# Algebraic Theory I

Thomas Fleming

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### Lecture 13 — Fri 24 Sep 2021 11:30

I originally missed this lecture so it is transcribed from a classmates notes.

### Lecture 14 — Fri 24 Sep 2021 11:30

Let $G$ be a group, and $Z_0(G) = \{1\}$ with $Z_1(G) = Z(G)$. Thus, $G/Z_1(G)$ is a group which has $Z(G/Z_1(G)) = \frac{Z_2(G)}{Z_1(G)}$ where $Z_2(G)$ is the preimage of $Z(G/Z_1(G))$, that being the subgroup of $G$ containing $Z_1(G)$. We see we may continue

$$\begin{aligned} Z_2(G)/Z_1(G) &= Z(G/Z_1(G)) \\ \text{then, } (G/Z_1(G))/(Z_2(G)/Z_1(G)) &\simeq G/Z_2(G) \\ \text{which has a center } Z(G/Z_2(G)) &= Z_3(G)/Z_2(G). \end{aligned}$$

**Definition 0.1** (Nilpotence)**.** We recursively define $Z_i(G)$ to be the subgroup such that $Z(G/Z_i(G)) = Z_i(G)/Z_{i-1}(G)$. This yields a growing sequence $Z_0(G) \trianglelefteq Z_1(G) \trianglelefteq Z_2(G) \trianglelefteq \ldots$. We say a group $G$ is **nilpotent** if $G = Z_n(G)$ for some $n \geq 0$. The minimal $n \geq 0$ for which this is the case is called the **nilpotent class** of $G$.

**Example.** The trivial group $\{1\}$ is nilpotent with class $c = 0$.
A nontrivial abelian group is nilpotent with class $c = 1$.

$\diamond$

**Theorem 0.1.** Every finite $p$-group is nilpotent.

*Proof.* We know the center of a nontrivial $p$-group to be nontrivial and its subgroups and quotient groups will also be $p$-groups. Hence $Z_1(G)$ is nontrivial except in the case $G$ is trivial. Hence we have that $Z_2(G)/Z_1(G)$ is nontrivial unless $Z_2(G) = G$. Hence either $Z_1 < Z_2$ or $Z_2 = G$. Now, denote $|G| = n$. Then either $1 = |Z_0| < |Z_1| < \ldots < |Z_n|$ hence $Z_n = G$ or $Z_i = G$ for some $i < n$, so $Z_n = G$. Hence, $G$ is nilpotent. $\square$

**Definition 0.2.** A subgroup $H \leq G$ is **characteristic** if for every automorphism of $G$, we have $\alpha(H) = H$. This is equivalent to $\alpha(H) \leq H$ for all automorphisms as $\alpha^{-1} : G \to G$ is also an automorphism, hence $H \leq \alpha(H)$, so equality holds. Since conjugation is always an automorphism, being characteristic implies normality.

**Proving vs. Using Characteristicness**

This means that in order to show that something is characteristic we need only show $\alpha(H) \leq H$, but when we use that something is characteristic we will often use the full equality.

**Lemma 0.1.** As we know $K \trianglelefteq H$ and $H \trianglelefteq G$ does not imply $K \trianglelefteq G$. On the other hand, $K$ being characteristic in $H$ and $H \trianglelefteq G$ does yield $K \trianglelefteq G$.

*Proof.* Let $\alpha_x : G \to G$ be the conjugation by $x$ map. We know this to be an automorphism of $G$, hence as $H$ is normal, we have $\alpha_x |_H : H \to H$ is an automorphism of $H$, and since $K$ is characteristic in $H$, we see an automorphism of $H$ fixed $K$, hence $\alpha_x(K) = xKx^{-1} = K$ for all $x \in G$, hence $K \trianglelefteq G$ . $\square$

**Lemma 0.2.** Let $G$ be a finite group with $p$ being prime and $P$ being a sylow $p$-group in $G$. Then, the following are equivalent

1. $P$ is the unique sylow $p$-group in $G$.
2. $P \trianglelefteq G$.
3. $P$ is characteristic in $G$.
4. Any subgroup generated by elements whose orders are each powers of $p$ is itself a $p$-group.

*Proof.*

1. We have already shown $1 \Leftrightarrow 2$.
2. As conjugation is always an automorphism, we see $2 \Leftarrow 3$ is trivial.
3. We show $1 \Rightarrow 3$. Let $\alpha : G \to G$ be an arbitrary automorphism of $G$. Then, $\alpha(P) \leq G$ and $|P| = |\alpha(P)|$. As $P$ is the unique sylow $p$-group, we see there is no distinct group of cardinality $|P|$, hence $\alpha(P) = P$.
4. Now we show $1 \Rightarrow 4$. Let $X$ be a set satisfying $\operatorname{ord}(x) = p^n$ for each $x \in X$. Then each $\langle x \rangle$ is contained in a $p$-group, and as there is a unique maximal $p$-group, we have that $\langle x \rangle \subseteq P$ for each $x \in X$. Hence, $\langle X \rangle \subseteq P$ and as $X$ is a $p$-group we have that $X = P$.

5. $4 \Rightarrow 1$. Let $X$ to be the union of all sylow $p$-groups in $G$. By hypothesis, $\langle X \rangle$ is a $p$-group and thus it is contained in some sylow $p$-group so WLOG, we have $\langle X \rangle \subseteq P$. But if there were distinct $p$-groups, $P' \neq P$ then $P' \subseteq X$ and $P \subset \langle P' \cup P \rangle \subseteq X \subseteq P$. $\lightning$. Hence $P$ is the unique sylow $p$-group.

$\square$