Analysis I

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Contents

We construct a cantor set.
First, suppose the interval $[0,1]$ and a series of sets $C_0, C_1, \ldots$ where $C_i = C_{i-1} \setminus D_i$ where D_i is just the set consisting of the middle thirds of each interval of C_{i-1}. Then, we let $C = \bigcap_{k\in\mathbb{N}} C_k$. We then define the nth partition of $[0,1] \setminus C_k$ to be $J_{k,n}$. We define $\mathscr{O} = \bigcup_{k,n\in\mathbb{N}} J_{k,n}$ and $\xi : \mathscr{O} \to \mathbb{R},\ x \in J_{k,n} \mapsto \frac{n}{2^k}$. We see this is well defined by an inductive argument.

Definition 0.1 (Cantor-Lebesque Function)**.** We define

$$\begin{aligned} \varphi : [0,1] &\longrightarrow \mathbb{R} \\ x &\longmapsto \varphi(x) = \begin{cases} 0, & x = 0 \\ \xi(x), & x \in \mathscr{O} \\ \sup\{\xi(y) : y \in \mathscr{O} \cap [0,x)\}, & x \in C \setminus \{0\} \end{cases} \end{aligned}$$

to be the **Cantor-Lebesque Function**

Proposition 0.1. φ is a continuous increasing function such that $\varphi([0,1]) = [0,1]$.

Proof. It is clear ξ is and this guarantees φ to be increasing.
Next, note $\varphi(0) = 0$ and $\varphi(1) = 1$. Hence, we have the intermediate value theorem guaranteeing the image is $[0,1]$ if φ is continuous.
We see φ is continuous on $\mathscr{O}$ since it is constant on each interval $J_{k,n}$. Now, we consider $x \in C \setminus \{0,1\}$. For a given ε, let $k \in \mathbb{N}$ such that $\frac{1}{2^k} < \varepsilon$. Then, there is $n \in \mathbb{N}$ such that $1 \le n \le 2^k - 2$ such that for all $u \in J_{k,n}$, $v \in J_{k,n+1}$ such that for all u, v we find $u < x < v$. Let $a_k \in J_{k,n}$ $b_k \in J_{k,n+1}$ then by monotinicity

of φ, for all $y \in [0,1]$ with $|x-y| < \delta = \min\{x - a_k, x + b_k\}$ we find

$$\begin{aligned} |\varphi(x) - \varphi(y)| &\leq \varphi(b_k) - \varphi(a_k) \\ &= \frac{n+1}{2^k} - \frac{n}{2^k} \\ &= \frac{1}{2^k} \\ &< \varepsilon. \end{aligned}$$

Finally, given $\varepsilon > 0$, we take $k \in \mathbb{N}$ such that $\frac{1}{2^k} < \varepsilon$ and let $c_k \in I_{k,1}$, $d_k \in I_{k,2^k-1}$. Then, for $o \leq y \leq c_k$, we find

$$\begin{aligned} |\varphi(0) - \varphi(y)| &= |\varphi(y)| \\ &\leq \varphi(c_k) \\ &= \frac{1}{2^k} \\ &< \varepsilon. \end{aligned}$$

Similairly, for $d_k < y \leq 1$, we find

$$\begin{aligned} |\varphi(1) - \varphi(y)| &\leq |1 - \varphi(d_k)| \\ &= 1 - \frac{2^k - 1}{2^k} \\ &= \frac{1}{2^k} \\ &< \varepsilon. \end{aligned}$$

□

Definition 0.2 (Modified Cantor-Lebesque Function)**.** Let $\psi = x + \varphi(x)$ be the **modified Cantor-Lebesque Function**. It is clear ψ is continuous, strictly increasing and has , $\psi([0,2]) = [0,2]$.

Proposition 0.2. The function ψ has the following properties

1. $\psi(C)$ is measurable with $\mu(\psi(C)) = 1$.
2. There is a measurable set $S \subseteq C$ such that $\psi(S)$ is not measurable.

Proof.
- Note that $[0,1] = C \cup \mathscr{O}$ and ψ is injective and continuous. Hence, we have $[0,2] = \psi(C) \cup \psi(\mathscr{O})$ with $\psi(C) \cap \psi(\mathscr{O}) = \varnothing$. Since ψ is strictly increasing, we know ψ^{-1} is well-defined and continuous. Hence, ψ is an open map and we see $\psi(\mathscr{O})$ is open in $[0,2]$, hence $\psi(C)$ is closed. Hence, both sets are measurable. We see $\psi(\mathscr{O})$ is the union of a countable collection of open disjoint intervals, $\{I_i : i \in \mathbb{N}\}$ such that $\varphi \mid J_i$ is constant by construction. Hence, we hve for each $i \in \mathbb{N}$ we find $\psi(I_n) = x_i + I_i$ where $x_i \in [0,1]$ is a constant. Since ψ is injective, we find it preserves

disjointness, hence the collection $\{\psi(I_i) : i \in \mathbb{N}\}$ is disjoint.
Then, by countable additivity and translation invariance of μ we find

$$\begin{aligned}\mu(\psi(\mathscr{O})) &= \mu\left(\bigcup_{i\in\mathbb{N}} I_i\right) \\ &= \bigcup_{i\in\mathbb{N}} \psi(I_i) \\ &= \sum_{i=1}^{\infty} \mu(\psi(I_i)) \\ &= \sum_{i=1}^{\infty} \ell(x_i + I_i) \\ &= \sum_{i=1}^{\infty} \ell(I_i) \\ &= \mu(\mathscr{O}).\end{aligned}$$

Since, $\mu(C) = 0$, we find

$$\mu(\mathscr{O}) = \mu([0,1] \setminus C) = \mu([0,1]) = 1.$$

Consequently, $\mu(\psi(\mathscr{O})) = 1 = \mu(\mathscr{O})$. Hence, we find $\mu(\psi(C)) = 1$.

Since $\psi(C)$ has positive measure, it contains a nonmeasurable subset T, however, we see $S = \psi^{-1}(T)$ is measurable as $S \subseteq C$ and $\mu(C) = 0$. □

Corollary 1. There is a measurable set $S \subseteq C$ such that S is not borel.

Proof. Since ψ has a continuous inverse, we see it maps borel sets to borel sets. Let S be a subset of C such that $\psi(S)$ is not measurable. Since $\psi(S)$ is not measurable, it is not a borel set. Hence S is not borel, but it was measurable with measure 0. □