Algebraic Theory I

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Lecture 16

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> **Corollary 1.** A finite abelian group is the direct product of its sylow groups.

This follows directly from the theorem from last class.

> **Corollary 2.** If G is a finite group such that for all $n \mid |G|$ such that there are at most n elements $x \in G$ with $x^n = 1$, then G is cyclic.

Proof. Let p be an arbitrary prime with $p \mid |G|$. Let P be a sylow p-group with $|P| = p^\alpha$. We know for any $x \in P$, we have $x^{|P|} = 1$, hence there are $|P| = p^\alpha$ elements $x \in P$ such that $x^{p^\alpha} = 1$. By hypothesis there is infact equality. If there was another distinct sylow p-group we would have elements $y \notin P$ such that $y^{p^\alpha} = 1$. Hence, P is unique. Hence, as every p-group is unique, so normal, we see G is the product of its P-groups.
Denote $G = P_1 \times P_2 \times \dots P_t$ with the P_is being the distinct sylow $_i$-groups of G. Also, if $|P_1| = p_1^{\alpha_1}$, then all $x \in P_1$ have $\operatorname{ord}(x) \mid p_1^{\alpha_1}$ and there are at most $p_1^{\alpha_1 - 1} < p_1^{\alpha_1}$ such x with $\operatorname{ord}(x) \mid p_1^{\alpha_1 - 1}$. Since $|P| < p_1^{\alpha_1 - 1}$ we see there is an $x \in P_1$ with $\operatorname{ord}(x) = p_1^{\alpha_1} = |P|$, hence $\langle x \rangle = P_1$. So, P_1 is cyclic. Likewise, all other P_i are shown cyclic by the same argument, with $P_i = \langle x_i \rangle$. Then, the element $x = \prod_{i=1}^t x_i$ is a generator of G, so G is cyclic. □

Lecture 15: Nilpotent Groups (2)

Tue 28 Sep 2021 17:46

> **Lemma 0.1.** If H, K are groups, then $Z(H \times K) = Z(H) \times Z(K)$.

Proof. Let $(x, y) \in H \times K$. If $(x, y) \in Z(H \times K)$ then

$$\underbrace{(a,1)(x,y)(a,1)^{-1}}_{=(axa^{-1},1)} = (x,y).$$

Hence, $x \in Z(H)$ and similarly, $y \in Z(K)$. Hence, $Z(H \times K) \subseteq Z(H) \times Z(K)$. The other direction of inclusion is trivial and left as an exercise. □

> **Lemma 0.2.** Let $\varphi : G \to G'$ be a homomorphism with $\ker(\varphi) = K$ and $H \leq G$ such that $K \leq H$. Then, $N_G(H) = f^{-1}(N_{G'}(\varphi(H)))$.

Proof. Let $x \in N_G(H)$, so $xHx^{-1} = H$. Hence,

$$\varphi(H) = \varphi\left(xHx^{-1}\right) = \varphi(x)\varphi(H)\varphi(x)^{-1}.$$

Thus,

$$\begin{aligned} \varphi(x) &\in N_{G'}(\varphi(H)) \\ \Rightarrow x &\in \varphi^{-1}(N_{G'}(\varphi(H))) \\ \Rightarrow N_G(H) &\subseteq \varphi^{-1}(N_{G'}(\varphi(H))). \end{aligned}$$

Conversely, let $x \in \varphi^{-1}(N_{G'}(\varphi(H)))$, hence $\varphi(x) \in N_{G'}(\varphi(H))$.
Then, we see

$$\begin{aligned} \varphi(H) &= \varphi(x)\varphi(H)\varphi\left(x^{-1}\right) \\ &= \varphi\left(xHx^{-1}\right) \\ \Rightarrow xHx^{-1} &\subseteq \varphi^{-1}(\varphi(H)) \\ &= \langle H, \ker(\varphi)\rangle \\ &= H \text{ as } \ker(\varphi) \subseteq H. \end{aligned}$$

Hence, $xHx^{-1} \subseteq H$, so $x \in N_G(H)$. This concludes the proof. $\square$

Now, recall that if G is a finite group with P being a sylow p-group, then TFAE

1. P is unique.
2. $P \trianglelefteq G$.
3. P is characteristic.
4. Any subgroup generated by elements whose orders are powers of p is itself a p-group.

> **Theorem 0.1.** If G is a finite group, then the following are equivalent:
>
> 1. G is nilpotent.
> 2. $H < G \Rightarrow H < N_G(H)$.
> 3. All sylow p-groups are normal.
> 4. G is the direct product of its sylow p-groups.

Proof.
- $(2 \Rightarrow 3)$. Let P be a sylow p-group of G. Assume P is not normal, then denote $N = N_G(P) \subset G$. Hence, by the preceding lemma, P is characteristic in N. Then, as $N \trianglelefteq N_G(N)$, we see $P \trianglelefteq N_G(N)$. But $N = N_G(P)$ was the largest subgroup in which P was normal, hence $N_G(P) = N_G(N)$. So, by contrapositive of the assumption, (2), we have $N = N_G(N)$, so $N = G$, hence $P \trianglelefteq G$.

- $(3 \Rightarrow 4)$.
- $(1 \Rightarrow 2)$. Let G be nilpotent. If G is abelian, then $N_G(A) = G$ for all $A \leq G$, hence any proper subgroup $H < G$ has $H < N_G(H) = G$. Hence, assume G is non-abelian and proceed by induction on $|G|$ with base case $|G| = p$ being already completed p-prime. Suppose indirectly that there is an $H < G$ such that $H = N_G(H)$.
  Now, we note that $Z(G) \leq N_G(H) = H$ by definition of $Z(G)$. That is, $Z(G) \leq H$. Let $\varphi : G \to G/Z(G)\,,\ x \mapsto \varphi(x) = xZ(G)$. Since G is nilpotent, $Z(G) = 1 \Leftrightarrow G = 1$, but we assumed G to be nonabelian, so this is not the case. Hence, we can assume $Z(G) = \{1\}$, hence $|G/Z(G)| < |G|$. As we know, G being nilpotent implies $G/Z(G)$ is nilpotent. Lastly, we note that $Z(G) \leq H < G$, so by the lattice theorem, we have $H/Z(G) < G/Z(G)$. Applying the induction hypothesis yields $H/Z(G) < N_{G/Z(G)}(H/Z(G))$. Recalling the lemma from last class, $\varphi^{-1}\left(N_{G/Z(G)}(H/Z(G))\right) = N_G(H)$. Then, we note

  $$\varphi^{-1}(\varphi(H)) < \varphi^{-1}\left(N_{\varphi(G)}(\varphi(H))\right) = N_G(H)\,.$$

  And as $\ker(\varphi) = Z(G) \leq H$, we have $H < N_G(H)$.

□