MATH 7237/8237

LECTURE Feb. 17, 2021

MULTIPARTITE GRAPHS

Let us restate the definition of a bipartite graph.

Definition A graph is called **bipartite** if its vertices can be partitioned into two independent sets**.**

We see that the number *two* does not seem to be exceptional in this definition, and are tempted to generalize it to any integer $r \geq 2$.

Definition A graph is called r-**partite** if its vertices can partitioned into r independent sets**.**

It turns out that r-partite graphs are of utmost importance in graph theory, but before going further, let us add a few details to the definition.

If a graph G is r-partite, the independent sets of its vertex partition are called the **color classes, vertex classes** or **partite sets** of G.

It is convenient to think the vertex classes of an r-partite graph as vertex colors, that is to say, each vertex is colored in one of r distinct colors so that the vertices of any edge are colored in different colors.

This terminology gives rise to an equivalent definition.

Definition A graph is called r-**colorable** if its vertices can be colored in r colors, so that the vertices of any edge are colored in different colors.

Convince yourself that a graph is r-partite if and only if it is r-colorable.

Note that the exact colors that may be used to color the vertices of an r-partite graph are irrelevant. All that matters is their number.

For example, to color the vertices of a bipartite graph, we may use black and white, or red and blue, or any other pair of colors.

The assignment of colors to vertices of a graph G such that the vertices of any edge are colored in different colors is called **vertex coloring** of G.

Bipartite graphs are rarely called 2-colorable, but for $r \geq 3$, the terms r-*partite* and r-*colorable* are in equal use.

Clearly every graph of order n can be colored with n colors, but most often fewer colors are sufficient.

Problem *Show that if G is a non-complete graph of order n, it is $(n-1)$-colorable.*

COLORING OF CYCLES

If you try to color the 5-cycle in two colors, you will soon find that this is impossible.

Moreover, you realize that no cycle of odd length can be colored in two colors.

Convince yourself that cycles of even length can be colored in two colors, and cycles of odd length need three colors.

While coloring cycles, you may notice another intriguing fact: even cycles can be colored in two colors in only one way, up to swapping colors.

However, the 7-cycle has at least two distinct colorings. Indeed let $v_1, v_2, v_3, v_4, v_5, v_6, v_7$ be the vertices of C_7.

(A) Color v_1, v_3, v_5 in blue, v_2, v_4, v_6 in red, and v_7 in green.

(B) Color v_1, v_4 in blue, v_3, v_6 in red, and v_2, v_5, v_7 in green.

These two colorings are distinct, as the color classes are of different sizes.

In fact, any odd cycle of length at least 7 can be colored into three colors in more than one way.

As we already know, cycle coloring helps to characterize bipartite graphs:

Theorem A graph is bipartite if and only if it does not have odd cycles..

r-COLORABLE GRAPHS

Unfortunately, for $r \geq 3$, r-colorable graphs cannot be characterized as simply as bipartite graphs.

Definition An r-partite graph is called **complete r-partite graph** if any two vertices of different colors are adjacent.

Clearly every r-partite graph can be extended to a complete r-partite graph by adding appropriate edges.

Earlier in this class, it was shown that a complete bipartite graph of order n has at most $n^2/4$ edges.

This fact can be generalized for r-colorable graphs.

MAXIMUM SIZE OF r-PARTITE GRAPHS

Theorem The number of edges of an r-partite graph of order n is at most
$$\left(1-\frac{1}{r}\right)\frac{n^2}{2}.$$

Indeed, let G be an r-partite graph of order n with maximum number of edges.

Clearly, G is a complete r-partite graph.

Let $V_1, \ldots, V_r$ be the color classes of G and set
$$n_i = V_i, \quad i = 1, \ldots, r.$$

Note that each edge of G belongs to a bipartite subgraph $G\left[V_i, V_j\right]$, for some i, j such that $1 \leq i < j \leq r$.

Hence

$$e(G) = \sum_{1 \le i < j \le r} |V_i| \left|V_j\right| = \sum_{1 \le i < j \le r} n_i n_j.$$

Next, using the fact

$$n_1 + \cdots + n_r = n$$

and the formula

$$(x_1 + \cdots + x_r)^2 = \sum_{i=1}^{r} x_i^2 + 2 \sum_{1 \le i < j \le r} x_i x_j,$$

which is valid for any real numbers $x_1, \ldots, x_r$, we find that

$$\begin{aligned} e(G) &= \frac{1}{2}\left((n_1 + \cdots + n_r)^2 - \sum_{i=1}^{r} n_i^2 \right) \\ &= \frac{1}{2} n^2 - \frac{1}{2} \sum_{i=1}^{r} n_i^2. \end{aligned}$$

Using the Mean Quadratic-Mean Arithmetic inequality, we get

$$\sum_{i=1}^{r} n_i^2 \geq \frac{(n_1 + \cdots + n_r)^2}{r} = \frac{n^2}{r}.$$

Putting everything together, we get

$$e(G) \leq \frac{1}{2}n^2 - \frac{n^2}{2r} = \left(1 - \frac{1}{r}\right)\frac{n^2}{2}.$$

One can easily see that if r divides n, then there is equality in the above inequality.

It is enough to take the complete r-partite graph of order n with all vertex classes of cardinality exactly n/r.

CHROMATIC NUMBER OF GRAPHS

Definition Given a graph G, its **chromatic number** is the minimum number r such that there is a vertex coloring of G in r colors. The chromatic number of G is denoted by $\chi(G)$.

In other words, the chromatic number of G is the minimum number of colors necessary to color its vertices so that no edge has its vertices of the same color.

This implies that if we color the vertices of G with fewer than $\chi(G)$ colors, there is always an edge with both vertices of the same color.

The chromatic number of graphs is an extremely important parameter and is even more difficult to study than the clique number.

Problem *Show that* $\chi(K_n) = n$.

Problem *If* H *is a subgraph of* G, *then* $\chi(H) \leq \chi(G)$.

Problem *For every graph* G, $\omega(G) \leq \chi(G)$.

Hint. If G contains a clique of order r, then G is not $(r-1)$-colorable.

Sometimes it is possible that $\omega(G) < \chi(G)$. For example, odd cycles of order at least 5 have clique number 2 and chromatic number 3.

Problem *If* G *is a graph of order* n, *then* $\alpha(G)\chi(G) \geq n$.

Indeed, note that G has $\chi(G)$ color classes, which are independent sets.

BROOKS'S THEOREM

Finding the chromatic number of graphs is a hard algorithmic problem, just like the maximal clique problem.

For that reason, there are innumerable inequalities which help to estimate the chromatic number in some cases.

One of them is Brooks's theorem:

Theorem (Brooks) *If G is a graph, then* $\chi(G) \leq \Delta(G) + 1$.
If G is connected, equality holds if and only if G is an odd cycle or a complete graph.

Proof We shall skip the proof of the cases of equality.

Let $\Delta = \Delta(G)$, and let us arrange the vertices of G by a simple procedure described next.

Let v_1 be a vertex of minimum degree in G.

Having chosen $v_1, \ldots, v_k$, choose v_{k+1} to be a vertex of minimum degree in $G - v_1 \cdots - v_k$.

Repeat this step until all vertices are arranged.

Note that the minimum degree in the graph $G - v_1 \ldots - v_k$ may be a lot larger than the minimum degree of G, but it is certainly not larger than $\Delta(G)$.

Together with the sequence $v_1, \ldots, v_n$, we obtain a sequence of graphs

$$G_1 \subset G_2 \subset \ \ldots \ \subset G_n,$$

where $G_n = G$ and

$$G_{n-k} = G - v_1 \ldots - v_k, \qquad k = 1, \ldots, n-1.$$

Observe that the order of G_i is precisely i, and that G_i is obtained from G_{i+1} by omitting the vertex v_{n-i}, which is of minimum degree in G_{i+1}.

We shall use induction on k to prove that that $\chi(G_k) \leq \Delta + 1$.

This inequality is obvious for $k = 1$, because G_1 has only one vertex.

Assume that $\chi(G_k) \leq \Delta + 1$, and color the vertices of G_k with $\Delta + 1$ colors.

Since the vertex v_{n-k} of G_{k+1} is joined to at most Δ vertices of G_k, it may be colored with a color that is not used for its neighbors.

We obtain a coloring of G_{k+1} with $\Delta + 1$ colors, and so $\chi(G_{k+1}) \leq \Delta + 1$, completing the induction step.

Therefore, $\chi(G) = \chi(G_n) \leq \Delta + 1$.

TWO PROPERTIES OF k-CHROMATIC GRAPHS

Theorem *Let $k \geq 2$ and G be a graph with $\chi(G) = k$. If G is colored in k colors, then there is a vertex of each color, joined to vertices of every other color.*

Let $V_1, \ldots, V_k$ be the color classes of G.

Assume for a contradiction, that for each $v \in V_k$, there is some $j_v \in \{1, \ldots, k-1\}$ such that v is not joined to any vertex from V_{j_v}.

Now, we can color each $v \in V_k$ with color j_v, which amounts effectively to moving v from V_k to V_{j_v}.

At the end, the graph G appears to be $(k-1)$-colorable, contrary to $\chi(G) = k$.

We proved the assertion for color k, but by symmetry it is true for any color.

▮

We get an obvious corollary.

Corollary Every k-chromatic graph has at least k vertices of degree at least $k-1$.

TRIANGLE–FREE GRAPHS

The chromatic number turns out to be very loosely connected to the clique number.

On the one-hand, we know that $\chi(G) \geq \omega(G)$, but inequalities going in the opposite direction are far from true.

Theorem (Mycielski) *For every* $k \geq 2$, *there exists a triangle-free graph* G_k *with* $\chi(G_k) = k + 1$.

It is immediate to see that the graph $G_1 = K_2$ is triangle-free and its chromatic number is 2. We shall construct G_k recursively.

Suppose that we have constructed a triangle-free graph G_k with $\chi(G_k) = k + 1$. Let $V = \{v_1, \ldots, v_n\}$ be its vertex set.

To construct G_{k+1}, first add a set of n new vertices $U = \{u_1, \ldots, u_n\}$ and join u_i to the neighbors of v_i for every $i = 1, \ldots, n$.

Finally add one more vertex w and join it to each $u_i \in U$.

(a) Check that G_{k+1} is triangle-free.

To check that G_{k+1} is triangle-free, note that the set U is independent and that w is joined only to the vertices of U. Hence, there is no triangle containing w or two vertices from U.

Since G_k is triangle-free, a triangle in G_{k+1} should have two vertices from V and one vertex from U. Suppose that u_i, v_j, v_l is such a triangle.

However, u_i and v_i are joined to the same vertices from V; hence v_i, v_j, v_l is also a triangle in G_k, contradicting the induction assumption.

Hence G_{k+1} is triangle-free.

(b) Check that $\chi\left(G_{k+1}\right)=k+2$.

We begin with the check of the inequality $\chi\left(G_{k+2}\right)\leq k+2$.

To prove this inequality, suppose that G_k is colored in $k+1$ colors, then color each $u_i \in U$ with the color of v_i, and finally color w in a new color.

We know now that either $\chi\left(G_{k+1}\right)=k+1$ or $\chi\left(G_{k+1}\right)=k+2$.

Assume that G_{k+1} is colored in $k+1$ colors. Then, for each of the colors, G_k has a vertex v_i that is joined to a vertex of each of the remaining k colors.

Therefore u_i is colored in the same color as v_i, and the set U turns out to be colored in $k+1$ colors.

Now for the vertex w we need to use a new color, contrary to the assumption that $k+1$ colors are enough.

THANK YOU