Analysis I

Thomas Fleming

November 5, 2021

Contents

Lecture 20: Derivatives (2)

Tue 02 Nov 2021 12:58

Recall. A monotone function on an interval has well defined limits at both its endpoints.

Definition 0.1 (Upper/Lower Derivatives)**.** Let $S \subseteq \mathbb{R}$, $f : S \to \mathbb{R}$

- We define $\overline{D}f(x) = \lim_{\tau \to 0} \sup\{\frac{f(x+h)-f(x)}{h} : 0 < |h| < \tau\}$ to be the **upper derivative**.
- We define $\underline{D}f(x) = \lim_{\tau \to 0} \inf\{\frac{f(x+h)-f(x)}{h} : 0 < |h| < \tau\}$ to be the **lower derivative**.
- If, for some $x \in \overset{\circ}{S}$, we find $\overline{D}f(x), \underline{D}f(x) \in \mathbb{R}$, with the upper and lower derivatives being equal, we say f is **differentiable** at x.
  We denote $f'(x) = \overline{D}f(x) = \underline{D}f(x)$.

We know, the limits of the upper and lower derivatives to be well defined as the supremum and infimum are monotone functions with respect to τ.

Proposition 0.1. Let $f : S \to \mathbb{R}$ and let $x \in \overset{\circ}{S}$. Then, f is differentiable at x if and only if

$$\lim_{y \to x} \frac{f(y) - f(x)}{y - x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \in \mathbb{R}.$$

That is, the classical derivative is equivalent to the lebesque derivative, so we will use the new definition for most proofs, but the old for most computations.

Theorem 0.1 (Mean-Value Theorem)**.** Let $f : [a,b] \to \mathbb{R}$ be continuous and differentiable at every $x \in (a,b)$. Then, there exists $\xi \in (a,b)$ so that $f(b) - f(a) = f'(\xi)(b-a)$.

Lemma 0.1. Let $f : [a, b] \to \mathbb{R}$ be increasing and suppose $\overline{D}f(x) = \underline{D}f(x)$ for almost every $x \in [a, b]$. Then, $\overline{D}f(x)$ and $\underline{D}f(x)$ are finite almost everywhere. Moreover, f is differentiable almost everywhere (on $[a, b]$). Furthermore, f' is an integrable function and

$$\int_{[a,b]} f' \leq f(b) - f(a).$$

Proof. Extend f to $[a, \infty)$ by letting $f(c) = f(b)$ for all $c \geq b$. Define a sequence (g_n), $g_n : [a, b] \to \overline{\mathbb{R}}$ with

$$x \mapsto n\left(f\left(x + \frac{1}{n}\right) - f(x)\right).$$

Then, b assumption, we know $(g_n(x))$ to be convergent in $\overline{\mathbb{R}}$ with limit $f'(x)$ for almost every $x \in (a, b)$. Each g_n is measurable, hence $\lim_{n\to\infty} g_n$ is increasing, we see $g(n) \geq 0$, hence $\overline{D}f \geq 0$.
Applying Fatou's lemma yields

$$\begin{aligned}
\int_{[a,b]} \overline{D}f &= \int_{[a,b]} \liminf_{n\to\infty} f_n \\
&\leq \liminf_{n\to\infty} \int_{[a,b]} g_n \\
&= \liminf_{n\to\infty} n\left(\int_{\left[a+\frac{1}{n},b+\frac{1}{n}\right]} f - \int_{[a,b]} f\right) \\
&= \liminf_{n\to\infty} \left(\underbrace{n\int_{\left[b,b+\frac{1}{n}\right]} f}_{=f(b)} - \underbrace{n\int_{\left[a,a+\frac{1}{n}\right]} f}_{\leq f(a)}\right) \\
&\leq f(b) - f(a).
\end{aligned}$$

We know the final inequality holds because f is constant on $\left[b, b + \frac{1}{n}\right]$ and though f is not constant, it is increasing on $\left[a, a + \frac{1}{n}\right]$ hence the upper bound of their difference is attained by $f(a)$.
Consequently, $\overline{D}f$ is integrable (so finite almost everywhere). And, since $\overline{D}f = \underline{D}f$, we find $f'(x)$ exists and equals $\overline{D}f(x)$ for almost every $x \in [a, b]$. □

Later, we will prove equality holds precisely in the case of absolute continuity.

Definition 0.2 (Vitali Covering)**.** Let $S \subseteq \mathbb{R}$. We call a collection of closed, bounded intervals (denoted $\mathscr{C}$) of positive length a **Vitali covering** of $S \subseteq \mathbb{R}$ if for every $x \in S$ and $\varepsilon > 0$ we find an $I \in \mathscr{C}$ such that $x \in I$ and $l(I) < \varepsilon$.

Example. A vitali covering of $S = [0, 1]$ goes as follows. Let $H = \mathbb{Q} \cap [0, 1]$, then $\mathscr{C} = \{[x, x + h] : h \in H, x \in [0, 1]\}$. ◇

Theorem 0.2 (Vitali Covering Lemma)**.** Let $\mathscr{C}$ be a Vitali covering of the set $S \subseteq \mathbb{R}$ with $m^*(S) < \infty$. Then, for every $\varepsilon > 0$ there is a finite, disjoint collection of intervals $\{I_k \in \mathscr{C} : 1 \leq k \leq n\}$ such that

$$m^*\left(S \setminus \bigcup_{k=1}^{n} I_k\right) < \varepsilon.$$

Theorem 0.3 (Lebesque's Theorem)**.** *Let $f : I \to \mathbb{R}$ be a monotone function on an interval $I \subseteq \mathbb{R}$. Then, f is differentiable at almost every $x \in I$ and f' is integrable on every interval $[a,b] \subseteq I$. In particular, if f is increasing, then*

$$\int_{[a,b]} f' \leq f(b) - f(a).$$

Proof. It suffices to show I is open and bounded, else we could replace I by $\mathring{I} \cap (-n,n)$ for $n \in N$ and we find $\mathring{I} = \bigcup_{n\in\mathbb{N}} \mathring{I} \cap (-n,n)$. Similarly, we can assume f to be increasing. Hence, for all $x \in I$, we have $0 \leq \underline{D}f(x) \leq \overline{D}f(x) \leq \infty$. So, we need only show $\overline{D}f(x) = \underline{D}f(x)$ with this quantity being finite for almost every $x \in I$.
For $p, q \in \mathbb{Q}$ and $p > q > 0$, define $E_{p,q} = \{x \in I : \underline{D}f(x) < q < p < \overline{D}f(x) < \infty\}$. Then,

$$\{x \in I : \underline{D}f(x) < \overline{D}f(x) < \infty\} = \bigcup_{p,q\in Q^+} E_{p,q}.$$

If f fails to be differentiable at $x \in I$, then either $x \in E_{p,q}$ for some $p, q \in \mathbb{Q}$ or $\overline{D}f(x) = \infty$. We know $\overline{D}f$ to be finite almost everywhere, so by subadditivity, we need only show the other component, $E_{p,q}$, has measure 0.
Fix $p, q \in \mathbb{Q}$ and suppose $m^*(E_{p,q} = m_0)$. Then, $m_0 \in [0,\infty)$ by the boundedness assumption. Given $\varepsilon > 0$ there is a nonempty open U such that $E_{p,q} \subseteq U$ and $m(U) < m_0 + \varepsilon$. Suppose $x \in E_{p,q}$. Since $\underline{D}f(x) < q$ by definition of $E_{p,q}$; for every $\delta > 0$ we find a $0 < h < \delta$ such that $[x, x+h] \subseteq U$ and $f(x+h) - f(x) < qh$ or $[x-h, x] \subseteq U$ and $f(x) - f(x-h) \leq qh$.
The collection $\mathscr{L}$ of all such intervals $[x, x+h]$ or $[x-h, x]$ for a fixed $\delta > 0$ and $x \in E_{p,q}$ forms a Vitali covering of $E_{p,q}$. We find all intervals $[a,b] \in \mathscr{L}$ have the property $f(b) - f(a) < q(b-a)$ by the earlier observation. Then, by the Vitali covering lemma, there is a finite, disjoint collection of intervals $\{I_n \in \mathscr{L} : 1 \leq n \leq N\}$ such that for $V = \bigcup_{n=1}^{N} I_n$, we have $m^*(E_{p,q} \setminus V) < \varepsilon$. Note that $m(V) < m_0 + \varepsilon$ since $V \subseteq U$. Since $m^*(E_{p,q} \setminus V) + m^*(E_{p,q} \cap V) \geq m_0$ since the two sets together contain $E_{p,q}$, we have $m^*(E_{p,q} \cap V) \geq m_0 - \varepsilon$.
Now, we follow a similair construction. If $x \in E_{p,q} \cap V$, then $p < \overline{D}f(x)$ implies for all $\delta > 0$ there is an $0 < h < \delta$ such that $[x, x+h] \subseteq V$ and $f(x+h) - f(x) \geq ph$ or $[x-h, x] \subseteq V$ and $f(x) - f(x-h) \geq ph$. The collection $\mathscr{U}$ of all such intervals $[x, x+h]$ or $[x-h, x]$ for a fixed $\delta > 0$ and $x \in E_{p,q} \cap V$ is a vitali covering of $E_{p,q} \cap V$. Moreover, if $[c,d] \in \mathscr{U}$, then $f(d) - f(c) \geq p(d-c)$. Applying Vitali Covering lemma yields a finite disjoint collection of intervals $\{I_k \in \mathscr{U} : 1 \leq k \leq K\}$ such that for $W = \bigcup_{k=1}^{K} J_k$, we have $m^*((E_{p,q} \cap V) \setminus W) < \varepsilon$. Since

$$m^*((E_{p,q} \cap V) \setminus W) + m(W) \geq m^*(E_{p,q} \cap V)$$

we have that $m(W) \geq m_0 - 2\varepsilon$.
We know each interval $J_k = [c_k, d_k]$ from W must be contained in V, furthermore it is contained in an interval $I_n = [a_n, b_n]$ of V. As each interval is disjoint and monotonic, we must have that

$$\sum_{k=1}^{K} (f(d_k) - f(c_k)) \leq \sum_{n=1}^{N} (f(b_n) - f(a_n)).$$

Now, since $I_n \in \mathscr{L}$ and $J_k \in \mathscr{U}$, we have

$$\begin{aligned} p \sum_{k=1}^{K} (d_k - c_k) &= pm\,(w) \\ &\leq qm\,(V) \\ &= q \sum_{n=1}^{N} (b_n - a_n) \end{aligned}$$

.

Hence, $p\,(m_0 - 2\varepsilon) \leq q\,(m_0 + \varepsilon)$ for each $\varepsilon > 0$, so $pm_0 \leq qm_0$ and as $p > q$, we must have $m_0 = 0$, so f is differentiable on all but sets of measure 0, so it is differentiable almost everywhere. □

Corollary 1. If the function $f : [a,b] \to \mathbb{R}$ is of bounded variation on the interval $[a,b] \subseteq \mathbb{R}$, then it is differentiable at almost every $x \in [a,b]$. Consequently, if f is absolutely continuous on $[a,b]$, then it is differentiable at almost every $x \in [a,b]$.

Proof. Bounded variation implies $f = g - h$ for increasing functions g, h. Applying lebesque's theorem yields g, h are differentiable almost everywhere, hence f is differentiable almost everywhere. □

Lecture 21: Fundamental Theorem of Calculus

Thu 04 Nov 2021 13:03

For the duration of this lecture, $[a,b]$ will denote a compact interval in $\mathbb{R}$, principally, it is not in $\overline{\mathbb{R}}$.

Lemma 0.2. Suppose $f : [a,b] \to \overline{\mathbb{R}}$ is integrable. Then, $f = 0$ almost everywhere if and only if $\int_{[a,x]} f = 0$ for all $x \in [a,b]$.

Proof. If $f = 0$ almost everywhere, then the integral must be 0 for all $x \in [a,b]$ so the forward implication holds.
Conversely, assume $\int_{[a,x]} f = 0$ for all $x \in [a,b]$. Then, let $E = \{x \in [a,b] : f\,(x) > 0\}$ and assume $m\,(E) > 0$. Then, there is a closed set $C \subset E$ so that $m\,(C) > 0$. Letting $O = (a,b) \setminus C$ (an open set) we see $\int_{[a,b]} f = \int_C f + \int_O f$ and as $\int_C f > 0$ as $C \subseteq E$ with $m\,(C) > 0$. Hence, we find $\int_O f \neq 0$. Hence, $m\,(O) > 0$, and there is an interval $(c,d) \subseteq O$ so that $\int_{[c,d]} \neq 0$. Since $\int_{[a,d]=0}$ by assumption, then we find $\int_{[a,d]} f = \int_{[a,c]} f + \int_{[c,d]} f$, hence $\int_{[a,c]} f \neq 0$ ↯. □

Proposition 0.2. Syppose $g : [a,b] \to \mathbb{R}$ is continuous. For every $x \in [a,b)$ and $\varepsilon > 0$ there is a δ with $0 < \delta < b - x$ such that

$$\left| \frac{1}{h} \int_{x,x+h} (g - g\,(x)) \right| < \varepsilon \text{ for } 0 < h < \delta\ .$$

Proof. Write $g(x) = g(x)\chi_{[x,x+h]}$. Then the claim immediately follows. $\square$

Theorem 0.4 (Fundamental Theorem of Calculus I)**.** Suppose $f : [a,b] \to \overline{\mathbb{R}}$ is integrable. Then the function

$$\begin{aligned} F : [a,b] &\longrightarrow \mathbb{R} \\ x &\longmapsto F(x) = \int_{[a,x]} f \end{aligned}$$

is absolutely continuous and differentiable almost everywhere with $F' = f$ almost everywhere.

Proof. It is clear that F is absolutely continuous and differentiable almost everywhere by a result from last lecture and the fact that absolute continuity $\Rightarrow$ bounded variation $\Rightarrow$ differentiable a.e.
Moreover, we can assume $f \geq 0$, otherwise replacing f by f^+ or f^-. We can temporarily assume f is bounded (though we will later remove this requirement). Let $f(x) \leq M$ for all $x \in [a,b]$. Then, extend f, F to functions on $[a,\infty)$ by letting $f(x) = f(b)$ for all $x \geq b$. Define the following sequence of continuous functions (g_n)

$$\begin{aligned} g_n : [a,b] &\longrightarrow \overline{\mathbb{R}} \\ x &\longmapsto g_n(x) = n\left(F\left(x+\frac{1}{n}\right) - F(x)\right) && = n\left(\int_{a,x+\frac{1}{n}} f - \int_{a,x} f\right) \\ & && = n\int_{\left[x,x+\frac{1}{n}\right]} f. \end{aligned}$$

Then, we find the sequence is pointwise convergent with limit $F'(x)$ for almost every $x \in [a,b]$. Furthermore, F' is measurable and $0 \leq g_m \leq M$ for all $x \in [a,b]$. So,applying dominated convergence and the previous proposition yields g_m is dominated by M with pointwise limit F', so $F' \leq M$ almost everywhere. So, F' is integrable and for all $x \in [a,b]$ we find

$$\begin{aligned} \int_{[a,x]} F' &= \lim_{n\to\infty} \int_{[a,x]} g_n \\ &= \lim_{n\to\infty} n(\int_{\left[a+\frac{1}{n},x+\frac{1}{n}\right]} F - \int_{[a,x]} F) \\ &= \lim_{n\to\infty} n\left(\int_{\left[x,x+\frac{1}{n}\right]} F - \int_{\left[a,a+\frac{1}{n}\right]} F\right) \\ &= F(x) - F(a) \\ &= F(x). \end{aligned}$$

Now, if f was unbounded, then define the sequences (f_n) and (F_n) with

$$\begin{aligned} f_n : [a,b] &\longrightarrow \overline{\mathbb{R}} \\ x &\longmapsto f_n(x) = \inf\{f(x), n\} \\ F_n : [a,b] &\longrightarrow \overline{\mathbb{R}} \\ x &\longmapsto F_n(x) = \int_{[a,x]} f_n . \end{aligned}$$

Since $f - f_n \geq 0$, we see $F - F_n$ is increasing for each n. Hence, $F - F_n$ is differentiable almost everywhere with $(F - F_n)' \geq 0$ almost everywhere. Consequently for $x \in [a,b]$ we see

$$\int_{[a,x]} F' \geq \int_{[a,x]} F'_n$$

for all $x \in [a,b]$. Since F_n is bounded for all n , we see $\int_{[a,x]} F'_n = F_n(x)$ by the bounded case. Thus, $\int_{[a,x]} F' \geq F_n(x)$ for all $x \in [a,b]$.
Now, applying MCT, we see (f_n) is a pointwise convergent sequence of functions which are increasing the F_ns also converge pointwise to F on $[a,b]$. Hence, $\int_{[a,x]} F' \geq F(x)$ for ever $x \in [a,b]$ by passing the earlier inequality to the limit. Since f is nonnegative, we see F is increasing, so we also have $\int_{[a,x]} F' \leq F(x) - F(a) = F(x)$. Hence $\int_{[a,x]} F' = F(x)$ since

$$\int_{[a,x]} (F' - f) = \int_{[a,x]} F' - \int_{[a,x]} f = \int_{[a,x]} F' - F(x) = 0 \text{ for a.e. } x \in [a,b] .$$

□

In order to prove the other part of the fundamental theorem of calculus, we will need the following lemma:

Lemma 0.3. If the function $f : [a,b] \to \mathbb{R}$ is absolutely continuous with $f' = 0$ almost everywhere then f is a constant function.

Proof. We will show $f(c) = f(a)$ for all $c \in (a,b]$. Fix $c \in (a,b]$ and let $E = \{x \in (a,c) : f' \text{ exists at } x, f'(x) = 0\}$.
By assumption, $m(E) = c - a > 0$, hence for $\varepsilon > 0$ choose $\delta > 0$ such that absolute continuity holds. For each $x \in E$ and $k > 0$, we see there is an $h \in (0,k)$ with either $[x, x+h] \subseteq [a,c]$ and $|f(x+h) - f(x)| < \varepsilon h$ or $[x-h, x] \subseteq [a,c]$ and $|f(x-h) - f(x)| < \varepsilon h$ (or both). Then, the collection $\mathscr{C}$ of these intervals for all $k > 0$ and $x \in E$ is a vitali covering of E. By the Vitali covering lemma, we find a finite disjoint collection $\{[x_k, y_k] \in \mathscr{C} : 1 \leq k \leq n\}$ so that $V = \bigcup_{k=1}^{N} [x_k, y_k]$ has $m(E \setminus V) < \delta$. Reindex these intervals such that $x_k < x_{k+1}$ for all k and let $y_0 = a$, $x_{n+1} = c$. Then, we see

$$a = y_0 \leq x_1 < y_1 < x_2 < y_2 < \ldots < x_n < y_n \leq x_{n+1} = c.$$

Hence, the set $P = \{x_i : 1 \le i \le n+1\} \cup \{y_i : 1 \le i \le n+1\}$ is a partition of $[a, c]$. Since

$$\sum_{k=1}^{n} (y_k - x_k) = m(V) > m(E) = c - a - \delta$$

we see the leftover pieces

$$\sum_{k=0}^{n} (x_{k+1} - y_k) \le m(E \setminus V) < \delta.$$

Since f is absolutely continuous, we see $\sum_{k=0}^{n} |f(x_{k+1}) - f(y_k)| < \varepsilon$. Consequently,

$$\begin{aligned} |f(c) - f(a)| &\le \sum_{k=1}^{n} |f(y_k) - f(x_k)| + \sum_{k=0}^{n} |f(x_{k+1} - f(y_k))| \\ &< \sum_{k=1}^{n} \varepsilon (y_k - x_k) + \varepsilon \\ &\le \varepsilon (c - a) + \varepsilon \end{aligned}$$

for all $\varepsilon > 0$, so we see $f(c) - f(a) = 0$ for all $c \in (a, b]$ and the claim follows. ☐

Theorem 0.5 (Fundamental Theorem of Calculus II)**.** Suppose the function $F : [a, b] \to \mathbb{R}$ is absolutely continuous. Then, F is differentiable almost everywhere and its derivative, F', is integrable with

$$\int_{[a,x]} F' = F(x) - F(a)$$

for all $x \in [a, b]$.

Proof. Since F is absolutely continuous, it is of bounded variation, so there are two increasing functions, $T, S : [a, b] \to \mathbb{R}$ with $F = T - S$. Moreover, the derivatives T', S' exist almost everywhere and are integrable.
Hence, F' exists almost everywhere and $F' = T' - S'$ almost everywhere, so it is integrable as well.
Then, letting $G(x) = \int_{[a,x]} F'$. We see G is absolutely continuous, so $F - G$ must be absolutely continuous. Then, by the FTC part 1, we see $(F - G)'$ exists almost everywhere and $(F - G)'(x) = 0$ for almost every $x \in [a, b]$. Hence $F - G$ is a constant function. So, we see $F(x) - G(x) = F(x) - \int_{[a,x]} F' = F(a)$ by letting $x = a$. ☐