Analysis I

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November 2, 2021

Contents

Lecture 19: End of Convergence, Functions of Bounded Variation, and Derivatives

Thu 28 Oct 2021 13:02

Recall we had the dominated convergence theorem. A similair version of the theorem makes use of convergence in measure as follows.

Theorem 0.1 (Dominated Convergence - Convergence in Measure)**.** Let (f_n) be a sequence of measurable functions $f_n : \mathbb{R} \to \overline{\mathbb{R}}$ and suppose there is an integrable function $g : \mathbb{R} \to \overline{\mathbb{R}}$ so that $|f_n| \leq g$ for all $n \in \mathbb{N}$. If $(f_n) \to f : \mathbb{R} \to \overline{\mathbb{R}}$ in measure, (with f measurable), then f is integrable and $\lim\limits_{n\to\infty} \int |f_n - f| = 0$ and $\lim\limits_{n\to\infty} \int f_n = f$.

Proof. First, note a subsequence of (f_n) converges to f pointwise almost everywhere. Hence, we find $|f| \leq g$ almost everywhere, so f is integrable. We cam assume $|f_n - f| \leq 2g$ (almost) everywhere. Then, we find a subsequence $(g_n) = (f_{n_k})$ such that $\limsup_{n\to\infty} |f_n - f| = \lim\limits_{n\to\infty} |g_k - f|$. Then, as $(g_k) \to f$ in measure, we find another subsequence $(h_j) = (g_{k_j}) = \left(f_{n_{k_j}}\right)$ which converges pointwise to f almost everywhere.
Applying dominated convergence theorem yields

$$\lim_{n\to\infty} \int |h_j - f| = 0.$$

Then, we find

$$\begin{aligned}\limsup_{n\to\infty} \int |f_n - f| &= \lim_{n\to\infty} \int |g_k - f| \\ &= \lim_{n\to\infty} |h_j - f| \\ &= 0.\end{aligned}$$

This completes the proof. □

1 Functions of Bounded Variation and Absolutely Continuous Functions

Remark. For this chapter $[a, b] \subseteq R$ will always denote a compact interval on $\mathbb{R}$.

Definition 1.1 (Partition)**.** A finite sequence $P = (x_k)_{k=n}^{N}$ with $n, N \in \mathbb{Z}$ and $n < N$ is called a **partition** of $[a, b]$ if $x_n = a$, $x_N = b$ and $x_{k-1} \leq x_k$ for $n < k \leq N$. We denote the collection of all partitions of $[a, b]$ to be $\mathscr{P}([a, b])$.

Definition 1.2. Let $f : [a, b] \to \mathbb{R}$ be a function. Then,

- For a partition $P = (x_k)_{k=n}^{N}$, we denote
$$V(f, P) = \sum_{k=n+1}^{N} \left| f(x_k) - f_{(x_{k-1})} \right|$$
to be the **variation of f with respect to P**.
- We define the quantity $\mathrm{TV}(f) = \sup\{V(f, P) : P \in \mathscr{P}([a, b])\}$ to be the **total variation of f** .

Remark. If $f : [a, b] \to \mathbb{R}$ and $c \in [a, b]$ with partitions $P_1 = (x_k)_{k=n}^{N}$ of $[a, c]$ and $P_2 = (x_k)_{k=N}^{K}$ of $[c, b]$. Then denote, $P = (x_k)_{k=n}^{K}$ to be a partition of $[a, b]$ and we find
$$V(f, P) = V\left(f \mid_{[a,c],P_1}\right) + V\left(f \mid_{[c,b]}, P_2\right).$$

Moreover,
$$\mathrm{TV}(f) = \mathrm{TV}\left(f \mid_{[a,c]}\right) + \mathrm{TV}\left(f \mid_{[c,b]}\right).$$

Definition 1.3 (Bounded Variation)**.** A function $f : \mathbb{R} \to \overline{\mathbb{R}}$ has **bounded variation** if $\mathrm{TV}(f) < \infty$.

Theorem 1.1 (Jordan's Theorem)**.** A function $f : [a, b] \to \mathbb{R}$ is of bounded variation if and only if there are increasing functions $g, h : [a, b] \to \mathbb{R}$ so that $f = g - h$.

Proof. Suppose $\mathrm{TV}(f) < \infty$ and let $x, y \in [a,b]$ with $x < y$. Then, we find

$$\begin{aligned}\mathrm{TV}\left(f\mid_{[a,y]}\right) &= \mathrm{TV}\left(f\mid_{[a,x]}\right) + \mathrm{TV}\left(f\mid_{[x,y]}\right)\\ &\geq \mathrm{TV}\left(f\mid_{[a,x]}\right) + \left|f(y) - f(x)\right|\\ &\geq \mathrm{TV}\left(f\mid_{[a,x]}\right) + f(x) - f(y).\end{aligned}$$

Furtheromre, $h : x \mapsto \mathrm{TV}\left(f\mid_{[a,x]}\right)$ and $g : x \mapsto \mathrm{TV}\left(f\mid_{[a,x]}\right) + f(x)$ are increasing. This fact is trivial for h and we find , adding $f(y)$ to both sides of the former inequality yields $g(y) \geq g(x)$ for arbitrary $y \geq x$, so this claim holds as well.
Taking the difference, $g - h = f$.
Conversely, suppose $f = g - h$ for increasing $g, h : [a,b] \to \mathbb{R}$. Then, let $x, y \in [a,b]$ with $y \geq x$. Then, we find

$$\begin{aligned}\left|f(y) - f(x)\right| &= \left|g(y) - g(x) + h(x) - h(y)\right|\\ &\leq \left|g(y) - g(x)\right| + \left|h(x) - h(y)\right|\\ &= g(y) - g(x) + h(y) - h(x).\end{aligned}$$

Hence, for a partition $P = (x_k)_{k=n}^{N}$, we find

$$\begin{aligned}V(f,P) &= \sum_{k=n+1}^{N} \left|f(x_k) - f(x_{k-1})\right|\\ &\leq \sum_{k=n+1}^{N} \left(g(x_k) - g(x_{k-1}) + h(x_k) - h(x_{k-1})\right) \quad = g(b) - g(a) + h(b) - h(a)\\ &< \infty.\end{aligned}$$

□

Definition 1.4 (Absolute Continuity)**.** A function $f : [a,b] \to \mathbb{R}$ is **absolutely continuous** if for each $\varepsilon > 0$ we find a $\delta > 0$ such that for every finite disjoint collection of nonempty intervals $\{(a_k, b_k) \subseteq [a,b] : 1 \leq k \leq K\}$ with $\sum_{k=1}^{K}(b_k - a_k) < \delta$, we have $\sum_{k=1}^{K}\left|f(a_k) - f(b_k)\right| < \varepsilon$.

Remark. Absolute continuity is stronger than uniform continuity, but weaker than lipschitz continuity.

Theorem 1.2. If a function $f : [a,b] \to \mathbb{R} \to$ is absolutely continuous, then f is continuous and f has bounded variation.

Proof. f is trivially continuous, taking a finite disjoint collection consisting only of 1 interval $\{(x,y)\}$ yields the definition of continuity.
Now we show bounded variation. For $\varepsilon = 1$, let $\delta > 0$ be the number such that the definition of absolute continuity holds for f.
Now fix $(x_k)_{k=n}^{N} \in \mathscr{P}([a,b])$ so that $x_k - x_{k-1} < \delta$ for all $n < k \leq N$. Then, if $P \in \mathscr{P}([x_{k-1}, x_k])$, we see $V\left(f\mid_{[x_{k-1},x_k]}, P\right) < 1$ by definition of absolute

continuity.
So, we have $\mathrm{TV}([x_{k-1}, x_k]) \leq 1$, so $\mathrm{TV}(f) = \sum_{k=n+1}^{N} \mathrm{TV}\left(f\mid_{[x_{k-1},x_k]}\right) \leq N - n$ by the ε assumption. $\square$

As it turns out, absolutely continuous functions have a relation to integrable functions, particularly, an integrable function f is simply the anti-integral of an absolutely continuous one.

Proposition 1.1. If $f : [a, b] \to \overline{\mathbb{R}}$ is integrable, then,

$$F : [a, b] \to \mathbb{R},\ x \mapsto \int_{[a,x]} f$$

is absolutely continuous.

This claim can be generalized into a sort of fundamental theorem of calculus for the lebesque integrals to characterize integrals and derivatives. For now, we only prove the weak version.

Proof. For $\varepsilon > 0$ there is a $\delta > 0$ such that $\int_S |f| < \varepsilon$ for every measurable set $S \subseteq [a, b]$ with $m(S) < \delta$.
Now, let $\{(a_k, b_k) : 1 \leq k \leq K\}$ be a disjoint collection of intervals in $[a, b]$ with $\sum_{k=1}^{K}(b_k - a_k) < \delta$. Fix $S = \bigcup_{k=1}^{K}(a_k, b_k)$. Then, since $m(S) < \delta$ and

$$\begin{aligned}\sum_{k=1}^{K}|F(b_k) - F(a_k)| &= \sum_{k=1}^{K}\left|\int_{[a_k,b_k]} f\right| \\ &\leq \sum_{k=1}^{K}\int_{[a_k,b_k]}|f| \\ &= \int_S |f| \\ &< \varepsilon \text{ by assumption.}\end{aligned}$$

Hence, absolute continuity holds. $\square$

2 Derivatives and Fundamental Theorem of Calculus

Proposition 2.1. Let $f : (a, b) \to \overline{\mathbb{R}}$ be monotone on $(a, b) \subseteq \mathbb{R}$ with $a, b \in \overline{\mathbb{R}}$ and $a < b$. Then,

$$\lim_{x\to a} f(x) = \inf\{f(x) : x \in (a, b)\}, \lim_{x\to b} f(x) = \sup\{f(x) : x \in (a, b)\}$$

are both well defined.

Lecture 20

Tue 02 Nov 2021 12:58