Combinatorics

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November 15, 2021

Contents

Lecture 31: Blowup Lemma

Fri 12 Nov 2021 16:22

Recall an ε-regular pair (A, B) admits an (ε, δ)-super-regular pair (A', B') with $A' \subseteq A$, $B' \subseteq B$.
Now, recall that for some $\varepsilon > 0$, if $n > \varepsilon^{-4}$, the the bipartite double of P_n is ε-regular. We construct a super-regular subpair from this bipartite double, denoted B_n with partite sets A, B. Applying the expander mixing lemma to two subsets $X \subseteq A$, $Y \subseteq B$ with $|X| > \varepsilon n$ $|Y| > \varepsilon n$, we find

$$\left(\frac{1}{2} - \varepsilon\right)|X|\,|Y| < e\,(X, Y) < \left(\frac{1}{2} + \varepsilon\right)|X|\,|Y|\,.$$

Then, inducing four subsets each of size $\sim \frac{n}{2}$, denoted A_1, A_2, B_1, B_2 of A and B respectively and completing the subgraphs (A_1, B_1) and (A_2, B_2) we see $d\,(A_1, B_1) = d\,(A_2, B_2) = 1$ and $d\,(A_1, B_2) \simeq d\,(A_2, B_1) \simeq \frac{1}{2}$. Collecting the densities, we find $d\,(A', B') = \frac{3}{4}$ where A', B' denote the sets A, B with the extra edges added between A_1, B_1 and A_2, B_2. From this, we can compute the new graph to be $\left(\varepsilon, \frac{1}{2+\kappa}\right)$-super regular for $\kappa > 0$.

Recall. We can obtain the blowup of a graph G on vertices $\{v_1, v_2, \ldots, v_r\}$ by replacing each vertex with a set $V_1, V_2, \ldots, V_r$ where each V_i is of equal cardinality. We construct the edges such that if $v_i \sim v_j$, then (V_i, V_j) is complete otherwise (V_i, V_j) is disconnected. Moreover if $|V_1| = \ldots = |V_r| = t$, then the blowup of this graph is $G \otimes J_t$.

Definition 0.1 (Generalized Blowup)**.** Let R be a graph with $V\,(R) = \{v_1, \ldots, v_r\}$. Then, we replace each vertex v_i with a set V_i of cardinality n_i and connect these sets in the same manner as a normal blowup. The induced graph is denoted $R\,(n_1, \ldots, n_r)$ and called the **generalized blowup**.

We modify this construction slightly. Let $\varepsilon, \delta \in (0, 1)$. Then we construct a new graph by applying the generalized blowup to G with numerical vector $(n_1, \ldots, n_r)$, but rather than each connected pair $v_i \sim v_j$ inducing a complete bipartite subgraph, we only connect sufficient edges in order for V_i, V_j to form an (ε, δ)-super regular pair. We denote this new graph $R_{\varepsilon,\delta}\,(n_1, \ldots, n_r)$.

Theorem 0.1 (Blowup Lemma)**.** Let R be a graph of order r with $\delta > 0$ and $\Delta \in \mathbb{N} \setminus \{1\}$. Then, there is $\varepsilon > 0$ so that if $H \subseteq R(n_1, n_2, \ldots, n_r)$ with $\Delta(H) \leq \Delta$, then $H \subseteq R_{\varepsilon,\delta}(n_1, \ldots, n_r)$

This lemma is espcially useful because it allows us to efficiently embed binary trees within these modified blowups. It is trivial to embed a binary tree into a complete generalized blowup, and $\Delta(T) = 3$ for a binary tree T, hence fixing a $\delta > 0$ we can find an ε so that the tree embeds in $R_{\varepsilon,\delta}(n_1, \ldots, n_r)$ as well.

Lecture 32

Mon 15 Nov 2021 10:24

Definition 0.2 (Cut Norm of Matrices)**.** Let A, be an $m \times n$ (possibly complex) matrix and define the **cut norm** of A to be

$$\|A\|_\square = \sup\{\left|\sum_{i\in S, j\in T} a_{ij}\right| : S \subseteq [m], T \subseteq [n]\}.$$

Remark. If $A \geq 0$ is a nonnegative real matrix, we find $\|A\|_\square = |A|_1 = \sum_{i\in[m],j\in[n]} a_{ij}$.
Similairly, for a nonpositive real matrix we find the cut norm to again be the modulus of the sum of entries.
Moreover