

Analysis I

Thomas Fleming

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Lecture 12: Negative Results of Measure Theory

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We construct a cantor set.

First, suppose the interval $[0, 1]$ and a series of sets C_0, C_1, \dots where $C_i = C_{i-1} \setminus D_i$ where D_i is just the set consisting of the middle thirds of each interval of C_{i-1} . Then, we let $C = \bigcap_{k \in \mathbb{N}} C_k$. We then define the n th partition of $[0, 1] \setminus C_k$ to be $J_{k,n}$. We define $\mathcal{O} = \bigcup_{k,n \in \mathbb{N}} J_{k,n}$ and $\xi : \mathcal{O} \rightarrow \mathbb{R}$, $x \in J_{k,n} \mapsto \frac{n}{2^k}$. We see this is well defined by an inductive argument.

Definition 0.1 (Cantor-Lebesgue Function). We define

$$\varphi : [0, 1] \longrightarrow \mathbb{R}$$
$$x \longmapsto \varphi(x) = \begin{cases} 0, & x = 0 \\ \xi(x), & x \in \mathcal{O} \\ \sup\{\xi(y) : y \in \mathcal{O} \cap [0, x)\}, & x \in C \setminus \{0\} \end{cases}$$

to be the **Cantor-Lebesgue Function**

Proposition 0.1. φ is a continuous increasing function such that $\varphi([0, 1]) = [0, 1]$.

Proof. It is clear ξ is and this guarantees φ to be increasing.

Next, note $\varphi(0) = 0$ and $\varphi(1) = 1$. Hence, we have the intermediate value theorem guaranteeing the image is $[0, 1]$ if φ is continuous.

We see φ is continuous on \mathcal{O} since it is constant on each interval $J_{k,n}$. Now, we consider $x \in C \setminus \{0, 1\}$. For a given ε , let $k \in \mathbb{N}$ such that $\frac{1}{2^k} < \varepsilon$. Then, there is $n \in \mathbb{N}$ such that $1 \leq n \leq 2^k - 2$ such that for all $u \in J_{k,n}$, $v \in J_{k,n+1}$ such that for all u, v we find $u < x < v$. Let $a_k \in J_{k,n}$, $b_k \in J_{k,n+1}$ then by monotonicity

of φ , for all $y \in [0, 1]$ with $|x - y| < \delta = \min\{x - a_k, x + b_k\}$ we find

$$\begin{aligned} |\varphi(x) - \varphi(y)| &\leq \varphi(b_k) - \varphi(a_k) \\ &= \frac{n+1}{2^k} - \frac{n}{2^k} \\ &= \frac{1}{2^k} \\ &< \varepsilon. \end{aligned}$$

Finally, given $\varepsilon > 0$, we take $k \in \mathbb{N}$ such that $\frac{1}{2^k} < \varepsilon$ and let $c_k \in I_{k,1}$, $d_k \in I_{k,2^k-1}$. Then, for $0 \leq y \leq c_k$, we find

$$\begin{aligned} |\varphi(0) - \varphi(y)| &= |\varphi(y)| \\ &\leq \varphi(c_k) \\ &= \frac{1}{2^k} \\ &< \varepsilon. \end{aligned}$$

Similarly, for $d_k < y \leq 1$, we find

$$\begin{aligned} |\varphi(1) - \varphi(y)| &\leq |1 - \varphi(d_k)| \\ &= 1 - \frac{2^k - 1}{2^k} \\ &= \frac{1}{2^k} \\ &< \varepsilon. \end{aligned}$$

□

Definition 0.2 (Modified Cantor-Lebesgue Function). Let $\psi = x + \varphi(x)$ be the **modified Cantor-Lebesgue Function**. It is clear ψ is continuous, strictly increasing and has $\psi([0, 2]) = [0, 2]$.

Proposition 0.2. The function ψ has the following properties

1. $\psi(C)$ is measurable with $\mu(\psi(C)) = 1$.
2. There is a measurable set $S \subseteq C$ such that $\psi(S)$ is not measurable.

Proof. • Note that $[0, 1] = C \cup \mathcal{O}$ and ψ is injective and continuous. Hence, we have $[0, 2] = \psi(C) \cup \psi(\mathcal{O})$ with $\psi(C) \cap \psi(\mathcal{O}) = \emptyset$. Since ψ is strictly increasing, we know ψ^{-1} is well-defined and continuous. Hence, ψ is an open map and we see $\psi(\mathcal{O})$ is open in $[0, 2]$, hence $\psi(C)$ is closed. Hence, both sets are measurable. We see $\psi(\mathcal{O})$ is the union of a countable collection of open disjoint intervals, $\{I_i : i \in \mathbb{N}\}$ such that $\varphi|_{J_i}$ is constant by construction. Hence, we have for each $i \in \mathbb{N}$ we find $\psi(I_i) = x_i + I_i$ where $x_i \in [0, 1]$ is a constant. Since ψ is injective, we find it preserves

disjointness, hence the collection $\{\psi(I_i) : i \in \mathbb{N}\}$ is disjoint. Then, by countable additivity and translation invariance of μ we find

$$\begin{aligned}
 \mu(\psi(\mathcal{O})) &= \mu\left(\bigcup_{i \in \mathbb{N}} I_i\right) \\
 &= \sum_{i \in \mathbb{N}} \mu(\psi(I_i)) \\
 &= \sum_{i=1}^{\infty} \mu(\psi(I_i)) \\
 &= \sum_{i=1}^{\infty} \ell(x_i + I_i) \\
 &= \sum_{i=1}^{\infty} \ell(I_i) \\
 &= \mu(\mathcal{O}).
 \end{aligned}$$

Since, $\mu(C) = 0$, we find

$$\mu(\mathcal{O}) = \mu([0, 1] \setminus C) = \mu([0, 1]) = 1.$$

Consequently, $\mu(\psi(\mathcal{O})) = 1 = \mu(\mathcal{O})$. Hence, we find $\mu(\psi(C)) = 1$.

Since $\psi(C)$ has positive measure, it contains a nonmeasurable subset T , however, we see $S = \psi^{-1}(T)$ is measurable as $S \subseteq C$ and $\mu(C) = 0$. \square

Corollary 1. There is a measurable set $S \subseteq C$ such that S is not borel.

Proof. Since ψ has a continuous inverse, we see it maps borel sets to borel sets. Let S be a subset of C such that $\psi(S)$ is not measurable. Since $\psi(S)$ is not measurable, it is not a borel set. Hence S is not borel, but it was measurable with measure 0. \square