Algebraic Theory I

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Lecture 22: Free Groups (5)

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Recall. Let G, H be groups with presentations $\varepsilon : F \to G$ and $\delta : F \to H$ for some free group F, If every relator of G is also a relator for H, then there is a surjective homomorphism $\varphi : G \to H,\ \varepsilon(x) \mapsto \delta(x)$.

Definition 0.1 (Reduced Word)**.** We define a word w to be **reduced** if no string xx^{-1} or $x^{-1}x$ occurs within w for any $x \in X$. We find any word is equivalent to some reduced word by applying our relations.

Theorem 0.1. Every word is equivalent to a unique reduced word.

Proof. We proceed fancily (he really said this). Let R be the set of reduced words on the alphabet X. For each $m \in X$, define a map

$$m' : R \to R,\ x_1^{\varepsilon_1} \dots x_\ell^{\varepsilon_\ell} \mapsto \begin{cases} m x_1^{\varepsilon_1} \dots x_\ell^{\varepsilon_\ell}, & m \neq x_1^{-\varepsilon_1} \\ x_2^{\varepsilon_2} \dots x_\ell^{\varepsilon_\ell}, & m = x_1^{-\varepsilon_1} \end{cases}$$

We see m' is a bijection as $\left(m^{-1}\right)' = m'^{-1}$. Hence, m' is simply a permutation of the set R.
Now, using the universal mapping property on $F(X)$, we define a homomorphism

$$\begin{aligned} \theta : F(X) &\longrightarrow \mathrm{Sym}(R) \\ [m] &\longmapsto m' \end{aligned}$$

where $\mathrm{Sym}(R)$ is simply the set of all permutations of R. Now, suppose $w = x_1^{\varepsilon_1} \dots x_\ell^{\varepsilon_\ell}$ and $w' = y_1^{\delta_1} \dots y_s^{\delta_s}$ are two reduced words that are equivalent, that is $[w] = [w']$. Then, we have $\theta([w]) = (x_1')^{\varepsilon_1} \dots (x_\ell')^{\varepsilon_\ell}$.Then, we see $\theta([w])(1) = w$. Hence, $\theta([w']) = \theta([w]) = y_1^{\delta_1} \dots y_s^{\delta_s}$. Hence, we see $x_1^{\varepsilon_1} \dots x_\ell^{\varepsilon_\ell} = y_1^{\delta_1} \dots y_s^{\delta_s}$ as words. Hence, there is at most one distinct reduced word in $[w]$. And, as there is always atleast 1 reduced word, we see this completes the proof.

□

Remark. We define $x^n = \underbrace{x \dots x}_{n \text{ times}}$ and $x^{-n} = \underbrace{x^{-1}x^{-1} \dots x^{-1}}_{n \text{ times}}$. Then, we see any reduced word has the form $x_1^{\ell_1} \dots x_s^{\ell_s}$ with $\ell_i \in \mathbb{Z} \setminus \{0\}$ and $x_i \neq x_{i-1}$ for all $1 \leq i \leq s$. This is called the normal form of a word.

Definition 0.2. With the normal form of a word, we define a **multiplicity function**. For $x \in X$ and a word $w = x_1^{\ell_1} \dots x_s^{\ell_s}$ we define $V_x(w) = \sum_{x_j = x} \ell_j$.

We note that if $w \sim w'$, we have $V_x(w) = V_x(w')$ for all $x \in X$. Furthermore, $V_x(w) = V_x\left(v^{-1}wv\right)$ for all $x \in X$ and words v, w. Moreover, $V_x(wv) = V_x(w) + V_x(v)$, so its a homormophism from $F(X) \to \mathbb{Z}$.

Definition 0.3 (Rank)**.** Recall that if $|X| = |Y|$, we had $F(X) \simeq F(Y)$. We define $(F(X)) = |X|$. We have yet to show this is well defined, but the next theorem will take care of this.

Theorem 0.2. If X and Y are sets with $F(X) \simeq F(Y)$, then $|X| = |Y|$.

We will prove this claim next class.

Lecture 23

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