Algebraic Theory I

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Contents

Lecture 16: Nilpotent Groups (3)

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Corollary 1. A finite abelian group is the direct product of its sylow groups.

This follows directly from the theorem from last class.

Corollary 2. If G is a finite group such that for all $n \mid |G|$ such that there are at most n elements $x \in G$ with $x^n = 1$, then G is cyclic.

Proof. Let p be an arbitrary prime with $p \mid |G|$. Let P be a sylow p-group with $|P| = p^\alpha$. We know for any $x \in P$, we have $x^{|P|} = 1$, hence there are $|P| = p^\alpha$ elements $x \in P$ such that $x^{p^\alpha} = 1$. By hypothesis there is infact equality. If there was another distinct sylow p-group we would have elements $y \notin P$ such that $y^{p^\alpha} = 1$. Hence, P is unique. Hence, as every p-group is unique, so normal, we see G is the product of its P-groups.
Denote $G = P_1 \times P_2 \times \ldots P_t$ with the P_is being the distinct sylow p_i
$) - groupsofG$. Also, if $|P_1| = p_1^{\alpha_1}$, then all $x \in P_1$ have $\operatorname{ord}(x) \mid p_1^{\alpha_1}$ and there are at most $p_1^{\alpha_1 - 1} < p_1^{\alpha_1}$ such x with $\operatorname{ord}(x) \mid p_1^{\alpha_1 - 1}$. Since $|P| < p_1^{\alpha_1 - 1}$ we see there is an $x \in P_1$ with $\operatorname{ord}(x) = p_1^{\alpha_1} = |P|$, hence $\langle x \rangle = P_1$. So, P_1 is cyclic. Likewise, all other P_i are shown cyclic by the same argument, with $P_i = \langle x_i \rangle$. Then, the element $x = \prod_{i=1}^{t} x_i$ is a generator of G, so G is cyclic. $\square$

Theorem 0.1 (Frattini's Argument)**.** Let G be a finite group , $H \trianglelefteq G$, $P \leq H$ being a sylow p-group in H. Then,

$$G = HN_G(P) \text{ and } |G : H| \mid |N_G(P)|.$$

Proof. Let $g \in G$, we wish to show $g \in HN_G(P)$. We know this to be a subgroup as $H \trianglelefteq G$. Let G act by conjugation on its sets. Now

$$\begin{aligned} P^g &= gPg^{-1} \\ &\leq H^g \\ &= gHg^{-1} \\ &= H \text{ by normality.} \end{aligned}$$

Then, we see as $|P^g| = |P|$, then P^g is another sylow p-group in H. And, as we know all sylow p-groups are conjugate. Hence, there is an $h \in H$ such that $P^h = P^g$. Hence, $P = P^{h^{-1}g}$, hence $h^{-1}g \in N_G(P)$. Then, we see $g \in hN_G(P) \subseteq HN_G(P)$. So, we see $G = HN_G(P)$
Now, we show the other result. Note that by the second isomorphism theorem, we have

$$G/H = (HN_G(P))/H \simeq \frac{N_G(P)}{H \cap N_G(P)}.$$

Thus, $|G : H| = |N_G(P) : H \cap N_G(P)|$. As we know this divides $|N_G(P)|$, hence $|G : H| \mid |N_G(P)|$. ☐

Theorem 0.2. if G is a finite group, then G is nilpotent if and only if every maximal subgroup in G is normal in G.

Lecture 15: Nilpotent Groups (2)

Tue 28 Sep 2021 17:46

Lemma 0.1. If H, K are groups, then $Z(H \times K) = Z(H) \times Z(K)$.

Proof. Let $(x, y) \in H \times K$. If $(x, y) \in Z(H \times K)$ then

$$\underbrace{(a, 1)(x, y)(a, 1)^{-1}}_{=(axa^{-1}, 1)} = (x, y).$$

Hence, $x \in Z(H)$ and similarly, $y \in Z(K)$. Hence, $Z(H \times K) \subseteq Z(H) \times Z(K)$. The other direction of inclusion is trivial and left as an exercise. ☐

Lemma 0.2. Let $\varphi : G \to G'$ be a homomorphism with $\ker(\varphi) = K$ and $H \leq G$ such that $K \leq H$. Then, $N_G(H) = f^{-1}(N_{G'}(\varphi(H)))$.

Proof. Let $x \in N_G(H)$, so $xHx^{-1} = H$. Hence,

$$\varphi(H) = \varphi\left(xHx^{-1}\right) = \varphi(x)\varphi(H)\varphi(x)^{-1}.$$

Thus,

$$\begin{aligned} &\varphi(x) \in N_{G'}(\varphi(H)) \\ &\Rightarrow x \in \varphi^{-1}(N_{G'}(\varphi(H))) \\ \Rightarrow &N_G(H) \subseteq \varphi^{-1}(N_{G'}(\varphi(H))). \end{aligned}$$

Conversely, let $x \in \varphi^{-1}\left(N_{G'}\left(\varphi\left(H\right)\right)\right)$, hence $\varphi\left(x\right) \in N_{G'}\left(\varphi(H)\right)$.
Then, we see

$$\begin{aligned}
\varphi\left(H\right) &= \varphi\left(x\right)\varphi\left(H\right)\varphi\left(x^{-1}\right) \\
&= \varphi\left(xHx^{-1}\right) \\
\Rightarrow xHx^{-1} &\subseteq \varphi^{-1}\left(\varphi\left(H\right)\right) \\
&= \left\langle H, \ker\left(\varphi\right)\right\rangle \\
&= H \text{ as } \ker\left(\varphi\right) \subseteq H.
\end{aligned}$$

Hence, $xHx^{-1} \subseteq H$, so $x \in N_G\left(H\right)$. This concludes the proof. □

Now, recall that if G is a finite group with P being a sylow p-group, then TFAE

1. P is unique.
2. $P \trianglelefteq G$.
3. P is characteristic.
4. Any subgroup generated by elements whose orders are powers of p is itself a p-group.

Theorem 0.3. If G is a finite group, then the following are equivalent:

1. G is nilpotent.
2. $H < G \Rightarrow H < N_G\left(H\right)$.
3. All sylow p-groups are normal.
4. G is the direct product of its sylow p-groups.

Proof.
- $(2 \Rightarrow 3)$. Let P be a sylow p-group of G. Assume P is not normal, then denote $N = N_G\left(P\right) \subset G$. Hence, by the preceding lemma, P is characteristic in N. Then, as $N \trianglelefteq N_G\left(N\right)$, we see $P \trianglelefteq N_G\left(N\right)$. But $N = N_G\left(P\right)$ was the largest subgroup in which P was normal, hence $N_G\left(P\right) = N_G\left(N\right)$. So, by contrapositive of the assumption, (2), we have $N = N_G\left(N\right)$, so $N = G$, hence $P \trianglelefteq G$.
- $(3 \Rightarrow 4)$.
- $(1 \Rightarrow 2)$. Let G be nilpotent. If G is abelian, then $N_G\left(A\right) = G$ for all $A \leq G$, hence any proper subgroup $H < G$ has $H < N_G\left(H\right) = G$. Hence, assume G is non-abelian and proceed by induction on $|G|$ with base case $|G| = p$ being already completed p-prime. Suppose indirectly that there is an $H < G$ such that $H = N_G\left(H\right)$.
  Now, we note that $Z\left(G\right) \leq N_G\left(H\right) = H$ by definition of $Z\left(G\right)$. That is, $Z\left(G\right) \leq H$. Let $\varphi : G \to G/Z\left(G\right),\ x \mapsto \varphi(x) = xZ\left(G\right)$. Since G is nilpotent, $Z\left(G\right) = 1 \Leftrightarrow G = 1$, but we assumed G to be nonabelian, so this is not the case. Hence, we can assume $Z\left(G\right) = \{1\}$, hence $|G/Z\left(G\right)| < |G|$. As we know, G being nilpotent implies $G/Z\left(G\right)$ is nilpotent. Lastly, we note that $Z\left(G\right) \leq H < G$, so by the lattice theorem, we have $H/Z\left(G\right) < G/Z\left(G\right)$. Applying the induction hypothesis yields $H/Z\left(G\right) < N_{G/Z(G)}\left(H/Z\left(G\right)\right)$.

Recalling the lemma from last class, $\varphi^{-1}\left(N_{G/Z(G)}\left(H/Z\left(G\right)\right)\right) = N_G\left(H\right)$. Then, we note

$$\varphi^{-1}\left(\varphi\left(H\right)\right) < \varphi^{-1}\left(N_{\varphi(G)}\left(\varphi\left(H\right)\right)\right) = N_G\left(H\right).$$

And as $\ker\left(\varphi\right) = Z\left(G\right) \leq H$, we have $H < N_G\left(H\right)$.

$\square$