

Algebraic Theory I

Thomas Fleming

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Lecture 16: Nilpotent Groups (3)

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Corollary 1. A finite abelian group is the direct product of its sylow groups.

This follows directly from the theorem from last class.

Corollary 2. If G is a finite group such that for all $n \mid |G|$ such that there are at most n elements $x \in G$ with $x^n = 1$, then G is cyclic.

Proof. Let p be an arbitrary prime with $p \mid |G|$. Let P be a sylow p -group with $|P| = p^\alpha$. We know for any $x \in P$, we have $x^{|P|} = 1$, hence there are $|P| = p^\alpha$ elements $x \in P$ such that $x^{p^\alpha} = 1$. By hypothesis there is infact equality. If there was another distinct sylow p -group we would have elements $y \notin P$ such that $y^{p^\alpha} = 1$. Hence, P is unique. Hence, as every p -group is unique, so normal, we see G is the product of its P -groups.

Denote $G = P_1 \times P_2 \times \dots \times P_t$ with the P_i s being the distinct sylow p_i groups of G . Also, if $|P_1| = p_1^{\alpha_1}$, then all $x \in P_1$ have $\text{ord}(x) \mid p_1^{\alpha_1}$ and there are at most $p_1^{\alpha_1-1} < p_1^{\alpha_1}$ such x with $\text{ord}(x) \mid p_1^{\alpha_1-1}$. Since $|P| < p_1^{\alpha_1-1}$ we see there is an $x \in P_1$ with $\text{ord}(x) = p_1^{\alpha_1} = |P|$, hence $\langle x \rangle = P_1$. So, P_1 is cyclic. Likewise, all other P_i are shown cyclic by the same argument, with $P_i = \langle x_i \rangle$. Then, the element $x = \prod_{i=1}^t x_i$ is a generator of G , so G is cyclic. \square

Theorem 0.1 (Frattini's Argument). Let G be a finite group, $H \trianglelefteq G$, $P \leq H$ being a sylow p -group in H . Then,

$$G = HN_G(P) \text{ and } |G : H| \mid |N_G(P)|.$$

Proof. Let $g \in G$, we wish to show $g \in HN_G(P)$. We know this to be a

subgroup as $H \trianglelefteq G$. Let G act by conjugation on its sets. Now

$$\begin{aligned} P^g &= gPg^{-1} \\ &\leq H^g \\ &= gHg^{-1} \\ &= H \text{ by normality.} \end{aligned}$$

Then, we see as $|P^g| = |P|$, then P^g is another sylow p -group in H . And, as we know all sylow p -groups are conjugate. Hence, there is an $h \in H$ such that $P^h = P^g$. Hence, $P = P^{h^{-1}g}$, hence $h^{-1}g \in N_G(P)$. Then, we see $g \in hN_G(P) \subseteq HN_G(P)$. So, we see $G = HN_G(P)$

Now, we show the other result. Note that by the second isomorphism theorem, we have

$$G/H = (HN_G(P))/H \simeq \frac{N_G(P)}{H \cap N_G(P)}.$$

Thus, $|G : H| = |N_G(P) : H \cap N_G(P)|$. As we know this divides $|N_G(P)|$, hence $|G : H| \mid |N_G(P)|$. \square

Theorem 0.2. if G is a finite group, then G is nilpotent if and only if every maximal subgroup in G is normal in G .

Lecture 17: Nilpotent Groups (3) and Solvable Groups

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Recall. We had a theorem that, for a finite group G , implied G was nilpotent if and only if all maximal subgroups are normal.

Proof. 1. (\Rightarrow). Let $M < G$ be a maximal subgroup, so $M < N \leq G$ implies $N = G$. Let $N_G(M)$ be the normalizer of M then $M < G$, hence $M < N_G(P)$ by the earlier characterization of finite nilpotent groups. Hence, $N_G(M) = G$. But $M < N_G(M)$ and M is maximal, hence $N_G(M)$ if and only if M is normal.

2. (\Leftarrow). Assume every maximal subgroup is normal. Note that it suffices to show that all sylow groups are normal in G by the earlier characterization. Let $P \leq G$ be an arbitrary sylow p -group and let $N = N_G(P)$. Let M be a maximal subgroup containing $N_G(P)$. We know such a group exists because if we assume indirectly that P is not normal, this implies $N_G(P) < G$ as every proper subgroup of a finite group is contained in a maximal subgroup.

We now have $P \leq N_G(P) \leq M < G$ and by hypothesis, we know $M \trianglelefteq G$. Since $P \leq M$ with P being a sylow group of G implies $P \leq M$ is a sylow group for M . But now we can applying the frattini argument. We see $G = N_G(P)M$ but $N_G(P) \leq M$, hence $G \subseteq MM = M < G$. \nmid

\square

Remark. If G is nilpotent, then recall $Z_0(G) < Z_1(G) < Z_2(G) < \dots < Z_i(G)$ is the upper central series where $Z_0(G) = \{1\}$, $Z_1(G) = Z(G)$ and $Z_i(G)/Z_{i-1}(G) = Z(G/Z_{i-1}(G))$.

There is an alternative characterization, let $G^0 = G$, $G^1 = [G, G] = \langle x^{-1}y^{-1}xy : x, y \in G \rangle$ and define recursively $G^i = [G, G^{i-1}] = \langle x^{-1}y^{-1}xy : x \in G, y \in G^{i-1} \rangle$ to be the lower central series. Then, G is nilpotent if and only if there is $c \geq 0$ such that $G^c = \{1\}$. Furthermore, we find $G^{c-i} \leq Z_i(G)$ for all $0 \leq i \leq c$, with the minimal constant c being the same in the upper and lower central series.

Definition 0.1 (Solvable Groups). A group G is **solvable** if there's a chain of subgroups

$$H_0 \triangleleft H_1 \triangleleft \dots \triangleleft H_n = G$$

such that H_i/H_{i-1} are abelian for $1 \leq i \leq n$.

As it turns out there is an equivalent chain condition for solvability closed to our characterizations of nilpotence. Define $G^{(0)} = G$, $G^{(1)} = [G, G] = G^1$. Now, define $G^{(i)} = [G^{(i-1)}, G^{(i-1)}] = \langle x^{-1}y^{-1}xy : x, y \in G^{(i-1)} \rangle$. So, $G^{(n)}$ is essentially the n -th iterated commutator of G . Then, we obtain a chain

$$G^{(0)} \geq G^{(1)} \geq \dots \geq G^{(c)} \geq \dots$$

If $G^{(c)} = 1$ for some $c \geq 1$, then G is solvable. We show these two conditions are equivalent. The proof will involve multiple invocations of the basic result that G/H is abelian if and only if $[G, G] \leq H$.

Proof. Assume G is solvable, and the 1st characterization is true with $1 = H_0 \triangleleft H_1 \triangleleft \dots \triangleleft H_n = G$ with H_i/H_{i-1} being abelian for all $1 \leq i \leq n$. We will show by induction that $G^{(i)} \leq H_{n-i}$ for all $1 \leq i \leq n$. For $i = 0$ we have $H_n = G$, hence $G^{(0)} = G$ and $G \leq G$, so the claim holds for $i = 0$. Now, note that

$$\begin{aligned} G^{(i)} &= [G^{(i-1)}, G^{(i-1)}] \\ &\leq [H_{n-(i-1)}, H_{n-(i-1)}] \text{ by inductive hypothesis} \\ &= [H_{n-i+1}, H_{n-i+1}] \end{aligned}$$

We also know that H_{n-i+1}/H_{n-i} is abelian, hence we have $G^{(i)} \leq [H_{n-i+1}, H_{n-i+1}] \leq H_{n-i}$ by the preceding lemma. This completes the induction. But, we have $G^{(n)} = G^{(n-n)} \leq H_{n-n} = H_0 = \{1\}$, so $G^{(n)}$ is trivial. \square