# Combinatorics

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### Lecture 24

Fri 22 Oct 2021 10:22

I originally missed this class.

### Lecture 25: Psuedo-Random Graphs

Fri 22 Oct 2021 10:23

# 1 Psuedo-Random Graphs

**Definition 1.1** (Psuedo-Random Graph)**.** A **psuedo-random graph** is a $d$-regular graph of order $n$ with $\sigma_2(G) \leq \lambda$ and $\lambda = o(d)$ . We denoted this $(n, d, \lambda)$

Let $G$ be a $\left(n, n^{\frac{2}{3}}, 2n^{\frac{1}{2}}\right)$. Then, we derive some nice conditions on the hamiltonicity of $G$.

**Proposition 1.1** (Expander Mixing Lemma)**.** Let $G$ be a $d$-regular graph of order $n$, then for every $X, Y \subseteq V(G)$, we find

$$\left| e(X, Y) - \frac{d}{n} |X| |Y| \right| \leq \sigma_2(G) \sqrt{|X| |Y|}.$$

*Proof.* Note here $e(X, Y)$ double counts edges in the intersection $X \cap Y$. Hence, $e(X, X) = 2e(X)$. Then, note $\frac{d}{n} = \frac{dn}{n^2} = \frac{e(G)}{n^2}$, the density of $G$.

Now, let $X \subseteq M$ and define a vector $j_X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ with $x_i = \begin{cases} 1, & i \in X \\ 0, & i \notin X \end{cases}$

Then, note that $\langle j_X, j_X \rangle = |X|$ and $\langle j_X, j_Y \rangle = |X \cap Y|$.
Then, letting $X \subseteq [n]$ to be a subsets of all numbers less than or equal to $n$, then we see letting $A$ be a $(0, 1)$ matrix which is $n \times n$, we have $\langle Aj_X, j_Y \rangle = e(Y, X) = e(X, Y)$. Lastly, $\langle J_n j_X, j_Y \rangle = |X| |Y|$ Defining $B = A - \frac{s}{n^2} J_n$, we

see $\frac{s}{n^2} = \frac{d}{n}$, hence $B = A - \frac{d}{n} J_n$. Then, note $\langle Bj_X, j_Y \rangle = e(X, Y) - \frac{d}{n} |X| |Y|$. Hence, we have reduced the left hand side to $\langle Bj_X, j_Y \rangle$. Then, note that $|\lambda_i| \leq \sigma_2(G) \leq \lambda$ for $2 \leq i \leq n$. Then, as $\lambda_1(G) = \lambda_1\left(\frac{d}{n} J\right) = d$, we see $\lambda_1(B) = 0$ and $\lambda_j(B) = \lambda_j(A)$ for all $2 \leq j \leq n$.Clearly $\square$