

Algebraic Theory I

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Contents

1	Ring Localization	1
2	Chinese Remainder Theorem	2

1 Ring Localization

Lecture 33: Localization of Rings

Wed 10 Nov 2021 17:33

Recall. Recall R denotes a commutative ring. If $S \subseteq R$ is a multiplicative subset, we see $x, y \in S$ implies $xy \in S$ and $0 \notin S$ but $1 \in S$.

Then, we define $S^{-1}R = \{X/s : x \in R, s \in S\}$. Then, we see $\frac{x_1}{s_1} = \frac{x_2}{s_2}$ if and only if there is an $s \in S$ so that $s(s_2x_1 - s_1x_2) = 0$. Of course, if R is an integral domain we see this implies $s_2x_1 - s_1x_2 = 0$, the normal definition of fraction equality.

Now, we turn this set into a ring. We define $\frac{x_1}{s_1} \cdot \frac{x_2}{s_2} := \frac{x_1x_2}{s_1s_2}$ and $\frac{x_1}{s_1} + \frac{x_2}{s_2} = \frac{s_2x_1 + s_1x_2}{s_1s_2}$. Now, we need to show that $+, \cdot$ are well defined (meaning they do not vary for different representatives of a given equivalence class). This fact is easily checked by symbolic manipulation so we omit the proof. For the addition case suppose $\frac{x_1}{s_1} = \frac{x'_1}{s'_1}$ and similarly for $\frac{x_2}{s_2}$ then take the multiplicative representation of the fraction and multiply the $\frac{x_1}{s_1}$ representation by $-s_2s'_2ts$ and the $\frac{x_2}{s_2}$ representation by $-s_1s'_1st$ and by adding together these representations we see terms cancel and we obtain that addition is in fact well defined. Moreover, it is trivial to check that the ring axioms hold.

Definition 1.1 (Ring Localization). We denote this new fraction ring $S^{-1}R$ to be the **localization of R** with additive identity $\frac{0}{1}$, multiplicative identity $\frac{1}{1}$ and $\frac{tx}{ts} = \frac{x}{s}$ for all $t \in S$.

Note that $s \in S$ is nonzero by definition, so $\frac{1}{s} \cdot \frac{s}{1} = \frac{1}{1} = 1_{S^{-1}R}$, so every element has an inverse.

Proposition 1.1. If R is a commutative ring with $S \subseteq R$ being a multiplicative subset. Then the map

$$\begin{aligned}\pi : R &\longrightarrow S^{-1}R \\ x &\longmapsto \pi(x) = \frac{x}{1}\end{aligned}$$

is a ring homomorphism. Moreover, if S has no zero-divisors, then π is an injection.

Proof. If $x, y \in R$ then $\pi(x \pm y) = \frac{x \pm y}{1} = \frac{x}{1} \pm \frac{y}{1} = \pi(x) \pm \pi(y)$. Furthermore $\pi(1) = \frac{1}{1} = 1$.

Lastly, $\pi(xy) = \frac{xy}{1} = \frac{x}{1} \frac{y}{1} = \pi(x) \pi(y)$. Hence, π is a ring homomorphism. Now consider $\ker(\pi) = \{x \in R : \frac{x}{1} = \frac{0}{1}\}$. We see this implies an $s \in S$ so that $s(1x - 1 \cdot 0) = sx = 0$, hence s is a zero divisor if $x \neq 0$. So, the kernel is trivial. \square

Example. If R is a commutative ring and $P \subseteq R$ is a prime ideal, then $S := R \setminus P$ is a multiplicative set. Moreover, $0 \in P$ so $0 \notin S$ and $P \subset R$ is proper, so $1 \in S$.

If $x, y \in S$ with $xy \notin S$, then $xy \in P$ so $x \in P$ or $y \in P$. So, S is closed under multiplication. Then the localization $S^{-1}R$ is often denoted R_P . This is the canonical example of localization which we will study more next class. \diamond

The use of this construction is that it allows us to embed an integral domain R in a field $R_{(0)}$ called the **field of fractions**.

2 Chinese Remainder Theorem

Lecture 34: Chinese Remainder Theorem

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Theorem 2.1 (Classical Chinese Remainder Theorem). If m_1, \dots, m_r are relatively prime integers, then for a_1, \dots, a_r we find an $x \in \mathbb{Z}$ so that $x \equiv a_i \pmod{m_i}$ for each $1 \leq i \leq r$.

Theorem 2.2 (Generalized Chinese Remainder Theorem). Let R be a commutative ring with $I_1, \dots, I_n \subseteq R$ being ideals so that $I_i + I_j = R$ for all $i \neq j$. That is, the I_i s are pairwise co-maximal. Then for any $x_1, \dots, x_n \in R$ we find an $x \in R$ so that $x \equiv x_i \pmod{I_i}$ for all $1 \leq i \leq n$.

Recall. $x \equiv x_i \pmod{I_i}$ if $x - x_i \in I_i$.

Proof. If $n = 1$ this is trivial. Of course, $x = x$.

For the case $n = 2$ we have $I_1 + I_2 = R$, hence $1 \in R = I_1 + I_2$. Hence, $1 = a_1 + a_2$ with $a_1 \in I_1, a_2 \in I_2$. Then, let $x = x_1 a_1 + x_2 a_2$, and we see $a_1 + a_2 = 1$ but $a_1 \equiv 0 \pmod{I_1}$ and likewise $a_2 \equiv 0 \pmod{I_2}$, hence $a_1 \equiv 1 \pmod{I_2}$ and $a_2 \equiv 1 \pmod{I_1}$.

$\text{mod } I_2$ and $a_2 \equiv 1 \pmod{I_1}$. Hence,

$$\begin{aligned} x &= x_1 a_2 + x_2 a_1 \\ &\equiv x_1 a_2 \pmod{I_1} \\ &\equiv x_1 \pmod{I_1} \\ \text{and } x &\equiv x_2 a_1 \\ &\equiv x_2 \pmod{I_2}. \end{aligned}$$

Hence, the claim holds for $n = 2$. Now, we induce on n .

Let $n \geq 3$ and suppose the case $n - 1$ to be true. Then, we find Then, we see $I_1 + I_i = R$ for all $i \geq 2$ by hypothesis. Hence, $1 = a_i + b_i$ with $a_i \in I_1$, $b_i \in I_i$. Then, we find

$$1 = \underbrace{1 \cdot \dots \cdot 1}_{n \text{ times}} = \prod_{i=1}^n (a_i + b_i) \in \prod_{i=1}^n (I_1 + I_i) \subseteq I_1 + \prod_{i=2}^n I_i.$$

Moreover, we know $I_1 + \prod_{i=2}^n I_i$ to be an ideal as the product and sum of ideals are still ideals.

Then applying the case $n = 2$, we find a $y \in R$ so that $y_1 \equiv 1 \pmod{I_1}$ and $y_1 \equiv 0 \pmod{\prod_{i=2}^n I_i}$. Repeating for each $1 \leq i \leq n$ yields a $y_j \in R$ so that $y_j \equiv 1 \pmod{I_j}$ and $y_j \equiv 0 \pmod{\prod_{1 \leq i \leq n; i \neq j} I_i}$. Now, define $x = \prod_{i=1}^n x_i y_i$. We see $y_j \in I_i$ for all $i \neq j$, hence $y_j x_j \equiv 0 \pmod{I_i}$ for all $i \neq j$. Hence $x \equiv x_i y_i \equiv x_i \pmod{I_i}$. \square

Note that in the preceding proof $\prod I_i$ denotes the ideal product as defined in the homework. In the next theorem we will use this symbol for the cartesian product, so ideal products will be written without product notation when the context is not necessarily clear.

Corollary 1 (Alternative Statement of the Chinese Remainder Theorem).

Let R be a commutative ring with $I_1, \dots, I_n \subseteq R$ being pairwise comaximal distinct ideals of R . Then the map

$$\begin{aligned} f : R &\longrightarrow \prod_{i=1}^n R/I_i \\ x &\longmapsto (x \pmod{I_i})_{1 \leq i \leq n} \end{aligned}$$

is a surjective ring homomorphism with kernel $\ker(f) = \bigcap_{i=1}^n I_i$. Specifically,

$$R / \left(\bigcap_{i=1}^n I_i \right) \simeq \prod_{i=1}^n (R/I_i).$$

Proof. It is easily confirmed that f is a ring homomorphism with the prescribed kernel. Hence, the only claim that remains to be shown is the surjectivity. For f to be surjective, we need to take an arbitrary congruence system $\hat{x} = (x_1 \pmod{I_1}, x_2 \pmod{I_2}, \dots, x_n \pmod{I_n})$ in the codomain of f and find a solution $x \in R$ so that $x \equiv x_i \pmod{I_i}$ for all $1 \leq i \leq n$ (that is $f(x) = \hat{x}$). We see the generalized remainder theorem yields such an x , so f is surjective. \square