Chapter 5

Posets

SECTION 5.1 Order Relation

1. Definition

Let's recall the defintiton of relation on a set. For a set X, a relation $\mathscr{R}$ on X is a subset of $X \times X$. For $(a, b) \in \mathscr{R}$ we write $a \sim_{\mathscr{R}} b$ and say "a is related to b."

In this section, we will focus on a relation called *order relation*.

Definition 5.1 *A relation $\mathscr{R}$ on X is an order relation if $\mathscr{R}$ satisfies the following:*

① ***reflexivity*** *for all $x \in X$, $((x, x) \in \mathscr{R})$,*

② ***antisymmetry*** *if $(x, y) \in \mathscr{R}$ and $(y, x) \in \mathscr{R}$, then $x = y$,*

③ ***transitivity*** *if $(x, y) \in \mathscr{R}$ and $(y, z) \in \mathscr{R}$, then $(x, z) \in \mathscr{R}$.*

This is an abstract version of inequality. Particularly, $a \leq_{\mathscr{R}} b$ denotes $a \sim_{\mathscr{R}} b$ for order relation $\mathscr{R}$. If there is no confusion, we simply write $a \leq b$ for $a \leq_{\mathscr{R}} b$. Then, the condition of Definition 5.1 could be rewrite as following:

① **reflexivity** for all $x \in X$, $(x \leq x)$,

② **antisymmetry** if $x \leq y$ and $y \leq x$ then $x = y$,

③ **transitivity** if $x \leq y$ and $y \leq z$ then $x \leq z$.

With this notation, we will also accept the notation of usual inequality of standard number systems.

Notation *① $b \geq a$ is often written for $a \leq b$.* *② $a < b$ means $a \leq b$ and $a \neq b$.*

Let's observe the set with an order relation.

Definition 5.2 *A set X with an order relation $\leq$ is called a* ***partially ordered set****. Simply, we also call it a poset. We will write $\langle X, \leq \rangle$ for a poset X.*

The term, partially ordered set, is origined from existence of elements that do not have relation between themselves. The definition of order relation does not guarantee that every element is always related to another element.

Definition 5.3 *Let X be a poset with order relation $\leq$. For $a, b \in X$, a and b are comparable if $a \leq b$ or $b \leq a$. Otherwise, we say that those are incomparable.*

For a poset X, X is a totally ordered set if a and b are comparable for all $a, b \in X$. Of course, real number system we usually use is a totally ordered set.

Example 5.1 Let $X = \mathbb{R}$. With the order relation of usual inequality, all $a, b \in X$ are comparable. In other words, there do not exist incomparable a, b in X.

2. Examples

Let's look at some other poset with some incomparable elements.

Definition 5.4 (Boolean Poset) *The Boolean poset $\mathcal{B}_n (= \mathscr{P}(\{1, 2, \ldots, n\}))$ is the collection of all subsets of $\{1, 2, \ldots, n\}$ with the order relation $\leq$ such that $S \leq T$ if and only if $S \subseteq T$ for $S, T \in \mathcal{B}_n$.*

First let's check that this relation (induced fron set inclusion) is an order relation.

① **reflexivity** $S \leq S$ since $S \subseteq S$ for all $S \in \mathcal{B}_n$.

② **antisymmetry** $S \leq T$ and $T \leq S$ means that $S \subseteq T$ and $T \subseteq S$. By elementary set theory, $S \leq T$ and $T \leq S$ implies $S = T$.

③ **transivity** $S \leq T$ and $T \leq R$ means that $S \subseteq T$ and $T \subseteq R$. By the definition of $\subseteq$, it implies

$$x \in S \implies x \in T \qquad \text{and} \qquad x \in T \implies x \in R.$$

Thus, it implies $x \in S \implies x \in R$ which means $S \subseteq R$, in other words, $S \leq R$.

Hence, $\leq$ induced by $\subseteq$ is an order relation. However, there exists incomparable elements in $\mathcal{B}_n$ for $n \geq 2$. For example, consider subsets A, B of $\mathcal{B}_n$ such that

$$A = \{1\} \qquad \text{and} \qquad B = \{2\}.$$

Let's look at the other examples of posets. Before we look at the other example, recall the equivalence relation on $\mathbb{Z}$ with the concept of $a \mid b$. In addition, we will also use $a \mid b$ to construct an order relation on $\mathbb{Z}^+$.

Example 5.2 For a set $X = \{1, 2, \ldots, n\}$, give an order relation such that $a \leq b$ if and only if $a \mid b$. Then, with this order relation, we write $D_n = \langle X, \leq \rangle$.

Let's show that this relation is an order relation.

① **reflexivity** $a \mid a$ since $a = 1 \cdot a$ for all $a \in X$. Thus, $a \leq a$.

② **antisymmetry** If $a \mid b$ and $b \mid a$, then $b = ap$, $a = bq$ for some $p, q \in \mathbb{Z}$. This implies

$$b = ap = bpq \implies pq = 1.$$

Since there exists one and only one pair $(p, q) = (1, 1) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ such that $p \cdot q = 1$, $b = ap = a$. Thus, $a \leq b$ and $b \leq a$ implies $a = b$.

③ **transivity** If $a \mid b$ and $b \mid c$, then $b = ap$ and $c = bq$ for some $p, q \in \mathbb{Z}$. Thus, $c = bq = apq$. Since there exists $d = pq \in \mathbb{Z}$ such that $c = ad$, $a \mid c$. Hence, $a \leq b$ and $b \leq c$ implies $a \leq c$.

Note that there exist incomparable elements in D_n for $n \geq 3$. For example, 2 and 3 are incomparable in D_3.

Definition 5.5 (Young's Poset) *Young's poset Y is the set of integer-partitions with $\lambda \leq \mu$ if and only if*

$$\lambda_j \leq \mu_j \quad \text{for all } j \geq 0$$

where $\lambda = (\lambda_1, \ldots, \lambda_k)$ and $\mu = (\mu_1, \ldots, \mu_k, \ldots, \mu_t)$.

We can check that Y is a poset by the similar way with the Boolean poset.

Example 5.3 For each of the following, verify that the given relation is indeed an order relation.

① The set of integers $\{1, 2, ..., n\}$ with the usual number inequality.

② $\mathcal{D}_n$, the set of all positive divisors of n with the relation of divisibility: $a \leq b$ if and only if $a \mid b$.

③ Π_n, the set of all set-partitions of $\{1, 2, ..., n\}$ with $A \leq B$ if and only if every part of A is a subset of a part of B: for each i,

$$A_i \subseteq B_j \quad \text{for some } j \text{ (depending on } i\text{)}$$

where A_i is a part of A and B_j is a part of B.

SECTION 5.2 FUNDAMENTAL CONCEPTS

Let's obverve the concepts about posets. First, we want to know about the mapping that preserves the structure of order relation.

DEFINITION 5.6 *For two posets $\langle P, \leq_1 \rangle$ and $\langle Q, \leq_2 \rangle$, a map*

$$\phi : P \longrightarrow Q$$

is order-preserving, if $x \leq_1 y$ implies that $\phi(x) \leq_2 \phi(y)$ for all $x, y \in P$.

Compare it with gorup homomorphism. Then, let's define the concept of isomorphism between posets.

DEFINITION 5.7 *Two posets are isomorphic if there is an order-preserving bijection ϕ whose inverse is also order-preserving. We call ϕ isomorphism.*

If two posets are isomorphic, we can consider that they have same structures with different notations.

Next, let's define subposets like subgroups or subspaces. However, for posets, there are two versions of subposets.

DEFINITION 5.8 (WEAK) *Q is a weak subposet of a poset P if Q satisfies the followings:*

① Q is a subset of P, ② Q in itself is a poset, ③ $x \leq y$ in Q implies $x \leq y$ in P for all $x, y \in Q$.

DEFINITION 5.9 *When Q is a weak subposet of P and $Q = P$ as a set, we say P is a refinement of Q.*

DEFINITION 5.10 (INDUCED) *Q is an induced subposet of a poset P if Q satisfies the followings:*

① Q is a subset of P, ② Q in itself is a poset, ③ $x \leq y$ in Q if and only if $x \leq y$ in P.

The term "subposet" usually means an induced subposet. Then, let's look at the abstract version of intervals with an order relation.

DEFINITION 5.11 *Let $\langle P, \leq \rangle$ be a poset. For $x \leq y$ in P, the closed interval is $[x, y] = \{u \in P : x \leq u \leq y\}$. Also, for $x \leq y$ in P, the open interval is $(x, y) = \{u \in P : x < u < y\}$.*

With the concept of intervals, we can define the finiteness locally.

DEFINITION 5.12 *A poset P is locally finite if every interval of P contains finitely many elements.*

$\mathbb{Z}$ is the example of a locally finite poset. Note that a locally finite poset is not always a finite poset. P is a finite poset if a poset P contains finitely many elements. Thus, if P is finite, then P is locally finite. For locally finite posets, we can always choose two elements such that there does not exist any elements between them.

1. Visualizing Poset

DEFINITION 5.13 *Let $\langle P, \leq \rangle$ be a poset. We say y covers x if $x, y \in P$ satisfies the followings:*
① $x < y$, ② there is no u such that $x < u < y$.
$x \lessdot y$ denotes y covers x.

When $x \lessdot y$, $[x, y] = \{x, y\}$.

DEFINITION 5.14 *We will define some notions.*

① $x \in P$ is the top (denoted by $\hat{1}$), if $x \geq y$ for all $y \in P$.

② $x \in P$ is maximal, if $x \leq y$ implies that $x = y$.

③ $x \in P$ is the bottom (denoted by $\hat{0}$), if $x \leq y$ for all $y \in P$.

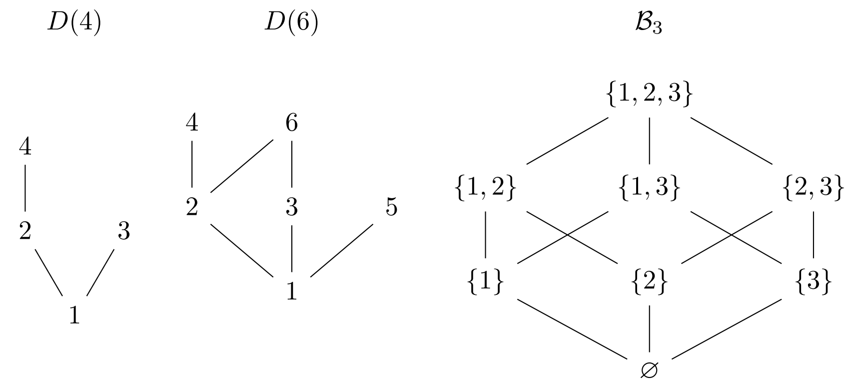
④ $x \in P$ is minimal, if $y \leq x$ implies that $x = y$.

DEFINITION 5.15 *A poset P is bounded, if it has both $\hat{0}$ and $\hat{1}$.*

For locally finite posets, we can visualize the posets using cover relations.

Definition 5.16 *The* <u>*Hasse diagram*</u> *of a poset P is a graph whose vertices are the elements of P and edges drawn upward from x to y whenever y covers x.*

Example 5.4 We will visualize D_4, D_6 and $\mathcal{B}_3$ with the Hasse diagram.



SECTION 5.3 Rank-Generating Functions

1. Chains

Definition 5.17 *Let P be a poset. A subset $C = \{c_0, c_1, ..., c_n\}$ of P is a* <u>*chain*</u> *(or totally/linearly ordered set) of length n, if*

$$c_0 < c_1 < \cdots < c_n$$

in P. A sequence $(c_0, c_1, ..., c_n)$ of P is called a <u>*multichain*</u>*, if*

$$c_0 \leq c_1 \leq \cdots \leq c_n$$

in P.

Definition 5.18 *For a poset P, a chain is* <u>*maximal*</u>*, if it is not contained in a larger chain of P.*

Definition 5.19 *For a poset P, an* <u>*antichain*</u> *of P is a subset S of P such that any two elements in S are incomparable.*

2. Rank-Generating Functions

Definition 5.20 *Let P be a poset. For all $x \in P$, let $C(x)$ be the set of all chains in P ending at x.*

① *P is* <u>*ranked*</u>*, if all maximal chains in $C(x)$ have the same length for all $x \in P$.*

② *For a ranked poset P, the* <u>*rank function*</u> *of P is a map*

$$r : P \longrightarrow \mathbb{Z}_{\geq 0}$$

assigning to x the length $r(x)$ of a maximal chain in $C(x)$.

③ *The* <u>*rank-generating function*</u> *of a ranked poset P is*

$$F(P, x) = \sum_{a \in P} x^{r(a)}.$$

Note that

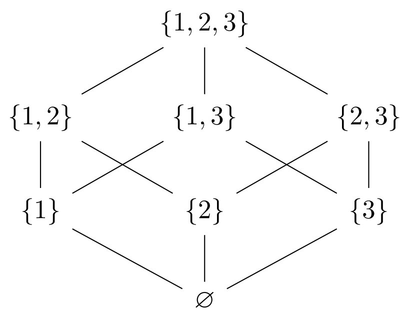
$$r(y) = \begin{cases} 0 & \text{if } y \text{ is a minimal element,} \\ r(x) + 1 & \text{if } x \lessdot y \end{cases}$$

and

$$F(P,x) = \sum_{n \geq 0} c_n x^n$$

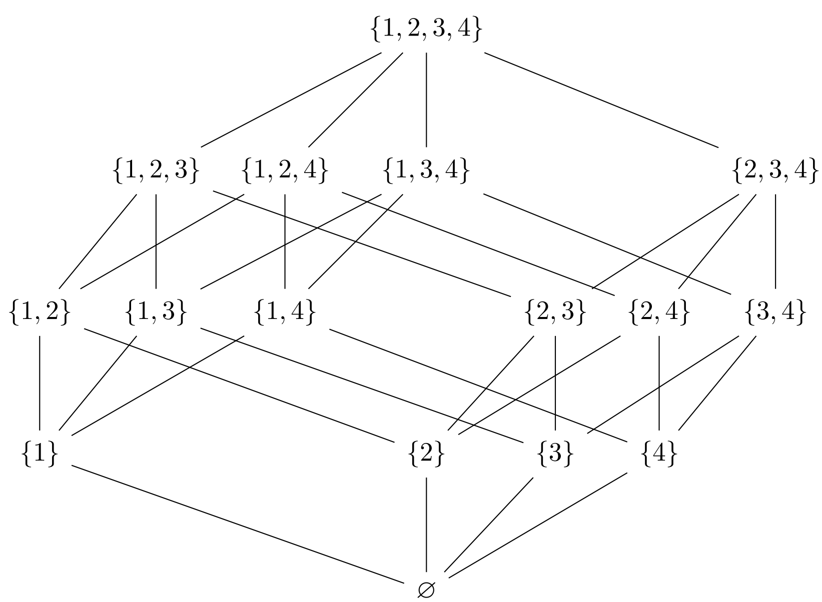
where c_n is the number of elements in P whose rank is n.

Example 5.5 (Boolean Poset) Let's draw the Hasse diagram of $\mathcal{B}_3$ and find the rank-generating function of it.

<table>
<tr><th>The Hasse Diagram of $\mathcal{B}_3$</th><th>The rank-generating function</th></tr>
<tr><td>


</td><td>$F(\mathcal{B}_3, x) = 1 \cdot x^3 + 3 \cdot x^2 + 3 \cdot x^1 + 1 \cdot x^0$</td></tr>
</table>

We can also do the same thing for $\mathcal{B}_4$.

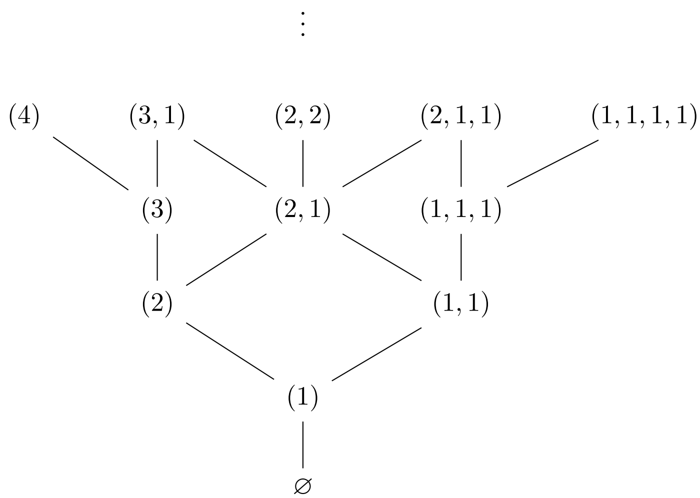
<table>
<tr><th>The Hasse Diagram of $\mathcal{B}_4$</th><th>The rank-generating function</th></tr>
<tr><td>


</td><td>$F(\mathcal{B}_4, x) = 1 \cdot x^4 + 4 \cdot x^3 + 6 \cdot x^2 + 4 \cdot x^1 + 1 \cdot x^0$</td></tr>
</table>

We can determine the rank-generating function of $\mathcal{B}_n$. For $A, B \in \mathcal{B}_n$, $A \lessdot B$ if and only if $(|B - A| = 1)$ and $(A \subset B)$. Thus, for $A \in \mathcal{B}_n$, the rank of A becomes $|A|$ and $\mathcal{B}_n$ is a ranked poset. Since the number of elements with rank r in $\mathcal{B}_n$ is $\binom{n}{r}$, we can write the rank-generating function of $\mathcal{B}_n$ as

$$F(\mathcal{B}_n, x) = \sum_{0 \leq r \leq n} \binom{n}{r} x^r.$$

Example 5.6 (Young's Poset) ❯ Consider Young's poset Y.

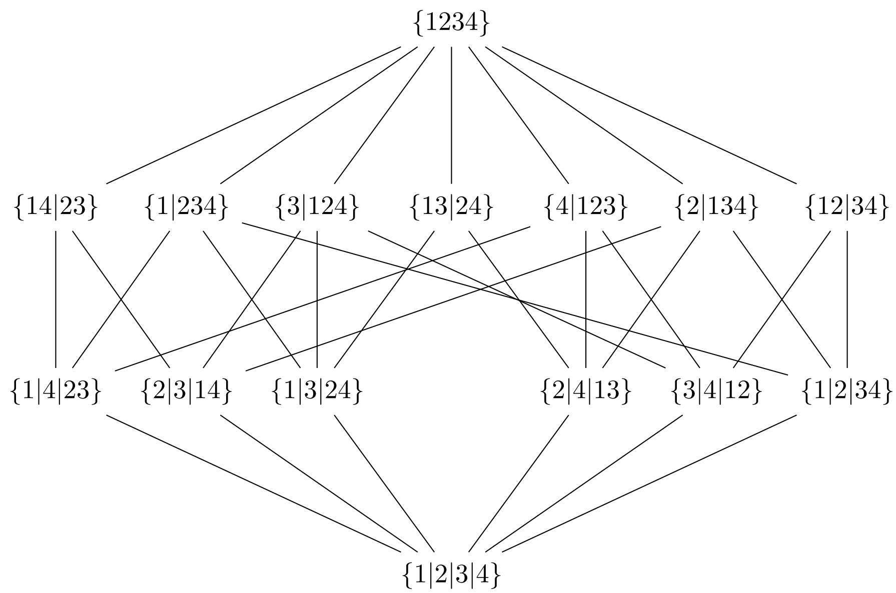
The Hasse Diagram of Y	The rank-generating function
	$F(Y,x) = \begin{array}{c} \vdots \\ 5\cdot x^4 \\ + \\ 3\cdot x^3 \\ + \\ 2\cdot x^2 \\ + \\ 1\cdot x^1 \\ \\ 1\cdot x^0 \end{array}$

Let $\lambda, \mu \in Y$ and let $\lambda \vdash n$ and $\mu \vdash m$. Then, $\lambda \lessdot \mu$ if and only if $\lambda \leq \mu$ and $m = n+1$. Thus, Y becomes a ranked poset and $r(\lambda) = n$ when $\lambda \vdash n$. Since $|\{\lambda \in Y : \lambda \vdash n\}| = p(n)$, the rank-generating function of Y is written as

$$F(Y,x) = \sum_{n \in \mathbb{Z}_{\geq 0}} p(n)\, x^n = \prod_{k \geq 1} \frac{1}{1 - x^k}.$$

We will study the rank-generating functions of the posets in Example 5.3.

Example 5.7 ❯ Consider Π_n in Example 5.3. Let's draw the Hasse diagram of Π_4 and find the rank-generating function of it.

Hasse Diagram of Π_4	generating function
	$F(\Pi_4, x) = 1 \cdot x^3 + 7 \cdot x^2 + 6 \cdot x^1 + 1 \cdot x^0$

We can generalize above results for Π_n. Let $A, B \in \Pi_n$. Then, $A \lessdot B$ if and only if $A_a \cup A_b = B_k$ for any $a \neq b$ and some k where A_i is a part of A and B_j is a part of B. Thus, for $A \in \Pi_n$, the rank of A becomes $|A|$ and Π is a ranked poset. Since the number of elements with rank $n-k$ in Π_n is $S(n,k)$, we can write the rank-generating function of Π_n as

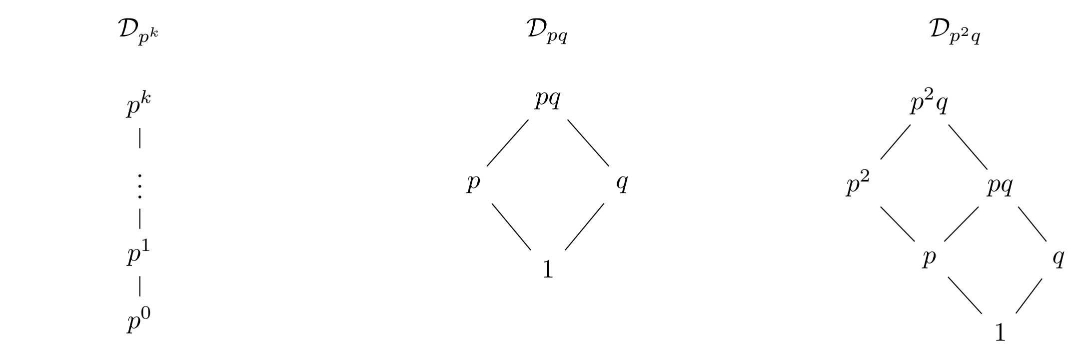
$$F(\Pi_n, x) = \sum_{1 \leq k \leq n} S(n,k) x^{n-k}.$$

Since there exists lot of unwanted edges and elements in D_n, we will change the focus to the subposet of it.

Example 5.8 Consider D_n in Example 5.2. D_n is a ranked poset with the rank function

$$r(m) = \sum_{i=1}^{k} \alpha_i$$

where $m = {p_1}^{\alpha_1} \times \cdots \times {p_k}^{\alpha_k}$ with distinct prime numbers p_i. However, D_n does not have $\hat{1}$ except $n = 1, 2$ and D_n contains incomparable elements for $n \geq 3$. Thus, we will focus on the subposet $\mathcal{D}_n$ of D_n. Then, we can draw the Hasse diagrams of $\mathcal{D}_{p^k}$, $\mathcal{D}_{pq}$ and $\mathcal{D}_{p^2q}$ where p and q are distinct prime numbers as following:



looks like a long chain looks like a lattice

We can easily check that

$$F(\mathcal{D}_{p^k}, x) = 1 + x + \cdots + x^k, \ F(\mathcal{D}_{pq}, x) = 1 + 2x + x^2, \text{ and } F(\mathcal{D}_{p^2q}, x) = 1 + 2x + 2x^2 + x^3.$$

Chapter 6

Lattice

We studied ranked posets such as $\mathcal{B}_n$, Y, D_n and Π_n. We have drawn the Hasse diagrams of these ranked posets and determined their rank-generating functions. In fact, they have some special algebraic structures.

SECTION 6.1 Join and Meet

Definition 6.1 *For x and y in a poset P, the join $x \vee y$ is the least upper bound of x and y; the meet $x \wedge y$ is the greatest lower bound of x and y. That is,*

① *$z = x \vee y$, if*

- *(a) $z \geq x$ and $z \geq y$,*
- *(b) For any $w \in P$ such that $w \geq x$ and $w \geq y$, we have $w \geq z$;*

② *$z = x \wedge y$, if*

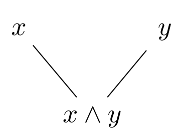
- *(a) $z \leq x$ and $z \leq y$,*
- *(b) For any $w \in P$ such that $w \leq x$ and $w \leq y$, we have $w \leq z$.*

Definition 6.2 *A lattice L is a poset in which any two elements in L have a unique join and a unique meet.*

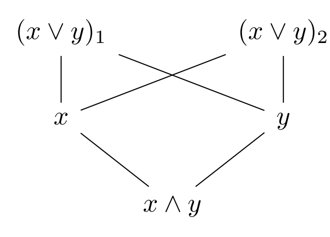
To obtain more clear intuition, let's look at some examples of posets that are not lattices.

Example 6.1 Let's look at the example of posets that $\vee$ is not well defined.

do not have $x \vee y$



$x \vee y$ is not unique



Then, let's observe $\vee$ and $\wedge$ for some familiar posets.

Example 6.2 (The Boolean Poset) For $A, B \in \mathcal{B}_n$, $A \leq A \cup B, B \leq A \cup B$ by the inclusion order. If $A \leq C$ and $B \leq C$, then $A \cup B \leq C$ which means that $A \cup B \subseteq C$ by simple set theory. Thus, by the definition of joins, $A \vee B = A \cup B$. In similary way, we can get $A \wedge B = A \cap B$.

Example 6.3 (The Young's Poset) For $\lambda, \mu \in Y$, put

$$\lambda = (\lambda_1, \ldots, \lambda_k) \text{ and } \mu = (\mu_1, \ldots, \mu_t)$$

and let $k \leq t$. Consider $\tau = (\max\{\lambda_1, \mu_1\}, \ldots, \max\{\lambda_k, \mu_k\}, \mu_{k+1}, \ldots, \mu_t) \in Y$. Then, by the definition of joins, $\tau = \lambda \vee \mu$. Simlarly, consider $\sigma = (\min\{\lambda_1, \mu_1\}, \ldots, \min\{\lambda_k, \mu_k\})$. Then, by the definition of joins, $\sigma = \lambda \wedge \mu$.

Question ❯ *Are $\mathcal{D}_n$ and Π_n lattices? If they are lattices, what are $\vee$ and $\wedge$ of them?*

Sol) 나중에 채웁시다.

Theorem 6.3 ❯ *If L is a lattice with $\vee$ and $\wedge$, the followings are true.*

For $x, y, z \in L$,

① $\begin{cases} x \vee x &= x \\ x \wedge x &= x \end{cases}$ ② $\begin{cases} x \vee y &= y \vee x \\ x \wedge y &= y \wedge x \end{cases}$

③ $\begin{cases} x \vee (y \vee z) &= (x \vee y) \vee z \\ x \wedge (y \wedge z) &= (x \wedge y) \wedge z \end{cases}$ ④ $\begin{cases} x \wedge (x \vee y) &= x \\ x \vee (x \vee y) &= x \end{cases}$

⑤ $x \wedge y = x \iff x \leq y \iff x \vee y = y$

Proof. Since 1 and 2 are trivial, we will prove 3.

Let $w = (x \vee y) \vee z$ and let $w' = x \vee (y \vee z)$. Since w is a upper bound of $x \vee y$ and z, we have

$$x \vee y \leq w \text{ and } z \leq w.$$

Because $x \leq x \vee y$ and $y \leq x \vee y$, we have $x \leq w$, $y \leq w$ and $z \leq w$. Since $y \vee z$ is the least upper bound of y and z, we get $y \vee z \leq w$. Hence, we have

$$w' = x \vee (y \vee z) \leq w$$

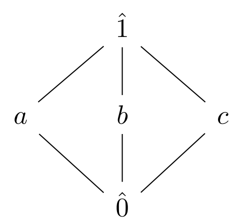
because w' is the least upper bound of x and $y \vee z$. In simlar way, we can show $w \leq w'$. Now, we conclude that $w = w'$.

The proof of 4 and 5 is not difficult but tedious. Thus, the proof is omitted. □

Although previous theorem implies commutative and associative laws of $\vee$ and $\wedge$ for a lattice holds, the distributive law of these operations does not always holds. Let's look at some examples of lattices that do not satisfy the distributive law.

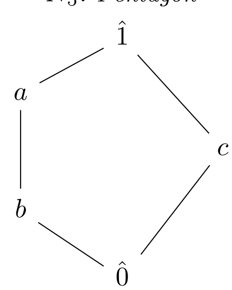
Notation ❯ *The diamond poset, M_3, is a poset with the following Hasse diagram.*

M_3: Diamond



The pentagon poset, N_5, is a poset with the following Hasse diagram.

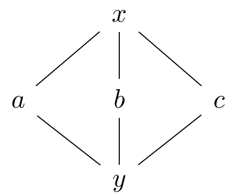
N_5: Pentagon



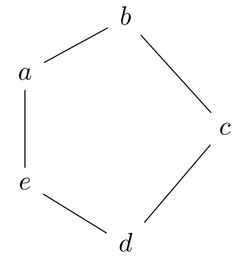
Example 6.4 ❯ M_3, the diamond poset, is a lattice but does not satisfy the distributive law, since $a \vee (b \wedge c) = a \vee y = a$ but $(a \vee b) \wedge (a \vee c) = x \wedge x = x$.

N_5, the pentagon poset, is a lattice but does not satisfy the distributive law, since $a \wedge (c \vee e) = a \wedge b = a$ but $(a \wedge c) \vee (a \wedge e) = d \vee e = e$.

M_3: Diamond



N_5: Pentagon



SECTION 6.2 Distributive Lattice

Now, we will focus on a lattice with distributive law.

Theorem 6.4 *Let P be a poset. The following conditions are equivalent:*

① $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ *for all* $x, y, z \in P$.

② $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ *for all* $x, y, z \in P$.

Proof. Suppose that ① is true. Then,

$$\begin{aligned}(x \wedge y) \vee (x \wedge z) &= ((x \wedge y) \vee x) \wedge ((x \wedge y) \vee z) \\ &= x \wedge (z \vee (x \wedge y)) \\ &= x \wedge ((z \vee x) \wedge (z \vee y)) \\ &= (x \wedge (z \vee x)) \wedge (y \vee z) \\ &= x \wedge (y \vee z)\end{aligned}$$

for all $x, y, z \in P$. Thus, ‘ $\implies a$. In similar way, we can show that $a \implies$ ‘. The proof is complete. □

Definition 6.5 *A lattice L is a distributive lattice, if the following two equivalent conditions hold:*

*D*① $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$,

*D*② $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$,

for all $x, y, z \in L$.

1. Characterizations

We can characterize distributive lattices by two method. The first method is using sublattices.

Definition 6.6 *For a lattice L, a subset S of L is a sublattice if it is closed under $\wedge$ and $\vee$.*

In Example 6.4, we verified that a lattice that contains diamond or pentagon as its sublattice is not distributive. In fact, the inverse of this statement is also true.

Theorem 6.7 *A lattice L is a distributive lattice if and only if L does not have a sublattice isomorphic to M_3 or N_5.*

Proof. (더 나아가기) □

The second method is using ring theoretic result. We can understand a distributive lattice with ring theory.

Theorem 6.8 *Let L be a finite lattice, $\mathcal{A} = \mathbb{C}[x_a : a \in L]$ be a polynomial algebra in indeterminates labeled by the elements of L, and I be the ideal of $\mathcal{A}$ generated by the set*

$$\{x_a x_b - x_{a \wedge b} x_{a \vee b} \mid a \text{ and } b \text{ are incomparable}\}.$$

The quotient ring $\mathcal{A}/I$ is an integral domain if and only if L is a distributive lattice.

Proof. (더 나아가기) □

We can verify that $\mathcal{B}_n$, Y, and $\mathcal{D}_n$ are distributive but Π_n is not by using Theorem 6.7.

2. Fundamental Theorem of Finite Distributive Lattices

Recall that the Boolean lattice, $\mathcal{B}_n$, is distributive. We can easily verify this result by elementary set theory. For $A, B \in \mathcal{B}_n$,

$$A \vee B = A \cup B \text{ and } A \wedge B = A \cap B.$$

In set theory,

$$X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z) \text{ and } X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$$

for all $X, Y, Z \subseteq \mathcal{U}$ where $\mathcal{U}$ is the universal set. Thus, $\mathcal{B}_n$ is distributive. We will prove Birkhoff's theorem, also known as the fundamental theorem of finite distributive lattices using some properties of $\mathcal{B}_n$.

Definition 6.9 ❯ *Let P be a poset.*

① *A subset I of P is an order-ideal(down-set, decreasing set), if $y \in I$ and $x \leq y$ in P imply that $x \in I$.*

② *A Subset J of P is a filter(up-set, increasing set), if $x \in J$ and $x \leq y$ in P imply that $y \in J$.*

Notation ❯ *For a poset P, $\mathcal{J}(P)$ denotes the set of all order-ideals of P.*

Proposition 6.10 ❯ *Under set-inclusion order, $\mathcal{J}(P)$ is a distributive lattice.*

Proof. Let $I, J \in \mathcal{J}(P)$. Suppose that $x \in I \cap J$ and $y \leq x$ in P. Then, $x \in I$ and $y \leq x$, so $y \in I$. Also, $x \in J$ and $y \leq x$, so $y \in J$. Thus, $y \in I \cap J$. Now, we have $I \cap J \in \mathcal{J}(P)$. Next, suppose that $x \in I \cup J$ and $y \leq x$ in P. Thus, $x \in I$ or $x \in J$. If $x \in I$, then $y \in I$. If $x \in J$, then $y \in J$. Hence, $y \in I \cup J$. Now, we have $I \cup J \in \mathcal{J}(P)$. Note that $I \wedge J = I \cap J$ and $I \vee J = I \cup J$. By set theory,

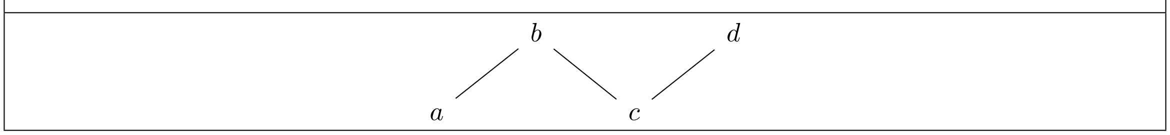
$$X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z) \text{ and } X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$$

for all $X, Y, Z \in JP$. The proof is complete. □

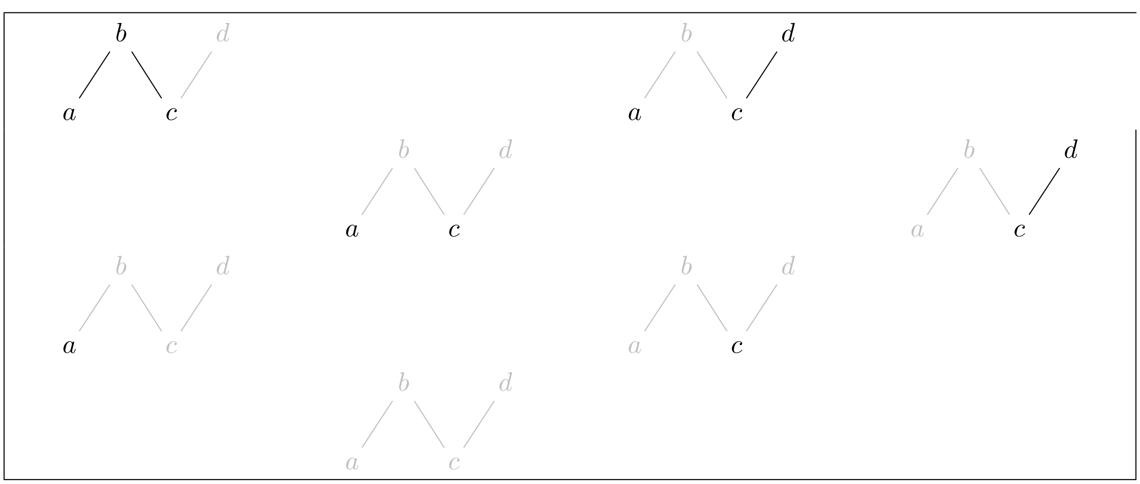
Now, we know that $\mathcal{J}(P)$ is also a distributive lattice with $\vee$ and $\wedge$ defined by $\cup$ and $\cap$ like the Boolean lattice.

Example 6.5 ❯ Let's see this simple poset P.

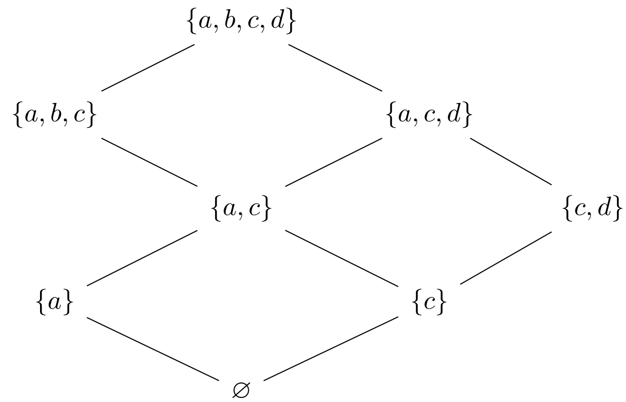
poset P



order-ideals of P



For this poset P, we can make the poset $\mathcal{J}(P)$ with set-inclusion order.



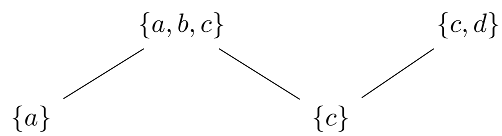
Then, $\mathcal{J}(P)$ becomes a distributive lattice as we see.

Definition 6.11 *Let L be a lattice. A nonbottom element u in L is join-irreducible, if $u = x \vee y$ implies that $u = x$ or $u = y$. Also, we let* $\mathrm{Irr}(L)$ *denote the subposet of L consisting of all the join-irreducible elements of L.*

Note that

$$\mathrm{Irr}(L) = \{x \in L : x \text{ is join-irreducible}\}.$$

Example 6.6 In Example 6.5, $\mathrm{Irr}(\mathcal{J}(P))$ becomes



Thus, we can check that $\mathrm{Irr}(\mathcal{J}(P))$ is isomorphic to P.

Theorem 6.12 (Birkoff's Theorem) *Let L be a finite distributive lattice. Then,*

$$L \cong \mathcal{J}(\mathrm{Irr}(L)).$$

Proof. Define $I_t = \{s \in \mathrm{Irr}(L) : s \le t \text{ in } L\}$ for $t \in L$. Note that t may not be in $\mathrm{Irr}(L)$. Let $t \in L$. Assume that $s \in I_t$ and $s' \le s$ in $\mathrm{Irr}(L)$. Then, $s' \in \mathrm{Irr}(L)$ and $s' \le s \le t$ in L, so $s' \in I_t$. Thus, $I_t \in \mathcal{J}(\mathrm{Irr}(L))$.

Consider $\phi : L \to \mathcal{J}(\mathrm{Irr}(L))$ defined by $\phi(t) = I_t$. We will show that ϕ is an order-isomorphism.

First, let's show that ϕ is order-preserving. Assume that $t \le t'$. We have

$$\phi(t) = I_t = \{s \in \mathrm{Irr}(L) : s \le t \text{ in } L\} \text{ and } \phi(t') = I_{t'} = \{s \in \mathrm{Irr}(L) : s \le t' \text{ in } L\}.$$

Let $s \in \phi(t)$. Then, $s \in \mathrm{Irr}(L)$ and $s \le t \le t'$. Thus, $s \in \phi(t')$. Hence, $\phi(t) \le \phi(t')$ by the set-inclusion order.

Second, we will show that ϕ is surjective. We need to show that, for each $I \in \mathcal{J}(P)$, there exist $t \in L$ such that $\phi(t) = I_t = I$. Let $I \in \mathcal{J}(\mathrm{Irr}(L))$ and define $t = \bigvee\limits_{s \in I} s$, the join of all elements in I.

(1) We need to show that $I \subseteq I_t$. Let $s \in I$. Then, we can easily check that $s \in \mathrm{Irr}(L)$. Also, $s \le t = \bigvee\limits_{s \in I} s$ since $s \in I$. Thus, $s \in I_t$.

(2) Next, we need to show that

$$\bigvee_{s \in I} s = \bigvee_{s \in I_t} s.$$

Since $I \subseteq I_t$, we have $t = \bigvee_{s \in I} s \le \bigvee_{s \in I_t} s$. Also, $t = \bigvee_{s \in I} s \ge \bigvee_{s \in I_t} s$ because every $s \in I_t$ satisfy $s \le t$.

(3) Finally, we have to show that $I_t \subseteq I$. Let $u \in I_t$. Then,

$$u \wedge \left(\bigvee_{s \in I} s \right) = \bigvee_{s \in I} (u \wedge s) = u \wedge \left(\bigvee_{s \in I_t} s \right) = \bigvee_{s \in I_t} (u \wedge s) = u.$$

The first equality and the third equality holds because L is distributive and the fourth equality holds because $u \in I_t$. Because u is join-irreduclble, $u = \bigvee_{s \in I} (u \wedge s)$ implies that $u = u \wedge s$ for some $s \in I$. Note that $u = u \wedge s$ implies that $u \leq s$. Since I is an order-ideal, $u \in I$.

Now, we conclude that $I = I_t = \phi(t)$. This proves the surjectivity of ϕ.

Third, we will show that ϕ is injective. Assume that $\phi(t) = \phi(t')$. Then, $I_t = I_{t'}$. Define $k = \bigvee_{s \in I_t} s$. Every $s \in I_t$ satisfy $s \leq t$. Thus, $k \leq t$.

(1) Suppose that t is join-irrducible. Then, $t \in I_t$. Thus, $t \leq k$.

(2) Suppose that t is not join-irreducible. Since L is finite, t is the join of finitely many join-irreducible elements. Thus, we have
$$t \vee k = k.$$
Hence, $t \leq k$.

Now, we have $k = t$. Then,

$$t = \bigvee_{s \in I_t} s = \bigvee_{s \in I_{t'}} s = t'.$$

Thus, ϕ is injective. We showed that ϕ is bijective.

Finally, we will show that ϕ^{-1} is order-preserving. Assume that $\phi(t) \subseteq \phi(t')$. Then, $I_t \subseteq I_{t'}$. We already show that $t = \bigvee_{s \in I_t} s$ and $t' = \bigvee_{s \in I_{t'} s}$. Thus,

$$t = \bigvee_{s \in I_t} s \leq \bigvee_{s \in I_{t'} s} = t'.$$

Now, we have ϕ^{-1} is order-preserving. The proof is complete. □

If P is a finite poset, then $\mathcal{J}(P)$ is isomorphic to some sublattice of the Boolean poset. FTFDL implies that every finite distributive lattice is isomorphic to some sublattice of the Boolean poset.