Analysis I

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Lecture 24: Riesz Representation Theorem

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Definition 0.1 (Signum Function)**.** We define the **sign function** to bf

$$\operatorname{sgn} \overline{\mathbb{R}} \longrightarrow \{-1, 0, 1\}$$
$$\operatorname{sgn}(x) \longmapsto \operatorname{sgn}(\operatorname{sgn}(x)) = \chi_{(0,\infty]}(x) - \chi_{[-\infty,0)}(x).$$

Note, if g is measurable, $\operatorname{sgn}(g)$ is measurable.

Remark. If $g : S \to \overline{\mathbb{R}}$ is measurable, then $\operatorname{sgn}(g^*)$ is simple. Moreover, $g \operatorname{sgn}(g) = |g|$.

Theorem 0.1. Let $S \subseteq \mathbb{R}$ be measurable with $1 \leq p \leq \infty$, and q being p's conjugate. For $g \in L^q(S)$, define the map

$$\varphi : L^p(S) \longrightarrow \mathbb{R}$$
$$f \longmapsto \varphi(f) = \int_S fg.$$

Then φ is a bounded linear functional on $L^p(S)$ with norm $\|\varphi\| = \|g\|_q$. In particular $\varphi(f) = 0$ for all $f \in L^p(S)$ if and only if $g = 0$ almost everywhere.

Proof. By Holder,

$$|\varphi(f)| \leq \int_S |fg| \leq \|f\|_p \cdot \|f\|_q.$$

Hence, φ is well defined, and since φ is linear we found it to be a bounded linear functional with $\|\varphi\| \leq \|g\|_q$.
It remains to be shown that $\|\varphi\| = \|g\|_q$.First, we can assume $\|g\|_q > 0$. Then, if $1 < p \leq \infty$ we have $1 \leq q < \infty$. Define

$$f = \|g\|_q^{1-q} |g|^{q-1} \operatorname{sgn}(g).$$

We see $f \in L^p(S)$ and $\|f\|_p = 1$. Moreover,

$$\varphi(f) = \int_S \|g\|^{1-q} |g|^{q-1} |g| \qquad = \|g\|_q^{1-q} \underbrace{\int_S |g|^q}_{\|g\|_q^q}$$
$$= \|g\|_q.$$

Hence, we see $\|g\|_q \leq \|\varphi\|$.
Lastly, consider the case $p = 1$ ($q = \infty$). For $\varepsilon > 0$, let $E_\varepsilon = \{x \in S : |g(x)| \geq \|\varphi\| + \varepsilon\}$ and define $f_n = \chi_{E_\varepsilon \cap (-n,n)} \operatorname{sgn}(g)$ for $n \in \mathbb{N}$. Then, $f_n \in L_1(S)$ and

$$\|f_n\|_1 = m(E_\varepsilon \cap (-n,n)).$$

Then, we see

$$\begin{aligned}\|f_n\|_1 \|\varphi\| &\geq \varphi(f_n)\\ &= \int_S g f_n\\ &\geq (\|p\| + \varepsilon) \cdot \|f_n\|_1.\end{aligned}$$

This implies $\|f_n\|_1 = 0$ as all other possibilities have already been ruled out. Then, we see $m(E_\varepsilon) = 0$. Then, letting $E = \bigcup_{k\in\mathbb{N}} E_{\frac{1}{k}} = \{x \in S : |g(x)| > \|\varphi\|\}$ we see $m(E) = 0$, so $|g(x)| \leq \|\varphi\|$ almost everywhere (i.e. $\|g\|_\infty \leq \|\varphi\|$ so the claim is shown.
The additional claim about $\varphi(f) = 0$ is then trivial. □

Lemma 0.1. Let $[a,b] \subseteq \mathbb{R}$ with $1 \leq p < \infty$ and q being p's conjugate. Suppose $g : [a,b] \to \overline{\mathbb{R}}$ is measurable and finite almost everywhere. If there is a $M \geq 0$ so that $\left|\int_{[a,b]} gs\right| \leq M\|s\|_p$ for every simple function $s \in L^p(\mathbb{R})$, then $g \in L^q([a,b])$ and $\|g\|_q \leq M$.

Proof. Consider $p = 1$ and let $E_\varepsilon = \{x \in [a,b] : |g(x)| \geq M + \varepsilon\}$ for some $\varepsilon > 0$. Define $f_\varepsilon = \chi_{E_\varepsilon} \operatorname{sgn}(g^*)$. Since E_ε is measurable and contained within $[a,b]$, then $m(E_\varepsilon) < \infty$ and f_ε is simple in $L^1(\mathbb{R})$ so that

$$Mm(E_\varepsilon) = M(\|f_\varepsilon\|_1) \geq \int_{[a,b]} g f_\varepsilon = \int_{E_\varepsilon} g f_\varepsilon \geq (M + \varepsilon)\, m(E\varepsilon).$$

Again, we find $m(E_\varepsilon) = 0$, so taking the union over all such $E_{\frac{1}{k}}$ yields $|g(x)| \leq M$ almost everywhere, hence the claim is shown.
For the case $1 < p < \infty$, we see g measurable implies a sequence of simple functions (s_n) so that $\lim_{n\to\infty} s_n(x) = |g^*(x)|$ for all $x \in \mathbb{R}$ and $0 \leq s_n \leq |g^*|$ for all n. Next, define a sequence of simple functions (t_n) with $t_n = s_n^{q-1} \operatorname{sgn}(g^*)$. Since $|g^*| \geq s_n \geq 0$, we find $t_n(x) = 0$ for $x \notin [a,b]$. Hence, $t_n \in L^p(\mathbb{R})$ with

$$\int |t_n|^p = \int |s_n|(pq - p) = \int |s_n|^q.$$

Moreover,

$$\begin{aligned}
\|s_n\|_q^q &= \int s_n^q \\
&= \int s_n^{q-1} s_n \qquad\qquad \leq \int_{[a,b]} s_n^{q-1}|g| \\
&= \int_{[a,b]} g \operatorname{sgn}(g)\, s_n^{q-1} \\
&= \int_{[a,b]} g t_n \\
&\leq M \cdot \|t_n\|_p \\
&= M \cdot \left(\int \underbrace{\left(s_n^{q-1}\right)^p}_{s_n^q} \right)^{\frac{1}{p}} \\
&= M \cdot \left(\int s_n^q \right)^{\frac{1}{q}\cdot\frac{q}{p}} \\
&= M \|s_n\|_q^{\frac{q}{p}}
\end{aligned}$$

. Hence, $\|s_n\|_q^q \leq M\|s_n\|_q^{\frac{q}{p}}$. Dividing yields

$$\|s_n\|_q^{q-\frac{q}{p}} = \|s_n\|_q \leq M.$$

Applying Fatous lemma

$$\begin{aligned}
\int_{[a,b]} |g| &= \int_{[a,b]} \left(\lim_{n\to\infty} s_n^q \right) \\
&\leq \liminf_{n\to\infty} \int_{[a,b]} s_n^q \\
&\leq M^q.
\end{aligned}$$

□

Theorem 0.2 (Riesz Representation Theorem)**.** Let $S \subseteq \mathbb{R}$ be measurable with $1 \leq p < \infty$ and q being p's conjugate. Then, for every bounded linear functional $\varphi : L^p(S) \to \mathbb{R}$ there is a unique $g \in L^q(S)$ so that

$$\pi(f) = \int_S fg \;\forall\; f \in L^p(S)$$

and $\|\varphi\| = \|g\|_q$.

Proof. Defining $\varphi^*(f) = \varphi(f\,|_S)$ for some $f \in L^p(\mathbb{R})$, we see φ is a bounded linear functional on $L^p(\mathbb{R})$ whiles preserving its norm. Hence, we can assume $S = \mathbb{R}$.

Let $[a, b] \subseteq R$ and define the following function

$$\begin{aligned} F : [a,b] &\longrightarrow \mathbb{R} \\ x &\longmapsto F(x) = \varphi\left(\chi_{[a,x]}\right). \end{aligned}$$

Given a finite disjoint collection $\{(a_k, b_k) : 1 \le k \le n\} \in [a, b]$ with each interval being nonempty ($a_k < b_k$). Define $s_k = \operatorname{sgn}(F(b_k) - F(a_k))$. Then, linearity yields

$$\begin{aligned} \varphi\left(\sum_{k=1}^{n} \delta_k \chi_{(a_k,b_k]}\right) &= \sum_{i=1}^{n} \delta_k \varphi\left(\chi_{(a_k,b_k]}\right) \\ &= \sum_{k=1}^{n} \delta_k \left(\varphi\left(\chi_{[a,b_k]} - \chi_{[a,a_k]}\right)\right. \\ &= \sum_{k=1}^{n} \left|F(b_k) - F(a_k)\right|. \end{aligned}$$

Since $\left|\sum_{k=1}^{n} \delta_k \chi_{(a_k,b_k]}\right|^p = \sum_{k=1}^{n} \chi_{(a_k,b_k]}$, we see

$$\begin{aligned} \sum_{k=1}^{n} \left|F(b_k) - F(a_k)\right| &\le \|\varphi\| \left(\int \sum_{k=1}^{n} \chi_{(a_k,b_k]}\right) \\ &= \|\varphi\| \left(\sum_{k=1}^{n} (b_k - a_k)\right)^{\frac{1}{p}}. \end{aligned}$$

Hence, we find F to be absolutely continuous.
Now, for $n \in \mathbb{N}$, define $I_n = [-n, n]$ and define the functions

$$\begin{aligned} F_n : I_n &\longrightarrow \mathbb{R} \\ x &\longmapsto F_n(x) = \varphi\left(\chi_{(-n,x]}\right). \end{aligned}$$

We see each F_n is absolutely continuous, so applying the fundamental theorem of calculus, we find a $g_n \in L^1(I_n)$ so that $F_n(x) = \int_{[-n,x]} g_n$ for $x \in I_n$ and $F_n' = g_n$ almost everywhere on I_n.
Since $F_{n+1}(x) = \varphi\left(\chi_{(-(n+1),n]}\right) + F_n(x)$ for $x \in I_n$. Since this differs only by a constant, we see $g_{n+1} = g_n$ almost everywhere on I_n.
Hence, the sequence of integrable functions (g_n^*) converges pointwise almost everywhere to a measurable $g : \mathbb{R} \to \overline{\mathbb{R}}$. Moreover, every bounded interval $I \subseteq \mathbb{R}$ has $\varphi(\chi_I) = \int_I (g) = \int g\chi_I$ since, there is an $N \in \mathbb{N}$ so that $g\chi_I = g_N^* \chi_I$ almost everywhere.
Hence linearity yields $\varphi(\psi) = \int g\psi$ for every step function ψ. Applying the density results from the previous lecture yields the result for all $f \in L^p(\mathbb{R})$. □

1 Seperability and Bounded Linear Functionals

Lecture 23: Seperability of L^p spaces

Thu 18 Nov 2021 13:57

Definition 1.1 (Step-Function)**.** A **step function**, $\psi : \mathbb{R} \to \mathbb{R}$ is a simple function of the form

$$x \mapsto \sum_{k=1}^{m} a_k \chi_{J_k}(x)$$

where every set J_k is a bounded interval.

Theorem 1.1. (22.4).

Proof. 1. For the case $p = \infty$, we have f bounded almost everywhere. By splitting f into functions f^+, f^- we can assume $f \geq 0$. Then, we see a sequence of simple functions (s_n) converging uniformly to f almost everywhere.
For $1 \leq p < \infty$ we find a sequence of simple functions (s_n) converging pointwise to f so that $|s_n| \leq |f|$. Consequently, we see

$$|f - s_n|^p \leq (|f| + |s|)^p \leq (2\,|f|)^p = 2^p\,|f|^p\,.$$

So, we see dominated convergence implies

$$\int |f - s_n|^p = 0.$$

2. Assuming the case 1, we see we can assume f simple. Moreover, we can assume $f = \chi_S$, a characteristic function in $L^p(\mathbb{R})$.
Then, we see S is measurable with $\int \chi_S = m(S) < \infty$, hence $\int \chi_S^p < \infty$. Applying littlewoods first princple and finxing $\varepsilon > 0$ we find a finite disjoint collection of open intervals $\{J_k : 1 \leq k \leq n\}$ so that for $U = \bigcup_{k=1}^{m} J_k$, we find $m(S \triangle U) < \varepsilon^p$.
Then, we see

$$\begin{aligned}\int |\chi_S - \chi_U|^p &= \int \chi_{S\triangle U}^p \\ &= m(S\triangle U) \\ &< \varepsilon^p.\end{aligned}$$

Since $m(U \setminus S) < \infty$, we see each interval J_k must be bounded (else U would be of infinite measure), so χ_U is a step function on the interval $[a, b] \supseteq U$ satisfying the required conditions.

3. Assuming 2 we see it suffices to show case for the step function $f = \chi_{[c,d]}$ with $c \leq d$. Then, fixing $\varepsilon > 0$ and considering the function

$$x \mapsto g(x) = \chi_{[c,d]} + \big(1 + \varepsilon^{-p}(x - c)\big)\,\chi_{\left(c - \frac{\varepsilon^p}{3},\right),c} + \big(1 - e^{-p}(x - d)\big)\,\chi\left(d, d + \frac{\varepsilon^p}{3}\right).$$

We see this functions is continuous as it is simply piecewise linear, being 1 on $[c, d]$ and a linear interpolation between 1 and 0 in a small interval either side of $[c, d]$. Importantly, $\int_{\left(c - \frac{1}{3}\varepsilon^p\right)} |g| \leq \frac{1}{3}\varepsilon^p$, the length of the

interval.
Hence, we find

$$\int \left|\chi_{[c,d]} - g\right|^p \leq (\frac{2}{3}\varepsilon^p)^p < \varepsilon^p.$$

This completes the proof.

□

Note that this proof essentially showed simple functions, step functions, and continuous functions are dense in $L^p(\mathbb{R})$ (given $1 \leq p < \infty$ for the last 2).

Definition 1.2 (Density)**.** Let $(X, \|\cdot\|)$ be a normed linear space. If $S \subseteq T \subseteq X$, then S is **dense** in T if for all $v \in T, \varepsilon > 0$ we find a vector $u \in S$ so that $\|v - u\| < \varepsilon$.

Definition 1.3 (Seperability)**.** A normed linear space $(X, \|\cdot\|)$ is **seperable** if it contains a countable, dense subset.

Theorem 1.2. For $1 \leq p < \infty$, $L^p(\mathbb{R})$ is seperable.

Proof. If $\varphi = c\chi_{[a,b]}$ with $a, b, c \in R$, then for any $\varepsilon > 0$ we find an interval $I = [c, d] \subseteq [a, b]$ with $c, d \in \mathbb{Q}$ and an $r \in \mathbb{Q}$ so that $\int |\varphi - r\chi_I|^p < \varepsilon^p$ (the function vanishes except on an arbitrarily small interval). Letting Ψ be the collection of all such step functions of the form $\psi = \sum_{i=1}^n c_k\chi_{I_k}$ with $c_k \in \mathbb{Q}$ and I_k having rational endpoints, then linearity combined with the preceding lemmas guarantees Ψ to be a countable dense subset, so $L^p(\mathbb{R})$ is seperable. □

2 Bounded Linear Functionals

Definition 2.1 (Functionals)**.**

- A function $\varphi : X \to \mathbb{R}$ on a linear space X is called a **linear functional** if the laws of linearity holds for φ.
- A linear functional $\varphi : X \to \mathbb{R}$ on a normed linear space $(X, \|\cdot\|)$ is called **bounded** if there is $M \geq 0$ so that $|\varphi(x)| \leq M\|x\|$ for all $x \in X$.
- If φ is a bounded linear functional, the quantity

  $$\|\varphi\| = \inf\{M \geq 0 : |\varphi(x)| \leq M\|x\| \ \forall\ x \in X\}$$

  is called the **norm** of φ.

Proposition 2.1. Let $\varphi : X \to \mathbb{R}$ be a bounded linear functional on a normed linear space $(X, \|\cdot\|)$. Then,

$$\|\varphi\| = \sup\{|\varphi(x)| : x \in X, \|x\| \leq 1\}.$$

Definition 2.2 (Continuity)**.** A linear functional $\varphi : X \to \mathbb{R}$ on $(X, \|\cdot\|)$ is **continuous at** x_0 if for every $\varepsilon > 0$ we find a $\delta > 0$ so that $|\varphi(x) - \varphi(x_0)| < \varepsilon$ if $\|x - x_0\| < \delta$.
If φ is continuous for all $x \in X$, then φ is **continuous.**

Proposition 2.2. Let $\varphi : X \to \mathbb{R}$ be a linear functional on $(X, \|\cdot\|)$. Then, the following are equivalent

- φ is continuous,
- φ is continuous at some $x_0 \in X$,
- φ is bounded.