Analysis I

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Lecture 18: General Lebesque Integral (2)

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Proposition 0.1. Let $f : \mathbb{R} \to \overline{\mathbb{R}}$ be integrable. Then for each $\varepsilon > 0$ there is a $\delta > 0$ such that each measurable $S \subseteq \mathbb{R}$ has $\int_S |f| < \varepsilon$ if $m(S < \delta)$.

Proof. Let $\varepsilon > 0$, then there is a $s \in \mathscr{S}(|f|)$ such that $\int (|f| - s) < \frac{\varepsilon}{2}$. Let $a\alpha = \sup\{s(x) : x \in \mathbb{R}\}$ and $\delta = \frac{\varepsilon}{2(\alpha+\varepsilon)}$. If S is measurable and $m(S) < \delta$, we find

$$\int_S |f| \leq \int s + \frac{\varepsilon}{2} \leq \alpha m(S) + \frac{\varepsilon}{2} < \varepsilon.$$

$\square$

Theorem 0.1 (Monotone Convergence Theorem)**.** Let (f_n) be a sequence of nonnegative measurable functions with $f_n : \mathbb{R} \to \overline{\mathbb{R}}$ such that $(f_n(x))$ is increasing for all $x \in \mathbb{R}$. Then, $f = \lim\limits_{n\to\infty} f_n$ is maesurable with $\int f = \lim\limits_{n\to\infty} \int f_n$.

Proof. Since $f = \limsup_{n\to\infty} f_n = \liminf_{n\to\infty} f_n$, we see f is measurable. Moreover, the sequence $\left(\int f_n\right)$ is increasing (as the f_ns are increasing). Hence, letting $L = \lim\limits_{n\to\infty} \int f_n$ exists with $L \in R_0^+$. Since $\int f_n \leq \int f$ for all n by monotonicity, we find $L \leq \int f$.
Let $s \in \mathscr{S}(f)$ and fix $c \in (0,1)$ and define $E_n = \{x \in \mathbb{R} : f_n(x) \geq cs(x)\}$. Then, we find $\{E_n : n \in \mathbb{N}\}$ is an ascending collection (again by monotonicity of (f_n)) of measurable sets with $\bigcup_{n\in\mathbb{N}} E_n = \mathbb{R}$ as $cs(x) < f_n(x) \leq f(x)$. Let $s = \sum_{k=1}^K a_k \chi_{S_k}$ and we see $cs\chi_{E_n}M = f_n\chi_{E_n} \leq f_n$, with

$$L \geq \int f_n \geq \int_{E_n} f_n \geq \int cs\chi_{E_n} = c\int_{E_n} s = c\sum_{k=1}^K a_k m(S_k \cap E_n).$$

Since $\lim_{n\to\infty} m(E_n \cap S_n) = m(S)$ for every measurable set S, we find $L \geq c\sum_{k=1}^{K} a_k m(S_k) = c\int s$. Since c was arbitrary, we see the inequality holds for all $c \in (0,1)$, hence we find $L \geq s$ (by taking supremums), but $s \in \mathscr{S}(f)$, hence $L \geq \int f$. So, $L = \int f$. ☐

Theorem 0.2 (Fatou's Lemma)**.** If (f_n) is a sequence of nonnegative measurable functions $f_n : \mathbb{R} \to \overline{\mathbb{R}}$, then $\int \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int f_n$.

Proof. For $x \in \mathbb{R}$, define $g_n(x) = \inf\{f_k(x) : k \geq n\}$ for $n \in \mathbb{N}$. Then, we find (g_n) is a nonnegative measurable sequence of functions with $(g_n(x))$ increasing for all fixed x and $g_n \leq f_n$ for all n. Consequently, $\int g_n \leq \int f_n$ and $\left(\int g_n\right)$ is increasing. As $\lim_{n\to\infty} g_n = \liminf_{n\to\infty} f_n$ is measurable by an earlier theorem, we find

$$\liminf_{n\to\infty} \int f_n \geq \liminf_{n\to\infty} \int g_n = \lim_{n\to\infty} \int g_n = \int \lim_{n\to\infty} g_n = \int \liminf_{n\to\infty} f_n.$$

☐

Proposition 0.2. For any integral function $f : \mathbb{R} \to \overline{\mathbb{R}}$, we find $\left|\int f\right| \leq \int |f|$.

Theorem 0.3 (Dominated Convergence Theorem)**.** Let (f_n) be a sequence of measurable functions $f_n : \mathbb{R} \to \overline{\mathbb{R}}$. Suppose there is an integrable function g with $|f_n| \leq g$ for all $n \in \mathbb{N}$. If (f_n) converges pointwise to a function $f : \mathbb{R} \to \overline{\mathbb{R}}$ almost everywhere, then f is integrable and

$$\lim_{n\to\infty} \int |f_n - f| = 0 \text{ and } \lim_{n\to\infty} \int f_n = \int f.$$

Proof. Since $f(x) = \lim_{n\to\infty} f_n(x)$ for almost all $x \in R$, we find f is measurable. Moreover, $|f_n| \leq g$ implies $|f| < g$ almost everywhere and since g is integrable (hence finite a.e) we find f, f_n are integrable (hence finite) almost everywhere. Now, define for each $n \in \mathbb{N}$

$$E_n = \{x \in \mathbb{R} : |f_n(x)|, |f(x)| < \infty, |f_n(x) - f(x)| \leq 2g(x)\}.$$

Since $R \setminus \bigcup_{n\in\mathbb{N}} E_n$ is a set of measure 0, we can assume $|f_n(x)|, |f(x)| < \infty$ and $|f_n() - f(x)| \leq 2g(x)$ for all $x \in \mathbb{R}$. Then, Fatou's lemma applies to the sequence on nonnegative measurable functions $(2g - |f_n - f|)$ yielding

$$\begin{aligned}\int 2g &\le \liminf_{n\to\infty} (2g - |f_n - f|)\\ &= \int 2g + \liminf_{n\to\infty}\left(-\int |f_n - f|\right)\\ &= \int 2g - \limsup_{n\to\infty}\int |f_n - f|\\ &\Rightarrow \limsup_{n\to\infty}\int |f_n - f| \le 0\\ &\Rightarrow \lim_{n\to\infty}\int |f_n - f| = 0.\end{aligned}$$

Hence, $\lim_{n\to\infty} \left|\int (f_n - f)\right| = 0$ by the earlier lemma. So, $\lim_{n\to\infty}\int f_n = \int f_n$. □

Definition 0.1 (Convergence in Measure)**.** Let (f_n) be a sequence of measurable functions $f_n : \mathbb{R} \to \overline{\mathbb{R}}$ and $f : \mathbb{R} \to \overline{\mathbb{R}}$ also be measurable. The sequence (f_n) **converges in measure** to f ($f_n \to f$ by measure) if each $f_n$is finite almost everywhere and for each $\varepsilon > 0$ there is a $N \in \mathbb{N}$ so that

$$m\left(\{x \in \mathbb{R} : |f_n(x) - f(x)| > \varepsilon\}\right) < \varepsilon$$

for $n \ge N$.

Theorem 0.4 (Riesz)**.** Let (f_n) be a sequence of measurable functions $f_n : \mathbb{R} \to \overline{\mathbb{R}}$ and $f : \mathbb{R} \to \overline{\mathbb{R}}$ also being measurable. If $(f_n) \to f$ in measure, then there is a subsequence (f_{n_k}) which converges pointwise almost everywhere to f.

Proof. First, we find a strictly increasing sequence of numbers (n_k) such that $m\left(\{x \in \mathbb{R} : |f_j(x) - f(x)| > 2^{-k}\}\right) < 2^{-k}$ if $j \ge n_k$. For $k \in \mathbb{N}$ denote

$$S_k = \{x \in \mathbb{R} : |f_{n_k} - f(x)| > 2^{-k}\}.$$

Then, $\sum_{k=1}^{\infty} m(S_k) \le \sum_{k=1}^{\infty} 2^{-k} < \infty$. Applying the Borel-Cantelli Lemma yields that almost every $x \in R$ does not belong to any infinite subcollections of (S_k). For such x, we find a $K \in \mathbb{N}$ such that $|f_{n_k}(x) - f(x)| \le 2^{-k}$ for $k \ge K$. Hence, f_{n_k} converges pointwise to f for all x not belonging to an infinite subcollection of (S_k), hence almost everywhere. □

Lecture 19: End of Convergence and Functions of Bounded Variation

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Recall we had the dominated convergence theorem. A similair version of the theorem makes use of convergence in measure as follows.

Theorem 0.5 (Dominated Convergence - Convergence in Measure)**.** Let (f_n) be a sequence of measurable functions $f_n : \mathbb{R} \to \overline{\mathbb{R}}$ and suppose there is an integrable function $g : \mathbb{R} \to \overline{\mathbb{R}}$ so that $|f_n| \leq g$ for all $n \in \mathbb{N}$. If $(f_n) \to f : \mathbb{R} \to \overline{\mathbb{R}}$ in measure, (with f measurable), then f is integrable and $\lim\limits_{n\to\infty} \int |f_n - f| = 0$ and $\lim\limits_{n\to\infty} \int f_n = f$.

Proof. First, note a subsequence of (f_n) converges to f pointwise almost everywhere. Hence, we find $|f| \leq g$ almost everywhere, so f is integrable. We cam assume $|f_n - f| \leq 2g$ (almost) everywhere. Then, we find a subsequence $(g_n) = (f_{n_k})$ such that $\limsup_{n\to\infty} |f_n - f| = \lim\limits_{n\to\infty} |g_k - f|$. Then, as $(g_k) \to f$ in measure, we find another subsequence $(h_j) = \left(g_{k_j}\right) = \left(f_{n_{k_j}}\right)$ which converges pointwise to f almost everywhere.
Applying dominated convergence theorem yields

$$\lim_{n\to\infty} \int |h_j - f| = 0.$$

Then, we find

$$\begin{aligned}\limsup_{n\to\infty} \int |f_n - f| &= \lim_{n\to\infty} \int |g_k - f| \\ &= \lim_{n\to\infty} |h_j - f| \\ &= 0.\end{aligned}$$

This completes the proof. □

1 Functions of Bounded Variation and Absolutely Continuous Functions

Remark. For this chapter $[a, b] \subseteq R$ will always denote a compact interval on $\mathbb{R}$.

Definition 1.1 (Partition)**.** A finite sequence $P = (x_k)_{k=n}^{N}$ with $n, N \in \mathbb{Z}$ and $n < N$ is called a **partition** of $[a, b]$ if $x_n = a$, $x_N = b$ and $x_{k-1} \leq x_k$ for $n < k \leq N$. We denote the collection of all partitions of $[a, b]$ to be $\mathscr{P}([a, b])$.

Definition 1.2. Let $f : [a, b] \to \mathbb{R}$ be a function. Then,

- For a partition $P = (x_k)_{k=n}^{N}$, we denote

$$V(f, P) = \sum_{k=n+1}^{N} \left| f(x_k) - f_{(x_{k-1})} \right|$$

  to be the **variation of f with respect to P**.
- We define the quantity $\mathrm{TV}(f) = \sup\{V(f, P) : P \in \mathscr{P}([a, b])\}$ to be the **total variation of f** .

Remark. If $f : [a, b] \to \mathbb{R}$ and $c \in [a, b]$ with partitions $P_1 = (x_k)_{k=n}^{N}$ of $[a, c]$ and $P_2 = (x_k)_{k=N}^{K}$ of $[c, b]$. Then denote, $P = (x_k)_{k=n}^{K}$ to be a partition of $[a, b]$ and we find

$$V(f, P) = V\left(f\mid_{[a,c],P_1}\right) + V\left(f\mid_{[c,b]}, P_2\right).$$

Moreover,

$$\mathrm{TV}(f) = \mathrm{TV}\left(f\mid_{[a,c]}\right) + \mathrm{TV}\left(f\mid_{[c,b]}\right).$$

Definition 1.3 (Bounded Variation)**.** A function $f : \mathbb{R} \to \overline{\mathbb{R}}$ has **bounded variation** if $\mathrm{TV}(f) < \infty$.

Theorem 1.1 (Jordan's Theorem)**.** A function $f : [a, b] \to \mathbb{R}$ is of bounded variation if and only if there are increasing functions $g, h : [a, b] \to \mathbb{R}$ so that $f = g - h$.

Proof. Suppose $\mathrm{TV}(f) < \infty$ and let $x, y \in [a, b]$ with $x < y$. Then, we find

$$\begin{aligned}
\mathrm{TV}\left(f\mid_{[a,y]}\right) &= \mathrm{TV}\left(f\mid_{[a,x]}\right) + \mathrm{TV}\left(f\mid_{[x,y]}\right) \\
&\geq \mathrm{TV}\left(f\mid_{[a,x]}\right) + |f(y) - f(x)| \\
&\geq \mathrm{TV}\left(f\mid_{[a,x]}\right) + f(x) - f(y).
\end{aligned}$$

Furtheromre, $h : x \mapsto \mathrm{TV}\left(f\mid_{[a,x]}\right)$ and $g : x \mapsto \mathrm{TV}\left(f\mid_{[a,x]}\right) + f(x)$ are increasing. This fact is trivial for h and we find , adding $f(y)$ to both sides of the former inequality yields $g(y) \geq g(x)$ for arbitrary $y \geq x$, so this claim holds as well.
Taking the difference, $g - h = f$.
Conversely, suppose $f = g - h$ for increasing $g, h : [a, b] \to \mathbb{R}$. Then, let $x, y \in [a, b]$ with $y \geq x$. Then, we find

$$\begin{aligned}
|f(y) - f(x)| &= |g(y) - g(x) + h(x) - h(y)| \\
&\leq |g(y) - g(x)| + |h(x) - h(y)| \\
&= g(y) - g(x) + h(y) - h(x).
\end{aligned}$$

Hence, for a partition $P = (x_k)_{k=n}^N$, we find

$$\begin{aligned} V(f, P) &= \sum_{k=n+1}^{N} |f(x_k) - f(x_{k-1})| \\ &\leq \sum_{k=n+1}^{N} (g(x_k) - g(x_{k-1}) + h(x_k) - h(x_{k-1})) \quad = g(b) - g(a) + h(b) - h(a) \\ &< \infty. \end{aligned}$$

□

Definition 1.4 (Absolute Continuity)**.** A function $f : [a, b] \to \mathbb{R}$ is **absolutely continuous** if for each $\varepsilon > 0$ we find a $\delta > 0$ such that for every finite disjoint collection of nonempty intervals $\{(a_k, b_k) \subseteq [a, b] : 1 \leq k \leq K\}$ with $\sum_{k=1}^{K} (b_k - a_k) < \delta$, we have $\sum_{k=1}^{K} |f(a_k) - f(b_k)| < \varepsilon$.

Remark. Absolute continuity is stronger than uniform continuity, but weaker than lipschitz continuity.

Theorem 1.2. If a function $f : [a, b] \to \mathbb{R} \to$ is absolutely continuous, then f is continuous and f has bounded variation.

Proof. f is trivially continuous, taking a finite disjoint collection consisting only of 1 interval $\{(x, y)\}$ yields the definition of continuity.
Now we show bounded variation. For $\varepsilon = 1$, let $\delta > 0$ be the number such that the definition of absolute continuity holds for f.
Now fix $(x_k)_{k=n}^N \in \mathscr{P}([a, b])$ so that $x_k - x_{k-1} < \delta$ for all $n < k \leq N$. Then, if $P \in \mathscr{P}([x_{k-1}, x_k])$, we see $V\left(f\,|_{[x_{k-1}, x_k]}, P\right) < 1$ by definition of absolute continuity.
So, we have $\mathrm{TV}([x_{k-1}, x_k]) \leq 1$, so $\mathrm{TV}(f) = \sum_{k=n+1}^{N} \mathrm{TV}\left(f\,|_{[x_{k-1}, x_k]}\right) \leq N - n$ by the ε assumption. □

As it turns out, absolutely continuous functions have a relation to integrable functions, particularly, an integrable function f is simply the anti-integral of an absolutely continuous one.

Proposition 1.1. If $f : [a, b] \to \overline{\mathbb{R}}$ is integrable, then,

$$F : [a, b] \to \mathbb{R}, \ x \mapsto \int_{[a,x]} f$$

is absolutely continuous.

This claim can be generalized into a sort of fundamental theorem of calculus for the lebesque integrals to characterize integrals and derivatives. For now, we only prove the weak version.

Proof. For $\varepsilon > 0$ there is a $\delta > 0$ such that $\int_S |f| < \varepsilon$ for every measurable set $S \subseteq [a, b]$ with $m(S) < \delta$.
Now, let $\{(a_k, b_k) : 1 \leq k \leq K\}$ be a disjoint collection of intervals in $[a, b]$ with $\sum_{k=1}^{K} (b_k - a_k) < \delta$. Fix $S = \bigcup_{k=1}^{K} (a_k, b_k)$. Then, since $m(S) < \delta$ and

$$\begin{aligned} \sum_{k=1}^{K} |F(b_k) - F(a_k)| &= \sum_{k=1}^{K} \left| \int_{[a_k, b_k]} f \right| \\ &\leq \sum_{k=1}^{K} \int_{[a_k, b_k]} |f| \\ &= \int_S |f| \\ &< \varepsilon \text{ by assumption.} \end{aligned}$$

Hence, absolute continuity holds. □

2 Derivatives and Fundamental Theorem of Calculus

Proposition 2.1. Let $f : [a, b] \to \overline{\mathbb{R}}$ be monotone on $(a, b) \subseteq \mathbb{R}$ with $a, b \in \overline{\mathbb{R}}$ and $a < b$. Then, the following limits are well defined: $\lim_{x \to a} f(x), \lim_{x \to b} f(x)$.