Chapter 1

기본용어 Terms in graph theory

A graph $G = (V, E)$ consists of the set of vertices and the collection of edges (cf. multiple edges, weighted graph, digraph).

1.1 Vertices and edges

1. If there is an edge e joins vertices x and y, then we say

   (a) x and y are adjacent; x is adjacent to y.

   (b) e is incident with x and y.

2. An edge joining a vertex x to itself is called a loop.

3. Two graphs are isomorphic, if there is a bijection

$$\phi : V_{G_1} \longrightarrow V_{G_2}$$

   such that for all pairs of vertices v, w in G_1, the number of edges between v and w in G_1 is equal to the number of edges between $\phi(v)$ and $\phi(w)$ in G_2.

4. A graph is simple, if it has no loops or multiple edges.

5. The degree $deg(v)$ of a vertex $v \in V(G)$ is the number of edges incident with a vertex v. If loops are allowed, count each loop twice.

6. If $deg(v) = 1$ then v is called end vertex or pendant.

7. Write $\delta(G)$ for the minimum degree and $\Delta(G)$ for the maximum degree.

8. A graph is regular, if $deg(v) = deg(w)$ for all $v, w \in V$. A graph is r-regular, if $deg(v) = r$ for all vertices v.

9. A graph is called a (p, q)-graph, if it has p vertices and q edges.

Theorem 1.1 (Handshaking lemma)**.** *For every graph $G = (V, E)$,*

$$\sum_{v \in V} deg(v) = 2|E|.$$

Proof. The identity easily follows from the observation that each edge contributes twice to the degree sum.

$$(+1)\bullet\!\!-\!\!-\!\!-\!\!-\!\!-\!\!-\!\!\bullet(+1)$$

Alternatively, we can use the double counting technique. Assume that G contains no loops. Let us count the cardinality of the following set in two different ways.

$$S = \{(v, e) \in V \times E \mid v \text{ is incident with } e\}$$

1. First,
$$|S| = \sum_{v \in V} (\text{for each } v \in V, \text{ the number of } e \text{ such that } (v, e) \in S),$$
which is $\sum_{v \in V} deg(v)$.

2. Second,
$$|S| = \sum_{e \in E} (\text{for each } e \in E, \text{ the number of } v \text{ such that } (v, e) \in S),$$
which is $\sum_{e \in E} 2 = 2|E|$.

3. Now we can attach loops to vertices and note that the above equality still holds.

□

Corollary 1.2. *The number of vertices with odd degree should be even.*

Example 1.3. *1. K_n: complete graph with n vertices. It is $(n-1)$-regular.*

2. P_n: path of length $n - 1$.

3. C_n: cycle of length n. It is 2-regular.

4. What is $|E|$ for K_n? Ans1: $\binom{n}{2}$, Ans2: It is $(n-1)$-regular. Use the handshaking lemma.

1.2 Subgraphs

1. A graph $H = (V_H, E_H)$ is a subgraph of $G = (V_G, E_G)$ if $V_H \subseteq V_G$ and $E_H \subseteq E_G$.

2. The complement of a simple graph G, denoted by $\bar{G}$, is the graph with $V_{\bar{G}} = V_G$ and $xy \in E_{\bar{G}}$ if and only if $xy \notin E_G$. In other words, $\bar{G}$ is a graph on the same vertices such that two distinct vertices of $\bar{G}$ are adjacent if and only if they are not adjacent in G.

3. For a vertex v of G, $G - v$ is the graph obtained from G by removing the vertex v and all the edges incident with v.

4. For an edge e of G, $G - e$ is the graph obtained from G by removing only the edge e.

5. A walk is a sequence of edges of the form

$$v_0 v_1, v_1 v_2, ..., v_{n-1} v_n.$$

   (if G has multiple edges then we need to specify edges). v_0 is called the initial vertex and v_n is called the final vertex. If G is simple then we can just write

$$v_0 \to v_1 \to \cdots \to v_n.$$

6. A trail is a walk with all distinct edges.

7. A path is a trail with all distinct vertices (except possibly v_0 and v_n).

8. A cycle is a closed path.

9. A graph with no cycle is acyclic.

1.3 Graphs

1. Recall that a graph is simple, if it has no loops or cycles.

2. A graph is bipartite, if its vertex set can be partitioned into two sets V_1 and V_2 in such a way that every edge of the graph joins a vertex in V_1 to a vertex in V_2.

3. $K_{m,n} = (V_m \cup V_n, E)$: complete bipartite graph (see §3.1).

4. A graph is planar, if it can be embedded in the plane, i.e., it can be drawn on the plane in such a way that its edges intersect only at their endpoints. In other words, it can be drawn in such a way that no edges cross each other. Such a drawing is called a plane graph or planar embedding of the graph.

5. A graph is connected if for every $v, w \in V$, there is a path in G from v to w. One can impose an equivalence relation on the vertex set as $v \sim w$ if there is a path from v to w. The equivalence classes derived from this relation are connected components of the graph.

6. A tree is a connected simple graph with no cycle.

7. A forest is a collection of one or more trees.

8. A leaf is an end vertex of a tree.