Chapter 5

색칠공부 Graph coloring

5.1 Vertex coloring

Definition 5.1. *A graph coloring (or proper vertex coloring) of $G = (V, E)$ is a graph labeling $f : V \to \{colors\}$ such that $f(x) \neq f(y)$ for all pairs of adjacent vertices x and y.*

$$f(x) \neq f(y) \qquad \text{for all } x \bullet - \bullet \, y$$

Since a vertex with a loop can never be properly colored, it is understood that graphs in this context are loopless. One can further assume that it is simple and connected.

그래프 색칠의 동기가 될만한 문제를 살펴봅시다. 지도 색칠 문제 (map coloring problem)는 지도 위의 나라들을 서로 구분되게 색칠하여 지도를 완성하기 위한 필요한 색깔의 최소 개수를 묻는 문제입니다. 인접한 두 나라는 다른 색깔로 칠해야해야 구분이 된다는 점에 주목해봅시다. 각 나라를 점으로, 두 점이 인접한 두 나라를 의미할때는 서로 선분으로 이어서 지도에 대응하는 평면그래프를 하나 얻을 수 있고, 이 맥락에서 지도 색칠 문제는 평면그래프의 색칠 문제가 됩니다.

Definition 5.2.

1. *A coloring using at most k colors is called a (proper) k-coloring.*
2. *The chromatic number $\chi(G)$ is the smallest number of colors needed to color a graph G. A graph that can be assigned a (proper) k-coloring is k-colorable, and it is k-chromatic, if its chromatic number is exactly k.*

A subset of vertices assigned to the same color is called a color class. Every such class forms an independent set (a set of vertices in a graph, no two of which are adjacent). Therefore, a k-coloring is the same as a partition of the vertex set into k independent sets, and the terms k-partite and k-colorable have the same meaning.

Example 5.3. *See Anderson §5.4.*

1. $\chi(K_n) = n$.
2. $\chi(C_n) = 2$ *if* n *even;* 3 *if* n *odd.*
3. $\chi(G) \leq \Delta + 1$. *(using greedy algorithm)*
4. $\chi(G) = 2$ *if and only if* G *is bipartite.*

Theorem 5.4. *Let* $G = (V, E)$ *be a simple planar graph. The average vertex degree is strictly less than* 6. *Also,* $\delta(G) \leq 5$, *i.e., for all* $v \in V$,

$$deg(v) \leq 5.$$

Proof. If $|E| < 2$ then it is clear. Assume that $|E| \geq 2$.

$$\delta \leq \text{(average of degrees)} = \frac{\sum_v deg(v)}{|V|} = \frac{2|E|}{|V|} \leq \frac{2(3|V|-6)}{|V|} = 6 - \frac{12}{|V|} < 6.$$

□

Theorem 5.5. *Every plane graph has a* 6*-coloring.*

Proof. Induction on $|V|$. Assume that G with $|V| = n$ is 6-colorable. Now for G with $|V| = n+1$, there is a vertex v with $deg(v) \leq 5$. We know that $G - v$ is 6-colorable by the induction hypothesis. Now we can assign to v a color already used for $G - v$. □

Theorem 5.6. *Every plane graph has a* 5*-coloring.*

Proof. See the proof of Theorem 2 in "An Update on the Four-Color Theorem" by Robin Thomas, Notices of the AMS, Vol 45, Number 7 (1998).

□

Example 5.7.

1. 꼭짓점의 수가 6인 평면 그래프 G에 대해 $\chi(G) \leq 4$임을 보여보자.
2. 꼭짓점의 수가 5인 평면 그래프 G에 대해 $\chi(G) \leq 4$임을 보여보자.

Theorem 5.8 (The Four Color Theorem)**.** *Every plane graph has a* 4*-coloring.*

The four color theorem has some equivalent formuations. See the article "An Update on the Four Color Theorem" mentioned above.

9				8		3		
			2	5		7		
	2		3					4
	9	4						
			7	3		5	6	
7		5		6		4		
		7	8		3	9		
		1						3
3								2

9	7	6	4	8	1	3	2	5
1	4	3	2	5	9	7	8	6
5	2	8	3	7	6	1	9	4
6	9	4	5	1	8	2	3	7
8	1	2	7	3	4	5	6	9
7	3	5	9	6	2	4	1	8
4	6	7	8	2	3	9	5	1
2	5	1	6	9	7	8	4	3
3	8	9	1	4	5	6	7	2

Figure 5.1: A sudoku and its solution

2	1	4	3
4	3	2	1
3	4	1	2
1	2	3	4

Figure 5.2: A shidoku solution

5.2 Representations of Sudoku

(스도쿠에 관한 설명)

5.2.1 Sudoku equations

Let us focus on shidoku.

16 variables and 40 equations

1. Each variable can take only $1, 2, 3$, or 4. We have 16 equations of the form

$$(x-1)(x-2)(x-3)(x-4) = 0$$

2. Four variables x, y, z, w in the same row or the same column or the same subblock take distinct values. This fact can be encoded

$$w + x + y = 10 \quad \text{and} \quad wxyz = 24.$$

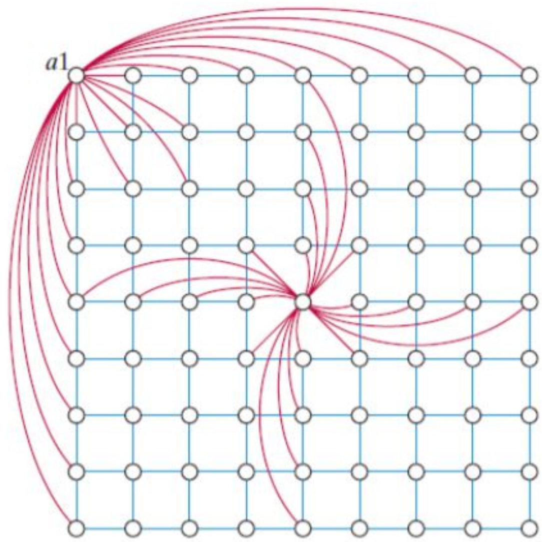

Figure 5.3: The sudoku graph

We have 12×2 more equations.

16 variables and 72 equations

1. This time, instead of $1, 2, 3, 4$, use $1, -1, i, -i$: each variable satisfies
$$x^4 = 1.$$

2. Then for each pair in the same row or the same column or the same subblock, $x \neq y$ can be obtained from $x^4 - y^4 = 0$ and thus
$$\frac{x^4 - y^4}{x - y} = (x + y)(x^2 + y^2) = 0.$$
How many such equations? The answer is 56 (see the graph coloring interpretation).

5.2.2 Sudoku graph

From Wikipedia: Sudoku graph. "The Sudoku graph is an undirected graph whose vertices represent the cells of a (blank) Sudoku puzzle and whose edges represent pairs of cells that belong to the same row, column, or block of the puzzle. The problem of solving a Sudoku puzzle can be represented as precoloring extension on this graph."

For Shidoku, the corresponding graph have 16 vertices and it is 7-regular, i.e., $deg(v) = 7$ for

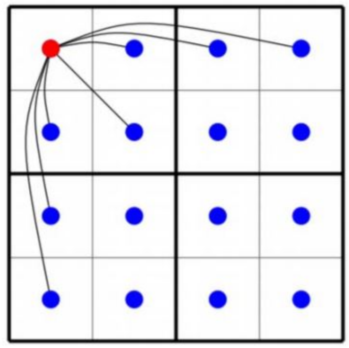

Figure 5.4: The shidoku graph

all v. From the degree-sum formula

$$2e = \sum_v deg(v)$$

we have $e = 56$. Is it planar? No, because its average vertex-degree is 7.

5.3 Edge coloring

앞서 공부한 그래프의 색칠은 정확히는 꼭짓점 색칠이었습니다. 이제 선분 색칠을 공부해봅시다.

Definition 5.9. 1. *An <u>edge coloring</u> of a graph is a proper edge labeling* $g : E \longrightarrow \{colors\}$, *if* $g(e) \neq g(e')$ *for all pairs of edges* e *and* e' *incident with the same vertex.*

$$g(e) \neq g(e') \qquad for \;\; e - \bullet - e'$$

2. *The smallest number of colors needed for an edge coloring of a graph* G *is the <u>chromatic index</u>, or edge chromatic number, denoted* $\chi'(G)$.

It is easy to see that $\chi'(G) \geq \Delta$, because $\chi'(G) \geq deg(v)$ for all $v \in V$. In fact, we have

Theorem 5.10 (Vizing's theorem)**.**

$$\chi'(G) = \Delta \;\; or \;\; \Delta + 1.$$

Theorem 5.11.

$$\chi'(K_n) = \begin{cases} n-1 \quad (= \Delta) & \textit{if } n \textit{ is even} \\ n \qquad (= \Delta + 1) & \textit{if } n \textit{ is odd} \end{cases}$$

Proof. From Anderson p100. "If n is odd, any matching in K_n can have at most $(n-1)$ edges. So at most $(n-1)/2$ edges can be given anyone colour. But there are $n(n-1)/2$ edges in K_n, so at least n colours are needed. Now we can colour the edges using n colours in the following way. Represent K_n as a regular n-gon, with all diagonals drawn. Colour the boundary edges by 1, ... , n; then colour each diagonal by the colour of the boundary edge parallel with it. This gives an edge colouring using n colours.

Now suppose n is even. Certainly $\chi'(K_n) \geq n-1$; we show how to use only $n-1$ colours. Since $n-1$ is odd, we can colour K_{n-l} using $n-1$ colours, as described above. Now take another vertex v and join each vertex of K_{n-1} to v, thus obtaining K_n . At each vertex of K_{n-1}, one colour has not been used. The colours missing at each vertex of K_{n-1} are all different, so we can use these $n-1$ colours to colour the added edges at v. This gives an edge colouring of K_n using $n-1$ colours." □

Theorem 5.12. *$\chi'(G) = \Delta$ for all bipartite graphs.*

Proof. From Anderson, p101. "Proceed by induction on q, the number of edges. The theorem is clearly true for graphs with $q = 1$; so suppose it is true for all bipartite graphs with k edges, and consider a bipartite graph G with maximum vertex degree Δ and with $k+1$ edges. Choose any edge vw of G, and remove it , thereby forming a new bipartite graph H. H has k edges and maximum vertex degree not more than Δ, so, by the induction hypothesis, H can be edge coloured using at most Δ colours. Now, in H, v and w both have degree not more than $\Delta - 1$, so there is at least one colour missing from the edges from v, and at least one missing from the edges of w. If there is a colour missing at both vertices then it can be used to colour edge vw. If there is no colour missing from both, then let C_1 be a colour missing at v, and C_2 a colour missing at w. Now there is some edge, say vu, coloured C_2; if there is an edge coloured C_1 from u, go along it, and continue along edges coloured C_1 and C_2 alternately as far as possible. The path so constructed will never reach w since if it did it would have to reach w along an edge coloured C_1 and so would be a path of even length, giving, with edge vw, an odd cycle in a bipartite graph. So the connected subgraph K, consisting of vertex v and all vertices and edges of H which can be reached by a path of edges coloured C_1 and C_2, does not contain w. So we can interchange the colours C_1 and C_2 in K without interfering with the colours in the rest of H. This gives a new edge colouring of H in which v and w have no edge coloured C_2, and we can use C_2 to colour vw." □

Chapter 6

클릭·독립집합·부합

6.1 Clique and independent set

Definition 6.1. 1. *A clique is a subset of the vertices such that every two distinct vertices are adjacent.*

2. *The clique number of a graph G, denoted by $\omega(G)$, is the number of vertices in a maximum clique of G. Equivalently, it is the size of a largest clique of G.*

A maximal clique is a clique that cannot be extended by including one more adjacent vertex, that is, a clique which does not exist exclusively within the vertex set of a larger clique. A maximum clique of a graph G is a clique such that there is no clique with more vertices. The clique cover problem concerns finding as few cliques as possible that include every vertex in the graph.

Theorem 6.2. 1. *Vertices in a clique cannot receive the same color. Thus,*

$$\omega(G) \;\leq\; \chi(G) \;\leq\; \Delta + 1.$$

Proof. □

Definition 6.3. 1. *An independent set (also called coclique or anticlique) is a set of vertices in a graph, no two of which are adjacent.*

2. *A maximum independent set is an independent set of largest possible size for a given graph G. This size is called the independence number of G, and denoted $\alpha(G)$.*

이 정의에 따르면, 그래프 G의 vertex coloring은 꼭짓점 집합 V를 독립집합으로 분할하는 것이 됩니다.

A maximal independent set is either an independent set such that adding any other vertex to the set forces the set to contain an edge or the set of all vertices with no edges. A maximum independent set is an independent set of largest possible size for a given graph G.

Theorem 6.4. *For a simple graph G with $|V| = n$,*

$$n \;\le\; \alpha(G)\,\chi(G) \;\le\; \alpha(G)\,(\Delta + 1).$$

Proof. Note that the number of vertices which can have the same color cannot be more than $\alpha(G)$.

$$\begin{aligned} n \;&=\; \sum_{i=1}^{\chi(G)} (\text{the number of vertices with the } i\text{th color}) \\ &\le\; \sum_{i=1}^{\chi(G)} \alpha(G) \;=\; \chi(G)\alpha(G). \end{aligned}$$

□

6.2 Clique number and independence number

Recall that for a simple graph the complement of G is the graph defined by

$$V(\bar{G}) = V(G) \quad \text{and} \quad E(\bar{G}) = \{xy : xy \notin E(G)\}.$$

For a simple graph G, note that a subset of V is a clique of G if and only if it is an independent set of $\bar{G}$. We have

$$\omega(G) = \alpha(\bar{G}) \quad \text{and} \quad \alpha(G) = \omega(\bar{G}).$$

Theorem 6.5 (Ramsey's theorem)**.** *Let G be a simple graph with at least six vertices. Then*

$$\omega(G) \ge 3 \quad \textit{or} \quad \alpha(G) \ge 3.$$

Proof. See Wikipedia: Ramsey's theorem, Example $R(3, 3) = 6$. □

Theorem 6.6. *For an arbitrary graph G with $|V| = n$, let d_i be the degree of graph vertex i.*

1. *Caro-Wei bound:*
$$\sum_{i=1}^{n} \frac{1}{d_i + 1} \;\le\; \alpha(G).$$

2. *Dual version of Caro-Wei bound*
$$\sum_{i=1}^{n} \frac{1}{n - d_i} \;\le\; \omega(G).$$

Proof. cf. The probabilistic method □

6.3 Matching and Hall's theorem

Definition 6.7.
1. *An independent edge set (also called a matching) of a graph G is a subset of the edges such that no two edges in the subset share a vertex in G.*
2. *A maximum independent edge set an independent edge set containing the largest possible number of edges among all independent edge sets for a given graph.*
3. *The size of a maximum independent edge set is known as the matching number or edge independence number, and denoted $\nu(G)$.*

그래프의 선분 채색edge coloring을 생각해보면, 동일한 색을 갖는 선분들은 점을 공유할 수 없으므로 matching을 구성하게 됩니다. 따라서, 그래프 $G = (V, E)$의 edge coloring은 선분 집합 E를 matching들로 분할한 것으로 이해할 수 있습니다.

Theorem 6.8. *For a simple graph G with $|E| = m$,*
$$m \le \nu(G)\,\chi'(G).$$

Proof. Note that the number of edges which can have the same color cannot be more than $\nu(G)$.

$$\begin{aligned} m &= \sum_{i=1}^{\chi'} (\text{the number of edges with the } i\text{th color}) \\ &\leq \sum_{i=1}^{\chi'} \nu(G) = \chi'(G)\nu(G). \end{aligned}$$

□

6.4 Chromatic polynomial

The chromatic polynomial counts the number of ways a graph can be colored using no more than a given number of colors. It is a function $P(G,t)$ that counts the number of t-colorings of G. For a given G the function is indeed a polynomial in t. A graph G is k-colorable if and only if $\chi(G) \leq k$ if and only if $P(G,k) > 0$. In particular,

$$\chi(G) = \min\{k : P(G,k) > 0\}.$$

Example 6.9. 1. *If G has n vertices with no edges, then $P(G,t) = t^n$.*

2. *For paths, $P(P_2,t) = t(t-1)$, $P(P_3,t) = t(t-1)^2$, and*

$$P(P_n,t) = t(t-1)^{n-1}.$$

3. *If $G = K_n$ then*

$$P(G,t) = t(t-1)\cdots(t-n+1).$$

Theorem 6.10. *1. If G is a disconnected graph with connected components $H_1, H_2, ..., H_c$ then*

$$P(G,t) = P(H_1,t)P(H_2,t)\cdots P(H_c,t).$$

2. For two non-adjacent vetices x and y,

$$P(G,t) = P(G_1,t) + P(G_2,t) \quad or \quad P(G_1,t) = P(G,t) - P(G_2,t)$$

where G_1 is obtained by adding an edge xy; G_2 is obtained by identifying x and y (contraction).

(figure 1) (figure 2)

Example 6.11. *1. Compute $P(G,t)$ for the graphs in Figure...*

2. $P(G,t)$ in the movie, "Good Will Hunting";

Theorem 6.12. *The chromatic polynomial of a tree T_n with n vertices is*

$$P(T_n,t) = t(t-1)^{n-1}.$$

Proof. Induction on n. It is clear that $P(T_1,t) = t = t(t-1)^{1-1}$. Assuming that the statement is true for T_n, let us consider T_{n+1}. Choose an edge e incident with an end vertex in T_{n+1}. Then,

1. $T_{n+1} - e$ is a tree with one isolated vertex and

2. T_{n+1}/e is a tree with n vertices.

Thus, □

Theorem 6.13. *The chromatic polynomial of a n-cycle, C_n, with $n \geq 3$ is*

$$P(C_n,t) = (t-1)^n + (-1)^n(t-1).$$

Proof. Let us us an induction on n. First, when $n = 3$,

$$\begin{aligned} P(C_3,t) =& P(C_3 - e, t) - P(C_3/e, t) \\ =& t(t-1)^2 - t(t-1) = (t-1)^3 - (t-1). \end{aligned}$$

Now assuming that the statement is true for C_n and consider C_{n+1}. □

Theorem 6.14 (Read conjecture 1968, June Huh 2012). *The coefficients of the chromatic polynomial of every graph is log-concave.*

See "The Work of June Huh" by Gil Kalai, ICM 2022.

cf. Matching polynomial, Tutte polynomial.