Combinatorics

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Contents

Lecture 11: Hadamard Matrices (4)

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Recall. A matrix was regular if all row sums are equal.

As it turns out, for regular real hadamard matrices regular also implies equal column sums.

Proof. Let H be hadamard regular and $n \times n$ with $\sum_{i=1}^{n} h_{i,j} = d$ for all j.

Then, note that $Hj = dj$ with $j = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$. Hence, d is an eigenvalue and as $H^*H = HH^*$, then we have that $H^*Hj = H^*dj$. Hence

$$nIj = dH^*j \text{ by hadamardness}$$

and as $Ij = j$ we have that $H^*j = \frac{n}{d}j$, hence $\frac{n}{d}$ is an eigenvalue of H^*, hence the row sums of H^* are all $\frac{n}{d}$, and as $H^* = H^T$ for real H, we see the column sums of H are $\frac{n}{d}$.
Additionally, if $d \neq 0$, then $\sum_{i=1}^{n} r_i(H) = \sum_{i=1}^{n} c_i(H)$, implying $nd = n \cdot \frac{n}{d}$, hence $n = d^2$ as we have proven earlier.
We have, of course, neglected the case where $d = 0$. In this case we have that $nj = \vec{0}$, but as $n \neq 0$ by assumption, and $cj \neq \vec{0}$ for $c \neq 0$, we have a contradiction. Hence $d \neq 0$. It is also true that the independence requirement of hadamard matrices implies this row sum cannot be 0. □

Proposition 0.1. Suppose H is a $n \times n$ matrix with entries $|h_{i,j}| = 1$ and singular values $\sigma_1 = \sigma_2 = \ldots = \sigma_n = \sqrt{n}$. Then, H is hadamard.

Proof. Recall from an earlier proposition, we know $\sum_{i=1}^{n} \sigma_i^2 = n^2$. Recall that a diagonal element of HH^* is $b_{i,i} = \sum_{k=1}^{n} a_{i,k} \cdot \overline{a_{i,k}} = \sum_{k=1}^{n} |a_{i,k}|^2 = n$ by construction. Hence, the diagonals are all $b_{i,i} = n$ for all $1 \leq i \leq n$. Next, we wish to see if there are any 0 entries in HH^*. Next, we take a principal submatrix $A_{i,j} = \begin{bmatrix} n & \overline{b_{i,j}} \\ b_{i,j} & n \end{bmatrix}$ (note this is as HH^* will is hermitian, so we know

opposing entries will be complex conjugates) Then, we see $\lambda_1(A_{i,j}) = n + |b_{i,j}|$ and $\lambda_2(A_{i,j}) = n - |b_{i,j}|$.
Now, we examine how the eigenvalues of a matrix and its principal sumbatrices are related. Let A be a $n \times n$ hermitian matrix and A' to be A with the i'th row and j'th column removed. Denoted the eigenvalues of A to be $\lambda_1, \lambda_2, \ldots, \lambda_n$ in decreasing order and eigenvalues of A' to be $\lambda'_1, \lambda'_2, \ldots, \lambda'_{n-1}$. Then, it is a theorem of Cauchy that $\lambda_1 \geq \lambda'_1 \geq \lambda_2 \geq \lambda'_2 \geq \ldots \geq \lambda'_{n-1} \geq \lambda_n$. Applying this again yields a matrix A'' with eigenvalues $\lambda_1 \geq \lambda'_1 \geq \lambda'_1{}'$ and $\lambda''_{n-2} \geq \lambda'_{n-1} \geq \lambda_n$. Returning to our original construction yields $\lambda_1(HH^*) \geq \lambda_1(A_{i,j}) \geq \lambda_2(A_{i,j}) \geq \lambda_n(HH^*)$ and as $\lambda_1(HH^*) = \sigma_1^2 = n$ and similarly, $\lambda_n(HH^*) = \sigma_n^2 = n$, hence $\lambda_1(A_{i,j}) = \lambda_2(A_{i,j}) = n$ implying $b_{i,j} = 0$ for all $j \neq i$ and $b_{i,i} = n$ so $HH^* = nI$. □

Recall. For a matrix H which is hadamard and has entries $h_{i,i} = \delta$ for all i, then the matrix $A = \frac{1}{n}(J - \delta H)$ is a square matrix with entries $0, 1$ and all 0s along the diagonal.

Proposition 0.2. If H is symmetric, then A is the adjacency matrix of a simple graph. If H is also regular with row sum d, then A is the adjacency matrix of a SRG with

$$\begin{aligned} n &= n \\ k &= \frac{n - \varepsilon\sqrt{n}}{2} \\ \lambda &= \frac{n - 2\varepsilon\sqrt{n}}{4} \\ \mu &= \frac{n - 2\varepsilon\sqrt{n}}{4} \end{aligned}$$

where $\varepsilon = \begin{cases} -1, & \delta d < 0 \\ 1, & \delta d > 0 \end{cases}$. It is of note that $\delta d \neq 0$ as $\delta = \pm 1$ and $d \neq 0$ by the earlier proof. Hence, $\varepsilon\sqrt{n} = \delta d$

Proof. First, we examine a few matrix products. Note that as $Hj = d$, we have $HJ = dJ$. Similarly, $JH = dJ$ and of course $H^2 = nI$.

Next, we examine A^2. By definition

$$\begin{aligned}
A^2 &= \frac{1}{4}\left(J-\delta H\right)^2 \\
&= \frac{1}{4}\left(J^2 - 2J\delta J + \delta^2 H^2\right) \\
&= \frac{1}{4}\left(nJ - 2\delta dJ + nI\right) \\
&= \frac{1}{4}\left(n-2\delta d\right)J + \frac{1}{4}nI \\
&= \frac{1}{4}\left(n-2\delta d\right)\left(J-I\right) + \frac{1}{4}\left(n-2\delta d\right)I + \frac{1}{4}nI \\
&= \frac{1}{4}\left(n-2\delta d\right)\left(J-I\right) + \frac{n-\delta d}{2}I.
\end{aligned}$$

Recalling our equation for the square of the adjacency matrix of a graph,

$$A^2 = \left(\lambda-\mu\right)A + \mu\left(J-I\right) + kI$$

yields $\lambda = \mu$, $\mu = \frac{n-2\delta d}{4} = \frac{n-2\varepsilon\sqrt{n}}{4} = \lambda$ and $k = \frac{n-\delta d}{2} = \frac{n-\varepsilon\sqrt{n}}{2}$. □

Lecture 12: Conference Matrices

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Recall. A Conference matrix is a matrix C with

1. C is $n \times n$
2. $c_{i,i} = 0$ for all $1 \le i \le n$
3. $c_{i,j} = \pm 1$ for $i \ne j$
4. $CC^T = \left(n-1\right)I$.

Just as with hadamard graphs, this implies the inner product of each pair of rows r_i, r_j with $i \ne j$ is $\langle r_i, r_j\rangle = 0$, hence the rows are orthogonal. This clearly also implies the columns are orthogonal. Furthermore, $n \equiv 0 \pmod 2$ by the same argument as hadamard matrices.
Just as with hadamard matrices, we look to determine which n have associated hadamard matrices.

Example. $\begin{bmatrix} 0 & + \\ + & 0 \end{bmatrix}$ is conference.
$\begin{bmatrix} 0 & + & + & + & + & + \\ + & 0 & + & - & - & + \\ + & + & 0 & + & - & - \\ + & - & + & 0 & + & - \\ + & - & - & + & 0 & + \\ + & + & - & - & + & 0 \end{bmatrix}$ is also conference.
Note that this matrix is simply the Seidel matrix of $\overline{C}_5$, that being the matrix $S\left(G\right)$ with $s_{i,j} = \begin{cases} -1, & i \sim j \\ 1, & i \not\sim j \text{ and } i \ne j \\ 0, & i = j \end{cases}$ for a matrix G, with an additional row and column of 1s along the top and left. This derives from the fact that C_5 is a conference graph. ◇

Similarly to hadamard, we can simultaneously transpose rows and columns of a conference matrix to obtain another conference matrix. Furthermore, we can negate any row or column while remaining hadamard. Hence,

Definition 0.1 (Normal Conference Matrix)**.** A conference matrix is called a **normal conference matrix** if it has $r_1 = \hat{j}_n = \begin{pmatrix} 0 & 1 & \dots & 1 \end{pmatrix}$ and $c_i = \hat{j}_n = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$.
We define the matrix $C' = S$ to be the remaining matrix when the first row and column are removed. This submatrix completely characterizes the normal conference matrix.

It is of note that the negation property makes every conference matrix normalizable.

Proposition 0.3. If $n \equiv (2 \mod 4)$, then S is symmetric. If $n \equiv 0 (\mod 4)$, then S is antisymmetric or skew symmetric ($A = -A^T$).

Remark. If A is antisymmetric, then iA is hermitian.

Proposition 0.4. If C is a conference matrix with $n \equiv 2 (\mod 4)$, then S is the seidel matrix of a conference graph.

Remark. All $2 (\mod 4)$ numbers up to $n = 22$ have been shown to have conference matrices of that order. For the case $n = 22$, we have a proof by Seidel vam Lint, that if n is the order of a conference matrix, then $n - 1$ is the sum of two squares. As 21 is not the sum of 2 perfect squares, there is no conference matrix of order 22. Similarly for 34 and 66. Note that all primes $p \equiv 1 (\mod 4)$ are the sums of two squares, so we need only check the composite cases.

Now, we introduce the Payley construction of conference matrices.
Let $q \equiv 3 (\mod 4)$ be a prime power. Then, there is a skew-symmetric conference matrix of order $q + 1$ and a hadamard matrix of order $q + 1$.

Proposition 0.5. Let $q = 1 (\mod 4)$, then there exists a symmetric conference matrix of order $q + 1$ and a symmetric hadamard matrix of order $2 (q + 1)$. We will introduce the construction of the hadmard matrix next lecture.