Analysis I: Homework 8 and 9

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Problem (36)**.** Our function will be φ, the cantor-lebesque function. We have already shown it to be continuous and increasing with $\varphi(1) = 1, \varphi(0) = 0$. Moreover, letting C be the cantor set, we see $[0,1] \setminus C \coloneqq C^c$ is open in $[0,1]$ so for all $x \in C^c$, there is an $\varepsilon > 0$ so that $(x-\varepsilon, x+\varepsilon) \subseteq C^c$. Then, since for all intervals I in the $[0,1]$ complement of the cantor set, we find $I \subseteq J_{n,k}$ for some $n, k \in \mathbb{N}$, we have $\xi(I) = \{\frac{n}{2^k}\}$, so

$$\overline{D}(\varphi(x)) = \lim_{r \to 0} \sup\{\frac{\varphi(x+h) - \varphi(x)}{h} : 0 < |h| < r\} = \lim_{r \to 0} \sup\{\frac{0}{h} : 0 < |h| < r\} = 0.$$

Similarly, we find $\underline{D}(\varphi(x)) = 0$. Hence, φ is differentiable at x and since $\varphi' = 0$ almost everywhere, yet φ is not constant by the initial claim, we find φ is not absolutely continuous.

Problem (38)**.** First, note that $\varphi : \mathbb{R} \to \overline{\mathbb{R}},\ x \mapsto \sqrt{1+x^2}$ is convex and since h is integrable, we see it is finite almost everywhere. Hence, discarding the points for which $h = \infty$, we see jensens inequality yields

$$\sqrt{1+A^2} \leq \int_{[0,1]} \sqrt{1+h^2}.$$

For the second inequality, note that since h is nonnegative and $\sqrt{\cdot}$ is an increasing function we have

$$\int_{[0,1]} \sqrt{1+h^2} \leq \int_{[0,1]} \sqrt{1+2h+h^2} \leq \int_{[0,1]} 1+h = 1+A.$$

Problem (39).

- Assume (f_n) does not converge to f in measure. That is, there is an $\varepsilon > 0$ so that for all $N \in \mathbb{N}$

$$m\left(\left\{x \in \mathbb{R} : \left|f_{n_N}(x) - f(x)\right| > \varepsilon\right\}\right) > \varepsilon$$

for some $n_N \geq N$. Denote this set A_N. Then, we see

$$\int \left|f_{n_N} - f\right| \geq \int_{A_N} \left|f_{n_N} - f\right| \geq \int \varepsilon \chi_{A_N} = \varepsilon m\left(A_N\right) \geq \varepsilon^2.$$

That is, for some $\varepsilon' = \varepsilon^2 > 0$, and all $N \in \mathbb{N}$ we find an $n_N \geq N$, so that $\int |f_n - f| \geq \varepsilon'$, so f_n does not converge to f in mean.

- First, note that if $x = 0$ or 1, then $f_n(x) = x$ for all $n \in \mathbb{N}$. Then, if $x \in (0,1)$, the ratio test proves $\sum_{i=1}^{\infty} nx^n < \infty$, hence $\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} nx^n = 0$.
To see that f_n converges to 0 in measure denote $E_{\varepsilon;n} = \{x \in [0,1] : nx^n < \varepsilon\}$. Then, suppose $c \in E_{\varepsilon;n}$, then either $c = 1$ or $\lim\limits_{n\to\infty} f_n(c) = 0$. We can exclude the first case as this happens only on a set of measure 0. Hence, fixing $\varepsilon > 0$ and assuming $c \in \left[0, 1 - \frac{\varepsilon}{2}\right)$ we see there is a $N \in \mathbb{N}$ so that $f_n(c) < \varepsilon$ for all $n \geq N$. So, we have $m\left(E_{\varepsilon;n}\right) \leq m\left(\left[1 - \frac{\varepsilon}{2}, 1\right]\right) < \varepsilon$ for all $n \geq N$, so f_n converges to 0 in measure.

- Finally, to show that f does not converge in measure take $\varepsilon = \frac{1}{100}$. Then, we define $a_n = 1 - \left(\frac{1}{100}\right)^{\frac{1}{n+1}}$ we define $s_n = f_n(a_n)\chi_{[a_n,1]}$. Then, we find f_n dominates s_n for every n, hence

$$\int f_n \geq \int s_n = n\left(\frac{1}{100}^{n+1} - \frac{1}{100}\right) \geq n\left(100^{-2} - \frac{1}{100}\right) = n\delta$$

for all $n \geq 1$. Since this grows linearly with n, we find for sufficiently large n, $n\delta > \varepsilon$. Hence, it is shown.

Problem (40).

- The first function will be $f_n = \chi_{(n,\infty)}$. We note that for all x, $x \notin (n, \infty)$ for all $n \geq \lceil x \rceil$, so (f_n) converges point wise. On the other hand for $\varepsilon = \frac{1}{2}$, we see $m\left(\left\{x \in \mathbb{R} : |f_n(x) - f(x)| > \frac{1}{2}\right\}\right) = m((n, \infty)) = \infty > \varepsilon$, so (f_n) does not converge in measure (hence not in mean).

- For the second function define the following sequence of intervals. $A_1 = [0,1]$, $A_{2^k} = \left[0, \frac{1}{2^k}\right]$ and $A_{2^k+c} = \left[\frac{c}{2^k}, \frac{c+1}{2^k}\right]$ for $c < 2^k$. This essentially enumerates all partitions with endpoints being a rational with denominators powers of 2 and consecutive numerators. Since the collection $\{A_{2^k+c} : 0 \leq c < 2^k\}$ covers $[0,1]$ for every $k \in \mathbb{N}$, we see for all $N \in \mathbb{N}$ and $x \in [0,1]$, the function $f_n = \chi_{A_n}$ will have $f_n(x) = 1$ for some (infinitely many) $n \geq N$, so it will not converge to 0 pointwise. On the other hand, we see $|f_n - 0| = f_n = \chi_{A_n}$, so $\int |f_n - 0| = m(A_n)$. Moreover, for all $k \in \mathbb{N}$ we find an $N = \lfloor \log_2(n) \rfloor$ so that $m(A_n) < \frac{1}{2^k}$ for all $n \geq N$, so f_n does in fact converge in mean and in measure.

- For the third function we adopt the same intervals from part 2, but we instead define the function $f_n = 2^n \chi_{A_n}$. Recalling that $m(A_n) \geq \frac{1}{2^n}$ for all n, we see $\int |f_n - 0| = \int 2^n \chi_{A_n} = 2^n m(A_n) \geq \frac{2^n}{2^n} = 1$ for all $n \in \mathbb{N}$. Hence for all $\varepsilon < 1$ we find convergence in mean to fail. Moreover, f_n still fails to converge pointwise. Lastly, recall for all $k \in \mathbb{N}$ there is a $N \in \mathbb{N}$ so that $m(A_n) \leq \frac{1}{2^k}$ for all $n \geq N$, hence for all $\varepsilon > \frac{1}{2^k}$ we find the convergence in measure criterion holds. Since there is a $k \in \mathbb{N}$ so that $0 < \frac{1}{2^k} < \varepsilon$ for all $\varepsilon > 0$, we see convergence in measure does in fact hold true.

Problem (41)**.** First, note that $\|g\|_1 = \int_S |g| \leq \operatorname{esssup}(g) \cdot m(S) = \|g\|_\infty$ as all values taken on a set of measure > 0 will be smaller in modulus than $\operatorname{esssup}(g)$. Then, we find

$$
\begin{aligned}
\int_S |f| \int_S |g| &= \|f\|_1 \|g\|_1 \\
&\geq \|f\|_1 \|g\|_\infty \text{ by the first result.} \\
&\geq \|fg\|_1 \text{ by holder's inequality.} \\
&= \int_S |fg| \\
&\geq \int_S 1 \text{ by assumption.} \\
&= 1
\end{aligned}
$$

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Problem (42).

1. First, we prove the case $s < \infty$. Let $f \in L^s(S)$. Then, we define r so that $\frac{1}{s} + \frac{1}{r} = \frac{1}{p}$ (hence $\frac{s}{p}$ and $\frac{r}{p}$ are conjugate). Then, as we aim to show $\|f\|_p$ finite, we see it suffices to show $\|f\|_p^p = \int_S |f|^p = \|f^p\|_1$ finite. We see

$$\begin{aligned}
\|f\|_p^p &= \|1f\|_p^p \\
&= \|1^p f^p\|_1 \\
&\leq \|1\|_{\frac{r}{p}} \|f^p\|_{\frac{s}{p}} \\
&= (\int_S 1^{\frac{r}{p}})^{\frac{p}{r}} \left(\int_S |f^p|^{\frac{s}{p}}\right)^{\frac{p}{s}} \\
&= \|1\|_p^{\frac{1}{r}} \|f\|_s^p \\
&= m(S)^{\frac{1}{r}} \|f\|_s^p \\
&< \infty.
\end{aligned}$$

We find this finite by assumption, hence $f \in L^p(S)$, so the claim is shown. Next, we show the case $s = \infty$. In this case $f \in L^p(S)$ is bounded almost everywhere (else its esssup would be infinite). Then, we see for $p < \infty$ $\int_S |f|^p \leq \int_S \operatorname{esssup}(f)^p = S\|f\|_\infty^p < \infty$ by assumption so the claim holds. It is clear that if $m(S) = \infty$ this does not hold. For an example, sake $S = [0, \infty]$ and $f = \frac{1}{x}$, we see $\|f\|_1 = \int_{[0,\infty]} \frac{1}{x} = \infty$, however $\|f\|_2 = (\int_{[0,\infty]} \frac{1}{x^2})^{\frac{1}{2}}$. As $\frac{1}{x^2}$ is integrable on $[0, \infty]$ we find its root to be finite, hence $f \in L_2([0, \infty])$ but $f \notin L_1([0, \infty])$.

2. Let $f \in L^r(S) \cap L^s(S)$. Denote the following sets, $A = \{x : x \in S, |f(x)| < 1\}$ and $B = \{x : x \in S, |f(x)| > 1\}$. It is clear $A \cup B = S$, with A, B being disjoint. Then, we see if $s \neq \infty$, we have

$$\begin{aligned}
\|f\|_p^p &= \int_S |f|^p \\
&= \int_A |f|^p + \int_B |f|^p \\
&\leq \int_A |f|^r + \int_B |f|^s \\
&\leq \int_S |f|^r + \int_S |f|^s \\
&= \|f\|_r^r + \|f\|_s^s \\
&< \infty.
\end{aligned}$$

In the other case where $s = \infty$ we apply the same logic as in 41, that being $|f| \leq \operatorname{esssup}(f)$ on all but a set of measure 0, hence they may be interchanged in the integral:

$$
\begin{aligned}
\|f\|_r^r &= \int_S |f|^r \\
&= \int_S |f|^p \, |f|^{r-p} \\
&\leq \int_S |f|^p \, \underbrace{[\operatorname{esssup}(f)]^{r-p}}_{\text{constant}} \\
&= \|f\|_\infty^{r-p} \|f\|_p^p < \infty \text{ by assumption.}
\end{aligned}
$$

Problem (43). • First, note that $\int_I \cos(nx) = \int_I \cos^+(nx) + \int_I \cos^-(nx)$. Since I is a bounded interval, we see for all but a set of measure 0 on its boundary, if $x \in I$, then there is an $\varepsilon > 0$ so that $(x-\varepsilon, x+\varepsilon) \in I$. Then, $\cos^-(nx) = \cos^+\left(n\left(x+\frac{\pi}{2n}\right)\right)$, so for almost every x, we find there is an $N \in \mathbb{N}$ so that $x + \frac{\pi}{2n} \in I$ for all $n \geq N$. Moreover it is bounded by $g = 1$ everywhere, so DCT proves it integrable. Then,

$$\begin{aligned}
\lim_{n\to\infty} \int_I \cos(nx) &= \lim_{n\to\infty} \int_I \cos^+(nx) - \lim_{n\to\infty} \int_I \cos^+\left(n\left(x+\frac{\pi}{2n}\right)\right) \\
&= \lim_{n\to\infty} \int_I \cos^+(nx) - \int_{I+\frac{\pi}{2n}} \cos^+(nx) \\
&= \lim_{n\to\infty} - \int_{\left(I+\frac{\pi}{2n}\right)\setminus I} \cos^+(nx) \\
&\geq - \int_{\left(I+\frac{\pi}{2n}\right)\setminus I} 1 \\
&= - \lim_{n\to\infty} \frac{\pi}{2n} \\
&= 0.
\end{aligned}$$

The same argument shows $\lim\limits_{n\to\infty} \int_I \cos(nx) \leq 0$ taking $\cos^-$ instead. Hence, $\lim\limits_{n\to\infty} \int_I \cos(nx) = 0$.

- First, note that an earlier exercise shows $f \in L^\infty(\mathbb{R})$. Next, note that

$$\begin{aligned}
\lim_{n\to\infty} \int f\cos(nx) &\leq \lim_{n\to\infty} \|f\cos(nx)\|_1 \\
&\leq \lim_{n\to\infty} \|f\|_\infty \|\cos(nx)\|_1 \\
&= \lim_{n\to\infty} \alpha \|\cos(nx)\|_1.
\end{aligned}$$

Similairly,

$$\begin{aligned}
\lim_{n\to\infty} \int f\cos(nx) &\geq \lim_{n\to\infty} -\|f\cos(nx)\|_1 \\
&\geq \lim_{n\to\infty} -\|f\|_\infty \|\cos(nx)\|_1 \\
&= -\alpha \|\cos(nx)\|_1.
\end{aligned}$$

Problem (44)**.** First, note that since $\int_{[a,b]} |f'|^p < \infty$, we find $\int_{[a,x]} |f'|^p + \int_{[x,y]} |f'|^p + \int_{[y,b]} |f'|^p < \infty$. Namely, $\int_{[x,y]} |f'|^p < \infty$, hence $f' \in L^p((a,b))$. Then, let q be p's conjugate and we find

$$\begin{aligned}
|f(x) - f(y)| &= \left| \int_{[x,y]} f' \right| \\
&\leq \int_{[x,y]} |f'| \\
&= \|1f'\|_1 \\
&\leq \|1\|_q \|f'\|_p \\
&\leq |x-y|^q \int_{[x,y]} f' \\
&\leq |x-y|^q \int_{[a,b]} f' \text{ since } f \text{ is increasing }.
\end{aligned}$$

Hence $L = \int_{[a,b]} f'$ and $\alpha = q$, namely the conjugate of p.