

# Algebraic Theory I

Thomas Fleming

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### 1 Free Groups

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### Lecture 18: Solvable Groups (2) and Free Groups

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**Recall.** A group is solvable if there exists a chain of subgroups

$$\{1\} \trianglelefteq H_0 \trianglelefteq H_1 \trianglelefteq \dots \trianglelefteq H_n = G$$

such that  $H_i/H_{i-1}$  is abelian.

We had that this is equivalent to the condition that  $G^{(n)} = \{1\}$  where  $G^{(0)} = G$  and  $G^{(i)} = [G^{(i-1)}, G^{(i-1)}]$  for some  $n \geq 0$ . We showed the forward implication, so now we show the reverse implication.

*Proof.* Suppose  $G^{(n)} = 1$  for some  $n \geq 0$ . Then, we have a chain

$$G = G^{(0)} \trianglelefteq G^{(1)} \trianglelefteq \dots \trianglelefteq G^{(n)} = \{1\}.$$

So, we have

$$\{1\} = G^{(n)} \trianglelefteq G^{(n-1)} \trianglelefteq \dots \trianglelefteq G^{(0)} = G.$$

Furthermore, we know the commutator of  $G^{(i)}$  is a characteristic subgroup, hence it is normal.

Then, define  $H_i = G^{(n-i)}$  for  $0 \leq i \leq n$ . We need only show the quotients to be abelian. We see  $H_i/H_{i-1} = G^{(n-i)}/G^{(n-i+1)}$ . But,  $G^{(n-i+1)} = [G^{(n-i)}, G^{(n-i)}]$  by definition. Hence,  $G^{(n-i)}/G^{(n-i+1)}$  is abelian by the lemma from last class. So, the chain condition holds and  $G$  is solvable.  $\square$

**Theorem 0.1.** Let  $G$  be a solvable group with  $H$  being a subgroup. Then,  $H$  is solvable.

*Proof.* We simply show  $H^{(n)} \leq G^{(n)}$  for all  $n$  by induction. For the base case we know  $H = H^{(0)} \leq G^{(0)} = G$ . Then, we note  $H^{(n)} = [H^{(n-1)}, H^{(n-1)}] \subseteq [G^{(n-1)}, G^{(n-1)}] = G^{(n)}$  by inductive hypothesis. Since  $G$  is solvable, we find a  $n \geq 0$  such that  $G^{(n)} = \{1\}$ . Then,  $H^{(n)} \leq G^{(n)} = \{1\}$ , so  $H^{(n)} = \{1\}$  hence  $H$  is solvable.  $\square$

**Theorem 0.2.** If  $G$  is solvable and  $\varphi : G \rightarrow G'$  is a homomorphism, then  $\varphi(G)$  is also solvable.

*Proof.* We see  $\varphi(G^{(0)}) = \varphi(G)^{(0)}$ . So,  $\varphi(G^{(0)}) = \varphi(G)^{(0)}$ . We induce on  $n$ . We see

$$\begin{aligned}
 \varphi(G^{(n)}) &= \varphi\left([G^{(n-1)}, G^{(n-1)}]\right) \\
 &= \varphi\left(\langle x^{-1}y^{-1}xy : x, y \in G^{(n-1)} \rangle\right) \\
 &= \langle \varphi(x^{-1}y^{-1}xy) : x, y \in G^{(n-1)} \rangle \\
 &= \langle \varphi(x)^{-1}\varphi(y)^{-1}\varphi(x)\varphi(y) : x, y \in G^{(n-1)} \rangle \\
 &= \langle \bar{x}^{-1}\bar{y}^{-1}\bar{x}\bar{y} : \bar{x}, \bar{y} \in \varphi(G^{(n-1)}) \rangle \\
 &= \langle \bar{x}^{-1}\bar{y}^{-1}\bar{x}\bar{y} : \bar{x}, \bar{y} \in \varphi(G)^{(n-1)} \rangle \text{ by the inductive hypothesis.} \\
 &= [\varphi(G)^{(n-1)}, \varphi(G)^{(n-1)}] \\
 &= \varphi(G)^{(n)}.
 \end{aligned}$$

Since  $G$  is solvable, we find an  $n \geq 0$  such that  $G^{(n)} = \{1\}$ . Hence,  $\varphi(G^{(n)}) = \varphi(\{1\}) = \{1\} = \varphi(G)^{(n)}$ , so  $\varphi(G)$  is solvable.  $\square$

**Theorem 0.3.** If  $G$  is a group with  $H \trianglelefteq G$ , then  $G$  is solvable if and only if  $H$  and  $G/H$  are solvable.

*Proof.* ( $\Rightarrow$ ). We know all subgroups and homomorphic images to be solvable, hence this direction is already proven.

( $\Leftarrow$ ). Assume  $H$  and  $G/H$  are solvable. As  $H$  is solvable it has a normal chain

$$H_0 \trianglelefteq H_1 \trianglelefteq \dots \trianglelefteq H_n = H$$

with  $H_i/H_{i-1}$  is abelian for all  $1 \leq i \leq n$ . Similarly, since  $G/H$  is solvable there is a normal chain

$$\{1\} = K_{n+0} \trianglelefteq K_{n+1} \trianglelefteq \dots \trianglelefteq K_{n+s} = G/H$$

With  $K_{n+i}/K_{n+i-1}$  being abelian for all  $i \geq 1$ . We know by the lattice theorem that there are groups  $H_{n+i}$  such that  $K_{n+i} = H_{n+i}/H$  for some  $H_{n+i} \leq G$  and  $H \leq H_{n+i}$ . Then, we have

$$\{1\} = H/H \trianglelefteq H_{n+1}/H \trianglelefteq \dots \trianglelefteq H_{n+s}/H = G/H.$$

Then, we have  $H_n = H$  and  $H_{n+s} = G$  and, as each contains the kernel, this correspondance preserves normality, hence we have

$$H_n = H \trianglelefteq H_{n+1} \trianglelefteq H_{n+2} \trianglelefteq \dots \trianglelefteq H_{n+s} = G.$$

Then, note that  $H_{n+i}/H_{n+i-1} = (H_{n+i}/H)/(H_{n+i-1}/H) = K_{n+i}/K_{n+i-1}$  which we know to be abelian. Hence all successive quotients are abelian. So,

$$\{1\} = H_0 \trianglelefteq H_1 \trianglelefteq \dots \trianglelefteq H_n \trianglelefteq H_{n+1} \trianglelefteq H_{n+2} \trianglelefteq \dots H_{n+s} = G.$$

with  $H_i/H_{i-1}$  being abelian, so  $G$  is solvable.  $\square$

**Remark.** Subgroups and quotients of nilpotent groups are nilpotent, but this converse does not hold in general for nilpotent groups.

## 1 Free Groups

**Recall.**  $\langle \alpha, \tau : \alpha^n = 1, \tau^2 = 1, \tau\alpha\tau = \alpha^{-1} \rangle = D_{2n}$  is the dihedral group of order  $2n$ . This is technically ill defined. In general, we have generators  $\alpha, \tau$  and a set of relations that allow us to say when products of generators are equal. Similarly, we find  $\langle \alpha : \alpha^n = 1, \alpha^{n+1} = 1 \rangle = \{1\}$ . We have not, however, ensured that these form groups. This problem motivates the definition of free groups.

If  $S$  is a set, then we let  $S^{-1}$  be a disjoint set of formal symbols with  $x \mapsto x^{-1}$ , so  $S = \{a, b, c\}$  and  $S^{-1} = \{a^{-1}, b^{-1}, c^{-1}\}$ . Then, let  $F(S)$  to be the set of all formal products of elements from  $S \cup S^{-1} \cup \{1\}$ . Next class we will define an equivalence relation which takes these products into a group.

### Lecture 19: Free Groups (2)

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Recall we had a set of letters  $X = \{a, b, c, \dots, a^{-1}, b^{-1}, c^{-1}, \dots, 1\}$ . Then, we define a word on the alphabet  $X$  to be a string  $\omega = x_1^{\varepsilon_1} x_2^{\varepsilon_2} \dots x_s^{\varepsilon_s}$  where  $x_1, x_2, \dots, x_s \in X$  and  $\varepsilon_i = \pm 1$ . For example with  $X = \{a, b, c\}$  we have a word  $x_1 x_2 x_3 x_1^{-1} x_1 x_3$  for example. Then, define 1 to be the empty product, that being a string with no symbols. Now, we define an equivalence relation on the words to induce a group.

We say two words  $\omega_1 \sim \omega_2$  if we can transform  $\omega_1$  into  $\omega_2$  with a finite sequence of the following operations

- Remove a sequential pair  $xx^{-1}$  or  $x^{-1}x$  from the string.
- Insert a substring  $xx^{-1}$  or  $x^{-1}x$  into the string.

So, we see  $x_1 x_2 x_3^{-1} x_4 \sim x_1 x_2 x_3^{-1} x_2 x_2^{-1} x_1^{-1} x_1 x_4$  and so on. It is trivial to verify this to be an equivalence relation, so we omit the proof. Henceforth, we will denote the equivalence class of a word  $\omega$  by  $[\omega]$ . So, we see if  $\omega_1 \sim \omega_2$ , we have  $[\omega_1] = [\omega_2]$ .

Now, let  $F(X)$  be the set of all equivalence classes on  $X$  and define  $[\omega_1][\omega_2] := [\omega_1\omega_2]$  with  $\omega_1\omega_2$  simply being the concatenation of the two words. First, we verify this to be well-defined. Suppose  $w' \sim w$  and  $v' \sim v$  are 4 words. Hence, there is a simple sequence taking  $v \mapsto v'$  and  $w \mapsto w'$ . It is easy to see then, that the same operations applied to their respective parts will take  $vw \mapsto v'w'$  and  $wv \mapsto w'v'$ , hence  $[vw] = [v'w']$ .

Next, we show this forms a group. We see  $[w][1] = [w \cdot 1] = [w]$  and likewise  $[1][w] = [w]$ , so 1 is the identity.

Next,

$$\begin{aligned}
 [w]([u][v]) &= [w][uv] \\
 &= [w(uv)] \\
 &= [(wu)v] \\
 &= [wu][v] \\
 &= ([w][u])[v]
 \end{aligned}$$

Hence,  $F(X)$  is associative. Lastly, we show inverses exist. Let  $w = x_1^{\varepsilon_1} \dots x_s^{\varepsilon_s}$ , then let  $w^{-1} = x_s^{-\varepsilon_s} \dots x_1^{-\varepsilon_1}$  and we see  $ww^{-1} \sim 1$ , so  $F(X)$  has inverses.

**Definition 1.1** (Free Group). For an alphabet  $X$ , we define  $F(X)$  to be the **Free Group on  $X$** . More generally, the free group  $F$  on  $X$  is a group  $F$  together with an injection  $\sigma : X \hookrightarrow F$  such that any  $\alpha : X \rightarrow G$ , with  $G$  being an arbitrary group, extends to a unique homomorphism  $\beta : F \rightarrow G$  such that  $\beta \circ \sigma = \alpha$ .

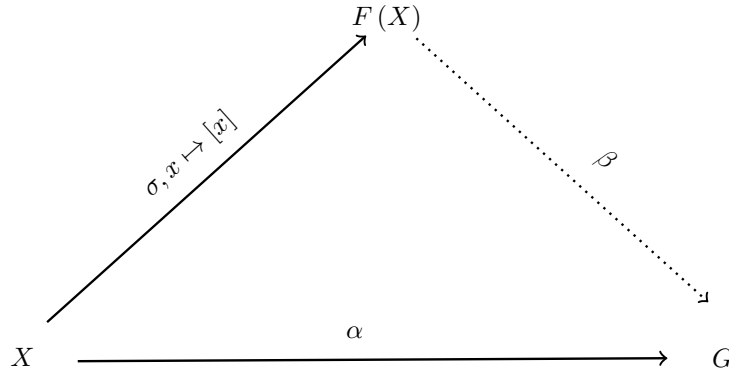


Figure 1: In this commutative diagram solid lines represent given maps and dotted lines represent maps that must then exist

Next, recall a homomorphism  $\varphi : H \rightarrow G$  is determined by the images of generators of  $H$ . Let  $H = \langle X \rangle$ . Then for an arbitrary  $h \in H$  with  $h = x_1^{\varepsilon_1} \dots x_n^{\varepsilon_n}$  we find  $\varphi(h) = \varphi(x_1)^{\varepsilon_1} \dots \varphi(x_n)^{\varepsilon_n}$  with  $x_i \in X$  and  $\varepsilon_i = \pm 1$ .

Now, let  $G$  be a group with  $\alpha : X \rightarrow G$  being a map and  $\sigma : X \hookrightarrow F$  be the inclusion map. Let  $w = x_1^{\varepsilon_1} \dots x_n^{\varepsilon_n}$  and let  $(w) = \alpha(x_1)^{\varepsilon_1} \dots \alpha(x_n)^{\varepsilon_n}$  with  $x_i \in X$  and  $\varepsilon_i = \pm 1$ . Then, we define  $\beta([w]) = [\beta(w)]$ . It is simple to check this is well defined as we may always insert or delete substrings of the form  $\alpha(x_i)^{\varepsilon_i} \alpha(x_i)^{-\varepsilon_i}$

in order to induce an equivalence. We see  $\beta$  is also a homomorphism as

$$\begin{aligned}\beta([w][v]) &= \beta([wv]) \\ &= \beta(wv) \\ &= \beta(w)\beta(v) \\ &= \beta([w])\beta([v]).\end{aligned}$$

Lastly, we see the map  $\beta$  is unique as a homomorphism is completely characterized by where it sends the generators.