Combinatorics

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Contents

Lecture 21: Quasi-Random Graphs (4)

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We complete the proof from last time. Recall our lemma that for orthornormal basis containing x_1 we have $\left|x_1 - j\right|_2 = o\left(1\right)$. We proceed

Proof. WLOG assume G to be a random graph of even order and $|S| = \frac{n}{2}$. Then, we define a vector $\vec{S}$ with $s_i = \begin{cases} \frac{1}{\sqrt{n}}, & i \in S \\ -\frac{1}{\sqrt{n}}, & i \in V \setminus S \end{cases}$ It is clear $|S|_2 = 1$ and we see

$$\langle S, j\rangle = \underbrace{\frac{1}{\sqrt{n}} \cdot \frac{1}{\sqrt{n}}}_{\frac{n}{2}\text{ times}} + \underbrace{\frac{1}{-\sqrt{n}} \cdot \frac{1}{\sqrt{n}}}_{\frac{n}{2}\text{ times}} = 0.$$

Then, we note $\langle S, x_1\rangle = \langle S, j\rangle + \langle S, x_1 - j\rangle = \langle S, x_1 - j\rangle$ and applying cauchy-shwartz yields

$$\langle S, x_1\rangle = \langle S, x_1 - j\rangle \leq |S|_2 \left|x_1 - j\right|_2 = 1 \cdot o\left(1\right) = o\left(1\right).$$

Now, define $Z = S - \langle S, x_1\rangle x_1)$. Then, we see

$$\langle Z, x_1\rangle = \langle S, x_1\rangle - \langle S, x_1\rangle \left|x_1\right|_2^2 = 0.$$

So, Z is orthogonal to x_1. Hence, there is a $n-1$ dimensional space, M, generated by $x_2, \ldots, x_n$ with eigenvalues $\lambda_2, \ldots, \lambda_n$ with largest eigenvalue $\max\{\lambda_2, |\lambda_n|\}$. Then, we find by the rayleigh quotient that $|\langle Ay, y\rangle| \leq \lambda_1\left(M\right)|y|_2^2 = \sigma_2 |y|_2^2$ for all $y \in M$. Similairly, we find

$$\lambda_n |y|_2^2 \leq \langle Ay, y\rangle \leq \lambda_2 |y|_2^2$$

for all $y \in M$. From this we get $\lambda_n |Z|_2^2 |\langle AZ, Z\rangle| \leq \lambda_2 |Z|_2^2$, and recalling $|Z|_2 \leq |S|_2 + |\langle S, x_1\rangle| \left|x_1\right|_2 = 1 + o\left(1\right)1 \leq 2$

$$|\langle AZ, Z\rangle| \leq \sigma_2 |Z|_2^2 \leq \sigma_2 |2|_2^2 = 4\sigma_2 = o\left(n\right).$$

Finally, we see

$$\begin{aligned}\langle AS, S\rangle &= \langle A\left(Z + \langle S, x_1\rangle x_1\right), Z + \langle S, x_1\rangle x_1\rangle \\ &= \underbrace{\langle AZ, Z\rangle}_{o(n)} + \underbrace{\langle S, x_1\rangle}_{o(1)} \underbrace{\langle AZ, x_1\rangle}_{=0} + \underbrace{\langle S, x_1\rangle}_{o(1)} \underbrace{\langle Ax_1, Z\rangle}_{0} + \underbrace{\langle S, x_1\rangle^2}_{o(1)} \langle Ax_1, x_1\rangle \\ &= o(n) + \langle S, x_1\rangle^2 \langle Ax_1, x_1\rangle \\ &= o(n) + \lambda_1 \\ &= o\left(n^2\right) \\ &.\end{aligned}$$

Recall we also know

$$\langle AS, S\rangle = 2e(S) + 2e(G \setminus S) - 2e(S, G \setminus S).$$

and $2e(S) + 2e(G \setminus S) + 2e(S, G \setminus S) = e(G) \geq \frac{1}{4}n^2 + o\left(n^2\right)$. Then, adding and dividing yields these identities yields $e(S) + e(G \setminus S) = \frac{n^2}{8} + o\left(n^2\right)$. Furthermore, $\sum_{i \in S} d_i == \frac{n^2}{4} + o\left(n^2\right) 2e(S) + e(S, G \setminus S)$ and $\sum_{i \in G \setminus S} d_i = \frac{n^2}{4} + o\left(n^2\right) = 2e(G \setminus S) + e(S, G \setminus S)$. Adding all of the identities thus far yields that $2e(S) - 2e(G \setminus S) = o\left(n^2\right)$, hence $e(S) = \frac{1}{16}n^2 + o\left(n^2\right)$. □

We are nearing the end of quasi-random graphs, but note we have always assumed a quasi-random graph to have density $\frac{1}{2}$. These properties are easily generalized to one of density p. We list the generalized properties.

Definition 0.1.

1. (P_2). A graph is P_2 if
   - $e(G) \geq \frac{pn^2}{2} + o\left(n^2\right)$
   - $\#CW_4 \leq p^4 n^4 + o\left(n^4\right)$.
2. (P_3). A graph is P_3 if
   - $e(G) \geq \frac{pn^2}{2} + o\left(n^2\right)$
   - $\lambda_1(G) = pn + o(n)$
   - $\sigma_2(G) = o(n)$.
3. (P_7) . A graph is P_7 if
   - $\sum_{1 \leq i,j \leq n} \left|\hat{d}_{ij} - p^2 n\right| = o\left(n^2\right)$.

Lecture 22: Quasi-Random Graphs (5)

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Recall. A quasi-random graph could be characterized as one with a gray adjacency matrix.

Example. A paley graph of order q is quasi-random.
For this graph G, we see

- $e(G) = \frac{1}{2}q\frac{q-1}{2} = \frac{1}{4}q^2 + o\left(q^2\right)$,

- $\lambda_1(G) = \frac{q-1}{2} = \frac{1}{2}q + o(q)$, and
- $\sigma_2(G) = \frac{1+\sqrt{q}}{2} = o(q)$.

Hence, G is P_3, so it is quasirandom.
We also have a conference graph $\mathrm{SRG}(4k+1, 2k, k-1, k)$ has

- $\lambda_1 = 2k = \frac{n}{2} + o(n)$,
- $\sigma_2 = \frac{1+\sqrt{n}}{2} = o(n)$, and
- $e(G) = k(4k+1) = \frac{1}{4}n^2 + o(n^2)$.

We also have $K_{n,n}$ and cK_n are trivially SRG, but not quasi-random. As it turns out these are the only SRG which are not quasi-random. ◇

Proposition 0.1. All nontrivial SRG (not $K_{n,n}$ or cK_n) are quasi-random.

Remark. A random graph of order n is quasi-random with probability 1 as $n \to \infty$.

Definition 0.2 (Perturbation)**.** Let G be a quasi-random graph of order n with adjacency matrix A. We may perturb G by choosing a set E of edges such that $|E| = o(n^2)$ and deleting them. From this we obtain a graph $G' = G - E$. We find G' is also quasi-random.

Proof. Let G' be the result of perturbing a quasi-random graph G having adjacency matrix A and let A' be the adjacency matrix of G'. Then, denote B to be the adjacency matrix containing only the deleted edges. So, we find $A' = A - B$. We wish to show $\lambda(A') = \lambda(A) + o(n)$ and $\sigma_2(A') = \sigma_2(A) + o(n) = o(n)$. Now employing Weyl's inequalities:

$$\lambda_i(A) + \inf\{\lambda_i(B) : 1 \le i \le n\} \le \lambda_i(A+B) \le \lambda_i(A) + \lambda_i(B)$$

yields

$$\lambda_i(A) + \lambda_{\min}(-B) \le \lambda_i(A') \le \lambda_i(A) + \lambda_1(-B).$$

We see it suffices to show $\lambda_{\min}(-B) = o(n)$ and $\lambda_1(-B) = o(n)$.
Recall that $\lambda_1^2(-B) + \ldots + \lambda_n^2(-B) = |-B|_2^2 = 2|E|$, hence $\lambda_1^2(-B) \le 2|E| = o(n)$ and likewise for $\lambda_{\min}^2(-B)$. Hence, we have $\lambda_i(B) = o(n)$, so

$$\lambda_1(A) + o(n) \le \lambda_1(A') \le \lambda_1(A) + o(n).$$

So, $\lambda_1(A')$ is desired. Similairly, WLOG we can assume $\lambda_2(A) = \sigma_2(A)$, so we see

$$\lambda_2(A) + o(n) \le \lambda_2(A') \le \lambda_2(A) + o(n).$$

and as $\lambda_2(A) = o(n)$ by quasi-randomness, we see $\lambda_2(A') = \sigma_2(A') = o(n)$.

□

Remark. This also clearly works with addition of $o(n^2)$ edges (provided they will fit). Furthermore, we can union a quasi-random graph with a graph of sufficiently small order and obtain a quasi-random graph.

Proposition 0.2. Let G be quasi-random with adjacency matrix A and construct the following matrix

$$J_2 \otimes A = \begin{bmatrix} A & A \\ A & A \end{bmatrix}.$$

Then, the graph G' obtained from this matrix is the blowup of G. We see for G being regular, we have G' is regular. It turns out G' is also quasi-random. However, we find G being SRG does not guarantee G' to be SRG.