Algebraic Theory I

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Lecture 25: Review of Test and Intro to Ring Theory

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Proof of question 6. Let $C_{105} \rtimes_{\alpha} C_5$ and define $\alpha : C_5 \to \operatorname{Aut}(C_{105})$. Recall, we need only show α is the trivial homomorphism. Recall $\operatorname{Aut}(C_{105}) = C_2 \times C_4 \times C_6$. Hence, $|\operatorname{Aut}(C_{105})| = 2 \cdot 4 \cdot 6$ and as $5 \nmid 2 \cdot 4 \cdot 6$, we see every element must map to 1. $\square$

1 Intro to Ring Theory

Definition 1.1 (Ring)**.** A **ring** R is a set equipped with two closed operations $+$ and $\times$ obeying the following properties

1. $(R, +)$ forms an abelian group with additive identity, 0 .
2. There is a multiplicative identity, 1 .
3. $0 \neq 1$. (This would guarantee the ring is trivial)
4. The multiplicative operation is associative : $(xy)\,z = x\,(yz)$ for all $x, y, z \in R$.
5. The distributive properties hold: $x\,(y+z) = xy + xz$ and $(x+y)\,z = xz + yz$ for all $x, y, z \in R$.

A ring for which the multiplication operation is also commutative: $xy = yx$, will be called a **commutative ring**.
In general not every element $x \in R$ has a multiplicative inverse. We define the special class of elements with inverses the **units** of R and we denote x^{-1} to denote the unique inverse of a unit x.
A (not necessarily commutative) ring in which every nonzero element is a unit is a **division ring**. A commutative ring for which every nonzero element is a unit is a **field**.

Remark. Technically, a ring need not have a multiplicative identity, but almost all of them will be equipped with one. Sometimes we denote a ring without identity to be a rng (no i).

Example. $\diamond$

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Recall. A ring is a set, an abelian addition and an associative multiplication with identity.

Definition 1.2 (Subring)**.** A **subring**, R' of R is a subset $R' \subseteq R$ such that R' is closed under its operations and $1 \in R'$.

This object turns out to be mostly uninteresting, so we introduce the following concept.

Definition 1.3 (Ideal)**.** A **left ideal** of the ring R is a nonempty subset $I \subseteq R$ so that $I \leq R$ under addition and $rI \subseteq I$ for all $r \in R$. This second condition is equivalent to for all $x \in I$, $r \in R \Rightarrow rx \in I$.
Right ideals follow the same first condition and for the second condition we have $Ir \subseteq I$ for all $r \in R$. A **(two-sided) ideal** is a set I which is both a left and a right ideal.

Example. $I = p\mathbb{Z}$ is an ideal of $\mathbb{Z}$.

$\diamond$

Ideals will play a similar role as that of normal subgroups.

Definition 1.4 (Ring Homomorphisms)**.** If R, R' are rings and $\psi : R \to R'$ is a map. ψ is a **ring homomorphism** if

- $\psi(x + y) = \psi(x) + \psi(y)$ for all $x, y \in R$,
- $\psi(xy) = \psi(x)\psi(y)$ for all $x, y \in R$,
- $\psi(1_R) = 1_{R'}$ (if R, R' are rings with identities).

Example. If $R = \mathbb{Z}/6\mathbb{Z}$. Consider the map $f : \mathbb{Z}/6\mathbb{Z} \to \mathbb{Z}/6\mathbb{Z},\ x \mapsto 3x$. We see the first two conditions hold under standard modular arithmetic, but the identity condition clearly fails, so we would consider this a ring homomorphism of rings without identity, but it is not a homomorphism of rings with identity.
$\diamond$

Definition 1.5. If R is a ring and $I \subseteq R$ is an ideal.
Then, we define $R/I = \{x+I : x \in R\}$, with $(x + I)+(y + I) \coloneqq (x + y)+I$ and $(x + I)(y + I) \coloneqq xy + I$, to be the ring quotient of $R \mod I$.