REAL VARIABLES I

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1. Basics

We assume throughout that the natural numbers $\mathbb{N}$, the integers $\mathbb{Z}$ and the rational numbers $\mathbb{Q}$ are known. X and Y will denote any sets.

A *function* (*map* or *mapping*) f from a set X to a set Y is a rule that assigns to each $x \in X$ a unique element $f(x) \in Y$. For such a function we often write $f : X \to Y$ or $x \mapsto f(x)$. The set X is the *domain* of f. The set Y is the *codomain* of f. The set $f(X) = \{f(x) \in Y \,|\, x \in X\}$ is the *range* of f. For $A \subset X$ the set $f(A) = \{f(x) \in Y \,|\, x \in A\}$ is the *image* under f of A. For $B \subset Y$ the set $f^{-1}(B) = \{x \in X \,|\, f(x) \in B\}$ is called the *preimage* of B. For $C \subset X$ the *restriction* of the function $f : X \to Y$ to C, denoted by $f\big|_C$, is the function from C to Y which assigns $f(x)$ to $x \in C$. If $f : X \to Y$ and $g : Y \to Z$ are two functions, then their *composition*, denoted by $g \circ f : X \to Z$, is the function from X to Z, given by $\left(g \circ f\right)(x) = g(f(x))$.

Definition 1.1.

(1) A function $f : X \to Y$ is called *onto* (or *surjective*) if $f(X) = Y$.
(2) A function $f : X \to Y$ is called *one-to-one* (or *injective*) if the equality $f(u) = f(v)$ for $u, v \in X$ implies $u = v$.

The function $\mathrm{id}_X : X \to X$, $x \mapsto \mathrm{id}_X(x) = x$ is called the *identity* on X. A function $f : X \to Y$ that is both one-to-one and onto is also called *bijective*. If $f : X \to Y$ is one-to-one, there is a unique function $g : f(X) \to X$ such $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_{f(X)}$. The function g is called the *inverse* of f, denoted by f^{-1}.

Definition 1.2.

(1) For $n \in \mathbb{N}$ a function $f : \{k \in \mathbb{N} \,|\, k \le n\} \to X$ is called a *finite sequence* of length n, denoted by $(x_k)_{1 \le k \le n}$ or $(x_k)_{k=1}^{n}$ with $x_k = f(k)$.
(2) A function $f : \mathbb{N} \to X$ is called an *(infinite) sequence*, denoted by (x_k) or $(x_k)_k$ or $(x_k)_{k=1}^{\infty}$ with $x_k = f(k)$.
(3) A *family* or *collection* $\mathscr{C}$ of subsets of X is a subset of the power set $\mathscr{P}(X) = \{A \,|\, A \subset X\}$ of X. A collection $\mathscr{C}'$ of subsets of X is a *subcollection* of the collection $\mathscr{C}$ if $\mathscr{C}' \subset \mathscr{C}$.
(4) A function f from a set Λ to the power set $\mathscr{P}(X)$ of X is called a *collection* of subsets of X for the *index set* Λ or an *indexed collection* of subsets of X, denoted by $\{A_\lambda \,|\, \lambda \in \Lambda\}$ (or $\{A_\lambda\}$ if Λ is clear) where $A_\lambda = f(\lambda)$.

Instead of $(x_k)_{1\le k\le n}$ we will also write $(x_1,\ldots,x_n)$. The notions of finite and infinite sequences can, of course, be generalized. If $m,n\in\mathbb{Z}$, $m\le n$, we may write $(x_k)_{m\le k\le n}$ or $(x_k)_{k=m}^{n}$ or $(x_m,\ldots,x_n)$ and $(x_k)_{k=m}^{\infty}$, respectively.

For A, $B\subset X$ we define the *intersection* $A\cap B=\left\{x\in X\,\middle|\,x\in A \text{ and } x\in B\right\}$, the *union* $A\cup B=\left\{x\in X\,\middle|\,x\in A \text{ or } x\in B\right\}$, the *difference* $A\setminus B=\left\{x\in X\,\middle|\,x\in A \text{ and } x\not\in B\right\}$, the *symmetric difference* $A\Delta B=(A\setminus B)\cup(B\setminus A)$, and the *complement* $A^c=X\setminus A$. If $A\cap B=\emptyset$, A and B are *disjoint*. A collection $\mathscr{C}$ of subsets of X is a *disjoint collection* if for any A, $B\in\mathscr{C}$, $A\cap B=\emptyset$. For a collection $\mathscr{C}$ of subsets of X we also define the *intersection*

$$\bigcap_{A\in\mathscr{C}} A=\left\{x\in X\,\middle|\,x\in A \text{ for all } A\in\mathscr{C}\right\}$$

and the *union*

$$\bigcup_{A\in\mathscr{C}} A=\left\{x\in X\,\middle|\,x\in A \text{ for some } A\in\mathscr{C}\right\}.$$

In the special case that $\mathscr{C}=\left\{A_k\,\middle|\,1\le k\le n\right\}$ or $\mathscr{C}=\left\{A_k\,\middle|\,k\in\mathbb{N}\right\}$, we will also write

$$\bigcap_{k=1}^{n} A_k \quad\text{or}\quad \bigcap_{k=1}^{\infty} A_k,\quad\text{respectively,}$$

instead of $\bigcap_{A\in\mathscr{C}} A$, and

$$\bigcup_{k=1}^{n} A_k \quad\text{or}\quad \bigcup_{k=1}^{\infty} A_k,\quad\text{respectively,}$$

instead of $\bigcup_{A\in\mathscr{C}} A$. Of course, we can (and shall) generalize this notation further.

Proposition 1.3. De Morgan's Laws
For $\mathscr{C}\subset\mathscr{P}(X)$

$$\left(\bigcup_{A\in\mathscr{C}} A\right)^c=\bigcap_{A\in\mathscr{C}} A^c \quad\textit{and}\quad \left(\bigcap_{A\in\mathscr{C}} A\right)^c=\bigcup_{A\in\mathscr{C}} A^c.$$

Proposition 1.4. Distributive Laws
For $\mathscr{C}\subset\mathscr{P}(X)$, $B\subset X$

$$B\cap\left(\bigcup_{A\in\mathscr{C}} A\right)=\bigcup_{A\in\mathscr{C}}(A\cap B)\quad\textit{and}\quad B\cup\left(\bigcap_{A\in\mathscr{C}} A\right)=\bigcap_{A\in\mathscr{C}}(A\cup B).$$

Proposition 1.5. *For $\mathscr{C} \subset \mathscr{P}(X)$, $\mathscr{C}' \subset \mathscr{P}(Y)$, $A \subset X$, $B \subset Y$ and $f : X \to Y$ the following holds true:*

(1) $$f\left(\bigcup_{A\in\mathscr{C}} A\right) = \bigcup_{A\in\mathscr{C}} f(A) \quad \textit{and} \quad f\left(\bigcap_{A\in\mathscr{C}} A\right) \subset \bigcap_{A\in\mathscr{C}} f(A),$$

(2) $$f^{-1}\left(\bigcup_{B\in\mathscr{C}'} B\right) = \bigcup_{B\in\mathscr{C}'} f^{-1}(B) \quad \textit{and} \quad f^{-1}\left(\bigcap_{B\in\mathscr{C}'} B\right) = \bigcap_{B\in\mathscr{C}'} f^{-1}(B),$$

(3) $$f^{-1}\left(B^c\right) = \left(f^{-1}(B)\right)^c,$$

(4) $$f\left(f^{-1}(B)\right) \subset B \quad \textit{and} \quad A \subset f^{-1}\left(f(A)\right).$$

Definition 1.6.

(1) A set X is called *finite* if it is empty or if there is a number $n \in \mathbb{N}$ and a bijective map $f : X \to \left\{k \in \mathbb{N} \,\middle|\, 1 \le k \le n\right\}$. A set that is not finite is called *infinite*.
(2) A set X is called *countably infinite* if there is a bijective map $f : X \to \mathbb{N}$.
(3) A set X is called *countable* if it is finite or countably infinite.
(4) A set X is called *uncountable* if it is not countable.

Proposition 1.7. $\mathbb{N}$, $\mathbb{Z}$ *and* $\mathbb{Q}$ *are countable.*

Proposition 1.8.

(1) *Every subset of a countable set is countable.*
(2) *The set of all finite sequences from a countable set is countable.*
(3) *The union of a countable collection of countable sets is countable.*

Definition 1.9.

(1) A collection $\mathscr{A}$ of subsets of X is called an *algebra* if $X \in \mathscr{A}$, $A^c \in \mathscr{A}$ for every $A \in \mathscr{A}$, and $A \cup B \in \mathscr{A}$ for every $A, B \in \mathscr{A}$.
(2) A collection $\mathscr{A}$ of subsets of X is called a σ*-algebra* if $X \in \mathscr{A}$, $A^c \in \mathscr{A}$ for every $A \in \mathscr{A}$, and $\bigcup_{k=1}^{\infty} A_k \in \mathscr{A}$ for every (countable) collection $\left\{A_k \,\middle|\, k \in \mathbb{N}\right\}$ of subset of X with $A_k \in \mathscr{A}$ for every k.

Proposition 1.10.

(1) *For every collection $\mathscr{C}$ of subsets of X there is a smallest algebra $\mathscr{A}$ containing $\mathscr{C}$ in the sense that any other algebra containing $\mathscr{C}$ also contains $\mathscr{A}$.*
(2) *For every collection $\mathscr{C}$ of subsets of X there is a smallest σ-algebra $\mathscr{A}$ containing $\mathscr{C}$ in the sense that any other σ-algebra containing $\mathscr{C}$ also contains $\mathscr{A}$.*

Proof. Let $\mathscr{F}$ be the collection of all algebras/σ-algebras containing $\mathscr{C}$. $\mathscr{F}$ is nonempty since it contains $\mathscr{P}(X)$. Let $\mathscr{A} = \bigcap_{\mathscr{B}\in\mathscr{F}} \mathscr{B}$. Then $\mathscr{C}$ is a subcollection of $\mathscr{A}$. Also, $X \in \mathscr{A}$ since $X \in \mathscr{B}$ for each $\mathscr{B} \in \mathscr{F}$. Similarly, if $A \in \mathscr{A}$, then $A, A^c \in \mathscr{B}$ for each $\mathscr{B} \in \mathscr{F}$, hence $A^c \in \mathscr{A}$. Finally, if $\{A_k\}$ is a finite/countable collection with $A_k \in \mathscr{A}$ for every k, then each set A_k belongs to $\mathscr{B}$ for each $\mathscr{B} \in \mathscr{F}$, hence the union of the sets A_k does as well. Consequently, this union also belongs to $\mathscr{A}$. □

A *choice function* on a collection $\mathscr{F}$ of nonempty sets is a map $f : \mathscr{F} \to \bigcup_{F \in \mathscr{F}} F$ such that $f(F) \in F$ for every $F \in \mathscr{F}$.

Axiom 1.11. Zermelo's Axiom of Choice
Every collection of nonempty sets has a choice function.

The Axiom of Choice is needed to define the *Cartesian product* of a collection $\{X_\lambda\}$ of (nonempty) sets indexed by an arbitrary (nonempty) index set Λ:

$$\prod_{\lambda \in \Lambda} X_\lambda = \left\{ f : \Lambda \to \bigcup_{\lambda \in \Lambda} X_\lambda \,\middle|\, f(\lambda) \in X_\lambda \text{ for each } \lambda \in \Lambda \right\}$$

If Λ is the finite index set $\left\{k \in \mathbb{N} \,\middle|\, 1 \le k \le n\right\}$, we often write $X_1 \times \cdots \times X_n$ for the Cartesian product and identify its elements with the finite sequences $(x_k)_{k=1}^n$, $x_k \in X_k$. In the special case where $X_k = X$ for all k, it is customary to write X^n instead of $X \times \cdots \times X$. If Λ is countably infinite, we identify the elements of the Cartesian product $\prod_{k \in \mathbb{N}} X_k$ with the infinite sequences (x_k), $x_k \in X_k$.

Given a nonempty set X, we call a subset R of $X \times X$ a *relation* on X and write $x\,R\,y$ if $(x, y) \in R$. A relation is *reflexive* if $x\,R\,x$ for all $x \in X$, *symmetric* if $x\,R\,y$ implies $y\,R\,x$, and *transitive* if $x\,R\,y$ and $y\,R\,z$ imply $x\,R\,z$.

Definition 1.12. A reflexive, symmetric, transitive relation R on a nonempty set X is called an *equivalence relation.*

Equivalence relations are often denoted by $\sim$ instead of R.
An equivalence relation $\sim$ on X gives rise to *equivalence classes,* defined for $x \in X$ by

$$[x] = \{y \in X \mid x \sim y\}$$

The set of all equivalence classes $\left\{[x] \,\middle|\, x \in X\right\}$ is denoted by $X/\sim$ and is called the *quotient set* of the relation $\sim$. Elements of an equivalence class are called *representatives* of the equivalence class.

Definition 1.13.

(1) A reflexive, transitive relation R on a nonempty set X is called a *partial ordering* if $x\,R\,y$ and $y\,R\,x$ imply $x = y$.
(2) A partial ordering R on a nonempty set X is called a *total ordering* if for all $x, y \in X$, $x\,R\,y$ or $y\,R\,x$ holds true. In this case X is called *(totally) ordered.*
(3) For a nonempty set X with partial ordering R we call $z \in X$ an *upper bound* of a subset $A \subset X$ if for all $x \in A$, $x\,R\,z$ holds true.
(4) For a nonempty set X with partial ordering R we call $z \in X$ a *maximal element* if $z\,R\,x$ implies $z = x$.

Lemma 1.14. Zorn's Lemma
Let X be a nonempty set with a partial ordering. If every totally ordered subset of X has an upper bound, then X has a maximal element.

Zorn's Lemma is a variant of the Axiom of Choice (i.e. it implies and is implied by the Axiom of Choice). Another variant is the *Hausdorff Maximality Theorem*.