Algebraic Theory I: Homework III

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Solution (1)**.** 1. First, we note that if $xK = yK$ for some $x \neq y$, then $\overline{\varphi}(xK) = \overline{x}H$ and $\overline{\varphi}(yK) = \overline{y}H$, hence we need to show $\overline{x}H = \overline{y}H$. We see $x^{-1}yK = K$, hence $\overline{\varphi}\left(x^{-1}yK\right) = \overline{x^{-1}y}H = \overline{x}^{-1}\overline{y}H = \overline{\varphi}(1K) = 1H$. So, $\overline{x}^{-1}\overline{y} \in H$, hence $\overline{y} \in \overline{x}H$ and similairly, $\overline{x} \in \overline{y}H$. So, $\overline{x}H = \overline{\varphi}(xK) = \overline{y}H = \overline{\varphi}(yK)$, so $\overline{\varphi}$ is well defined.
Now,

$$\begin{aligned}
\overline{\varphi}(xKyK) &= \overline{\varphi}(xyKK) \\
&= \overline{xy}H \\
&= \varphi(xy)H \\
&= \varphi(x)\varphi(y)H \\
&= \overline{xy}H \\
&= \overline{xy}HH \text{ as } H = HH \text{ by closure} \\
&= \overline{x}H\overline{y}H \\
&= \overline{\varphi}(xK)\overline{\varphi}(yK).
\end{aligned}$$

Furthermore, we see $\overline{\varphi}(1K) = \overline{1}H = 1H$ as $\varphi(1) = \overline{1} = 1$ by homomorphism properties.

2. First, note that $Z_0\left(\overline{G}\right) = \{1\} = \overline{Z_0(G)}$. Now, we induce on n and we see $Z_{n-1}(G) \trianglelefteq G$ and $Z_{n-1}\left(\overline{G}\right) \trianglelefteq \overline{G}$ with $\overline{Z_{n-1}(G)} \leq Z_{n-1}\left(\overline{G}\right)$ by inductive hypothesis, so $\overline{\varphi} : Z_n(G)/Z_{n-1}(G) \to Z_n\left(\overline{G}\right)/Z_{n-1}\left(\overline{G}\right)$ is a well defined homomorphism. Hence, letting $\overline{x} \in \overline{Z_n(G)}$, hence $x \in Z_n(G)$ and hence $xZ_{n-1}(G) \in Z_n(G)/Z_{n-1}(G)$ implies $\overline{\varphi}xZ_{n-1}(G) = \overline{x}Z_{n-1}\left(\overline{G}\right) \in Z_n\left(\overline{G}\right)/Z_{n-1}\left(\overline{G}\right)$. Hence, we find $\overline{x} \in Z_n\left(\overline{G}\right)$. This completes the induction.

3. Suppose G is nilpotent and let n be the nilpotence class of G. Then, we see $\overline{Z_n(G)} = \overline{G} \leq Z_n\left(\overline{G}\right)$. Hence, $\overline{G}$ is of nilpotence class at most n, so we see $\overline{G}$ is nilpotent.

4. Let $\overline{G} = H$ with $\varphi : G \to H$ being the restriction to H homomoprhism. That being $\varphi(x) = \begin{cases} x, & x \in H \\ 1, & x \notin H \end{cases}$ It is clear this is well defined

5. Suppose n is the nilpotence class of G. Then $Z_n(G) \cap H = G \cap H = H \leq Z_n(H)$, so H is of nilpotence class at most n, hence H is nilptent.

Lemma 0.1. Automorphisms preserve maximality of subgroups.
Let $\varphi : G \to G$ be an automorphism and let $M < G$ be a maximal subgroup. Suppose $\varphi(M) = M'$ is not maximal. That is, there is a $\overline{M}'$ such that $M' < \overline{M}' < G$. Then, we find

$$\begin{aligned}\varphi^{-1}\left(\overline{M}'\right) &= \varphi^{-1}\left(M' \cup \left(\overline{M}' \setminus M'\right)\right)\\ &= \varphi^{-1}(M') \cup \varphi^{-1}\left(\overline{M}' \setminus M'\right)\\ &= M \cup \{\varphi^{-1}(m) : m \in \overline{M}' \setminus M'\}\\ &> M.\end{aligned}$$

Furthermore, $\overline{M}' < G$ by assumption, hence $M < \overline{M}' < G$. $\lightning$.

Solution (2)**.** *Proof.* Now, let $\alpha : G \to G$ be an automorphism of G and denote $\alpha(M) = M'$. Then, we see

$$\begin{aligned}\alpha(\Phi(G)) &= \alpha\left(\bigcap_{\substack{M<G\\ M \text{ is maximal}}} M\right)\\ &= \bigcap_{\substack{M<G\\ M \text{ is maximal}}} \alpha(M)\\ &= \bigcap_{\substack{M<G\\ M \text{ is maximal}}} M'\end{aligned}$$

.

Then, as M' is maximal and α is an injection, we see if $N \neq M$ are both maximal subgroups, we have $\alpha(N) \neq \alpha(M)$, hence

$$\{M : \underset{M \text{ is maximal}}{M < G}\} = \{M' : \underset{M \text{ is maximal}}{M < G}\}.$$

So, we have

$$\alpha(\Phi(G)) = \bigcap_{\substack{M<G\\ M \text{ is maximal}}} M' = \bigcap_{\substack{M<G\\ M \text{ is maximal}}} M = \Phi(G).$$

□

Solution (3).

Lemma 0.2. $[M,M]$ and $\langle x^p : x \in M \rangle$ are characteristic in M.
Let $\alpha : M \to M$ be an automorphism. Then, denote $\alpha(x) = x'$ for $x \in M$ and we see,

$$\begin{aligned}
\alpha([M,M]) &= \alpha\left(\left\langle xyx^{-1}y^{-1} : x,y \in M \right\rangle\right) \\
&= \left\langle \alpha\left(xyx^{-1}y^{-1}\right) x,y \in M \right\rangle \\
&= \left\langle \alpha(x)\,\alpha(y)\,\alpha(x)^{-1}\,\alpha(y)^{-1} : x,y \in M \right\rangle \\
&= \left\langle x'y'x'^{-1}y'^{-1} : x,y \in M \right\rangle \\
&\leq \left\langle x'y'x'^{-1}y'^{-1} : x',y' \in M \right\rangle \\
&= [M,M].
\end{aligned}$$

Similairly,

$$\begin{aligned}
\alpha(\langle x^p : x \in M \rangle) &= \langle \alpha(x^p) : x \in M \rangle \\
&= \langle \alpha(x)^p : x \in M \rangle \\
&= \langle x'^p : x \in M \rangle \\
&\leq \langle x'^p : x' \in M \rangle && = \langle x^p : x \in M \rangle
\end{aligned}$$

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Then, we see as $M \trianglelefteq G$ and these two groups are characteristic we also have $\langle x^p : x \in M \rangle \trianglelefteq G$ and $[M,M] \trianglelefteq G$. Furthermore, we note that as $xyx^{-1}y^{-1} \in M$ for $x,y \in M$ we have $\{M,M\} \leq \langle x^p : x \in M \rangle$. Now, Suppose M is not an elementary abelian p-group. Then, we find either $[M,M] > 1$ or there is an element x of order $q \neq p$.

Solution (5).