Analysis I

Thomas Fleming

November 30, 2021

Contents

Lecture 24: Riesz Representation Theorem

Tue 23 Nov 2021 14:02

Definition 0.1 (Signum Function)**.** We define the **sign function** to bf

$$\operatorname{sgn} \overline{\mathbb{R}} \longrightarrow \{-1, 0, 1\}$$
$$\operatorname{sgn}(x) \longmapsto \operatorname{sgn}(\operatorname{sgn}(x)) = \chi_{(0,\infty]}(x) - \chi_{[-\infty,0)}(x).$$

Note, if g is measurable, $\operatorname{sgn}(g)$ is measurable.

Remark. If $g : S \to \overline{\mathbb{R}}$ is measurable, then $\operatorname{sgn}(g^*)$ is simple. Moreover, $g \operatorname{sgn}(g) = |g|$.

Theorem 0.1. Let $S \subseteq \mathbb{R}$ be measurable with $1 \le p \le \infty$, and q being p's conjugate. For $g \in L^q(S)$, define the map

$$\varphi : L^p(S) \longrightarrow \mathbb{R}$$
$$f \longmapsto \varphi(f) = \int_S fg.$$

Then φ is a bounded linear functional on $L^p(S)$ with norm $\|\varphi\| = \|g\|_q$. In particular $\varphi(f) = 0$ for all $f \in L^p(S)$ if and only if $g = 0$ almost everywhere.

Proof. By Holder,

$$|\varphi(f)| \le \int_S |fg| \le \|f\|_p \cdot \|f\|_q.$$

Hence, φ is well defined, and since φ is linear we found it to be a bounded linear functional with $\|\varphi\| \le \|g\|_q$.
It remains to be shown that $\|\varphi\| = \|g\|_q$.First, we can assume $\|g\|_q > 0$. Then, if $1 < p \le \infty$ we have $1 \le q < \infty$. Define

$$f = \|g\|_q^{1-q} |g|^{q-1} \operatorname{sgn}(g).$$

We see $f \in L^p(S)$ and $\|f\|_p = 1$. Moreover,

$$\varphi(f) = \int_S \|g\|^{1-q} |g|^{q-1} |g| \qquad = \|g\|_q^{1-q} \underbrace{\int_S |g|^q}_{\|g\|_q^q}$$
$$= \|g\|_q.$$

Hence, we see $\|g\|_q \leq \|\varphi\|$.
Lastly, consider the case $p = 1$ ($q = \infty$). For $\varepsilon > 0$, let $E_\varepsilon = \{x \in S : |g(x)| \geq \|\varphi\| + \varepsilon\}$ and define $f_n = \chi_{E_\varepsilon \cap (-n,n)} \operatorname{sgn}(g)$ for $n \in \mathbb{N}$. Then, $f_n \in L_1(S)$ and

$$\|f_n\|_1 = m(E_\varepsilon \cap (-n,n)).$$

Then, we see

$$\begin{aligned} \|f_n\|_1 \|\varphi\| &\geq \varphi(f_n) \\ &= \int_S g f_n \\ &\geq (\|p\| + \varepsilon) \cdot \|f_n\|_1. \end{aligned}$$

This implies $\|f_n\|_1 = 0$ as all other possibilities have already been ruled out. Then, we see $m(E_\varepsilon) = 0$. Then, letting $E = \bigcup_{k \in \mathbb{N}} E_{\frac{1}{k}} = \{x \in S : |g(x)| > \|\varphi\|\}$ we see $m(E) = 0$, so $|g(x)| \leq \|\varphi\|$ almost everywhere (i.e. $\|g\|_\infty \leq \|\varphi\|$ so the claim is shown.
The additional claim about $\varphi(f) = 0$ is then trivial. $\square$

Lemma 0.1. Let $[a,b] \subseteq \mathbb{R}$ with $1 \leq p < \infty$ and q being p's conjugate. Suppose $g : [a,b] \to \overline{\mathbb{R}}$ is measurable and finite almost everywhere. If there is a $M \geq 0$ so that $\left|\int_{[a,b]} gs\right| \leq M\|s\|_p$ for every simple function $s \in L^p(\mathbb{R})$, then $g \in L^q([a,b])$ and $\|g\|_q \leq M$.

Proof. Consider $p = 1$ and let $E_\varepsilon = \{x \in [a,b] : |g(x)| \geq M + \varepsilon\}$ for some $\varepsilon > 0$. Define $f_\varepsilon = \chi_{E_\varepsilon} \operatorname{sgn}(g^*)$. Since E_ε is measurable and contained within $[a,b]$, then $m(E_\varepsilon) < \infty$ and f_ε is simple in $L^1(\mathbb{R})$ so that

$$Mm(E_\varepsilon) = M(\|f_\varepsilon\|_1) \geq \int_{[a,b]} g f_\varepsilon = \int_{E_\varepsilon} g f_\varepsilon \geq (M + \varepsilon) m(E\varepsilon).$$

Again, we find $m(E_\varepsilon) = 0$, so taking the union over all such $E_{\frac{1}{k}}$ yields $|g(x)| \leq M$ almost everywhere, hence the claim is shown.
For the case $1 < p < \infty$, we see g measurable implies a sequence of simple functions (s_n) so that $\lim\limits_{n \to \infty} s_n(x) = |g^*(x)|$ for all $x \in \mathbb{R}$ and $0 \leq s_n \leq |g^*|$ for all n. Next, define a sequence of simple functions (t_n) with $t_n = s_n^{q-1} \operatorname{sgn}(g^*)$. Since $|g^*| \geq s_n \geq 0$, we find $t_n(x) = 0$ for $x \notin [a,b]$. Hence, $t_n \in L^p(\mathbb{R})$ with

$$\int |t_n|^p = \int |s_n| (pq - p) = \int |s_n|^q.$$

Moreover,

$$\begin{aligned}
\|s_n\|_q^q &= \int s_n^q \\
&= \int s_n^{q-1} s_n \qquad\qquad \leq \int_{[a,b]} s_n^{q-1} |g| \\
&= \int_{[a,b]} g \operatorname{sgn}(g)\, s_n^{q-1} \\
&= \int_{[a,b]} g t_n \\
&\leq M \cdot \|t_n\|_p \\
&= M \cdot \left(\int \underbrace{\left(s_n^{q-1}\right)^p}_{s_n^q} \right)^{\frac{1}{p}} \\
&= M \cdot \left(\int s_n^q \right)^{\frac{1}{q} \cdot \frac{q}{p}} \\
&= M \|s_n\|_q^{\frac{q}{p}}
\end{aligned}$$

. Hence, $\|s_n\|_q^q \leq M \|s_n\|_q^{\frac{q}{p}}$. Dividing yields

$$\|s_n\|_q^{q - \frac{q}{p}} = \|s_n\|_q \leq M.$$

Applying Fatous lemma

$$\begin{aligned}
\int_{[a,b]} |g| &= \int_{[a,b]} \left(\lim_{n\to\infty} s_n^q \right) \\
&\leq \liminf_{n\to\infty} \int_{[a,b]} s_n^q \\
&\leq M^q.
\end{aligned}$$

□

Theorem 0.2 (Riesz Representation Theorem)**.** Let $S \subseteq \mathbb{R}$ be measurable with $1 \leq p < \infty$ and q being p's conjugate. Then, for every bounded linear functional $\varphi : L^p(S) \to \mathbb{R}$ there is a unique $g \in L^q(S)$ so that

$$\pi(f) = \int_S fg \ \forall\ f \in L^p(S)$$

and $\|\varphi\| = \|g\|_q$.

Proof. Defining $\varphi^*(f) = \varphi(f\,|_S)$ for some $f \in L^p(\mathbb{R})$, we see φ is a bounded linear functional on $L^p(\mathbb{R})$ whiles preserving its norm. Hence, we can assume $S = \mathbb{R}$.

Let $[a,b] \subseteq R$ and define the following function

$$\begin{aligned} F : [a,b] &\longrightarrow \mathbb{R} \\ x &\longmapsto F(x) = \varphi\left(\chi_{[a,x]}\right). \end{aligned}$$

Given a finite disjoint collection $\{(a_k, b_k) : 1 \leq k \leq n\} \in [a,b]$ with each interval being nonempty $(a_k < b_k)$. Define $s_k = \operatorname{sgn}\left(F\left(b_k\right) - F\left(a_k\right)\right)$. Then, linearity yields

$$\begin{aligned} \varphi\left(\sum_{k=1}^{n} \delta_k \chi_{(a_k,b_k]}\right) &= \sum_{i=1}^{n} \delta_k \varphi\left(\chi_{(a_k,b_k]}\right) \\ &= \sum_{k=1}^{n} \delta_k \left(\varphi\left(\chi_{[a,b_k]} - \chi_{[a,a_k]}\right)\right. \\ &= \sum_{k=1}^{n} \left|F\left(b_k\right) - F\left(a_k\right)\right|. \end{aligned}$$

Since $\left|\sum_{k=1}^{n} \delta_k \chi_{(a_k,b_k]}\right|^p = \sum_{k=1}^{n} \chi_{(a_k,b_k]}$, we see

$$\begin{aligned} \sum_{k=1}^{n} \left|F\left(b_k\right) - F\left(a_k\right)\right| &\leq \|\varphi\| \left(\int \sum_{k=1}^{n} \chi_{(a_k,b_k]}\right) \\ &= \|\varphi\| \left(\sum_{k=1}^{n} \left(b_k - a_k\right)\right)^{\frac{1}{p}}. \end{aligned}$$

Hence, we find F to be absolutely continuous.
Now, for $n \in \mathbb{N}$, define $I_n = [-n, n]$ and define the functions

$$\begin{aligned} F_n : I_n &\longrightarrow \mathbb{R} \\ x &\longmapsto F_n(x) = \varphi\left(\chi_{(-n,x]}\right). \end{aligned}$$

We see each F_n is absolutely continuous, so applying the fundamental theorem of calculus, we find a $g_n \in L^1\left(I_n\right)$ so that $F_n\left(x\right) = \int_{[-n,x]} g_n$ for $x \in I_n$ and $F_n' = g_n$ almost everywhere on I_n.
Since $F_{n+1}\left(x\right) = \varphi\left(\chi_{(-(n+1),n]}\right) + F_n\left(x\right)$ for $x \in I_n$. Since this differs only by a constant, we see $g_{n+1} = g_n$ almost everywhere on I_n.
Hence, the sequence of integrable functions $\left(g_n^*\right)$ converges pointwise almost everywhere to a measurable $g : \mathbb{R} \to \overline{\mathbb{R}}$. Moreover, every bounded interval $I \subseteq \mathbb{R}$ has $\varphi\left(\chi_I\right) = \int_I \left(g\right) = \int g\chi_I$ since, there is an $N \in \mathbb{N}$ so that $g\chi_I = g_N^* \chi_I$ almost everywhere.
Hence linearity yields $\varphi\left(\psi\right) = \int g\psi$ for every step function ψ. Applying the density results from the previous lecture yields the result for all $f \in L^p\left(\mathbb{R}\right)$. □

Lecture 25

Tue 30 Nov 2021 14:16