Combinatorics

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October 6, 2021

Contents

Lecture 18: Semi-Circle Law Corrections and Quasi-Random Graphs

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Let G be a random graph of order n and denote $N(x)$ to be the number of eigenvalues λ such that $\frac{\lambda}{\sqrt{n}} \leq x W_n(x) = \frac{1}{n} N(x)$. Then, we find the sequesnce of functions approaches

$$W(x) = \begin{cases} 0, & x \leq -1 \\ \frac{2}{\pi} \int_{-1}^{x} \sqrt{1-x^2} dx, & -1 < x < 1 \\ 1, & x \geq 1 \end{cases}.$$

Furthermore, we even find W to be continuous in the hole real line and $W_h(x)$ converges to $W(x)$ earlier.

1 Quasi-Random Graphs

Definition 1.1. Let G be a graph of order n with M being an arbitrary subgraph of K_n. We define $N_G^*(M)$ to be the number of labeled induced copies of M in G. Equivalently,

$$N_G^*(M) = |\{\alpha : \ \alpha : V(M) \to V(G)\}|$$

with each α preserving adjacenc and $\alpha(V(M))$ being isomorphic to M.

Example. $N_G^*(K_2 = 2e(G))$.
$N_G^*(C_4) = \frac{1}{64} n^4 + o(n^4)$. This is because every copy of K_4 in G has 8 copies isomorphic to C_4. Furthermore there are 3 symmetries of a K_4 copy, so altogether we get $\frac{1}{24}\binom{n}{4} \cdot \frac{1}{2^6} = \frac{n^4}{64} + o(n^4)$. $\diamond$

Definition 1.2 (Graph Properties)**.** The following are equivalent:

- We define an infinite family of graphs with arbitrary orders $\mathscr{G}$ to have property $P_1(s)$ or **property I** with power s if for all graphs M of order s, we find $N_G^*(M) = \frac{n^s}{2^{\binom{n}{2}}} + o(n^s)$ for each $G \in \mathscr{G}$ having order n.
- A family $\mathscr{G}$ has property P_2 or **property II** if $e(G) \geq \frac{n^2}{4} + o(n^2)$ and the number of closed walks of order 4, $CW_4(G) \leq \frac{n^4}{16} + o(n^4)$ for each $G \in \mathscr{G}$ of order n.
- A family $\mathscr{G}$ has property P_3 or **property III** if $e(G) \geq \frac{n^2}{4} + o(n^2)$, $\lambda_1(G) = \frac{n}{2} + o(n)$ and $\sigma_2(G) = o(n)$ for all $G \in \mathscr{G}$ of order n.
- A family $\mathscr{G}$ has property P_4 or **property IV** if for all sets S we have $\left| e(S) - \frac{1}{4}^{|S|^2} \right| = o(n^2)$ for all $G \in \mathscr{G}$ of order n.
- A family $\mathscr{G}$ has property P_5 or **property V** if for all sets S of order $\left\lfloor \frac{n}{2} \right\rfloor$ we find $\left| e(S) - \frac{1}{16}n^2 \right| = o(n^2)$ for all $G \in \mathscr{G}$ of order n.
- A family $\mathscr{G}$ has property P_7 or **property VII** if $\sum_{1 \leq i,j \leq n} \left| \hat{d}(v_i, v_j) - \frac{n}{4} \right| = o(n^3)$ for $G \in \mathscr{G}$ of order n and $v_i, v_j \in V(G)>$

We find

$$P_2 \Rightarrow P_1(s) \Rightarrow P_3 \Rightarrow P_4 \Rightarrow P_5 \Rightarrow P_7 \Rightarrow P_2.$$

Example. It is trivial to find that in order for G to be $P_1(2)$ it must have $e(G) = \frac{n^2}{4} + o(n^2)$.
We see if $|S| = \frac{1}{2}n$ we obtain P_5 from P_4.
Random graphs and Payley graphs are P_5. ◇

Lecture 19: Quasi-Random Graphs (2)

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Recall we had many equivalent conditions, cleverly names properties *I*-*VII*. We prove the are equivalent.

$P_2 \Leftrightarrow P_3$.

- $(P_2 \Rightarrow P_3)$. Recall $\frac{n^4}{16} + o(n^4) = CW_4(G) = \operatorname{tr}(A^4)$. We know

$$\begin{aligned} \operatorname{tr}(A^4) &= \sum_{i=1}^{n} \lambda_i^4 \\ \Rightarrow \lambda_1^4 &\leq \frac{n^4}{16} + o(n^4) \\ \Rightarrow \lambda_1 &\leq \frac{n}{2} + o(n). \end{aligned}$$

From this, we also know

$$\begin{aligned} \sum_{i=1}^{n} \lambda_i^4 &= \lambda_1^4 + \sum_{i=2}^{n} \lambda_i^4 \\ \Rightarrow \sum_{i=2}^{n} \lambda_i^4 &= o\left(n^4\right) \\ \Rightarrow \lambda_i &= o\left(n\right) \\ \Rightarrow \sigma_2 &= o\left(n\right). \end{aligned}$$

- ($P_3 \Rightarrow P_2$). Again, we know

$$\begin{aligned} CW_4 &= \sum_{i=1}^{n} \lambda_i^4 \\ &= \lambda_1^4 + \sum_{i=2}^{n} \lambda_i^4 \\ &= \frac{n^4}{16} + o\left(n^4\right) \\ \Rightarrow \lambda_1^4 = \frac{n^4}{16}. \end{aligned}$$

Similarly, we find $\sum_{i=2}^{n} \lambda_i^4 \leq \sigma_2^2 \sum_{i=2}^{n} \lambda_i^2$
Then, we have $\sum_{i=2}^{4} \lambda_i^2 = 2e\left(G\right) - \lambda_1^2 \leq o\left(n^2\right) n^2 = o\left(n^4\right)$. $P_2 \Leftrightarrow P_3$.

□

Remark. Sometimes, we wish to only have 2 conditions to check for P_3, and we find that there is an equivalent statement of P_3 such that a family $\mathscr{G}$ follows

- $e\left(G\right) \geq \frac{n^2}{4} + o\left(n^2\right)$.
- $\left|\lambda_n\left(G\right)\right| + \left|\lambda_n\left(\overline{G}\right)\right| = o\left(n\right)$.

$P_3 \Leftrightarrow P_7$.

- ($P_3 \Rightarrow P_7$). As we have P_3, then we have $CW_4 = \frac{n^4}{16} + o(n^4)$. Then, recall $\sum_{1 \leq i,j, \leq n} \binom{\hat{d}_{ij}}{2} = 2\#C_4 = \frac{CW_4}{4} + o\left(n^4\right) = \frac{n^4}{64} + o(n^4)$ where $\#C_4$ is simply the number of four cycles in G. Hence, with some intermediate theorems, we find

$$\sum_{1 \leq i,j \leq n} \hat{d}_{i,j}^2 = \frac{n^4}{32} + o\left(n^4\right).$$

Hence,

$$\sum_{1 \leq i,j \leq n} \left(\hat{d}_{ij} - \frac{n^2}{16}\right) = o\left(n^4\right).$$

Then, we see as $\sum_{1 \leq i,j \leq n} \hat{d}_{i,j} = \sum_{i=1}^{n} \binom{d_i}{2} = \sum_{i=1}^{n} \frac{d_i^2}{2} \leq \frac{n}{2}\lambda_1^2 = \frac{n^3}{8} + o\left(n^3\right)$. Then, applying subadditivity yields the desired value of $\sum_{1 \leq i,j \leq n} \left|\hat{d}_{i,j} - \frac{n}{4}\right| = o\left(n^3\right)$.

□

Proposition 1.1. Let G be random on n-vertices with all degrees about $\frac{n}{2}$ and codegrees about $\frac{n}{4}$. Then, we ask how likely is it that by changing at most $o\left(n^2\right)$ edges, we find a conference graph.