Algebraic Theory I: Homework III

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Solution (1)**.**

Lemma 0.1. Automorphisms preserve maximality of subgroups.
Let $\varphi : G \to G$ be an automorphism and let $M < G$ be a maximal subgroup. Suppose $\varphi(M) = M'$ is not maximal. That is, there is a $\overline{M}'$ such that $M' < \overline{M}' < G$. Then, we find

$$\begin{aligned}
\varphi^{-1}\left(\overline{M}'\right) &= \varphi^{-1}\left(M' \cup \left(\overline{M}' \setminus M'\right)\right) \\
&= \varphi^{-1}(M') \cup \varphi^{-1}\left(\overline{M}' \setminus M'\right) \\
&= M \cup \{\varphi^{-1}(m) : m \in \overline{M}' \setminus M'\} \\
&> M.
\end{aligned}$$

Furthermore, $\overline{M}' < G$ by assumption, hence $M < \overline{M}' < G$. $\lightning$.

Solution (2)**.** *Proof.* Now, let $\alpha : G \to G$ be an automorphism of G and denote $\alpha(M) = M'$. Then, we see

$$\begin{aligned}
\alpha(\Phi(G)) &= \alpha\left(\bigcap_{\substack{M<G \\ M \text{ is maximal}}} M\right) \\
&= \bigcap_{\substack{M<G \\ M \text{ is maximal}}} \alpha(M) \\
&= \bigcap_{\substack{M<G \\ M \text{ is maximal}}} M'
\end{aligned}$$

.

Then, as M' is maximal and α is an injection, we see if $N \neq M$ are both maximal subgroups, we have $\alpha(N) \neq \alpha(M)$, hence

$$\{M : \underset{M \text{ is maximal}}{M < G}\} = \{M' : \underset{M \text{ is maximal}}{M < G}\}.$$

So, we have

$$\alpha(\Phi(G)) = \bigcap_{\substack{M<G \\ M \text{ is maximal}}} M' = \bigcap_{\substack{M<G \\ M \text{ is maximal}}} M = \Phi(G).$$

$\square$

Solution (3).

Solution (4).

Solution (5).