Algebraic Theory I

Thomas Fleming

October 8, 2021

Contents

Lecture 19: Free Groups (2)

Wed 06 Oct 2021 11:33

Recall we had a set of letters $X = \{a, b, c, \ldots, a^{-1}, b^{-1}, c^{-1}, \ldots, 1\}$. Then, we define a word on the alphabet X to be a string $\omega = x_1^{\varepsilon_1} x_2^{\varepsilon_2} \ldots, x_s^{\varepsilon_s}$ where $x_1, x_2, \ldots, x_s \in X$ and $\varepsilon_i = \pm 1$. For example with $X = \{x_1, x_2, x_3\}$ we have a word $x_1 x_1 x_2 x_1^{-1} x_1 x_3$ for example. Then, define 1 to be the empty product, that being a string with no symbols. Now, we define an equivalence relation on the words to induce a group.
We say two words $\omega_1 \sim \omega_2$ if we can transform ω_1 into ω_2 with a finite sequence of the following operations

- Remove a sequential pair xx^{-1} or $x^{-1}x$ from the string.
- Insert a substring xx^{-1} or $x^{-1}x$ into the string.

So, we see $x_1 x_2 x_3^{-1} x_4 \sim x_1 x_2 x_3^{-1} x_2 x_2^{-1} x_1^{-1} x_1 x_4$ and so on. It is trivial to verify this to be an equivalence relation, so we omit the proof. Henceforth, we will denote the equivalence class of a word ω by $[\omega]$. So, we see if $\omega_1 \sim \omega_2$, we have $[\omega_1] = [\omega_2]$.
Now, let $F(X)$ be the set of all equivalence classes on X and define $[\omega_1][\omega_2] \coloneqq [\omega_1 \omega_2]$ with $\omega_1 \omega_2$ simply being the concatenation of the two words. First, we verify this to be well-defined. Suppose $w' \sim w$ and $v' \sim v$ are 4 words. Hence, there is a simple sequence taking $v \mapsto v'$ and $w \mapsto w'$. It is easy to see then, that the same operations applied to their respective parts will take $vw \mapsto v'w'$ and $wv \mapsto w'v'$, hence $[vw] = [v'w']$.
Next, we show this forms a group. We see $[w][1] = [w \cdot 1] = [w]$ and likewise $[1][w] = [w]$, so 1 is the identity.
Next,

$$\begin{aligned}
[w]([u][v]) &= [w][uv] \\
&= [w(uv)] \\
&= [(wu)v] \\
&= [wu][v] \\
&= ([w][u])[v]
\end{aligned}$$

.

Hence, $F(X)$ is associative. Lastly, we show inverses exist. Let $w = x_1^{\varepsilon_1} \ldots x_s^{\varepsilon_s}$, then let $w^{-1} = x_s^{-\varepsilon_s} \ldots x_1^{-\varepsilon_1}$ and we see $ww^{-1} \sim 1$, so $F(X)$ has inverses.

Definition 0.1 (Free Group)**.** For an alphabet X, we define $F(X)$ to be the **Free Group on** X. More generally, the free group F on X is a group F together with an injection $\sigma : X \hookrightarrow F$ such that any $\alpha : X \to G$, with G being an arbitrary group, extends to a unique homomorphism $\beta : F \to G$ such that $\beta \circ \sigma = \alpha$.

$$\begin{tikzcd}
& F(X) \arrow[dr, dotted, "\beta"] & \\
X \arrow[ur, "{\sigma, x \mapsto [x]}"] \arrow[rr, "\alpha"] & & G
\end{tikzcd}$$

Figure 1: In this commutative diagram solid lines represent given maps and dotted lines represent maps that must then exist

Next, recall a homomprhism $\varphi : H \to G$ is determined by the images of generators of H. Let $H = \langle X \rangle$. Then for an arbitrary $h \in H$ with $h = x_1^{\varepsilon_1} \ldots x_n^{\varepsilon_n}$ we find $\varphi(h) = \varphi(x_1)^{\varepsilon_1} \ldots \varphi(x_n)^{\varepsilon_n}$ with $x_i \in X$ and $\varepsilon_i = \pm 1$.
Now, let G be a group with $\alpha : X \to G$ being a map and $\sigma : X \hookrightarrow F$ be the inclusion map. Let $w = x_1^{\varepsilon_1} \ldots x_n^{\varepsilon_n}$ and let $(w) = \alpha(x_1)^{\varepsilon_1} \ldots \alpha(x_n)^{\varepsilon_n}$ with $x_i \in X$ and $\varepsilon_i = \pm 1$. Then, we define $\beta([w]) = [\beta(w)]$. It is simple to check this is well defined as we may always insert or delete substrings of the form $\alpha(x_i)^{\varepsilon_i} \alpha(x_i)^{-\varepsilon_i}$ in order to induce an equivalence. We see β is also a homomoprhism as

$$\begin{aligned}
\beta([w][v]) &= \beta([wv]) \\
&= \beta(wv) \\
&= \beta(w)\beta(v) \\
&= \beta([w])\beta([v]).
\end{aligned}$$

Lastly, we see the map β is unique as a homomoprhism is completely characterized by where it sends the generators.

Lecture 20: Free Groups (3)

Fri 08 Oct 2021 11:26

Recall. F is a free group on the set X when there is an injection $\sigma : X \overset{F}{\hookrightarrow}$ such that for all maps $\alpha : X \to G$, there is a homomorphism $\beta : F \to G$ such that $\beta \circ \sigma = \alpha$.

Remark. F is also a free group on $\sigma(X) \subseteq F$, using a similair inclusion map, so often we will assume $X \subseteq F$.

Theorem 0.1. If F_1 is free on X_1 and F_2 is free on X_2 and $|X_1| = |X_2|$, then $F_1 \simeq F_2$.

Proof. Since $|X_1| = |X_2|$ we find a bijection $\alpha : X_1 \to X_2$ and we can assume WLOG that $X_1 \subseteq F_1$ and $X_2 \subseteq F_2$. Then, the free property of F_1 implies there is a unique homomorphism $\beta : F_1 \to F_2$ such that $\beta(x) = \alpha(x)$ for all $x \in X_1$. Similairly, thee is a unique map $\gamma : F_2 \to F_1$ extending $\alpha^{-1} : X_2 \to X_1$ such that $\gamma(y) = \alpha^{-1}(y)$ for all $y \in X_2$. So, we see

$$\begin{aligned} \beta\,|_{X_1} : X_1 &\longrightarrow X_2 \\ x &\longmapsto \beta(x) = \alpha(x) \end{aligned}$$

and

$$\begin{aligned} \gamma\,|_{X_2} : X_2 &\longrightarrow X_1 \\ y &\longmapsto \gamma(y) = \alpha^{-1}(y) \end{aligned}$$

are inverses.
Hence, we have β and γ are a pair of inverse homomorphisms as X_1 generates F_1 and likewise X_2 generates F_2 .
Then, for an arbitrary element in F of the form $x = x_1^{\varepsilon_1} \dots x_\ell^{\varepsilon_\ell}$ with $\varepsilon_i \in \mathbb{Z}$ and $x_i \in X_1$, then we see $\gamma(\beta(x)) = x$, hence this completes the proof. $\square$

Theorem 0.2. Let F be a free group with H, G being groups. Suppose $\alpha : F \to H$ is a homomorphism and $\beta : G \to H$ is a surjective homomorphism. Then, there is a $\gamma : F \to G$ such that $\beta\gamma = \alpha$.

Proof. Let F be free on $X \subseteq F$. Then, each $x \in X$ has $\alpha(x) \in H = \operatorname{Im}(\beta)$. Then, there is some $g_x \in G$ such that $\beta(g_x) = \alpha(x)$. By the universal mapping property of F , we have the map $X \to G, x \mapsto g_x$ extends to a homomorphism

$$\begin{aligned} \gamma : F &\longrightarrow G \\ x &\longmapsto \gamma(x) = g_x. \end{aligned}$$

Then, for $x \in X$ we see $\beta(\gamma(x)) = \beta(g_x) = \alpha(x)$, so $\beta \circ \gamma = \alpha$ on X which generates F , so $\beta \circ \gamma = \alpha$ on F as $\beta \circ \gamma, \alpha$ are homomorphisms. $\square$

Definition 0.2 (Group Presentations)**.** Any group G is a homomorphic image of a free group F. An explicit homomorphism $\alpha : F \to G$ with F is called a **presentation** of G. Its kernel $N = \ker(\alpha) \trianglelefteq F$ has $F/N \simeq G$. So, we may write $\langle X : Y \rangle = G$ where F is a free group on X and $Y \subseteq F$ has normal closure, $\bigcap_{H \trianglelefteq G, Y \leq H} H = N$.

Example. $D_{2n} = \langle \alpha, \tau : \alpha^n, \tau^2, \tau\alpha\tau\alpha \rangle$. Here, we see F is free on the set $\{\alpha, \tau\}$ and N is the normal closure of $\langle \alpha^n, \tau^2, \tau\alpha\tau\alpha \rangle$, that being the smallest normal subgroup of F containing these three elements.
In general if $H \leq G$, then $\bigcap_{N \trianglelefteq G, H \leq N} N \trianglelefteq G$ is the normal closure of H. $\diamond$

Remark. In general, a group of relations can generate other relations that we may not account for, so it is good to know what elements in the normal closure look like. If $X \subseteq G$, we find elements in the normal closure N of $\langle X \rangle$ in G include inverses and products of elements from X. Furthermore, arbitrary conjugates and their products/inverses will be in N. We see this yields

$$N \supseteq \{\prod_{i=1}^{\ell} \left(g_i x_i g_i^{-1}\right) : \ell \geq 0, g_i \in G, x_i \in X \cup X^{-1}\}.$$

Furthermore, we see this set is in fact a normal subgroup itself, so equality holds.