Analysis I

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Contents

1 Intro to Functional Analysis

Lecture 22: L^p spaces

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I skipped a chapter on supporting lines and Jensen's inequality because the material was rather simple and well explained in Hagen's notes.

Definition 1.1 (Essential Supremum)**.** Let $f : S \to \overline{\mathbb{R}}$ be measurable. Then, we denote the quantity

$$\operatorname{esssup} f = \inf\{M \in \overline{\mathbb{R}} : m\left(\{x \in S : f\left(x\right) > M\}\right) = 0\}$$

is called the **essentail supremum** of f. Note that $f \leq \operatorname{esssup} f$ almost everywhere.

Definition 1.2 (Lp space)**.** Let $f : S \to \overline{\mathbb{R}}$ be measurable ,then

- For $1 \leq p \leq$ we define $\|f\|_p = \left(\int_S |f|^p\right)^{\frac{1}{p}}$ to be the L^p **norm** of f.
- $\|f\|_\infty = \operatorname{esssup} |f|$ is the L^∞ **norm** of f.

Definition 1.3 (Equivalent functions)**.** For $1 \leq p \leq \infty$ let $V_p\left(s\right)$ be the set of all measurable functions $f : S \to \overline{\mathbb{R}}$ so that $\|_p < \infty$. Then, functions $f, g \in V_p\left(S\right)$ are **equivalent**, denoted $f \sim g$, if $f = g$ almost everywhere in S.
The set of all equivalence classes $V_p\left(S\right) / \sim$ is denoted $L^p\left(S\right)$ and called the **Lebesque space**.

Remark. If $f \sim g$ in $L_P(S)$, then $f = g$ almost everywhere (on S) hence $\|f - g\|_p = 0$. Hence the L^p norm can be extended to norms on equivalence classes by simply denoting $\|\,[f]\,\|_p = \|f\|$ for some equivalence class $[f] \in L^p(S)$.

Theorem 1.1 (Minkowski's Inequality)**.** Suppose $f, g \in L^p(S)$ for a $1 \leq p \leq \infty$. Then, $\|f + g\|_p \leq \|f\|_p + \|g\|_p$.
Moreover, if $1 < p < \infty$, then $\|f + g\|_p = \|f\|_p + \|g\|_p$ if and only if there is a $c \geq 0$ so that $f = cg$ almost everywhere.

Proof. Let $x = \|f\|_p$, $s = \|g\|_p$. Then, we see the claim is trivial true if $r = 0$, $s = 0$, or $p = \infty$. Hence, define $\lambda = \frac{r}{r+s}$ and we may assume f, g are finite by definition of L^p space. Since $t \mapsto |t|^p$ is convex on $\mathbb{R}$ and $\lambda \in (0, 1)$, we see

$$\begin{aligned}
|f + g|^p &= \left|\lambda \frac{f}{\lambda} + (1 - \lambda)\frac{g}{1 - \lambda}\right|^p \\
&\leq \lambda \left|\frac{f}{\lambda}\right|^p + (1 - \lambda)\left|\frac{g}{1 - \lambda}\right|^p \\
\Rightarrow \|f + g\|_p &\leq \lambda \|\frac{f}{\lambda}\|_p^p + (1 - \lambda)\|\frac{g}{1 - \lambda}\|_p^p \\
&= \lambda (r + s)^p + (1 - \lambda)(r + s)^p \\
&= (\|f\|_p + \|g\|_p)^p
\end{aligned}$$

.

Note that this last step comes from appealing to the definition of lambda and noting $r^p = \int |f|^p$ and similarly for g. Now, we note that $t \mapsto |t|^p$ is strictly convex for $1 < p < \infty$, so equality occurs if and only if $\frac{f}{\lambda} = \frac{g}{1-\lambda}$ (almost everywhere if f, g are functions and not equivalence classes) hence f is a multiple of g. ☐

Remark. Note that this implies $L^p(S)$ is closed under addition, and constant multiplication (this part is trivial), so it is a linear space.

Definition 1.4 (Normed Linear Space)**.** A linear space V is a **normed linear space** if there is a function $\|.\| : V \to \mathbb{R}$ called the **norm of** V so that the following hold

- $\|v\| \geq 0$ for all $v \in V$,
- $\|v\| = 0$ if and only if $v = 0$,
- $\|\lambda v\| = |\lambda|\,\|v\|$ for all $\lambda \in R$, $v \in V$,
- $\|v + w\| \leq \|v\| + \|w\|$ for all $v, w \in V$.

Remark. $V_p(S)$ is not itself a normed linear space as the function $f(x) = \begin{cases} 0, & x \notin \mathbb{Q} \\ 1, & x \in Q \end{cases}$ has $\|f\| = 0$ even though f is not the zero function. We rule out this possibility by considering only the equivalence classes, in which case $f \sim 0$, so $L^p(S)$ is in fact a normed metric space.

Definition 1.5 (Conjugate)**.** For $p \in [1, \infty]$ we define the **conjugate** of p to be the extended real number $q \in [1, \infty]$ so that $\frac{1}{p} + \frac{1}{q} = 1$.

Lemma 1.1 (Young's Inequality)**.** Suppose $p \in (1, \infty)$ with q its conjugate and $a, b \in \mathbb{R}$ with $a, b \geq 0$. Then, $ab \leq \frac{a^p}{p} + \frac{q^p}{p}$. Moreover equality holds if and only if $a^p = b^q$.
Specifically $\sqrt{ab} \leq \frac{a+b}{2}$, that is the geometric mean is at most the arithmetic mean.

Proof. It suffices to assume a, b positive as the 0 case is trivial. Then, define $F(t) = a^{p(1-t)} b^{qt} = a^p \left(\frac{b^q}{a^p}\right)^t$. We see F is convex on $\mathbb{R}$ as it is exponential. Hence,

$$\begin{aligned} ab &= F\left(\frac{1}{p} \cdot 0 + \left(1 - \frac{1}{p}\right) q\right) \\ &\leq \frac{1}{p} F(0) + \left(1 - \frac{1}{p}\right) F(1) \\ &= \frac{a^p}{p} + \frac{b^q}{q}. \end{aligned}$$

As F is strictly convex (except in the case $\frac{b^q}{a^p} = 1$), we see equality will not arrive except in this exceptional case. □

2 Seperability and Bounded Linear Functionals

Lecture 23: Seperability of L^p spaces

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Definition 2.1 (Step-Function)**.** A **step function**, $\psi : \mathbb{R} \to \mathbb{R}$ is a simple function of the form

$$x \mapsto \sum_{k=1}^{m} a_k \chi_{J_k}(x)$$

where every set J_k is a bounded interval.

Theorem 2.1. (22.4).

Proof. 1. For the case $p = \infty$, we have f bounded almost everywhere. By splitting f into functions f^+, f^- we can assume $f \geq 0$. Then, we see a sequence of simple functions (s_n) converging uniformly to f almost everywhere.
For $1 \leq p < \infty$ we find a sequence of simple functions (s_n) converging pointwise to f so that $|s_n| \leq |f|$. Consequently, we see

$$|f - s_n|^p \leq (|f| + |s|)^p \leq (2\,|f|)^p = 2^p\,|f|^p .$$

So, we see dominated convergence implies

$$\int |f - s_n|^p = 0.$$

2. Assuming the case 1, we see we can assume f simple. Moreover, we can assume $f = \chi_S$, a characteristic function in $L^p(\mathbb{R})$.
Then, we see S is measurable with $\int \chi_S = m(S) < \infty$, hence $\int \chi_S^p < \infty$. Applying littlewoods first princple and finxing $\varepsilon > 0$ we find a finite disjoint collection of open intervals $\{J_k : 1 \leq k \leq n\}$ so that for $U = \bigcup_{k=1}^m J_k$, we find $m(S \triangle U) < \varepsilon^p$.
Then, we see
$$\begin{aligned}\int |\chi_S - \chi_U|^p &= \int \chi_{S\triangle U}^p \\ &= m(S\triangle U) \\ &< \varepsilon^p.\end{aligned}$$
Since $m(U \setminus S) < \infty$, we see each interval J_k must be bounded (else U would be of infinite measure), so χ_U is a step function on the interval $[a, b] \supseteq U$ satisfying the required conditions.

3. Assuming 2 we see it suffices to show case for the step function $f = \chi_{[c,d]}$ with $c \leq d$. Then, fixing $\varepsilon > 0$ and considering the function
$$x \mapsto g(x) = \chi_{[c,d]} + \big(1 + \varepsilon^{-p}(x - c)\big)\chi_{\left(c - \frac{\varepsilon^p}{3},\right),c} + \big(1 - e^{-p}(x - d)\big)\chi\left(d, d + \frac{\varepsilon^p}{3}\right).$$
We see this functions is continuous as it is simply piecewise linear, being 1 on $[c, d]$ and a linear interpolation between 1 and 0 in a small interval either side of $[c, d]$. Importantly, $\int_{\left(c - \frac{1}{3}\varepsilon^p\right)} |g| \leq \frac{1}{3}\varepsilon^p$, the length of the interval.
Hence, we find
$$\int \left|\chi_{[c,d]} - g\right|^p \leq (\frac{2}{3}\varepsilon^p)^p < \varepsilon^p.$$
This completes the proof.

□

Note that this proof essentially showed simple functions, step functions, and continuous functions are dense in $L^p(\mathbb{R})$ (given $1 \leq p < \infty$ for the last 2).

Definition 2.2 (Density)**.** Let $(X, \|\cdot\|)$ be a normed linear space. If $S \subseteq T \subseteq X$, then S is **dense** in T if for all $v \in T, \varepsilon > 0$ we find a vector $u \in S$ so that $\|v - u\| < \varepsilon$.

Definition 2.3 (Seperability)**.** A normed linear space $(X, \|\cdot\|)$ is **seperable** if it contains a countable, dense subset.

Theorem 2.2. For $1 \leq p < \infty$, $L^p(\mathbb{R})$ is seperable.

Proof. If $\varphi = c\chi_{[a,b]}$ with $a, b, c \in R$, then for any $\varepsilon > 0$ we find an interval $I = [c, d] \subseteq [a, b]$ with $c, d \in \mathbb{Q}$ and an $r \in \mathbb{Q}$ so that $\int |\varphi - r\chi_I|^p < \varepsilon^p$ (the function vanishes except on an arbitrarily small interval). Letting Ψ be the collection of all such step functions of the form $\psi = \sum_{i=1}^{n} c_k \chi_{I_k}$ with $c_k \in \mathbb{Q}$ and I_k having rational endpoints, then linearity combined with the preceding lemmas guarantees Ψ to be a countable dense subset, so $L^p(\mathbb{R})$ is seperable. $\square$

3 Bounded Linear Functionals

Definition 3.1 (Functionals)**.**

- A function $\varphi : X \to \mathbb{R}$ on a linear space X is called a **linear functional** if the laws of linearity holds for φ.
- A linear functional $\varphi : X \to \mathbb{R}$ on a normed linear space $(X, \|\cdot\|)$ is called **bounded** if there is $M \geq 0$ so that $|\varphi(x)| \leq M\|x\|$ for all $x \in X$.
- If φ is a bounded linear functional, the quantity

$$\|\varphi\| = \inf\{M \geq 0 : |\varphi(x)| \leq M\|x\| \ \forall\, x \in X\}$$

is called the **norm** of φ.

Proposition 3.1. Let $\varphi : X \to \mathbb{R}$ be a bounded linear functional on a normed linear space $(X, \|\cdot\|)$. Then,

$$\|\varphi\| = \sup\{|\varphi(x)| : x \in X, \|x\| \leq 1\}.$$

Definition 3.2 (Continuity)**.** A linear functional $\varphi : X \to \mathbb{R}$ on $(X, \|\cdot\|)$ is **continuous at** x_0 if for every $\varepsilon > 0$ we find a $\delta > 0$ so that $|\varphi(x) - \varphi(x_0)| < \varepsilon$ if $\|x - x_0\| < \delta$.
If φ is continuous for all $x \in X$, then φ is **continuous.**

Proposition 3.2. Let $\varphi : X \to \mathbb{R}$ be a linear functional on $(X, \|\cdot\|)$. Then, the following are equivalent

- φ is continuous.
- φ is continuous at some $x_0 \in X$.
- φ is bounded.