Analysis I

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Proposition 0.1. Let $f : \mathbb{R} \to \overline{\mathbb{R}}$ be integrable. Then for each $\varepsilon > 0$ there is a $\delta > 0$ such that each measurable $S \subseteq \mathbb{R}$ has $\int_S |f| < \varepsilon$ if $m(S < \delta)$.

Proof. Let $\varepsilon > 0$, then there is a $s \in \mathscr{S}(|f|)$ such that $\int (|f| - s) < \frac{\varepsilon}{2}$. Let $a\alpha = \sup\{s(x) : x \in \mathbb{R}\}$ and $\delta = \frac{\varepsilon}{2(\alpha+\varepsilon)}$. If S is measurable and $m(S) < \delta$, we find

$$\int_S |f| \leq \int s + \frac{\varepsilon}{2} \leq \alpha m(S) + \frac{\varepsilon}{2} < \varepsilon.$$

□

Theorem 0.1 (Monotone Convergence Theorem)**.** Let (f_n) be a sequence of nonnegative measurable functions with $f_n : \mathbb{R} \to \overline{\mathbb{R}}$ such that $(f_n(x))$ is increasing for all $x \in \mathbb{R}$. Then, $f = \lim\limits_{n\to\infty} f_n$ is maesurable with $\int f = \lim\limits_{n\to\infty} \int f_n$.

Proof. Since $f = \limsup_{n\to\infty} f_n = \liminf_{n\to\infty} f_n$, we see f is measurable. Moreover, the sequence $\left(\int f_n\right)$ is increasing (as the f_ns are increasing). Hence, letting $L = \lim\limits_{n\to\infty} \int f_n$ exists with $L \in R_0^+$. Since $\int f_n \leq \int f$ for all n by monotonicity, we find $L \leq \int f$.
Let $s \in \mathscr{S}(f)$ and fix $c \in (0,1)$ and define $E_n = \{x \in \mathbb{R} : f_n(x) \geq cs(x)\}$. Then, we find $\{E_n : n \in \mathbb{N}\}$ is an ascending collection (again by monotonicity of (f_n)) of measurable sets with $\bigcup_{n\in\mathbb{N}} E_n = \mathbb{R}$ as $cs(x) < f_n(x) \leq f(x)$. Let $s = \sum_{k=1}^{K} a_k \chi_{S_k}$ and we see $cs\chi_{E_n} M = f_n \chi_{E_n} \leq f_n$, with

$$L \geq \int f_n \geq \int_{E_n} f_n \geq \int cs\chi_{E_n} = c\int_{E_n} s = c\sum_{k=1}^{K} a_k m(S_k \cap E_n).$$

Since $\lim\limits_{n\to\infty} m\left(E_n \cap S_n\right) = m\left(S\right)$ for every measurable set S, we find $L \geq c\sum_{k=1}^{K} a_k m\left(S_k\right) = c\int s$. Since c was arbitrary, we see the inequality holds for all $c \in (0,1)$, hence we find $L \geq s$ (by taking supremums), but $s \in \mathscr{S}\left(f\right)$, hence $L \geq \int f$. So, $L = \int f$. □

Theorem 0.2 (Fatou's Lemma)**.** If (f_n) is a sequence of nonnegative measurable functions $f_n : \mathbb{R} \to \overline{\mathbb{R}}$, then $\int \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int f_n$.

Proof. For $x \in \mathbb{R}$, define $g_n\left(x\right) = \inf\{f_k\left(x\right) : k \geq n\}$ for $n \in \mathbb{N}$. Then, we find (g_n) is a nonnegative measurable sequence of functions with $\left(g_n\left(x\right)\right)$ increasing for all fixed x and $g_n \leq f_n$ for all n. Consequently, $\int g_n \leq \int f_n$ and $\left(\int g_n\right)$ is increasing. As $\lim\limits_{n\to\infty} g_n = \liminf_{n\to\infty} f_n$ is measurable by an earlier theorem, we find

$$\liminf_{n\to\infty} \int f_n \geq \liminf_{n\to\infty} \int g_n = \lim_{n\to\infty} \int g_n = \int \lim_{n\to\infty} g_n = \int \liminf_{n\to\infty} f_n.$$

□

Proposition 0.2. For any integral function $f : \mathbb{R} \to \overline{\mathbb{R}}$, we find $\left|\int f\right| \leq \int |f|$.

Theorem 0.3 (Dominated Convergence Theorem)**.** Let (f_n) be a sequence of measurable functions $f_n : \mathbb{R} \to \overline{\mathbb{R}}$. Suppose there is an integrable function g with $|f_n| \leq g$ for all $n \in \mathbb{N}$. If (f_n) converges pointwise to a function $f : \mathbb{R} \to \overline{\mathbb{R}}$ almost everywhere, then f is integrable and

$$\lim_{n\to\infty} \int |f_n - f| = 0 \text{ and } \lim_{n\to\infty} \int f_n = \int f.$$

Proof. Since $f\left(x\right) = \lim\limits_{n\to\infty} f_n\left(x\right)$ for almost all $x \in R$, we find f is measurable. Moreover, $|f_n| \leq g$ implies $|f| < g$ almost everywhere and since g is integrable (hence finite a.e) we find f, f_n are integrable (hence finite) almost everywhere. Now, define for each $n \in \mathbb{N}$

$$E_n = \{x \in \mathbb{R} : \left|f_n\left(x\right)\right|, \left|f\left(x\right)\right| < \infty, \left|f_n\left(x\right) - f\left(x\right)\right| \leq 2g\left(x\right)\}.$$

Since $R \setminus \bigcup_{n\in\mathbb{N}} E_n$ is a set of measure 0, we can assume $\left|f_n\left(x\right)\right|, \left|f\left(x\right)\right| < \infty$ and $\left|f_n\left(\right) - f\left(x\right)\right| \leq 2g\left(x\right)$ for all $x \in \mathbb{R}$. Then, Fatou's lemma applies to the sequence on nonnegative measurable functions $(2g - |f_n - f|)$ yielding

$$\begin{aligned}\int 2g &\leq \liminf_{n\to\infty} (2g - |f_n - f|) \\ &= \int 2g + \liminf_{n\to\infty} \left(-\int |f_n - f|\right) \\ &= \int 2g - \limsup_{n\to\infty} \int |f_n - f| \\ &\Rightarrow \limsup_{n\to\infty} \int |f_n - f| \leq 0 \\ &\Rightarrow \lim_{n\to\infty} \int |f_n - f| = 0.\end{aligned}$$

Hence, $\lim_{n\to\infty} \left|\int (f_n - f)\right| = 0$ by the earlier lemma. So, $\lim_{n\to\infty} \int f_n = \int f_n$. □

Definition 0.1 (Convergence in Measure)**.** Let (f_n) be a sequence of measurable functions $f_n : \mathbb{R} \to \overline{\mathbb{R}}$ and $f : \mathbb{R} \to \overline{\mathbb{R}}$ also be measurable. The sequence (f_n) **converges in measure** to f ($f_n \to f$ by measure) if each $f_n$is finite almost everywhere and for each $\varepsilon > 0$ there is a $N \in \mathbb{N}$ so that

$$m\left(\{x \in \mathbb{R} : |f_n(x) - f(x)| > \varepsilon\}\right) < \varepsilon$$

for $n \geq N$.

Theorem 0.4 (Riesz)**.** Let (f_n) be a sequence of measurable functions $f_n : \mathbb{R} \to \overline{\mathbb{R}}$ and $f : \mathbb{R} \to \overline{\mathbb{R}}$ also being measurable. If $(f_n) \to f$ in measure, then there is a subsequence (f_{n_k}) which converges pointwise almost everywhere to f.

Proof. First, we find a strictly increasing sequence of numbers (n_k) such that $m\left(\{x \in \mathbb{R} : |f_j(x) - f(x)| > 2^{-k}\}\right) < 2^{-k}$ if $j \geq n_k$. For $k \in \mathbb{N}$ denote

$$S_k = \{x \in \mathbb{R} : |f_{n_k} - f(x)| > 2^{-k}\}.$$

Then, $\sum_{k=1}^{\infty} m(S_k) \leq \sum_{k=1}^{\infty} 2^{-k} < \infty$. Applying the Borel-Cantelli Lemma yields that almost every $x \in R$ does not belong to any infinite subcollections of (S_k). For such x, we find a $K \in \mathbb{N}$ such that $|f_{n_k}(x) - f(x)| \leq 2^{-k}$ for $k \geq K$. Hence, f_{n_k} converges pointwise to f for all x not belonging to an infinite subcollection of (S_k), hence almost everywhere. □