Analysis I

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Lecture 14: Measurable Functions (2)

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Recall. A function $f : S \to \mathbb{R}$ was measurable if S is measurable and $f^{-1}((c, \infty])$ is measurable for all $c \in \mathbb{R}$. There was an equivalent definition using the extended borel σ-algebra that we will use occasionally.

Proposition 0.1. Suppose $f : S \to \overline{\mathbb{R}}$ is continuous on the measurable set S, then f is measurable.

Proof. Let H be an extending function, then we must show $H \circ f$ is continuous. We see any subray , $f(X_0) = (c, \infty]$ will have $(H \circ f)(X_0) = (\hat{c}, 1]$. We know the preimage of this to be open in S, hence measurable. □

Proposition 0.2. Let $S \subseteq \mathbb{R}$. Suppose $f : S \to \mathbb{R}$ is measurable. and let $g : B \to \mathbb{R}$ with $B \in \overline{\mathscr{B}}$ and $f(S) \subseteq B$. Then, $g \circ f : S \to \mathbb{R}$ is measurable.

Proof. For $c \in \mathbb{R}$, we note that $(g \circ f)^{-1}((c, \infty]) = f^{-1}\left(g^{-1}((c, \infty])\right)$. By continuity of g, we know $g^{-1}((c, \infty]) \in \overline{\mathscr{B}}$. And, since f is measurable, we find $f^{-1}\left(g^{-1}((c, \infty])\right)$. □

Corollary 1. Let $S \subseteq \mathbb{R}$ and $f : S \to \mathbb{R}$ to be a measurable function. Then, for every $\alpha \in \mathbb{R}$ and $0 < \rho < \infty$, we find αf and $|f|^\rho$ are measurable.

Proof. We see the functions $g(u) = \alpha u$ on $\overline{\mathbb{R}}$ and $h(u) = |u|^\rho$ on $\overline{\mathbb{R}}$ to be the corresponding functions. We see the case h is clearly continuous and well defined. On the other hand g may be poorly defined if $\alpha = 0$ and $f(x) = \infty$. Recall, however, we had $0 \cdot \pm\infty = 0$ so g is just the zero functions and we see continuity holds. □

Definition 0.1 (Almost-everywhere)**.** Let S be measurable, then a property is said to hold true **almost everywhere** on S or **for almost all** $x \in S$ if there is a set T with $\mu(T) = 0$ and the property holds on all of $S \setminus T$.

Proposition 0.3. Let $S \subseteq \mathbb{R}$ and suppose $f, g : S \to \overline{\mathbb{R}}$ such that f is measurable and $g = f$ almost everywhere on S, then g is measurable.

Proof. Let $T = \{x \in S : f(x) \neq g(x)\}$. Fix $c \in \mathbb{R}$ and let $F = f^{-1}((c, \infty]) \setminus T$ and $G = f^{-1}((c, \infty]) \cup T$. Clearly, both F and G are measurable. Furthermore, $F \subseteq G$ and $\mu(G \setminus F) = \mu(T) = 0$. Since, $F \subseteq g^{-1}((c, \infty]) \subseteq G$. And, by an earlier characterization we recall that a set X is measurable if and only if there were nested sets around it with a difference of measure 0 . Hence, g is measurable. □

Remark. Suppose $f : S \to \overline{\mathbb{R}}$ is a measurable set and $S \subseteq X \subseteq \mathbb{R}$. If $\mu(X \setminus S) = 0$ and $h : X \to \overline{\mathbb{R}}$ is any extension of f , then h is measurable since $h^{-1}((c, \infty]) = f^{-1}((c, \infty]) \cup \{x \in X \setminus S : h(x) \in (c, \infty]\}$. This is the union of a measurable set with a set of measure 0, so we see h is measurable.

Notation. Instead of saying that every extension of a measurable function $f : S \to \overline{\mathbb{R}}$ to a function $h : X \to \overline{\mathbb{R}}$, we often just say f is measurable on X as long as it is defined almost everywhere on X and is measurable on that set.

Proposition 0.4. Suppose $f : I \to \overline{\mathbb{R}}$ is monotone on $I \subseteq \mathbb{R}$. Then, the set of all points in I where f fails to be continuous is countable, hence measure 0. Another characterization is that f is continuous almost everywhere, hence f is measurable.

Proof. It suffices to consider the case f is increasing and I open. Let E be the set of all $x \in I$ where f fails to be continuous. For $x \in E$ let $\alpha_x = \sup(\{f(z) : z < x\} z \in J)$ and $\beta_x = \inf(\{f(z) : z > x\} z \in J)$. Since f is not continuous at x, we find the interval $(\alpha_x, \beta_x) = I_x$ to be nonempty. Also, if $x, y \in E$ are distinct with $x < y$ we find $\beta_x <= \alpha_y$. Hence, we find $I_x \cap I_y = \varnothing$. Since each interval I_x for $x \in E$ contains a rational number, we see E is countable. Hence, $\mu(E) = 0$ and we see $f|_{I \setminus E}$ is continuous on $I \setminus E$ which is measurable, hence the restriction is measurable and as f coincides with its restriction almost everywhere, we see f is measurable. □

Definition 0.2 (Finite Functions)**.**

- Let $S \subseteq \mathbb{R}$. A function $f : S \to \overline{\mathbb{R}}$ is called **finite on** S if $|f(x)| < \infty$ for all $x \in S$.
- Let $f, g : S \to \overline{\mathbb{R}}$ Then we say $f < g$ if $f(x) < g(x)$ for all $x \in S$. Similarly for all other inequalities.
- f is called **nonnegative** if $f \geq 0$ and **positive** if the inequality is strict.

Proposition 0.5. Let $f, g : S \to \overline{\mathbb{R}}$ be measurable and finite almost everywhere. Then, $f+g, f-g, f \cdot g$ are measurable.
If $g(x) \neq 0$ for almost every $x \in S$, then $\frac{f}{g}$ is measurable.

Proof.

1. First, we prove addition. We may assume f, g are finite on S . Then, $h = f + g$ is well defined. Since for $x \in S$, we have $h(x) > q$ for $c \in R$ if and only if there is a $q \in \mathbb{Q}$ such that $f(x) > q$ and $g(x) > c - q$, we have

$$\begin{aligned} h^{-1}((c, \infty]) &= h^{-1}((c, \infty)) \text{ by finiteness.} \\ &= \bigcup_{q \in \mathbb{Q}} f^{-1}((q, \infty)) \cup g^{-1}(c-q, \infty). \end{aligned}$$

   Hence, h as measurable as these are all measurable sets. If f, g are measurable, then so are $f, -g$, hence $f + (-g) = f - g$

2. With addition, subtraction is completely trivial,

3. Now multiplication, Let h be any measurable finite function on S. Consider $(h)^2$. If $c \geq 0$, we have

$$\left((h)^2\right)^{-1}((c, \infty)) = h^{-1}\left((-\infty, \sqrt{c})\right) \cup h^{-1}\left((\sqrt{c}, \infty)\right).$$

   If $c < o$, then

$$\left((h)^2\right)^{-1}((c, \infty)) = h^{-1}(\mathbb{R}) = S.$$

   As in either case we had the preimage being measurable, we see $(h)^2$ is measurable. Since $f \cdot g = \frac{1}{2}(f+g)^2 - \frac{1}{2}(f)^2 - \frac{1}{2}\left(g^2\right)$ being the sum, constant multiple and square of measurable functions yields $f \cdot g$ to be measurable.

4. Lastly, let $h = \frac{1}{g}$, and note we can assume g is nonzero for all S , hence h is well defined on S and h is finite. If $c > 0$ we see $h^{-1}((c, \infty)) = g^{-1}(\left(0, \frac{1}{c}\right))$. As this interval is open and borel, we see $g^{-1}\left(\left(0, \frac{1}{c}\right)\right)$ is borel, hence $h^{-1}((c, \infty))$ is measurable.
   Similairly, if $c = 0$, we see $h^{-1}((0, \infty)) = g^{-1}((0, \infty))$.
   Lastly, if $c < 0$ we have $h^{-1}(c, \infty) = g^{-1}(\left(-\infty, \frac{1}{c}\right)) \cup g^{-1}((0, \infty)) = g^{-1}\left(\left[\frac{1}{c}, 0\right)^c\right)$ hence measurable. This completes the proof.

$\square$

Lecture 15: Measurable Functions (3)

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