MATH 8237

LECTURE Mar. 26, 2021

PERMUTATIONS OF ROWS AND COLUMNS OF MATRICES

Let G be a graph with adjacency matrix A. Suppose that p is a permutation of the vertices of G.

As a result of the permutation, the adjacency matrix of G gets its rows and columns permuted as well. How the permuted matrix can be represented?

Definition Given a permutation

$$p = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix},$$

the **permutation matrix** $P = \left[p_{j,j}\right]$ of p is defined as

$$p_{i,j} = \begin{cases} 1, & \text{if } j = \sigma(i), \\ 0, & \text{otherwise.} \end{cases}$$

It turns out that for any $n \times n$ matrix A, permuting the columns of A amounts to post-multiplication of A by P.

Likewise permuting the rows of A amounts to pre-multiplication of A by P^{-1}.

Observation If G is a graph and $V(X)$ are permuted by a permutation p, the adjacency matrix of the resulting graph is given by

$$P^{-1}AP,$$

where P is the permutation matrix of p.

Since an automorphism of a graph is a permutation of its vertices, we see the following fact:

Observation If G is a graph with adjacency matrix A and p is an automorphism of G, then

$$P^{-1}AP = A,$$

where P is the permutation matrix of p.

EIGENVECTORS TO THE LARGEST EIGENVALUE AND AUTOMORPHISMS

Knowing the symmetries of a graph G can be quite useful in finding the spectral radius of $\lambda_1(G)$.

Definition We say that u and v are **equivalent in** G, if there exists an automorphism $p : G \to G$ such that $p(u) = v$.

Vertex equivalence implies useful properties of eigenvectors to $\lambda_1(G)$:

Proposition 2 *Let G be a connected graph of order n, and let u and v be equivalent vertices in G.*
If $(x_1, \ldots, x_n)$ is an eigenvector to $\lambda_1(G)$, then $x_u = x_v$.

Proof Let G be a connected graph with adjacency matrix A, and let $\mathbf{x} := (x_1, \ldots, x_n)$ be an eigenvector to $\lambda_1(G)$.

Without loss of generality, we suppose that $\mathbf{x}$ is the Perron vector of G.

Let $p : G \to G$ be an automorphism of G such that $p(u) = v$.

Note that p is a permutation of $V(G)$, and let P be the permutation matrix of p.

Since p is an automorphism, we have

$$P^{-1}AP = A.$$

Hence,

$$P^{-1}AP\mathbf{x} = A\mathbf{x} = \lambda_1\mathbf{x},$$

and left-multiplying with P, we get

$$AP\mathbf{x} = \lambda_1 P\mathbf{x}.$$

Now we see that

$$A(P\mathbf{x}) = \lambda_1 (P\mathbf{x}),$$

and therefore, $P\mathbf{x}$ is an eigenvector to A. Moreover, $P\mathbf{x}$ is a permutation of $\mathbf{x}$ and therefore it is also a Perron vector.

Since G is connected, $\mathbf{x}$ is unique, implying that $P\mathbf{x} = \mathbf{x}$, and so

$$x_u = x_v.$$

Remark Note that eigenvector entries corresponding to equivalent vertices need not be equal for disconnected graphs.

For example, let G is a union of two disjoint copies of a graph H. There is a natural automorphism that exchanges the two copies of H, and we can always choose a Perron vector that is zero over one of the copies, because $\lambda_1(G) = \lambda_1(H)$.

Proposition 2 implies the following practical statement:

Corollary 3 *If G is a connected graph and $V(G)$ is partitioned into equivalence classes by the relation "u is equivalent to v", then every eigenvector to $\lambda_1(G)$ is constant within each equivalence class.*

SPECTRAL RADIUS AND PERRON VECTOR OF $K_{p,q}$

Let us now use Corollary 3 to calculate the spectral radius and the Perron vector of the complete bipartite graph $K_{p,q}$.

Proposition 4 *For the graph $K_{p,q}$ we have:*

(1)

$$\lambda_1\left(K_{p,q}\right) = \sqrt{pq}.$$

(2) The Perron vector of $K_{p,q}$ is

$$(\overbrace{\frac{1}{\sqrt{2p}},\ldots,\frac{1}{\sqrt{2p}}}^{p},\overbrace{\frac{1}{\sqrt{2q}},\ldots,\frac{1}{\sqrt{2q}}}^{q})$$

Proof Write X and Y for the vertex classes of $K_{p,q}$ and suppose that $|X| = p$ and $|Y| = q$.

Before starting the proof, it is necessary to realize that every two vertices from X are equivalent in $K_{p,q}$, and so are every two vertices from Y.

Let $X = \{1, \ldots, p\}$, $Y = \{p+1, \ldots, p+q\}$, and let $(x_1, \ldots, x_n)$ be a Perron vector of $K_{p,q}$.

Corollary 3 implies that

$$x_1 = \cdots = x_p,$$
$$x_{p+1} = \cdots = x_{p+q}.$$

Let $x = x_1$ and $y = x_{p+1}$.

The eigenequations for vertices 1 and $p+1$ give

$$\lambda_1 x = qy,$$
$$\lambda_1 y = px.$$

Multiplying these equations and cancelling xy, which is nonzero, we get

$$\lambda_1^2 = pq,$$

thus proving clause (1) of Proposition 4.

For clause (2) note that the eigenequation for vertex 1, together with $\lambda_1 = \sqrt{pq}$, gives

$$\sqrt{pq}x = qy.$$

Cancelling $\sqrt{q}$ and squaring both sides, we get

$$px^2 = qy^2.$$

On the other hand,

$$px^2 + qy^2 = 1,$$

because $(x_1, \ldots, x_n)$ is a Perron vector.

Now we see that

$$px^2 = qy^2 = 1/2,$$

and (2) follows.

JOIN OF GRAPHS

Next we introduce an important graph operation, which belongs to graph theory in general.

Definition Suppose that H and G are two vertex disjoint graphs. The **join** of H and G is a graph obtained by taking the union of H and G, and joining every vertex for H to every vertex of G.

The join of H and G is denoted by $H \vee G$.

Let us right away note that $H \vee G = G \vee H$.

In addition, graph join is an associative operation, that is

$$F \vee (G \vee H) = (F \vee G) \vee H.$$

So we can simply write $F \vee G \vee H$ for $F \vee (G \vee H)$.

Examples of joins of graphs

(1)

$$K_n = K_s \vee K_{n-s}.$$

(2)

$$K_{p,q} = \overline{K}_p \vee \overline{K}_q.$$

(3) A complete 3-partite graph $K_{a,b,c}$ can be described as

$$K_{a,b,c} = \overline{K}_a \vee \overline{K}_b \vee \overline{K}_c.$$

(4) Recall that $T_r(n)$ stands for the r-partite Turán graph of order n. We have

$$T_r(n) = T_{r-1}(n - \lfloor n/r \rfloor) \vee \overline{K}_{\lfloor n/r \rfloor}.$$

(5) If G and H are disjoint graphs, their union $G \cup H$ satisfies

$$\overline{G \cup H} = \overline{G} \vee \overline{H}.$$

THE SPECTRAL RADIUS OF A JOIN OF REGULAR GRAPHS

Theorem (Finck, Grohman) *Let* G_1 *be a* r_1*-regular graph of order* n_1 *and* G_2 *be a* r_2*-regular graph of order* n_2*.*

Then $\lambda_1 (G_1 \vee G_2)$ *is the larger solution of the equation*

$$x^2 - (r_1 + r_2)\, x + r_1 r_2 - n_1 n_2 = 0. \tag{1}$$

This formula is useful in numerous situations, but the actual goal of our proof is to introduce yet another technique for solving such problems.

Proof When one looks at $G_1 \vee G_2$, it seems very likely that its Perron vector must be constant within G_1 and within G_2.

This turns out to be indeed the case, but a proof is not immediately seen.

However, based on that expectation, we may try to come up with an appropriate nonnegative vector that is an eigenvector to some eigenvalue λ of $G_1 \vee G_2$.

Then by the last Perron Frobenius theorem it would turn out that

$$\lambda_1 (G_1 \vee G_2) = \lambda.$$

Take a $(n_1 + n_2)$-vector with n_1 entries equal to x and n_2 entries equal to y, where x and y are to be determined. Assign the x-entries to the vertices of G_1 and the y-entries to the vertices of G_2.

We use this vector to devise one equation for all vertices of G_1 and another one for all vertices of G_2 :

$$\begin{aligned} \lambda x &= r_1 x + n_2 y \\ \lambda y &= r_2 y + n_1 x. \end{aligned} \tag{2}$$

This system is equivalent to

$$\begin{aligned}(\lambda - r_1)\, x &= n_2 y\\ (\lambda - r_2)\, y &= n_1 x.\end{aligned}$$

- and eventually we get

$$\lambda^2 - (r_1 + r_2)\,\lambda + r_1 r_2 - n_1 n_2 = 0.$$

- The larger root of this equation is

$$\lambda = \frac{r_1 + r_2 + \sqrt{(r_1 - r_2)^2 + 4 n_1 n_2}}{2}.$$

- It is not hard to see that $\lambda > r_1$ and $\lambda > r_2$.

- Consequently, the system (2) has a solution for some positive x and y.

So we have found that λ is an eigenvalue to $G_1 \vee G_2$ with a positive eigenvector.

Since $G_1 \vee G_2$ is always connected (even if G_1 or G_2 are not) the Perron-Frobenius theorem implies that

$$\lambda = \lambda_1 \left(G_1 \vee G_2 \right).$$

THANK YOU