

Combinatorics

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September 29, 2021

Contents

Lecture 16

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First, we examine some more random graphs. For a random graph G , it is a trivial result of probability theory that the number of four cycles is precisely $\frac{1}{2} \sum_{u,v \in V(G); v \neq u} \binom{\hat{d}(u,v)}{2}$. Then, applying our estimation $\hat{d}(u,v) = \frac{n}{4} + o(n)$ yields $\binom{n}{2}$ possible pairs u, v and $\hat{d} \approx \frac{n}{4}$, hence the number of four cycles is

$$\frac{1}{n} \binom{\frac{n}{4}}{2} \binom{n}{2} = \frac{n^4}{128} + o(n^4).$$

Now, we examine the k -walks.

Definition 0.1 (Walks). A k -walk is a k -path $v_1, v_2, v_3, \dots, v_k$.
A closed k -walk is a k -cycle, $v_1, v_2, \dots, v_k, v_1$.

Remark. Walks need not have all vertices distinct, hence a graph of order 2 where one simply oscillates between the vertices to produce a degenerate $2n$ -walk. Similarly, one can traverse a triangle to induce a 4-walk as well. Overall this yields 14 possible 4-walks on a graph of order 4.

Now, we examine the number of closed 4-walks on a random graph of order n . We see nondegenerate 4-walks are just 4-cycles of which we know there to be $\frac{n^4}{128}$ with 8 possible permutations of directions and starting point yields $8 \cdot \frac{n^4}{128}$. Similarly, we note that $4 \cdot \sum_{v \in V} \binom{d_i(v)}{2} = 4n \binom{n}{2} = \frac{1}{2}n^3 + o(n^3) = o(n^4)$ degenerate graphs on 3 vertices exist. Lastly, the number of degenerate graphs on 2-vertices is clearly, $2 \cdot e(g) = o(n^4)$. Hence, the number of 4-walks is just $\frac{n^4}{16} + o(n^4)$.

Proposition 0.1. $\text{tr} \left(A(G)^k \right) = \sum_{i=1}^n \lambda_i^k$ is the number of closed k -walks in a graph G of order n .

From this, we arrive at $6k_3(G) = \text{tr}(A^3) = \sum_{i=1}^3 \lambda_i^3$.
We also see the number of closed walks of order 4 is

$$\begin{aligned} CW_4 &= \sum_{i=1}^n \lambda_i^4 \\ \frac{n^4}{16} + o(n^4) &= \lambda_1^4 + \sum_{i=2}^n \lambda_i^4 \\ \Rightarrow \sum_{i=2}^n \lambda_i^4 &= o(n^4). \end{aligned}$$

Similarly, we find $\sigma_2(G) = o(n)$ and $O(\sqrt{n})$.

Definition 0.2 (Local Density). The **local density** of a graph is simply $e(U)$ for some graph $U \subseteq V$.

Remark. Local density is highly variable. For instance in $K_{n,n}$ we find U being one of the partite sets yields 0 local density and U being a set of half the vertices in each partite set yields $\frac{1}{4}e(G)$ local density.

Proposition 0.2. Suppose G is a random graph of order n and let U be a set with $|U| > 502 \log(n)$. Then, $\left| e(U) - \frac{1}{2} \binom{|U|}{2} \right| < \binom{|U|}{2} \left(\frac{3.5 \log n}{|U|} \right)^{\frac{1}{2}}$.

Proposition 0.3. There exists a function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that almost every graph of order n has clique number $f(n)$ or $f(n+1)$.

This function is approximated by

$$f(n) \approx 2 \log_2(n).$$

Remark. There is clearly also such a function for the independence number.

Furthermore, more investigation yields $\chi(G) \approx \frac{n}{2 \log_2(n)}$ for almost all graphs G .

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