MATH 7237/8237

LECTURE Mar. 1, 2020

EXTREMAL GRAPH THEORY

In this lecture we shall study some extremal problems in graph theory.

Before defining the scope of extremal graph theory, let us discuss a concrete problem.

Recall that the 3-cliques in a graph are often called *triangles*.

Definition A graph G is called **triangle-free** if G does not contain a 3-clique.

Note that bipartite graphs are triangle-free.

However the property "being triangle-free" is much more complex than "being bipartite."

Indeed, the 5-cycle is triangle-free, but is not bipartite.

It is clear that if we take a triangle-free graph of order $n \geq 3$, and start adding new edges to it, sooner or later a triangle is going to appear.

This intuition leads us to a natural question:

Question *What is the maximum number of edges in a triangle-free graph of order n?*

Let us try to understand what this question is about.

We want to determine a function of n, say $f(n)$, such that if a graph G of order n has more than $f(n)$ edges, then G surely contains a triangle.

At the same time, we want $f(n)$ to be best possible, that is to say, there exists a triangle-free graph of order n with $f(n)$ edges.

MANTEL'S THEOREM

The answer to the above question was given by Mantel in 1907.

Theorem (Mantel) *If a graph G of order $n \geq 3$ has more than $\lfloor n^2/4 \rfloor$ edges, then G contains a triangle. For every $n \geq 3$, there is a triangle-free graph of order n and $\lfloor n^2/4 \rfloor$ edges.*

Looking at Mantel's theorem, we see that it consists of two rather distinct statements, which are logically independent and need therefore separate proofs.

For us it would be easier to prove the second statement, namely: *for every $n \geq 3$ there is a triangle-free graph of order n and $\lfloor n^2/4 \rfloor$ edges.*

The prove such statements, one starts with a search in one's long-term memory for graphs that possibly fit the required conditions.

THE EXTREMAL GRAPH

In our case, we may remember that some time ago we found the maximum number of edges in a bipartite graph of order n, which is precisely $\lfloor n^2/4 \rfloor$ and is attained on the complete bipartite graph $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$.

Since $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ is triangle-free, the proof of the second part of Mantel's theorem is completed.

For what follows, it is helpful to realize that

$$\left\lfloor n^2/4 \right\rfloor = \begin{cases} n^2/4, & \text{if } n \text{ is even;} \\ \left(n^2 - 1\right)/4, & \text{if } n \text{ is odd.} \end{cases}$$

In addition, note that the cardinalities of the vertex classes of the complete bipartite graph $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ are equal if n is even, and differ by 1 if n is odd.

Before continuing with the proof of the first statement of Mantel's theorem, let us prove a simple lemma, similar to Dirac's theorem:

Lemma *If* G *is a graph of order* $n \geq 3$*, and* $\delta(G) > n/2$*, then* G *contains a triangle.*

Indeed, let $\{u, v\}$ be an edge of G.
Recall that $N(u)$ is the set of neighbors of u and $N(v)$ is the set of neighbors of v.

We see that

$$|N(u)| = d(u) > n/2 \quad \text{and} \quad |N(v)| = d(v) > n/2.$$

Hence, $N(v)$ and $N(u)$ have a vertex in common, as otherwise we get

$$n \geq |N(u) \cup N(v)| = |N(u)| + |N(v)| > n,$$

which is a contradiction.

Let w belong to both $N(u)$ and $N(v)$, that is, w is joined to both u and v.

Clearly u, v, w is a triangle in G, completing the proof of the lemma.

PROOF OF THE FIRST PART OF MANTEL'S THEOREM

To finish the proof of Mantel's theorem, it remains to prove the following statement:

- **Theorem** *If a graph G of order $n \geq 3$ has more than $\lfloor n^2/4 \rfloor$ edges, then G contains a triangle.*

- The proof is by induction on n. For $n = 3$, we see that $\lfloor 3^2/4 \rfloor = 2$. So G has at least 3 edges, and therefore G is a triangle.

- Assume that $n \geq 4$ and that the statement holds for graphs with fewer than n vertices.

- If $\delta(G) > n/2$, then the above lemma implies that G contains a triangle, so we may suppose that $\delta(G) \leq n/2$.

Since $\delta(G)$ is an integer, we see that, in fact, $\delta(G) \leq \lfloor n/2 \rfloor$

Suppose that u is a vertex such that

$$d(u) = \delta(G) \leq \lfloor n/2 \rfloor .$$

Remove the vertex u and write G' for the graph $G - u$.

The graph G' is of order $n - 1$, and moreover

$$\begin{aligned} e(G') &= e(G) - d(u) \\ &> \left\lfloor \frac{n^2}{4} \right\rfloor - \left\lfloor \frac{n}{2} \right\rfloor . \end{aligned}$$

Our next goal is to prove that

$$\left\lfloor \frac{n^2}{4} \right\rfloor - \left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{(n-1)^2}{4} \right\rfloor .$$

Consider separately the cases n odd and n even.

If $n = 2k$, then

$$\left\lfloor \frac{n^2}{4} \right\rfloor - \left\lfloor \frac{n}{2} \right\rfloor = k^2 - k = \frac{(2k-1)^2 - 1}{4} = \left\lfloor \frac{(n-1)^2}{4} \right\rfloor .$$

If $n = 2k+1$, then

$$\left\lfloor \frac{n^2}{4} \right\rfloor - \left\lfloor \frac{n}{2} \right\rfloor = k(k+1) - k = \frac{(2k)^2}{4} = \left\lfloor \frac{(n-1)^2}{4} \right\rfloor .$$

Hence, $e(G') > \left\lfloor (n-1)^2 / 4 \right\rfloor$. By the induction assumption, G' contains a triangle, and so does G.

This completes the induction step and the proof of Mantel's theorem.

There are other proofs of Mantels's theorem. The one given above has a few points that can be used in other similar situations.

THE TURÁN GRAPH

Triangles are 3-cliques, so it is not hard to think of a possible generalization of Mantel's theorem to r-cliques for $r \geq 4$.

Indeed such a generalization was given by the great Hungarian mathematician Paul Turán.

Let us first define the Turán graph $T_r(n)$.

For $r = 2$, the Turán graph $T_r(n)$ is just the familiar complete bipartite graph $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$.

For $r \geq 3$, the Turán graph $T_r(n)$ is a complete r-partite graph of order n, such that the cardinalities of its vertex classes are as close as possible.

The definition implies that the size of each vertex class of $T_r(n)$ is either $\lfloor n/r \rfloor$ or $\lceil n/r \rceil$ (Convince yourself.)

In particular, if r divides n, then all vertex classes of $T_r(n)$ contain exactly n/r vertices.

The Turán graph $T_r(n)$ is remarkable in many respects.

For example, $T_r(n)$ has the largest number of edges among all r-partite graphs.

Obviously $T_r(n)$ contains no $(r+1)$-cliques (being r-partite.)

Definition A graph G is called K_r-**free** if G does not contain r-cliques.

TURÁN'S THEOREM

In 1940, Turán came up with an important generalization of Mantel's theorem:

Theorem (Turán) *If a graph G of order $n \geq r$ has more edges than $T_r(n)$, then G contains an $(r+1)$-clique.*

Another logically equivalent way of stating Turán's theorem is the following one:

Theorem (Turán) *If G is a K_{r+1}-free graph of order $n \geq r$, then*

$$e(G) \leq e(T_r(n)).$$

Note that using the number $e(T_r(n))$ instead of an explicit expression of r and n has advantages, but gives no info as to how large $e(T_r(n))$ is indeed.

In fact, the quantity $e\left(T_r\left(n\right)\right)$ can be expressed as a function of n and r, but the expression is somewhat clumsy for $r > 2$:

If s is the remainder of n mod r, then

$$e\left(T_r\left(n\right)\right) = \left(1 - \frac{1}{r}\right)\frac{n^2}{2} - \frac{s\left(r-s\right)}{2r}.$$

It is not hard to deduce bounds on $e\left(T_r\left(n\right)\right)$:

$$\left(1 - \frac{1}{r}\right)\frac{n^2}{2} - \frac{r}{8} \leq e\left(T_r\left(n\right)\right) \leq \left(1 - \frac{1}{r}\right)\frac{n^2}{2},$$

In particular if r divides n, then

$$e\left(T_r\left(n\right)\right) = \left(1 - \frac{1}{r}\right)\frac{n^2}{2}.$$

Often these relations are used to give an approximate version of Turán's theorem:

THE CONCISE TURÁN'S THEOREM

Theorem *If G is a graph of order $n \geq r$ and*

$$e(G) > \left(1 - \frac{1}{r}\right)\frac{n^2}{2},$$

then G contains an $(r+1)$-clique.

An equivalent and even more concise statement is:

Theorem *If G is a graph of order n with clique number ω, then*

$$e(G) \leq \left(1 - \frac{1}{\omega}\right)\frac{n^2}{2}.$$

EXTREMAL GRAPH THEORY

Turán's theorem has deeply influenced the development of graph theory.

In particular, a new area in graph theory was born, called **Extremal Graph Theory**.

The central type of problem in extremal graph theory is the following one:

Problem *Let H be a fixed graph. For every n, find the maximum number of edges in a graph G of order n that does not contain a subgraph isomorphic to H.*

In the above context, H is usually called a **forbidden** subgraph, and G is called an H-**free** graph..

This problem has been solved, at least approximately, for many graphs H.

FORBIDDEN PATHS AND CYCLES

For forbidden paths and cycles the following two results have been proved by Erdős and Gallai:

- **Theorem (Erdős-Gallai)** *Let $k \geq 2$. If G is a graph of order n with no P_k, then*
$$e(G) \leq \frac{k-2}{2} n.$$
Equality is possible if and only if G is a union of disjoint $(k-1)$-cliques.

- **Theorem (Erdős-Gallai)** *Let $k \geq 3$. If G is a graph of order n with no cycle longer than $k-1$, then*
$$e(G) \leq \frac{k-1}{2}(n-1).$$
Equality is possible if G is a union of $(k-1)$-cliques sharing a common vertex.

FORBIDDEN CYCLES

Note that Erdős-Gallai's result does not imply that if

$$e(G) > \frac{k-1}{2}(n-1),$$

then G contains C_k. All we can claim in such a situation that G contains a cycle longer than $k-1$.

More complicated theorems exist for forcing cycles of specified length.

Theorem (Bollobás) *If G is a graph of order n and more than $\lfloor n^2/4 \rfloor$ edges, then G contains C_l for every l such that $3 \leq l \leq (n+1)/2$.*

For even cycles the question is less well understood.

Theorem (Klein) *If G is a C_4-free graph of order n, then*

$$e(G) \leq \frac{n}{4}\left(1+\sqrt{4n-3}\right)$$

Equality holds for infinitely many values of n.

NONBIPARTITE FORBIDDEN GRAPHS

A fairly general theorem settles asymptotically the problem for all nonbipartite graphs.

Theorem (Erdős-Stone-Simonovits) *Let H be a graph with $\chi(H) \geq 3$. For every ε, there exists $n(\varepsilon)$ such that if G is a H-free graph of order $n > n(\varepsilon)$, then*

$$e(G) \leq \left(1 - \frac{1}{\chi - 1} + \varepsilon\right) \frac{n^2}{2}.$$

In addition, for many specific graphs H, there are exact results, e.g., **books** (a few triangles sharing an edge) and **fans** (a few triangles sharing a vertex.).

For bipartite forbidden graphs the problem is more difficult in general.

THE THEOREM OF MOTZKIN AND STRAUS

Let G be a graph of order n and let us assume that $V(G) = \{1, \ldots, n\}$.

Suppose that $\mathbf{x} := (x_1, \ldots, x_n)$ is a real n-vector and consider the function

$$f_G(\mathbf{x}) = \sum_{\{i,j\} \in E(G)} x_i x_j.$$

Suppose further that the entries $x_1, \ldots, x_n$ are nonnegative and in addition

$$x_1 + \cdots + x_n = 1.$$

The set of all such n-vectors is called **the standard** $(n-1)$**-dimensional simplex** and is denoted by Δ_{n-1}.

Note that Δ_{n-1} is a bounded and closed subset of $\mathbb{R}^n$, and therefore it is a compact set.

On the other hand, for any fixed graph G, the function $f_G(\mathbf{x})$ is continuous in $\mathbf{x}$.

Hence $f_G(\mathbf{x})$ attains both a maximum and a minimum over the compact set Δ_{n-1}.

It is interesting to find

$$\min_{\mathbf{y}\in\Delta_{n-1}} f_G(\mathbf{y}) \text{ and } \max_{\mathbf{y}\in\Delta_{n-1}} f_G(\mathbf{y}).$$

The former case is trivial: For any graph G

$$\min_{\mathbf{y}\in\Delta_{n-1}} f_G(\mathbf{y}) = 0.$$

Indeed, it is enough to take

$$x_1 = 1, \quad x_2 = \cdots = x_n = 0.$$

Motzkin and Straus solved the case of maximum of $f_G(\mathbf{y})$:

Theorem (Motzkin and Straus) If G is a graph of order n with clique number ω, then

$$\max_{\mathbf{y}\in\Delta_{n-1}} f_G(\mathbf{y}) = \frac{1}{2}\left(1-\frac{1}{\omega}\right).$$

Before getting to the proof of this theorem, let us note an important application:

Letting

$$x_1 = \cdots = x_n = \frac{1}{n},$$

we see that

$$f_G(\mathbf{x}) = \frac{e(G)}{n^2}.$$

Since $(x_1, \ldots, x_n) \in \Delta_{n-1}$, the theorem of Motzkin and Straus implies that

$$e(G) \leq \left(1 - \frac{1}{\omega}\right) \frac{n^2}{2},$$

which is the concise Turán theorem.

Surprisingly, the theorem of Motzkin and Straus can in turn be deduced from the concise Turán theorem.

However, we shall follow the original proof given by Motzkin and Straus.

Let $\mathbf{x} := (x_1, \ldots, x_n) \in \Delta_{n-1}$ be such that

$$f_G(\mathbf{x}) = \max_{\mathbf{y} \in \Delta_{n-1}} f_G(\mathbf{y}).$$

It is possible that some of the entries of $\mathbf{x}$ are 0, so let us further assume that $\mathbf{x}$ has minimum number of nonzero entries.

We claim that every two vertices corresponding to nonzero entries of $\mathbf{x}$ are adjacent.

Assume that this is not the case, and let u and v be two non-adjacent vertices such that $x_u > 0$ and $x_v > 0$.

Note that

$$f_G(\mathbf{x}) = x_u \sum_{w \in N(u)} x_w + x_v \sum_{w \in N(v)} x_w + \sum_{\{i,j\} \in E(G-u-v)} x_i x_j.$$

Our first goal is to show that

$$\sum_{w \in N(u)} x_w = \sum_{w \in N(v)} x_w.$$

Indeed, assume that this is not the case and suppose by symmetry that

$$\sum_{w \in N(u)} x_w > \sum_{w \in N(v)} x_w.$$

Now, define $\mathbf{z} := (z_1, \ldots, z_n) \in \Delta_{n-1}$ by

$$z_i := \begin{cases} x_u + x_v, & \text{if } i = u; \\ 0, & \text{if } i = v; \\ x_i, & \text{if } i \neq u \text{ and } i \neq v. \end{cases} .$$

We see that

$$\begin{aligned} f_G(\mathbf{z}) &= (x_u + x_v) \sum_{w \in N(u)} x_w + \sum_{\{i,j\} \in E(G-u-v)} x_i x_j \\ &> x_u \sum_{w \in N(u)} x_w + x_v \sum_{w \in N(v)} x_w + \sum_{\{i,j\} \in E(G-u-v)} x_i x_j \\ &= f_G(\mathbf{x}). \end{aligned}$$

However, this inequality contradicts the fact that $f_G(\mathbf{y})$ attains maximum at $\mathbf{x}$. Hence,

$$\sum_{w \in N(u)} x_w = \sum_{w \in N(v)} x_w.$$

Now, defining

$$\mathbf{z} := (z_1, \ldots, z_n) \in \Delta_{n-1}$$

as above, we find that

$$f_G(\mathbf{z}) = f_G(\mathbf{x}).$$

But $\mathbf{z}$ has fewer nonzero entries than $\mathbf{x}$, contrary to our assumption that $\mathbf{x}$ has minimum number of nonzero entries.

Hence, every two vertices corresponding to nonzero entries of $\mathbf{x}$ are adjacent.

In other words, there exists a clique of G, say H, such that

$$f_G(\mathbf{x}) = \sum_{i,j \in V(H),\ i \neq j} x_i x_j.$$

Write k for the order of H and note that

$$
\begin{aligned}
2 \sum_{i,j \in V(H),\, i \neq j} x_i x_j &= \left(\sum_{i \in V(H)} x_i \right)^2 - \sum_{i \in V(H)} x_i^2 \\
&\leq \left(\sum_{i \in V(H)} x_i \right)^2 - \frac{1}{k} \left(\sum_{i \in V(H)} x_i \right)^2 \\
&\leq 1 - \frac{1}{k} \leq 1 - \frac{1}{\omega}.
\end{aligned}
$$

We just proved that

$$
\max_{\mathbf{y} \in \Delta_{n-1}} f_G(\mathbf{y}) \leq \frac{1}{2} \left(1 - \frac{1}{\omega} \right).
$$

To show the opposite inequality, take a maximal clique H of G, and define

$$\mathbf{z} := (z_1, \ldots, z_n) \in \Delta_{n-1}$$

by

$$z_i := \begin{cases} 1/\omega, & \text{if } i \in V(H); \\ 0, & \text{if } i \notin V(H). \end{cases}$$

We see that

$$f_G(\mathbf{z}) = \frac{e(H)}{\omega^2} = \frac{\omega(\omega-1)}{2\omega^2} = \frac{1}{2}\left(1 - \frac{1}{\omega}\right).$$

Hence,

$$\max_{\mathbf{y} \in \Delta_{n-1}} f_G(\mathbf{y}) \geq \frac{1}{2}\left(1 - \frac{1}{\omega}\right),$$

completing the proof of the theorem of Motzkin and Straus.

THANK YOU