

Combinatorics

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October 4, 2021

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Lecture 17: Semi-circle Law

Fri 01 Oct 2021 10:20

Recall that for eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ we have $\lambda_1 = \frac{n}{2} + \sqrt{n \log(n)} = o(n)$. Additionally, we know $\sigma_1 = \lambda_1$ and $\sigma_2, \sigma_3, \dots, \sigma_n$ correspond to $|\lambda_2|, |\lambda_3|, \dots, |\lambda_n|$. Further, it is known by Furedi and Kowlos that $\sigma_2 = O(\sqrt{n})$.

Theorem 0.1. For a randomly chosen graph of order n , with eigenvalues $\lambda_2 \geq \lambda_3 \geq \dots \geq \lambda_n$. Define $W_n(x) : \mathbb{R} \rightarrow \mathbb{Z}^+$ to be the number of eigenvalues λ_i , such that $\frac{\lambda_i}{\sqrt{n}} \leq x$, divided by n . Then, we find the function which

$W_n(x)$ tends to pointwise, $W(x)$ has $W(x) = \begin{cases} \frac{2}{\pi} \sqrt{1-x^2}, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases}$

Here recall that $\sqrt{1-x^2}$ is an upper half semicircle of radius 1 and the factor $\frac{2}{\pi}$ compresses it into an ellipse. This fact essentially characterizes the distribution of eigenvalues of a random graph. That is, plurality of eigenvalues will be 0 and we find the number of eigenvalues of a given magnitude decreases as $\lambda \rightarrow \sqrt{n}$. We note that the leading $\frac{2}{\pi}$ is to normalize the area such that this is a probability density function. Then, we note $E[x^2 W(x)] = \int_{-1}^1 \frac{2}{\pi} x^2 \sqrt{1-x^2} dx = \frac{1}{4}$. Hence, we find $\frac{1}{n^2} \sum_{i=2}^n \lambda_i^2 \approx \frac{1}{4}$.

It is a well known result that $\sum_{i=1}^n |\lambda_i| = \sum_{i=1}^{\infty} \sigma_i \leq \frac{1}{2} n^{\frac{3}{2}} \leq 2(n-1)$. Applying our integral formula from earlier yields $\sum_{i=1}^{\infty} |\lambda_i| = \int_{-1}^1 |x| \sqrt{1-x^2} dx = 2 \int_0^1 x \sqrt{1-x^2} dx$.

At this point, Runze found a contradiction in the argument and we ended class early.

Lecture 18: Semi-Circle Law Corrections and Quasi-Random Graphs

Mon 04 Oct 2021 10:21

Let G be a random graph of order n and denote $N(x)$ to be the number of eigenvalues λ such that $\frac{\lambda}{\sqrt{n}} \leq x$. $N(x) = \frac{1}{n} W_n(x)$. Then, we find the sequence

of functions approaches

$$W(x) = \begin{cases} 0, & x \leq -1 \\ \frac{2}{\pi} \int_{-1}^x \sqrt{1-x^2} dx, & -1 < x < 1 \\ 1, & x \geq 1 \end{cases}.$$

Furthermore, we even find W to be continuous in the whole real line and $W_h(x)$ converges to $W(x)$ earlier.

1 Quasi-Random Graphs