# Algebraic Theory I

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## Contents

### Lecture 20: Free Groups (3)

Fri 08 Oct 2021 11:26

**Recall.** $F$ is a free group on the set $X$ when there is an injection $\sigma : X \xhookrightarrow{F}$ such that for all maps $\alpha : X \to G$ , there is a homomorphism $\beta : F \to G$ such that $\beta \circ \sigma = \alpha$.

**Remark.** $F$ is also a free group on $\sigma(X) \subseteq F$ , using a similair inclusion map, so often we will assume $X \subseteq F$.

> **Theorem 0.1.** If $F_1$ is free on $X_1$ and $F_2$ is free on $X_2$ and $|X_1| = |X_2|$ , then $F_1 \simeq F_2$.

*Proof.* Since $|X_1| = |X_2|$ we find a bijection $\alpha : X_1 \to X_2$ and we can assume WLOG that $X_1 \subseteq F_1$ and $X_2 \subseteq F_2$. Then, the free property of $F_1$ implies there is a unique homomorphism $\beta : F_1 \to F_2$ such that $\beta(x) = \alpha(x)$ for all $x \in X_1$ . Similairly, thee is a unique map $\gamma : F_2 \to F_1$ extending $\alpha^{-1} : X_2 \to X_1$ such that $\gamma(y) = \alpha^{-1}(y)$ for all $y \in X_2$ . So, we see

$$\begin{aligned} \beta|_{X_1} : X_1 &\longrightarrow X_2 \\ x &\longmapsto \beta(x) = \alpha(x) \end{aligned}$$

and

$$\begin{aligned} \gamma|_{X_2} : X_2 &\longrightarrow X_1 \\ y &\longmapsto \gamma(y) = \alpha^{-1}(y) \end{aligned}$$

are inverses.
Hence, we have $\beta$ and $\gamma$ are a pair of inverse homomorphisms as $X_1$ generates $F_1$ and likewise $X_2$ generates $F_2$ .
Then, for an arbitrary element in $F$ of the form $x = x_1^{\varepsilon_1} \dots x_\ell^{\varepsilon_\ell}$ with $\varepsilon_i \in \mathbb{Z}$ and $x_i \in X_1$ , then we see $\gamma(\beta(x)) = x$ , hence this completes the proof. $\square$

**Theorem 0.2.** Let $F$ be a free group with $H, G$ being groups. Suppose $\alpha : F \to H$ is a homomorphism and $\beta : G \to H$ is a surjective homomorphism. Then, there is a $\gamma : F \to G$ such that $\beta\gamma = \alpha$ .

*Proof.* Let $F$ be free on $X \subseteq F$ . Then, each $x \in X$ has $\alpha(x) \in H = \mathrm{Im}(\beta)$ . Then, there is some $g_x \in G$ such that $\beta(g_x) = \alpha(x)$ . By the universal mapping property of $F$ , we have the map $X \to G, x \mapsto g_x$ extends to a homomorphism

$$\begin{aligned} \gamma : F &\longrightarrow G \\ x &\longmapsto \gamma(x) = g_x. \end{aligned}$$

Then, for $x \in X$ we see $\beta(\gamma(x)) = \beta(g_x) = \alpha(x)$ , so $\beta \circ \gamma = \alpha$ on $X$ which generates $F$ , so $\beta \circ \gamma = \alpha$ on $F$ as $\beta \circ \gamma, \alpha$ are homomorphisms. $\square$

**Definition 0.1** (Group Presentations)**.** Any group $G$ is a homomorphic image of a free group $F$. An explicit homomorphism $\alpha : F \to G$ with $F$ is called a **presentation** of $G$. Its kernel $N = \ker(\alpha) \trianglelefteq F$ has $F/N \simeq G$. So, we may write $\langle X : Y \rangle = G$ where $F$ is a free group on $X$ and $Y \subseteq F$ has normal closure, $\bigcap_{H \trianglelefteq G, Y \leq H} H = N$.

**Example.** $D_{2n} = \left\langle \alpha, \tau : \alpha^n, \tau^2, \tau\alpha\tau\alpha \right\rangle$. Here, we see $F$ is free on the set $\{\alpha, \tau\}$ and $N$ is the normal closure of $\left\langle \alpha^n, \tau^2, \tau\alpha\tau\alpha \right\rangle$, that being the smallest normal subgroup of $F$ containing these three elements.
In general if $H \leq G$ , then $\bigcap_{N \trianglelefteq G, H \leq N} N \trianglelefteq G$ is the normal closure of $H$. $\diamond$

**Remark.** In general, a group of relations can generate other relations that we may not account for, so it is good to know what elements in the normal closure look like. If $X \subseteq G$ , we find elements in the normal closure $N$ of $\langle X \rangle$ in $G$ include inverses and products of elements from $X$. Furthermore, arbitrary conjugates and their products/inverses will be in $N$. We see this yields

$$N \supseteq \{\prod_{i=1}^{\ell} \left(g_i x_i g_i^{-1}\right) : \ell \geq 0, g_i \in G, x_i \in X \cup X^{-1}\}.$$

Furthermore, we see this set is in fact a normal subgroup itself, so equality holds.

## Lecture 21: Homework and Free Groups (4)

Wed 13 Oct 2021 11:23

Homework II

We spent the majority of class reviewing homework problems.

**Theorem 0.3.** Let $G = \langle X : R \rangle$ and $H = \langle X : R' \rangle$ be groups generated by $X$ following relations $R$ and $R'$ . Suppose all generators for $H$ satisfy all defining relations for $G$. That is, $R$ is a subset of $R'$. Then, we find $H$ is a homomorphic image of $G$.

*Proof.* Recall $G = F(X)/N$ where $N$ is the normal closure of $R$ in $F(X)$ and $H = F(X)/N'$ where $N'$ is the normal closure of $R'$ in $F(X)$ . But, since all relations on $R$ are satisfied by $H$ , we have $N \leq N'$ . Then, since $F(X)/N' = (F(X)/N)/(N'/N) = G/(N'/N)$, hence $H$ is a homomorphic image of $G$. $\square$