Algebraic Theory I: Homework II

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Problem (1)**.** Let G_1, G_2 be finite groups with $\gcd(|G_1|, |G_2|) = 1$. Show that $\operatorname{Aut}(G_1 \times G_2) \simeq \operatorname{Aut}(G_1) \times \operatorname{Aut}(G_2)$.

Solution.

Problem (2)**.** Let $n \geq 1$ be an integer. For $x \in \mathbb{Z}$, denote $\overline{x} = x + n\mathbb{Z} \in \mathbb{Z}/n\mathbb{Z}$ and let $(\mathbb{Z}/n\mathbb{Z})^{\times} = \{\overline{x} : x \in \mathbb{Z}, \gcd(x, n) = 1\}$.

1. Show that $(\mathbb{Z}/n\mathbb{Z})^{\times}$ is an abelian multiplicative group.
2. Show that $\operatorname{Aut}(\mathbb{Z}/n\mathbb{Z}) \simeq (\mathbb{Z}/n\mathbb{Z})^{\times}$.

Solution.

Problem (3)**.** Let $H = \langle x \rangle \simeq C_2$ and $N = \langle y \rangle \simeq C_{15}$ be cyclic groups generated by $x \in H$ and $y \in N$ respectively.

1. Show that $\mathrm{Aut}\,(C_{15}) \simeq C_2 \times C_4$.
2. Let $\alpha : H \to \mathrm{Aut}\,(N)$ be a homomorphism and let $\alpha\,(x)\,(y) = y^r$ with $r \in \{0, 1, \ldots, 14\}$. What possible values can r take?
3. For each possible value of α from item 2 determine which of the following four groups is isomoprhic to $N \rtimes_\alpha H$: $C_{30}, D_{15}, C_3 \times D_5, C_5 \times S_3$.

Solution.

Problem (4)**.** Show there is no simple group of order 5103.

Solution. First, note that $5103 = 3^6 \cdot 7$ and denote n_3, n_7 to be the number of sylow 3-groups and 7-groups respectively. Then, we note by sylows theorms that $n_7 \mid 3^6$ and $n_7 \equiv 1 \pmod 7$. Note that the only numbers dividing 3^6 are $1, 3, 3^2, 3^3, 3^4, 3^5$, and 3^6, with

$$\begin{array}{llll} 1 \equiv 1 \pmod 7 & 3 \equiv 3 \pmod 7 & 3^2 \equiv 2 \pmod 7 & 3^3 \equiv 6 \pmod 7 \\ 3^4 \equiv 4 \pmod 7 & 3^5 \equiv 5 \pmod 7 & 3^6 \equiv 1 \pmod 7 & \end{array}$$

If $n_7 = 1$, then there is a unique normal sylow 7-group, so let us assume $n_7 = 3^6$. Similairly, $n_3 \mid 7$ and $n_3 \equiv 1 \pmod 3$, hence $n_3 = 1$ or 7. If $n_3 = 1$ then there is a unique normal sylow 3-group, hence let us assume $n_3 = 7$. Then, we have 3^6

Problem (5)**.** Show there is no simple group of order 4851.

Solution.