Algebraic Theory I

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November 10, 2021

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Recall. Recall R denotes a commutative ing. If $S \subseteq R$ is a multiplicative subset, we see $x, y \in S$ implies $xy \in S$ and $0 \notin S$ but $1 \in S$.
Then, we define $S^{-1}R = \{X/s : x \in R, s \in S\}$. Then, we see $\frac{x_1}{s_1} = \frac{x_2}{s_2}$ if and only if there is an $s \in S$ so that $s\left(s_2x_1 - s_1x_2\right) = 0$. Of course, if R is an integral domain we see this iplies $s_2x_1 - s_1x_2 = 0$, the normal definition of fraction equality.

Now, we turn this set into a ring. We define $\frac{x_1}{s_1} \cdot \frac{x_2}{s_2} := \frac{x_1x_2}{s_1s_2}$ and $\frac{x_1}{s_2} + \frac{x_2}{s_2} = \frac{s_2x_1}{s_1s_2} + \frac{s_1x_2}{s_1s_2} = \frac{s_2x_1+s_1x_2}{s_1s_2}$. Now, we need to show that $+, \cdot$ are well defined (meaining they do not vary for different representatives of a given equivalence class). This fact is easily checked by symbolic manipulation so we omit the proof. For the addition case suppose $\frac{x_1}{s_1} = \frac{x_1'}{s_1'}$ and similairly for $\frac{x_2}{s_2}$ then take the multiplicative representation of the fraction and multiply the $\frac{x_1}{s_1}$ representation by $-s_2s_2'ts$ and the $\frac{x_2}{s_2}$ representation by $-s_1s_1'st$ and by adding together these representations we see terms cancel and we obtain that addition is in fact well defined. Moreover, it is trivial to check that the ring axioms hold.

> **Definition 0.1** (Ring Localization)**.** We denote this new fraction ring $S^{-1}R$ to be the **localization of** R with additive identity $\frac{0}{1}$, multiplicative identity $\frac{1}{1}$ and $\frac{tx}{ts} = \frac{x}{s}$ for all $t \in S$.

Note that $s \in S$ is nonzero by definition, so $\frac{1}{s} \cdot \frac{s}{1} = \frac{1}{1} = 1_{S^{-1}R}$, so every element has an inverse.

Proposition 0.1. If R is a commutative ring with $S \subseteq R$ being a mult-plicative subset. Then the map

$$\begin{aligned} \pi : R &\longrightarrow S^{-1}R \\ x &\longmapsto \pi(x) = \frac{x}{1} \end{aligned}$$

is a ring homomorphism. Moreover, if S has no zero-divisors, then π is an injection.

Proof. If $x, y \in R$ then $\pi(x \pm y) = \frac{x \pm y}{1} = \frac{x}{1} \pm \frac{y}{1} = \pi(x) \pm \pi(y)$. Furthermore $\pi(1) = \frac{1}{1} = 1$.
Lastly, $\pi(xy) = \frac{xy}{1} = \frac{x}{1}\frac{y}{1} = \pi(x)\pi(y)$. Hence, π is a ring homomorphism. Now consider $\ker(\pi) = \{x \in R : \frac{x}{1} = \frac{0}{1}\}$. We see this implies an $s \in S$ so that $s(1x - 1 \cdot 0) = sx = 0$, hence s is a zero divisor if $x \neq 0$ $\lightning$. So, the kernel is trivial. $\square$

Example. If R is a commutative ring and $P \subseteq R$ is a prime ideal, then $S := R \setminus P$ is a multiplicative set. Moreover, $0 \in P$ so $0 \notin S$ and $P \subset R$ is proper, so $1 \in S$.
If $x, y \in S$ with $xy \notin S$, then $xy \in P$ so $x \in P$ or $y \in P$ $\lightning$. So, S is closed under multiplication. Then the localization $S^{-1}R$ is often denoted R_P. This is the canonical example of localization which we will study more next class. $\diamond$

The use of this construction is that it allows us to embed an integral domain R in a field $R_{(0)}$ called the **field of fractions**.