Combinatorics

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October 4, 2021

Contents

Lecture 1: Strongly Regular Graphs (1)

Mon 23 Aug 2021 09:10

1 Strongly Regular Graphs

Recall (Regular Graphs)**.** A k-regular graph is a graph with all vertices of equal degree (namely $d(v) = k$ for $v \in G$ implies G is k-regular). Examples of k-regular graphs are cycles with common degree 2 and cubes with common degree 3.

Definition 1.1 (Codegree)**.** We define the **codegree** of two vertices u, v as the number of common neighbors of u, v. We denote this $\hat{d}(u, v) = |N(u) \cap N(v)|$ where N is simply the neighborhood of connected vertices to a given vertex. Note this implies $|N(i)| = d(i)$.

Remark. Excluding loops, we see that u, v cannot be in their own common neighborhoods.

Definition 1.2 (Strongly Regular Graph)**.** A graph G is **SRG (Strongly Regular Graph)** if G is regular, $i \sim j$ (i adjacent to j) implies $\hat{d}(i, j)$ is equal for all adjacent pairs $i, j \in V(G)$. Lastly, for all nonadjacent $i \neq j$, $\hat{d}(i, j)$ is also equal for all nonadjacent pairs $i, j \in V(G)$. We denote this by G is $\mathrm{SRG}(n, k, \lambda, \mu)$. Here, n is the order of G, k is the degree for which G is k-regular, λ is the codegree of all pairs of adjacent vertices, μ is the codegree of all pairs of nonadjacent unique vertices.

Example. The most trivial SRG is the union of two disjoint edges, denoted $2K_2 = SRG(4, 1, 0, 0)$.
K_n is an edge case but is not considered a SRG as we cannot meaningfully

define μ as there are no nonadjacent vertices. $2K_3 = \mathrm{SRG}\,(6, 2, 1, 0)$.
$2K_n = \mathrm{SRG}\,(2n, n-1, n-2, 0)$.
$C_5 = \mathrm{SRG}\,(5, 2, 0, 1)$.
$K_{n,n} = \mathrm{SRG}\,(2n, n, 0, n)$.
$K_{m,n}, m \neq n$ is not SRG.
$C_n, n > 5$ is not SRG.

$\diamond$

Problem. Prove the complete multipartite graph which is regular is strongly regular. Here we define the complete multipartite graph $K_{m_1,m_2,\ldots,m_n}$ to be a graph with n partite sets $A_1, A_2, \ldots, A_n$ for which all A_i are independent and for any u, v such that $u, v \notin A_i$ for a particular A_i implies $u \sim v$.

Solution.

Proposition 1.1. A SRG is disconnected if and only if it is isomorphic to mK_r, $1 < m, r$.

Proposition 1.2. For a given SRG (n, k, λ, μ), we have $k(k - \lambda - 1) = (n - k - 1)\mu$.

Proof. First, note that for a given u, the graph induced by $N(u)$ (simply denoted $N(u)$) is a λ-regular graph. Furthermore, the graph induced by $\overline{N(u)} - u$ (denoted $\overline{N(u)}$) will be μ-regular.
Now, let $v \in N(u)$ and note that this has $k - \lambda - 1$ neighbors in $\overline{N(u)}$ as v has k neighbors by regularity, λ of which are shared with u (hence in $N(u)$ and 1 of which is u itself. Now, for a $w \in \overline{N(u)}$, note that w has μ neighbors in $N(G)$ as $\hat{d}(w, u) = \lambda$ by definition. So, we can partition $G - u$ into a bipartite graph with sets $A = N(u)$, $B = \overline{N(u)}$ and we see each $u \in A$ has $n - \lambda - 1$ neighbors in B and $v \in B$ has μ neighbors in A. We will use double counting on $e(G - u)$, counting the number of edges between A and B yields that there are k members in A each with $k - \lambda - 1$ edges, hence $e(G - u) = k(k - \lambda - 1)$. Similarly, there are $n - k - 1$ members in B (as there are n vertices overall, k of which are neighbors of u, hence not in $\overline{N(u)}$ and 1 of which is u itself) each with μ neighbors in A, hence $e(G - u) = (n - k - 1)\mu$. As these two quantities are equal, this completes the proof by double counting. □

Remark. This implies that an arbitrary combination of parameters may not yield a proper graph. Hence, it is unknown whether SRG $(99, 14, 1, 2)$ exists. Furthermore, these parameters do not uniquely define a graph, for a given set of parameters one may be able to find an arbitrarily large number of SRG's with these parameters which are non-isomorphic.

Lecture 2: Strongly Regular Graphs (2)

Wed 25 Aug 2021 10:16

Proposition 1.3. Let G be SRG (n, k, λ, mu). Then, $\overline{G}$ is SRG $(n, n - k - 1, n - 2k + \mu - 2, n - 2k + \lambda)$

Proof. Let G be SRG (n, k, λ, μ). It is clear $\overline{G}$ will have n vertices. Furthermore, we have already shown a k-regular graph is $(n - k - 1)$-regular. We will verify the nonadjacent co-regularity condition and the adjacent co-regularity will be left as an exercise. Let u, v be two adjacent vertices in G. Then, they have codegree λ. We see u, v are nonadjacent in $\overline{G}$, hence we must who they have codegree $n - 2k - \lambda$. Let us divide the remaining vertices into 4 groups, the neighbors of only u, neighbors of only v, neighbors of both, and neighbors of neither. Let us define the set which is adjacent to both (in G) to be A and the set which is adjacent to neither (in G) to be B. In more precise terms, we have $B = V(G) \setminus (N(u) \cup N(v))$. Hence, we see

$$
\begin{aligned}
|A| &= n - |N(u) \cup N(v)| \\
&= n - |N(u)| - |N(v)| + |N(u) \cap N(v)| \\
&= n - 2k + \lambda.
\end{aligned}
$$

As A is precisely the set of vertices which have both u or v as common neighbors in G, it is the set which has neither u nor v as common neighbors in $\overline{G}$, hence as u, v were arbitrary we see this completes this portion of the proof. □

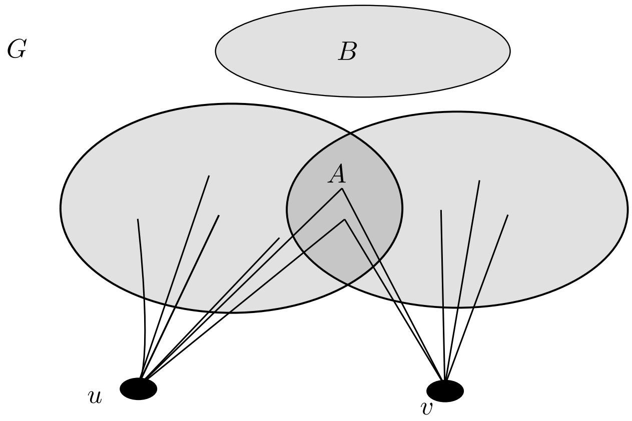

Figure 1: The partitioning of G into 4 sets with A and B labeled.

Problem. Complete the proof of the third parameter.

Proof. $\square$

Remark. It is a general strategy in graph theoretic proofs to take two vertices, u and v and split the remaining vertices into 4 sets, those adjacent only to u, those adjacent only to v, those adjacent to both u and v, and those adjacent to neither u or v.

Proposition 1.4. If G is connected and SRG, then $\operatorname{diam}(G) = 2$.

Proof. As G is not a complete graph (by SRG assumption), we have that $\operatorname{diam}(G) \geq 2$, hence we must only show that $\operatorname{diam}(G) < 3$. WLOG, assume u, v are two points of distance 3 (if the diameter is > 3 we may always choose 2 points at distance 3 on the longest path). Now, label the internal vertices of the path from u to v to be x and y, that is, u, x, y, v forms a path. We see u, v are nonadjacent and $\hat{d}(u, v) = 0$, but u, y are also nonadjacent and $\hat{d}(u, y) = 1$. $\lightning$. $\square$

If G is r-regular, of order n, and $\operatorname{diam}(G) = 2$ then $n \leq r^2 + 1$.

Proof. First, let us partition the graph into 3 parts. The first part will contain u, the second part will be $N(u)$, labeled A, and the third part will be the remaining vertices, $V(G) \setminus (N(u) \cup \{u\})$, labeled B. We see that every point in B will be connected to a point in A, as it must have a path of length 2 to u. Now, let us examine the bipartite graph, $G[A, B]$ generated by A and B. We see $|A| = r$ by r-regularity and $|B| = n - r - l$ as this is how many remain. Next, we will double count the number of edges in this graph, denoted by $e(G[A, B]) = e(A, B)$. First, let us assume the maximal case, that being that each of the r vertices in A has no internal connections within A and $r - 1$ neighbors in B. Then, we see the $e(A,B) \leq \max(e(A,B)) = r(r-1)$. However, we know B has precisely $n-r-1$ vertices, and as each vertex has degree 1 within the induced bipartite graph (by construction), then we have $e(A,B) = n-r-1$. Hence, this yields the inequality $n-r-1 \leq r(r-1)$. Rearranging yields $n \leq r^2+1$. □

Remark. In the case of equality, we have precisely this maximal case, that being each neighbor of u has no common neighbors with u and each nonadjacent vertex to u has precisely 1 common neighbor, so G is $\mathrm{SRG}\left(r^2+1, r, 0, 1\right)$.

Example. Graphs with this quality are rare, with only 3 nontrivial examples known and 1 theorized but not yet proven, these are:

C_5 $(\mathrm{SRG}(5,2,0,1))$

The Peterson Graph $(\mathrm{SRG}(10,3,0,1)\ )$

The Hoffman-Singelton Graph $(\mathrm{SRG}(50,7,0,1)\ )$

The theorized but yet-unproven $\mathrm{SRG}(3250,57,0,1)$.

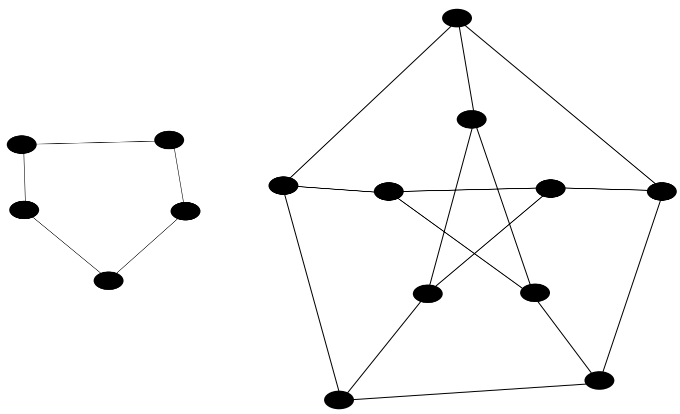

Figure 2: C_5 (left) and the Peterson Graph (right)

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These graphs are essentially the minimally connected r-regular graphs for each r. All SRG's have a minimum degree of $\sqrt{n}$ where n is the order.

Lecture 3: Strongly Regular Graphs (3)

Fri 27 Aug 2021 09:31

Recall. Even if a SRG's parameters satisfy whatever equations we desire, we still cannot say anything about their existence. Hence, graphs of a certain type tend to be named after people as their may only be one or two which satisfy a particular criterion.

Definition 1.3 (Line Graph)**.** Given a graph $G = (V, E)$, we may produce a new graph, called the **line graph**, $\mathscr{L}(G) = \left(E, \hat{E}\right)$. Here we consider two vertices of $\mathscr{L}(G)$ to be adjacent if the corresponding edges in G intersect (meet at a common vertex).

Example. $\mathscr{L}(P_n) = P_{n-1}$.
$\mathscr{L}(C_n) = C_n$.
$\mathscr{L}(K_{1,n}) = K_n$. $\mathscr{L}(K_q) = \text{SRG}\left(\binom{q}{2}, 2(q-2), q-2, 4\right)$, see the picture below for an illustration of this. We call $\mathscr{L}(K_q)$ a triangular graph. $\mathscr{L}(K_{q,q}) = \text{SRG}\left(q^2, 2(q-1), q-2, 2\right)$, we call this the Lattice 2 graph, denoted $L_2(q)$. $\diamond$

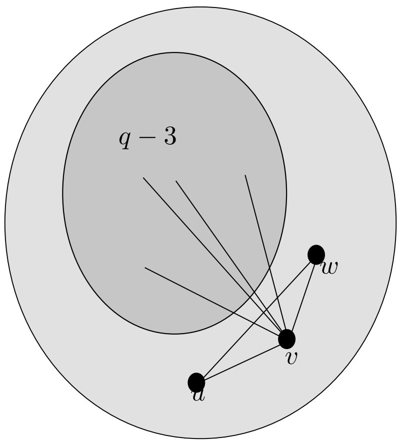

Figure 3: We see uv and wv have $n-3$ neighboring edges with the rest of the graph and uw forms the last neighboring edge. A similar diagram illustrates the final parameter.

Definition 1.4 (Triangular Graph)**.** We define $\mathscr{L}(K_q) = T(q)$ to be the **triangular graph** of order q. We know this to be an infinite family of SRGs, so it is of particular interest. Notably, the first nontrivial triangular graph is equivalent to a usual graph of interest, $K_{2,2,2}$.

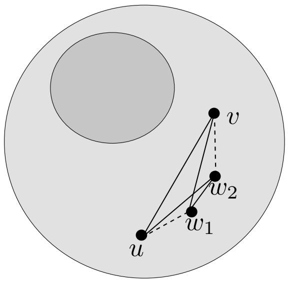

Figure 4: We see the disjoint edges uw_1 and vw_2 (dotted) have exactly 4 edges in common.

Example. $T(4) = \mathrm{SRG}(6, 4, 2, 4)$. The fact that this is a 4-regular graph of order 6 makes this undesirable to work in, so we often examine the complement (which we know to be also SRG). $\overline{T(4)}$ is a 1-regular graph, in other words the union of disjoint edges. Further examination yields that $\overline{T(4)}$ has 3 disjoint edges, and hence its complement graph is the complete tripartite regular graph. Thus, $T(4) = K_{2,2,2}$. $\diamond$

Definition 1.5 (Lattice 2-Graph)**.** We denote $\mathscr{L}(K_{q,q}) = L_2(q)$ to be the **lattice 2-graph**. This is known to be SRG $\left(q^2, 2(q-1), q-2, 2\right)$ by the following diagrams.

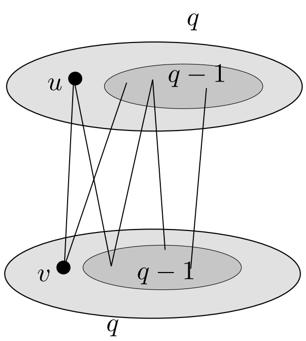

Figure 5: We see each edge has precisely $q-1$ edges in its partite set and $q-1$ edges in the other partite set, hence $2(q-1)$.

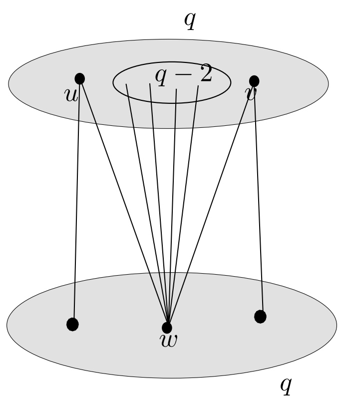

Figure 6: We see any intersection of two edges with a common vertex v must be in the opposing partite set.

Problem. Determine what is $\mathscr{L}(K_{3,3}) = L_2(3)$.

Remark. An equivalent construction of $L_2(q)$ is to take the $q \times q$ matrix and connect all vertices with a common column or row. We call this the greek graph.

Definition 1.6 (Latin Square Graph)**.** A **Latin square** consists of a $q \times q$ matrix where the elements of the matrix consist of a set of q symbols, normally $\{1, 2, \ldots, q\}$. The only requirement of the matrix is that every row and column should contain exactly 1 copy of each number. This is similar to a generalized sudoku grid (without the square requirement).

Example. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$ and $\begin{bmatrix} 4 & 1 & 2 & 3 \\ 1 & 4 & 3 & 2 \\ 2 & 3 & 4 & 1 \\ 3 & 2 & 1 & 4 \end{bmatrix}$.

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Remark. There is a latin square of order q for every q. To obtain such a latin square we can copy the construction of the first example, that is starting with the ordered row and shifting the following row right (or left) by 1 unit.

Problem. Show if L is a symmetric latin ($\ell_{i,j} = \ell_{j,i}$ for all matrix entries $\ell_{i,j}$) square of odd order, then if a symbol shows in the bottom diagonal , it must show in the top diagonal. In other words, each symbol should show an even number of times in the bottom partition, an equal (even) number of times in the top partition, and once on the diagonal.

Lecture 4: Strongly Regular Graphs (4)

Mon 30 Aug 2021 10:20

Recall. Recall a Latin square of order q is a $q \times q$ matrix with no duplicate entries in any given row or column. The number of Latin squares of size q grows even faster than $q!$.

Definition 1.7 (Back-Circulant Latin Square)**.** We define a **back-circulant Latin Square** to be a $q \times q$ matrix,L, defined such that $\ell_{i,j} = (i + j) \bmod 4\,(+1)$, where the $+1$ is optional. This yields entries $0, 1, 2, \ldots, q - 1$ or $1, 2, \ldots, q$ in the case of $(+1)$. It is clear this is symmetric and satisfies the q symbols requirement of a Latin square. Lastly, we must show that this produces the unique column and row requirement, this will be left as an exercise.

Problem. Check that the back-circulant Latin square satisfies uniqueness of columns/rows.

The main use of these Latin squares is that a Latin square of order q can generate a SRG $\left(q^2, 3\,(q-1)\,, q, 6\right)$. This graph we define will be an extension of the lattice graph on q^2. To generate this graph we define the vertices to be the entries of the matrix, L, where L is a Latin square (hence there are q^2). Two vertices, $\ell_{i,j}$ and $\ell_{p,q}$, are joined if one of the following holds

1. $i = p$
2. $j = q$
3. $\ell_{i,j} = \ell_{p,q}$.

We see this identifies elements of the same row, column, and symbol. Let us check that this is indeed a regular graph of order $3(q-1)$. Let $\ell_{i,j}$ be an entry in L and note that as L is $q \times q$ matrix there are $q-1$ entries in row i and $q-1$ entries in column j. Furthermore, by construction of Latin squares, we have that every column contains exactly one entry, $\ell_{p,q}$ with $\ell_{i,j} = \ell_{p,q}$, so excluding $\ell_{i,j}$ itself yields another $(q-1)$ entries, which must be distinct from the entries sharing a column or row by construction. Hence the graph generated by L is $(q-1)$-regular.

Definition 1.8 (Conference Graphs)**.** A graph is called a **conference graph** if it is SRG $\left(n, \frac{n-1}{2}, \frac{n-5}{4}, \frac{n-1}{4}\right)$. We see this definition necessitates $n = 1 \mod 4$. Less obvious, we must have that $n = q^2 + p^2$ for $p, q \in \mathbb{Z}$. This graph also has the special property that for a conference graph G which is $\left(\frac{n-1}{2}\right)$-regular, $\overline{G}$ is also $\left(\frac{n-1}{2}\right)$-regular.

Example. C_5 is a conference graph.
Paley graphs are conference graphs.

◇

Definition 1.9 (Paley Graphs)**.** Let $p^n = q \cong 1 \mod 4$ for a prime p and $n \in Z$. Consider the finite field $\mathrm{GF}(q)$. We say a graph G is a **paley graph** if $\mathrm{V}(G) = \mathrm{GF}(q)$, the elements of the finite field, and two elements $x, y \in \mathrm{GF}(q)$ have $x \sim y$ if $x - y = k^2$ for some $k \in \mathrm{GF}(q) \setminus \{0\}$ (hence $x \neq y$). The reason we restrict $q \cong 1 \mod 4$ is that this implies the relation symmetric ($x \sim y \Leftrightarrow y \sim x$). If we let $q \cong 3 \mod 4$ this property does not hold and we get a directed graph instead.

Example. $\mathrm{GF}(5)$ consists of elements $0, 1, 2, 3, 4$ and $1, 4$ are defined as squares. So the graph has edges $\{4,3\}, \{3,2\}, \{2,1\}\{1,0\}, \ldots$.

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Remark. Take a quadratic nonresidue $a \in \mathrm{GF}(q)$ and define $x \mapsto ax$. We see this is a bijection of $GF(q)$ with itself. Furthermore, if $x - y$ is a square in $\mathrm{GF}(q)$, then $a(x-y)$ is not a square by the fact that a is a quadratic nonresidue. Similairly, this maps a nonsquare to a square. Hence, this mapping takes the paley graph of $\mathrm{GF}(q)$ to its complement.

Remark. If we take a SRG with n vertices which is $\sim cn$-regular for $0 < c < 1$, then the graph will be approximately SRG $\left(n, \sim cn, c^2 n, c^2 n\right)$. We see this in the definition of the conference graph, which is about $\frac{n}{2}$-regular and is approximatly SRG $\left(n, \frac{n}{2}, \frac{n}{4}, \frac{n}{4}\right)$. We call such graphs **quasi-random**.

Remark. Delsarte-Goethals and Turyn have desribed these graphs.

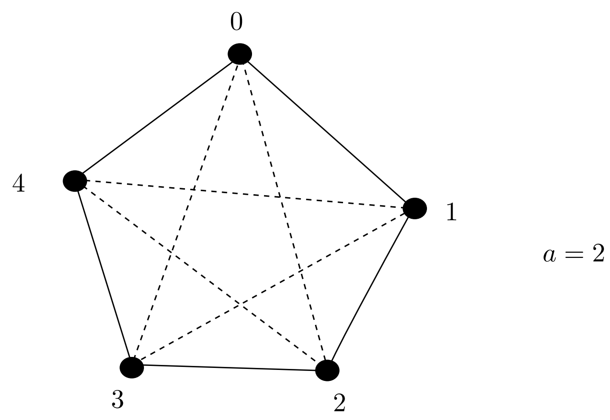

Figure 7: We see the payley graph of $\mathrm{GF}(5)$, C_5, (solid) and the mapping $x \mapsto 2x$ yields the complement (dotted)

Definition 1.10. Take a 2-dimensional linear space over $\mathrm{GF}(q)$ denoted X. This is simply ordered pairs of elements of $\mathrm{GF}(q)$. Then, there are $q+1$ distinct lines passing through the origin. Taking an arbitrary set S, of these lines allows us to define a graph G with $\mathrm{V}(G)$ being the points in X and we say $x \sim y$ if $x - y$ is a point on an element of S. This graph is $SRG\left(q^2, s(q-1), (s-1)(s-2)+q-2, s(s-1)\right)$ where $|S| = s$.

Lecture 5: SRG (5) and Graph Matrices

Wed 01 Sep 2021 10:18

Recall (Delsarte-Goethals-Turyn Graphs)**.** Let $n = q^2$ for q being a prime power. Then, let $1 \leq s \leq q$. From this, we may construct a graph $\mathrm{SRG}\left(q^2, s(q-1), (s-1)(s-2)+q-2, s(s-1)\right)$. Such a graph is part of a family of graphs called psuedo-Latin square graphs.

Problem. Is there a graph which is $\mathrm{SRG}\left(n, \frac{n}{3}, \sim \frac{n}{9}, \sim \frac{n}{9}\right)$.

Solution. If we let $n = q^2$ where q is a prime power and $s = \left\lfloor \frac{q}{3} \right\rfloor$, we obtain

$$\begin{aligned}
n &= q^2 \\
k &= \left\lfloor \frac{q}{3} \right\rfloor (q-1) \\
\lambda &= \left(\left\lfloor \frac{q}{3} \right\rfloor - 1\right)\left(\left\lfloor \frac{q}{3} \right\rfloor - 2\right) + q - 2 = \left(\left\lfloor \frac{q}{3} \right\rfloor\right)^2 - 3\left\lfloor \frac{q}{3} \right\rfloor + q \\
\mu &= \left\lfloor \frac{q}{3} \right\rfloor \left(\left\lfloor \frac{q}{3} \right\rfloor - 1\right) \simeq \frac{q^2}{9} = \frac{n}{9}.
\end{aligned}$$

We see this yields such an approximate construction.

Remark. The parameter $s = \frac{q-1}{2}$ yields a conference graph. This is useful for a problem posted on the website.

2 Graph Matrices and Eigenvalues

Definition 2.1 (Eigenvalues)**.** Recall, if A is a $n \times n$ square matrix and $x \neq \vec{0}$ is a n-vector, then x is a **eigenvector** of A if $Ax = \lambda x$ for some $\lambda \in \mathbb{R}$.
For symmetric (real) matrices A, we have that all eigenvalues are real. Furthermore, if we arrange eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ in descending order ($\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$), then we have $\lambda_1 = \max\{\langle Ax, x\rangle : \|x\|_2 = 1\}$ (the ℓ_2 norm) where $\langle Ax, x\rangle = \sum_{i=1}^n \sum_{j=1}^n a_{ij} x_j x_i$.
Similarly, $\lambda_n = \min\{\langle Ax, x\rangle : \|x\|_2 = 1\}$
Lastly, we also have that $\sum_{i=1}^n \lambda_i = \operatorname{tr}(A)$.

Remark. It is trivial that λ^2 is an eigenvalue to A^2, and generally λ^n is an eigenvalue to A^n.

Definition 2.2 (Adjacency Matrix)**.** Let $G = (V, E)$, where $V = \{1, 2, \ldots, n\}$ WLOG, then the **adjacency matrix**, A (G) is defined such that the columns and rows are indexed by $1, 2, \ldots, n$ (so $A(G)$ is $n \times n$) and the entry $a_{ij} = \begin{cases} 1, & i \sim j \\ 0, & i \nsim j \end{cases}$

Remark. It is clear that the entry $a_{i,i} = 0$ for all i.

As for the eigenvalues, we can say that $\lambda_1 \geq 0$ and $\lambda_1 + \lambda_2 + \ldots + \lambda_n = 0$ by the earlier remark on the trace. Lastly, as we need to have $\lambda_1 > 0$ for nontrivial graphs, then we can say $\lambda_n < 0$, and furthermore we even have that $\lambda_n \leq -1$. This can be shown in the following way:

Proof. Let i, j be adjacent vertices, $x_i = -\frac{1}{\sqrt{2}}$ and $x_j = \frac{1}{\sqrt{2}}$ with $x_k = 0$ for all $k \neq i, j$. Then, we see $a_{i,j} = a_{j,i} = 1$ in the adjacency matrix. Hence taking $\langle Ax, x\rangle = a_{i,j}\left(-\frac{1}{\sqrt{2}}\right)\frac{1}{\sqrt{2}} + a_{j,i}\left(-\frac{1}{\sqrt{2}}\right)\frac{1}{\sqrt{2}} = -1$, hence the smallest eigenvalue must be less than -1. $\square$

Let A, B be $n \times n$ with entries $a_{i,j}$ and $b_{i,j}$ being their entries respectively. Remember $AB = C$ is $n \times n$ with entries $c_{i,j} = \sum_{k=1}^n a_{i,k} b_{k,j}$.

Remark. Let A be the adjacency matrix of an arbitrary graph, and denote $A^2 = B$. We see $\sum_{k=1}^n a_{i,k} a_{k,j} = b_{i,j}$ are the entries of B. Furthermore, each product will either be 0 or 1 as each element was either 0 or 1. Hence, we have $b_{i,j}$ is equal to the number of pairs $a_{i,k}, a_{k,j}$ are both 1. We see this implies $k \sim i$ and $k \sim j$, in other words, $b_{i,j}$ is the number of vertices which are adjacent to both i and j, hence $b_{i,j} = \hat{d}(i, j)$. Thus the entries of A^2 are equal to the codegree of the corresponding vertices.

Lecture 6: Graph Eigenvalues

Fri 03 Sep 2021 10:22

Recall. An entry $b_{i,j} \in A^2$ where A is the adjacency matrix of a graph has $b_{i,j} = \hat{d}(i, j)$ and $b_{i,i} = d(i)$. This allows us to formalize SRG's further as we must have $b_{i,i} = k$, $b_{i,j} = \lambda$ for $i \sim j$ and $b_{i,j} = \mu$ for $i \not\sim j$. Hence, the entries of A^2 for a given SRG will take only 3 possible values.

Let us now examine the matrix λA. We see for adjacent entries, those being entries $a_{i,j} = 1$ this yields $\lambda a_{i,j} = \lambda = \hat{d}(i,j)$. Hence, for the purposes of adjacent edges this yields the same entries as A^2. We wish to take this even further. Recall that the complement of a SRG is also SRG, this gives a hint as to where to proceed. Note that the adjacency matrix for a graph complement $\overline{G}$ has $\text{Adj}\left(\overline{G}\right) = [c_{i,j}]$ has

$$c_{i,j} = \begin{cases} 1, & i \not\sim j \text{ and } i \neq j \\ 0, & i \sim j \text{ or } i = j \end{cases}$$

Hence, we see $\text{Adj}\left(\overline{G}\right) = J_n - I_n - \text{Adj}(G)$ where J_n is the matrix with all entries 1 and I_n is the standard $n \times n$ identity matrix. This equation essentially takes the "complement" of the identity matrix with the exception of the diagonal, which should still be 0 (as it is in all adjacency matrices for non-loop graphs). With this revelation, we can now define

$$A^2 = \lambda A + \mu (J_n - I_n - A) + kI_n. \tag{2.1}$$

This guarantees we have k along the main diagonal, μ for nonadjacent vertices and λ for adjacent vertices, hence it characterizes the SRG. The following re-arrangement yields $A^2 - (\lambda - \mu) A - (k - \mu) I = \mu J$. We see this is essentially quadratic in nature, and as matrices forms a ring, we can perform the normal arithmetic operations to search for possible solutions.

Remark. If G is k-regular, then k is the largest eigenvalue of G with $j_n = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ as its corresponding eigenvector.

This is a foundational result in spectral graph theory. The proof essentially relies on the result of the largest eigenvalue being equal to the maximum of the inner product of Ax with x over all normal x. This proof also relies on the two facts $\lambda_1 \geq \frac{2\,\mathrm{e}(G)}{\mathrm{v}(G)}$ and $\lambda_1 \leq \Delta(G)$.

Proposition 2.1. The eigenvectors of a graph G (generally any real symmetric matrix) can be chosen to be an orthogonal basis of R^n where $n = v(G)$.

Recall. $x, y \in R^n$ are orthogonal if the inner product $\langle x, y \rangle = 0$.

Now, let us take a basis of orthogonal eigenvectors of G with G being an SRG satisfying (2.1). Denote these to be $j_n, \overline{v_2}, \ldots, \overline{v_n}$ and let the corresponding eigenvalues to be $k, \lambda_2, \ldots, \lambda_n$.

Recall $j_n = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ and denote $\overline{v}_j = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$, then note by the orthogonality, we have $\overline{v_i} \cdot j = \sum_{i=1}^n a_i \cdot 1 = \sum_{i=1}^n a_i = 0$. From this it is clear that $J\overline{v}_i = 0$ for all

$\overline{v}_i$. With this fact, let us take equation 1 and multiply by $\overline{v}_i$ for some i. This yields

$$\begin{aligned}
0 &= \left(A^2 - (\lambda - \mu) A - (k - \mu) I\right) \overline{v}_i \\
&= A^2\overline{v_i} - (\lambda - \mu)A\overline{v_i} - (k - \mu) I\overline{v}_i) \\
&= \lambda_i^2\overline{v_i} - (\lambda - \mu) \lambda_i\overline{v_i} - (k - \mu) \overline{v_i} \\
&= \overline{v_i} \left(\lambda_i^2 - (\lambda - \mu) \lambda_i - (k - \mu)\right) \\
&\Rightarrow \lambda_i^2 - (\lambda - \mu) \lambda_i - (k - \mu) = 0 \text{ as } v_i \text{ is an eigenvector, so nonzero.}
\end{aligned}$$

This implies all eigenvalues of a SRG (n, k, λ, μ) are

1. k
2. $r = \frac{(\lambda-\mu)+\sqrt{(\lambda-\mu)^2+4(k-\mu)}}{2}$
3. $s = \frac{(\lambda-\mu)-\sqrt{(\lambda-\mu)^2+4(k-\mu)}}{2}$

Additionally, we generally define the multiplicity of r to be f and the multiplicity of s to be g. This yields eigenvalues corresponding to $j_n, \overline{v_2}, \ldots, \overline{v_n}$ to be $k, \underbrace{r, \ldots, r}_{f \text{ times}}, \underbrace{s, \ldots, s}_{g \text{ times}}$, so $f + g = n - 1$, and as we know the sum of all eigenvalues is 0, we have $k + fr + gs = 0$.

Remark. This quadratic expression of the eigenvalues guarantees all SRGs of equal parameters to have exactly the same eigenvalues.

As usual, we define the discriminant to be $\Delta = (\lambda - \mu)^2 + 4(k - \mu)$, not to be confused with the maximum degree.
It turns out that the multiplicity of the eigenvalues are also completely characterized by the parameters of the SRG as follows,

$$\begin{aligned}
f &= \frac{1}{2}\left(n - 1 - \frac{(\lambda - \mu)(n - 1) + 2k}{\sqrt{\Delta}}\right) \\
g &= \frac{1}{2}\left(n - 1 + \frac{(\lambda - \mu)(n - 1) + 2k}{\sqrt{\Delta}}\right).
\end{aligned}$$

As f and g must be integers, this additionally guarantees either $\sqrt{\Delta} \in \mathbb{Z}$ or $(\lambda - \mu)(n - 1) = -2k$ (hence the top is zero). Further investigation yields that if $f \neq g$, then Δ is a perfect square and r, s are integers as the roots of an algebraic equation with integral coefficients must be integral (if it is rational which is guaranteed by Δ being a perfect square). On the other hand, if $f = g$, then we have G is a conference graph.

Example. The eigenvalues of C_5 are $r = \frac{\sqrt{5}-1}{2}$ and $s = \frac{-\sqrt{5}-1}{2}$. ◇

Lecture 7: Conclusion of SRG and Graph Eigenvalues

Wed 08 Sep 2021 10:19

Recall. The eigenvalues of a connected SRG take only 3 distinct values. Let $G = \text{SRG}(n, k, \lambda, \mu)$ Then the eigenvalues are k with multiplicity 1, r with multiplicity f and $s < 0$ with multiplicity g. Furthermore, either $r, s \in \mathbb{Z}$,

$(f \neq g)$, or G is a conference graph, $f = g$. The values values of all these parameters are so interconnected that a given triple of parameters almost always completely characterizes the rest of the parameters.

Theorem 2.1 (Seidel)**.**

$$n \leq \frac{f(f+3)}{2}$$
$$n \leq \frac{g(g+3)}{2}.$$

From these statements, we can extract the following inequalities:

$$f \geq \sqrt{n}$$
$$g \leq \sqrt{n}.$$

Remark. Taking a sufficiently large SRG and extracting a given number of vertices small in proportion to the size, yields an approximately random graph.

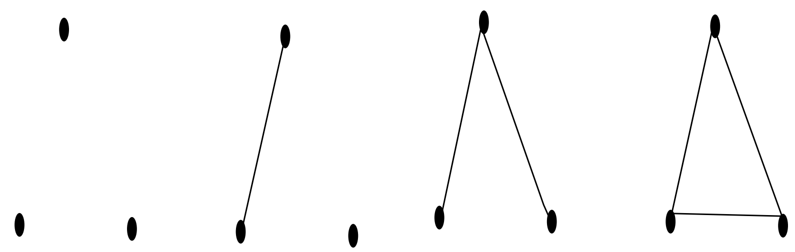

Figure 8: All possible Payley graphs of order 3

Recall (Triangular Graphs)**.** $T(q) = \mathrm{SRG}\left(\binom{q}{2}, 2(q-1), q-2, 4\right)$.

This yields the following eigenvalues.

- $k = 2(q-2)$
- $r = q - 4$ with $f = q - 1$
- $s = -2$ with $g = \frac{q(q-3)}{2}$.

We illustrate these eigenvalues in the following diagram: These proportions of graph are essentially guaranteed by Seidel's inequalities.

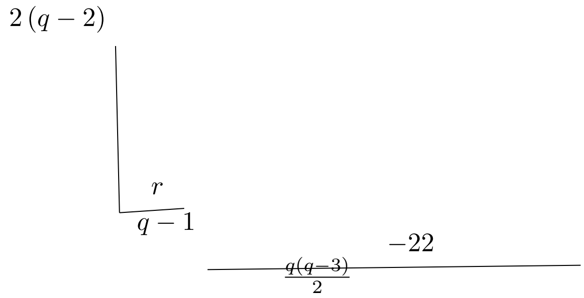

Figure 9: In this graph the length is proportional to multiplicity and the height is proportional to eigenvalue

Recall (Lattice Graphs)**.** $L_2(q) = \left(q^2, 2(q-1), q-2, 2\right)$.

This yields eigenvalues:

- $k = 2(q-1)$
- $r = q-2$ with $f = 2(q-1)$
- $s = -2$ with $g = (q-1)^2$

The triangular and lattice graphs are the only families of graph with $s = -2$.

Recall (Latin Square Graphs)**.** $L_3(q) = \mathrm{SRG}\left(q^2, 3(q-1), q, 6\right)$.

This yields eigenvalues:

- $k = 3(q-1)$
- $r = q-3$ with $f = 3(q-1)$
- $s = -3$ with $g = q^2 - 3q + 2$

Lastly, recall

Recall (Conference Graphs)**.** A graph G is conference if it is SRG $\left(n, \frac{n-1}{2}, \frac{n-5}{4}, \frac{n-1}{4}\right)$.

This yields Eigenvalues

- $k = \frac{n-1}{2}$
- $r = \frac{\sqrt{n}-1}{2}$ with $f = \frac{n-1}{2}$
- $s = -\frac{\sqrt{n}+1}{2}$ with $g = \frac{n-1}{2}$

A rare counterexample where the proportions of the eigenvalue diagram are not equal is seen in the conference graph

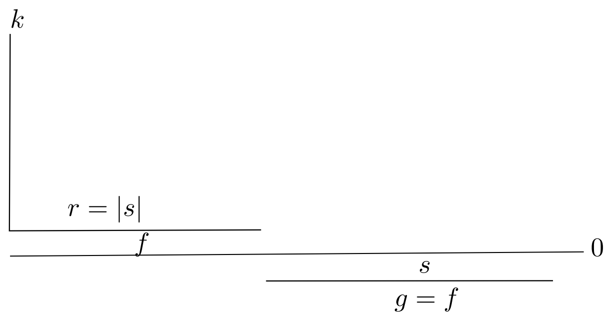

Figure 10: A graph is conference precisely when $f = g$. Furthermore, any graph with $f \neq g$ will have $r \neq |s|$

Recall (Quasi-random Graphs)**.** A quasi-random graph is one with parameters approximately SRG $\left(n, \sim \frac{n}{2}, \sim \frac{n}{4}, \sim \frac{n}{4}\right)$. These graphs have the quality that for a sufficiently large quasi-random graph we can find an induced subgraph equivalent to any sufficently smaller graph.

Definition 2.3 (Taylor Graphs)**.** Let q be an odd prime power, then the parameters of a **Taylor SRG** are

- $n = q^3$
- $k = \frac{(q-1)\left(q^2+1\right)}{2}$
- $\lambda = \frac{(q-1)^3}{4} - 1 \simeq \frac{q^3}{4}$
- $\mu = \frac{(q-1)\left(q^2+1\right)}{4} \simeq \frac{q^3}{4}$

These imply

- $r = \frac{(q-1)}{2}$ with $f = (q-1)\left(q^2+1\right)$
- $s = -\frac{q^2+1}{2}$ with $g = q\,(q-1)$

This case of $f \sim q^3$ is nearly the maximal ratio of f to g.

3 Hadamard Matrices

Let A be a complex matrix. Then the **adjoint matrix**$A^* = \overline{(A^T)}$. Furthermore, a **hermitian matrix** is a matrix for which $A^* = A$.

Example. $A = \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$ is a hermitian matrix. ⋄

Let $A \in M[\mathbb{C}]$. We find $A \cdot A^\star$ is always hermitian. Moreover, its eigenvalues are real and nonnegative.

> **Definition 3.1** (Singular Values)**.** The square roots of the eigenvalues of AA^* are called **singular values** of A.

Lecture 8: Hadamard Matrices

Fri 10 Sep 2021 10:19

Recall. For a complex matrix A, $A^* = \overline{(A^T)}$ is the adjoint and $A \cdot A^*$ is a (square) hermitian matrix (an extension of symmetric real matrices in the complex case). Note, a matrix is hermitian if $A = A^*$. Furthermore, all eigenvalues of $A \cdot A^*$ are nonnegative and these eigenvalues are the squares of the singular values of a (eigenvalues λ yields singular values $\sqrt{\lambda}$). We generally denote these singular values $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_m >=\geq 0$

Let A be a $n \times n$ symmetric matrix. Clearly, $A^* = A$, hence $AA^* = A^2$. We know for eigenvalues of A, we have

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n.$$

This yields eigenvalues of A^2, $\lambda_1^2 \geq \lambda_2^2, \ldots$, hence the singular values $\sigma = |\lambda|$. We cannot however say that $\sigma_i = |\lambda_i|$ as the λ_i may be negative.
We wish to prove the following theorem

> **Proposition 3.1.** The singular values of a hermitian matrix are also $|\lambda_1|, |\lambda_2|, \ldots, |\lambda_n|$.

If G is a graph, with eigenvalues $\lambda_1 \geq \ldots \geq \lambda_n$ with $\lambda_1 > 0$. Again, we cannot say $\sigma_1 = \lambda_1$, for example take hypothetical eigenvalues $(1, -10)$, hence $\sigma = 10$.

> **Definition 3.2** (Spectral Radius)**.** We define the largest absolute eigenvalue of a graph to be the **spectral radius**.

Further investigation yields a theorem from spectral graph theory which proves $\lambda_1 = \sigma_1$ after all.

Example. The spectrum of $K_{n,n} = \{n, 0, \ldots, 0, -n\}$. Hence its spectral radius is $\lambda_1 = n$. ⋄

> **Singular values vs eigenvalues**
>
> A 3×5 matrix trivially has no eigenvalues, but it will still have singular values.

Consider such a 3×5 matrix and note that AA^* is a 3×3 matrix yielding 3 singular values which are not necessarily distinct.

Example. $J_{5,3}^* = J_{3,5}$ hence $J_{5,3}J_{3,5} = \begin{bmatrix} 5 & 5 & 5 \\ 5 & 5 & 5 \\ 5 & 5 & 5 \end{bmatrix}$. It is of note that J_3 has eigenvalues $3, 0, 0$, hence our matrix above has eigenvalues $15, 0, 0$. Hence its singular values are $\sqrt{15}, 0, 0$ ◇

In general the singular values of $J_{n,m}$ are $\sigma_1 = \sqrt{nm}$ and $\sigma_i = 0$ for $2 \leq i \leq n$. However, it is accepted convention to index only up to $\min\{n, m\}$ as all eigenvalues past this point will be guaranteed 0. This is due to the fact that A and A^* have the same singular values. Additionally, the rank of any matrix is equal to the number of nonzero singular values of a.
Let A be a $n \times m$ matrix. We attempt to produce a formula for the sum of squares of its singular values, $\sigma_1^2 + \ldots + \sigma_n^2$. But, as we know these are just the eigenvalues of AA^*, and for a matrix B, $\lambda_1 + \lambda_2 + \ldots + \lambda_i = \operatorname{tr}(B)$ where λ_i is an eigenvalues of B. Hence, $\sigma_1^2 + \ldots + \sigma_i^2 = \operatorname{tr} AA^*)$. We see for entries $a_{i,j} \in A$. Then, $\operatorname{tr}(AA^*) = \sum_{i=1}^n a_{i,k}\overline{a}_{i,k} = \sum_{i=1}^n |a_{i,k}|^2$.

Definition 3.3 (Hadamard Matrix)**.** A matrix H is called **hadamard** if

- It is square ($n \times n$).
- Its entries $a_{i,j}$ have $|a_{i,j}| = 1$ for all i, j.
- $HH^* = nI_n$.

Example. $H = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ is hadamard as $HH^* = 2I_2$.
$H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$ is also hadamard.
As all real values will take on either 1 or -1, so we often denote then $H = \begin{bmatrix} + & + \\ + & - \end{bmatrix}$. ◇

As it turns out, there is an abundance of complex examples, but these are lesser studied. For now, we will mostly consider real hadamard matrices.
Take H to be a $n \times n$ matrix with $|h_{i,j}| \leq 1$ for all $1 \leq i, j \leq n$. We ask how large can $|\det(H)|$ be. As it turns out the maximum possible size is $h^{\frac{h}{2}}$. The proof follows: We know the eigenvalues of $|\det(H)| = \lambda_1\lambda_2 \ldots \lambda_n = \sigma_1\sigma_2 \ldots \sigma_n$ and the earlier found follows form there. The maximal case is precisely in the hadamard matrices.
We can even determine that a 3×3 case does not exist. And we wonder is there always a $4k \times 4k$ hadamard matrix? This question is yet unsolved but has been checked to a high bound manually.

Lecture 9: Hadamard Matrices (2) and Kroenikker Product

Mon 13 Sep 2021 10:22

Recall. A Hadamard matrix is a square matrix H with entries $|h_{i,j}| = 1$ and $HH^* = nI_n$. Furthermore, its rows are orthogonal.

Let H, H^T be real symmetric matrices, hence $H^T = \overline{(H^T)}$. Then, take rows, r_i

and r_j. We that for an entry $h_{i,j} \in HH^*$ has $h_{i,j} = \langle r_i, r_j \rangle$.

Proposition 3.2. If H is hadamard, then H^* is hadamard.

Proof. We know $HH^* = nI_n$, implying $H \cdot \frac{1}{n}H^* = I$, hence $\frac{1}{n}H^* = H^{-1}$ and as inverses commute, we see $\frac{1}{n}H^*H = I$, hence $H^*H = nI_n$. □

From this fact, we an also say H^T is hadamard and $\overline{H}$ is hadamard. This also clearly yields that the columns of H are orthogonal. Furthermore, any interchanging of rows or columns will also leave a matrix hadamard. Also, negating any rows or columns preserves hadamardness.
These simple facts imply that we can always construct a hadamard matrix with all positives in the first row(column). This is the general case we will use becuase of the following theorem.

Proposition 3.3. For a normalized $n \times n$ hadamard matrix , H, with all entries in first row being $+1$, we must have that n is even and each row after the first will have an equal number of $+1$s and -1s.

Proof. For the case $n = 2$ it is clear the matrix must be of the form $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.
For the case $n \geq 3$ we have that $\langle r_1, r_2 \rangle = \sum_{i=1}^{n}(1 + a_{2,i}) = 0$, hence half of the r_2 entries are $+1$ and the other half are -1. The same argument follows for row r_i □

From this, we may see that for row r_2, r_3 we must have exactly $\frac{n}{2}$ positions in agreement by a similar argument. Let the entries of r_2, r_3 be partitioned into those where both entries are $+1$, with cardinality denoted a, those where both are -1, with cardinality denoted d, those where r_2 is $+1$ and r_3 is -1, with cardinality denoted b, and the opposite, with cardinality denoted c.Then, $a+b = \frac{n}{2}$ and $c + d = \frac{n}{2}$ by $r_1 \perp r_2$. Similarly, $a + c = \frac{n}{2}$ and $b + d = \frac{n}{2}$ by $r_1 \perp r_3$. This yields $a + d = \frac{n}{2}$ and $b + c = \frac{n}{2}$ by $r_2 \perp r_3$, hence a quick calculation yields $a = b = c = d = \frac{n}{4}$.

Next, we examine the columns or rows of a hadamard. An alternative definition is to note that all possible rows of a $n \times n$ hadamard matrix is equivalent to $\{-1, 1\}^n$ yielding 2^n possible vectors. Then, a hadamard matrix H is simply any orthogonal set of these vectors.

Definition 3.4 (Kroenkker product)**.** Let A be a $m \times n$ matrix and B be a $p \times q$ matrix. The **kronecker product/tensor product** is denoted by $A \otimes B$, which is an $mp \times nq$ matrix such that $A \otimes B = \begin{bmatrix} a_{1,1}B & a_{1,2}B & \dots & a_{1,n}B \\ \vdots & \vdots & \vdots & \vdots \\ a_{n,1}B & \dots & \dots & a_{n,n}B \end{bmatrix}$

Generally speaking $A \otimes B \neq B \otimes A$, but their rows and columns may be rearranged to yields equivalent matrices.

Theorem 3.1. Let $m = n$ and let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues of A and let $p = q$ and $\mu_1, \mu_2, \dots, \mu_p$ be the eigenvalues of P. Then, the eigenvalues of $A \otimes B$ are $\lambda_i \mu_j$ where $1 \leq i \leq n$ and $1 \leq j \leq p$.

Theorem 3.2. Let A be $m \times n$ and $B = p \times q$ with $\lambda_1, \dots, \lambda_n$ being singular values of A and $\mu_1, \dots, \mu_n$ be singular values of B, then the singular values of $A \otimes B$ are $\lambda_i \mu_j$ with $1 \leq i \leq n$ and $1 \leq j \leq p$.

Proposition 3.4. For two hadamard matrices, A, B, we have $A \otimes B$ is hadamard.

Definition 3.5 (Blowup of Graph)**.** Let G be a graph and replace each vertex with a group of t vertices. We define the blowup of G to have these new vertices after replacement and two edges are connected if their generating vertices were connected. Hence, any two connected vertices from G yield a bipartite relationship (as all possible combinations of their degenerate vertices are connected).

We wonder what is $A\left(G^{(t)}\right)$. We see, grouping respective independent sets, then each 1 or 0 in the adjacency matrix of G will be a $t \times t$ block of 0 or 1 in $\left(G^{(t)}\right)$. This yields eigenvalues of $G^{(t)} = t\lambda_1, t\lambda_2, \dots, t\lambda_n, \underbrace{0, \dots, 0}_{n(t-1)}$.

This yields $v\left(G^{(t)}\right) = tn$, $e\left(G^{(t)}\right) = t^2 m$ and $K_3\left(G^{(t)}\right) = t^3 K_3(G)$, Many other graph parameters, such as clique numbers, are also preserved in some form.

Lecture 10: Hadamard Matrices (3)

Wed 15 Sep 2021 10:20

Recall. The tensor product of matrices A, B is $A \otimes B$ and this preserves hadamardness.

Example.

$$\underbrace{\begin{bmatrix} + & + \\ + & - \end{bmatrix}}_{=H} \otimes \begin{bmatrix} + & + \\ + & - \end{bmatrix} = \begin{bmatrix} + & + & + & + \\ + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{bmatrix} = H \otimes H.$$

Furthermore, $H \otimes H \otimes H$ will be an 8×8 hadamard matrix. And the arbitrary $\bigotimes_{i=1}^{n} H$ yields a hadamard matrix of order 2^n. $\diamond$

A natural question arises, what are the singular values of an arbitrary hadamard H?

Recall. Singular values are the square roots of the eigenvalues of AA^*.

Other definitions also arise, for example the largest singular value of A, denoted σ_1 is equal to the operator norm on A. Similarly, we can change the matrix slightly to remove singular value σ_1 and this yields σ_2 is the operator norm on the modified $\hat{A}$.

For now, we return to the original definition, and we note that as $HH^* = nI$, we have eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ with $\lambda_i = n$ and corresponding eigenvactor $e_i = \begin{pmatrix} 0 \\ \vdots \\ \underbrace{1}_{\text{position } i} \\ \vdots \\ 0 \end{pmatrix}$. Hence the singular values of A are all $\sqrt{n}$.

Proposition 3.5. Let $H = [h_{i,j}]$ with $|h_{i,j}| = 1$ for all i, j. Then, the following are equivalent

- $HH^* = nI$
- All singular values are equal to $\sqrt{n}$
- For singular values σ_i, $1 \leq i \leq n$, the sum $\sum_{i=1}^{n} \sigma_i = n\sqrt{n}$.

Definition 3.6 (Nuclear Norm)**.** For a matrix A, we define the **nuclear norm or trace norm** to be $\|A\|_* = \sum_{i=1}^{n} \sigma_i (A)$.

Remark. If A is $n \times n$ with $|a_{i,j}| \leq 1$ then $\|A\|_* \leq n\sqrt{n}$. Furthermore, equality holds if and only if A is hadamard.

Now, let A be $m \times n$ with $m \leq n$. Then, $\|A\|_* \leq m\sqrt{n}$. Equality holds if and only if A is a **partial hadamard matrix** meaining $AA^* = nI_m$.

Definition 3.7 (Regular Matrix)**.** For a matrix A we say A is **regular** if all row sums are equal.

We examine the properties of a regular hadamard matrix.
It is clear, as we may switch rows and columns and multiply by ± 1 for each row, that these row sums are fragile, and occasionally we may even induce a regular hadamard matrix from a nonregular one this way.

Example. $\begin{bmatrix} + & + & + & - \\ + & + & - & + \\ + & - & + & + \\ - & + & + & + \end{bmatrix}$ is a regular hadamard matrix induced by the 4×4 hadamard from earlier. $\diamond$

Remark. A regular matrix need not be symmetric. For example $\begin{bmatrix} + & + & + & - \\ + & - & + & + \\ - & + & + & + \\ + & + & - & + \end{bmatrix}$ is regular and nonsymmetric.

Note that a real symmetric hadamard matrix has real eigenvalues.

Proposition 3.6. Suppose H is a $n \times n$ symmetric and regular (row sum d). Then, $n = d^2$.

Proof. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of H and note that $|\lambda_i| = \sqrt{n}$ for $1 \le i \le n$ as the eigenvalues of a real symmetric matrix are precisely the singular values.
Next, we note that there is atleast one $\lambda_i = \sqrt{n}$ and one $\lambda_j = -\sqrt{n}$. Otherwise, suppose WLOG all $\lambda_i = \sqrt{n}$, then $\sum_{i=1}^n \lambda_i = n\sqrt{n} = \operatorname{tr}(H)$, but the trace can be atmost n by an earlier theorem. Hence, $\lambda_1 = \sqrt{n}$ and $\lambda_n = -\sqrt{n}$. Then, note that $Hj = dj$ for $j = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$. Hence, d is an eigenvalue with j as its eigenvector.
Hence $d = -\sqrt{n}$ or $d = \sqrt{n}$. Hence either case yields $n = d^2$. $\square$

Definition 3.8 (Constant Diagonal)**.** A hadamard matrix H is said to have a **constant diagonal** if $h_{1,1} = h_{2,2} = \ldots = h_{n,n}$.

This property can always be ensured for a hadamard matrix with just elementary transformations. Furthermore, if H is a $n \times n$ constant diagonal hadamard matrix with $\delta = h_{1,1}$. Then, δH has a constant diagonal of 1 and we define $A = \frac{1}{2}(J_n - \delta H)$, hence the diagonal of A is constant 0. Next, note that δH is a hadamard matrix and for an element $h_{i,j} = 1$, we see $\delta h_{i,j} = \delta$. Similairly if $h_{i,j} = -1$ we have $\delta h_{i,j} = -\delta$. Hence the entries of A are $a_{i,j} = 0$ if $\delta h_{i,j} = 1$ and $a_{i,j} = 1$ if $\delta h_{i,j} = -1$. So, this matrix has all entries 0 and 1, something we call a **digraph matrix**. Furthermore, if H is regular, the graph induced by A is a strongly regular graph.

Lecture 11: Hadamard Matrices (4)

Fri 17 Sep 2021 10:19

Recall. A matrix was regular if all row sums are equal.

As it turns out, for regular real hadamard matrices regular also implies equal column sums.

Proof. Let H be hadamard regular and $n \times n$ with $\sum_{i=1}^{n} h_{i,j} = d$ for all j.

Then, note that $Hj = dj$ with $j = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$. Hence, d is an eigenvalue and as $H^*H = HH^*$, then we have that $H^*Hj = H^*dj$. Hence

$$nIj = dH^*j \text{ by hadamardness}$$

and as $Ij = j$ we have that $H^*j = \frac{n}{d}j$, hence $\frac{n}{d}$ is an eigenvalue of H^*, hence the row sums of H^* are all $\frac{n}{d}$, and as $H^* = H^T$ for real H, we see the column sums of H are $\frac{n}{d}$.
Additionally, if $d \neq 0$, then $\sum_{i=1}^{n} r_i(H) = \sum_{i=1}^{n} c_i(H)$, implying $nd = n \cdot \frac{n}{d}$, hence $n = d^2$ as we have proven earlier.
We have, of course, neglected the case where $d = 0$. In this case we have that $nj = \vec{0}$, but as $n \neq 0$ by assumption, and $cj \neq \vec{0}$ for $c \neq 0$, we have a contradiction. Hence $d \neq 0$. It is also true that the independence requirement of hadamard matrices implies this row sum cannot be 0. ☐

> **Proposition 3.7.** Suppose H is a $n \times n$ matrix with entries $|h_{i,j}| = 1$ and singular values $\sigma_1 = \sigma_2 = \ldots = \sigma_n = \sqrt{n}$. Then, H is hadamard.

Proof. Recall from an earlier proposition, we know $\sum_{i=1}^{n} \sigma_i^2 = n^2$. Recall that a diagonal element of HH^* is $b_{i,i} = \sum_{k=1}^{n} a_{i,k} \cdot \overline{a_{i,k}} = \sum_{k=1}^{n} |a_{i,k}|^2 = n$ by construction. Hence, the diagonals are all $b_{i,i} = n$ for all $1 \leq i \leq n$. Next, we wish to see if there are any 0 entries in HH^*. Next, we take a principal submatrix $A_{i,j} = \begin{bmatrix} n & \overline{b_{i,j}} \\ b_{i,j} & n \end{bmatrix}$ (note this is as HH^* will is hermitian, so we know opposing entries will be complex conjugates) Then, we see $\lambda_1(A_{i,j}) = n + |b_{i,j}|$ and $\lambda_2(A_{i,j}) = n - |b_{i,j}|$.
Now, we examine how the eigenvalues of a matrix and its principal sumbatrices are related. Let A be a $n \times n$ hermitian matrix and A' to be A with the i'th row and j'th column removed. Denoted the eigenvalues of A to be $\lambda_1, \lambda_2, \ldots, \lambda_n$ in decreasing order and eigenvalues of A' to be $\lambda_1', \lambda_2', \ldots, \lambda_{n-1}'$. Then, it is a theorem of Cauchy that $\lambda_1 \geq \lambda_1' \geq \lambda_2 \geq \lambda_2' \geq \ldots \geq \lambda_{n-1}' \geq \lambda_n$. Applying this again yields a matrix A'' with eigenvalues $\lambda_1 \geq \lambda_1' \geq \lambda_1'{}'$ and $\lambda_{n-2}'' \geq \lambda_{n-1}' \geq \lambda_n$. Returning to our original construction yields $\lambda_1(HH^*) \geq \lambda_1(A_{i,j}) \geq \lambda_2(A_{i,j}) \geq \lambda_n(HH^*)$ and as $\lambda_1(HH^*) = \sigma_1^2 = n$ and similarly, $\lambda_n(HH^*) = \sigma_n^2 = n$, hence $\lambda_1(A_{i,j}) = \lambda_2(A_{i,j}) = n$ implying $b_{i,j} = 0$ for all $j \neq i$ and $b_{i,i} = n$ so $HH^* = nI$. ☐

Recall. For a matrix H which is hadamard and has entries $h_{i,i} = \delta$ for all i, then the matrix $A = \frac{1}{n}(J - \delta H)$ is a square matrix with entries $0, 1$ and all 0s along the diagonal.

Proposition 3.8. If H is symmetric, then A is the adjacency matrix of a simple graph. If H is also regular with row sum d, then A is the adjacency matrix of a SRG with

$$\begin{aligned} n &= n \\ k &= \frac{n - \varepsilon\sqrt{n}}{2} \\ \lambda &= \frac{n - 2\varepsilon\sqrt{n}}{4} \\ \mu &= \frac{n - 2\varepsilon\sqrt{n}}{4} \end{aligned}$$

where $\varepsilon = \begin{cases} -1, & \delta d < 0 \\ 1, & \delta d > 0 \end{cases}$. It is of note that $\delta d \neq 0$ as $\delta = \pm 1$ and $d \neq 0$ by the earlier proof. Hence, $\varepsilon\sqrt{n} = \delta d$

Proof. First, we examine a few matrix products. Note that as $Hj = d$, we have $HJ = dJ$. Similarly, $JH = dJ$ and of course $H^2 = nI$.
Next, we examine A^2. By definition

$$\begin{aligned} A^2 &= \frac{1}{4}(J - \delta H)^2 \\ &= \frac{1}{4}\left(J^2 - 2J\delta J + \delta^2 H^2\right) \\ &= \frac{1}{4}(nJ - 2\delta dJ + nI) \\ &= \frac{1}{4}(n - 2\delta d)J + \frac{1}{4}nI \\ &= \frac{1}{4}(n - 2\delta d)(J - I) + \frac{1}{4}(n - 2\delta d)I + \frac{1}{4}nI \\ &= \frac{1}{4}(n - 2\delta d)(J - I) + \frac{n - \delta d}{2}I. \end{aligned}$$

Recalling our equation for the square of the adjacency matrix of a graph,

$$A^2 = (\lambda - \mu)A + \mu(J - I) + kI$$

yields $\lambda = \mu$, $\mu = \frac{n-2\delta d}{4} = \frac{n-2\varepsilon\sqrt{n}}{4} = \lambda$ and $k = \frac{n-\delta d}{2} = \frac{n-\varepsilon\sqrt{n}}{2}$. ☐

Lecture 12: Conference Matrices

Mon 20 Sep 2021 10:21

Recall. A Conference matrix is a matrix C with

1. C is $n \times n$
2. $c_{i,i} = 0$ for all $1 \leq i \leq n$
3. $c_{i,j} = \pm 1$ for $i \neq j$
4. $CC^T = (n-1)I$.

Just as with hadamard graphs, this implies the inner product of each pair of rows r_i, r_j with $i \neq j$ is $\langle r_i, r_j \rangle = 0$, hence the rows are orthogonal. This clearly also implies the columns are orthogonal. Furthermore, $n \equiv 0 \pmod 2$ by the same argument as hadamard matrices.
Just as with hadamard matrices, we look to determine which n have associated hadamard matrices.

Example. $\begin{bmatrix} 0 & + \\ + & 0 \end{bmatrix}$ is conference.
$\begin{bmatrix} 0 & + & + & + & + & + \\ + & 0 & + & - & - & + \\ + & + & 0 & + & - & - \\ + & - & + & 0 & + & - \\ + & - & - & + & 0 & + \\ + & + & - & - & + & 0 \end{bmatrix}$ is also conference.
Note that this matrix is simply the Seidel matrix of $\overline{C}_5$, that being the matrix $S(G)$ with $s_{i,j} = \begin{cases} -1, & i \sim j \\ 1, & i \not\sim j \text{ and } i \neq j \\ 0, & i = j \end{cases}$ for a matrix G, with an additional row and column of 1s along the top and left. This derives from the fact that C_5 is a conference graph. ◇

Similarly to hadamard, we can simultaneously transpose rows and columns of a conference matrix to obtain another conference matrix. Furthermore, we can negate any row or column while remaining hadamard. Hence,

Definition 3.9 (Normal Conference Matrix)**.** A conference matrix is called a **normal conference matrix** if it has $r_1 = \hat{j}_n = \begin{pmatrix} 0 & 1 & \dots & 1 \end{pmatrix}$ and $c_i = \hat{j}_n = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$.
We define the matrix $C' = S$ to be the remaining matrix when the first row and column are removed. This submatrix completely characterizes the normal conference matrix.

It is of note that the negation property makes every conference matrix normalizable.

Proposition 3.9. If $n \equiv (2 \mod 4)$, then S is symmetric. If $n \equiv 0 \pmod 4$, then S is antisymmetric or skew symmetric ($A = -A^T$).

Remark. If A is antisymmetric, then iA is hermitian.

Proposition 3.10. If C is a conference matrix with $n \equiv 2 \pmod 4$, then S is the seidel matrix of a conference graph.

Remark. All $2 \pmod 4$ numbers up to $n = 22$ have been shown to have conference matrices of that order. For the case $n = 22$, we have a proof by Seidel vam Lint, that if n is the order of a conference matrix, then $n-1$ is the sum of two squares. As 21 is not the sum of 2 perfect squares, there is no conference matrix of order 22. Similarly for 34 and 66. Note that all primes $p \equiv 1 \pmod 4$ are the sums of two squares, so we need only check the composite cases.

Now, we introduce the Payley construction of conference matrices.
Let $q \equiv 3 \pmod 4$ be a prime power. Then, there is a skew-symmetric conference matrix of order $q+1$ and a hadamard matrix of order $q+1$.

Proposition 3.11. Let $q = 1 \pmod 4$, then there exists a symmetric conference matrix of order $q+1$ and a symmetric hadamard matrix of order $2(q+1)$. We will introduce the construction of the hadmard matrix next lecture.

Lecture 13

Fri 24 Sep 2021 10:20

I originally missed this lecture, so this is transcribed from a classmates notes.

Lecture 14: Quasi-Random Graphs

Fri 24 Sep 2021 10:21

Remark. Random graphs have applications in ramsey theory. For instance, if $p \geq 2$, $q \geq 2$, then there is a ramsey number $R(p,q)$ such that for a graph G of order at least $R(p,q)$, then either G constains a p-clique or an independent set of q vertices.
It is a well known result that $R(3,3) = 6$.
A counterexample to a graph of order 5 is K_5.

We can even obtain an upper bound $R(p,q) \leq \binom{p+q-2}{q-1}$. This is obtained from the trivial fact that $R(p,q) \leq R(p,q-1) + R(p-1,q)$.
Now, we examine the diagonal case, $R(k,k) \leq \binom{2k-2}{k-1} \leq \frac{4^{k-1}}{\sqrt{k}}$. From this we obtain, $\sqrt[k]{R(k,k)} \leq 4$, and a probabalistic argument from erdos yields

$$\sqrt{2} \leq \sqrt[k]{R(k,k)} \leq 4$$

Proposition 3.12. For all k and n with $n \leq \sqrt{2}^k$, there is a graph of order n such that G has no k-clique and no independent set of order k.

Halmos. Fix n vertices, $1, 2, \ldots, n$ and consider all labeled graphs , denoted LG. Now, recall there are $2^{\binom{n}{2}}$ labeled graphs of order n. Next, denote $k_k(G)$ to be the number of k-cliques in a graph G and we see an independent set is simply a clique of $\overline{G}$, so we see we need only consider $k_k(G) + k_k(\overline{G})$, hence the total number of graphs with either a k-clique or k-independent set of order

n are $S = \sum_{g \in LG(n)} k_k(G) + k_k(\overline{G}) = 2 \cdot \binom{n}{k} 2^{\binom{n}{2} - \binom{k}{2}}$.
The leading $\binom{n}{k}$ is due to the fact that there are $\binom{n}{k}$ subsets of order k in a set of order n and the exponent comes from the total amount of possible edges outside of the k-clique.
Now we construct a bipartite graph G with $A = LG(n)$ and B being the set of all possible k-cliques. We see each $a \in A$ is a labeled graph, so it may have differing numbers of k-cliques, each $b \in B$ is a k-clique, and all k-cliques participate in the same number of labeled graphs of order n hence B is regular to A.
Taking our earlier definition of S and manipulating yields

$$S \leq 2^{\binom{n}{2}} \left(\frac{2\binom{n}{k}}{2^{\binom{k}{2}}} \right).$$

Hence,

$$\frac{S}{2^{\binom{n}{2}}} \leq \frac{2\binom{n}{k}}{2^{\binom{k}{2}}} < 1.$$

Assuming $k \geq 3$ and applying definitions yields

$$\begin{aligned} \frac{2\binom{n}{k}}{2^{\binom{k}{2}}} &< \frac{2n^k}{k!2^{k\frac{k-1}{2}}} \\ \binom{n}{k} &= \frac{n(n-1)\dots(n-k+1)}{k!} \\ &\leq \frac{2 \cdot 2^{\frac{k^2}{2}}}{k!2^{\left(\frac{k(k-1)}{2}\right)}} \\ \text{taking } k = \sqrt{2}^k \text{ yields } & \frac{2\left(\sqrt{2}\right)^k}{k!}. \end{aligned}$$

Remark. Note that after $\binom{n}{2}$ flips of a fair coin, one obtains a graph in $LG(n)$. Take a subset M of cardinality k in the set of all such graphs and note that there is a $\frac{1}{2^{\binom{k}{2}}}$ probability this will be a k-clique. Hence the total probability summed over all subsets M is $\binom{n}{k}\frac{2}{2^{\binom{k}{2}}}$. Applying the subbadditivity of probability yields that this is strictly less than 1. Hence, there is such a graph not containing a k-clique or independent set of order 20.

$\square$

Lecture 15

Wed 29 Sep 2021 10:26

Lecture 16: Random Graphs

Wed 29 Sep 2021 10:27

First, we examine some more random graphs. For a random graph G, it is a trivial rseult of probability theory that the number of four cycles is precisely $\frac{1}{2}\sum_{u,v \in V(G); v \neq u} \binom{\hat{d}(u,v)}{2}$. Then, applying our estimation $\hat{d}(u,v) = \frac{n}{4} + o(u)$ yields $\binom{n}{2}$ possible pairs u, v and $\hat{d} \approx \frac{n}{4}$, hence the number of four cycles is

$$\frac{1}{n}\binom{\frac{n}{4}}{2}\binom{n}{2} = \frac{n^4}{128} + o\left(n^4\right).$$

Now, we examine the k-walks.

Definition 3.10 (Walks)**.** A **k-walk** is a k-path $v_1, v_2, v_3, \ldots, v_k$.
A **closed k-walk** is a k-cycle, $v_1, v_2, \ldots, v_k, v_1$.

Remark. Walks need not have all vertices distinct, hence a graph of order 2 where one simply oscillates between the vertices to produce a degenerate $2n$-walk. Similairly, one can traverse a triangle to induce a 4-walk as well. Overall this yields 14 possible 4-walks on a graph of order 4.

Now, we examine the number of closed 4-walks on a random graph of order n. We see nondegenerate 4-walks are just 4-cycles of which we know there to be $\frac{n^4}{128}$ with 8 possible permutations of directions and starting point yields $8 \cdot \frac{n^4}{128}$. Similairly, we note that $4 \cdot \sum_{v \in V} \binom{d_i(v)}{2} = 4n\binom{n}{2} = \frac{1}{2}n^3 + o\left(n^3\right) = o\left(n^4\right)$ degenerate graphs on 3 vertices exist. Lastly, the number of degenerate graphs on 2-vertices is clearly, $2 \cdot e\left(g\right) = o\left(n^4\right)$. Hence, the number of $4-walks$ is just $\frac{n^4}{16} + o\left(n^4\right)$.

Proposition 3.13. $\operatorname{tr}\left(A\left(G\right)^k\right) = \sum_{i=1}^{n} \lambda_i^k$ is the number of closed k-walks in a graph G of order n.

From this, we arrive at $6k_3\left(G\right) = \operatorname{tr}\left(A^3\right) = \sum_{i=1}^{3} \lambda_i^3$.
We also see the number of closed walks of order 4 is

$$\begin{aligned} CW_4 &= \sum_{i=1}^{n} \lambda_i^4 \\ \frac{n^4}{16} + o\left(n^4\right) &= \lambda_1^4 + \sum_{i=2}^{n} \lambda_i^4 \\ \Rightarrow \sum_{i=2}^{n} \lambda_i^4 &= o\left(n^4\right). \end{aligned}$$

Similarly, we find $\sigma_2\left(G\right) = o\left(n\right)$ and $O\left(\sqrt{n}\right)$.

Definition 3.11 (Local Density)**.** The **local density** of a graph is simply $e\left(U\right)$ for some graph $U \subseteq V$.

Remark. Local density is highly variable. For instance in $K_{n,n}$ we find U being one of the partite sets yields 0 local density and U being a set of half the vertices in each partite set yields $\frac{1}{4}e\left(G\right)$ local density.

Proposition 3.14. Suppose G is a random graph of order n and let U be a set with $|U| > 502\log\left(n\right)$. Then, $\left|e\left(U\right) - \frac{1}{2}\binom{|U|}{2}\right| < \binom{|U|}{2}\left(\frac{3.5\log n}{|U|}\right)^{\frac{1}{2}}$.

Proposition 3.15. There exists a function $f:\mathbb{N}\to\mathbb{N}$ such that almost every graph of order n as clique number $f(n)$ or $f(n+1)$.

This function is approximated by

$$f(n)\approx 2\log_2(n).$$

Remark. There is clearly also such a function for the independence number.

Furthermore, more investigation yields $\chi(G)\approx\frac{n}{2\log_2(n)}$ for almost all graphs G.

Lecture 17: Semi-circle Law

Fri 01 Oct 2021 10:20

Recall that for eigenvalues $\lambda_1,\lambda_2,\ldots,\lambda_n$ we have $\lambda_1=\frac{n}{2}+\sqrt{n\log(n)}=o(n)$. Additionally, we know $\sigma_1=\lambda_1$ and $\sigma_2,\sigma_3,\ldots,\sigma_n$ correspond to $|\lambda_2|,|\lambda_3|,\ldots,|\lambda_n|$. Further, it is known by Furedi and Kowlos that $\sigma_2=O(\sqrt{n})$.

Theorem 3.3. For a randomly chosen graph of order n, with eigenvalues $\lambda_2\geq\lambda_3\geq\ldots\geq\lambda_n$.Define $W_n(x):\mathbb{R}\to\mathbb{Z}^+$ to be the number of eigenvalues λ_i, such that $\frac{\lambda_i}{\sqrt{n}}\leq x$, divided by n. Then, we find the function which $W_n(x)$ tends to pointwise, $W(x)$ has $W(x)=\begin{cases}\frac{2}{\pi}\sqrt{1-x^2}, & |x|\leq 1\\ 0, & |x|>1\end{cases}$

Here recall that $\sqrt{1-x^2}$ is an upper half semicircle of radius 1 and the factor $\frac{2}{\pi}$ compresses it into an ellipse. This fact essentially characterizes the distribution of eigenvalues of a random graph. That is, plurality of eigenvalues will be 0 and we find the number of eigenvalues of a given magnitude decreases as $\lambda\to\sqrt{n}$. We note that the leading $\frac{2}{\pi}$ is to normalize the area such that this is a probability density function. Then, we note $E\left[x^2W(x)\right]=\int_{-1}^{1}\frac{2}{\pi}x^2\sqrt{1-x^2}dx=\frac{1}{4}$. Hence, we find $\frac{1}{n^2}\sum_{i=2}^{n}\lambda_i^2\approx\frac{1}{4}$.
It is a well known result that $\sum_{i=1}^{n}|\lambda_i|=\sum_{i=1}^{\infty}\sigma_i\leq\frac{1}{2}n^{\frac{3}{2}}\leq 2(n-1)$. Applying our integral formula from earlier yields $\sum_{i=1}^{\infty}|\lambda_i|=\int_{-1}^{1}|x|\sqrt{1-x^2}=2\int_0^1 x\sqrt{1-x^2}$.
At this point, Runze found a contradiction in the argument and we ended class early.

Lecture 18

Mon 04 Oct 2021 10:21