MATH 7237/8237

LECTURE Jan. 22, 2021

VERTEX DELETION

Given a graph G there are two basic ways to obtain smaller graphs from G :

- **vertex deletion;**
- **edge deletion.**

Deleting a vertex v in a graph G amounts to removing the vertex from $V(G)$ and removing all edges containing the vertex v from $E(G)$.

Vertex deletions can be iterated as long as there are vertices in the graph.

When deleting multiple vertices it doesn't matter in what order the deletions are performed, as the result is always the same.

The graph obtained by deleting the vertex v from G is denoted by $G - v$.

EDGE DELETION

Deleting an edge $\{u, v\}$ in a graph G amounts to removing the edge from $E(G)$. The vertices u and v remain in $V(G)$.

Edge deletions can be iterated as long as there are edges in the graph.

When deleting multiple edges it doesn't matter in what order the deletions are performed, as the result is always the same.

The graph obtained by deleting the edge e from G is denoted by $G - e$.

EXAMPLES OF DELETION

Deleting an end-vertex from P_n produces either P_{n-1} or an isolated vertex if $n = 2$.

- Deleting any vertex from C_n produces P_{n-1}.

- Deleting any edge from C_n produces P_n.

- Deleting any vertex from K_n produces K_{n-1} $(n \geq 2.)$

- Deleting any edge from $K_{1,n}$ produces a graph with one isolated vertex.

SUBGRAPHS

Definition A graph H is called a **subgraph** of a graph G if $V(H) \subset V(G)$ and $E(H) \subset E(G)$.

Equivalently, we may say: G **contains** H or H **is contained** in G.

A subgraph H of G can be thought as obtained by deleting all vertices that are not in $V(H)$ and all edges that are not in $E(H)$.

Definition A graph H is called an **induced subgraph** of a graph G if $V(H) \subset V(G)$ and $E(H)$ is the set of all edges of G that join vertices within H.

An induced subgraph H of G can be thought as obtained by deleting all vertices that are not in $V(H)$.

EXAMPLES OF SUBGRAPHS

- Every graph of order at most n is a subgraph of K_n.
- Every induced subgraph of K_n is a complete subgraph.
- The path P_n is an induced subgraph of the path P_{n+1}.
- The path P_n is an induced subgraph of the cycle C_{n+1}.
- The path P_n is a subgraph of the cycle C_n.
- The star $K_{1,n}$ is an induced subgraph of $K_{1,n+1}$.
- Every graph is an induced subgraph of itself.

SPANNING SUBGRAPHS

Definition If H is a subgraph of G and $V(H) = V(G)$, then H is called a **spanning** subgraph of G.

A spanning subgraph H of G can be thought as obtained by edge deletions: deleted are the edges that are not in $E(H)$.

Examples:

The n-cycle contains a spanning path P_n.

The complete graph K_n contains a spanning star $K_{1,n-1}$.

The even cycle C_{2k} contains a spanning graph consisting of k disjoint edges (take every other edge.)

BIPARTITE GRAPHS

Definition A graph is called **bipartite** if its vertices can be split into two disjoint sets A and B such that all edges of the graph have one vertex from A and one vertex from B. Sometimes A and B are called **partite** sets.

Equivalently, we may say: a graph G is **bipartite** if its vertices can be split into two disjoint sets A and B such that there are no edges joining two vertices from A or two vertices from B.

A third way of defining bipartite graphs is through coloring of the vertices: a graph G is **bipartite** if its vertices can be colored with two colors so that every vertex is colored in one color and the vertices of every edge are colored with distinct colors.

EXAMPLES OF BIPARTITE GRAPHS

Example Every path is a bipartite graph.

Indeed, let $v_1, \ldots, v_n$ be a path. Color the odd numbered vertices in red and the even numbered vertices in blue. Since each edge joins two consecutive vertices, one of them is colored in red and the other one in blue.

▮

Example Every cycle of even order is bipartite.

Let $v_1, \ldots, v_{2n}$ be a cycle. Color the odd numbered vertices in red and the even numbered vertices in blue. Clearly the edge $\{v_{2n}, v_1\}$ is correctly colored. All other edges join two consecutive vertices, so one of them is colored in red and the other one in blue.

▮

Example The star $K_{1,n}$ is a bipartite graph for every n.

COUNTING EDGES IN BIPARTITE GRAPHS

Proposition *If G is a bipartite graph and A and B are its partite sets, then*

$$e(G) = \sum_{u \in A} d(u)$$
$$e(G) = \sum_{u \in B} d(u)$$

- Indeed, every edge contain one and only one vertex from A, and so each edge is counted exactly once in the sum $\sum_{u \in A} d(u)$.
- Likewise each edge is counted exactly once in the sum $\sum_{u \in B} d(u)$.

REGULAR BIPARTITE GRAPHS

Proposition *If a nonempty graph is bipartite and regular, then its partite sets have equal cardinality.*

Indeed, let G be an r-regular bipartite graph, and let A and B be its partite sets.

By the previous proposition,

$$e(G) = \sum_{u \in A} d(u) = r|A|$$
$$e(G) = \sum_{u \in B} d(u) = r|B|.$$

Hence $|A| = |B|$.

Corollary *Cycles of odd order are nonbipartite.*

Indeed, every cycle is 2-regular, so if it is bipartite, its partite sets have equal cardinality and its order must be even.

COMPLETE BIPARTITE GRAPHS

Definition Let G be a bipartite graph with partite sets A and B. The graph is called **complete bipartite** if it contains all possible edges $\{u, v\}$, where $u \in A$ and $v \in B$.

A complete bipartite graph with partite sets A and B is denoted by $K_{a,b}$ where $a = |A|$ and $b = |B|$.

A complete bipartite graph with partite sets A and B has $|A| |B|$ edges, because each edge is determined uniquely by a vertex $u \in A$ and a vertex $v \in B$.

Examples of complete bipartite graphs

Every star is a complete bipartite graph.

The 4-cycle is isomorphic to $K_{2,2}$, but no other cycle is complete bipartite.

MAXIMAL SIZE OF BIPARTITE GRAPHS

Let us discuss the following typical graph theoretical problem:

Problem *What is the maximum number of edges that a bipartite graph of order n can have?*

This question must have an answer for every n, because there are finitely many graphs of order n.

Note that if G is a bipartite graph of order n with maximum number of edges, then G must be complete bipartite. If not, we could add edges to G without increasing its order.

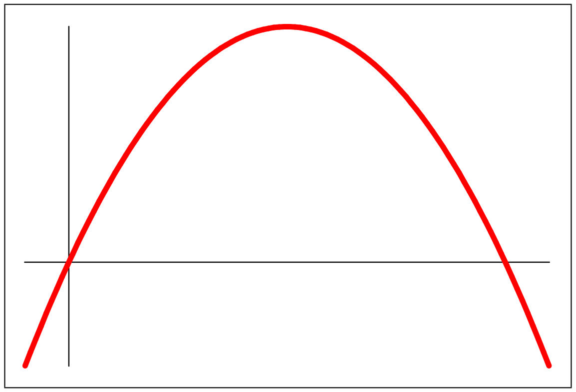
So we must search for the solution within the complete bipartite graphs $K_{a,n-a}$, where $1 \leq a \leq n-1$.

MAXIMAL SIZE OF BIPARTITE GRAPHS

The problem boils down to finding

$$\max_{1 \le a \le n-1} a(n-a).$$

This is a simple analytical problem, but we need a to be an integer.



Clearly the function $f(a) = a(n-a)$ is increasing for $1 \le a \le \lfloor n/2 \rfloor$ and decreasing for $\lceil n/2 \rceil \le a \le n-1$. Therefore,

$$\max_{1 \le a \le n-1} a(n-a) = \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil = \left\lfloor \frac{n^2}{4} \right\rfloor.$$

GRAPH COMPLEMENT

Definition Let $G = (V, G)$ be a graph. The **complement** of G is a graph $\overline{G}$ whose vertex set is also V and the edges are the two-element subsets of V that are not in $E(G)$.

Remarks

- The complement of the complement of G is G;

- The complement of K_n is the edgeless graph of order n;

- If G has order n and size m, then $\overline{G}$ has order n and size $\binom{n}{2} - m$;

- For every $u \in V$, we have $d_G(u) + d_{\overline{G}}(u) = n - 1$.

  In particular,

$$\delta(G) + \Delta(\overline{G}) = n - 1 \quad \text{and} \quad \Delta(G) + \delta(\overline{G}) = n - 1.$$

DISCONNECTED GRAPHS

Definition A graph G is called disconnected, if $V(G)$ can be split into two disjoint nonempty sets A and B such that there are no edges G that join a vertex from A to a vertex from B.

- **Examples**
  The complement of K_n for $n \geq 2$ consists of n isolated vertices and is disconnected.

- The complement of C_4 consists of two disjoint edges and is disconnected.

- The complement of the complete bipartite graph $K_{a,b}$ consists of disjoint K_a and K_b and is disconnected.

CONNECTED GRAPHS

Definition A graph that is not disconnected is called **connected.**

Theorem *A graph G is connected if and only if for every two vertices u and v, G contains a path with end vertices u and v.*

Examples: Paths, cycles, stars, complete graphs, complete bipartite graphs.

Proposition *If a graph G contains a connected spanning subgraph, then G is connected itself.*

Indeed, for every two vertices u and v of G there is a path P in H that joins u to v.

But H is a subgraph of G, so P is a path in G that joins u and v.

PROPERTIES OF CONNECTED GRAPHS

Theorem *If G is disconnected, then its complement is connected.*

- Indeed, let $V(G)$ be partitioned into disjoint nonempty sets A and B such that there is no edge joining vertex in A to a vertex in B.

- Clearly for every $u \in A$ and $v \in B$, the set $\{u, v\}$ is an edge of $\overline{G}$. Hence $\overline{G}$ contains the complete bipartite graph with partition sets A and B; write H for this graph.

- Since $A \cup B = V(\overline{G})$, the graph H is spanning; hence, $\overline{G}$ is connected.

- Note that if G is connected, then $\overline{G}$ may or may not be disconnected.

PROPERTIES OF CONNECTED GRAPHS

Remark A connected graph of order at least two does not have isolated vertices.

Theorem *If u is a vertex in a graph G such that every vertex $v \in V(G)$ can be joined to u by a path, then G is connected.*

Theorem *If G and H are connected graphs that have a common vertex, their union is a connected graph.*

Corollary *Let G be a connected graph. Add a new vertex to G and join it to a vertex of G. The resulting graph is connected.*

Theorem *A connected graph of order n has at least $n-1$ edges.*

We shall use induction on n.

If $n = 2$, the graph K_2 is the only connected graph of order 2, and it has one edge.

Assume that $n \geq 3$ and the statement holds for all graphs with fewer than n vertices.

Next, note that G does not have isolated vertices, whence $\delta(G) \geq 1$.

If $\delta(G) \geq 2$, then

$$2e(G) \geq \sum_{v \in V(G)} d(u) \geq 2n > 2(n-1),$$

and the assertion follows.

Thus, it remain the case $\delta(G) = 1$. Take a vertex u with $d(u) = 1$, and write v for the single neighbor of u.

Let $w \in V(G) \setminus \{u, v\}$. As G is connected, there is a path joining u to w in G.

Note that every path starting at u must contain v as a second vertex. Therefore, there is a path joining v to w in $G - u$.

Hence $G - u$ is connected, and by the induction assumption we have

$$e(G - v) \geq n - 2.$$

Hence,

$$e(G) \geq n - 2 + 1 = n - 1.$$

This completes the induction step and the proof.

DISTANCE IN GRAPHS

Let G be a connected graph.

Definition For every two vertices u and v, the length of the shortest path joining u to v is called the **distance** between u and v, and is denoted by $dist\,(u,v)$.

The distance function $dist\,(u,v)$ has three basic properties:

(a) $dist\,(u,u) = 0$;

(b) $dist\,(u,v) = dist\,(v,u)$;

(c) $dist\,(u,v) + dist\,(v,w) \geq dist\,(u,w)$.

Remark Two vertices u and v are adjacent if and only if

$$dist\,(u,v) = 1.$$

DIAMETER OF A GRAPH

Definition Given a connected graph G, the **diameter** of G is defined as the largest distance between two vertices in G, and is denoted by $diam\,(G)$.

Examples

$$diam\,(K_n) = 1,$$

$$diam\,(K_{a,b}) = 2,$$

$$diam\,(C_n) = \lfloor n/2 \rfloor\,,$$

$$diam\,(P_n) = n - 1.$$

COMPONENTS

Definition Given a graph G, a **component** of G is a maximal connected subgraph of G.

Let us note that a subgraph H is maximal connected subgraph if there is no connected subgraph F, such that H is a proper subgraph of F.

Hence, a component C is either the whole G (if G is connected) or a proper induced subgraph such that no vertex of C is joined to any of the vertices that are not in C.

Indeed, if G contains an edge e that joins two vertices of C but is not in $E(C)$, adding e to $E(C)$ keeps C connected and augments C. Hence C is not maximal, contradicting its definition.

We see that any disconnected graph can be partitioned into connected subgraphs such there is no edge joining vertices from distinct subgraphs.

THANK YOU