Combinatorics

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Contents

Lecture 19: Quasi-Random Graphs (2)

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Recall we had many equivalent conditions, cleverly names properties I-VII. We prove the are equivalent.

$P_2 \Leftrightarrow P_3$.

- $(P_2 \Rightarrow P_3)$. Recall $\frac{n^4}{16} + o\left(n^4\right) = CW_4\left(G\right) = \operatorname{tr}\left(A^4\right)$. We know

$$\begin{aligned} \operatorname{tr}\left(A^4\right) &= \sum_{i=1}^{n} \lambda_i^4 \\ \Rightarrow \lambda_1^4 &\leq \frac{n^4}{16} + o\left(n^4\right) \\ \Rightarrow \lambda_1 &\leq \frac{n}{2} + o\left(n\right). \end{aligned}$$

  From this, we also know

$$\begin{aligned} \sum_{i=1}^{n} \lambda_i^4 &= \lambda_1^4 + \sum_{i=2}^{n} \lambda_i^4 \\ \Rightarrow \sum_{i=2}^{n} \lambda_i^4 &= o\left(n^4\right) \\ \Rightarrow \lambda_i &= o\left(n\right) \\ \Rightarrow \sigma_2 &= o\left(n\right). \end{aligned}$$

- $(P_3 \Rightarrow P_2)$. Again, we know

$$\begin{aligned} CW_4 &= \sum_{i=1}^{n} \lambda_i^4 \\ &= \lambda_1^4 + \sum_{i=2}^{n} \lambda_i^4 \\ &= \frac{n^4}{16} + o\left(n^4\right) \\ \Rightarrow \lambda_1^4 = \frac{n^4}{16}. \end{aligned}$$

Similarly, we find $\sum_{i=2}^{n} \lambda_i^4 \leq \sigma_2^2 \sum_{i=2}^{n} \lambda_i^2$
Then, we have $\sum_{i=2}^{4} \lambda_i^2 = 2e\left(G\right) - \lambda_1^2 \leq o\left(n^2\right) n^2 = o\left(n^4\right)$. $P_2 \Leftrightarrow P_3$.

□

Remark. Sometimes, we wish to only have 2 conditions to check for P_3, and we find that there is an equivalent statement of P_3 such that a family $\mathscr{G}$ follows

- $e\left(G\right) \geq \frac{n^2}{4} + o\left(n^2\right)$.
- $\left|\lambda_n\left(G\right)\right| + \left|\lambda_n\left(\overline{G}\right)\right| = o\left(n\right)$.

$P_3 \Leftrightarrow P_7$.

- ($P_3 \Rightarrow P_7$). As we have P_3, then we have $CW_4 = \frac{n^4}{16} + o(n^4)$. Then, recall $\sum_{1\leq i,j,\leq n} \binom{\hat{d}_{ij}}{2} = 2\#C_4 = \frac{CW_4}{4} + o\left(n^4\right) = \frac{n^4}{64} + o(n^4)$ where $\#C_4$ is simply the number of four cycles in G. Hence, with some intermediate theorems, we find

$$\sum_{1\leq i,j\leq n} \hat{d}_{i,j}^2 = \frac{n^4}{32} + o\left(n^4\right).$$

Hence,

$$\sum_{1\leq i,j\leq n} \left(\hat{d}_{ij} - \frac{n^2}{16}\right) = o\left(n^4\right).$$

Then, we see as $\sum_{1\leq i,j\leq n} \hat{d}_{i,j} = \sum_{i=1}^{n} \binom{d_i}{2} = \sum_{i=1}^{n} \frac{d_i^2}{2} - 1/2 \sum_{i=1}^{n} d_i \leq \frac{n}{2}\lambda_1^2 = \frac{n^3}{8} + o\left(n^3\right)$. Then, applying subadditivity yields the desired value of $\sum_{1\leq i,j\leq n} \left|\hat{d}_{i,j} - \frac{n}{4}\right| = o\left(n^3\right)$.

□

Proposition 0.1. Let G be random on n-vertices with all degrees about $\frac{n}{2}$ and codegrees about $\frac{n}{4}$. Then, we ask how likely is it that by changing at most $o\left(n^2\right)$ edges, we find a conference graph.

Lecture 20: Quasi-Random Graphs (3)

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We complete the proof from last time.

Proof. Take m values $x_1, x_2, \ldots, x_m$ and let $\overline{x}$ be their arithmetic mean. Then, recall that $\sum_{i=1}^{m} \left(x_i - \overline{x}\right)^2 = \sum_{i=1}^{m} x_i^2 - n\overline{x}^2$This is simply the definition of variance.
Then, letting $m = \binom{n}{2}$, $\hat{d}_{ij} = x_k$ and the mean codegree to be mcd $= \frac{1}{\binom{n}{2}} \sum_{1\leq i,j\leq n} \hat{d}_{ij} =$

$\frac{1}{\binom{n}{2}}\left(\frac{1}{8}n^3+o\left(n^3\right)\right)=\frac{n}{4}+o\left(n\right)\right)$. Then, we have

$$\begin{aligned}\sum_{1\leq i,j\leq n}\left(\hat{d}_{ij}-\mathrm{mcd}\right)^2 &= \sum_{1\leq i,j\leq n}\hat{d}_{ij}^2-\binom{n}{2}\mathrm{mcd}\\ &= \frac{1}{32}n^4+o\left(n^4\right)-\frac{1}{32}n^4+o\left(n^4\right)\\ &= o\left(n^4\right).\end{aligned}$$

Hence, we obtain $\sum_{1\leq i,j\leq n}\left(\hat{d}_{ij}-\mathrm{mcd}\right)^2=o\left(n^4\right)$. Then, letting $y_i=\left|\hat{d}_{ij}-\mathrm{mcd}\right|$ we see by cauchy shwartz that $\frac{1}{m}\sum_{i=1}^n y_i\leq\sqrt{\frac{1}{m}\sum_{i=1}^n y_i}$, hence $\sum_{i=1}^n x_i\leq\sqrt{m\sum_{i=1}^n y_i}$. Hence, we have $\sum_{1\leq i,j\leq n}\left|\hat{d}_{ij}-\mathrm{mcd}\right|\leq\sqrt{\binom{n}{2}\sum_{1\leq i,j\leq n}\left(\hat{d}_{ij}-\mathrm{mcd}\right)^2}=o\left(n^3\right)$. Hence,

$$\sum_{1\leq i,j\leq 2}\left|\hat{d}_{ij}-\mathrm{mcd}\right|=o\left(n^3\right).$$

Then triangle inequality yields

$$\begin{aligned}\sum_{1\leq i,j\leq n}\left|\hat{d}_{ij}-\frac{n}{4}\right| &\leq \sum_{1\leq i,j\leq n}\left|\hat{d}_{ij}-\mathrm{mcd}\right|+\left|\mathrm{mcd}-\frac{n}{4}\right|\\ &= o\left(n^3\right)+o\left(n^3\right)\\ &= o\left(n^3\right).\end{aligned}$$

□

Now, we proceed to prove some more implications, but first we state a lemma.

Lemma 0.1. Let $x_1,x_2,\ldots,x_n$ be an orthornormal basis with associated eigenvalues $\lambda_1,\ldots,\lambda_n$. Then for $j=\frac{1}{\sqrt{n}}\begin{pmatrix}1&\ldots&1\end{pmatrix}$, we find $\left|x_1-j\right|_2=o\left(1\right)$.

Proof. $\left(P_3\Rightarrow P_5\right)$. Let x_1 be a unit eigenvector of G corresponding to λ_1. Then, let $j=\frac{1}{\sqrt{n}}\begin{pmatrix}1&\ldots&1\end{pmatrix}$, then by lemma we have $\left|x_1-j\right|_2=o\left(1\right)$. □