1120 Class Activity

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1. Let graph $G = G_1 \cup G_2$ where $G_1 = C_3$ and $G_2 = P_3$.
Find all eigenvalues of L_G, the Laplacian matrix of G.
2. Above situation, let $G_2 = C_3$. Find all eigenvalues of L_G.

Proof. 1. We want to find the adjacent matrix of G.
We know that the adjacent matrices of C_3 and P_3.

$$A_{C_3} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}; A_{P_3} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Therefore, the adjacent matrix of G is $A_G = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$.

We can find the diagonal matrix $\triangle_G$ where $(\triangle_G)_{ii} = \deg(v_i)$ using A_G.

$$\triangle_G = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

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Therefore, the Laplacian matrix of G,

$$L_G = \triangle_G - A_G = \begin{bmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

Using matrix calculator, we can find all eigenvalues of L_G.
$(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6) = (0, 0, 1, 3, 3, 3)$.

2. Using A_{C_3}, the adjacent matrix of G is $A_G = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$.

Also, the diagonal matrix $\triangle_G$ is $\triangle_G = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$.

The Laplacian matrix of G is $L_G = \triangle_G - A_G = \begin{bmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & -1 & 2 \end{bmatrix}$.

Using matrix calculator, we can find all eigenvalues of L_G.
$(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6) = (0, 0, 3, 3, 3, 3)$.
We can check that both two cases have $\lambda_2 = 0$, so, this is not connected graph. So, $\lambda_2(L_G), \lambda_2(L_{\tilde{G}})$, the algebraic connectivity of G and $\tilde{G}$ holds below inequality.

$$\lambda_2(L_G) \leq \lambda_2(L_{\tilde{G}}) \leq \lambda_2(L_G) + 2$$

$\square$