Combinatorics

Thomas Fleming

October 13, 2021

Contents

Lecture 20: Quasi-Random Graphs (3)

Fri 08 Oct 2021 10:13

We complete the proof from last time.

Proof. Take m values $x_1, x_2, \ldots, x_m$ and let $\overline{x}$ be their arithmetic mean. Then, recall that $\sum_{i=1}^{m} (x_i - \overline{x})^2 = \sum_{i=1}^{m} x_i^2 - n\overline{x}^2$This is simply the definition of variance.
Then, letting $m = \binom{n}{2}$, $\hat{d}_{ij} = x_k$ and the mean codegree to be $\text{mcd} = \frac{1}{\binom{n}{2}} \sum_{1 \leq i,j \leq n} \hat{d}_{ij} = \frac{1}{\binom{n}{2}} \left(\frac{1}{8} n^3 + o\left(n^3\right) \right) = \frac{n}{4} + o\left(n\right) \big)$. Then, we have

$$\begin{aligned}
\sum_{1 \leq i,j \leq n} \left(\hat{d}_{ij} - \text{mcd} \right)^2 &= \sum_{1 \leq i,j \leq n} \hat{d}_{ij}^2 - \binom{n}{2} \text{mcd} \\
&= \frac{1}{32} n^4 + o\left(n^4\right) - \frac{1}{32} n^4 + o\left(n^4\right) \\
&= o\left(n^4\right).
\end{aligned}$$

Hence, we obtain $\sum_{1 \leq i,j \leq n} \left(\hat{d}_{ij} - \text{mcd} \right)^2 = o\left(n^4\right)$. Then, letting $y_i = \left| \hat{d}_{ij} - \text{mcd} \right|$ we see by cauchy shwartz that $\frac{1}{m} \sum_{i=1}^{n} y_i \leq \sqrt{\frac{1}{m} \sum_{i=1}^{n} y_i}$, hence $\sum_{i=1}^{n} x_i \leq \sqrt{m \sum_{i=1}^{n} y_i}$. Hence, we have $\sum_{1 \leq i,j \leq n} \left| \hat{d}_{ij} - \text{mcd} \right| \leq \sqrt{\binom{n}{2} \sum_{1 \leq i,j \leq n} \left(\hat{d}_{ij} - \text{mcd} \right)^2} = o\left(n^3\right)$. Hence,

$$\sum_{1 \leq i,j \leq 2} \left| \hat{d}_{ij} - \text{mcd} \right| = o\left(n^3\right).$$

Then triangle inequality yields

$$\begin{aligned}
\sum_{1 \leq i,j \leq n} \left| \hat{d}_{ij} - \frac{n}{4} \right| &\leq \sum_{1 \leq i,j \leq n} \left| \hat{d}_{ij} - \text{mcd} \right| + \left| \text{mcd} - \frac{n}{4} \right| \\
&= o\left(n^3\right) + o\left(n^3\right) \\
&= o\left(n^3\right).
\end{aligned}$$

□

Now, we proceed to prove some more implications, but first we state a lemma.

Lemma 0.1. Let $x_1, x_2, \ldots, x_n$ be an orthornormal basis with associated eigenvalues $\lambda_1, \ldots, \lambda_n$. Then for $j = \frac{1}{\sqrt{n}}\begin{pmatrix} 1 & \ldots & 1 \end{pmatrix}$, we find $|x_1 - j|_2 = o(1)$.

Proof. $(P_3 \Rightarrow P_5)$. Let x_1 be a unit eigenvector of G corresponding to λ_1. Then, let $j = \frac{1}{\sqrt{n}}\begin{pmatrix} 1 & \ldots & 1 \end{pmatrix}$, then by lemma we have $|x_1 - j|_2 = o(1)$. □

Lecture 21: Quasi-Random Graphs (4)

Wed 13 Oct 2021 10:17

We complete the proof from last time. Recall our lemma that for orthornormal basis containing x_1 we have $|x_1 - j|_2 = o(1)$. We proceed

Proof. WLOG assume G to be a random graph of even order and $|S| = \frac{n}{2}$. Then, we define a vector $\vec{S}$ with $s_i = \begin{cases} \frac{1}{\sqrt{n}}, & i \in S \\ -\frac{1}{\sqrt{n}}, & i \in V \setminus S \end{cases}$ It is clear $|S|_2 = 1$ and we see

$$\langle S, j \rangle = \underbrace{\frac{1}{\sqrt{n}} \cdot \frac{1}{\sqrt{n}}}_{\frac{n}{2}\text{ times}} + \underbrace{\frac{1}{-\sqrt{n}} \cdot \frac{1}{\sqrt{n}}}_{\frac{n}{2}\text{ times}} = 0.$$

Then, we note $\langle S, x_1 \rangle = \langle S, j \rangle + \langle S, x_1 - j \rangle = \langle S, x_1 - j \rangle$ and applying cauchy-shwartz yields

$$\langle S, x_1 \rangle = \langle S, x_1 - j \rangle \le |S|_2 \, |x_1 - j|_2 = 1 \cdot o(1) = o(1) .$$

Now, define $Z = S - \langle S, x_1 \rangle x_1)$. Then, we see

$$\langle Z, x_1 \rangle = \langle S, x_1 \rangle - \langle S, x_1 \rangle |x_1|_2^2 = 0.$$

So, Z is orthogonal to x_1. Hence, there is a $n-1$ dimensional space, M, generated by $x_2, \ldots, x_n$ with eigenvalues $\lambda_2, \ldots, \lambda_n$ with largest eigenvalue $\max\{\lambda_2, |\lambda_n|\}$. Then, we find by the rayleigh quotient that $|\langle Ay, y \rangle| \le \lambda_1(M) |y|_2^2 = \sigma_2 |y|_2^2$ for all $y \in M$. Similairly, we find

$$\lambda_n |y|_2^2 \le \langle Ay, y \rangle \le \lambda_2 |y|_2^2$$

for all $y \in M$. From this we get $\lambda_n |Z|_2^2 |\langle AZ, Z \rangle| \le \lambda_2 |Z|_2^2$, and recalling $|Z|_2 \le |S|_2 + |\langle S, x_1 \rangle| \, |x_1|_2 = 1 + o(1) 1 \le 2$

$$|\langle AZ, Z \rangle| \le \sigma_2 |Z|_2^2 \le \sigma_2 |2|_2^2 = 4\sigma_2 = o(n) .$$

Finally, we see

$$\begin{aligned} \langle AS, S \rangle &= \langle A(Z + \langle S, x_1 \rangle x_1), Z + \langle S, x_1 \rangle x_1 \rangle \\ &= \underbrace{\langle AZ, Z \rangle}_{o(n)} + \underbrace{\langle S, x_1 \rangle}_{o(1)} \underbrace{\langle AZ, x_1 \rangle}_{=0} + \underbrace{\langle S, x_1 \rangle}_{o(1)} \underbrace{\langle Ax_1, Z \rangle}_{0} + \underbrace{\langle S, x_1 \rangle^2}_{o(1)} \langle Ax_1, x_1 \rangle \\ &= o(n) + \langle S, x_1 \rangle^2 \langle Ax_1, x_1 \rangle \\ &= o(n) + \lambda_1 \\ &= o(n^2) \end{aligned}$$

.

Recall we also know

$$\langle AS, S\rangle = 2e\left(S\right) + 2e\left(G\setminus S\right) - 2e\left(S, G\setminus S\right).$$

and $2e\left(S\right) + 2e\left(G\setminus S\right) + 2e\left(S, G\setminus S\right) = e\left(G\right) \geq \frac{1}{4}n^2 + o\left(n^2\right)$. Then, adding and dividing yields these identities yields $e\left(S\right) + e\left(G\setminus S\right) = \frac{n^2}{8} + o\left(n^2\right)$. Furthermore, $\sum_{i\in S} d_i == \frac{n^2}{4} + o\left(n^2\right) 2e\left(S\right) + e\left(S, G\setminus S\right)$ and $\sum_{i\in G\setminus S} d_i = \frac{n^2}{4} + o\left(n^2\right) = 2e\left(G\setminus S\right) + e\left(S, G\setminus S\right)$. Adding all of the identities thus far yields that $2e\left(S\right) - 2e\left(G\setminus S\right) = o\left(n^2\right)$, hence $e\left(S\right) = \frac{1}{16}n^2 + o\left(n^2\right)$. $\square$

We are nearing the end of quasi-random graphs, but note we have always assumed a quasi-random graph to have density $\frac{1}{2}$. These properties are easily generalized to one of density p. We list the generalized properties.

Definition 0.1.

1. (P_2). A graph is P_2 if
   - $e\left(G\right) \geq \frac{pn^2}{2} + o\left(n^2\right)$
   - $\#CW_4 \leq p^4n^4 + o\left(n^4\right)$.
2. (P_3). A graph is P_3 if
   - $e\left(G\right) \geq \frac{pn^2}{2} + o\left(n^2\right)$
   - $\lambda_1\left(G\right) = pn + o\left(n\right)$
   - $\sigma_2\left(G\right) = o\left(n\right)$.
3. (P_7) . A graph is P_7 if
   - $\sum_{1\leq i,j\leq n}\left|\hat{d}_{ij} - p^2n\right| = o\left(n^2\right)$.