Algebraic Theory I

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Lecture 17: Nilpotent Groups (4) and Solvable Groups

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Recall. We had a theorem that, for a finite group G, implied G was nilpotent if and only if all maximal subgroups are normal.

Proof.

1. ($\Rightarrow$). Let $M < G$ be a maximal subgroup, so $M < N \leq G$ implies $N = G$. Let $N_g(M)$ be the normalizer of $M<$ then $M < G$, hence $M < N_G(P)$ by the earlier characterization of finite nilpotent groups. Hence, $N_G(M) = G$. But $M < N_G(M)$ and M ix maximal, hence $N_G(M)$ if and only if M is normal.

2. ($\Leftarrow$). Assume every maximal subgroup is normal. Note that it suffies to show that all sylow groups are normal in G by the earlier characterization. Let $P \leq G$ be an arbitrary sylow p-group and let $N = N_G(P)$. Let M be a maximal subgroup containing $N_G(P)$. We know such a group exists because if we assume indirectly that P is not normal, this implies $N_G(P) < G$ as every proper subgroup of a finite group is contained in a maximal subgroup.
   We now have $P \leq N_G(P) \leq M < G$ and by hypothesis, we know $M \trianglelefteq G$. Since $P \leq M$with P being a sylow group of G implies $P \leq M$ is a sylow group for M. But now we can applying the frattini argument. We see $G = N_G(P)M$ but $N_G(P) \leq M$, hence $G \subseteq MM = M < G$. $\lightning$.

$\square$

Remark. If G is nilpotent, then recall $Z_0(G) < Z_1(G) < Z_2(G) < \ldots < Z_i(G)$ is the upper central series where $Z_0(G) = \{1\}$, $Z_1(G) = Z(G)$ and $Z_i(G)/Z_{i-1}(G) = Z(G/Z_{i-1}(G))$.
There is an alternative characterization, let $G^0 = G$, $G^1 = [G,G] = \left\langle x^{-1}y^{-1}xy : x, y \in G\right\rangle$ and define recursively $G^i = \left[G, G^{i-1}\right] = \left\langle x^{-1}y^{-1}xy : x \in G, y \in G^{i-1}\right\rangle$ to be the lower central series. Then, G is nilpotent if and only if there is $c \geq 0$ such that $G^c = \{1\}$. Furthermore, we find $G^{c-i} \leq Z_i(G)$ for all $0 \leq i \leq c$, with the minimal constant c being the same in the upper and lower central series.

1 Solvable Groups

Definition 1.1 (Solvable Groups)**.** A group G is **solvable** if there's a chain of subgroups

$$H_0 \lhd H_1 \lhd \ldots \lhd H_n = G$$

such that H_i/H_{i-1} are abelian for $1 \leq i \leq n$.

As it turns out there is an equivalent chain condition for solvability closed to our characterizations of nilpotence. Define $G^{(0)} = G$, $G^{(1)} = [G,G] = G^1$, Now, define $G^{(i)} = \left[G^{(i-1)}, G^{(i-1)}\right] = \left\langle x^{-1}y^{-1}xy : x, y \in G^{(i-1)}\right\rangle$. So, $G^{(n)}$ is essentially the n-th iterated commutator of G. Then, we obtain a chain

$$G^{(0)} \geq G^{(1)} \geq \ldots \geq G^{(c)} \geq \ldots.$$

If $G^{(c)} = 1$ for some $c \geq 1$, then G is solvable. We show these two conditions are equivalent. The proof will involve multiple invocations of the basic result that G/H is abelian if and only if $[G,G] \leq H$.

Proof. Assume G is solvable, and the 1st characterization is true with $1 = H_0 \trianglelefteq H_1 \trianglelefteq \ldots \trianglelefteq H_n = G$ with H_i/H_{i-1} being abelian for all $1 \leq i \leq n$. We will show by induction that $G^{(i)} \leq H_{n-i}$ for all $1 \leq i \leq n$. For $i = 0$ we have $H_n = G$, hence $G^{(0)=G}$ and $G \leq G$, so the claim holds for $i = 0$. Now, note that

$$\begin{aligned} G^{(i)} &= \left[G^{(i-1)}, G^{(i-1)}\right] \\ &\leq \left[H_{n-(i-1)}, H_{n-(i-1)}\right] \text{ by inductive hypothesis} \\ &= [H_{n-i+1}, H_{n-i+1}] \end{aligned}$$

.

We also know that H_{n-i+1}/H_{n-i} is abelian, hence we have $G^{(i)} \leq [H_{n-i+1}, H_{n-i+1}] \leq H_{n-i}$ by the preceding lemma. This completes the induction. But, we have $G^{(n)} \leq H_{n-n} = H_0 = \{1\}$, so $G^{(n)}$ is trivial. □

Lecture 18: Solvable Groups (2)

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Recall. A group is solvable if there exists a chain of subgroups

$$\{1\} \trianglelefteq H_0 \trianglelefteq H_1 \trianglelefteq \ldots \trianglelefteq H_n = G$$

such that H_i/H_{i-1} is abelian.
We had that this is equivalent to the condition that for $G^{(n)} = 1$ where $G^{(0)} = G$ and $G^{(n)} = \left[G^{n-1}, G^{n-1}\right]$. We showed the forward implication, so now we show the reverse implication.

Proof. Suppose $G^{(n)} = 1$ for some $n \geq 0$. Then, we have a chain

$$G = G^{(0)} \trianglelefteq G^{(1)} \trianglelefteq \ldots \trianglelefteq G^{(n)} = \{1\}.$$

So, we have

$$\{1\} = G^{(n)} \trianglerighteq G^{(n-1)} \trianglerighteq \ldots \trianglerighteq G^{(0)} = G.$$

Furthermore, we know the commutator of $G^{(i)}$is a characteristic subgroup, hence it is normal.
Then, define $H_i = G^{(n-i)}$ for $0 \leq i \leq n$. We need only show the quotients to be abelian. We see $H_i/H_{i-1} = G^{(n-i)}/G^{(n-i+1)}$. But, $G^{(n-i+1)} = \left[G^{(n-i)}, G^{(n-i)}\right]$ by definition. Hence, $G^{(n-i)}/G^{(n-i+1)}$ is abelian by the lemma from last class. So, the chain condition holds and G is solvable. $\square$

Theorem 1.1. Let G be a solvable group with H being a subgroup. Then, H is solvable.

Proof. We simply show $H^{(n)} \leq G^{(n)}$ for all n by induction. For the base case we know $H = H^{(0)} \leq G^{(0)} = G$. Then, we note $H^{(n)} = \left[H^{(n-1)}, H^{(n-1)}\right] \subseteq \left[G^{(n-1)}, G^{(n-1)}\right] = G^{(n)}$ by inductive hypothesis. Since G is solvable, we find a $n \geq 0$ such that $G^{(n)} = \{1\}$. Then, $H^{(n)} \leq G^{(n)} = \{1\}$, so $H^{(n)} = \{1\}$ hence H is solvable. $\square$

Theorem 1.2. If G is solvable and $\varphi : G \to G'$ is a homomorphism, then $\varphi(G)$ is also solvable.

Proof. We see $\varphi\left(G^{(0)}\right) = \varphi(G)^{(0)}$. So, $\varphi\left(G^{(0)}\right) = \varphi(G)^{(0)}$. We induce on n. We see

$$\begin{aligned}
\varphi\left(G^{(n)}\right) &= \varphi\left(\left[G^{(n-1)}, G^{(n-1)}\right]\right) \\
&= \varphi\left(\left\langle x^{-1}y^{-1}xy : x, y \in G^{(n-1)}\right\rangle\right) \\
&= \left\langle \varphi\left(x^{-1}y^{-1}xy : x, y \in G^{(n-1)}\right)\right\rangle \\
&= \left\langle \varphi(x)^{-1}\varphi(y)^{-1}\varphi(x)\varphi(y) : x, y \in G^{(n-1)}\right\rangle \\
&= \left\langle \overline{x}^{-1}\overline{y}^{-1}\overline{xy} : \overline{x}, \overline{y} \in \varphi\left(G^{(n-1)}\right)\right\rangle \\
&= \left\langle \overline{x}^{-1}\overline{y}^{-1}\overline{xy} : \overline{x}, \overline{y} \in \varphi(G)^{(n-1)}\right\rangle \text{ by the inductive hypothesis.} \\
&= \left[\varphi(G)^{(n-1)}, \varphi(G)^{(n-1)}\right] \\
&= \varphi(G)^{(n)}.
\end{aligned}$$

Since G is solvable, we find an $n \geq 0$ such that $G^{(n)} = \{1\}$. Hence, $\varphi\left(G^{(n)}\right) = \varphi(\{1\}) = \{1\} = \varphi(G)^{(n)}$, so $\varphi(G)$ is solvable. $\square$

Theorem 1.3. If G is a group with $H \trianglelefteq G$, then G is solvable if and only if H and G/H are solvable.

Proof. $(\Rightarrow)$. We know all subgroups and homomorphic images to be solvable, hence this direction is already proven.

($\Leftarrow$). Assume H and G/H are solvable. As H is solvable it has a normal chain

$$H_0 \trianglelefteq H_1 \trianglelefteq \ldots \trianglelefteq H_n = H$$

with H_i/H_{i-1} is abelian for all $1 \leq i \leq n$. Similarly, since G/H is solvable there is a normal chain

$$\{1\} = K_{n+0} \trianglelefteq K_{n+1} \trianglelefteq \ldots K_{n+s} = G/H$$

With K_{n+i}/K_{n+i-1} being abelian for all $i \geq 1$. We know by the lattice theorem that there are groups H_{n+i} such that $K_{n+i} = H_{n+i}/H$ for some $H_{n+i} \leq G$ and $H \leq H_{n+i}$. Then, we have

$$\{1\} = H/H \trianglelefteq H_{n+1}/H \trianglelefteq \ldots \trianglelefteq H_{n+s}/H = G/H.$$

Then, we have $H_n = H$ and $H_{n+s} = G$ and, as each contains the kernel, this correspondance preserves normality, hence we have

$$H_n = H \trianglelefteq H_{n+1} \trianglelefteq H_{n+2} \trianglelefteq \ldots H_{n+s} = G.$$

Then, note that $H_{n+i}/H_{n+i-1} = (H_{n+i}/H)\,/\,(H_{n+i-1}/H) = K_{n+i}/K_{n+i-1}$ which we know to be abelian. Hence all successive quotients are abelian. So,

$$\{1\} = H_0 \trianglelefteq H_1 \trianglelefteq \ldots \trianglelefteq H_n \trianglelefteq H_{n+1} \trianglelefteq H_{n+2} \trianglelefteq \ldots H_{n+s} = G.$$

with H_i/H_{i-1} being abelian, so G is solvable. $\square$

Remark. Subgroups and quotients of nilpotent groups are nilpotent, but this converse does not hold in general for nilpotent groups.