Analysis I

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1 Measurable Functions **3**

Lecture 13: Negative Results (2) and Measurable Functions

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We construct a cantor set.
First, suppose the interval $[0,1]$ and a series of sets $C_0, C_1, \ldots$ where $C_i = C_{i-1} \setminus D_i$ where D_i is just the set consisting of the middle thirds of each interval of C_{i-1}. Then, we let $C = \bigcap_{k\in\mathbb{N}} C_k$. We then define the nth partition of $[0,1] \setminus C_k$ to be $J_{k,n}$. We define $\mathscr{O} = \bigcup_{k,n\in\mathbb{N}} J_{k,n}$ and $\xi : \mathscr{O} \to \mathbb{R},\ x \in J_{k,n} \mapsto \frac{n}{2^k}$. We see this is well defined by an inductive argument.

Definition 0.1 (Cantor-Lebesque Function)**.** We define

$$\begin{aligned} \varphi : [0,1] &\longrightarrow \mathbb{R} \\ x &\longmapsto \varphi(x) = \begin{cases} 0, & x = 0 \\ \xi\left(x\right), & x \in \mathscr{O} \\ \sup\{\xi\left(y\right) : y \in \mathscr{O} \cap [0,x)\}, & x \in C \setminus \{0\} \end{cases} \end{aligned}$$

to be the **Cantor-Lebesque Function**

Proposition 0.1. φ is a continuous increasing function such that $\varphi\left([0,1]\right) = [0,1]$.

Proof. It is clear ξ is and this guarantees φ to be increasing.
Next, note $\varphi\left(0\right) = 0$ and $\varphi\left(1\right) = 1$. Hence, we have the intermediate value theorem guaranteeing the image is $[0,1]$ if φ is continuous.
We see φ is continuous on $\mathscr{O}$ since it is constant on each interval $J_{k,n}$. Now, we consider $x \in C \setminus \{0,1\}$. For a given ε, let $k \in \mathbb{N}$ such that $\frac{1}{2^k} < \varepsilon$. Then, there is $n \in \mathbb{N}$ such that $1 \leq n \leq 2^k - 2$ such that for all $u \in J_{k,n}$, $v \in J_{k,n+1}$ such that for all u, v we find $u < x < v$. Let $a_k \in J_{k,n}$ $b_k \in J_{k,n+1}$ then by monotinicity

of φ, for all $y \in [0,1]$ with $|x-y| < \delta = \min\{x-a_k, x+b_k\}$ we find

$$\begin{aligned}|\varphi(x) - \varphi(y)| &\leq \varphi(b_k) - \varphi(a_k) \\ &= \frac{n+1}{2^k} - \frac{n}{2^k} \\ &= \frac{1}{2^k} \\ &< \varepsilon.\end{aligned}$$

Finally, given $\varepsilon > 0$, we take $k \in \mathbb{N}$ such that $\frac{1}{2^k} < \varepsilon$ and let $c_k \in I_{k,1}$, $d_k \in I_{k,2^k-1}$. Then, for $o \leq y \leq c_k$, we find

$$\begin{aligned}|\varphi(0) - \varphi(y)| &= |\varphi(y)| \\ &\leq \varphi(c_k) \\ &= \frac{1}{2^k} \\ &< \varepsilon.\end{aligned}$$

Similairly, for $d_k < y \leq 1$, we find

$$\begin{aligned}|\varphi(1) - \varphi(y)| &\leq |1 - \varphi(d_k)| \\ &= 1 - \frac{2^k-1}{2^k} \\ &= \frac{1}{2^k} \\ &< \varepsilon.\end{aligned}$$

□

Definition 0.2 (Modified Cantor-Lebesque Function)**.** Let $\psi = x + \varphi(x)$ be the **modified Cantor-Lebesque Function**. It is clear ψ is continuous, strictly increasing and has , $\psi([0,2]) = [0,2]$.

Proposition 0.2. The function ψ has the following properties

1. $\psi(C)$ is measurable with $\mu(\psi(C)) = 1$.
2. There is a measurable set $S \subseteq C$ such that $\psi(S)$ is not measurable.

Proof.
- Note that $[0,1] = C \cup \mathscr{O}$ and ψ is injective and continuous. Hence, we have $[0,2] = \psi(C) \cup \psi(\mathscr{O})$ with $\psi(C) \cap \psi(\mathscr{O}) = \varnothing$. Since ψ is strictly increasing, we know ψ^{-1} is well-defined and continuous. Hence, ψ is an open map and we see $\psi(\mathscr{O})$ is open in $[0,2]$, hence $\psi(C)$ is closed. Hence, both sets are measurable. We see $\psi(\mathscr{O})$ is the union of a countable collection of open disjoint intervals, $\{I_i : i \in \mathbb{N}\}$ such that $\varphi \mid J_i$ is constant by construction. Hence, we hve for each $i \in \mathbb{N}$ we find $\psi(I_n) = x_i + I_i$ where $x_i \in [0,1]$ is a constant. Since ψ is injective, we find it preserves

disjointness, hence the collection $\{\psi(I_i) : i \in \mathbb{N}\}$ is disjoint.
Then, by countable additivity and translation invariance of μ we find

$$\begin{aligned}\mu(\psi(\mathscr{O})) &= \mu\left(\bigcup_{i\in\mathbb{N}} I_i\right)\\ &= \bigcup_{i\in\mathbb{N}} \psi(I_i)\\ &= \sum_{i=1}^{\infty} \mu(\psi(I_i))\\ &= \sum_{i=1}^{\infty} \ell(x_i + I_i)\\ &= \sum_{i=1}^{\infty} \ell(I_i)\\ &= \mu(\mathscr{O}).\end{aligned}$$

Since, $\mu(C) = 0$, we find

$$\mu(\mathscr{O}) = \mu([0,1] \setminus C) = \mu([0,1]) = 1.$$

Consequently, $\mu(\psi(\mathscr{O})) = 1 = \mu(\mathscr{O})$. Hence, we find $\mu(\psi(C)) = 1$.

Since $\psi(C)$ has positive measure, it contains a nonmeasurable subset T, however, we see $S = \psi^{-1}(T)$ is measurable as $S \subseteq C$ and $\mu(C) = 0$. □

Corollary 1. There is a measurable set $S \subseteq C$ such that S is not borel.

Proof. Since ψ has a continuous inverse, we see it maps borel sets to borel sets. Let S be a subset of C such that $\psi(S)$ is not measurable. Since $\psi(S)$ is not measurable, it is not a borel set. Hence S is not borel, but it was measurable with measure 0. □

1 Measurable Functions

Definition 1.1 (Measurable Functions)**.** A function $f : S \to \overline{\mathbb{R}}$ is **Lebesque-measurable** on S if $S \subseteq \mathbb{R}$ is measurable and $f^{-1}((c,\infty])$ is a measurable set for every $c \in \mathbb{R}$. This is equivalent to the condition that $f^{-1}(B)$ is measurable for all $B \in \overline{\mathscr{B}}$, the extended borel σ-algebra.

Proposition 1.1. Let $S \subseteq \mathbb{R}$ be measurable, then a function $f : S \to \overline{\mathbb{R}}$ is measurable if and only if one of the following holds for all $c \in \mathbb{R}$:

- $f^{-1}([c,\infty])$ is measurable,
- $f^{-1}([-\infty,c])$ is measurable,
- $f^{-1}([-\infty,c))$ is measurable.

Definition 1.2. The extended Borel σ-algebra, $\overline{\mathscr{B}}$ consists of all subsets $B \subseteq \overline{\mathbb{R}}$ such that $B \setminus \{-\infty, \infty\} \in \mathscr{B}$.

Remark. It is clear $\overline{\mathscr{B}}$ is the smallest σ-algebra containing all open subsets of $\overline{\mathbb{R}}$.

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Recall. A function $f : S \to \mathbb{R}$ was measurable if S is measurable and $f^{-1}((c, \infty])$ is measurable for all $c \in \mathbb{R}$. There was an equivalent definition using the extended borel σ-algebra that we will use occasionally.

Proposition 1.2. Suppose $f : S \to \overline{\mathbb{R}}$ is continuous on the measurable set S, then f is measurable.

Proof. Let H be an extending function, then we must show $H \circ f$ is continuous. We see any subray , $f(X_0) = (c, \infty]$ will have $(H \circ f)(X_0) = (\hat{c}, 1]$. We know the preimage of this to be open in S, hence measurable. ☐

Proposition 1.3. Let $S \subseteq \mathbb{R}$. Suppose $f : S \to \mathbb{R}$ is measurable. and let $g : B \to \mathbb{R}$ with $B \in \overline{\mathscr{B}}$ and $f(S) \subseteq B$. Then, $g \circ f : S \to \mathbb{R}$ is measurable.

Proof. For $c \in \mathbb{R}$, we note that $(g \circ f)^{-1}((c, \infty]) = f^{-1}\left(g^{-1}((c, \infty])\right)$. By continuity of g, we know $g^{-1}((c, \infty]) \in \overline{\mathscr{B}}$. And, since f is measurable, we find $f^{-1}\left(g^{-1}((c, \infty])\right)$. ☐

Corollary 2. Let $S \subseteq \mathbb{R}$ and $f : S \to \mathbb{R}$ to be a measurable function. Then, for every $\alpha \in \mathbb{R}$ and $0 < \rho < \infty$, we find αf and $|f|^\rho$ are measurable.

Proof. We see the functions $g(u) = \alpha u$ on $\overline{\mathbb{R}}$ and $h(u) = |u|^\rho$ on $\overline{\mathbb{R}}$ to be the corresponding functions. We see the case h is clearly continuous and well defined. On the other hand g may be poorly defined if $\alpha = 0$ and $f(x) = \infty$. Recall, however, we had $0 \cdot \pm\infty = 0$ so g is just the zero functions and we see continuity holds. ☐

Definition 1.3 (Almost-everywhere)**.** Let S be measurable, then a property is said to hold true **almost everywhere** on S or **for almost all** $x \in S$ if there is a set T with $\mu(T) = 0$ and the property holds on all of $S \setminus T$.

Proposition 1.4. Let $S \subseteq \mathbb{R}$ and suppose $f, g : S \to \overline{\mathbb{R}}$ such that f is measurable and $g = f$ almost everywhere on S, then g is measurable.

Proof. Let $T = \{x \in S : f(x) \neq g(x)\}$. Fix $c \in \mathbb{R}$ and let $F = f^{-1}((c, \infty]) \setminus T$ and $G = f^{-1}((c, \infty]) \cup T$. Clearly, both F and G are measurable. Furthermore, $F \subseteq G$ and $\mu(G \setminus F) = \mu(T) = 0$. Since, $F \subseteq g^{-1}((c, \infty]) \subseteq G$. And, by an earlier characterization we recall that a set X is measurable if and only if there were nested sets around it with a difference of measure 0 . Hence, g is measurable. □

Remark. Suppose $f : S \to \overline{\mathbb{R}}$ is a measurable set and $S \subseteq X \subseteq \mathbb{R}$. If $\mu(X \setminus S) = 0$ and $h : X \to \overline{\mathbb{R}}$ is any extension of f , then h is measurable since $h^{-1}((c, \infty]) = f^{-1}((c, \infty]) \cup \{x \in X \setminus S : h(x) \in (c, \infty]\}$. This is the union of a measurable set with a set of measure 0, so we see h is measurable.

Notation. Instead of saying that every extension of a measurable function $f : S \to \overline{\mathbb{R}}$ to a function $h : X \to \overline{\mathbb{R}}$, we often just say f is measurable on X as long as it is defined almost everywhere on X and is measurable on that set.

Proposition 1.5. Suppose $f : I \to \overline{\mathbb{R}}$ is monotone on $I \subseteq \mathbb{R}$. Then, the set of all points in I where f fails to be continuous is countable, hence measure 0. Another characterization is that f is continuous almost everywhere, hence f is measurable.

Proof. It suffices to consider the case f is increasing and I open. Let E be the set of all $x \in I$ where f fails to be continuous. For $x \in E$ let $\alpha_x = \sup(\{f(z) : z < x\} z \in J)$ and $\beta_x = \inf(\{f(z) : z > x\} z \in J)$. Since f is not continuous at x, we find the interval $(\alpha_x, \beta_x) = I_x$ to be nonempty. Also, if $x, y \in E$ are distinct with $x < y$ we find $\beta_x <= \alpha_y$. Hence, we find $I_x \cap I_y = \varnothing$. Since each interval I_x for $x \in E$ contains a rational number, we see E is countable. Hence, $\mu(E) = 0$ and we see $f|_{I \setminus E}$ is continuous on $I \setminus E$ which is measurable, hence the restriction is measurable and as f coincides with its restriction almost everywhere, we see f is measurable. □

Definition 1.4 (Finite Functions).

- Let $S \subseteq \mathbb{R}$. A function $f : S \to \overline{\mathbb{R}}$ is called **finite on** S if $|f(x)| < \infty$ for all $x \in S$.
- Let $f, g : S \to \overline{\mathbb{R}}$ Then we say $f < g$ if $f(x) < g(x)$ for all $x \in S$. Similarly for all other inequalities.
- f is called **nonnegative** if $f \geq 0$ and **positive** if the inequality is strict.

Proposition 1.6. Let $f, g : S \to \overline{\mathbb{R}}$ be measurable and finite almost everywhere. Then, $f + g, f - g, f \cdot g$ are measurable.
If $g(x) \neq 0$ for almost every $x \in S$, then $\frac{f}{g}$ is measurable.

Proof.

1. First, we prove addition. We may assume f, g are finite on S . Then, $h = f + g$ is well defined. Since for $x \in S$, we have $h(x) > q$ for

$c \in R$ if and only if there is a $q \in \mathbb{Q}$ such that $f(x) > q$ and $g(x) > c - q$, we have

$$h^{-1}((c,\infty]) = h^{-1}((c,\infty)) \text{ by finiteness.}$$
$$= \bigcup_{q\in\mathbb{Q}} f^{-1}((q,\infty)) \cup g^{-1}(c-q,\infty).$$

Hence, h as measurable as these are all measurable sets. If f, g are measurable, then so are $f, -g$, hence $f + (-g) = f - g$

2. With addition, subtraction is completely trivial,

3. Now multiplication, Let h be any measurable finite function on S. Consider $(h)^2$. If $c \geq 0$, we have

$$\left((h)^2\right)^{-1}((c,\infty)) = h^{-1}\left(\left(-\infty,\sqrt{c}\right)\right) \cup h^{-1}\left(\left(\sqrt{c},\infty\right)\right).$$

If $c < o$, then

$$\left((h)^2\right)^{-1}((c,\infty)) = h^{-1}(\mathbb{R}) = S.$$

As in either case we had the preimage being measurable, we see $(h)^2$ is measurable. Since $f \cdot g = \frac{1}{2}(f+g)^2 - \frac{1}{2}(f)^2 - \frac{1}{2}\left(g^2\right)$ being the sum, constant multiple and square of measurable functions yields $f \cdot g$ to be measurable.

4. Lastly, let $h = \frac{1}{g}$, and note we can assume g is nonzero for all S , hence h is well defined on S and h is finite. If $c > 0$ we see $h^{-1}((c,\infty)) = g^{-1}(\left(0,\frac{1}{c}\right))$. As this interval is open and borel, we see $g^{-1}\left(\left(0,\frac{1}{c}\right)\right)$ is borel, hence $h^{-1}((c,\infty))$ is measurable.
Similairly, if $c = 0$, we see $h^{-1}((0,\infty)) = g^{-1}((0,\infty))$.
Lastly, if $c < 0$ we have $h^{-1}(c,\infty) = g^{-1}(\left(-\infty,\frac{1}{c}\right)) \cup g^{-1}((0,\infty)) = g^{-1}\left(\left[\frac{1}{c},0\right)^c\right)$ hence measurable. This completes the proof.

□