Combinatorics

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Contents

Lecture 17: Semi-circle Law

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Recall that for eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ we have $\lambda_1 = \frac{n}{2} + \sqrt{n \log(n)} = o(n)$. Additionally, we know $\sigma_1 = \lambda_1$ and $\sigma_2, \sigma_3, \ldots, \sigma_n$ correspond to $|\lambda_2|, |\lambda_3|, \ldots, |\lambda_n|$. Further, it is known by Furedi and Kowlos that $\sigma_2 = O(\sqrt{n})$.

Theorem 0.1. For a randomly chosen graph of order n, with eigenvalues $\lambda_2 \geq \lambda_3 \geq \ldots \geq \lambda_n$.Define $W_n(x) : \mathbb{R} \to \mathbb{Z}^+$ to be the number of eigenvalues λ_i, such that $\frac{\lambda_i}{\sqrt{n}} \leq x$, divided by n. Then, we find the function which $W_n(x)$ tends to pointwise, $W(x)$ has $W(x) = \begin{cases} \frac{2}{\pi}\sqrt{1-x^2}, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases}$

Here recall that $\sqrt{1-x^2}$ is an upper half semicircle of radius 1 and the factor $\frac{2}{\pi}$ compresses it into an ellipse. This fact essentially characterizes the distribution of eigenvalues of a random graph. That is, plurality of eigenvalues will be 0 and we find the number of eigenvalues of a given magnitude decreases as $\lambda \to \sqrt{n}$. We note that the leading $\frac{2}{\pi}$ is to normalize the area such that this is a probability density function. Then, we note $E\left[x^2 W(x)\right] = \int_{-1}^{1} \frac{2}{\pi} x^2 \sqrt{1-x^2} dx = \frac{1}{4}$. Hence, we find $\frac{1}{n^2} \sum_{i=2}^{n} \lambda_i^2 \approx \frac{1}{4}$.
It is a well known result that $\sum_{i=1}^{n} |\lambda_i| = \sum_{i=1}^{\infty} \sigma_i \leq \frac{1}{2} n^{\frac{3}{2}} \leq 2(n-1)$. Applying our integral formula from earlier yields $\sum_{i=1}^{\infty} |\lambda_i| = \int_{-1}^{1} |x| \sqrt{1-x^2} = 2\int_0^1 x\sqrt{1-x^2}$.
At this point, Runze found a contradiction in the argument and we ended class early.

Lecture 18: Semi-Circle Law Corrections and Quasi-Random Graphs

Mon 04 Oct 2021 10:21

Let G be a random graph of order n and denote $N(x)$ to be the number of eigenvalues λ such that $\frac{\lambda}{\sqrt{n}} \leq x W_n(x) = \frac{1}{n} N(x)$. Then, we find the sequesnce of functions approaches

$$W\left(x\right)=\begin{cases}0, & x\leq -1\\ \frac{2}{\pi}\int_{-1}^{x}\sqrt{1-x^{2}}dx, & -1<x<1\\ 1, & x\geq 1\end{cases}.$$

Furthermore, we even find W to be continuous in the hole real line and $W_h\left(x\right)$ converges to $W\left(x\right)$ earlier.

1 Quasi-Random Graphs

Definition 1.1. Let G be a graph of order n with M being an arbitrary subgraph of K_n. We define $N_G^*\left(M\right)$ to be the number of labeled induced copies of M in G. Equivalently,

$$N_G^*\left(M\right)=\left|\left\{\alpha:\ \alpha:V\left(M\right)\to V\left(G\right)\right\}\right|$$

with each α preserving adjacenc and $\alpha\left(V\left(M\right)\right)$ being isomorphic to M.

Example. $N_G^*\left(K_2=2e\left(G\right)\right)$.
$N_G^*\left(C_4\right)=\frac{1}{64}n^4+o\left(n^4\right)$. This is because every copy of K_4 in G has 8 copies isomorphic to C_4. Furthermore there are 3 symmetries of a K_4 copy, so altogether we get $\frac{1}{24}\binom{n}{4}\cdot\frac{1}{2^6}=\frac{n^4}{64}+o\left(n^4\right)$. ◇

Definition 1.2 (Graph Properties)**.** The following are equivalent:

- We define an infinite family of graphs with arbitrary orders $\mathscr{G}$ to have property $P_1\left(s\right)$ or **property I** with power s if for all graphs M of order s, we find $N_G^*\left(M\right)=\frac{n^s}{2^{\binom{n}{2}}}+o\left(n^s\right)$ for each $G\in\mathscr{G}$ having order n.
- A family $\mathscr{G}$ has property P_2 or **property II** if $e\left(G\right)\geq\frac{n^2}{4}+o\left(n^2\right)$ and the number of closed walks of order 4, $CW_4\left(G\right)\leq\frac{n^4}{16}+o\left(n^4\right)$ for each $G\in\mathscr{G}$ of order n.
- A family $\mathscr{G}$ has property P_3 or **property III** if $\lambda_1\left(G\right)=\frac{n}{2}+o\left(n\right)$ and $\sigma_2\left(G\right)=o\left(n\right)$ for all $G\in\mathscr{G}$ of order n.
- A family $\mathscr{G}$ has property P_4 or **property IV** if for all sets S we have $\left|e\left(S\right)-\frac{1}{4}^{\left|S\right|^2}\right|=o\left(n^2\right)$ for all $G\in\mathscr{G}$ of order n.
- A family $\mathscr{G}$ has property P_5 or **property V** if for all sets S of order $\left\lfloor\frac{n}{2}\right\rfloor$ we find $\left|e\left(S\right)-\frac{1}{16}n^2\right|=o\left(n^2\right)$ for all $G\in\mathscr{G}$ of order n.
- A family $\mathscr{G}$ has property P_7 or **property VII** if $\sum_{1\leq i,j\leq n}\left|\hat{d}\left(v_i,v_j\right)-\frac{n}{4}\right|=o\left(n^3\right)$ for $G\in\mathscr{G}$ of order n and $v_i,v_j\in V\left(G\right)$>

We find

$$P_2\Rightarrow P_1\left(s\right)\Rightarrow P_3\Rightarrow P_4\Rightarrow P_5\Rightarrow P_7\Rightarrow P_2.$$

Example. It is trivial to find that in order for G to be $P_1(2)$ it must have $e(G) = \frac{n^2}{4} + o(n^2)$.
We see if $|S| = \frac{1}{2}n$ we obtain P_5 from P_4.
Random graphs and Payley graphs are P_5. $\diamond$