Analysis I: Homework 7

Thomas Fleming

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Problem (32)**.**

Problem (33)**.** First, basic limits show $\lim_{n\to\infty} h_n(x) = \begin{cases} 3, & x \in (-1,1) \\ 2, & x = -1 \text{ or } x = 1 \\ 1, & x \in (-\infty,-1) \cup (1,\infty) \end{cases}$
Moreover, $h_n(x)$ is continuous for every $n \in \mathbb{N}$, hence measurable. So, we see $h_n{\cdot}f$ is measurable for every $n \in \mathbb{N}$. Then, $\lim_{n\to\infty} (h_n \cdot f)(x) = \begin{cases} 3f(x), & x \in (-1,1) \\ 2f(x), & x = \pm 1 \\ f(x), & x \in (-\infty,-1) \cup (1,\infty) \end{cases}$.
Hence, we see $|h_n \cdot f| \le 3|f|$ with $3|f|$ being integrable (since f is integrable).
Applying dominated convergence yields

$$\lim_{n\to\infty} \int h_n{\cdot}f = \int \lim_{n\to\infty} h_n{\cdot}f = \int_{[-\infty,-1]} f + \int_{[-1,1]} 3f + \int_{[1,\infty]} f = \int f \,\mathrm{dx} + 2\int_{[-1,1]} f \,\mathrm{dx}\,.$$

Problem (34)**.** First, basic limits again show $\lim_{n\to\infty} e^{-\frac{x}{n}} = 1$. Moreover, fixing x, we see $e^{-\frac{x}{n}} < e^{-\frac{x}{n+1}}$, so we see $e^{-\frac{x}{n}} |f| \leq e^{-\frac{x}{n+1}} |f|$. Then, denoting $e^{-\frac{x}{n}} |f| = f_n$, we see $\lim_{n\to\infty} f_n = \lim_{n\to\infty} e^{-\frac{x}{n}} \lim_{n\to\infty} |f| = \lim_{n\to\infty} |f|$ with each f_n being measurable (as it is the product of continous functions) and increasing, hence passing to the 0-extension and applying monotone convergence yields

$$1 \geq \lim_{n\to\infty} \int_{(0,\infty)} f_n = \lim_{n\to\infty} \int f_n^* = \int \lim_{n\to\infty} f_n^* = \int (|f|)^* = \int_{(0,\infty)} |f| \, .$$

Since f is continuous, we see it is measurable, and since it is absolutely integrable on $(0, \infty)$, we have f being integrable on $(0, \infty)$.

Problem (35)**.** First, recall $\sum_{i=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$.
Then, define $g_n = \sum_{i=1}^{n} f_i^2$ and note that $g_n \le g_{n+1}$ as each term is finite. Moreover g_n is the sum of measurable functions, so it is measurable. Lastly, define $\lim\limits_{n\to\infty} g_n(x) = g(x) = \sum_{i=1}^{\infty} f_n^2(x)$ Then, monotone convergence and zero extensions yield

$$\begin{aligned}
\int_{[0,1]} g = \lim_{n\to\infty} \int_{[0,1]} g_n &= \lim_{n\to\infty} \int g^* \\
&= \lim_{n\to\infty} \int (\sum_{i=1}^{n} f_n^2)^* \\
&= \lim_{n\to\infty} \int_{[0,1]} \sum_{i=1}^{n} f_n^2 \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \int_{[0,1]} f_n^2 \\
&\le \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n^4} \\
&= \frac{\pi^4}{90}
\end{aligned}$$

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Moreover, $0 \le \int_{[0,1]} f_n^2$ as the integrand is always non-negative. Hence, as the sum is bounded and strictly increasing, we see the terms tend to 0. That is $\lim\limits_{n\to\infty} \int_{[0,1]} f_n^2 = 0$.

Problem (36)**.** Our function will be φ, the cantor-lebesque function. We have already shown it to be continuous and increasing with $\varphi(1) = 1, \varphi(0) = 0$. Moreover, letting C be the cantor set, we see $[0,1] \setminus C \coloneqq C^c$ is open in $[0,1]$ so for all $x \in C^c$, there is an $\varepsilon > 0$ so that $(x - \varepsilon, x + \varepsilon) \subseteq C^c$. Then, since for all intervals I in the $[0,1]$ complement of the cantor set, we find $I \subseteq J_{n,k}$ for some $n, k \in \mathbb{N}$, we have $\xi(I) = \{\frac{n}{2^k}\}$, so

$$\overline{D}(\varphi(x)) = \lim_{r \to 0} \sup\{\frac{\varphi(x+h) - \varphi(x)}{h} : 0 < |h| < r\} = \lim_{r \to 0} \sup\{\frac{0}{h} : 0 < |h| < r\} = 0.$$

Similarly, we find $\underline{D}(\varphi(x)) = 0$. Hence, φ is differentiable at x and since $\varphi' = 0$ almost everywhere, yet φ is not constant by the initial claim, we find φ is not absolutely continuous.