Combinatorics

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Contents

Lecture 16

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First, we examine some more random graphs. For a random graph G, it is a trivial rseult of probability theory that the number of four cycles is precisely $\frac{1}{2}\sum_{u,v\in V(G);v\neq u}\binom{\hat{d}(u,v)}{2}$. Then, applying our estimation $\hat{d}(u,v) = \frac{n}{4} + o(u)$ yields $\binom{n}{2}$ possible pairs u, v and $\hat{d} \approx \frac{n}{4}$, hence the number of four cycles is

$$\frac{1}{n}\binom{\frac{n}{4}}{2}\binom{n}{2} = \frac{n^4}{128} + o\left(n^4\right).$$

Now, we examine the k-walks.

Definition 0.1 (Walks)**.** A k**-walk** is a k-path $v_1, v_2, v_3, \ldots, v_k$.
A **closed** k**-walk** is a k-cycle, $v_1, v_2, \ldots, v_k, v_1$.

Remark. Walks need not have all vertices distinct, hence a graph of order 2 where one simply oscillates between the vertices to produce a degenerate $2n$-walk. Similairly, one can traverse a triangle to induce a 4-walk as well. Overall this yields 14 possible 4-walks on a graph of order 4.

Now, we examine the number of closed 4-walks on a random graph of order n. We see nondegenerate 4-walks are just 4-cycles of which we know there to be $\frac{n^4}{128}$ with 8 possible permutations of directions and starting point yields $8 \cdot \frac{n^4}{128}$. Similairly, we note that $4 \cdot \sum_{v\in V}\binom{d_i(v)}{2} = 4n\binom{n}{2} = \frac{1}{2}n^3 + o\left(n^3\right) = o\left(n^4\right)$ degenerate graphs on 3 vertices exist. Lastly, the number of degenerate graphs on 2-vertices is clearly, $2 \cdot e(g) = o\left(n^4\right)$. Hence, the number of $4 - walks$ is just $\frac{n^4}{16} + o\left(n^4\right)$.

Proposition 0.1. $\operatorname{tr}\left(A(G)^k\right) = \sum_{i=1}^{n}\lambda_i^k$ is the number of closed k-walks in a graph G of order n.

From this, we arrive at $6k_3(G) = \operatorname{tr}(A^3) = \sum_{i=1}^{3} \lambda_i^3$.
We also see the number of closed walks of order 4 is

$$CW_4 = \sum_{i=1}^{n} \lambda_i^4$$
$$\frac{n^4}{16} + o(n^4) = \lambda_1^4 + \sum_{i=2}^{n} \lambda_i^4$$
$$\Rightarrow \sum_{i=2}^{n} \lambda_i^4 = o(n^4).$$

Similarly, we find $\sigma_2(G) = o(n)$ and $O(\sqrt{n})$.

Definition 0.2 (Local Density)**.** The **local density** of a graph is simply $e(U)$ for some graph $U \subseteq V$.

Remark. Local density is highly variable. For instance in $K_{n,n}$ we find U being one of the partite sets yields 0 local density and U being a set of half the vertices in each partite set yields $\frac{1}{4}e(G)$ local density.

Proposition 0.2. Suppose G is a random graph of order n and let U be a set with $|U| > 502\log(n)$. Then, $\left|e(U) - \frac{1}{2}\binom{|U|}{2}\right| < \binom{|U|}{2}\left(\frac{3.5\log n}{|U|}\right)^{\frac{1}{2}}$.

Proposition 0.3. There exists a function $f : \mathbb{N} \to \mathbb{N}$ such that almost every graph of order n as clique number $f(n)$ or $f(n+1)$.

This function is approximated by

$$f(n) \approx 2\log_2(n).$$

Remark. There is clearly also such a function for the independence number.

Furthermore, more investigation yields $\chi(G) \approx \frac{n}{2\log_2(n)}$ for almost all graphs G.

Lecture 17: Semi-circle Law

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Recall that for eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ we have $\lambda_1 = \frac{n}{2} + \sqrt{n\log(n)} = o(n)$. Additionally, we know $\sigma_1 = \lambda_1$ and $\sigma_2, \sigma_3, \ldots, \sigma_n$ correspond to $|\lambda_2|, |\lambda_3|, \ldots, |\lambda_n|$. Further, it is known by Furedi and Kowlos that $\sigma_2 = O(\sqrt{n})$.

> **Theorem 0.1.** For a randomly chosen graph of order n, with eigenvalues $\lambda_2 \geq \lambda_3 \geq \ldots \geq \lambda_n$.Define $W_n(x) : \mathbb{R} \to \mathbb{Z}^+$ to be the number of eigenvalues λ_i, such that $\frac{\lambda_i}{\sqrt{n} \leq x}$, divided by n. Then, we find the function which $W_n(x)$ tends to pointwise, $W(x)$ has $W(x) = \begin{cases} \frac{2}{\pi}\sqrt{1-x^2}, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases}$

Here recall that $\sqrt{1-x^2}$ is an upper half semicircle of radius 1 and the factor $\frac{2}{\pi}$ compresses it into an ellipse. This fact essentially characterizes the distribution of eigenvalues of a random graph. That is, plurality of eigenvalues will be 0 and we find the number of eigenvalues of a given magnitude decreases as $\lambda \to \sqrt{n}$. We note that the leading $\frac{2}{\pi}$ is to normalize the area such that this is a probability density function. Then, we note $E\left[x^2 W(x)\right] = \int_{-1}^{1} \frac{2}{\pi} x^2 \sqrt{1-x^2} dx = \frac{1}{4}$. Hence, we find $\frac{1}{n^2} \sum_{i=2}^{n} \lambda_i^2 \approx \frac{1}{4}$.
It is a well known result that $\sum_{i=1}^{n} |\lambda_i| = \sum_{i=1}^{\infty} \sigma_i \leq \frac{1}{2} n^{\frac{3}{2}} \leq 2(n-1)$. Applying our integral formula from earlier yields $\sum_{i=1}^{\infty} |\lambda_i| = \int_{-1}^{1} |x| \sqrt{1-x^2} = 2\int_0^1 x\sqrt{1-x^2}$.