

# Analysis I

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## Lecture 19: End of Convergence and Functions of Bounded Variation

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Recall we had the dominated convergence theorem. A similar version of the theorem makes use of convergence in measure as follows.

**Theorem 0.1** (Dominated Convergence - Convergence in Measure). Let  $(f_n)$  be a sequence of measurable functions  $f_n : \mathbb{R} \rightarrow \overline{\mathbb{R}}$  and suppose there is an integrable function  $g : \mathbb{R} \rightarrow \overline{\mathbb{R}}$  so that  $|f_n| \leq g$  for all  $n \in \mathbb{N}$ . If  $(f_n) \rightarrow f : \mathbb{R} \rightarrow \overline{\mathbb{R}}$  in measure, (with  $f$  measurable), then  $f$  is integrable and  $\lim_{n \rightarrow \infty} \int |f_n - f| = 0$  and  $\lim_{n \rightarrow \infty} \int f_n = \int f$ .

*Proof.* First, note a subsequence of  $(f_n)$  converges to  $f$  pointwise almost everywhere. Hence, we find  $|f| \leq g$  almost everywhere, so  $f$  is integrable. We can assume  $|f_n - f| \leq 2g$  (almost) everywhere. Then, we find a subsequence  $(g_n) = (f_{n_k})$  such that  $\limsup_{n \rightarrow \infty} |f_n - f| = \lim_{n \rightarrow \infty} |g_n - f|$ . Then, as  $(g_n) \rightarrow f$  in measure, we find another subsequence  $(h_j) = (g_{k_j}) = (f_{n_{k_j}})$  which converges pointwise to  $f$  almost everywhere.

Applying dominated convergence theorem yields

$$\lim_{n \rightarrow \infty} \int |h_j - f| = 0.$$

Then, we find

$$\begin{aligned} \limsup_{n \rightarrow \infty} \int |f_n - f| &= \lim_{n \rightarrow \infty} \int |g_k - f| \\ &= \lim_{n \rightarrow \infty} \int |h_j - f| \\ &= 0. \end{aligned}$$

This completes the proof.  $\square$

# 1 Functions of Bounded Variation and Absolutely Continuous Functions

**Remark.** For this chapter  $[a, b] \subseteq \mathbb{R}$  will always denote a compact interval on  $\mathbb{R}$ .

**Definition 1.1** (Partition). A finite sequence  $P = (x_k)_{k=n}^N$  with  $n, N \in \mathbb{Z}$  and  $n < N$  is called a **partition** of  $[a, b]$  if  $x_n = a$ ,  $x_N = b$  and  $x_{k-1} \leq x_k$  for  $n < k \leq N$ . We denote the collection of all partitions of  $[a, b]$  to be  $\mathcal{P}([a, b])$ .

**Definition 1.2.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be a function. Then,

- For a partition  $P = (x_k)_{k=n}^N$ , we denote

$$V(f, P) = \sum_{k=n+1}^N |f(x_k) - f(x_{k-1})|$$

to be the **variation of  $f$  with respect to  $P$** .

- We define the quantity  $\text{TV}(f) = \sup\{V(f, P) : P \in \mathcal{P}([a, b])\}$  to be the **total variation of  $f$** .

**Remark.** If  $f : [a, b] \rightarrow \mathbb{R}$  and  $c \in [a, b]$  with partitions  $P_1 = (x_k)_{k=n}^N$  of  $[a, c]$  and  $P_2 = (x_k)_{k=N}^K$  of  $[c, b]$ . Then denote,  $P = (x_k)_{k=n}^K$  to be a partition of  $[a, b]$  and we find

$$V(f, P) = V(f|_{[a, c]}, P_1) + V(f|_{[c, b]}, P_2).$$

Moreover,

$$\text{TV}(f) = \text{TV}(f|_{[a, c]}) + \text{TV}(f|_{[c, b]}).$$

**Definition 1.3** (Bounded Variation). A function  $f : \mathbb{R} \rightarrow \overline{\mathbb{R}}$  has **bounded variation** if  $\text{TV}(f) < \infty$ .

**Theorem 1.1** (Jordan's Theorem). A function  $f : [a, b] \rightarrow \mathbb{R}$  is of bounded variation if and only if there are increasing functions  $g, h : [a, b] \rightarrow \mathbb{R}$  so that  $f = g - h$ .

*Proof.* Suppose  $\text{TV}(f) < \infty$  and let  $x, y \in [a, b]$  with  $x < y$ . Then, we find

$$\begin{aligned} \text{TV}(f|_{[a,y]}) &= \text{TV}(f|_{[a,x]}) + \text{TV}(f|_{[x,y]}) \\ &\geq \text{TV}(f|_{[a,x]}) + |f(y) - f(x)| \\ &\geq \text{TV}(f|_{[a,x]}) + f(x) - f(y). \end{aligned}$$

Furthermore,  $h : x \mapsto \text{TV}(f|_{[a,x]})$  and  $g : x \mapsto \text{TV}(f|_{[a,x]}) + f(x)$  are increasing. This fact is trivial for  $h$  and we find, adding  $f(y)$  to both sides of the former inequality yields  $g(y) \geq g(x)$  for arbitrary  $y \geq x$ , so this claim holds as well.

Taking the difference,  $g - h = f$ .

Conversely, suppose  $f = g - h$  for increasing  $g, h : [a, b] \rightarrow \mathbb{R}$ . Then, let  $x, y \in [a, b]$  with  $y \geq x$ . Then, we find

$$\begin{aligned} |f(y) - f(x)| &= |g(y) - g(x) + h(x) - h(y)| \\ &\leq |g(y) - g(x)| + |h(x) - h(y)| \\ &= g(y) - g(x) + h(y) - h(x). \end{aligned}$$

Hence, for a partition  $P = (x_k)_{k=n}^N$ , we find

$$\begin{aligned} V(f, P) &= \sum_{k=n+1}^N |f(x_k) - f(x_{k-1})| \\ &\leq \sum_{k=n+1}^N (g(x_k) - g(x_{k-1}) + h(x_k) - h(x_{k-1})) = g(b) - g(a) + h(b) - h(a) \\ &< \infty. \end{aligned}$$

□

**Definition 1.4** (Absolute Continuity). A function  $f : [a, b] \rightarrow \mathbb{R}$  is **absolutely continuous** if for each  $\varepsilon > 0$  we find a  $\delta > 0$  such that for every finite disjoint collection of nonempty intervals  $\{(a_k, b_k) \subseteq [a, b] : 1 \leq k \leq K\}$  with  $\sum_{k=1}^K (b_k - a_k) < \delta$ , we have  $\sum_{k=1}^K |f(a_k) - f(b_k)| < \varepsilon$ .

**Remark.** Absolute continuity is stronger than uniform continuity, but weaker than Lipschitz continuity.

**Theorem 1.2.** If a function  $f : [a, b] \rightarrow \mathbb{R}$  is absolutely continuous, then  $f$  is continuous and  $f$  has bounded variation.

*Proof.*  $f$  is trivially continuous, taking a finite disjoint collection consisting only of 1 interval  $\{(x, y)\}$  yields the definition of continuity.

Now we show bounded variation. For  $\varepsilon = 1$ , let  $\delta > 0$  be the number such that the definition of absolute continuity holds for  $f$ .

Now fix  $(x_k)_{k=n}^N \in \mathcal{P}([a, b])$  so that  $x_k - x_{k-1} < \delta$  for all  $n < k \leq N$ . Then, if  $P \in \mathcal{P}([x_{k-1}, x_k])$ , we see  $V(f|_{[x_{k-1}, x_k]}, P) < 1$  by definition of absolute

continuity.

So, we have  $\text{TV}([x_{k-1}, x_k]) \leq 1$ , so  $\text{TV}(f) = \sum_{k=n+1}^N \text{TV}(f|_{[x_{k-1}, x_k]}) \leq N - n$  by the  $\varepsilon$  assumption.  $\square$

As it turns out, absolutely continuous functions have a relation to integrable functions, particularly, an integrable function  $f$  is simply the anti-integral of an absolutely continuous one.

**Proposition 1.1.** If  $f : [a, b] \rightarrow \overline{\mathbb{R}}$  is integrable, then,

$$F : [a, b] \rightarrow \mathbb{R}, \quad x \mapsto \int_{[a, x]} f$$

is absolutely continuous.

This claim can be generalized into a sort of fundamental theorem of calculus for the lebesgue integrals to characterize integrals and derivatives. For now, we only prove the weak version.

*Proof.* For  $\varepsilon > 0$  there is a  $\delta > 0$  such that  $\int_S |f| < \varepsilon$  for every measurable set  $S \subseteq [a, b]$  with  $m(S) < \delta$ .

Now, let  $\{(a_k, b_k) : 1 \leq k \leq K\}$  be a disjoint collection of intervals in  $[a, b]$  with  $\sum_{k=1}^K (b_k - a_k) < \delta$ . Fix  $S = \bigcup_{k=1}^K (a_k, b_k)$ . Then, since  $m(S) < \delta$  and

$$\begin{aligned} \sum_{k=1}^K |F(b_k) - F(a_k)| &= \sum_{k=1}^K \left| \int_{[a_k, b_k]} f \right| \\ &\leq \sum_{k=1}^K \int_{[a_k, b_k]} |f| \\ &= \int_S |f| \\ &< \varepsilon \text{ by assumption.} \end{aligned}$$

Hence, absolute continuity holds.  $\square$

## 2 Derivatives and Fundamental Theorem of Calculus

**Proposition 2.1.** Let  $f : (a, b) \rightarrow \overline{\mathbb{R}}$  be monotone on  $(a, b) \subseteq \mathbb{R}$  with  $a, b \in \overline{\mathbb{R}}$  and  $a < b$ . Then,

$$\lim_{x \rightarrow a} f(x) = \inf\{f(x) : x \in (a, b)\}, \quad \lim_{x \rightarrow b} f(x) = \sup\{f(x) : x \in (a, b)\}$$

are both well defined.