# MATH 7237/8237

## LECTURE Feb. 10, 2021

# WEIGHTED GRAPHS

In many applications of graphs we need to assign numbers to vertices or edges.

For example if $G$ is a graph whose vertices are cities and the edges represent roads connecting them, it is natural to assign to each edge the length of the corresponding road.

To tackle such situations let us define **weighted graphs:**

**Definition** A graph $G$ is called a **weighted graph** if some real function $w : E(G) \to \mathbb{R}$ is given explicitly or is understood.

For simplicity, for any edge $uv \in E(G)$, we write $w_{uv}$ for $w(uv)$ and call $w_{uv}$ the **weight** of the edge $uv$.

The function $w$ is called the **weight function** of $G$.

# A TOY PROBLEM

Let us consider the following practical problem.

- Some 20 villages want to build a common water supply pipeline of minimum cost. For every two villages it is known what would be the cost of a pipeline joining them.

- This problem can be translated as a problem about weighted graphs.

- Indeed let us represent each village as a vertex of the complete graph $K_{20}$.

- Now, to any edge of $K_{20}$ assign weight equal to the cost of the pipeline between the villages corresponding to the vertices of the edge.

The purpose is to find a **connected spanning subgraph** $G$ of $K_{20}$ such that the sum of its edge weights is minimum.

Note that the graph $G$ should be connected, as the supply line is common to all villages, and is spanning because all villages have access to it. Once $G$ is found, it specifies which villages have to be joined by pipelines so that the common supply has minimum cost.

Moreover, any connected spanning subgraph of $K_{20}$ determines a valid supply pipeline, although it may not have minimum cost.

Given that edge weights are positive, it is clear that $G$ must be a tree, as otherwise we may omit an edge and the remaining graph will still be connected and spanning, but the sum of its edge weights would be smaller.

# MINIMUM SPANNING TREES

We thus arrive at the following purely graph-theoretic problem:

**Problem** *Given a connected weighted graph $G$, find a spanning tree of $G$ with minimum sum of edge weights.*

This problems can be applied to many different practical situations and is one of the few problems in graph theory that is completely solved.

Since the number of spanning trees of $G$ is finite, there is one or maybe several of them with minimum sum of edge weights. Hence, the problem is how to find a solution most efficiently.

For simplicity, a spanning tree with minimum sum of edge weights will be called a **minimum spanning tree**.

# KRUSKAL'S ALGORITHM

One of the several known algorithms for finding a minimum spanning tree is the algorithm of Kruskal:

Let $G$ be a connected weighted graph of order $n$. The algorithm constructs a minimum spanning tree of $G$, by consecutively adding edges to a set $X$.

- **Step 1.** Sort (arrange) the edges of $G$ in ascending order (that is, edges with smaller weights come first.)

- **Step 2.** Initialize $X$ to consist of an edge of smallest weight.

- **Step 3.** From the edges in $E(G) \setminus X$ choose an edge with smallest weight that does not make a cycle with the edges in $X$, and add it to $X$. Repeat Step 3 until there are $n-1$ edges in $X$.

## KRUSKAL'S ALGORITHM YIELDS A SPANNING TREE

Since $X$ is acyclic by construction, its components are trees. Let

$$X_1, \ \dots\ , \ X_k$$

be the components of $X$. Since

$$e(X_1) + \ \dots\ + e(X_k) = n - 1$$

and

$$v(X_i) = e(X_i) + 1 \text{ for } i = 1, \dots, \ k,$$

we see that

$$n \geq v(X_1) + \ \dots\ + v(X_k) = n - 1 + k,$$

implying that $k = 1$ and $v(X) = n$. Thus $X$ is a connected spanning graph.

Because $X$ is acyclic it is a spanning tree.

It can be shown that $X$ is a minimum spanning tree, but we shall omit this proof.

Moreover, it can be shown that if all edge weights of $G$ are distinct, then $G$ has only one minimum spanning tree.

Note that in Kruskal's algorithm, the order of the graph $F(X)$ determined by the edges in $X$ grows unpredictably, only the number of its edges increases always by one..

**Question** *Why step 3 of Kruskal's algorithm can be repeated $n-2$ times?*

In other words, can the execution of step 3 stop because of lack of suitable edges? Consider two cases: $v(F(X)) < n$ and $v(F(X)) = n$. In the latter case, $F(X)$ must be disconnected if $e(F(X)) < n-1$.

# HAMILTON PATHS AND CYCLES

Let $G$ be a connected graph.

**Definition** A spanning path in $G$ is called a **Hamilton path** in G.

**Definition** A spanning cycle in $G$ is called a **Hamilton cycle** in G.

Note that a path or cycle can be called Hamilton only with respect to a given graph $G$ that contains them.

**Definition** A graph is called **Hamilton graph** if it contains a spanning cycle.

# EXAMPLES OF HAMILTON AND NONHAMILTON GRAPHS

*Prove that $K_n$ contains $n!/2$ distinct Hamilton paths.*

Indeed, every $n$-path in $K_n$ is a permutation of the vertices of $K_n$. However, a permutation $v_1, \ldots, v_n$ and its inverse $v_n, \ldots, v_1$ determine the same path so there are $n!/2$ Hamilton paths in $K_n$.

*$K_{n,n}$ is Hamilton graph if $n > 1$.*

Choose in arbitrary sequence of $2n$ vertices alternating between the vertex classes of $K_{n,n}$ and end with the first vertex.

*$K_{3,4}$ is not Hamilton.*

Indeed a Hamilton cycle in $K_{3,4}$ would have order $7$, and be bipartite, which is a contradiction, because only even cycles are bipartite.

# EXAMPLES OF HAMILTON AND NONHAMILTON GRAPHS

However, the above argument fails here:

- *Prove that $K_{4,6}$ is not Hamilton.*

- Suppose that $C$ is a Hamilton cycle in $K_{4,6}$. Then clearly $C$ is bipartite.

- Because $C$ is a $2$-regular graph, its vertices must be equally split between the vertex classes, that is, $C$ must have $5$ vertices in each class

- This is impossible because one of the classes of $K_{4,6}$ has only $4$ vertices.

- The same proof can be applied to generalize the statement.

**Proposition** *$K_{m,n}$ is Hamilton if and only if $m = n$ and $n > 1$.*

Along the same lines one can prove the following slightly more difficult statement.

**Proposition** *$K_{m,n}$ has a Hamilton path if and only if $|m-n| \leq 1$.*

Obviously $K_{n,n}$ and $K_{n,n+1}$ have Hamilton paths.

Suppose that $m \geq n$, and assume that $K_{m,n}$ contains $P_{m+n}$.

Note that $P_{m+n}$ is bipartite and the sizes of its vertex classes are equal to $\lfloor (n+m)/2 \rfloor$ and $\lceil (n+m)/2 \rceil$.

Hence,

$$m = \lceil (n+m)/2 \rceil \quad \text{and} \quad n = \lfloor (n+m)/2 \rfloor,$$

implying that $|m-n| \leq 1$.

**Question** *Can a connected graph with a cut edge be Hamilton?*

# THE TRAVELLING SALESMAN PROBLEM

Finding if a graph is Hamilton is a difficult problem, in the sense that the known algorithms require very large number of steps.

- Nonetheless Hamilton cycles are important in practical problems.
- Suppose a salesman is responsible for a company sales in some 20 cities.
- Starting at one of these cities, the salesman must visit each of them exactly once and return back to where he started.
- The travel cost between any two cities is known.
- The purpose is to find a route of minimum cost.

# THE TRAVELLING SALESMAN PROBLEM

With our experience in weighted graphs, we see that the above problem can be reduced to the following graph-theoretic one:

- **Problem** *Let the vertices of* $K_{20}$ *represent each of the 20 cities and the edge weights of* $K_{20}$ *be the travel cost between the corresponding cities. Find a Hamilton cycle with minimum sum of the edge weights.*

- Since the number of Hamilton cycles in $K_{20}$ is

$$\frac{19!}{2} = 60,822,550,204,416,000,$$

it is clear that a brute force solution is not feasible.

As before, in a weighted graph, a **minimum** Hamilton cycle is a Hamilton cycle with minimum sum of edge weights.

The general Traveling Salesman Problem is formulated as follows:

**Problem** *Let $G$ be a weighted graph. Find a minimum Hamilton cycle in $G$.*

Unfortunately, even the best known algorithms are not efficient for this problem.

It is equally hard just to determine if a given graph is Hamilton or not.

# DIRAC'S THEOREM FOR HAMILTON CYCLES

Mathematicians have provided numerous sufficient conditions for a graph to be Hamilton. The best-known results of this type are Dirac's theorem and Ore's theorem.

**Theorem (Dirac)** *If $G$ is a graph of order $n \geq 3$, and $\delta(G) \geq n/2$, then $G$ is Hamilton.*

First, note that $G$ is connected, because any two vertices are either adjacent, or have a common neighbor.

Let $P = v_1, \cdots, v_k$ be a path of maximum length in $G$. We shall show that $G$ contains a cycle $C$ of order $k$.

Since $P$ has maximum length, the neighbors of $v_1$ and $v_k$ are among the vertices $\{v_1, \ldots, v_k\}$.

In particular, the neighbors of $v_1$ are among the vertices $\{v_2, \ldots, v_{k-1}\}$. Let the neighbors of $v_1$ be

$$v_{i_1}, \ldots, v_{i_p},\ i_1 < \cdots < i_p,\ p = d(v_1).$$

Note that each of the vertices $v_{i_1-1}, \ldots, v_{i_p-1}$ belongs to $\{v_1, \ldots, v_k\}$

If $v_k$ is not joined to any of the vertices $v_{i_1-1}, \ldots, v_{i_p-1}$, we find that

$$N(v_k) \cap \left\{v_{i_1-1}, \ldots, v_{i_p-1}\right\} = \varnothing.$$

Since

$$v_k \notin N(v_k) \cup \left\{v_{i_1-1}, \ldots, v_{i_p-1}\right\},$$

we see that

$$d(v_k) + d(v_1) = |N(v_k)| + p \leq k - 1 \leq n - 1,$$

contradicting that

$$d(v_1) + d(v_k) \geq 2\delta(G) \geq n.$$

Hence, $v_k$ is joined to some of the vertices $v_{i_1-1}, \ldots, v_{i_p-1}$, say to $v_{i_j-1}$.
Now, we construct a $k$-cycle $C$ as follows

$$v_1, v_{i_j}, v_{i_{j+1}}, \ldots, v_k, v_{i_j-1}, v_{i_j-1}, \ldots, v_2.$$

If $k = n$, then $G$ is Hamilton.

If $k < n$, then there is an edge joining a vertex of $C$ to a vertex outside $C$, because $G$ is connected.

By symmetry we can assume that

$$C = v_1, \ldots, v_k$$

and $v_k$ is connected to a vertex $u$ outside $C$.

Clearly the sequence

$$v_1, \ldots, v_k, u$$

is a path that is longer than $P$, contrary to the choice of $P$.

# ORE'S THEOREM FOR HAMILTON CYCLES

The following theorem clearly enhances Dirac's theorem:

- **Theorem (Ore)** *If $G$ is a graph of order $n \geq 3$, and if*

$$d(u) + d(v) \geq n$$

*for every two nonadjacent vertices $u$ and $v$, then $G$ is Hamilton.*

- The proof of Ore's theorem is almost identical to that of Dirac's theorem.

- The graph $K_{n,n+1}$ shows that neither Dirac's theorem, nor Ore's theorem can be improved significantly.

- Just in case, a Hamilton graph of order $n$ may have minimum degree much lower than $n/2$; clearly $C_n$ is Hamilton with $\delta(C_n) = 2$.

# ANOTHER THEOREM OF ORE

**Theorem (Ore)** *Every graph of order $n \geq 3$ and at least $\binom{n-1}{2} + 2$ edges is Hamilton.*

Indeed, if $G$ is a graph of order $n \geq 3$ and at least $\binom{n-1}{2} + 2$ edges, then $\overline{G}$ has at least

$$\binom{n}{2} - \binom{n-1}{2} - 2 = n - 3$$

edges.

Let $u$ and $v$ be any two nonadjacent in $G$ vertices. We see that

$$\begin{aligned} d_G(u) + d_G(v) &= 2n - 2 - d_{\overline{G}}(u) - d_{\overline{G}}(v) \\ &\geq 2n - 2 - (n-2) = n, \end{aligned}$$

because

$$d_{\overline{G}}(u) + d_{\overline{G}}(v) \leq e(\overline{G}) + 1 \leq n - 2.$$

Hence, the previous theorem of Ore implies that $G$ is Hamilton.

THANK YOU