

# Analysis I

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## Lecture 18: General Lebesgue Integral (2)

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**Proposition 0.1.** Let  $f : \mathbb{R} \rightarrow \overline{\mathbb{R}}$  be integrable. Then for each  $\varepsilon > 0$  there is a  $\delta > 0$  such that each measurable  $S \subseteq \mathbb{R}$  has  $\int_S |f| < \varepsilon$  if  $m(S) < \delta$ .

*Proof.* Let  $\varepsilon > 0$ , then there is a  $s \in \mathcal{S}(|f|)$  such that  $\int (|f| - s) < \frac{\varepsilon}{2}$ . Let  $\alpha = \sup\{s(x) : x \in \mathbb{R}\}$  and  $\delta = \frac{\varepsilon}{2(\alpha + \varepsilon)}$ . If  $S$  is measurable and  $m(S) < \delta$ , we find

$$\int_S |f| \leq \int s + \frac{\varepsilon}{2} \leq \alpha m(S) + \frac{\varepsilon}{2} < \varepsilon.$$

□

**Theorem 0.1** (Monotone Convergence Theorem). Let  $(f_n)$  be a sequence of nonnegative measurable functions with  $f_n : \mathbb{R} \rightarrow \overline{\mathbb{R}}$  such that  $(f_n(x))$  is increasing for all  $x \in \mathbb{R}$ . Then,  $f = \lim_{n \rightarrow \infty} f_n$  is measurable with  $\int f = \lim_{n \rightarrow \infty} \int f_n$ .

*Proof.* Since  $f = \limsup_{n \rightarrow \infty} f_n = \liminf_{n \rightarrow \infty} f_n$ , we see  $f$  is measurable. Moreover, the sequence  $(\int f_n)$  is increasing (as the  $f_n$ s are increasing). Hence, letting  $L = \lim_{n \rightarrow \infty} \int f_n$  exists with  $L \in \mathbb{R}_0^+$ . Since  $\int f_n \leq \int f$  for all  $n$  by monotonicity, we find  $L \leq \int f$ .

Let  $s \in \mathcal{S}(f)$  and fix  $c \in (0, 1)$  and define  $E_n = \{x \in \mathbb{R} : f_n(x) \geq cs(x)\}$ . Then, we find  $\{E_n : n \in \mathbb{N}\}$  is an ascending collection (again by monotonicity of  $(f_n)$ ) of measurable sets with  $\bigcup_{n \in \mathbb{N}} E_n = \mathbb{R}$  as  $cs(x) < f_n(x) \leq f(x)$ . Let  $s = \sum_{k=1}^K a_k \chi_{S_k}$  and we see  $cs \chi_{E_n} = f_n \chi_{E_n} \leq f_n$ , with

$$L \geq \int f_n \geq \int_{E_n} f_n \geq \int cs \chi_{E_n} = c \int_{E_n} s = c \sum_{k=1}^K a_k m(S_k \cap E_n).$$

Since  $\lim_{n \rightarrow \infty} m(E_n \cap S_n) = m(S)$  for every measurable set  $S$ , we find  $L \geq c \sum_{k=1}^K a_k m(S_k) = c \int s$ . Since  $c$  was arbitrary, we see the inequality holds for all  $c \in (0, 1)$ , hence we find  $L \geq s$  (by taking supremums), but  $s \in \mathcal{S}(f)$ , hence  $L \geq \int f$ . So,  $L = \int f$ .  $\square$

**Theorem 0.2** (Fatou's Lemma). If  $(f_n)$  is a sequence of nonnegative measurable functions  $f_n : \mathbb{R} \rightarrow \overline{\mathbb{R}}$ , then  $\int \liminf_{n \rightarrow \infty} f_n \leq \liminf_{n \rightarrow \infty} \int f_n$ .

*Proof.* For  $x \in \mathbb{R}$ , define  $g_n(x) = \inf\{f_k(x) : k \geq n\}$  for  $n \in \mathbb{N}$ . Then, we find  $(g_n)$  is a nonnegative measurable sequence of functions with  $(g_n(x))$  increasing for all fixed  $x$  and  $g_n \leq f_n$  for all  $n$ . Consequently,  $\int g_n \leq \int f_n$  and  $(\int g_n)$  is increasing. As  $\lim_{n \rightarrow \infty} g_n = \liminf_{n \rightarrow \infty} f_n$  is measurable by an earlier theorem, we find

$$\liminf_{n \rightarrow \infty} \int f_n \geq \liminf_{n \rightarrow \infty} \int g_n = \lim_{n \rightarrow \infty} \int g_n = \int \lim_{n \rightarrow \infty} g_n = \int \liminf_{n \rightarrow \infty} f_n.$$

$\square$

**Proposition 0.2.** For any integral function  $f : \mathbb{R} \rightarrow \overline{\mathbb{R}}$ , we find  $|\int f| \leq \int |f|$ .

**Theorem 0.3** (Dominated Convergence Theorem). Let  $(f_n)$  be a sequence of measurable functions  $f_n : \mathbb{R} \rightarrow \overline{\mathbb{R}}$ . Suppose there is an integrable function  $g$  with  $|f_n| \leq g$  for all  $n \in \mathbb{N}$ . If  $(f_n)$  converges pointwise to a function  $f : \mathbb{R} \rightarrow \overline{\mathbb{R}}$  almost everywhere, then  $f$  is integrable and

$$\lim_{n \rightarrow \infty} \int |f_n - f| = 0 \text{ and } \lim_{n \rightarrow \infty} \int f_n = \int f.$$

*Proof.* Since  $f(x) = \lim_{n \rightarrow \infty} f_n(x)$  for almost all  $x \in R$ , we find  $f$  is measurable. Moreover,  $|f_n| \leq g$  implies  $|f| < g$  almost everywhere and since  $g$  is integrable (hence finite a.e) we find  $f, f_n$  are integrable (hence finite) almost everywhere. Now, define for each  $n \in \mathbb{N}$

$$E_n = \{x \in \mathbb{R} : |f_n(x)|, |f(x)| < \infty, |f_n(x) - f(x)| \leq 2g(x)\}.$$

Since  $R \setminus \bigcup_{n \in \mathbb{N}} E_n$  is a set of measure 0, we can assume  $|f_n(x)|, |f(x)| < \infty$  and  $|f_n(x) - f(x)| \leq 2g(x)$  for all  $x \in \mathbb{R}$ . Then, Fatou's lemma applies to the

sequence on nonnegative measurable functions  $(2g - |f_n - f|)$  yielding

$$\begin{aligned}
 \int 2g &\leq \liminf_{n \rightarrow \infty} (2g - |f_n - f|) \\
 &= \int 2g + \liminf_{n \rightarrow \infty} \left( - \int |f_n - f| \right) \\
 &= \int 2g - \limsup_{n \rightarrow \infty} \int |f_n - f| \\
 &\Rightarrow \limsup_{n \rightarrow \infty} \int |f_n - f| \leq 0 \\
 &\Rightarrow \lim_{n \rightarrow \infty} \int |f_n - f| = 0.
 \end{aligned}$$

Hence,  $\lim_{n \rightarrow \infty} \int (f_n - f) = 0$  by the earlier lemma. So,  $\lim_{n \rightarrow \infty} \int f_n = \int f$ .  $\square$

**Definition 0.1** (Convergence in Measure). Let  $(f_n)$  be a sequence of measurable functions  $f_n : \mathbb{R} \rightarrow \mathbb{R}$  and  $f : \mathbb{R} \rightarrow \mathbb{R}$  also be measurable. The sequence  $(f_n)$  **converges in measure** to  $f$  ( $f_n \rightarrow f$  by measure) if each  $f_n$  is finite almost everywhere and for each  $\varepsilon > 0$  there is a  $N \in \mathbb{N}$  so that

$$m(\{x \in \mathbb{R} : |f_n(x) - f(x)| > \varepsilon\}) < \varepsilon$$

for  $n \geq N$ .

**Theorem 0.4** (Riesz). Let  $(f_n)$  be a sequence of measurable functions  $f_n : \mathbb{R} \rightarrow \mathbb{R}$  and  $f : \mathbb{R} \rightarrow \mathbb{R}$  also being measurable. If  $(f_n) \rightarrow f$  in measure, then there is a subsequence  $(f_{n_k})$  which converges pointwise almost everywhere to  $f$ .

*Proof.* First, we find a strictly increasing sequence of numbers  $(n_k)$  such that  $m(\{x \in \mathbb{R} : |f_j(x) - f(x)| > 2^{-k}\}) < 2^{-k}$  if  $j \geq n_k$ . For  $k \in \mathbb{N}$  denote

$$S_k = \{x \in \mathbb{R} : |f_{n_k}(x) - f(x)| > 2^{-k}\}.$$

Then,  $\sum_{k=1}^{\infty} m(S_k) \leq \sum_{k=1}^{\infty} 2^{-k} < \infty$ . Applying the Borel-Cantelli Lemma yields that almost every  $x \in \mathbb{R}$  does not belong to any infinite subcollections of  $(S_k)$ . For such  $x$ , we find a  $K \in \mathbb{N}$  such that  $|f_{n_k}(x) - f(x)| \leq 2^{-k}$  for  $k \geq K$ . Hence,  $f_{n_k}$  converges pointwise to  $f$  for all  $x$  not belonging to an infinite subcollection of  $(S_k)$ , hence almost everywhere.  $\square$

## Lecture 19: End of Convergence and Functions of Bounded Variation

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Recall we had the dominated convergence theorem. A similar version of the theorem makes use of convergence in measure as follows.

**Theorem 0.5** (Dominated Convergence - Convergence in Measure). Let  $(f_n)$  be a sequence of measurable functions  $f_n : \mathbb{R} \rightarrow \overline{\mathbb{R}}$  and suppose there is an integrable function  $g : \mathbb{R} \rightarrow \overline{\mathbb{R}}$  so that  $|f_n| \leq g$  for all  $n \in \mathbb{N}$ . If  $(f_n) \rightarrow f : \mathbb{R} \rightarrow \overline{\mathbb{R}}$  in measure, (with  $f$  measurable), then  $f$  is integrable and  $\lim_{n \rightarrow \infty} \int |f_n - f| = 0$  and  $\lim_{n \rightarrow \infty} \int f_n = \int f$ .

*Proof.* First, note a subsequence of  $(f_n)$  converges to  $f$  pointwise almost everywhere. Hence, we find  $|f| \leq g$  almost everywhere, so  $f$  is integrable. We can assume  $|f_n - f| \leq 2g$  (almost) everywhere. Then, we find a subsequence  $(g_n) = (f_{n_k})$  such that  $\limsup_{n \rightarrow \infty} |f_n - f| = \lim_{n \rightarrow \infty} |g_n - f|$ . Then, as  $(g_n) \rightarrow f$  in measure, we find another subsequence  $(h_j) = (g_{k_j}) = (f_{n_{k_j}})$  which converges pointwise to  $f$  almost everywhere.

Applying dominated convergence theorem yields

$$\lim_{n \rightarrow \infty} \int |h_j - f| = 0.$$

Then, we find

$$\begin{aligned} \limsup_{n \rightarrow \infty} \int |f_n - f| &= \lim_{n \rightarrow \infty} \int |g_k - f| \\ &= \lim_{n \rightarrow \infty} \int |h_j - f| \\ &= 0. \end{aligned}$$

This completes the proof.  $\square$

## 1 Functions of Bounded Variation and Absolutely Continuous Functions

**Remark.** For this chapter  $[a, b] \subseteq \mathbb{R}$  will always denote a compact interval on  $\mathbb{R}$ .

**Definition 1.1** (Partition). A finite sequence  $P = (x_k)_{k=n}^N$  with  $n, N \in \mathbb{Z}$  and  $n < N$  is called a **partition** of  $[a, b]$  if  $x_n = a$ ,  $x_N = b$  and  $x_{k-1} \leq x_k$  for  $n < k \leq N$ . We denote the collection of all partitions of  $[a, b]$  to be  $\mathcal{P}([a, b])$ .

**Definition 1.2.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be a function. Then,

- For a partition  $P = (x_k)_{k=n}^N$ , we denote

$$V(f, P) = \sum_{k=n+1}^N |f(x_k) - f(x_{k-1})|$$

to be the **variation of  $f$  with respect to  $P$** .

- We define the quantity  $\text{TV}(f) = \sup\{V(f, P) : P \in \mathcal{P}([a, b])\}$  to be the **total variation of  $f$** .

**Remark.** If  $f : [a, b] \rightarrow \mathbb{R}$  and  $c \in [a, b]$  with partitions  $P_1 = (x_k)_{k=n}^N$  of  $[a, c]$  and  $P_2 = (x_k)_{k=N}^K$  of  $[c, b]$ . Then denote,  $P = (x_k)_{k=n}^K$  to be a partition of  $[a, b]$  and we find

$$V(f, P) = V(f|_{[a, c]}, P_1) + V(f|_{[c, b]}, P_2).$$

Moreover,

$$\text{TV}(f) = \text{TV}(f|_{[a, c]}) + \text{TV}(f|_{[c, b]}).$$

**Definition 1.3** (Bounded Variation). A function  $f : \mathbb{R} \rightarrow \overline{\mathbb{R}}$  has **bounded variation** if  $\text{TV}(f) < \infty$ .

**Theorem 1.1** (Jordan's Theorem). A function  $f : [a, b] \rightarrow \mathbb{R}$  is of bounded variation if and only if there are increasing functions  $g, h : [a, b] \rightarrow \mathbb{R}$  so that  $f = g - h$ .

*Proof.* Suppose  $\text{TV}(f) < \infty$  and let  $x, y \in [a, b]$  with  $x < y$ . Then, we find

$$\begin{aligned} \text{TV}(f|_{[a, y]}) &= \text{TV}(f|_{[a, x]}) + \text{TV}(f|_{[x, y]}) \\ &\geq \text{TV}(f|_{[a, x]}) + |f(y) - f(x)| \\ &\geq \text{TV}(f|_{[a, x]}) + f(x) - f(y). \end{aligned}$$

Furthermore,  $h : x \mapsto \text{TV}(f|_{[a, x]})$  and  $g : x \mapsto \text{TV}(f|_{[a, x]}) + f(x)$  are increasing. This fact is trivial for  $h$  and we find, adding  $f(y)$  to both sides of the former inequality yields  $g(y) \geq g(x)$  for arbitrary  $y \geq x$ , so this claim holds as well.

Taking the difference,  $g - h = f$ .

Conversely, suppose  $f = g - h$  for increasing  $g, h : [a, b] \rightarrow \mathbb{R}$ . Then, let  $x, y \in [a, b]$  with  $y \geq x$ . Then, we find

$$\begin{aligned} |f(y) - f(x)| &= |g(y) - g(x) + h(x) - h(y)| \\ &\leq |g(y) - g(x)| + |h(x) - h(y)| \\ &= g(y) - g(x) + h(y) - h(x). \end{aligned}$$

Hence, for a partition  $P = (x_k)_{k=n}^N$ , we find

$$\begin{aligned} V(f, P) &= \sum_{k=n+1}^N |f(x_k) - f(x_{k-1})| \\ &\leq \sum_{k=n+1}^N (g(x_k) - g(x_{k-1}) + h(x_k) - h(x_{k-1})) = g(b) - g(a) + h(b) - h(a) \\ &< \infty. \end{aligned}$$

□

**Definition 1.4** (Absolute Continuity). A function  $f : [a, b] \rightarrow \mathbb{R}$  is **absolutely continuous** if for each  $\varepsilon > 0$  we find a  $\delta > 0$  such that for every finite disjoint collection of nonempty intervals  $\{(a_k, b_k) \subseteq [a, b] : 1 \leq k \leq K\}$  with  $\sum_{k=1}^K (b_k - a_k) < \delta$ , we have  $\sum_{k=1}^K |f(a_k) - f(b_k)| < \varepsilon$ .

**Remark.** Absolute continuity is stronger than uniform continuity, but weaker than lipschitz continuity.

**Theorem 1.2.** If a function  $f : [a, b] \rightarrow \mathbb{R}$  is absolutely continuous, then  $f$  is continuous and  $f$  has bounded variation.

*Proof.*  $f$  is trivially continuous, taking a finite disjoint collection consisting only of 1 interval  $\{(x, y)\}$  yields the definition of continuity.

Now we show bounded variation. For  $\varepsilon = 1$ , let  $\delta > 0$  be the number such that the definition of absolute continuity holds for  $f$ .

Now fix  $(x_k)_{k=n}^N \in \mathcal{P}([a, b])$  so that  $x_k - x_{k-1} < \delta$  for all  $n < k \leq N$ . Then, if  $P \in \mathcal{P}([x_{k-1}, x_k])$ , we see  $V(f|_{[x_{k-1}, x_k]}, P) < 1$  by definition of absolute continuity.

So, we have  $\text{TV}([x_{k-1}, x_k]) \leq 1$ , so  $\text{TV}(f) = \sum_{k=n+1}^N \text{TV}(f|_{[x_{k-1}, x_k]}) \leq N - n$  by the  $\varepsilon$  assumption. □

As it turns out, absolutely continuous functions have a relation to integrable functions, particularly, an integrable function  $f$  is simply the anti-integral of an absolutely continuous one.

**Proposition 1.1.** If  $f : [a, b] \rightarrow \overline{\mathbb{R}}$  is integrable, then,

$$F : [a, b] \rightarrow \mathbb{R}, \quad x \mapsto \int_{[a, x]} f$$

is absolutely continuous.

This claim can be generalized into a sort of fundamental theorem of calculus for the lebesgue integrals to characterize integrals and derivatives. For now, we only prove the weak version.

*Proof.* For  $\varepsilon > 0$  there is a  $\delta > 0$  such that  $\int_S |f| < \varepsilon$  for every measurable set  $S \subseteq [a, b]$  with  $m(S) < \delta$ .

Now, let  $\{(a_k, b_k) : 1 \leq k \leq K\}$  be a disjoint collection of intervals in  $[a, b]$  with  $\sum_{k=1}^K (b_k - a_k) < \delta$ . Fix  $S = \bigcup_{k=1}^K (a_k, b_k)$ . Then, since  $m(S) < \delta$  and

$$\begin{aligned} \sum_{k=1}^K |F(b_k) - F(a_k)| &= \sum_{k=1}^K \left| \int_{[a_k, b_k]} f \right| \\ &\leq \sum_{k=1}^K \int_{[a_k, b_k]} |f| \\ &= \int_S |f| \\ &< \varepsilon \text{ by assumption.} \end{aligned}$$

Hence, absolute continuity holds. □

## 2 Derivatives and Fundamental Theorem of Calculus

**Proposition 2.1.** Let  $f : [a, b] \rightarrow \overline{\mathbb{R}}$  be monotone on  $(a, b) \subseteq \mathbb{R}$  with  $a, b \in \overline{\mathbb{R}}$  and  $a < b$ . Then, the following limits