Real Variables I: Homework I

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Problem (1)**.** Let $f : X \to Y$.

1. Show that for $A \subseteq X$, $B \subseteq Y$, $f\left(f^{-1}(B)\right) \subseteq B$ and $A \subseteq f^{-1}(f(A))$.
2. Give examples to show that the set inclusions can be proper.

Solution.

1. Let $b \in f\left(f^{-1}(B)\right)$ and note that, as b is in the image of $f^{-1}(B)$, there is $a \in f^{-1}(B)$ such that $f(a) = b$. As $a \in f^{-1}(B)$, we see $f(a) \in B$. As $f(a) = b \in B$ this completes the proof.

   Now, let $a \in A$. We see $f(a) \in f(A)$ by definition, and as $f(a) \in f(A)$ we see that for all $b \in A$ such that $f(b) = f(a) \in f(A)$, we have $b \in f^{-1}(f(A))$. It is clear that a is one such element, so $a \in f^{-1}(f(A))$. This completes the proof.

2. Let $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto f(x) = x^2$ and denote $B = [-1, 1]$. We see $f^{-1}(B) = [-1, 1]$ and $f([-1, 1]) = [0, 1]$. Hence, $f\left(f^{-1}(B)\right) = [0, 1] \subset [-1, 1] = B$.

   Now, let $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto f(x) = 0$ and denote $A = [0, 1]$. We see $f(A) = \{0\}$ and $f^{-1}(\{0\}) = \mathbb{R}$ as the function is zero everywhere. Hence $f^{-1}(f(A)) = \mathbb{R} \supset [0, 1] = A$.

Problem (2)**.** Let $A, B \subseteq X$. Prove or disprove

1. $A \triangle B = \varnothing \Leftrightarrow A = B$.
2. $A \triangle B = X \Leftrightarrow A = B^c$.

Solution. 1. Suppose $A \triangle B = \varnothing$. Then, $(A \setminus B) \cup (B \setminus A) = \varnothing$, hence $A \setminus B = \varnothing$ and $B \setminus A = \varnothing$. If $A = \varnothing$, then $B \setminus \varnothing = B = \varnothing$, hence $A = B$. Similarly, if $B = \varnothing$, $A \setminus \varnothing = A = \varnothing$, hence we can assume $A, B \neq \varnothing$. Now, as $A \setminus B = \varnothing$, we see $A \subseteq B$, similarly as $B \setminus A = \varnothing$ we have $B \subseteq A$, hence $A = B$.
Conversely,

$$\begin{aligned} A \triangle B &= A \triangle A \\ &= (A \setminus A) \cup (A \setminus A) \\ &= \varnothing \cup \varnothing \\ &= \varnothing \end{aligned}$$

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2. Now, suppose $A \triangle B = X$. If $A = \varnothing <$ we have $A \triangle B = (\varnothing \setminus B) \cup (B \setminus \varnothing) = B = X$, hence $A = B^c$. Hence, we can assume A is nonempty and $B \subset X$ is proper, hence B^c is nonempty.
Then, let $a \in A$, then we have $a \in A \triangle B$ and $a \notin B \setminus A$ by definition, hence $a \in A \setminus B$, so $a \notin B$, hence $a \in B^c$, so $A \subseteq B^c$.
Similairly, if $b \in B^c$, then we have $b \in A \triangle B$ and $b \notin B \setminus A$ by definition, so $b \in A \setminus B$, hence $b \in A$, so $A = B^c$.
Conversely,

$$\begin{aligned} A \triangle B &= B^c \triangle B \\ &= (B^c \setminus B) \cup (B \setminus B^c) \\ &= B^c \cup B \\ &= X. \end{aligned}$$

Problem (3)**.** Suppose $f : X \to Y$ and $g : Y \to Z$ are functions.

1. Show that $f : X \to Y$ is injective if and only if there is a map $g : Y \to X$ such that $g \circ f$ is the identity on X. If such a map g exists is it necessarily unique, injective, or surjective.
2. Show that f is onto if and only if there is a map $g : Y \to X$ such that $f \circ g$ is the identity on Y.

Solution. 1. Let $g : X \to Y$ be a map such that $g \circ f$ is the identity on X. Then, suppose f is not injective. Let $x, y \in X$ such that $x \neq y$ and $f(x) = f(y)$. Then $g(f(x)) = x$ element. WLOG, suppose $g(f(x)) = g(f(y)) = x$. Then, $g(f(y)) = x$ contradicts the assumption that $g \circ f$ was the identity.

Now, let $f : [0,2] \to [0,5]\,,\ x \mapsto x^2$. This is clearly injective, and their exists a fuction $g : [0,5] \to [0,2]\,, y\ \mapsto g(y) = \begin{cases} \sqrt{y}, & 0 \leq y \leq 4 \\ 0, & 4 < y \leq 5 \end{cases}$. Then, note that for $x \in [0,2]$, $f(x) \in |0,4|$ hence $g(f(x)) = \sqrt{x^2} = x$. Hence, $g \circ f$ is the identity on $[0,2]$. Clearly, g is noninjective as $g(0) = g(5) = 0$. Furthermore the function

$$\hat{g} : [0,5] \to [0,2] : x \mapsto \hat{g}(x) = \begin{cases} \sqrt{x}, & 0 \leq x \leq 4 \\ 1, & 4 < x \leq 5 \end{cases}$$

is another function such that $\hat{g} \circ f$ is the idenity on X, so g need not be unique.
Lastly we show it is surjective. Let $f : X \to Y$ be injective with $g : X \to Y$ being a map such that $g \circ f$ is the identity and let $x \in X$. Then, note that $f(x) \in Y$, hence $g(f(x)) = x \in g(Y)$. Hence $X \subseteq g(Y)$, so g is surjective.

2. Suppose $g : Y \to X$ is a map such that $f \circ g$ is the identity on Y. Then, we have for each $y \in Y$, $f(g(y)) = y$, so, we see

$$\bigcup_{y \in Y} f(g(\{y\})) = f\left(g\left(\bigcup_{y \in Y} \{y\}\right)\right) = f(g(Y)) = \bigcup_{y \in Y} \{y\} = Y.$$

That is, $f(g(Y)) = Y$, hence the image of f is Y, so f is a surjection.

Now, suppose f is onto, hence $\left|f^{-1}(\{y\})\right| \geq 1$ for each $y \in Y$. Let $h : \{f^{-1}(\{y\}) : y \in Y\} \to \bigcup_{y \in Y} f^{-1}(\{y\})$ be a choice function mapping each singleton preinverse to an element within it.
Let $k : Y \to \{f^{-1}(y) : y \in Y\},\ y \mapsto f^{-1}(\{y\})$. Then denote $h \circ k = g$. Note that $g(y) \in f^{-1}(\{y\})$ by properties of choice function. Hence, $(f \circ g)(y) = y$ for each $y \in Y$, so it is the inverse on y.

Problem (4)**.** Prove or disprove the following. If $\mathscr{A}$ is a σ-Algebra of subsets of Y and $f : X \to Y$ is a function, then the collection $\{f^{-1}(A) : A \in \mathscr{A}\}$ is a σ-Algebra of subsets of X.

Solution. First, denote the collection $\{f^{-1}(A) : A \in \mathscr{A}\} = \mathscr{B}$. We show all three conditions:

1. As $Y \in \mathscr{A}$ and $f(X) \subseteq Y$ necessarily, we see $X \subseteq f^{-1}(Y)$ (as X is the whole of the domain, we can even say $X = f^{-1}(Y)$). Hence, $f^{-1}(Y) = X \in \mathscr{B}$.
2. Let $A \in \mathscr{A}$, then $A^c \in \mathscr{A}$. As $f^{-1}(A^c) = \left[f^{-1}(A)\right]^c$, we see $\left[f^{-1}(A)\right]^c \in \mathscr{B}$ (for all $f^{-1}(A) \in \mathscr{B}$).
3. Lastly, let $B_1, B_2, \ldots \in \mathscr{B}$ be a countable collection of elements with each $B_i = f^{-1}(A_i)$ for $A_i \in \mathscr{A}$ and define $\bigcup_{n\in\mathbb{N}} B_n = B$. We see $\bigcup_{n\in\mathbb{N}} A_n \in \mathscr{A}$ by hypothesis, hence

$$\begin{aligned}
\mathscr{B} &\ni f^{-1}(A) \\
&= f^{-1}\left(\bigcup_{n\in\mathbb{N}} A_n\right) \text{ by construction of } \mathscr{B} \\
&= \bigcup_{n\in\mathbb{N}} f^{-1}(A_n) \\
&= \bigcup_{n\in\mathbb{N}} B_n \\
&= B \in \mathscr{B}.
\end{aligned}$$

Hence $\mathscr{B}$ is a σ-Algebra.

Problem (5)**.** Prove the set of all polynomials with rational coefficients is rational.

Solution. Let $f = \sum_{i=0}^{n} a_i x^i$ be an arbitrary polynomial and define the finite sequence $(f_k)_{i=0}^{n}$ such that $f_k = a_k$ for each k and each polynomial f. Next, define $\mathscr{F} = \{(f_k)_{k=1}^{n} : n \text{ is finite}, f \in P_{\mathbb{Q}}(n)\}$ where $P_{\mathbb{Q}}(n)$ is the set of all rational polynomials of degree at most n. We see $\mathscr{F}$ contains a sequence corresponding to each finite polynomial with rational coefficients, hence as $\mathbb{Q}$ is countable, and $\mathscr{F}$ is a subset of the set of all finite sequences from $\mathbb{Q}$ (which is countable by a proposition in class), we see $\mathscr{F}$ is countable. As each rational polynomial of finite length, f, has a corresponding sequence $(f_k) \in \mathscr{F}$, we see the set $\{f : f \text{ is a rational polynmomial of finite length}\} \subseteq \mathscr{F}$. Hence, this set is also countable.

Problem (6)**.** Prove the set of all infinite sequences (x_k) with $x_k \in \{0,1\}$ is uncountable.

Solution. Assume indirectly that such a set is countable. Let $f : \mathbb{N} \to \{(x_k)_{k\in\mathbb{N}} : x_k \in \{0,1\}\}$, $n \mapsto f(n) = (x_{n,k})_k$. Now define a sequence (y_k) such that

$$y_k = \begin{cases} 0, & x_{k,k} = 1 \\ 1, & x_{k,k} = 0 \end{cases}$$

We see (y_k) differs from each sequence , $f(n) = (x_{n,k})_k$ in the n-th position. Hence, f is not surjective, so there is no bijection from $\{(x_k) : x_k \in \{0,1\}\} \to \mathbb{N}$, so the set is not countable.

Problem (7)**.** Let A be a set and $B = \{0,1\}$. Prove there exists a bijection from $\mathscr{P}(A)$ to the set of all functions from A to B.

Solution. Define the set of all functions from A to B as $\mathscr{F}(A,B)$. Define a function $f : \mathscr{P}(A) \to \mathscr{F}(A,B)$ such that for $X \in \mathscr{P}(A)$, $f(X) = g : A \to B$ such that for $a \in A$,

$$g(a) = \begin{cases} 0, & a \in X \\ 1, & a \notin X \end{cases}.$$

This is clearly a function as each element $X \in \mathscr{P}(A)$ has either $a \in X$ or $a \notin X$ for every $a \in A$. Now, we check that it it bijective.
Suppose $X, Y \in \mathscr{P}(A)$ such that $f(X) = g = f(Y)$. Then, we see for each element $a \in A$, $f(X)(a) = g(a) = f(Y)(a)$, hence if $a \in X$, then $a \in Y$. Similarly, if $a \notin X$, then $a \notin Y$. Hence, as every $a \in X$ has $a \in Y$ and every $a \notin X$ has $a \notin Y$, we see $X = Y$, so f is an injection. Now, we wish to show that $f(\mathscr{P}(A)) = \mathscr{F}(A,B)$. As we already know $f(\mathscr{P}(A)) \subseteq \mathscr{F}(A,B)$, we must only show the reverse containment holds. Let $g \in \mathscr{F}(A,B)$. Then, for each element $a \in A$, $g(a) = 0$ or 1. Define a new set J such that

$$\begin{cases} a \in J, & g(a) = 1 \\ a \notin J, & g(a) \neq 1 \end{cases}.$$

We see $J \subseteq A$, hence $J \in \mathscr{P}(A)$ as it contains some (perhaps all) of the elements of A, and

$$\begin{aligned} f(J)(a) &= \begin{cases} 1, & a \in J \\ 0, & a \notin J \end{cases} \\ &= g(a) \end{aligned}$$

so $g \in f(\mathscr{P}(A))$. Hence $f(\mathscr{P}(A)) = \mathscr{F}(A,B)$, so f is a bijection.

Problem. Let $X = Z \times (Z \setminus \{0\})$. Define a relation $\sim$ on X such that $(p,q) \sim (u,v)$ if $pv = qu$.

1. Show that $\sim$ is an equivalence relation on X.

2. Show that there exists a bijection $f : (X/\sim) \to \mathbb{Q}$.

Solution.

1. First we show is reflexive. Note that $pq = pq$, hence $(p,q) \sim (p,q)$.
   Now, we show it is symmetric. Note that if $pv = qu$, then $uq = vp$, hence $(p,q) \sim (u,v) \Rightarrow (u,v) \sim (p,q)$.
   Lastly, we show transitivity. Suppose $(a,b) \sim (c,d)$ and $(c,d) \sim (e,f)$. Then this implies $ad = bc$ and $cf = de$ dividing through by the (guaranteed) nonzero term in both equations yields $\frac{a}{b} = \frac{c}{d}$ and $\frac{c}{d} = \frac{e}{f}$, hence $\frac{a}{b} = \frac{e}{f}$ so $af = eb$, so $(a,b) \sim (e,f)$.
   Hence the relation is an equivalence relation.

2. Now, we wish to induce a bijection between $(X/\sim)$ and $\mathbb{Q}$, this will follow directly from the proof of transitivity. For each equivalence class $[(a,b)] \in (X/\sim)$ define $f([(a,b)]) = \frac{a}{b}$ (we know this is well defined as the second element is guaranteed to be nonzero). Now, we wish to show that the choice of representative is unimportant, so let $(a,b),(c,d) \in [(a,b)]$ (hence $(a,b) \sim (c,d)$). From the previous proof, we see that dividing by the nonzero term yields $\frac{a}{b} = f([(a,b)]) = \frac{c}{d} = f([(c,d)])$ hence the choice of representative produces the same rational.
   Now, we wish to show this mapping is injective. Suppose two different equivalence classes, $x, y \in (X/\sim)$ have $f(x) = f(y)$. Let $(x_1, x_2) \in x$ and $(y_1, y_2) = y$ be representatives of each equivalence class. Then, this implies $f(x) = \frac{x_1}{x_2} = \frac{y_1}{y_2} = f(y)$. Multiplying through by the denominators yields $x_1 y_2 = y_1 x_2$, hence $(x_1, x_2) \sim (y_1, y_2)$, so $x = y$. Lastly, we wish to show this is a surjection. Let $\frac{p}{q} \in \mathbb{Q}$ be a rational. Then, by definition, $p \in \mathbb{Z}$ and $q \in Z \setminus \{0\}$, so $[(p,q)] \in (X/\sim)$ and $f([(p,q)]) = \frac{p}{q}$, so there is an equivalence class that produces each rational.
   Hence, the mapping $f : (X/\sim) \to \mathbb{Q} : [x_1, x_2] \mapsto f([(x_1, x_2)]) = \frac{x_1}{x_2}$ is a bijection.