Analysis I

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Contents

Lecture 10: Measure Theory (2) and Lebesque Measure

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Definition 0.1. A set $S \subseteq R$ is **measureable/Lebesque mesaurable** if for every $A \subseteq \mathbb{R}$,

$$\mu^* (A) = \mu^* (A \cap S) + \mu^* (A \cap S^c).$$

It actually suffices to show only

$$\mu^* (A) \geq \mu^* (A \cap S) + \mu^* (A \cap S^c).$$

Proposition 0.1. Every set $S \subseteq \mathbb{R}$ with $\mu^* (S) = 0$ is measurable.

Proof. For every $A \subseteq \mathbb{R}$, $\mu^* (A \cap S) \leq \mu^* (S) = 0$. Similairly, $\mu^* (A \cap S^c) = 0$. $\square$

Definition 0.2. A set $S \subseteq \mathbb{R}$ with $\mu^* (S) = 0$ is said to have measure 0.

Lemma 0.1. For each $a \in \mathbb{R}$, (a, ∞) is measurable.

. Given $A \subseteq \mathbb{R}$ and $\varepsilon > 0$, we fine $\{I_n : n \in N\} \in J(A)$ such that

$$\mu^* (A) \geq \sum_{n=1}^{\infty} \ell (I_n) - \varepsilon.$$

Since $A \cap (a, \infty) \subseteq \bigcup_{n \in \mathbb{N}} (I_n \cap (a, \infty))$ and

$$A \cap (a, \infty)^c \subseteq \left(\bigcup_{n \in \mathbb{N}} (I_n \cap (-\infty, a)) \right) \cup (a - \varepsilon, a + \varepsilon).$$

It follows that $\mu^* (A \cap (a, \infty)) \leq \sum_{n=1}^{\infty} \ell (I_n \cap (a, \infty))$ and $\mu^* (A \cap (a, \infty)^c) \leq \sum_{n=1}^{\infty} \ell (I_n \cap (-\infty, a)) + 2\varepsilon$. As $\ell (I_n) = \ell (I_n \cap (a, \infty)) + \ell (I_n \cap (-\infty, a))$ as the singular point a will not change the length. Hence,

$$\begin{aligned}\mu^*(A) &\geq \sum_{n=1}^{\infty} \ell(I_n \cap (a,\infty)) + \sum_{n=1}^{\infty} \ell(I_n \cap (-\infty), a) - \varepsilon \\ &\geq \mu^*(A \cap (a,\infty)) + \mu^*(A \cap (a,\infty)^c) - 3\varepsilon.\end{aligned}$$

□

Proposition 0.2. The collection of Lebesque measurable sets in $\mathbb{R}$ is a σ-algebra $\mathscr{L}$ containing all Borel sets.

Proof. If the measurable sets form of σ-algebra $\mathscr{L}$, then $\mathscr{L}$ must contain all open and closed subsets of $\mathbb{R}$, since it contains all intervals of the form (a,∞). To show that the measurable sets form a σ-algebra $\mathscr{L}$ we first note that $(a,a) = \varnothing$ and the complement of each measurable set are both measurable sets. This is due to the symmetry in the definition of measurbility

$$\mu^*(A) \geq \mu^*(A \cap S) + \mu^*(A \cap S^c).$$

Now, suppose $\{S_n : n \in \cap\}$ is a countable collection of measurable sets. Let $S = \bigcup_{n\in\mathbb{N}} S_n$, then we need only show S is measurable.
Given $A \subseteq \mathbb{R}$, we define a sequence with $A_1 = A$, $A_{n+1} = A \cap (\bigcap_{k=1}^{n} S_k^c)$. Hence, $A_2 = A \cap S_1^c$, $A_3 = A \cap (S_1^c \cap S_2^c)$. Now, note that $A_{n+1} = A_n \cap S_n^c$, hence the sequence is decreasing in size. And $A \cap S = \bigcup_{k\in\mathbb{N}} (A_k \cap S_k)$. We present a short proof of this claim.
Note that for $x \in A \cap S$, there is a smallest positive integer k such that $x \in S_k$. If $k = 1$, then $x \in A_1 \cap S_1$, if $k > 1$, then $x \notin S_n$ for any $n < k$, consequently $x \in A_k$ by construction. Hence, $x \in A_k \cap S_k$, so $A \cap S \subseteq \bigcup_{k\in\mathbb{N}} (A_k \cap S_k)$.
Now, $\bigcup_{k\in\mathbb{N}} (A_k \cap S_k) \subseteq A \cap S$, as each $A_k \in A$ and $S_k \in S$, hence their intersection and subsequent union are also contained. Hence the equality is shown

$$A \cap S = \bigcup_{k\in\mathbb{N}} (A_k \cap S_k).$$

By measurability of S_n, we know any set A has

$$\mu^*(A_n) = \mu^*(A_n \cap S_n) + \underbrace{\mu^*(A_n \cap S_n^c)}_{A_{n+1}}$$

.
Hence, by induction, we have $\mu^*(A) = \mu^*(A_1) = \sum_{k=1}^{n} \mu^*(A_k \cap S_k) + \mu^*(A_{n+1})$. Since $A \cap \left(\bigcap_{k\in\mathbb{N}} S_k^c\right) = A \cap S^c \subseteq A_{n+1}$ for any n.
Hence,

$$\mu^*(A) \geq \sum_{k=1}^{n} \mu^*(A_i \cap S_i) + \mu^*(A \cap S^c).$$

Finally, as $\bigcup_{k\in\mathbb{N}} (A_k \cap S_k) = A \cap S$ and since μ^* is contably subadditive, we obtain

$$\begin{aligned}
\mu^*(A) &\geq \sum_{k=1}^{\infty} \mu^*(A_k \cap S_k) + \mu^*(A \cap S^c) \\
&\geq \mu^*\left(\bigcup_{k\in\mathbb{N}} (A_k \cap S_k)\right) + \mu^*(A \cap S^c) \\
&= \mu^*(A \cap S) + \mu^*(A \cap S^c).
\end{aligned}$$

□

Definition 0.3 (Lebesque Measure)**.** The **Lebseque Measure** of a measurable set $S \subseteq \mathbb{R}$, denoted by $\mu^*(S)$ is defined by $\mu(S) = \mu^*(S)$. The set function $\mu : \mathscr{L} \to [0, \infty]$ is called the **Lebesque Measure**.

Theorem 0.1. The Lebesque measure μ is a measure on $\mathscr{L}$ such that

- $\mu(I) = \ell(I)$ for every interval $I \subseteq \mathbb{R}$.
- μ is translation invariant.
- μ is countablly additive.

Proof.

1. As μ^* has the interval property, μ trivially inherits this,
2. Similarly, as μ^* was translationaly invariant, we see μ inherits this.
3. Let $\{S_k : k \in \mathbb{N}\}$ be a countable, disjoint collection of measurable sets and define $T_n = \bigcup_{k=n}^{\infty} S_k$ for $n \in \mathbb{N}$.
   Since, $T_{n+1} = T_n \cap S_n^c$ we have

   $$\mu(T_n) = \mu(T_n \cap S_n) + \mu\left(\underbrace{T_n + S_n^c}_{=T_{n+1}}\right)$$

   by measurability of S_n.

   Consequently, $\mu(T_1) = \sum_{k=1}^{n} \mu\left(\underbrace{T_k \cap S_k}_{=S_n}\right) + \mu(T_{n+1}) \geq \sum_{k=1}^{n} \mu(S_k)$ for every $n \in \mathbb{N}$. Thus $T_1 = \bigcup_{k\in\mathbb{N}} S_k$ gives $\mu\left(\bigcup_{k\in\mathbb{N}} S_k\right) \geq \sum_{k=1}^{n} \mu(S_k)$. And as we already know the inequality goes in the other direction by subadditivity of μ^*, we see equality holds.

□

Corollary 1. Every countable set of real numbers is measurable with measure 0.

Proof. Let C be our countable sets and note that $C = \bigcup_{k \in \mathbb{N}} \{x_k\}$ with $x_k \neq x_m$ for $k \neq m$. Then, we see that

$$\mu \left(\bigcup_{k \in \mathbb{N}} \{x_k\} \right) = \sum_{k=1}^{\infty} \mu \left(\{x_k\} \right) = 0.$$

□

Theorem 0.2 (Properties of Lebesque Measure)**.** Let $S \subseteq \mathbb{R}$, the following are equivalent

1. S is measurable.
2. For each $\varepsilon > 0$, there is an open set O and a closed set C such that $C \subseteq S \subseteq O$ and $\mu (O \setminus C) < \varepsilon$.
3. There is a G_δ set G and a F_σ set F such that $F \subseteq S \subseteq G$ and $\mu (G \setminus F) = 0$.
4. For each $\varepsilon > 0$, there are measurable sets G and F such that $F \subseteq S \subseteq G$ and $\mu (G \setminus F) < \varepsilon$.

We will prove this result next time, though it is completely trivial that $3 \Rightarrow 4$, so we will primarily focus on proving $1 \Rightarrow 2$ and $4 \Rightarrow 1$.

Lecture 11: Measure Theory (3)

Tue 28 Sep 2021 13:00

We prove the final theorem from last lecture.

Proof.

- ($1 \Rightarrow 2$). There are 2 cases, S being bounded and S being unbounded.

  If S is bounded, there is an interval $(a, b) \supseteq S$, $a, b \in \mathbb{R}$. Then for any given $\varepsilon > 0$, we find $\{I_k : k \in \mathbb{N}\} \in J(S)$ and $\{J_k : k \in \mathbb{N}\} \in J([a, b] \setminus S)$ such that $\mu(s) \geq \sum_{k=1}^{\infty} \ell(I_k) - \frac{\varepsilon}{3}$ and $\mu([a, b] \setminus S) \geq \sum_{k=1}^{\infty} \ell(J_k) - \frac{\varepsilon}{3}$. Let $O = \bigcup_{k \in \mathbb{N}} I_k$, $U = \bigcup_{k \in \mathbb{N}} J_k$ and $C = [a, b] \setminus U$. Then, $C \subseteq S \subseteq O$. Note that O, U are open and C is closed. Then,

  $$\begin{aligned} \mu(S) &\geq \mu(O) - \frac{\varepsilon}{3} \\ \mu([a, b] \setminus S) &\geq \mu(U) - \frac{\varepsilon}{3}.. \end{aligned}$$

  Furthermore, U, C are disjoint and $\mu(U) < \infty$ (as it is an interval minus a measurable set) and $[a, b] \subseteq U \cup C$. Hence,

  $$\begin{aligned} \mu(C) &\geq \mu([a, b]) \setminus \mu(U) \\ &= b - a - \mu(U). \end{aligned}$$

Then, since $\mu(C) \leq \mu(S) < \infty$, we have

$$\begin{aligned}\mu(O \setminus C) &= \mu(O) - \mu(C) \\ &\leq \frac{\varepsilon}{3} + \underbrace{\mu(S) - (b-a)}_{=-\mu([a,b\setminus S])} + \mu(U) \\ &= \frac{\varepsilon}{3} - \mu([a,b] \setminus S) + \mu(U) \\ &\leq \frac{2\varepsilon}{3} \\ &< \varepsilon.\end{aligned}$$

For a general S, let $S_n = S \cap [n, n+1]$, $n \in \mathbb{Z}$. Then, there are open O_n and closed C_n such that $C_n \subseteq S_n \subseteq O_n$ and $\mu(O_n \setminus C_n) < \frac{\varepsilon}{3 \cdot 2^{|n|}}$.
Let $O = \bigcup_{n\in\mathbb{Z}} O_n$ and $C = \bigcap_{n\in\mathbb{Z}} C_n$. Then, O is open and C is closed by definition and we see $O \setminus C = \bigcup_{n\in\mathbb{Z}} (O_n \setminus C_n)$ by demorgen and we have $C \subseteq S \subseteq O$. Then,

$$\begin{aligned}\mu(O \setminus C) &\leq \sum_{n\in\mathbb{Z}} \mu(O_n \setminus C_n) \\ &< \sum_{n\in\mathbb{Z}} \frac{\varepsilon}{3 \cdot 2^{|n|}} \\ &= \varepsilon \text{ by geometric summation.}\end{aligned}$$

- $(2 \Rightarrow 3)$. For each $n \in \mathbb{N}$, there are closed C_n and open O_n such that $C_n \subseteq S \subseteq O_n$ and $\mu(O_n \setminus C_n) < \frac{1}{n}$. Let $F = \bigcup_{n\in\mathbb{N}} C_n$ and $G = \bigcap_{O_n}$. Then, F is a F_σ set and G is a G_δ set. Then, we have $F \subseteq S \subseteq G$ and $\mu(G \setminus F) \leq \mu(O_n \; C_n) < \frac{1}{n}$ for all $n \in \mathbb{N}$. Hence, $\mu(G \setminus F) = 0$.
- $(3 \Rightarrow 4)$. This is immediately obvious as F_σ and G_δ sets are measurable.
- $(4 \Rightarrow 1)$. Let $A \subseteq \mathbb{R}$ and $\varepsilon > 0$. Then $S^c \subseteq G \cup (G \cap F^c)$. Then, $A \cap S^c \subseteq (A \cap G^c) \cup (G \cap F^c)$.
  Hence,

$$\begin{aligned}\mu^*(A \cap S^c) &\leq \mu^*(A \cap G^c) + \underbrace{\mu^*(G \cap F^c)}_{<\varepsilon} \\ &\leq \mu^*(A \cap G^c) + \varepsilon.\end{aligned}$$

And, as G is measurable, we have

$$\mu^*(A) = \mu^*(A \cap G) + \mu^*(A \cap G^c) \geq \mu^*(A \cap S) + \mu^*(A \cap S^c) - \varepsilon$$

. Hence, in the infimum we have

$$\mu^*(A) \geq \mu^*(A \cap S) + \mu^*(A \cap S^c).$$

So, S is measurable.

□

Definition 0.4 (Nested Sets)**.** A countable collection of sets $\{S_k : k \in \mathbb{N}\}$ is called

1. **ascending** if $S_k \subseteq S_{k+1}$ for all k.
2. **descending** if $S_{k+1} \subseteq S_k$ for all k.

Lemma 0.2.

1. If $\{S_k : k \in \mathbb{N}\}$ is an ascending collection of measurable sets, then $\mu\left(\bigcup_{k\in\mathbb{N}} S_k\right) = \lim_{k\to\infty} \mu(S_k)$.
2. If $\{S_k : k \in \mathbb{N}\}$ is a descending collection of measurable sets and $\mu(S_1) < \infty$. Then, $\mu\left(\bigcap_{k\in\mathbb{N}} S_k\right) = \lim_{k\to\infty} \mu(S_k)$.

Proof.

1. It suffices to consider the case $\mu(S_k) < \infty$ for all k, else the union and limit both trivially have measure ∞. Define $S_0 = \varnothing$, $X_n = S_n \setminus S_{n-1}$. Then, $\{X_k : k \in \mathbb{N}\}$ is a disjoint collection of measurable sets such that $\bigcup_{k\in\mathbb{N}} X_k = \bigcup_{k\in\mathbb{N}} S_k$. Hence, as we know the lebesque measure to be countably additive, we have

$$\begin{aligned}
\mu\left(\bigcup_{k\in\mathbb{N}} X_k\right) &= \sum_{k=1}^{\infty} \mu(X_k) \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \mu(X_k) \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \left(\mu(S_k) - \mu(S_{k-1})\right) \\
&= \lim_{k\to\infty} \mu(S_k).
\end{aligned}$$

2. Let $X_n = S_1\ S_n$. Then, $\{X_k : k \in \mathbb{N}\}$ is an ascending collection of measurable sets such that $\bigcup_{k\in\mathbb{N}} X_k = S_1 \setminus \left(\bigcap_{k\in\mathbb{N}} S_k\right)$. Since $S_k \subseteq S_1$ and $\mu(S_1) < \infty$ we have by the first lemma that

$$\begin{aligned}
\mu(S_1) - \mu\left(\bigcap_{k\in\mathbb{N}} S_k\right) &= \mu\left(\bigcup_{k\in\mathbb{N}} X_k\right) \\
&= \lim_{k\to\infty} \mu(X_k) \\
&= \mu(S_1) - \lim_{k\to\infty} \mu(S_k).
\end{aligned}$$

As $\mu(S_1)$ is finite we know this to be well defined, hence

$$\mu\left(\bigcap_{k\in\mathbb{N}} S_k\right) = \lim_{k\to\infty} \mu(S_k).$$

□

Theorem 0.3 (Borel-Cantelli Lemma)**.** Suppose $\{S_k : k \in \mathbb{N}\}$ is a countable collection of measurable sets such that $\sum_{k=1}^{\infty} \mu(S_k) < \infty$. Then, the set of all $x \in \mathbb{R}$ which belong to an infinite subcollection of $\{S_k : k \in \mathbb{N}\}$ has measure 0.

Proof. Note that x belongs to an infinite subcollection of $\{S_k : k \in \mathbb{N}\}$ if and only if $x \in \bigcap_{k\in\mathbb{N}} \bigcup_{n=k}^{\infty} S_n$.
Then, the collection $\{\bigcup_{n=k}^{\infty} S_n : k \in \mathbb{N}\}$ is descending and $\mu\left(\bigcup_{n\in\mathbb{N}} S_n\right) \leq \sum_{n=1}^{\infty} \mu(S_n) < \infty$. Hence, by the preceding lemma, we have

$$\begin{aligned}\mu\left(\bigcap_{k\in\mathbb{N}} \bigcup_{n=k}^{\infty} S_n\right) &= \lim_{k\to\infty} \mu\left(\bigcup_{n=k}^{\infty} S_n\right)\\ &\leq \lim_{k\to\infty} \sum_{n=k}^{\infty} \mu(S_n)\\ &= 0\end{aligned}$$

.

This final equality is because for all $\varepsilon > 0$ there is a $K \in \mathbb{N}$ such that for $k \geq K$ we have

$$\left|\sum_{i=1}^{\infty} \mu(S_i) - \sum_{i=1}^{k-1} \mu(S_i)\right| < \varepsilon.$$

□

Problem.

1. Is every set measurable?
2. Is every set of measure 0 countable?
3. Is every measurable set Borel?