

Algebraic Theory I: Homework III

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Solution (1). 1. First, we note that

$$\begin{aligned}
 \overline{\varphi}(xKyK) &= \overline{\varphi}(xyKK) \\
 &= \overline{xy}H \\
 &= \varphi(xy)H \\
 &= \varphi(x)\varphi(y)H \\
 &= \overline{xy}H \\
 &= \overline{xy}HH \text{ as } H = HH \text{ by closure} \\
 &= \overline{xH}\overline{y}H \\
 &= \overline{\varphi}(xK)\overline{\varphi}(yK).
 \end{aligned}$$

Furthermore, we see $\overline{\varphi}(1K) = \overline{1}H = 1H$ as $\varphi(1) = \overline{1} = 1$ by homomorphism properties.

- First, note that $Z_0(\overline{G}) = \{1\} = \overline{Z_0(G)}$. Now, we induce on n and we see $Z_{n-1}(G) \trianglelefteq G$ and $Z_{n-1}(\overline{G}) \trianglelefteq \overline{G}$ with $\overline{Z_{n-1}(G)} \leq Z_{n-1}(\overline{G})$ by inductive hypothesis, so $\overline{\varphi} : Z_n(G)/Z_{n-1}(G) \rightarrow Z_n(\overline{G})/Z_{n-1}(\overline{G})$ is a well defined homomorphism. Hence, letting $\overline{x} \in \overline{Z_n(G)}$, hence $x \in Z_n(G)$ and hence $xZ_{n-1}(G) \in Z_n(G)/Z_{n-1}(G)$ implies $\overline{\varphi}xZ_{n-1}(G) = \overline{x}Z_{n-1}(\overline{G}) \in Z_n(\overline{G})/Z_{n-1}(\overline{G})$. Hence, we find $\overline{x} \in Z_n(\overline{G})$. This completes the induction.
- Suppose G is nilpotent and let n be the nilpotence class of G . Then, we see $\overline{Z_n(G)} = \overline{G} \leq Z_n(\overline{G})$. Hence, \overline{G} is of nilpotence class at most n , so we see \overline{G} is nilpotent.
- Let $\overline{G} = H$ with $\varphi : G \rightarrow H$ being the restriction to H homomorphism. That being $\varphi(x) = \begin{cases} x, & x \in H \\ 1, & x \notin H \end{cases}$ It is clear this is well defined
- Suppose n is the nilpotence class of G . Then $Z_n(G) \cap H = G \cap H = H \leq Z_n(H)$, so H is of nilpotence class at most n , hence H is nilptent.

Lemma 0.1. Automorphisms preserve maximality of subgroups.

Let $\varphi : G \rightarrow G$ be an automorphism and let $M < G$ be a maximal subgroup. Suppose $\varphi(M) = M'$ is not maximal. That is, there is a \overline{M}' such that $M' < \overline{M}' < G$. Then, we find

$$\begin{aligned}\varphi^{-1}(\overline{M}') &= \varphi^{-1}\left(M' \cup (\overline{M}' \setminus M')\right) \\ &= \varphi^{-1}(M') \cup \varphi^{-1}(\overline{M}' \setminus M') \\ &= M \cup \{\varphi^{-1}(m) : m \in \overline{M}' \setminus M'\} \\ &> M.\end{aligned}$$

Furthermore, $\overline{M}' < G$ by assumption, hence $M < \overline{M}' < G$. \nmid

Solution (2). *Proof.* Now, let $\alpha : G \rightarrow G$ be an automorphism of G and denote $\alpha(M) = M'$. Then, we see

$$\begin{aligned}\alpha(\Phi(G)) &= \alpha\left(\bigcap_{\substack{M < G \\ M \text{ is maximal}}} M\right) \\ &= \bigcap_{\substack{M < G \\ M \text{ is maximal}}} \alpha(M) \\ &= \bigcap_{\substack{M < G \\ M \text{ is maximal}}} M'\end{aligned}$$

Then, as M' is maximal and α is an injection, we see if $N \neq M$ are both maximal subgroups, we have $\alpha(N) \neq \alpha(M)$, hence

$$\{M : \substack{M < G \\ M \text{ is maximal}}\} = \{M' : \substack{M < G \\ M \text{ is maximal}}\}.$$

So, we have

$$\alpha(\Phi(G)) = \bigcap_{\substack{M < G \\ M \text{ is maximal}}} M' = \bigcap_{\substack{M < G \\ M \text{ is maximal}}} M = \Phi(G).$$

□

Solution (3).

Lemma 0.2. $[M, M]$ and $\langle x^p : x \in M \rangle$ are characteristic in M .
Let $\alpha : M \rightarrow M$ be an automorphism. Then, denote $\alpha(x) = x'$ for $x \in M$ and we see,

$$\begin{aligned}
\alpha([M, M]) &= \alpha(\langle xyx^{-1}y^{-1} : x, y \in M \rangle) \\
&= \langle \alpha(xyx^{-1}y^{-1}) : x, y \in M \rangle \\
&= \langle \alpha(x)\alpha(y)\alpha(x)^{-1}\alpha(y)^{-1} : x, y \in M \rangle \\
&= \langle x'y'x'^{-1}y'^{-1} : x, y \in M \rangle \\
&\leq \langle x'y'x'^{-1}y'^{-1} : x', y' \in M \rangle \\
&= [M, M].
\end{aligned}$$

Similarly,

$$\begin{aligned}
\alpha(\langle x^p : x \in M \rangle) &= \langle \alpha(x^p) : x \in M \rangle \\
&= \langle \alpha(x)^p : x \in M \rangle \\
&= \langle x'^p : x \in M \rangle \\
&\leq \langle x'^p : x' \in M \rangle.
\end{aligned}$$

Solution (5).