# Algebraic Theory I

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## Contents

### Lecture 22: Free Groups (5)

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**Recall.** Let $G, H$ be groups with presentations $\varepsilon : F \to G$ and $\delta : F \to H$ for some free group $F$, If every relator of $G$ is also a relator for $H$, then there is a surjective homomorphism $\varphi : G \to H,\ \varepsilon(x) \mapsto \delta(x)$.

**Definition 0.1** (Reduced Word)**.** We define a word $w$ to be **reduced** if no string $xx^{-1}$ or $x^{-1}x$ occurs within $w$ for any $x \in X$. We find any word is equivalent to some reduced word by applying our relations.

**Theorem 0.1.** Every word is equivalent to a unique reduced word.

*Proof.* We proceed fancily (he really said this). Let $R$ be the set of reduced words on the alphabet $X$. For each $m \in X$, define a map

$$m' : R \to R,\ x_1^{\varepsilon_1} \dots x_\ell^{\varepsilon_\ell} \mapsto \begin{cases} mx_1^{\varepsilon_1} \dots x_\ell^{\varepsilon_\ell}, & m \neq x_1^{-\varepsilon_1} \\ x_2^{\varepsilon_2} \dots x_\ell^{\varepsilon_\ell}, & m = x_1^{-\varepsilon_1} \end{cases}$$

We see $m'$ is a bijection as $\left(m^{-1}\right)' = m'^{-1}$. Hence, $m'$ is simply a permutation of the set $R$.
Now, using the universal mapping property on $F(X)$, we define a homomorphism

$$\begin{aligned} \theta : F(X) &\longrightarrow \mathrm{Sym}(R) \\ [m] &\longmapsto m' \end{aligned}$$

where $\mathrm{Sym}(R)$ is simply the set of all permutations of $R$. Now, suppose $w = x_1^{\varepsilon_1} \dots x_\ell^{\varepsilon_\ell}$ and $w' = y_1^{\delta_1} \dots y_s^{\delta_s}$ are two reduced words that are equivalent, that is $[w] = [w']$. Then, we have $\theta([w]) = (x_1')^{\varepsilon_1} \dots (x_\ell')^{\varepsilon_\ell}$.Then, we see $\theta([w])(1) = w$. Hence, $\theta([w']) = \theta([w]) = y_1^{\delta_1} \dots y_s^{\delta_s}$. Hence, we see $x_1^{\varepsilon_1} \dots x_\ell^{\varepsilon_\ell} = y_1^{\delta_1} \dots y_s^{\delta_s}$ as words. Hence, there is at most one distinct reduced word in $[w]$. And, as there is always atleast 1 reduced word, we see this completes the proof.

□

**Remark.** We define $x^n = \underbrace{x \ldots x}_{n \text{ times}}$ and $x^{-n} = \underbrace{x^{-1}x^{-1} \ldots x^{-1}}_{n \text{ times}}$. Then, we see any reduced word has the form $x_1^{\ell_1} \ldots x_s^{\ell_s}$ with $\ell_i \in \mathbb{Z} \setminus \{0\}$ and $x_i \neq x_{i-1}$ for all $1 \leq i \leq s$. This is called the normal form of a word.

**Definition 0.2.** With the normal form of a word, we define a **multiplicity function**. For $x \in X$ and a word $w = x_1^{\ell_1} \ldots x_s^{\ell_s}$ we define $V_x(w) = \sum_{x_j = x} \ell_j$.

We note that if $w \sim w'$, we have $V_x(w) = V_x(w')$ for all $x \in X$. Furthermore, $V_x(w) = V_x\left(v^{-1}wv\right)$ for all $x \in X$ and words $v, w$. Moreover, $V_x(wv) = V_x(w) + V_x(v)$ , so its a homormophism from $F(X) \to \mathbb{Z}$.

**Definition 0.3** (Rank)**.** Recall that if $|X| = |Y|$, we had $F(X) \simeq F(Y)$. We define $\operatorname{Rank}(F(X)) = |X|$. We have yet to show this is well defined, but the next theorem will take care of this.

**Theorem 0.2.** If $X$ and $Y$ are sets with $F(X) \simeq F(Y)$, then $|X| = |Y|$.

We will prove this claim next class.

## Lecture 23: Free Groups (6)

Mon 18 Oct 2021 11:26

Recall, we defined the rank of a free group to be the size of its underlying alphabet. In order to ensure this was well defined, we needed to prove the following claim

**Proposition 0.1.** If $F(X) \simeq F(Y)$ via the isomoprhism $\varphi$, then $|X| = |Y|$ .

*Proof.* Denote $G = F(X)$ and $G' = F(Y)$ and let $H = \left\langle g^2 : g \in F(X) \right\rangle$. We know this to be a characteristic subgroup by the homework problem. Hence, we have $H \trianglelefteq F(X)$. Consider $G/H$ and note that $\varphi(H) = H' = \{h^2 : h \in F(Y)\}$. Since, $\varphi(H) = \{\varphi\left(g^2\right) = \varphi(g)^2 : g \in F(X)\} = \{h^2 : h \in \varphi(F(X)) = F(Y)\}$. Hence, $G/H \simeq \varphi(G)/\varphi(H) \simeq G'/H'$ as $\varphi$ is an isomorphism. We show that $G/H \simeq \underbrace{\mathbb{Z}/2\mathbb{Z} + \ldots + \mathbb{Z}/2\mathbb{Z}}_{|X| \text{ times}} \simeq (\mathbb{Z}/2\mathbb{Z})^{|X|}$.

First, note $xyxy = (xy)^2 = 1$ in $G/H$ for all $x, y \in G/H$ by definition. Hence, $xyx^{-1}y^{-1} = xyxy$ as $x^2 = y^2 = 1$ for every $x, y \in G/H$. Hence, $xyx^{-1} = y$, so $G/H$ is an abelian 2-group. Now, note that $\langle xH : x \in X \rangle = G/H$ and denote $xH = \overline{x}$ for each $x \in G$. Then $G/H = \{\overline{x} : x \in X\}$. Note that an element $g \in G/H$ has

$$\overline{x_1 x_2} \ldots \overline{x_\ell}$$

with all $\overline{x_1}, \ldots, \overline{x_\ell}$ being distinct.
Suppose $\overline{x_1} \ldots \overline{x_\ell} = \overline{y_1} \ldots \overline{y_s}$. We claim that $\ell = s$ and there is a permutation such that $x_i = y_i$ for all $i$. Suppose the contrary, so WLOG $x_1 \notin \{y_1, \ldots, y_\ell\}$.

Hence, $w = \overline{x_1} \dots \overline{x_\ell y_s} \dots \overline{y_1} = 1$, so $w \in H$. Furthermore, we find $V_{x_1}(w) = 1$. But, for any generator $g^2 \in H$, we have $V_{x_1}(g^2) = 2n$ for some $n \geq 0$. So, we must have $V_{x_1}(w) = \sum_{i=1}^{m} V_{x_1}(g_i^2) = 2\hat{n}$ for generators $g_i$ and some $\hat{n} \geq 0$. $\lightning$.
Hence there is a unique representation in $G/H$.
This shows that

$$\begin{aligned} G/H &= \langle \overline{x} : x \in X \rangle \\ &= \bigoplus_{x \in X} \langle x \rangle \end{aligned}$$

with each $\langle \overline{x} \rangle \in \mathbb{Z}/2\mathbb{Z}$ as $\operatorname{ord}(\overline{x}) = 2$. Hence,

$$G/H = \sum_{i=1}^{|X|} \mathbb{Z}/2\mathbb{Z}.$$

We know this to be a vector space over a 2 element field, $\mathbb{F}_2$, consisting of elements $(\varepsilon_x)_{x \in X} \mapsto \prod_{x \in X} \overline{x}^{\varepsilon_x}$ with almost all (finitely many) $\varepsilon_x = 0$ and $\dim_{\mathbb{F}_2}(G/H) = |X|$ as $\overline{X}$ is a basis for $G/H$. As $G/H \simeq G'/H'$, we see $\dim_{\mathbb{F}_2}(G'/H') = |X|$. But by the same argument, we see $\dim_{\mathbb{F}_2}(G'/H') = |Y|$ as well. Hence, $|X| = |Y|$. $\square$

**Remark.** If $F \simeq F(X)$ is free and $H \leq F$, then $H$ is free. Similairly, if $|F : H| = m < \infty$ then $\operatorname{Rank}(H) = \operatorname{Rank}(F) \cdot m + (1 - m)$ for some $m \geq 0$.

**Midterm**

The test Wednesday will be proofs of $\sim 4$ (choose 2 out of 4) theorems, propositions, lemmas we proved in class. There will be a second part consisting of short answers consisting of applying theorems, lemmas, ... from class to prove simple or concrete results.