Combinatorics

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Lecture 24

Fri 22 Oct 2021 10:22

I originally missed this class.

Lecture 23: Quasi-Random Graphs (6)

Mon 18 Oct 2021 10:21

We prove the preservation of Regularity and Quasi-Randomness and provide a counterexample for SRG from last time.

Proof. First, we prove regularity. If G is k-regular, then we see all rowsums are k. Hence, we find all row sums of G' to $2k$, so G' is $2k$-regular.
For quasi-randomness, denote our adjacency matrix of G' to be $B = J_2 \otimes A$ and recall the eigenvalues of this product are simply the products of the eigenvalues of the factors. Hence, our eigenvalues are $2\lambda_1, 2\lambda_2, \ldots, 2\lambda_n, 0, \ldots, 0$. Furthermore, as G is quasi-random, we have that $\lambda_1 = \frac{1}{2}n + o(n)$ and $|\lambda_i| = o(n)$ for $n \geq 2$. Applying this yields $2\lambda_1 = n + o(n)$ and $|2\lambda_i| = o(n)$, $i \geq 2$. Hence, G' is quasirandom.

□

Remark. In general $J_i \otimes A$ preserves regularity and quasi-randomness of A by the same argument.

Proposition 0.1. If G, H are quasi-random graphs with adjacency matrices A, B we have $A \otimes B$ induces a quasi-random graph.

Proof. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of G and $\mu_1, \ldots, \mu_n$ to be the eigenvalues of H. Then, the eigenvalues of $A \otimes B$ would have eigenvalues $\lambda_i \mu_j$ and we see $\lambda_1 \mu_1$ is the largest eigenvalue. For the second largest (in magnitude) eigenvalues, we see there are four potential candidates, $\lambda_1 \mu_2$, λ_1, μ_n, $\mu_1 \lambda_2$, $\mu_1 \lambda_n$. Then, we know $\lambda_1 \leq n - 1$ and $\mu_2 = o(m)$, hence $|\lambda_1 \mu_2| = o(nm)$. Similair constructions follow for the other candidates to prove that $G \otimes H$ is infact quasi-random. □

Proposition 0.2. Let A_{ij}, $1 \leq i, j \leq k$ be the adjacency matrices of quasi-random graphs of order n and $e\left(A_{ij}\right) = \frac{1}{4}n^2 + o\left(n^2\right)$ with $A_{ij} = A_{ji}$. We arrange these matrices in a $kn \times kn$ matrix

$$B = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1k} \\ A_{21} & A_{22} & \dots & A_{2k} \\ \vdots & \vdots & \vdots & \vdots \\ A_{k1} & A_{k2} & \dots & A_{kk} \end{bmatrix}.$$

Then, we find the graph induced by B to be quasi-random.

Definition 0.1 (Bipartite Quasi-Random Graph)**.** A bipartite graph, $G\left(A, B\right)$ with $|A| = |B|$ and density p , is **Bipartite Quasi-Random** if it obeys one of the following (equivalent) tweaked quasi-random properties

- (P_2). $e\left(G\right) \geq pn^2 + o\left(n^2\right)$ and $\#CW_4 \leq p^4n^4 + o\left(n^4\right)$.
- (P_3). $e\left(G\right) \geq pn^2 + o\left(n^2\right)$ and $\lambda_1 = pn + o\left(n\right)$ and $\lambda_2 = o\left(n\right)$.
- (P_4). For all $X \subseteq A$, $Y \subseteq B$, we find $\left|e\left(X, Y\right) - p\left|X\right|\left|Y\right|\right| \leq o\left(n^2\right)$.

Recall. G is bipartite on two sets of size k if and only if the eigenvalues of G are $\lambda_1, \lambda_2, \dots, \lambda_k, -\lambda_k, -\lambda_{k-1}, \dots, -\lambda_1$.

Definition 0.2 (Bipartite Double)**.** We define the **Bipartite Double** of a graph G with adjacency matrix A to be the graph induced by

$$B = \begin{bmatrix} 0_{n\times n} & A \\ A & 0_{n\times n} \end{bmatrix}.$$

Essnetially, this splits G into two graphs G, G' such that a vertex $x \in G$ is connected to all of its neighbors, but in G' and similairly, a $x' \in G'$ will be connected to all of its neighbors, but in G. Hence, this induces a bipartite graph yielding some interesting properties.

Example. If G is regular, we find the bipartite double of G to be regular.
Furthermore, the bipartite double of C_3 is C_6.
Similarly, the bipartite double of K_3 is $K_{3,3}$.
The bipartite double of a graph which is already bipartite is simply 2 independent of the original graph.
For example, the double of $K_{2,2}$ is $2K_{2,2}$. ◇

Using the bipartite double we can construct new bipartite quasi-random graphs.

Proposition 0.3. If G is quasi-random and A is its adjacency matrix, then the bipartite double induced by

$$\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}$$

is bipartite quasi-random.

Problem. Prove that P_3 (for a general quasi-random graph) implies the existence of a subgraph isomorphic to C_k with $k \geq n + o\,(n)$.