Analysis I: Homework III

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Problem (14)**.** Let (x_n) be a sequence. A point x^* is called an accumulation point of (x_n) if for each $\varepsilon ?0$ and each $N \in \mathbb{N}$ there is a $n \in \mathbb{N}$ with $n \geq N$ such that $|x_n - x^*| < \varepsilon$. Show the set of all accumulation points is closed.

Solution. Denote the set of all accumulation points X of (x_n) and let $x \in \overline{X}$. Then, for all $\varepsilon > 0$, we have $X \cap (x - \varepsilon, x + \varepsilon) \neq \varnothing$. Hence, for every $\frac{\varepsilon}{2} > 0$ there is an accumulation point $x^* \in X$ such that $x^* \in \left(x - \frac{\varepsilon}{2}, x + \frac{\varepsilon}{2}\right)$. Thus, $|x - x^*| < \frac{\varepsilon}{2}$. Furthermore, for each $\frac{\varepsilon}{2} > 0$ and $N \in \mathbb{N}$ there is an $n \in \mathbb{N}$ such that $|x^* - x_n| < \frac{\varepsilon}{2}$. Combining these yields for each $\varepsilon > 0$ and $N \in \mathbb{N}$, a $n \in \mathbb{N}$ with $n \geq N$ such that

$$\begin{aligned}
|x_n - x| &= |x_n - x^* - (x - x^*)| \\
&\leq |x_n - x^*| + |x - x^*| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Hence, x is an accumulation point, so $X \subseteq \overline{X} \subseteq X$, so $X = \overline{X}$ and X is closed. ■

Problem (15)**.** Let S be a set of nonnegative real numbers. Define $\sum_{x\in S} x = \sup\{\sum_{x\in S_0} x : S_0 \subseteq S \text{ is finite}\}$. Prove if $\sum_{x\in S} x < \infty$, then S is countable.

Solution. We induce a countable covering of S by finite sets. Note that for each $n \in \mathbb{N}$ we must have at most finitely many $x \in S$ such that $x \geq \frac{1}{n}$. Otherwise, there would be a family of sets $\hat{S}_i$ where $\hat{S}_i$ contains i elements $x \geq \frac{1}{n}$, hence $\sum_{x\in S_i} x \geq \frac{i}{n}$ for all $i \in \mathbb{N}$, hence we would have

$$\sup\{\sum_{x\in S_0} x : S_0 \subseteq S \text{ is finite}\} \geq \sup\{\sum_{x\in \hat{S}_i} x : i \in \mathbb{N}\} \geq \sup\{\frac{i}{n} : i \in \mathbb{N}\} > M$$

for all $M \in \mathbb{R}$, hence our sum would be unbounded, so $\sum_{x\in S} x \not< \infty$ $\lightning$.
Thus, the set $\{x \in S : x \geq \frac{1}{n}\}$ is finite for all $n \in \mathbb{N}$. Then, we have that

$$\bigcup_{n\in\mathbb{N}} \{x \in S : x \geq \frac{1}{n}\} = (0, \infty) \cap S = S \setminus \{0\} \text{ by nonnegative assumption.}$$

Hence, we have a countable covering of $S \setminus \{0\}$ by finite sets, so $S \setminus \{0\}$ is countable. Thus, S is countable. ■

Problem (16)**.** For a collection $\mathscr{S}$ of subsets of X, denote the smallest σ-algebra containing $\mathscr{S}$ by $\sigma(\mathscr{S})$. Let $\mathscr{C}$ be a collection of subsets of X and let $\mathscr{U}$ be the collection of all countable subcollections $\mathscr{F} \subseteq \mathscr{C}$. Hence, each subcollection $\mathscr{F}$ contains only countable many subsets of X. Prove $\bigcup_{\mathscr{F} \in \mathscr{U}} \sigma(\mathscr{F})$ is a σ-algebra which is equal with $\sigma(\mathscr{C})$.

Solution. First, we show $\bigcup_{\mathscr{F} \in \mathscr{U}} \sigma(\mathscr{F})$ is a σ-algebra. As each $\sigma(\mathscr{F})$ is a σ-algebra, we have that $X \in \sigma(\mathscr{F})$ so $X \in \bigcup_{\mathscr{F} \in \mathscr{U}} \sigma(\mathscr{F})$.
Next, let $A \in \bigcup_{\mathscr{F} \in \mathscr{U}} \sigma(\mathscr{F})$. Then, $A \in \sigma(\mathscr{F})$ for some $\mathscr{F} \in \mathscr{U}$, hence $A^c \in \sigma(\mathscr{F})$, so $A^c \in \bigcup_{\mathscr{F} \in \mathscr{U}} \sigma(\mathscr{F})$.
Lastly, let $(A_k)_{k\in\mathbb{N}}$ be a countable collection of elements $A_k \in \bigcup_{\mathscr{F} \in \mathscr{U}} \sigma(\mathscr{F})$. Then, each $A_k \in \sigma(\mathscr{F}_k)$ for some $\mathscr{F}_k \in \mathscr{U}$. As each $\mathscr{F}_k$ is countable, then $\bigcup_{k\in\mathbb{N}} \mathscr{F}_k$ is countable, hence $\bigcup_{k\in\mathbb{N}} \mathscr{F}_k \in \mathscr{U}$ by definition of $\mathscr{U}$. Thus, $\sigma\left(\bigcup_{k\in\mathbb{N}} \mathscr{F}_k\right) \subseteq \bigcup_{\mathscr{F} \in \mathscr{U}} \sigma(\mathscr{F})$ and as $\bigcup_{k\in\mathbb{N}} A_k \in \sigma\left(\bigcup_{k\in\mathbb{N}} \mathscr{F}_k\right)$, we see $\bigcup_{k\in\mathbb{N}} A_k \in \bigcup_{\mathscr{F} \in \mathscr{U}} \sigma(\mathscr{F})$.

Note that it is clear as each $\mathscr{F} \subseteq \mathscr{C}$ that each $\sigma(\mathscr{F}) \subseteq \sigma(\mathscr{C})$ hence $\bigcup_{\mathscr{F} \in \mathscr{U}} \sigma(\mathscr{F}) \subseteq \sigma(\mathscr{C})$.
Now, we show equality. Let $A \in \sigma(\mathscr{C})$. If $A \in \mathscr{C}$, then $A \in \mathscr{F}$ for some $\mathscr{F} \in \mathscr{U}$ hence $A \in \bigcup_{\mathscr{F} \in \mathscr{U}} \sigma(\mathscr{F})$. Otherwise, we may have that $A = B^c$ for some $B \in \mathscr{C}$. Hence $B \in \mathscr{F}^*$ for some $\mathscr{F}^* \in \mathscr{U}$, so $A^c \in \sigma(\mathscr{F}^*)$ hence it is in the union. Else, as $\sigma(\mathscr{C})$ is the minimal σ-algebra containing $\mathscr{C}$ we must have that $A = \bigcup_{i=1}^{\infty} B_i$ for $B_i \in \mathscr{C}$. Hence, each $B_i \in \mathscr{F}_i$ for some $\mathscr{F}_i \in \mathscr{U}$, hence every $B_i \in \bigcup_{\mathscr{F} \in \mathscr{U}} \sigma(\mathscr{F})$, so $\bigcup_{i\in\mathbb{N}} B_i = A \in \bigcup_{\mathscr{F} \in \mathscr{U}} \sigma(\mathscr{F})$ as this is a σ algebra. Hence, we see $A \in \bigcup_{\mathscr{F} \in \mathscr{U}} \sigma(\mathscr{F})$ for all $A \in \sigma(\mathscr{C})$, so $\sigma(\mathscr{C}) = \bigcup_{\mathscr{F} \in \mathscr{U}} \sigma(\mathscr{F})$. ■