1204 Class Activity

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Let n be a positive integer. Let S_n be the set consisting of sequences $[i_1, i_2, \cdots, i_k]$ of positive integers such that $0 < i_1 < i_2 < \cdots < i_k < n+1$ (thus the length k of a sequence in this set should be between 1 and n).
On this set S_n, we impose the following order (you can assume that it is indeed an order relation on S_n):
$[i_1, i_2, \cdots, i_p]$ is less than or equal to $[j_1, j_2, \cdots, j_q]$, if
(1) p is less than or equal to q, and
(2) i_k is less than or equal to j_k for all k between 1 and q.

Question: Prove that for any positive integer n, the poset S_n is a distributive lattice.

Proof. First, we want to show that S_n is a lattice.
So, take any two elements in S_n, call it I and J. Let $I = [i_1, i_2, \cdots, i_p]$ and $J = [j_1, j_2, \cdots, j_q]$ and let $p \leq q$.
Consider
$$A = [\max(i_1, j_1), \max(i_2, j_2), \cdots, \max(i_p, j_p), j_{p+1}, \cdots, j_q]$$
and
$$B = [\min(i_1, j_1), \min(i_2, j_2), \cdots, \min(i_p, j_p)]$$

Recall the definition of the join and the meet. It is obvious that A is larger than or equal to I and J, B is less than or equal to I and J
Consider $C, D \in S_n$ such that C is larger than or equal to A and B, D is less than or equal to A and B. Then, the length of C is obviously larger than or equal to the length of A. Also, all integers constitute C are larger than or equal to the integers constitute A. Similarly, the length of D is less than or equal to B. Also, all integers constitute D are less than or equal to the integers constitute B. Therefore, they are the join and the meet of I and J. Uniqueness is obvious. Hence, we showed that S_n is a lattice.
Second, we want to show that S_n has a distributive property. i.e. $I \vee (J \wedge K) = (I \vee J) \wedge (I \vee K)$.
Let I and J are the same elements we defined above, and $K = [k_1, k_2, \cdots, k_r]$ where $p \leq q \leq r$.
Compute the left-hand side.
$$I \vee (J \wedge K) = [i_1, \cdots, i_p] \vee [\min(j_1, k_1), \cdots, \min(j_q, k_q)]$$
$$= [\max(i_1, \min(j_1, k_1)), \cdots, \max(i_p, \min(j_p, k_p)), \cdots, \min(j_q, k_q)]$$

Compute the right-hand side.

$$(I \vee J) \wedge (I \vee K)$$

$$= [\max(i_1, j_1), \cdots, \max(i_p, j_p), \cdots, j_q] \wedge [\max(i_1, k_1), \cdots, \max(i_p, k_p), \cdots, k_r]$$

$$= [\min(\max(i_1, j_1), \max(i_1, k_1)), \cdots, \min(\max(i_p, j_p), \max(i_p, k_p)), \cdots, \min(j_q, k_q)]$$

So, we want to show that for any i, j, k in positive integers, $\max(i, \min(j, k)) = \min(\max(i, j), \max(i, k))$.
If i is maximal element among i, j, k, then,

$$\max(i, \min(j, k)) = i$$

$$\min(\max(i, j), \max(i, k)) = \min(i, i) = i$$

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If j is maximal element among i, j, k, then,

$$\max(i, \min(j, k)) = \max(i, k)$$

$$\min(\max(i, j), \max(i, k)) = \min(j, \max(i, k)) = \max(i, k)$$

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If k is maximal element among i, j, k, then,

$$\max(i, \min(j, k)) = \max(i, j)$$

$$\min(\max(i, j), \max(i, k)) = \min(\max(i, j), k) = \max(i, j)$$

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Therefore, distributive property works well in this lattice. □