Algebraic Theory I

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Contents

Lecture 32
Wed 10 Nov 2021 17:32

Lecture 33
Wed 10 Nov 2021 17:33

Recall. Recall R denotes a commutative ing. If $S \subseteq R$ is a multiplicative subset, we see $x, y \in S$ implies $xy \in S$ and $0 \notin S$ but $1 \in S$.
Then, we define $S^{-1}R = \{X/s : x \in R, s \in S\}$. Then, we see $\frac{x_1}{s_1} = \frac{x_2}{s_2}$ if and only if there is an $s \in S$ so that $s\left(s_2 x_1 - s_1 x_2\right) = 0$. Of course, if R is an integral domain we see this iplies $s_2 x_1 - s_1 x_2 = 0$, the normal definition of fraction equality.

Now, we turn this set into a ring. We define $\frac{x_1}{s_1} \cdot \frac{x_2}{s_2} := \frac{x_1 x_2}{s_1 s_2}$ and $\frac{x_1}{s_2} + \frac{x_2}{s_2} = \frac{s_2 x_1}{s_1 s_2} + \frac{s_1 x_2}{s_1 s_2} = \frac{s_2 x_1 + s_1 x_2}{s_1 s_2}$. Now, we need to show that $+, \cdot$ are well defined (meaining they do not vary for different representatives of a given equivalence class). This fact is easily checked by symbolic manipulation so we omit the proof. For the addition case suppose $\frac{x_1}{s_1} = \frac{x_1'}{s_1'}$ and similairly for $\frac{x_2}{s_2}$ then take the multiplicative representation of the fraction and multiply the $\frac{x_1}{s_1}$ representation by $-s_2 s_2' ts$ and the $\frac{x_2}{s_2}$ representation by $-s_1 s_1' st$ and by adding together these representations we see terms cancel and we obtain that addition is in fact well defined. Moreover, it is trivial to check that the ring axioms hold.

Definition 0.1 (Ring Localization)**.** We denote this new fraction ring $S^{-1}R$ to be the **localization of** R with additive identity $\frac{0}{1}$, multiplicative identity $\frac{1}{1}$ and $\frac{tx}{ts} = \frac{x}{s}$ for all $t \in S$.

Note that $s \in S$ is nonzero by definition, so $\frac{1}{s} \cdot \frac{s}{1} = \frac{1}{1} = 1_{S^{-1}R}$, so every element has an inverse.