Mathematics 3 Lecture Notes

Kieran Hughes

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Chapter 1

Coordinate Geometry in 2 Dimensions

1.1 Brief History of Coordinate Geometry

René Descartes (1596-1650)
French mathematician,
natural scientist, and
philosopher.

Coordinate geometry, also known as *analytic geometry*, is a branch of mathematics that studies geometric figures through the use of a coordinate system. This discipline bridges the gap between algebra and geometry, allowing for the description of geometric shapes using algebraic equations.
The foundation of coordinate geometry was laid by the French mathematician René Descartes in the 17th century. In 1637, Descartes published his work *La Géométrie*, where he introduced the Cartesian coordinate system, named after him. This system allows the representation of points in a plane using pairs of numbers (coordinates), which correspond to the distances from two fixed perpendicular lines called the axes (x-axis and y-axis).
Descartes' innovation provided a powerful tool for the mathematical description of curves and surfaces, paving the way for further developments in calculus and differential geometry. The ability to translate geometric problems into algebraic equations marked a significant advancement in mathematical thought, leading to more complex and abstract applications in various fields.

1.2 Applications of Coordinate Geometry in Computer Science

Coordinate geometry has a wide range of applications in computer science, especially in the fields of graphics, image processing, machine learning, and robotics. Some key applications include:

1. **Computer Graphics:** Coordinate geometry is fundamental to computer graphics, where it is used to model and render shapes, objects, and scenes. The transformation of shapes (e.g., translation, rotation, scaling) can be described using matrices, which are an extension of coordinate geometry. Algorithms for rendering 2D and 3D graphics rely heavily on the principles of coordinate geometry to project 3D scenes onto a 2D screen, handle perspective, and simulate light and shadows.

2. **Image Processing:** In image processing, coordinate geometry is used to manipulate and analyze digital images. Operations such as rotation, scaling, and translation of images are based on coordinate transformations. Additionally, edge detection and object recognition algorithms often employ coordinate-based methods to identify and classify features within an image.

3. **Robotics:** Robotics relies on coordinate geometry for tasks such as navigation, path planning, and manipulation of objects. Robots use coordinate systems to understand their environment, plot trajectories, and interact with

objects. Kinematics, which deals with the motion of robots, involves the use of coordinate transformations to describe the position and orientation of robot components.

4. **Machine Learning:** In machine learning, coordinate geometry is applied in the visualization and interpretation of high-dimensional data. Techniques like Principal Component Analysis (PCA) use concepts from coordinate geometry to reduce the dimensionality of data while preserving important features, making it easier to visualize and analyze.

5. **Game Development:** Game development extensively uses coordinate geometry to create virtual worlds, define collision boundaries, and simulate physical interactions. Characters, objects, and environments within a game are often modeled using coordinates, allowing for realistic movement and interaction within the game space.

1.3 The Cartesian Coordinate System

In this chapter, we will explore the fundamental concepts of coordinate geometry in two dimensions, which forms the basis for many areas of mathematics and applications in computer science and engineering.

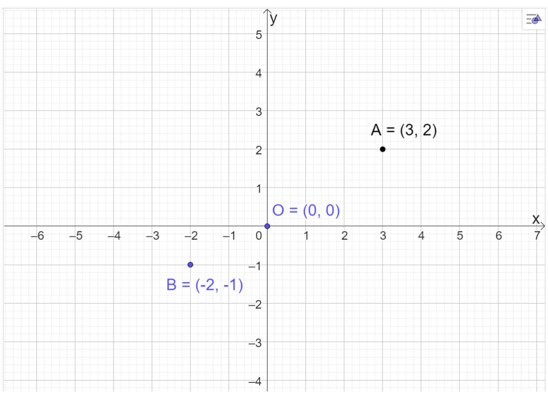
Definition 1.3.1. In coordinate geometry, a **point** is a fundamental, dimensionless entity that represents a specific location. It is defined by an ordered pair of numbers (x, y).

- x: The first number in the pair, representing the horizontal position.
- y: The second number in the pair, representing the vertical position.

For instance, the point $P = (4, 5)$ corresponds to a position where $x = 4$ and $y = 5$. This pair of numbers uniquely identifies the point's location.

Definition 1.3.2. The **plane** in 2-dimensional coordinate geometry is the set of all possible points. Each point is uniquely identified by an ordered pair of real numbers (x, y), so the plane can be represented by the set $\{(x, y) \mid x, y \in \mathbb{R}\}$. This set includes every possible pair (x, y), where x and y are real numbers and nothing else.

The Cartesian coordinate system is a mathematical framework used to uniquely determine the position of points in a plane. It consists of two perpendicular axes: the horizontal axis, known as the x-axis, and the vertical axis, known as the y-axis. These axes intersect at a point called the origin, denoted as $O = (0, 0)$.
Any point can be represented by an ordered pair of numbers (x, y), where x indicates the position along the x-axis and y indicates the position along the y-axis. For example, the point $A = (3, 2)$ is located 3 units to the right of the origin along the x-axis and 2 units above the origin along the y-axis. Similarly, the point $B = (-2, -1)$ is located 2 units to the left and 1 unit below the origin.



Points of the Cartesian plane.

1.3.1 Subsets of the plane

The geometric shapes in 2-D coordinate geometry such as lines, line segments, rays, circles, rectangles etc are subsets of the plane. In each of these cases the subset contains an infinite number of points. When drawing a line you don't use a infinite number of dots. However, mathematically it is useful to think of a line as an infinite collection of points.

1.4 The distance between 2 points of the plane and the midpoint

Let $A = (x_1, y_1)$ and let $B = (x_2, y_2)$. Then the distance between the points A and B is given by the following formula.

$$|AB| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{1.1}$$

The midpoint of the line segment $[A, B]$ is given by the formula

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right). \tag{1.2}$$

Example 1.4.1. Let $A = (-2, 5)$ and let $B = (6, -2)$. plot the points A and B on the Cartesian plane. Calculate the distance between A and B and also plot the midpoint of the line segment $[A, B]$.

Answer: To plot the point $(-2, 5)$ we start at the origin $(0, 0)$ and go left 2 units and go up 5 units. To plot the point $(6, -2)$ we start at the origin $(0, 0)$ and go right 6 units and go down 2 units.

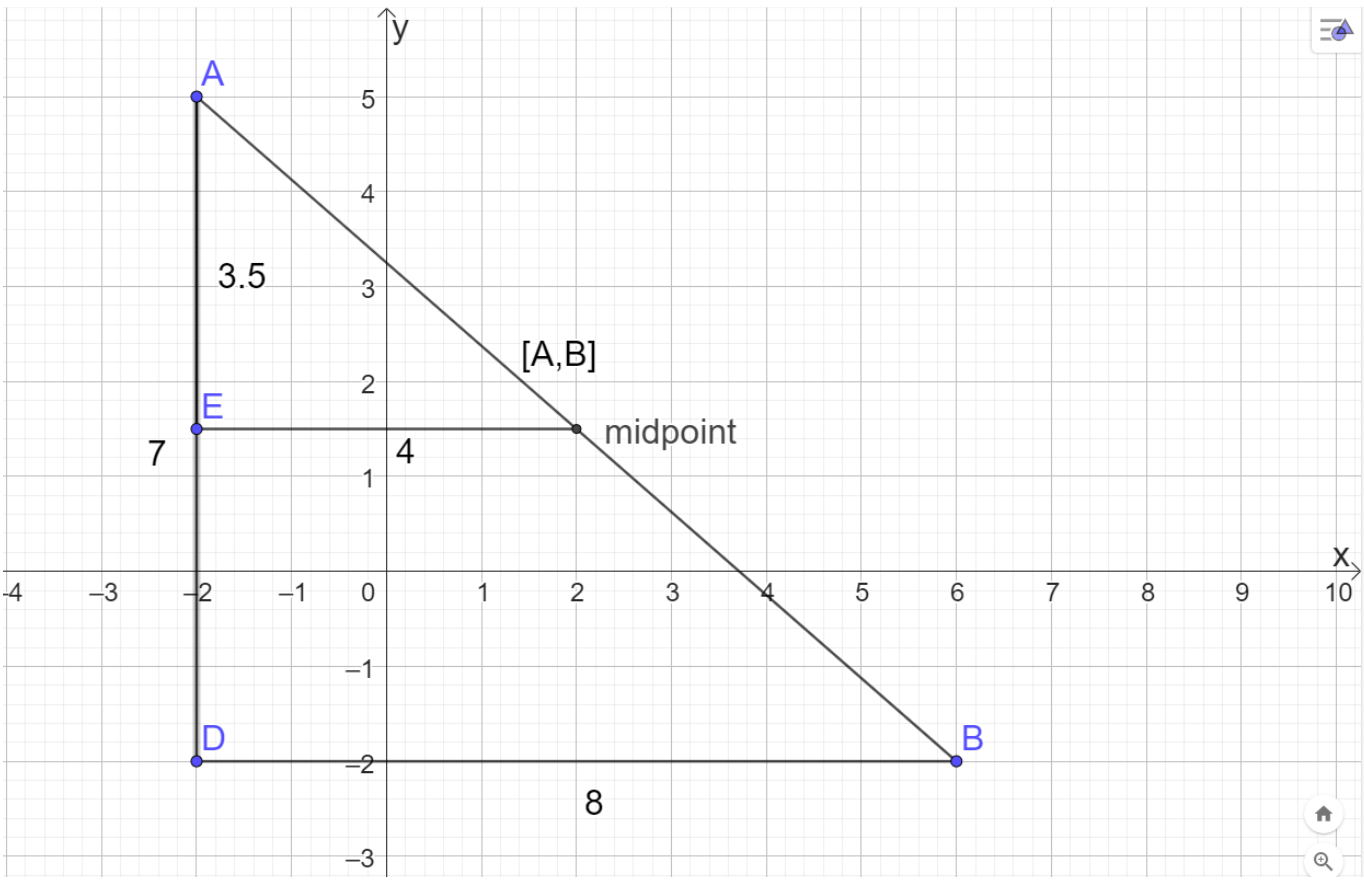
We label the points A and B as $A = (\underset{x_1}{-2}, \underset{y_1}{5})$ and $B = (\underset{x_2}{6}, \underset{y_2}{-2})$. We now calculate the distance between A and B using the formula (1.1).

$$|AB| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(6 - (-2))^2 + (-2 - 5)^2} = \sqrt{8^2 + (-7)^2} = \sqrt{64 + 49} = \sqrt{113}.$$

Note that this formula is an application of Pythagoras' Theorem.

The midpoint is calculated using the formula (1.2).

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{-2 + 6}{2}, \frac{5 + (-2)}{2}\right) = \left(\frac{4}{2}, \frac{3}{2}\right) = \left(2, \frac{3}{2}\right).$$



Exercises 1.4.2. Let $A = (2, -4)$, $B = (11, 7)$ and $C = (0, 3)$. Plot the points A, B and C on a Cartesian plane. Then calculate each of the following:

(a) $|AB|$

(b) The midpoint of $[A, B]$

(c) $|AC|$

(d) The midpoint of $[A, C]$

(e) $|BC|$

(f) The midpoint of $[C, B]$

(g) The coordinates of the point D such that A is the midpoint of $[B, D]$.

(h) The coordinates of the point E such that the points A, B, C and E form a rectangle.

1.5 The slope of a Line

Given any two distinct points $A = (x_1, y_1)$ and $B = (x_2, y_2)$ on the Cartesian plane, there is exactly one line that passes through both points. The slope m of this line is a measure of its steepness and is given by the following formula:

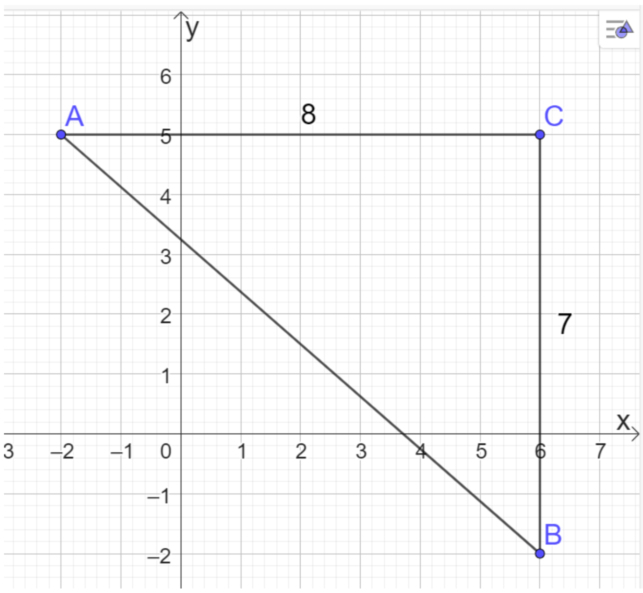
$$m = \frac{y_2 - y_1}{x_2 - x_1} \tag{1.3}$$

However, not all lines have a defined slope. For the slope to exist, the difference in the x-coordinates (known as the "run") must be non-zero ($x_2 \neq x_1$). If $x_2 = x_1$, the line is vertical, and its slope is undefined.

Example 1.5.1. Let $A = (-2, 5)$, $B = (6, -2)$, and $C = (6, 5)$. Calculate the slopes of the lines passing through each pair of these points.

Answer: We label the points A, B, and C as follows:

$$A = (\underset{x_1}{-2}, \underset{y_1}{5}),$$
$$B = (\underset{x_2}{6}, \underset{y_2}{-2}),$$
$$C = (\underset{x_3}{6}, \underset{y_3}{5}).$$



The slope of the line passing through A and B is:

$$m_{AB} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2 - 5}{6 - (-2)} = \frac{-7}{8}.$$

The slope of the line passing through B and C is:

$$m_{BC} = \frac{y_3 - y_2}{x_3 - x_2} = \frac{5 - (-2)}{6 - 6} = \frac{7}{0}.$$

Since the denominator is zero, the slope of the line passing through B and C is undefined. Do not use ∞ to describe the slope use the correct term undefined. The slope of a line is undefined if and only if the line is a vertical line.
The slope of the line passing through A and C is:

$$m_{AC} = \frac{y_3 - y_1}{x_3 - x_1} = \frac{5 - 5}{6 - (-2)} = \frac{0}{8} = 0.$$

A slope of 0 indicates that the line passing through A and C is horizontal.

Exercises 1.5.2. Calculate the slope (where possible) of the line passing through each pair of points. If the slope is undefined, use the correct term undefined.

(a) $P = (1, 2)$ and $Q = (3, 6)$

(b) $R = (-4, -2)$ and $S = (0, 0)$

(c) $T = (-3, 5)$ and $U = (2, -7)$

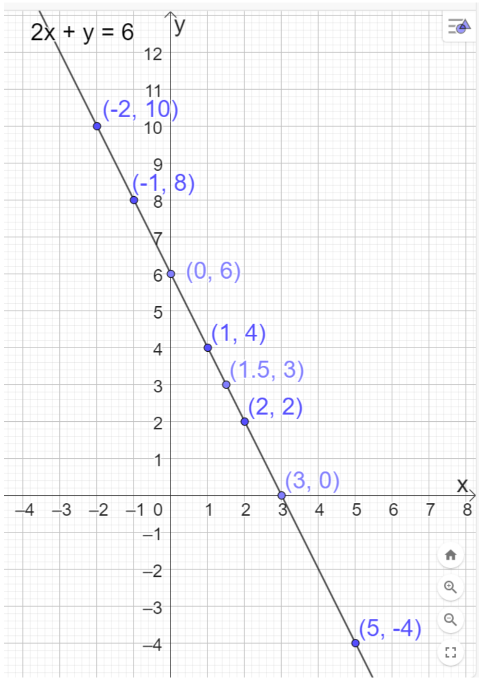
(d) $V = (5, 4)$ and $W = (-1, -8)$

(e) $X = (2, 3)$ and $Y = (2, -5)$

(f) $M = (-3, 2)$ and $N = (5, 2)$

1.6 The Equation of a Line

A line is an infinite set of points and can be represented by an equation. The line drawn on the graph is given by the equation $2x + y = 6$. But how do we interpret the equation of a line. The equation of the line will tell us how the x and y coordinates of the points of the line relate to one another. In other words, given a point (x_1, y_1), this point will be on the line $2x + y = 6$ if and only if $2x_1 + y_1 = 6$. For example, we can see from the graph that the point $(-2, 10)$ is on the line and we can confirm this algebraically as follows:



$$\begin{aligned} 2x_1 + y_1 &= 6 \\ 2(-2) + (10) &= 6 \\ -4 + 10 &= 6 \\ 6 &= 6. \end{aligned}$$

Given the equation of a line, to determine whether a point is on the line substitute the x coordinate of the point in for x and the y coordinate of the point in for y into the equation. If the resulting equation is true, then the point is on the line. If the resulting equation is false the point is not on the line.

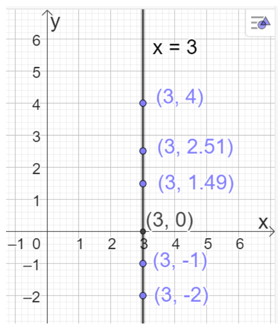
In coordinate geometry, lines can be categorised into two types: those with a defined slope and those without. Lines with a defined slope are non-vertical, while vertical lines have an undefined slope.

1.6.1 Vertical Lines

A vertical line has an undefined slope and is represented by the equation:

$$x = a, \tag{1.4}$$

where a is the x-coordinate of every point on the line. This line is parallel to the y-axis and does not intersect it. Also any line with undefined slope is vertical. In the image on the right the line $x = 3$ is drawn. Every point on the line has an x-coordinate of 3 and furthermore all points with an x-coordinate of 3 are on the line.



1.6.2 Non-Vertical Lines

For lines with a defined slope, the equation can be written in one of the following forms:

The slope-intercept form of the equation of a line is:

$$y = mx + c, \quad \text{where } m \text{ represents the slope of the line and } c \text{ is the } y\text{-intercept.} \tag{1.5}$$

The point-slope form of the equation of a line is:

$$y - y_1 = m(x - x_1), \quad \text{where } m \text{ represents the slope of the line and } (x_1, y_1) \text{ is any point on the line.} \tag{1.6}$$

1.6.3 Different Starting Information

The equation of a line can be determined from various starting points:

- The slope m and the y-intercept c.
- A point (x_1, y_1) on the line and the slope m.
- Two distinct points (x_1, y_1) and (x_2, y_2) on the line.

Example 1.6.1. Given Slope and y-Intercept

Find the equation of a line with a slope of $m = \frac{2}{3}$ and y-intercept $c = -1$.

Answer: We are given the slope and the y-intercept and so we use the slope-intercept form of the equation of a line (1.5).

$$\begin{aligned} y &= mx + c \\ y &= \frac{2}{3}x - 1 \end{aligned}$$

Alternative Answer: However, you can if you like use the point-slope form of the equation of a line (1.6). The y-intercept, $c = -1$ implies that the point of intersection between the line and the y-axis is $(0, -1)$. Therefore the point $(0, -1)$ is on the line and so

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 0 &= \frac{2}{3}(x - 0) - 1 \\ y &= \frac{2}{3}x - 1. \end{aligned}$$

Example 1.6.2. Given a Point and Slope

Find the equation of a line passing through the point $(2, 3)$ with a slope of $m = -\frac{1}{2}$.

Answer: We are given a point on the line and the slope and so we use point-slope form of the equation of a line (1.6).

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 3 &= -\frac{1}{2}(x - 2) \\ y - 3 &= -\frac{1}{2}x + 1 \\ y &= -\frac{1}{2}x + 4 \end{aligned}$$

Alternative Answer: However, you can if you like use the slope-intercept form of the equation of a line (1.5).

$$\begin{aligned} y &= mx + c \\ y &= -\frac{1}{2}x + c. \end{aligned}$$

Now substitute the point $(2,3)$ in for (x,y) and solve for c.

$$\begin{aligned} y &= -\frac{1}{2}x + c \\ 3 &= -\frac{1}{2}(2) + c \\ 3 &= -1 + c \\ 4 &= c. \end{aligned}$$

Therefore the equation of the line is $y = -\frac{1}{2}x + 4$.

Example 1.6.3. Given Two Points

Find the equation of a line passing through the points $(1,2)$ and $(-3,6)$.

Answer: First, calculate the slope m:

$$m = \frac{6-2}{-3-1} = \frac{4}{-4} = -1.$$

Using Point-Slope Form: Sub $(1,2)$ as (x_1, y_1) gives

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 2 &= -1(x - 1) \\ y - 2 &= -1x + 1 \\ y &= -x + 3 \end{aligned}$$

Using Slope-Intercept Form:

$$\begin{aligned} y &= mx + c \\ y &= -x + c. \end{aligned}$$

Now substitute the point $(1,2)$ in for (x,y) and solve for c.

$$\begin{aligned} y &= -x + c \\ 2 &= -(1) + c \\ 3 &= c. \end{aligned}$$

Therefore the equation of the line is $y = -x + 3$.

Example 1.6.4. Vertical Line

Find the equation of the line passing through the points $(4,-2)$ and $(4,7)$.

Answer: Note that the question does not mention the word vertical. However we can still try to calculate the slope m:

$$m = \frac{7-(-2)}{4-4} = \frac{9}{0}.$$

The slope is undefined as you can't divide by 0 and therefore the line is a vertical line. The equation of a vertical line is given by (1.4), that is, $x = a$, where a is the x-coordinate of every point on the line. In this case $a = 4$ and so the equation of the line is $x = 4$.

Exercises 1.6.5. Write the equation of the line for each of the following situations.

(a) Slope $m = 3$ and y-intercept $c = -2$

(b) Points $(2,-3)$ and $(2,1)$

(c) Slope $m = -\frac{1}{4}$ and point $(0,5)$

(d) Points $(2,-3)$ and $(4,1)$

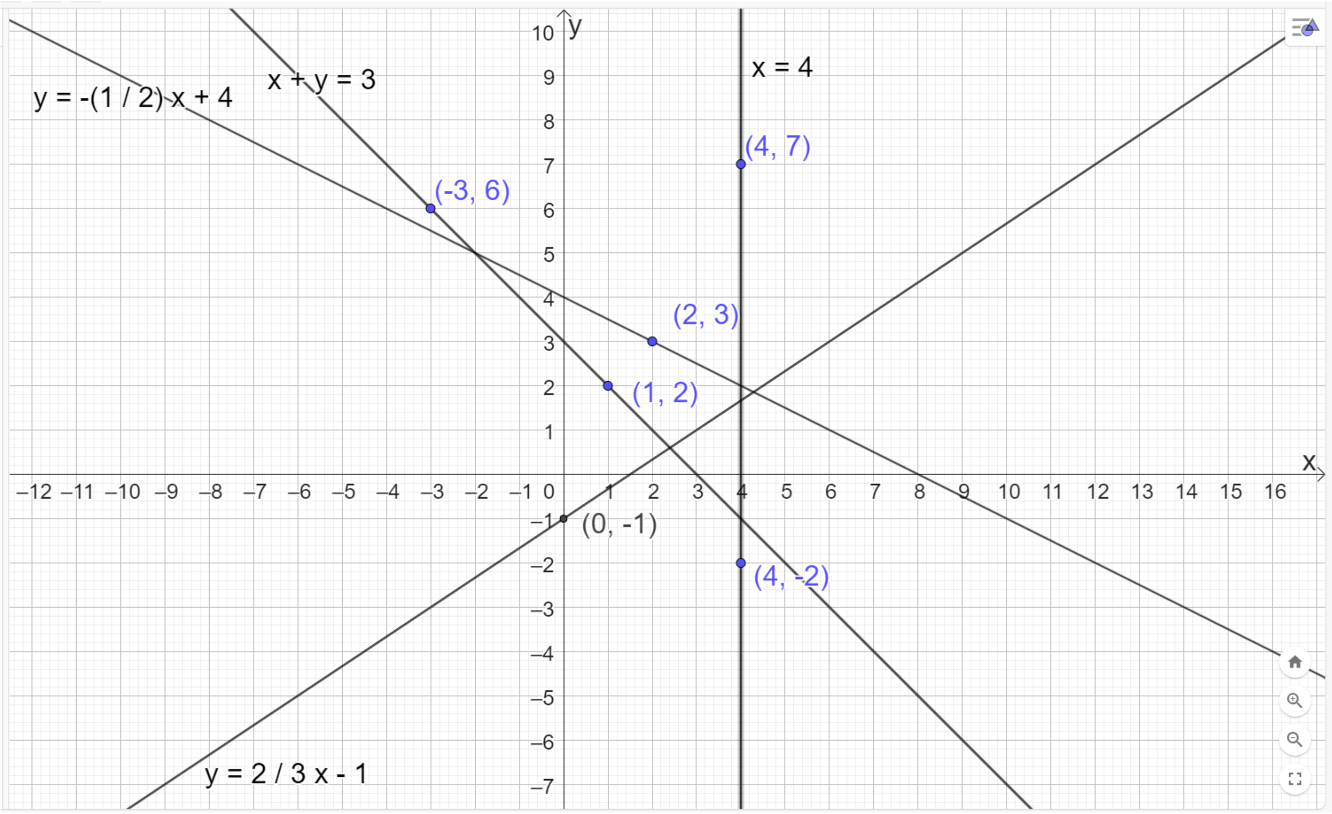


Figure 1.3: The graph containing the lines in the previous 4 examples

(e) Slope $m = -\frac{1}{2}$ and y-intercept $c = -5$

(f) Points $(-1, 4)$ and $(3, 4)$

(g) Points $(-3, 3)$ and $(-3, 6)$

(h) Slope $m = -\frac{2}{7}$ and point $(-5, 6)$

1.7 Parallel and Perpendicular Lines

1.7.1 Finding the Slope from the Equation of a Line

Given the equation of a line in standard form, $Ax + By = C$, the slope m of the line can be found by rearranging the equation into the slope-intercept form (1.5), $y = mx + c$. The slope m is then given by:

$$m = -\frac{A}{B}$$

provided $B \neq 0$. If $B = 0$, then the equation represents a vertical line of the form $x =$ constant, and the slope is undefined.

Example 1.7.1. Find the slope of the line with equation $3x - 4y = 12 - 2x$.

Answer: To find the slope, we rewrite it in the slope-intercept form:

$$\begin{aligned} 3x - 4y &= 12 - 2x \\ -4y &= -5x + 12 \\ 4y &= 5x - 12 \\ y &= \frac{5}{4}x - 3 \\ y &= mx + c \end{aligned}$$

Therefore, the slope of the line is $m = \frac{5}{4}$.

Exercises 1.7.2. Find the slope of the lines given by the following equations:

(a) $2x + 3y = 6$

(b) $5x - 4y = 10$

(c) $7x + y = -14$

(d) $-2y = 9$

(e) $x - 7 = 0$

(f) $x - 2 + y = 4 + 3y - 6x$

1.7.2 Parallel Lines

Vertical lines are parallel only with vertical lines. Two (non-vertical) lines having slopes m_1 and m_2, are parallel if and only if:

$$m_1 = m_2. \tag{1.7}$$

This means that that 2 non-vertical parallel lines have the same slope, and conversely, if two lines have the same slope, then the lines are parallel. Recall that if a line is vertical, that is the equation is in the form of (1.4), then the line does not have a defined slope and so you cannot apply Equation (1.7).

Example 1.7.3. Determine whether the lines represented by the equations $L_1 : y = 2x + 3$ and $L_2 : -4x = -2y + 7$ are parallel.

Answer: Comparing the equation of the line $L_1 : y = 2x + 3$ with the standard form $y = mx + c$ shows that the slope of the line L_1 is $m = 2$. Rearranging the equation of L_2 gives

$$\begin{aligned} -4x &= -2y + 7 \\ 0 &= 4x - 2y + 7 \\ 2y &= 4x + 7 \\ y &= 2x + \frac{7}{2} \\ y &= mx + c. \end{aligned}$$

Comparing the equation of the line with the standard form shows that the slope of the line L_2 is $m = 2$. Therefore both lines have a slope of 2. Since their slopes are equal, the lines are parallel.

Exercises 1.7.4. Determine whether the following pairs of lines are parallel:

(a) Line 1: $y = 3x + 7$, Line 2: $y = 3x - 2$

(b) Line 1: $2x - 4y = 8$, Line 2: $x + 2y = 1$

(c) Line 1: $4x + 2y = 10$, Line 2: $2x + y = 5$

(d) Line 1: $y = -\frac{1}{2}x + 6$, Line 2: $2y - x = 3$

(e) Line 1: $x = 6$, Line 2: $y = 6$

(f) Line 1: $2x = -\frac{1}{2}$, Line 2: $\pi - x = 0$

1.7.3 Perpendicular Lines

Two lines that both have defined slopes are said to be **perpendicular** if the product of their slopes is -1. If two lines have slopes m_1 and m_2, they are perpendicular if and only if:

$$m_1 \times m_2 = -1$$

This can also be written as:

$$m_2 = -\frac{1}{m_1}$$

provided that $m_1 \neq 0$ and $m_2 \neq 0$.

In the case where one of the lines does not have a defined slope, then that line is vertical and is perpendicular to horizontal lines, that is lines with slope $m = 0$. Note that the above formula does not apply if one line is vertical, since the slope is undefined. The only lines that are perpendicular to vertical lines are horizontal lines (slope is 0). For example, the line $x = 2$ is perpendicular to the line $y = 3$, but their slopes do not satisfy the formula $m_1 \times m_2 = -1$.

Example 1.7.5. Determine whether the lines represented by the equations $y = \frac{1}{2}x + 1$ and $y = -2x - 3$ are perpendicular.

Answer: The slope of the first line is $m_1 = \frac{1}{2}$, and the slope of the second line is $m_2 = -2$. Since:

$$m_1 \times m_2 = \frac{1}{2} \times -2 = -1,$$

the lines are perpendicular.

Exercises 1.7.6. Determine whether the following pairs of lines are perpendicular:

(a) Line 1: $y = 3x - 5$, Line 2: $y = -\frac{1}{3}x + 2$

(b) Line 1: $4x + y = 8$, Line 2: $x - 4y = 4$

(c) Line 1: $y = \frac{2}{3}x - 7$, Line 2: $3y = 2x + 5$

(d) Line 1: $2x - 3y = 6$, Line 2: $5x + 2y = 12$

(e) Line 1: $y = 7$, Line 2: $7 = x$

(f) Line 1: $x = 0$, Line 2: $y = x$

1.8 Points of Intersection of Two Lines

1.8.1 Finding the Point of Intersection

Consider two lines L_1 and L_2 that intersect at a unique point. To find this point, we need to solve the system of linear equations representing the two lines.

Given two lines with equations:

$$L_1 : a_1x + b_1y = c_1$$
$$L_2 : a_2x + b_2y = c_2$$

The coordinates of the intersection point (x, y) can be found by solving the following system of linear equations:

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$$

This can be done using either substitution or elimination methods.

Example Using Substitution

Find the point of intersection of the lines given by $2x + 3y = 6$ and $x - y = 1$.

Answer: First we will make one of the variables x or y the subject of one of the equations. Here we will choose to make x the subject of the second equation:

$$x = y + 1$$

We now make a substitution for x (since we choose x) in the first equation (since we choose the second equation):

$$2(y + 1) + 3y = 6$$

Simplify and solve for y:

$$\begin{aligned} 2y + 2 + 3y &= 6 \\ 5y + 2 &= 6 \\ 5y &= 4 \\ y &= \frac{4}{5} \end{aligned}$$

Substitute $y = \frac{4}{5}$ back into either original equation $x = y + 1$ to find x:

$$x = \frac{4}{5} + 1 = \frac{9}{5}$$

Thus, the point of intersection is $\left(\frac{9}{5}, \frac{4}{5}\right)$.

Example Using Elimination

Find the point of intersection of the lines given by $2x + 3y = 6$ and $x - y = 1$.

Answer: To eliminate one of the variables, we first multiply the second equation by 2 so that the coefficients of x in both equations are the same:

$$\begin{aligned} 2(x - y) &= 2(1) \\ 2x - 2y &= 2 \end{aligned}$$

Now, subtract (change the signs and add) the second equation from the first equation to eliminate x:

$$\begin{aligned} 2x + 3y &= 6 \\ -2x + 2y &= -2 \\ \hline 5y &= 4 \end{aligned}$$

Solve for y:

$$y = \frac{4}{5}$$

Substitute $y = \frac{4}{5}$ back into either original equation, here we choose $x - y = 1$ to find x:

$$\begin{aligned} x - \frac{4}{5} &= 1 \\ x &= 1 + \frac{4}{5} \\ x &= \frac{9}{5} \end{aligned}$$

Thus, the point of intersection is $\left(\frac{9}{5}, \frac{4}{5}\right)$.

1.8.2 Parallel Lines

Definition 1.8.1. Parallel lines are lines that satisfy either of the following conditions:

- They have the same slope or
- They are both vertical, that is the slopes are undefined.

In other words there are 2 cases to consider:

- **Parallel lines with a defined slope**: These are non-vertical lines that have the same slope.
- **Parallel lines with an undefined slope**: These are vertical lines, where the slope is undefined.

Given this definition, parallel lines can either (a) have the same slope or (b) both have undefined slopes. Parallel lines can be further classified based on whether they intersect. Parallel lines either never intersect or they are coincident, that is they are the same line, and so intersect in a infinite number of points.

Parallel Lines without Intersection

Two non-vertical lines are parallel without intersection if and only if they have the same slope but different y-intercepts. These lines will never intersect because they maintain a constant distance from each other.

Example 1.8.2. Find the points of intersection of the lines $2x + 3y = 5$ and $4x + 6y = 7$.

Answer: First, convert both equations to the slope-intercept form to compare their slopes:

$$L_1 : 2x + 3y = 5 \quad \Rightarrow \quad y = -\frac{2}{3}x + \frac{5}{3}$$

$$L_2 : 4x + 6y = 7 \quad \Rightarrow \quad y = -\frac{2}{3}x + \frac{7}{6}$$

Both lines have the same slope $-\frac{2}{3}$ but different intercepts ($\frac{5}{3}$ and $\frac{7}{6}$). Therefore, the lines are parallel and do not intersect.

Example 1.8.3. Find the points of intersection of the lines $x = 5$ and $x = 7$.

Answer: Both lines are vertical and so they are parallel. The also have different x-intercepts and so they do not intersect.

Coincident Lines

Two lines are coincident if they have the same set of points, that is they are the same line and thus intersect at infinitely many points.

Example 1.8.4. Find the points of intersection of the lines $2x + 3y = 5$ and $4x + 6y = 10$.

Answer: First, simplifying the second equation by dividing by 2 gives $2x + 3y = 5$. That is both equations represent the same line which are called coincident, meaning they overlap entirely and so intersect at infinitely many points. Every point on the line is a point of intersection. The set of points of intersection is $\{(x, y) \mid 2x + 3y = 5\}$.

Example 1.8.5. Find the points of intersection of the lines $x = -5$ and $x + 5 = 0$.

Answer: Again both equations represent the same line and so every point on the line is a point of intersection. The set of Therefore the set of points of intersection is $\{(x, y) \mid x = -5\}$.

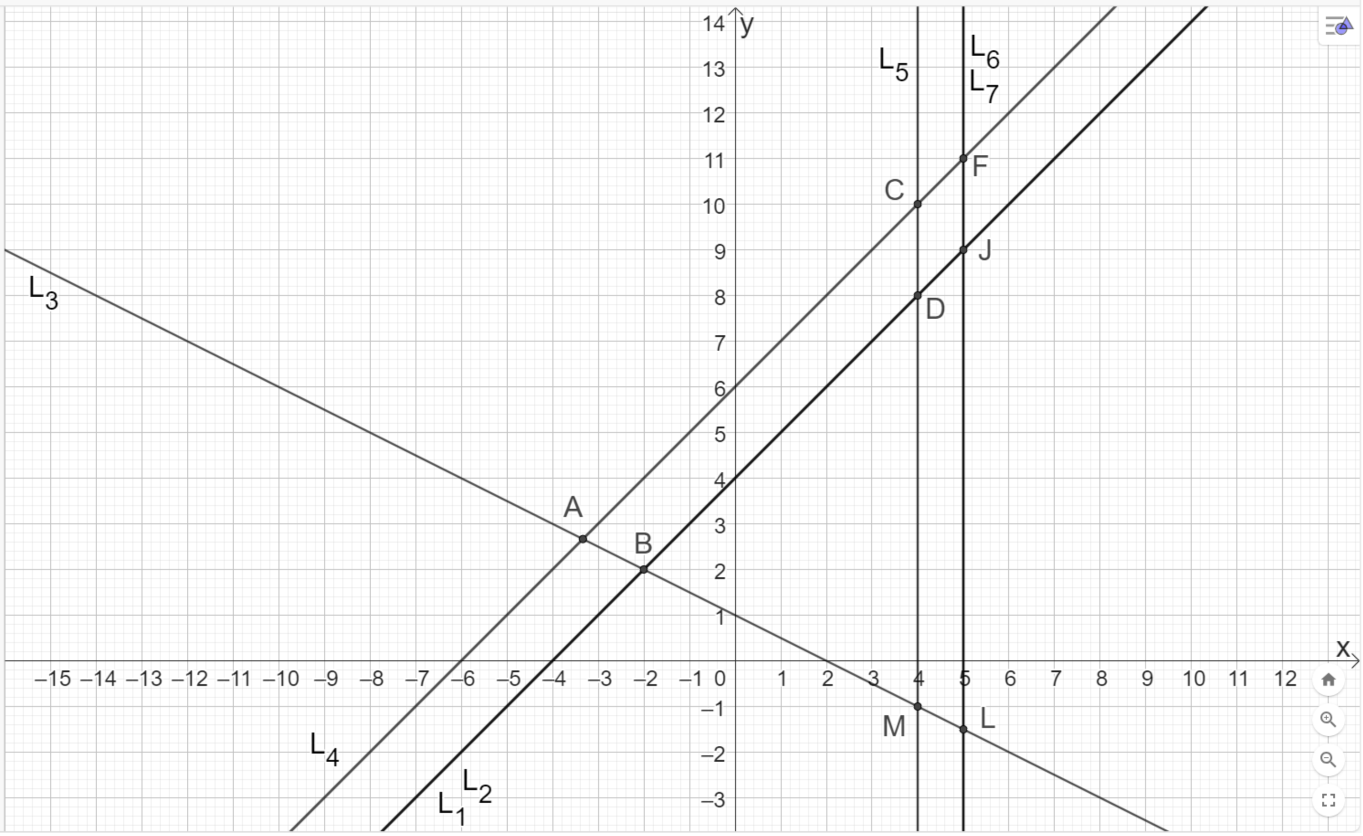


Figure 1.4: Graph of various lines with their points of intersection

Exercises 1.8.6. Determine the points of intersection of the following pairs of lines:

(a) $L_1 : y = 3x - 4, \quad L_2 : y = -3x + 2$

(b) $L_1 : y = 3, \quad L_2 : y = -3$

(c) $L_1 : 2x + 3y = 7, \quad L_2 : 4x - y = 5$

(d) $L_1 : 2y = x - 3, \quad L_2 : -2x = -4y - 6$

(e) $L_1 : x - 5 = 1, \quad L_2 : y + 6 = 0$

(f) $L_1 : 2x = 3 - x, \quad L_2 : 7x = 7$

1.9 Shortest Distance Between a Point and a Line

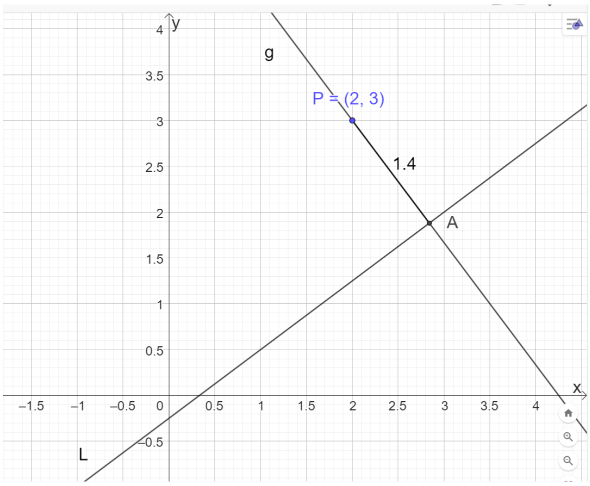
1.9.1 Introduction

The shortest distance between a point and a line is the perpendicular distance from the point to the line. This distance is crucial in various applications, such as geometry, physics, and optimisation problems. Given a point $P = (x_1, y_1)$ and a line $L : ax + by + c = 0$, the shortest distance d from the point to the line is given by the formula:

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

This formula calculates the perpendicular distance by considering the coefficients of the line equation and the coordinates of the point.

1.9.2 Example Calculation



Find the shortest distance between the point $P = (2, 3)$ and the line L given by the equation $4y = 3x - 1$.
Answer: First we write the equation in standard form, that is $3x - 4y - 1 = 0$. Using the distance formula we get:

$$d = \frac{|3(2) - 4(3) - 1|}{\sqrt{3^2 + (-4)^2}} = \frac{|6 - 12 - 1|}{\sqrt{9 + 16}} = \frac{|-7|}{5} = \frac{7}{5}$$

So, the shortest distance between the point $P = (2, 3)$ and the line $4y = 3x - 1$ is $\frac{7}{5}$ units.

1.9.3 Special Cases

- **Point on the Line**: If the point $P = (x_1, y_1)$ lies on the line L, then the shortest distance $d = 0$.
- **Vertical or Horizontal Lines**: The distance formula still applies to vertical and horizontal lines, though it can be computed directly using the difference in the relevant coordinate (x for vertical, y for horizontal).

Exercises 1.9.1. Determine the shortest distance between the given points and lines:

(a) $P = (-2, 7)$, $L : y = 3$

(b) $P = (1, 2)$, $L : x + y = 3$

(c) $P = (-5, 8)$, $L : x = 3$

(d) $P = (-2, 5)$, $L : 2x - y + 1 = 0$

(e) $P = (0, 0)$, $L : 4x + 3y = 7$

(f) $P = (3, -1)$, $L : 5x + 2y - 4 = 0$

1.10 Circles

1.10.1 Introduction to the Equation of a Circle

The equation of a circle in its standard form is derived from the definition of a circle: a set of all points in a plane that are equidistant from a fixed point called the centre. If the centre of the circle is (h, k) and the radius is r, then the equation of the circle is:

$$(x - h)^2 + (y - k)^2 = r^2 \tag{1.8}$$

This equation represents a circle with centre (h, k) and radius r.

Note that if the centre is the origin then the equation simplifies to $x^2 + y^2 = r^2$.

Example 1.10.1. Write down the centre and the radius of the circle represented by the equation $(x+7)^2+(y-3)^2 = 23$.

Answer: The centre point of the circle is $(-7, 3)$ and the radius is $\sqrt{23}$.

Exercises 1.10.2. Write down the centre and the radius of the following circles

(a) $x^2 + y^2 = 16$

(b) $(x - 5)^2 + (y + 9)^2 - 10 = 0$

1.10.2 Finding the Equation of a Circle

Given the centre and Radius

To find the equation of a circle given the centre (h, k) and radius r, substitute the values into the standard form:

$$(x-h)^2 + (y-k)^2 = r^2$$

Example 1.10.3. Find the equation of the circle with centre $(3, -2)$ and radius 5.

Answer:

$$(x-3)^2 + (y+2)^2 = 25$$

Given the centre and a Point on the Circle

If the centre (h, k) and a point (x_1, y_1) on the circle are known, first calculate the radius using the distance formula:

$$r = \sqrt{(x_1-h)^2 + (y_1-k)^2}$$

Then, substitute the centre and the radius into the standard form:

$$(x-h)^2 + (y-k)^2 = r^2$$

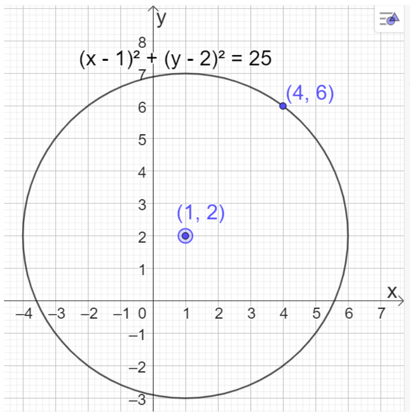
Example 1.10.4. Find the equation of the circle with centre $(1, 2)$ that passes through the point $(4, 6)$.

Answer: We shall use the distance between 2 points formula (1.1) to calculate the radius.

$$r = \sqrt{(4-1)^2 + (6-2)^2} = \sqrt{9+16} = 5.$$

The equation of the circle is:

$$(x-1)^2 + (y-2)^2 = 25$$



Exercises 1.10.5. Write down the equation of the circle with the following information.

(a) radius $= 4$ and centre $= (-3, 0)$

(b) radius $= \sqrt{13}$ and centre $= (-1, 2)$

(c) centre $= (-4, -5)$ and contains the point $(1, 7)$

(d) centre $= (1, -3)$ and contains the point $(2, 4)$

1.10.3 Intersection of a Line and a Circle

To find the point(s) of intersection between a line and a circle, you can follow the following procedure. The procedure assumes that the variables are x and y.

1. Write one of the variables (x or y you choose) as the subject of the equation representing the line.
2. Use the rearranged linear equation from (1) to substitute in for the variable that is the subject of the equation into the equation of the circle. This results in a quadratic equation in the variable that was not chosen in (1) to be the subject.

3. Solve this quadratic equation for the non-subject variable either by factorising or by using the "$-b$"-formula.

4. Substitute the value(s) found in (3) for the non-subject variable into the rearranged linear equation in (1) to obtain the corresponding subject variable's values.

5. The corresponding values are the coordinates of the point(s) of intersection.

Example 1.10.6. Find the point(s) of intersection between the circle $(x-3)^2+(y+2)^2 = 25$ and the line $2x-y-3 = 0$.

Answer: We will follow the procedure above.

1. We choose to make y the subject of $2x-y-3 = 0$ and so $y = 2x-3$

2. We substitute $2x-3$ in for y into the equation of the circle and simplify.

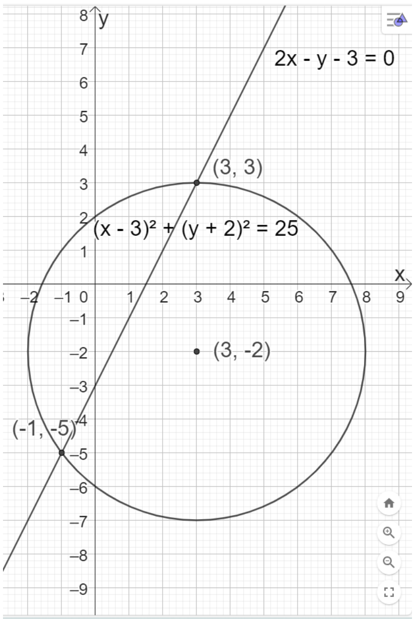
$$\begin{aligned}(x-3)^2+\big((2x-3)+2\big)^2 &= 25\\ (x-3)^2+(2x-1)^2 &= 25\\ x^2-3x-3x+9+4x^2-2x-2x+1-25 &= 0\\ 5x^2-10x-15 &= 0\\ x^2-2x-3 &= 0\end{aligned}$$

   This results in a quadratic equation in x (the variable that was not chosen in (1) to be the subject).

3. Solving $x^2-2x-3 = 0$ by factorising gives $(x-3)(x+1) = 0$ and so $x = 3$ or $x = -1$.

4. Substitute $x = 3$ or $x = -1$ into $y = 2x-3$ gives $y = 3$ and $y = -5$ respectively.

5. Therefore the points of intersection are $(3, 3)$ and $(-1, -5)$.



Exercises 1.10.7. Find the point(s) of intersection between the circle C and the line L in each of the following cases.

(a) $C : x^2+y^2 = 25$ and $L : x = 4$

(b) $C : x^2+y^2 = 10$ and $L : y = x$

(c) $C : (x-5)^2+(y+3)^2 = 20$ and $L : -2x+y = -13$

(d) $C : (x-5)^2+(y+3)^2 = 20$ and $L : 3+y = 2x$

1.10.4 The Position of a Point Relative to a Circle

Any point must lie either inside, on, or outside a circle. The position of a point relative to a circle is determined by its distance d to the centre of the circle.

- The point lies **inside the circle:** if and only if $d < r$.
- The point lies **on the circle:** if and only if $d = r$.
- The point lies **outside the circle:** if and only if $d > r$.

Note that both d and r are positive values and so $d < r$ if and only if $d^2 < r^2$. Similarly $d = r$ if and only if $d^2 = r^2$ and $d > r$ if and only if $d^2 > r^2$. This fact can be used to simplify the process of determining the position of a point relative to a circle as follows.

To determine the position of the point (x_1, y_1) relative to the circle with equation $(x-h)^2 + (y-k)^{=} r^2$ we substitute the point (x_1, y_1) in for (x, y) in the equation of the circle.

$$
\begin{aligned}
(x-h)^2 + (y-k)^2 &= r^2 \\
(x_1-h)^2 + (y_1-k)^2 &= r^2
\end{aligned}
$$

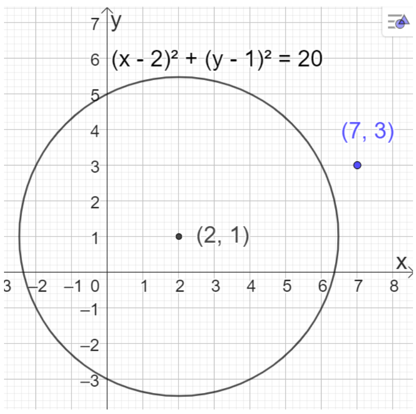
Let d be the distance from the point (x_1, y_1) to the centre of the circle (h, k). Then the value on the left hand side (LHS) of the equation is d^2, and the value on the right hand side (RHS) of the equation is r^2. Therefore we can just compare the values on each side of this equation to determine the relative position.

- The point lies **inside the circle:** if and only if $d^2(LHS) < r^2(RHS)$
- The point lies **on the circle:** if and only if $d^2(LHS) = r^2(RHS)$
- The point lies **outside the circle:** if and only if $d^2(LHS) > r^2(RHS)$.

Example 1.10.8. Determine the position of the point $(7, 3)$ relative to the circle $(x-2)^2 + (y-1)^2 = 20$.

Answer: Substitute the point $(7, 3)$ in for (x, y) in the equation of the circle

$$
\begin{aligned}
(x-2)^2 + (y-1)^2 &= 20 \\
(7-2)^2 + (3-1)^2 &= 20 \\
(5)^2 + (2)^2 &= 20 \\
29 &= 20
\end{aligned}
$$



$d^2 = 29 > 20 = r^2$ and so the point lies outside the circle.

Exercises 1.10.9. Determine the position of the point P relative to the circle C in each of the following cases. That is, determine whether the point P lies inside, on or outside the circle C.

(a) $P = (1, -2)$ and $C : (x+3)^2 + (y-1)^2 = 21$

(b) $P = (3, 2)$ and $C : (x-6)^2 + (y-5)^2 = 19$

(c) $P = (-4, 2)$ and $C : (x-7)^2 + (y-8)^2 = 157$

(d) $P = (11, -24)$ and $C : (x-12)^2 + (y-20)^2 = 17$

1.10.5 Verifying a Diameter

To verify that two points (x_1, y_1) and (x_2, y_2) are endpoints of a diameter of a given circle, you can check the following 2 criteria:

- The centre of the circle is the midpoint between the points.
- that the distance between the two points equals the diameter $2r$, where r is the radius of the circle.

Note that alternatively you could check if both points are on the circle and the midpoint is the centre.

Example 1.10.10. Verify that the points $(-12, 9)$ and $(0, -3)$ are endpoints of a diameter of the circle represented by the equation $(x+6)^2 + (y-3)^2 = 72$.

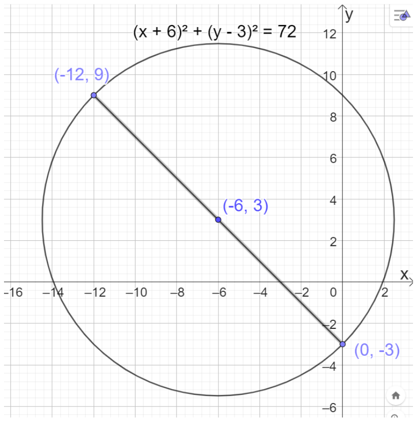
Answer: Using Equation (1.2), the midpoint P of the line segment with endpoints $(-12,9)$ and $(0,-3)$ is given by

$$P = \left(\frac{-12+0}{2}, \frac{9-3}{2}\right) = (-6,3),$$

which is the centre of the circle.
Using Equation (1.1), the distance d between the points is given by

$$d = \sqrt{(0-(-12))^2 + (-3-9)^2} = \sqrt{144+144} = 2\sqrt{72},$$

which is twice the radius of the circle. Therefore the points $(-12,9)$ and $(0,-3)$ are endpoints of a diameter of the circle $(x+6)^2 + (y-3)^2 = 72$.



Exercises 1.10.11. Determine whether the points P and Q are endpoints of a diameter of the circle C in each of the following cases.

(a) $P = (4,-5)$, $Q = (-2,3)$ and $C : (x+1)^2 + (y-1)^2 = 10$

(b) $P = (1,0)$, $Q = (-7,4)$ and $C : (x+3)^2 + (y-2)^2 = 20$

(c) $P = (5,6)$, $Q = (-1,2)$ and $C : (x-7)^2 + (y+13)^2 = 31$

(d) $P = (5,6)$, $Q = (-1,2)$ and $C : (x-2)^2 + (y-4)^2 = 52$

1.10.6 Equation of a Tangent to a Circle

When the Equation of the Circle is in Standard Form

Suppose that the equation of the circle is given in standard form (Equation (1.8)), that is $(x-h)^2 + (y-k)^2 = r^2$. Then the equation of the tangent to a circle with centre (h,k) and radius r at the point (x_1,y_1) is given by:

$$(x-h)(x_1-h) + (y-k)(y_1-k) = r^2$$

Example 1.10.12. Find the equation of the tangent to the circle $(x-3)^2 + (y+2)^2 = 25$ at the point $(6,2)$.

Answer:

$$\begin{aligned}(x-h)(x_1-h) + (y-k)(y_1-k) &= r^2\\ (x-3)(6-3) + (y+2)(2+2) &= 25\\ 3(x-3) + 4(y+2) &= 25\\ 3x + 4y &= 26.\end{aligned}$$

When the Equation of the Circle is in General Form

For a circle with centre $(-g,-f)$ and radius $r = \sqrt{g^2+f^2-c}$, the equation of the circle is given by

$$x^2 + y^2 + 2gx + 2fy + c = 0, \tag{1.9}$$

and the equation of the tangent at the point (x_1,y_1) is given by:

$$xx_1 + yy_1 + g(x+x_1) + f(y+y_1) + c = 0 \tag{1.10}$$

Example 1.10.13. Let C be the circle given by the equation $x^2 + y^2 - 9x + 8y + 9 = 0$. Find the centre and the radius of the circle C. Also find the equation of the tangent to the circle C at the point $(6, 1)$.

You are not expected to verify that the point $(6, 1)$ is on the circle. The question does not make sense if the point is not on the circle. However, for this example we will verify this fact now by substituting $(6, 1)$ in for (x, y) in the equation of the circle. $x^2 + y^2 - 9x + 8y + 9 = (6)^2 + (1)^2 - 9(6) + 8(1) + 9 = 0$.

Answer:

Comparing the equation $x^2 + y^2 - 9x + 8y + 9 = 0$ to Equation (1.9) we see that $g = -\frac{9}{2}$, $f = 4$, and $c = 9$. Therefore the centre is $(-g, -f) = (\frac{9}{2}, -4)$ and the radius $r = \sqrt{g^2 + f^2 - c} = \sqrt{\left(-\frac{9}{2}\right)^2 + (4)^2 - (9)} = \frac{\sqrt{109}}{2}$.

Using Equation (1.10), the equation of the tangent to C at the point $(6, 1)$ is given by:

$$\begin{aligned}
xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c &= 0 \\
6x + y + \left(-\frac{9}{2}\right)(x + 6) + 4(y + 1) + 9 &= 0 \\
6x + y + -\frac{9}{2}x - 27 + 4y + 4 + 9 &= 0 \\
\frac{3}{2}x + 5y - 14 &= 0 \\
3x + 10y - 28 &= 0.
\end{aligned}$$

Exercises 1.10.14. Find the equation of the tangent to the circle C at the point P in each of the following cases.

(a) $P = (4, 6)$ and $C : (x - 1)^2 + (y - 2)^2 = 25$

(b) $P = (-11, -2)$ and $C : (x + 5)^2 + (y - 6)^2 = 100$

(c) $P = (4, 1)$ and $C : x^2 + y^2 + 4x - 8y - 25 = 0$

(d) $P = (-2, -1)$ and $C : x^2 + y^2 - 6x + 10y - 7 = 0$.

Chapter 2

Transformation Geometry in 2D

2.1 Introduction and Application in Computer Science

Transformation geometry is a fundamental concept in computer science, with wide-ranging applications in various fields such as computer graphics, computer game design, and robotics.

2.1.1 Applications in Computer Science

Computer Game Design: In computer games, especially those involving 2D environments, transformation geometry is used extensively to manipulate characters, objects, and backgrounds. For example, when a character moves across the screen, this movement is typically achieved through a translation transformation. Similarly, when a character rotates or turns, a rotation transformation is applied. Combining these transformations allows for complex movements and interactions, creating a more dynamic and engaging gaming experience. Additionally, these transformations are essential for collision detection, ensuring that characters and objects interact correctly within the game world.

Computer Graphics: Transformation geometry forms the backbone of computer graphics, where it is used to position and manipulate images, shapes, and models. Operations such as scaling, rotating, and translating objects within a scene are crucial for rendering 2D and 3D graphics. In 2D graphics, transformations help in tasks like animating characters, creating effects, and designing user interfaces. These transformations are applied using matrices, which allow for efficient computation and seamless integration of multiple transformations.

Robotics: In robotics, transformation geometry is used to control the movement and orientation of robotic arms and other mechanical components. By applying translations and rotations, robots can be programmed to perform precise movements, such as picking up objects, assembling parts, or navigating through environments. In autonomous systems, these transformations are often combined with sensor data to make real-time decisions about movement and positioning.

Image Processing: Transformation geometry is also critical in image processing, where it is used to manipulate digital images. Common operations include rotating an image, scaling it to different sizes, and translating it to different positions within a frame. These operations are vital for tasks like image registration, where multiple images are aligned, or in computer vision, where objects within images need to be identified and tracked.

2.1.2 Overview of the Chapter

In this chapter, we will explore the basic transformations in 2D geometry, focusing on translations and rotations. We will also discuss how these transformations can be combined to achieve more complex manipulations of geometric objects. By understanding these concepts, you will be able to apply them in various computer science applications, from game development to robotics and beyond.

2.2 Linear Transformations and Matrix Representation

A **linear transformation** is a function that maps points from one vector space to another, preserving the operations of vector addition and scalar multiplication. For our purposes in this chapter, you can think of a linear transformation simply as a function that can be represented by a matrix. In 2D geometry, linear transformations include operations such as rotations, reflections, scalings, and shears. These transformations can be represented using matrices, which allow for efficient computation and manipulation of points and geometric objects.

Let $\mathbf{T}$ be the linear transformation represented by the matrix $\begin{pmatrix} 1 & 5 \\ 2 & 1 \end{pmatrix}$. $\mathbf{T}$ is a shear transformation and scaling. Also, let P be the point $(2, 8)$. Then the image P' of the point P under the transformation $\mathbf{T}$ is calculated by the following matrix multiplication .

$$P' = \mathbf{T}P = \begin{pmatrix} 1 & 5 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 8 \end{pmatrix} = \begin{pmatrix} 1(2) + 5(8) \\ 2(2) + 1(8) \end{pmatrix} = \begin{pmatrix} 42 \\ 12 \end{pmatrix}.$$

Note that when asked to find the image of an object under a linear transformation you must use the method of matrix multiplication. We are learning the technique, if you decide to answer the question using a different method you will receive no marks even if you apply the alternative method correctly.

2.3 Translations

A **translation** in 2D space involves shifting a geometric object from one location to another without rotating or resizing it. Unlike the linear transformations discussed earlier, a translation in 2D is not a linear transformation because it does not preserve the origin (i.e., it does not map the origin to the origin unless the translation vector is zero).

However, by moving from 2D to 3D space, we can represent a 2D translation as a linear transformation. This is achieved by using homogeneous coordinates, where each point (x, y) in 2D is represented as $(x, y, 1)$ in 3D. A translation by (t_x, t_y) can then be represented by the following 3×3 matrix:

$$\text{Translation matrix: } \mathbf{T} = \begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix} \tag{2.1}$$

To obtain the image of a point (x, y) under this translation, we perform the following matrix multiplication:

$$\mathbf{T} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x + t_x \\ y + t_y \\ 1 \end{pmatrix}$$

The resulting vector represents the translated point in homogeneous coordinates, and the first two components $(x', y') = (x + t_x, y + t_y)$ give the coordinates of the translated point in 2D space.

This approach allows us to apply translations as part of a sequence of linear transformations, making it easier to combine translations with other linear transformations like rotations and scalings in a unified framework.

Example 2.3.1. Translate the point $A = (2, 3)$ by 4 units in the x-direction and 5 units in the y-direction.

Answer: Let T be the translation. Then using $t_x = 4$ and $t_y = 5$ in Equation (2.1) gives $T = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{pmatrix}$. Let A' be the image of the point A under T. Then $A' = (6, 8)$ since

$$\begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 2+4 \\ 3+5 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \\ 1 \end{pmatrix}.$$

Note that we multiplied T by the representation of the point $(2, 3)$ in homogeneous coordinates and the result was the representation of the point $A' = (6, 8)$ in homogeneous coordinates.

Exercises 2.3.2. Translate the following points by the given translations.

(a) $B = (5, 7)$ by $t_x = -3$, $t_y = -2$

(b) $C = (-4, 2)$ by $t_x = -6$, $t_y = 5$

(c) $D = (0, 0)$ by $t_x = 12$, $t_y = \pi$

(d) $E = (-3, -1)$ by $t_x = 7$, $t_y = -4$

2.4 Rotations

A rotation in 2D space involves rotating a geometric object around a fixed point, usually the origin. The amount of rotation is specified by an angle θ. A rotation in 2D space is a linear transformation and so we can represent it by a 2×2 matrix. However, we will want to combine a rotation with a translation which is represented by a 3×3 matrix, and so we will also use a 3×3 matrix to represent a rotation in 2D space. Note that in Mathematics anti-clockwise is the positive direction and we shall use degrees as the unit of measurement, so a rotation of 50 about the origin means an anti-clockwise rotation of 50 degrees about the origin .

Rotation Matrix

An anti-clockwise rotation R by an angle θ about the origin is represented by the matrix:

$$R = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.2}$$

The new coordinates (x', y') are obtained by multiplying the rotation matrix by the original coordinates $(x, y, 1)$.

Example 2.4.1. Rotate the point $A = (2, 3)$ by 90° counterclockwise about the origin.

Answer: Let R be the matrix representing the rotation and Let A' be the image of A under R. The angle $\theta = 90^\circ$ and so $\cos(90^\circ) = 0$ and $\sin(90^\circ) = 1$. Using these values in Equation (2.2) gives

$$R = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We now multiple R by the point A in homogeneous coordinates.

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}.$$

Therefore the rotated point is $A' = (-3, 2)$.

Exercises 2.4.2. Rotate the following points by the given angles about the origin.

(a) $B = (5, 7)$ by $45°$

(b) $C = (-4, 2)$ by $180°$

(c) $D = (0, 0)$ by $213°$ clockwise

(d) $E = (-3, -3)$ by $90°$ clockwise

2.5 Finding the Image of 2D Shapes under Transformations

In this section, we explore how to find the image of simple 2D shapes under transformations such as translations and rotations. A key property of affine transformations, which include translations and rotations, is that they preserve the relative structure of geometric shapes. This property, known as **affine invariance**, allows us to find the image of a complex shape by only calculating the images of a subset of its key points.

2.5.1 Affine Invariance

Affine invariance means that under an affine transformation, straight lines remain straight, parallel lines remain parallel, and the ratios of lengths along parallel lines are preserved. Consequently, When finding the image of a 2D shape under a transformation like a translation or a rotation, it is often sufficient to calculate the image of specific key points that define the shape. Below is a summary of these key points for different 2D shapes:

In the list below the image of a point P under the transformation is denoted by P'.

- The image of the line segment $[A, B]$, is the line segment $[A', B']$.
- The image of the line passing through A and B, is the line passing through A' and B'.
- The image of the triangle ABC, is the triangle $A'B'C'$.
- The image of the rectangle $ABCD$, is the rectangle $A'B'C'D'$.
- The image of the circle with centre P and radius r, is the circle with centre P' and radius r.

Example 2.5.1. let $A = (1, 3)$, $B = (-2, 5)$ and let T be a translation by 2 units in the x-direction and -3 units in the y-direction. To find the image of the line segment $[A, B]$ under the transformation T.

Answer: We calculate the image of the points A and B under the translation by matrix multiplication. However by one of the many nice properties of matrices we can perform this calculation as 1 matrix multiplication. Recall that we need to write the points in homogeneous coordinates prior to the multiplication.

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 3 & 5 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \\ 1 & 1 \end{pmatrix}.$$

Therefore the image is the line segment $[A', B']$, where $A' = (3, 0)$ and $B' = (0, 2)$.

Example 2.5.2. Let $A = (-5, 4)$, $B = (1, -3)$ and $C = (4, 2)$ and let R be a rotation $40°$ about the origin. Find the image of the triangle ABC under R.

Answer:

$$\begin{pmatrix} \cos(40) & -\sin(40) & 0 \\ \sin(40) & \cos(40) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -5 & 1 & 4 \\ 4 & -3 & 2 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0.766 & -0.643 & 0 \\ 0.643 & 0.766 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -5 & 1 & 4 \\ 4 & -3 & 2 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -6.402 & 2.695 & 1.778 \\ -0.151 & -1.655 & 4.104 \\ 1 & 1 & 1 \end{pmatrix}.$$

Therefore the image is the triangle $A'B'C'$, where $A' = (-6.402, -0.151)$, $B' = (2.695, -1.655)$ and $C' = (1.778, 4.104)$.

2.5.2 Exercises

Let $A = (1, -2)$, $B = (6, -2)$, $C = (6, 5)$ and $D = (1, 5)$. Find the image of the following shapes under the given transformations.

a) The line segment $[A, B]$ under a translation of -3 units in the x-direction and 2 units in the y-direction.

b) The triangle ABC under a rotation by 90° counterclockwise about the origin.

c) The rectangle $ABCD$ under a translation of 6 units in the x-direction and -3 units in the y-direction.

d) A circle with centre at D and radius $\sqrt{19}$ under a rotation of 50° about the origin.

2.6 Combinations of Translations and Rotations

2.6.1 Introduction to Combinations of Transformations

In many applications, it is necessary to apply a sequence of transformations to a geometric object. For example, a geometric object might be rotated, translated, and then rotated again. These combined transformations can be represented by multiplying the corresponding matrices. This implies the fact that combining a number of linear transformations on $\mathbb{R}^3$ results in a linear transformation on $\mathbb{R}^3$, since multiplying a number of 3×3 matrices results in a 3×3 matrix.

Matrix Representation of Combined Transformations

Let R_1 and R_2 be rotations and let by T be a translation. Then the transformation of R_1 then T and then R_2 is represented by the matrix $R_2 T R_1$. It is really important to note the order of the matrices. Recall that matrix multiplication is not commutative. The rotation R_1 is performed first and it is positioned on the right. The translation T is performed second and it is positioned second from the right. The rotation R_2 is performed third and it is positioned third from the right.

A Rotation R_P of θ° about a point P is the same as applying the following 3 transformations in the correct order.

1. First, The translation $T_{P\to O}$ that takes the point P to the origin.

2. Then the rotation R_O of θ° about the origin.

3. Finally, the translation $T_{O\to P}$ that takes the origin to the point P.

That is

$$R_P = T_{O\to P} R_O T_{P\to O}. \tag{2.3}$$

Again, it is really important to note the order of the matrices. The translation $T_{P\to O}$ is performed first and it is positioned on the right. The rotation R_O is performed second and it is positioned second from the right. The translation $T_{O\to P}$ is performed third and it is positioned third from the right.

Although the exam questions on this topic typically involve rotations, the following example does not simply to ease the calculations as a first example of combining linear transformations.

Example 2.6.1. Let $A = (11, -7)$ and let M be a transformation represented by $M = \begin{pmatrix} 2 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Let T_1 be a translation of $+2$ in the x-direction and -3 in the y-direction and let T_2 be a translation of -5 in the x-direction and

+6 in the y-direction. Write down the 3×3 matrix S representing the transformation of applying T_1 first then M and then T_2. Find the image of A under the transformation S.

Answer:

$$T_1 = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix} \qquad M = \begin{pmatrix} 2 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad T_2 = \begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{pmatrix}$$

The transformation $S = T_2MT_1$ and so we calculate that now.

$$MT_1 = \begin{pmatrix} 2 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 5 \\ -1 & 1 & -5 \\ 0 & 0 & 1 \end{pmatrix}$$

$$S = T_2MT_1 = \begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 5 \\ -1 & 1 & -5 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

We now convert the point A into a 3×1 matrix P. Then the image of the point A can be found by calculating $P' = SP$.

$$SP = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 11 \\ -7 \\ 1 \end{pmatrix} = \begin{pmatrix} 22 \\ -17 \\ 1 \end{pmatrix}.$$

Therefore the image of A under the transformation S is the point $(22, -17)$.

Example 2.6.2. Let $A = (2, 7)$. Rotate the point A by $25°$ anti-clockwise about the point $P = (-2, 3)$.

Answer:

First we shall write down the relevant matrices

$$T_{O \to P} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_O = \begin{pmatrix} \cos(25) & -\sin(25) & 0 \\ \sin(25) & \cos(25) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad T_{P \to O} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}.$$

We now calculate the matrix R_P by matrix multiplication

$$\begin{aligned} R_P = T_{O \to P} R_O T_{P \to O} &= \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0.906 & -0.423 & 0 \\ 0.423 & 0.906 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0.906 & -0.423 & 3.081 \\ 0.423 & 0.906 & -1.872 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0.906 & -0.423 & 1.081 \\ 0.423 & 0.906 & 1.128 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

We now calculate the image point A' of A under R_P by multiplying R_P by the the representation of A in homogeneous coordinates.

$$\begin{pmatrix} 0.906 & -0.423 & 1.081 \\ 0.423 & 0.906 & 1.128 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 7 \\ 1 \end{pmatrix} = \begin{pmatrix} -0.068 \\ 8.316 \\ 1 \end{pmatrix}.$$

Therefore the image point is $A' = (-0.068, 8.316)$. Note that we have used rounding in the Matrix R_O and so if you perform this calculation you could get a slightly different answer.

Exercises 2.6.3. In each of the following questions:

(i) Express each transformation as a matrix.
(ii) Find the overall transformation matrix.
(iii) Hence, find the image.

(a) Rotate $B = (3,4)$ by 60° about the origin, then translate by $(t_x = 2, t_y = 3)$, and then rotate by 120° clockwise.

(b) Translate $C = (-1,-2)$ by $(t_x = -3, t_y = 4)$, then rotate by 45° anti-clockwise, and then translate by $(t_x = 1, t_y = 1)$.

(c) Rotate the circle $(x-2)^2+(y-3)^2 = 23$ by 90° clockwise about the origin, then translate by $(t_x = 0, t_y = -5)$, and then rotate by 10° about the origin.

(d) Rotate the triangle XYZ by 70° counterclockwise about the point $B = (3,4)$, where $X = (-5,1)$, $Y = (7,7)$ and $Z = (0,10)$.

Chapter 3

Differentiation

3.1 Introduction to Differentiation

Differentiation is a fundamental concept in calculus that deals with the rate at which a quantity changes. It is the process of finding the derivative of a function, which represents the slope of the function at any given point for single-variable functions. For multivariable functions, the concept generalises to the gradient, which is a vector indicating the direction and rate of the steepest increase in the function's value.

3.1.1 Brief History

The concept of differentiation has its roots in the development of calculus, which emerged in the 17th century. The two mathematicians most famously associated with the creation of calculus are Sir Isaac Newton and Gottfried Wilhelm Leibniz. While Newton focused on the physical interpretation of calculus in terms of motion and change, Leibniz developed the formal rules and notation that are still used today. Despite initial disputes over the credit for these discoveries, both Newton and Leibniz are now recognised as the co-founders of calculus.
Differentiation was initially applied to problems in physics, particularly in understanding motion, but its utility quickly spread to other fields of mathematics and science.

3.1.2 Applications in Computer Science

Differentiation has a wide range of applications in computer science, particularly in the following areas:

- **Optimisation:** In machine learning, differentiation is used to optimise models by minimising or maximising objective functions. The gradients, which are derivatives of the objective functions, guide the update of model parameters during training.
- **Numerical Methods:** Numerical differentiation techniques are used in various algorithms that approximate derivatives, especially when analytical solutions are difficult or impossible to obtain.

- **Computer Graphics:** In computer graphics, differentiation is employed in rendering techniques, such as calculating normals for shading, which require the derivative of surface functions.
- **Algorithmic Differentiation:** This is a set of techniques to automatically compute derivatives of functions specified by computer programs. It is particularly useful in fields like deep learning and scientific computing.
- **Signal Processing:** Differentiation is used in the analysis of signals, where the derivative of a signal can provide information about its frequency content or other characteristics.

Differentiation is a powerful tool that underpins much of modern computational theory and practice, providing essential methods for analysing and optimising complex systems.

3.2 Differentiation of Functions

3.2.1 Introduction to Differentiation

Differentiation is a fundamental concept in calculus used to determine how a function changes as its input changes. For a function $f(x)$ of a single real variable x, the derivative $f'(x)$ represents the rate of change of the function with respect to x. The derivative at a point gives the slope of the tangent line to the function's graph at that point. Formally, the derivative of $f(x)$ is defined by:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

This limit, if it exists, provides the slope of the tangent line to the function at the point x.

3.3 Different Notations for Differentiation

In calculus, different notations are used to represent the derivative of a function. These notations are equivalent and describe the same concept: the rate of change of a function with respect to its variable. Below is a table summarising the different notations for differentiation.

Function	Derivative	Explanation
$f(x)$	$\frac{\mathrm{d}}{\mathrm{d}x}\big(f(x)\big)$	The derivative of the function f with respect to x, using Leibniz's notation.
$f(x)$	$f'(x)$	The derivative of the function f with respect to x, using the prime notation.
y	$\frac{\mathrm{d}y}{\mathrm{d}x}$	The derivative of the variable y with respect to x, using Leibniz's notation.
y	y'	The derivative of the variable y with respect to x, using the prime notation.
$f'(x)$	$f''(x)$	The second derivative of the function f with respect to x, using the prime notation.
$\frac{\mathrm{d}y}{\mathrm{d}x}$	$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$	The second derivative of the variable y with respect to x, using Leibniz's notation.
$f^{(n-1)}(x)$	$f^{(n)}(x)$	The n^{th} derivative of the function f with respect to x, using the prime notation.
$\frac{\mathrm{d}^{n-1}y}{\mathrm{d}x^{n-1}}$	$\frac{\mathrm{d}^ny}{\mathrm{d}x^n}$	The n^{th} derivative of the variable y with respect to x, using Leibniz's notation.

These notations are interchangeable and convey the same mathematical operation. The choice of notation often depends on context or preference, but the underlying concept remains consistent across all forms.

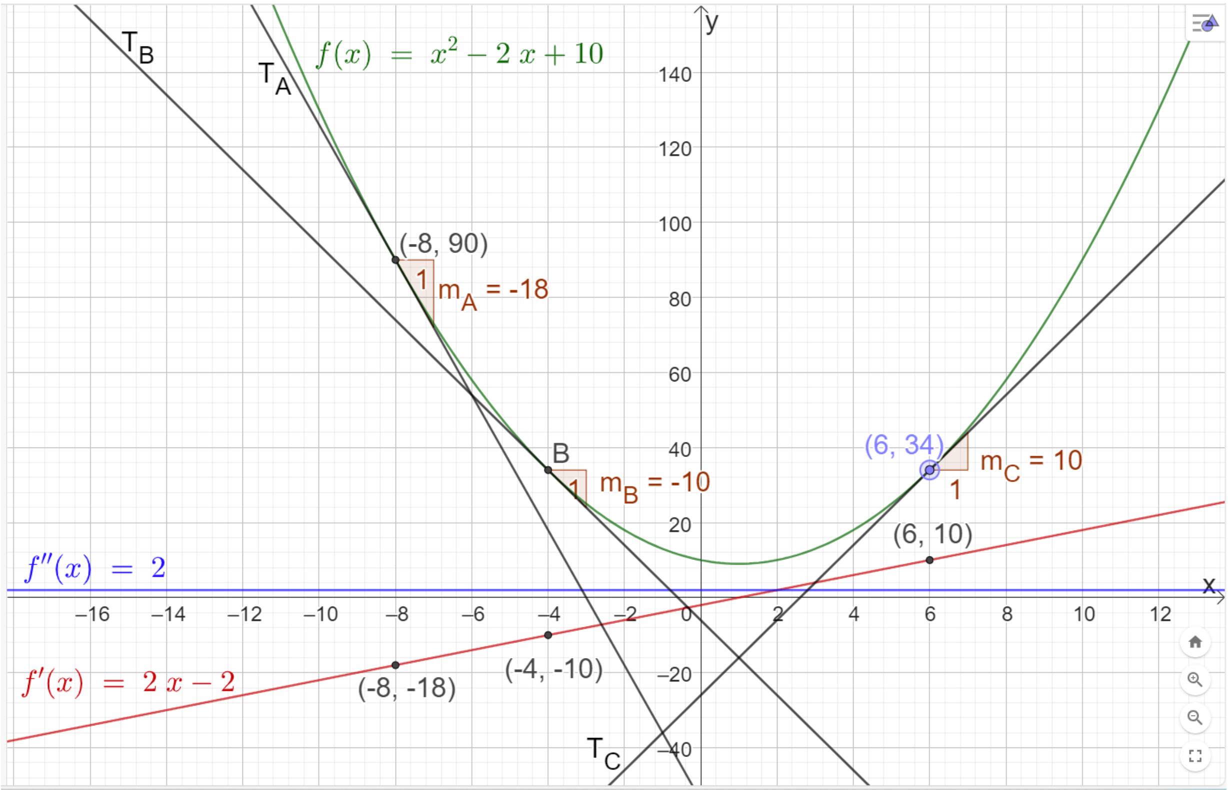


Figure 3.1: A graph showing the relationship between a function and its derivative

3.4 Basic Derivatives

Below is a table of basic derivatives for common functions. The derivatives listed in this table are useful for differentiating more complex functions, and their usefulness is enhanced by a key property: the derivative operation is linear, meaning it is additive and multiplicative with respect to constants. This means that for functions $f(x)$ and $g(x)$, and constants a and b, the following rules hold:

$$\frac{d}{dx}\big(af(x) + bg(x)\big) = af'(x) + bg'(x) \tag{3.1}$$

This additivity property allows us to differentiate functions that are sums of simpler functions by simply differentiating each term separately and then adding the results. Thus, by knowing the derivatives of basic functions, we can easily handle more complex expressions formed by combining these basic functions.

These tables provide the derivatives of common functions which are essential for solving calculus problems. Remember that these tables cover only basic derivatives. For more complex functions, breaking them down into simpler functions for which you know the derivatives can be useful.

Function	Derivative
x^n	nx^{n-1}
$\ln(x)$	$\frac{1}{x}$
e^x	e^x
e^{ax}	ae^{ax}
a^x	$a^x \ln(a)$

Function	Derivative
$\cos(x)$	$-\sin(x)$
$\sin(x)$	$\cos(x)$
$\tan(x)$	$\sec^2(x)$
$\cos^{-1}\left(\frac{x}{a}\right)$	$-\frac{1}{\sqrt{a^2-x^2}}$
$\sin^{-1}\left(\frac{x}{a}\right)$	$\frac{1}{\sqrt{a^2-x^2}}$
$\tan^{-1}\left(\frac{x}{a}\right)$	$\frac{a}{a^2+x^2}$

How to Use These Tables

To find the derivative of a function, locate the function in the left column of the appropriate table. The corresponding derivative is listed in the right column. For functions not listed, you may need to use rules such as the product rule, quotient rule, or chain rule to determine the derivative.

Example 3.4.1. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$, where $y = 3x^4 - 5e^{2x} + \sin(x) + \ln(x) - 2x^{\frac{-3}{2}} + sin^{-1}\left(\frac{x}{5}\right)$.

Answer:

We know that the derivative is a linear operation, see Equation (3.1). This means that we can apply the derivative to each term separately and also that constants can "pass through" the derivative. Applying this y we get

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 3\frac{\mathrm{d}}{\mathrm{d}x}(x^4) - 5\frac{\mathrm{d}}{\mathrm{d}x}(e^{2x}) + \frac{\mathrm{d}}{\mathrm{d}x}(\sin(x)) + \frac{\mathrm{d}}{\mathrm{d}x}(\ln(x)) - 2\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{\frac{-3}{2}}\right) + \frac{\mathrm{d}}{\mathrm{d}x}\left(sin^{-1}\left(\frac{x}{5}\right)\right)\\ &= 3(4x^3) - 5(2e^{2x}) + \cos(x) + \frac{1}{x} - 2\left(\frac{-3}{2}x^{\frac{-5}{2}}\right) + \frac{1}{\sqrt{25-x^2}}\\ &= 12x^3 - 10e^{2x} + \cos(x) + \frac{1}{x} + 3x^{\frac{-5}{2}} + \frac{1}{\sqrt{25-x^2}}.\end{aligned}$$

Exercises 3.4.2. Differentiate the following functions with respect to x:

(a) $f(x) = x^3 - 2$

(b) $g(x) = 7x^{-3} - 2\ln(x) + e^{3x} + \pi$

(c) $h(x) = \frac{7}{x} - 2\ln(x) + e^{3x}$

(d) $i(x) = \frac{1}{x^2} + 2\sin(x) + 5\sqrt{x}$

(e) $j(x) = 5e^{3x} - \tan^{-1}(x)$

(f) $k(x) = \cos^{-1}\left(\frac{x}{3}\right) - \tan(x)$

3.5 The Product Rule

In many situations, we need to differentiate functions that are products of two or more functions. For instance, if we have a function that is the product of two functions, say $u(x)$ and $v(x)$, the differentiation process is not as straightforward as differentiating each function independently. This is because although differentiation is additive, it is not multiplicative. To find the derivative of such a product, we use the Product Rule.

The Product Rule states that if $y = u(x)v(x)$, where both u and v are differentiable functions of x, then the derivative of y with respect to x is given by:

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}. \tag{3.2}$$

This formula accounts for the fact that when differentiating the product uv, both functions contribute to the change in the product.

Example 3.5.1. Find the derivative of $y = (3x^2 + 2x)(5x - 7)$.

Answer:

Let $u = 3x^2 + 2x$ and $v = 5x - 7$.

To use the Product Rule, we first find the derivatives of u and v:

$$\frac{du}{dx} = \frac{d}{dx}(3x^2 + 2x) = 6x + 2, \qquad \text{and} \qquad \frac{dv}{dx} = \frac{d}{dx}(5x - 7) = 5.$$

Applying the Product Rule:

$$\begin{aligned}\frac{dy}{dx} &= u\frac{dv}{dx} + v\frac{du}{dx}\\ &= (3x^2 + 2x) \cdot 5 + (5x - 7) \cdot (6x + 2)\\ &= 15x^2 + 10x + (30x^2 + 10x - 42x - 14)\\ &= 15x^2 + 10x + 30x^2 - 32x - 14\\ &= 45x^2 - 22x - 14.\end{aligned}$$

Exercises 3.5.2. In each of the following questions, use the Product Rule to find the derivative of the given function:

(a) $f(x) = (2x^3 - x)(4x^2 + 3)$.

(b) $g(x) = (x^2 + 1)(\sin(x))$.

(c) $h(x) = (e^x)(\ln(x))$.

(d) $i(x) = (x^2 + 3x)(\tan(x))$.

(e) $j(x) = e^{3x}3^x$.

(f) $k(x) = (x + 2)(x + 3)(x + 4)$.

3.6 The Quotient Rule

When dealing with functions that are ratios of two functions, such as $\frac{u(x)}{v(x)}$, the differentiation process is slightly more complex than differentiating each function independently. This is because differentiation, while additive, is not straightforwardly applicable to division. To handle this, we use the Quotient Rule.

The Quotient Rule states that if $y = \frac{u(x)}{v(x)}$, where both u and v are differentiable functions of x, then the derivative of y with respect to x is given by:

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}. \tag{3.3}$$

This formula accounts for the fact that when differentiating the quotient $\frac{u}{v}$, both the numerator and the denominator contribute to the change in the quotient.

Example 3.6.1. Find the derivative of $y = \frac{3x^2 + 2x}{5x - 7}$.

Answer:

Let $u = 3x^2 + 2x$ and $v = 5x - 7$.

To use the Quotient Rule, we first find the derivatives of u and v:

$$\frac{du}{dx} = \frac{d}{dx}(3x^2 + 2x) = 6x + 2, \qquad \text{and} \qquad \frac{dv}{dx} = \frac{d}{dx}(5x - 7) = 5.$$

Applying the Quotient Rule:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \\
&= \frac{(5x-7)(6x+2) - (3x^2+2x)(5)}{(5x-7)^2} \\
&= \frac{(30x^2+10x-42x-14) - (15x^2+10x)}{(5x-7)^2} \\
&= \frac{30x^2-32x-14-15x^2-10x}{(5x-7)^2} \\
&= \frac{15x^2-42x-14}{(5x-7)^2}.
\end{aligned}$$

Exercises 3.6.2. In each of the following questions, use the Quotient Rule to find the derivative of the given function:

(a) $f(x) = \dfrac{2x^3 - x}{4x^2+3}$.

(b) $g(x) = \dfrac{x^2+1}{\sin(x)}$.

(c) $h(x) = \dfrac{e^x}{\ln(x)}$.

(d) $i(x) = \dfrac{x^2+3x}{\tan(x)}$.

(e) $j(x) = \dfrac{3^x}{e^{3x}}$.

(f) $k(x) = \dfrac{(x+2)(x+3)}{x+4}$.

3.7 The Chain Rule

In many cases, we encounter functions that are composed of other functions. For instance, if we have a function $f(x)$ that is itself a function of another function $v(x)$, such as $f(x) = u(v(x))$, the differentiation process requires a different approach than differentiating a function of a single variable. To differentiate such composite functions, we use the Chain Rule.

The Chain Rule states that if $f(x) = u(v(x))$, where u and v are differentiable functions of x, then the derivative of f with respect to x is given by:

$$\frac{df}{dx} = \frac{du}{dx} = \frac{du}{dv}\cdot\frac{dv}{dx}. \tag{3.4}$$

This formula accounts for the fact that the derivative of a composite function is the product of the derivative of the outer function evaluated at the inner function and the derivative of the inner function.

Example 3.7.1. Find the derivative of $y = \sin(3x^2+1)$ with respect to x.

Answer:

Let $u(v) = \sin(v)$ and $v(x) = 3x^2+1$. Then $y = u(v)$.

To use the Chain Rule, we first need to find the derivatives of u with respect to v and v with respect to x:

$$\frac{du}{dv} = \frac{d}{dv}(\sin(v)) = \cos(v), \quad \text{and} \quad \frac{dv}{dx} = \frac{d}{dx}(3x^2+1) = 6x.$$

Applying the Chain Rule:

$$\begin{aligned}\frac{dy}{dx} &= \frac{du}{dv} \cdot \frac{dv}{dx} \\ &= \cos(v) \cdot 6x \\ &= \cos(3x^2 + 1) \cdot 6x.\end{aligned}$$

Therefore, the derivative of $y = \sin(3x^2 + 1)$ with respect to x is $6x \cos(3x^2 + 1)$.

Exercises 3.7.2. In each of the following questions, use the Chain Rule to find the derivative of the given function:

(a) $f(x) = (x^2 - 5x + 7)^8$.

(b) $g(x) = \cos(2x^3 - 5x)$.

(c) $h(x) = e^{\sin(x)}$.

(d) $i(x) = \sqrt{2x^2 + 3}$.

(e) $j(x) = \ln(4x^2 + 1)$.

(f) $k(x) = \dfrac{1}{\sqrt{3x^2 + 2}}$.

(g) $l(x) = (x^2 + 1) \sin(x^2 + 1)$.

(h) $m(x) = \dfrac{e^{x^2 - x + 1}}{(2x - 1)^3}$.

3.8 Higher-Order Derivatives

In addition to the first derivative, which provides information about the rate of change of a function, higher-order derivatives offer further insights into the behaviour of functions. The second derivative tells us about the curvature or concavity of a function, the third derivative indicates the rate of change of the curvature, and the nth derivative provides deeper insights into the function's behaviour.

Second Derivative

The second derivative of a function $f(x)$ is denoted by $f''(x)$ or $\dfrac{d^2 f}{dx^2}$. It is the derivative of the first derivative $f'(x)$. The second derivative provides information about the concavity of the function and can be used to identify points of inflection.

Third Derivative

The third derivative of a function $f(x)$ is denoted by $f'''(x)$ or $\dfrac{d^3 f}{dx^3}$. It is the derivative of the second derivative $f''(x)$. The third derivative can provide insights into the rate of change of concavity and is useful for understanding more complex behaviours of the function.

nth Derivative

The nth derivative of a function $f(x)$ is denoted by $f^{(n)}(x)$ or $\dfrac{d^n f}{dx^n}$. It is the derivative of the $(n - 1)$th derivative. Higher-order derivatives are used to study the function's behaviour in more detail, particularly in fields like physics and engineering where detailed analysis is required.

Example 3.8.1. Consider an object moving on a full HD (1080p) computer screen with 1920×1080 pixels. The pixel location of the object's centre of mass at time t (in seconds) is given by the coordinates $(x(t), y(t))$. The top-left corner of the screen is the origin $(0, 0)$. If $x(t) < 0$ or $1920 < x(t)$ or $y(t) < 0$ or $1080 < y(t)$, then the object is deemed to be off the screen.

Let the object's position be modelled by the following functions:

$$x(t) = \left(\frac{t}{5} - 5\right)^2 + 15, \quad \text{and} \quad y(t) = \left(\frac{t}{10} - 3\right)^3 + 30,$$

where $x(t)$ rounded to the nearest integer gives the horizontal position (in pixels) and $y(t)$ rounded to the nearest integer gives the vertical position (in pixels) of the object at time t (measured in seconds).

Questions:

1. Calculate the position of the object at $t = 5s$.
2. Calculate the speed that the object is travelling at in the horizontal direction at $t = 5s$.
3. Calculate the speed that the object is travelling at in the vertical direction at $t = 5s$.
4. At $t = 5s$ is the object moving left, right, upwards, downwards or neither on the screen.
5. What is the furthest left position that the object occupies and at what time does this occur.
6. What is the acceleration of the object in the horizontal direction.
7. What is the acceleration of the object in the vertical direction.
8. Does the object leave the screen within the first 100 seconds.

Answers:

1. **Position at $t = 5s$:**

$$x(5) = \left(\frac{5}{5} - 5\right)^2 + 15 = (1 - 5)^2 + 15 = (-4)^2 + 15 = 16 + 15 = 31 \text{ pixels},$$
$$y(5) = \left(\frac{5}{10} - 3\right)^3 + 30 = (0.5 - 3)^3 + 30 = (-2.5)^3 + 30 = -15.625 + 30 = 14.375 \approx 14 \text{ pixels}.$$

   So, the position of the object at $t = 5s$ is $(31, 14)$ pixels.

2. **Speed in the horizontal direction at $t = 5s$:**

$$\frac{dx}{dt} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\left(\frac{t}{5} - 5\right)^2 + 15\right) = 2\left(\frac{t}{5} - 5\right) \cdot \frac{1}{5},$$
$$\text{At } t = 5s, \quad \frac{dx}{dt} = 2\left(\frac{5}{5} - 5\right) \cdot \frac{1}{5} = 2(-4) \cdot \frac{1}{5} = -\frac{8}{5} \text{ pixels/second}.$$

3. **Speed in the vertical direction at $t = 5s$:**

$$\frac{dy}{dt} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\left(\frac{t}{10} - 3\right)^3 + 30\right) = 3\left(\frac{t}{10} - 3\right)^2 \cdot \frac{1}{10},$$
$$\text{At } t = 5s, \quad \frac{dy}{dt} = 3\left(\frac{5}{10} - 3\right)^2 \cdot \frac{1}{10} = 3(-2.5)^2 \cdot \frac{1}{10} = \frac{18.75}{10} = 1.875 \text{ pixels/second}.$$

4. **Direction of motion at $t = 5s$:**
   The object is moving to the left since $\frac{dx}{dt}$ is negative. The object is also moving downwards since $\frac{dy}{dt}$ is positive. Remember that the origin is in the top left corner of the screen and a position of $(0, 10)$ is 10 pixels vertically down from the top left corner.

5. **Furthest left position and time:**

   To find when the object reaches its furthest left position, we set $\frac{dx}{dt} = 0$.

$$\frac{dx}{dt} = 2\left(\frac{t}{5} - 5\right) \cdot \frac{1}{5} = 0, \text{which implies } \frac{t}{5} - 5 = 0 \quad \Rightarrow \quad \frac{t}{5} = 5 \quad \Rightarrow \quad t = 25s.$$

   The furthest left position occurs at $t = 25s$, and the position is given by:

$$\begin{aligned} x(25) &= \left(\frac{25}{5} - 5\right)^2 + 15 = 0^2 + 15 = 15 \text{ pixels}, \\ y(25) &= \left(\frac{25}{10} - 3\right)^3 + 30 = \left(-\frac{1}{2}\right)^3 + 30 = 29.875 \approx 30 \text{ pixels}. \end{aligned}$$

   Therefore the furthest left position that the object attains is $(15, 30)$.

6. **Acceleration in the horizontal direction:**

$$\frac{d^2x}{dt^2} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{2}{5}\left(\frac{t}{5} - 5\right)\right) = \frac{2}{25} \text{ pixels/seconds}^2.$$

7. **Acceleration in the vertical direction:**

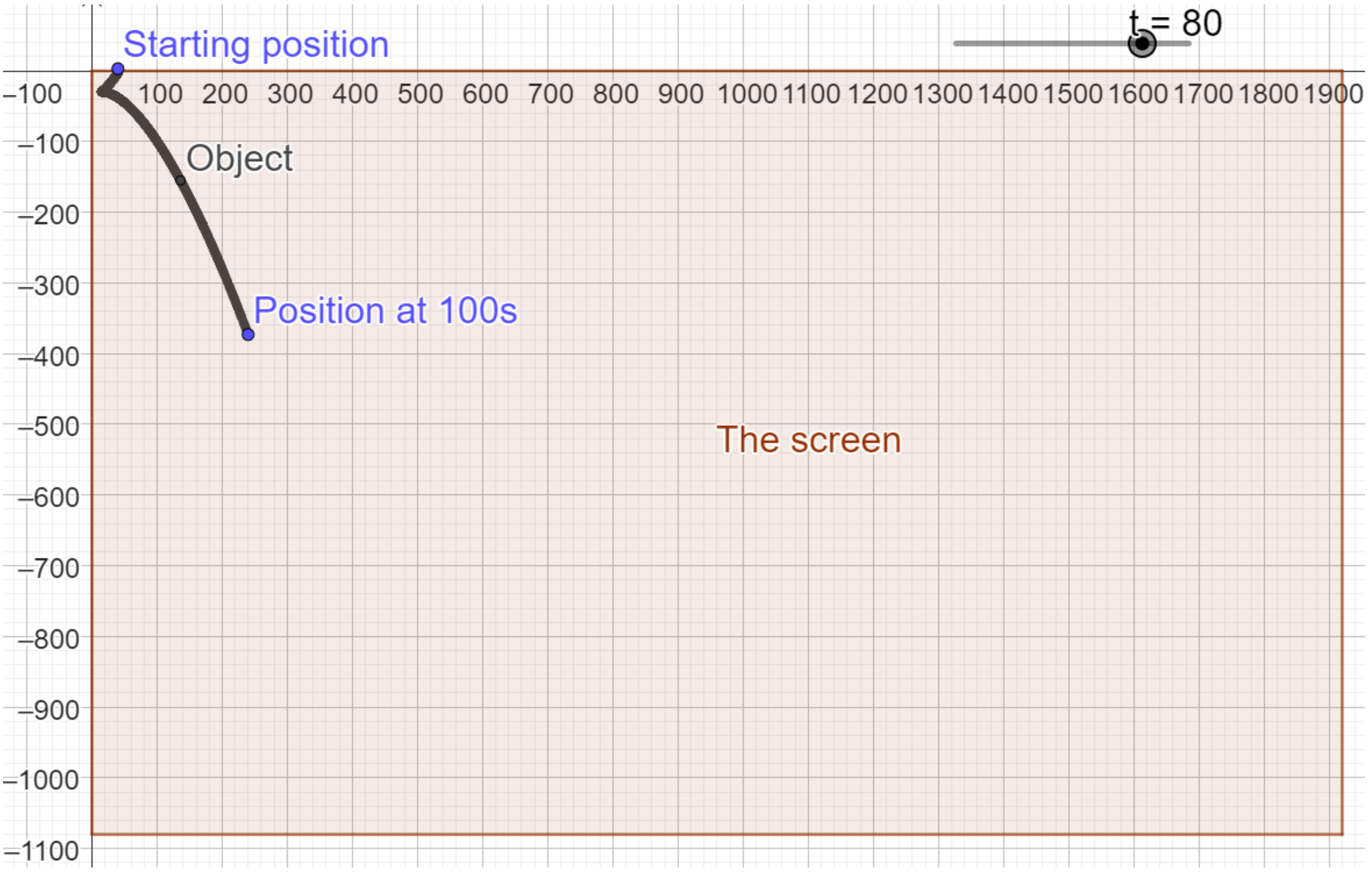
$$\frac{d^2y}{dt^2} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{3}{10}\left(\frac{t}{10} - 3\right)^2\right) = \frac{6}{10}\left(\frac{t}{10} - 3\right)\frac{1}{10} = \frac{3}{50}\left(\frac{t}{10} - 3\right) \text{ pixels/seconds}^2.$$

8. **Does the object leave the screen within the first 100 seconds:** We know that the object will not leave the screen on the left and that after $t = 25$ it continues to travel to the right. Also we know that $\frac{\mathrm{d}y}{\mathrm{d}t}$ is always positive and so the object is always travelling downwards.

9. **Position at $t = 100s$:**

$$\begin{aligned} x(100) &= \left(\frac{100}{5} - 5\right)^2 + 15 = (20 - 5)^2 + 15 = (15)^2 + 15 = 240 \text{ pixels}, \\ y(100) &= \left(\frac{100}{10} - 3\right)^3 + 30 = (10 - 3)^3 + 30 = (7)^3 + 30 = 373 \text{ pixels}. \end{aligned}$$

   $240 < 1920$ and $373 < 1080$ and so the object remains on the screen throughout the 100 seconds.



Exercises 3.8.2. In each of the following questions, find the first, second and third derivatives of the given functions:

(a) $f(x) = 4x^4 - 3x^3 + 2x^2 - x$.

(b) $g(x) = \sqrt{x}$.

(c) $h(x) = sin(2x - 1)$.

(d) $i(x) = e^x(x^2 - 1)$.

3.9 Maximum and Minimum Turning Points and Points of Inflection

In calculus, turning points, maximum and minimum points, and points of inflection are critical in understanding the behaviour of functions. These concepts help us determine where a function changes direction, reaches extreme values, or changes concavity.

3.9.1 Turning Points

A turning point of a function is where the function changes direction from increasing to decreasing or vice versa. To identify turning points, we examine the first and second derivatives of the function.

Finding Turning Points

1. **First Derivative:**

   - A turning point occurs where the first derivative $\dfrac{dy}{dx}$ changes sign. We solve $\dfrac{dy}{dx} = 0$ to find potential turning points.

2. **Second Derivative:**
   - To determine whether a turning point is a local maximum or minimum, we use the second derivative $\frac{d^2y}{dx^2}$. If $\frac{d^2y}{dx^2} > 0$, it is a local minimum; if $\frac{d^2y}{dx^2} < 0$, it is a local maximum.

Points of Inflection

A point of inflection is where the function changes its concavity. To identify points of inflection, we use the second derivative test. Points of inflection occur where the second derivative $\frac{d^2y}{dx^2}$ changes sign. We solve $\frac{d^2y}{dx^2} = 0$ and check for a change in concavity.

Example 3.9.1. Consider a function $P(x)$ that models the profit in thousands of euros of a company based on the number of units x sold, where

$$P(x) = -\frac{1}{500{,}000}x^3 + \frac{1}{100}x^2 - 10x - 100, \text{ for } 0 \leq x \leq 4{,}000.$$

This function was chosen to be somewhat realistic.

- $-\frac{1}{500{,}000}x^3$: This term represents the diminishing returns on profit as the number of units x increases. As production scales up, the company might face inefficiencies or higher costs per unit. This cubic term ensures that the profit function reflects decreasing returns or increasing marginal costs at higher production levels.
- $+\frac{1}{100}x^2$: This term accounts for the increasing profit due to economies of scale. Initially, as production increases, the profit grows at a quadratic rate, reflecting the benefit of producing more units up to a certain point.
- $-10x$: This term accounts for the fact that it is not cost effective to produce very small quantities.
- -100: This term represents the fixed cost. Companies still have costs even when producing no units. For example, staff will still need to be paid.

Questions :

(i). Find the maximum profit and maximum loss that the company can make according to the function P.

(ii). Does the function P have any points of inflection?

Answer:

(i). To find the turning points of the function $P(x)$ we calculate the first derivative and then set it to zero.

$$\begin{aligned}\frac{dP}{dx} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(-\frac{1}{500{,}000}x^3 + \frac{1}{100}x^2 - 10x - 100\right)\\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left(-\frac{1}{500{,}000}x^3\right) + \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{100}x^2\right) - \frac{\mathrm{d}}{\mathrm{d}x}(10x) - \frac{\mathrm{d}}{\mathrm{d}x}(100)\\ &= -\frac{3}{500{,}000}x^2 + \frac{1}{50}x - 10.\end{aligned}$$

Setting $\frac{dP}{dx}$ to zero gives

$$\begin{aligned} -\frac{3}{500{,}000}x^2 + \frac{1}{50}x - 10 &= 0, \quad \times -500{,}000 \\ 3x^2 - 10{,}000x + 5{,}000{,}000 &= 0. \end{aligned}$$

Solving this quadratic equation using the "-b"-formula gives

$$x = \frac{-(-10{,}000) \pm \sqrt{(-10{,}000)^{-}4(3)(5{,}000{,}000)}}{2(3)}.$$

Therefore rounded to the nearest unit the turning points are at $x = 613$ and $x = 2721$. To determine whether these local turning points are a maximum or minimum we calculate the second derivative.

$$\frac{\mathrm{d}^2P}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(-\frac{3}{500{,}000}x^2 + \frac{1}{50}x - 10\right) = -\frac{6}{500{,}000}x + \frac{1}{50}.$$

Evaluating the second derivative at $x = 613$ and $x = 2721$ gives

$$\begin{aligned} \frac{\mathrm{d}^2P}{\mathrm{d}x^2} \text{ at } x = 613 \quad &= -\frac{6}{500{,}000}(613) + \frac{1}{50} = 0.0126 \text{ and} \\ \frac{\mathrm{d}^2P}{\mathrm{d}x^2} \text{ at } x = 2721 \quad &= -\frac{6}{500{,}000}(2721) + \frac{1}{50} = -0.0126. \end{aligned}$$

Therefore the turning point at $x = 613$ is a local minimum and the turning point at $x = 2721$ is a local maximum. However, in order to confirm that these points are the global max or min we must also evaluate the function at the end points.

$$\begin{aligned} P(x) \text{ at } x = 0 \quad &= -\frac{1}{500{,}000}(0)^3 + \frac{1}{100}(0)^2 - 10(0) - 100 = -100, \\ P(x) \text{ at } x = 613 \quad &= -\frac{1}{500{,}000}(613)^3 + \frac{1}{100}(613)^2 - 10(613) - 100 = -2{,}933, \\ P(x) \text{ at } x = 2721 \quad &= -\frac{1}{500{,}000}(2721)^3 + \frac{1}{100}(2721)^2 - 10(2721) - 100 = 6{,}437, \text{ and} \\ P(x) \text{ at } x = 4000 \quad &= -\frac{1}{500{,}000}(4000)^3 + \frac{1}{100}(4000)^2 - 10(4000) - 100 = -8{,}100. \end{aligned}$$

Therefore the maximum profit is €6,437,000 and the maximum loss is €8,100,000.

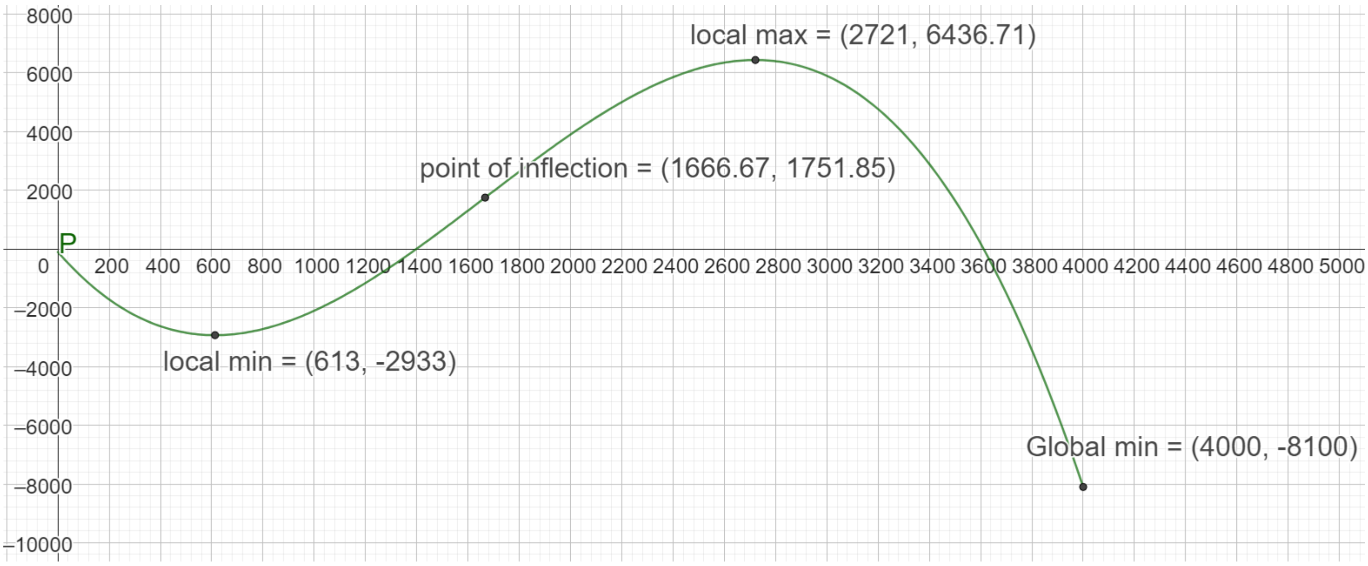
(ii). The points of inflection of P, if there are any, are where $\frac{\mathrm{d}^2P}{\mathrm{d}x^2}$ changes sign. If this happens it will happen when $\frac{\mathrm{d}^2P}{\mathrm{d}x^2} = 0$. We know from part (i). that $\frac{\mathrm{d}^2P}{\mathrm{d}x^2} = -\frac{6}{500{,}000}x + \frac{1}{50}$. Therefore

$$\begin{aligned} \frac{\mathrm{d}^2P}{\mathrm{d}x^2} &= 0 \\ -\frac{6}{500{,}000}x + \frac{1}{50} &= 0, && \times 500{,}000 \\ -6x + 10{,}000 &= 0, && +6x \\ 6x &= 10{,}000, && \div 6 \\ x &= \frac{10{,}000}{6} \approx 1666.6667. \end{aligned}$$

Finally we must check that there is a change of sign of $\frac{\mathrm{d}^2P}{\mathrm{d}x^2}$ at the point $x = 1666.6667$, so we evaluate $\frac{\mathrm{d}^2P}{\mathrm{d}x^2}$ at a value just less than and just greater than $x = 1666.6667$.

$$\begin{aligned}\frac{\mathrm{d}^2P}{\mathrm{d}x^2} \text{ at } x = 1666 &= -\frac{6}{500{,}000}(1666) + \frac{1}{50} = +\frac{1}{125{,}000},\\ \frac{\mathrm{d}^2P}{\mathrm{d}x^2} \text{ at } x = 1667 &= -\frac{6}{500{,}000}(1667) + \frac{1}{50} = -\frac{1}{250{,}000}\end{aligned}$$

Therefore the only point of inflection of P is at $x = \frac{5{,}000}{3} \approx 1666.6667$.



Exercises 3.9.2. Solve the following problems to find turning points and points of inflection in real-world contexts:

a) A company's revenue R (in thousands of euros) is given by $R(x) = -\frac{1}{10}x^3 + 3x^2 - 20x + 100$, where x is the number of units sold. Find the turning points and determine if they are local maxima or minima. Also, identify any points of inflection.

b) A landscape architect models the height h (in metres) of a hill as a function of distance d (in metres) from a specific point using the function $h(d) = -\frac{1}{10}d^3 + 2d^2 - 5d + 10$. Determine the turning points and their nature (maximum or minimum). Check for any points of inflection.

c) The displacement $s(t)$ (in metres) from a fixed point O of a particle moving along a straight line is given by $s(t) = -2t^3 + 6t^2 + 12t + 5$, where t is the time in seconds.
   1. Find the expressions for the velocity $v(t)$ and acceleration $a(t)$ of the particle.
   2. What is the furthest away that the particle gets from O.
   3. what is the fastest speed attained by the particle.

3.10 Implicit Differentiation

In many cases, the relationship between variables is given implicitly, rather than explicitly. This means that the dependent variable is not isolated on one side of the equation. That is, neither variable is the subject of the equation. For example, the equation of the circle, $x^2 + y^2 = 25$, defines y implicitly as a function of x. However, in general given an equation in x and y, it is not easy (or sometimes possible) to solve for y explicitly in terms of x. That is it may not be possible or easy to make y the subject of the equation. When we need to differentiate such implicitly defined functions, we use a technique called *implicit differentiation*. The idea is to differentiate both sides of the equation with respect to x, treating y as an implicit function of x.

3.10.1 The Process of Implicit Differentiation

The chain rule (Section 3.7) plays an important role in implicit differentiation. Let y be a function on x. How do we find out the rate at which y^2 is changing in x, that is how do we find $\frac{\mathrm{d}}{\mathrm{d}x}(y^2)$. The solution is to apply the chain rule. Using the chain rule we can write $\frac{\mathrm{d}}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}x}$, which means that differentiating with respect to x is equivalent to differentiating with respect to y and multiplying by $\frac{\mathrm{d}y}{\mathrm{d}x}$. Therefore

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(y^2) &= \frac{\mathrm{d}}{\mathrm{d}y}(y^2)\frac{\mathrm{d}y}{\mathrm{d}x}\\ &= 2y\frac{\mathrm{d}y}{\mathrm{d}x}.\end{aligned}$$

Example 3.10.1. Let x and n be positive real numbers and let y be a function of x. Differentiate y^n with respect to x. Hence find, $\frac{\mathrm{d}y^2}{\mathrm{d}x}, \frac{\mathrm{d}y^3}{\mathrm{d}x}, \frac{\mathrm{d}y^4}{\mathrm{d}x}, \frac{\mathrm{d}y^5}{\mathrm{d}x}$ and $\frac{\mathrm{d}y^6}{\mathrm{d}x}$.

Answer:

Using the chain rule we can write $\frac{\mathrm{d}}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}x}$. Applying this to y^n we get

$$\frac{\mathrm{d}}{\mathrm{d}x}(y^n) = \frac{\mathrm{d}}{\mathrm{d}y}(y^n)\frac{\mathrm{d}y}{\mathrm{d}x} = ny^{n-1}\frac{\mathrm{d}y}{\mathrm{d}x}. \tag{3.5}$$

Letting $n = 2, 3, \ldots, 6$ in Equation (3.5) we get

$f(x)$	$f'(x)$
y^2	$2y\frac{\mathrm{d}y}{\mathrm{d}x}$
y^3	$3y^2\frac{\mathrm{d}y}{\mathrm{d}x}$
y^4	$4y^3\frac{\mathrm{d}y}{\mathrm{d}x}$
y^5	$5y^4\frac{\mathrm{d}y}{\mathrm{d}x}$
y^6	$6y^5\frac{\mathrm{d}y}{\mathrm{d}x}$

Consider an equation involving x and y, for example:

$$F(x, y) = 0.$$

To differentiate this equation implicitly with respect to x, follow these steps:

1. Differentiate both sides of the equation with respect to x. When differentiating terms involving y, apply the chain rule, treating y as a function of x. For instance, the derivative of y^2 with respect to x is $2y\dfrac{dy}{dx}$.

2. Collect all terms involving $\dfrac{dy}{dx}$ on one side of the equation.

3. Solve for $\dfrac{dy}{dx}$.

Example 3.10.2. Find $\dfrac{dy}{dx}$, given the equation $x^2 + y^2 = 25$.

Answer:

Differentiate both sides with respect to x:

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25).$$

Applying the additive property of differentiation:

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(25).$$

Applying the chain rule:

$$2x + 2y\frac{dy}{dx} = 0.$$

Now, solve for $\dfrac{dy}{dx}$:

$$\begin{aligned} 2y\frac{dy}{dx} &= -2x, \\ \frac{dy}{dx} &= -\frac{x}{y}. \end{aligned}$$

Example 3.10.3. Given $\ln(xy) = x^2 - y^2$, find $\dfrac{dy}{dx}$.

Answer:

$$\begin{aligned}
\ln(xy) &= x^2 - y^2 \\
\ln(x) + \ln(y) &= x^2 - y^2 && \text{using the logarithmic identity } \log(AB) = \log(A) + \log(B) \\
\frac{\mathrm{d}}{\mathrm{d}x}(\ln(x)) + \frac{\mathrm{d}}{\mathrm{d}x}(\ln(y)) &= \frac{\mathrm{d}}{\mathrm{d}x}(x^2) - \frac{\mathrm{d}}{\mathrm{d}x}(y^2) && \text{applying } \frac{\mathrm{d}}{\mathrm{d}x} \text{ to both sides of the equation} \\
\frac{1}{x} + \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} &= 2x - 2y\frac{\mathrm{d}y}{\mathrm{d}x} && \text{differentiating and using the chain rule for the terms involving } y \\
\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} + 2y\frac{\mathrm{d}y}{\mathrm{d}x} &= 2x - \frac{1}{x} && \text{grouping the } \frac{\mathrm{d}y}{\mathrm{d}x} \text{ terms on the left and other terms on the right} \\
\frac{\mathrm{d}y}{\mathrm{d}x}\left(\frac{1}{y} + 2y\right) &= 2x - \frac{1}{x} && \text{factorising the left hand side of the equation} \\
\frac{\mathrm{d}y}{\mathrm{d}x}\left(\frac{2y^2+1}{y}\right) &= \frac{2x^2-1}{x} && \text{Using a common denominator} \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{2x^2-1}{x} \div \left(\frac{2y^2+1}{y}\right) && \text{divide across by } \left(\frac{2y^2+1}{y}\right) \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{2x^2-1}{x} \times \left(\frac{y}{2y^2+1}\right) && \text{dividing by a fraction is same as multiplying by the reciprocal.}
\end{aligned}$$

Therefore $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{(2x^2-1)y}{x(2y^2+1)}$.

Exercises 3.10.4. Use implicit differentiation to find $\frac{dy}{dx}$, given the equations in each of the following questions:

(a) $x^2 + y^2 = 16$.

(b) $2x^3 + y^3 = 3x^2y$.

(c) $\sin(x + y) = x^2 - y^2$.

(d) $e^x + e^y = xy$.

3.11 The Newton-Raphson Method

The Newton-Raphson method is an iterative technique used for finding the roots of a real-valued function. It is one of the most efficient and widely used methods for solving equations numerically, particularly when the function is differentiable.

3.11.1 The Idea Behind the Method

The Newton-Raphson method is based on the idea of linear approximation. If $f(x)$ is a differentiable function, then near a point x_0, the function $f(x)$ can be approximated by its tangent line at x_0. The equation of the tangent line is given by:

$$y = f(x_0) + f'(x_0)(x - x_0).$$

Setting $y = 0$ (since we are interested in finding the root where $f(x) = 0$), we get:

$$0 = f(x_0) + f'(x_0)(x - x_0).$$

Solving for x, we obtain the next approximation x_1 as:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

This formula is the core of the Newton-Raphson method. Starting with an initial guess x_0, the method generates a sequence $\{x_n\}$ which ideally converges to the root of $f(x) = 0$.

3.11.2 The Newton-Raphson Iteration

Given an initial guess x_0, the Newton-Raphson iteration is defined by:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

This process is repeated until the difference between successive approximations is sufficiently small, indicating convergence to the root.

Example 3.11.1. Approximate the root of the equation $f(x) = x^2 - 2$ using 2 iterations of the Newton-Raphson method, starting with an initial guess $x_0 = 2$.

Answer:

First, we compute the derivative of the function:

$$f(x) = x^2 - 2, \quad \text{so} \quad f'(x) = 2x.$$

Now, applying the Newton-Raphson formula:

$$x_{n+1} = x_n - \frac{x_n^2 - 2}{2x_n}.$$

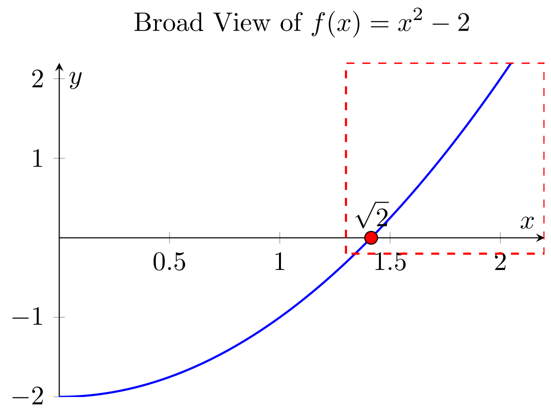
Starting with $x_0 = 2$:

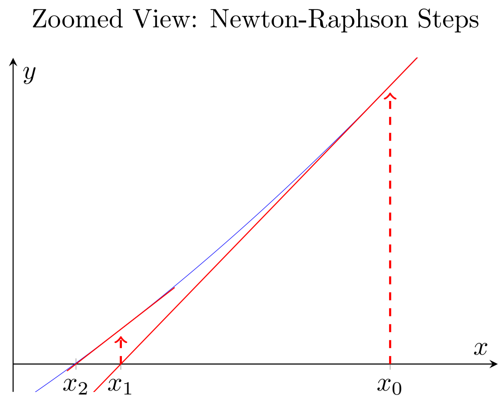
$$x_1 = 2 - \frac{2^2 - 2}{2 \times 2} = 2 - \frac{2}{4} \approx 1.5.$$

Repeating the iteration:

$$x_2 = 1.5 - \frac{1.5^2 - 2}{2 \times 1.5} \approx 1.4167.$$

Therefore our approximation of the root of $f(x) = x^2 - 2$ using 2 iterations of the Newton-Raphson method, starting with an initial guess $x_0 = 2$ is $x = 1.4167$. This approximation is very close to the actual root which is $\sqrt{2} \approx 1.4142136$.





3.11.3 Convergence of the Newton-Raphson Method

The Newton-Raphson method converges quadratically near the root, meaning that the number of correct digits roughly doubles with each iteration. However, convergence is guaranteed only if the initial guess is sufficiently close to the actual root and $f'(x)$ is not zero at the root. If these conditions are not met, the method may fail to converge or may converge to a different root.

Exercises 3.11.2. Approximate the root of the followings equations using 2 iterations of the Newton-Raphson method, using the initial guess x_0 provided.

(a) $f(x) = x^3 - 4x + 1$, with $x_0 = \frac{3}{2}$.

(b) $f(x) = 3\cos(x) - x$, with $x_0 = \frac{7}{10}$.

(c) $f(x) = e^x - 3x^2$, with $x_0 = \frac{1}{2}$.

(d) $f(x) = \ln(x) + x^2 - 3$, with $x_0 = \frac{1}{3}$.

3.12 Partial Differentiation

Partial differentiation is a technique used to find the rate at which a function of multiple variables changes with respect to one of the variables, while keeping the other variables constant. This method is particularly useful in multivariable calculus and optimisation problems.

3.12.1 The Concept of Partial Differentiation

Given a function $f(x, y)$ of two variables, the partial derivative of f with respect to x, denoted by $\frac{\partial f}{\partial x}$, measures how f changes as x changes, while y is held constant. Similarly, the partial derivative of f with respect to y, denoted by

$\dfrac{\partial f}{\partial y}$, measures how f changes as y changes, while x is held constant.

3.12.2 Computing Partial Derivatives

To compute the partial derivative of $f(x, y)$ with respect to x, differentiate f with respect to x while treating y as a constant. Similarly, to compute the partial derivative with respect to y, differentiate f with respect to y while treating x as a constant.

Example 3.12.1. Compute $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$, when $f(x, y) = x^2 - y^3 + 3xy^2 - 7$.

Answer:

We will first calculate $\dfrac{\partial f}{\partial x}$.

$$\begin{aligned}
f(x,y) &= x^2 - y^3 + 3xy^2 - 7, && \text{apply } \frac{\partial}{\partial x}\\
\frac{\partial}{\partial x}(f(x,y)) &= \frac{\partial}{\partial x}(x^2 - y^3 + 3xy^2 - 7), && \text{use the additive property of differentiation}\\
\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial x}(y^3) + \frac{\partial}{\partial x}(3xy^2) - \frac{\partial}{\partial x}(7), && \text{treat } y \text{ like a constant}\\
\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}(x^2) - y^3\frac{\partial}{\partial x}(1) + 3y^2\frac{\partial}{\partial x}(x) - 7\frac{\partial}{\partial x}(1), && \text{differentiate with respect to } x\\
\frac{\partial f}{\partial x} &= 2x - y^3(0) + 3y^2(1) - 7(0), && \text{tidy up}\\
\frac{\partial f}{\partial x} &= 2x + 3y^2.
\end{aligned}$$

We now calculate $\dfrac{\partial f}{\partial y}$.

$$\begin{aligned}
f(x,y) &= x^2 - y^3 + 3xy^2 - 7, && \text{apply } \frac{\partial}{\partial y}\\
\frac{\partial}{\partial y}(f(x,y)) &= \frac{\partial}{\partial y}(x^2 - y^3 + 3xy^2 - 7), && \text{use the additive property of differentiation}\\
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(x^2) - \frac{\partial}{\partial y}(y^3) + \frac{\partial}{\partial y}(3xy^2) - \frac{\partial}{\partial y}(7), && \text{treat } x \text{ like a constant}\\
\frac{\partial f}{\partial y} &= x^2\frac{\partial}{\partial y}(1) - \frac{\partial}{\partial y}(y^3) + 3x\frac{\partial}{\partial y}(y^2) - 7\frac{\partial}{\partial y}(1), && \text{differentiate with respect to } y\\
\frac{\partial f}{\partial y} &= x^2(0) - 3y^2 + 3x(2y) - 7(0), && \text{tidy up}\\
\frac{\partial f}{\partial y} &= -3y^2 + 6xy,
\end{aligned}$$

3.12.3 Higher-Order Partial Derivatives

In addition to the first-order partial derivatives, we can compute second-order partial derivatives. These include:

1. The second partial derivative with respect to x: $\dfrac{\partial^2 f}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial x}\right)$.

2. The second partial derivative with respect to y: $\dfrac{\partial^2 f}{\partial y^2} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial y}\right)$.

3. The mixed partial derivatives: $\dfrac{\partial^2 f}{\partial x\,\partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right)$ and $\dfrac{\partial^2 f}{\partial y\,\partial x} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right)$.

For functions with continuous second-order partial derivatives, the mixed partial derivatives are equal by Clairaut's theorem:

$$\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 f}{\partial y\,\partial x}.$$

Example 3.12.2. Consider the function $f(x,y) = x^3 - 3xy^2$. Compute the following partial derivatives.

(a) $\dfrac{\partial f}{\partial x}$ (b) $\dfrac{\partial f}{\partial y}$ (c) $\dfrac{\partial^2 f}{\partial x^2}$

(d) $\dfrac{\partial^2 f}{\partial y^2}$ (e) $\dfrac{\partial^2 f}{\partial y\,\partial x}$ (f) $\dfrac{\partial^2 f}{\partial x\,\partial y}$

Answer:

(a) $\dfrac{\partial f}{\partial x} = \dfrac{\partial}{\partial x}(x^3 - 3xy^2) = \dfrac{\partial}{\partial x}(x^3) - \dfrac{\partial}{\partial x}(3xy^2) = \dfrac{\partial}{\partial x}(x^3) - 3y^2\dfrac{\partial}{\partial x}(x) = 3x^2 - 3y^2(1) = 3x^2 - 3y^2.$

(b) $\dfrac{\partial f}{\partial y} = \dfrac{\partial}{\partial y}(x^3 - 3xy^2) = \dfrac{\partial}{\partial y}(x^3) - \dfrac{\partial}{\partial y}(3xy^2) = x^3\dfrac{\partial}{\partial y}(1) - 3x\dfrac{\partial}{\partial y}(y^2) = x^3(0) - 3x(2y) = -6xy.$

(c) $\dfrac{\partial^2 f}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial}{\partial x}\left(3x^2 - 3y^2\right) = \dfrac{\partial}{\partial x}(3x^2) - \dfrac{\partial}{\partial x}(3y^2) = 3\dfrac{\partial}{\partial x}(x^2) - 3y^2\dfrac{\partial}{\partial x}(1) = 3(2x) - 3y^2(0) = 6x.$

(d) $\dfrac{\partial^2 f}{\partial y^2} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial}{\partial y}(-6xy) = -6x\dfrac{\partial}{\partial y}(y) = -6x(1) = -6x.$

(e) $\dfrac{\partial^2 f}{\partial y\,\partial x} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial}{\partial y}(3x^2 - 3y^2) = \dfrac{\partial}{\partial y}(3x^2) - \dfrac{\partial}{\partial y}(3y^2) = 3x^2\dfrac{\partial}{\partial y}(1) - 3\dfrac{\partial}{\partial y}(y^2) = 3x^2(0) - 3(2y) = -6y.$

(f) $\dfrac{\partial^2 f}{\partial x\,\partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial}{\partial x}(-6xy) = -6y\dfrac{\partial}{\partial x}(x) = -6y(1) = -6y.$

Note that the mixed partial derivatives are equal, which is consistent with Clairaut's theorem.

3.12.4 Significance of Mixed Partial Derivatives

The equality of mixed partial derivatives $\dfrac{\partial^2 f}{\partial xy^2}$ and $\dfrac{\partial^2 f}{\partial yx^2}$ reflects the symmetry in the second-order behaviour of the function with respect to the variables x and y. This property is crucial in optimisation problems and in determining the nature of critical points. For functions where the mixed partial derivatives are not equal, the function might exhibit different behaviours or discontinuities in its second-order effects, which can be significant in applied contexts.

Exercises 3.12.3. For each of the following functions, compute the partial derivatives

$$(i)\ \frac{\partial}{\partial x} \quad (ii)\ \frac{\partial}{\partial y} \quad (iii)\ \frac{\partial^2}{\partial x^2} \quad (iv)\ \frac{\partial^2}{\partial y^2} \quad (v)\ \frac{\partial^2}{\partial x\,\partial y} \quad (vi)\ \frac{\partial^2}{\partial y\,\partial x}:$$

a) $f(x, y) = x^2y^2 + xy + 5$, with respect to x and y, including second-order partials.

b) $g(x, y) = e^{xy} + \ln(x + y)$, with respect to x and y, including second-order partials.

c) $h(x, y) = \cos(x)\cos(y)$, with respect to x and y, including second-order partials.

d) $k(x, y) = x^4 - 4x^2y^2 + y^4$, with respect to x and y, including second-order partials.

Chapter 4

Vectors

4.1 Introduction to Vectors

Vectors are mathematical objects used to represent quantities that have both magnitude and direction. In this chapter, we will explore the fundamental concepts of vectors, including their components, operations, properties, and applications in computing. A vector in three-dimensional space, denoted as $\mathbb{R}^3$, is typically expressed in terms of its components along the three coordinate axes: the x-axis, y-axis, and z-axis. Each vector can be represented as a combination of its components along these axes, with the standard unit vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$ corresponding to the x-, y-, and z-axes, respectively. In this chapter, we will focus on vectors in $\mathbb{R}^3$, but the concepts introduced here extend naturally to higher dimensions and to more general vector spaces.

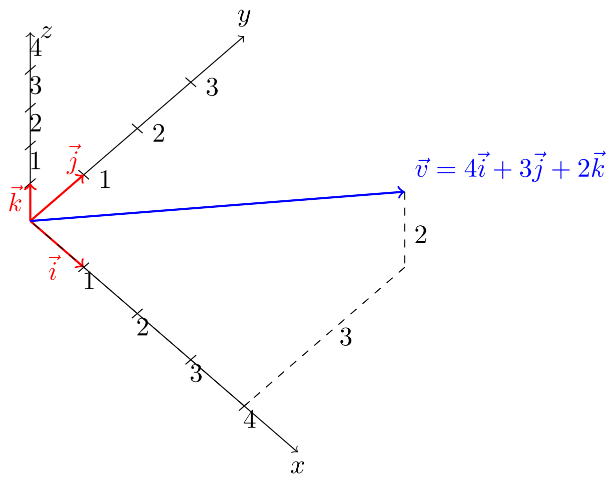


Figure 4.1: A 3D vector $\vec{v}$ and its components in $\mathbb{R}^3$.

4.2 Vector Operations

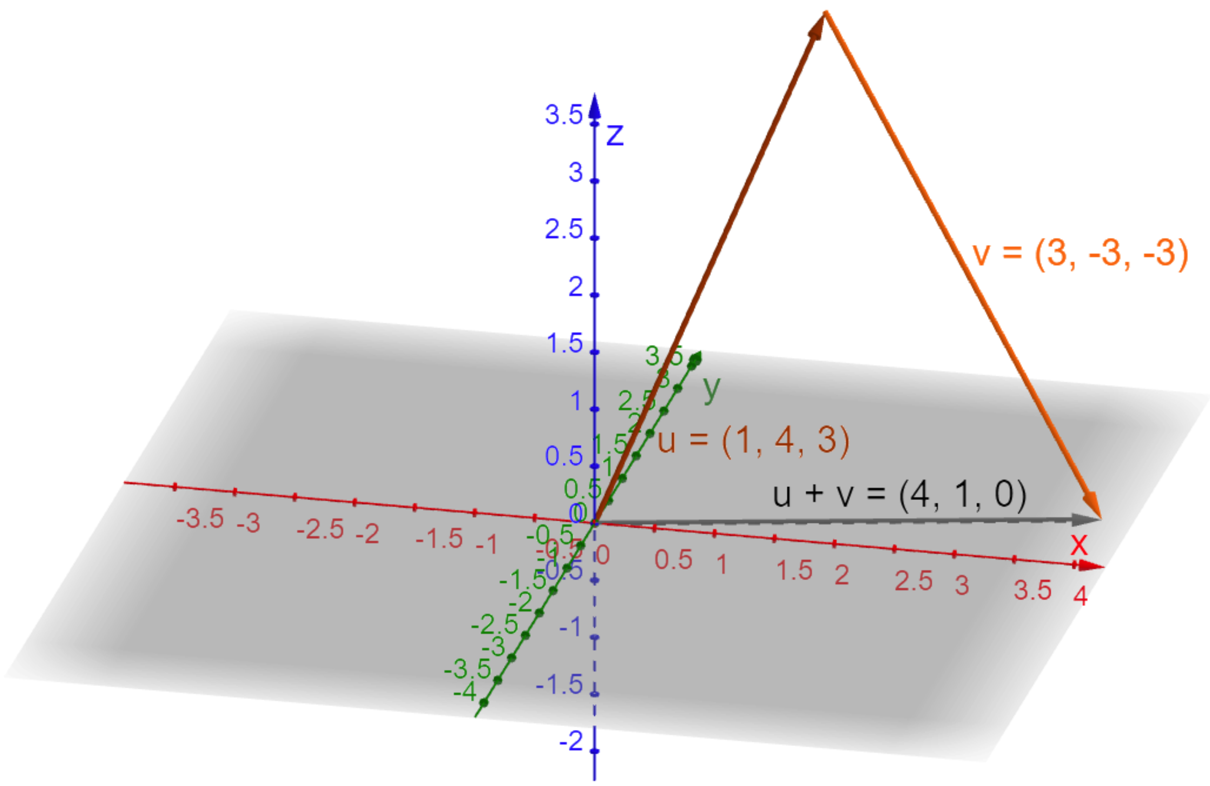
Vector operations such as addition, subtraction, and scalar multiplication are foundational concepts in vector algebra. These operations allow us to manipulate and combine vectors in various ways to obtain new vectors.

4.2.1 Addition of Vectors

Given two vectors $\vec{a}$ and $\vec{b}$, their sum $\vec{a}+\vec{b}$ is the vector obtained by adding the corresponding components:

$$\vec{a}+\vec{b} = (a_1+b_1)\vec{i} + (a_2+b_2)\vec{j} + (a_3+b_3)\vec{k} \tag{4.1}$$

When an operation is performed on each of the components separately, like it is here, then the operation is said to be done **componentwise**. Note that by Equation (4.1), we can see that vector addition is commutative. Vector addition is also associative.



Example 4.2.1. Let $\vec{a} = 3\vec{i}+2\vec{j}-4\vec{k}$ and $\vec{b} = 6\vec{i}-3\vec{j}+2\vec{k}$. Compute $\vec{a}+\vec{b}$ and $(\vec{a}+\vec{b})+\vec{b}$.

Answer:

$$\begin{aligned} \vec{a} &= 3\vec{i}+2\vec{j}-4\vec{k} \\ \vec{b} &= 6\vec{i}-3\vec{j}+2\vec{k} \\ \hline \vec{a}+\vec{b} &= (3+6)\vec{i}+(2-3)\vec{j}+(-4+2)\vec{k} \\ \vec{a}+\vec{b} &= 9\vec{i}-\vec{j}-2\vec{k} \end{aligned}$$

$$\begin{aligned} \vec{a}+\vec{b} &= 9\vec{i}-1\vec{j}-2\vec{k} \\ \vec{b} &= 6\vec{i}-3\vec{j}+2\vec{k} \\ \hline \vec{a}+\vec{b}+\vec{b} &= (9+6)\vec{i}+(-1-3)\vec{j}+(-2+2)\vec{k} \\ \vec{a}+\vec{b}+\vec{b} &= 15\vec{i}-4\vec{j}+0\vec{k} \end{aligned}$$

4.2.2 Scalar Multiplication of Vectors

Scalar multiplication of vectors, is the process of changing the magnitude of a vector without changing its direction (except when scaling by zero). Scalar multiplication is done componentwise. Given a vector $\vec{a} = a_1\vec{i}+a_2\vec{j}+a_3\vec{k}$ and a scalar (real number) r, the vector $r\vec{a}$, is given by

$$r\vec{a} = ra_1\vec{i}+ra_2\vec{j}+ra_3\vec{k} \tag{4.2}$$

Note that by Equation (4.2), we can see that scalar multiplication is commutative.

Example 4.2.2. Let $\vec{a} = 3\vec{i} + 2\vec{j} - 4\vec{k}$ and $\vec{b} = 6\vec{i} - 3\vec{j} + 2\vec{k}$. Compute $3\vec{a}$ and $\vec{a} - \vec{b}$.

Answer:

$$\begin{aligned} \vec{a} &= 3\vec{i} + 2\vec{j} - 4\vec{k} \\ 3\vec{a} &= 3(3)\vec{i} + 3(2)\vec{j} + 3(-4)\vec{k} \\ 3\vec{a} &= 9\vec{i} + 6\vec{j} - 12\vec{k}. \end{aligned}$$

$$\begin{aligned} \vec{a} &= 3\vec{i} + 2\vec{j} - 4\vec{k} \\ -\vec{b} &= -6\vec{i} + 3\vec{j} - 2\vec{k} \\ \hline \vec{a} - \vec{b} &= (3-6)\vec{i} + (2+3)\vec{j} + (-4-2)\vec{k} \\ \vec{a} - \vec{b} &= -3\vec{i} + 5\vec{j} - 6\vec{k}. \end{aligned}$$

As you can see from the example subtracting a vector is simply scaling it by -1 and then adding, so $\vec{a} - \vec{b} = \vec{a} + -1\vec{b}$.

Exercises 4.2.3. Lat $\vec{a} = 5\vec{i} - \vec{j} + 3\vec{k}$, $\vec{b} = 3\vec{i} - \vec{j} + 4\vec{k}$ and $\vec{c} = -\vec{i} + 0\vec{j} - 2\vec{k}$. Compute

(i) $\vec{a} + \vec{b}$ (ii) $\vec{a} - \vec{c}$ (iii) $-3\vec{a} + 2\vec{b}$ (iv) $4\vec{c} - 3(\vec{a} - 3\vec{a})$.

4.3 Magnitude of a Vector and Unit Vectors

The magnitude of a vector is a measure (nonnegative real number) of its size. If a vector is represented by an arrow in 3D-space its magnitude is the length of the arrow. Let $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$, then the magnitude of $\vec{a}$, $\|\vec{a}\|$ is given by:

$$\|\vec{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2} \tag{4.3}$$

Example 4.3.1. Compute the magnitude of the vector $\vec{a} = 4\vec{i} - 3\vec{j} + 12\vec{k}$.

Answer:

$$\|\vec{a}\| = \sqrt{4^2 + (-3)^2 + 12^2} = \sqrt{16 + 9 + 144} = \sqrt{169} = 13.$$

4.3.1 Unit Vectors

A **unit vector** is a vector whose magnitude is equal to 1. As examples, the vectors $\vec{i}, \vec{j}$, and $\vec{k}$ are all unit vectors. Given a vector $\vec{v}$, the unit vector in the direction of $\vec{v}$ is $\dfrac{\vec{v}}{\|\vec{v}\|}$, i.e. the vector $\vec{v}$ scaled by the reciprocal of its magnitude. The **direction of cosines** of a vector $\vec{v}$, is a tuple (an ordered list) of the components of the vector $\dfrac{\vec{v}}{\|\vec{v}\|}$.

Example 4.3.2. Let $\vec{a} = 4\vec{i} - 3\vec{j} + 12\vec{k}$. Find the unit vector in the direction of $\vec{a}$ and the direction of cosines of $\vec{a}$.

Answer: The unit vector in the direction of $\vec{a}$ is given by

$$\frac{\vec{a}}{\|\vec{a}\|} = \frac{1}{\|\vec{a}\|}\vec{a} = \frac{1}{13}\vec{a} = \frac{4}{13}\vec{i} - \frac{3}{13}\vec{j} + \frac{12}{13}\vec{k}.$$

Thus, the direction of cosines of $\vec{a}$ is $\left(\dfrac{4}{13}, -\dfrac{3}{13}, \dfrac{12}{13}\right)$.

Exercises 4.3.3. Let $\vec{a} = 4\vec{i} + 0\vec{j} - 3\vec{k}$, $\vec{b} = 1\vec{i} - 1\vec{j} + 1\vec{k}$ and $\vec{c} = 2\vec{i} - \vec{j} + 2\vec{k}$. Compute

(i) $\|\vec{a}\|$ (ii) A unit vector in the direction of $\vec{a}$

(iii) The magnitude of $\vec{b}$ (iv) A unit vector in the direction of $\vec{b}$

(v) $\|\vec{c}\|$ (vi) $\dfrac{\vec{c}}{\|\vec{c}\|}$.

4.4 The Dot (Scalar) Product

The dot product, also known as the scalar product, is an operation that takes two vectors and returns a scalar value. This scalar value provides important information about the relationship between the two vectors.

4.4.1 Formula and Definition

Given two vectors $\vec{a}$ and $\vec{b}$ in $\mathbb{R}^3$:

$$\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$$

$$\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$$

The dot product $\vec{a} \bullet \vec{b}$ is defined as:

$$\vec{a} \bullet \vec{b} = a_1b_1 + a_2b_2 + a_3b_3 \tag{4.4}$$

Alternatively, it can be expressed using the magnitudes of the vectors and the angle θ between them:

$$\vec{a} \bullet \vec{b} = \|\vec{a}\|\|\vec{b}\| \cos(\theta) \tag{4.5}$$

where:

- $\|\vec{a}\|$ and $\|\vec{b}\|$ are the magnitudes (lengths) of the vectors $\vec{a}$ and $\vec{b}$, respectively.
- θ is the angle between $\vec{a}$ and $\vec{b}$.

4.4.2 Physical Interpretation

Projection

The dot product measures the magnitude of the projection of one vector onto another. If you project vector $\vec{a}$ onto vector $\vec{b}$, the dot product $\vec{a} \bullet \vec{b}$ tells you how much of $\vec{a}$ lies in the direction of $\vec{b}$.

Angle Between Vectors

The dot product is closely related to the cosine of the angle between the two vectors. When θ is 0° (the vectors are in the same direction), $\cos\theta = 1$, and the dot product is maximized. When θ is 90° (the vectors are perpendicular), $\cos\theta = 0$, and the dot product is zero. When θ is 180° (the vectors are in opposite directions), $\cos\theta = -1$, and the dot product is negative.

Example 4.4.1. Let $\vec{u} = \frac{1}{2}\vec{i} + \frac{1}{2}\vec{j} + \frac{1}{\sqrt{2}}\vec{k}$ and $\vec{v} = \frac{7}{10}\vec{i} + \frac{7}{10}\vec{j} + \sqrt{1 - 2\left(\frac{7}{10}\right)^2}\,\vec{k}$. Compute the magnitude of both $\vec{u}$ and $\vec{v}$, the dot product $\vec{u} \bullet \vec{v}$ and the angle θ between the vectors $\vec{u}$ and $\vec{v}$.

Answer: We will compute the magnitude of each vector using Equation (4.3).

$$\|\vec{u}\| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{\frac{1}{4} + \frac{1}{4} + \frac{1}{2}} = \sqrt{1} = 1.$$

$$\begin{aligned}\|\vec{v}\| &= \sqrt{\left(\frac{7}{10}\right)^2 + \left(\frac{7}{10}\right)^2 + \left(\sqrt{1 - 2\left(\frac{7}{10}\right)^2}\right)^2} \\ &= \sqrt{\left(\frac{7}{10}\right)^2 + \left(\frac{7}{10}\right)^2 + 1 - 2\left(\frac{7}{10}\right)^2} = \sqrt{1} = 1.\end{aligned}$$

We will now compute the dot product using Equation (4.4)

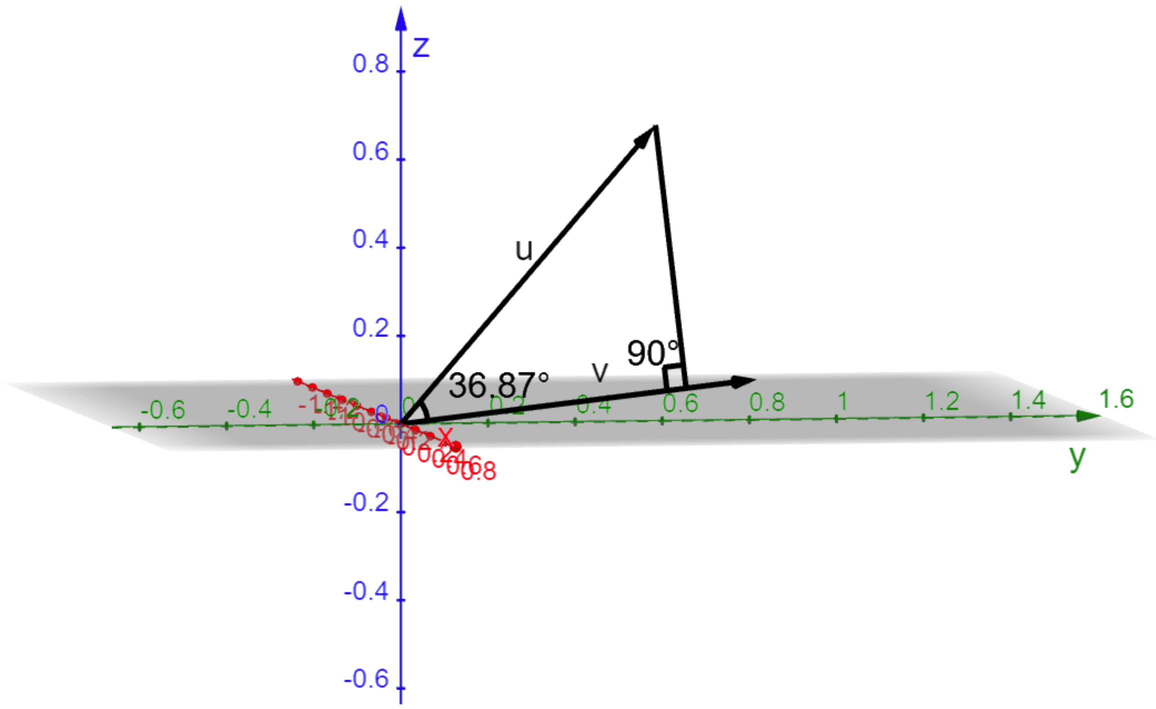
$$\vec{u} \bullet \vec{v} = \frac{1}{2}\left(\frac{7}{10}\right) + \frac{1}{2}\left(\frac{7}{10}\right) + \frac{1}{\sqrt{2}}\left(\sqrt{1 - 2\left(\frac{7}{10}\right)^2}\right) = \frac{4}{5}.$$

To calculate the angle θ between the vectors we will use Equation (4.5)

$$\begin{aligned}\vec{u} \bullet \vec{v} &= \|\vec{u}\|\|\vec{v}\| \cos(\theta) \\ \frac{4}{5} &= 1(1)\cos(\theta)\end{aligned}$$

Therefore $\theta = \cos^{-1}\left(\frac{4}{5}\right) = 36.87°$.

Below is a simple TikZ diagram that illustrates the dot product and the angle between the two vectors in the previous example. Both $\vec{u}$ and $\vec{v}$ are unit vectors and their dot product is 0.8. Therefore the projection of the end point of $\vec{u}$ onto $\vec{v}$ is a point 80% along $\vec{v}$.



Exercises 4.4.2. Let $\vec{a} = -4\vec{i} + 4\vec{j} - 2\vec{k}$, $\vec{b} = \vec{i} - 5\vec{j} + 3\vec{k}$ and $\vec{c} = -2\vec{i} + 7\vec{j} - 5\vec{k}$. Compute the following:

(i) $\vec{a} \cdot \vec{b}$ (ii) $\vec{b} \cdot \vec{c}$ (iii) $(\vec{a} \cdot \vec{c})\vec{c}$

(iv) $(\vec{a} + \vec{b}) \cdot \vec{c}$ (v) The angle between $\vec{a}$ and $\vec{b}$ (vi) The angle between $\vec{a}$ and $\vec{c}$

4.5 The Cross Product of Two Vectors

The cross product of two vectors in three-dimensional space is another vector that is perpendicular to both original vectors. This is significant because, while there are many vectors that are perpendicular to two given vectors, the cross product is unique due to its magnitude and orientation.

4.5.1 Uniqueness of the Cross Product

Given two vectors $\vec{a}$ and $\vec{b}$ in three-dimensional space, their cross product $\vec{a} \times \vec{b}$ is the unique vector that is:

- **Perpendicular** to both $\vec{a}$ and $\vec{b}$.
- Has a **magnitude** equal to the area of the parallelogram formed by $\vec{a}$ and $\vec{b}$.
- Follows the **right-hand rule** to determine its direction.

4.5.2 Direction and Orientation

The direction of the cross product vector is determined by the right-hand rule. If you point the index finger of your right hand in the direction of $\vec{a}$ and your middle finger in the direction of $\vec{b}$, your thumb will point in the direction of $\vec{a} \times \vec{b}$. This rule ensures that the cross product has a well-defined orientation.

4.5.3 Magnitude of the Cross Product

The magnitude of the cross product is given by:

$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin(\theta)$$

where θ is the angle between $\vec{a}$ and $\vec{b}$. This magnitude represents the area of the parallelogram spanned by the two vectors.

4.5.4 Definition and Calculation of Cross Product

For two vectors $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ and $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$, the cross product $\vec{a} \times \vec{b}$ is defined as:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Example 4.5.1. Let $\vec{a} = 2\vec{i} - 3\vec{j} + 6\vec{k}$ and $\vec{b} = \vec{i} + 4\vec{j} - \vec{k}$. Compute $\vec{a} \times \vec{b}$.

Answer:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -3 & 6 \\ 1 & 4 & -1 \end{vmatrix} = \vec{i}\,((-3)(-1) - (4)(6)) - \vec{j}\,((2)(-1) - (1)(6)) + \vec{k}\,((2)(4) - (1)(-3))$$

$$\vec{a} \times \vec{b} = \vec{i}(3 - 24) - \vec{j}(-2 - 6) + \vec{k}(8 + 3) = -21\vec{i} + 8\vec{j} + 11\vec{k}$$

Exercises 4.5.2. Let $\vec{a} = 3\vec{i} + \vec{j} - 2\vec{k}$, $\vec{b} = 2\vec{i} - \vec{j} + \vec{k}$ and $\vec{c} = 2\vec{i} + 5\vec{j} - 6\vec{k}$. Calculate:

(i) $\vec{a} \times \vec{b}$ (ii) $\vec{b} \times \vec{a}$ (iii) $\vec{a} \times \vec{c}$ (iv) $\vec{a} \cdot (\vec{b} \times \vec{c})$.

4.6 Differentiation of Vectors

Differentiation of vectors is a fundamental concept in vector calculus, extending the familiar operations of differentiation to vector-valued functions. If a vector function $\vec{r}(t)$ depends on a parameter t, its derivative with respect to t is another vector that represents the rate of change of $\vec{r}$ with respect to t. The derivative provides both the magnitude and direction of this change.

4.6.1 Definition of the Derivative of a Vector Function

Given a vector function $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$, its derivative with respect to t is defined as:

$$\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \frac{\mathrm{d}x(t)}{\mathrm{d}t}\vec{i} + \frac{\mathrm{d}y(t)}{\mathrm{d}t}\vec{j} + \frac{\mathrm{d}z(t)}{\mathrm{d}t}\vec{k}$$

This derivative vector represents the velocity of a particle whose position is given by $\vec{r}(t)$.

4.6.2 Linearity of the Derivative

Let $\vec{r}(t)$ and $\vec{s}(t)$ be vector functions and let c and k be scalars. The derivative of a vector function is a linear operation, which means that:

$$\frac{\mathrm{d}}{\mathrm{d}t}(c\vec{r}(t) + k\vec{s}(t)) = c\frac{\mathrm{d}\vec{r}(t)}{\mathrm{d}t} + k\frac{\mathrm{d}\vec{s}(t)}{\mathrm{d}t}.$$

4.6.3 Physical Interpretation

In physical applications, if $\vec{r}(t)$ represents the position of a particle at time t, then:

- $\dfrac{\mathrm{d}\vec{r}(t)}{\mathrm{d}t}$ is the velocity vector, representing the direction and speed of the particle's motion.
- The magnitude of the velocity vector $\left\| \dfrac{\mathrm{d}\vec{r}(t)}{\mathrm{d}t} \right\|$ represents the speed of the particle.

Example 4.6.1. Let $\vec{r}(t) = 5t^2\vec{i} + 6\vec{j} + sin(t)\vec{k}$ and $\vec{s}(t) = ln(t)\vec{i} - 7t^3\vec{j} + 6t^4\vec{k}$. Compute the following:

(i) $\dfrac{\mathrm{d}\vec{r}(t)}{\mathrm{d}t}$ (ii) $\dfrac{\mathrm{d}\vec{s}(t)}{\mathrm{d}t}$ (iii) $\left\| \dfrac{\mathrm{d}\vec{r}(t)}{\mathrm{d}t} \right\|$ (iv) $\dfrac{\mathrm{d}\vec{r}(t)}{\mathrm{d}t} \cdot \dfrac{\mathrm{d}\vec{s}(t)}{\mathrm{d}t}$

Answer:

(i) The derivative of $\vec{r}(t)$ with respect to t is:

$$\frac{\mathrm{d}\vec{r}(t)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(5t^2)\vec{i} + \frac{\mathrm{d}}{\mathrm{d}t}(6)\vec{j} + \frac{\mathrm{d}}{\mathrm{d}t}(\sin(t))\vec{k} = 10t\vec{i} + 0\vec{j} + \cos(t)\vec{k}$$

(ii) The derivative of $\vec{s}(t)$ with respect to t is:

$$\frac{\mathrm{d}\vec{s}(t)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(ln(t))\vec{i} - \frac{\mathrm{d}}{\mathrm{d}t}(7t^3)\vec{j} + \frac{\mathrm{d}}{\mathrm{d}t}(6t^4)\vec{k} = \frac{1}{t}\vec{i} - 21t^2\vec{j} + 24t^3\vec{k}.$$

(iii) The magnitude of the derivative of $\vec{r}(t)$ is:

$$\left\|\frac{\mathrm{d}\vec{r}(t)}{\mathrm{d}t}\right\| = \sqrt{(10t)^2 + (0)^2 + (\cos(t))^2} = \sqrt{100t^2 + \cos^2(t)}.$$

(iv) The dot product of $\frac{\mathrm{d}\vec{r}(t)}{\mathrm{d}t}$ and $\frac{\mathrm{d}\vec{s}(t)}{\mathrm{d}t}$ is:

$$\begin{aligned}\frac{\mathrm{d}\vec{r}(t)}{\mathrm{d}t} \bullet \frac{\mathrm{d}\vec{s}(t)}{\mathrm{d}t} &= (10t\vec{i} + 0\vec{j} + \cos(t)\vec{k}) \bullet (\frac{1}{t}\vec{i} - 21t^2\vec{j} + 24t^3\vec{k}) \\ &= 10t\left(\frac{1}{t}\right) + 0(-21t^2) + \cos(t)(24t^3) = 10 + 24t^3\cos(t).\end{aligned}$$

Exercises 4.6.2. Let $\vec{f}(t) = -7t^2\vec{i} + 6t\vec{j} - 5t^3\vec{k}$ and $\vec{g}(t) = t\vec{i} - 2t\vec{j} - t^2\vec{k}$. Compute the following:

(i) $\frac{\mathrm{d}\vec{f}(t)}{\mathrm{d}t}$ (ii) $\frac{\mathrm{d}\vec{g}(t)}{\mathrm{d}t}$ (iii) $\frac{\mathrm{d}^2\vec{f}(t)}{\mathrm{d}t^2}$ (iv) $\frac{\mathrm{d}^2\vec{g}(t)}{\mathrm{d}t^2}$

(v) $\left\|\frac{\mathrm{d}\vec{g}(t)}{\mathrm{d}t}\right\|$ (vi) $\left\|\frac{\mathrm{d}^2\vec{g}(t)}{\mathrm{d}t^2}\right\|$ (vii) $\frac{\mathrm{d}^2\vec{r}(t)}{\mathrm{d}t^2} \bullet \frac{\mathrm{d}^2\vec{s}(t)}{\mathrm{d}t^2}$ (viii) $\frac{\mathrm{d}^2\vec{r}(t)}{\mathrm{d}t^2} \times \frac{\mathrm{d}^2\vec{s}(t)}{\mathrm{d}t^2}$.

Chapter 5

Coordinate Geometry in 3D

In a Chapter 1, we explored the principles of coordinate geometry in two dimensions. Many of the concepts and formulas from 2D coordinate geometry extend naturally to three dimensions, allowing us to analyse geometric problems in $\mathbb{R}^3$. In this chapter, we will build on our understanding of vectors in 3D, using them to explore distances between points, the equations of planes, and the equations of spheres in three-dimensional space. While the basic ideas remain similar, the additional dimension introduces new challenges and opportunities for deeper analysis.

5.1 Distance between Two Points in Space and the Midpoint

Let $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ be points in $\mathbb{R}^3$. The distance between points P and Q is given by the formula:

$$|PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}. \tag{5.1}$$

The midpoint of the line segment $[P, Q]$ is given by the formula:

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right). \tag{5.2}$$

Example 5.1.1. Let $P = (2, -5, 1)$ and $Q = (4, 9, -3)$. Calculate the distance between P and Q, and find the midpoint of the line segment $[P, Q]$.

Answer: We label the points as $P = (\underset{x_1}{2}, \underset{y_1}{-5}, \underset{z_1}{1})$ and $Q = (\underset{x_2}{4}, \underset{y_2}{9}, \underset{z_2}{-3})$.

The distance between P and Q is calculated using the formula (5.1):

$$\begin{aligned}
|PQ| &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \\
&= \sqrt{(4-2)^2 + (9-(-5))^2 + (-3-1)^2} \\
&= \sqrt{2^2 + 14^2 + (-4)^2} \\
&= \sqrt{4 + 196 + 16} = \sqrt{216} = 6\sqrt{6}.
\end{aligned}$$

The midpoint is calculated using the formula (5.2):

$$\begin{aligned}\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right) &= \left(\frac{2+4}{2}, \frac{-5+9}{2}, \frac{1-3}{2}\right)\\ &= \left(\frac{6}{2}, \frac{4}{2}, \frac{-2}{2}\right)\\ &= (3, 2, -1).\end{aligned}$$

Exercises 5.1.2. Let $A = (2, -4, 3)$, $B = (11, -1, 7)$ and $C = (0, 3, -7)$. Calculate each of the following:

(a) $|AB|$

(b) The midpoint of $[A, B]$

(c) $|AC|$

(d) The midpoint of $[A, C]$

(e) $|BC|$

(f) The midpoint of $[C, B]$

(g) The coordinates of the point D such that A is the midpoint of $[B, D]$.

5.2 Equation of a Sphere

The equation of a sphere with centre (h, k, l) and radius r is given by:

$$(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2. \tag{5.3}$$

Example 5.2.1. Write down the centre and the radius length of the sphere with equation $(x-3)^2+y^2+(z+4)^2 = 29$.

Answer: The centre is $(3, 0, -4)$ and the radius length is $\sqrt{29}$.

Example 5.2.2. The points $P = (-6, 2, 2)$ and $Q = (6, 5, 8)$ are the end points of a diameter of a sphere S. Calculate the midpoint of $[PQ]$ and the length of $[PQ]$. Hence, find the equation of the sphere S.

Answer: The midpoint of $[P, Q]$ is found using the formula (5.2):

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right) = \left(\frac{-6+6}{2}, \frac{2+5}{2}, \frac{2+8}{2}\right) = \left(0, \frac{7}{2}, 5\right).$$

The length of $[PQ]$ is calculated using the formula (5.1):

$$|PQ| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2} = \sqrt{(6-(-6))^2+(5-2)^2+(8-2)^2} = \sqrt{144+9+36} = \sqrt{189}.$$

Since PQ is the diameter, the radius length r is half of $|PQ|$:

$$r = \frac{\sqrt{189}}{2}.$$

Therefore the sphere has centre $\left(0, \frac{7}{2}, 5\right)$ and radius $r = \frac{\sqrt{189}}{2}$ and so using Equation (5.3), the equation of the sphere is given by:

$$\begin{aligned}(x-0)^2 + \left(y-\frac{7}{2}\right)^2 + (z-5)^2 &= \left(\frac{\sqrt{189}}{2}\right)^2\\ x^2 + \left(y-\frac{7}{2}\right)^2 + (z-5)^2 &= \frac{189}{4}.\end{aligned}$$

Exercises 5.2.3. Let $A = (1, -2, -3)$, $B = (1, 0, 3)$ and $C = (0, 3, -6)$. In each of the following questions, write down the equation of the sphere with:

(a) centre A and diameter 5.

(b) centre C and radius $\sqrt{23}$.

(c) centre A and contains the point B.

(d) centre B and contains the point C.

(e) $[B, A]$ as a diameter.

(f) $[A, C]$ as a diameter.

5.3 Equation of a Plane

A plane in three-dimensional space is a flat, two-dimensional surface that extends infinitely in all directions within that space. It can be uniquely determined either by a point on the plane and a normal vector (which is perpendicular to the plane), or by three distinct points that lie on the plane. The equation of a plane can be derived using either of these methods, providing a foundation for analysing geometric relationships in $\mathbb{R}^3$.

5.3.1 Equation of a Plane Using a Normal Vector and a Point

The equation of a plane in $\mathbb{R}^3$ with a normal vector $\vec{n} = a\vec{i} + b\vec{j} + c\vec{k}$ that contains the point (x_0, y_0, z_0) is given by:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0. \tag{5.4}$$

Example 5.3.1. Find the equation of the plane with normal vector $-2\vec{i} + 3\vec{j} + 7\vec{k}$ that contains the point $(-8, 0, 1)$.

Answer: Using the formula (5.4), we substitute $a = -2$, $b = 3$, $c = 7$, and the point $(x_0, y_0, z_0) = (-8, 0, 1)$:

$$-2(x - (-8)) + 3(y - 0) + 7(z - 1) = 0,$$

Simplifying we get

$$\begin{aligned} -2x - 16 + 3y + 7z - 7 &= 0 \\ -2x + 3y + 7z &= 23. \end{aligned}$$

5.3.2 Equation of a Plane Using Three Points

A plane in $\mathbb{R}^3$ can also be uniquely determined by three distinct points that lie on the plane. Let the points $A = (x_1, y_1, z_1)$, $B = (x_2, y_2, z_2)$, and $C = (x_3, y_3, z_3)$ be the three given points. To find the equation of the plane passing through these points, we first find two vectors that lie on the plane, such as $\overrightarrow{AB}$ and $\overrightarrow{AC}$. The cross product of these two vectors gives a normal vector $\vec{n}$ to the plane. The equation of the plane is then derived using this normal vector and one of the points on the plane.

Given $\overrightarrow{AB} = (x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j} + (z_2 - z_1)\vec{k}$ and $\overrightarrow{AC} = (x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j} + (z_2 - z_1)\vec{k}$, the normal vector $\vec{n}$ is:

$$\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC}. \tag{5.5}$$

The equation of the plane is then given by:

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0, \tag{5.6}$$

where $\vec{n} = a\vec{i} + b\vec{j} + c\vec{k}$.

Example 5.3.2. Find the equation of the plane that passes through the points $A = (1, 2, 3)$, $B = (4, 0, -1)$, and $C = (2, -1, 5)$.

Answer: First, calculate the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$:

$$\begin{aligned}\overrightarrow{AB} &= (4-1)\vec{i} + (0-2)\vec{j} + (-1-3)\vec{k} = 3\vec{i} - 2\vec{j} - 4\vec{k},\\ \overrightarrow{AC} &= (2-1)\vec{i} + (-1-2)\vec{j} + (5-3)\vec{k} = \vec{i} - 3\vec{j} + 2\vec{k}.\end{aligned}$$

Next, find the cross product $\overrightarrow{AB} \times \overrightarrow{AC}$:

$$\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -2 & -4 \\ 1 & -3 & 2 \end{vmatrix} = \Big((-2)(2) - (-4)(-3)\Big)\vec{i} - \Big((3)(2) - (-4)(1)\Big)\vec{j} + \Big((3)(-3) - (-2)(1)\Big)\vec{k}.$$

Simplifying, we get:

$$\begin{aligned}\vec{n} &= (-4 - 12)\vec{i} - (6 + 4)\vec{j} + (-9 + 2)\vec{k}\\ &= -16\vec{i} - 10\vec{j} - 7\vec{k}.\end{aligned}$$

Therefore, using Equation (5.4), the equation of the plane is

$$-16(x - 1) - 10(y - 2) - 7(z - 3) = 0,$$

which simplifies to:

$$\begin{aligned}-16x + 16 - 10y + 20 - 7z + 21 &= 0\\ -16x - 10y - 7z &= -57\\ 16x + 10y + 7z &= 57.\end{aligned}$$

Exercises 5.3.3. In each of the following questions, write down the equation of the plane with:

(a) normal vector $2\vec{i} - 3\vec{j} + 4\vec{k}$ and contains the point $(0, 7, -4)$.

(b) normal vector $-4\vec{i} - 5\vec{j} + \vec{k}$ and contains the point $(1, 1, -5)$.

(c) contains the points $(0, 1, 3)$, $(-1, 1, -2)$ and $(6, 0, 6)$.

(d) contains the points $(5, -5, -3)$, $(2, 1, 2)$ and $(0, 1, 8)$.