Analysis I

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November 23, 2021

Contents

1 Seperability and Bounded Linear Functionals

Lecture 23: Seperability of L^p spaces

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Definition 1.1 (Step-Function)**.** A **step function**, $\psi : \mathbb{R} \to \mathbb{R}$ is a simple function of the form

$$x \mapsto \sum_{k=1}^{m} a_k \chi_{J_k}(x)$$

where every set J_k is a bounded interval.

Theorem 1.1. (22.4).

Proof. 1. For the case $p = \infty$, we have f bounded almost everywhere. By splitting f into functions f^+, f^- we can assume $f \geq 0$. Then, we see a sequence of simple functions (s_n) converging uniformly to f almost everywhere.
For $1 \leq p < \infty$ we find a sequence of simple functions (s_n) converging pointwise to f so that $|s_n| \leq |f|$. Consequently, we see

$$|f - s_n|^p \leq (|f| + |s|)^p \leq (2\,|f|)^p = 2^p\,|f|^p\,.$$

So, we see dominated convergence implies

$$\int |f - s_n|^p = 0.$$

2. Assuming the case 1, we see we can assume f simple. Moreover, we can assume $f = \chi_S$, a characteristic function in $L^p(\mathbb{R})$.
Then, we see S is measurable with $\int \chi_S = m(S) < \infty$, hence $\int \chi_S^p < \infty$. Applying littlewoods first princple and finxing $\varepsilon > 0$ we find a finite

disjoint collection of open intervals $\{J_k : 1 \leq k \leq n\}$ so that for $U = \bigcup_{k=1}^{m} J_k$, we find $m(S \triangle U) < \varepsilon^p$.
Then, we see

$$\begin{aligned}\int |\chi_S - \chi_U|^p &= \int \chi_{S\triangle U}^p \\ &= m(S \triangle U) \\ &< \varepsilon^p.\end{aligned}$$

Since $m(U \setminus S) < \infty$, we see each interval J_k must be bounded (else U would be of infinite measure), so χ_U is a step function on the interval $[a, b] \supseteq U$ satisfying the required conditions.

3. Assuming 2 we see it suffices to show case for the step function $f = \chi_{[c,d]}$ with $c \leq d$. Then, fixing $\varepsilon > 0$ and considering the function

$$x \mapsto g(x) = \chi_{[c,d]} + \left(1 + \varepsilon^{-p}(x - c)\right)\chi_{\left(c - \frac{\varepsilon^p}{3},\right),c} + \left(1 - e^{-p}(x - d)\right)\chi\left(d, d + \frac{\varepsilon^p}{3}\right).$$

We see this functions is continuous as it is simply piecewise linear, being 1 on $[c, d]$ and a linear interpolation between 1 and 0 in a small interval either side of $[c, d]$. Importantly, $\int_{\left(c - \frac{1}{3}\varepsilon^p\right)} |g| \leq \frac{1}{3}\varepsilon^p$, the length of the interval.
Hence, we find

$$\int \left|\chi_{[c,d]} - g\right|^p \leq (\frac{2}{3}\varepsilon^p)^p < \varepsilon^p.$$

This completes the proof.

$\square$

Note that this proof essentially showed simple functions, step functions, and continuous functions are dense in $L^p(\mathbb{R})$ (given $1 \leq p < \infty$ for the last 2).

Definition 1.2 (Density)**.** Let $(X, \|\cdot\|)$ be a normed linear space. If $S \subseteq T \subseteq X$, then S is **dense** in T if for all $v \in T, \varepsilon > 0$ we find a vector $u \in S$ so that $\|v - u\| < \varepsilon$.

Definition 1.3 (Seperability)**.** A normed linear space $(X, \|\cdot\|)$ is **seperable** if it contains a countable, dense subset.

Theorem 1.2. For $1 \leq p < \infty$, $L^p(\mathbb{R})$ is seperable.

Proof. If $\varphi = c\chi_{[a,b]}$ with $a, b, c \in R$, then for any $\varepsilon > 0$ we find an interval $I = [c, d] \subseteq [a, b]$ with $c, d \in \mathbb{Q}$ and an $r \in \mathbb{Q}$ so that $\int |\varphi - r\chi_I|^p < \varepsilon^p$ (the function vanishes except on an arbitrarily small interval). Letting Ψ be the collection of all such step functions of the form $\psi = \sum_{i=1}^{n} c_k \chi_{I_k}$ with $c_k \in \mathbb{Q}$ and I_k having rational endpoints, then linearity combined with the preceding lemmas guarantees Ψ to be a countable dense subset, so $L^p(\mathbb{R})$ is seperable. $\square$

2 Bounded Linear Functionals

Definition 2.1 (Functionals)**.**
- A function $\varphi : X \to \mathbb{R}$ on a linear space X is called a **linear functional** if the laws of linearity holds for φ.
- A linear functional $\varphi : X \to \mathbb{R}$ on a normed linear space $(X, \|\cdot\|)$ is called **bounded** if there is $M \geq 0$ so that $|\varphi(x)| \leq M\|x\|$ for all $x \in X$.
- If φ is a bounded linear functional, the quantity

$$\|\varphi\| = \inf\{M \geq 0 : |\varphi(x)| \leq M\|x\| \ \forall\, x \in X\}$$

is called the **norm** of φ.

Proposition 2.1. Let $\varphi : X \to \mathbb{R}$ be a bounded linear functional on a normed linear space $(X, \|\cdot\|)$. Then,

$$\|\varphi\| = \sup\{|\varphi(x)| : x \in X, \|x\| \leq 1\}.$$

Definition 2.2 (Continuity)**.** A linear functional $\varphi : X \to \mathbb{R}$ on $(X, \|\cdot\|)$ is **continuous at** x_0 if for every $\varepsilon > 0$ we find a $\delta > 0$ so that $|\varphi(x) - \varphi(x_0)| < \varepsilon$ if $\|x - x_0\| < \delta$.
If φ is continuous for all $x \in X$, then φ is **continuous.**

Proposition 2.2. Let $\varphi : X \to \mathbb{R}$ be a linear functional on $(X, \|\cdot\|)$. Then, the following are equivalent

- φ is continuous,
- φ is continuous at some $x_0 \in X$,
- φ is bounded.

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Definition 2.3 (Signum Function)**.** We define the **sign function** to bf

$$\begin{aligned} \operatorname{sgn} \overline{\mathbb{R}} &\longrightarrow \{-1, 0, 1\} \\ \operatorname{sgn}(x) &\longmapsto \operatorname{sgn}(\operatorname{sgn}(x)) = \chi_{(0,\infty]}(x) - \chi_{[-\infty,0)}(x). \end{aligned}$$

Note, if g is measurable, $\operatorname{sgn}(g)$ is measurable.

Remark. If $g : S \to \overline{\mathbb{R}}$ is measurable, then $\operatorname{sgn}(g^*)$ is simple. Moreover, $g \operatorname{sgn}(g) = |g|$.

Theorem 2.1. Let $S \subseteq \mathbb{R}$ be measurable with $1 \leq p \leq \infty$, and q being p's conjugate. For $g \in L^q(S)$, define the map

$$\begin{aligned} \varphi : L^p(S) &\longrightarrow \mathbb{R} \\ f &\longmapsto \varphi(f) = \int_S fg. \end{aligned}$$

Then φ is a bounded linear functional on $L^p(S)$ with norm $\|\varphi\| = \|g\|_q$. In particular $\varphi(f) = 0$ for all $f \in L^p(S)$ if and only if $g = 0$ almost everywhere.