Analysis I: Homework 8 and 9

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Problem (36)**.** Our function will be φ, the cantor-lebesque function. We have already shown it to be continuous and increasing with $\varphi(1) = 1, \varphi(0) = 0$. Moreover, letting C be the cantor set, we see $[0,1] \setminus C \coloneqq C^c$ is open in $[0,1]$ so for all $x \in C^c$, there is an $\varepsilon > 0$ so that $(x-\varepsilon, x+\varepsilon) \subseteq C^c$. Then, since for all intervals I in the $[0,1]$ complement of the cantor set, we find $I \subseteq J_{n,k}$ for some $n, k \in \mathbb{N}$, we have $\xi(I) = \{\frac{n}{2^k}\}$, so

$$\overline{D}(\varphi(x)) = \lim_{r\to 0} \sup\{\frac{\varphi(x+h) - \varphi(x)}{h} : 0 < |h| < r\} = \lim_{r\to 0} \sup\{\frac{0}{h} : 0 < |h| < r\} = 0.$$

Similarly, we find $\underline{D}(\varphi(x)) = 0$. Hence, φ is differentiable at x and since $\varphi' = 0$ almost everywhere, yet φ is not constant by the initial claim, we find φ is not absolutely continuous.

Problem (38)**.** First, note that $\varphi : \mathbb{R} \to \overline{\mathbb{R}},\ x \mapsto \sqrt{1+x^2}$ is convex and since h is integrable, we see it is finite almost everywhere. Hence, discarding the points for which $h = \infty$, we see jensens inequality yields

$$\sqrt{1+A^2} \leq \int_{[0,1]} \sqrt{1+h^2}.$$

For the second inequality, note that since h is nonnegative and $\sqrt{\cdot}$ is an increasing function we have

$$\int_{[0,1]} \sqrt{1+h^2} \leq \int_{[0,1]} \sqrt{1+2h+h^2} \leq \int_{[0,1]} 1+h = 1+A.$$

Problem (39).

- Assume (f_n) does not converge to f in measure. That is, there is an $\varepsilon > 0$ so that for all $N \in \mathbb{N}$

$$m\left(\{x \in \mathbb{R} : |f_{n_N}(x) - f(x)| > \varepsilon\}\right) > \varepsilon$$

for some $n_N \geq N$. Denote this set A_N. Then, we see

$$\int |f_{n_N} - f| \geq \int_{A_N} |f_{n_N} - f| \geq \int \varepsilon \chi_{A_N} = \varepsilon m(A_N) \geq \varepsilon^2.$$

That is, for some $\varepsilon' = \varepsilon^2 > 0$, and all $N \in \mathbb{N}$ we find an $n_N \geq N$, so that $\int |f_n - f| \geq \varepsilon'$, so f_n does not converge to f in mean.

- First, note that if $x = 0$ or 1, then $f_n(x) = x$ for all $n \in \mathbb{N}$. Then, if $x \in (0, 1)$, for all $\varepsilon > 0$, there is an $N \in \mathbb{N}$ so that $x^n <$

Problem (40).

- The first function will be $f_n = \chi_{(n,\infty)}$. We note that for all x, $x \notin (n,\infty)$ for all $n \geq \lceil x \rceil$, so (f_n) converges point wise. On the other hand for $\varepsilon = \frac{1}{2}$, we see $m\left(\left\{x \in \mathbb{R} : |f_n(x) - f(x)| > \frac{1}{2}\right\}\right) = m((n,\infty)) = \infty > \varepsilon$, so (f_n) does not converge in measure (hence not in mean).

- For the second function define the following sequence of intervals. $A_1 = [0,1]$, $A_{2^k} = \left[0, \frac{1}{2^k}\right]$ and $A_{2^k+c} = \left[\frac{c}{2^k}, \frac{c+1}{2^k}\right]$ for $c < 2^k$. This essentially enumerates all partitions with endpoints being a rational with denominators powers of 2 and consecutive numerators. Since the collection $\{A_{2^k+c} : 0 \leq c < 2^k\}$ covers $[0,1]$ for every $k \in \mathbb{N}$, we see for all $N \in \mathbb{N}$ and $x \in [0,1]$, the function $f_n = \chi_{A_n}$ will have $f_n(x) = 1$ for some (infinitely many) $n \geq N$, so it will not converge to 0 pointwise. On the other hand, we see $|f_n - 0| = f_n = \chi_{A_n}$, so $\int |f_n - 0| = m(A_n)$. Moreover, for all $k \in \mathbb{N}$ we find an $N = \lfloor \log_2(n) \rfloor$ so that $m(A_n) < \frac{1}{2^k}$ for all $n \geq N$, so f_n does in fact converge in mean and in measure.

- For the third function we adopt the same intervals from part 2, but we instead define the function $f_n = 2^n \chi_{A_n}$. Recalling that $m(A_n) \geq \frac{1}{2^n}$ for all n, we see $\int |f_n - 0| = \int 2^n \chi_{A_n} = 2^n m(A_n) \geq \frac{2^n}{2^n} = 1$ for all $n \in \mathbb{N}$. Hence for all $\varepsilon < 1$ we find convergence in mean to fail. Moreover, f_n still fails to converge pointwise. Lastly, recall for all $k \in \mathbb{N}$ there is a $N \in \mathbb{N}$ so that $m(A_n) \leq \frac{1}{2^k}$ for all $n \geq N$, hence for all $\varepsilon > \frac{1}{2^k}$ we find the convergence in measure criterion holds. Since there is a $k \in \mathbb{N}$ so that $0 < \frac{1}{2^k} < \varepsilon$ for all $\varepsilon > 0$, we see convergence in measure does in fact hold true.