2. The Real Numbers: Construction and Ordering

We will construct the real numbers from the rational numbers, assuming that the properties of $\mathbb{Q}$ are known.

A sequence of rational numbers (a_k) is called a *rational Cauchy sequence* if for each rational $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$|a_n - a_m| < \epsilon \quad \text{if } n, m \geq N.$$

Denote the set of all rational Cauchy sequences by $\mathrm{CS}(\mathbb{Q})$. It follows easily from this definition that rational Cauchy sequences are bounded and that sums, differences, and products of rational Cauchy sequences are rational Cauchy sequences. We say that two rational Cauchy sequences (a_k), (b_k) are equivalent and write $(a_k) \sim (b_k)$ if for each rational $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$|a_n - b_n| < \epsilon \quad \text{if } n \geq N.$$

Clearly, $\sim$ is an equivalence relation on $\mathrm{CS}(\mathbb{Q})$ (reflexivity and symmetry are obvious, while transitivity follows from the triangle inequality). We now consider the quotient set $\mathrm{CS}(\mathbb{Q})/\sim$, consisting of all equivalence classes $[(a_k)]$ of rational Cauchy sequences (a_k). A rational number $q \in \mathbb{Q}$ can be identified with the equivalence class $[(q)] \in \mathrm{CS}(\mathbb{Q})/\sim$. Hence $\mathbb{Q}$ can be identified with a subset of $\mathrm{CS}(\mathbb{Q})/\sim$.

We now intend to extend the total ordering $\leq$ on $\mathbb{Q}$ to $\mathrm{CS}(\mathbb{Q})/\sim$. To this end, we define the relation $\preccurlyeq$ on $\mathrm{CS}(\mathbb{Q})/\sim$ as follows: For x and $y \in \mathrm{CS}(\mathbb{Q})/\sim$, we write $x \preccurlyeq y$ if for every $(x_k) \in x$, $(y_k) \in y$ and every rational $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$x_n \leq y_n + \epsilon \quad \text{if } n \geq N.$$

Proposition 2.1. *Suppose $x, y \in \mathrm{CS}(\mathbb{Q})/\sim$ and there are $(x_k) \in x$ and $(y_k) \in y$ with the property that for every rational $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that*

$$x_n \leq y_n + \epsilon \quad \textit{if } n \geq N.$$

Then $x \preccurlyeq y$.

Proof. Let $\epsilon > 0$ be rational. Then there exists $N \in \mathbb{N}$ such that $x_n \leq y_n + \frac{\epsilon}{3}$ if $n \geq N$. For $(\tilde{x}_k) \in x$ and $(\tilde{y}_k) \in y$, there exists $K \in \mathbb{N}$ such that $|\tilde{x}_n - x_n| < \frac{\epsilon}{3}$ and $|\tilde{y}_n - y_n| < \frac{\epsilon}{3}$ if $n \geq K$. Hence if $n \geq \max\{N, K\}$,

$$\tilde{x}_n < x_n + \frac{\epsilon}{3} \leq y_n + \frac{2\epsilon}{3} < \tilde{y}_n + \epsilon.$$

□

Lemma 2.2. *The relation $\preccurlyeq$ is a total ordering on $\mathrm{CS}(\mathbb{Q})/\sim$.*

Proof. The relation $\preccurlyeq$ is certainly reflexive and transitive. Suppose $x, y \in \mathrm{CS}(\mathbb{Q})/\sim$ are such that $x \preccurlyeq y$ and $y \preccurlyeq x$. Given $(x_k) \in x$, $(y_k) \in y$, for any rational $\epsilon > 0$, we can find $N \in \mathbb{N}$ such that $x_n \leq y_n + \frac{\epsilon}{2}$ and $y_n \leq x_n + \frac{\epsilon}{2}$ for $n \geq N$. Consequently, $|x_n - y_n| \leq \frac{\epsilon}{2} < \epsilon$ if $n \geq N$, proving that $x = y$ and that $\preccurlyeq$ is a partial ordering on $\mathrm{CS}(\mathbb{Q})/\sim$. Hence it suffices to show that for any $x, y \in \mathrm{CS}(\mathbb{Q})/\sim$, $x \preccurlyeq y$ or $y \preccurlyeq x$ holds true. Suppose $x \preccurlyeq y$ is wrong. Then for some $(x_k) \in x$, $(y_k) \in y$ there is a rational $\epsilon > 0$ such that $x_n > y_n + \epsilon$ for infinitely many $n \in \mathbb{N}$. Also, there exists $K \in \mathbb{N}$ such that for k, $m \geq K$, $|x_k - x_m| < \frac{\epsilon}{2}$ and $|y_k - y_m| < \frac{\epsilon}{2}$. Hence if $n \geq K$ is such that $x_n > y_n + \epsilon$, we obtain for $k \geq K$

$$x_k + \frac{\epsilon}{2} > x_k + (x_n - x_k) = x_n > y_n + \epsilon = y_k + (y_n - y_k) + \epsilon > y_k + \frac{\epsilon}{2}.$$

Consequently, $y_k < x_k$ for all $k \geq K$. Thus $y \preccurlyeq x$. □

Next, we introduce addition, subtraction and multiplication on $\mathrm{CS}(\mathbb{Q})/\sim$. For x, $y \in \mathrm{CS}(\mathbb{Q})/\sim$ with $(x_k) \in x$ and $(y_k) \in y$ define

$$x + y = [(x_k + y_k)], \quad x - y = [(x_k - y_k)] \quad \text{and} \quad x\, y = [(x_k\, y_k)].$$

Note that addition, subtraction and multiplication are well-defined, i.e. $(x_k + y_k)$, $(x_k - y_k)$ and $(x_k\, y_k)$ are rational Cauchy sequences and the definitions are independent of the particular representatives (x_k) and (y_k).

We now define the set of *real numbers*, denoted by $\mathbb{R}$, to be $\mathrm{CS}(\mathbb{Q})/\sim$ with the total ordering $\preccurlyeq$ and addition, subtraction and multiplication as given above. We say that a real number x is a rational number/integer/natural number if there exists a rational number/integer/natural number z such that $[(z)] = x$. In this way, we may just write z for this real number instead of $[(z)]$. Observe that for any two rational numbers p and q,

$$p \leq q \quad \text{if and only if} \quad p \preccurlyeq q.$$

Hence the total ordering $\leq$ on $\mathbb{Q}$ extends to the total ordering $\preccurlyeq$ on $\mathbb{R}$. It is thus convenient to stop distinguishing between $\leq$ and $\preccurlyeq$ and and to just write $\leq$ from now on. We may now also use the order relations $<$, $>$ and $\geq$ on $\mathbb{R}$, defined as always.

Lemma 2.3. The Archimedean Property
For any two real numbers x, y with $x > 0$ there exists a natural number m such that $m\,x > y$.

Note: The result is certainly true if x and y are rational numbers.

Proof. Suppose the claim is wrong. Then $m\,x \leq y$ for all $m \in \mathbb{N}$. Let $(x_k) \in x$ and $(y_k) \in y$ be representatives of their equivalence classes and let $M \in \mathbb{Q}$ be a number such that $y_n \leq M$ for all n (M exists because a rational Cauchy sequence is bounded). Since $\mathbb{Q}$ is Archimedean, for given rational $\epsilon > 0$ there exists $m \in \mathbb{N}$ such that $\frac{M+1}{m} < \epsilon$. Also, there is $N \in \mathbb{N}$ such that $0 \leq x_n + \frac{\epsilon}{2}$ and $m\,x_n \leq y_n + 1$ for $n \geq N$. The latter inequality implies that $x_n \leq \frac{M+1}{m} < \epsilon$ for all $n \geq N$, while the first proves $-\epsilon < x_n$ for all $n \geq N$. Hence $|x_n| < \epsilon$ for $n \geq N$ and thus $x = 0$ in contradiction to the assumptions. □

Corollary 2.4. *For any two real numbers x, y with $x < y$ there exists a rational number q such that $x < q < y$.*

Proof. By the Archimedean property, there exists $m \in \mathbb{N}$ such that $m(y-x) > 1$. Also, there exist $j \in \mathbb{N}$ such that $j > -mx$. Let $l \in \mathbb{N}$ be the smallest number such that $l > mx + j$ (here we are using the well-ordering principle of $\mathbb{N}$ guaranteeing the existence of a smallest natural number l). Thus $my + j > mx + 1 + j \geq l > mx + j$ or $my > l - j > mx$. Now let $(x_k) \in x$ and $(y_k) \in y$ be representatives of their equivalence classes. Then for every rational $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$l - j \leq m\, y_n + m\epsilon \quad \text{and} \quad m\, x_n \leq l - j + m\epsilon \quad \text{if } n \geq N.$$

Consequently, with $q = \frac{l-j}{m}$ we have $x \leq q \leq y$. If q were equal to x or y, then we would obtain $mx = l - j$ or $my = l - j$, respectively. Hence we must have $x < q < y$. □

For $x \in \mathbb{R}$ we call a real number y *inverse* to x if

$$x\,y = 1.$$

Proposition 2.5. *Every real number $x \neq 0$ has a unique inverse.*

Proof. If $x \neq 0$ and $(x_k) \in x$, the preceding corollary guarantees that there exist $q \in \mathbb{Q}$ such that either $x < -2q < 0$ or $x > 2q > 0$. Hence there exists $N \in \mathbb{N}$ such that $2q \leq |x_n| + q$, i.e. $|x_n| \geq q > 0$ for $n \geq N$. Now we define a rational sequence (y_k) by setting

$$y_n = \frac{1}{x_n} \quad \text{if } n \geq N \quad \text{and} \quad y_n = 1 \quad \text{if } 1 \leq n < N.$$

Then it readily follows that (y_k) is a rational Cauchy sequence and that $[(x_k\, y_k)] = 1$. Hence $y = [(y_k)]$ is inverse to x. Finally, if (z_k) is a rational Cauchy sequence such that $[(x_k\, z_k)] = 1$, then $(x_k\, z_k) \sim (x_k\, y_k)$. Since $|x_n| \geq q$ for $n \geq N$,

$$|y_n - z_n| = \frac{1}{|x_n|}\,|x_n\, y_n - x_n\, z_n| \leq \frac{1}{q}\,|x_n\, y_n - x_n\, z_n| \quad \text{if } n \geq N.$$

Consequently, $[(y_k)] = [(z_k)]$, hence the inverse is unique. □

The notation x^{-1} is used to denote the unique inverse of the real number $x \neq 0$. We have now all the key ingredients for the following straightforward (but somewhat tedious) result:

Theorem 2.6. $\mathbb{R}$ *is an ordered field.*

The reader is referred to texts on abstract algebra for the precise definition of a *field*.

Finally, let us define division of two real numbers x and y, provided that $y \neq 0$, by

$$\frac{x}{y} = x\,y^{-1}.$$