Analysis I

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Lecture 19: End of Convergence, Functions of Bounded Variation, and Derivatives

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Recall we had the dominated convergence theorem. A similair version of the theorem makes use of convergence in measure as follows.

Theorem 0.1 (Dominated Convergence - Convergence in Measure)**.** Let (f_n) be a sequence of measurable functions $f_n : \mathbb{R} \to \overline{\mathbb{R}}$ and suppose there is an integrable function $g : \mathbb{R} \to \overline{\mathbb{R}}$ so that $|f_n| \leq g$ for all $n \in \mathbb{N}$. If $(f_n) \to f : \mathbb{R} \to \overline{\mathbb{R}}$ in measure, (with f measurable), then f is integrable and $\lim\limits_{n\to\infty} \int |f_n - f| = 0$ and $\lim\limits_{n\to\infty} \int f_n = f$.

Proof. First, note a subsequence of (f_n) converges to f pointwise almost everywhere. Hence, we find $|f| \leq g$ almost everywhere, so f is integrable. We cam assume $|f_n - f| \leq 2g$ (almost) everywhere. Then, we find a subsequence $(g_n) = (f_{n_k})$ such that $\limsup_{n\to\infty} |f_n - f| = \lim\limits_{n\to\infty} |g_k - f|$. Then, as $(g_k) \to f$ in measure, we find another subsequence $(h_j) = \left(g_{k_j}\right) = \left(f_{n_{k_j}}\right)$ which converges pointwise to f almost everywhere.
Applying dominated convergence theorem yields

$$\lim_{n\to\infty} \int |h_j - f| = 0.$$

Then, we find

$$\begin{aligned}\limsup_{n\to\infty} \int |f_n - f| &= \lim_{n\to\infty} \int |g_k - f| \\ &= \lim_{n\to\infty} |h_j - f| \\ &= 0.\end{aligned}$$

This completes the proof. $\square$

1 Functions of Bounded Variation and Absolutely Continuous Functions

Remark. For this chapter $[a, b] \subseteq R$ will always denote a compact interval on $\mathbb{R}$.

Definition 1.1 (Partition)**.** A finite sequence $P = (x_k)_{k=n}^{N}$ with $n, N \in \mathbb{Z}$ and $n < N$ is called a **partition** of $[a, b]$ if $x_n = a$, $x_N = b$ and $x_{k-1} \leq x_k$ for $n < k \leq N$. We denote the collection of all partitions of $[a, b]$ to be $\mathscr{P}([a, b])$.

Definition 1.2. Let $f : [a, b] \to \mathbb{R}$ be a function. Then,

- For a partition $P = (x_k)_{k=n}^{N}$, we denote

$$V(f, P) = \sum_{k=n+1}^{N} \left| f(x_k) - f_{(x_{k-1})} \right|$$

  to be the **variation of f with respect to** P.
- We define the quantity $\mathrm{TV}(f) = \sup\{V(f, P) : P \in \mathscr{P}([a, b])\}$ to be the **total variation of f** .

Remark. If $f : [a, b] \to \mathbb{R}$ and $c \in [a, b]$ with partitions $P_1 = (x_k)_{k=n}^{N}$ of $[a, c]$ and $P_2 = (x_k)_{k=N}^{K}$ of $[c, b]$. Then denote, $P = (x_k)_{k=n}^{K}$ to be a partition of $[a, b]$ and we find

$$V(f, P) = V\left(f\mid_{[a,c],P_1}\right) + V\left(f\mid_{[c,b]}, P_2\right).$$

Moreover,

$$\mathrm{TV}(f) = \mathrm{TV}\left(f\mid_{[a,c]}\right) + \mathrm{TV}\left(f\mid_{[c,b]}\right).$$

Definition 1.3 (Bounded Variation)**.** A function $f : \mathbb{R} \to \overline{\mathbb{R}}$ has **bounded variation** if $\mathrm{TV}(f) < \infty$.

Theorem 1.1 (Jordan's Theorem)**.** A function $f : [a, b] \to \mathbb{R}$ is of bounded variation if and only if there are increasing functions $g, h : [a, b] \to \mathbb{R}$ so that $f = g - h$.

Proof. Suppose $\mathrm{TV}(f) < \infty$ and let $x, y \in [a,b]$ with $x < y$. Then, we find

$$\begin{aligned}\mathrm{TV}\left(f\mid_{[a,y]}\right) &= \mathrm{TV}\left(f\mid_{[a,x]}\right) + \mathrm{TV}\left(f\mid_{[x,y]}\right)\\ &\geq \mathrm{TV}\left(f\mid_{[a,x]}\right) + \left|f(y) - f(x)\right|\\ &\geq \mathrm{TV}\left(f\mid_{[a,x]}\right) + f(x) - f(y).\end{aligned}$$

Furtheromre, $h : x \mapsto \mathrm{TV}\left(f\mid_{[a,x]}\right)$ and $g : x \mapsto \mathrm{TV}\left(f\mid_{[a,x]}\right) + f(x)$ are increasing. This fact is trivial for h and we find , adding $f(y)$ to both sides of the former inequality yields $g(y) \geq g(x)$ for arbitrary $y \geq x$, so this claim holds as well.
Taking the difference, $g - h = f$.
Conversely, suppose $f = g - h$ for increasing $g, h : [a,b] \to \mathbb{R}$. Then, let $x, y \in [a,b]$ with $y \geq x$. Then, we find

$$\begin{aligned}\left|f(y) - f(x)\right| &= \left|g(y) - g(x) + h(x) - h(y)\right|\\ &\leq \left|g(y) - g(x)\right| + \left|h(x) - h(y)\right|\\ &= g(y) - g(x) + h(y) - h(x).\end{aligned}$$

Hence, for a partition $P = (x_k)_{k=n}^{N}$, we find

$$\begin{aligned}V(f, P) = &\sum_{k=n+1}^{N} \left|f(x_k) - f(x_{k-1})\right|\\ \leq &\sum_{k=n+1}^{N} \left(g(x_k) - g(x_{k-1}) + h(x_k) - h(x_{k-1})\right) \quad = g(b) - g(a) + h(b) - h(a)\\ < &\infty.\end{aligned}$$

□

Definition 1.4 (Absolute Continuity)**.** A function $f : [a,b] \to \mathbb{R}$ is **absolutely continuous** if for each $\varepsilon > 0$ we find a $\delta > 0$ such that for every finite disjoint collection of nonempty intervals $\{(a_k, b_k) \subseteq [a,b] : 1 \leq k \leq K\}$ with $\sum_{k=1}^{K} (b_k - a_k) < \delta$, we have $\sum_{k=1}^{K} \left|f(a_k) - f(b_k)\right| < \varepsilon$.

Remark. Absolute continuity is stronger than uniform continuity, but weaker than lipschitz continuity.

Theorem 1.2. If a function $f : [a,b] \to \mathbb{R} \to$ is absolutely continuous, then f is continuous and f has bounded variation.

Proof. f is trivially continuous, taking a finite disjoint collection consisting only of 1 interval $\{(x,y)\}$ yields the definition of continuity.
Now we show bounded variation. For $\varepsilon = 1$, let $\delta > 0$ be the number such that the definition of absolute continuity holds for f.
Now fix $(x_k)_{k=n}^{N} \in \mathscr{P}([a,b])$ so that $x_k - x_{k-1} < \delta$ for all $n < k \leq N$. Then, if $P \in \mathscr{P}([x_{k-1}, x_k])$, we see $V\left(f\mid_{[x_{k-1},x_k]}, P\right) < 1$ by definition of absolute

continuity.
So, we have $\mathrm{TV}([x_{k-1}, x_k]) \leq 1$, so $\mathrm{TV}(f) = \sum_{k=n+1}^{N} \mathrm{TV}\left(f\mid_{[x_{k-1},x_k]}\right) \leq N-n$ by the ε assumption. ☐

As it turns out, absolutely continuous functions have a relation to integrable functions, particularly, an integrable function f is simply the anti-integral of an absolutely continuous one.

Proposition 1.1. If $f : [a,b] \to \overline{\mathbb{R}}$ is integrable, then,

$$F : [a,b] \to \mathbb{R},\ x \mapsto \int_{[a,x]} f$$

is absolutely continuous.

This claim can be generalized into a sort of fundamental theorem of calculus for the lebesque integrals to characterize integrals and derivatives. For now, we only prove the weak version.

Proof. For $\varepsilon > 0$ there is a $\delta > 0$ such that $\int_S |f| < \varepsilon$ for every measurable set $S \subseteq [a,b]$ with $m(S) < \delta$.
Now, let $\{(a_k, b_k) : 1 \leq k \leq K\}$ be a disjoint collection of intervals in $[a,b]$ with $\sum_{k=1}^{K} (b_k - a_k) < \delta$. Fix $S = \bigcup_{k=1}^{K} (a_k, b_k)$. Then, since $m(S) < \delta$ and

$$\begin{aligned}\sum_{k=1}^{K} |F(b_k) - F(a_k)| &= \sum_{k=1}^{K} \left| \int_{[a_k,b_k]} f \right| \\ &\leq \sum_{k=1}^{K} \int_{[a_k,b_k]} |f| \\ &= \int_S |f| \\ &< \varepsilon \text{ by assumption.}\end{aligned}$$

Hence, absolute continuity holds. ☐

2 Derivatives and Fundamental Theorem of Calculus

Proposition 2.1. Let $f : (a,b) \to \overline{\mathbb{R}}$ be monotone on $(a,b) \subseteq \mathbb{R}$ with $a, b \in \overline{\mathbb{R}}$ and $a < b$. Then,

$$\lim_{x \to a} f(x) = \inf\{f(x) : x \in (a,b)\}, \lim_{x \to b} f(x) = \sup\{f(x) : x \in (a,b)\}$$

are both well defined.

Lecture 20: Derivatives (2)

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Recall. A monotone function on an interval has well defined limits at both its endpoints.

Definition 2.1 (Upper/Lower Derivatives)**.** Let $S \subseteq \mathbb{R}$, $f : S \to \mathbb{R}$

- We define $\overline{D}f(x) = \lim_{\tau \to 0} \sup\{\frac{f(x+h)-f(x)}{h} : 0 < |h| < \tau\}$ to be the **upper derivative**.
- We define $\underline{D}f(x) = \lim_{\tau \to 0} \inf\{\frac{f(x+h)-f(x)}{h} : 0 < |h| < \tau\}$ to be the **lower derivative**.
- If, for some $x \in \overset{\circ}{S}$, we find $\overline{D}f(x), \underline{D}f(x) \in \mathbb{R}$, with the upper and lower derivatives being equal, we say f is **differentiable** at x.
  We denote $f'(x) = \overline{D}f(x) = \underline{D}f(x)$.

We know, the limits of the upper and lower derivatives to be well defined as the supremum and infimum are monotone functions with respect to τ.

Proposition 2.2. Let $f : S \to \mathbb{R}$ and let $x \in \overset{\circ}{S}$. Then, f is differentiable at x if and only if

$$\lim_{y \to x} \frac{f(y) - f(x)}{y - x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \in \mathbb{R}.$$

That is, the classical derivative is equivalent to the lebesque derivative, so we will use the new definition for most proofs, but the old for most computations.

Theorem 2.1 (Mean-Value Theorem)**.** Let $f : [a,b] \to \mathbb{R}$ be continuous and differentiable at every $x \in (a,b)$. Then, there exists $\xi \in (a,b)$ so that $f(b) - f(a) = f'(\xi)(b-a)$.

Lemma 2.1. Let $f : [a,b] \to \mathbb{R}$ be increasing and suppose $\overline{D}f(x) = \underline{D}f(x)$ for almost every $x \in [a,b]$. Then, $\overline{D}f(x)$ and $\underline{D}f(x)$ are finite almost everywhere. Moreover, f is differentiable almost everywhere (on $[a,b]$). Furthermore, f' is an integrable function and

$$\int_{[a,b]} f' \leq f(b) - f(a).$$

Proof. Extend f to $[a,\infty)$ by letting $f(c) = f(b)$ for all $c \geq b$. Define a sequence (g_n), $g_n : [a,b] \to \overline{\mathbb{R}}$ with

$$x \mapsto n\left(f\left(x + \frac{1}{n}\right) - f(x)\right).$$

Then, b assumption, we know $(g_n(x))$ to be convergent in $\overline{\mathbb{R}}$ with limit $f'(x)$ for almost every $x \in (a,b)$. Each g_n is measurable, hence $\lim_{n\to\infty} g_n$ is increasing,

we see $g(n) \geq 0$, hence $\overline{D}f \geq 0$.
Applying Fatou's lemma yields

$$\begin{aligned}
\int_{[a,b]} \overline{D}f &= \int_{[a,b]} \liminf_{n\to\infty} f_n \\
&\leq \liminf_{n\to\infty} \int_{[a,b]} g_n \\
&= \liminf_{n\to\infty} n \left(\int_{\left[a+\frac{1}{n}, b+\frac{1}{n}\right]} f - \int_{[a,b]} f \right) \\
&= \liminf_{n\to\infty} \left(\underbrace{n \int_{\left[b, b+\frac{1}{n}\right]} f}_{=f(b)} - \underbrace{n \int_{\left[a, a+\frac{1}{n}\right]} f}_{\leq f(a)} \right) \\
&\leq f(b) - f(a).
\end{aligned}$$

We know the final inequality holds because f is constant on $\left[b, b+\frac{1}{n}\right]$ and though f is not constant, it is increasing on $\left[a, a+\frac{1}{n}\right]$ hence the upper bound of their difference is attained by $f(a)$.
Consequently, $\overline{D}f$ is integrable (so finite almost everywhere). And, since $\overline{D}f = \underline{D}f$, we find $f'(x)$ exists and equals $\overline{D}f(x)$ for almost every $x \in [a,b]$. $\square$