Combinatorics

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Contents

Lecture 22: Quasi-Random Graphs (5)

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Recall. A quasi-random graph could be characterized as one with a gray adjacency matrix.

Example. A paley graph of order q is quasi-random.
For this graph G, we see

- $e(G) = \frac{1}{2}q\frac{q-1}{2} = \frac{1}{4}q^2 + o\left(q^2\right)$,
- $\lambda_1(G) = \frac{q-1}{2} = \frac{1}{2}q + o(q)$, and
- $\sigma_2(G) = \frac{1+\sqrt{q}}{2} = o(q)$.

Hence, G is P_3, so it is quasirandom.
We also have a conference graph $\mathrm{SRG}(4k+1, 2k, k-1, k)$ has

- $\lambda_1 = 2k = \frac{n}{2} + o(n)$,
- $\sigma_2 = \frac{1+\sqrt{n}}{2} = o(n)$, and
- $e(G) = k(4k+1) = \frac{1}{4}n^2 + o\left(n^2\right)$.

We also have $K_{n,n}$ and cK_n are trivially SRG, but not quasi-random. As it turns out these are the only SRG which are not quasi-random. $\diamond$

Proposition 0.1. All nontrivial SRG (not $K_{n,n}$ or cK_n) are quasi-random.

Remark. A random graph of order n is quasi-random with probability 1 as $n \to \infty$.

Definition 0.1 (Perturbation)**.** Let G be a quasi-random graph of order n with adjacency matrix A. We may perturb G by choosing a set E of edges such that $|E| = o\left(n^2\right)$ and deleting them. From this we obtain a graph $G' = G - E$. We find G' is also quasi-random.

Proof. Let G' be the result of perturbing a quasi-random graph G having adjacency matrix A and let A' be the adjacency matrix of G'. Then, denote B to be the adjacency matrix containing only the deleted edges. So, we find $A' = A - B$. We wish to show $\lambda(A') = \lambda(A) + o(n)$ and $\sigma_2(A') = \sigma_2(A) + o(n) = o(n)$. Now employing Weyl's inequalities:

$$\lambda_i(A) + \inf\{\lambda_i(B) : 1 \le i \le n\} \le \lambda_i(A+B) \le \lambda_i(A) + \lambda_i(B)$$

yields

$$\lambda_i(A) + \lambda_{\min}(-B) \le \lambda_i(A') \le \lambda_i(A) + \lambda_1(-B).$$

We see it suffices to show $\lambda_{\min}(-B) = o(n)$ and $\lambda_1(-B) = o(n)$.
Recall that $\lambda_1^2(-B) + \ldots + \lambda_n^2(-B) = |-B|_2^2 = 2|E|$, hence $\lambda_1^2(-B) \le 2|E| = o(n)$ and likewise for $\lambda_{\min}^2(-B)$. Hence, we have $\lambda_i(B) = o(n)$, so

$$\lambda_1(A) + o(n) \le \lambda_1(A') \le \lambda_1(A) + o(n).$$

So, $\lambda_1(A')$ is desired. Similairly, WLOG we can assume $\lambda_2(A) = \sigma_2(A)$, so we see

$$\lambda_2(A) + o(n) \le \lambda_2(A') \le \lambda_2(A) + o(n).$$

and as $\lambda_2(A) = o(n)$ by quasi-randomness, we see $\lambda_2(A') = \sigma_2(A') = o(n)$.

□

Remark. This also clearly works with addition of $o\left(n^2\right)$ edges (provided they will fit). Furthermore, we can union a quasi-random graph with a graph of sufficiently small order and obtain a quasi-random graph.

Proposition 0.2. Let G be quasi-random with adjacency matrix A and construct the following matrix

$$J_2 \otimes A = \begin{bmatrix} A & A \\ A & A \end{bmatrix}.$$

Then, the graph G' obtained from this matrix is the blowup of G. We see for G being regular, we have G' is regular. It turns out G' is also quasi-random. However, we find G being SRG does not guarantee G' to be SRG.

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We prove the preservation of Regularity and Quasi-Randomness and provide a counterexample for SRG from last time.

Proof. First, we prove regularity. If G is k-regular, then we see all rowsums are k. Hence, we find all row sums of G' to $2k$, so G' is $2k$-regular.
For quasi-randomness, denote our adjacency matrix of G' to be $B = J_2 \otimes A$ and recall the eigenvalues of this product are simply the products of the eigenvalues of the factors. Hence, our eigenvalues are $2\lambda_1, 2\lambda_2, \ldots, 2\lambda_n, 0, \ldots, 0$. Furthermore, as G is quasi-random, we have that $\lambda_1 = \frac{1}{2}n + o(n)$ and $|\lambda_i| = o(n)$

for $n \geq 2$. Applying this yields $2\lambda_1 = n + o(n)$ and $|2\lambda_i| = o(n)$, $i \geq 2$. Hence, G' is quasirandom.

$\square$

Remark. In general $J_i \otimes A$ preserves regularity and quasi-randomness of A by the same argument.

Proposition 0.3. If G, H are quasi-random graphs with adjacency matrices A, B we have $A \otimes B$ induces a quasi-random graph.

Proof. Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of G and $\mu_1, \dots, \mu_n$ to be the eigenvalues of H. Then, the eigenvalues of $A \otimes B$ would have eigenvalues $\lambda_i \mu_j$ and we see $\lambda_1 \mu_1$ is the largest eigenvalue. For the second largest (in magnitude) eigenvalues, we see there are four potential candidates, $\lambda_1 \mu_2$, λ_1, μ_n, $\mu_1 \lambda_2$, $\mu_1 \lambda_n$. Then, we know $\lambda_1 \leq n - 1$ and $\mu_2 = o(m)$, hence $|\lambda_1 \mu_2| = o(nm)$. Similair constructions follow for the other candidates to prove that $G \otimes H$ is infact quasi-random. $\square$

Proposition 0.4. Let A_{ij}, $1 \leq i, j \leq k$ be the adjacency matrices of quasi-random graphs of order n and $e(A_{ij}) = \frac{1}{4}n^2 + o(n^2)$ with $A_{ij} = A_{ji}$. We arrange these matrices in a $kn \times kn$ matrix

$$B = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1k} \\ A_{21} & A_{22} & \dots & A_{2k} \\ \vdots & \vdots & \vdots & \vdots \\ A_{k1} & A_{k2} & \dots & A_{kk} \end{bmatrix}.$$

Then, we find the graph induced by B to be quasi-random.

Definition 0.2 (Bipartite Quasi-Random Graph)**.** A bipartite graph, $G(A, B)$ with $|A| = |B|$ and density p , is **Bipartite Quasi-Random** if it obeys one of the following (equivalent) tweaked quasi-random properties

- (P_2). $e(G) \geq pn^2 + o(n^2)$ and $\#CW_4 \leq p^4 n^4 + o(n^4)$.
- (P_3). $e(G) \geq pn^2 + o(n^2)$ and $\lambda_1 = pn + o(n)$ and $\lambda_2 = o(n)$.
- (P_4). For all $X \subseteq A$, $Y \subseteq B$, we find $|e(X, Y) - p|X||Y|| \leq o(n^2)$.

Recall. G is bipartite on two sets of size k if and only if the eigenvalues of G are $\lambda_1, \lambda_2, \dots, \lambda_k, -\lambda_k, -\lambda_{k-1}, \dots, -\lambda_1$.

Definition 0.3 (Bipartite Double)**.** We define the **Bipartite Double** of a graph G with adjacency matrix A to be the graph induced by

$$B = \begin{bmatrix} 0_{n\times n} & A \\ A & 0_{n\times n} \end{bmatrix}.$$

Essnetially, this splits G into two graphs G, G' such that a vertex $x \in G$ is connected to all of its neighbors, but in G' and similairly, a $x' \in G'$ will be connected to all of its neighbors, but in G. Hence, this induces a bipartite graph yielding some interesting properties.

Example. If G is regular, we find the bipartite double of G to be regular.
Furthermore, the bipartite double of C_3 is C_6.
Similarly, the bipartite double of K_3 is $K_{3,3}$.
The bipartite double of a graph which is already bipartite is simply 2 independent of the original graph.
For example, the double of $K_{2,2}$ is $2K_{2,2}$. $\diamond$

Using the bipartite double we can construct new bipartite quasi-random graphs.

Proposition 0.5. If G is quasi-random and A is its adjacency matrix, then the bipartite double induced by

$$\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}$$

is bipartite quasi-random.

Problem. Prove that P_3 (for a general quasi-random graph) implies the existence of a subgraph isomorphic to C_k with $k \geq n + o\,(n)$.