Algebraic Theory I

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Lecture 20: Free Groups (3)

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Recall. F is a free group on the set X when there is an injection $\sigma : X \overset{F}{\hookrightarrow}$ such that for all maps $\alpha : X \to G$, there is a homomorphism $\beta : F \to G$ such that $\beta \circ \sigma = \alpha$.

Remark. F is also a free group on $\sigma(X) \subseteq F$, using a similair inclusion map, so often we will assume $X \subseteq F$.

Theorem 0.1. If F_1 is free on X_1 and F_2 is free on X_2 and $|X_1| = |X_2|$, then $F_1 \simeq F_2$.

Proof. Since $|X_1| = |X_2|$ we find a bijection $\alpha : X_1 \to X_2$ and we can assume WLOG that $X_1 \subseteq F_1$ and $X_2 \subseteq F_2$. Then, the free property of F_1 implies there is a unique homomorphism $\beta : F_1 \to F_2$ such that $\beta(x) = \alpha(x)$ for all $x \in X_1$. Similairly, thee is a unique map $\gamma : F_2 \to F_1$ extending $\alpha^{-1} : X_2 \to X_1$ such that $\gamma(y) = \alpha^{-1}(y)$ for all $y \in X_2$. So, we see

$$\begin{aligned} \beta|_{X_1} : X_1 &\longrightarrow X_2 \\ x &\longmapsto \beta(x) = \alpha(x) \end{aligned}$$

and

$$\begin{aligned} \gamma|_{X_2} : X_2 &\longrightarrow X_1 \\ y &\longmapsto \gamma(y) = \alpha^{-1}(y) \end{aligned}$$

are inverses.
Hence, we have β and γ are a pair of inverse homomorphisms as X_1 generates F_1 and likewise X_2 generates F_2 .
Then, for an arbitrary element in F of the form $x = x_1^{\varepsilon_1} \ldots x_\ell^{\varepsilon_\ell}$ with $\varepsilon_i \in \mathbb{Z}$ and $x_i \in X_1$, then we see $\gamma(\beta(x)) = x$, hence this completes the proof. $\square$

Theorem 0.2. Let F be a free group with H, G being groups. Suppose $\alpha : F \to H$ is a homomorphism and $\beta : G \to H$ is a surjective homomorphism. Then, there is a $\gamma : F \to G$ such that $\beta\gamma = \alpha$.

Proof. Let F be free on $X \subseteq F$. Then, each $x \in X$ has $\alpha(x) \in H = \operatorname{Im}(\beta)$. Then, there is some $g_x \in G$ such that $\beta(g_x) = \alpha(x)$. By the universal mapping property of F , we have the map $X \to G, x \mapsto g_x$ extends to a homomorphism

$$\begin{aligned} \gamma : F &\longrightarrow G \\ x &\longmapsto \gamma(x) = g_x. \end{aligned}$$

Then, for $x \in X$ we see $\beta(\gamma(x)) = \beta(g_x) = \alpha(x)$, so $\beta \circ \gamma = \alpha$ on X which generates F , so $\beta \circ \gamma = \alpha$ on F as $\beta \circ \gamma, \alpha$ are homomorphisms. $\square$

Definition 0.1 (Group Presentations)**.** Any group G is a homomorphic image of a free group F. An explicit homomorphism $\alpha : F \to G$ with F is called a **presentation** of G. Its kernel $N = \ker(\alpha) \trianglelefteq F$ has $F/N \simeq G$. So, we may write $\langle X : Y \rangle = G$ where F is a free group on X and $Y \subseteq F$ has normal closure, $\bigcap_{H \trianglelefteq G, Y \leq H} H = N$.

Example. $D_{2n} = \left\langle \alpha, \tau : \alpha^n, \tau^2, \tau\alpha\tau\alpha \right\rangle$. Here, we see F is free on the set $\{\alpha, \tau\}$ and N is the normal closure of $\left\langle \alpha^n, \tau^2, \tau\alpha\tau\alpha \right\rangle$, that being the smallest normal subgroup of F containing these three elements.
In general if $H \leq G$, then $\bigcap_{N \trianglelefteq G, H \leq N} N \trianglelefteq G$ is the normal closure of H. $\diamond$

Remark. In general, a group of relations can generate other relations that we may not account for, so it is good to know what elements in the normal closure look like. If $X \subseteq G$, we find elements in the normal closure N of $\langle X \rangle$ in G include inverses and products of elements from X. Furthermore, arbitrary conjugates and their products/inverses will be in N. We see this yields

$$N \supseteq \{\prod_{i=1}^{\ell} \left(g_i x_i g_i^{-1}\right) : \ell \geq 0, g_i \in G, x_i \in X \cup X^{-1}\}.$$

Furthermore, we see this set is in fact a normal subgroup itself, so equality holds.

Lecture 21

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