Algebraic Theory I

Thomas Fleming

October 1, 2021

Contents

Lecture 16: Nilpotent Groups (3)

Wed 29 Sep 2021 11:25

Corollary 1. A finite abelian group is the direct product of its sylow groups.

This follows directly from the theorem from last class.

Corollary 2. If G is a finite group such that for all $n \mid |G|$ such that there are at most n elements $x \in G$ with $x^n = 1$, then G is cyclic.

Proof. Let p be an arbitrary prime with $p \mid |G|$. Let P be a sylow p-group with $|P| = p^\alpha$. We know for any $x \in P$, we have $x^{|P|} = 1$, hence there are $|P| = p^\alpha$ elements $x \in P$ such that $x^{p^\alpha} = 1$. By hypothesis there is infact equality. If there was another distinct sylow p-group we would have elements $y \notin P$ such that $y^{p^\alpha} = 1$. Hence, P is unique. Hence, as every p-group is unique, so normal, we see G is the product of its P-groups.
Denote $G = P_1 \times P_2 \times \ldots P_t$ with the P_is being the distinct sylow p_i
$) - groupsofG$. Also, if $|P_1| = p_1^{\alpha_1}$, then all $x \in P_1$ have $\operatorname{ord}(x) \mid p_1^{\alpha_1}$ and there are at most $p_1^{\alpha_1 - 1} < p_1^{\alpha_1}$ such x with $\operatorname{ord}(x) \mid p_1^{\alpha_1 - 1}$. Since $|P| < p_1^{\alpha_1 - 1}$ we see there is an $x \in P_1$ with $\operatorname{ord}(x) = p_1^{\alpha_1} = |P|$, hence $\langle x \rangle = P_1$. So, P_1 is cyclic. Likewise, all other P_i are shown cyclic by the same argument, with $P_i = \langle x_i \rangle$. Then, the element $x = \prod_{i=1}^{t} x_i$ is a generator of G, so G is cyclic. □

Theorem 0.1 (Frattini's Argument)**.** Let G be a finite group , $H \trianglelefteq G$, $P \leq H$ being a sylow p-group in H. Then,

$$G = HN_G(P) \text{ and } |G : H| \mid |N_G(P)|.$$

Proof. Let $g \in G$, we wish to show $g \in HN_G(P)$. We know this to be a subgroup as $H \trianglelefteq G$. Let G act by conjugation on its sets. Now

$$\begin{aligned} P^g &= gPg^{-1} \\ &\leq H^g \\ &= gHg^{-1} \\ &= H \text{ by normality.} \end{aligned}$$

Then, we see as $|P^g| = |P|$, then P^g is another sylow p-group in H. And, as we know all sylow p-groups are conjugate. Hence, there is an $h \in H$ such that $P^h = P^g$. Hence, $P = P^{h^{-1}g}$, hence $h^{-1}g \in N_G(P)$. Then, we see $g \in hN_G(P) \subseteq HN_G(P)$. So, we see $G = HN_G(P)$

Now, we show the other result. Note that by the second isomorphism theorem, we have

$$G/H = (HN_G(P))/H \simeq \frac{N_G(P)}{H \cap N_G(P)}.$$

Thus, $|G : H| = |N_G(P) : H \cap N_G(P)|$. As we know this divides $|N_G(P)|$, hence $|G : H| \mid |N_G(P)|$. ☐

> **Theorem 0.2.** if G is a finite group, then G is nilpotent if and only if every maximal subgroup in G is normal in G.

Lecture 17: Nilpotent Groups (3) and Solvable Groups

Fri 01 Oct 2021 11:28

Recall. We had a theorem that, for a finite group G, implied G was nilpotent if and only if all maximal subgroups are normal.

Proof.

1. ($\Rightarrow$). Let $M < G$ be a maximal subgroup, so $M < N \leq G$ implies $N = G$. Let $N_g(M)$ be the normalizer of $M<$ then $M < G$, hence $M < N_G(P)$ by the earlier characterization of finite nilpotent groups. Hence, $N_G(M) = G$. But $M < N_G(M)$ and M ix maximal, hence $N_G(M)$ if and only if M is normal.

2. ($\Leftarrow$). Assume every maximal subgroup is normal. Note that it suffices to show that all sylow groups are normal in G by the earlier characterization. Let $P \leq G$ be an arbitrary sylow p-group and let $N = N_G(P)$. Let M be a maximal subgroup containing $N_G(P)$. We know such a group exists because if we assume indirectly that P is not normal, this implies $N_G(P) < G$ as every proper subgroup of a finite group is contained in a maximal subgroup.
   We now have $P \leq N_G(P) \leq M < G$ and by hypothesis, we know $M \trianglelefteq G$. Since $P \leq M$with P being a sylow group of G implies $P \leq M$ is a sylow group for M. But now we can applying the frattini argument. We see $G = N_G(P)M$ but $N_G(P) \leq M$, hence $G \subseteq MM = M < G$. $\lightning$.

☐

Remark. If G is nilpotent, then recall $Z_0(G) < Z_1(G) < Z_2(G) < \ldots < Z_i(G)$ is the upper central series where $Z_0(G) = \{1\}$, $Z_1(G) = Z(G)$ and $Z_i(G)/Z_{i-1}(G) = Z(G/Z_{i-1}(G))$.

There is an alternative characterization, let $G^0 = G$, $G^1 = [G, G] = \left\langle x^{-1}y^{-1}xy : x, y \in G \right\rangle$ and define recursively $G^i = \left[G, G^{i-1}\right] = \left\langle x^{-1}y^{-1}xy : x \in G, y \in G^{i-1} \right\rangle$ to be the lower central series. Then, G is nilpotent if and only if there is $c \geq 0$ such that $G^c = \{1\}$. Furthermore, we find $G^{c-i} \leq Z_i\left(G\right)$ for all $0 \leq i \leq c$, with the minimal constant c being the same in the upper and lower central series.

Definition 0.1 (Solvable Groups)**.** A group G is **solvable** if there's a chain of subgroups

$$H_0 \lhd H_1 \lhd \ldots \lhd H_n = G$$

such that H_i/H_{i-1} are abelian for $1 \leq i \leq n$.

As it turns out there is an equivalent chain condition for solvability closed to our characterizations of nilpotence. Define $G^{(0)} = G$, $G^{(1)} = [G, G] = G^1$, Now, define $G^{(i)} = \left[G^{(i-1)}, G^{(i-1)}\right] = \left\langle x^{-1}y^{-1}xy : x, y \in G^{(i-1)} \right\rangle$. So, $G^{(n)}$ is essentially the n-th iterated commutator of G. Then, we obtain a chain

$$G^{(0)} \geq G^{(1)} \geq \ldots \geq G^{(c)} \geq \ldots.$$

If $G^{(c)} = 1$ for some $c \geq 1$, then G is solvable. We show these two conditions are equivalent. The proof will involve multiple invocations of the basic result that G/H is abelian if and only if $[G, G] \leq H$.

Proof. Assume G is solvable, and the 1st characterization is true with $1 = H_0 \trianglelefteq H_1 \trianglelefteq \ldots \trianglelefteq H_n = G$ with H_i/H_{i-1} being abelian for all $1 \leq i \leq n$. We will show by induction that $G^{(i)} \leq H_{n-i}$ for all $1 \leq i \leq n$. For $i = 0$ we have $H_n = G$, hence $G^{(0)=G}$ and $G \leq G$, so the claim holds for $i = 0$. Now, note that

$$\begin{aligned} G^{(i)} &= \left[G^{(i-1)}, G^{(i-1)}\right] \\ &\leq \left[H_{n-(i-1)}, H_{n-(i-1)}\right] \text{ by inductive hypothesis} \\ &= \left[H_{n-i+1}, H_{n-i+1}\right] \\ &. \end{aligned}$$

We also know that H_{n-i+1}/H_{n-i} is abelian, hence we have $G^{(i)} \leq [H_{n-i+1}, H_{n-i+1}] \leq H_{n-i}$ by the preceding lemma. This completes the induction. But, we have $G^{(n)} \leq H_{n-n} = H_0 = \{1\}$, so $G^{(n)}$ is trivial. $\square$