MATH 8237

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MORE ON QUADRATIC FORMS

Let G be a graph of order n and let λ_1 be its largest eigenvalue. The Rayleigh principle implies that

$$\lambda_1 = 2\max\left\{ \sum_{\{i,j\}\in E(G)} x_i x_j \;:\, x_1^2+\cdots+x_n^2 = 1 \right\}$$

Here is practical consequence of this equation:

Proposition 1 *Let G is a graph of order n, and A be its adjacency matrix. For any $\mathbf{x} := (x_1,\ldots,x_n)$, we have*

$$2\sum_{\{i,j\}\in E(G)} x_i x_j \leq \lambda_1 \left(x_1^2+\cdots+x_n^2\right) = \lambda_1 \left|\mathbf{x}\right|_2^2 \tag{1}$$

or in matrix form

$$\langle A\mathbf{x},\mathbf{x}\rangle \leq \lambda_1 \left|\mathbf{x}\right|_2^2.$$

Likewise, for λ_n we have

$$2 \sum_{\{i,j\} \in E(G)} x_i x_j \geq \lambda_n \left(x_1^2 + \cdots + x_n^2 \right) \tag{2}$$

or in matrix form

$$\langle A\mathbf{x}, \mathbf{x} \rangle \geq \lambda_n \left| \mathbf{x} \right|_2^2 .$$

Proof of (1) Indeed, if $\mathbf{x}$ is the zero vector, we obviously have equality in (1).

If $\mathbf{x}$ is nonzero, then $\left|\mathbf{x}\right|_2 > 0$, so we can define the n-vector $\mathbf{y}$ by

$$\mathbf{y} := \frac{1}{\left|\mathbf{x}\right|_2} \mathbf{x}.$$

Clearly

$$\left|\mathbf{y}\right|_2 = \frac{1}{\left|\mathbf{x}\right|_2} \left|\mathbf{x}\right|_2 = 1.$$

Hence, Rayleigh's principle implies that

$$\begin{aligned}\lambda_1 &\geq 2 \sum_{\{i,j\} \in E(G)} y_i y_j = 2 \sum_{\{i,j\} \in E(G)} \frac{1}{|\mathbf{x}|_2} x_i \frac{1}{|\mathbf{x}|_2} x_j \\ &= \frac{2}{|\mathbf{x}|_2^2} \sum_{\{i,j\} \in E(G)} x_i x_j,\end{aligned}$$

and so,

$$2 \sum_{\{i,j\} \in E(G)} x_i x_j \leq \lambda \, |\mathbf{x}|_2^2 = \lambda_1 \left(x_1^2 + \cdots + x_n^2 \right).$$

∎

The proof of inequality (2) is very similar and is omitted.

Here is one more detail, which is given with no proof:

Corollary 2 *Let G be a graph of order n and $(x_1, \ldots, x_n)$ be a nonzero vector. If*

$$2 \sum_{\{i,j\} \in E(G)} x_i x_j = \lambda_1 \left(x_1^2 + \cdots + x_n^2 \right),$$

then $(x_1, \ldots, x_n)$ is an eigenvector to λ_1.

Likewise, if

$$2 \sum_{\{i,j\} \in E(G)} x_i x_j = \lambda_n \left(x_1^2 + \cdots + x_n^2 \right),$$

then $(x_1, \ldots, x_n)$ is an eigenvector to λ_n.

SPECTRA OF DISCONNECTED GRAPHS

Recall that the spectrum of a graph G is the multiset of the roots of its characteristic polynomial, hereafter denoted by $Sp\left(G\right)$.

Let G be a graph that is union of two disjoint graphs G_1 and G_2. Hence the adjacency matrix of G can be written as a block matrix:

$$A\left(G\right)=\begin{bmatrix} A\left(G_1\right) & 0 \\ 0 & A\left(G_2\right) \end{bmatrix}.$$

For the characteristic polynomial $\phi\left(G\right)$ we get

$$\phi\left(G\right)=\det\left[xI-A\left(G\right)\right]=\det\begin{bmatrix} xI'-A\left(G_1\right) & 0 \\ 0 & xI''-A\left(G_2\right) \end{bmatrix}.$$

A short inspection leads us to the conclusion that

$$\phi\left(G\right)=\phi\left(G_1\right)\phi\left(G_2\right).$$

Therefore,

$$Sp\left(G\right) = Sp\left(G_1\right) \sqcup Sp\left(G_2\right),$$

where $\sqcup$ stands for the union of multisets.

In particular for $\lambda_1\left(G\right)$ and $\lambda_{\min}\left(G\right)$ we see that

Corollary 3 If G is a union of two disjoint graphs G_1 and G_2, then

$$\lambda_1\left(G\right) = \max\left\{\lambda_1\left(G_1\right), \lambda_1\left(G_2\right)\right\}$$

and

$$\lambda_{\min}\left(G\right) = \min\left\{\lambda_{\min}\left(G_1\right), \lambda_{\min}\left(G_2\right)\right\}.$$

MONOTONICITY OF THE SPECTRAL RADIUS

It turns out that $\lambda_1(G)$ is monotone with respect to taking subgraphs:

Proposition 4 If $H \subset G$, then

$$\lambda_1(H) \leq \lambda_1(G). \tag{3}$$

Indeed, let $\mathbf{x} := (x_1, \ldots, x_p)$ be a nonnegative unit eigenvector to $\lambda_1(H)$.

If G has more vertices than H, let us extend $\mathbf{x}$ by adding zero entries to $\mathbf{x}$ for each of the extra vertices of G. Write $\mathbf{x}' = (x'_1, \ldots, x'_n)$ for the resulting vector.

Clearly, $|\mathbf{x}'|_2 = |\mathbf{x}|_2 = 1$, and (3) follows by Rayleigh's principle

$$\lambda_1(H) = 2 \sum_{\{i,j\} \in E(H)} x_i x_j \leq 2 \sum_{\{i,j\} \in E(G)} x'_i x'_j \leq \lambda_1(G).$$

PARTIAL MONOTONICITY OF THE SMALLEST EIGENVALUE

For the smallest eigenvalue $\lambda_{\min}(G)$ the situation is somewhat different.

Proposition 5 If H is an *induced* subgraph of G, then

$$\lambda_{\min}(G) \leq \lambda_{\min}(H). \tag{4}$$

Indeed, let $\mathbf{x} := (x_1, \ldots, x_p)$ be a unit eigenvector to $\lambda_{\min}(H)$.
If G has more vertices than H, let us extend $\mathbf{x}$ by adding zero entries to $\mathbf{x}$ for each of the extra vertices of G. Write $\mathbf{x}' = (x'_1, \ldots, x'_n)$ for the resulting vector.

Clearly, $|\mathbf{x}'|_2 = |\mathbf{x}|_2 = 1$, and (4) follows by Rayleigh's principle

$$\lambda_{\min}(H) = 2 \sum_{\{i,j\} \in E(H)} x_i x_j = 2 \sum_{\{i,j\} \in E(G)} x'_i x'_j \geq \lambda_{\min}(G).$$

Easy examples show that the inequality $\lambda_1(H) \leq \lambda_1(G)$ cannot be improved in general.

Example

Let G be the union of two disjoint copies of K_3. Then $K_3 \subset G$.

On the other hand, both K_3 and G are 2-regular. Hence,

$$\lambda_1(K_3) = 2 = \lambda_1(G).$$

Remark on the eigenvectors of $\lambda_1(G)$. Suppose that the vertices of the two copies of K_3 are $\{1, 2, 3\}$ and $\{4, 5, 6\}$.

We know that $(1, 1, 1, 1, 1, 1)$ is an eigenvector to $\lambda_1(G)$. Using Corollary 2 one can check that the vectors

$$(1, 1, 1, -1, -1, -1)$$
$$(1, 1, 1, 0, 0, 0)$$

are also eigenvectors to $\lambda_1(G)$.

BASIC FACTS FORM PERRON-FROBENIUS THEORY

The Perron-Frobenius theory has been developed for the largest eigenvalue of square nonnegative matrices, and therefore it has implications for graphs as well.

Below we shall spell the basic statements of this theory for graphs:

- **Theorem 6** If G is a *connected* graph and $\mathbf{x}$ is an eigenvector to $\lambda_1(G)$, then all entries of $\mathbf{x}$ are nonzero and have the same sign.

- **Theorem 7** If G is a connected graph, then $\lambda_1(G)$ has multiplicity one.

- **Corollary 8** If G is a connected graph, then $\lambda_1(G)$ has a unique up to scaling nonnegative eigenvector, which is positive.

- **Theorem 9** If G is a connected graph and λ is an eigenvalue with a nonnegative eigenvector, then $\lambda = \lambda_1(G)$.

Remark Note that if a graph G has two nontrivial components G_1 and G_2 such that

$$\lambda_1(G_1) > \lambda_1(G_2),$$

then $\lambda_1(G) = \lambda_1(G_1)$, and therefore $\lambda_1(G_2)$ is an eigenvalue of G with nonnegative eigenvector (why) such that

$$\lambda_1(G_2) < \lambda_1(G).$$

Definition Let G be a graph. Any nonnegative unit eigenvector to $\lambda_1(G)$ is called a **Perron vector** of G.

If G is connected, then G has a unique Perron vector, which is positive.

If G is not connected it may not have a positive Perron vector.

Proposition 10 *If G is the disjoint union of K_2 and K_3, then G has no positive Perron vector.*

Indeed, suppose that the vertices of K_3 are $\{1, 2, 3\}$ and the vertices of K_2 are $\{4, 5\}$.

Note that

$$\lambda_1(G) = \max\{\lambda_1(K_2), \lambda_1(K_3)\} = \max\{1, 2\} = 2.$$

Let $(x_1, x_2, x_3, x_4, x_5)$ be a positive vector with $x_1^2 + \cdots + x_5^2 = 1$. We shall show that

$$2 \sum_{\{i,j\} \in E(G)} x_i x_j < 2 = \lambda_1(G).$$

Indeed, using Proposition 1, we see that

$$\begin{aligned} 2 \sum_{\{i,j\} \in E(G)} x_i x_j &= 2x_1x_2 + 2x_2x_3 + 2x_3x_1 + 2x_4x_5 \\ &\leq \lambda_1(K_3)\left(x_1^2 + x_2^2 + x_3^2\right) + \lambda_1(K_2)\left(x_4^2 + x_5^2\right) \\ &< \lambda_1(K_3)\left(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2\right) \\ &= 2. \end{aligned}$$

IMPROVING THE MONOTONICITY THEOREM

The Perron-Frobenius's theorems allow us to strengthen the monotonicity theorem:

- **Theorem 11** If H is a proper subgraph of a connected graph G, then

$$\lambda_1(H) < \lambda_1(G).$$

- Indeed, assume for a contradiction that $\lambda_1(H) = \lambda_1(G)$ and let $\mathbf{x} = (x_1, \ldots, x_p)$ be a nonnegative unit eigenvector to $\lambda_1(H)$.

- If G has more vertices than H, let us extend $\mathbf{x}$ by adding zero entries to $\mathbf{x}$ for each of the extra vertices of G.

  Write $\mathbf{x}' = (x'_1, \ldots, x'_n)$ for the resulting vector.

We see that $|\mathbf{x}'|_2 = |\mathbf{x}|_2 = 1$, and

$$\lambda_1(H) = 2 \sum_{\{i,j\} \in E(H)} x_i x_j \leq 2 \sum_{\{i,j\} \in E(G)} x'_i x'_j \leq \lambda_1(G) = \lambda_1(H).$$

Hence, equalities hold above and $\mathbf{x}'$ is an eigenvector to $\lambda_1(G)$.

Since G is connected, $\mathbf{x}'$ must be positive, that is $V(H) = V(G)$ and $\mathbf{x}' = \mathbf{x}$.

Since H is a proper subgraph of G, there is an edge $\{p, q\} \in E(G)$ that does not belong to $E(H)$. Hence,

$$\begin{aligned} \lambda_1(H) = 2 \sum_{\{i,j\} \in E(H)} x_i x_j &< 2x_p x_q + 2 \sum_{\{i,j\} \in E(H)} x_i x_j \\ &\leq 2 \sum_{\{i,j\} \in E(G)} x_i x_j \leq \lambda_1(G), \end{aligned}$$

contradicting the assumption that $\lambda_1(H) = \lambda_1(G)$.

THANK YOU