Combinatorics

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May 7, 2018

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Lecture 32

Mon 15 Nov 2021 10:24

> **Definition 0.1** (Cut Norm of Matrices)**.** Let A, be an $m \times n$ (possibly complex) matrix and define the **cut norm** of A to be
>
> $$\|A\|_\square = \sup\{\left|\sum_{i\in S, j\in T} a_{ij}\right| : S \subseteq [m], T \subseteq [n]\}.$$

Remark. If $A \geq 0$ is a nonnegative real matrix, we find

$$\|A\|_\square = |A|_1 = \sum_{i\in[m], j\in[n]} a_{ij}.$$

Similairly, for a nonpositive real matrix we find the cut norm to again be the modulus of the sum of entries.
Moreover, the cut norm is in fact a norm, as it is always nonnegative, it is only zero in the case of a zero matrix, it behaves linearly with real multiplication, and with a bit of derivation we find it obeys the triangle inequality.

Example. $\| \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \|_\square = 1$ as any rectangle yields a sum 0 and the square consisting of just a_{11} yields a sum 1.
$\| \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \|_\square = 2$ taking either a_{11}, a_{12}, or a_{11}, a_{21}, or simply summing over the whole matrix.

$$\|J_n - 2I_n\|_\square = \| \begin{bmatrix} -1 & 1 & \dots & 1 \\ 1 & -1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & -1 \end{bmatrix} \|_\square = n\,(n-2)$$

by taking the whole matrix. It is simple to show that if $|S|$ or $|T| \leq n-2$, then the sum over their entries must be strictly less than $n\,(n-2)$. Then, this leaves only four possibilities, the possible permutations of sets of size $n-1$ and n. If $|S| = n-1$ and $|T| = n$ (WLOG) we see the sum of entries is at most

$(n-1)(n-2) < n(n-2)$. Lastly, if $|T| = |S| = n-1$, then we have exactly one row and one column missing, so the sum of their entries will be

$$n(n-2) - r_i - c_j + 1 \leq n(n-2).$$

Hence, we have the claim is shown. $\diamond$

Proposition 0.1. The $2n \times 2n$ matrix $J_n \otimes (J_2 - 2I_2) = A$ has

$$\|A\|_\Box = \|J_n \otimes (J_2 - 2I_2)\|_\Box = \| \begin{bmatrix} -J_n & J_n \\ J_n & -J_n \end{bmatrix} \|_\Box = n^2.$$

Proof. Note that for any individual row of length ℓ, we find the row sum $r_i \leq \begin{cases} \ell, & \ell \leq n \\ \ell - 2(\ell - n), & \ell > n \end{cases}$ and similairly for the column sums. Denoting $|S| = a$, $|T| = b$ for sets S, T which maximize the element sums, we first,note one of $a, b > n$ else the sum would be less than n^2. Hence, we find $\|A\|_\Box \leq \inf\{a(n-2(b-n)), b(n-2(b-n))\}$. Moreover, we find both a and $b > n$, hence we can assume WLOG $a \geq b > n$ and the solution follows by minimizing the two quadratic upper bounds. $\Box$

Remark. We wish to examine the cut norm of a hadamard matrix. We will show a hadamard matrix H has $\|H\|_\Box \leq n^{\frac{3}{2}} = n\sqrt{n}$.
The key to this proof is to let x, y be the indicator vectors for the sets S, T on which the maximum is obtained respectively. Then we find $\|H\|_\Box = |\langle Hy, x\rangle| \leq \sigma_1(H)\|x\|_2\|y\|_2$ (this is true for any value). Applying the fact that $\sigma_1(H) = \sqrt{n}$ and $\|x\|, \|y\| \leq \sqrt{n}$ as H is hadamard and x, y are indicator vectors of length n and from this we obtain the earlier upper bound.

We can generalize the first steps of this argument to any matrix A in the following way:

Proposition 0.2. For an arbitrary $m \times n$ matrix A, we find

$$\|A\|_\Box \leq \sigma_1(A)\sqrt{mn}.$$

Lecture 33

Mon 07 May 2018 03:04

Let A be a $m \times n$ matrix with $\vec{x} \in \mathbb{R}^n$, $\vec{y} \in \mathbb{R}^m$ and $|\vec{x}|_\infty \leq 1$ and $|\vec{y}|_\infty \leq 1$. Then, we consider $\max|\langle A\vec{x}, \vec{y}\rangle| = \|A\|_\pi$.

Proposition 0.3. We claim

$$\|A\|_\Box \leq \|A\|_\pi.$$

Proof. If S, T are submatrices inducing $\|A\|_\Box$. That is

$$\left| \sum_{i \in T, j \in S} a_{i,j} \right| = \|A\|_\Box.$$

Letting $\vec{x}, \vec{y}$ be indicator vectors for S, T respectively, we see this sum is simply

$$\left| \sum_{i \in T, j \in S} a_{ij} \right| = \left| \langle Ax, y \rangle \right| \leq \max \left| \langle Ax, y \rangle \right| .$$

It is also possible to set an upper bound, $\|A\|_\pi \leq 4\|A\|_\square$. $\square$