Algebraic Theory I: Homework III

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Solution (1)**.**

1. First, we note that if $xK = yK$ for some $x \neq y$, then $\overline{\varphi}(xK) = \overline{x}H$ and $\overline{\varphi}(yK) = \overline{y}H$, hence we need to show $\overline{x}H = \overline{y}H$. We see $x^{-1}yK = K$, hence $\overline{\varphi}\left(x^{-1}yK\right) = \overline{x^{-1}y}H = \overline{x}^{-1}\overline{y}H = \overline{\varphi}(1K) = 1H$. So, $\overline{x}^{-1}\overline{y} \in H$, hence $\overline{y} \in \overline{x}H$ and similairly, $\overline{x} \in \overline{y}H$. So, $\overline{x}H = \overline{\varphi}(xK) = \overline{y}H = \overline{\varphi}(yK)$, so $\overline{\varphi}$ is well defined.
   Now,
   $$\begin{aligned}\overline{\varphi}(xKyK) &= \overline{\varphi}(xyKK)\\ &= \overline{xy}H\\ &= \varphi(xy)H\\ &= \varphi(x)\varphi(y)H\\ &= \overline{x}\overline{y}H\\ &= \overline{x}\overline{y}HH \text{ as } H = HH \text{ by closure}\\ &= \overline{x}H\overline{y}H\\ &= \overline{\varphi}(xK)\overline{\varphi}(yK).\end{aligned}$$
   Furthermore, we see $\overline{\varphi}(1K) = \overline{1}H = 1H$ as $\varphi(1) = \overline{1} = 1$ by homomorphism properties. So, $\overline{\varphi}$ is a homomorphism.

2. First, note that $Z_0\left(\overline{G}\right) = \{1\} = \overline{Z_0(G)}$. Now, we induce on n and we see $Z_{n-1}(G) \trianglelefteq G$ and $Z_{n-1}\left(\overline{G}\right) \trianglelefteq \overline{G}$ with $\overline{Z_{n-1}(G)} \leq Z_{n-1}\left(\overline{G}\right)$ by inductive hypothesis, so $\overline{\varphi} : Z_n(G)/Z_{n-1}(G) \to Z_n\left(\overline{G}\right)/Z_{n-1}\left(\overline{G}\right)$ is a well defined homomorphism. Hence, letting $\overline{x} \in \overline{Z_n(G)}$, hence $x \in Z_n(G)$ and hence $xZ_{n-1}(G) \in Z_n(G)/Z_{n-1}(G)$ implies $\overline{\varphi}xZ_{n-1}(G) = \overline{x}Z_{n-1}\left(\overline{G}\right) \in Z_n\left(\overline{G}\right)/Z_{n-1}\left(\overline{G}\right)$. Hence, we find $\overline{x} \in Z_n\left(\overline{G}\right)$. This completes the induction.

3. Suppose G is nilpotent and let n be the nilpotence class of G. Then, we see $\overline{Z_n(G)} = \overline{G} \leq Z_n\left(\overline{G}\right)$. Hence, $\overline{G}$ is of nilpotence class at most n, so we see $\overline{G}$ is nilpotent.

4.

5. Suppose n is the nilpotence class of G. Then $Z_n(G) \cap H = G \cap H = H \leq Z_n(H)$, so H is of nilpotence class at most n, hence H is nilptent.

Lemma 0.1. Automorphisms preserve maximality of subgroups.
Let $\varphi : G \to G$ be an automorphism and let $M < G$ be a maximal subgroup. Suppose $\varphi(M) = M'$ is not maximal. That is, there is a $\overline{M}'$ such that $M' < \overline{M}' < G$. Then, we find

$$\begin{aligned}\varphi^{-1}\left(\overline{M}'\right) &= \varphi^{-1}\left(M' \cup \left(\overline{M}' \setminus M'\right)\right) \\ &= \varphi^{-1}(M') \cup \varphi^{-1}\left(\overline{M}' \setminus M'\right) \\ &= M \cup \{\varphi^{-1}(m) : m \in \overline{M}' \setminus M'\} \\ &> M.\end{aligned}$$

Furthermore, $\overline{M}' < G$ by assumption, hence $M < \overline{M}' < G$. $\lightning$.

Solution (2)**.** *Proof.* Now, let $\alpha : G \to G$ be an automorphism of G and denote $\alpha(M) = M'$. Then, we see

$$\begin{aligned}\alpha(\Phi(G)) &= \alpha\left(\bigcap_{\substack{M<G \\ M \text{ is maximal}}} M\right) \\ &= \bigcap_{\substack{M<G \\ M \text{ is maximal}}} \alpha(M) \\ &= \bigcap_{\substack{M<G \\ M \text{ is maximal}}} M'\end{aligned}$$
.

Then, as M' is maximal and α is an injection, we see if $N \neq M$ are both maximal subgroups, we have $\alpha(N) \neq \alpha(M)$, hence

$$\{M : \underset{M \text{ is maximal}}{M < G}\} = \{M' : \underset{M \text{ is maximal}}{M < G}\}.$$

So, we have

$$\alpha(\Phi(G)) = \bigcap_{\substack{M<G \\ M \text{ is maximal}}} M' = \bigcap_{\substack{M<G \\ M \text{ is maximal}}} M = \Phi(G).$$

$\square$

Solution (3). 1. Let P be a sylow p-group in $\Phi(G) = \Phi$. Then, we have $\Phi \trianglelefteq G$ and $P \trianglelefteq G$, so applying frattini's argument yields

$$G = \Phi N_G(P).$$

Suppose P is not normal, so $N_G(P) < G$. Then, as there is some maximal subgroup M with $N_G(P) \leq M$ and $\Phi \leq M$ for all maximal M, we find $\Phi N_G(P) = G \leq M$. $\lightning$ as M was maximal. Hence, we must have $N_G(P) = G$.

2. As all P-groups of Φ are normal in Φ , we have by characterization of nilpotence that Φ is nilpotent.

Lemma 0.2. $[M, M]$ and $\langle x^p : x \in M \rangle$ are characteristic in M.
Let $\alpha : M \to M$ be an automorphism. Then, denote $\alpha(x) = x'$ for $x \in M$ and we see,

$$\begin{aligned}
\alpha([M, M]) &= \alpha\left(\left\langle xyx^{-1}y^{-1} : x, y \in M \right\rangle\right) \\
&= \left\langle \alpha\left(xyx^{-1}y^{-1}\right) x, y \in M \right\rangle \\
&= \left\langle \alpha(x)\,\alpha(y)\,\alpha(x)^{-1}\,\alpha(y)^{-1} : x, y \in M \right\rangle \\
&= \left\langle x'y'x'^{-1}y'^{-1} : x, y \in M \right\rangle \\
&\leq \left\langle x'y'x'^{-1}y'^{-1} : x', y' \in M \right\rangle \\
&= [M, M].
\end{aligned}$$

Similairly,

$$\begin{aligned}
\alpha(\langle x^p : x \in M \rangle) &= \langle \alpha(x^p) : x \in M \rangle \\
&= \langle \alpha(x)^p : x \in M \rangle \\
&= \langle x'^p : x \in M \rangle \\
&\leq \langle x'^p : x' \in M \rangle \\
&= \langle x^p : x \in M \rangle.
\end{aligned}$$

Then, we see as $M \trianglelefteq G$ and these two groups are characteristic we also have $\langle x^p : x \in M \rangle \trianglelefteq G$ and $[M, M] \trianglelefteq G$. Furthermore, we note that as $xyx^{-1}y^{-1} \in M$ for $x, y \in M$ we have $\{M, M\} \leq \langle x^p : x \in M \rangle$. Now, Suppose M is not an elementary abelian p-group. Then, we find either $[M, M] > 1$ or there is an element x of order $q \neq p$.

Solution (4)**.** Now, as $[M, M]$ is characteristic in $M \trianglelefteq G$, we see $[M, M] \trianglelefteq G$. Hence as $[M, M] \leq M$ we have $[M, M] = \{1\}$ or $[M, M] = M$. If $[M, M] = M$, then $M^{(1)} = M$. We induce on the superscript. Suppose $M^{(n)} = M$. Then, $M^{(n+1)} = \left[M^{(n)}, M^{(n)}\right] = [M, M] = M$, hence $M^{(n)} = M$ for all $n \in \mathbb{N}$, hence M is not solvable. $\lightning$. So, we see $[M, M] = \{1\}$, hence M is abelian.
Now, let $M = p_1^{\varepsilon_1} \dots p_n^{\varepsilon_n}$ for primes $p_1, \dots, p_n$ and $\varepsilon_i \in \mathbb{N}$. Then, let P be a sylow p_i-group in M and we see M is abelian, hence $P \trianglelefteq M$, hence P is characteristic in M, so we see $P \trianglelefteq G$ and so $P = M$ as P is assumed nontrivial. So, M is a sylow p_i-group, so $|M| = p_i^{\varepsilon_i}$.
Lastly, note that there is an element $x \in M$ with $\operatorname{ord}(x) = p$, hence $x^p = 1 = 1^p$, so there is no bijection between $\langle x^p : x \in M \rangle$ and M and we see $|\langle x^p : x \in M \rangle| \neq |M|$. Thus, as $\langle x^p : x \in M \rangle \leq M$, we see $\langle x^p : x \in M \rangle < M$, but as this subgroup is characteristic within $M \trianglelefteq G$, we find $\langle x^p : x \in M \rangle \trianglelefteq G$, hence $\langle x^p : x \in M \rangle = \{1\}$. Thus, all elements $x \in M$ have $\operatorname{ord}(x) = p$.

Solution (5)**.** Let G be finite and solvable and let $|G| = p_1^{\alpha_1} \dots p_\ell^{\alpha_\ell}$ and define $\alpha = \sum_{i=1}^{\ell} \alpha_i$ to be the sum of all powers in the prime factorization of $|G|$. If $\alpha = 1$, then $|G| = p$, so $G \simeq \mathbb{Z}/p\mathbb{Z}$. Since $\mathbb{Z}/p\mathbb{Z}$ has no proper nontrivial subgroups, we see its sole maximal subgroup is $\{1\}$ and $|G : \{1\}| = p$.
We induce on α. Suppose the case $\alpha = n$ true and observe the case $\alpha = n+1$. Let N be a minimal normal subgroup of G. If $N = G$, then we find G has no normal subgroups hence it is simple. Furthermore, G being solvable implies a chain

$$\{1\} = H_0 \trianglelefteq H_1 \trianglelefteq \dots \trianglelefteq H_c = G$$

and as G is simple, we see $H_1 = G$ up to prepending copies of $\{1\}$ to the chain. Hence, $H_1/\{1\} \simeq G$ is abelian so $G \simeq \mathbb{Z}/q\mathbb{Z}$ for some prime q. Hence, $|G| = q$, so $\alpha = 1$ $\lightning$.
So, we see $N < G$. Let $|N| = p_1^{\beta_1} \dots p_r^{\beta_r}$ for some $r \le \ell$ and $\beta_i \le \alpha_i$ and define $\beta = \sum_{i=1}^{r} \beta_i$ be the sum of powers of primes in $|N|$. Then, we see as $\{1\} \neq N$ by assumption, we have $\beta > 0$, so atleast one $\beta_i > 0$. Hence, we find $|G/N| = p_1^{\alpha_1 - \beta_1} \dots p_\ell^{\alpha_\ell - \beta_\ell} < p_1^{\alpha_1} \dots p_\ell^{\alpha_\ell} = |G|$, hence as the prime bases are the same, we have $\sum_{i=1}^{\ell} \alpha_i > \sum_{i=1}^{\ell} \alpha_i - \beta_i$, so G/N has a sum of prime powers in $|G/N|$, denoted $\kappa = \sum_{i=1}^{\ell} \alpha_\ell - \beta_\ell$, at most k. Furthermore, letting $\varphi : G \to G/N,\ x \mapsto xN$, a surjective homomorphism, we see G/N is solvable and as homomorphisms preserve maximality by the earlier lemma, we have a maximal subgroup $M \le G$ having a direct correspondence with the maximal subgroup $M/N \le G/N$. Then, as $|G/N : M/N| = p^m$ for some $m \ge 1$ by the inductive hypothesis. And, as $(G/N)\,/\,(M/N) \simeq G/M$, we see $|G : M| = p^m$.