Analysis I: Homework III

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Solution (18)**.** As $[a, b]$ is compact, we see f is uniformly continuous. Hence, there is a $\delta > 0$ such that for all $\varepsilon > 0$ and $x, y \in [a, b]$ we find $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$.
Fix $\varepsilon > 0$, and define the following sequence. Let $y_0 = a$ and $y_i = \max\{a + \delta \cdot i, b\}$ for $i \geq 0$. Then, we see $\{[y_{i-1}, y_i] : i \in \mathbb{N}\}$ is a cover and there is a $n \geq 0$ such that $y_n = b$, hence $y_m = b$ for $m \geq n$ and we see $\{[y_{i-1}, y_i] : 1 \leq i \leq n\}$ is a finite subcover. Define

$$\begin{aligned} g : [a, b] &\longrightarrow \mathbb{R} \\ x \longmapsto g(x) &= \frac{f(y_i) - f(y_{i-1})}{y_i - y_{i-1}} (x - y_i) + f(y_i), \qquad \text{for } x \in [y_{i-1}, y_i]. \end{aligned}$$

We see g is simply the piecewise linear interpolation of f on the y_i's and it is well defined (the endpoints agree for each closed interval). Hence, for all $x \in [a, b]$ there is an $i \geq 1$ such that $x \in [y_{i-1}, y_i] = [y_{i-1}, y_{i-1} + \delta] = [y_i - \delta, y_i]$, hence $|y_{i-1} - x| < \delta$ and $|y_i - x| < \delta$ so we see $|f(y_{i-1}) - f(x)| < \frac{\varepsilon}{3}$ and $|f(y_i) - f(x)| < \frac{\varepsilon}{3}$. Then, either $f(y_{i-1}) \leq g(x) \leq f(y_i)$ or $f(y_i) \leq g(x) \leq f(y_{i-1})$ as g is the linear interpolation between these two points. Then, we see $|f(y_i) - g(x)| \leq |f(y_i) - f(y_{i-1})|$. Hence, we find

$$\begin{aligned} |g(x) - f(x)| &\leq |f(y_i) - g(x)| + |f(x) - f(y_i)| \\ &\leq |f(y_i) - f(y_{i-1})| + \frac{\varepsilon}{3} \\ &\leq |f(y_i) - f(x)| + |f(x) - f(y_{i-1})| + \frac{\varepsilon}{3} \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \varepsilon. \end{aligned}$$

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Solution (22). 1. It suffices to assume $m(S) < \infty$, because for all sets of infinite measure, we can choose a subset of finite measure $S' \subseteq S$ and $S \cap (a, b) \supseteq S' \cap (a, b)$, so $m(S \cap (a, b)) \geq m(S' \cap (a, b))$.
Then assuming $m(S)$ finite, for $\varepsilon = \frac{1}{3} m(S)$, we find an open U with $S \subseteq U$ and $m(U \setminus S) < \varepsilon = \frac{1}{3} m(S)$. Hence, $m(U) < \frac{4}{3} m(S)$. As U is open it is the countable union of disjoint intervals (a_i, b_i) and $m(U) = \sum_{i=1}^{\infty} (b_i - a_i) < \frac{4}{3} m(S)$. Hence,

$$\sum_{i=1}^{\infty} \frac{3}{4} (b_i - a_i) < m(S).$$

Suppose $m(S \cap (a_i, b_i)) \leq \frac{3}{4} (b_i - a_i)$ for all the intervals (a_i, b_i) . Then,

$$\begin{aligned} m(S) &= \sum_{i=1}^{\infty} m(S \cap (a_i, b_i)) \\ &\leq \sum_{i=1}^{\infty} \frac{3}{4} m(a_i, b_i) \\ &= \sum_{i=1}^{\infty} \frac{3}{4} (b_i - a_i) \\ &< m(S) \lightning. \end{aligned}$$

Hence, we have atleast one (a_i, b_i) such that $m(S \cap (a_i, b_i)) > \frac{3}{4} (b_i - a_i)$.

2. First, note that $S \cap (r + S) = \{s - r \in S : s \in S\}$, and suppose $S \cap (r + S) \cap (a, b) = \varnothing$.
That is, for all $s \in S \cap (a, b)$, we have $s + r \notin S \cap (a, b) \subseteq (a, b)$. Hence, $s \in (b - r, b) \subseteq \left(b - \frac{1}{4} (b - a), b\right) = \left(\frac{1}{4} a + \frac{3}{4} b, b\right)$. But, we see $m\left(\left(\frac{1}{4} a + \frac{3}{4} b, b\right)\right) = \frac{1}{4}(b - a) < \frac{3}{4} (b - a)$.
So, we have $S \cap (a, b) \subseteq \left(\frac{1}{4} a + \frac{3}{4} b, b\right)$, $\lightning$.
Hence there is a $s \in S \cap \left(a, a + \frac{3}{4}(b - a)\right)$, so $s + r \in (a, b)$, so

$$S \cap (r + S) \cap (a, b) \neq \varnothing.$$

For each $x \in \left[-\frac{1}{4} (b - a), \frac{1}{4} (b - a)\right]$, note that we have some $s \in S$ such that $s + x \in S$ or $s - x \in S$ since $S \cap (r + S)$ is nonempty, $0 \leq r \leq \frac{1}{4} (b - a)$. Denote $s + x = \overline{s}$ and $s - x = \hat{s}$. If $\overline{s} \in S$, then $\overline{s} - s = x \in S - S$. Otherwise, if $\hat{s} \in S$, then $s - \hat{s} = x \in S - S$. Hence, $\left[-\frac{1}{4} (b - a), \frac{1}{4} (b - a)\right] \subseteq S - S$.

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