automorphisms α (though some could give rise to isomorphic groups). It turns out that there are 4 such automorphisms, yielding nonisomorphic groups $C_{30}, D_{30}, C_3 \times D_{10}, C_5 \times S_3$.
We now introduce a second trick for inducing normal subgroups by exploiting low-index subgroups.

Proof. Assume G is finite and $H \leq G$ with $|G : H| = k$, k being sufficiently small. Let G act on the left H-cosets by left multiplications. This is of course transitive as $aH \mapsto bH$ by ba^{-1}.
Let $\alpha : G \to S_k$ be the associated homomorphism. If $\ker(\alpha) = G$, then there is a $g \in G$ such that $x^g = 1$ hence $k = 1$ by transitivity, hence $\ker(\alpha) = G \Leftrightarrow H = G$.

Similairly, if $\ker(\alpha) = \{1\}$, then α is an injection. Thus, $G \leq S_k$ up to isomorphism. Hence, knowledge of the subgroups of S_k may yield that $G \trianglelefteq S_k$, hence a contradiction. If we have a contradiction, then $\{1\} < \ker(\alpha) < G$, so we have a nontrivial normal subgroup.
One easy way to exploit this is to compare $|G|$ and $|S_k| = k!$. Clearly, $|G| \mid k!$ or $G \not\leq S_k$. So, if $|G| \mid k!$ we have the kernel is nontrivial so there is a proper nontrivial subgroup $K = \ker(\alpha) \trianglelefteq G$. $\square$

Example. Recall that $n_p = |G : N_G(P)|$ where P is a sylow p-group. Hence, if n_p is small (but larger than 1), we can use $N_G(P)$ to be our group of small index. $\diamond$