Stability analysis of discrete time control structures

1 Theoretical aspects

- Stability analysis (Jury's stability criterion)
- Root Locus for discrete – time systems
- Grid of constant damping factors and natural frequencies
- Matlab functions: c2d, rlocus, zgrid, feedback, pzmap

2 Negative feedback structures with data sampling

Classic rigid negative feedback control structure involving digital control using zero order hold method, is presented in Figure 2.1. The transfer function of the zero-order hold element can be readily recognized, $H_{zoh}(s)=\frac{1-e^{-sT}}{s}$, with the sampling period T (given in seconds). The considered process is an unstable one (one pole in the right half plane)

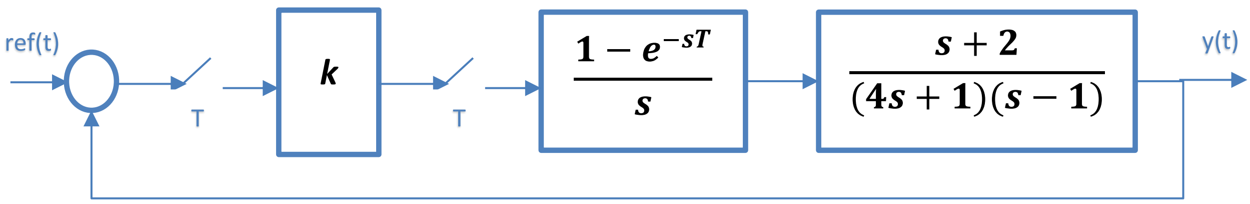


Figure 2.1 Sampled data control systems

2.1 Stability analysis depending on the proportional gain, k

As in the continuous time approach, the design of the proportional gain k, can be faster accomplished using root locus method. The next MATLAB script can be used for stability analysis of structure in in Figure 2.1, when $k\in(0,\infty)$. The control problem can be expressed as: "is it a simple proportional gain sufficiently enough to stabilize the closed loop".

```
%% S1: -    Stability analysis depending on k

clear;% erase all the existing workspace variables
num=[1 2];den=[4 -3 -1];% the nominator and denominator of the process
Hp=tf(num,den);% the process transfer function
T=0.2;% the sampling period
Hdes=c2d(Hp,T,'zoh');% the open loop discrete time transfer function
subplot(121);rlocus(Hp);title('Continuous-time approach');
text(-8,2,'$H_{des}(s)=\frac{s+2}{(4s+1)(s-1)}$','Interpreter','Latex','FontSize',18)
subplot(122);rlocus(Hdes);title('Discrete-time approach')
text(0.2,0.8,'$H_{des}(z)=\frac{0.064563 (z-0.6655)}{(z-1.221) (z-
0.9512)}$','Interpreter','Latex','FontSize',18)
```

Replacing the analog proportional gain by a digital one, will affect the domain values of k for which the closed loop becomes stable. Evaluating the results in Figure 2.2, for continuous time approach the values of k leading to a stable closed loop are in the interval $(3,\infty)$; for discrete-time approach or for digital control algorithm implementation, the interval narrows down to $(3.76, 40.3)$

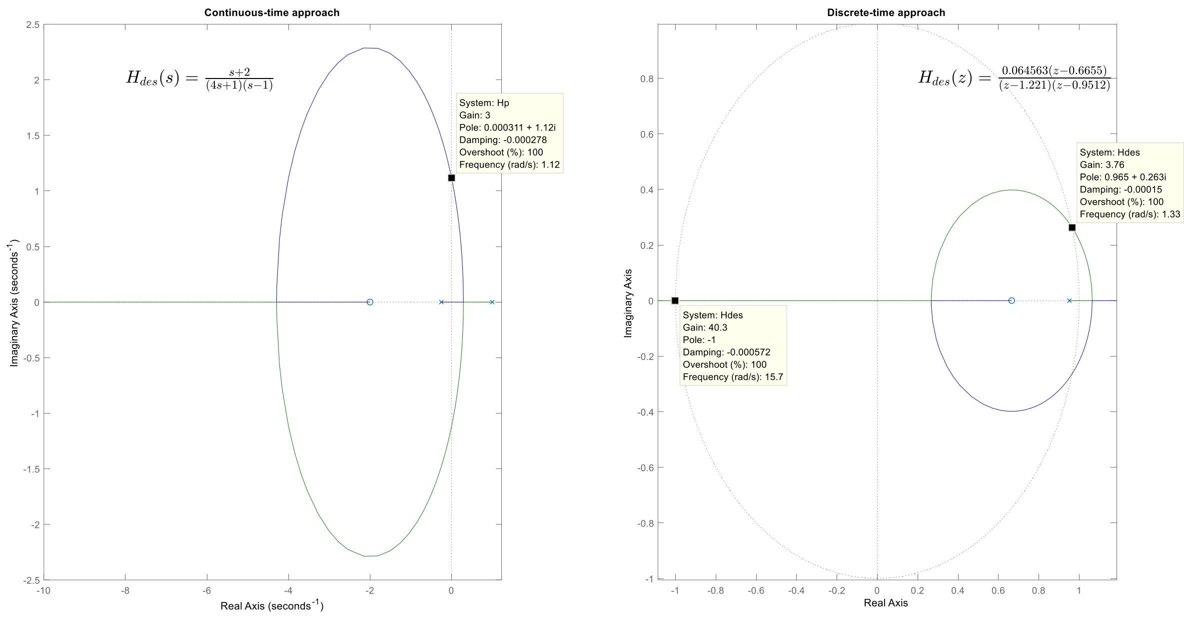


Figure 2.2 Stability comparison of continuous and discrete-time approach

2.2 Dynamic behaviors depending on the proportional gain k

Evaluating the pole location (see Figure 2.3, Table 1) depending on the gain k, varied inside the stable interval $(3.76, 40.3)$, the next types can be obtained:

- $k = 3.76$, $\zeta = 0$, UNDAMPED system, both poles are located exactly on the unity circle;
- $k \in (3.76, 25.4)$, $\zeta \in (0,1)$, UNDERDAMPED system, pair of complex conjugated poles, inside the unity circle;
- $k = 25.4$, $\zeta = 1$, CRITICALY DAMPED system, real pole with multiplicity order 2;
- $k \in (25.4, 40.3)$, $\zeta > 1$, OVERDAMPED system, two real and distinctive poles, inside the unity circle;
- $k = 40.3$, $\zeta = 0$, UNDAMPED system, one pole located exactly on the unity circle (in -1).

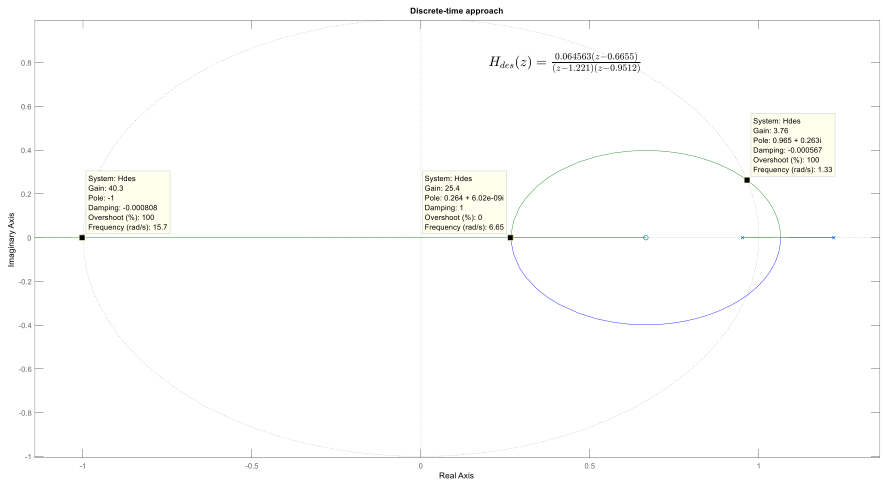
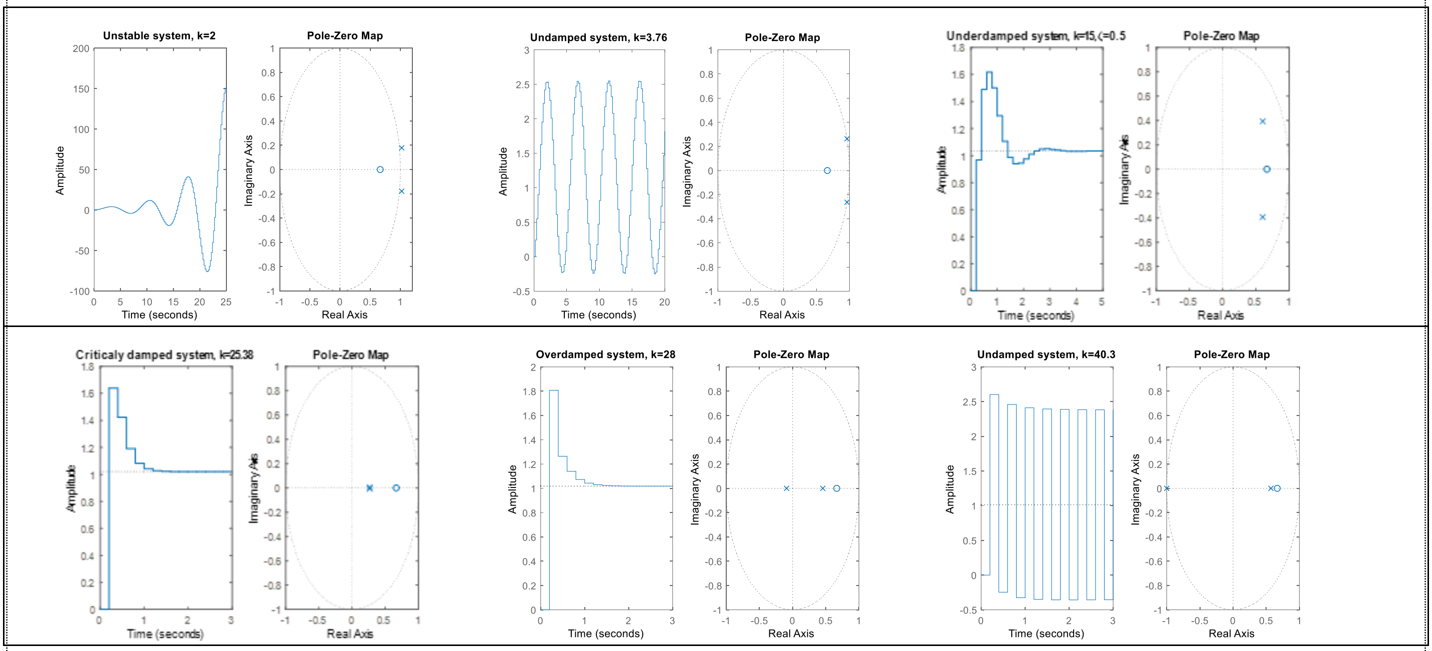


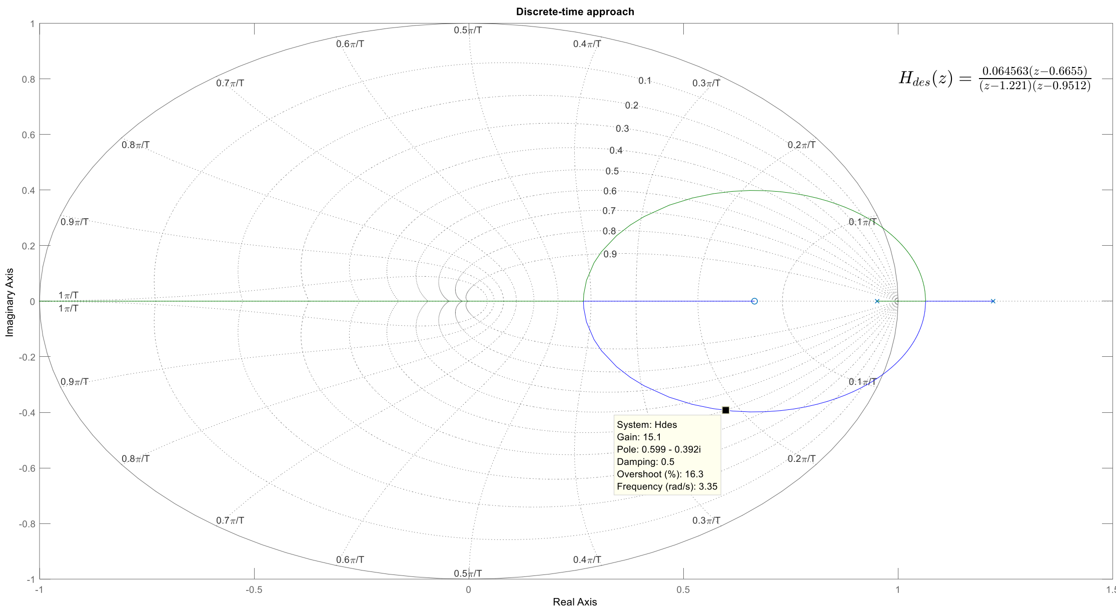
Figure 2.3 Behaviors depending on the proportional gain k

Table 1 Closed loop step response depending on k



2.3 Damping ratio (ζ) constant locus

In a similar manner as in the continuous time approach, the overshoot of the closed loop step response can be evaluated from the root locus if drawing the damping ratio constant line (locus). Making the grid ”lines” active in the root locus plot, curves of constant damping factor, and natural frequency become visible. In the next figure, the gain k=15.1 will lead to a damping factor $\zeta = 0.5$ and implicitly to an overshoot around 16%



The `sgrid`, respectively `zgrid` function can be used for faster graphical results and easier graphical evaluations, as in the next script:

```
%% S2: -    damping ratio constant locus ζ = 0.5 (σ ≈ 16%)
subplot(121);rlocus(Hp);title('Continuous-time approach');sgrid(0.5,[]);
subplot(122);rlocus(Hdes);title('Discrete-time approach');zgrid(0.5,[])
```

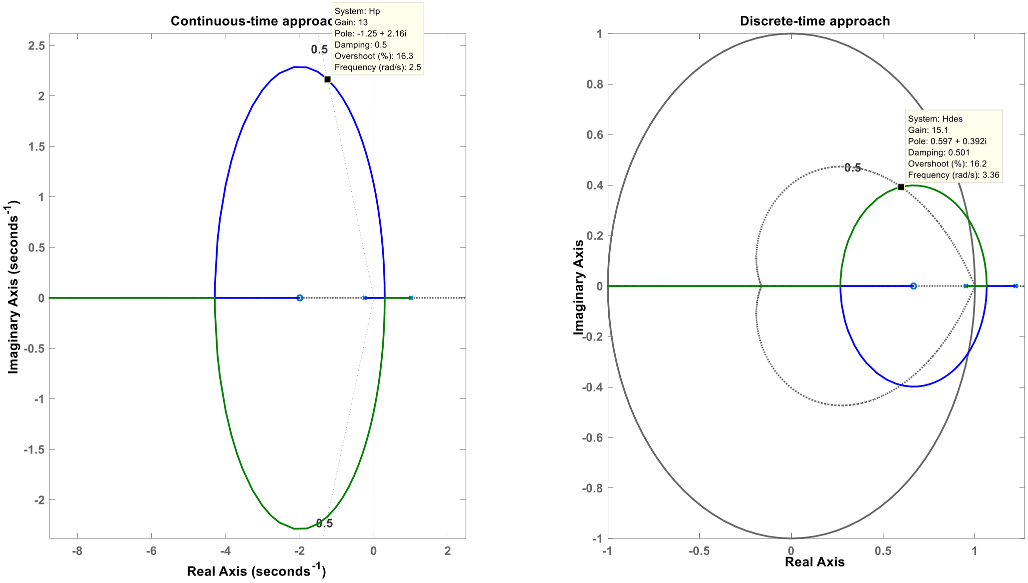


Figure 2.4 Damping factor constant locus of 0.5

2.4 Stability analysis depending on the sampling period

In most cases, not only the variation of the proportional gain affects the stability of the closed loop but also the value of the sampling period. In the next table, successive increasing values for the sampling period are proposed. Using root locus, it can be found the domain of k for which the closed loop remains stable. For such situations, the Jury's stability test can be used to establish the highest value for the sampling period.

From Figure 2.5, the domain of k for 3 different sampling period values can be evaluated.

Tabel 2.1 Closed loop stability domain of proportional gain k depending on the sampling period

T	0	0.01	0.02	0.05	0.1	0.2	0.5	1
k	$(3,\infty)$	$(3.02, 801)$	$(3.06, 401)$	$(3.07, 160)$	$(3.3, 80.2)$	$(3.76, 40.3)$	$(6.24, 16.8)$	Ø
k (critically damped)	37.3	36.5	35.8	33.6	30.5	25.4	16.1	NaN

A short and rough analysis of how the interval is narrowing down with the increase of the sampling period: "the right limit of the interval is inverse proportional to the value of the sampling period"

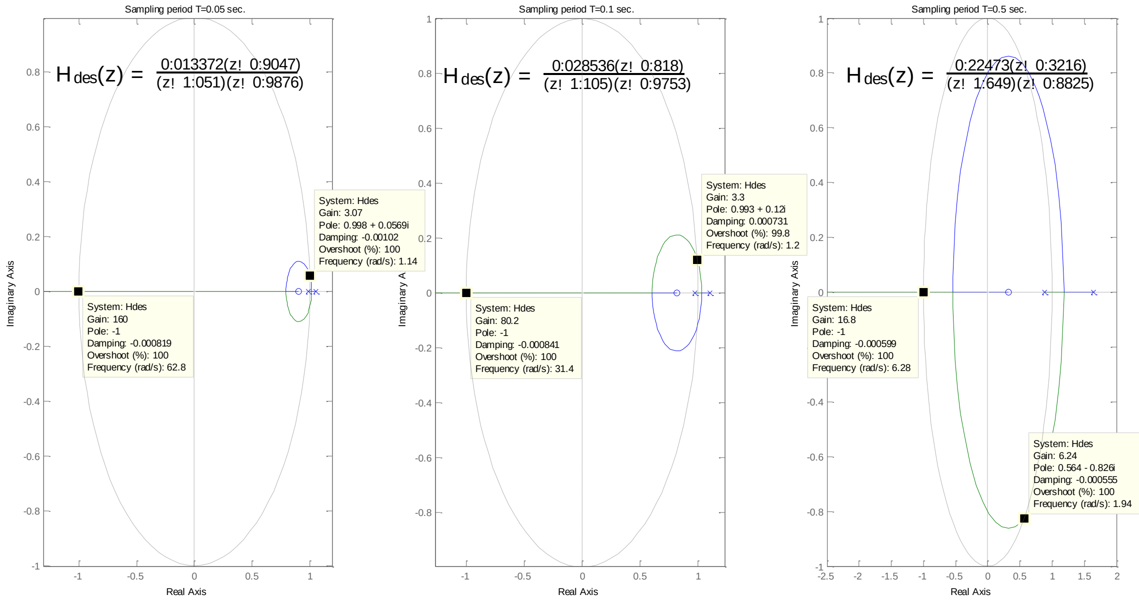


Figure 2.5 Root Locus for different sampling periods (T=0.05, T=0.1; T=0.5)

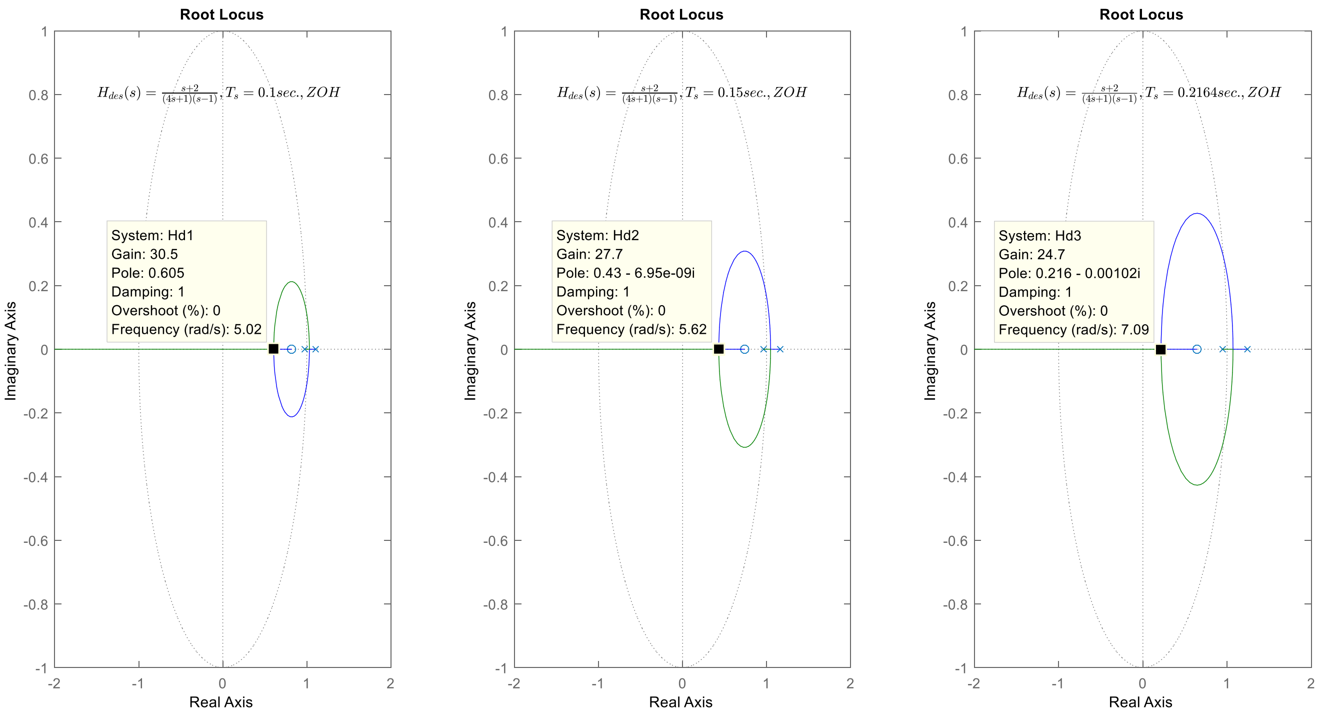


Figure 2.6 Root Locus for different sampling periods, with the critically damped behaviors

3 Problems

3.1 For the next discrete time control structure with the sampling period T=0.05 sec.:(v2)

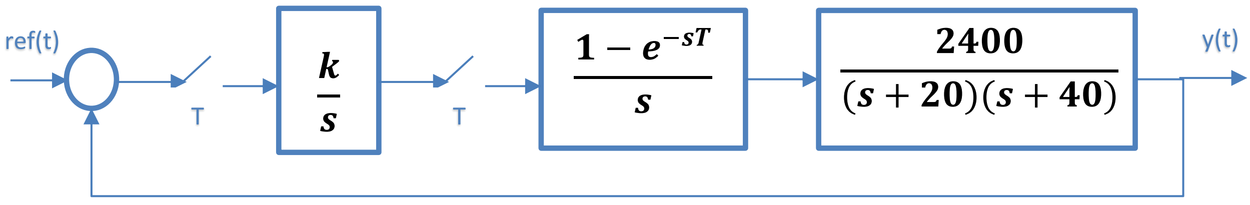


Figure 3.1 Sampled data control systems

1. Discretize the controller ($H_c(s) = \frac{k}{s}$) using tustin (bilinear) transformation.
2. Discretize the process ($H_p(s) = \frac{2400}{(s+20)(s+40)}$) using zero order hold method;
3. Use `zpk` function to write on your notebook the open loop transfer function.
4. Analyze the stability of the closed loop system depending on $k \in (0, \infty)$; draw on your notebook the root locus and mention the obtained values of k directly on the graphics.
5. Analyze closed loop behavior depending on $k \in (0, \infty)$; use Matlab to generate the closed loop step response for all different behaviors that result depending on k; give suggestive titles to the generated plots.
6. Use the zgrid function to obtain the value of k for which the overshoot of the closed loop is below 10%.
7. Generate the step response of the closed loop when the damping ratio is 0.5.

3.2 For the next discrete time control structure with the sampling period T=0.1 sec.: (v3)

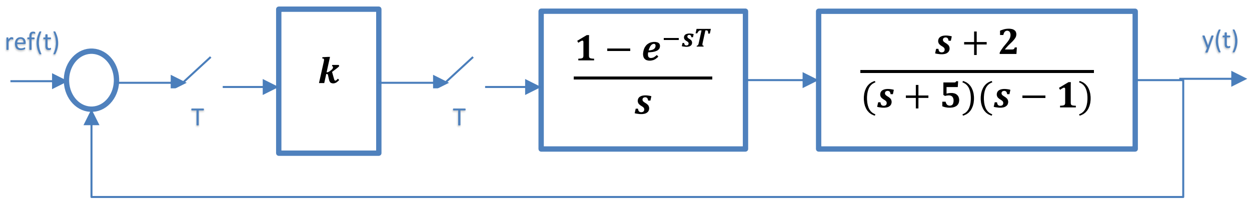


Figure 3.2 Sampled data control systems

Follow the same steps as in the solved problem.

3.3 For the next discrete time control structure with the sampling period T=0.02 sec.:(v4)

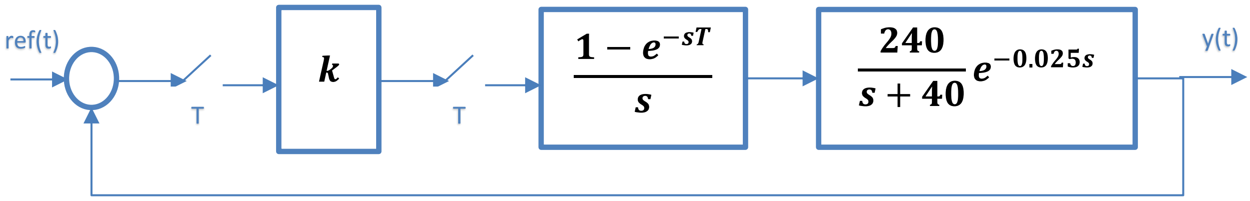


Figure 3.3 Sampled data control systems

Follow the same steps as in the solved problem.