

# Algebraic Theory I

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## 1 Chinese Remainder Theorem

### Lecture 34: Chinese Remainder Theorem

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**Theorem 1.1** (Classical Chinese Remainder Theorem). If  $m_1, \dots, m_r$  are relatively prime integers, then for  $a_1, \dots, a_r$  we find an  $x \in \mathbb{Z}$  so that  $x \equiv a_i \pmod{m_i}$  for each  $1 \leq i \leq r$ .

**Theorem 1.2** (Generalized Chinese Remainder Theorem). Let  $R$  be a commutative ring with  $I_1, \dots, I_n \subseteq R$  being ideals so that  $I_i + I_j = R$  for all  $i \neq j$ . That is, the  $I_i$ s are pairwise co-maximal. Then for any  $x_1, \dots, x_n \in R$  we find an  $x \in R$  so that  $x \equiv x_i \pmod{I_i}$  for all  $1 \leq i \leq n$ .

**Recall.**  $x \equiv x_i \pmod{I_i}$  if  $x - x_i \in I_i$ .

*Proof.* If  $n = 1$  this is trivial. Of course,  $x = x$ .

For the case  $n = 2$  we have  $I_1 + I_2 = R$ , hence  $1 \in R = I_1 + I_2$ . Hence,  $1 = a_1 + a_2$  with  $a_1 \in I_1, a_2 \in I_2$ . Then, let  $x = x_1 a_1 + x_2 a_2$ , and we see  $a_1 + a_2 = 1$  but  $a_1 \equiv 0 \pmod{I_1}$  and likewise  $a_2 \equiv 0 \pmod{I_2}$ , hence  $a_1 \equiv 1 \pmod{I_2}$  and  $a_2 \equiv 1 \pmod{I_1}$ . Hence,

$$\begin{aligned} x &= x_1 a_2 + x_2 a_1 \\ &\equiv x_1 a_2 \pmod{I_1} \\ &\equiv x_1 \pmod{I_1} \\ \text{and } x &\equiv x_2 a_1 \\ &\equiv x_2 \pmod{I_2}. \end{aligned}$$

Hence, the claim holds for  $n = 2$ . Now, we induce on  $n$ .

Let  $n \geq 3$  and suppose the case  $n - 1$  to be true. Then, we find Then, we see

$I_1 + I_i = R$  for all  $i \geq 2$  by hypothesis. Hence,  $1 = a_i + b_i$  with  $a_i \in I_1$ ,  $b_i \in I_i$ . Then, we find

$$1 = \underbrace{1 \cdot \dots \cdot 1}_{n \text{ times}} = \prod_{i=1}^n (a_i + b_i) \in \prod_{i=1}^n (I_1 + I_i) \subseteq I_1 + \prod_{i=2}^n I_i.$$

Moreover, we know  $I_1 + \prod_{i=2}^n I_i$  to be an ideal as the product and sum of ideals are still ideals.

Then applying the case  $n = 2$ , we find a  $y \in R$  so that  $y_1 \equiv 1 \pmod{I_1}$  and  $y_1 \equiv 0 \pmod{\prod_{i=2}^n I_i}$ . Repeating for each  $1 \leq i \leq n$  yields a  $y_j \in R$  so that  $y_j \equiv 1 \pmod{I_j}$  and  $y_j \equiv 0 \pmod{\prod_{1 \leq i \leq n; i \neq j} I_i}$ . Now, define  $x = \prod_{i=1}^n x_i y_i$ . We see  $y_j \in I_i$  for all  $i \neq j$ , hence  $y_j x_j \equiv 0 \pmod{I_i}$  for all  $i \neq j$ . Hence  $x \equiv x_i y_i \equiv x_i \pmod{I_i}$ .  $\square$

Note that in the preceding proof  $\prod I_i$  denotes the ideal product as defined in the homework. In the next theorem we will use this symbol for the cartesian product, so ideal products will be written without product notation when the context is not necessarily clear.

**Corollary 1** (Alternative Statement of the Chinese Remainder Theorem). Let  $R$  be a commutative ring with  $I_1, \dots, I_n \subseteq R$  being pairwise comaximal distinct ideals of  $R$ . Then the map

$$\begin{aligned} f : R &\longrightarrow \prod_{i=1}^n R/I_i \\ x &\longmapsto (x \pmod{I_i})_{1 \leq i \leq n} \end{aligned}$$

is a surjective ring homomorphism with kernel  $\ker(f) = \bigcap_{i=1}^n I_i$ . Specifically,

$$R / \left( \bigcap_{i=1}^n I_i \right) \simeq \prod_{i=1}^n (R/I_i).$$

*Proof.* It is easily confirmed that  $f$  is a ring homomorphism with the prescribed kernel. Hence, the only claim that remains to be shown is the surjectivity. For  $f$  to be surjective, we need to take an arbitrary congruence system  $\hat{x} = (x_1 \pmod{I_1}, x_2 \pmod{I_2}, \dots, x_n \pmod{I_n})$  in the codomain of  $f$  and find a solution  $x \in R$  so that  $x \equiv x_i \pmod{I_i}$  for all  $1 \leq i \leq n$  (that is  $f(x) = \hat{x}$ ). We see the generalized remainder theorem yields such an  $x$ , so  $f$  is surjective.  $\square$

## 2 Polynomial Rings

### Lecture 35: Polynomials

Mon 15 Nov 2021 11:32

**Definition 2.1** (Polynomial Ring). Let  $R$  be a commutative ring and we define  $R[X]$  to be the ring of polynomials in the variable  $x$  with coefficients from  $R$  defined as follows.

An element  $f \in R[X]$  has the form

$$f = a_0 + a_1x + \dots + a_nx^n$$

for some  $n \geq 0$  and each  $a_i \in R$ . This is a formal sum in the sense that two polynomials

$$\begin{aligned} f &= a_0 + a_1x + \dots + a_nx^n \\ g &= b_0 + b_1x + \dots + b_mx^m \end{aligned}$$

have  $f = g$  if and only if  $a_i = b_i$  for every  $i$ .

For the polynomial  $f$ , we call  $a_0$  the **constant term** and  $a_n$  to be the **leading coefficient** and  $n$  to be the **degree**, denoted  $\deg(f) = n$ .

For the polynomial  $f = 0$ , we specifically define  $\deg(f) = -1$ . For all other constant polynomials  $g$ , we define  $\deg(g) = 0$ .

**Remark.** Occasionally, we will write  $f = \sum_{i=0}^{\infty} a_i x^i$  with almost every  $a_i = 0$ . With this form we see elements of  $R[X]$  are in a bijective correspondence with finite support tuples from  $R^{\mathbb{N}}$ .

We see  $R[X]$  forms a ring with two polynomials  $f, g \in R[X]$  as defined earlier having sum

$$(f + g) = \sum_{i=0}^{\infty} (a_i + b_i) x^i$$

and

$$fg = \sum_{i=0}^{\infty} a_i x^i \sum_{j=0}^{\infty} b_j x^j = \sum_{n=0}^{\infty} \sum_{\substack{i,j \\ i+j=n}} a_i b_j x^n.$$

**Definition 2.2** (Multivariate Polynomial Rings). We define a **multivariate polynomial ring**  $R[x_1, \dots, x_n] = (R[x_1, \dots, x_{n-1}])[x_n]$  with addition and multiplication defined similarly.