

Analysis I

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Contents

Lecture 18: General Lebesgue Integral (2)

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Proposition 0.1. Let $f : \mathbb{R} \rightarrow \overline{\mathbb{R}}$ be integrable. Then for each $\varepsilon > 0$ there is a $\delta > 0$ such that each measurable $S \subseteq \mathbb{R}$ has $\int_S |f| < \varepsilon$ if $m(S) < \delta$.

Proof. Let $\varepsilon > 0$, then there is a $s \in \mathcal{S}(|f|)$ such that $\int (|f| - s) < \frac{\varepsilon}{2}$. Let $\alpha = \sup\{s(x) : x \in \mathbb{R}\}$ and $\delta = \frac{\varepsilon}{2(\alpha + \varepsilon)}$. If S is measurable and $m(S) < \delta$, we find

$$\int_S |f| \leq \int s + \frac{\varepsilon}{2} \leq \alpha m(S) + \frac{\varepsilon}{2} < \varepsilon.$$

□

Theorem 0.1 (Monotone Convergence Theorem). Let (f_n) be a sequence of nonnegative measurable functions with $f_n : \mathbb{R} \rightarrow \overline{\mathbb{R}}$ such that $(f_n(x))$ is increasing for all $x \in \mathbb{R}$. Then, $f = \lim_{n \rightarrow \infty} f_n$ is measurable with $\int f = \lim_{n \rightarrow \infty} \int f_n$.

Proof. Since $f = \limsup_{n \rightarrow \infty} f_n = \liminf_{n \rightarrow \infty} f_n$, we see f is measurable. Moreover, the sequence $(\int f_n)$ is increasing (as the f_n s are increasing). Hence, letting $L = \lim_{n \rightarrow \infty} \int f_n$ exists with $L \in \mathbb{R}_0^+$. Since $\int f_n \leq \int f$ for all n by monotonicity, we find $L \leq \int f$.

Let $s \in \mathcal{S}(f)$ and fix $c \in (0, 1)$ and define $E_n = \{x \in \mathbb{R} : f_n(x) \geq cs(x)\}$. Then, we find $\{E_n : n \in \mathbb{N}\}$ is an ascending collection (again by monotonicity of (f_n)) of measurable sets with $\bigcup_{n \in \mathbb{N}} E_n = \mathbb{R}$ as $cs(x) < f_n(x) \leq f(x)$. Let $s = \sum_{k=1}^K a_k \chi_{S_k}$ and we see $cs \chi_{E_n} = f_n \chi_{E_n} \leq f_n$, with

$$L \geq \int f_n \geq \int_{E_n} f_n \geq \int cs \chi_{E_n} = c \int_{E_n} s = c \sum_{k=1}^K a_k m(S_k \cap E_n).$$

Since $\lim_{n \rightarrow \infty} m(E_n \cap S_k) = m(S_k)$ for every measurable set S , we find $L \geq c \sum_{k=1}^K a_k m(S_k) = c \int s$. Since c was arbitrary, we see the inequality holds

for all $c \in (0, 1)$, hence we find $L \geq s$ (by taking supremums), but $s \in \mathcal{S}(f)$, hence $L \geq \int f$. So, $L = \int f$. \square

Theorem 0.2 (Fatou's Lemma). If (f_n) is a sequence of nonnegative measurable functions $f_n : \mathbb{R} \rightarrow \overline{\mathbb{R}}$, then $\int \liminf_{n \rightarrow \infty} f_n \leq \liminf_{n \rightarrow \infty} \int f_n$.

Proof. For $x \in \mathbb{R}$, define $g_n(x) = \inf\{f_k(x) : k \geq n\}$ for $n \in \mathbb{N}$. Then, we find (g_n) is a nonnegative measurable sequence of functions with $(g_n(x))$ increasing for all fixed x and $g_n \leq f_n$ for all n . Consequently, $\int g_n \leq \int f_n$ and $(\int g_n)$ is increasing. As $\lim_{n \rightarrow \infty} g_n = \liminf_{n \rightarrow \infty} f_n$ is measurable by an earlier theorem, we find

$$\liminf_{n \rightarrow \infty} \int f_n \geq \liminf_{n \rightarrow \infty} \int g_n = \lim_{n \rightarrow \infty} \int g_n = \int \lim_{n \rightarrow \infty} g_n = \int \liminf_{n \rightarrow \infty} f_n.$$

\square

Proposition 0.2. For any integral function $f : \mathbb{R} \rightarrow \overline{\mathbb{R}}$, we find $|\int f| \leq \int |f|$.

Theorem 0.3 (Dominated Convergence Theorem). Let (f_n) be a sequence of measurable functions $f_n : \mathbb{R} \rightarrow \overline{\mathbb{R}}$. Suppose there is an integrable function g with $|f_n| \leq g$ for all $n \in \mathbb{N}$. If (f_n) converges pointwise to a function $f : \mathbb{R} \rightarrow \overline{\mathbb{R}}$ almost everywhere, then f is integrable and

$$\lim_{n \rightarrow \infty} \int |f_n - f| = 0 \text{ and } \lim_{n \rightarrow \infty} \int f_n = \int f.$$

Proof. Since $f(x) = \lim_{n \rightarrow \infty} f_n(x)$ for almost all $x \in R$, we find f is measurable. Moreover, $|f_n| \leq g$ implies $|f| < g$ almost everywhere and since g is integrable (hence finite a.e) we find f, f_n are integrable (hence finite) almost everywhere. Now, define for each $n \in \mathbb{N}$

$$E_n = \{x \in \mathbb{R} : |f_n(x)|, |f(x)| < \infty, |f_n(x) - f(x)| \leq 2g(x)\}.$$

Since $R \setminus \bigcup_{n \in \mathbb{N}} E_n$ is a set of measure 0, we can assume $|f_n(x)|, |f(x)| < \infty$ and $|f_n(x) - f(x)| \leq 2g(x)$ for all $x \in \mathbb{R}$. Then, Fatou's lemma applies to the

sequence on nonnegative measurable functions $(2g - |f_n - f|)$ yielding

$$\begin{aligned}
 \int 2g &\leq \liminf_{n \rightarrow \infty} (2g - |f_n - f|) \\
 &= \int 2g + \liminf_{n \rightarrow \infty} \left(- \int |f_n - f| \right) \\
 &= \int 2g - \limsup_{n \rightarrow \infty} \int |f_n - f| \\
 &\Rightarrow \limsup_{n \rightarrow \infty} \int |f_n - f| \leq 0 \\
 &\Rightarrow \lim_{n \rightarrow \infty} \int |f_n - f| = 0.
 \end{aligned}$$

Hence, $\lim_{n \rightarrow \infty} \int (f_n - f) = 0$ by the earlier lemma. So, $\lim_{n \rightarrow \infty} \int f_n = \int f$. \square

Definition 0.1 (Convergence in Measure). Let (f_n) be a sequence of measurable functions $f_n : \mathbb{R} \rightarrow \overline{\mathbb{R}}$ and $f : \mathbb{R} \rightarrow \overline{\mathbb{R}}$ also be measurable. The sequence (f_n) **converges in measure** to f ($f_n \rightarrow f$ by measure) if each f_n is finite almost everywhere and for each $\varepsilon > 0$ there is a $N \in \mathbb{N}$ so that

$$m(\{x \in \mathbb{R} : |f_n(x) - f(x)| > \varepsilon\}) < \varepsilon$$

for $n \geq N$.

Theorem 0.4 (Riesz). Let (f_n) be a sequence of measurable functions $f_n : \mathbb{R} \rightarrow \overline{\mathbb{R}}$ and $f : \mathbb{R} \rightarrow \overline{\mathbb{R}}$ also being measurable. If $(f_n) \rightarrow f$ in measure, then there is a subsequence (f_{n_k}) which converges pointwise almost everywhere to f .

Proof. First, we find a strictly increasing sequence of numbers (n_k) such that $m(\{x \in \mathbb{R} : |f_j(x) - f(x)| > 2^{-k}\}) < 2^{-k}$ if $j \geq n_k$. For $k \in \mathbb{N}$ denote

$$S_k = \{x \in \mathbb{R} : |f_{n_k}(x) - f(x)| > 2^{-k}\}.$$

Then, $\sum_{k=1}^{\infty} m(S_k) \leq \sum_{k=1}^{\infty} 2^{-k} < \infty$. Applying the Borel-Cantelli Lemma yields that almost every $x \in \mathbb{R}$ does not belong to any infinite subcollections of (S_k) . For such x , we find a $K \in \mathbb{N}$ such that $|f_{n_k}(x) - f(x)| \leq 2^{-k}$ for $k \geq K$. Hence, f_{n_k} converges pointwise to f for all x not belonging to an infinite subcollection of (S_k) , hence almost everywhere. \square

Lecture 19

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