Chapter 10

라플라스 행렬

In this chapter, we consider only simple graphs $G = (V, E)$ with n vertices and m edges.

$$|V| = n, \quad |E| = m.$$

10.1 Positivity of Laplacian matrix

Definition 10.1. *Let Δ_G be the $n \times n$ diagonal matrix such that $(\Delta_G)_{ii} = \deg v_i$ for $1 \le i \le n$. The* <u>*Laplacian matrix*</u> *L_G of G is*

$$L_G = \Delta_G - A_G.$$

Definition 10.2. *The* <u>*oriented incidence matrix*</u> *B_G^o of G is an $n \times m$ matrix such that*

$$(B_G^o)_{ij} = \begin{cases} -1 & \textit{if } e_i \textit{ is incident with } v_a \textit{ and } v_b, \textit{ and } j = min(a,b) \\ 1 & \textit{if } e_i \textit{ is incident with } v_a \textit{ and } v_b, \textit{ and } j = max(a,b) \\ 0 & \textit{otherwise.} \end{cases}$$

Theorem 10.3. *The Laplacian matrix L_G of any simple graph G is positive semi-definite. Thus, all the eigenvalues of L_G are greater than or equal to 0.*

Proof. By Theorem 8.10, it is enough to show that $L_G = (B_G^o)^T B_G^o$. Note that the (i, j) entry of the product $(B_G^o)^T B_G^o$ is the dot product of the ith column and jth column of B_G^o. Thus,

$$((B_G^o)^T B_G^o)_{ij} = \begin{cases} \deg v_i & \text{if } i = j \\ 0 & \text{if } v_i \text{ and } v_j \text{ are not adjacent,} \\ -1 & \text{if } v_i \text{ and } v_j \text{ are adjacent.} \end{cases}$$

Therefore, $(B_G^o)^T B_G^o = \Delta_G - A_G$, which shows that $L_G = (B_G^o)^T B_G^o$. □

We let $\mathbf{1}_n$ be the vector in $\mathbb{R}^n$ all whose entries are 1:

$$\mathbf{1}_n = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{R}^n.$$

Lemma 10.4. *The vector $\mathbf{1}_n$ is an eigenvector of L_G corresponding to eigenvalue 0. That is,*

$$L_G \mathbf{1}_n = 0\,\mathbf{1}_n = \mathbf{0}_n.$$

Proof. For any $n \times n$ matrix A, observe that the ith entry of the column vector $A\mathbf{1}_n$ is the sum of entries in the ith row of A. □

Recall that the null space of a $n \times n$ matrix A is

$$\mathrm{Null}(A) = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}.$$

Then, the above lemma shows that for any simple graph G, $\mathbf{1}_n \in \mathrm{Null}(L_G)$.

Proposition 10.5. *Let G be a simple graph.*

1. *The smallest eigenvalue of L_G is 0.*
2. *The null space $\mathrm{Null}(L_G)$ of L_G is nontrivial, thus the nullity of L_G is greater than or equal to 1.*

Proof. The first statement follows from the above lemma and Theorem 10.3. The second statement follows from $\mathbf{1}_n \in \mathrm{Null}(L_G)$. □

10.2 Nullity of Laplacian matrix

Theorem 10.6. *Let $G = (V, E)$ be a simple graph such that $|V| = n$. The nullity of L_G is equal to the number of connected components of G.*

Proof. For each connected component G_i of G, we attach the vector $\mathbf{v}_i$ in $\mathbb{R}^n$ whose kth entry is

$$(\mathbf{v}_i)_k = \begin{cases} 1 & \text{if vertex } v_k \text{ is in } G_i \\ 0 & \text{otherwise.} \end{cases}$$

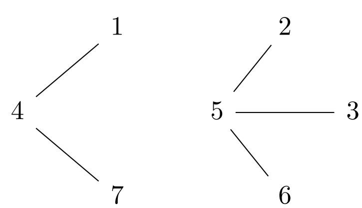
Then we claim that

1. vectors $\mathbf{v}_i$ span $\mathrm{Null}(L_G)$,

2. vectors $\mathbf{v}_i$ are linearly independent,

and thus they form a basis for the null space. $\square$

As an illustration of the proof, let us consider G with two connected components G_1 and G_2.



Then, the following vectors form a basis of $\mathrm{Null}(L_G)$

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

and thus the nullity of L_G is equal to the number of connected components in G.

10.3 The second smallest eigenvalue

Unlike the case of adjacency matrices, we list the eigenvalues of the Laplacian matrix L_G in weakly increasing order. Then, from Proposition 10.5, we know that $\lambda_1 = 0$.

$$0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$$

Since we already know that $\mathbf{1}_n$ is an eigenvector corresponding to $\lambda_1 = 0$ and thus an element in $\mathrm{Null}(L_G)$, if the nullity of L_G is strictly bigger than 1 then it means that there is a vector $\mathbf{v}$ linearly independent with $\mathbf{1}_n$ such that

$$L_G\mathbf{v} = \mathbf{0} = 0\mathbf{v},$$

which shows that $\mathbf{v}$ is an eigenvector of L_G corresponding to $\lambda_2 = 0$. With this discussion, it is easy to show the following result.

Proposition 10.7. *Let G be a simple graph and L_G be its Laplacian matrix. Then the following are equivalent.*

1. $\mathrm{Null}(L_G) > 1$.
2. *G has more than one connected components.*
3. *G is disconnected.*
4. $\lambda_2 = 0$.

10.4 Algebraic connectivity

From Proposition 10.7, we know that λ_2, the second smallest eigenvalue of L_G, gives some information about the connectivity of G.

Definition 10.8. *Let G be a simple graph. The second smallest eigenvalue of L_G, denoted by $\lambda_2(G)$, is called the algebraic connectivity of G.*

Lemma 10.9. *Let G be a simple graph with n vertices and*

$$S_G = \{(a,b) : v_a \text{ and } v_b \text{ are adjacent in } G \text{ and } a \le b\}.$$

Then,

$$\mathbf{x}^T L_G \mathbf{x} = \sum_{(a,b)\in S_G} (x_a - x_b)^2 \qquad \text{for all} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n.$$

Proof. Note that

$$\mathbf{x}^T L_G \mathbf{x} = (B_G^0 \mathbf{x}) \cdot (B_G^0 \mathbf{x}) = \sum_{(a,b)\in S} (-x_a + x_b)^2$$

□

Proposition 10.10. *Let G be a simple graph with n vertices, and $0 = \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ be the eigenvalues of L_G. Then,*

$$\mathbf{x}^T L_G \mathbf{x} \ge \lambda_2 \qquad \text{for any unit vector } \mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \text{ such that } \sum_{i=1}^n x_i = 0.$$

The equality holds for a unit eigenvector $\mathbf{x}$ corresponding to λ_2.

Proof. Since L_G is symmetric, it is orthogonally diagonalizable. Let us choose an orthonormal eigenbasis

$$\mathcal{B} = \{\mathbf{u}_1, \dots, \mathbf{u}_n\}$$

where $\mathbf{u}_i$ is a unit eigenvector corresponding to λ_i and $\mathbf{u}_1 = (1/\sqrt{n})\mathbf{1}$.

Let $\mathbf{x}$ be a unit vector such that $\sum_i x_i = 0$. Then,

$$\mathbf{x} \cdot \mathbf{u}_1 = \mathbf{x} \cdot \frac{1}{\sqrt{n}}\,\mathbf{1} = \frac{1}{\sqrt{n}} \sum_{i=1}^n x_i = 0.$$

On the other hand, if we express $\mathbf{x}$ as a linear combination $\mathbf{x} = \sum_{i=1}^n a_i \mathbf{u}_i$ for some $a_i \in \mathbb{R}$, then

$$\mathbf{x} \cdot \mathbf{u}_1 = \left(\sum_{i=1}^n a_i \mathbf{u}_i \right) \cdot \mathbf{u}_1 = a_1 \mathbf{u}_1 \cdot \mathbf{u}_1 = a_1.$$

Therefore, $a_1 = 0$ and we can write $\mathbf{x} = \sum_{i=2}^n a_i \mathbf{u}_i$.

Next, from $\|\mathbf{x}\|^2 = 1$, we have

$$\left(\sum_{i=2}^{n} a_i u_i\right) \cdot \left(\sum_{i=2}^{n} a_i u_i\right) = \sum_{i=2}^{n} {a_i}^2 = 1$$

and then we have

$$\begin{aligned}
\mathbf{x}^T L_G \mathbf{x} &= \left(\sum_{i=2}^{n} a_i \mathbf{u}_i\right)^T L_G \left(\sum_{j=2}^{n} a_j u_j\right) = \left(\sum_{i=2}^{n} a_i \mathbf{u}_i\right)^T \left(\sum_{j=2}^{n} a_j \lambda_j u_j\right) \\
&= \left(\sum_{i=2}^{n} a_i \mathbf{u}_i\right) \cdot \left(\sum_{j=2}^{n} a_j \lambda_j \mathbf{u}_j\right) = \sum_{i=2}^{n} a_i^2 \lambda_i \\
&= \lambda_2 \left(\sum_{i=2}^{n} a_i^2\right) + \sum_{i=3}^{n} {a_i}^2 (\lambda_i - \lambda_2) = \lambda_2 + \sum_{i=3}^{n} {a_i}^2 (\lambda_i - \lambda_2) \geq \lambda_2.
\end{aligned}$$

Lastly, if $\mathbf{x}$ is a unit eigenvector of L_G corresponding to λ_2, then, since $L_G\mathbf{x} = \lambda_2 \mathbf{x}$ and $\|x\|^2 = 1$. we have

$$\mathbf{x}^T L_G \mathbf{x} = \mathbf{x}^T \lambda_2 \mathbf{x} = \lambda_2 \|x\|^2 = \lambda_2.$$

□

Theorem 10.11. *Let G be a simple graph, and let $\tilde{G}$ be a simple graph obtained from G by adding a new edge. Then,*

$$\lambda_2(G) \leq \lambda_2(\tilde{G}) \leq \lambda_2(G) + 2.$$

Proof. Suppose that $\tilde{G}$ is obtained from G by adding the edge incident with v_r and v_s. Let $\mathbf{x}$ be a unit eigenvector of $L_{\tilde{G}}$ corresponding to $\lambda_2(\tilde{G})$. Furthermore, we can assume that $\mathbf{x}$ is orthogonal to the eigenvector $\mathbf{1}_n$ corresponding to eigenvalue $\lambda_1 = 0$. Let us write x_k for the kth entry of $\mathbf{x}$. Then, using the identity in Proposition 10.10, we have

$$\begin{aligned}
\lambda_2(\tilde{G}) = \mathbf{x}^T L_{\tilde{G}} \mathbf{x} &= \sum_{(a,b) \in S_{\tilde{G}}} (x_a - x_b)^2 \\
&= \sum_{(a,b) \in S_G} (x_a - x_b)^2 + (x_r - x_s)^2 \\
&\geq \sum_{(a,b) \in S_G} (x_a - x_b)^2 = \mathbf{x}^T L_G \mathbf{x}
\end{aligned}$$

where the last equality is by Lemma 10.9. Since $\mathbf{x} \cdot \mathbf{1}_n = 0$ and thus $\sum_i x_i = 0$, using the inequality in Proposition 10.10, we have

$$\sum_{(a,b)\in S_G} (x_a - x_b)^2 = \mathbf{x}^T L_G \mathbf{x} \geq \lambda_2(G).$$

Therefore, we obtain the first inequality.

Next, let $\mathbf{y}$ be a unit eigenvector of L_G corresponding to $\lambda_2(G)$ and $\mathbf{y} \cdot \mathbf{1} = 0$. Let us write y_k for the kth entry of $\mathbf{y}$. Then, by the inequality in Proposition 10.10,

$$\begin{aligned}\lambda_2(\tilde{G}) \leq \mathbf{y}^T L_{\tilde{G}} \mathbf{y} = \sum_{(a,b)\in S_{\tilde{G}}} (y_a - y_b)^2 &= \sum_{(a,b)\in S_G} (y_a - y_b)^2 + (y_r - y_s)^2 \\ &= \lambda_2(G) + y_r^2 - 2y_r y_s + y_s^2 \leq \lambda_2(G) + (y_r^2 + y_s^2) + |2y_r y_s|\end{aligned}$$

Here, we used Lemma 10.9 for the first equality and the equality in Proposition 10.10 for the third equality. Since $1 = \|y\|^2 \geq y_r^2 + y_s^2 \geq |2y_r y_s|$, we finally have

$$\lambda_2(\tilde{G}) \leq \lambda_2(G) + (y_r^2 + y_s^2) + |2y_r y_s| \leq \lambda_2(G) + 2.$$

□