MATH 7237/8237

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Quite often, in practical and theoretical problems, we have to deal with a set of discrete similar objects, some pairs of which are related in a particular way.

For example, the objects may be the guests at a party, and the relation between two guests is "knowing each other".

Another example is the set of cities in a state and the relation is existence of a road between two cities.

Such situations can be conveniently described on paper by representing each object by a point, and each existing relation by a line joining the corresponding points.

It should be understood that such models do not depend on the positioning of the points or on the shape of the joining lines.

ABSTRACT GRAPH

In fact, we can come up with an even more abstract model:

Definition A **graph** is an ordered pair (V, E) of two sets

– a set V of elements called **vertices;**

– a set E of two-element subsets of vertices, called **edges**.

We write $V(G)$ and $E(G)$ for the sets of vertices and edges of a graph G.

Usually we shall assume that $V(G)$ is the set of the first natural numbers $\{1, 2, \ldots\}$.

The assignment of the vertices may be arbitrary, and is usually chosen for convenience.

Vertices are denoted by lower case letters like u, v, w, $\ldots$, but we may also use subscripts like

$$v_1, v_2, \ldots$$

An edge consisting of the vertices u and v is denoted by $\{u, v\}$.

For simplicity, $\{u, v\}$ may be reduced to uv in writing.

There are different ways to put in words the fact that a graph has an edge $\{u, v\}$. We can use either of the following phrases:

– u is **joined** to v;

– u is **adjacent** to v;

– u and v are **adjacent**;

– u is a **neighbor** to v (v is a neighbor to u.)

The number of vertices of a graph G is called its **order** and is denoted by $v(G)$.

The number of edges of a graph G is called its **size** and is denoted by $e(G)$.

Question *What is the maximum number of edges in a graph of order* n?

This is the number of all 2-element subsets of the vertex set, which is

$$\frac{n(n-1)}{2} = \binom{n}{2}.$$

Definition The unique graph of order n and size $\binom{n}{2}$ is called **complete** graph of order n and is denoted by K_n.

GRAPH DRAWING

There are many different ways to draw the edges of a graph. It doesn't matter if we use straight or curved line segments. What matters is the logical state of being joined or not being joined.

Edges may intersect at points other than the vertices..This is acceptable as long as the intersection points are clearly distinct from the vertices.

Sometimes intersections are avoidable as in K_4, sometimes they are not, as in K_5.

Later we shall study a theorem of Kuratowski that characterizes which graphs can be drawn in the plane without intersections.

Note that representing a graph in the space is always possible without intersections.

PATHS

Definition A **path** of order n is a graph whose vertices can be indexed as

$$v_1, v_2, \ldots, v_n$$

so that the edges of the graph are

$$\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}.$$

A path of order n is denoted by P_n.

Clearly, P_n has $n-1$ edges. We say that P_n has **length** $n-1$.

Definition A path has two **end vertices** (ends). The other vertices of the path are called **internal vertices.**

Definition If u and v are the ends of a path P, we say that P **joins** u to v.

Definition Let G be a graph and $P = u_1, \ldots, u_k$ be a path in G. The path P is called **maximal** if G does not contain a path

$$u_0, u_1, \ldots, u_k$$

or

$$u_1, \ldots, u_k, u_{k+1}.$$

Note that if $u_1, \ldots, u_k$ is a maximal path in G, then all neighbors of u_1 and u_k belong to the set $\{u_1, \ldots, u_k\}$

Caution. A maximal path may not be the longest path in a graph.

CYCLES

Let $n \geq 3$.

Definition A **cycle** of order n is a graph whose vertices can be indexed as

$$v_1, v_2, \ldots, v_n$$

so that the edges of the graph are

$$\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}, \{v_n, v_1\}.$$

A cycle of order n is denoted by C_n, and is sometimes referred to as n-cycle.

Clearly, C_n has n edges. We say that C_n has **length** n.

Cycles of order 3 are usually called **triangles**.

STARS

Definition A **star** of order n is a graph whose vertices can be indexed as

$$v_1, v_2, \ldots, v_n$$

so that the edges of the graph are

$$\{v_1, v_2\}, \{v_1, v_3\}, \ldots, \{v_1, v_n\}.$$

- A star of order n is denoted by $K_{1,n-1}$.
- Clearly, $K_{1,n-1}$ has $n-1$ edges.
- Note that

$$K_{1,1} = K_2 = P_2,$$
$$K_{1,2} = P_3.$$

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GRAPH ISOMORPHISM

Since graphs can be described in different ways, we need a formal definition when two graphs are considered equivalent.

Definition Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are called **isomorphic** if there exists a bijection

$$\varphi : V_1 \rightarrow V_2$$

such that $\{u, v\} \in E_1$ if and only if $\{\varphi(u), \varphi(v)\} \in E_2$.

The bijection φ is called **isomorphism.**

From the point of view of graph theory, isomorphic graphs are identical.

Finding whether two given graphs are isomorphic is a difficult algorithmic problem.

GRAPH AUTOMORPHISMS

Many graphs possess obvious symmetries. The concept of automorphism formalizes symmetry.

Definition Let $G = (V, E)$ be a graph. A permutation $\varphi : V \longrightarrow V$ is called an **automorphism** of G if $uv \in E$ if and only if $\varphi(u)\,\varphi(v) \in E$.

The set of all automorphisms of a graph is a group, which is a subgroup of the symmetric group of order n. The automorphism group of G is denoted by $\mathrm{Aut}(G)$.

In particular, every permutation of the vertices of the complete graph K_n is an automorphism, and therefore K_n has $n!$ automorphisms.

It is known that almost all graphs do not have non-trivial automorphisms, that is, almost all graphs are asymmetric..

LABELED GRAPHS

Let V be a nonempty finite set whose elements are assigned unique labels (say, unique numbers.)

Definition A **labeled** graph with vertex set V is a set of two-element subsets of V.

Two labeled graphs may be isomorphic, but are considered distinct if their edge sets are different.

Hence, if V has n elements there are

$$2^{n(n-1)/2}$$

labeled graphs with vertex set V.

Note that the vertices of any graph G of order n can be labeled in $n!$ possible ways, but the resulting graphs may not always be distinct from each other.

Example: All labellings of a complete graph result in the same labeled graph.

Note that any labelling of a graph is just a permutation of its vertices.

Using some basic group theory arguments, one can prove the following statement.

Proposition *The number of labeled graphs isomorphic to a given graph G of order n is equal to*

$$\frac{n!}{|\mathrm{Aut}\,(G)|}.$$

Let us sum this equality over all graphs of order n. We see that

$$\sum_{v(G)=n} \frac{n!}{|\mathrm{Aut}\,(G)|} = 2^{n(n-1)/2}.$$

Note that the right side is exactly the number of all labeled graphs of order n.

Hence the number $\Gamma\,(n)$ of all graphs of order n satisfies

$$\Gamma\,(n) = \sum_{v(G)=n} 1 \geq \sum_{v(G)=n} \frac{1}{|\mathrm{Aut}\,(G)|} = \frac{2^{n(n-1)/2}}{n!}.$$

Since for almost all graphs it is known that $|\mathrm{Aut}\,(G)| = 1$, it turns out that

$$\Gamma\,(n) \approx \frac{2^{n(n-1)/2}}{n!}.$$

DEGREES AND NEIGHBORHOODS

Let G be a graph and v be a vertex of G.

Definition The **degree** of v in G is the number of edges containing v. The degree of v is denoted by $d_G(v)$, or $d(v)$ if G is understood.

Neighborhoods are sets that are closely related to degrees.

Definition The **neighborhood** of v is the set of all vertices of G that are joined to v. The neighborhood of v is denoted by $N_G(u)$, or simply $N(u)$ if G is understood.

Obviously,

$$d_G(v) = |N_G(v)|.$$

DEGREES

Proposition *If G is a graph with vertex set V and size m, then*

$$\sum_{u \in V} d_G(u) = 2m.$$

Indeed, the sum in the left side counts each edge $\{u, v\}$ precisely twice: once in $d_G(u)$ and once in $d_G(v)$.

Corollary *The number of vertices of odd degree in G is even.*

Some statements require more work

Proposition *Every graph of order at least 2 has two vertices of the same degree.*

MINIMUM AND MAXIMUM DEGREES

Definition The **minimum degree** $\delta(G)$ of a graph G is defined as

$$\delta(G) = \min_{u \in V} \{d_G(u)\}.$$

The **maximum degree** $\Delta(G)$ of G is defined as

$$\Delta(G) = \max_{u \in V} \{d_G(u)\},$$

Clearly if G is a graph of order n, then

$$0 \leq \delta(G) \leq \Delta(G) \leq n - 1.$$

Definition If u is a vertex with $d_G(u) = 0$, then u is called an **isolated vertex** of G.

DEGREE BASED CONCEPTS

Definition If u is a vertex with $d_G(u) = |V(G)| - 1$, then u is called a **dominating vertex** of G.

Note that the star $K_{1,n-1}$ has one dominating vertex (if $n > 2$), and in K_n all vertices are dominating.

Definition A graph is called **regular** if all its degrees are the same.

For short, a regular graph of degree r is called an r-**regular** graph.

Examples Cycles are 2-regular graphs.

The complete graph K_n is $(n-1)$-regular graph.

REGULAR GRAPHS

Remark If G is an r-regular graph of order n, then G has

$$\frac{nr}{2}$$

edges.

Indeed,

$$2e(G) = \sum_{u \in V(G)} d(u) = \sum_{u \in V(G)} r = nr$$

Definition Often, 3-regular graphs are called **cubic**.

Question *Is there a cubic graph of order* 101?

THANK YOU