Algebraic Theory I

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Recall R will be a commutative ring unless otherwise noted.

Definition 0.1 (Prime Ideal)**.** Recall an ideal $P \subseteq R$ is a **prime ideal** when $xy \in P$ implies one of $x \in P$ or $y \in P$. This is equivalent to the statement that R/P is an integral domain.

Definition 0.2 (Maximal Ideals)**.** A proper ideal $M \subseteq R$ is maximal if it is not strictly contained in any other proper ideal. That is, the only ideals containing M are M and R. Equivalently, an ideal I is maximal if and only if R/I is a field.

We prove these two definitions to be equivalent.

Proof. First, assume I maximal. Then, note that an ideal in R/I has the form J/I with $I \subseteq J \subseteq R$ and J being an ideal in R. Hence, as I is maximal, we find $J = I$ or $J = R$. Hence, R/I is a field by prior characterization.
Now assume R/I is a field for some ideal I. Then, the only ideals of R/I are $\{0\}$ and R/I. Suppose I nonmaximal, then we find a $I \subset J \subset R$ corresponding to a proper nontrivial ideal $J/I \subseteq R/I$, $\lightning$ as R/I is a field. $\square$

Proposition 0.1. In a commutative ring R any maximal ideal is prime.

Proof. Since $M \subset R$ and R/M is a field (hence integral domain), we find M to be a prime ideal by the quotient characterization. $\square$

Example. If $R = \mathbb{Z}$, then (0) is a prime ideal, but it is obviously not maximal. $\diamond$

In order to prove some theorems concerning maximal ideals, we need to state some results from basic set theory.

Definition 0.3. If $(X, \preceq)$ is a poset (partially ordered set), with a totally ordered subset $Y \subseteq X$, then an **upper bound** of Y is an element $x \in X$ so that $y \leq x$ for all $y \in Y$. A **maximal element** of X is a $x \in X$ so that for all $y \in X$, $x \leq y$ implies $x = y$.

Law 1 (Zorn's Lemma)**.** If $(X, \preceq)$ is a nonempty poset, with every totally ordered subset having an upper bound, then we find a maximal element $x \in X$.

Of course, this is equivalent to axiom of choice, so we must take it as an axiom. Using Zorn's lemma, we find that every ideal is contained in a maximal ideal (as with subgroups).

Theorem 0.1. If R is a commutative ring with $I \subset R$ being a proper ideal. Then there is a maximal ideal $M \subset R$ with $I \subseteq M$.

Proof. Let $(X, \subseteq)$ be the set of all proper ideals of R which contain I partially ordered by inclusion. As I is proper, we see $I \subseteq I$ hence $I \in X$, so $X \neq \varnothing$. Any maximal element $m \in X$ will be a maximal ideal of R containing I. Hence, we need only show the existence of a maximal element.
Let $(I_\alpha)_{\alpha \in \Omega}$ by a nonempty totally ordered subset of X. Hence, each I_α is a proper ideal containing I with either $I \subseteq I_\alpha \subseteq I_\beta$ or $I \subseteq I_\beta \subseteq I_\alpha$ for all $\alpha, \beta \in \Omega$. Let $J = \bigcup_{\alpha \in \Omega} I_\alpha$, clearly, $I_\alpha \subseteq J$ for all $\alpha \in \Omega$, so we need only show $J \in X$. Clearly, $I \subseteq I_\alpha \subseteq J$, so J is nonempty and contains I. Now, let $x, y \in J$ with $x \in I_\alpha$, $y \in I_\beta$. By total ordering WLOG, let $I_\alpha \subseteq I_\beta$. Hence, $x, y \in I_\beta$. Hence, $x - y \in I_\beta \subseteq J$ as this is an ideal and $rx \in I_\beta \subseteq J$ for all $r \in R$, hence J is an ideal. Finally, suppose $J = R$, then $1 \in J$, so $1 \in I_\alpha$ for some $\alpha \in \Omega$ ↯, as I_α is assumed proper. Hence, $J \in X$ is an upper bound of $(I_\alpha)_{\alpha \in \Omega}$, so there is a maximal element $M \in X$ which is clearly a maximal ideal. $\square$